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University of Calcutta
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MATRICES

AND

DETERMINOIDS

BY

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PREFACE

THE present work is an amplification of a course of lectures given for the University of Calcutta in the winter of 1909–10. Its chief feature is that it deals with *rectangular matrices* and *determinoids* as distinguished from *square matrices* and *determinants*, the determinoid of a rectangular matrix being related to it in the same way as a determinant is related to a square matrix. An attempt is made to set forth a complete and consistent theory or calculus of rectangular matrices and determinoids.

The first volume contains the most fundamental portions of the theory, and concludes with the solution of any system of linear algebraic equations, which is treated as a special case of the solution of a matrix equation of the first degree.

A second volume, which is nearly ready, will contain further developments of the general theory, including a discussion of matrix equations of the second degree. It will also contain a large number of applications to Algebra and to the Analytical Geometry of space of two, three and n dimensions.

A third volume, if opportunity should occur for its completion, would deal chiefly with applications to Vector Analysis and the Theory of Invariants. The complete exposition was in fact undertaken with a view to these last-mentioned applications.

A first attempt of this kind must necessarily contain many imperfections. In particular it may be pointed out that in the present volume no complete account has been given of the properties of the reciprocal of a rectangular matrix, and that the discussion of extravagant solutions of a system of linear algebraic equations is incomplete.

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MATHEMATICS

Owing to the omission of an article at the beginning of Chapter II, the *affects* of the elements of a matrix have not been formally defined. This omission is briefly supplied here:

§ 5*a*. Affects of the elements of a matrix.

When we pass from the leading element a_{11} of the matrix

$$[a]_m^n = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1y} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2y} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{x1} & a_{x2} & \dots & a_{xy} & \dots & a_{xn} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{my} & \dots & a_{mn} \end{bmatrix}$$

to any other element a_{xy} by vertical and horizontal forward steps, the number of vertical steps must always be $x-1$, and the number of horizontal steps must always be $y-1$.

The *affect of the element* a_{xy} of the matrix $[a]_m^n$ is defined to be the number ω given by the equation

$$\omega = (x-1)+(y-1).$$

The numbers $x-1$ and $y-1$ are called respectively the *vertical affect* and the *horizontal affect* of a_{xy}, these being respectively the number of vertical steps and the number of horizontal steps which are taken. Their sum, the number ω, is called the *total affect* or simply the *affect* of the element a_{xy}.

The *sign determined by the affect* ω is $+$ or $-$ according as $(-1)^\omega = +1$ or -1, i.e. according as ω is even or odd.

Further $(-1)^\omega a_{xy}$ will be called the *affected element* corresponding to the *unaffected element* a_{xy}.

If in passing from a_{11} to a_{xy} both forward and backward steps are permissible, then the number of vertical steps taken differs from $x-1$ by an even number, and the number of horizontal steps taken differs from $y-1$ by an even number; also the total number of steps taken differs from ω by an even number.

My thanks are due to Sir Asutosh Mukhopadhyay, Vice-Chancellor of the University of Calcutta, for encouragement in the prosecution of this work; to the Syndicate of the University of Calcutta for their liberality in defraying the cost of publication; and to the officials of the Cambridge University Press for the great care which has been exercised in the printing.

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PRESIDENCY COLLEGE,
CALCUTTA.

August 1912.

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CHAPTER I.

INTRODUCTION OF RECTANGULAR MATRICES AND DETERMINOIDS.

[This Chapter contains an introductory account of rectangular matrices and determinoids and a description of various abbreviated notations which will be used in connection with them.]

§ 1. Rectangular Matrices.

1. *Definition.*

If m and n are positive integers, an aggregate of mn quantities or *elements* arranged in m horizontal rows and n vertical rows will be called a rectangular matrix. For any such matrix A we may use the notation

$$A = [a]_m^n = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{.......................(1).}$$

The numbers m and n will be called the *orders* of the matrix, m being the *horizontal order* and n the *vertical order*. The smaller of these two numbers (or either of them if they are equal) will be called the effective order or the *efficiency* of the matrix.

If m and n are unequal, either a vertical row contains more elements than a horizontal row, or a horizontal row contains more elements than a vertical row. Those rows, horizontal or vertical, which contain the greater number of elements will be called *long rows*, and the other rows will be called *short rows*. The number of long rows is equal to the efficiency of the matrix.

In the special case in which m and n are equal the matrix will be called a *square matrix*. A square matrix has only one order, which is also its efficiency, and in it either set of parallel rows may be regarded as long rows.

By the *leading element* of the matrix A is meant the element a_{11} in the top left-hand corner. The elements $a_{11}, a_{22}, \dots a_{ii}, \dots$ are said to form the *leading line* of the matrix. In the case of a square matrix the leading line becomes the *leading diagonal*.

The vertical row on the extreme left in A will be called the *leading vertical row* of A, and any set of vertical rows lying to the left of all other vertical rows will be said to occupy a *leading position* in A. So the topmost horizontal row will be called the *leading horizontal row*, and any set of horizontal rows lying above all other horizontal rows occupy a *leading position* in A.

It will be assumed that the elements of a matrix are scalar numbers or letters denoting scalar numbers. The equality of matrices and the addition, subtraction and multiplication of matrices are defined in Chapter VI. In the first five chapters, which deal with properties of a single given matrix, these latter definitions will not be required.

Ex. i. In the matrix

$$[a]_3^5 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

the long rows are horizontal and the short rows vertical. The horizontal and vertical orders are 3 and 5 respectively. The efficiency is 3. The leading line is that occupied by the elements a_{11}, a_{22}, a_{33}.

Ex. ii. In the matrix

$$[a/c]_{1234} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$$

the long rows are vertical and the short rows horizontal. The horizontal and vertical orders are 4 and 3 respectively. The efficiency is 3. The leading line is that occupied by the elements a_1, b_2, c_3.

2. *Derived Matrices.*

A matrix formed from A by re-arranging the horizontal rows amongst themselves in any manner and also re-arranging the vertical rows amongst themselves in any manner will be called a *deranged matrix* of A or a *derangement* of A.

Any matrix A' formed from A by simply striking out some of its horizontal and vertical rows and leaving the retained horizontal and vertical rows in the same relative orders as in A will be called a *corranged minor matrix* of A. In particular by striking out all horizontal rows below a certain horizontal row and all vertical rows to the right of a certain vertical row we form a corranged minor matrix occupying a *leading position* in A.

Any matrix A'' formed from a corranged minor matrix A' by re-arranging the retained horizontal rows amongst themselves in any manner and also re-arranging the retained vertical rows amongst themselves in any manner will be called a *deranged minor matrix* of A.

A minor matrix formed from A by striking out rows of one kind only, horizontal or vertical, is called a *simple minor matrix*.

The corranged and deranged minor matrices of A and the derangements of A together constitute all the *derived matrices* of A.

Every minor matrix of A, whether corranged or deranged, has at least one of its orders less than the corresponding order of A, and the elements of any one of its horizontal or vertical rows all lie in a parallel row of A. Thus the most general form of a minor matrix of A is

$$A'' = [a_{pq}]_{\mu}^{\nu} = \begin{bmatrix} a_{p_1 q_1} & a_{p_1 q_2} \dots a_{p_1 q_\nu} \\ a_{p_2 q_1} & a_{p_2 q_2} \dots a_{p_2 q_\nu} \\ \dots\dots & \dots\dots\dots \\ a_{p_\mu q_1} & a_{p_\mu q_2} \dots a_{p_\mu q_\nu} \end{bmatrix} \quad \dots\dots\dots\dots\dots\dots\dots (2),$$

where $p_1, p_2, \dots p_\mu$ is some arrangement of μ of the numbers $1, 2, \dots m$, and $q_1, q_2, \dots q_\nu$ is some arrangement of ν of the numbers $1, 2, \dots n$.

A matrix will be said to *contain* each of its minor matrices.

Ex. iii. With respect to the matrix

$$[abcd]_{123} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

the following three matrices are minor matrices:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}, \quad \begin{bmatrix} a_1 & c_1 & e_1 \\ a_2 & c_2 & e_2 \end{bmatrix}, \quad \begin{bmatrix} c_2 & a_2 & d_2 \\ c_3 & a_3 & d_3 \\ c_1 & a_1 & d_1 \end{bmatrix}.$$

The first two of these are corranged minor matrices, and the first occupies a leading position. The third is a deranged simple minor matrix.

3. *Derived products.*

If as large a number as possible of elements $\alpha, \beta, \gamma, \delta, \dots$ are selected from the matrix A in such a manner that no two of the selected elements occur in the same horizontal row and no two in the same vertical row, their product $\alpha\beta\gamma\delta\dots$ will be called a *complete derived product* belonging to the matrix. The number of elements in each complete derived product is equal to the efficiency of the matrix; and the total number of such products, when the order of arrangement of the factors is disregarded, is

$$n(n-1)(n-2)\dots(n-m+1) \quad \text{or} \quad m(m-1)(m-2)\dots(m-n+1)$$

according as n is greater or less than m. To form any particular complete derived product each long row contributes just one factor, but (except in the case of a square matrix) there are short rows which do not contribute any factor.

More generally we will define a *derived product of order* r belonging to the matrix A to be a product $\alpha\beta\gamma\delta\ldots$ formed with r elements $\alpha, \beta, \gamma, \delta, \ldots$ selected from the matrix A in such a manner that no two of the elements occur in the same horizontal row and no two in the same vertical row. Then r cannot be greater than the efficiency of the matrix, and the derived product is *complete* or *incomplete* according as r is equal to or less than the efficiency. The most general form of a derived product P of order r belonging to the matrix A is

$$P = a_{p_1 q_1} a_{p_2 q_2} \ldots a_{p_r q_r} \quad\ldots\ldots\ldots\ldots(3),$$

where $p_1, p_2, \ldots p_r$ is any arrangement of r of the numbers $1, 2, \ldots n$,

and $q_1, q_2, \ldots q_r$ is any arrangement of r of the numbers $1, 2, \ldots m$.

The total number of derived products of order r belonging to the matrix A is

$$r!\binom{m}{r}\binom{n}{r} = \frac{m(m-1)(m-2)\ldots(m-r+1).n(n-1)(n-2)\ldots(n-r+1)}{r!}.$$

Ex. iv. In the matrix

$$[abcde]_{1234} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{bmatrix}$$

$c_1 d_2 b_3 e_4$ is a complete derived product;

$d_4 b_2 a_3$ is an (incomplete) derived product of order 3.

4. *Steps.*

The passage from any element of the matrix to an adjacent element in the same horizontal row will be called a *forward* or *backward horizontal step* according as the second element lies to the right or to the left of the first. Similarly the passage from any element to an adjacent element in the same vertical row will be called a *forward* or *backward vertical step* according as the second element lies below or above the first. When the word *step* is used without qualification it will be understood to mean a forward step.

Ex. v. In the matrix

$$[abcde]_{1234} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{bmatrix}$$

the passage from b_3 to c_3 is a (forward) horizontal step, and the passage from d_2 to d_3 is a (forward) vertical step. The passage from a_1 to c_4 requires two horizontal steps and three vertical steps.

5. *Conjugate matrices.*

Two matrices A and A' are said to be *conjugate* to one another when the horizontal and vertical rows of the one taken in order are respectively the vertical and horizontal rows of the other taken in order. The conjugate matrix A' of the matrix A defined by equation (1) above will be denoted by

$$A' = \overset{\frown m}{\underset{\smile n}{a}} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \quad \text{.........................(4).}$$

A *self-conjugate matrix* is a square matrix which is symmetrical with respect to its leading diagonal. Thus the matrix $[a]_m^m$ is self-conjugate if $a_{ij} = a_{ji}$, where i and j are any two of the numbers 1, 2, ... m.

Ex. vi. The two matrices

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix}$$

are conjugate to one another.

Ex. vii. The matrix $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a self-conjugate matrix.

§ 2. Abbreviated notations for a matrix.

In all the following defining formulae A and A' are a pair of mutually conjugate matrices.

1. *Standard double-suffix notation.*

$$\left.\begin{aligned} A = [a]_m^n &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \\ A' = \overset{\frown m}{\underset{\smile n}{a}} &= \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \end{aligned}\right\} \quad \text{.....................(A).}$$

Here $[a]_m^n$, $\overline{\underline{a}}{}_n^m$ are to be regarded as abbreviations for the matrices following them on the right. This is the notation which will most commonly be employed in proving general theorems relating to matrices. The general symbol for an element of either matrix is a_{xy}, and the various individual elements are obtained from this by giving to x the values 1, 2, ... m and to y the values 1, 2, ... n. With respect to the matrix $[a]_m^n$, the suffixes x and y will be called the *vertical* and *horizontal suffixes* respectively of the element a_{xy}, since they indicate its vertical and horizontal positions relative to the leading element a_{11}. With respect to the matrix $\overline{\underline{a}}{}_n^m$ these terms must be reversed.

Ex. i. $[a]_2^3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $\overline{\underline{a}}{}_3^2 = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$.

2. *More general double-suffix notations.*

In order to represent the minor matrices of a matrix $[a]_\mu^\nu$, other more elaborate notations are necessary. All such minors can be represented by symbols $[a_{pq}]_m^n$, $\overline{\underline{a_{pq}}}{}_n^m$ defined as follows:

$$\left.\begin{aligned} A = [a_{pq}]_m^n &= \begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} & \dots & a_{p_1q_n} \\ a_{p_2q_1} & a_{p_2q_2} & \dots & a_{p_2q_n} \\ \dots & \dots & \dots & \dots \\ a_{p_mq_1} & a_{p_mq_2} & \dots & a_{p_mq_n} \end{bmatrix} \\ A' = \overline{\underline{a_{pq}}}{}_n^m &= \begin{bmatrix} a_{p_1q_1} & a_{p_2q_1} & \dots & a_{p_mq_1} \\ a_{p_1q_2} & a_{p_2q_2} & \dots & a_{p_mq_2} \\ \dots & \dots & \dots & \dots \\ a_{p_1q_n} & a_{p_2q_n} & \dots & a_{p_mq_n} \end{bmatrix} \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (B).$$

Here the general symbol for an element of either matrix is $a_{p_xq_y}$, and the various individual elements are obtained from this by giving to x the values 1, 2, ... m and to y the values 1, 2, ... n. Keeping x constant we obtain all the elements of the xth horizontal row of A, which is the xth vertical row of A'. So keeping y constant we obtain all the elements of the yth vertical row of A, which is the yth horizontal row of A'. The suffixes p_x, q_y will be called respectively the vertical and the horizontal suffix of the element $a_{p_xq_y}$ in the matrix A, and they will be called respectively the horizontal and the vertical suffix of that element in the matrix A'.

We will also define symbols $[a_{p1}]^n_m$, $\overline{\underline{a_{p1}}}{}^m_n$ by the equations

$$\left.\begin{aligned} A = [a_{p1}]^n_m &= \begin{bmatrix} a_{p_1 1} & a_{p_1 2} & \dots & a_{p_1 n} \\ a_{p_2 1} & a_{p_2 2} & \dots & a_{p_2 n} \\ \dots & \dots & \dots & \dots \\ a_{p_m 1} & a_{p_m 2} & \dots & a_{p_m n} \end{bmatrix} \\ A' = \overline{\underline{a_{p1}}}{}^m_n &= \begin{bmatrix} a_{p_1 1} & a_{p_2 1} & \dots & a_{p_m 1} \\ a_{p_1 2} & a_{p_2 2} & \dots & a_{p_m 2} \\ \dots & \dots & \dots & \dots \\ a_{p_1 n} & a_{p_2 n} & \dots & a_{p_m n} \end{bmatrix} \end{aligned}\right\} \quad \dots\dots\dots(C).$$

Here the general symbol for an element of either matrix is $a_{p_x y}$, and the various individual elements are obtained from this by giving to x the values 1, 2, ... m, and to y the values 1, 2, ... n. This notation can be conveniently employed to represent the various corranged simple minor matrices obtained from a matrix $[a]^n_\mu$ by striking out vertical rows.

Similarly we will define symbols $[a_{1q}]^n_m$, $\overline{\underline{a_{1q}}}{}^m_n$ by the equations

$$\left.\begin{aligned} A = [a_{1q}]^n_m &= \begin{bmatrix} a_{1q_1} & a_{1q_2} & \dots & a_{1q_n} \\ a_{2q_1} & a_{2q_2} & \dots & a_{2q_n} \\ \dots & \dots & \dots & \dots \\ a_{mq_1} & a_{mq_2} & \dots & a_{mq_n} \end{bmatrix} \\ A' = \overline{\underline{a_{1q}}}{}^m_n &= \begin{bmatrix} a_{1q_1} & a_{2q_1} & \dots & a_{mq_1} \\ a_{1q_2} & a_{2q_2} & \dots & a_{mq_2} \\ \dots & \dots & \dots & \dots \\ a_{1q_n} & a_{2q_n} & \dots & a_{mq_n} \end{bmatrix} \end{aligned}\right\} \quad \dots\dots\dots(D).$$

Here the general symbol for an element of either matrix is a_{xq_y} and the various individual elements are obtained from this by giving to x the values 1, 2, ... m and to y the values 1, 2, ... n. This is a convenient notation for the corranged simple minor matrices of a matrix $[a]^\nu_m$ obtained by striking out horizontal rows.

Ex. ii. $[a_{pq}]^3_2 = \begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} & a_{p_1q_3} \\ a_{p_2q_1} & a_{p_2q_2} & a_{p_2q_3} \end{bmatrix}$, $\quad \overline{\underline{a_{pq}}}{}^2_3 = \begin{bmatrix} a_{p_1q_1} & a_{p_2q_1} \\ a_{p_1q_2} & a_{p_2q_2} \\ a_{p_1q_3} & a_{p_2q_3} \end{bmatrix}$.

Ex. iii. $[a_{p1}]^3_2 = \begin{bmatrix} a_{p_1 1} & a_{p_1 2} & a_{p_1 3} \\ a_{p_2 1} & a_{p_2 2} & a_{p_2 3} \end{bmatrix}$, $\quad \overline{\underline{a_{p1}}}{}^2_3 = \begin{bmatrix} a_{p_1 1} & a_{p_2 1} \\ a_{p_1 2} & a_{p_2 2} \\ a_{p_1 3} & a_{p_2 3} \end{bmatrix}$.

3. *Most general double-suffix notation.*

Other symbols $\begin{bmatrix} uv\ldots w \\ a \\ pq\ldots r \end{bmatrix}$, $\underline{\overline{\begin{matrix} pq\ldots r \\ a \\ uv\ldots w \end{matrix}}}$ will be defined by the equations

$$\left.\begin{aligned} A = \begin{bmatrix} uv\ldots w \\ a \\ pq\ldots r \end{bmatrix} &= \begin{bmatrix} a_{pu} & a_{pv} & \ldots & a_{pw} \\ a_{qu} & a_{qv} & \ldots & a_{qw} \\ \ldots & \ldots & \ldots & \ldots \\ a_{ru} & a_{rv} & \ldots & a_{rw} \end{bmatrix} \\ A' = \underline{\overline{\begin{matrix} pq\ldots r \\ a \\ uv\ldots w \end{matrix}}} &= \begin{bmatrix} a_{pu} & a_{qu} & \ldots & a_{ru} \\ a_{pv} & a_{qv} & \ldots & a_{rv} \\ \ldots & \ldots & \ldots & \ldots \\ a_{pw} & a_{qw} & \ldots & a_{rw} \end{bmatrix} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(E)}.$$

All the matrices occurring in formulae (A), (B), (C), (D) can be represented in this form.

Ex. iv. $[a]_m^n = \begin{bmatrix} 12\ldots n \\ a \\ 12\ldots m \end{bmatrix}$, $[a_{pq}]_m^n = \begin{bmatrix} q_1q_2\ldots q_n \\ a \\ p_1p_2\ldots p_m \end{bmatrix}$, $[a_{1q}]_m^n = \begin{bmatrix} q_1q_2\ldots q_n \\ a \\ 12\ldots m \end{bmatrix}$.

4. *Double-suffix notation for augmented matrices.*

Symbols $[a, b]_m^{n, r}$, $\begin{bmatrix} a \\ b \end{bmatrix}_{m, r}^n$ will be defined by

$$[a, b]_m^{n, r} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_{11} & b_{12} & \ldots & b_{1r} \\ a_{21} & a_{22} & \ldots & a_{2n} & b_{21} & b_{22} & \ldots & b_{2r} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_{m1} & b_{m2} & \ldots & b_{mr} \end{bmatrix}, \quad \begin{bmatrix} a \\ b \end{bmatrix}_{m, r}^n = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \\ b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{r1} & b_{r2} & \ldots & b_{rn} \end{bmatrix}$$

$$\ldots\ldots\ldots\ldots\text{(F)}.$$

These matrices will be called *augmented matrices* of $[a]_m^n$. The first is formed by placing the vertical rows of $[b]_m^r$ to the right of the vertical rows of $[a]_m^n$. The second is formed by placing the horizontal rows of $[b]_r^n$ below the horizontal rows of $[a]_m^n$.

The conjugate matrices of $[a, b]_m^{n, r}$, $\begin{bmatrix} a \\ b \end{bmatrix}_{m, r}^n$ will be denoted by

$$\underline{\overline{\begin{matrix} a \\ b \end{matrix}}}\,{}_{n, r}^{m} \quad \text{and} \quad \underline{\overline{a,\ b}}\,{}_{n}^{m, r}.$$

Similarly we will write

$$[a_{pq},\ b_{pu}]_m^{n,\,r} = \begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} & \dots & a_{p_1q_n} & b_{p_1u_1} & b_{p_1u_2} & \dots & b_{p_1u_r} \\ a_{p_2q_1} & a_{p_2q_2} & \dots & a_{p_2q_n} & b_{p_2u_1} & b_{p_2u_2} & \dots & b_{p_2u_r} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{p_mq_1} & a_{p_mq_2} & \dots & a_{p_mq_n} & b_{p_mu_1} & b_{p_mu_2} & \dots & b_{p_mu_r} \end{bmatrix} \quad \dots\dots(G),$$

$$\begin{bmatrix} a_{pq} \\ b_{vq} \end{bmatrix}_{m,\,r}^{n} = \begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} & \dots & a_{p_1q_n} \\ a_{p_2q_1} & a_{p_2q_2} & \dots & a_{p_2q_n} \\ \dots & \dots & \dots & \dots \\ a_{p_mq_1} & a_{p_mq_2} & \dots & a_{p_mq_n} \\ b_{v_1q_1} & b_{v_1q_2} & \dots & b_{v_1q_n} \\ b_{v_2q_1} & b_{v_2q_2} & \dots & b_{v_2q_n} \\ \dots & \dots & \dots & \dots \\ b_{v_rq_1} & b_{v_rq_2} & \dots & b_{v_rq_n} \end{bmatrix} \quad \dots\dots\dots\dots\dots\dots(H).$$

Also $[a_{1q},\ b_{1u}]_m^{n,\,r}$ will denote the matrix formed by placing the vertical rows of $[b_{1u}]_m^r$ to the right of the vertical rows of $[a_{1q}]_m^n$, and $\begin{bmatrix} a_{p1} \\ b_{v1} \end{bmatrix}_{m,\,r}^{n}$ will denote the matrix formed by placing the horizontal rows of $[b_{v1}]_r^n$ below the horizontal rows of $[a_{p1}]_m^n$. Thus $[a_{1q},\ b_{1u}]_m^{n,\,r}$ is formed from $[a_{pq},\ b_{pu}]_m^{n,\,r}$ by putting $p_1, p_2, \dots p_m = 1, 2, \dots m$, and $\begin{bmatrix} a_{p1} \\ b_{v1} \end{bmatrix}_{m,\,r}^{n}$ is formed from $\begin{bmatrix} a_{pq} \\ b_{vq} \end{bmatrix}_{m,\,r}^{n}$ by putting $q_1, q_2, \dots q_n = 1, 2, \dots n$.

The conjugates of $[a_{pq},\ b_{pu}]_m^{n,\,r}$, $\begin{bmatrix} a_{pq} \\ b_{vq} \end{bmatrix}_{m,\,r}^{n}$ may be denoted by

$$\overline{\underline{\begin{matrix} a_{pq} \\ b_{pu} \end{matrix}}}\,{}_{n,\,r}^{m} \quad \text{and} \quad \overline{\underline{a_{pq},\ b_{vq}}}\,{}_{n}^{m,\,r}.$$

Ex. v. $$[a,\ b]_3^{2,\,3} = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

Ex. vi. $$\begin{bmatrix} a_{p1} \\ b_{q1} \end{bmatrix}_{2,\,2}^{3} = \begin{bmatrix} a_{p_11} & a_{p_12} & a_{p_13} \\ a_{p_21} & a_{p_22} & a_{p_23} \\ b_{q_11} & b_{q_12} & b_{q_13} \\ b_{q_21} & b_{q_22} & b_{q_23} \end{bmatrix}.$$

5. *Standard single-suffix notation.*

$$\left.\begin{aligned} A = [ab \ldots k]_{12 \ldots m} &= \begin{bmatrix} a_1 & b_1 & \ldots & k_1 \\ a_2 & b_2 & \ldots & k_2 \\ \ldots & \ldots & \ldots & \ldots \\ a_m & b_m & \ldots & k_m \end{bmatrix} \\ A' = \begin{bmatrix} a \\ b \\ \vdots \\ k \end{bmatrix}_{12 \ldots m} &= \begin{bmatrix} a_1 & a_2 & \ldots & a_m \\ b_1 & b_2 & \ldots & b_m \\ \ldots & \ldots & \ldots & \ldots \\ k_1 & k_2 & \ldots & k_m \end{bmatrix} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{I}).$$

This is a convenient notation when we are considering a matrix whose orders are given small numbers or when we are considering the simple minor matrices of a given matrix. When no ambiguity will be thereby caused, the suffixes in the abbreviated symbols can be omitted. In particular they will often be omitted when the matrix is a square matrix.

Ex. vii. The matrix

$$[abc]_{123} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

will often be denoted simply by $[abc]$.

Ex. viii. The corranged simple minors of the matrix

$$[abcd]_{123} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

obtained by striking out two vertical rows are

$$[ab]_{123}, \quad [ac]_{123}, \quad [ad]_{123}, \quad [bc]_{123}, \quad [bd]_{123}, \quad [cd]_{123}.$$

These will often be denoted simply by

$$[ab], \ [ac], \ [ad], \ [bc], \ [bd], \ [cd].$$

6. *Most general single-suffix notation.*

In order to represent all the minors, or even all the simple minors, of a matrix of the form $[ab \ldots k]_{12 \ldots m}$, a more general notation is required. Accordingly we shall define symbols

$$[ab \ldots k]_{\alpha\beta \ldots \kappa}, \quad \begin{bmatrix} a \\ b \\ \vdots \\ k \end{bmatrix}_{\alpha\beta \ldots \kappa}$$

by the equations

$$\left.\begin{aligned} A = [ab \ldots k]_{\alpha\beta\ldots\kappa} &= \begin{bmatrix} a_\alpha & b_\alpha & \ldots & k_\alpha \\ a_\beta & b_\beta & \ldots & k_\beta \\ \ldots & \ldots & \ldots & \ldots \\ a_\kappa & b_\kappa & \ldots & k_\kappa \end{bmatrix} \\ A' = \begin{bmatrix} a \\ b \\ \vdots \\ k \end{bmatrix}_{\alpha\beta\ldots\kappa} &= \begin{bmatrix} a_\alpha & a_\beta & \ldots & a_\kappa \\ b_\alpha & b_\beta & \ldots & b_\kappa \\ \ldots & \ldots & \ldots & \ldots \\ k_\alpha & k_\beta & \ldots & k_\kappa \end{bmatrix} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(J)}.$$

Ex. ix. $[abcd]_{243} = \begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}, \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{241} = \begin{bmatrix} a_2 & a_4 & a_1 \\ b_2 & b_4 & b_1 \\ c_2 & c_4 & c_1 \end{bmatrix}.$

Ex. x. The corranged simple minors of $[abc]_{1234}$ obtained by striking out two horizontal rows are

$$[abc]_{12},\ [abc]_{13},\ [abc]_{14},\ [abc]_{23},\ [abc]_{24},\ [abc]_{34}.$$

7. *Notation for one-rowed matrices.*

Symbols $[a]_m$, $\overline{\underline{a}}_m$ will be defined by

$$A = [a]_m = [a_1 a_2 \ldots a_m]; \quad A' = \overline{\underline{a}}_m = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \quad \ldots\ldots\ldots\ldots\text{(K)}.$$

Further symbols $[a_x]_m$, $\overline{\underline{a_x}}_m$ will be defined by

$$A = [a_x]_m = [a_{x_1} a_{x_2} \ldots a_{x_m}], \quad A' = \overline{\underline{a_x}}_m = \begin{bmatrix} a_{x_1} \\ a_{x_2} \\ \vdots \\ a_{x_m} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\text{(L)}.$$

Ex. xi. $[a]_4 = [a_1 a_2 a_3 a_4]; \quad \overline{\underline{a}}_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$

Ex. xii. $[a_x]_4 = [a_{x_1} a_{x_2} a_{x_3} a_{x_4}]; \quad \overline{\underline{a_x}}_3 = \begin{bmatrix} a_{x_1} \\ a_{x_2} \\ a_{x_3} \end{bmatrix}.$

8. *Special notation for a unit matrix.*

A square matrix of order m in which every element of the leading diagonal has the value 1 and every other element has the value 0 will be called the unit matrix of order m. We shall use for it the abbreviated notation $[1]_m^m$, so that

$$[1]_m^m = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \quad \text{.........................(M)},$$

where in the matrix on the right there are m horizontal rows and m vertical rows.

9. *Vertical and horizontal suffixes.*

When any single-suffix or double-suffix notation whatever is used for a matrix, suffixes which are each common to all the elements of a horizontal row and which indicate the horizontal rows to which elements belong will be called *vertical suffixes*. Similarly suffixes which are each common to all the elements of a vertical row and which indicate the vertical rows to which elements belong will be called *horizontal suffixes*. Thus in the matrix A of sub-article 6 the suffixes $\alpha, \beta, \dots \kappa$ are vertical suffixes. In the matrix A' they are horizontal suffixes.

§ 3. Determinoids.

The determinoid of any matrix A will be defined to be the algebraical sum of all its complete derived products, when each product is provided with a positive or negative sign in accordance with the following *rule of signs.*

Rule of Signs. Let $\alpha\beta\gamma\delta\dots$ be any complete derived product, in which the order of the factors is immaterial. In the matrix A count the total number of horizontal and vertical steps from the leading element to the element α; let this number be α'. Strike out in A the horizontal row and the vertical row in which α occurs, and let the matrix formed by the remaining horizontal and vertical rows be called A_1.

In the matrix A_1 count the total number of horizontal and vertical steps from the leading element to the element β. Let this number be β'. Strike out in A_1 the horizontal row and the vertical row in which β occurs, and let the matrix formed by the remaining horizontal and vertical rows be called A_2.

In the matrix A_2 count the total number of horizontal and vertical steps from the leading element to the element γ. Let this number be γ'. Strike out in A_2 the horizontal row and the vertical row in which γ occurs, and let the matrix formed by the remaining horizontal and vertical rows be called A_3. Continue this process as long as possible.

Then the sign to be attached to the product $\alpha\beta\gamma\delta\ldots$ is the same as the sign of $(-1)^{\alpha'+\beta'+\gamma'+\delta'+\ldots}$. In other words the product $\alpha\beta\gamma\delta\ldots$ is to be provided with a positive or negative sign according as $\alpha'+\beta'+\gamma'+\delta'+\ldots$, the total number of steps counted, is even or odd.

The number $\alpha'+\beta'+\gamma'+\delta'+\ldots$ will be called the *affect* of the product $\alpha\beta\gamma\delta\ldots$.

It is to be understood that a product formed with given elements of the matrix never occurs more than once in the algebraical sum which constitutes the determinoid.

In order to show that the above definition is legitimate, it is necessary to prove that the sign of any complete derived product $\alpha\beta\gamma\delta\ldots$, determined as above, is independent of the order of arrangement of its factors. Accordingly we proceed to prove that the above rule of signs is self-consistent.

Proof of the self-consistency of the Rule of Signs.

It will be sufficient to show that the sign of any complete derived product, determined as above, is not affected by interchanging two consecutive factors of the product.

Let $$P = \alpha\beta\ldots\kappa\lambda\mu\nu\ldots\rho$$

be any complete derived product in which λ and μ are consecutive factors.

Let $$Q = \alpha\beta\ldots\kappa\mu\lambda\nu\ldots\rho$$

be the same complete product after the two factors λ and μ have been interchanged.

In proceeding according to the rule, the values of $\alpha', \beta', \ldots\kappa', \nu', \ldots\rho'$ will be the same whether it be the sign of P or the sign of Q which we are determining. It is only in the cases of λ' and μ' than any differences in value can occur.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

be the original matrix, and let

$$B = \begin{bmatrix} b_{11} & b_{12} & . & . & . & . \\ b_{21} & b_{22} & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}$$

be the matrix which remains after the horizontal and vertical rows of A in which $\alpha, \beta, \ldots\kappa$ occur have been struck out.

It will be convenient to consider separately four cases, which are the only cases possible.

CASE I. *Let μ be below and to the right of λ in the matrix B*, which accordingly has the form

$$B = \begin{bmatrix} b_{11} & b_{12} & . & . & . & . & . \\ b_{21} & b_{22} & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & \lambda & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & \mu & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

In this case we may write

$$\lambda = b_{xy}, \quad \mu = b_{x+p,\, y+q}$$

where x, y, p, q are positive integers.

Following the rule of signs, we have

for P,
$$\lambda' = (x-1) + (y-1),$$
$$\mu' = (x+p-2) + (y+q-2),$$
$$\lambda' + \mu' = 2(x+y) + (p+q) - 6;$$

and for Q,
$$\mu' = (x+p-1) + (y+q-1),$$
$$\lambda' = (x-1) + (y-1),$$
$$\mu' + \lambda' = 2(x+y) + (p+q) - 4.$$

Therefore the number $\alpha' + \beta' + \ldots + \mu' + \lambda' + \ldots + \rho'$ for Q
exceeds the number $\alpha' + \beta' + \ldots + \lambda' + \mu' + \ldots + \rho'$ for P by 2.

CASE II. *Let λ be below and to the right of μ in the matrix B.*

This case is derivable from Case I by interchanging λ and μ.

Hence we may write $\mu = b_{xy}, \quad \lambda = b_{x+p,\, y+q}$

where x, y, p, q are positive integers, and we have

for Q, $\mu' + \lambda' = 2(x+y) + (p+q) - 6$

and for P, $\lambda' + \mu' = 2(x+y) + (p+q) - 4.$

Thus the number $\alpha' + \beta' + \ldots + \lambda' + \mu' + \ldots + \rho'$ for P
exceeds the number $\alpha' + \beta' + \ldots + \mu' + \lambda' + \ldots + \rho'$ for Q by 2.

CASE III. *Let μ be below and to the left of λ in the matrix B*, so that B has the form

$$B = \begin{bmatrix} b_{11} & b_{12} & . & . & . & . & . \\ b_{21} & b_{22} & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & \lambda & . \\ . & . & . & . & . & . & . \\ . & . & . & \mu & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

In this case we may write

$$\lambda = b_{x,\, y+q}, \quad \mu = b_{x+p,\, y}$$

where x, y, p, q are positive integers.

CHAPTER IX.

RANK OF A MATRIX AND CONNECTIONS BETWEEN THE ROWS OF A MATRIX.

CHAPTER X.

MATRIX EQUATIONS OF THE FIRST DEGREE.

Ex. i. $\det[a] = |a| = a.$

Ex. ii. $\det\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{vmatrix} a \\ b \\ c \end{vmatrix} = a - b + c.$

Ex. iii. $\det[abcd] = |abcd| = a - b + c - d.$

Ex. iv. $\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$

$$\begin{aligned} = \; & a_1b_2c_3 - a_1b_2c_4 - a_1b_3c_2 + a_1b_3c_4 + a_1b_4c_2 - a_1b_4c_3 \\ & - a_2b_1c_3 + a_2b_1c_4 + a_2b_3c_1 - a_2b_3c_4 - a_2b_4c_1 + a_2b_4c_3 \\ & + a_3b_1c_2 - a_3b_1c_4 - a_3b_2c_1 + a_3b_2c_4 + a_3b_4c_1 - a_3b_4c_2 \\ & - a_4b_1c_2 + a_4b_1c_3 + a_4b_2c_1 - a_4b_2c_3 - a_4b_3c_1 + a_4b_3c_2. \end{aligned}$$

Ex. v. $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \end{vmatrix}$ is the algebraical sum of 6.5.4.3 or 360 terms.

One of these is $+a_{31}a_{26}a_{42}a_{15}$; for the steps counted to the successive factors of the product $a_{31}a_{26}a_{42}a_{15}$ in following the rule of signs are respectively 2, 5, 1, 2, and therefore the sign of this product is the same as the sign of $(-1)^{2+5+1+2}$ or $(-1)^{10}$.

Such terms as *orders, long* and *short rows, leading element, leading line, leading position, derived products, forward and backward steps* will be applied to a determinoid just as they have been applied to a matrix.

Also the *derived determinoids,* and the *minor determinoids* of a given determinoid will be defined in exactly the same way as the derived matrices and the minor matrices of a given matrix.

If A is any matrix, the determinoid of any minor matrix of A or of any derived matrix of A will often be called a minor determinoid or a derived determinoid of A.

Similarly a derived matrix or a minor matrix of a determinoid will be understood to mean the corresponding derived matrix or minor matrix of the matrix of that determinoid.

The value of a determinoid is unaltered by interchanging its horizontal and vertical rows, i.e. by making its horizontal rows vertical and its vertical rows horizontal. For this interchange changes vertical steps into horizontal steps and horizontal steps into vertical steps, and accordingly leaves unaltered the total number of steps counted to each element of a product in following the rule of signs. Thus *two conjugate matrices have determinoids of equal value.* It follows from this that a determinoid can always be written with its long rows horizontal.

Ex. vi. The matrices $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$, $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$ are different, but the determinoids $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$, $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$ are equal in value and can be equated.

§ 4. Abbreviated notations for a determinoid.

In § 2 we introduced symbols

$$[a]_m^n, \quad [a_{pq}]_m^n, \quad [a_{p1}]_m^n, \quad [a_{1q}]_m^n, \quad \begin{bmatrix} uv\ldots w \\ a \\ pq\ldots r \end{bmatrix}$$

and symbols

$$[abc\ldots k]_{123\ldots m}, \quad [abc\ldots k]_{\alpha\beta\gamma\ldots\kappa}$$

to serve as abbreviated notations for a matrix of orders m and n.

We now proceed to define corresponding symbols

$$(a)_m^n, \quad (a_{pq})_m^n, \quad (a_{p1})_m^n, \quad (a_{1q})_m^n, \quad \begin{pmatrix} uv\ldots w \\ a \\ pq\ldots r \end{pmatrix}$$

and symbols

$$(abc\ldots k)_{123\ldots m}, \quad (abc\ldots k)_{\alpha\beta\gamma\ldots\kappa}$$

which will be used as abbreviated notations for a determinoid of orders m and n.

1. *Standard double-suffix notation.*

If $$A = [a]_m^n,$$

then $$\det A = (a)_m^n = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{vmatrix} \quad \text{..................(A).}$$

No new notation will be required for the determinoid of the conjugate matrix of A, for

$$\det \overleftarrow{a}^{\,m}_{\,\underleftarrow{\,n\,}} = \det [a]_m^n = (a)_m^n.$$

2. *More general double-suffix notations.*

If $$A = [a_{pq}]_m^n,$$

then $$\det A = (a_{pq})_m^n = \begin{vmatrix} a_{p_1q_1} & a_{p_1q_2} & \ldots & a_{p_1q_n} \\ a_{p_2q_1} & a_{p_2q_2} & \ldots & a_{p_2q_n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{p_mq_1} & a_{p_mq_2} & \ldots & a_{p_mq_n} \end{vmatrix} \quad \text{...............(B).}$$

If $$A = [a_{p1}]_m^n,$$

then $$\det A = (a_{p1})_m^n = \begin{vmatrix} a_{p_1 1} & a_{p_1 2} & \dots & a_{p_1 n} \\ a_{p_2 1} & a_{p_2 2} & \dots & a_{p_2 n} \\ \dots & \dots & \dots & \dots \\ a_{p_m 1} & a_{p_m 2} & \dots & a_{p_m n} \end{vmatrix} \quad \text{.................(C).}$$

If $$A = [a_{1q}]_m^n,$$

then $$\det A = (a_{1q})_m^n = \begin{vmatrix} a_{1q_1} & a_{1q_2} & \dots & a_{1q_n} \\ a_{2q_1} & a_{2q_2} & \dots & a_{2q_n} \\ \dots & \dots & \dots & \dots \\ a_{mq_1} & a_{mq_2} & \dots & a_{mq_n} \end{vmatrix} \quad \text{.................(D).}$$

Similarly the determinoids of the matrices

$$[a, b]_m^{n, r}, \quad [a_{pq}, b_{pu}]_m^{n, r}, \quad [a_{1q}, b_{1u}]_m^{n, r}, \quad \begin{bmatrix} a \\ b \end{bmatrix}_{m, r}^n, \quad \begin{bmatrix} a_{pq} \\ b_{vq} \end{bmatrix}_{m, r}^n, \quad \begin{bmatrix} a_{p1} \\ b_{v1} \end{bmatrix}_{m, r}^n$$

will be denoted by

$$(a, b)_m^{n, r}, \quad (a_{pq}, b_{pu})_m^{n, r}, \quad (a_{1q}, b_{1u})_m^{n, r}, \quad \begin{pmatrix} a \\ b \end{pmatrix}_{m, r}^n, \quad \begin{pmatrix} a_{pq} \\ b_{vq} \end{pmatrix}_{m, r}^n, \quad \begin{pmatrix} a_{p1} \\ b_{v1} \end{pmatrix}_{m, r}^n.$$

3. *Most general double-suffix notation.*

If $$A = \begin{bmatrix} {}^{uv\dots w} \\ a \\ {}_{pq\dots r} \end{bmatrix},$$

then $$\det A = \begin{pmatrix} {}^{uv\dots w} \\ a \\ {}_{pq\dots r} \end{pmatrix} = \begin{vmatrix} a_{pu} & a_{pv} & \dots & a_{pw} \\ a_{qu} & a_{qv} & \dots & a_{qw} \\ \dots & \dots & \dots & \dots \\ a_{ru} & a_{rv} & \dots & a_{rw} \end{vmatrix} \quad \text{.....................(E).}$$

Ex. i. $(a)_m^n = \begin{pmatrix} {}^{12\dots n} \\ a \\ {}_{12\dots m} \end{pmatrix}$.

Ex. ii. $(a_{pq})_m^n = \begin{pmatrix} {}^{q_1 q_2 \dots q_n} \\ a \\ {}_{p_1 p_2 \dots p_m} \end{pmatrix}, \quad (a_{p1})_m^n = \begin{pmatrix} {}^{12\dots n} \\ a \\ {}_{p_1 p_2 \dots p_m} \end{pmatrix}$.

4. *Standard single-suffix notation.*

If $$A = [ab \dots k]_{12\dots m},$$

then $$\det A = (ab \dots k)_{12\dots m} = \begin{vmatrix} a_1 & b_1 & \dots & k_1 \\ a_2 & b_2 & \dots & k_2 \\ \dots & \dots & \dots & \dots \\ a_m & b_m & \dots & k_m \end{vmatrix} \quad \text{...............(F).}$$

In using this notation the suffixes will often be omitted in the abbreviated symbol when doing so will cause no ambiguity. In particular they will often be omitted in the case of a determinant.

Ex. iii. The determinant

$$(abc)_{123} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

will often be denoted simply by (abc).

Ex. iv. The corranged simple minor determinoids of the determinoid

$$(abcd)_{123} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

obtained by striking out two vertical rows are

$$(ab)_{123},\quad (ac)_{123},\quad (ad)_{123},\quad (bc)_{123},\quad (bd)_{123},\quad (cd)_{123}.$$

These will often be denoted simply by

$$(ab),\quad (ac),\quad (ad),\quad (bc),\quad (bd),\quad (cd).$$

5. *Most general single-suffix notation.*

If
$$A = [ab \ldots k]_{\alpha\beta\ldots\kappa},$$

then
$$\det A = (ab \ldots k)_{\alpha\beta\ldots\kappa} = \begin{vmatrix} a_\alpha & b_\alpha & \ldots & k_\alpha \\ a_\beta & b_\beta & \ldots & k_\beta \\ \ldots & \ldots & \ldots & \ldots \\ a_\kappa & b_\kappa & \ldots & k_\kappa \end{vmatrix} \quad \ldots\ldots\ldots\ldots\text{(G)}.$$

Ex. v.
$$(abcd)_{315} = \begin{vmatrix} a_3 & b_3 & c_3 & d_3 \\ a_1 & b_1 & c_1 & d_1 \\ a_5 & b_5 & c_5 & d_5 \end{vmatrix}.$$

§ 5. Some simple properties of determinoids.

1. *A determinoid is a homogeneous linear function of the elements of any long row.*

Hence if $m < n$, we have such results as

$$\begin{vmatrix} a_{11}, & a_{12}, & \ldots & a_{1n} \\ a_{21}+b_{21}, & a_{22}+b_{22}, & \ldots & a_{2n}+b_{2n} \\ a_{31}, & a_{32}, & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1}, & a_{m2}, & \ldots & a_{mn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{vmatrix},$$

$$\begin{vmatrix} a_{11}, & a_{12}, & \ldots & a_{1n} \\ ka_{21}, & ka_{22}, & \ldots & ka_{2n} \\ a_{31}, & a_{32}, & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1}, & a_{m2}, & \ldots & a_{mn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{vmatrix}.$$

The general result corresponding to the second equation is that *multiplying every element of any long row of a determinoid by a number k is equivalent to multiplying the determinoid by k.*

2. *If two consecutive long rows are interchanged in any matrix, the determinoid of the matrix is changed in sign but is unaltered in magnitude.*

In other words, if the interchange of two *consecutive* long rows changes the matrix A into A', then $\det A' = -\det A$. To prove this let P be any complete derived product, and let

$$P = \alpha\beta \dots \lambda \dots \mu \dots \rho,$$

where $\alpha, \beta, \dots \lambda, \dots \mu, \dots \rho$ are elements of A, and in particular λ and μ are elements belonging to the two interchanged rows.

Then in determining the sign of P by the rule of signs of § 3, the number of steps counted to each element of P is the same in A' as in A except in the case of the element λ, which is that one of the two elements λ and μ which occurs first in P. In the case of this element λ, there is one step more or one step less in A' than in A according as λ belongs to the earlier or the later of the two rows of A which are interchanged. Thus the total number of steps counted in A' is either greater by one or less by one than the total number of steps counted in A.

Accordingly every complete derived product P has the opposite sign in A' to that which it has in A, and therefore

$$\det A' = -\det A.$$

3. *If any two long rows of a matrix are interchanged, the determinoid of the matrix is changed in sign but is unaltered in magnitude.*

Let A be any rectangular matrix, and let A' be the matrix obtained from it by interchanging the ith and the $(i+r)$th long rows. Then A' can be obtained from A by first interchanging the $(i+r)$th long row in turn with each of the r long rows immediately preceding it, and then interchanging the row which was the ith long row in turn with each of the $r-1$ long rows immediately following it. Thus A' can be derived from A by $2r-1$ interchanges of *consecutive* long rows. By the previous theorem it follows that

$$\det A' = (-1)^{2r-1} \det A = -\det A.$$

4. *If two long rows of a matrix are identically the same, the determinoid of the matrix is identically equal to zero.*

This is an immediate consequence of the last theorem.

5. *The value of a determinoid is unaltered by adding to the elements of any long row the corresponding elements of any other long row each multiplied by the same quantity.*

For example, if $m > n$, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{vmatrix} = \begin{vmatrix} a_{11} + ka_{13}, & a_{12}, & a_{13}, & \dots & a_{1n} \\ a_{21} + ka_{23}, & a_{22}, & a_{23}, & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} + ka_{m3}, & a_{m2}, & a_{m3}, & \dots & a_{mn} \end{vmatrix}.$$

This theorem follows from sub-articles 1 and 4 above.

The properties of determinoids will be considered in greater detail in the following chapters. It is sufficient for the present to observe that, as regards long rows, the properties of determinoids are closely analogous to the properties of determinants, which are of course a special class of determinoids.

CHAPTER II.

AFFECTS OF THE ELEMENTS AND DERIVED PRODUCTS OF A MATRIX OR DETERMINOID.

[In §§ 6, 7 and 8 these affects are defined and various ways of obtaining them are described. The four following articles, §§ 9—12, contain an investigation of the effect of interchanging two parallel rows or two suffixes on the value of the affect of a derived product. In § 13 it is shown that the sign of a derived product is independent of the order of its factors. In § 14 it is shown that the affect of any derived product of a matrix is equal to the number of forward moves by which the product can be brought to the leading position in the matrix. The last two articles, §§ 15 and 16, contain the results of inverting the orders of the rows in a matrix or determinoid.]

§ 6. Derived products of a matrix. Complete, incomplete, extended and completed products.

If $z_1, z_2, \dots z_r$ are any r elements of a matrix A selected in such a manner that no two of them occur in the same horizontal row and no two in the same vertical row, the product

$$P = z_1 z_2 \dots z_r$$

has been called a *derived product of order* r belonging to the matrix A. The product is *complete* or *incomplete* according as r is equal to or less than the efficiency of the matrix.

If P is an incomplete product we may form from it an *extended product*

$$P' = z_1 z_2 \dots z_r z_{r+1} z_{r+2} \dots z_k \dots$$

by postfixing other elements $z_{r+1}, z_{r+2}, \dots z_k, \dots$ so chosen that z_{r+1} is the leading element in the matrix A_r which remains when all the horizontal and vertical rows of A in which $z_1, z_2, \dots z_r$ occur have been struck out, and generally z_k is the leading element in the matrix A_{k-1} which remains when all the horizontal and vertical rows of A in which $z_1, z_2, \dots z_{k-1}$ occur have been struck out.

If we extend an incomplete product until it becomes complete, we obtain the corresponding *completed product*.

Ex. i. In the matrix

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}_{1234567} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{bmatrix}$$

b_2a_5 is an incomplete product;
$b_2a_5c_1$ and $b_2a_5c_1d_3$ are successive extensions of it;
$b_2a_5c_1d_3a_4$ is the corresponding completed product.

Ex. ii. In the matrix

$$[a]_5^8 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \end{bmatrix}$$

one of the incomplete derived products is $a_{43}a_{26}$.

If we extend this product once, twice, and complete it, we obtain in succession the products

$$a_{43}a_{26}, \quad a_{43}a_{26}a_{11}, \quad a_{43}a_{26}a_{11}a_{32}, \quad a_{43}a_{26}a_{11}a_{32}a_{54}.$$

When the standard double-suffix notation is employed, the extensions and the completion of a given derived product can be written down immediately without reference to the matrix.

For if
$$A = [a]_m^n$$
and if
$$z_1 = a_{x_1y_1}, \quad z_2 = a_{x_2y_2}, \ldots z_k = a_{x_ky_k}, \ldots$$
and if the product $P = z_1z_2 \ldots z_r$ is extended s times, then $x_{r+1}, x_{r+2}, \ldots$ are the first s of the numbers $1, 2, \ldots m$ which do not occur amongst $x_1, x_2, \ldots x_r$ when these are arranged in ascending order of magnitude, and similarly $y_{r+1}, y_{r+2}, \ldots$ are the first s of the numbers $1, 2, \ldots n$ which do not occur amongst $y_1, y_2, \ldots y_r$ when these are arranged in ascending order of magnitude. In other words, the vertical (or horizontal) suffix of each successive added factor is the smallest vertical (or horizontal) suffix which has not already occurred in the product.

A similar rule can be enunciated for use with the standard single-suffix notation.

Ex. iii. Consider, as in Ex. ii, the incomplete product $a_{43}a_{26}$ of the matrix $[a]_5^8$.

Taking the suffixes which do not occur in the product $a_{43}a_{26}$ and arranging them in ascending order of magnitude, we have

the vertical suffixes 1, 3, 5,
and the horizontal suffixes 1, 2, 4, 5, 7, 8.

Hence the successive added factors are

$$a_{11}, \quad a_{32}, \quad a_{54}.$$

§ 7. Affect of any derived product of a matrix.

Let P be any derived product of order r of a matrix A, and let

$$P = z_1 z_2 \dots z_k \dots z_r.$$

Then the *affect of the product* P in the matrix A will be defined to be the number ω given by the equation

$$\omega = \omega_1 + \omega_2 + \dots + \omega_k + \dots + \omega_r \quad\text{......(1)},$$

where ω_k is the affect of the element z_k in the matrix A_{k-1} obtained from A by striking out all the horizontal and vertical rows which contain the elements $z_1, z_2, \dots z_{k-1}$ preceding z_k in the product P.

This number ω can clearly be obtained by counting steps just as in the Rule of Signs of § 3.

Let ω_k', ω_k'' be the vertical and horizontal affects of z_k in A_{k-1}, so that

$$\omega_k = \omega_k' + \omega_k'' \quad\text{......(2)}.$$

Then using such results as (2) in (1), we have

$$\omega = \omega' + \omega'' \quad\text{......(3)},$$

where

$$\omega' = \Sigma\omega_k', \quad \omega'' = \Sigma\omega_k'' \quad\text{......(4)}.$$

We shall call ω_k *the affect of* z_k *in* A *relative to the product* P, or briefly *the relative affect of the factor* z_k.

Similarly we shall call ω_k', ω_k'' *the relative vertical affect* and *the relative horizontal affect* of the factor z_k.

Then equation (1) can be expressed as follows:

The affect of P *is the sum of the relative affects of all its factors*......(5).

Further if we call ω' and ω'' the *vertical affect* and the *horizontal affect* of the product P in A, equations (3) and (4) lead to the following results:

The vertical affect of P *is the sum of the relative vertical affects of all its factors* ......(6).

The horizontal affect of P *is the sum of the relative horizontal affects of all its factors* ......(7).

The total affect of P *is the sum of its vertical and horizontal affects*...(8).

If P is any derived product and ω its affect, the sign of $(-1)^\omega$, whose value is always either $+1$ or -1, will be called *the sign determined by the affect of* P. It is + or − according as ω is even or odd. Also $(-1)^\omega P$ will be called *the product* P *with the sign determined by its affect*. Briefly we shall refer to P as an *unaffected product* and to $(-1)^\omega P$ as the corresponding *affected product*.

From § 3 it is clear that the determinoid of a matrix is the algebraical sum of all its affected *complete* derived products.

Ex. i. If
$$A=[a]_8^5=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} \end{bmatrix},$$
and
$$P=a_{43}a_{35}a_{61}a_{84},$$
it can easily be seen by counting steps as in the Rule of Signs of § 3 that
$$\text{the affect of } P \text{ in } A=5+5+3+5=18.$$

Here 5, 5, 3, 5 are the relative affects of the factors a_{43}, a_{35}, a_{61}, a_{84} respectively.

Ex. ii. If
$$A=[abcde]_{1234}=\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{bmatrix},$$
and
$$P=c_2a_4e_1,$$
it can be seen by counting steps that
$$\text{the affect of } P \text{ in } A=3+2+2=7.$$

Here 3, 2, 2 are the relative affects of the factors c_2, a_4, e_1 respectively.

§ 8. Rules for determining the affect of a derived product of a matrix.

1. *Standard double-suffix notation.*

When the standard double-suffix notation is employed, the affect of any derived product can be written down from an inspection of the product, without striking out rows and counting steps in the matrix.

Let the matrix be
$$A=[a]_m^n,$$
and let the derived product be
$$P=z_1z_2\ldots z_k\ldots z_r,$$
where
$$z_1=a_{x_1y_1},\quad z_2=a_{x_2y_2},\ \ldots\ z_r=a_{x_ry_r}.$$

Then the relative vertical affect of z_k
$$=(x_k-1)-\xi_k \quad\ldots\ldots(1),$$
where ξ_k = the number of the suffixes $x_1, x_2, \ldots x_{k-1}$ which are less than x_k

= the number of the suffixes $x_1, x_2, \ldots x_r$ which are less than x_k and precede x_k in the product P

= the number of vertical suffixes occurring in the product P which are less than the suffix x_k and precede x_k in P.

Also the relative horizontal affect of z_k

$$= (y_k - 1) - \eta_k \qquad \text{.....................} (2),$$

where $\eta_k =$ the number of the suffixes $y_1, y_2, \ldots y_{k-1}$ which are less than y_k

= the number of the suffixes $y_1, y_2, \ldots y_r$ which are less than y_k and precede y_k in the product P

= the number of horizontal suffixes occurring in the product P which are less than the suffix y_k and precede y_k in P.

Thus if ω is the affect of P in A, we have

$$\omega = \sum_{k=1}^{k=r} \{(x_k - 1) - \xi_k\} + \sum_{k=1}^{k=r} \{(y_k - 1) - \eta_k\} \qquad \text{..............} (3),$$

where the first term is the vertical affect and the second term is the horizontal affect of P in $\boldsymbol{A}$.

Ex. i. As in Ex. i of § 7 let

$$A = [a]_8^5, \quad \text{and} \quad P = a_{43} a_{35} a_{61} a_{84}.$$

The vertical suffixes occurring in the product P are

4, 3, 6, 8.

In this series of numbers there are

none which are less than 4 and precede 4,
none which are less than 3 and precede 3,
two which are less than 6 and precede 6,
three which are less than 8 and precede 8.

Accordingly by formula (1) the relative vertical affects of the successive factors of P are

$$(4-1)-0, \quad (3-1)-0, \quad (6-1)-2, \quad (8-1)-3$$

or

$$3, \; 2, \; 3, \; 4.$$

The vertical affect of P, being the sum of these, is 12.

Again the horizontal suffixes occurring in the product P are

3, 5, 1, 4.

In this series of numbers there are

none which are less than 3 and precede 3,
one which is less than 5 and precedes 5,
none which are less than 1 and precede 1,
two which are less than 4 and precede 4.

Accordingly by formula (2) the relative horizontal affects of the successive factors of P are

$$(3-1)-0, \quad (5-1)-1, \quad (1-1)-0, \quad (4-1)-2$$

or

$$2, \; 3, \; 0, \; 1.$$

The horizontal affect of P, being the sum of these, is 6.

Finally the total affect of P, being the sum of its vertical and horizontal affects, is

$$12 + 6 \text{ or } 18.$$

Ex. ii. Let $A=[a]_8^5, \quad P=a_{54}a_{13}a_{81}a_{35}$.

The vertical affect of P $=4+0+5+1=10$,

where the four terms are the relative vertical affects of the successive factors of P.

The horizontal affect of P $=3+2+0+1=6$,

where the four terms are the relative horizontal affects of the successive factors of P.

The total affect of P $=10+6=16$.

Ex. iii. Let $A=[a]_8^5, \quad P=a_{71}a_{34}a_{55}a_{13}$.

The vertical affect of P $=6+2+3+0=11$.

The horizontal affect of P $=0+2+3+1=6$.

Therefore the affect of P in A $=11+6=17$.

2. *Complete derived products.*

The determinoid of a matrix is the algebraical sum of all its complete derived products, each with the sign determined by its affect. But it has been shown in § 3 that the sign of a complete derived product, as determined by its affect, is independent of the order in which its factors are arranged. This property, it may be observed, is true for all derived products whether complete or incomplete. In fact the proof given in § 3 is valid for incomplete products as well as for complete products.

Accordingly in developing a determinoid it will generally be convenient to arrange the factors of each complete product in such an order that the first factor belongs to the first long row, the second factor to the second long row, and generally the kth factor to the kth long row. Then if long rows are horizontal, all horizontal affects will be zero, and it will only be necessary to determine the vertical affects. Similarly if long rows are vertical, all vertical affects will be zero, and it will only be necessary to determine the horizontal affects. Further it is immaterial whether the determinoid is written with its long rows vertical or with its long rows horizontal.

The same arrangement of factors can be used with advantage whenever it is required to determine the *sign* of any complete derived product.

Ex. iv. If $A=[a]_m^n$

is a matrix in which long rows are horizontal, and if ω is the affect of the complete product

$$P=a_{1y_1}a_{2y_2}\dots a_{my_m},$$

in A, we have

$$\omega=\sum_{k=1}^{k=m}\{(y_k-1)-\eta_k\},$$

where $\eta_k=$ the number of the vertical suffixes occurring in the product P which are less than the suffix y_k and precede y_k in P.

Ex. v. If $$A = \begin{bmatrix} a \\ b \\ \vdots \\ k \end{bmatrix}_{12\dots n} = \begin{bmatrix} a_1 & a_2 \dots a_n \\ b_1 & b_2 \dots b_n \\ \dots & \dots \\ k_1 & k_2 \dots k_n \end{bmatrix}$$

is a matrix with m horizontal long rows and n vertical short rows, and if ω is the affect in A of the complete derived product

$$P = a_{x_1} b_{x_2} \dots k_{x_m},$$

we have $$\omega = \sum_{s=1}^{s=m} \{(x_s - 1) - \xi_s\},$$

where ξ_s = the number of the suffixes $x_1, x_2, \dots x_m$ which are less than x_s and precede x_s in P.

Ex. vi. If $$A = [ab \dots k]_{12\dots m} = \begin{bmatrix} a_1 & b_1 \dots k_1 \\ a_2 & b_2 \dots k_2 \\ \dots & \dots \\ a_m & b_m \dots k_m \end{bmatrix}$$

is a matrix with m horizontal short rows and n vertical long rows, and if ω is the affect in A of the complete derived product

$$P = a_{x_1} b_{x_2} \dots k_{x_n},$$

we have $$\omega = \sum_{s=1}^{s=n} \{(x_s - 1) - \xi_s\},$$

where ξ_s = the number of the suffixes $x_1, x_2, \dots x_n$ which are less than x_s and precede x_s in P.

Ex. vii. Consider the determinoid $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{vmatrix}$ and its complete derived product $a_3 b_1 c_5 d_2$.

The relative affects of the successive factors found by counting *horizontal* steps or by using the formula of Ex. v are 2, 0, 2, 0. Thus the total affect of the product is 4 and the corresponding term in the development of the determinoid is $+a_3 b_1 c_5 d_2$.

Similarly every term in the development of the determinoid can be found with its appropriate sign.

Ex. viii. The development of the determinoid $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}$ is

$$a_1 b_2 - a_1 b_3 + a_1 b_4 - a_2 b_1 + a_2 b_3 - a_2 b_4 + a_3 b_1 - a_3 b_2 + a_3 b_4 - a_4 b_1 + a_4 b_2 - a_4 b_3.$$

Here the affects of the successive products are

0, 1, 2, 1, 2, 3, 2, 3, 4, 3, 4, 5.

3. *Rules applicable with any notation.*

In the formulae of sub-article 1, the number $x_k - 1$ can be interpreted to be the number of horizontal rows of the matrix A which lie above the element z_k, and the number $y_k - 1$ can be interpreted to be the number of vertical rows of the matrix A which lie to the left of the element z_k.

We can therefore enunciate the following rules for determining the affect ω of any derived product

$$P = z_1 z_2 \dots z_r$$

of any matrix A.

(i) The relative vertical affect of the factor z_k

= (the number of horizontal rows which lie above z_k in the matrix A) − (the number of elements which precede z_k in the product P and also lie above z_k in the matrix A).

The vertical affect of P is the sum of the relative vertical affects of all its factors.

(ii) The relative horizontal affect of the factor z_k

= (the number of vertical rows which lie to the left of z_k in the matrix A) − (the number of elements which precede z_k in the product P and also lie to the left of z_k in the matrix A).

The horizontal affect of P is the sum of the relative horizontal affects of all its factors.

(iii) The affect ω is the sum of the vertical and horizontal affects of P.

By the use of these rules the affects of all the derived products of a matrix can be found without striking out rows.

Ex. ix. We will consider the matrix

$$M = \begin{bmatrix} a & b & c & d \\ p & q & r & s \\ x & y & z & w \end{bmatrix},$$

and denote by ω the affect in M of the product

$$P = sxb.$$

The relative vertical affects of the successive factors are

$$(1-0), \quad (2-1), \quad (0-0) \quad \text{or} \quad 1, 1, 0.$$

The relative horizontal affects of the successive factors are

$$(3-0), \quad (0-0), \quad (1-1) \quad \text{or} \quad 3, 0, 0.$$

Thus $$\omega = (1+1+0)+(3+0+0) = 5.$$

4. *Extended and completed products.*

No change is made in the affect of a derived product when it is extended or completed; for each added factor is the leading element in the matrix in which steps are counted to determine its relative affect, and therefore the relative affect of every added factor is zero.

It follows that the determination of the affects of all derived products can be reduced to the determination of the affects of *complete* derived products.

Further the sign of any derived product P is the same as the sign of its completed product P'; and in determining the sign of P' we can re-arrange its factors and use the simpler rules of sub-article 2.

Ex. x. Let $$A=[a]_5^8 \text{ and } P=a_{43}a_{26}.$$

The successive extensions of P are

$$P_1=a_{43}a_{26}a_{11}, \quad P_2=a_{43}a_{26}a_{11}a_{32}, \quad P_3=a_{43}a_{26}a_{11}a_{32}a_{54}.$$

The relative affects of the successive factors are

for P, 5, 5
for P_1, 5, 5, 0
for P_2, 5, 5, 0, 0
for P_3, 5, 5, 0, 0, 0.

Thus all these products have the same affect 10.

Again the sign of P is the same as the sign of P_3, and the sign of P_3 is the same as the sign of

$$P_3'=a_{11}a_{26}a_{32}a_{43}a_{54}.$$

The relative affects of the successive factors of P_3' are

$$(1-1)+0, \quad (6-1)-1, \quad (2-1)-1, \quad (3-1)-2, \quad (4-1)-3$$

or $$0, \; 4, \; 0, \; 0, \; 0.$$

Thus the affect of P_3' is 4 and its sign is $+$.

Accordingly the sign of P is $+$.

§ 9. Changes in the affect and sign of a derived product caused by the interchange of two consecutive parallel rows of the matrix to which it belongs.

The results which will be obtained in this article can be summarised as follows:

Change in affect.

I. *If the product contains no element belonging to the interchanged rows, its affect is unaltered.*

II. *If the product contains two elements belonging to the interchanged rows, then its affect is*

(1) *increased by* 1 *if these elements have the same relative order in the product as the rows to which they belong have in the matrix,*

(2) *diminished by* 1 *if they have the reverse order.*

III. *If the product contains only one element belonging to the two interchanged rows, then its affect is*

(1) *increased by* 1 *if that element belongs to the earlier of the two interchanged rows,*

(2) *diminished by* 1 *if it belongs to the later of the two interchanged rows.*

Note. The third case can be reduced to the second case by adding extra rows to the matrix after those already existing, and by extending the product, neither of which processes alters the affect.

We can therefore regard Rule III as included in Rule II. The first case can also be similarly reduced to the second case.

Change in sign.

A. *If no element belonging to the interchanged rows occurs in the product, the sign of the product as determined by its affect is unchanged.*

B. *If any element belonging to the two interchanged rows occurs in the product (i.e. if either one or two such elements occur), the sign of the product is changed.*

The rules regarding change in sign are immediate consequences of the rules regarding change in affect. We shall therefore content ourselves with proving the first set of rules. Also we shall suppose that the two interchanged rows are vertical rows, the argument being exactly similar when they are horizontal rows.

Let the original matrix M and the matrix M' obtained from it by interchanging the ith and $(i+1)$th vertical rows be

$$M = \begin{bmatrix} a_1 & a_2 \ldots a_i & a_{i+1} \ldots a_n \\ b_1 & b_2 \ldots b_i & b_{i+1} \ldots b_n \\ \cdots & \cdots & \cdots \\ k_1 & k_2 \ldots k_i & k_{i+1} \ldots k_n \end{bmatrix}, \quad M' = \begin{bmatrix} a_1 & a_2 \ldots a_{i+1} & a_i \ldots a_n \\ b_1 & b_2 \ldots b_{i+1} & b_i \ldots b_n \\ \cdots & \cdots & \cdots \\ k_1 & k_2 \ldots k_{i+1} & k_i \ldots k_n \end{bmatrix}.$$

We will compare the affects of any derived product P in the two matrices M and M'.

The relative vertical affects of the various factors of P are clearly the same in M' as in M. We may therefore confine ourselves to a consideration of the relative horizontal affects.

Let λ_s be *any one* of the factors of P, so that λ is one of the letters $a, b, \ldots k$, and s is one of the numbers $1, 2, \ldots n$. Let S be the matrix in which steps are counted when we are determining the relative affect of the factor λ_s in M, and let S' be the matrix in which steps are counted when we are determining the relative affect of λ_s in M'. S and S' are formed from M and M' by striking out those horizontal and vertical rows which contain the factors of P preceding λ_s.

CASE I. *Let* $P = \alpha_p \beta_q \ldots \lambda_s \ldots \sigma_r$, *and contain no element belonging to the two interchanged rows.*

Here $\alpha, \beta, \ldots \lambda, \ldots \sigma$ are certain of the letters $a, b, \ldots k$; and $p, q, \ldots s, \ldots r$ are certain of the numbers $1, 2, \ldots n$, not including i or $i+1$.

In this case neither of the two interchanged rows is struck out from M and M' in the formation of S and S'. The forms of S and S' are therefore

$$S = \begin{bmatrix} a_u' & a_v' & \dots & a_i' & a'_{i+1} & \dots & a_w' \\ b_u' & b_v' & \dots & b_i' & b'_{i+1} & \dots & b_w' \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ k_u' & k_v' & \dots & k_i' & k'_{i+1} & \dots & k_w' \end{bmatrix}, \quad S' = \begin{bmatrix} a_u' & a_v' & \dots & a'_{i+1} & a_i' & \dots & a_w' \\ b_u' & b_v' & \dots & b'_{i+1} & b_i' & \dots & b_w' \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ k_u' & k_v' & \dots & k'_{i+1} & k_i' & \dots & k_w' \end{bmatrix},$$

where $a', b', \dots k'$ are certain letters selected from $a, b, \dots k$ without disarrangement of relative order, and $u, v, \dots i, i+1, \dots w$ are certain of the suffixes $1, 2, \dots n$ arranged in ascending order of magnitude, and including both i and $i+1$. The vertical rows not shown are the same in S' as in S.

The element λ_s occurs in both S and S', but it is in neither of the vertical rows whose suffixes are i and $i+1$. It occupies exactly the same position in both these matrices.

Hence its horizontal affects in S and S' are the same.

Consequently the relative horizontal affect of the factor λ_s is the same in M' as in M. This being true for every factor of P, we see that the affect of P in M' is the same as the affect of P in M.

Thus Rule I is proved.

CASE II. *Let $P = \alpha_p \beta_q \dots x_i \dots y_{i+1} \dots \sigma_r$, containing an element x_i from the ith vertical row of M and an element y_{i+1} from the $(i+1)$th vertical row of M, the latter occurring later than the former in P.*

As before let λ_s be any one of the factors of P.

(1) *If λ_s occurs before x_i in P*, neither of the two interchanged rows is struck out from M and M' in the formation of S and S'. The forms of S and S' are the same as in Case I above, and it follows as in that case that λ_s has the same relative affect in M' as in M.

(2) *If λ_s is the element x_i*, the forms of S and S' are still as before, but now x_i or λ_s lies in S' one step further to the right than in S. We see then that the relative affect of the factor x_i in M' is greater by 1 than its relative affect in M.

(3) *If λ_s occurs after x_i in P*, then S and S' are identical. Either they both contain one only of the two interchanged rows, viz. the row in which the suffix is $i+1$, or they both contain neither of the two interchanged rows.

The element λ_s occupies exactly the same position in S' as in S. Its affects in S and S', i.e. its relative affects in M and M', are therefore the same.

In this case then the relative affect of the factor x_i in M' is greater by 1 than its relative affect in M. All other factors of P have the same relative affects in M and M'. Consequently the affect of P in M' exceeds the affect of P in M by 1.

Thus the first part of Rule II is proved.

The second part of Rule II can be proved in a similar way by assuming that

$$P = \alpha_p \beta_q \dots x_{i+1} \dots y_i \dots \sigma_r.$$

CASE III. *Let $P = \alpha_p \beta_q \dots y_{i+1} \dots \sigma_r$, containing an element y_{i+1} from the $(i+1)$th vertical row of M, but no element from the ith vertical row.*

Let λ_s as before be any one of the factors of P.

(1) *If λ_s occurs before y_{i+1} in P*, then S and S' have the forms given in Case I, and λ_s, which does not belong to either of the two interchanged rows, has the same horizontal affect in S' as in S, and therefore its relative affects in M and M' are the same.

(2) *If λ_s is the element y_{i+1}*, the forms of S and S' are still the same, but y_{i+1} lies in S one step further to the right than in S'. The horizontal affect of y_{i+1} in S' is therefore less by 1 than its horizontal affect in S, i.e. the relative affect of y_{i+1} in M' is less by 1 than its relative affect in M.

(3) *If λ_s occurs after y_{i+1} in P*, then S and S' are identical, and λ_s occupies identical positions in these two matrices. Consequently λ_s has the same relative affect in M' as in M.

In this case then the relative affect of the factor y_{i+1} in M' is less by 1 than its relative affect in M, and every other factor of P has the same affect in M' as in M. Consequently the affect of P in M' is smaller by 1 than the affect of P in M.

Thus the second part of Rule III is proved.

The first part of Rule III can be proved in a similar manner by assuming that

$$P = \alpha_p \beta_q \dots x_i \dots \sigma_r.$$

Note 1. *Rule III can be deduced from Rule II* as follows.

Let $P = \alpha_p \beta_q \dots y_{i+1} \dots \sigma_r$, containing only one element y_{i+1} belonging to the two interchanged rows. Let Q be the completion of P in M. It is also the completion of P in M'. Accordingly the affects of Q in M and M' are equal respectively to the affects of P in M and M'.

If Q contains two elements belonging to the interchanged rows, then by Rule II

$$\text{the affect of } Q \text{ in } M' = \text{the affect of } Q \text{ in } M - 1$$

and therefore

$$\text{the affect of } P \text{ in } M' = \text{the affect of } P \text{ in } M - 1.$$

If Q still contains only the one element y_{i+1} belonging to the interchanged rows, the long rows of M are horizontal.

Let M be converted into a square matrix M_1 by the addition of new horizontal rows after those already occurring in it, and let M_1' be obtained from M_1 by interchanging the ith and the $(i+1)$th vertical rows. Let R be the completion of P in M_1. It is also the completion of P in M_1'. Accordingly the affects of R in M_1 and M_1' are equal respectively to the affects of P in M and M'.

Now R has necessarily the form

$$R = a_p\beta_q \ldots y_{i+1} \ldots \sigma_r \ldots x_i \ldots.$$

Consequently by Rule II

$$\text{the affect of } R \text{ in } M_1' = \text{the affect of } R \text{ in } M_1 - 1,$$

and therefore

$$\text{the affect of } R \text{ in } M' = \text{the affect of } R \text{ in } M - 1.$$

Thus the second part of Rule III is proved.

The first part of Rule III can be proved in a similar manner by assuming that

$$P = a_p\beta_q \ldots x_i \ldots \sigma_r.$$

Note 2. Changes in relative affects.

In the course of the foregoing proofs, the following properties of the relative affects have been established.

The only factor of P whose relative affect is changed is that factor belonging to the two interchanged rows which occurs first in P. The relative affect of that one factor is increased or diminished by 1 *according as it belongs to the earlier or the later of the two rows of M which are interchanged.*

Ex. i. Let $M = [abcde]_{12345678}, \quad M' = [abcde]_{12346578},$

so that M' is obtained from M by interchanging the fifth and sixth horizontal rows.

The following five examples illustrate in turn a case falling under each of the Rules I, II (1), II (2), III (1), III (2). Elements belonging to the interchanged rows are marked by asterisks.

Rule I. $P = d_3e_8a_2$.

	For P in M	For P in M'
Relative horizontal affects	3, 3, 0	3, 3, 0
Relative vertical affects	2, 6, 1	2, 6, 1
Total affect	15	15

Rule II (1). $P = b_3 \overset{*}{d_5} e_1 \overset{*}{c_6}$.

	For P in M	For P in M'
Relative horizontal affects	1, 2, 2, 1	1, 2, 2, 1
Relative vertical affects	2, 3, 0, 2	2, 4, 0, 2
Total affect	13	14

Rule II (2). $P = \overset{*}{b_6} e_2 c_8 \overset{*}{a_5} d_7$.

	For P in M	For P in M'
Relative horizontal affects	1, 3, 1, 0, 0	1, 3, 1, 0, 0
Relative vertical affects	5, 1, 5, 3, 3	4, 1, 5, 3, 3
Total affect	22	21

Rule III (1). $P = a_2 \overset{*}{c_5} d_4 e_8$.

	For P in M	For P in M'
Relative horizontal affects	0, 1, 1, 1	0, 1, 1, 1
Relative vertical affects	1, 3, 2, 4	1, 4, 2, 4
Total affect	13	14

Rule III (2). $P = a_2 \overset{*}{c_6} d_4 e_8$.

	For P in M	For P in M'
Relative horizontal affects	0, 1, 1, 1	0, 1, 1, 1
Relative vertical affects	1, 4, 2, 4	1, 3, 2, 4
Total affect	11	13

Ex. ii. *The interchange of two consecutive long rows in a determinoid is equivalent to a change in the sign of the determinoid.*

For every complete derived product contains elements belonging to the interchanged rows, and therefore the sign of every complete derived product is changed.

§ 10. Changes in the affect and sign of a derived product caused by the interchange (in the product) of two suffixes of the same kind which are consecutive in the matrix.

It may be supposed that a standard double-suffix or single-suffix notation is employed. The two interchanged suffixes will then be consecutive integers, and will be either both vertical suffixes or both horizontal suffixes. The result of the interchange is expressed by rules similar to those obtained in § 9.

Change in affect.

I. *If neither of the interchanged suffixes occurs in the product, the affect of the product is unaltered.*

II. *If both the interchanged suffixes occur in the product, then the affect of the product is*

(1) *increased by* 1 *if they have the same relative order in the product as in the matrix,*

(2) *diminished by* 1 *if they have the reverse relative order in the product.*

III. *If only one of the two interchanged suffixes occurs in the product, then the affect of the product is*

(1) *increased by* 1 *if that suffix is the smaller one or the one which occurs earlier in the matrix,*

(2) *diminished by* 1 *if that suffix is the larger one or the one which occurs later in the matrix.*

Change in sign.

A. *If neither of the two interchanged suffixes occurs in the product, the sign of the product, as determined by its affect, is unaltered.*

B. *If the product contains either one or both of the interchanged suffixes, its sign is changed.*

We can establish these rules most easily by deducing them from the rules of § 9. It will be sufficient to consider the case in which the interchanged suffixes are horizontal suffixes.

Let any matrix M and the matrix derived from it by interchanging the ith and $(i+1)$th vertical rows be

$$M = \begin{bmatrix} a_1 & a_2 \dots a_i & a_{i+1} \dots a_n \\ b_1 & b_2 \dots b_i & b_{i+1} \dots b_n \\ \dots & \dots & \dots \\ k_1 & k_2 \dots k_i & k_{i+1} \dots k_n \end{bmatrix}, \quad M' = \begin{bmatrix} a_1 & a_2 \dots a_{i+1} & a_i \dots a_n \\ b_1 & b_2 \dots b_{i+1} & b_i \dots b_n \\ \dots & \dots & \dots \\ k_1 & k_2 \dots k_{i+1} & k_i \dots k_n \end{bmatrix}.$$

Let P be any derived product of M, and let the interchange of the suffixes i and $i+1$ change P into P'.

Thus if $P = \alpha_p \beta_q \dots x_i \dots y_{i+1} \dots \sigma_r$, then $P' = \alpha_p \beta_q \dots x_{i+1} \dots y_i \dots \sigma_r$, and if $P = \alpha_p \beta_q \dots x_i \dots \sigma_r$, then $P' = \alpha_p \beta_q \dots x_{i+1} \dots \sigma_r$.

We have to compare the affects of P and P' in M.

If the operation of interchanging the suffixes i and $i+1$ is applied simultaneously to P' and M, it converts them into P and M' respectively. Consequently the positions occupied by the successive factors of P' in M are identical with the positions occupied by the successive factors of P in M', and therefore the affect of P' in M is the same as the affect of P in M'. Accordingly the results of comparing the affect of P' in M with the affect of P in M are the same as the results of comparing the affect of P in M' with the affect of P in M.

Thus the rules of this article are consequences of the rules of § 9.

Note 1. *Changes in relative affects.*

Let the relative affect of the kth factor of a derived product be called the relative affect of the kth place. Then by comparing the relative affects of the same places in P and P' we can obtain the following result.

There is only one place in P where the relative affect is changed, viz. the first place in P where a factor having one of the two interchanged suffixes occurs. The relative affect of that place is increased or diminished by 1 according as the suffix of the factor occurring there in P is the earlier or the later of the two interchanged suffixes in M.

We have already shown that the successive elements of P' occupy the same positions in M as the successive elements of P in M'. It follows that the relative affect of the kth factor of P' in M is equal to the relative affect of the kth factor of P in M'.

Let ω be the relative affect of *any factor* e of P in M, and ω' the relative affect of the similarly placed factor of P' in M.

Then ω' is also the relative affect of the similarly placed factor of P in M', i.e. ω' is also the relative affect of the factor e of P in M'.

Therefore by Case I, ω' is equal to ω except when e is that element having one of the two interchanged suffixes which occurs first in P. In that one case $\omega' = \omega + 1$ or $\omega' = \omega - 1$ according as the suffix of e belongs to the earlier or the later of the two interchanged suffixes in M.

This result can be expressed in the form given above.

If e is x_i, the similarly placed element of P' is x_{i+1}.

If e is y_{i+1}, the similarly placed element of P' is y_i.

In every other case the similarly placed element of P' is also e.

Note 2. *Direct proof.*

We may if we please prove the results of the present article directly by the method used in § 3, and then deduce the results of § 9.

Ex. i. Let

$$M = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}_{12345678} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{bmatrix},$$

and let the derived product P become P' when the two consecutive suffixes 5 and 6 are interchanged. Each of the five values of P considered below illustrates one case of the rules for change in affect.

Rule	Product	Relative affects in M	Total affect in M
I	$P = d_3 e_8 a_2$ $P' = d_3 e_8 a_2$	5, 9, 1 5, 9, 1	15 15
II (1)	$P = b_3 \overset{*}{d_5} e_1 \overset{*}{c_6}$ $P' = b_3 d_6 e_1 c_5$	3, 5, 2, 3 3, 6, 2, 3	13 14
II (2)	$P = \overset{*}{b_6} e_2 c_8 \overset{*}{a_5} d_7$ $P' = b_5 e_2 c_8 a_6 d_7$	6, 4, 6, 3, 3 5, 4, 6, 3, 3	22 21
III (1)	$P = a_2 \overset{*}{c_5} d_4 e_8$ $P' = a_2 c_6 d_4 e_8$	1, 4, 3, 5 1, 5, 3, 5	13 14
III (2)	$P = a_2 \overset{*}{c_6} d_4 e_8$ $P' = a_2 c_5 d_4 e_8$	1, 5, 3, 5 1, 4, 3, 5	14 13

Ex. ii. When a single-suffix notation is employed exactly similar results can be enunciated concerning the changes caused by the interchange in the derived product of two *letters* which are consecutive in the matrix.

§ 11. Changes in the affect and sign of a derived product caused by the interchange of any two parallel rows in the matrix to which it belongs.

Let the two interchanged rows be the ith and jth vertical rows or the ith and jth horizontal rows, where $i < j$. These may be called respectively the earlier interchanged row and the later interchanged row.

When the interchanged rows are vertical we shall use the single-suffix notation with horizontal suffixes and denote the original matrix M and the matrix M' derived from it by interchanging the ith and jth vertical rows by

$$M = \begin{bmatrix} a_1 & a_2 \dots a_i \dots a_j \dots a_n \\ b_1 & b_2 \dots b_i \dots b_j \dots b_n \\ \dots & \dots \\ k_1 & k_2 \dots k_i \dots k_j \dots k_n \end{bmatrix}, \quad M' = \begin{bmatrix} a_1 & a_2 \dots a_j \dots a_i \dots a_n \\ b_1 & b_2 \dots b_j \dots b_i \dots b_n \\ \dots & \dots \\ k_1 & k_2 \dots k_j \dots k_i \dots k_n \end{bmatrix}.$$

All vertical rows not shown are then the same in M' as in M.

When the two interchanged rows are horizontal we shall use the single-suffix notation with vertical suffixes and denote the original matrix M and the matrix M' derived from it by interchanging the ith and jth horizontal rows by

$$M = \begin{bmatrix} a_1 & b_1 \dots k_1 \\ a_2 & b_2 \dots k_2 \\ \dots & \dots \\ a_i & b_i \dots k_i \\ \dots & \dots \\ a_j & b_j \dots k_j \\ \dots & \dots \\ a_n & b_n \dots k_n \end{bmatrix}, \quad M' = \begin{bmatrix} a_1 & b_1 \dots k_1 \\ a_2 & b_2 \dots k_2 \\ \dots & \dots \\ a_j & b_j \dots k_j \\ \dots & \dots \\ a_i & b_i \dots k_i \\ \dots & \dots \\ a_n & b_n \dots k_n \end{bmatrix}.$$

All horizontal rows not shown are then the same in M' as in M.

Let P be any derived product belonging to the matrix M.

It is required to compare the affect and sign of P in M' with the affect and sign of P in M.

Let r be the number of parallel rows lying between the two interchanged rows. Let these rows be called *inner rows*, and let elements belonging to them be called *inner elements*.

Let rows parallel to the two interchanged rows which are neither interchanged rows nor inner rows be called *outer rows*, and let elements belonging to the outer rows be called *outer elements*.

The product P may contain an element belonging to the earlier interchanged row; if so, let that element be denoted by x_i. So let y_j denote that element, if any, occurring in P which belongs to the later interchanged row.

When x_i occurs in P, let ξ be the number of inner elements which precede it in P.

When y_j occurs in P, let η be the number of inner elements which precede it in P.

When x_i and y_j both occur in P, let ρ be the number of inner elements which lie between them in P, so that $\rho = \eta - \xi$ or $\xi - \eta$ according as x_i

precedes or follows y_j in P. We might describe ρ as being the number of elements which are inner elements both of M and of P.

Then we can enunciate the following rules.

I. *If P contains neither x_i nor y_j, the affect of P is unaltered, and therefore also the sign of P is unaltered.*

II. *If x_i occurs earlier than y_j in P, so that P has the form*

$$P = \alpha_p \beta_q \ldots x_i \ldots y_j \ldots \sigma_m,$$

but may terminate before y_j is reached, then:

(1) *The relative affect of x_i is increased by $r + 1 - \xi$.*

(2) *The relative affect of y_j is diminished by $r - \eta$.*

(3) *The relative affect of each inner element occurring in P after x_i but before y_j is increased by* 1.

(4) *The relative affects of all other factors of P are unaltered.*

(5) *If both x_i and y_j occur in P, the affect of P is increased by $2\rho + 1$, and therefore its sign is changed.*

III. *If y_j occurs earlier than x_i in P, so that P has the form*

$$P = \alpha_p \beta_q \ldots y_j \ldots x_i \ldots \sigma_m,$$

but may terminate before x_i is reached, then

(1) *The relative affect of y_j is diminished by $r + 1 - \eta$.*

(2) *The relative affect of x_i is increased by $r - \xi$.*

(3) *The relative affect of each inner element occurring in P after y_j but before x_i is diminished by* 1.

(4) *The relative affects of all other factors of P are unaltered.*

(5) *If both y_j and x_i occur in P, the affect of P is diminished by $2\rho + 1$, and therefore its sign is changed.*

Two very simple conclusions concerning the change in sign are contained in the above rules:

A. *If P contains no element belonging to the interchanged rows, its sign is unaltered.*

B. *If both the interchanged rows supply elements to P, the sign of P is changed, i.e. the sign of P in M' is opposite to the sign of P in M.*

In Case II the elements belonging to the two interchanged rows which occur in P may be said to have the same relative order in P as in M.

In Case III the elements belonging to the two interchanged rows which occur in P may be said to have reverse relative orders in P and in M.

Rule I is of course included in Rules II and III.

We will proceed to give a direct proof of Rule II only, the direct proof of Rule III being similar. Further, in proving Rule II we shall assume that the two interchanged rows are vertical. The treatment in the case in which they are horizontal is similar.

Proof of Rule II.

Let $$P = a_p\beta_q \ldots x_i \ldots y_j \ldots \sigma_m.$$

It is to be understood that P may terminate before x_i is reached, or it may contain x_i and terminate before y_j is reached, or it may contain both x_i and y_j.

Let λ_s be *any one* of the factors of P.

Let S be the matrix in which steps are counted in order to determine the relative affect of λ_s in M.

Let S' be the matrix in which steps are counted in order to determine the relative affect of λ_s in M'.

S and S' are formed from M and M' respectively by striking out those horizontal and vertical rows which contain the factors of P preceding λ_s.

(1) *If λ_s occurs before x_i in P*, the rows struck out from M' to form S' are the same as those struck out from M to form S, and the two interchanged rows occur in both S and S'. Accordingly the matrices S and S' have the forms

$$S = \begin{bmatrix} a_u' & a_v' \ldots a_i' \ldots a_j' \ldots a_w' \\ b_u' & b_v' \ldots b_i' \ldots b_j' \ldots b_w' \\ \ldots & \ldots\ldots\ldots\ldots\ldots \\ k_u' & k_v' \ldots k_i' \ldots k_j' \ldots k_w' \end{bmatrix}, \quad S' = \begin{bmatrix} a_u' & a_v' \ldots a_j' \ldots a_i' \ldots a_w' \\ b_u' & b_v' \ldots b_j' \ldots b_i' \ldots b_w' \\ \ldots & \ldots\ldots\ldots\ldots\ldots \\ k_u' & k_v' \ldots k_j' \ldots k_i' \ldots k_w' \end{bmatrix},$$

where $a', b', \ldots k'$ are certain letters selected from $a, b, \ldots k$ without change of relative order; $u, v, \ldots i, \ldots j, \ldots w$ are certain of the numbers $1, 2, \ldots n$ arranged in ascending order of magnitude; and S' is formed from S by interchanging the two rows whose suffixes are i and j. Since λ_s does not belong to either of the two interchanged rows, it occupies exactly the same position in S' as in S. Consequently the affects of λ_s in S and S' are the same, and therefore the factor λ_s of P has the same relative affect in M' as in M.

(2) *If λ_s is the element x_i*, then S and S' have the same forms as in the previous case, but the positions of x_i or λ_s in S and S' are, as regards vertical rows, not the same. The vertical rows which have been struck out from M and M' to form S and S' are the same, and ξ of these rows lay between the two interchanged rows, i.e. were inner rows. Consequently there are $r - \xi$ vertical rows between the two interchanged rows in S and S', i.e. there are $r + 1 - \xi$ vertical rows before x_i in S' which come after x_i in S. We conclude that the affect of x_i in S' exceeds the affect of x_i in S by $r + 1 - \xi$, or that the relative affect of x_i in M' exceeds the relative affect of x_i in M by $r + 1 - \xi$.

(3) *If λ_s lies between x_i and y_j in P and is not an inner element*, then the vertical rows with suffix i are struck out from M and M' in the process of forming S and S'. The other rows struck out from M and M' are rows occupying identical positions in those two matrices. Therefore the forms of S and S' are now

$$S = \begin{bmatrix} a_u' & a_v' \ldots\ldots a_j' \ldots a_w' \\ b_u' & b_v' \ldots\ldots b_j' \ldots b_w' \\ \ldots & \ldots\ldots\ldots\ldots \\ k_u' & k_v' \ldots\ldots k_j' \ldots k_w' \end{bmatrix}, \quad S' = \begin{bmatrix} a_u' & a_v' \ldots a_j' \ldots\ldots a_w' \\ b_u' & b_v' \ldots b_j' \ldots\ldots b_w' \\ \ldots & \ldots\ldots\ldots\ldots \\ k_u' & k_v' \ldots k_j' \ldots\ldots k_w' \end{bmatrix}.$$

where S' can be formed from S by moving the row with suffix j so many places towards the left as will bring it into the position which the row with suffix i would have occupied if it had been retained, or by moving the row with suffix j just to the left of all the retained inner rows.

Thus each retained inner row lies one place further to the right in S' than in S, and each retained outer row occupies the same position in S' as in S. Now λ_s, being an outer element, belongs to one of the retained outer rows, and occupies exactly the same position in S' as in S. Hence in this case the relative affects of λ_s in M and M' are the same.

(4) *If λ_s lies between x_i and y_j in P and is an inner element*, the forms of S and S' are the same as in the previous case, but now λ_s, belonging to one of the retained inner rows, lies one step further to the right in S' than in S. Therefore in this case the relative affect of the factor λ_s of P is greater by 1 in M' than in M.

(5) *If λ_s is the element y_j* the forms of S and S' are still the same as in Case (3). But there are now $r-\eta$ retained inner rows, and these all lie to the left of the row with suffix j in S, and to the right of the row with suffix j in S'. Therefore the number of horizontal steps to y_j in S' is less by $r-\eta$ than the number of horizontal steps to y_j in S.

Accordingly the relative affect of y_j in M' is less by $r-\eta$ than the relative affect of y_j in M.

(6) *If λ_s lies beyond y_j in P*, both the interchanged rows are struck out from M and M' in the formation of S and S'. Consequently S and S' are identical, and λ_s occupies exactly the same position in both these matrices. Thus in this case the affects of λ_s in S and S' are the same, and therefore the relative affect of λ_s in M' is the same as its relative affect in M.

Thus Rule II, including Rule I, is completely proved.

The standard single-suffix notations have been used merely for the sake of convenience. It will be clear that the above rules hold whatever notation is adopted for the matrix M, it being understood that x_i, y_j are the elements occurring in the product P which belong respectively to the earlier and the later of the two interchanged rows in M.

From these general considerations it is clear that Rule III can be deduced from Rule II.

For in Case III, let ω be the relative affect of any factor of P in the matrix M, and let ω' be the relative affect of the same factor of P in the matrix M'. Then $\omega'-\omega$ is the change in the relative factor of that element considered in Rule II. Now the matrix M can be obtained from the matrix M' by interchanging the rows with suffixes j and i. Consequently $\omega-\omega'$ can be found by Rule II, and then $\omega'-\omega$ is known, being obtained from $\omega-\omega'$ by a change of sign.

Ex. i. The only factors of P whose relative affects are changed are those which are elements belonging to the two interchanged rows, and those which are inner elements both of M and of P.

Ex. ii. Let

$$M = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix}_{123456789}, \quad M' = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix}_{127456389},$$

so that M' is derived from M by interchanging the third and seventh vertical rows.

In illustration of the foregoing rules, the relative and total affects in M and M' are given in the table below for five different values of the derived product P. Elements belonging to the interchanged rows are marked by asterisks and inner elements by dots. The relative affects of such elements are similarly marked.

	P	Affect of P in M	Affect of P in M'	Increase
(1)	$\dot{b}_5 c_8 g_2 e_1 a_9$	$\dot{5}+7+5+2+4 = 23$	$\dot{5}+7+5+2+4 = 23$	0
(2)	$c_1 \overset{*}{f_3} \dot{b}_1 g_8 \overset{*}{a_7} d_2$	$4+\overset{*}{5}+\dot{2}+7+\overset{*}{3}+1 = 22$	$4+\overset{*}{9}+\dot{3}+7+\overset{*}{1}+1 = 25$	3
(3)	$\overset{*}{b_7} d_2 \dot{e}_5 \dot{a}_1 \overset{*}{g_3} c_8$	$\overset{*}{7}+3+\dot{5}+\dot{2}+\overset{*}{3}+2 = 22$	$\overset{*}{3}+3+\dot{4}+\dot{1}+\overset{*}{4}+2 = 17$	5
(4)	$\dot{d}_1 \overset{*}{f_3} \dot{a}_5 e_8 c_2$	$\dot{6}+\overset{*}{6}+\dot{2}+5+2 = 21$	$\dot{6}+\overset{*}{9}+\dot{3}+5+2 = 25$	4
(5)	$\dot{a}_5 \dot{d}_1 \overset{*}{c_7} g_9 \dot{b}_6$	$\dot{4}+\dot{5}+\overset{*}{6}+8+\dot{3} = 26$	$\dot{4}+\dot{5}+\overset{*}{4}+8+\dot{2} = 23$	3

In (2), $x_i = f_3$, $y_j = a_7$, $r = 3$, $\xi = 0$, $\eta = 1$, $\rho = 1$,
$r+1-\xi = 4$, $r-\eta = 2$, $2\rho+1 = 3$.

In (3), $x_i = a_3$, $y_j = b_7$, $r = 3$, $\xi = 2$, $\eta = 0$, $\rho = 2$,
$r+1-\eta = 4$, $r-\xi = 1$, $2\rho+1 = 5$.

In (4), $x_i = f_3$, $r = 3$, $\xi = 1$, $r+1-\xi = 3$.

In (5), $y_j = c_7$, $r = 3$, $\eta = 2$, $r+1-\eta = 2$.

Ex. iii. *The interchange of any two long rows in a determinoid is equivalent to a change in the sign of the determinoid.*

Let Δ be any determinoid, and let Δ be the determinoid into which Δ is converted by the interchange of any two long rows

Any derived product of Δ is also a derived product of Δ', and conversely any derived product of Δ' is also a derived product of Δ.

If P is a *complete* derived product of Δ, it contains an element belonging to every long row, and therefore it contains elements belonging to both the interchanged rows.

Accordingly its sign in Δ' is opposite to its sign in Δ.

Thus the affected derived products of Δ' are equal in magnitude but opposite in sign to the affected derived products of Δ.

Consequently $$\Delta' = -\Delta.$$

For example $$(abcde)_{1234} = -(abcde)_{1423}.$$

If two *short rows* of the determinoid are interchanged, then some of the complete derived products are changed in sign and some are unchanged.

Ex. iv. *If two long rows of a determinoid are identical, the determinoid vanishes identically.*

§ 12. Changes in the affect and sign of a derived product caused by the interchange (in the product) of any two suffixes of the same kind.

It is supposed that some double-suffix or single-suffix notation is employed for a matrix M, and that the interchanged suffixes are either both horizontal suffixes or both vertical suffixes.

Let the interchanged suffixes be i and j, of which i occurs earlier than j in M.

Let P be any derived product belonging to M, and let P be converted into P' when the suffixes i and j are interchanged. *It is required to compare the affect and sign of P' in M with the affect and sign of P in M.*

Let the rows of M which are parallel to the rows with suffixes i and j and lie between these be called *inner rows*. Let the parallel rows other than the inner rows and the rows with suffixes i and j be called *outer rows*. Let elements of M belonging to inner and outer rows be called *inner elements* and *outer elements* respectively.

Let r be the number of inner rows in M.

If P contains amongst its factors an element having the suffix i, let that element be denoted by x_i, and let the number of inner elements which precede x_i in P be ξ.

If P contains amongst its factors an element having the suffix j, let that element be denoted by y_j, and let the number of inner elements which precede y_j in P be η.

When x_i and y_j both occur in P, let ρ be the number of inner elements lying between them in P.

If the change in passing from the relative affect of the kth factor of P in M to the relative affect of the kth factor of P' in M is called the change of the relative affect in the kth place of P, we can enunciate the following rules.

I. *If neither of the interchanged suffixes occurs in P, so that x_i and y_j are both absent from P, then the affect of P in M is unaltered, and therefore also the sign of P is unaltered; further the relative affect in every place of P is unaltered. In fact in this case P' is the same as P.*

II. *If x_i occurs before y_j in P, then*

(1) *The relative affect in the place of P occupied by x_i is increased by $r+1-\xi$.*

(2) *The relative affect in the place of P occupied by y_j is diminished by $r-\eta$.*

(3) *The relative affect in the place of each inner element occurring in P after x_i but before y_j is increased by 1.*

(4) *The relative affect in every other place of P is unaltered.*

(5) *If both x_i and y_j occur in P, the total affect of P is increased by $2\rho+1$.*

III. *If y_j occurs before x_i in P, then*

(1) *The relative affect in the place of P occupied by y_j is diminished by $r+1-\eta$.*

(2) *The relative affect in the place of P occupied by x_i is increased by $r-\xi$.*

(3) *The relative affect in the place of each inner element occurring in P after y_j but before x_i is diminished by 1.*

(4) *The relative affect in every other place of P is unaltered.*

(5) *If both y_j and x_i occur in P, the total affect of P is diminished by $2\rho+1$.*

Regarding the change in sign, we have the following simple results.

A. *If neither of the two interchanged suffixes occurs in P, the sign of P is unchanged. In fact P' is the same as P.*

B. *If both the interchanged suffixes occur in P, the sign of P is changed, i.e. the sign of P' in M is opposite to the sign of P in M.*

Case II can be described as the case in which the two interchanged suffixes have the same relative order in P as in M.

Case III can be described as the case in which the two interchanged suffixes have reverse relative orders in P and in M.

It is to be understood that it is not necessary in either case that both the interchanged suffixes shall occur in P.

Case I can be regarded as included both in Case II and in Case III.

Proof. Let M' be the matrix obtained from M by interchanging the rows with suffixes i and j.

If the operation of interchanging the suffixes i and j is applied simultaneously to P' and M, they are converted into P and M' respectively. Hence the positions which the successive factors of P' occupy in M are the same as the positions which the successive factors of P occupy in M'. It follows that the relative affect of the kth factor of P' in M is equal to the relative affect of the kth factor of P in M'.

Consequently the change in passing

from the relative affect of the kth factor of P in M

to the relative affect of the kth factor of P' in M

is the same as the change in passing

from the relative affect of the kth factor of P in M

to the relative affect of the kth factor of P in M'.

But this last change has already been investigated in § 11. Accordingly all the results stated above are immediately deducible from the corresponding results in § 11.

Note. We may, if we please, prove the results of the present article directly by a method similar to that used in § 3, and then deduce the results of § 11.

Ex. i. The only places of P in which the relative affects are changed are those occupied by elements having the interchanged suffixes and those occupied by elements which are inner elements both of M and of P.

Ex. ii. Let $$M = [abcdefg]_{123456789},$$

and let any derived product P belonging to M be converted into P' by the interchange of the two suffixes 3 and 7. In the following table the affects of P and P' in M are compared for five different values of P. Elements possessing one of the interchanged suffixes are marked by asterisks, and inner elements are marked by dots. The relative affects of such elements are similarly marked.

	Products	Affects in M	Increase
(1)	$P = \dot{b}_5 c_8 g_2 e_1 a_9$	$\dot{5}+7+5+2+4=23$	0
	$P' = \dot{b}_5 c_8 g_2 e_1 a_9$	$\dot{5}+7+5+2+4=23$	
(2)	$P = e_1 \overset{*}{f_3} \dot{b}_4 g_8 \overset{*}{a_7} d_2$	$4+\overset{*}{5}+\dot{2}+7+\overset{*}{3}+1=22$	3
	$P' = e_1 \overset{*}{f_7} \dot{b}_4 g_8 \overset{*}{a_3} d_2$	$4+\overset{*}{9}+\dot{3}+7+\overset{*}{1}+1=25$	
(3)	$P = \overset{*}{b_7} d_2 \dot{e}_5 \dot{a}_4 \overset{*}{g_3} e_8$	$\overset{*}{7}+3+\dot{5}+\dot{2}+\overset{*}{3}+2=22$	-5
	$P' = \overset{*}{b_3} d_2 \dot{e}_5 \dot{a}_4 \overset{*}{g_7} e_8$	$\overset{*}{3}+3+\dot{4}+\dot{1}+\overset{*}{4}+2=17$	
(4)	$P = \dot{d}_1 \overset{*}{f_3} \dot{a}_5 e_8 e_2$	$\dot{6}+\overset{*}{6}+\dot{2}+5+2=21$	4
	$P' = \dot{d}_1 \overset{*}{f_7} \dot{a}_5 e_8 e_2$	$\dot{6}+\overset{*}{9}+\dot{3}+5+2=25$	
(5)	$P = \dot{a}_5 \dot{d}_4 \overset{*}{e_7} g_9 \dot{b}_6$	$\dot{4}+\dot{5}+\overset{*}{6}+8+\dot{3}=26$	-3
	$P' = \dot{a}_5 \dot{d}_4 \overset{*}{e_3} g_9 \dot{b}_6$	$\dot{4}+\dot{5}+\overset{*}{4}+8+\dot{2}=23$	

In (2),
$$x_i = f_3,\quad y_j = a_7,\quad r=3,\quad \xi=0,\quad \eta=1,\quad \rho=1,$$
$$r+1-\xi=4,\quad r-\eta=2,\quad 2\rho+1=3.$$

In (3),
$$x_i = g_3,\quad y_j = b_7,\quad r=3,\quad \xi=2,\quad \eta=0,\quad \rho=2,$$
$$r+1-\eta=4,\quad r-\xi=1,\quad 2\rho+1=5.$$

In (4),
$$x_i = f_3,\quad r=3,\quad \xi=1,\quad r+1-\xi=3.$$

In (5),
$$y_j = e_7,\quad r=3,\quad \eta=2,\quad r+1-\eta=2.$$

(Compare with Ex. ii of § 11.)

Ex. iii. When a single-suffix notation is used, similar results are true regarding the changes caused by the interchange of any two *letters* of the matrix in the derived product.

§ 13. Invariance of the sign of a derived product.

Let
$$A = [a]_m^n \text{ ....} (1)$$

be any matrix, and let
$$P = a_{x_1 y_1} a_{x_2 y_2} \dots a_{x_i y_i} \dots a_{x_j y_j} \dots a_{x_r y_r} \text{ ....} (2)$$

be any derived product, complete or incomplete, belonging to A.

Let any two factors $a_{x_iy_i}$, $a_{x_jy_j}$ of P be interchanged, and let P then become P', so that

$$P' = a_{x_1y_1}a_{x_2y_2}\ldots a_{x_jy_j}\ldots a_{x_iy_i}\ldots a_{x_ry_r} \quad \text{..............(3).}$$

P' can be obtained from P by interchanging the two vertical suffixes x_i, x_j with one another and also interchanging the two horizontal suffixes y_i, y_j.

In each of these interchanges both the interchanged suffixes occur in the product P. Therefore by § 12 each of these interchanges introduces a change of sign, i.e. increases or diminishes the affect of the product by an odd number. Consequently the signs of P and P', as determined by their affects in A, are the same, and their affects differ by an even number.

Since then the interchange of any two factors of P does not alter the sign of P, we conclude that *the sign of any derived product is independent of the order in which its factors are arranged.*

This property has already been proved in § 3 for complete derived products, and the proof there given is valid also for incomplete derived products.

It follows that if r is any number not greater than the efficiency of a matrix, the algebraical sum of all the affected derived products of order r belonging to the matrix is a definite function of the elements of the matrix. In the special case in which r is equal to the efficiency, this function is called the determinoid of the matrix.

Ex. If $$A = [abcde]_{1234},$$
the affects in A of the products

$$b_3a_2e_1,\quad b_3e_1a_2,\quad a_2b_3e_1,\quad a_2e_1b_3,\quad e_1a_2b_3,\quad e_1b_3a_2$$

are respectively 6, 6, 4, 4, 4, 6.

Thus all these products have the same sign.

§ 14. Reduction of any derived product of a matrix M to a leading product by forward moves.

By re-arranging the horizontal rows of a matrix M amongst themselves in any manner, and also re-arranging the vertical rows amongst themselves in any manner, we can form other derived matrices of M.

Every such re-arrangement can be effected by a succession of transpositions or interchanges of pairs of adjacent parallel rows.

A transposition or interchange of two adjacent parallel rows in M or in any derived matrix of M will be called a *move.*

The move will be called a *horizontal move* when the two adjacent rows are vertical, and a *vertical move* when the two adjacent rows are horizontal.

Further the move will be called a *forward move* when the two adjacent parallel rows which are interchanged have the same relative order before the change as they had in the original matrix M. In the contrary case it will be called a *backward move*.

Ex. i. Let $M=[abcd]_{123}=\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ be the original matrix.

Then $M'=[bdac]_{213}=\begin{bmatrix} b_2 & d_2 & a_2 & c_2 \\ b_1 & d_1 & a_1 & c_1 \\ b_3 & d_3 & a_3 & c_3 \end{bmatrix}$ is a derived matrix.

A single forward horizontal move and a single backward vertical move will convert

M' into $\begin{bmatrix} b_2 & d_2 & c_2 & a_2 \\ b_1 & d_1 & c_1 & a_1 \\ b_3 & d_3 & c_3 & a_3 \end{bmatrix}$, and into $\begin{bmatrix} b_1 & d_1 & a_1 & c_1 \\ b_2 & d_2 & a_2 & c_2 \\ b_3 & d_3 & a_3 & c_3 \end{bmatrix}$ respectively.

Now if P is any derived product of a matrix M, it is clearly possible by forward vertical moves to bring the horizontal rows which occur in P (i.e. which contribute elements to P) to the leading position in the matrix, arranged in the same relative order as in P, without at any stage disturbing the relative order of the horizontal rows which do not occur in P. Also the vertical rows which occur in P can be similarly brought to the leading position in the matrix by forward horizontal moves.

Hence it must always be possible by forward moves (some horizontal and some vertical) to bring the horizontal and vertical rows which occur in P to the leading position in the matrix arranged in the same relative orders as in P, *without at any stage disturbing the relative orders of the horizontal and vertical rows which do not occur in P*.

Every such move consists in a transposition of two adjacent parallel rows, of which one at least contains an element occurring in P. Therefore by § 9 every such move either increases or diminishes the affect of P by 1. Since the move is a forward move, the affect is *diminished* by 1 each time.

In the final matrix the product P occupies the leading position, and its affect is zero.

Since each such move diminishes the affect by 1, and since the final affect is 0, it follows that *the total number of such moves is equal to the affect of P in M.*

There are many sets of moves, by which this reduction can be effected, but the number of moves in each set is always equal to the affect of P in M.

Similar reasoning will show that the total number of vertical moves is equal to the vertical affect of P in M, and that the total number of horizontal moves is equal to the horizontal affect of P in M.

The completed product of P in M appears in the final matrix as the leading complete product.

Ex. ii. Let

$$A=[a]_5^6=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \end{bmatrix},$$

and

$$P=a_{26}a_{51}a_{45}.$$

The vertical, horizontal and total affects of P are respectively 6, 8, and 14.

The completed product of P is

$$P'=a_{26}a_{51}a_{45}a_{12}a_{33}.$$

By means of 6 vertical forward moves which leave the relative order of the first and third horizontal rows unaltered and 8 horizontal forward moves which leave the relative order of the second, third and fourth vertical rows unaltered, the matrix A can be converted into

$$A'=\begin{bmatrix} a_{26} & a_{21} & a_{25} & a_{22} & a_{23} & a_{24} \\ a_{56} & a_{51} & a_{55} & a_{52} & a_{53} & a_{54} \\ a_{46} & a_{41} & a_{45} & a_{42} & a_{43} & a_{44} \\ a_{16} & a_{11} & a_{15} & a_{12} & a_{13} & a_{14} \\ a_{36} & a_{31} & a_{35} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

In A' the product P occupies the leading position, and the product P' is the leading line.

Ex. iii. If M is a square matrix and P any one of its complete derived products, the sign determined by the affect of P in M is the same as the sign of the product P in the development of the determinant of M, as ordinarily defined. Accordingly *the determinoid of a square matrix is identical with the determinant of that matrix as ordinarily defined.*

§ 15. Changes in the sign of a derived product caused by inversions in the orders of arrangement of the rows of a matrix.

Let a matrix with m horizontal and n vertical rows be denoted by

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Let A_1', A_2', A_3' be the matrices obtained from A by inverting the order of the horizontal rows, by inverting the order of the vertical rows, and by inverting the orders of both sets of rows respectively, so that

$$A_1' = \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}, \quad A_2' = \begin{bmatrix} a_{1n} & \cdots & a_{12} & a_{11} \\ a_{2n} & \cdots & a_{22} & a_{21} \\ \cdots & \cdots & \cdots & \cdots \\ a_{mn} & \cdots & a_{m2} & a_{m1} \end{bmatrix}, \quad A_3' = \begin{bmatrix} a_{mn} & \cdots & a_{m2} & a_{m1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{2n} & \cdots & a_{22} & a_{21} \\ a_{1n} & \cdots & a_{12} & a_{11} \end{bmatrix}.$$

Let $$P = a_{x_1 y_1} a_{x_2 y_2} \dots a_{x_k y_k} \dots a_{x_r y_r}$$

be any derived product of A of order r whose affect in A is ω.

Let A' denote each of the matrices A_1', A_2', A_3' in turn, and let ω' denote the affect of P in A'.

Then the following three results can be proved:

I. When $A' = A_1'$, $\quad \omega' \equiv \omega + r(m-1) - \frac{1}{2}r(r-1) \pmod{2}$.

II. When $A' = A_2'$, $\quad \omega' \equiv \omega + r(n-1) - \frac{1}{2}r(r-1) \pmod{2}$.

III. When $A' = A_3'$, $\quad \omega' \equiv \omega + r(m+n) \pmod{2}$.

Proof of I. Let α, α' be the vertical and β, β' the horizontal affects of P in A and A_1' respectively.

The vertical rows to which the successive factors of P belong occupy the same positions in A_1' as in A_1.

Therefore $$\beta' = \beta.$$

Let ξ_k be the number of vertical suffixes occurring in P which are less than x_k and precede x_k in P, and let ξ_k' be the number of vertical suffixes occurring in P which are greater than x_k and precede x_k in P.

Then the relative vertical affects of the kth factor of P in A and A_1' are $(x_k - 1) - \xi_k$ and $(m - x_k) - \xi_k'$ respectively.

Therefore
$$\alpha = \sum_{k=1}^{k=r} \{(x_k - 1) - \xi_k\}, \qquad \alpha' = \sum_{k=1}^{k=r} \{(m - x_k) - \xi_k'\},$$

$$\alpha + \alpha' = \sum_{k=1}^{k=r} \{(m-1) - (\xi_k + \xi_k')\}.$$

Now $\xi_k + \xi_k' =$ the total number of factors which precede the kth factor in P

$= k - 1$.

Accordingly $$\alpha + \alpha' = r(m-1) - \tfrac{1}{2}r(r-1).$$

Moreover $$\beta + \beta' = 2\beta.$$

Adding these two results, we have

$$\omega + \omega' = r(m-1) - \tfrac{1}{2}r(r-1) + 2\beta.$$

This equation determines ω' and the result I is immediately deducible from it.

Proof of II. In this case it can be shown in the same way that

$$\alpha' = \alpha,$$

$$\beta + \beta' = r(n-1) - \tfrac{1}{2}r(r-1).$$

Therefore $$\omega + \omega' = r(n-1) - \tfrac{1}{2}r(r-1) + 2\alpha.$$

Proof of III. In this case the horizontal and vertical affects of the product are both changed, and we have

$$\alpha + \alpha' = r(m-1) - \tfrac{1}{2}r(r-1),$$

$$\beta + \beta' = r(n-1) - \tfrac{1}{2}r(r-1).$$

Therefore $$\omega + \omega' = r(m+n) - r(r-1) - 2r,$$

where $r(r-1)$ is even.

It should be observed that the results I, II, III as enunciated are the same for all the derived products of a given order r. The changes in affects are in general different for different derived products of the same order, but the changes in sign are the same for all.

Note. If A' is the conjugate matrix of A, every derived product P, whether complete or incomplete, has the same total affect in A' as in A.

For with the notation used above, we have

$$\alpha' = \beta, \quad \beta' = \alpha, \quad \alpha' + \beta' = \alpha + \beta,$$

i.e. we have $$\omega' = \omega.$$

§ 16. Inversion of the orders of arrangement of the rows in a determinoid.

Let Δ be a determinoid with m long rows and n short rows, and let it be written with the long rows horizontal so that

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix} = (a)_m^n.$$

Let Δ' be a determinoid derived from Δ by inverting the order of arrangement of the long rows, or by inverting the order of arrangement of the short rows, or by inverting the orders of arrangement of both sets of rows.

Let P be any complete derived product of Δ, i.e. any derived product of order m, and let the affects of P in Δ and Δ' be ω and ω' respectively.

In all three cases we have $\Delta' = \pm \Delta$. The signs appropriate to the separate cases are given below.

Case I. *Inversion of long rows.*

$$\Delta' = \left| \begin{matrix} a_{m1} & a_{m2} & \dots & a_{mn} \\ \dots & \dots & \dots & \dots \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{11} & a_{12} & \dots & a_{1n} \end{matrix} \right| .$$

In this case we have

$$\Delta' = \Delta \times (-1)^{\frac{1}{2}m(m-1)} \qquad \text{.........................(1).}$$

For putting $r = m$ in the result I of § 15, we see that

$$\omega' \equiv \omega + \tfrac{1}{2} m(m-1) \quad (\text{mod. } 2).$$

Accordingly

$$\Delta' = \Sigma (-1)^{\omega'} P = (-1)^{\frac{1}{2}m(m-1)} . \Sigma (-1)^{\omega} P$$
$$= (-1)^{\frac{1}{2}m(m-1)} . \Delta.$$

Case II. *Inversion of short rows.*

$$\Delta' = \left| \begin{matrix} a_{1n} & \dots & a_{12} & a_{11} \\ a_{2n} & \dots & a_{22} & a_{21} \\ \dots & \dots & \dots & \dots \\ a_{mn} & \dots & a_{m2} & a_{m1} \end{matrix} \right| .$$

In this case we have

$$\Delta' = \Delta \times (-1)^{m(n-m) + \frac{1}{2}m(m-1)} \qquad \text{....................(2).}$$

This is seen by putting $r = m$ in the result II of § 15.

Case III. *Inversion of both sets of rows.*

$$\Delta' = \left| \begin{matrix} a_{mn} & \dots & a_{m2} & a_{m1} \\ \dots & \dots & \dots & \dots \\ a_{2n} & \dots & a_{22} & a_{21} \\ a_{1n} & \dots & a_{12} & a_{11} \end{matrix} \right| .$$

In this case

$$\Delta' = \Delta \times (-1)^{m(n-m)} \qquad \text{.........................(3).}$$

This is seen by putting $r = m$ in the result III of § 15.

The first case can easily be deduced from § 9, Ex. ii or from § 11, Ex. iii.

We can bring the long row with suffix m in Δ to the leading position by transposing it in succession with each of the $m-1$ long rows preceding it. We can then bring the long row with suffix $m-1$ to the second position by transposing it in succession with each of the $m-2$ long rows preceding it. Proceeding in this way, we can convert Δ into Δ' by $\frac{1}{2}m(m-1)$ interchanges of pairs of long rows.

Since at each of these interchanges the sign of the determinoid is changed, we conclude that

$$\Delta' = \Delta \times (-1)^{\frac{1}{2}m(m-1)}.$$

The second case can be deduced, though less easily, from the results of § 11.

CHAPTER III.

SEQUENCES AND THE AFFECTS OF DERIVED SEQUENCES.

[The first two articles, §§ 17 and 18, contain definitions and a description of various ways of obtaining the affects of derived sequences. The next article, § 19, contains a number of simple theorems concerning the affects of derived sequences. Most of these are introduced to facilitate the proofs of theorems in Chapter IV and the following chapters, and will be referred to as occasion arises. The alteration in the affect of a derived sequence produced by the interchange of any two elements is investigated in §§ 20 and 21. In § 22 the properties of the affects of derived products are shown to be deducible from the properties of the affects of derived sequences.]

§ 17. Definitions.

A *sequence* is any linear arrangement of *elements*, which are letters or numbers, such as

$$A = [a_1 a_2 \dots a_i \dots a_n] \quad \dots\dots\dots\dots(1).$$

It can be regarded as a matrix with only one long row.

The passages from any one element a_i of the sequence to the adjacent element on the right or left of it are called respectively a *forward step* and a *backward step*.

The *affect of any element* a_i of the sequence A is the number of forward steps from the *leading element* a_1 up to the element a_i, or it is the number of elements which occur in A before a_i.

The affects of the successive elements of A are therefore $0, 1, 2, \dots (n-1)$.

Let A be regarded as a *fundamental sequence*, and let

$$B = [b_1 b_2 \dots b_k \dots b_r] \quad \dots\dots\dots\dots(2)$$

be any sequence of r of the elements of A, arranged in any order, and let B be regarded as a *derived sequence*.

We may call B a *corranged* or a *deranged* derived sequence according as the elements which occur in it have or have not the same relative order as in A. If B contains all the elements of A, it will be called a *complete derived sequence*, or a *derangement* of A; if it does not contain all the elements of A, it will be called an *incomplete derived sequence* or a *minor sequence*.

If B is a minor sequence, it can be *extended* by adding to the right of the last of its elements one or more of the remaining elements of A arranged in the same relative order as in A. It is said to be *completed* when *all* the remaining elements of A are added to it in this manner.

The *affect of the derived sequence B in the fundamental sequence A* can be defined as follows:

(1) Count the number of forward steps in A from the leading element a_1 up to the element b_1.

(2) Strike out the element b_1 from A, and in the sequence that remains count the number of forward steps from the leading element up to the element b_2.

(3) Strike out the elements b_1 and b_2 from A, and in the sequence that remains count the number of forward steps from the leading element up to the element b_3.

. . . .

(k) Strike out the elements $b_1, b_2, \dots b_{k-1}$ from A, and in the sequence that remains count the number of forward steps from the leading element up to the element b_k.

. . . .

(r) Strike out the elements $b_1, b_2, \dots b_{r-1}$ from A, and in the sequence that remains count the number of forward steps from the leading element up to the element b_r.

The sum of the r numbers thus obtained, i.e. the total number of steps counted, is called the affect of B in A.

If the affect of b_k in the sequence A_{k-1} formed from A by striking out the elements $b_1, b_2, \dots b_{k-1}$ is called *the affect of b_k in A relative to B*, then the affect of B in A is the sum of the relative affects of all the elements $b_1, b_2, \dots b_r$ occurring in B.

It is useful to observe that *the affect of a derived sequence is not altered when it is extended or completed.* For if

$$B' = [b_1 b_2 \dots b_k \dots b_r \, c_1 c_2 \dots c_s] \dots\dots\dots\dots\dots\dots\dots (3)$$

is the sth extension of B in A, the relative affect of b_k when we are finding the affect of B' in A is the same as the relative affect of b_k when we are finding the affect of B in A, and the relative affects of the added elements $c_1, c_2, \dots c_s$ in the former case are all zero.

Two corranged derived sequences are said to be *complementary* with respect to the fundamental sequence A, when each of them is derived from A by striking out the elements which occur in the other.

Ex. i. Let $A=[abcdef]$, $B=[dfb]$.

The successive extensions and the completion of B in A are

$$B_1=[dfba],\quad B_2=[dfbac],\quad B_3=[dfbace].$$

The affect of B in A $=3+4+1$ $=8$.

The affect of B_1 in A $=3+4+1+0$ $=8$.

The affect of B_2 in A $=3+4+1+0+0$ $=8$.

The affect of B_3 in A $=3+4+1+0+0+0=8$.

Ex. ii. The sequences $[bdf]$, $[ace]$ are complementary corranged minor sequences of the fundamental sequence $[abcdef]$.

A transposition or interchange of two *consecutive* elements of a derived sequence is called a *forward* or a *backward move* according as the relative order of the two elements before transposition is or is not the same as their relative order in the fundamental sequence.

Ex. iii. Let $[abcdef]$ be the fundamental sequence

and $[deac]$ be a derived sequence.

The transposition of the elements c and a in the derived sequence, which converts it into $[dace]$, is a backward move.

Again, the transposition of the elements a and e in the derived sequence, which converts it into $[dcea]$, is a forward move.

§ 18. Expressions for the affect of a derived sequence.

1. *Standard suffix notation.*

If a standard suffix notation is employed, we may denote the fundamental sequence by

$$A=[a_1a_2\dots a_n]=[a]_n$$

and the derived sequence by

$$B=[a_{x_1}a_{x_2}\dots a_{x_k}\dots a_{x_r}]=[a_x]_r$$

where $x_1, x_2, \dots x_r$ is any arrangement of r of the suffixes $1, 2, \dots n$.

Let ξ_k be the number of the suffixes $x_1, x_2, \dots x_r$ occurring in B which are less than x_k and precede x_k in B.

Then the relative affect of the element a_{x_k} is

$$(x_k-1)-\xi_k,$$

and therefore the affect of B in A is

$$\sum_{k=1}^{k=r}\{(x_k-1)-\xi_k\}\ \dots\dots\dots\dots\dots\dots\dots\dots\dots(A).$$

By means of this formula, the affect of B in A can be at once determined from an inspection of the suffixes occurring in B.

Ex. i. Let $A=[a_1a_2a_3a_4a_5a_6]$, $B=[a_3a_5a_2a_6]$.

Of the suffixes 3, 5, 2, 6 occurring in B there are

none which are smaller than 3 and precede 3,
one which is smaller than 5 and precedes 5,
none which are smaller than 2 and precede 2,
three which are smaller than 6 and precede 6.

Hence the relative affects of the successive elements of B in A are

$$(3-1)-0, \quad (5-1)-1, \quad (2-1)-0, \quad (6-1)-3,$$

or

$$2, \ 3, \ 1, \ 2.$$

Accordingly the total affect of B in A is

$$2+3+1+2=8.$$

2. *Any notation.*

Let the fundamental sequence be

$$A=[a_1a_2 \dots a_n]=[a]_n,$$

and the derived sequence be

$$B=[b_1b_2 \dots b_k \dots b_r]=[b]_r,$$

so that $b_1, b_2, \dots b_r$ is some arrangement of r of the elements $a_1, a_2, \dots a_n$.

Then the relative affect of b_k

= (*the number of elements which precede* b_k *in* A)

− (*the number of elements which precede* b_k *both in* A *and in* B) ......(B).

Adding all the relative affects, we obtain the total affect of B in A.

The formula (B) can be used in conjunction with any notation for the elements of the sequence.

Ex. ii. Let $A=[abcdef]$, $B=[dcfae]$.

The relative affects of the successive elements of B in A are

$$3-0, \quad 2-0, \quad 5-2, \quad 0-0, \quad 4-3.$$

Therefore the affect of B in A is

$$3+2+3+0+1=9.$$

If B is a *corranged* minor sequence of A, then any element which precedes b_k in B must also precede b_k in A.

Hence the above formula can in this case be put into the simpler form:

The relative affect of b_k

$=$ (*the number of elements which precede* b_k *in* A)

$-$ (*the number of elements which precede* b_k *in* B) ....(C).

Ex. iii. Let $A=[abcdef]$, $B=[bde]$.

The relative affects of the successive elements of B in A are

$$1-0, \quad 3-1, \quad 4-2.$$

Therefore the affect of B in A is

$$1+2+2=5.$$

3. *General suffix notation.*

When the most general suffix notation is employed, we may denote the fundamental sequence by

$$A=[a_{p_1}a_{p_2}\dots a_{p_n}]=[a_p]_n,$$

and the derived sequence by

$$B=[a_{x_1}a_{x_2}\dots a_{x_r}]=[a_x]_r,$$

where $x_1, x_2, \dots x_r$ is any arrangement of r of the suffixes $p_1, p_2, \dots p_n$.

Since the affect of B in A depends only on the positions of its successive elements relative to the successive elements of A, it is clear that the affect of B in A is the same as the affect of $[x_1 x_2 \dots x_r]$ in $[p_1 p_2 \dots p_n]$.

Hence in determining the affect of B in A, we can replace the sequences A and B by the sequences formed by their suffixes........................(D).

Ex. iv. The affect of $[a_8a_5a_6]$ in $[a_2a_5a_6a_8a_9]$

$=$the affect of $[8\,5\,6]$ in $[2\,5\,6\,8\,9]$

$=3+1+1=5.$

4. *Complete derived sequences.*

Let the fundamental sequence be

$$A=[a_1 a_2 \dots a_n]=[a]_n,$$

and let any *complete* derived sequence be

$$B=[b_1 b_2 \dots b_k \dots b_n]=[b]_n.$$

In this case the elements occurring in B are identical with the elements occurring in A. Hence referring to formula (B) we see that *the relative affect of b_k is the number of elements which precede b_k in A and follow b_k in B.*

Any two elements whose relative order in B is the reverse of their relative order in A will be said to form an *inversion* in B.

Then the relative affect of b_k is equal to the number of inversions formed by b_k and the elements which follow it in B.

Also the total affect of B is equal to the total number of inversions occurring in B. ....(E).

Since the affect of a derived sequence remains the same when it is completed, we see that *the affect of any derived sequence can be found by completing it and then counting the number of inversions.*

Ex. v. Let $A=[abcde]$, and $B=[dcaeb]$.

The inversions in B are

$$(dc),\ (da),\ (db),\ (ca),\ (cb),\ (eb).$$

The number of these being six, the affect of B in A is 6.

Again, the inversions formed with c and the elements following it in B are (ca), (cb). The number of these being two, the relative affect of c in A is 2.

Ex. vi. Let $A=[abcdef]$, and $B=[dcfae]$.

The completed sequence of B is $B'=[dcfaeb]$.

The inversions in B' are

$$(dc),\ (da),\ (db),\ (ca),\ (cb),\ (fa),\ (fe),\ (fb),\ (eb).$$

Their number is nine. Therefore the affect of B in A is 9.

5. *Sequences of natural numbers.*

Let the fundamental sequence S be the sequence of the first n natural numbers, so that

$$S=[1\ 2\ 3\ \dots\ n],$$

and let the derived sequence be

$$X=[x_1\, x_2 \dots x_k \dots x_r]=[x]_r,$$

so that $x_1, x_2, \dots x_r$ is any arrangement of r of the numbers $1, 2, \dots n$.

If ξ_k is the number of integers less than x_k which precede x_k in X, then the relative affect of x_k is

$$(x_k-1)-\xi_k,$$

and the total affect of X in S is

$$\sum_{k=1}^{k=r}\{(x_k-1)-\xi_k\} \text{ ....(F).}$$

The fundamental and derived sequences can always be reduced to the above forms in determining the affect, since the affects depend only on the positions of the elements in the two sequences and not on the symbols used for them. To reduce the fundamental and derived sequences $A=[a_1 a_2 \dots a_n]$, $B=[b_1 b_2 \dots b_r]$ to these forms, we may substitute 1 for a_1, 2 for a_2, ... n for a_n in both sequences; or we may at once replace A by $[1\ 2\ 3 \dots n]$, and in B replace each element by the number denoting its position in A.

Ex. vii. The affect of [2 4 1 5] in [1 2 3 4 5 6]

$$=(2-1-0)+(4-1-1)+(1-1-0)+(5-1-3)$$
$$=1+2+0+1=4.$$

For in the series 2, 4, 1, 5, there are

no integers less than 2 and preceding 2,
one integer less than 4 and preceding 4,
no integers less than 1 and preceding 1,
three integers less than 5 and preceding 5.

Ex. viii. The affect of [dca] in [$abcde$]

$$= \text{the affect of } [4\ 3\ 1] \text{ in } [1\ 2\ 3\ 4\ 5]$$
$$= 3+2+0=5.$$

Here we substitute 1, 2, 3, 4, 5 for a, b, c, d, e respectively in both sequences; or we observe that d, c, a are respectively the 4th, 3rd, and 1st elements in [$abcde$].

Ex. ix. The affect of [2 8 6 1 3] in [1 6 3 2 8]

$$= \text{the affect of } [4\ 5\ 2\ 1\ 3] \text{ in } [1\ 2\ 3\ 4\ 5]$$
$$= 3+3+1+0+0=7.$$

Ex. x. The affect of $[a_3 a_5 a_2 a_6]$ in $[a_1 a_2 a_3 a_4 a_5 a_6]$

$$= \text{the affect of } [3\ 5\ 2\ 6] \text{ in } [1\ 2\ 3\ 4\ 5\ 6]$$
$$= 2+3+1+2=8.$$

When the derived sequence X above is a *corranged* minor of S, we have

$$\xi_k = k-1.$$

Therefore the affect of a corranged minor X in S

$$= \sum_{k=1}^{k=r} \{(x_k-1)-(k-1)\} = \sum_{k=1}^{k=r} (x_k - k)$$
$$= (x_1+x_2+\dots+x_r) - \tfrac{1}{2}r(r+1) \dots\dots\dots\dots\dots\dots (G).$$

It is often required to find the signs determined by the affects of all corranged derived sequences of order r. After the sign has been found for any one sequence, it can be found for any other sequence by simply noticing whether $x_1+x_2+\dots+x_r$ is even or odd.

Ex. xi. The corranged minors of order 3 of the sequence $[a_1 a_2 a_3 a_4 a_5]$ are

$[a_1 a_2 a_3]$, $[a_1 a_2 a_4]$, $[a_1 a_2 a_5]$, $[a_1 a_3 a_4]$, $[a_1 a_3 a_5]$,
$[a_1 a_4 a_5]$, $[a_2 a_3 a_4]$, $[a_2 a_3 a_5]$, $[a_2 a_4 a_5]$, $[a_3 a_4 a_5]$.

The affect of the first of these minors is 0, the corresponding sign is +, and the sum of the suffixes is even. The sign determined by the affect of any other minor is + or − according as the sum of its suffixes is even or odd.

Hence the signs determined by the affects of the above minors are

$$\begin{matrix} + & - & + & + & - \\ + & - & + & - & +. \end{matrix}$$

Ex. xii. Let the fundamental sequence be $[a_1 a_2 a_3 a_4 a_5 a_6]$.

The affect of the minor $[a_1 a_2 a_3 a_4]$ is 0, an even number, and the sum of the suffixes in it is odd.

We conclude that the affects of the corranged minors $[a_2 a_4 a_5 a_6]$, $[a_1 a_3 a_4 a_6]$ are respectively even and odd, since the sums of their suffixes are respectively odd and even.

§ 19. Theorems concerning the affects of derived sequences.

Theorem Ia. *If ω is the affect of a complete derived sequence B in the fundamental sequence A, then A can be converted into B by forward moves, and the total number of such moves is always ω.*

Let $A = [a_1\, a_2 \dots a_n]$, and $B = [b_1\, b_2 \dots b_n]$,

so that $b_1, b_2, \dots b_n$ is some other arrangement of the elements $a_1, a_2, \dots a_n$.

Starting with $\boldsymbol{A}$, we can bring b_1 to the leading position by a succession of forward moves. We can then bring b_2 to the second position by a succession of forward moves. We can then proceed to bring b_3 to the third position by forward moves, b_4 to the fourth position by forward moves, and so on. This is one way of converting A into B by moves which are all forward. There are clearly in general many other ways in which this can be done.

Now whatever procedure is adopted, we always start with the sequence A, which has affect 0 in A and therefore no inversions, and finish with the sequence B, which has affect ω in A and therefore ω inversions. Further each forward move increases the number of inversions by 1. Therefore the total number of forward moves must always be equal to ω.

Ex. i. Let $A = [abcde]$, and $B = [dcaeb]$.

The affect of B in A is then 6.

The results of six successive forward moves converting A into B are

$$[acbde],\ [acdbe],\ [acdeb],\ [cadeb],\ [cdaeb],\ [dcaeb].$$

Theorem Ib. *If ω is the affect of a minor sequence B in the fundamental sequence A, then B can be brought to the leading position in A by forward moves without at any stage disturbing the relative order of those elements of A which do not occur in B, and the total number of such moves is always ω.*

It will be observed that each of the applied moves consists in the transposition of two adjacent elements, at least one of which belongs to B. There is no transposition of two elements, neither of which occurs in B.

Let $A=[a_1 a_2 \dots a_n]$, and $B=[b_1 b_2 \dots b_r]$.

Also let $C=[b_1 b_2 \dots b_r c_1 c_2 \dots c_{n-r}]$

be the sequence obtained by completing B in A, so that $c_1, c_2, \dots c_{n-r}$ are those elements of A which do not occur in B, arranged in the same relative order as in A.

Starting with A, we can bring b_1 to the leading position by forward moves without disturbing the relative order of $c_1, c_2, \dots c_{n-r}$. We can then bring b_2 to the second position by forward moves without disturbing the relative order of the c's. We can then proceed to bring b_3 to the third position, b_4 to the fourth position, ..., and finally b_r to the rth position without disturbing the relative order of the c's. This is one way of effecting the required transformation of A, and clearly many other ways are possible.

However we proceed, we always start with the sequence A, which has affect 0 in A and no inversions, and finish with the sequence C, which has affect ω in A and therefore ω inversions. Now each forward move increases the number of inversions by 1. Therefore the total number of forward moves employed must always be equal to ω.

Ex. ii. Let $A=[1\,2\,3\,4\,5\,6]$, and $B=[3\,2\,5]$.

Then $C=[3\,2\,5\,1\,4\,6]$; and the affects of B and C in A are 5.

The results of five successive forward moves bringing B to the leading position in A are

$$[1\,2\,3\,5\,4\,6],\quad [1\,3\,2\,5\,4\,6],\quad [3\,1\,2\,5\,4\,6],\quad [3\,2\,1\,5\,4\,6],\quad [3\,2\,5\,1\,4\,6].$$

Theorem II a. *If ω is the affect of a complete derived sequence B in the fundamental sequence A, and if A is converted into B by forward and backward moves, then:*

(1) *The least number of moves by which the conversion can be effected is ω.*

(2) *The total number of moves, forward and backward, always exceeds ω by an even positive number, which may be 0.*

(3) *The total number of forward moves always exceeds the total number of backward moves by ω.*

The following considerations serve to establish the above results:

The first move is necessarily a forward move.

A forward move always introduces one, and only one, new inversion.

A backward move always takes away one, and only one, existing inversion.

Since there are ω inversions in B, at least ω forward moves must be employed.

If a forward move is employed which introduces an inversion not occurring in B, then this inversion must be subsequently removed by a backward move.

If a backward move removes an inversion occurring in B, then that inversion must subsequently be reintroduced by a forward move.

Ex. iii. Let $A=[abcde]$, and $B=[dcaeb]$.

Then the affect of B in A is 6.

The results of fourteen successive moves transforming A into B are

$$[acbde],\ [acbed],\ [abced],\ [abecd],\ [aebcd],\ [aecbd],\ [aecdb],$$
$$[eacdb],\ [ecabd],\ [ceadb],\ [cedab],\ [cdeab],\ [dceab],\ [dcaeb].$$

Here we have used ten forward moves corresponding to the transpositions

$$(bc),\ (de),\ (ce),\ (be),\ (bc),\ (bd),\ (ae),\ (ac),\ (ad),\ (cd),$$

and four backward moves corresponding to the transpositions

$$(cb),\ (ec),\ (ed),\ (ea).$$

Theorem II b. *If ω is the affect of a minor sequence B in the fundamental sequence **A**, and if B is brought to the leading position in A by forward and backward moves without causing any final alteration in the relative order of the elements of **A** which do not occur in B, then:*

(1) *The least number of moves by which this can be done is ω.*

(2) *The total number of moves, forward and backward, always exceeds ω by an even positive number, which may be 0.*

(3) *The total number of forward moves always exceeds the total number of backward moves by ω.*

This can be deduced from Theorem II*a* by replacing B by its completed sequence B'; for to bring B to the leading position in A in the prescribed manner, it is necessary and sufficient to convert A into B' by moves applied to the elements of A.

Ex. iv. Let $A=[abcde]$, and $B=[dea]$,

so that $\omega=5$.

Then $B'=[dcabe]$.

The required transformation can be effected by the eleven moves corresponding to the successive transpositions

$$(cd),\ (ce),\ (ab),\ (ad),\ (bd),\ (ae),\ (ac),\ (ec),\ (bc),\ (ea),\ (ba).$$

Here there are eight forward moves and three backward moves corresponding respectively to the transpositions

$$(cd),\ (ce),\ (ab),\ (ad),\ (bd),\ (ae),\ (ac),\ (bc)$$

and $(ec),\ (ea),\ (ba)$.

The various stages of the transformation are

$$[abcde],\ [abdce],\ [abdec],\ [badec],\ [bdaec],\ [dbaec],$$
$$[dbeac],\ [dbeca],\ [dbcea],\ [dcbea],\ [dcbae],\ [dcabe].$$

Theorem III. *In determining the affect of any derived sequence we can associate its elements together in any manner so as to form subsidiary sequences, and express the affect as the sum of the partial affects of the various subsidiary sequences.*

Let the fundamental sequence be

$$A = [a_1 a_2 \dots a_n],$$

and let the derived sequence be

$$S = [b_1 b_2 \dots b_p c_1 c_2 \dots c_q d_1 d_2 \dots d_r \dots],$$

so that $b_1, b_2, \dots c_1, c_2, \dots d_1, d_2, \dots$ is some arrangement of elements selected from $a_1, a_2, \dots a_n$.

Then the affect of S in A

= the affect of $[b_1 b_2 \dots b_p]$ in A

+ the affect of $[c_1 c_2 \dots c_q]$ in the sequence obtained from A by striking out the elements $b_1, b_2, \dots b_p$

+ the affect of $[d_1 d_2 \dots d_r]$ in the sequence obtained from A by striking out the elements $b_1, b_2, \dots b_p$ and the elements $c_1, c_2, \dots c_q$

+ etc.

This follows immediately from the definition of the affect of S in A.

Ex. v. The affect of $[dhefjgbe]$ in $[abcdefghij]$ is equal to the sum of the affects of $[dh]$ in $[abcdefghij]$, $[cfj]$ in $[abcefgij]$, g in $[abegi]$, and $[be]$ in $[abei]$.

In fact $24 = 9 + 10 + 3 + 2$.

Ex. vi. Let a fundamental sequence and any one of its derived sequences be

$$A = [a_1 a_2 \dots a_n], \text{ and } B = [b_1 b_2 \dots b_r].$$

Let the completed sequence of B be

$$B' = [b_1 b_2 \dots b_r c_1 c_2 \dots c_{n-r}].$$

Then the affect of B' in A

= the affect of $[b_1 b_2 \dots b_r]$ in $[a_1 a_2 \dots a_n]$

+ the affect of $[c_1 c_2 \dots c_{n-r}]$ in $[c_1 c_2 \dots c_{n-r}]$

= (the affect of B in A) + 0

= the affect of B in A.

Theorem IV. *If B is any corranged derived sequence of a fundamental sequence A, the relative affects of the remaining elements of B are unaltered when any of the elements occurring in B are struck out in both A and B.*

That this is true when any one common element of A and B is struck out is at once clear from formula (C) of § 18.

It follows that it is true when any number of common elements are struck out.

Conversely if $B=[b_1b_2\ldots b_r]$ is any derived sequence of $A=[a_1a_2\ldots a_n]$, and if an additional element x, not occurring in A, is inserted between b_i and b_{i+1} both in B and in A, thus forming new sequences B' and A', then the relative affect of every element of B' except x when we are finding the affect of B' in A' is the same as its relative affect when we are finding the affect of B in A. Consequently the affect of B' in A' exceeds the affect of B in A by the affect of x in A' relative to B'.

Ex. vii. The affect of $[bcf]$ in $[abcdefg]$

$$= 1+1+3 = 5.$$

The affect of $[bcxf]$ in $[abcdxefg]$

$$= 1+1+2+3 = 5+2 = 7.$$

The affect of $[bcxyf]$ in $[abcdxeyfg]$

$$= 1+1+2+3+3 = 7+3 = 10.$$

Theorem V a. *If B is any corranged derived sequence of the fundamental sequence A, and if B' is any derangement of B, then*

$$\textit{aff. } B' \textit{ in } A = \textit{aff. } B' \textit{ in } B + \textit{aff. } B \textit{ in } A.$$

Here *aff. B in A* is used as an abbreviation for *the affect of B in A*.

Let $A=[a_1a_2\ldots a_n]$, $B=[b_1b_2\ldots b_r]$, $B'=[\beta_1\beta_2\ldots\beta_r]$.

Then B is the sequence which remains when $n-r$ elements are struck out from A, and B' is formed by re-arranging the elements of B.

Let ω, ω', η be respectively the affects of B in A, of B' in A, and of B' in B.

We have to show that $\omega' = \omega + \eta$.

Let $[c_1c_2\ldots c_{n-r}]$ be the sequence which is left when the elements of B (or B') are struck out from A.

By Theorems Ia and Ib, since forward moves with respect to B are forward moves with respect to A,

ω forward moves change $[a_1a_2\ldots a_n]$ into $[b_1b_2\ldots b_r c_1c_2\ldots c_{n-r}]$;

also η forward moves change $[b_1b_2\ldots b_r]$ into $[\beta_1\beta_2\ldots\beta_r]$,

and therefore change $[b_1b_2\ldots b_r c_1c_2\ldots c_{n-r}]$ into $[\beta_1\beta_2\ldots\beta_r c_1c_2\ldots c_{n-r}]$.

Thus $\omega+\eta$ forward moves change $[a_1a_2\ldots a_n]$ into $[\beta_1\beta_2\ldots\beta_r c_1c_2\ldots c_{n-r}]$.

Consequently $\omega+\eta$ is the affect of $[\beta_1\beta_2\ldots\beta_r c_1c_2\ldots c_{n-r}]$ in $[a_1a_2\ldots a_n]$.

But $[\beta_1\beta_2\ldots\beta_r c_1c_2\ldots c_{n-r}]$ is the completed sequence of B' in A, and has the same affect in A as B'.

Therefore $\omega + \eta$ is the affect of B' in A,

i.e.

$$\omega + \eta = \omega'.$$

Ex. viii. Let $A=[abcdef]$, $B=[acde]$, $B'=[dcea]$.

Then the affect of B' in B is 4, the affect of B in A is 3, and the affect of B' in A is 7.

Since

$$7=3+4,$$

the equation of the theorem is satisfied.

Theorem V b. *If B and B' are any two derived sequences formed of the same elements differently arranged, and if A is the fundamental sequence, then*

$$\text{aff. } B' \text{ in } B + \text{aff. } B \text{ in } A = \text{aff. } B' \text{ in } A + 2k$$

where k is a positive integer.

Let $A=[a_1a_2\dots a_n]$, $B=[b_1b_2\dots b_r]$, $B'=[\beta_1\beta_2\dots\beta_r]$,

where the elements of B' are the same as the elements of B, but differently arranged, and let $[c_1c_2\dots c_{n-r}]$ be the sequence which is left when the elements occurring in B and B' are struck out from A.

Let ω, ω' and η be respectively the affects of B in A, of B' in A, and of B' in B.

We can convert $[a_1a_2\dots a_n]$ into $[b_1b_2\dots b_r c_1c_2\dots c_{n-r}]$ by ω moves, all of which are forward moves with respect to A.

We can convert $[b_1b_2\dots b_r]$ into $[\beta_1\beta_2\dots\beta_r]$, and therefore

$$[b_1b_2\dots b_r c_1c_2\dots c_{n-r}] \text{ into } [\beta_1\beta_2\dots\beta_r c_1c_2\dots c_{n-r}]$$

by η moves all of which are forward moves with respect to B. These moves will in general be some forward and some backward with respect to A.

Hence we can convert $[a_1a_2\dots a_n]$ into $[\beta_1\beta_2\dots\beta_r c_1c_2\dots c_{n-r}]$ by $\omega+\eta$ moves, some forward and some backward with respect to A.

Now since $[\beta_1\beta_2\dots\beta_r c_1c_2\dots c_{n-r}]$ is the completion of B' in A, the affect of $[\beta_1\beta_2\dots\beta_r c_1c_2\dots c_{n-r}]$ in $[a_1a_2\dots a_n]$ is ω'.

Therefore by Theorem IIa, $\omega+\eta$ exceeds ω' by an even integer, and we have

$$\omega + \eta = \omega' + 2k$$

where k is a positive integer.

Ex. ix. Let $A=[abcde]$, $B=[caed]$, so that $\omega=5$.

If $B'=[daec]$, then $\eta=5$, $\omega'=6$, and therefore $\omega+\eta=\omega'+4$.

If $B'=[deca]$, then $\eta=4$, $\omega'=7$, and therefore $\omega+\eta=\omega'+2$.

If $B'=[ecad]$, then $\eta=2$, $\omega'=7$, and therefore $\omega+\eta=\omega'$.

Theorem VI. *If the elements of a sequence* $A=[a_1a_2\ldots a_n]$ *are divided in any manner into three groups of corranged elements, so as to form three corranged sequences* $U=[u_1u_2\ldots u_\lambda]$, $V=[v_1v_2\ldots v_\mu]$, $W=[w_1w_2\ldots w_\nu]$; *and if moreover a corranged sequence* $P=[p_1p_2\ldots p_{\lambda+\mu}]$ *is formed with the elements of* U *and* V *taken together, and a corranged sequence* $Q=[q_1q_2\ldots q_{\mu+\nu}]$ *is formed with the elements of* V *and* W *taken together; then*

$$\text{aff. } P \text{ in } A + \text{aff. } U \text{ in } P = \text{aff. } U \text{ in } A + \text{aff. } V \text{ in } Q.$$

A forward move with respect to any one of these sequences is a forward move with respect to A.

Let the affect of P in $A=\omega_1$, the affect of U in $P=\omega_2$,
the affect of U in $A=\eta_1$, the affect of V in $Q=\eta_2$.

Then

ω_1 forward moves convert A into $[p_1p_2\ldots p_{\lambda+\mu}\, w_1w_2\ldots w_\nu]$,

and ω_2 forward moves convert this into $[u_1u_2\ldots u_\lambda\, v_1v_2\ldots v_\mu\, w_1w_2\ldots w_\nu]$.

Therefore

$\omega_1+\omega_2$ forward moves convert A into $[u_1u_2\ldots u_\lambda\, v_1v_2\ldots v_\mu\, w_1w_2\ldots w_\nu]$.

Again

η_1 forward moves convert A into $[u_1u_2\ldots u_\lambda\, q_1q_2\ldots q_{\mu+\nu}]$,

and η_2 forward moves convert this into $[u_1u_2\ldots u_\lambda\, v_1v_2\ldots v_\mu\, w_1w_2\ldots w_\nu]$.

Therefore

$\eta_1+\eta_2$ forward moves convert A into $[u_1u_2\ldots u_\lambda\, v_1v_2\ldots v_\mu\, w_1w_2\ldots w_\nu]$.

Thus both $\omega_1+\omega_2$ and $\eta_1+\eta_2$ are equal to the affect of

$$[u_1u_2\ldots u_\lambda\, v_1v_2\ldots v_\mu\, w_1w_2\ldots w_\nu]$$

in A, and therefore $\omega_1+\omega_2=\eta_1+\eta_2.$

This proves the theorem.

The above equation simply expresses the fact that the number of forward moves which must be applied to A to bring first W to the rear position and then U to the leading position is equal to the number of forward moves which must be applied to A to bring first U to the leading position and then W to the rear position.

Ex. x. Let $A=[1\,2\,3\,4\,5\,6\,7\,8\,9]$,

$U=[2\,6\,8]$, $V=[1\,3\,5\,9]$, $W=[4\,7]$.

Then $P=[1\,2\,3\,5\,6\,8\,9]$, $Q=[1\,3\,4\,5\,7\,9]$.

In this case $\omega_1=\text{aff. } P \text{ in } A=6$, $\eta_1=\text{aff. } U \text{ in } A=10$,

$\omega_2=\text{aff. } U \text{ in } P=7$, $\eta_2=\text{aff. } V \text{ in } Q=3$.

Thus $\omega_1+\omega_2=\eta_1+\eta_2=13.$

Theorem VII a. *If A and B are two sequences formed by different arrangements of the same elements, then*

$$\textit{aff. } B \textit{ in } A = \textit{aff. } A \textit{ in } B.$$

To prove this it is sufficient to observe that if a pair of elements have (or have not) the same relative order in B as in A, then they have (or have not) the same relative order in A as in B. Thus the number of inversions in B when A is fundamental is equal to the number of inversions in A when B is fundamental.

Ex. xi. Let $A=[abcdef]$, $B=[dfaceb]$.

Then aff. B in A = aff. A in B = 9.

Theorem VII b. *If A and B are two sequences formed by different arrangements of the same elements, then the affect of either sequence in the other is unaltered when both sequences are reversed.*

Let $b_1, b_2, \ldots b_n$ be the same elements as $a_1, a_2, \ldots a_n$ differently arranged, and let

$$A = [a_1\, a_2 \ldots a_n], \quad B = [b_1\, b_2 \ldots b_n].$$

By reversing these sequences we obtain

$$A' = [a_n\, a_{n-1} \ldots a_1], \quad B' = [b_n\, b_{n-1} \ldots b_1].$$

The theorem states that

$$\text{aff. } B' \text{ in } A' = \text{aff. } B \text{ in } A.$$

To prove this theorem we observe that if the relative order of two elements a_r, a_s in B is the reverse of (or the same as) their relative order in A, then the relative order of the elements a_s, a_r in B' is the reverse of (or the same as) their relative order in A'. Thus the number of inversions in B' when A' is the fundamental sequence is the same as the number of inversions in B when A is the fundamental sequence. Consequently the affect of B' in A' is equal to the affect of B in A.

The theorem can also be proved as follows. When A is the fundamental sequence we can by forward moves, starting with the sequence A, first bring b_n to the last position, then bring b_{n-1} to the last position but one, then bring b_{n-2} to the last position but two, and proceed in this way until A is converted into B. Each of these moves consists in the transposition of two adjacent elements. If we start with A' and in the same succession transpose exactly the same pairs of elements, we shall convert A' into B'. Also these latter transpositions will now be all forward moves when A' is regarded as the fundamental sequence. Thus the number of forward moves required to convert A' into B' when A' is the fundamental sequence is equal to the number of forward moves required to convert A into B when A is the fundamental sequence. That is, the affect of B' in A' is equal to the affect of B in A.

Ex. xii. Let $A = [abcd]$, $B = [dacb]$.

Then $A' = [dcba]$, $B' = [bcad]$.

In this case aff. B in A = aff. B' in $A' = 4$.

When A is fundamental the inversions in B are

$$(da),\quad (dc),\quad (db),\quad (cb).$$

When A' is fundamental the inversions in B' are

$$(bc),\quad (bd),\quad (cd),\quad (ad).$$

Again, when A is fundamental the results of four successive forward moves are

$$[acbd],\quad [acdb],\quad [adcb],\quad [dacb].$$

Also, when A' is fundamental the results of the corresponding moves are

$$[dbca],\quad [bdca],\quad [bcda],\quad [bcad].$$

Theorem VIII a. *The sum of the affects of two complementary corranged minor sequences is equal to the product of their orders.*

For the purpose of determining the affects we may by sub-article 5 of § 18 reduce the fundamental sequence A and the two complementary minor sequences X and Y to the forms

$$A = [1\ 2\ 3\ \dots\ n],\quad X = [x_1\, x_2 \dots x_i \dots x_r],\quad Y = [y_1\, y_2 \dots y_j \dots y_{r'}].$$

Then X is any sequence of r of the numbers 1, 2, ... n arranged in ascending order of magnitude, and Y is the sequence formed from A by striking out the numbers which occur in X, and consists of r' of the numbers 1, 2, ... n arranged in ascending order of magnitude, where

$$r + r' = n \qquad\text{.......................(1).}$$

If ω and ω' are the affects of X and Y in $\boldsymbol{A}$, we have to show that

$$\omega + \omega' = rr' \qquad\text{.......................(2).}$$

Now by formula (C) of § 18, the affect of x_i in A relative to X

= (the number of integers which precede x_i in A)

− (the number of integers which precede x_i in X)

= $(x_i - 1) - (i - 1) = x_i - i$.

Summing for all values of i from 1 to r, or using formula (G) of § 18, we see that

$$\omega = (x_1 + x_2 + \dots + x_r) - \tfrac{1}{2} r (r + 1).$$

Similarly we have

$$\omega' = (y_1 + y_2 + \dots + y_{r'}) - \tfrac{1}{2} r' (r' + 1).$$

Since
$$(x_1+x_2+\ldots+x_r)+(y_1+y_2+\ldots+y_{r'})$$
$$=1+2+3+\ldots+n=\tfrac{1}{2}n(n+1),$$
it follows that
$$\omega+\omega'=\tfrac{1}{2}n(n+1)-\tfrac{1}{2}r(r+1)-\tfrac{1}{2}r'(r'+1).$$

Making use of the relation (1), we have
$$\omega+\omega'=\tfrac{1}{2}(r+r')(r+r'+1)-\tfrac{1}{2}r(r+1)-\tfrac{1}{2}r'(r'+1)$$
$$=rr'.$$

Ex. xiii. [*acdg*] and [*bef*] are complementary corranged minors of orders 4 and 3 of the sequence [*abcdefg*].

The affect of [*acdg*] in [*abcdefg*]

= the affect of [1 3 4 7] in [1 2 3 4 5 6 7] = 5.

The affect of [*bef*] in [*abcdefg*]

= the affect of [2 5 6] in [1 2 3 4 5 6 7] = 7.

The sum of these two affects = 12 = 4 × 3.

Theorem VIII b. *If B and C are two complementary corranged minor sequences of the fundamental sequence A, and if A′ and C′ are the sequences obtained from A and C by reversing the orders of their elements, then*

$$\textit{aff. } B \textit{ in } A=\textit{aff. } C' \textit{ in } A'.$$

We shall make use of the theorem that the affect of a minor sequence is not altered by completing the sequence, and also of Theorem VII *b*.

Let
$$A=[a_1\,a_2\ldots a_n],\quad B=[b_1\,b_2\ldots b_r],\quad C=[c_1\,c_2\ldots c_{n-r}],$$
so that
$$A'=[a_n\ldots a_2\,a_1],\quad C'=[c_{n-r}\ldots c_2\,c_1].$$

Then the affect of B in A

= the affect of $[b_1\,b_2\ldots b_r]$ in $[a_1\,a_2\ldots a_n]$

= the affect of $[b_1\,b_2\ldots b_r\,c_1\,c_2\ldots c_{n-r}]$ in $[a_1\,a_2\ldots a_n]$

= the affect of $[c_{n-r}\ldots c_2\,c_1\,b_r\ldots b_2\,b_1]$ in $[a_n\ldots a_2\,a_1]$

= the affect of $[c_{n-r}\ldots c_2\,c_1]$ in $[a_n\ldots a_2\,a_1]$

= the affect of C' in A'.

Ex. xiv. [*acdg*], [*bef*] are complementary corranged minors of the sequence [*abcdefg*].

The affect of [*acdg*] in [*abcdefg*] = 5,

and the affect of [*feb*] in [*gfedcba*] = 5.

Again, the affect of [*bef*] in [*abcdefg*] = 7,

and the affect of [*gdca*] in [*gfedcba*] = 7.

§ 20. Change in the affect of a derived sequence B caused by the interchange of two elements which are consecutive in the fundamental sequence A.

It may be observed that the results of this and the following article are contained in §§ 9—12, since a sequence is a matrix with one long row. The proofs given there, depending on the method of counting steps, could be repeated here with the appropriate simplifications. We shall however give independent proofs depending on the method of counting inversions explained in sub-article 4 of § 18.

Let the fundamental sequence be

$$A = [a_1 a_2 \dots a_i a_{i+1} \dots a_n],$$

and let a_i, a_{i+1} be the two consecutive elements which are interchanged.

CASE I. *Let the operation of interchanging a_i and a_{i+1} be applied to the fundamental sequence A but not to the derived sequence B.*

Let A become A' when a_i and a_{i+1} are interchanged, so that

$$A' = [a_1 a_2 \dots a_{i+1} a_i \dots a_n].$$

We have to find the change in passing from the affect of B in A to the affect of B in A′.

The results which will be obtained are as follows:

(1) *If neither a_i nor a_{i+1} occurs in B,*

then the affect of B in A is unaltered; moreover the relative affect of every element of B is unaltered.

(2) *If a_i occurs in B and a_{i+1} either does not occur or occurs later than a_i,*

then the affect of B in A is increased by 1; moreover the relative affect of a_i is increased by 1, and the relative affect of every other element of B is unaltered.

(3) *If a_{i+1} occurs in B and a_i either does not occur or occurs later than a_{i+1},*

then the affect of B in A is diminished by 1; moreover the relative affect of a_{i+1} is diminished by 1, and the relative affect of every other element of B is unaltered.

In (2) the interchanged elements have the same relative order in the completed sequence of B as in A.

In (3) the interchanged elements have reverse relative orders in the completed sequence of B and in A.

Proof of (1). In this case the successive elements of B occupy the same positions in A' as in A. Consequently each element of B has the same relative affect in A' as in A, and the total affect of B in A' is equal to the total affect of B in A.

Proof of (2). The sequence obtained by completing B is in this case the same when A' is the fundamental sequence as it is when A is the fundamental sequence. Let this completed sequence be C. Then C contains the elements a_i, a_{i+1} in the same relative order as A.

Now every pair of elements belonging to A (or C), except the pair (a_i, a_{i+1}), have the same relative order in A' as in A. Therefore every two such elements have or have not the same relative order in C as in A' according as they have or have not the same relative order in C as in A. That is, they do or do not form an inversion in C when A' is the fundamental sequence according as they do or do not form an inversion in C when A is the fundamental sequence. Accordingly any difference in the inversions in C in the two cases can only arise from the pair of elements (a_i, a_{i+1}). As regards these two elements, they have reverse relative orders in C and A', and the same relative orders in C and A. They form an inversion in C when A' is the fundamental sequence, but not when A is the fundamental sequence. Thus C has one more inversion, that formed by the elements (a_i, a_{i+1}), when A' is the fundamental sequence than it has when A is the fundamental sequence. Therefore the affect of C in A' is greater by 1 than the affect of C in A, and consequently the affect of B in A' is greater by 1 than the affect of B in A.

Again, if x is any element of B except a_i, it appears from the above that the inversions in C formed with x and the elements occurring later than x in C are the same when A' is the fundamental sequence as when A is the fundamental sequence. Remembering that C is the completion of B both in A and in A', it follows from sub-article 4 of § 18 that the relative affect of x in A' is the same as the relative affect of x in A.

As for the element a_i of B, the element a_{i+1} occurs later than it in C, and the pair of elements (a_i, a_{i+1}) form an inversion in C when A' is the fundamental sequence but not when A is the fundamental sequence. With this exception, the inversions formed with a_i and the elements occurring later than a_i in C are the same in the two cases. Thus the total number of inversions formed with a_i and the elements following a_i in C is greater by 1 when A' is the fundamental sequence than when A is the fundamental sequence. That is, the relative affect of the element a_i of B is greater by 1 in A' than in A.

Proof of (3). A similar argument holds good in this case. Let C be the sequence obtained by completing B in A. It is also the sequence obtained by completing B in A'.

The two elements (a_{i+1}, a_i), of which a_i occurs later than a_{i+1} in C, form an inversion in C when A is the fundamental sequence, but not when A' is the fundamental sequence. With this exception the inversions in C are the same in both cases. Hence the relative affect of the element a_{i+1} of B is less by 1 in A' than in A, and every other element of B has the same relative affect in A' as in A.

Note. If (1) is proved in the same way as (2) and (3), it must be observed that in that case the completed sequences of B are different according as A or A' is regarded as the fundamental sequence.

Ex. i. The only element of B whose relative affect is changed is that one of the two interchanged elements which occurs first in B. The relative affect of that one element is increased or diminished by 1 according as it is the earlier or the later of the two interchanged elements in B.

Ex. ii. Let $A=[ab\overset{*}{c}\overset{*}{d}efg]$, $A'=[ab\overset{*}{d}\overset{*}{c}efg]$,

so that A' is derived from A by interchanging the two consecutive elements c and d.

The affects of a derived sequence B in A and in A' are given in the table below for five values of B illustrative of the five different cases which can occur.

B	Affect of B in A	Affect of B in A'
$agfe$	$0+5+4+3=12$	$0+5+4+3=12$
$f\overset{*}{c}e\overset{*}{d}b$	$5+\overset{*}{2}+3+\overset{*}{2}+1=13$	$5+\overset{*}{3}+3+\overset{*}{2}+1=14$
$f\overset{*}{c}eb$	$5+\overset{*}{2}+3+1=11$	$5+\overset{*}{3}+3+1=12$
$g\overset{*}{d}fa\overset{*}{c}$	$6+\overset{*}{3}+4+0+\overset{*}{1}=14$	$6+\overset{*}{2}+4+0+\overset{*}{1}=13$
$g\overset{*}{d}fa$	$6+\overset{*}{3}+4+0=13$	$6+\overset{*}{2}+4+0=12$

CASE II. *Let the operation of interchanging a_i and a_{i+1} be applied to the derived sequence B but not to the fundamental sequence A.*

Let B' be the sequence into which B is converted when the two elements a_i and a_{i+1} are interchanged.

We have to find the change in passing from the affect of B in **A** *to the affect of B' in A.*

If the change in passing from the relative affect of the kth element of B in A to the relative affect of the kth element of B' in A is called the change of the relative affect in the kth place of B, we can enunciate the following rules:

(1) *If neither a_i nor a_{i+1} occurs in B,*

then the affect of B in A is unaltered; moreover the relative affect in each place of B is unaltered.

(2) *If a_i occurs in B, and a_{i+1} either does not occur or occurs later than a_i,*

then the affect of B in A is increased by 1; *moreover the relative affect in the place of B occupied by a_i is increased by* 1, *and the relative affect in every other place of B is unaltered.*

(3) *If a_{i+1} occurs in B, and a_i either does not occur or occurs later than a_{i+1},*

then the affect of B in A is diminished by 1; *moreover the relative affect in the place of B occupied by a_{i+1} is diminished by* 1, *and the relative affect in every other place of B is unaltered.*

In (2) the interchanged elements have the same relative order in the completed sequence of B as in A.

In (3) the interchanged elements have reverse relative orders in the completed sequence of B and in A.

Proof. Let A' be the sequence obtained by interchanging the elements a_i and a_{i+1} in A.

If the operation of interchanging a_i and a_{i+1} is applied simultaneously to B' and to A, these are converted into B and A' respectively. Hence the successive elements of B' occupy in A the same positions as the successive elements of B occupy in A'.

It follows from this property that the affect of B' in A is equal to the affect of B in A'. Consequently the change in passing

from the affect of B in A to the affect of B' in A

is the same as the change in passing

from the affect of B in A to the affect of B in A'.

The last change is known, having been found in Case I.

Again, it follows from the property mentioned above that the relative affect of the kth element of B' in A is equal to the relative affect of the kth element of B in A'. Consequently the change in passing

from the relative affect of the kth element of B in A
to the relative affect of the kth element of B' in A

is the same as the change in passing

from the relative affect of the kth element of B in A
to the relative affect of the kth element of B in A'.

This last change is known, having been found in Case I.

Thus the rules stated above follow immediately from the corresponding rules obtained in Case I.

Ex. iii. There is only one place in which the relative affect is changed, viz. the place which is occupied in B by that one of the two interchanged elements which occurs first in B. The relative affect in that place is increased or diminished by 1 according as the element which occupies it in B is the earlier or the later of the two interchanged elements in the fundamental sequence.

Ex. iv. Let

$$A=[ab\overset{*}{c}\overset{*}{d}efg],$$

and let the derived sequence B be converted into B' when the two elements c and d are interchanged.

The affects of B and B' in A are given in the table below for five representative values of B.

Derived sequences	Affects in A
$B\ = agfe$	$0+5+4+3=12$
$B' = agfe$	$0+5+4+3=12$
$B\ = f\overset{*}{e}e\overset{*}{d}b$	$5+\overset{*}{2}+3+\overset{*}{2}+1=13$
$B' = f\overset{*}{d}e\overset{*}{c}b$	$5+\overset{*}{3}+3+\overset{*}{2}+1=14$
$B\ = f\overset{*}{e}eb$	$5+\overset{*}{2}+3+1=11$
$B\ = f\overset{*}{d}eb$	$5+\overset{*}{3}+3+1=12$
$B\ = g\overset{*}{d}fa\overset{*}{e}$	$6+\overset{*}{3}+4+0+\overset{*}{1}=14$
$B' = g\overset{*}{e}fa\overset{*}{d}$	$6+\overset{*}{2}+4+0+\overset{*}{1}=13$
$B\ = g\overset{*}{d}fa$	$6+\overset{*}{3}+4+0=13$
$B' = g\overset{*}{e}fa$	$6+\overset{*}{2}+4+0=12$

Comparison should be made with Ex. ii.

§ 21. Change in the affect of a derived sequence caused by the interchange of any two elements of the fundamental sequence.

Let the fundamental sequence be

$$A = [a_1 a_2 \ldots a_i \ldots a_j \ldots a_n],$$

where $i < j$, so that the element a_i occurs earlier in A than the element a_j, and let a_i, a_j be the two elements which are interchanged.

Let those elements of A which lie between a_i and a_j be called *inner elements*, and let those which lie to the left or to the right of both a_i and a_j be called *outer elements*.

Let r be the number of inner elements.

Let the derived sequence be B. It may contain neither, or one only, or both of the interchanged elements a_i, a_j.

If a_i occurs in B, let s_i be the number of inner elements which precede it in B.

If a_j occurs in B, let s_j be the number of inner elements which precede it in B.

If a_i and a_j both occur in B, let ρ be the number of inner elements lying between them in B, so that $\rho = s_j - s_i$ or $\rho = s_i - s_j$ according as a_i precedes or follows a_j in B.

CASE I. *Let the operation of interchanging a_i and a_j be applied to the fundamental sequence A but not to the derived sequence B.*

Let the sequence obtained by interchanging a_i and a_j in A be A', so that

$$A' = [a_1 a_2 \dots a_j \dots a_i \dots a_n],$$

where the elements not shown are the same in A' as in A.

We have to find the change in passing from the affect of B in A to the affect of B in A'.

The results which will be obtained are as follows:

(1) *If neither a_i nor a_j occurs in B,*
then the affect of B in A is unaltered, and the relative affect of every element of B is unaltered.

(2) *If a_i occurs in B, and a_j either does not occur or occurs later than a_i, then:*

(i) *The relative affect of a_i is increased by $r + 1 - s_i$.*

(ii) *The relative affect of a_j is diminished by $r - s_j$.*

(iii) *The relative affect of each inner element which occurs in B after a_i and before a_j is increased by 1.*

(iv) *The relative affects of all other elements of B are unaltered.*

(v) *If both a_i and a_j occur in B, the total affect of B in A is increased by $2\rho + 1$.*

(3) *If a_j occurs in B, and a_i either does not occur or occurs later than a_j, then:*

(i) *The relative affect of a_j is diminished by $r + 1 - s_j$.*

(ii) *The relative affect of a_i is increased by $r - s_i$.*

(iii) *The relative affect of each inner element which occurs in B after a_j and before a_i is diminished by 1.*

(iv) *The relative affects of all other elements of B are unaltered.*

(v) *If both a_j and a_i occur in B, the total affect of B in A is diminished by $2\rho + 1$.*

In (2) the two interchanged elements have the same relative order in the completed sequence of B as in A.

In (3) the two interchanged elements have reverse relative orders in the completed sequence of B and in A.

It may be observed that $r - s_i$ is the number of inner elements which do not precede a_i in B, or the number of inner elements which follow a_i in the completed sequence of B.

Similarly $r - s_j$ is the number of inner elements which do not precede a_j in B, or the number of inner elements which follow a_j in the completed sequence of B.

We may regard (1) as contained both in (2) and in (3).

Proof of (1). In this case the successive elements of B occupy the same positions in A' as in A. Therefore the affect of B in A' is equal to the affect of B in A. Also each element of B has the same relative affect in A' as in A.

Proof of (2). Let $$B = [b_1 b_2 \dots a_i \dots b_m].$$

The element a_j, if it occurs in B, occurs later than a_i.

Let the results of completing B with respect to A and A' be respectively

$$C = [b_1 b_2 \dots a_i \dots b_m \, c_1 c_2 \dots c_{n-m}],$$
$$C' = [b_1 b_2 \dots a_i \dots b_m \, c_1' c_2' \dots c'_{n-m}].$$

The elements $c_1', c_2', \dots c'_{n-m}$ are the same as the elements $c_1, c_2, \dots c_{n-m}$, but there is a difference in their arrangement in the two cases if a_j occurs amongst them.

Let x be any element of B, and let y be any element of C *which occurs later than* x *in* C. Then y is also an element of C' occurring later than x in C'.

When A is the fundamental sequence, the affect of x relative to B is equal to the affect of x relative to C and is the number of inversions of the type (xy) in C.

When A' is the fundamental sequence, the affect of x relative to B is equal to the affect of x relative to C' and is the number of inversions of the type (xy) in C'. This again is equal to the affect of x relative to C and is the number of inversions of the type (xy) in C.

Hence the change in the relative affect of x when A' is made fundamental instead of A is *the change in the number of inversions of the type* (xy) *in* C, y being any element occurring later than x in C.

This is what we shall proceed to determine.

For given values of x and y, an inversion (xy) is either introduced or removed in C when x and y have different relative orders in A and A', but there is no addition or removal of an inversion (xy) when x and y have the same relative order in A' as in A.

Now the two elements x, y have the same relative order in A' as in A when one of them is an outer element and when they are both inner elements. Again, the two elements x, y have reverse relative orders in A and A' when and only when one at least of them is one of the elements a_i, a_j and the other is either a_i or a_j or an inner element.

Hence there is either an addition or a removal of an inversion (xy) in C when and only when the elements x and y satisfy the last restriction.

Let x occur before a_i in B.

If x is an outer element, no inversion of the type (xy) can be introduced or removed in C, and there is therefore no change in the relative affect of x.

If x is an inner element, an inversion of the type (xy) is either introduced or removed in C when and only when y is one of the elements a_i, a_j. One inversion (xa_i) is removed and one inversion (xa_j) is introduced. For (xa_i) is an inversion in C when A is fundamental, but not when A' is fundamental. Also (xa_j) is an inversion in C when A' is fundamental but not when A is fundamental. We see then that when x is an inner element, there is no change in the total number of inversions of the type (xy) occurring in C, and that there is therefore no change in the relative affect of x.

Thus in every case the relative affect of x is unchanged.

Let x occur after a_j in B.

In this case neither x nor y can be either a_i or a_j. Consequently no inversion of the type (xy) is either introduced or removed in C, and the relative affect of x is unchanged.

Let x occur after a_i and before a_j in B.

If x is an outer element, no inversion of the type (xy) is either introduced or removed in C, and the relative affect of x is unaltered.

If x is an inner element, one inversion (xa_j) is introduced into C, and no other inversion of the type (xy) is either introduced or removed in C; consequently the relative affect of x is increased by 1.

Let x be the element a_i.

If y is an outer element, no inversion (xy) is either introduced or removed in C.

If y is the element a_j, the inversion (xy) or (xa_j) is introduced in C.

If y is an inner element (occurring after a_i in C), an inversion (xy) is introduced in C. The number of such introduced inversions is equal to the number of inner elements occurring in C after a_i, i.e. is equal to $r-s_i$.

Thus altogether $r+1-s_i$ inversions of the type (xy) are introduced and no inversions of the type (xy) are removed in C.

Accordingly in this case the relative affect of x is increased by $r+1-s_i$.

Let x be the element a_j.

If y is an outer element, no inversion (xy) is either introduced or removed in C.

The element y cannot be a_i, since a_i occurs before a_j in C.

If y is an inner element (occurring after a_j in C), an inversion (xy) or (a_jy) is removed from C. The number of such removed inversions is equal to the number of inner elements occurring in C after a_j, i.e. is equal to $r-s_j$.

Thus in this case $r-s_j$ inversions of the type (xy) are removed from C, and no inversion of the type (xy) is introduced into C.

Accordingly the relative affect of x is diminished by $r-s_j$.

We have now proved the results (i), (ii), (iii), (iv); and the result (v) can be immediately deduced from these. Thus (2) is completely proved.

Proof of (3). A proof similar to the proof of (2) can be used. But it is easier to proceed as follows.

In Case (3) let ω be the algebraic increase in the relative affect of any element x of B when we pass from A as fundamental sequence to A' as fundamental sequence, and let ω' be the algebraic increase of the element x of B when we pass from A' as fundamental sequence to A as fundamental sequence.

Then $$\omega = -\omega'.$$

Now ω' can be found from the rules (2). And when ω' is known, ω is known.

Thus the results (3) can be immediately deduced from the results (2).

Ex. i. The only elements whose relative affects are changed are the interchanged elements and those elements which are inner elements both of the fundamental sequence and of the derived sequence.

Ex. ii. Let the fundamental sequence be

$$A=[ab\overset{*}{c}\dot{d}\dot{e}\dot{f}\overset{*}{g}h]$$

and let this be converted by the interchange of the elements c and g into

$$A'=[ab\overset{*}{g}\dot{d}\dot{e}\dot{f}\overset{*}{c}h].$$

In the following table the affects of the derived sequence B in A and A' are given for five representative values of B. The interchanged elements are marked by asterisks and the inner elements by dots.

B	Affect of B in A	Affect of B in A'	r	s_i	s_j	ρ
$\dot{d}\dot{f}bh\dot{e}$	$\dot{3}+\dot{4}+1+4+\dot{2}$ $=14$	$\dot{3}+\dot{4}+1+4+\dot{2}$ $=14$	3	—	—	—
$\dot{f}\overset{*}{c}a\dot{d}\overset{*}{g}b$	$\dot{5}+\overset{*}{2}+0+\dot{1}+\overset{*}{2}+0$ $=10$	$\dot{5}+\overset{*}{5}+0+\dot{2}+\overset{*}{1}+0$ $=13$	3	1	2	1
$\dot{f}b\overset{*}{c}a\dot{e}h$	$\dot{5}+1+\overset{*}{1}+0+\dot{1}+2$ $=10$	$\dot{5}+1+\overset{*}{4}+0+\dot{2}+2$ $=14$	3	1	—	—
$b\overset{*}{g}\dot{e}\dot{d}\overset{*}{c}\dot{f}$	$1+\overset{*}{5}+\dot{3}+\dot{2}+\overset{*}{1}+\dot{1}$ $=13$	$1+\overset{*}{1}+\dot{2}+\dot{1}+\overset{*}{2}+\dot{1}$ $=8$	3	2	0	2
$b\dot{d}\overset{*}{g}\dot{e}h\dot{f}$	$1+\dot{2}+\overset{*}{4}+\dot{2}+3+\dot{2}$ $=14$	$1+\dot{2}+\overset{*}{1}+\dot{1}+3+\dot{1}$ $=9$	3	—	1	—

Case II. *Let the operation of interchanging a_i and a_j be applied to the derived sequence B but not to the fundamental sequence A.*

Let B' be the sequence into which B is converted when the elements a_i and a_j are interchanged.

We have to find the change in passing from the affect of B in A to the affect of B' in A.

If the change in passing from the relative affect of the kth element of B in A to the relative affect of the kth element of B' in A is called the change of the relative affect in the kth place of B, we can enunciate the following rules:

(1) *If neither a_i nor a_j occurs in B,*
then the affect of B in A is unaltered, and the relative affect in every place of B is unaltered.

(2) *If a_i occurs in B, and a_j either does not occur or occurs later than a_i in B, then:*

(i) *The relative affect in the place of B occupied by a_i is increased by $r+1-s_i$.*

(ii) *The relative affect in the place of B occupied by a_j is diminished by $r-s_j$.*

(iii) *The relative affect in the place of each inner element occurring in B after a_i and before a_j is increased by 1.*

(iv) *The relative affect in every other place of B is unaltered.*

(v) *If both a_i and a_j occur in B, the total affect of B is increased by $2\rho+1$.*

(3) *If a_j occurs in B, and a_i either does not occur or occurs later than a_j in B, then:*

(i) *The relative affect in the place of B occupied by a_j is diminished by $r+1-s_j$.*

(ii) *The relative affect in the place of B occupied by a_i is increased by $r-s_i$.*

(iii) *The relative affect in the place of each inner element occurring in B after a_j and before a_i is diminished by 1.*

(iv) *The relative affect in every other place of B is unaltered.*

(v) *If both a_j and a_i occur in B, the total affect of B is diminished by $2\rho+1$.*

In (2) the interchanged elements have the same relative order in the completed sequence of B as in A.

In (3) they have reverse relative orders in the completed sequence of B nd in A.

Proof. Let A' be the sequence obtained by interchanging a_i and a_j in A. Then if the operation of interchanging a_i and a_j is applied to both B' and A, they are converted into B and A' respectively. Hence the successive elements of B' occupy in A the same positions as the successive elements of B occupy in A'. Therefore the relative affect of the kth element of B' in A is equal to the relative affect of the kth element of B in A'. Consequently the change in passing

from the relative affect of the kth element of B in A
to the relative affect of the kth element of B' in A

is the same as the change in passing

from the relative affect of the kth element of B in A
to the relative affect of the kth element of B in A'.

Since this latter change has already been investigated in Case I, the rules just given follow immediately from the results obtained in Case I.

Ex. iii. The only places in which the relative affects are changed are the places of B occupied by the interchanged elements and the places of B occupied by those elements which are inner elements both of A and of B.

Ex. iv. Let the fundamental sequence be

$$A=[ab\overset{*}{c}\dot{d}\dot{e}\dot{f}\overset{*}{g}h]$$

and let the derived sequence B be converted into B' when the elements c and g are interchanged.

In the table below the affects of B and B' in A are given for five representative values of B.

The interchanged elements are marked by asterisks and the inner elements by dots.

	Affects in A	r	s_i	s_j	ρ
$B=\dot{d}\dot{f}bh\dot{e}$	$\dot{3}+\dot{4}+1+4+\dot{2}=14$	3		—	—
$B'=\dot{d}\dot{f}bh\dot{e}$	$\dot{3}+\dot{4}+1+4+\dot{2}=14$				
$B=\dot{f}\overset{*}{c}a\dot{d}\overset{*}{g}b$	$\dot{5}+\overset{*}{2}+0+\dot{1}+\overset{*}{2}+0=10$	3	1	2	1
$B'=\dot{f}\overset{*}{g}a\dot{d}\overset{*}{c}b$	$\dot{5}+\overset{*}{5}+0+\dot{2}+\overset{*}{1}+0=13$				
$B=\dot{f}b\overset{*}{c}a\dot{e}h$	$\dot{5}+1+\overset{*}{1}+0+\dot{1}+2=10$	3	1	—	—
$B'=\dot{f}b\overset{*}{g}a\dot{e}h$	$\dot{5}+1+\overset{*}{4}+0+\dot{2}+2=14$				
$B=b\overset{*}{g}\dot{e}\dot{d}\overset{*}{c}\dot{f}$	$1+\overset{*}{5}+\dot{3}+\dot{2}+\overset{*}{1}+\dot{1}=13$	3	2	0	2
$B'=b\overset{*}{c}\dot{e}\dot{d}\overset{*}{g}\dot{f}$	$1+\overset{*}{1}+\dot{2}+\dot{1}+\overset{*}{2}+\dot{1}=8$				
$B=b\dot{d}\overset{*}{g}\dot{e}h\dot{f}$	$1+\dot{2}+\overset{*}{4}+\dot{2}+3+\dot{2}=14$	3	—	1	—
$B'=b\dot{d}\overset{*}{c}\dot{e}h\dot{f}$	$1+\dot{2}+\overset{*}{1}+\dot{1}+3+\dot{1}=9$				

Comparison should be made with Ex. ii.

§ 22. Reduction of the affects of derived products to the affects of sequences.

Let
$$A = [a]_m^n = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

be any matrix, and let

$$P = [a_{x_1 y_1} a_{x_2 y_2} \dots a_{x_k y_k} \dots a_{x_r y_r}]$$

be any one of its derived products.

With the notation of § 9.1 the affect of P in A is

$$\sum_{k=1}^{k=r} \{(x_k - 1) - \xi_k\} + \sum_{k=1}^{k=r} \{(y_k - 1) - \eta_k\}.$$

Here the first term is the vertical affect of P in A, and the second term is the horizontal affect of P in A.

Now by § 19.5 the first term

= the affect of the sequence $[x_1 x_2 \dots x_r]$ in the sequence $[1\ 2 \dots m]$

= the affect of the sequence $[a_{x_1 1} a_{x_2 1} \dots a_{x_r 1}]$ in the sequence

$$[a_{11} a_{21} \dots a_{m1}].$$

Also the second term

= the affect of the sequence $[y_1 y_2 \dots y_r]$ in the sequence $[1\ 2 \dots n]$

= the affect of the sequence $[a_{1 y_1} a_{1 y_2} \dots a_{1 y_r}]$ in the sequence

$$[a_{11} a_{12} \dots a_{1n}].$$

From the first results in each case we see that

the vertical affect of P in A is the affect of the sequence formed by the vertical suffixes of P in the sequence formed by the vertical suffixes of A,

and that

the horizontal affect of P in A is the affect of the sequence formed by the horizontal suffixes of P in the sequence formed by the horizontal suffixes of A ....(A).

The second results in each case can be expressed in a form independent of the notation used for the elements of the matrix.

Let V be the sequence formed from P by replacing each element in it by the element of the leading vertical row of A which lies in the same horizontal row as that element.

Let H be the sequence formed from P by replacing each element in it by the element of the leading horizontal row of A which lies in the same vertical row as that element.

Then

the vertical affect of P in A is the affect of the sequence V in the sequence formed by the leading vertical row of A,

and

the horizontal affect of P in A is the affect of the sequence H in the sequence formed by the leading horizontal row of A ....(B).

We may call V *the projection of the derived product P on the leading vertical row of A,*

and we may call H *the projection of the derived product P on the leading horizontal row of A.*

It will be clear that

the relative vertical affect of any element of P in A is equal to the relative affect of the corresponding element of the sequence V in the sequence of the leading vertical row of A,

and that

the relative horizontal affect of any element of P in A is equal to the relative affect of the corresponding element of the sequence H in the sequence of the leading horizontal row of A ....(C).

By means of these results, all properties of the affects of derived products can be deduced from the properties of the affects of sequences. In particular the properties contained in §§ 9—12 can be deduced from the properties contained in §§ 20 and 21.

Ex. i. If $A = [a]_8^5$, and $P = a_{13}a_{35}a_{61}a_{84}$,

then the vertical affect of P in A

$= $ the affect of $[a_{41}a_{31}a_{61}a_{81}]$ in $[a_{11}a_{21}a_{31}a_{41}a_{51}a_{61}a_{71}a_{81}]$

$= $ the affect of $[4\,3\,6\,8]$ in $[1\,2\,3\,4\,5\,6\,7\,8]$

$= 12$,

and the horizontal affect of P in A

$= $ the affect of $[a_{13}a_{15}a_{11}a_{14}]$ in $[a_{11}a_{12}a_{13}a_{14}a_{15}]$

$= $ the affect of $[3\,5\,1\,4]$ in $[1\,2\,3\,4\,5]$

$= 6$.

Therefore the total affect of P in $A = 12 + 6 = 18$.

Ex. ii. The affect of $[a_{44}a_{81}a_{73}a_{15}]$ in $[a]_8^5$

$= $ the affect of $[4\,8\,7\,1]$ in $[1\,2\,3\,4\,5\,6\,7\,8]$

$+ $ the affect of $[4\,1\,3\,5]$ in $[1\,2\,3\,4\,5]$

$= 14 + 5 = 19$.

Ex. iii. If $$A = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{bmatrix}, \text{ and } P = d_3 a_2 b_1 e_4,$$

then the vertical affect of P in A

= the affect of $[a_3 a_2 a_1 a_4]$ in $[a_1 a_2 a_3 a_4]$
= the affect of $[3\ 2\ 1\ 4]$ in $[1\ 2\ 3\ 4] = 3$,

and the horizontal affect of P in A

= the affect of $[d_1 a_1 b_1 e_1]$ in $[a_1 b_1 c_1 d_1 e_1]$
= the affect of $[dabe]$ in $[abcde]$
= the affect of $[4\ 1\ 2\ 5]$ in $[1\ 2\ 3\ 4\ 5] = 4$.

Thus the total affect of P in $A = 3 + 4 = 7$.

Ex. iv. If $$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}, \text{ and } P = gid,$$

then the affect of P in A

= the affect of $[eia]$ in $[aei]$ + the affect of $[cad]$ in $[abcd]$
= the affect of $[2\ 3\ 1]$ in $[1\ 2\ 3]$ + the affect of $[3\ 1\ 4]$ in $[1\ 2\ 3\ 4]$
$= 2 + 3 = 5$.

CHAPTER IV.

AFFECTS OF DERIVED MATRICES AND DERIVED DETERMINOIDS.

[The first two articles, §§ 23 and 24, contain the essential properties of the affects of derived matrices and determinoids. They are deducible from the properties of the affects of derived sequences. The last two articles, §§ 25 and 26, contain a number of theorems concerning such affects. Most of these are introduced to facilitate the proofs of theorems in Chapter V and the following chapters, and will be referred to as occasion arises.]

§ 23. Definitions.

Any matrix (or any determinoid) will be called a *fundamental matrix* (or a *fundamental determinoid*) with respect to the derived matrices and the derived determinoids belonging to it.

Let
$$A = [a]_m^n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{.........................(1)}$$

be a fundamental matrix, and let

$$B = [a_{xy}]_\mu^\nu = \begin{bmatrix} a_{x_1y_1} & a_{x_1y_2} & \cdots & a_{x_1y_\nu} \\ a_{x_2y_1} & a_{x_2y_2} & \cdots & a_{x_2y_\nu} \\ \cdots & \cdots & \cdots & \cdots \\ a_{x_\mu y_1} & a_{x_\mu y_2} & \cdots & a_{x_\mu y_\nu} \end{bmatrix} \quad \text{..................(2)}$$

be any derived matrix belonging to it.

Then the *affect of the derived matrix B in the fundamental matrix A* will be defined to be the quantity ω given by the equations

ω' = *the affect of the sequence* $[x_1\ x_2 \ldots x_\mu]$ *in the sequence* $[1\ 2 \ldots m]$

= *the affect of the sequence of the vertical suffixes of B in the sequence of the vertical suffixes of A*

= *the affect of the sequence determined by the horizontal rows of B in the sequence determined by the horizontal rows of A;*

ω'' = *the affect of the sequence* $[y_1\, y_2 \dots y_\nu]$ *in the sequence* $[1\ 2 \dots n]$

= *the affect of the sequence of the horizontal suffixes of B in the sequence of the horizontal suffixes of A*

= *the affect of the sequence determined by the vertical rows of B in the sequence determined by the vertical rows of A*;

$\omega = \omega' + \omega''$ ....(A).

The number ω' will be called the *vertical affect of B in A*, or the *affect of the horizontal rows of B in the horizontal rows of A*.

The number ω'' will be called the *horizontal affect of B in A*, or the *affect of the vertical rows of B in the vertical rows of* **A**.

The number ω will be called the *total affect of B in* **A** or simply the *affect of B in A*.

Thus the total affect of B in A is the sum of the vertical and horizontal affects of B in A.

Since the derived matrix B is formed from the fundamental matrix **A** by striking out certain of the vertical and horizontal rows and re-arranging the retained rows amongst themselves, it follows that to each given vertical (or horizontal) row of B there corresponds one of the vertical (or horizontal) rows of A which has not been struck out. It is that row of A which contains all the elements of the given row of B. With respect to the given row of B, it may be called the *corresponding* row of A. Thus if $[a_{x_i y_1}\, a_{x_i y_2} \dots a_{x_i y_\nu}]$ is any given horizontal row of B, the corresponding horizontal row of A is $[a_{x_i 1}\, a_{x_i 2} \dots a_{x_i n}]$. The foregoing definitions can then be expressed in the following forms which are independent of the notation used for the elements of the matrices:

(i) *If there are m horizontal rows in the fundamental matrix A, and if the horizontal rows of B taken in order correspond to the* x_1*th*, x_2*th*, ... x_μ*th horizontal rows of A, then the vertical affect of B in A is equal to the affect of*

$$[x_1\, x_2 \dots x_\mu] \text{ in } [1\ 2 \dots m].$$

(ii) *If there are n vertical rows in the fundamental matrix A, and if the vertical rows of B taken in order correspond to the* y_1*th*, y_2*th*, ... y_ν*th vertical rows of A, then the horizontal affect of B in* **A** *is equal to the affect of*

$$[y_1\, y_2 \dots y_\nu] \text{ in } [1\ 2 \dots n].$$

(iii) *The total affect of B in A is the sum of the vertical and horizontal affects of B in A* ....(B).

Again since a horizontal or vertical row of A or B is completely determined by two elements lying in it, the row in A which corresponds to a given row of B may in a certain sense be called the *same* row of A. Then by § 19.5, the definitions of the affects may be worded as follows:

(i) *The vertical affect of B in A is the affect of the derived sequence formed by any vertical row of B in the fundamental sequence formed by the same (or corresponding) vertical row of A.*

(ii) *The horizontal affect of B in A is the affect of the derived sequence formed by any horizontal row of B in the fundamental sequence formed by the same (or corresponding) horizontal row of A.*

(iii) *The total affect of B in A is the sum of its vertical and horizontal affects in A* ....(C).

The *projection* of a given horizontal (or vertical) row of B on the leading vertical (or horizontal) row of A will be understood to be that element of the leading row of A which lies in A in the same horizontal (or vertical) row as the elements of the given row of B, or it is that element of the leading vertical (or horizontal) row of A which lies in the row of A corresponding to the given horizontal (or vertical) row of B. Accordingly the definitions of the affects can be expressed in the following very useful forms.

(i) *The vertical affect of B in A is the affect of the derived sequence formed by the projections of the successive horizontal rows of B on the leading vertical row of A in the fundamental sequence formed by the leading vertical row of A.*

(ii) *The horizontal affect of B in A is the affect of the derived sequence formed by the projections of the successive vertical rows of B on the leading horizontal row of A in the fundamental sequence formed by the leading horizontal row of A.*

(iii) *The total affect of B in A is the sum of its vertical and horizontal affects in A* ....(D).

The definition of the affect of B in A remains unaltered when A is a fundamental matrix and B a derived determinoid: also when A is a fundamental determinoid and B is a derived matrix or a derived determinoid.

It should be observed that the affect of a derived matrix B is not in general equal to the affect of the leading derived product of B. For if $A = [a]_m^n$, $B = [a_{xy}]_\mu^\nu$ as in (1) and (2) above, and if ξ_k is the number of vertical suffixes occurring in B which are less than x_k and precede x_k in B, and if η_k is the number of horizontal suffixes occurring in B which are less than y_k and precede y_k in B:

then the affect of B in A

$$=\sum_{k=1}^{k=\mu}\{(x_k-1)-\xi_k\}+\sum_{k=1}^{k=\nu}\{(y_k-1)-\eta_k\},$$

whereas the affect of the leading derived product of B in A

$$=\sum_{k=1}^{k=\mu}\{(x_k-1)-\xi_k\}+\sum_{k=1}^{k=\mu}\{(y_k-1)-\eta_k\}$$

$$\text{or}\quad \sum_{k=1}^{k=\nu}\{(x_k-1)-\xi_k\}+\sum_{k=1}^{k=\nu}\{(y_k-1)-\eta_k\}$$

according as $\mu<\nu$ or $\nu<\mu$.

Thus the two affects are in general different except when $\mu=\nu$. They are equal when $\mu=\nu$, and also in the case considered in § 24, Ex. ii.

A square matrix derived from a given fundamental matrix is completely determined, i.e. every one of its elements is known, when its leading product is known. Also its affect is the same as the affect of its leading product. Hence it is often convenient to represent a square matrix symbolically by its leading product. Similarly a determinant can be completely represented by its leading derived product.

Ex. i. Let $$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\end{bmatrix},\quad B=\begin{bmatrix} a_{32} & a_{33}\\ a_{12} & a_{13}\\ a_{22} & a_{23}\end{bmatrix},$$

and let ω', ω'', ω be the vertical, horizontal and total affects of B in A.

Then ω' = the affect of [3 1 2] in [1 2 3] = 2;

ω'' = the affect of [2 3] in [1 2 3 4] = 2;

$\omega = 2+2=4$.

Ex. ii. Let

$$A=\begin{bmatrix} a\\ b\\ c\\ d\\ e\end{bmatrix}_{123456}=\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6\\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6\\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6\\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6\\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6\end{bmatrix},\quad B=\begin{bmatrix} e\\ b\\ d\end{bmatrix}_{4632}=\begin{bmatrix} e_4 & e_6 & e_3 & e_2\\ b_4 & b_6 & b_3 & b_2\\ d_4 & d_6 & d_3 & d_2\end{bmatrix}.$$

Starting from formula (D),

ω' = the affect of $[e_1b_1d_1]$ in $[a_1b_1c_1d_1e_1]$

= the affect of $[ebd]$ in $[abcde]$

= the affect of [5 2 4] in [1 2 3 4 5] = 7;

ω'' = the affect of $[a_1a_6a_3a_2]$ in $[a_1a_2a_3a_4a_5a_6]$

= the affect of [1 6 3 2] in [1 2 3 4 5 6] = 10;

$\omega=\omega'+\omega''=7+10=17$.

Starting from formula (C),

$$\omega' = \text{the affect of } [e_4 b_4 d_4] \text{ in } [a_4 b_4 c_4 d_4 e_4]$$
$$= \text{the affect of } [ebd] \text{ in } [abcde] = 7;$$
$$\omega'' = \text{the affect of } [e_4 e_6 e_3 e_2] \text{ in } [e_1 e_2 e_3 e_4 e_5 e_6]$$
$$= \text{the affect of } [4\,6\,3\,2] \text{ in } [1\,2\,3\,4\,5\,6] = 10.$$

If we make use of formula (B), we first observe that the successive horizontal rows of B are (or correspond to) the 5th, 2nd and 4th horizontal rows of A, and conclude that

$$\omega' = \text{the affect of } [5\,2\,4] \text{ in } [1\,2\,3\,4\,5] = 7.$$

We then observe that the successive vertical rows of B are (or correspond to) the 4th, 6th, 3rd, and 2nd vertical rows of A, and conclude that

$$\omega'' = \text{the affect of } [4\,6\,3\,2] \text{ in } [1\,2\,3\,4\,5\,6] = 10.$$

Ex. iii. Let

$$A = \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \end{bmatrix}, \quad B = \begin{bmatrix} q & t & s \\ b & e & d \\ g & j & i \end{bmatrix}.$$

Starting from formula (C),

$$\omega' = \text{the affect of } [qbg] \text{ in } [bglq]$$
$$= \text{the affect of } [4\,1\,2] \text{ in } [1\,2\,3\,4] = 3;$$
$$\omega'' = \text{the affect of } [bed] \text{ in } [abcde]$$
$$= \text{the affect of } [2\,5\,4] \text{ in } [1\,2\,3\,4\,5] = 6;$$
$$\omega = \omega' + \omega'' = 9.$$

If we apply formula (B), we notice that the horizontal rows of B are the 4th, 1st and 2nd horizontal rows of A, and that the vertical rows of B are the 2nd, 5th and 4th vertical rows in A, and conclude at once that

$$\omega' = \text{the affect of } [4\,1\,2] = [1\,2\,3\,4] = 3;$$
$$\omega'' = \text{the affect of } [2\,5\,4] \text{ in } [1\,2\,3\,4\,5] = 6;$$
$$\omega = \omega' + \omega'' = 9.$$

Ex. iv. In the last example, B is a square matrix.

Accordingly the affect of B in A

$= \text{the affect of the derived product } qei \text{ in } A$

$= \text{the affect of } [pqf] \text{ in } [afkp] + \text{the affect of } [bed] \text{ in } [abcde]$

$= \text{the affect of } [4\,1\,2] \text{ in } [1\,2\,3\,4] + \text{the affect of } [2\,5\,4] \text{ in } [1\,2\,3\,4\,5]$

$= 3 + 6 = 9.$

§ 24. Extended and completed derived matrices and determinoids.

Any derived matrix B of a fundamental matrix A is *completed* by inserting in it the vertical and horizontal rows which were omitted from A in the formation of B, and placing them respectively to the right and below the rows which were retained, each set of added rows having the same relative order as in A.

Any derived matrix B of a fundamental matrix A is *extended* by successive additions of vertical and horizontal rows of A not already occurring in it. Each vertical row as it is added must be the earliest vertical row of A which is still omitted, and must be placed to the right of all the existing vertical rows. Similarly each horizontal row as it is added must be the highest horizontal row of A which is still omitted, and must be placed below all the existing horizontal rows.

A derived determinoid can be extended and completed in the same way.

The affect of a derived matrix is not altered when the derived matrix is extended or completed. For the sequences determined by its horizontal and vertical rows respectively are simultaneously extended or completed in the sequences determined by the horizontal and vertical rows of the fundamental matrix. Accordingly the vertical and horizontal affects of the derived matrix are both unaltered.

Similarly the extension or completion of a derived determinoid leaves the affect of the derived determinoid unaltered.

Ex. i. Let $A=[abcde]_{1234567}$, $B=[ebd]_{4631}$.

Then $B_1=[ebda]_{463125}$ is one of the extensions of B with respect to A,

and $B'=[ebdac]_{4631257}$ is the completed matrix of B with respect to A.

The vertical affects of B, B_1 and B' in A are respectively equal to the affects of [4 6 3 1], [4 6 3 1 2 5] and [4 6 3 1 2 5 7] in [1 2 3 4 5 6 7]; and these are all three the same.

The horizontal affects of B, B_1 and B' in A are respectively equal to the affects of [ebd], [$ebda$] and [$ebdac$] in [$abcde$]; and these are all three the same.

Ex. ii. If a derived matrix is the extension of a square derived matrix, its affect is equal to the affect of its leading derived product.

§ 25. Theorems concerning the affects of derived matrices.

Where the theorems of the present article are enunciated or proved only for derived matrices of a fundamental matrix, they evidently remain true when the fundamental matrix is replaced by a fundamental determinoid, or the derived matrices are replaced by derived determinoids.

Theorem I a. *If ω is the affect of a derived matrix B in a fundamental matrix A, it is possible to bring B to the leading position in A by forward horizontal and vertical moves applied to the rows of A without at any stage altering the relative orders of the horizontal and vertical rows of A which do not occur in B; and the total number of such forward moves is always equal to ω.*

At the end of such a process A will have been converted into the completed matrix of B with respect to A.

Let $$A = [ab \dots c]_{12 \dots n}, \qquad B = [\alpha\beta \dots \gamma]_{pq \dots r},$$

so that $\alpha, \beta, \dots \gamma$ is some arrangement of letters selected from $a, b, \dots c$, and $p, q, \dots r$ is some arrangement of numbers selected from $1, 2, \dots n$.

Let ω' be the vertical and ω'' the horizontal affect of B in A.

Let $[\lambda\mu \dots \nu]$ be the corranged sequence which remains when $\alpha, \beta, \dots \gamma$ are struck out from the sequence $[a, b, \dots c]$; and let $[u, v, \dots w]$ be the corranged sequence which remains when $p, q, \dots r$ are struck out from the sequence $[1\ 2 \dots n]$.

Then the completed matrix of B is the matrix B', where
$$B' = [\alpha\beta \dots \gamma\, \lambda\mu \dots \nu]_{pq \dots r\, uv \dots w}.$$

To convert A into B' in the prescribed manner it is necessary and sufficient to apply to the horizontal rows of A such forward moves as will convert the sequence $[1\ 2 \dots n]$ into $[pq \dots r\, uv \dots w]$ without at any stage altering the relative order of the numbers $u, v, \dots w$, and to the vertical rows of A such forward moves as will convert the sequence $[ab \dots c]$ into $[\alpha\beta \dots \gamma\, \lambda\mu \dots \nu]$ without at any stage altering the relative order of the letters $\lambda, \mu, \dots \nu$.

By the properties of sequences (see § 19, Theorem I b), this conversion can always be effected. The moves applied to the sequence $[1\ 2 \dots n]$ must be always ω' in number, since ω' is the affect of $[pq \dots r]$ in $[1\ 2 \dots n]$; and the moves applied to the sequence $[ab \dots c]$ must be always ω'' in number, since ω'' is the affect of $[\alpha\beta \dots \gamma]$ in $[ab \dots c]$.

Thus it is always possible to convert A into B' in the prescribed manner, and the conversion, however effected in the prescribed manner, always requires exactly ω' forward vertical moves and exactly ω'' forward horizontal moves. The total number of moves is therefore always equal to ω.

Ex. i. *If B is simply a derangement of A, then A can be converted into B by forward moves.*

However this is done, exactly ω' forward vertical moves and exactly ω'' forward horizontal moves must be applied. Accordingly *the total number of moves must always be equal to ω.*

This is merely a particular case of Theorem I a. It can be proved independently with the help of Theorem I a of § 19.

Ex. ii. Let $$A = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{bmatrix}, \quad B = \begin{bmatrix} d_3 & b_3 & e_3 \\ d_1 & b_1 & e_1 \\ d_4 & b_4 & e_4 \end{bmatrix}.$$

Then $$\omega' = 3, \quad \omega'' = 6, \quad \omega = 9.$$

We can bring the horizontal rows of B to the leading position in A by applying to A in succession the forward vertical moves (2 3), (1 3), (3 4).

And we can then bring the vertical rows of B to the leading position by the forward horizontal moves

$$(cd), \quad (bd), \quad (ad), \quad (ab), \quad (ce), \quad (ae).$$

Theorem I b. *If ω is the affect of a derived matrix B in a fundamental matrix A, and if B is brought to the leading position in A by forward and backward horizontal and vertical moves applied to the rows of **A** without causing any final alteration in the relative orders of the horizontal and vertical rows of A, then:*

(1) *The least number of moves by which the transformation of A can be effected is ω.*

(2) *The total number of moves, forward and backward, always exceeds ω by an even number, which may be 0.*

(3) *The total number of forward moves always exceeds the total number of backward moves by ω.*

Let ω' be the vertical and ω'' the horizontal affect of B in A, and let B' be the completed matrix of B with respect to A.

Using the same notation as in the proof of Theorem I a, we write

$$A = [ab \ldots c]_{12 \ldots n}, \qquad B = [\alpha\beta \ldots \gamma]_{pq \ldots r}, \qquad B' = [\alpha\beta \ldots \gamma\, \lambda\mu \ldots \nu]_{pq \ldots r\, uv \ldots w}.$$

We now have to convert A into B' by forward and backward horizontal and vertical moves applied to the rows of A.

To do this it is necessary and sufficient to apply to the horizontal rows of A such vertical moves as will convert the sequence $[1\,2 \ldots n]$ into $[pq \ldots r\, uv \ldots w]$, and to apply to the vertical rows of A such horizontal moves as will convert the sequence $[ab \ldots c]$ into $[\alpha\beta \ldots \gamma\, \lambda\mu \ldots \nu]$.

By Theorem II b of § 19, the total number of vertical moves must always be $\omega' + 2s$, where s is an arbitrary positive integer which may be 0, and the total number of forward vertical moves must always exceed the total number of backward vertical moves by ω'.

Also by the same theorem, the total number of horizontal moves must always be $\omega'' + 2t$, where t is an arbitrary positive integer which may be 0, and the total number of forward horizontal moves must always exceed the total number of backward horizontal moves by ω''.

Hence the total number of moves must always be $\omega + 2k$, where k is an arbitrary positive integer which may be 0, and the total number of forward moves must always exceed the total number of backward moves by ω.

When the conversion is effected in the least possible number of moves, ω moves, all forward moves, must be employed, viz. ω' forward vertical moves and ω'' forward horizontal moves.

Ex. iii. The above theorem remains true in the special case in which B is simply a derangement of A. If A is converted into B by forward and backward horizontal and vertical moves, the moves employed are subject to the same restrictions as in the general theorem.

This special case can be proved independently by means of Theorem II *a* of § 19.

Ex. iv. Let $A=[abcde]_{1234}$, $B=[dbe]_{314}$ as in Ex. ii.

Then $\omega'=3,\quad \omega''=6,\quad \omega=9.$

Also in this case $B'=[dbeac]_{3142}.$

We can convert A into B' by the seven successive vertical moves

$$\underline{(12)},\quad \underline{(13)},\quad \underline{(14)},\quad \underline{(23)},\quad \underline{(24)},\quad (21),\quad (41),$$

and the eight successive horizontal moves

$$\underline{(bc)},\quad (bd),\quad \underline{(cd)},\quad \underline{(ad)},\quad (cb),\quad \underline{(ce)},\quad \underline{(ab)},\quad \underline{(ae)}.$$

Here the forward moves are underlined, and there are in all twelve forward moves and three backward moves.

Theorem II a. *If B is any corranged minor of the fundamental matrix A, and if B' is any derangement of B, then*

$$\text{aff. } B' \text{ in } A = \text{aff. } B' \text{ in } B + \text{aff. } B \text{ in } A.$$

Let $$A=[a]_m^n,\quad B=[a_{xy}]_\mu^\nu,\quad B'=[a_{pq}]_\mu^\nu,$$

so that

$[x_1x_2\dots x_\mu]$ is a corranged minor of the sequence $[1\ 2\dots m]$,

$[y_1y_2\dots y_\nu]$ is a corranged minor of the sequence $[1\ 2\dots n]$,

$[p_1p_2\dots p_\mu]$ is a derangement of the sequence $[x_1x_2\dots x_\mu]$,

$[q_1q_2\dots q_\mu]$ is a derangement of the sequence $[y_1y_2\dots y_\nu]$.

We shall make use of Theorem V *a* of § 19.

The vertical affect of B' in A

$$\begin{aligned}
&= \text{the affect of } [p_1p_2\dots p_\mu] \text{ in } [1\ 2\dots m]\\
&= \text{the affect of } [p_1p_2\dots p_\mu] \text{ in } [x_1x_2\dots x_\mu]\\
&\quad + \text{the affect of } [x_1x_2\dots x_\mu] \text{ in } [1\ 2\dots m]\\
&= (\text{the vertical affect of } B' \text{ in } B)\\
&\quad + (\text{the vertical affect of } B \text{ in } A).
\end{aligned}$$

Similarly the horizontal affect of B' in A

= the affect of $[q_1 q_2 \dots q_\nu]$ in $[1\,2 \dots n]$

= the affect of $[q_1 q_2 \dots q_\nu]$ in $[y_1 y_2 \dots y_\nu]$

+ the affect of $[y_1 y_2 \dots y_\nu]$ in $[1\,2 \dots n]$

= (the horizontal affect of B' in B)

+ (the horizontal affect of B' in A).

Adding these two results, we obtain the theorem enunciated above.

The theorem can also be easily deduced from Theorem I*a* of the present article.

Ex. v. Let $A=[abcde]_{1234}$, $B=[bce]_{24}$, $B'=[ebc]_{42}$.

In this case

aff. B' in $B=3$; aff. B in $A=7$; aff. B' in $A=10=3+7$.

Theorem II b. *If B is any deranged minor of the fundamental matrix A, and if B' is any derangement of B, then*

$$\textit{aff. } B' \textit{ in } B + \textit{aff. } B \textit{ in } A = \textit{aff. } B' \textit{ in } A + 2k,$$

where k is a positive integer.

The same notation will be used as in the proof of the previous theorem; but now

$[x_1 x_2 \dots x_\mu]$ is a *deranged* minor of the sequence $[1\,2 \dots m]$,

$[y_1 y_2 \dots y_\nu]$ is a *deranged* minor of the sequence $[1\,2 \dots n]$.

By Theorem V*b* of § 19, the affect of $[p_1 p_2 \dots p_\mu]$ in $[x_1 x_2 \dots x_\mu]$ + the affect of $[x_1 x_2 \dots x_\mu]$ in $[1\,2 \dots m]$ exceeds the affect of $[p_1 p_2 \dots p_\mu]$ in $[1\,2 \dots m]$ by an even positive number: that is (the vertical affect of B' in B) + (the vertical affect of B in A) exceeds (the vertical affect of B' in A) by an even positive integer.

Also by the same theorem, the affect of $[q_1 q_2 \dots q_\nu]$ in $[y_1 y_2 \dots y_\nu]$ + the affect of $[y_1 y_2 \dots y_\nu]$ in $[1\,2 \dots n]$ exceeds the affect of $[q_1 q_2 \dots q_\nu]$ in $[1\,2 \dots n]$ by an even positive number; that is (the horizontal affect of B' in B) + (the horizontal affect of B in A) exceeds (the horizontal affect of B' in A) by an even positive integer.

Adding these two results, we obtain the theorem enunciated above.

The theorem can also be easily deduced from Theorem I*b* of the present article.

Ex. vi. Let $A=[abcde]_{123456}$, $B=[dae]_{3651}$, $B'=[eda]_{1635}$.

In this case

aff. B' in $B=6$; aff. B in $A=13$; aff. B' in $A=11=(6+13)-8$.

Theorem III a. *If D is any corranged square matrix (or any corranged determinant) derived from the fundamental matrix A, and if P is any complete derived product of D, then*

$$\text{aff. } P \text{ in } A = \text{aff. } P \text{ in } D + \text{aff. } D \text{ in } A.$$

This follows immediately from Theorem IIa if we remember that the affect of P is equal to the affect of the derived square matrix (or determinant) which has P for its leading derived product.

We can also prove the theorem independently with the help of Theorem Va of § 19.

Let $\quad A = [a]_m^n, \quad D = [a_{xy}]_r^r, \quad P = a_{\xi_1\eta_1} a_{\xi_2\eta_2} \dots a_{\xi_r\eta_r},$

so that

$[x_1 x_2 \dots x_r]$ is a corranged minor of the sequence $[1\ 2 \dots m]$,

$[y_1 y_2 \dots y_r]$ is a corranged minor of the sequence $[1\ 2 \dots n]$,

$[\xi_1 \xi_2 \dots \xi_r]$ is a derangement of the sequence $[x_1 x_2 \dots x_r]$,

$[\eta_1 \eta_2 \dots \eta_r]$ is a derangement of the sequence $[y_1 y_2 \dots y_r]$.

Then the vertical affect of P in A

= the affect of $[\xi_1 \xi_2 \dots \xi_r]$ in $[1\ 2 \dots m]$

= the affect of $[\xi_1 \xi_2 \dots \xi_r]$ in $[x_1 x_2 \dots x_r]$

+ the affect of $[x_1 x_2 \dots x_r]$ in $[1\ 2 \dots m]$

= (the vertical affect of P in D)

+ (the vertical affect of D in A).

Similarly the horizontal affect of P in A

= the affect of $[\eta_1 \eta_2 \dots \eta_r]$ in $[1\ 2 \dots n]$

= the affect of $[\eta_1 \eta_2 \dots \eta_r]$ in $[y_1 y_2 \dots y_r]$

+ the affect of $[y_1 y_2 \dots y_r]$ in $[1\ 2 \dots n]$

= (the horizontal affect of P in D)

+ (the horizontal affect of D in A).

The theorem is now obtained by adding these two results.

Ex. vii. Let $\quad A = [abcde]_{1234}, \quad D = [bce]_{134}, \quad P = e_3 b_1 c_4.$

In this case

$$\text{aff. } P \text{ in } D = 3\,; \quad \text{aff. } D \text{ in } A = 6\,; \quad \text{aff. } P \text{ in } A = 9 = 3 + 6.$$

Theorem III b. *If D is any deranged square matrix (or any deranged determinant) derived from the fundamental matrix* **A**, *and if P is any complete derived product of D, then*

$$\text{aff. } P \text{ in } D + \text{aff. } D \text{ in } A = \text{aff. } P \text{ in } A + 2k,$$

where k is a positive integer.

This follows immediately from Theorem IIb if we remember that the affect of P is equal to the affect of the derived square matrix (or determinant) which has P for its leading derived product.

To prove the theorem independently we make use of Theorem Vb of § 19.

We use the same notation as in the proof of the previous theorem, but now

$[\xi_1 \xi_2 \dots \xi_r]$ is a deranged minor of the sequence $[1\ 2 \dots m]$,

$[\eta_1 \eta_2 \dots \eta_r]$ is a deranged minor of the sequence $[1\ 2 \dots n]$.

If we confine ourselves in the first instance to the vertical affects, we have

$$\begin{aligned} &\text{aff. } P \text{ in } D + \text{aff. } D \text{ in } A \\ &\quad = \text{the affect of } [\xi_1 \xi_2 \dots \xi_r] \text{ in } [x_1 x_2 \dots x_r] \\ &\qquad + \text{the affect of } [x_1 x_2 \dots x_r] \text{ in } [1\ 2 \dots m] \\ &\quad = \text{the affect of } [\xi_1 \xi_2 \dots \xi_r] \text{ in } [1\ 2 \dots m] + 2s \\ &\quad = \text{aff. } P \text{ in } A + 2s, \end{aligned}$$

where s is a positive integer.

Thus the theorem is true as regards the vertical affects.

Similarly it can be shown to be true as regards the horizontal affects.

It follows by addition that the theorem is true as regards the total affects.

It may be remarked that Theorems IIIa and IIIb are not true *in general* when D is a *rectangular* derived matrix and P is a complete derived product of D.

Ex. viii. Let $A = [abcde]_{1234}$, $D = [ceb]_{314}$, $P = b_4 e_3 c_1$.

In this case

$$\text{aff. } P \text{ in } D = 5;\ \text{aff. } D \text{ in } A = 9;\ \text{aff. } P \text{ in } A = 10 = (5+9) - 4.$$

Theorem III c. *If B is any matrix (or determinoid) formed from a fundamental matrix A by omitting certain rows of one kind only and re-arranging the retained rows of that kind, and if the retained rows of that* *kind are long rows in B, and if further P is any complete derived product of B, then*

$$\text{aff. } P \text{ in } B + \text{aff. } B \text{ in } A = \text{aff. } P \text{ in } A + 2k,$$

where k is a positive integer.

In particular when B is corranged, then $k = 0$.

Let

$$A = [a]_m^n = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad B = [a_{x1}]_\mu^n = \begin{bmatrix} a_{x_1 1} & a_{x_1 2} & \dots & a_{x_1 n} \\ a_{x_2 1} & a_{x_2 2} & \dots & a_{x_2 n} \\ \dots & \dots & \dots & \dots \\ a_{x_\mu 1} & a_{x_\mu 2} & \dots & a_{x_\mu n} \end{bmatrix},$$

and
$$P = a_{\xi_1\eta_1} a_{\xi_2\eta_2} \dots a_{\xi_\mu\eta_\mu},$$

where it is supposed that the long rows of B are horizontal.

Then in general

$$\begin{aligned} &[x_1x_2 \dots x_\mu] \text{ is a deranged minor of } [1\ 2 \dots m], \\ &[\xi_1\xi_2 \dots \xi_\mu] \text{ is a derangement of } [x_1x_2 \dots x_\mu], \\ &[\eta_1\eta_2 \dots \eta_\mu] \text{ is a deranged minor of } [1\ 2 \dots n], \end{aligned}$$

and by Theorem V*b* of § 19,

$$\begin{aligned} &\text{the affect of } [\xi_1\xi_2 \dots \xi_\mu] \text{ in } [x_1x_2 \dots x_\mu] \\ &\qquad + \text{the affect of } [x_1x_2 \dots x_\mu] \text{ in } [1\ 2 \dots m] \\ &\qquad = \text{the affect of } [\xi_1\xi_2 \dots \xi_\mu] \text{ in } [1\ 2 \dots m] + 2k, \end{aligned}$$

where k is a positive integer.

Now

$$\begin{aligned} \text{aff. } P \text{ in } B &= \text{the affect of } [\xi_1\xi_2 \dots \xi_\mu] \text{ in } [x_1x_2 \dots x_\mu] \\ &\quad + \text{the affect of } [\eta_1\eta_2 \dots \eta_\mu] \text{ in } [1\ 2 \dots n], \\ \text{aff. } B \text{ in } A &= \text{the affect of } [x_1x_2 \dots x_\mu] \text{ in } [1\ 2 \dots m], \\ \text{aff. } P \text{ in } A &= \text{the affect of } [\xi_1\xi_2 \dots \xi_\mu] \text{ in } [1\ 2 \dots m] \\ &\quad + \text{the affect of } [\eta_1\eta_2 \dots \eta_\mu] \text{ in } [1\ 2 \dots n]. \end{aligned}$$

Utilising the preceding result it follows that

$$\text{aff. } P \text{ in } B + \text{aff. } B \text{ in } A = \text{aff. } P \text{ in } A + 2k.$$

When B is corranged, then

$$[x_1x_2 \dots x_\mu] \text{ is a corranged minor of } [1\ 2 \dots m],$$

and applying Theorem V*a* of § 19 instead of Theorem V*b*, we see that $k = 0$.

A similar proof can be given when the long rows of B are vertical.

With the terminology of § 31, B is an *inferior simple minor* of A.

Ex. ix. Let $A=[abcde]_{1234567}$, $B=[abcde]_{3751}$, $P=b_7e_3a_1c_5$.

In this case B is *deranged*, and we have

$$\text{aff. } P \text{ in } B=2+4, \quad \text{aff. } B \text{ in } A=10, \quad \text{aff. } P \text{ in } A=10+4.$$

Thus $$\text{aff. } P \text{ in } B + \text{aff. } B \text{ in } A = \text{aff. } P \text{ in } A + 2.$$

Again, let $A=[abcde]_{1234567}$, $B=[bdea]_{1234567}$, $P=e_2a_5d_7b_3$.

In this case also B is *deranged*, and we have

$$\text{aff. } P \text{ in } B=5+9, \quad \text{aff. } B \text{ in } A=5, \quad \text{aff. } P \text{ in } A=6+9.$$

Thus $$\text{aff. } P \text{ in } B + \text{aff. } B \text{ in } A = \text{aff. } P \text{ in } A+4.$$

Ex. x. Let $A=[abcde]_{1234567}$, $B=[abcde]_{1357}$, $P=b_7e_3a_1c_5$.

In this case B is *corranged*, and we have

$$\text{aff. } P \text{ in } B=4+4, \quad \text{aff. } B \text{ in } A=6, \quad \text{aff. } P \text{ in } A=4+10.$$

Thus $$\text{aff. } P \text{ in } B + \text{aff. } B \text{ in } A = \text{aff. } P \text{ in } A.$$

Theorem III d. *If B is any matrix (or determinoid) formed from a fundamental matrix A by omitting certain rows of one kind only and rearranging the retained rows of that kind, and if the retained rows of that kind are long rows in B, and if further D is any simple minor determinant of B (corranged or deranged), then*

$$\textit{aff. } D \textit{ in } B + \textit{aff. } B \textit{ in } A = \textit{aff. } D \textit{ in } A + 2k,$$

where k is a positive integer.

In particular when B is corranged, then $k=0$.

Let $$A=[a]_m^n, \quad B=[a_{\alpha 1}]_\mu^n, \quad D=(a_{xy})_\mu^\mu,$$

where it is assumed that the long rows of B are horizontal.

Then in general

$[\alpha_1\alpha_2 \ldots \alpha_\mu]$ is a deranged minor of $[1\ 2 \ldots m]$,

$[x_1x_2 \ldots x_\mu]$ is a derangement of $[\alpha_1\alpha_2 \ldots \alpha_\mu]$,

$[y_1y_2 \ldots y_\mu]$ is a deranged minor of $[1\ 2 \ldots n]$,

and by Theorem V *b* of § 19,

the affect of $[x_1x_2 \ldots x_\mu]$ in $[\alpha_1\alpha_2 \ldots \alpha_\mu]$ + the affect of $[\alpha_1\alpha_2 \ldots \alpha_\mu]$ in $[1\ 2 \ldots m]$
= the affect of $[x_1x_2 \ldots x_\mu]$ in $[1\ 2 \ldots m] + 2k$,

where k is a positive integer.

Now aff. D in B = the affect of $[x_1x_2 \ldots x_\mu]$ in $[\alpha_1\alpha_2 \ldots \alpha_\mu]$
+ the affect of $[y_1y_2 \ldots y_\mu]$ in $[1\ 2 \ldots n]$,

aff. B in A = the affect of $[\alpha_1\alpha_2 \ldots \alpha_\mu]$ in $[1\ 2 \ldots m]$,

aff. D in A = the affect of $[x_1x_2 \ldots x_\mu]$ in $[1\ 2 \ldots m]$
+ the affect of $[y_1y_2 \ldots y_\mu]$ in $[1\ 2 \ldots n]$.

Utilising the preceding result, it follows that

$$\text{aff. } D \text{ in } B + \text{aff. } B \text{ in } A = \text{aff. } D \text{ in } A + 2k.$$

When B is corranged, then

$$[\alpha_1 \alpha_2 \ldots \alpha_\mu] \text{ is a corranged minor of } [1\ 2 \ldots m],$$

and applying Theorem V *a* of § 19 instead of Theorem V *b*, we see that $k=0$.

A similar proof can be given when the long rows of B are vertical.

Ex. xi. Let $A=[abcde]_{1234567}$, $B=[abcde]_{625}$, $D=(cea)_{526}$.

In this case B is deranged, and we have

$$\text{aff. } D \text{ in } B=3+5, \quad \text{aff. } B \text{ in } A=9, \quad \text{aff. } D \text{ in } A=8+5.$$

Thus $\quad \text{aff. } D \text{ in } B + \text{aff. } B \text{ in } A = \text{aff. } D \text{ in } A + 4.$

Ex. xii. Let $A=[abcde]_{1234567}$, $B=[abcde]_{256}$, $D=(cea)_{526}$.

In this case B is corranged, and we have

$$\text{aff. } D \text{ in } B=1+5, \quad \text{aff. } B \text{ in } A=7, \quad \text{aff. } D \text{ in } A=8+5.$$

Thus $\quad \text{aff. } D \text{ in } B + \text{aff. } B \text{ in } A = \text{aff. } D \text{ in } A.$

Theorem IV a. *If D is a square matrix (or a determinant), D' any derangement of D, and P any complete derived product of D, then*

$$\textit{aff. } P \textit{ in } D' + \textit{aff. } D' \textit{ in } D = \textit{aff. } P \textit{ in } D + 2k,$$

where k is a positive integer.

This is a special case of Theorem III *b*, and can be proved independently in the same way as that theorem.

Ex. xiii. Let

$$D=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad D'=\begin{bmatrix} b_3 & b_1 & b_2 \\ c_3 & c_1 & c_2 \\ a_3 & a_1 & a_2 \end{bmatrix}, \quad P=b_3 a_1 c_2.$$

Then $\quad \text{aff. } P \text{ in } D'=1, \quad \text{aff. } D' \text{ in } D=4, \quad \text{aff. } P \text{ in } D=3.$

Thus $\quad \text{aff. } P \text{ in } D' + \text{aff. } D' \text{ in } D = \text{aff. } P \text{ in } D + 2.$

Theorem IV b. *If D is any matrix (or determinoid), D' any matrix (or determinoid) formed from D by derangements of the long rows only, and P any complete derived product of D, then*

$$\textit{aff. } P \textit{ in } D' + \textit{aff. } D' \textit{ in } D = \textit{aff. } P \textit{ in } D + 2k,$$

where k is a positive integer

This is a special case of Theorem III *c*, and can be proved independently in the same way as that theorem.

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Ex. xiv. Let $D=[abcde]_{1234}$, $D'=[abcde]_{4132}$, $P=e_2c_3d_4a_1$.

Then aff. P in $D'=8+5$, aff. D' in $D=4$, aff. P in $D=8+3$.

Thus aff. P in D' + aff. D' in D = aff. P in $D+6$.

Theorem V a. *If D is any determinant (corranged or deranged) derived from a fundamental matrix A, and D' any derangement of D, and if ω and ω' are the affects of D and D' respectively in A, then*

$$(-1)^{\omega'} D' = (-1)^{\omega} D.$$

Let η be the affect of D' in D. By Theorem VII *a* of § 19, η is also the affect of D in D'.

Let P be any complete derived product of D, and let its affects in D and D' be σ and σ' respectively.

Then $D = \Sigma(-1)^{\sigma} P$, $(-1)^{\omega} D = \Sigma(-1)^{\omega+\sigma} P$;

$D' = \Sigma(-1)^{\sigma'} P$, $(-1)^{\omega'} D' = \Sigma(-1)^{\omega'+\sigma'} P$.

Now by Theorem II *b*, $\omega' \equiv \omega + \eta \pmod{2}$.

Also by Theorem IV *a* or III *b*, $\sigma' \equiv \sigma + \eta \pmod{2}$.

Hence $\omega' + \sigma' \equiv \omega + \sigma \pmod{2}$,

and therefore $(-1)^{\omega'} D' = (-1)^{\omega} D$.

We see then that *if any derived determinant of a given matrix is provided with the sign determined by its affect, the result is independent of the orders of arrangement of the horizontal and vertical rows of the determinant.*

Ex. xv. *If D' is any derangement of a determinant D, and if η is the affect of D' in D, then*

$$D' = (-1)^{\eta} D.$$

This is a particular case of the above theorem obtained by taking the matrix of D as the fundamental matrix A. It can also be deduced from the equation of the theorem by substituting for $(-1)^{\omega'}$ its value $(-1)^{\omega+\eta}$.

Ex. xvi. Let $A=[abcde]_{1234}$, $D=(bde)_{413}$, $D'=(edb)_{134}$.

In this case

aff. D in $A=\omega=9$; aff. D' in $A=\omega'=10$; aff. D' in $D=\eta=5$.

Accordingly $(-1)^{10} D' = (-1)^{9} D$, or $D' = (-1)^{5} D$.

Theorem V b. *If D is any determinoid (corranged or deranged) derived from a fundamental matrix A, and D' any determinoid formed by deranging the long rows only of D, and if ω and ω' are the affects of D and D' respectively in A, then*

$$(-1)^{\omega'} D' = (-1)^{\omega} D.$$

Let η be the affect of D' in D: it is also the affect of D in D'.

Let P be any complete derived product of D, and let its affects in D and D' be σ and σ' respectively.

Then $\qquad D = \Sigma(-1)^{\sigma} P, \quad (-1)^{\omega} D = \Sigma(-1)^{\omega+\sigma} P;$

$$D' = \Sigma(-1)^{\sigma'} P, \quad (-1)^{\omega'} D = \Sigma(-1)^{\omega'+\sigma'} P.$$

Now by Theorem II *b*, $\qquad \omega' \equiv \omega + \eta \quad (\text{mod. } 2).$

Also by Theorem IV *b* or III *c*, $\qquad \sigma' \equiv \sigma + \eta \quad (\text{mod. } 2).$

Hence $\qquad \omega' + \sigma' \equiv \omega + \sigma \quad (\text{mod. } 2),$

and therefore $\qquad (-1)^{\omega'} D' = (-1)^{\omega} D.$

Thus *if any derived determinoid of a given matrix is provided with the sign determined by its affect, the result is independent of the order of arrangement of the long rows of the determinoid.*

Ex. xvii. *If D' is a determinoid derived from the determinoid D by deranging the long rows only of D, and if η is the affect of D' in D, then*

$$D' = (-1)^{\eta} D.$$

This is a particular case of the above theorem obtained by taking the matrix of D as the fundamental matrix A. The result can also be deduced from the equation of the theorem by substituting for $(-1)^{\omega'}$ its value $(-1)^{\omega+\eta}$.

Ex. xviii. Let $A = [abcde]_{1234}$, $D = (bdea)_{431}$, $D' = (bdea)_{143}$.

In this case

$$\text{aff. } D \text{ in } A = \omega = 10; \quad \text{aff. } D' \text{ in } A = \omega' = 8; \quad \text{aff. } D' \text{ in } D = \eta = 2.$$

Accordingly $\qquad (-1)^8 D' = (-1)^{10} D, \quad \text{or} \quad D' = (-1)^2 D.$

§ 26. Theorems concerning the affects of complementary derived matrices.

Two corranged minor matrices are said to be complementary to one another when each is formed from the fundamental matrix by striking out the vertical and horizontal rows which occur in the other.

Complementary determinoids are defined in a similar way.

The two theorems given in Chapter III regarding complementary sequences (Theorems VIII *a* and VIII *b* of § 19) can be at once extended to complementary matrices and determinoids, and lead to the two theorems which follow.

Theorem I. *If ω and ω' are the affects of two complementary corranged minor matrices (or determinoids) whose orders are respectively (μ, ν) and (μ', ν'), then*

$$\omega + \omega' = \mu\mu' + \nu\nu'.$$

The theorem will be proved for matrices only.

Let $$B=[a_{xy}]_{\mu}^{\nu},\quad B'=[a_{pq}]_{\mu}^{\nu'}$$

be two complementary corranged minors of the fundamental matrix

$$A=[a]_m^n,$$

so that

$[x_1 x_2 \dots x_\mu]$, $[p_1 p_2 \dots p_{\mu'}]$ are complementary corranged minors of the sequence $[1\ 2 \dots m]$,

and

$[y_1 y_2 \dots y_\nu]$, $[q_1 q_2 \dots q_{\nu'}]$ are complementary corranged minors of the sequence $[1\ 2 \dots n]$,

and further $$\mu+\mu'=m,\quad \nu+\nu'=n.$$

Let ω_1, ω_2, ω be the vertical, horizontal and total affects of B in A, and ω_1', ω_2', ω' be the vertical, horizontal and total affects of B' in A.

Then ω_1 is the affect of $[x_1 x_2 \dots x_\mu]$ in $[1\ 2 \dots m]$,

and ω_1' is the affect of $[p_1 p_2 \dots p_{\mu'}]$ in $[1\ 2 \dots m]$.

Therefore, by Theorem VIII a of § 19,

$$\omega_1+\omega_1'=\mu\mu'.$$

Again ω_2 is the affect of $[y_1 y_2 \dots y_\nu]$ in $[1\ 2 \dots n]$,

and ω_2' is the affect of $[q_1 q_2 \dots q_{\nu'}]$ in $[1\ 2 \dots n]$.

Therefore, by the same theorem

$$\omega_2+\omega_2'=\nu\nu'.$$

Adding these two results, we see that

$$\omega+\omega'=\mu\mu'+\nu\nu'.$$

Ex. i. Let
$$A=\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & g_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & g_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & g_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 & g_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 & f_5 & g_5 \\ a_6 & b_6 & c_6 & d_6 & e_6 & f_6 & g_6 \end{bmatrix}$$

be a fundamental matrix. Then

$$B=\begin{vmatrix} b_2 & e_2 \\ b_4 & e_4 \\ b_6 & e_6 \end{vmatrix},\quad \text{and}\quad B'=\begin{vmatrix} a_1 & c_1 & d_1 & f_1 & g_1 \\ a_3 & c_3 & d_3 & f_3 & g_3 \\ a_5 & c_5 & d_5 & f_5 & g_5 \end{vmatrix}$$

are complementary minor determinoids.

The sum of the vertical affects of B and $B'=6+3=3\times 3$.

The sum of the horizontal affects of B and $B'=4+6=2\times 5$.

The sum of the total affects of B and $B'=19=3\times 3+2\times 5$.

Ex. ii. If ω and ω' are the affects of two complementary corranged minor *determinants* of a square matrix, then

$$\omega' \equiv \omega \quad (\text{mod. } 2).$$

Consequently the signs of two complementary corranged minor determinants of a square matrix, as determined by their affects, are the same.

Theorem II. *If B and C are two complementary corranged minor matrices (or determinoids) of the fundamental matrix A, and if A' and C' are the matrices (or determinoids) obtained from A and C by reversing the orders of both their horizontal rows and their vertical rows, then*

$$\text{aff. } B \text{ in } A = \text{aff. } C' \text{ in } A'.$$

By Theorem VIII *b* of § 19,

the vertical affect of B in A = the vertical affect of C' in A'

and the horizontal affect of B in A = the horizontal affect of C' in A'.

Adding these two results, we obtain the result stated in the above enunciation.

Ex. iii. Let
$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{bmatrix}, \qquad A' = \begin{bmatrix} d_5 & d_4 & d_3 & d_2 & d_1 \\ c_5 & c_4 & c_3 & c_2 & c_1 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ a_5 & a_4 & a_3 & a_2 & a_1 \end{bmatrix},$$

$$B = \begin{bmatrix} a_2 & a_4 & a_5 \\ d_2 & d_4 & d_5 \end{bmatrix}, \quad C = \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}, \quad C' = \begin{bmatrix} c_3 & c_1 \\ b_3 & b_1 \end{bmatrix}.$$

Then aff. B in $A = 2+5=7$; aff. C' in $A' = 2+5=7$.

CHAPTER V.

EXPANSIONS OF A DETERMINOID.

[The expansions of a given determinoid Δ in terms of the elements of any given long row of Δ, in terms of the simple minor *determinants* of Δ, and in terms of the simple minor *determinants* of any given long-cut minor matrix of Δ, are first considered in §§ 27—32. Then in §§ 33—36 the algebraical sum of the affected simple minor *determinoids* of Δ of any given reduced order is evaluated. In the case of superior short-cut simple minors considered in § 33, this sum is a certain numerical multiple of Δ. Finally, all the preceding results are generalised in §§ 37 and 38, which contain an investigation of the algebraical sum of the products obtained by multiplying each affected simple minor *determinoid* of given reduced order of a given simple minor matrix of Δ by its corranged complement or co-factor. In Case 1 of § 38 this sum is a certain numerical multiple of Δ.]

§ 27. Expansion of a determinoid in terms of the elements of any long row.

The chief result of the present article is contained in the following theorem:

Theorem I. *A determinoid is the algebraical sum of the products which can be obtained by multiplying each affected element of any given long row by the corranged minor determinoid complementary to it* ....(A).

Let
$$\Delta = \left| \begin{matrix} a_1 & a_2 \ldots a_s \ldots a_n \\ b_1 & b_2 \ldots b_s \ldots b_n \\ \cdots & \cdots \\ e_1 & e_2 \ldots e_s \ldots e_n \\ \cdots & \cdots \\ k_1 & k_2 \ldots k_s \ldots k_n \end{matrix} \right|$$
be a determinoid with m long rows (horizontal), and n short rows.

Since every complete derived product of the determinoid contains amongst its factors one element belonging to each long row, it is clear that Δ can be expressed as a homogeneous linear function of the elements of any assigned long row. We will suppose it to be expressed in the form

$$\Delta = e_1 E_1 + e_2 E_2 + \ldots + e_s E_s + \ldots + e_n E_n \text{ ....(1)},$$

and we will determine E_s, the coefficient of e_s in this expansion.

Let Δ_s be the corranged minor determinoid formed from Δ by striking out the horizontal and vertical rows containing e_s, so that Δ_s is the corranged determinoid complementary to e_s in Δ.

If we take any complete derived product of Δ_s and insert in it the additional factor e_s, we obtain one of those complete derived products of Δ in which e_s occurs as a factor. Also if we take any complete derived product of Δ in which e_s occurs, and strike out from it the factor e_s, we obtain one of the complete derived products of Δ_s. Hence the complete derived products of Δ which contain e_s are identical with the products which can be formed by multiplying each of the complete derived products of Δ_s by e_s.

Now let $a_p b_q \ldots k_u$ be any complete derived product of Δ_s whose affect in Δ_s is σ', so that

$$\Delta_s = \Sigma(-1)^{\sigma'} a_p b_q \ldots k_u \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and let σ be the affect of $e_s a_p b_q \ldots k_u$ in Δ.

It follows from the above reasoning that

$$e_s E_s = \Sigma(-1)^{\sigma} e_s a_p b_q \ldots k_u \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Let ω_s be the affect of e_s in Δ. Then by Theorem III of § 19

$$\begin{aligned} \sigma &= \text{the affect of } e_s a_p b_q \ldots k_u \text{ in } \Delta \\ &= \text{the affect of } e_s \text{ in } \Delta + \text{the affect of } a_p b_q \ldots k_u \text{ in } \Delta_s \\ &= \omega_s + \sigma'. \end{aligned}$$

Substituting this value for σ in (3), we have

$$\begin{aligned} e_s E_s &= \Sigma(-1)^{\omega_s+\sigma'} e_s a_p b_q \ldots k_u \\ &= (-1)^{\omega_s} e_s \Sigma(-1)^{\sigma'} a_p b_q \ldots k_u = (-1)^{\omega_s} e_s \Delta_s. \end{aligned}$$

We see then that

$$E_s = (-1)^{\omega_s} \Delta_s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and that

$$\Delta = (-1)^{\omega_1} e_1 \Delta_1 + (-1)^{\omega_2} e_2 \Delta_2 + \ldots + (-1)^{\omega_s} e_s \Delta_s + \ldots + (-1)^{\omega_n} e_n \Delta_n \ldots(5).$$

Equation (5) is equivalent to the theorem (A).

It states that Δ can be expressed as a homogeneous linear function of the elements $e_1, e_2, \ldots e_s, \ldots e_n$ of any long row, and that the coefficient of any element e_s in this expansion is the corranged determinoid complementary to e_s provided with the sign determined by the affect of e_s in Δ.

Since the signs determined by the successive affects are alternately positive and negative, only one of the affects needs to be actually found by counting steps.

The quantity E_s determined by equation (4) will be called the *co-factor* of the element e_s in Δ, or in the matrix of Δ, and we can replace (A) by the following equivalent statement:

Alternative Form of Theorem I. *A determinoid is the algebraical sum of all the products which can be formed by multiplying each element of any given long row by its co-factor* ....(B).

The complete evaluation of a determinoid is usually most easily effected by successive applications of the expansion just described.

Ex. i. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_1 \,|\, b_2\, b_3 \,| - a_2 \,|\, b_1\, b_3 \,| + a_3 \,|\, b_1\, b_2 \,|$

$$= a_1 (b_2 - b_3) - a_2 (b_1 - b_3) + a_3 (b_1 - b_2)$$

$$= a_1 b_2 - a_2 b_1 + a_2 b_3 - a_3 b_2 + a_3 b_1 - a_1 b_3.$$

Ex. ii. $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = -a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 & d_1 \\ a_3 & c_3 & d_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \end{vmatrix} + d_2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= -a_2 \{b_1 (c_3 - d_3) - c_1 (b_3 - d_3) + d_1 (b_3 - c_3)\} + b_2 \{a_1 (c_3 - d_3) - c_1 (a_3 - d_3) + d_1 (a_3 - c_3)\}$$
$$- c_2 \{a_1 (b_3 - d_3) - b_1 (a_3 - d_3) + d_1 (a_3 - b_3)\} + d_2 \{a_1 (b_3 - c_3) - b_1 (a_3 - c_3) + c_1 (a_3 - b_3)\}$$

$$= -a_2 b_1 c_3 + a_2 b_1 d_3 + a_2 c_1 b_3 - a_2 c_1 d_3 - a_2 d_1 b_3 + a_2 d_1 c_3$$
$$+ b_2 a_1 c_3 - b_2 a_1 d_3 - b_2 c_1 a_3 + b_2 c_1 d_3 + b_2 d_1 a_3 - b_2 d_1 c_3$$
$$- c_2 a_1 b_3 + c_2 a_1 d_3 + c_2 b_1 a_3 - c_2 b_1 d_3 - c_2 d_1 a_3 + c_2 d_1 b_3$$
$$+ d_2 a_1 b_3 - d_2 a_1 c_3 - d_2 b_1 a_3 + d_2 b_1 c_3 + d_2 c_1 a_3 - d_2 c_1 b_3.$$

Ex. iii. If the elements of any one *long row* are all zeros, the determinoid is identically equal to zero.

Ex. iv. *Two determinoids of the same orders whose elements are the same except those lying in two correspondingly situated long rows can be added by adding corresponding elements of those two long rows.*

Let $\Delta = \begin{vmatrix} a_1 & a_2 \ldots a_i \ldots a_n \\ b_1 & b_2 \ldots b_i \ldots b_n \\ \ldots & \ldots \\ k_1 & k_2 \ldots k_i \ldots k_n \end{vmatrix}, \quad \Delta' = \begin{vmatrix} a_1 & a_2 \ldots \alpha_i \ldots a_n \\ b_1 & b_2 \ldots \beta_i \ldots b_n \\ \ldots & \ldots \\ k_1 & k_2 \ldots \kappa_i \ldots k_n \end{vmatrix},$

be two such determinoids whose *long rows are vertical*, and which differ only in the elements of their ith long rows; and let $A_i, B_i, \ldots K_i$ be the co-factors of $a_i, b_i, \ldots k_i$ in Δ.

Then
$$\Delta = a_i A_i + b_i B_i + \ldots + k_i K_i,$$
$$\Delta' = \alpha_i A_i + \beta_i B_i + \ldots + \kappa_i K_i,$$
$$\Delta + \Delta' = (a_i + \alpha_i) A_i + (b_i + \beta_i) B_i + \ldots + (k_i + \kappa_i) K_i.$$

Consequently $\Delta + \Delta' = \begin{vmatrix} a_1 & a_2 \ldots (a_i + \alpha_i) \ldots a_n \\ b_1 & b_2 \ldots (b_i + \beta_i) \ldots b_n \\ \ldots & \ldots \\ k_1 & k_2 \ldots (k_i + \kappa_i) \ldots k_n \end{vmatrix}.$

Ex. v. *If the elements of any one long row of a determinoid are all multiplied by the same scalar quantity* λ, *the determinoid itself is multiplied by* λ.

Let *long rows be vertical*, and let

$$\Delta = \begin{vmatrix} a_1 & a_2 \ldots a_i \ldots a_n \\ b_1 & b_2 \ldots b_i \ldots b_n \\ \cdots & \cdots \\ k_1 & k_2 \ldots k_i \ldots k_n \end{vmatrix}, \quad \Delta' = \begin{vmatrix} a_1 & a_2 \ldots \lambda a_i \ldots a_n \\ b_1 & b_2 \ldots \lambda b_i \ldots b_n \\ \cdots & \cdots \\ k_1 & k_2 \ldots \lambda k_i \ldots k_n \end{vmatrix}.$$

Then
$$\Delta = a_i A_i + b_i B_i + \ldots + k_i K_i,$$
$$\Delta' = \lambda a_i A_i + \lambda b_i B_i + \ldots + \lambda k_i K_i = \lambda\Delta.$$

Ex. vi. If, with the notation of the text, η_s is the affect of Δ_s in Δ, then by Theorem II of § 26,

$$\omega_s + \eta_s = m + n.$$

Hence
$$\Sigma(-1)^{\eta_s} e_s \Delta_s = (-1)^{m+n} \Sigma(-1)^{\omega_s} e_s \Delta_s = (-1)^{m+n} \Delta.$$

Thus if we multiply each element of a given long row of a determinoid Δ by its affected complement, the algebraical sum of such products is equal to $+\Delta$ or $-\Delta$ according as the sum (or difference) of the two orders of the determinoid is even or odd.

If Δ is a determinant, the sum is always equal to $+\Delta$.

The fact that a determinoid vanishes identically when two of its long rows are the same (see § 5.4 or § 11, Ex. iv) leads to the following second theorem:

Theorem II. *If we select any two long rows of a determinoid, occupying different positions in the determinoid, and multiply each element of one of these rows by the co-factor of the correspondingly situated element of the other row, the algebraical sum of the products so formed is identically equal to zero* ...(C).

To prove this, let $[e_1 e_2 \ldots e_n]$, $[h_1 h_2 \ldots h_n]$ be any two differently situated long rows of the determinoid Δ considered above. Then

$$h_1 E_1 + h_2 E_2 + \ldots + h_n E_n$$

is what

$$e_1 E_1 + e_2 E_2 + \ldots + e_n E_n$$

becomes when in it $e_1, e_2, \ldots e_n$ are replaced by $h_1, h_2, \ldots h_n$. Now the latter sum is identically equal to Δ; and therefore $h_1 E_1 + h_2 E_2 + \ldots + h_n E_n$ is what the determinoid Δ becomes when in it $e_1, e_2 \ldots e_n$ are replaced by $h_1, h_2 \ldots h_n$. But Δ then becomes a determinoid in which the long row $[h_1 h_2 \ldots h_n]$ occurs twice in different positions, and we know that such a determinoid vanishes identically. It follows that $h_1 E_1 + h_2 E_2 + \ldots + h_n E_n$ vanishes identically.

Ex. vii. Let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix},$$

and let A_2, B_2, C_2, D_2 be the co-factors of a_2, b_2, c_2, d_2.

Then $$a_2 A_2 + b_2 B_2 + c_2 C_2 + d_2 D_2$$

$$= -a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 & d_1 \\ a_3 & c_3 & d_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \end{vmatrix} + d_2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \Delta.$$

Consequently $$a_3 A_2 + b_3 B_2 + c_3 C_2 + d_3 D_2$$

$$= -a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & c_1 & d_1 \\ a_3 & c_3 & d_3 \end{vmatrix} - c_3 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \end{vmatrix} + d_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0.$$

Similarly $$a_1 A_2 + b_1 B_2 + c_1 C_2 + d_1 D_2 = 0.$$

Ex. viii. *The value of a determinoid is unaltered when to the elements of any long row there are added the corresponding elements of any other long row, each multiplied by the same scalar quantity* λ.

In the determinoids

$$\Delta = \begin{vmatrix} a_1 & a_2 \dots a_i \dots a_j \dots a_n \\ b_1 & b_2 \dots b_i \dots b_j \dots b_n \\ \dots & \dots \\ k_1 & k_2 \dots k_i \dots k_j \dots k_n \end{vmatrix}, \quad \Delta' = \begin{vmatrix} a_1 & a_2 \dots (a_i + \lambda a_j) \dots a_j \dots a_n \\ b_1 & b_2 \dots (b_i + \lambda b_j) \dots b_j \dots b_n \\ \dots & \dots \\ k_1 & k_2 \dots (k_i + \lambda k_j) \dots k_j \dots k_n \end{vmatrix},$$

let long rows be vertical.

Then
$$\Delta = a_i A_i + b_i B_i + \dots + k_i K_i,$$
$$\Delta' = (a_i + \lambda a_j) A_i + (b_i + \lambda b_j) B_i + \dots + (k_i + \lambda k_j) K_i$$
$$= \Delta + \lambda (a_j A_i + b_j B_i + \dots + k_j K_i).$$

Now by Theorem II, the last bracket vanishes.

Thus $$\Delta' = \Delta.$$

Ex. ix. *The value of a determinoid is unaltered when to the elements of any long row there are added constant multiples of the corresponding elements of any number of other long rows, the multipliers having given values for each one of those other long rows.*

Let Δ be the determinoid of Ex. viii and let Δ' be the determinoid formed from it when we replace

$$a_i \text{ by } a_i' = a_i + \lambda_1 a_1 + \dots + \lambda_{i-1} a_{i-1} + \lambda_{i+1} a_{i+1} + \dots + \lambda_n a_n,$$
$$b_i \text{ by } b_i' = b_i + \lambda_1 b_1 + \dots + \lambda_{i-1} b_{i-1} + \lambda_{i+1} b_{i+1} + \dots + \lambda_n b_n,$$
$$\dots\dots\dots$$
$$k_i \text{ by } k_i' = k_i + \lambda_1 k_1 + \dots + \lambda_{i-1} k_{i-1} + \lambda_{i+1} k_{i+1} + \dots + \lambda_n k_n.$$

Then
$$\Delta' = a_i' A_i + b_i' B_i + \dots + k_i' K_i$$
$$= (a_i A_i + b_i B_i + \dots + k_i K_i) + \Sigma \lambda_j (a_j A_i + b_j B_i + \dots + k_j K_i),$$

where j assumes all the values $1, 2, \dots n$ except i.

Now by Theorems I and II,

$$a_i A_i + b_i B_i + \dots + k_i K_i = \Delta, \quad a_j A_i + b_j B_i + \dots + k_j K_i = 0.$$

Thus $$\Delta' = \Delta.$$

§ 28. Reciprocal Matrices and Reciprocal Determinoids.

Let
$$[a]_m^n = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad \text{.......................(1)}$$
be any matrix, and let a_{ij} be any one of its elements.

Also let A_{ij} be defined by the equation
$$A_{ij} = (-1)^{\omega_{ij}} \Delta_{ij} \qquad \text{.............................(2),}$$
where ω_{ij} is the affect of a_{ij} in $[a]_m^n$, and Δ_{ij} is the corranged minor determinoid complementary to a_{ij} in $[a]_m^n$. Thus Δ_{ij} is the determinoid formed from $(a)_m^n$ by striking out the horizontal row and the vertical row in which a_{ij} occurs, and A_{ij} is the co-factor of the element a_{ij} in the determinoid $(a)_m^n$, or the co-factor of the element a_{ij} in the matrix $[a]_m^n$.

The matrix
$$[A]_m^n = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \qquad \text{.......................(3)}$$
will be called the *reciprocal matrix* of the matrix $[a]_m^n$.

Also the determinoid $(A)_m^n = \det [A]_m^n$ will be called the *reciprocal determinoid* of the determinoid $(a)_m^n = \det [a]_m^n$.

In accordance with these definitions the reciprocal matrix of $[a]_m^n$ is formed from $[a]_m^n$ by replacing each element by its co-factor; also the reciprocal determinoid of $(a)_m^n$ is formed from $(a)_m^n$ by replacing each element by its co-factor.

Ex. i. *If we multiply each element of the ith long row of a given matrix by the corresponding element of the jth long row of the reciprocal matrix, the sum of the products so formed is equal to zero when $j \neq i$ and is equal to the determinoid of the given matrix when $j = i$.*

This is another statement of the results (A), (B) and (C) of § 27.

Let $[a]_m^n$ be any matrix and $[A]_m^n$ its reciprocal matrix.

If the long rows of $[a]^n_m$ are horizontal, then

$$a_{i1}A_{j1}+a_{i2}A_{j2}+\dots+a_{in}A_{jn}=0 \text{ or } (a)^n_m$$

according as $j\neq i$ or $j=i$.

If the long rows of $[a]^n_m$ are vertical, then

$$a_{1i}A_{1j}+a_{2i}A_{2j}+\dots+a_{mi}A_{mj}=0 \text{ or } (a)^n_m$$

according as $j\neq i$ or $j=i$.

Ex. ii. Let $[ab\dots k]_{12\dots n}$ be a matrix whose long rows are horizontal and let $[AB\dots K]_{12\dots m}$ be its reciprocal matrix. Then

$$a_iA_j+b_iB_j+\dots+k_iK_j=0 \text{ or } (ab\dots k)_{12\dots m}$$

according as $j\neq i$ or $j=i$.

Ex. iii. *The reciprocal of the conjugate of a given matrix is identical with the conjugate of the reciprocal of the given matrix.*

Let $[a]^n_m$ be any matrix and $[A]$ its reciprocal matrix.

The conjugate of $[a]^n_m$ is $\overline{\underline{a}}\,^m_n$.

Now any element a_{ij} has the same total affect in $\overline{\underline{a}}\,^m_n$ as in $[a]^n_m$.

And the corranged minor determinoid complementary to a_{ij} in $\overline{\underline{a}}\,^m_n$ is the conjugate of the corranged minor determinoid complementary to a_{ij} in $[a]^n_m$.

Since any determinoid has the same value as its conjugate determinoid, it follows that the co-factor of a_{ij} in $\overline{\underline{a}}\,^m_n$ is equal to A_{ij}, the co-factor of a_{ij} in $[a]^n_m$.

Consequently the reciprocal of $\overline{\underline{a}}\,^m_n$ is $\overline{\underline{A}}\,^m_n$, which is the conjugate of $[A]^n_m$.

The conjugate of the reciprocal of a given matrix will be called the *conjugate reciprocal* of that matrix.

§ 29. Properties of the short rows of a determinoid.

Let
$$\Delta = \begin{array}{c} a_1\ a_2 \dots a_s \dots a_n \\ b_1\ b_2 \dots b_s \dots b_n \\ \dots\dots\dots\dots\dots \\ k_1\ k_2 \dots k_s \dots k_n \end{array}$$

be a determinoid with m long rows and n short rows, the long rows being horizontal; and let the co-factors of $a_s, b_s, \dots k_s$ be $A_s, B_s, \dots K_s$.

We cannot in general express Δ as a *homogeneous* linear function of the elements of any given short row, for Δ has complete derived products no factor of which belongs to the given short row. In fact, since the terms of Δ

which contain $a_s, b_s, \dots k_s$ respectively can be obtained as in § 27, and since the terms of Δ which contain no element from the sth short row are unaltered in value by putting $a_s = 0, b_s = 0, \dots k_s = 0$, we have

$$\Delta = a_s A_s + b_s B_s + \dots + k_s K_s + \begin{vmatrix} a_1 & a_2 \dots 0, \dots a_n \\ b_1 & b_2 \dots 0, \dots b_n \\ \dots & \dots \\ k_1 & k_2 \dots 0, \dots k_n \end{vmatrix},$$

where the last determinant is formed from Δ by replacing the sth vertical row by a row of 0's. Thus Δ is in general a *non-homogeneous* linear function of $a_s, b_s, \dots k_s$.

So if we make any selection of short rows less than $n - m + 1$ in number, Δ in its expanded form contains terms no factor of which belongs to the selected short rows. We conclude that Δ does not necessarily vanish when the elements of short rows less than $n - m + 1$ in number all vanish.

If however we make any selection of $n - m + 1$ short rows, then every complete product belonging to Δ contains at least one factor belonging to the selected rows. It must therefore be possible to express Δ as a sum of terms each of which contains at least one factor belonging to the selected rows, or we can expand Δ in terms of the elements of any given $n - m + 1$ short rows. Further Δ vanishes identically when any $n - m + 1$ short rows are rows of zeros.

The following two results are useful in obtaining expansions of the kind just mentioned.

$$\begin{vmatrix} a_1 & a_2 \dots a_s \dots a_n & 0 \\ b_1 & b_2 \dots b_s \dots b_n & 0 \\ \dots & \dots & \\ k_1 & k_2 \dots k_s \dots k_n & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \dots a_s \dots a_n \\ b_1 & b_2 \dots b_s \dots b_n \\ \dots & \dots \\ k_1 & k_2 \dots k_s \dots k_n \end{vmatrix} \quad \dots\dots\dots\dots\dots(A),$$

$$\begin{vmatrix} 0 & a_1 & a_2 \dots a_s \dots a_n \\ 0 & b_1 & b_2 \dots b_s \dots b_n \\ \dots & \dots & \dots \\ 0 & k_1 & k_2 \dots k_s \dots k_n \end{vmatrix} = (-1)^m \begin{vmatrix} a_1 & a_2 \dots a_s \dots a_n \\ b_1 & b_2 \dots b_s \dots b_n \\ \dots & \dots \\ k_1 & k_2 \dots k_s \dots k_n \end{vmatrix} \quad \dots\dots\dots(B).$$

To prove formula (A) we observe that the two determinoids have the same non-vanishing complete derived products, and that every such product has equal affects in the two determinoids.

To prove formula (B) we observe that the horizontal affect of every non-vanishing complete derived product of the determinoid on the left is greater by m than the horizontal affect of that same product in the determinoid on the right.

From formula (A) we see that:

The value of a determinoid is not altered by inserting in it an additional short row of 0's after the existing short rows.

All the results of the present article are consequences of the general result of § 30.

Ex. i. *To expand the determinoid* $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{vmatrix}$ *in terms of the elements of the first, third and fourth short rows.*

$$\Delta = \text{ sum of terms containing } a_1,\ b_1,\ c_1 \text{ in } \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{vmatrix}$$

$$+ \text{ sum of terms containing } a_3,\ b_3,\ c_3 \text{ in } \begin{vmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & b_2 & b_3 & b_4 & b_5 \\ 0 & c_2 & c_3 & c_4 & c_5 \end{vmatrix}$$

$$+ \text{ sum of terms containing } a_4,\ b_4,\ c_4 \text{ in } \begin{vmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & b_2 & 0 & b_4 & b_5 \\ 0 & c_2 & 0 & c_4 & c_5 \end{vmatrix}.$$

Thus Δ is expressible as a sum of nine terms each of which can be obtained as in long row expansions.

Ex. ii. *To expand the same determinoid in terms of the elements of the last three short rows.*

In this case

$$\Delta = \text{ sum of terms containing } a_5,\ b_5,\ c_5 \text{ in } \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{vmatrix}$$

$$+ \text{ sum of terms containing } a_4,\ b_4,\ c_4 \text{ in } \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

$$+ \text{ sum of terms containing } a_3,\ b_3,\ c_3 \text{ in } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

§ 30. Expansion of a determinoid in terms of its simple minor determinants.

Let Δ be any determinoid with m long rows and n short rows. Let $D_1, D_2, \ldots D_i, \ldots D_\nu$ be the determinants which can be formed from Δ by omissions of short rows. These are the corranged simple minors of Δ which are determinants. Their number is ν, where

$$\nu = \frac{n!}{m!\,(n-m)!} = \binom{n}{m}.$$

Let $\omega_1, \omega_2, \ldots \omega_i, \ldots \omega_\nu$ be the affects of $D_1, D_2, \ldots D_i, \ldots D_\nu$ in Δ.

Let S_i be the algebraical sum of all the complete derived products of D_i, when each product has the sign determined by its affect in Δ. Since every complete derived product of Δ belongs to one and only one of the determinants $D_1, D_2, \ldots D_i, \ldots D_\nu$, we see that

$$\Delta = S_1 + S_2 + \ldots + S_i + \ldots + S_\nu \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let P be any one of the complete derived products belonging to D_i, and let η and η' be the affects of P in Δ and in D_i.

Then $$S_i = \Sigma(-1)^\eta P, \quad D_i = \Sigma(-1)^{\eta'} P \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now by Theorem III a of § 25,

$$\eta = \eta' + \omega_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Therefore

$$S_i = \Sigma(-1)^{\eta'+\omega_i} P = (-1)^{\omega_i}\Sigma(-1)^{\eta'} P = (-1)^{\omega_i} D_i \ldots\ldots\ldots(4).$$

Consequently

$$\Delta = (-1)^{\omega_1} D_1 + (-1)^{\omega_2} D_2 + \ldots + (-1)^{\omega_\nu} D_\nu \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

This result can be stated as follows:

Theorem I. *A determinoid is the algebraical sum of all its corranged simple minor determinants, when each of these determinants has the sign determined by its affect in the original determinoid* ....(A).

Ex. i. Let $$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{vmatrix}.$$

If for the sake of brevity the determinant $\begin{vmatrix} a_p & a_q & a_r \\ b_p & b_q & b_r \\ c_p & c_q & c_r \end{vmatrix}$, which has the same value as $(abc)_{pqr}$, is denoted by (pqr), we have

$$\Delta = (1\,2\,3) - (1\,2\,4) + (1\,2\,5) + (1\,3\,4) - (1\,3\,5) + (1\,4\,5) - (2\,3\,4) + (2\,3\,5) - (2\,4\,5) + (3\,4\,5).$$

The sign of the leading determinant $(1\,2\,3)$ is positive.

The signs of all the other determinants can be immediately found (see § 19.5) by noticing that the determinant (pqr) has one sign when $p+q+r$ is even, and the opposite sign when $p+q+r$ is odd.

Ex. ii. Let $$\Delta = (abcdef)_{1234}.$$

If for the sake of brevity the determinant $(\alpha\beta\gamma\delta)_{1234}$ is denoted by $(\alpha\beta\gamma\delta)$, we have

$$\begin{aligned}\Delta = \; & (abcd) - (abce) + (abcf) + (abde) - (abdf) \\ & + (abef) - (acde) + (acdf) - (acef) + (adef) \\ & + (bcde) - (bcdf) + (bcef) - (bdef) + (cdef).\end{aligned}$$

Equation (5) remains true when the simple minor determinants D_1, D_2, ... D_i, ... D_ν are deranged in any way. We then make use of Theorem III*b* of § 25 and replace equation (3) by

$$\eta \equiv \eta' + \omega_i \quad (\text{mod. } 2).$$

In all other respects the reasoning is exactly as before. Accordingly we can replace (A) by the following more general statement:

Theorem II. *A determinoid is the algebraical sum of all its distinct simple minor determinants, corranged or deranged, when each of these determinants has the sign determined by its affect in the original determinoid* ....(B).

Here two determinants are regarded as distinct when one is not simply a derangement of the other, i.e. when one has a row which does not occur in the other.

The more general statement (B) can also be deduced from (A) with the help of Theorem V*a* of § 25.

Let D_1', D_2', ... D_i', ... D_ν' be any derangements of the corranged simple minor determinants D_1, D_2, ... D_i, ... D_ν, and let ω_1', ω_2', ... ω_i', ... ω_ν' be the affects of these deranged determinants in Δ. Then by Theorem V*a* of § 25,

$$(-1)^{\omega_i} D_i = (-1)^{\omega_i'} D_i'.$$

Accordingly from equation (5) it follows that

$$\Delta = (-1)^{\omega_1'} D_1' + (-1)^{\omega_2'} D_2' + \dots + (-1)^{\omega_\nu'} D_\nu',$$

which is the result embodied in statement (B).

Ex. iii. If
$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{vmatrix},$$

then
$$\Delta = (bac)_{213} + (bca)_{421} - (acb)_{125} - (abc)_{431} + (bca)_{315}$$
$$+ (cba)_{154} + (cab)_{243} - (acb)_{352} + (cba)_{245} - (bac)_{534}.$$

§ 31. Classification of simple minor determinoids.

1. *Long-cut and short-cut simple minors.*

A *simple* minor determinoid Δ' of a given fundamental determinoid Δ is a derived determinoid formed from Δ by striking out rows of one kind only, i.e. by striking out horizontal rows only or by striking out vertical rows only. It is understood that the retained rows may be re-arranged in any manner.

The simple minor determinoid Δ' will be called a *long-cut* or a *short-cut* simple minor determinoid according as the rows struck out from Δ are long rows or short rows of Δ. The number of rows retained of the kind struck out will be called the *reduced order* of the simple minor determinoid.

Let Δ have m long rows and n short rows.

If Δ' is formed from Δ by retaining any μ of the long rows and striking out the other long rows, then Δ' is a long-cut simple minor determinoid whose reduced and unreduced orders are μ and n.

If Δ' is formed from Δ by retaining any ν of the short rows and striking out the other short rows, then Δ' is a short-cut simple minor determinoid whose reduced and unreduced orders are ν and m.

2. *Inferior and superior minors.*

The simple minor determinoid Δ' will be called an *inferior* or a *superior* minor according as its reduced order is or is not less than the efficiency of the fundamental determinoid Δ. If Δ' is a superior minor, its efficiency is equal to the efficiency of Δ. If Δ' is an inferior minor, its efficiency is less than the efficiency of Δ.

All long-cut simple minor determinoids are inferior minors. A short-cut simple minor determinoid may be either an inferior or a superior minor.

The terms *long-cut, short-cut, inferior* and *superior* may also be applied with similar significations to the *simple minor matrices* of a fundamental matrix.

§ 32. Expansion of a determinoid in terms of the simple minor determinants of a given long-cut minor matrix.

The theorem to be now proved can be enunciated as follows:

Theorem I. *A determinoid is the algebraical sum of all the products which are obtained when each affected simple minor determinant of any given long-cut minor matrix is multiplied by its corranged complementary determinoid* ....(A).

An affected simple minor determinant of the given long-cut minor matrix is here understood to be a simple minor determinant of that matrix provided with the sign determined by its affect in the given determinoid.

Let
$$\Delta = (a)_m^n = \left| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix} \right|$$

be a determinoid with m horizontal long rows and n vertical short rows.

Let
$$U = [a_{u1}]_{\mu}^{n} = \begin{bmatrix} a_{u_1 1} & a_{u_1 2} & \dots & a_{u_1 n} \\ a_{u_2 1} & a_{u_2 2} & \dots & a_{u_2 n} \\ \dots & \dots & \dots & \dots \\ a_{u_\mu 1} & a_{u_\mu 2} & \dots & a_{u_\mu n} \end{bmatrix}$$

be the *corranged* minor matrix formed from Δ by retaining the u_1th, u_2th, ... u_μth long rows and striking out the remaining long rows.

Then

$[u_1 u_2 \dots u_\mu]$ is some corranged minor of the sequence $[1\ 2 \dots n]$.

Let
$$B = (b)_{\mu}^{\mu}$$

be a corranged determinant of order μ formed from U by striking out $n-\mu$ short rows, so that

$[b_{11} b_{12} \dots b_{1\mu}]$ is a corranged minor of $[a_{u_1 1} a_{u_1 2} \dots a_{u_1 n}]$.

Let
$$C = (c)_{\rho}^{\sigma}$$

be the determinoid complementary to B in Δ. It is a corranged minor determinoid of Δ whose long rows are horizontal, and ρ and σ are given by

$$\rho = m - \mu, \quad \sigma = n - \mu.$$

Let ω be the affect of B in Δ.

Then we will prove (A) by showing that

$$\Delta = \Sigma (-1)^{\omega} BC \text{ ....(1)},$$

where the various terms in the sum are obtained by choosing the determinant B from the fixed matrix U in all possible ways.

The number of terms in the sum is r, where

$$r = \frac{n!}{\mu!(n-\mu)!} = \binom{n}{\mu}.$$

If the various determinants which B can be are denoted by $B_1, B_2, \dots B_r$, and if the determinoids complementary to them in Δ are $C_1, C_2, \dots C_r$, and if the affects of $B_1, B_2, \dots B_r$ in Δ are $\omega_1, \omega_2, \dots \omega_r$, the result to be proved can be expressed more fully in the form

$$\Delta = (-1)^{\omega_1} B_1 C_1 + (-1)^{\omega_2} B_2 C_2 + \dots + (-1)^{\omega_r} B_r C_r \text{ ....(2)}.$$

Let S be the algebraical sum of all the complete derived products of Δ which occur in the expanded product BC, each derived product having the sign determined by its affect in Δ, and let the corresponding sums for the products $B_1C_1, B_2C_2, \dots B_rC_r$ be denoted respectively by $S_1, S_2, \dots S_r$.

Every complete derived product of Δ occurs in one and only one of the sums $S_1, S_2, \dots S_i, \dots S_r$; for if P is any given complete derived product of Δ, there is one and only one of the determinants $B_1, B_2, \dots B_i, \dots B_r$ which contain *all* those factors of P which occur in the matrix U, and if B_i is the determinant which contains them, then the remaining factors of P must all occur in the determinoid C_i, and therefore P occurs as a term in the expanded product B_iC_i. Moreover the terms of the sums $S_1, S_2, \dots S_r$ are all of them complete derived products of Δ with the signs determined by their affects in Δ. It follows that

$$\Delta = S_1 + S_2 + \dots + S_r \qquad\qquad (3).$$

The result which we wish to prove will therefore be established if we can show that

$$S_1 = (-1)^{\omega_1} B_1C_1, \quad S_2 = (-1)^{\omega_2} B_2C_2, \dots S_r = (-1)^{\omega_r} B_rC_r \qquad (4).$$

This we proceed to do. It will be sufficient to prove that

$$S = (-1)^{\omega} BC \qquad\qquad (5).$$

Let $b_1b_2 \dots b_\mu$ be any complete derived product of B, and let β be its affect in B.

Let $c_1c_2 \dots c_\rho$ be any complete derived product of C, and let γ be its affect in C.

Let $b_1b_2 \dots b_\mu c_1c_2 \dots c_\rho$, which is a complete derived product of Δ, have affect α in Δ.

Then
$$B = \Sigma (-1)^{\beta} \quad b_1b_2 \dots b_\mu,$$
$$C = \Sigma (-1)^{\gamma} \quad c_1c_2 \dots c_\rho,$$
$$BC = \Sigma (-1)^{\beta+\gamma} b_1b_2 \dots b_\mu c_1c_2 \dots c_\rho \qquad (6),$$
$$S = \Sigma (-1)^{\alpha} \quad b_1b_2 \dots b_\mu c_1c_2 \dots c_\rho \qquad (7).$$

Now by Theorem III of § 19, the affect of $b_1b_2 \dots b_\mu c_1c_2 \dots c_\rho$ in Δ
= the affect of $b_1b_2 \dots b_\mu$ in Δ + the affect of $c_1c_2 \dots c_\rho$ in C.

Also by Theorem III*a* of § 25, the affect of $b_1b_2 \dots b_\mu$ in Δ
= the affect of $b_1b_2 \dots b_\mu$ in B + the affect of B in Δ.

Therefore the affect of $b_1b_2 \dots b_\mu c_1c_2 \dots c_\rho$ in Δ
= the affect of $b_1b_2 \dots b_\mu$ in B + the affect of $c_1c_2 \dots c_\rho$ in C
+ the affect of B in Δ.

The equivalent of this result in symbols is

$$\alpha = \beta + \gamma + \omega.$$

Substituting this value for α in (7), we have

$$S = \Sigma (-1)^{\beta+\gamma+\omega} \quad b_1b_2 \dots b_\mu c_1c_2 \dots c_\rho$$
$$= (-1)^{\omega} \Sigma (-1)^{\beta+\gamma} b_1b_2 \dots b_\mu c_1c_2 \dots c_\rho$$
$$= (-1)^{\omega} BC.$$

We have now proved (5) or (4), and this with the help of (3) leads to equation (2) or (1), which is equivalent to (A).

Referring to Theorem V*a* of § 25, it appears that the result obtained remains true when the matrix U and the determinants $B_1, B_2, \dots B_r$ are deranged, provided only that the determinoids $C_1, C_2, \dots C_r$ are corranged.

It has been assumed in the course of the proof that the long rows of the determinoid are horizontal, but this is clearly not essential. The same result is true when the long rows are vertical.

We will define the *co-factor of any minor determinant* B (corranged or deranged) of a determinoid (or matrix) Δ to be the corranged minor determinoid complementary to B in Δ provided with the sign determined by the affect of B in Δ. Then the result which we have obtained can also be stated as follows:

Alternative Form of Theorem I. *A determinoid is the algebraical sum of all the products which are obtained when every distinct simple minor determinant of a given long-cut minor matrix is multiplied by its co-factor* ......(B).

This theorem can be regarded as a generalisation of *Laplace's* development of a determinant.

Ex. i. Let
$$\Delta=(abcde)_{1234}=\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \end{vmatrix}.$$

Expanding this determinoid in terms of the simple minor determinants of the matrix of the first and second long rows, we have

$$\begin{aligned}\Delta= & (ab)_{12}(cde)_{34}-(ac)_{12}(bde)_{34}+(ad)_{12}(bce)_{34}-(ae)_{12}(bcd)_{34}+(bc)_{12}(ade)_{34}\\ & -(bd)_{12}(ace)_{34}+(be)_{12}(acd)_{34}+(cd)_{12}(abe)_{34}-(ce)_{12}(abd)_{34}+(de)_{12}(abc)_{34}\\ = & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.\begin{vmatrix} c_3 & d_3 & e_3 \\ c_4 & d_4 & e_4 \end{vmatrix}-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.\begin{vmatrix} b_3 & d_3 & e_3 \\ b_4 & d_4 & e_4 \end{vmatrix}+\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}.\begin{vmatrix} b_3 & c_3 & e_3 \\ b_4 & c_4 & e_4 \end{vmatrix}\\ & -\begin{vmatrix} a_1 & e_1 \\ a_2 & e_2 \end{vmatrix}.\begin{vmatrix} b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}+\text{etc.}\end{aligned}$$

If we expand it in terms of the simple minor determinants of the matrix of the first, third and fourth long rows, we obtain

$$\begin{aligned}\Delta= & (abc)_{134}(de)_2-(abd)_{134}(ce)_2+(abe)_{134}(cd)_2+(acd)_{134}(be)_2-(ace)_{134}(bd)_2\\ & +(ade)_{134}(bc)_2-(bcd)_{134}(ae)_2+(bce)_{134}(ad)_2-(bde)_{134}(ac)_2+(cde)_{134}(ab)_2\\ = & \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}.\,d_2e_2-\begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}.\,c_2e_2+\text{etc.}\\ = & \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}(d_2-e_2)-\begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}(c_2-e_2)+\text{etc.}\end{aligned}$$

Ex. ii. Let $$\Delta=(abcd)_{12345}=\left|\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \\ a_5 & b_5 & c_5 & d_5 \end{matrix}\right|.$$

Expanding the determinoid in terms of the simple minor determinants of the matrix of the second and fourth long rows, we have

$$\Delta=-(bd)_{12}(ac)_{345}+(bd)_{13}(ac)_{245}-(bd)_{14}(ac)_{235}+(bd)_{15}(ac)_{234}-(bd)_{23}(ac)_{145}$$
$$+(bd)_{24}(ac)_{135}-(bd)_{25}(ac)_{134}-(bd)_{34}(ac)_{125}+(bd)_{35}(ac)_{124}-(bd)_{45}(ac)_{123}.$$

Ex. iii. $$\left|\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}\right|=0.$$

This is seen by expanding the determinoid in terms of the simple minor determinants of the matrix of the first three long rows. The co-factor of every determinant

$$=\pm|1,\ 1|=\pm(1-1)=0.$$

Ex. iv. $$\left|\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}\right|=\left|\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{matrix}\right|.$$

For if we denote the first determinoid by Δ, and the second by Δ', and if D is any corranged simple minor determinant of Δ' and ω is the affect of D in Δ or Δ', we have

$$\Delta=\Sigma(-1)^{\omega}D.|1,\ 1,\ 1|=\Sigma(-1)^{\omega}D(1-1+1)=\Sigma(-1)^{\omega}D=\Delta'.$$

Ex. v. Generally if $$\Delta=\left|\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \\ 1 & 1 & \dots & 1 \end{matrix}\right|, \text{ where } n>m,$$

then $$\Delta=0 \quad\text{or}\quad \Delta=(a)^n_m$$

according as $n-m$ is even or odd.

Thus if an additional long row of 1's is inserted after the existing long rows in a determinoid with m long rows and n short rows, the value of the determinoid is unaltered if $n-m$ is odd and is reduced to 0 if $n-m$ is even.

Let U and U' be two similar corranged long-cut simple minor matrices of Δ, both composed of the same number of long rows, and let B and B' be two similarly formed or *corresponding* simple minor determinants (corranged or deranged) of U and U'; also let C and C' be the corranged minor determinoids of Δ complementary to B and B' respectively. Then if ω is the affect of B in Δ, the value of the algebraical sum $\Sigma(-1)^{\omega}BC'$ can be found by means of Theorem I.

Let V and V' be the corranged long-cut simple minor matrices of Δ complementary to U and U' respectively, and let Δ' be the determinoid formed from Δ when the long rows of V are replaced by the long rows of V'. Then by Theorem I

$$\Sigma (-1)^{\omega} BC' = \Delta' \qquad \text{.............................(8).}$$

If U' is the same as U, then C' is the same as C, the corranged minor determinoid complementary to B in Δ, and Δ' is the same as Δ, and equation (8) is simply the equation given by Theorem I.

If U' is not the same as U, then U has long rows which do not occur in U' and which therefore do occur in V'. In this case Δ', which contains the long rows of U and V', has at least one pair of identical long rows and vanishes; and therefore in this case

$$\Sigma (-1)^{\omega} BC' = 0.$$

Thus $\qquad \Sigma (-1)^{\omega} BC' = \Delta$ or 0 ....(9),

according as U' is or is not the same as U.

Now let

ω' = the affect of B' in Δ,

ϵ = the affect of B in U = the affect of B' in U',

η = the affect of U in Δ, $\qquad \eta'$ = the affect of U' in Δ.

Then by Theorem III*d* of § 25,

$$\omega = \epsilon + \eta, \quad \omega' = \epsilon + \eta', \text{ and therefore } \omega' = \omega + \eta' - \eta \text{ ......(10).}$$

If U' is the same as U, then B' is the same as B, C' is the same as C, $\omega' = \omega$, and

$$\Sigma (-1)^{\omega'} BC' = \Sigma (-1)^{\omega} BC = \Delta.$$

If U' is not the same as U, then

$$\Sigma (-1)^{\omega'} BC' = (-1)^{\eta'-\eta} \Sigma (-1)^{\omega} BC' = 0.$$

Thus $\qquad \Sigma (-1)^{\omega'} BC' = \Delta$ or 0 ....(11),

according as U' is or is not the same as U.

If we understand an *affected* simple minor determinant of U to be a simple minor determinant of U provided with the sign determined by its affect in Δ, then equation (9) gives rise to the following second theorem in which Theorem I is included:

Theorem II. *If U and U' are any two given similar corranged long-cut simple minor matrices of a given determinoid Δ, the algebraical sum of the products obtained by multiplying each affected simple minor determinant of U by the corranged complement of the corresponding minor determinant of U' is equal to Δ or 0 according as U' is or is not the same as U* ....(C).

In equation (11), $(-1)^{\omega'} C'$ is the co-factor of B' in Δ. Hence by means of this equation we can express Theorem II in another form.

Alternative Form of Theorem II. *If U and U' are any two given similar corranged long-cut simple minor matrices of a given determinoid Δ, the algebraical sum of the products obtained by multiplying each simple minor determinant of U (corranged or deranged) by the co-factor in Δ of the corresponding minor determinant of U' is equal to Δ or 0 according as U' is or is not the same as U* ....(D).

Ex. vi. Let $\Delta=[abcde]_{12345}$, $U=[abcde]_{123}$, $U'=[abcde]_{234}$. Then

$$(abc)_{123}(de)_{15}-(abd)_{123}(ce)_{15}+(abe)_{123}(cd)_{15}+(acd)_{123}(be)_{15}-(ace)_{123}(bd)_{15}$$
$$+(ade)_{123}(bc)_{15}-(bcd)_{123}(ae)_{15}+(bce)_{123}(ad)_{15}-(bde)_{123}(ac)_{15}+(cde)_{123}(ab)_{15}=0.$$

Ex. vii. When the standard double suffix notation is employed, we can give formulae which are equivalent to Theorems I and II.

Let $\qquad \Delta=(a)_m^n \quad$ be any determinoid.

If $m<n$, let

$[a_1 a_2 \dots a_\mu]$, $[\beta_1\beta_2\dots\beta_{m-\mu}]$ be fixed corranged minors of the sequence $[1\,2\dots m]$,

$[x_1 x_2 \dots x_\mu]$, $[y_1 y_2 \dots y_{n-\mu}]$ be complementary corranged minors of $[1\,2\dots n]$,

each minor sequence having the fixed order shown by its last suffix.

Then if ω is the affect of $(a_{\alpha x})_\mu^\mu$ in $(a)_m^n$, we have

$$\sum_x (-1)^\omega (a_{\alpha x})_\mu^\mu (a_{\beta y})_{m-\mu}^{n-\mu} = (a)_m^n \text{ or } 0 \;\dots\dots\dots\dots\dots\dots\dots\dots (\text{E}),$$

according as the minor sequences $[a_1 a_2 \dots a_\mu]$, $[\beta_1\beta_2\dots\beta_{m-\mu}]$ are or are not complementary, i.e. according as they do not or do contain common elements.

If $m>n$, let

$[a_1 a_2 \dots a_\nu]$, $[\beta_1\beta_2\dots\beta_{n-\nu}]$ be fixed corranged minors of the sequence $[1\,2\dots n]$,

$[x_1 x_2 \dots x_\nu]$, $[y_1 y_2 \dots y_{m-\nu}]$ be complementary corranged minors of $[1\,2\dots m]$,

each minor sequence having the fixed order shown by its last suffix.

Then if ω is the affect of $(a_{x\alpha})_\nu^\nu$ in $(a)_m^n$, we have

$$\sum_x (-1)^\omega (a_{x\alpha})_\nu^\nu (a_{y\beta})_{m-\nu}^{n-\nu} = (a)_m^n \text{ or } 0 \;\dots\dots\dots\dots\dots\dots\dots\dots (\text{F}),$$

according as the minor sequences $[a_1 a_2 \dots a_\nu]$, $[\beta_1\beta_2\dots\beta_{n-\nu}]$ are or are not complementary, i.e. according as they do not or do contain common elements.

Although all minor sequences have been supposed to be corranged, it is clear that the minor sequences occurring in the determinantal factor of each term may be either corranged or deranged.

The summation in (E) extends over all distinct values of the minor sequence $[x_1 x_2 \dots x_\mu]$, and the summation in (F) extends over all distinct values of the minor sequence $[x_1 x_2 \dots x_\nu]$.

Ex. viii. Let m, p, q be positive integers such that $p>q$.

Let $[x_1 x_2 \dots x_m]$, $[y_1 y_2 \dots y_p]$ be complementary corranged minors of the sequence $[1\,2\,3 \dots (m+p)]$, and let ω be the affect of $[x_1 x_2 \dots x_m]$ in $[1\,2\,3 \dots (m+p)]$.

Then $$\Sigma (-1)^{\omega} (a_{1x})_m^m (b_{1y})_q^p = \begin{bmatrix} a \\ b \end{bmatrix}_{m,\,q}^{m+p} \quad \text{.............................(G)},$$

$$\Sigma (-1)^{\omega} (a_{x1})_m^m (b_{y1})_p^q = [a,\ b]_{m+p}^{m,\,q} \quad \text{..........................(H)}.$$

Ex. ix. Let U and U'' be any two long-cut simple minor matrices of the determinoid Δ *whose long rows only are deranged in any manner*, and let other letters have the same meanings as in the text.

Then if U' does not consist of the same long rows of Δ as U, we have

$$\Sigma (-1)^{\omega} BC'' = 0, \qquad \Sigma (-1)^{\omega'} BC' = 0 \quad \text{.......................(12)}.$$

The first of these results is proved as in the text.

To prove the second result, we replace equations (10) by

$$\omega \equiv \epsilon + \eta \pmod{2}, \qquad \omega' \equiv \epsilon + \eta' \pmod{2}, \qquad \omega' \equiv \omega + \eta' - \eta \pmod{2},$$

and then proceed as in the text.

If U'' consists of the same long rows as U, differently arranged, and if Ω is the affect of U'' in U, so that by Theorem II*b* of § 25

$$\Omega \equiv \eta' - \eta \pmod{2},$$

we have $$\Sigma (-1)^{\omega} BC' = (-1)^{\Omega} \Delta, \qquad \Sigma (-1)^{\omega'} BC' = \Delta \quad \text{....................(13)}.$$

We deduce the first of these results from equation (8). Since we can convert U' into U by Ω interchanges of pairs of long rows, we can also convert Δ' into Δ by Ω interchanges of pairs of long rows, and therefore

$$\Delta' = (-1)^{\Omega} \Delta.$$

Giving this value to Δ' in (8), we obtain the first of equations (13).

Then to obtain the second equation, we have

$$\Sigma (-1)^{\omega'} BC'' = (-1)^{\eta' - \eta} . \Sigma (-1)^{\omega} BC'' = (-1)^{\omega} . (-1)^{\omega} \Delta = \Delta.$$

From the second results in (12) and (13) we deduce the following generalisation of (D):

If U and U' are any two given similar long-cut simple minor matrices of a determinoid Δ whose long rows only are deranged, the algebraical sum of the products obtained by multiplying each simple minor determinant of U (corranged or deranged) by the co-factor of the similarly formed simple minor determinant of U' is equal to Δ or 0 according as U' does or does not consist of the same long rows of Δ as U ....(I).

§ 33. Algebraical sum of superior short-cut simple minor determinoids of given reduced order.

We shall prove the following theorem:

Theorem. *The algebraical sum S_ν of all the corranged superior simple minor determinoids of reduced order ν derived from a fundamental determinoid Δ with m long rows and n short rows, when each minor has the sign determined by its affect in Δ, is given by*

$$S_\nu = \Delta \times Q_{\nu\ \ m}^{n-m},$$

where $Q^{n-m}_{\nu-m} = \Sigma(-1)^k$, *the values of k being the affects of all possible corranged minor sequences of* $\nu - m$ *elements derived from a fundamental sequence of* $n - m$ *elements* ....(A).

We may without loss of generality confine ourselves to the case in which the long rows of Δ are horizontal, and will therefore suppose that

$$\Delta = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ \dots & \dots & \dots & \dots \\ c_1 & c_2 & \dots & c_n \end{vmatrix},$$

where there are m horizontal long rows and n vertical short rows.

Let Δ_ν be any *corranged* determinoid derived from Δ by retaining ν of the short rows and striking out the other short rows, ν being not less than m, and let ω be the affect of Δ_ν in Δ.

Then $$S_\nu = \Sigma(-1)^\omega \Delta_\nu \quad \text{..............................(1).}$$

In the special case in which $\nu = m$, it has been already proved in § 30 that $S_\nu = \Delta$. This special case will be included in (A) if we make the convention that $Q^{n-m}_0 = 1$. In future we shall assume that ν is greater than m.

It is clear that S_ν can be expressed as a homogeneous linear function of the simple minor determinants of order m which can be derived from Δ by omissions of short rows, for by § 31 every one of the determinoids Δ_ν can be so expressed.

To evaluate S_ν we shall find the coefficient of every such determinant in S_ν.

Let $[\alpha_1 \alpha_2 \dots \alpha_m]$ be any corranged sequence of m integers selected from the fundamental sequence $[1\ 2 \dots n]$ of the suffixes occurring in Δ.

Let $[\beta_1 \beta_2 \dots \beta_{\nu-m}]$ be any corranged sequence of $\nu - m$ integers selected from the remaining $n - m$ integers.

Let $[\gamma_1 \gamma_2 \dots \gamma_{n-\nu}]$ be the corranged sequence which can be formed with the $n - \nu$ integers which are still left.

Let $[p_1 p_2 \dots p_\nu]$ be the corranged sequence which can be formed with the integers $\alpha_1, \alpha_2, \dots \alpha_m, \beta_1, \beta_2, \dots \beta_{\nu-m}$ taken together.

Let $[q_1 q_2 \dots q_{n-m}]$ be the corranged sequence which can be formed with the integers $\beta_1, \beta_2, \dots \beta_{\nu-m}, \gamma_1, \gamma_2, \dots \gamma_{n-\nu}$ taken together.

In each of these sequences the integers which occur are arranged in ascending order of magnitude.

Let $$D = \begin{vmatrix} a_{\alpha_1} & a_{\alpha_2} \dots a_{\alpha_m} \\ b_{\alpha_1} & b_{\alpha_2} \dots b_{\alpha_m} \\ \dots\dots & \dots\dots \\ c_{\alpha_1} & c_{\alpha_2} \dots c_{\alpha_m} \end{vmatrix}, \quad \Delta_\nu' = \begin{vmatrix} a_{p_1} & a_{p_2} \dots a_{p_\nu} \\ b_{p_1} & b_{p_2} \dots b_{p_\nu} \\ \dots\dots & \dots\dots \\ c_{p_1} & c_{p_2} \dots c_{p_\nu} \end{vmatrix},$$

so that D is any simple minor determinant of order m derived from Δ by striking out short rows, and Δ_ν' is any one of the determinoids Δ_ν which contains D as a minor.

Let ω' = the affect of Δ_ν' in Δ = the affect of $[p_1 p_2 \dots p_\nu]$ in $[1\ 2\ \dots\ n]$,

η = the affect of D in Δ = the affect of $[\alpha_1 \alpha_2 \dots \alpha_m]$ in $[1\ 2\ \dots\ n]$,

η' = the affect of D in Δ_ν' = the affect of $[\alpha_1 \alpha_2 \dots \alpha_m]$ in $[p_1 p_2 \dots p_\nu]$,

k = the affect of $[\beta_1 \beta_2 \dots \beta_{\nu-m}]$ in $[q_1 q_2 \dots q_{n-m}]$.

The terms of S_ν, as given by equation (1), are the terms such as $(-1)^{\omega'} \Delta_\nu'$, where Δ_ν' has the value just assigned.

Now the coefficient of D in the expansion of Δ_ν' in terms of its simple minor determinants is $(-1)^{\eta'}$.

Therefore the coefficient of D in $(-1)^{\omega'} \Delta_\nu'$ is $(-1)^{\eta'+\omega'}$.

Hence the coefficient of D in S_ν, when S_ν is expanded in terms of the simple minor determinants of Δ, is $\Sigma(-1)^{\eta'+\omega'}$, where the various terms of the sum are obtained by keeping $\alpha_1, \alpha_2, \dots \alpha_m$ fixed and selecting $\beta_1, \beta_2, \dots \beta_{\nu-m}$ in all possible ways.

But by Theorem VI of § 19,

$$\omega' + \eta' = \eta + k \quad\dots\dots(2).$$

Hence the coefficient of D in $S_\nu = \Sigma(-1)^{\eta+k} = (-1)^\eta \Sigma(-1)^k$.

We now write

$$Q_{\nu-m}^{n-m} = \Sigma(-1)^k \quad\dots\dots(3),$$

where the different values assumed by k are the different values of the affect of $[\beta_1 \beta_2 \dots \beta_{\nu-m}]$ in $[q_1 q_2 \dots q_{n-m}]$, the latter sequence being a given fundamental sequence of $n-m$ elements, and the former sequence being any corranged minor sequence of $\nu - m$ elements derived from it.

Then the coefficient of D in S_ν is $(-1)^\eta Q_{\nu-m}^{n-m}$.

It follows that

$$S_\nu = Q_{\nu-m}^{n-m} \{(-1)^{\eta_1} D_1 + (-1)^{\eta_2} D_2 + (-1)^{\eta_3} D_3 + \dots\} \quad\dots\dots(4),$$

where $D_1, D_2, D_3, \dots$ are the different corranged determinants of order m which can be derived from Δ, and $\eta_1, \eta_2, \eta_3, \dots$ are their affects in Δ.

By § 30 the expression inside the large brackets in (4) is equal to Δ.

Accordingly $$S_\nu = Q^{n-m}_{\nu-m}\,\Delta \qquad\qquad (5).$$

Thus the theorem (A) is proved.

When D is the leading simple minor determinant of Δ, $\eta = 0$. The coefficient of this determinant in S_ν is therefore $Q^{n-m}_{\nu-m}$. Thus we could have defined $Q^{n-m}_{\nu-m}$ as being the coefficient of the leading simple minor determinant of Δ in S_ν.

We could also have defined it as being the sum $\Sigma(-1)^k$ when the values assumed by k are the affects of all corranged minors of a determinoid with $n-m$ rows which can be formed by retaining only $\nu - m$ of those rows.

Ex. i. Let $$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}.$$

Then $$S_3 = -\begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= 0.$$

In this case $n=4,\quad m=2,\quad \nu=3,\quad \nu-m=1,\quad n-m=2.$

Hence $$S_3 = Q^2_1\,\Delta.$$

Also $Q^2_1 = \Sigma(-1)^k$ where the values of k are the affects of [1] and [2] in [1 2], which are 0 and 1.

Thus $$Q^2_1 = (-1)^0 + (-1)^1 = 1 - 1 = 0,$$

and therefore $$S_3 = 0.$$

The value of Q^2_1 can also be seen by picking out the coefficient of the leading determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ or of its leading product $a_1 b_2$ in S_3.

If the determinoids of S_3 are expanded in terms of their minor determinants of order 2, the terms containing the leading determinant are

$$-\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \text{ and } +\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Therefore the coefficient of the leading determinant in S_3 is 0.

Ex. ii. Let $$\Delta = (abc)_{1234567}.$$

For the sake of brevity let the determinoid $(abc)_{pqrst}$ formed with the horizontal rows whose suffixes are p, q, r, s, t be denoted by $(pqrst)$.

Then
$$\begin{aligned} S_5 = {} & (12345) - (12346) + (12347) + (12356) - (12357) + (12367) - (12456) \\ & + (12457) - (12467) + (12567) + (13456) - (13457) + (13467) - (13567) \\ & + (14567) - (23456) + (23457) - (23467) + (23567) - (24567) + (34567) \\ = {} & 2\Delta. \end{aligned}$$

To obtain the value of S_5 we observe that

$$n=7, \quad m=3, \quad \nu=5, \quad \nu-m=2, \quad n-m=4.$$

Accordingly $$S_5=Q_2^4\Delta.$$

Now $Q_2^4=\Sigma(-1)^k$, where the values of k are the affects of [1 2], [1 3], [1 4], [2 3], [2 4], [3 4] in [1 2 3 4], i.e. the values of k are 0, 1, 2, 2, 3, 4.

Thus $$Q_2^4=(-1)^0+(-1)^1+(-1)^2+(-1)^2+(-1)^3+(-1)^4=2,$$

and therefore $$S_5=2\Delta.$$

Ex. iii. Theorem (A) remains true when the *long rows* of the simple minor determinoids are not corranged.

For by Theorem V *b* of § 25, if Δ_ν is any *corranged* superior short-cut simple minor determinoid of reduced order ν derived from Δ, and if Δ_ν' is formed from Δ_ν by derangements of its long rows, and if ω and ω' are the affects of Δ_ν and Δ_ν' in Δ, we have

$$(-1)^\omega\Delta_\nu=(-1)^{\omega'}\Delta_\nu',$$

and therefore $$S_\nu=\Sigma(-1)^\omega\Delta_\nu=\Sigma(-1)^{\omega'}\Delta_\nu'.$$

§ 34. Properties of the number Q_m^n.

It is supposed that m and n are positive integers such that $m \not> n$.

1. *Various forms of the definition.*

(i) $Q_m^n=\Sigma(-1)^\omega$, where the values ascribed to ω are the affects of all corranged minor sequences of m elements in a fundamental sequence of n elements.

This is the definition of Q_m^n given in § 33.

(ii) $Q_m^n=\Sigma(-1)^\omega$, where ω is the affect of $[p_1 p_2 \ldots p_m]$ in $[1\,2 \ldots n]$, and $p_1, p_2, \ldots p_m$ are any m different integers selected from 1, 2, ... n and arranged in ascending order of magnitude.

This definition is equivalent to (i) above.

(iii) Q_m^n is the algebraical sum of the coefficients when a determinoid with m long rows and n short rows is expanded in terms of its simple minor determinants of order m.

This statement is again equivalent to (i) above.

(iv) Q_m^n is the coefficient of the leading simple minor determinant in the algebraical sum of the short-cut simple minor determinoids of reduced order $m+r$ derived from a fundamental determinoid with r long rows and $n+r$ short rows.

This has been proved in § 33.

(v) $Q_m^n=(-1)^{\frac{1}{2}m(m+1)}\Sigma(-1)^{p_1+p_2+\ldots+p_m}$, where $p_1, p_2, \ldots p_m$ are any m different integers selected from 1, 2, ... n and *arranged in ascending order of magnitude.*

This follows at once from (ii) above, for if ω is the affect of $[p_1 p_2 \dots p_m]$ in $[1\ 2 \dots n]$, we have

$$\omega = \sum_{k=1}^{k=m} \{(p_k - 1) - (k - 1)\} = (p_1 + p_2 + \dots + p_m) - \tfrac{1}{2}m(m+1).$$

(vi) $Q^n_m = \frac{1}{m!}(-1)^{\frac{1}{2}m(m+1)} \Sigma (-1)^{p_1+p_2+\dots+p_m}$, where $p_1, p_2, \dots p_m$ are *all possible arrangements of m unequal numbers* selected from $1, 2, \dots n$.

This is obtained from (v) by removing the restriction as to the order of arrangement of $p_1, p_2, \dots p_m$.

2. *Special properties.*

(i) $Q^n_n = 1$.

This is an immediate consequence of the first definition of Q^n_m.

(ii) $Q^n_m = Q^n_{n-m}$.

To prove this, let $[b_1 b_2 \dots b_m]$ be any corranged minor sequence of order m of the fundamental sequence $[a_1 a_2 \dots a_n]$, and let $[c_1 c_2 \dots c_{n-m}]$ be the complementary corranged minor sequence formed with the remaining elements of the fundamental sequence.

Let ω be the affect of $[b_1 b_2 \dots b_m]$ in $[a_1 a_2 \dots a_n]$,

and let ω' be the affect of $[c_{n-m} \dots c_2 c_1]$ in $[a_n \dots a_2 a_1]$.

Then $$Q^n_m = \Sigma(-1)^{\omega}, \quad Q^n_{n-m} = \Sigma(-1)^{\omega'}.$$

But by Theorem VIII *b* of § 19, $\omega' = \omega$.

Consequently $$Q^n_{n-m} = Q^n_m.$$

(iii) $Q^n_0 = 1$.

This is the conventional definition of Q^n_0 given in § 33. That it is the only possible definition can be seen by putting $m=0$ in (iv) of sub-article 1. We can also obtain it from (i) and (ii) above, for from these we have

$$Q^n_0 = Q^n_n = 1.$$

(iv) $Q^0_0 = 1$.

For by (iv) of sub-article 1, Q^0_0 is the coefficient of $(a)^r_r$ in the algebraical sum of the simple minor determinoids of reduced order r derived from the fundamental determinoid $(a)^r_r$.

3. *Formulae of reduction for* Q^n_m.

Various formulae of reduction can be found by means of which the values of Q^n_m can be determined for all values of m and n, m being $\not> n$. Four of the most useful of these formulae are given below.

(i) *First pair of formulae.*

$$Q^{m+r}_m = Q^{m+r-1}_{m-1} + (-1)^m Q^{m+r-2}_{m-1} + (-1)^{2m} Q^{m+r-3}_{m-1} + \dots + (-1)^{rm} Q^{m-1}_{m-1} \quad \dots(A),$$

$$Q^{m+r}_m = Q^{m+r-1}_r + (-1)^m Q^{m+r-2}_{r-1} + (-1)^{2m} Q^{m+r-3}_{r-2} + \dots + (-1)^{rm} Q^{m-1}_0 \quad \dots(B).$$

The second of these can be deduced from the first by means of the result

$$Q_m^{m+n} = Q_n^{m+n}.$$

We proceed to prove formula (A).

We have in the first place by formula (G) of § 18

$$Q_m^{m+r} = \Sigma(-1)^{(p_1+p_2+\ldots+p_m)-\frac{1}{2}m(m+1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $[p_1\, p_2 \ldots p_m]$ is any corranged minor of order m of $[1\ 2 \ldots (m+r)]$.

The various terms in the sum are obtained by choosing for $p_1, p_2, \ldots p_m$ any m of the integers 1, 2, ... $(m+r)$ arranged in ascending order of magnitude. The smallest integer p_1 can have any one of the values 1, 2, ... k, ... $(r+1)$.

Let S_1 be the sum of the terms in which p_1 has the value 1; let S_2 be the sum of the terms in which p_1 has the value 2; and generally let S_k be the sum of the terms in which p_1 has the value k; so that

$$Q_m^{m+r} = S_1 + S_2 + \ldots + S_k + \ldots + S_{r+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

We will proceed to evaluate S_k.

We have from (1),

$$S_k = \Sigma(-1)^{p_2+p_3+\ldots+p_m-\frac{1}{2}m(m+1)+k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where $[p_2 p_3 \ldots p_m]$ is any corranged minor of order $m-1$ of the sequence

$$[(k+1),\ (k+2),\ \ldots (m+r)].$$

Writing $p_2 - k = q_1$, $p_3 - k = q_2$, ... $p_m - k = q_{m-1}$, and observing that

$$\begin{aligned} &p_2+p_3+\ldots+p_m-\tfrac{1}{2}m(m+1)+k \\ &= (p_2-k)+(p_3-k)+\ldots+(p_m-k)-\tfrac{1}{2}m(m+1)+k+(m-1)k \\ &= q_1+q_2+\ldots+q_{m-1}-\tfrac{1}{2}(m-1)m+(k-1)m, \end{aligned}$$

we see that $\quad S_k = (-1)^{(k-1)m}\,\Sigma(-1)^{(q_1+q_2+\ldots+q_{m-1})-\frac{1}{2}(m-1)m} \quad \ldots\ldots\ldots\ldots(4),$

where now $[q_1 q_2 \ldots q_{m-1}]$ is any corranged minor of order $m-1$ of the sequence

$$[1\ 2 \ldots (m+r-k)].$$

Referring to (1), we see that this result is equivalent to

$$S_k = (-1)^{(k-1)m}\, Q_{m-1}^{m+r-k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Substituting in (2) the values of $S_1, S_2, \ldots S_{r+1}$ given by (5), we obtain formula (A).

(ii) *Second pair of formulae.*

$$Q_m^{m+r} = Q_0^{r-1} + (-1)^1 Q_1^r + \ldots + (-1)^k Q_k^{k+r-1} + \ldots + (-1)^m Q_m^{m+r-1} \quad \ldots(C),$$

$$Q_r^{m+r} = Q_{r-1}^{r-1} + (-1)^1 Q_{r-1}^r + \ldots + (-1)^k Q_{r-1}^{k+r-1} + \ldots + (-1)^m Q_{r-1}^{m+r-1} \quad \ldots(D).$$

The second of these formulae can be deduced from the first by means of the result

$$Q_m^{m+n} = Q_n^{m+n}.$$

We proceed to prove formula (C).

We know that $$Q_m^{m+r} = \Sigma(-1)^\omega \quad\text{...(6)},$$

where the values of ω are the affects of all possible corranged minor sequences $[p_1 p_2 \ldots p_m]$ of order m derived from the fundamental sequence $[1\,2\,3 \ldots (m+r)]$.

There will be one term in the sum (6) corresponding to each such minor sequence.

Now the minor sequences of order m can be classified as follows:

Class 1. The sequence $[1, 2, 3, \ldots m]$.

Class 2. The sequences $[1, 2, 3, \ldots (m-1), q_1]$, where q_1 is any element of the sequence $[(m+1), (m+2), \ldots (m+r)]$.

Class 3. The sequences $[1, 2, 3, \ldots (m-2), q_1, q_2]$, where $[q_1 q_2]$ is a corranged minor of order 2 of the sequence $[m, (m+1), \ldots (m+r)]$.

. . . .

Class $(k+1)$. The sequences $[1, 2, 3, \ldots (m-k), q_1, q_2, \ldots q_k]$, where $[q_1 q_2 \ldots q_k]$ is a corranged minor of order k of the sequence

$$[(m-k+2), (m-k+3), \ldots (m+r)].$$

. . . .

Class $(m+1)$. The sequences $[q_1 q_2 \ldots q_m]$, where $[q_1 q_2 \ldots q_m]$ is a corranged minor of order m of the sequence $[2, 3, \ldots (m+r)]$.

Let S_k be the sum of the terms in (6) which correspond to the minor sequences belonging to Class $(k+1)$.

Then $$Q_m^{m+r} = S_0 + S_1 + S_2 + \ldots + S_k + \ldots + S_m \quad\text{..............................(7)}.$$

We will proceed to evaluate S_k.

For a minor sequence belonging to Class $(k+1)$, we have

$$\begin{aligned}
\omega = {} & \text{the affect of } [1, 2, 3, \ldots (m-k), q_1, q_2, \ldots q_k] \text{ in } [1, 2, 3, \ldots (m+r)] \\
= {} & \text{the affect of } [1, 2, 3, \ldots (m-k)] \text{ in } [1, 2, 3, \ldots (m+r)] \\
& + \text{the affect of } [q_1 q_2 \ldots q_k] \text{ in } [(m-k+1), (m-k+2), \ldots (m+r)] \\
= {} & \text{the affect of } [q_1 q_2 \ldots q_k] \text{ in } [(m-k+1), (m-k+2), \ldots (m+r)] \\
= {} & k + \text{the affect of } [q_1 q_2 \ldots q_k] \text{ in } [(m-k+2), (m-k+3), \ldots (m+r)];
\end{aligned}$$

for the relative affect of each element of $[q_1 q_2 \ldots q_k]$ is less by 1 in

$$[(m-k+2), (m-k+3), \ldots (m+r)]$$

than in $$[(m-k+1), (m-k+2), (m-k+3), \ldots (m+r)],$$

there being one additional element $(m-k+1)$ before it to be counted in the latter sequence.

Thus for the minor sequences of this class, we have

$$\omega = k + \omega',$$

where

$$\begin{aligned}
\omega' = {} & \text{the affect of } [q_1 q_2 \ldots q_k] \text{ in } [(m-k+2), (m-k+3), \ldots (m+r)] \\
= {} & \text{the affect of any corranged minor sequence of order } k \text{ in a given fundamental} \\
& \text{sequence of order } k+r-1.
\end{aligned}$$

When $q_1, q_2, \ldots q_k$ receive all possible values permissible in this class of minor sequences,

$$\Sigma(-1)^{\omega'} = Q_k^{k+r-1},$$

and $$\Sigma(-1)^{\omega} = \Sigma(-1)^{k+\omega'} = (-1)^k \Sigma(-1)^{\omega'} = (-1)^k Q_k^{k+r-1}.$$

We have now proved that $$S_k = (-1)^k Q_k^{k+r-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Substituting in (7) the values of $S_0, S_1, S_2, \ldots S_m$ given by (8), we obtain the formula (C).

4. *Tabular representation of the values of* Q_m^n.

With the help of the formula of reduction (A) the following table of values can be constructed for Q_m^n.

Values of m	0	1	2	3	4	5	6	7	8	9	10	11	12
$n=0$	1	—	—	—	—	—	—	—	—	—	—	—	—
$n=1$	1	1	—	—	—	—	—	—	—	—	—	—	—
$n=2$	1	0	1	—	—	—	—	—	—	—	—	—	—
$n=3$	1	1	1	1	—	—	—	—	—	—	—	—	—
$n=4$	1	0	2	0	1	—	—	—	—	—	—	—	—
$n=5$	1	1	2	2	1	1	—	—	—	—	—	—	—
$n=6$	1	0	3	0	3	0	1	—	—	—	—	—	—
$n=7$	1	1	3	3	3	3	1	1	—	—	—	—	—
$n=8$	1	0	4	0	6	0	4	0	1	—	—	—	—
$n=9$	1	1	4	4	6	6	4	4	1	1	—	—	—
$n=10$	1	0	5	0	10	0	10	0	5	0	1	—	—
$n=11$	1	1	5	5	10	10	10	10	5	5	1	1	—
$n=12$	1	0	6	0	15	0	20	0	15	0	6	0	1

The value of Q^n_m is the number common to the mth vertical column and the nth horizontal column. For example, the table shows that $Q^7_5=3$ and that $Q^{11}_7=10$.

Since $Q^n_0 = 1$, the numbers in the first vertical column are all 1's. This being known the following vertical columns can be filled up in turn by repeated applications of the formula of reduction (A). Any one of the other three formulae of reduction may also be used to construct the table.

5. *Relations between the numbers Q^n_m and binomial coefficients.*

From the table just constructed we can form the following subsidiary tables, in which m and n are positive integers.

Table I. Values of Q^{2n}_{2m}.

Values of m	0	1	2	3	4
$n=0$	1	—	—	—	—
$n=1$	1	1	—	—	—
$n=2$	1	2	1	—	—
$n=3$	1	3	3	1	—
$n=4$	1	4	6	4	1

Table II. Values of Q^{2n+1}_{2m+1}.

Values of m	0	1	2	3	4
$n=0$	1	—	—	—	—
$n=1$	1	1	—	—	—
$n=2$	1	2	1	—	—
$n=3$	1	3	3	1	—
$n=4$	1	4	6	4	1

Table III. Values of Q^{2n+1}_{2m}.

Values of m	0	1	2	3	4
$n=0$	1	—	—	—	—
$n=1$	1	1	—	—	—
$n=2$	1	2	1	—	—
$n=3$	1	3	3	1	—
$n=4$	1	4	6	4	1

Table IV. Values of Q^{2n}_{2m+1}.

Values of m	0	1	2	3	4
$n=0$	0	—	—	—	—
$n=1$	0	0	—	—	—
$n=2$	0	0	0	—	—
$n=3$	0	0	0	0	—
$n=4$	0	0	0	0	0

Let the coefficient of x^m in the expansion of $(1+x)^n$ be denoted by $B(m, n)$, so that $B(m, n)$ is the number of combinations of n things taken m at a time, and

$$B(m, n) = \frac{n!}{m!\,(n-m)!} = \binom{n}{m}.$$

The above tables suggest that

$$Q^{2n}_{2m} = Q^{2n+1}_{2m+1} = Q^{2n+1}_{2m} = B(m, n), \quad Q^{2n}_{2m+1} = 0 \quad \text{.....................(E)}.$$

We shall proceed to prove the correctness of these results.

We shall first show that

$$Q^n_m = Q^{n-2}_m + Q^{n-2}_{m-2}, \quad m \text{ being } \not> n-2 \quad \text{..........................(9)}.$$

Let $P^n_m = \Sigma(-1)^{p_1+p_2+\dots+p_m}$, where $[p_1p_2\dots p_m]$ is any corranged minor of $[1\,2\,3\dots n]$, so that

$$Q^n_m = (-1)^{\frac{1}{2}m(m+1)}\,P^n_m \quad \text{...............................(10)}.$$

The possible sequences $[p_1p_2\dots p_m]$ may be classified as follows:

Class 1. *Those in which* $p_1, p_2, \dots p_m$ *all belong to the sequence* $[1\,2\,3\dots(n-2)]$.

For these $$\Sigma(-1)^{p_1+p_2+\dots+p_m} = P^{n-2}_m.$$

Class 2. *Those in which* $p_m = n-1$, *and* $p_1, p_2, \dots p_{m-1}$ *all belong to the sequence* $[1\,2\,3\dots(n-2)]$.

For these

$$\Sigma(-1)^{p_1+p_2+\dots+p_m} = (-1)^{n-1}\,\Sigma(-1)^{p_1+p_2+\dots+p_{m-1}} = (-1)^{n-1}\,P^{n-2}_{m-1}.$$

Class 3. *Those in which* $p_m = n$, *and* $p_1, p_2, \dots p_{m-1}$ *all belong to the sequence* $[1\,2\,3\dots(n-2)]$.

For these

$$\Sigma(-1)^{p_1+p_2+\dots+p_m} = (-1)^n\,\Sigma(-1)^{p_1+p_2+\dots+p_{m-1}} = (-1)^n\,P^{n-2}_{m-1}.$$

Class 4. *Those in which* $p_{m-1} = n-1$, $p_m = n$, *and* $p_1, p_2, \dots p_{m-2}$ *all belong to the sequence* $[1\,2\,3\dots(n-2)]$.

For these

$$\Sigma(-1)^{p_1+p_2+\dots+p_m} = (-1)^{2n-1}\,\Sigma(-1)^{p_1+p_2+\dots+p_{m-2}} = -P^{n-2}_{m-2}.$$

Adding these four results to obtain $\Sigma(-1)^{p_1+p_2+\dots+p_m}$ for all possible sequences $[p_1p_2\dots p_m]$, we see that

$$P^n_m = P^{n-2}_m - P^{n-2}_{m-2} \text{.....................................(11)}.$$

Utilising (10) we at once deduce the truth of (9), and from (9) we can pass on to the proof of formulae (E).

To verify that $Q^{2n}_{2m} = B(m, n)$, let

$$Q^{2n}_{2m} = f(m, n).$$

Then the result to be proved is

$$f(m, n) = B(m, n), \text{ where } m \ngtr n \quad \text{.........................}(12).$$

From (9) we have

$$f(m, n) = f(m, n-1) + f(m-1, n-1), \text{ where } m \ngtr n-1 \text{..............}(13).$$

And by a known property of binomial coefficients, we have

$$B(m, n) = B(m, n-1) + B(m-1, n-1), \text{ where } m \ngtr n-1 \quad \text{...........}(14).$$

Now suppose that there exists an integer $r-1$ such that (12) is true for all values of n up to $r-1$ and for all values of m not exceeding m up to $r-1$. Then (13) and (14) show that (12) is also true for all values of n up to r and for all values of m not exceeding n up to $r-1$. But (12) is true when $m=r$ and $n=r$, for

$$f(r, r) = Q_{2r}^{2r} = 1 = B(r, r).$$

It follows that (12) is true for all values of n up to r and all values of m not exceeding n up to r. By successive repetitions of the same reasoning it follows that (12) is true for all values of n and all values of m not exceeding n.

Such an integer $r-1$ certainly does exist, for (12) is clearly true for small integral values of n whenever $m \ngtr n$.

Hence equation (12) is true generally, and we conclude that the result

$$Q_{2m}^{2n} = B(m, n)$$

is true generally.

The remaining results included in (E) can be proved similarly by writing

$$Q_{2m+1}^{2n+1} = f(m, n), \quad \text{or} \quad Q_{2m}^{2n+1} = f(m, n), \quad \text{or} \quad Q_{2m+1}^{2n} = f(m, n).$$

§ 35. Algebraical sum of inferior short-cut simple minor determinoids of given reduced order.

We shall prove the following theorem:

Theorem. *The algebraical sum S_ν of all the corranged inferior short-cut simple minor determinoids of reduced order ν derived from a fundamental determinoid Δ, each minor having the sign determined by its affect in Δ, is equal to the algebraical sum of all the minor determinants of order ν of Δ, each determinant having the sign determined by its affect in Δ. It is therefore also equal to the algebraical sum of all the derived products of order ν of Δ, each derived product having the sign determined by its affect in Δ* ......(A).

Let

$$\Delta = (a)_m^n,$$

having m horizontal long rows and n vertical short rows.

Let Δ_ν be any *corranged* determinoid derived from Δ by retaining ν of the short rows and striking out the other short rows, ν being less than m, and let ω be the affect of Δ_ν in Δ.

Then the algebraical sum S_ν may be defined by the equation

$$S_\nu = \Sigma (-1)^\omega \Delta_\nu \quad \text{.................................(1)}.$$

It is clear that S_ν can be expressed as a homogeneous linear function of the minor determinants of order ν of Δ, since Δ_ν can be so expressed by the expansion of § 30. We will find the coefficient of every such determinant in S_ν.

Let $$D = (a_{xy})_\nu^\nu$$

be any corranged minor determinant of order ν derived from Δ.

This determinant D is contained as a minor in one and only one of the determinoids Δ_ν, viz. in the determinoid

$$\Delta_\nu' = (a_{1y})_m^\nu.$$

Let
$$\omega' = \text{the affect of } \Delta_\nu' \text{ in } \Delta,$$
$$\eta = \text{the affect of } D \text{ in } \Delta,$$
$$\eta' = \text{the affect of } D \text{ in } \Delta_\nu'.$$

Then the coefficient of D in S_ν, being the coefficient of D in $(-1)^{\omega'} \Delta_\nu'$, is $(-1)^{\omega'+\eta'}$.

Now the horizontal affect of D in Δ = the affect of D in $\Delta_\nu' = \eta'$,

and the vertical affect of D in Δ = the affect of Δ_ν' in $\Delta = \omega'$.

By addition, or by Theorem III*d* of § 25, it follows that

$$\eta = \omega' + \eta'.$$

Accordingly D occurs in S_ν with the coefficient $(-1)^\eta$, and therefore

$$S_\nu = \Sigma (-1)^\eta D \quad \text{..............................(2)}.$$

This last result remains true when the determinants D are deranged in any manner, for it has been shown in Theorem V*a* of § 25 that $(-1)^\eta D$ is unaltered by any derangement of D.

Thus S_ν is the algebraical sum of all the distinct minor determinants of order ν, these determinants being either corranged or deranged, and each having the sign determined by its affect in Δ.

Again let P be any derived product of Δ of order ν whose affect in Δ is σ. Then P is a derived product of one and only one of the minor determinants of order ν. Suppose that these minor determinants are all corranged and that P is a derived product of D having affect σ' in D.

Then P occurs in the expansion of D with the coefficient $(-1)^{\sigma'}$, and we see from (2) that it occurs in S_ν with the coefficient $(-1)^{\sigma'+\eta}$. But by Theorem III*a* of § 25, $\eta = \sigma' + \eta$.

Thus $$S_\nu = \Sigma(-1)^\sigma P \quad \text{.............................(3)}.$$

The order of arrangement of the factors of P is of course immaterial.

Equations (2) and (3) are together equivalent to (A).

We can also deduce (3) directly from (1) by observing that P occurs as a derived product in one and only one of the determinoids Δ_ν.

Ex. i. If $$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}.$$

then

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \\ c_1 & c_4 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \\ c_2 & c_4 \end{vmatrix} + \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \\ c_3 & c_4 \end{vmatrix}$$

= the algebraical sum of all determinants of order 2 derived from Δ

= the algebraical sum of all derived products of order 2 belonging to Δ.

Ex. ii. *Theorem (A) remains true when the long rows of the simple minor determinoids are not corranged.*

For if Δ_ν and ω have the same meanings as in equation (1), and if Δ_ν' is formed from Δ_ν by derangements of its long rows, and if ω' is the affect of the deranged determinoid Δ_ν' in Δ, we know by Theorem V*b* of § 25 that

$$(-1)^\omega \Delta_\nu = (-1)^{\omega'} \Delta_\nu'.$$

Consequently $$S_\nu = \Sigma(-1)^\omega \Delta_\nu = \Sigma(-1)^{\omega'} \Delta_\nu'.$$

§ 36. Algebraical sum of (inferior) long-cut simple minor determinoids of given reduced order.

We shall prove the following theorem:

Theorem. *The algebraical sum S_μ of all the corranged (inferior) long-cut simple minor determinoids of reduced order μ derived from a fundamental determinoid Δ, each minor having the sign determined by its affect in Δ, is equal to the algebraical sum of all the minor determinants of order μ of Δ, each determinant having the sign determined by its affect in Δ. It is therefore also equal to the algebraical sum of all the derived products of order μ belonging to Δ, each derived product having the sign determined by its affect in Δ*..(A).

Let $$\Delta = (a)_m^n,$$

having m horizontal long rows and n vertical short rows.

Let Δ_μ be any corranged determinoid derived from Δ by retaining μ long rows and striking out the other long rows, μ being less than m, and let ω be the affect of Δ_μ in Δ.

Then the algebraical sum S_μ may be defined by the equation

$$S_\mu = \Sigma (-1)^\omega \Delta_\mu \quad \text{.............................(1)}.$$

Since Δ_μ can, by the expansion of § 30, be expressed in terms of its minor determinants of order μ, it is clear that S_μ can be expressed as a homogeneous linear function of the minor determinants of order μ derived from Δ. We will find the coefficient of every such determinant in S_μ.

Let $$D = (a_{xy})_\mu^\mu$$

be any corranged minor determinant of order μ derived from Δ. This determinant is contained as a minor in one and only one of the determinoids Δ_μ, viz. in the determinoid

$$\Delta_\mu' = (a_{x1})_\mu^n.$$

Let
$$\omega' = \text{the affect of } \Delta_\mu' \text{ in } \Delta,$$
$$\eta = \text{the affect of } D \text{ in } \Delta,$$
$$\eta' = \text{the affect of } D \text{ in } \Delta_\mu'.$$

Then the coefficient of D in S_μ is $(-1)^{\omega'+\eta'}$.

Now the horizontal affect of D in Δ = the affect of D in $\Delta_\mu' = \eta'$,

and the vertical affect of D in Δ = the affect of Δ_μ' in $\Delta = \omega'$.

By addition, or by Theorem IIId of § 25, it follows that

$$\eta = \omega' + \eta'.$$

Accordingly D occurs in S_μ with the coefficient $(-1)^\eta$, and therefore

$$S_\mu = \Sigma (-1)^\eta D \quad \text{.............................(2)}.$$

This result remains true when the determinants D are deranged in any manner.

If P is any derived product of Δ of order μ whose affect in Δ is σ, we deduce from (2) as in § 35 that

$$S_\mu = \Sigma (-1)^\sigma P \quad \text{.............................(3)}.$$

Equations (2) and (3) are together equivalent to (A).

Comparing the results of the present article with the results of the preceding article, we see that in any determinoid:

The algebraical sum of the inferior short-cut simple minor determinoids of reduced order r or efficiency r

= *the algebraical sum of the (inferior) long-cut simple minor determinoids of reduced order r or efficiency r*

= *the algebraical sum of the minor determinants of order r*

= *the algebraical sum of the derived products of order r* ....(B).

Ex. i. From Theorem V*b* of § 25 it follows that Theorem (A) above remains true when the *long rows* only of the simple minor determinoids are deranged in any manner.

Also in Theorem (B) the simple minor determinoids must be either corranged or have only their *long rows* deranged.

Ex. ii. If
$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix},$$

then
$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$
$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \\ c_1 & c_4 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \\ c_2 & c_4 \end{vmatrix} + \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \\ c_3 & c_4 \end{vmatrix}$$

= the algebraical sum of all determinants of order 2 derived from Δ

= the algebraical sum of all derived products of order 2 belonging to Δ.

§ 37. Algebraical sum of the products obtained by multiplying each affected simple minor determinant of a given short-cut matrix of a given determinoid by the complementary corranged minor determinoid lying in the complementary short-cut matrix.

Here an affected simple minor determinant of the given short-cut minor matrix is understood to be a simple minor determinant (corranged or deranged) of that matrix provided with the sign determined by its affect *in the given determinoid.* By Theorem V*a* of § 25, it is immaterial whether these simple minor determinants are corranged or deranged.

The value of the algebraical sum is clearly the same whether the given short-cut minor matrix and its complementary minor matrix are or are not corranged. For convenience it will be assumed that they are both corranged.

The algebraical sum can also be described (see § 32) as that obtained by multiplying each simple minor determinant (corranged or deranged) of the given short-cut matrix by its co-factor in the given determinoid.

Let
$$\Delta = (a)_m^n$$
be a determinoid with m horizontal long rows and n vertical short rows.

Let
$$A_m^\nu = [a_{1\alpha}]_m^\nu, \qquad A_m^{n-\nu} = [a_{1\beta}]_m^{n-\nu}$$
be two given complementary corranged short-cut matrices of Δ.

It is supposed that $\nu < m$, so that A_m^ν contains simple minor determinants whose complements in Δ are contained in the complementary matrix $A_m^{n-\nu}$.

Let
$$A_\nu^\nu = (a_{x\alpha})_\nu^\nu$$
be any one of the corranged simple minor determinants formed from A_m^ν by the omission of short rows.

Let $$A^{n-\nu}_{m-\nu} = (a_{y\beta})^{n-\nu}_{m-\nu}$$

be the corranged determinoid complementary to A^{ν}_{ν} in Δ. It is a simple minor determinoid of the matrix $A^{n-\nu}_{m}$.

Let ω be the affect of the determinant A^{ν}_{ν} in Δ.

Then $$S = \Sigma_x (-1)^{\omega} A^{\nu}_{\nu} A^{n-\nu}_{m-\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots\ldots(1)$$

is the algebraical sum which it is required to evaluate.

With this notation

$[a_1 a_2 \ldots a_\nu]$, $[\beta_1 \beta_2 \ldots \beta_{n-\nu}]$ are complementary corranged minors of $[1\ 2 \ldots n]$;
$[x_1 x_2 \ldots x_\nu]$, $[y_1 y_2 \ldots y_{m-\nu}]$ are complementary corranged minors of $[1\ 2 \ldots m]$.

The summation in (1) is to extend over all the corranged minor sequences $[x_1 x_2 \ldots x_\nu]$ of ν elements derived from $[1\ 2 \ldots m]$.

Let ω_1 = the affect of $[a_1 a_2 \ldots a_\nu]$ in $[1\ 2 \ldots n]$,
ω_2 = the affect of $[x_1 x_2 \ldots x_\nu]$ in $[1\ 2 \ldots m]$,

so that $$\omega = \omega_1 + \omega_2.$$

Then $$S = (-1)^{\omega_1} \Sigma_x (-1)^{\omega_2} A^{\nu}_{\nu} A^{n-\nu}_{m-\nu}.$$

Since $n-\nu > m-\nu$, the long rows of $A^{n-\nu}_{m-\nu}$ are horizontal.

Expanding $A^{n-\nu}_{m-\nu}$ in terms of its simple minor determinants, we have

$$A^{n-\nu}_{m-\nu} = \Sigma (-1)^{\eta} (a_{yq})^{m-\nu}_{m-\nu},$$

where $[q_1 q_2 \ldots q_{m-\nu}]$ is any corranged minor of order $m-\nu$ of $[\beta_1 \beta_2 \ldots \beta_{n-\nu}]$ and η is the affect of $[q_1 q_2 \ldots q_{m-\nu}]$ in $[\beta_1 \beta_2 \ldots \beta_{n-\nu}]$.

Consequently $$S = (-1)^{\omega_1} \Sigma_x \Sigma_q (-1)^{\omega_2 + \eta} (a_{x\alpha})^{\nu}_{\nu} (a_{yq})^{m-\nu}_{m-\nu}$$

where Σ_q denotes summation for the different values of $[q_1 q_2 \ldots q_{m-\nu}]$.

Here the two summations are independent, and their order can be reversed.

When we sum with respect to $[x_1 x_2 \ldots x_\nu]$ keeping $[q_1 q_2 \ldots q_{m-\nu}]$ fixed, η also remains fixed, and by *Laplace's* development of determinants, included in § 32,

$$\Sigma_x (-1)^{\omega_2} (a_{x\alpha})^{\nu}_{\nu} (a_{yq})^{m-\nu}_{m-\nu} = \begin{pmatrix} a_1 a_2 \ldots a_\nu\, q_1 q_2 \ldots q_{m-\nu} \\ a \\ 1\ 2 \ldots m \end{pmatrix} = A^{m}_{m}.$$

Therefore $$S = \Sigma_q (-1)^{\omega_2 + \eta} A^{m}_{m}.$$

Now the affect of A^{m}_{m} in Δ

= the affect of $[a_1 a_2 \ldots a_\nu\, q_1 q_2 \ldots q_{m-\nu}]$ in $[1\ 2 \ldots n]$
= the affect of $[a_1 a_2 \ldots a_\nu]$ in $[1\ 2 \ldots n]$
+ the affect of $[q_1 q_2 \ldots q_{m-\nu}]$ in $[\beta_1 \beta_2 \ldots \beta_{n-\nu}]$ $= \omega_2 + \eta$.

Hence S is the algebraical sum of all such determinants as A_m^m, when each determinant has the sign determined by its affect in Δ.

Let $[s_1 s_2 \ldots s_m]$ be the corranged minor of $[1\,2 \ldots n]$ formed with the elements of $[a_1 a_2 \ldots a_\nu]$, $[q_1 q_2 \ldots q_{m-\nu}]$ taken together.

Then by Theorem V *b* of § 25

$$S = \Sigma(-1)^\sigma (a_{1s})_m^m,$$

where σ is the affect of the determinant $(a_{1s})_m^m$ in Δ.

We see then that

S is the algebraical sum of all those simple minor determinants of Δ which contain the given short-cut matrix, each of these determinants having the sign determined by its affect in Δ ....(A).

By Theorem III *a* of § 25, it follows from this that

S is the algebraical sum of all those complete derived products of Δ which contain factors from every one of the short rows of Δ which belong to the given short-cut matrix, each derived product having the sign determined by its affect in Δ ......(B).

The corresponding theorem for a given long-cut matrix of the determinoid is contained in § 32.

COROLLARY. *The algebraical sum of the products obtained by multiplying each affected element of a given short row of the determinoid Δ by its complementary determinoid (or by multiplying each element of the short row by its co-factor) is equal to the algebraical sum of all those affected simple minor determinants of Δ which contain the given short row. It is also equal to the algebraical sum of all those affected complete derived products of Δ which contain a factor belonging to the given short row*..............................(C).

This result is deduced by taking the given short-cut matrix to consist of one short row only.

The corresponding result for a long row of the determinoid is that contained in Theorem I of § 27.

§ 38. Algebraical sum of the products obtained by multiplying each affected simple minor determinoid of given reduced order of a given simple minor matrix of a given determinoid by the complementary corranged minor determinoid lying in the complementary simple minor matrix.

1. *Formation of the algebraical sum.*

In the above algebraical sum an affected simple minor determinoid of the given simple minor matrix is a simple minor determinoid of that matrix which is either corranged with respect to the given determinoid or has only its long rows deranged and which is provided with the sign determined by its affect *in the given determinoid*. By Theorem V*b* of § 25 it is immaterial whether these simple minor determinoids are corranged or have their long rows only deranged.

This being understood, the value of the algebraical sum is the same whether the given simple minor matrix and its complement are corranged or deranged. In order however that we may be able to state the result of evaluating the sum in a simple form in all cases, it will be assumed that the two complementary simple minor matrices are both corranged.

If D is any minor determinoid of a given fundamental determinoid Δ, and if D is either corranged or has only its long rows deranged, we will define the co-factor of D in Δ to be a determinoid D' such that $D' = (-1)^{\omega} D_0'$, where D_0' is the corranged minor determinoid of Δ complementary to D, and ω is the affect of D in Δ. Then by Theorem Vb of § 25, the product DD' of the determinoid D and its co-factor is independent of the order of arrangement of the long rows of D. In particular the product of a corranged minor determinoid and its co-factor is equal to the product of the corresponding affected minor determinoid and its co-factor. More generally if D' is the co-factor of D, then $(-1)^{\eta} D'$ is the co-factor of $(-1)^{\eta} D$.

Using this definition of a co-factor, we see that each term in the above algebraical sum is obtained by multiplying an affected simple minor determinoid of the given simple minor matrix (whose long rows only may be deranged) by its co-factor; and that the same term can be obtained by replacing the affected simple minor determinoid by the corresponding corranged minor determinoid and multiplying this latter determinoid by its co-factor.

2. *Statement of results.*

For the sake of clearness we shall suppose that *the two fixed complementary simple minor matrices are composed of vertical rows of the given determinoid.*

Let
$$\Delta = (a)_m^n$$
be the given determinoid.

Let
$$A_m^{\nu} = [a_{1\alpha}]_m^{\nu}, \quad A_m^{n-\nu} = [a_{1\beta}]_m^{n-\nu}$$
be two fixed complementary *corranged* simple minor matrices of Δ formed from Δ by striking out vertical rows, so that
$$[\alpha_1 \alpha_2 \dots \alpha_{\nu}], \quad [\beta_1 \beta_2 \dots \beta_{n-\nu}]$$
are two fixed complementary corranged minors of the sequence $[1\,2 \dots n]$.

Let
$$A_{\mu}^{\nu} = (a_{x\alpha})_{\mu}^{\nu}, \quad A_{m-\mu}^{n-\nu} = (a_{y\beta})_{m-\mu}^{n-\nu}$$
be any two complementary corranged minor determinoids of Δ of given orders

belonging respectively to the matrix A_m^ν and to the matrix $A_m^{n-\nu}$, so that

$$[x_1 x_2 \ldots x_\mu], \quad [y_1 y_2 \ldots y_{m-\mu}]$$

are any two complementary corranged minors of the sequence $[1\,2 \ldots m]$ of given orders μ and $m-\mu$.

Let $\omega =$ the affect of A_μ^ν in Δ.

Then the sum to be evaluated is

$$S = \sum_x (-1)^\omega (a_{x\alpha})_\mu^\nu (a_{y\beta})_{m-\mu}^{n-\nu} = \sum_x (-1)^\omega A_\mu^\nu A_{m-\mu}^{n-\nu} \quad \ldots\ldots\ldots\ldots(1),$$

the summation extending over all possible corranged minors $[x_1 x_2 \ldots x_\mu]$ of the sequence $[1\,2 \ldots m]$ of the given order μ.

Let $\omega_1 =$ the affect of $[\alpha_1 \alpha_2 \ldots \alpha_\nu]$ in $[1\,2 \ldots n]$,

and $\omega_2 =$ the affect of $[x_1 x_2 \ldots x_\mu]$ in $[1\,2 \ldots m]$,

so that $\omega = \omega_1 + \omega_2$.

Since ω_1 remains constant during the summation, we can replace (1) by

$$S = (-1)^{\omega_1} \sum_x (-1)^{\omega_2} A_\mu^\nu A_{m-\mu}^{n-\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The determinoid A_μ^ν may be either a superior or an inferior simple minor determinoid of the fixed matrix A_m^ν.

So the determinoid $A_{m-\mu}^{n-\nu}$ may be a superior or an inferior simple minor determinoid of the fixed matrix $A_m^{n-\nu}$.

The result of the summation depends on whether A_μ^ν, $A_{m-\mu}^{n-\nu}$ are superior or inferior simple minors of their respective matrices.

To express the result in a convenient form some additional notation will be introduced.

Let $\epsilon_1 =$ the efficiency of the determinoid A_μ^ν,

$\epsilon_2 =$ the efficiency of the determinoid $A_{m-\mu}^{n-\nu}$.

Thus ϵ_1 is the smaller of the two numbers μ and ν, and ϵ_2 is the smaller of the two numbers $m-\mu$, $n-\nu$.

Let $\delta =$ the difference of the orders of the first determinoid A_μ^ν,

$\epsilon =$ the efficiency of the second determinoid $A_{m-\mu}^{n-\nu}$,

in each product $A_\mu^\nu A_{m-\mu}^{n-\nu}$.

Then $\delta = \mu - \nu$ or $\nu - \mu$ according as $\mu > \nu$ or $\nu > \mu$;

also $\epsilon = \epsilon_2$, and is the smaller of the two numbers $m-\mu$, $n-\nu$.

We shall show that the evaluation of the sum S leads to the following results:

Case I. *When both the determinoids A^{ν}_{μ}, $A^{n-\nu}_{m-\mu}$ are superior simple minors of their respective matrices*, then

$$S = (-1)^{\delta\epsilon}\, Q^{m-n}_{\mu-\nu}\, \Delta \quad \text{..........................(A)}.$$

Thus S is a certain numerical multiple of the given determinoid. This case, which includes § 32, can be regarded as a still further generalisation of *Laplace's* development of a determinant.

Case II. *When one of the determinoids A^{ν}_{μ}, $A^{n-\nu}_{m-\mu}$ is a superior and the other is an inferior simple minor*, then

$$S = (-1)^{\delta\epsilon}\, \Sigma\, (-1)^{\sigma}\, \Delta_{\epsilon_1+\epsilon_2} \quad \text{.......................(B)},$$

where $\Delta_{\epsilon_1+\epsilon_2}$ is any corranged simple minor determinoid of Δ formed with ϵ_1 vertical rows of the first matrix A^{ν}_{m} and ϵ_2 vertical rows of the second matrix $A^{n-\nu}_{m}$, and σ is the affect of $\Delta_{\epsilon_1+\epsilon_2}$ in Δ.

Thus $\Delta_{\epsilon_1+\epsilon_2}$ contains *all* the vertical rows of that one of the two fixed matrices A^{ν}_{m}, $A^{n-\nu}_{m}$ to which the superior determinoid belongs, and it contains as many vertical rows of the other matrix as are equal in number to the efficiency of the inferior determinoid belonging to it.

In this case S is expressed as an algebraical sum of certain simple minor determinoids of Δ of reduced order $\epsilon_1 + \epsilon_2$.

Case III. *When both the determinoids A^{ν}_{μ}, $A^{n-\nu}_{m-\mu}$ are inferior simple minors of their respective matrices*, then

$$S = (-1)^{\delta\epsilon}\, \Sigma\, (-1)^{\sigma+\sigma'}\, \Delta_{\epsilon_1+\epsilon_2} \quad \text{....................(C)},$$

where $\Delta_{\epsilon_1+\epsilon_2}$ is any corranged simple minor determinant of Δ formed with ϵ_1 vertical rows of the first matrix A^{ν}_{m} and ϵ_2 vertical rows of the second matrix $A^{n-\nu}_{m}$, σ is the affect of $\Delta_{\epsilon_1+\epsilon_2}$ in Δ, and σ' is the affect of the corranged sequence formed with the vertical rows of A^{ν}_{m} omitted in $\Delta_{\epsilon_1+\epsilon_2}$ in the corranged sequence formed with the vertical rows of Δ omitted in $\Delta_{\epsilon_1+\epsilon_2}$.

Since $\epsilon_1 + \epsilon_2 = m$, $\Delta_{\epsilon_1+\epsilon_2}$ is a determinant of order m.

In this case S is expressed as an algebraical sum of certain simple minor determinants of Δ of order $\epsilon_1 + \epsilon_2$, but the signs of these determinants are in general not those determined by the affects of the determinants in Δ.

NOTE. *Corresponding results for matrices composed of horizontal rows.*

We write
$$\Delta=(a)^n_m,$$
$$A^n_\mu=[a_{\alpha 1}]^n_\mu,\qquad A^n_{m-\mu}=[a_{\beta 1}]^n_{m-\mu},$$
$$A^\nu_\mu=(a_{\alpha x})^\nu_\mu,\qquad A^{n-\nu}_{m-\mu}=(a_{\beta y})^{n-\nu}_{m-\mu},$$
where $[\alpha_1\alpha_2\ldots\alpha_\mu]$, $[\beta_1\beta_2\ldots\beta_{m-\mu}]$ are two fixed corranged complementary minors of the sequence $[1\,2\ldots m]$, and $[x_1x_2\ldots x_\nu]$, $[y_1y_2\ldots y_{n-\nu}]$ are any two corranged complementary minors of the sequence $[1\,2\ldots n]$ of given orders ν and $n-\nu$.

Then
$$S=\sum_x(-1)^\omega(a_{\alpha x})^\nu_\mu(a_{\beta y})^{n-\nu}_{m-\mu}=\sum_x(-1)^\omega A^\nu_\mu A^{n-\nu}_{m-\mu},$$
where ω is the affect of A^ν_μ in Δ.

The values of S in the various cases are clearly obtained from those just given by interchanging the words *horizontal* and *vertical*, interchanging the letters m and n, and interchanging the letters μ and ν.

Before proceeding to the proofs of formulae (A), (B) and (C), we shall give some examples in illustration of them.

Ex. i. Let
$$\Delta=(abcde)_{123456789},\quad A^\nu_m=[ace]_{123456789},\quad A^{n-\nu}_m=[bd]_{123456789}.$$
Also let
$$S=-(ace)_{12345}(bd)_{6789}+(ace)_{12346}(bd)_{5789}\ldots$$
$$\ldots+(ace)_{46789}(bd)_{1235}-(ace)_{56789}(bd)_{1234}.$$
There are 126 terms in S.

In this case $(ace)_{12345}$ and $(bd)_{6789}$ are both superior simple minors of their respective matrices; also
$$m=9,\quad n=5,\quad \mu=5,\quad \nu=3,\quad \delta=2,\quad \epsilon=2.$$
Therefore formula (A) gives $\quad S=(-1)^4Q^4_2\Delta=2\Delta.$

Ex. ii. Let
$$\Delta=(abcde)_{123456},\quad A^\nu_m=[bce]_{123456},\quad A^{n-\nu}_m=[ad]_{123456}.$$
Also let
$$S=(bce)_{12}(ad)_{3456}-(bce)_{13}(ad)_{2456}+(bce)_{14}(ad)_{2356}\ldots$$
$$\ldots-(bce)_{46}(ad)_{1235}+(bce)_{56}(ad)_{1234}.$$
There are 15 terms in S.

In this case $(bce)_{12}$ and $(ad)_{3456}$ are inferior and superior simple minors of their respective matrices; also
$$\epsilon_1=2,\quad \epsilon_2=2,\quad \delta=1,\quad \epsilon=2,\quad (-1)^{\delta\epsilon}=(-1)^2=1.$$
Therefore formula (B) gives
$$S=(abcd)_{123456}+(abde)_{123456}-(acde)_{123456}.$$

Ex. iii. Let

$$\Delta=(abcde)_{123456},\quad A^{\nu}_{m}=[ad]_{123456},\quad A^{n-\nu}_{m}=[bce]_{123456}.$$

Also let

$$S=(ad)_{1234}(bce)_{56}-(ad)_{1235}(bce)_{46}+(ad)_{1236}(bce)_{45}\dots$$
$$\dots-(ad)_{2456}(bce)_{13}+(ad)_{3456}(bce)_{12}.$$

In this case $(ad)_{1234}$ and $(bce)_{56}$ are superior and inferior simple minors of their respective matrices; also

$$\epsilon_1=2,\quad \epsilon_2=2,\quad \delta=2,\quad \epsilon=2,\quad (-1)^{\delta\epsilon}=(-1)^4=1.$$

Therefore formula (B) gives

$$S=(abcd)_{123456}+(abde)_{123456}-(acde)_{123456}.$$

Ex. iv. Let $\Delta=(abcdef)_{1234},\quad A^{\nu}_{m}=[bef]_{1234},\quad A^{n-\nu}_{m}=[acd]_{1234}.$

Also let

$$S=-(bef)_{12}(acd)_{34}+(bef)_{13}(acd)_{24}-(bef)_{14}(acd)_{23}$$
$$-(bef)_{23}(acd)_{14}+(bef)_{24}(acd)_{13}-(bef)_{34}(acd)_{12}.$$

In this case $(bef)_{12}$ and $(acd)_{34}$ are both inferior simple minors of their respective matrices; also

$$\delta=1,\quad \epsilon=2,\quad (-1)^{\delta\epsilon}=1.$$

Therefore formula (C) gives

$$S=(abce)-(abde)-(bcde)-(abcf)+(abdf)+(bcdf)-(acef)+(adef)-(cdef),$$

where $(abce)$ is short for $(abce)_{1234}$, and so on.

The values of σ' for the successive determinants are equal to the affects of f in $[df]$, f in $[cf]$, f in $[af]$, e in $[de]$, e in $[ce]$, e in $[ae]$, b in $[bd]$, b in $[bc]$, b in $[ab]$.

Thus for the successive determinants

the values of σ are 1, 2, 4, 2, 3, 5, 5, 6, 8,

and the values of σ' are 1, 1, 1, 1, 1, 1, 0, 0, 1.

Ex. v. When $m=n$ and $\mu=\nu$, each of the formulae (A), (B), (C) gives *Laplace's* development of a determinant.

Ex. vi. When $\mu=\nu$, each of the formulae (A) and (B) gives the generalisation of *Laplace's* development contained in § 32.

Ex. vii. When $\mu=\nu=1$, we obtain the theorems contained in § 27 and in the Corollary of § 37.

Ex. viii. In order that the results of §§ 30, 32, 33, 35 and 36 may be included in formulae (A), (B), and (C), we must regard the corranged complement of any simple minor determinoid of a given determinoid as being a determinoid whose value is 1.

3. **Proof of Formula (A) in Case I.**

This case occurs when $\mu>\nu$ and $m-\mu>n-\nu$.

A necessary consequence of these inequalities is $m>n$.

Hence this case can only occur when the two fixed complementary simple minor matrices are long-cut minors of Δ.

To obtain formula (A), we commence by expanding A^{ν}_{μ} in terms of its simple minor determinants by § 30. The expansion is

$$A^{\nu}_{\mu} = (a_{x\alpha})^{\nu}_{\mu} = \sum_{p} (-1)^{\eta} (a_{p\alpha})^{\nu}_{\nu}$$

where $[p_1 p_2 \dots p_\nu]$ is any corranged minor of order ν of $[x_1 x_2 \dots x_\mu]$,

and η is the affect of $[p_1 p_2 \dots p_\nu]$ in $[x_1 x_2 \dots x_\mu]$.

The summation extends over all such values of $[p_1 p_2 \dots p_\nu]$.

Inserting this value of A^{ν}_{μ} in (1) or (2), we have

$$S = (-1)^{\omega_1} \sum_{x\,p}\sum (-1)^{\omega_2+\eta} (a_{p\alpha})^{\nu}_{\nu} (a_{y\beta})^{n-\nu}_{m-\mu} \quad \dots\dots\dots\dots\dots\dots(3).$$

Here we have first to sum for all corranged minors $[p_1 p_2 \dots p_\nu]$ of order ν of the fixed sequence $[x_1 x_2 \dots x_\mu]$, and then to sum for all corranged minors $[x_1 x_2 \dots x_\mu]$ of order μ of the sequence $[1\ 2 \dots m]$.

Let $[q_1 q_2 \dots q_{m-\nu}]$ be the (corranged) complement of $[p_1 p_2 \dots p_\nu]$ in $[1\ 2 \dots m]$.

Then $[y_1 y_2 \dots y_{m-\mu}]$ is a corranged minor of $[q_1 q_2 \dots q_{m-\nu}]$.

In forming the terms of (3), $[p_1 p_2 \dots p_\nu]$, $[q_1 q_2 \dots q_{m-\nu}]$ may be any two complementary corranged minors of $[1\ 2 \dots m]$ of orders ν and $m-\nu$, and when these are given, $[y_1 y_2 \dots y_{m-\mu}]$ may be any corranged minor of order $m-\mu$ of $[q_1 q_2 \dots q_{m-\nu}]$.

Hence the double sum (3) can be evaluated by first summing for all corranged minors $[y_1 y_2 \dots y_{m-\mu}]$ of order $m-\mu$ of the fixed sequence $[q_1 q_2 \dots q_{m-\nu}]$, and then summing for all corranged minors $[q_1 q_2 \dots q_{m-\nu}]$ of order $m-\nu$ of $[1\ 2 \dots m]$.

When we sum in this way, we have

$$S = (-1)^{\omega_1} \sum_{q\,y}\sum (-1)^{\omega_2+\eta} (a_{p\alpha})^{\nu}_{\nu} (a_{y\beta})^{n-\nu}_{m-\mu} \quad \dots\dots\dots\dots\dots\dots(4).$$

Let $[u_1 u_2 \dots u_{\mu-\nu}]$ be the (corranged) complement of $[y_1 y_2 \dots y_{m-\mu}]$ in $[q_1 q_2 \dots q_{m-\nu}]$, which is also the (corranged) complement of $[p_1 p_2 \dots p_\nu]$ in $[x_1 x_2 \dots x_\mu]$.

Let σ = the affect of $[y_1 y_2 \dots y_{m-\mu}]$ in $[q_1 q_2 \dots q_{m-\nu}]$,

σ' = the affect of $[u_1 u_2 \dots u_{\mu-\nu}]$ in $[q_1 q_2 \dots q_{m-\nu}]$,

ω_2' = the affect of $[p_1 p_2 \dots p_\nu]$ in $[1\ 2 \dots n]$.

Thus by Theorem VI of § 19

$$\omega_2 + \eta = \omega_2' + \sigma';$$

for $[1\ 2 \dots m]$ can be converted into $[p_1 p_2 \dots p_\nu u_1 u_2 \dots u_{\mu-\nu} y_1 y_2 \dots y_{m-\mu}]$ both by ω_2 forward moves followed by η forward moves, and by ω_2' forward moves followed by σ' forward moves.

Also by Theorem VIII *a* of § 19

$$\sigma + \sigma' = (\mu - \nu)(m - \mu).$$

Therefore $\omega_2 + \eta \equiv (\mu - \nu)(m - \mu) + \omega_2' + \sigma \pmod{2}$.

Accordingly

$$S = (-1)^{(\mu-\nu)(m-\mu)} \sum_{q\,y}\sum (-1)^{\omega_1+\omega_2'+\sigma} (a_{p\alpha})^{\nu}_{\nu} (a_{y\beta})^{n-\nu}_{m-\mu} \quad \dots\dots\dots\dots(5).$$

When $[q_1 q_2 \dots q_{m-\nu}]$ is kept constant, $(a_{p\alpha})^{\nu}_{\nu}$, ω_1 and ω_2' remain constant, and by § 33

$$\sum_{y} (-1)^{\sigma} (a_{y\beta})^{n-\nu}_{m-\mu} = (a_{q\beta})^{n-\nu}_{m-\nu} \times Q^{m-n}_{(m-n)-(\mu-\nu)}$$

$$= (a_{q\beta})^{n-\nu}_{m-\nu} \times Q^{m-n}_{\mu-\nu}.$$

Thus $$S = (-1)^{(\mu-\nu)(m-\mu)} Q^{m-n}_{\mu-\nu} \sum_{p} (-1)^{\omega_1+\omega_2'} (a_{p\alpha})^{\nu}_{\nu} (a_{q\beta})^{n-\nu}_{m-\nu} \quad\text{..............(6).}$$

Here we have replaced $\sum_{q}$ by its equivalent $\sum_{p}$.

Now $\omega_1 + \omega_2'$ is the affect of $(a_{p\alpha})^{\nu}_{\nu}$ in Δ.

Therefore, by § 32, the result (6) is equivalent to

$$S = (-1)^{(\mu-\nu)(m-\mu)} Q^{m-n}_{\mu-\nu} \Delta \quad\text{..............................(7).}$$

Again the equality $\quad (-1)^{(\mu-\nu)(m-\mu)} = (-1)^{(\mu-\nu)(n-\nu)}$

is true except when $\quad (\mu-\nu)(m-\mu) - (\mu-\nu)(n-\nu)$ is odd,

or $\quad (\mu-\nu)\{(m-n)-(\mu-\nu)\}$ is odd,

or $\quad \mu-\nu$ is odd and $(m-n)-(\mu-\nu)$ is odd,

i.e. except when $\quad \mu-\nu$ is odd and $m-n$ is even.

But in this exceptional case, $\quad Q^{m-n}_{\mu-\nu} = 0.$

Therefore in all cases

$$(-1)^{(\mu-\nu)(m-\mu)} Q^{m-n}_{\mu-\nu} = (-1)^{(\mu-\nu)(n-\nu)} Q^{m-n}_{\mu-\nu}.$$

It follows that we can replace (7) by

$$S = (-1)^{(\mu-\nu)(m-\mu)} Q^{m-n}_{\mu-\nu} \Delta = (-1)^{(\mu-\nu)(n-\nu)} Q^{m-n}_{\mu-\nu} \Delta \quad\text{..............(8).}$$

Since in the present case $\delta = \mu - \nu$, $\epsilon = n - \nu$, we see that

$$S = (-1)^{\delta\epsilon} Q^{m-n}_{\mu-\nu} \Delta \quad\text{....................................(9),}$$

and this is the formula (A).

It may be observed that if we replace $A^{n-\nu}_{m}$ by any similar minor matrix $B^{n-\nu}_{m}$ of Δ which is not complementary to A^{ν}_{m}, and if we also replace the minor determinoid $A^{n-\nu}_{m-\mu}$ of $A^{n-\nu}_{m}$ by the corresponding minor determinoid $B^{n-\nu}_{m-\mu}$ of $B^{n-\nu}_{m}$, the sum S thus obtained has the value zero. Thus we could enunciate a more general theorem than (A) analogous to Theorem II of § 32.

4. Proof of Formula (B) in Case II.

(i) We will first suppose that A^{ν}_{μ} is an inferior simple minor of A^{ν}_{m}, and $A^{n-\nu}_{m-\mu}$ a superior simple minor of $A^{n-\nu}_{m}$.

This occurs when $\quad \mu < \nu$ and $m - \mu > n - \nu$.

Since these inequalities can be satisfied both when $m > n$ and when $m < n$, we see that the present case can occur both when the two fixed complementary simple minor matrices are long-cut minors and when they are short-cut minors.

To prove formula (B), we commence by expanding the inferior minor determinoid A^{ν}_{μ} in terms of its simple minor determinants. The expansion is

$$A^{\nu}_{\mu} = (a_{x\alpha})^{\nu}_{\mu} = \sum_{p} (-1)^{\eta} (a_{xp})^{\mu}_{\mu},$$

where $[p_1 p_2 \ldots p_\mu]$ is any corranged minor of order μ of $[a_1 a_2 \ldots a_\nu]$,

and η is the affect of $[p_1 p_2 \ldots p_\mu]$ in $[a_1 a_2 \ldots a_\nu]$.

The summation extends over all such values of the sequence $[p_1 p_2 \ldots p_\mu]$.

Inserting this value of A^{ν}_{μ} in (1) and (2), we have

$$S = (-1)^{\omega_1} \sum_{x\,p}\sum (-1)^{\omega_2 + \eta} (a_{xp})^{\mu}_{\mu} (a_{y\beta})^{n-\nu}_{m-\mu} \quad \ldots\ldots(10).$$

The summations with respect to $[x_1 x_2 \ldots x_\mu]$ and $[p_1 p_2 \ldots p_\mu]$ are independent of one another and can be performed in either order.

When we keep $[p_1 p_2 \ldots p_\mu]$ constant and sum for the different values of $[x_1 x_2 \ldots x_\mu]$, η remains constant, and by § 32

$$\sum_{x} (-1)^{\omega_2} (a_{xp})^{\mu}_{\mu} (a_{y\beta})^{n-\nu}_{m-\mu} = \begin{pmatrix} p_1 p_2 \ldots p_\mu \beta_1 \beta_2 \ldots \beta_{n-\nu} \\ a \\ 1\ 2 \ldots m \end{pmatrix} = A^{n+\mu-\nu}_{m} \quad \ldots\ldots(11).$$

Since $m > n + \mu - \nu$, the long rows of $A^{n+\mu-\nu}_{m}$ are vertical.

It is on account of this fact that we can apply § 32 to obtain the result (11).

It now follows that $$S = \sum_{p} (-1)^{\omega_1 + \eta} A^{n+\mu-\nu}_{m} \quad \ldots\ldots(12).$$

Let $[q_1 q_2 \ldots q_{\nu-\mu}]$ be the (corranged) complement of $[p_1 p_2 \ldots p_\mu]$ in $[a_1 a_2 \ldots a_\nu]$,

and let ω_1' be the affect of $[p_1 p_2 \ldots p_\mu \beta_1 \beta_2 \ldots \beta_{n-\nu}]$ in $[1\ 2 \ldots n]$.

Then ω_1 forward moves convert $[1\ 2 \ldots n]$ into $[a_1 a_2 \ldots a_\nu \beta_1 \beta_2 \ldots \beta_{n-\nu}]$,

and η forward moves convert this into $[p_1 p_2 \ldots p_\mu q_1 q_2 \ldots q_{\nu-\mu} \beta_1 \beta_2 \ldots \beta_{n-\nu}]$.

Also by moving each of the q's in turn to the right of the β's, commencing with $q_{\nu-\mu}$, we can clearly convert the last sequence into $[p_1 p_2 \ldots p_\mu \beta_1 \beta_2 \ldots \beta_{n-\nu} q_1 q_2 \ldots q_{\nu-\mu}]$ by $(\nu-\mu)(n-\nu)$ moves.

Thus we can convert $[1\ 2 \ldots n]$ into $[p_1 p_2 \ldots p_\mu \beta_1 \beta_2 \ldots \beta_{n-\nu} q_1 q_2 \ldots q_{\nu-\mu}]$ by $(\nu-\mu)(n-\nu) + \omega_1 + \eta$ forward and backward moves.

Therefore by Theorem 11 *b* of § 19

$$(\nu-\mu)(n-\nu) + \omega_1 + \eta \equiv \omega_1' \quad (\text{mod. } 2),$$

and therefore $$\omega_1 + \eta \equiv (\nu-\mu)(n-\nu) + \omega_1' \quad (\text{mod. } 2).$$

We see then that $$S = (-1)^{(\nu-\mu)(n-\nu)} \sum_{p} (-1)^{\omega_1'} A^{n+\mu-\nu}_{m} \quad \ldots\ldots(13).$$

Here ω_1' is the affect of $A^{n+\mu-\nu}_{m}$ in Δ.

Let $[s_1 s_2 \ldots s_{n+\mu-\nu}]$ be the corranged minor sequence of $[1\ 2 \ldots n]$ formed with the elements of $[p_1 p_2 \ldots p_\mu]$, $[\beta_1 \beta_2 \ldots \beta_{n-\nu}]$ taken together, so that

$$\Delta_{n+\mu-\nu} = (a_{1s})^{n+\mu-\nu}_{m}$$

is the corranged minor determinoid of Δ of which $A_m^{n+\mu-\nu}$ is a derangement, and let σ be the affect of $\Delta_{n+\mu-\nu}$ in Δ.

Since the long rows of $A_m^{n+\mu-\nu}$ are vertical and ω_1' is the affect of $A_m^{n+\mu-\nu}$ in Δ, it follows from Theorem V*b* of § 25 that

$$(-1)^{\omega_1'} A_m^{n+\mu-\nu} = (-1)^{\sigma} \Delta_{n+\mu-\nu}.$$

Thus from (13) we deduce that

$$S = (-1)^{(\nu-\mu)(n-\nu)} \Sigma (-1)^{\sigma} \Delta_{n+\mu-\nu} \quad \text{.......................(14).}$$

Now in the case under consideration

$$\epsilon_1 = \mu, \quad \epsilon_2 = n-\nu, \quad \delta = \nu - \mu, \quad \epsilon = n - \nu.$$

Also $\Delta_{n+\mu-\nu}$, being a derangement of $A_m^{n+\mu-\nu}$ which is given by (11), is any corranged simple minor determinoid of Δ which contains μ (or ϵ_1) of the vertical rows of A_m^{ν} and all the $n-\nu$ (or ϵ_2) vertical rows of $A_m^{n-\nu}$.

Accordingly (14) can be written in the form

$$S = (-1)^{\delta\epsilon} \Sigma (-1)^{\sigma} \Delta_{\epsilon_1+\epsilon_2} \quad \text{.................................(15),}$$

where every symbol has the same meaning as in formula (B).

Thus formula (B) is proved on the supposition that A_μ^{ν} is an inferior minor and $A_{m-\mu}^{n-\nu}$ a superior minor.

(ii) Next let us suppose that A_μ^{ν} is a superior simple minor of A_m^{ν} and that $A_{m-\mu}^{n-\nu}$ is an inferior simple minor of $A_m^{n-\nu}$, so that

$$\mu > \nu \quad \text{and} \quad m - \mu < n - \nu.$$

Let ω' be the affect of $A_{m-\mu}^{n-\nu}$ in Δ, ω denoting as before the affect of A_μ^{ν} in Δ, and let

$$S' = \Sigma (-1)^{\omega'} A_{m-\mu}^{n-\nu} A_\mu^{\nu}.$$

Then S' can be evaluated by the case already considered.

Using (15), we have

$$S' = (-1)^{(n-\nu-m-\mu)\nu} \Sigma (-1)^{\sigma} \Delta_{m+\nu-\mu},$$

where $\Delta_{m+\nu-\mu}$ is any corranged simple minor determinoid of Δ formed with $m-\mu$ of the vertical rows of $A_m^{n-\nu}$ and all the ν vertical rows of A_m^{ν}.

Now $$\omega + \omega' = \mu(m-\mu) + \nu(n-\nu).$$

Therefore
$$\begin{aligned} S &= \Sigma (-1)^{\omega} A_\mu^{\nu} A_{m-\mu}^{n-\nu} \\ &= (-1)^{\mu(m-\mu)+\nu(n-\nu)} \Sigma (-1)^{\omega'} A_\mu^{\nu} A_{m-\mu}^{n-\nu} \\ &= (-1)^{\mu(m-\mu)+\nu(n-\nu)} S' \\ &= (-1)^{(\mu-\nu)(m-\mu)} \Sigma (-1)^{\sigma} \Delta_{m+\nu-\mu} \quad \text{..........................(16).} \end{aligned}$$

In the present case $\epsilon_1 = \nu,\quad \epsilon_2 = m-\mu,\quad \delta = \mu-\nu,\quad \epsilon = m-\mu.$

Therefore equation (16) can be written in the form

$$S = (-1)^{\delta\epsilon}\,\Sigma(-1)^{\sigma}\,\Delta_{\epsilon_1+\epsilon_2} \quad\text{.................................(17)},$$

where every symbol has the same meaning as in formula (B).

Thus formula (B) is always true in Case II.

Ex. ix. From formula (B) we can deduce that in Case II

$$S = (-1)^{\delta\epsilon}\,\Sigma(-1)^{\varpi}\,P_{\epsilon_1+\epsilon_2} \quad\text{.................................(B}'\text{)},$$

where $P_{\epsilon_1+\epsilon_2}$ is any one of the derived products of Δ of order $\epsilon_1+\epsilon_2$ which contains ϵ_1 factors belonging to ϵ_1 of the vertical rows of the first matrix A_m^ν and ϵ_2 factors belonging to ϵ_2 of the vertical rows of the second matrix $A_m^{n-\nu}$, and ϖ is its affect in Δ.

All the products $P_{\epsilon_1+\epsilon_2}$ contain factors from every one of the vertical rows of the matrix to which the superior determinoids belong.

To prove formula (B′) we observe that every one of the determinoids $\Delta_{\epsilon_1+\epsilon_2}$ can be expanded in terms of products of the type $P_{\epsilon_1+\epsilon_2}$, and that every such product occurs in one and only one of the determinoids $\Delta_{\epsilon_1+\epsilon_2}$.

Let P be any one of the products $P_{\epsilon_1+\epsilon_2}$ which occurs in the particular determinoid $\Delta_{\epsilon_1+\epsilon_2}$ and let ϖ' and ϖ be the affects of P in $\Delta_{\epsilon_1+\epsilon_2}$ and Δ.

Then $\qquad \Delta_{\epsilon_1+\epsilon_2} = \Sigma(-1)^{\varpi'} P, \qquad (-1)^{\sigma}\Delta_{\epsilon_1+\epsilon_2} = \Sigma(-1)^{\varpi'+\sigma} P.$

Now by Theorem III*c* of § 25

$$\varpi = \varpi' + \sigma.$$

Thus $\qquad (-1)^{\sigma}\Delta_{\epsilon_1+\epsilon_2} = \Sigma(-1)^{\varpi} P.$

Substituting this value in (B), we obtain (B′).

Formula (B′) expresses S as the algebraical sum of certain derived products of Δ of order $\epsilon_1+\epsilon_2$.

Ex. x. It can also be deduced that in Case II

$$S = (-1)^{\delta\epsilon}\,\Sigma(-1)^{\rho}\,D_{\epsilon_1+\epsilon_2} \quad\text{.................................(B}''\text{)},$$

where $D_{\epsilon_1+\epsilon_2}$ is any one of the minor determinants of Δ of order $\epsilon_1+\epsilon_2$ in which ϵ_1 of the vertical rows of the first matrix A_m^ν and ϵ_2 of the vertical rows of the second matrix $A_m^{n-\nu}$ occur, and ρ is its affect in Δ.

In each of the determinants $D_{\epsilon_1+\epsilon_2}$ every one of the vertical rows of the matrix containing the superior determinoids occurs.

Formula (B″) can be deduced from the formula (B) by expanding each of the determinoids $\Delta_{\epsilon_1+\epsilon_2}$ in terms of its simple minor determinants and using Theorem III*d* of § 25.

It expresses S as the algebraical sum of certain minor determinants of Δ of order $\epsilon_1+\epsilon_2$.

5. Proof of Formula (C) in Case III.

This case occurs when $\mu < \nu$ and $m-\mu < n-\nu$.

A necessary consequence of these inequalities is $m < n$.

Hence this case can only occur when the two fixed complementary simple minor matrices are both short-cut minors.

To obtain the formula (C), we expand each of the inferior minor determinoids A_μ^ν, $A_{m-\mu}^{n-\nu}$ in terms of its simple minor determinants.

The expansions are

$$A_\mu^\nu = \sum_p (-1)^{\eta_1} (a_{xp})_\mu^\mu, \quad A_{m-\mu}^{n-\nu} = \sum_q (-1)^{\eta_2} (a_{yq})_{m-\mu}^{m-\mu},$$

where $[p_1 p_2 \dots p_\mu]$ is a corranged minor of $[a_1 a_2 \dots a_\nu]$,

$[q_1 q_2 \dots q_{m-\mu}]$ is a corranged minor of $[\beta_1 \beta_2 \dots \beta_{n-\nu}]$,

η_1 is the affect of $[p_1 p_2 \dots p_\mu]$ in $[a_1 a_2 \dots a_\nu]$,

η_2 is the affect of $[q_1 q_2 \dots q_{m-\mu}]$ in $[\beta_1 \beta_2 \dots \beta_{n-\nu}]$.

Inserting these values of A_μ^ν, $A_{m-\mu}^{n-\nu}$ in (1) or (2), we have

$$S = (-1)^{\omega_1} \sum_{x\,p\,q}\sum\sum (-1)^{\omega_2+\eta_1+\eta_2} (a_{xp})_\mu^\mu (a_{yq})_{m-\mu}^{m-\mu} \quad \dots\dots\dots\dots(18).$$

The three summations are independent and can be performed in any order.

When we keep $[p_1 p_2 \dots p_\mu]$, $[q_1 q_2 \dots q_{m-\mu}]$ fixed, η_1 and η_2 remain fixed, and by § 32

$$\sum_x (-1)^{\omega_2} (a_{xp})_\mu^\mu (a_{yq})_{m-\mu}^{m-\mu} = \begin{pmatrix} p_1 p_2 \dots p_\mu q_1 q_2 \dots q_{m-\mu} \\ a \\ 1\,2 \dots m \end{pmatrix} = A_m^m \quad \dots\dots\dots(19).$$

Thus

$$S = \sum_{p\,q}\sum (-1)^{\omega_1+\eta_1+\eta_2} A_m^m \quad \dots\dots\dots\dots(20).$$

Let

$[u_1 u_2 \dots u_{\nu-\mu}]$ be the complement of $[p_1 p_2 \dots p_\mu]$ in $[a_1 a_2 \dots a_\nu]$,

$[v_1 v_2 \dots v_{n-\nu-m+\mu}]$ be the complement of $[q_1 q_2 \dots q_{m-\mu}]$ in $[\beta_1 \beta_2 \dots \beta_{n-\nu}]$,

$[r_1 r_2 \dots r_m]$ be the corranged minor of $[1\,2 \dots n]$ formed with the elements $p_1, p_2, \dots q_1, q_2, \dots$,

$[w_1 w_2 \dots w_{n-m}]$ be the corranged minor of $[1\,2 \dots n]$ formed with the elements $u_1, u_2, \dots v_1, v_2, \dots$.

Also let ω' be the affect of $[p_1 p_2 \dots p_\mu q_1 q_2 \dots q_{m-\mu}]$ in $[1\,2 \dots n]$,

and σ' be the affect of $[u_1 u_2 \dots u_{\nu-\mu}]$ in $[w_1 w_2 \dots w_{n-m}]$.

Then

ω_1 forward moves can convert $[1\,2 \dots n]$ into $[a_1 a_2 \dots \beta_1 \beta_2 \dots]$,

$\omega_1+\eta_1$ forward moves can convert it into $[p_1 p_2 \dots u_1 u_2 \dots \beta_1 \beta_2 \dots]$,

$\omega_1+\eta_1+\eta_2$ forward moves can convert it into $[p_1 p_2 \dots u_1 u_2 \dots q_1 q_2 \dots v_1 v_2 \dots]$.

Consequently $[1\,2 \dots n]$ can be converted into $[p_1 p_2 \dots u_1 u_2 \dots q_1 q_2 \dots v_1 v_2 \dots]$ by $\omega_1+\eta_1+\eta_2+(\nu-\mu)(m-\mu)$ forward and backward moves.

The same conversion can be effected by $\omega'+\sigma'$ forward moves.

Therefore $\omega_1+\eta_1+\eta_2+(\nu-\mu)(m-\mu)\equiv\omega'+\sigma' \pmod{2}$.

Thus $$S=(-1)^{(\nu-\mu)(m-\mu)}\sum_{pq}\sum(-1)^{\omega'+\sigma'}A_m^m \quad\text{.........................(21)}.$$

Let σ be the affect in Δ of the corranged minor determinant

$$\Delta_m=\begin{pmatrix} r_1\, r_2 \ldots r_m \\ a \\ 1\, 2 \ldots m\end{pmatrix}.$$

Since ω' is the affect of A_m^m in Δ, it follows from Theorem V a of § 25 that

$$(-1)^{\omega'}A_m^m=(-1)^{\sigma}\Delta_m.$$

Hence $$S=(-1)^{(\nu-\mu)(m-\mu)}\Sigma(-1)^{\sigma+\sigma'}\Delta_m \quad\text{..........................(22)}.$$

Now in the present case

$$\epsilon_1=\mu,\quad \epsilon_2=m-\mu,\quad \delta=\nu-\mu,\quad \epsilon=m-\mu.$$

Observing the form of A_m^m, which is a derangement of Δ_m, in (19), we see that (22) is equivalent to

$$S=(-1)^{\delta\epsilon}\Sigma(-1)^{\sigma+\sigma'}\Delta_{\epsilon_1+\epsilon_2},$$

where every symbol has the same meaning as in the formula (C).

Thus the formula (C) is always true in Case III.

CHAPTER VI.

PROPERTIES OF A PRODUCT FORMED BY A CHAIN OF MATRIX FACTORS.

[The equality of matrices, the addition and subtraction of matrices, and the multiplication of a matrix by a scalar number are first considered in §§ 39—41. Then in §§ 42—46 a product of two matrices is defined and the chief properties of such a product are discussed. The remaining articles, §§ 47—55, deal with a product formed by any chain of matrix factors. The active and passive rows and the standard form of the product are defined. It is shown that such a product is always associative and distributive but in general not commutative. Finally the chief properties of such products, including the properties of active and passive rows, are discussed.]

§ 39. Law of equality for matrices.

1. *Similar and inversely similar matrices.*

Two matrices will be said to be *similar* when each has the same number of horizontal rows as the other and also the same number of vertical rows as the other, i.e. when the vertical and horizontal orders of the one are the same respectively as the vertical and horizontal orders of the other.

They will be said to be *inversely similar* when the horizontal and vertical rows of the one are the same in number respectively as the vertical and horizontal rows of the other.

Ex. i. $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ and $\begin{bmatrix} p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \end{bmatrix}$ are similar matrices.

Ex. ii. $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ and $\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{bmatrix}$ are inversely similar matrices.

2. *Equality of similar matrices. Identical equality.*

Two similar matrices are equal when every element of the one matrix is equal to the correspondingly situated element of the other matrix. In this case the two matrices are *identically equal* to one another.

Ex. iii. If
$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix},$$
then
$$x_1 = a_1, \quad y_1 = b_1, \quad z_1 = c_1, \quad x_2 = a_2, \quad y_2 = b_2, \quad z_2 = c_2.$$

3. *Equality of dissimilar matrices. Conventional equality.*

Two dissimilar matrices will be said to be equal when the similar matrices to which they can be reduced by the addition of rows of 0's after existing rows are equal. This definition is equivalent to the following two statements:

(i) If x is any element of one matrix and y a correspondingly situated element of the other matrix, then $x = y$.

(ii) If x is any element of one matrix, and if the other matrix has no correspondingly situated element, then $x = 0$.

In this case the two matrices may be said to be *conventionally equal* to one another.

According to the above definition, two matrices which differ only by final rows of 0's are equal.

Again when two dissimilar matrices are equal, all rows which one matrix has in excess of the other must be rows of 0's; and if we strike out from each matrix the rows which it has in excess of the other, we reduce the two matrices to a pair of equal similar matrices.

The sign of equality connecting two matrices will in general denote a conventional equality: but if the two matrices which it connects are similar, as will in practice almost invariably be the case, then it denotes also an identical equality.

Ex. iv.
$$\begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{bmatrix} \quad \text{if} \quad \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & 0 & 0 \\ a_4 & b_4 & 0 & 0 \end{bmatrix};$$
i.e. if
$$z_1 = z_2 = z_3 = 0, \quad w_1 = w_2 = w_3 = 0, \quad a_4 = b_4 = 0,$$
and
$$x_1 = a_1, \quad y_1 = b_1, \quad x_2 = a_2, \quad y_2 = b_2, \quad x_3 = a_3, \quad y_3 = b_3;$$
or if
$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix},$$
and all other elements vanish.

The two matrices then reduce respectively to

$$\begin{bmatrix} x_1 & y_1 & 0 & 0 \\ x_2 & y_2 & 0 & 0 \\ x_3 & y_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

Ex. v.
$$\begin{bmatrix} x_1 & y_1 & z_1 & 0 & 0 \\ x_2 & y_2 & z_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}.$$

Here the equality is conventional but not identical.

Ex. vi. If $A = B$, and A and B are similar matrices, then $\det A = \det B$. But this conclusion cannot be drawn when A and B are dissimilar matrices.

4. *Zero matrices.*

A matrix, every one of whose elements is zero, will be called a zero matrix.

If A is any matrix, the equation $A = 0$ will be interpreted to mean that A is a zero matrix. With this exception a matrix cannot be equated to a scalar number, being essentially different in character. This is shown in § 46, Ex. ix.

The equation $A = [k]$, where A is a matrix and k a scalar number, means that the leading element of A is k, and that every other element of A is 0; but the equation $A = k$ cannot occur.

Ex. vii. If $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = 0$, then $a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 0$.

§ 40. Addition and subtraction of matrices.

1. *Similar matrices.*

If $A = [a]_m^n$, $B = [b]_m^n$ are two similar matrices, then $A \pm B$ is defined to be the similar matrix $C = [c]_m^n$ which is such that $c_{ij} = a_{ij} \pm b_{ij}$ for all values of i and j belonging to the sequences $[1\ 2\ \ldots\ m]$, $[1\ 2\ \ldots\ n]$ respectively.

So if $A = [a]_m^n$, $B = [b]_m^n$, ..., $K = [k]_m^n$ are any number of similar matrices, then $\pm A \pm B \pm \ldots \pm K$ is defined to be the similar matrix $S = [s]_m^n$, where

$$s_{ij} = \pm a_{ij} \pm b_{ij} \pm \ldots \pm k_{ij}.$$

The additions and subtractions thus defined are clearly both commutative and associative.

Ex. i. $$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} = \begin{bmatrix} (a_1+a_1), & (a_2+a_2) \\ (b_1+\beta_1), & (b_2+\beta_2) \\ (c_1+\gamma_1), & (c_2+\gamma_2) \end{bmatrix}.$$

2. *Dissimilar matrices.*

If A and B are two dissimilar matrices, we reduce them to a pair of similar matrices A', B' by insertions of additional final rows of 0's. Then $A \pm B$ is defined to be the same matrix as $A' \pm B'$.

So if $A, B, \ldots K$ are any number of dissimilar matrices, we reduce them all to similar matrices $A', B', \ldots K'$, and define $\pm A \pm B \pm \ldots \pm K$ to be the same matrix as $\pm A' \pm B' \pm \ldots \pm K'$.

Additions and subtractions are thus defined as both commutative and associative.

It will be observed that in order to find an algebraical sum of a number of dissimilar matrices, we replace each of the matrices by a certain other matrix conventionally equal to it before performing the additions and subtractions. Further if S is an algebraical sum of any number of matrices, and if S' is the algebraical sum of other matrices conventionally equal one by one to the former matrices, then S' is conventionally equal to S. Thus in performing additions and subtractions we can consistently regard conventionally equal matrices as equivalent.

Ex. ii. $$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix} - \begin{bmatrix} a_1 & \beta_1 \\ a_2 & \beta_2 \\ a_3 & \beta_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} a_1 & \beta_1 & 0 & 0 \\ a_2 & \beta_2 & 0 & 0 \\ a_3 & \beta_3 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} (a_1-a_1), & (b_1-\beta_1), & c_1, & d_1 \\ (a_2-a_2), & (b_2-\beta_2), & c_2, & d_2 \\ -a_3\ , & -\beta_3\ , & 0, & 0 \end{bmatrix}.$$

Ex. iii. $$[a_1 b_1] + \begin{bmatrix} a_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} (a_1+a_1), & b_1 \\ \beta_1\ , & 0 \end{bmatrix}.$$

$$\begin{bmatrix} a_1 & b_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ \beta_1 & 0 \end{bmatrix} = \begin{bmatrix} (a_1+a_1), & b_1, & 0 \\ \beta_1\ , & 0\ , & 0 \end{bmatrix} = \begin{bmatrix} (a_1+a_1), & b_1 \\ \beta_1\ , & 0 \end{bmatrix}.$$

Here the two sums are conventionally equal.

3. *Manipulation of matrix equations.*

If A, B, C are matrices, then from the equation $A = B$ it follows that $A + C = B + C$, and that $A - C = B - C$.

Hence terms can be transferred from one side to the other of a matrix equation according to the same rules as in the case of an algebraic equation.

Such conclusions are only true in general when the sign of equality stands for conventional equality. But if at any stage in the reduction of an equation by this process two similar matrices are equated, the equality is then necessarily an identical equality.

Ex. iv. If $A+B=C$, then $C-B=A$.

Ex. v. If $A+B=A+C$, then $B=C$.

§ 41. Multiplication of a matrix by a scalar number.

If
$$A=[a]_m^n=\begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots a_{mn} \end{bmatrix}$$

is any matrix and k is any scalar number, the product kA or $k[a]_m^n$ is defined to be the matrix given by the identical equation

$$kA=k[a]_m^n=\begin{bmatrix} ka_{11} & ka_{12} & \dots ka_{1n} \\ ka_{21} & ka_{22} & \dots ka_{2n} \\ \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots ka_{mn} \end{bmatrix}.$$

The product Ak is defined to be the same as kA.

Thus to multiply a matrix by a scalar number k, every element of the matrix must be multiplied by k.

Such products are clearly distributive; for example, $k(A+B)=kA+kB$; also

$$(k_1+k_2)(A+B)=k_1A+k_1B+k_2A+k_2B.$$

Ex. i. $3\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}=\begin{bmatrix} 3a_1 & 3b_1 & 3c_1 \\ 3a_2 & 3b_2 & 3c_2 \end{bmatrix}$.

Ex. ii. If A is a matrix with m long rows and k is any scalar number, then

$$\det kA=k^m \det A.$$

Ex. iii. If A and B are any two matrices, similar or dissimilar, and if k is any scalar number except 0, then each of the equations

$$A=B, \quad kA=kB$$

is a necessary consequence of the other.

Thus in any matrix equation we can multiply or divide both sides by any scalar number which is not zero.

§ 42. Product of two matrices.

1. *Active and passive rows.*

If A and B are any two matrices, the product AB is defined below to be a certain third matrix which is completely known when A and B are known. If this third matrix is denoted by C, we have the identical equation

$$AB = C.$$

A and B will be called the *factor matrices*, and AB or its equivalent C will be called the *product matrix*.

In the product AB we shall always regard the factor matrix on the left as the operator or multiplier and the factor matrix on the right as the operand or multiplicand.

The horizontal rows of A and the vertical rows of B will be called *active rows*. The remaining rows of A and B, viz. the vertical rows of A and the horizontal rows of B, will be called *passive rows*. The number of active rows and the number of passive rows in either factor matrix will be called respectively the *activity* and the *passivity* of that factor matrix. The activities and passivities of the two factor matrices will also be called the activities and passivities of the product.

The product AB will be said to be a product in *the standard form* or a *standard product* when the passivities of A and B are equal, i.e. when the number of passive rows in one factor matrix is equal to the number of passive rows in the other factor matrix.

When the standard double-suffix notation is employed, then for a standard product AB has the form $[a]_m^r\,[b]_r^n$, and for other products AB has the form $[a]_m^r\,[a]_s^n$, where $r \neq s$.

2. *Product of two matrices whose passivities are equal.*

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{r1} & b_{r2} & \dots & b_{rn} \end{bmatrix}$$

be two matrices such that the number of vertical rows in A is equal to the number of horizontal rows in B.

The product AB will be defined to be the matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix},$$

in which

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ir}b_{rj} = \sum_u a_{iu}b_{uj},$$

u receiving all integral values from 1 to r ....(A).

This definition of the product is equivalent to the following two statements:

The product matrix has the same number of horizontal rows as the first factor matrix and the same number of vertical rows as the second factor matrix ....(B).

The element c_{ij} common to the ith horizontal row and the jth vertical row of the product matrix is the sum of the products which can be obtained by multiplying each element of the ith horizontal row of the first factor matrix by the corresponding element of the jth vertical row of the second factor matrix ....(C).

As a particular case of such a product we have

$$[a_{i1}a_{i2} \ldots a_{ir}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix} = [a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ir}b_{rj}] = [c_{ij}].$$

Since $c_{ij} = \det [c_{ij}]$, it follows that

$$c_{ij} = \det [a_{i1}a_{i2} \ldots a_{ir}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix}.$$

Thus the element c_{ij} common to the ith horizontal row and the jth vertical row of the product matrix is the determinoid of the product formed by multiplying the matrix of the ith horizontal or active row of the first factor matrix into the matrix of the jth vertical or active row of the second factor matrix ....(D).

If then the matrices of the 1st, 2nd, 3rd, ... active rows of A are denoted by $a_1, a_2, a_3, \ldots$, and the matrices of the 1st, 2nd, 3rd, ... active rows of B are denoted by $b_1, b_2, b_3, \ldots$, we have $c_{ij} = \det a_ib_j$, and

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1r} \\ a_{21} & a_{22} & \ldots & a_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mr} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{r1} & b_{r2} & \ldots & b_{rn} \end{bmatrix} = \begin{bmatrix} \det a_1b_1, & \det a_1b_2, & \ldots & \det a_1b_n \\ \det a_2b_1, & \det a_2b_2, & \ldots & \det a_2b_n \\ \cdots & \cdots & \cdots & \cdots \\ \det a_mb_1, & \det a_mb_2, & \ldots & \det a_mb_n \end{bmatrix}$$

...........(E).

When the abbreviated standard double-suffix notation is employed, the definition of a standard product takes the form:

$$[a]_m^r [b]_r^n = [c]_m^n,$$

where

$$c_{ij} = det\, a_i b_j = det\,[a_{i1} a_{i2} \dots a_{ir}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix} = \sum_u a_{iu} b_{uj},$$

u receiving all integral values from 1 *to* r ....(F).

Ex. i. $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix}$

$$= \begin{bmatrix} \det\,[a_1 b_1 c_1]\begin{bmatrix} a_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}, & \det\,[a_1 b_1 c_1]\begin{bmatrix} a_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix}, & \det\,[a_1 b_1 c_1]\begin{bmatrix} a_3 \\ \beta_3 \\ \gamma_3 \end{bmatrix}, & \det\,[a_1 b_1 c_1]\begin{bmatrix} a_4 \\ \beta_4 \\ \gamma_4 \end{bmatrix} \\ \det\,[a_2 b_2 c_2]\begin{bmatrix} a_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}, & \det\,[a_2 b_2 c_2]\begin{bmatrix} a_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix}, & \det\,[a_2 b_2 c_2]\begin{bmatrix} a_3 \\ \beta_3 \\ \gamma_3 \end{bmatrix}, & \det\,[a_2 b_2 c_2]\begin{bmatrix} a_4 \\ \beta_4 \\ \gamma_4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (a_1 a_1 + b_1\beta_1 + c_1\gamma_1), & (a_1 a_2 + b_1\beta_2 + c_1\gamma_2), & (a_1 a_3 + b_1\beta_3 + c_1\gamma_3), & (a_1 a_4 + b_1\beta_4 + c_1\gamma_4) \\ (a_2 a_1 + b_2\beta_1 + c_2\gamma_1), & (a_2 a_2 + b_2\beta_2 + c_2\gamma_2), & (a_2 a_3 + b_2\beta_3 + c_2\gamma_3), & (a_2 a_4 + b_2\beta_4 + c_2\gamma_4) \end{bmatrix}.$$

The ith horizontal or active row of the first factor matrix will be said to *correspond* to the ith horizontal row of the product matrix.

The ith vertical or active row of the second factor matrix will be said to *correspond* to the ith vertical row of the product matrix.

The ith vertical or passive row in the first factor matrix and the ith horizontal or passive row in the second factor matrix will be said to *correspond* to one another.

Thus the passive rows of one factor matrix correspond one by one to the passive rows of the other factor matrix; and the active rows of the two factor matrices correspond one by one to the rows of the product matrix.

In forming the product matrix we multiply together *matrices of active rows* and *elements of corresponding passive rows.* In illustration of the second point, we observe that the element a_{ij} of the jth passive row of the first factor matrix only occurs in the elements of the product matrix in the forms $a_{ij}b_{j1}$, $a_{ij}b_{j2}$, ... $a_{ij}b_{jn}$, i.e. it only occurs after it has been multiplied by one of the elements of the corresponding jth passive row of the second factor matrix.

Generally wherever any element belonging to a particular passive row of either factor matrix occurs in the product matrix, it is multiplied by one of the elements of the *corresponding* passive row of the other factor matrix.

3. *Product of two matrices whose passivities are not equal.*

Let A and B be two matrices such that the number of vertical rows in A is not equal to the number of horizontal rows in B.

In this case to obtain the product matrix AB, we will

> *insert additional final passive rows of* 0's *in the factor matrix which has the smaller number of passive rows,*

and so reduce the product AB to a product $A'B'$ in which A' and B' are two matrices whose passivities are equal. The product AB will be defined to be the product $A'B'$, which can be found as in sub-article 2.

Thus in order to define the product AB in this case, we replace one of the factor matrices by another conventionally equal to it.

Ex. ii.
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix}$$
$$= \begin{bmatrix} (a_1\alpha_1+b_1\beta_1), & (a_1\alpha_2+b_1\beta_2), & (a_1\alpha_3+b_1\beta_3), & (a_1\alpha_4+b_1\beta_4) \\ (a_2\alpha_1+b_2\beta_1), & (a_2\alpha_2+b_2\beta_2), & (a_2\alpha_3+b_2\beta_3), & (a_2\alpha_4+b_2\beta_4) \end{bmatrix}$$
$$= \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix}.$$

Ex. iii.
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} (a_1\alpha_1+b_1\beta_1), & (a_1\alpha_2+b_1\beta_2), & (a_1\alpha_3+b_1\beta_3), & (a_1\alpha_4+b_1\beta_4) \\ (a_2\alpha_1+b_2\beta_1), & (a_2\alpha_2+b_2\beta_2), & (a_2\alpha_3+b_2\beta_3), & (a_2\alpha_4+b_2\beta_4) \end{bmatrix}$$
$$= \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix}.$$

The elements of those passive rows in the unaltered factor matrix which correspond to the added passive rows of 0's in the other factor matrix only contribute zero terms to the expressions for the elements c_{ij} of the product matrix, since each of them wherever it occurs in those expressions is multiplied by 0. Hence instead of inserting additional final passive rows of 0's in the factor matrix with the smaller number of passive rows, we may

> *strike out the redundant final passive rows which occur in the matrix with the larger number of passive rows,*

and so in a second way reduce the product AB to a product $A'B'$ in which A' and B' are two matrices whose passivities are equal. Then the product AB is also identical with this product $A'B'$.

This is illustrated by the final expressions given in Exs. ii and iii. In Ex. ii we strike out one final redundant passive row in the second factor matrix. In Ex. iii we strike out two redundant final passive rows in the first factor matrix.

We see then that there are two ways of reducing a product of any two matrices to the standard form, and we have the following theorem:

A product AB of any two matrices can be reduced to the standard form

(1) *by adding final passive rows of 0's to the matrix with the smaller number of passive rows;*

(2) *by striking out the redundant final passive rows in the matrix with the larger number of passive rows* ....(G).

The second method is the more convenient to adopt in practice.

If now
$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{s1} & b_{s2} & \dots & b_{sn} \end{bmatrix}$$
are any two matrices whatever, we have the following result:

The product AB is by definition the matrix
$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix},$$
in which
$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{iu}b_{uj} + \dots = \sum_u a_{iu}b_{uj},$$
u receiving all integral values from 1 to the smaller of the two numbers r and s ....(H).

Denoting as before the matrix of the ith active or horizontal row of A by a_i and the matrix of the jth active or vertical row of B by b_j, we have as a particular case of such a product
$$a_i b_j = [a_{i1} a_{i2} \dots a_{ir}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{sj} \end{bmatrix} = [a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{iu}b_{uj} + \dots] = [c_{ij}],$$
and therefore
$$c_{ij} = \det a_i b_j.$$

The results (B), (C), (D) are still true of this most general product, but in the case of (C) we must understand that where no corresponding element exists, a zero element must be supplied.

In place of (E) we have

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{s1} & b_{s2} & \dots & b_{sn} \end{bmatrix} = \begin{bmatrix} \det a_1 b_1, & \det a_1 b_2, & \dots & \det a_1 b_n \\ \det a_2 b_1, & \det a_2 b_2, & \dots & \det a_2 b_n \\ \dots & \dots & \dots & \dots \\ \det a_m b_1, & \det a_m b_2, & \dots & \det a_m b_n \end{bmatrix}$$

.........(I).

When the abbreviated standard double-suffix notation is employed, the general definition of a product takes the form:

$$[a]_m^r \, [b]_s^n = [c]_m^n,$$

where

$$c_{ij} = \det a_i b_j = \det [a_{i1} a_{i2} \dots a_{ir}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{sj} \end{bmatrix} = \sum_u a_{iu} b_{uj},$$

u receiving all integral values from 1 *to the smaller of the two numbers r and s* ....(J).

Ex. iv. $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix}$

$$= \begin{bmatrix} \det [a_1 b_1] \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}, & \det [a_1 b_1] \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix}, & \det [a_1 b_1] \begin{bmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{bmatrix}, & \det [a_1 b_1] \begin{bmatrix} \alpha_4 \\ \beta_4 \\ \gamma_4 \end{bmatrix} \\ \det [a_2 b_2] \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}, & \det [a_2 b_2] \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix}, & \det [a_2 b_2] \begin{bmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{bmatrix}, & \det [a_2 b_2] \begin{bmatrix} \alpha_4 \\ \beta_4 \\ \gamma_4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (a_1 \alpha_1 + b_1 \beta_1), & (a_1 \alpha_2 + b_1 \beta_2), & (a_1 \alpha_3 + b_1 \beta_3), & (a_1 \alpha_4 + b_1 \beta_4) \\ (a_2 \alpha_1 + b_2 \beta_1), & (a_2 \alpha_2 + b_2 \beta_2), & (a_2 \alpha_3 + b_2 \beta_3), & (a_2 \alpha_4 + b_2 \beta_4) \end{bmatrix}.$$

There is the same one-one correspondence as before between the active rows of the first factor matrix and the horizontal rows of the product matrix; there is also the same one-one correspondence as before between the active rows of the second factor matrix and the vertical rows of the product matrix. We shall still say that the ith vertical or passive row of the first factor matrix corresponds to the ith horizontal or passive row of the second factor matrix so long as there is an ith passive row in both factor matrices. But the factor matrix with the larger number of passive rows will now have passive rows whose corresponding rows in the other factor matrix are absent. Such rows are *redundant final passive rows*, the elements of which do not occur in the product matrix. We can omit some or all of these redundant final passive rows or replace some or all of them by other rows of arbitrary elements without in any way altering the product matrix. If we speak of a row *corresponding* to a redundant final passive row we must understand it to be a row of 0's.

§ 43. Properties of the passive rows in a product of two matrices.

1. *The product matrix is the sum of the products of the matrices of every pair of corresponding passive rows.*

Let $$[a]_m^r\,[b]_s^n = [c]_m^n,$$

and let C_u be the product of the matrices of the uth pair of corresponding passive rows in the factor matrices, so that

$$C_u = \begin{bmatrix} a_{1u} \\ a_{2u} \\ \vdots \\ a_{mu} \end{bmatrix} [b_{u1} b_{u2} \dots b_{un}].$$

Then C_u is a matrix similar to $[c]_m^n$ in which the element common to the ith horizontal row and the jth vertical row is $a_{iu} b_{uj}$.

Therefore when u receives all integral values from 1 to the smaller of the two integers r and s, we see that

$$C_1 + C_2 + \dots + C_u + \dots$$

is a matrix similar to $[c]_m^n$ in which the element common to the ith horizontal row and the jth vertical row

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{iu} b_{uj} + \dots = c_{ij}.$$

Therefore $$C_1 + C_2 + \dots + C_u + \dots = [c]_m^n.$$

Ex. i. $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [\alpha_1 \alpha_2 \alpha_3 \alpha_4] + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} [\beta_1 \beta_2 \beta_3 \beta_4].$$

Ex. ii. $$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [\alpha_1 \alpha_2 \alpha_3 \alpha_4] + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} [\beta_1 \beta_2 \beta_3 \beta_4] + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} [\gamma_1 \gamma_2 \gamma_3 \gamma_4].$$

Here there are only three pairs of corresponding passive rows, the last two passive rows of the first factor matrix being redundant and having no rows (or only rows of 0's) corresponding to them.

2. *The elements of a given passive row of either factor matrix only occur in the elements of the product matrix after each of them has been multiplied by an element of the corresponding passive row of the other factor matrix.*

This is an immediate consequence of the previous property. It can also be seen from the expression

$$c_{ij} = \sum_u a_{iu} b_{uj}$$

for any element of the product matrix. Wherever an element a_{pk} from the kth vertical row of the first factor matrix appears in such expressions, it is multiplied by some element b_{kq} from the kth horizontal row of the second factor matrix. So wherever an element b_{kq} belonging to the kth horizontal row of the second factor matrix occurs in such expressions, it is multiplied by some element a_{pk} belonging to the kth vertical row of the first factor matrix.

3. *The product matrix is unaffected by the insertion or the omission of final passive rows of 0's in either factor matrix.*

In fact the only effect of such a change is to add zero terms to or subtract zero terms from the various elements of the product matrix.

Ex. iii.
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For denoting the first and second product matrices by $[x]_2^4$, and $[y]_2^4$ respectively, we have

$$x_{23} = \det[a_2 b_2 c_2 d_2 e_2] \begin{bmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{bmatrix} = a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 = \det[a_2 b_2 c_2 d_2 e_2] \begin{bmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \\ 0 \end{bmatrix} = y_{23},$$

and similarly for the other elements of the product matrices.

4. *If either factor matrix contains redundant final passive rows, we may omit all or any number of them without in any way altering the product matrix.*

Conversely we can insert arbitrary redundant final passive rows in either factor matrix without in any way altering the product matrix.

In fact elements of redundant final passive rows do not occur at all in the product matrix. The product matrix is entirely independent of such redundant rows.

Ex. iv.
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & g_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & g_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & g_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{bmatrix}.$$

5. *If any non-redundant passive row in either factor matrix is a row of 0's, we can strike out that row and the corresponding passive row in the other factor matrix without in any way altering the product matrix.*

Let $$[a]_m^r [b]_s^n = [c]_m^n,$$

and let the product of the matrices obtained from $[a]_m^r$, $[b]_s^n$ by striking out the uth passive row in each be $[d]_m^n$.

Then
$$c_{ij} = a_{i1} b_{1j} + \dots + a_{i,u-1} b_{u-1,j} + a_{iu} b_{uj} + a_{i,u+1} b_{u+1,j} + \dots,$$
and $$d_{ij} = a_{i1} b_{1j} + \dots + a_{i,u-1} b_{u-1,j} + a_{i,u+1} b_{u+1,j} + \dots,$$
where the terms not shown are the same in both series.

If the uth passive row in either $[a]_m^r$ or $[b]_s^n$ is a row of 0's, then $a_{iu} b_{uj} = 0$, and therefore $c_{ij} = d_{ij}$.

We conclude that
$$[d]_m^n = [c]_m^n.$$

This theorem also follows immediately from sub-article 1. For with the notation used there, we have
$$[c]_m^n = [d]_m^n + C_u.$$

If one of the uth passive rows is a row of 0's, $C_u = 0$, and therefore
$$[d]_m^n = [c]_m^n.$$

Ex. v. $$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{bmatrix}.$$

For denoting the first and second product matrices by $[x]_3^4$ and $[y]_3^4$, we have
$$x_{23} = \det[a_2 b_2 c_2 d_2] \begin{bmatrix} \alpha_3 \\ \beta_3 \\ 0 \\ \delta_3 \end{bmatrix} = a_2 \alpha_3 + b_2 \beta_3 + d_2 \delta_3 = \det[a_2 b_2 d_2] \begin{bmatrix} \alpha_3 \\ \beta_3 \\ \delta_3 \end{bmatrix} = y_{23},$$
and similarly for the other elements of the product matrices.

Ex. vi. $$\begin{bmatrix} a_1 & 0 & c_1 & d_1 & e_1 \\ a_2 & 0 & c_2 & d_2 & e_2 \\ a_3 & 0 & c_3 & d_3 & e_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} = \begin{bmatrix} a_1 & c_1 & d_1 & e_1 \\ a_2 & c_2 & d_2 & e_2 \\ a_3 & c_3 & d_3 & e_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix}.$$

6. *Conversely we can insert any additional non-redundant passive row of arbitrary elements in any position in either of the factor matrices and an*

additional corresponding passive row of 0's in the corresponding position in the other factor matrix without in any way altering the product matrix.

Ex. vii. $$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & 0 & c_1 & d_1 \\ a_2 & b_2 & 0 & c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \\ x_1 & x_2 \end{bmatrix}.$$

Other illustrations of this theorem are given by Exs. v and vi when the equations occurring in them are read backwards.

7. *The product matrix is unaltered when the sign of every element in two corresponding passive rows is changed.*

This is an immediate consequence of sub-article 2.

Ex. viii. $$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} = \begin{bmatrix} a_1, & -b_1, & c_1 \\ a_2, & -b_2, & c_2 \end{bmatrix} \begin{bmatrix} a_1, & a_2 \\ -\beta_1, & -\beta_2 \\ \gamma_1, & \gamma_2 \end{bmatrix}.$$

8. *The product matrix is unaltered when the non-redundant passive rows of one factor matrix are re-arranged in any manner and the corresponding passive rows of the other factor matrix are re-arranged in exactly the same manner.*

This is an immediate consequence of sub-article 1.

Ex. ix. $$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} = \begin{bmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \beta_1 & \beta_2 & \beta_3 \\ a_1 & a_2 & a_3 \end{bmatrix},$$

for each product $= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} [a_1 a_2 a_3] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} [\beta_1 \beta_2 \beta_3] + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} [\gamma_1 \gamma_2 \gamma_3]$.

Ex. x. $$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} = \begin{bmatrix} c_1 & a_1 & b_1 & d_1 & e_1 \\ c_2 & a_2 & b_2 & d_2 & e_2 \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 \\ a_1 & a_2 \\ \beta_1 & \beta_2 \end{bmatrix},$$

for each product $= \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} [a_1 a_2] + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} [\beta_1 \beta_2] + \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} [\gamma_1 \gamma_2]$.

9. *If we strike out any number of pairs of corresponding passive rows from the two factor matrices, we obtain a partial product. Any sum of such partial products in which every pair of corresponding passive rows occurs once and once only is equal to the complete product matrix.*

If we express each partial product as the sum of the products of the matrices of pairs of corresponding passive rows, it appears that the sum of all such partial products is the sum of the products of the matrices of all pairs of corresponding passive rows in the original product, and this by sub-article 1 is equal to the complete product matrix.

Ex. xi.
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} [\gamma_1 \gamma_2].$$

For each side $= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [\alpha_1 \alpha_2] + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} [\beta_1 \beta_2] + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} [\gamma_1 \gamma_2]$.

Ex. xii.
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \end{bmatrix}$$
$$= \begin{bmatrix} a_1 & f_1 \\ a_2 & f_2 \\ a_3 & f_3 \end{bmatrix} [\alpha_1 \alpha_2 \alpha_3] + \begin{bmatrix} b_1 & e_1 & f_1 \\ b_2 & e_2 & f_2 \\ b_3 & e_3 & f_3 \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \end{bmatrix} + \begin{bmatrix} c_1 & d_1 & f_1 \\ c_2 & d_2 & f_2 \\ c_3 & d_3 & f_3 \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{bmatrix}.$$

§ 44. Properties of the active rows in a product of two matrices.

1. *The elements of any given row in the product matrix are homogeneous linear functions of the elements of the active row which corresponds to it in the factor matrices.*

If
$$[a]_m^r \, [b]_s^n = [c]_m^n,$$
we know that
$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{iu} b_{uj} + \dots.$$

Regarding c_{ij} as an element of the ith horizontal row in the product matrix, we see that it is a homogeneous linear function of $a_{i1}, a_{i2}, \dots a_{iu}, \dots$, which are the elements of the corresponding active row in the first factor matrix. Again regarding c_{ij} as an element of the jth vertical row in the product matrix, we see that it is a homogeneous linear function of
$$b_{1j}, \ b_{2j}, \ \dots \ b_{uj}, \ \dots,$$
which are the elements of the corresponding active row in the second factor matrix.

2. *The elements of any given active row in either factor matrix occur in the product matrix only in the row corresponding to that active row.*

For with the notation used above an element a_{ip} where p is variable occurs only in such elements of the product matrix as c_{iq}, where q is variable. That is, the elements of the ith horizontal or active row of the first factor matrix occur only in the ith horizontal row of the product matrix. Again an element b_{pj} where p is variable occurs only in such elements of the product matrix as c_{qj} where q is variable. That is, the elements of the jth vertical or active row of the second factor matrix occur only in the jth vertical row of the product matrix.

3. *If any active row in either of the factor matrices is a row of 0's, the corresponding row in the product matrix is a row of 0's.*

This is an immediate consequence of the preceding two properties.

Ex. i.
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & \gamma_1 \\ \alpha_2 & 0 & \gamma_2 \end{bmatrix} = \begin{bmatrix} (a_1\alpha_1+b_1\alpha_2), & 0, & (a_1\gamma_1+b_1\gamma_2) \\ (a_2\alpha_1+b_2\alpha_2), & 0, & (a_2\gamma_1+b_2\gamma_2) \\ (a_3\alpha_1+b_3\alpha_2), & 0, & (a_3\gamma_1+b_3\gamma_2) \end{bmatrix}.$$

Ex. ii.
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} = \begin{bmatrix} (a_1\alpha_1+b_1\alpha_2+c_1\alpha_3), & (a_1\beta_1+b_1\beta_2+c_1\beta_3) \\ (a_2\alpha_1+b_2\alpha_2+c_2\alpha_3), & (a_2\beta_1+b_2\beta_2+c_2\beta_3) \\ 0 \quad , & 0 \end{bmatrix}.$$

4. *We may strike out any active rows in one or both factor matrices provided that we strike out the corresponding rows in the product matrix.*

This follows immediately from sub-articles 1 and 2. It also follows from the general expression for an element c_{ij} of the product matrix.

Ex. iii. If
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \epsilon_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \epsilon_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 & \epsilon_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 & \epsilon_4 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 & s_1 & t_1 \\ p_2 & q_2 & r_2 & s_2 & t_2 \\ p_3 & q_3 & r_3 & s_3 & t_3 \end{bmatrix},$$

then
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \gamma_1 & \epsilon_1 \\ \alpha_2 & \gamma_2 & \epsilon_2 \\ \alpha_3 & \gamma_3 & \epsilon_3 \\ \alpha_4 & \gamma_4 & \epsilon_4 \end{bmatrix} = \begin{bmatrix} p_1 & r_1 & t_1 \\ p_3 & r_3 & t_3 \end{bmatrix}.$$

Here we have struck out the second horizontal rows from the first factor matrix and the product matrix, and we have also struck out the second and fourth vertical rows from the second factor matrix and the product matrix.

This result can be proved as follows:

Let
$$[abcd]_{13}\,[\alpha\gamma\epsilon]_{1234} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}.$$

Then
$$y_2 = \det\,[a_3 b_3 c_3 d_3] \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix} = r_3.$$

Similarly $\quad x_1 = p_1, \quad y_1 = r_1, \quad z_1 = t_1, \quad x_2 = p_3, \quad z_2 = t_3.$

The general theorem can be proved in a similar manner.

Ex. iv. If
$$[abcdef]_{123}[\alpha\beta\gamma\delta\epsilon]_{1234}=[pqrst]_{123},$$
then
$$[abcdef]_{13}[\alpha\gamma\epsilon]_{1234}=[prt]_{13}.$$

Ex. v. If
$$[a]^r_m\,[b]^n_s=[c]^n_m,$$
then
$$[c_{ij}]=[a_{i1}a_{i2}\dots a_{ir}]\begin{bmatrix} b_{1j}\\ b_{2j}\\ \vdots\\ b_{sj}\end{bmatrix},$$
and therefore, equating the determinoids of both sides,
$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\dots+a_{iu}b_{uj}+\dots.$$

Here we have retained only the ith horizontal rows in the first factor matrix and the product matrix and only the jth vertical rows in the second factor matrix and the product matrix.

Thus the property under consideration provides us, as a particular case, with convenient means for finding the elements of the product matrix.

5. *We may replace any active rows in one or both factor matrices by rows of 0's, provided that we replace the corresponding rows in the product matrix by rows of 0's.*

This again follows immediately from sub-articles 1 and 2.

Ex. vi. If
$$\begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\\ a_4 & b_4 & c_4\end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \epsilon_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \epsilon_2\end{bmatrix}=\begin{bmatrix} p_1 & q_1 & r_1 & s_1 & t_1\\ p_2 & q_2 & r_2 & s_2 & t_2\\ p_3 & q_3 & r_3 & s_3 & t_3\\ p_4 & q_4 & r_4 & s_4 & t_4\end{bmatrix},$$
then
$$\begin{bmatrix} a_1 & b_1 & c_1\\ 0 & 0 & 0\\ a_3 & b_3 & c_3\\ a_4 & b_4 & c_4\end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & \epsilon_1\\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \epsilon_2\end{bmatrix}=\begin{bmatrix} p_1 & q_1 & r_1 & 0 & t_1\\ 0 & 0 & 0 & 0 & 0\\ p_3 & q_3 & r_3 & 0 & t_3\\ p_4 & q_4 & r_4 & 0 & t_4\end{bmatrix}.$$

6. *We may insert additional active rows of 0's anywhere in either or both factor matrices, provided that we at the same time insert similarly placed corresponding rows of 0's in the product matrix.*

This follows from sub-articles 1, 2 and 3.

Ex. vii. If
$$\begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{bmatrix}=\begin{bmatrix} p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\end{bmatrix},$$
then
$$\begin{bmatrix} a_1 & b_1 & c_1\\ 0 & 0 & 0\\ a_2 & b_2 & c_2\end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 & 0 & \gamma_1\\ \alpha_2 & \beta_2 & 0 & \gamma_2\\ \alpha_3 & \beta_3 & 0 & \gamma_3\end{bmatrix}=\begin{bmatrix} p_1 & q_1 & 0 & r_1\\ 0 & 0 & 0 & 0\\ p_2 & q_2 & 0 & r_2\end{bmatrix}.$$

7. *We can re-arrange the active rows in one or both factor matrices in any manner, provided that the corresponding rows in the product matrix are subjected to exactly the same re-arrangements.*

This will be obvious from formula (I) of § 42.

Ex. viii. If
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \end{bmatrix},$$

then
$$\begin{bmatrix} a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \end{bmatrix} \begin{bmatrix} \delta_1 & \alpha_1 & \gamma_1 & \beta_1 \\ \delta_2 & \alpha_2 & \gamma_2 & \beta_2 \\ \delta_3 & \alpha_3 & \gamma_3 & \beta_3 \end{bmatrix} = \begin{bmatrix} s_2 & p_2 & r_2 & q_2 \\ s_1 & p_1 & r_1 & q_1 \end{bmatrix}.$$

8. *If $AB = C$ is the product of any two matrices A and B, we can insert anywhere in A an arbitrary active row whose matrix is x, provided that we insert in the same situation in C the corresponding row whose matrix is xB; also we can insert anywhere in B an arbitrary active row whose matrix is y, provided that we insert in the same situation in C the corresponding row whose matrix is Ay.*

This can be seen from formula (I) of § 42, from which it appears that the matrix of the ith horizontal row of the product matrix is a_iB and that the matrix of the jth vertical row of the product matrix is Ab_j.

Ex. ix. Let
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{bmatrix}, \quad [xyz] \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = [\xi\eta\zeta].$$

Then
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ x & y & z \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 \\ \xi & \eta & \zeta \\ p_2 & q_2 & r_2 \end{bmatrix}.$$

Ex. x. Let
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix}, \quad \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix},$$

then
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & y_1 & \beta_1 & \gamma_1 \\ \alpha_2 & y_2 & \beta_2 & \gamma_2 \\ \alpha_3 & y_3 & \beta_3 & \gamma_3 \end{bmatrix} = \begin{bmatrix} p_1 & \eta_1 & q_1 & r_1 \\ p_2 & \eta_2 & q_2 & r_2 \\ p_3 & \eta_3 & q_3 & r_3 \end{bmatrix}.$$

Ex. xi. Let $[abc]_{12}[\alpha\beta\gamma]_{123} = [pqr]_{12}, \quad [abc]_{12}[\delta\epsilon]_{123} = [st]_{12}.$

Then
$$[abc]_{12}[\alpha\beta\gamma\delta\epsilon]_{123} = [pqrst]_{12}.$$

9. *We can multiply all the elements of any active row in either factor matrix by a scalar number k, provided that we also multiply all the elements of the corresponding row in the product matrix by k.*

This follows immediately from sub-articles 1 and 2.

Ex. xii. If $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \end{bmatrix}$,

then $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1, & \beta_1, & k\gamma_1, & \delta_1 \\ \alpha_2, & \beta_2, & k\gamma_2, & \delta_2 \\ \alpha_3, & \beta_3, & k\gamma_3, & \delta_3 \end{bmatrix} = \begin{bmatrix} p_1, & q_1, & kr_1, & s_1 \\ p_2, & q_2, & kr_2, & s_2 \\ p_3, & q_3, & kr_3, & s_3 \end{bmatrix}$;

also $\begin{bmatrix} a_1, & b_1, & c_1, & d_1 \\ ka_2, & kb_2, & kc_2, & kd_2 \\ a_3, & b_3, & c_3, & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \end{bmatrix} = \begin{bmatrix} p_1, & q_1, & r_1, & s_1 \\ kp_2, & kq_2, & kr_2, & ks_2 \\ p_3, & q_3, & r_3, & s_3 \end{bmatrix}$.

10. *We can add to the elements of any active row in either factor matrix the corresponding elements of any other active row of the same factor matrix each multiplied by the scalar number k, provided that we deal similarly with the corresponding rows in the product matrix.*

Ex. xiii. If $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \end{bmatrix}$,

then $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1, & (\beta_1+k\delta_1), & \gamma_1, & \delta_1 \\ \alpha_2, & (\beta_2+k\delta_2), & \gamma_2, & \delta_2 \\ \alpha_3, & (\beta_3+k\delta_3), & \gamma_3, & \delta_3 \end{bmatrix} = \begin{bmatrix} p_1, & (q_1+ks_1), & r_1, & s_1 \\ p_2, & (q_2+ks_2), & r_2, & s_2 \\ p_3, & (q_3+ks_3), & r_3, & s_3 \end{bmatrix}$.

To prove this, let the second product $=[xyzw]_{123}$.

Then if i is any one of the numbers 1, 2, 3 we have

$$y_i = a_i(\beta_1+k\delta_1)+b_i(\beta_2+k\delta_2)+c_i(\beta_3+k\delta_3)$$
$$=(a_i\beta_1+b_i\beta_2+c_i\beta_3)+k(a_i\delta_1+b_i\delta_2+c_i\delta_3)=q_i+ks_i.$$

Also $x_i=a_i\alpha_1+b_i\beta_1+c_i\gamma_1=p_i$, and similarly $z_i=r_i$, $w_i=s_i$.

Thus $[xyzw]_{123}$ is identically equal to the matrix on the right in the second equation.

The result can also be obtained by postfixing the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & k & 0 & 1 \end{bmatrix}$$

on both sides of the first equation.

§ 45. Other properties of a product of two matrices.

1. *Equivalence of conventionally equal matrices.*

If either factor matrix in a product of two matrices is replaced by another conventionally equal to it, the product matrix is either unaltered or is replaced by a matrix conventionally equal to it. The same thing is true when both factor matrices are replaced by matrices conventionally equal to them. For if additional final passive rows of 0's are inserted in one or both factor matrices, or if existing final passive rows of 0's are removed from one or both factor matrices, the product matrix is unaltered. And if additional final active rows of 0's are inserted in one or both factor matrices, or if existing final active rows of 0's are removed from one or both factor matrices, the new product matrix is obtained from the original product matrix by inserting or removing corresponding rows of 0's, i.e. the original product matrix is replaced by one conventionally equal to it.

Conversely if a matrix C is given as the product AB of two factor matrices, any matrix conventionally equal to C can be expressed as the product of two matrices conventionally equal to A and B. For the identical equation $C = AB$ remains true when we insert any final rows of 0's in C and corresponding final active rows of 0's in A and B.

From the first statement we conclude that in multiplying matrices, as well as in adding and subtracting them, we can consistently regard conventionally equal matrices as equivalent. The sign of equality must then be understood to denote conventional equality.

2. *Distributive character of a product of two matrices.*

The product of two matrices taken in specified order is distributive.

To show this it will be sufficient to prove that

$$\text{(i)}\quad [a]_m^r\,\{[b]_\lambda^p + [c]_\mu^q\} = [a]_m^r\,[b]_\lambda^p + [a]_m^r\,[c]_\mu^q,$$

$$\text{(ii)}\quad \{[a]_\lambda^p + [b]_\mu^q\}\,[c]_s^n = [a]_\lambda^p\,[c]_s^n + [b]_\mu^q\,[c]_s^n.$$

That (i) and (ii) are identical equalities is at once seen by expressing both sides as single matrices.

Proof of (i). Let n be the larger of the two numbers p and q, and let s be the larger of the two numbers λ and μ.

Then $$[b]_\lambda^p = [b]_s^n,$$

where $$b_{ij} = 0 \text{ if } i > \lambda \text{ or } j > p,$$

and $$[c]_\mu^q = [c]_s^n,$$

where $$c_{ij} = 0 \text{ if } i > \mu \text{ or } j > q.$$

Also $$[b]_\lambda^p + [c]_\mu^q = [b]_s^n + [c]_s^n = [d]_s^n,$$
where $$d_{ij} = b_{ij} + c_{ij}.$$

Therefore the left-hand side of (i) $= [a]_m^r [d]_s^n = [x]_m^n$,

where $$x_{ij} = \sum_u a_{iu} d_{uj} = \sum_u a_{iu}(b_{uj} + c_{uj})$$
$$= \sum_u a_{iu} b_{uj} + \sum_u a_{iu} c_{uj},$$

u receiving all integral values from 1 to the smaller of the two numbers r and s.

Again $$[a]_m^r [b]_\lambda^p = [a]_m^r [b]_s^n = [\beta]_m^n,$$
$$[a]_m^r [c]_\mu^q = [a]_m^r [c]_s^n = [\gamma]_m^n,$$

where $$\beta_{ij} = \sum_u a_{iu} b_{uj}, \quad \gamma_{ij} = \sum_u a_{iu} c_{uj},$$

u receiving the same values as before.

Therefore the right-hand side of (i) $= [\beta]_m^n + [\gamma]_m^n = [y]_m^n$,

where $$y_{ij} = \beta_{ij} + \gamma_{ij} = \sum_u a_{iu} b_{uj} + \sum_u a_{iu} c_{uj}.$$

Thus the left-hand side is equal to a matrix $[x]_m^n$ and the right-hand side to a matrix $[y]_m^n$ such that
$$[x]_m^n = [y]_m^n.$$

The equalities in the above reasoning are conventional, but the last two matrices being similar are identically equal.

Proof of (ii). If we denote the larger of the two numbers λ and μ by m, and the larger of the two numbers p and q by r, then by similar reasoning it can be shown that the left-hand side and the right-hand side of (ii) are respectively equal to two matrices $[x]_m^n$, $[y]_m^n$ such that
$$[x]_m^n = [y]_m^n.$$

3. *Non-commutative character of a product of two matrices.*

A *product of two matrices is in general not commutative.*

By multiplying out the two products $[a]_m^r [b]_s^n$, $[b]_s^n [a]_m^r$, it is at once seen that their product matrices are not equal except in certain very special cases. In fact the orders of the two product matrices are in general different, being m and n in the one case and s and r in the other case. Even when m, n, s and r are all equal, the two product matrices are still different except in certain special cases.

For if $[a]_m^m [b]_m^m = [x]_m^m$, and $[b]_m^m [a]_m^m = [y]_m^m$, we have
$$x_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{im} b_{mj},$$
$$y_{ij} = b_{i1} a_{1j} + b_{i2} a_{2j} + \ldots + b_{im} a_{mj}.$$

Thus in general x_{ij} and y_{ij} are not equal, and therefore $[x]_m^m \neq [y]_m^m$.

NOTE. *Special commutative products.*

In § 46. 6, it is shown that a product of a square matrix and its conjugate reciprocal is commutative.

In Ex. i of § 67, it is shown that when a product of two square matrices of the same order is a non-zero scalar matrix, then the product is commutative.

4. *Conjugate of a product of two matrices.*

Theorem. *The conjugate of the product of two matrices taken in a given order is identical with the product of the conjugates of the two factor matrices taken in the reverse order.*

To prove this we have to show that

$$\text{if}\quad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{s1} & b_{s2} & \dots & b_{sn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix},$$

$$\text{then}\quad \begin{bmatrix} b_{11} & b_{21} & \dots & b_{s1} \\ b_{12} & b_{22} & \dots & b_{s2} \\ \dots & \dots & \dots & \dots \\ b_{1n} & b_{2n} & \dots & b_{sn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1r} & a_{2r} & \dots & a_{mr} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & \dots & c_{m1} \\ c_{12} & c_{22} & \dots & c_{m2} \\ \dots & \dots & \dots & \dots \\ c_{1n} & c_{2n} & \dots & c_{mn} \end{bmatrix};$$

or that if $\qquad [a]_m^r\,[b]_s^n = [c]_m^n$, then $\underset{n}{\overset{s}{\overline{b}}}\;\underset{r}{\overset{m}{\overline{a}}} = \underset{n}{\overset{m}{\overline{c}}}$.

Supposing then that $[a]_m^r\,[b]_s^n = [c]_m^n$, let $\underset{n}{\overset{s}{\overline{b}}}\;\underset{r}{\overset{m}{\overline{a}}} = [x]_n^m$.

Then

$$x_{ij} = \det\,[b_{1i} b_{2i} \dots b_{si}] \begin{bmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jr} \end{bmatrix} = \det\,[a_{j1} a_{j2} \dots a_{jr}] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{si} \end{bmatrix} = c_{ji}.$$

It follows that $[x]_n^m$ is the conjugate of $[c]_m^n$,

or that $\qquad [x]_n^m = \underset{n}{\overset{m}{\overline{c}}}$.

Thus the theorem is proved.

Ex. i. $$\begin{bmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha_1 a_1 + b_1\beta_1 + c_1\gamma_1), & (\alpha_1 a_2 + b_1\beta_2 + c_1\gamma_2), & (\alpha_1 a_3 + b_1\beta_3 + c_1\gamma_3) \\ (\alpha_2 a_1 + b_2\beta_1 + c_2\gamma_1), & (\alpha_2 a_2 + b_2\beta_2 + c_2\gamma_2), & (\alpha_2 a_3 + b_2\beta_3 + c_2\gamma_3) \end{bmatrix}.$$

$$\begin{bmatrix} a_1 & \beta_1 & \gamma_1 & \delta_1 \\ a_2 & \beta_2 & \gamma_2 & \delta_2 \\ a_3 & \beta_3 & \gamma_3 & \delta_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} = \begin{bmatrix} (\alpha_1 a_1 + b_1\beta_1 + c_1\gamma_1), & (\alpha_2 a_1 + b_2\beta_1 + c_2\gamma_1) \\ (\alpha_1 a_2 + b_1\beta_2 + c_1\gamma_2), & (\alpha_2 a_2 + b_2\beta_2 + c_2\gamma_2) \\ (\alpha_1 a_3 + b_1\beta_3 + c_1\gamma_3), & (\alpha_2 a_3 + b_2\beta_3 + c_2\gamma_3) \end{bmatrix}.$$

Thus $$[abc]_{12}\begin{bmatrix} a \\ \beta \\ \gamma \\ \delta \end{bmatrix}_{123} = [a\beta\gamma\delta]_{123}\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{12}.$$

Ex. ii. As a particular case of the above theorem we have

$$[a_1 a_2 \dots a_r]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{bmatrix} = [b_1 b_2 \dots b_s]\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix}.$$

For in this case the product matrix is a matrix with one element only, viz. the matrix $[e]$ where $e = \Sigma a_u b_u$, and such a matrix is necessarily self-conjugate.

5. *Manipulation of matrix equations.*

Let A, B, M be any matrices.

If $$A = 0 \qquad\qquad (1),$$
it is clearly a necessary consequence that
$$AM = 0 \text{ and } MA = 0 \qquad\qquad (2).$$

But when either of the equations (2) is given, we cannot in general conclude that (1) is true. The cases in which this conclusion can be drawn are considered in Theorems III of §§ 81 and 82 and summarised in § 84.

Thus when a product of two matrices vanishes, it does not necessarily follow that one of the factor matrices must vanish.

Ex. iii. $$\begin{bmatrix} 1, & 3, & 2, & 1 \\ 2, & 3, & 1, & 5 \end{bmatrix}\begin{bmatrix} 3, & 3, & 0 \\ 3, & 0, & 3 \\ -5, & -1, & -4 \\ -2, & -1, & -1 \end{bmatrix} = 0.$$

Here the product matrix vanishes, but neither of the factor matrices vanishes.

Again if $$A = B \qquad\qquad (3),$$
it is clearly a necessary consequence that
$$AM = BM, \text{ and } MA = MB \qquad\qquad (4).$$

But when either of the equations (4) is given, we cannot conclude that (3) is true. The cases in which this conclusion can be drawn are summarised in § 84.

Thus we can prefix or postfix any additional factor matrix on both sides of a given matrix equation, but we cannot in general cancel a factor matrix which is common to both sides of a given equation.

It has been shown in § 41, Ex. iii that we can multiply both sides of a given matrix equation by any scalar number and that we can divide both sides by any scalar number which is not 0.

Ex. iv.
$$\begin{bmatrix} 1, & 3, & 2, & 1 \\ 2, & 3, & 1, & 5 \end{bmatrix}\begin{bmatrix} 4, & -2, & 1 \\ 3, & 1, & 2 \\ -3, & 2, & -5 \\ 1, & 2, & -3 \end{bmatrix} = \begin{bmatrix} 1, & 3, & 2, & 1 \\ 2, & 3, & 1, & 5 \end{bmatrix}\begin{bmatrix} 1, & -5, & 1 \\ 0, & 1, & -1 \\ 2, & 3, & -1 \\ 3, & 3, & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 8, & 7, & -6 \\ 19, & 11, & -12 \end{bmatrix}.$$

Here the two products are equal, and have a common factor matrix, but the second factor matrices in the two products are not equal.

§ 46. Special cases of a product of two matrices.

1. *One factor a unit matrix.*

A square matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

in which every element of the leading diagonal is equal to 1 and every other element is equal to 0, is called a *unit matrix*.

As in § 2.8 we shall use $[1]_m^m$ as an abbreviated notation for the unit matrix of order m. Thus the fact that $[a]_m^m$ is a unit matrix of order m is expressed by the equation

$$[a]_m^m = [1]_m^m.$$

We now have the following obvious theorem:

If either factor matrix in a standard product of two matrices is a unit matrix, it leaves the other factor matrix unaltered, i.e. it is equivalent to the scalar multiplier 1.

Ex. i.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

Ex. ii. $[1]_m^m [a]_m^n = [a]_m^n [1]_n^n = [a]_m^n$.

Here the equality is identical.

Ex. iii. $$[1]_r^r [a]_s^n = [a]_m^n,$$

where m is the smaller of the two numbers r and s.

Here the equality is conventional.

Ex. iv. $$[a]_m^r [1]_s^s = [a]_m^n,$$

where n is the smaller of the two numbers r and s.

Here again the equality is conventional.

2. *One factor a scalar matrix.*

A square matrix of the form

$$\begin{bmatrix} k & 0 & 0 & \dots & 0 \\ 0 & k & 0 & \dots & 0 \\ 0 & 0 & k & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & k \end{bmatrix}$$

in which every element of the leading diagonal is equal to the scalar number k and every other element is equal to 0, will be called a *scalar matrix with argument* k.

We shall use $k[1]_m^m$ as an abbreviated notation for the scalar matrix of order m with argument k, for that matrix is clearly the product of the scalar number k and the unit matrix $[1]_m^m$.

The fact that $[a]_m^m$ is a scalar matrix of order m with argument k is expressed by the equation

$$[a]_m^m = k[1]_m^m.$$

We can now state the following obvious theorem:

If either factor matrix in a standard product of two matrices is a scalar matrix with argument k, it is equivalent to the scalar multiplier k, i.e. it causes each element of the other factor matrix to be multiplied by k.

Ex. v. $$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}$$

$$= k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} k.$$

Ex. vi. $$k[1]_m^m \cdot [a]_m^n = [a]_m^n \cdot k[1]_n^n = k[a]_m^n.$$

Ex. vii. $$k[1]_r^r \cdot [a]_s^n = k \cdot [1]_r^r [a]_s^n = k[a]_m^n,$$

where m is the smaller of the two numbers r and s.

Ex. viii. $$[a]^r_m \,.\, k[1]^s_s = k \,.\, [a]^r_m [1]^s_s = k[a]^n_m,$$

where n is the smaller of the two numbers r and s.

Ex. ix. *There is no matrix which can be regarded as being always equivalent to the non-zero scalar number k for purposes of multiplication.*

If there were a matrix $[x]^n_m$ which could be regarded as equivalent to the non-zero number k, we should have

$$[x]^n_m [1]^{m+r}_{m+r} = k[1]^{m+r}_{m+r}.$$

Now the product matrix on the left has only m horizontal rows and cannot be conventionally equal to the matrix on the right which has $m+r$ horizontal rows, none of which are rows of 0's. Accordingly the above equation is impossible.

On the other hand the scalar multiplier k can in every particular case be replaced by *some* scalar matrix with argument k.

3. *One factor a quasi-scalar matrix.*

A square matrix of the form

$$\begin{bmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & c & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & k \end{bmatrix}$$

occurring as one of the factor matrices in a standard product of two matrices multiplies the passive rows of the other factor matrix by $a, b, c, \dots k$ respectively.

Such a matrix may be called a *quasi-scalar matrix*.

Ex. x. $$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} = \begin{bmatrix} a\alpha_1 & a\alpha_2 \\ b\beta_1 & b\beta_2 \\ c\gamma_1 & c\gamma_2 \end{bmatrix}.$$

Ex. xi. $$\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a\alpha_1 & b\beta_1 & c\gamma_1 \\ a\alpha_2 & b\beta_2 & c\gamma_2 \end{bmatrix}.$$

4. *Product of two inversely similar matrices.*

A product of two inversely similar matrices is always a square matrix.

For the product of the two matrices $[a]^r_m \,.\, [b]^m_r$ is of the form

$$[a]^r_m [b]^m_r = [c]^m_m.$$

Ex. xii. $$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} = \begin{bmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1, & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1, & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 \end{bmatrix}.$$

Ex. xiii. $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix} = \begin{bmatrix} (a_1\alpha_1+b_1\beta_1), & (a_1\alpha_2+b_1\beta_2), & (a_1\alpha_3+b_1\beta_3) \\ (a_2\alpha_1+b_2\beta_1), & (a_2\alpha_2+b_2\beta_2), & (a_2\alpha_3+b_2\beta_3) \\ (a_3\alpha_1+b_3\beta_1), & (a_3\alpha_2+b_3\beta_2), & (a_3\alpha_3+b_3\beta_3) \end{bmatrix}.$

5. *Products of a matrix and its conjugate reciprocal.*

Let $[a]_m^n$ be any matrix, and let $[A]_m^n$ be its reciprocal matrix. Also let Δ be the determinoid of $[a]_m^n$, so that $\Delta = (a)_m^n$. Then the *conjugate reciprocal* of $[a]_m^n$ is $\overline{\underline{A}}{}_n^m$. We know by § 28, Ex. iii that $\overline{\underline{A}}{}_n^m$ is both the conjugate of the reciprocal of $[a]_m^n$ and the reciprocal of the conjugate of $[a]_m^n$. Clearly a matrix and its conjugate reciprocal are inversely similar matrices.

If $m < n$, the equations of § 28, Ex. i are together equivalent to the single matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{m1} \\ A_{12} & A_{22} & \dots & A_{m2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{mn} \end{bmatrix} = \begin{bmatrix} \Delta & 0 & \dots & 0 \\ 0 & \Delta & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta \end{bmatrix},$$

or $$[a]_m^n \overline{\underline{A}}{}_n^m = \Delta [1]_m^m \quad \text{.............................(1).}$$

They are also together equivalent to the single matrix equation

$$[A]_m^n \overline{\underline{a}}{}_n^m = \Delta [1]_m^m \quad \text{.............................(2).}$$

If $m > n$, the equations of § 28, Ex. i are together equivalent to the single matrix equation

$$\begin{bmatrix} A_{11} & A_{21} & \dots & A_{m1} \\ A_{12} & A_{22} & \dots & A_{m2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \Delta & 0 & \dots & 0 \\ 0 & \Delta & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta \end{bmatrix},$$

or $$\overline{\underline{A}}{}_n^m [a]_m^n = \Delta [1]_n^n \quad \text{.............................(3).}$$

They are also together equivalent to the single matrix equation

$$\overline{\underline{a}}{}_n^m [A]_m^n = \Delta [1]_n^n \quad \text{.............................(4).}$$

It may be observed that equation (2) can be deduced from equation (1) by equating the conjugate matrices of both sides and making use of § 45.4. Equation (4) can be deduced from equation (3) in the same way.

The results (1) and (3) lead to the following theorem:

Theorem. *Of the two products MM', $M'M$ which can be formed with a matrix M and its conjugate reciprocal matrix M' that one in which long rows are active rows is a scalar matrix with argument Δ, where Δ is the determinoid of M* ....(A).

The product in question is that one in which the passivity exceeds (or is not less than) the activity.

Equations (2) and (4) as well as equations (1) and (3) are included in (A), for $[A]^{n}_{m}$ is the conjugate reciprocal of $\overline{\underline{a}}^{\,m}_{\,n}$ and Δ is also the determinoid of $\overline{\underline{a}}^{\,m}_{\,n}$.

When $[a]^{n}_{m}$ is a square matrix, so that $n = m$, the four products considered above are all equal, and we have the following theorem:

Theorem. *If $[a]^{m}_{m}$ is a square matrix and if $[A]^{m}_{m}$ is its reciprocal matrix, then*

$$[a]^{m}_{m}\overline{\underline{A}}^{\,m}_{\,m} = \overline{\underline{A}}^{\,m}_{\,m}[a]^{m}_{m} = \overline{\underline{a}}^{\,m}_{\,m}[A]^{m}_{m} = [A]^{m}_{m}\overline{\underline{a}}^{\,m}_{\,m} = \Delta\,[1]^{m}_{m},$$

where $\Delta = (a)^{m}_{m}$ ....(B).

Thus the product $[a]^{m}_{m}\overline{\underline{A}}^{\,m}_{\,m}$ is *commutative* and in it we can interchange the two factors or replace each factor by its conjugate.

Ex. xiv. Let $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{bmatrix}$ and Δ be the reciprocal and the determinoid of $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$.

Then $\begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \\ C_1 & C_2 \end{bmatrix}$ and Δ are the reciprocal and the determinoid of $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$.

In this case we have

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{bmatrix}\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}.$$

Ex. xv. The reciprocal matrix and the determinoid of

$$\begin{bmatrix} 1, & 2 \\ 3, & 4 \\ 2, & 0 \end{bmatrix} \text{ are } \begin{bmatrix} 4, & -1 \\ -2, & -1 \\ -2, & 2 \end{bmatrix} \text{ and } -6.$$

Accordingly

$$\begin{bmatrix} 1, & 3, & 2 \\ 2, & 4, & 0 \end{bmatrix} \begin{bmatrix} 4, & -1 \\ -2, & -1 \\ -2, & 2 \end{bmatrix} = \begin{bmatrix} 4, & -2, & -2 \\ -1, & -1, & 2 \end{bmatrix} \begin{bmatrix} 1, & 2 \\ 3, & 4 \\ 2, & 0 \end{bmatrix} = \begin{bmatrix} -6, & 0 \\ 0, & -6 \end{bmatrix}.$$

Ex. xvi. The reciprocal and the determinoid of

$$\begin{bmatrix} 2, & 5, & 3 \\ 3, & 1, & 2 \\ 1, & 2, & 1 \end{bmatrix} \text{ are } \begin{bmatrix} -3, & -1, & 5 \\ 1, & -1, & 1 \\ 7, & 5, & -13 \end{bmatrix} \text{ and } 4.$$

Accordingly

$$\begin{bmatrix} 2, & 5, & 3 \\ 3, & 1, & 2 \\ 1, & 2, & 1 \end{bmatrix} \begin{bmatrix} -3, & 1, & 7 \\ -1, & -1, & 5 \\ 5, & 1, & -13 \end{bmatrix} = \begin{bmatrix} -3, & 1, & 7 \\ -1, & -1, & 5 \\ 5, & 1, & -13 \end{bmatrix} \begin{bmatrix} 2, & 5, & 3 \\ 3, & 1, & 2 \\ 1, & 2, & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2, & 3, & 1 \\ 5, & 1, & 2 \\ 3, & 2, & 1 \end{bmatrix} \begin{bmatrix} -3, & -1, & 5 \\ 1, & -1, & 1 \\ 7, & 5, & -13 \end{bmatrix} = \begin{bmatrix} -3, & -1, & 5 \\ 1, & -1, & 1 \\ 7, & 5, & -13 \end{bmatrix} \begin{bmatrix} 2, & 3, & 1 \\ 5, & 1, & 2 \\ 3, & 2, & 1 \end{bmatrix} = \begin{bmatrix} 4, & 0, & 0 \\ 0, & 4, & 0 \\ 0, & 0, & 4 \end{bmatrix}.$$

Ex. xvii. If $[a]_m^n$, $[b]_n^m$ are any two inversely similar matrices such that

$$[a]_m^n \, [b]_n^m = k\,[1]_m^m,$$

then also

$$\overline{\underline{b}}{}_m^n \; \overline{\underline{a}}{}_n^m = k\,[1]_m^m.$$

This result is obtained by equating the conjugate matrices of both sides and observing that the scalar matrix $k\,[1]_m^m$ is self-conjugate.

Ex. xviii. If a product of any two square matrices of the same order is a non-zero scalar matrix, then the product is commutative.

Consequently if $\qquad [a]_m^m \, [b]_m^m = k\,[1]_m^m, \qquad$ where $k \neq 0$,

then

$$[a]_m^m \, [b]_m^m = [b]_m^m \, [a]_m^m = \overline{\underline{a}}{}_m^m \; \overline{\underline{b}}{}_m^m = \overline{\underline{b}}{}_m^m \; \overline{\underline{a}}{}_m^m.$$

This is proved in § 67, Ex. i.

6. *Products of a matrix and its inverse matrix.*

Let $[a]_m^n$ be any matrix, and let $[A]_m^n$ be its reciprocal matrix.

Also let Δ be the determinoid of $[a]_m^n$, so that $\Delta = (a)_m^n$.

If $\Delta \neq 0$, then the matrix

$$\overline{\underline{\alpha}}{}_n^m = \frac{1}{\Delta}\,\overline{\underline{A}}{}_n^m$$

will be called the inverse matrix of $[a]_m^n$.

Since
$$\alpha_{ij} = \frac{1}{\Delta} A_{ij},$$
the inverse matrix is obtained from the conjugate reciprocal matrix by dividing each element by Δ.

Dividing both sides of equations (1), (2), (3) and (4) of the previous sub-article by the scalar quantity Δ, we have

$$[a]^n_m \, \overline{\underline{\alpha}}{}^m_n = [\alpha]^n_m \, \overline{\underline{a}}{}^m_n = [1]^m_m, \text{ when } m < n \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$\overline{\underline{\alpha}}{}^m_n \, [a]^n_m = \overline{\underline{a}}{}^m_n \, [\alpha]^n_m = [1]^n_n, \text{ when } m > n \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

The first equations in (5) and (6) lead to the following theorem:

Theorem. *Of the two products which can be formed with a matrix and its inverse, that product in which long rows are active rows is equal to a unit matrix*..(C).

Similarly by dividing throughout by the scalar quantity Δ in (B), we obtain the following theorem:

Theorem. *If* $[a]^m_m$ *is a square matrix and if* $\overline{\underline{\alpha}}{}^m_m$ *is its inverse matrix, then*

$$[a]^m_m \, \overline{\underline{\alpha}}{}^m_m = \overline{\underline{\alpha}}{}^m_m \, [a]^m_m = \overline{\underline{a}}{}^m_m \, [\alpha]^m_m = [\alpha]^m_m \, \overline{\underline{a}}{}^m_m = [1]^m_m \quad \ldots\ldots\ldots(D).$$

Note. More generally if A and B are two matrices such that one of the two products AB, BA is a unit matrix, we may say that the two matrices are inverse to one another. When AB is a unit matrix, B may be called an inverse post-factor of A. When BA is a unit matrix, B may be called an inverse pre-factor of A.

It will be shown in §§ 81 and 82 that the following results are true except in special cases:

(i) If $m < n$, then $[a]^n_m$ has an infinite number of inverse post-factors but no inverse pre-factor.

(ii) If $m > n$, then $[a]^n_m$ has an infinite number of inverse pre-factors, but no inverse post-factor.

(iii) The square matrix $[a]^m_m$ has a unique inverse matrix which is both an inverse pre-factor and an inverse post-factor.

Thus AB cannot be a unit matrix unless the long rows of A and B are active rows.

The inverse matrix defined in the text may be called the *principal inverse matrix* to distinguish it from other inverse matrices.

Ex. xix. The principal inverse matrix of $\begin{bmatrix} 1, & 2 \\ 3, & 4 \\ 2, & 0 \end{bmatrix}$ is $\begin{bmatrix} -\frac{2}{3}, & \frac{1}{3}, & \frac{1}{3} \\ \frac{1}{6}, & \frac{1}{6}, & -\frac{1}{3} \end{bmatrix}$.

Accordingly

$$\begin{bmatrix} 1, & 3, & 2 \\ 2, & 4, & 0 \end{bmatrix} \begin{bmatrix} -\frac{2}{3}, & \frac{1}{6} \\ \frac{1}{3}, & \frac{1}{6} \\ \frac{1}{3}, & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}, & \frac{1}{3}, & \frac{1}{3} \\ \frac{1}{6}, & \frac{1}{6}, & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1, & 2 \\ 3, & 4 \\ 2, & 0 \end{bmatrix} = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}.$$

Another inverse matrix is $\begin{bmatrix} 2, & -1, & 1 \\ \frac{19}{2}, & -\frac{9}{2}, & 2 \end{bmatrix}$.

§ 47. Associative and distributive character of a product of three matrices.

1. *Reduction to a product of standard form.*

When A, B, C are any matrices, let $A\,.\,BC$ mean the product obtained by multiplying the matrix A into the matrix BC, and let $AB\,.\,C$ mean the product obtained by multiplying the matrix AB into the matrix C.

Then any product of three matrices must be of one of the forms

$$[a]_m^\rho \,.\, [b]_r^s \, [c]_\sigma^n, \quad [a]_m^\rho \, [b]_r^s \,.\, [c]_\sigma^n.$$

Each of these products will be said to be *of standard form* or to be a *standard product* when $\rho = r$ and $\sigma = s$. This is the case when in both pairs of adjacent factor matrices the number of vertical rows in the matrix on the left is equal to the number of horizontal rows in the matrix on the right.

A standard product of three matrices must have one of the forms

$$[a]_m^r \,.\, [b]_r^s \, [c]_s^n, \quad [a]_m^r \, [b]_r^s \,.\, [c]_s^n.$$

By utilising the properties of a product of two matrices any product whatever of three matrices can be at once reduced to a product of standard form. The reduction can be effected equally well by the insertion of additional final passive rows of 0's, or by the omission of redundant final passive rows, or by operations of both kinds.

When we make use of both kinds of operations, we can always effect the reduction without altering the middle matrix in any way. When we confine ourselves to the omission of redundant final passive rows, no rows of 0's are introduced. In practice this last method is most convenient, as it gives the factor matrices in their simplest forms without any superfluous rows.

The manner of effecting the reduction is illustrated in the following examples. To abbreviate the reasoning we will denote the matrix formed by adding two additional final vertical rows of 0's to $[ab \ldots k]_{12\ldots n}$ by $[ab \ldots k\,0\,0]_{12\ldots n}$, and the matrix formed by adding two additional final horizontal rows of 0's to $[ab \ldots k]_{12\ldots n}$ by $[ab \ldots k]_{12\ldots n\,0\,0}$. Similar notations will be used for any number of added final rows of 0's.

Ex. i. *To reduce the product* $P=[abcde]_{12}\cdot[pqr]_{1234}[\alpha\beta\gamma\delta]_{12345}$ *to a standard product.*

Let $$[pqr]_{1234}[\alpha\beta\gamma\delta]_{12345}=[xyzw]_{1234},$$

so that $$P=[abcde]_{12}[xyzw]_{1234}.$$

First Method. Insertion of rows of 0's.

We have in succession:

(1). $P=[abcde]_{12}[xyzw]_{1234}=[abcde]_{12}[xyzw]_{12340}.$

(2). $[xyzw]_{1234}=[pqr]_{1234}[\alpha\beta\gamma\delta]_{12345}=[pqr00]_{1234}[\alpha\beta\gamma\delta]_{12345}.$

(3). $[xyzw]_{12340}=[pqr00]_{12340}[\alpha\beta\gamma\delta]_{12345}.$

(4). $P=[abcde]_{12}\cdot[pqr00]_{12340}[\alpha\beta\gamma\delta]_{12345}.$

The first and second steps are obtained by the properties of passive rows (see § 43. 3), and the third step by the properties of active rows (see § 44. 6).

The final expression for P has the standard form.

Second Method. Omission of redundant rows.

Making use of the properties of passive rows (see § 43. 4), we have in succession:

(1). $P=[abcde]_{12}[xyzw]_{1234}=[abcd]_{12}[xyzw]_{1234}.$

(2). $[xyzw]_{1234}=[pqr]_{1234}[\alpha\beta\gamma\delta]_{12345}=[pqr]_{1234}[\alpha\beta\gamma\delta]_{123}.$

(3). $P=[abcd]_{12}\cdot[pqr]_{1234}[\alpha\beta\gamma\delta]_{123}.$

Third Method. Operations of both kinds.

We have in succession:

(1). $P=[abcde]_{12}[xyzw]_{1234}=[abcde]_{12}[xyzw]_{12340}.$

(2). $[xyzw]_{1234}=[pqr]_{1234}[\alpha\beta\gamma\delta]_{12345}=[pqr]_{1234}[\alpha\beta\gamma\delta]_{123}.$

(3). $[xyzw]_{12340}=[pqr]_{12340}[\alpha\beta\gamma\delta]_{123}.$

(4). $P=[abcde]_{12}\cdot[pqr]_{12340}[\alpha\beta\gamma\delta]_{123}.$

To leave the middle matrix unaltered we must leave the suffixes of the second matrix unaltered at the first step. In the present case we then fall back on the second method.

Ex. ii. *To reduce the product* $P=[abcde]_{12}[pqr]_{1234}\cdot[\alpha\beta\gamma]_{12}$ *to a standard form by omission of redundant rows.*

Let $$[abcde]_{12}[pqr]_{1234}=[xyz]_{12},$$

so that $$P=[xyz]_{12}[\alpha\beta\gamma]_{12}.$$

Then we have in succession:

(1). $P=[xyz]_{12}[\alpha\beta\gamma]_{12}=[xy]_{12}[\alpha\beta\gamma]_{12}.$

(2). $[xyz]_{12}=[abcde]_{12}[pqr]_{1234}=[abcd]_{12}[pqr]_{1234}.$

(3). $[xy]_{12}=[abcd]_{12}[pq]_{1234}.$

(4). $P=[abcd]_{12}[pq]_{1234}\cdot[\alpha\beta\gamma]_{12}.$

Here the first two steps follow from the properties of passive rows and the third step from the properties of active rows.

Note. A more general treatment of the reduction to a product of standard form is given in § 50.

2. *Proof of the associative property.*

We proceed to show if A, B, C are any three matrices

$$A \,.\, BC = AB \,.\, C \quad \text{(A)}.$$

Since each of these products can be reduced to a standard product, it will be sufficient to prove that

$$[a]_m^r \,.\, [b]_r^s \, [c]_s^n = [a]_m^r \, [b]_r^s \,.\, [c]_s^n \quad \text{(B)}.$$

Let $$[b]_r^s \, [c]_s^n = [p]_r^n, \quad [a]_m^r \, [p]_r^n = [x]_m^n \quad \text{(1)},$$

so that $$[a]_m^r \,.\, [b]_r^s \, [c]_s^n = [x]_m^n \quad \text{(2)},$$

and let x_{ij} be any element of the matrix $[x]_m^n$.

Since $$[x]_m^n = [a]_m^r \, [p]_r^n,$$

it follows by the properties of active rows that

$$[x_{ij}] = [a_{i1} a_{i2} \dots a_{ir}] \begin{bmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{rj} \end{bmatrix}.$$

Again since $$[p]_r^n = [b]_r^s \, [c]_s^n,$$

it follows by the properties of active rows that

$$\begin{bmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{rj} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \dots b_{1s} \\ b_{21} & b_{22} \dots b_{2s} \\ \dots & \dots \\ b_{r1} & b_{r2} \dots b_{rs} \end{bmatrix} \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{sj} \end{bmatrix}.$$

Thus $$[x_{ij}] = [a_{i1} a_{i2} \dots a_{ir}] \,.\, \begin{bmatrix} b_{11} & b_{12} \dots b_{1s} \\ b_{21} & b_{22} \dots b_{2s} \\ \dots & \dots \\ b_{r1} & b_{r2} \dots b_{rs} \end{bmatrix} \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{sj} \end{bmatrix}$$

$$= [a_{i1} a_{i2} \dots a_{ir}] \begin{bmatrix} b_{11} c_{1j} + b_{12} c_{2j} + \dots + b_{1s} c_{sj} \\ b_{21} c_{1j} + b_{22} c_{2j} + \dots + b_{2s} c_{sj} \\ \dots \\ b_{r1} c_{1j} + b_{r2} c_{2j} + \dots + b_{rs} c_{sj} \end{bmatrix}$$

$$= [a_{i1} (b_{11} c_{1j} + b_{12} c_{2j} + \dots + b_{1s} c_{sj}) + a_{i2} (b_{21} c_{1j} + b_{22} c_{2j} + \dots + b_{2s} c_{sj}) + \dots$$
$$\dots + a_{ir} (b_{r1} c_{1j} + b_{r2} c_{2j} + \dots + b_{rs} c_{sj})].$$

On the left-hand side we have a matrix with a single element x_{ij}.

On the right-hand side we have a matrix with a single element

$$\sum_u \sum_v a_{iu} b_{uv} c_{vj},$$

where u receives all integral values from 1 to r, and v receives independently all integral values from 1 to s.

But when similar matrices are equal, their corresponding elements are equal.

Therefore
$$x_{ij} = \sum_u \sum_v a_{iu} b_{uv} c_{vj} \quad \text{.............................(3).}$$

Similarly if

$$[a]_m^r [b]_r^s = [q]_m^s, \quad [q]_m^s [c]_s^n = [y]_m^n \quad \text{.................(4),}$$

so that

$$[a]_m^r [b]_r^s \,.\, [c]_s^n = [y]_m^n \quad \text{.........................(5),}$$

we have

$$[y_{ij}] = [q_{i1} q_{i2} \ldots q_{is}] \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{sj} \end{bmatrix} = [a_{i1} a_{i2} \ldots a_{ir}] [b]_r^s \,.\, \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{sj} \end{bmatrix}$$

$$= \begin{bmatrix} a_{i1} b_{11} + a_{i2} b_{21} + \ldots + a_{ir} b_{r1} \\ a_{i1} b_{12} + a_{i2} b_{22} + \ldots + a_{ir} b_{r2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{i1} b_{1s} + a_{i2} b_{2s} + \ldots + a_{ir} b_{rs} \end{bmatrix} \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{sj} \end{bmatrix},$$

from which we conclude that

$$y_{ij} = \sum_u \sum_v a_{iu} b_{uv} c_{vj} \quad \text{............................(6),}$$

the values received by u and v being the same as before.

From (3) and (6), it follows that

$$[x]_m^n = [y]_m^n \quad \text{..............................(7).}$$

Thus (B), and therefore (A), is proved.

If then A, B, C are any three matrices, we may write

$$A \,.\, BC = AB \,.\, C = ABC \quad \text{.........................(8),}$$

and call ABC the *product matrix* or the *product* of the three matrices A, B, C taken in this order.

In evaluating the product ABC either pair of adjacent factor matrices can be associated together and replaced by their product matrix. This result may be stated as follows:

A product ABC of any three matrices is always associative ....(C).

It is clear from the corresponding property of a product of two matrices that *a product of three matrices is not commutative*, i.e. the order of the factors cannot in general be changed without altering the value of the product matrix.

3. *Proof of the distributive property.*

We shall now prove the following theorem:

A product ABC of any three matrices is always distributive ....(D).

First let $A = X + Y$, where X and Y are matrices.

Then $$(X+Y)BC = (X+Y).BC = X.BC + Y.BC$$
$$= XBC + YBC \quad \text{.........(9).}$$

We have here used the facts that a product of three matrices is associative and that a product of two matrices is distributive.

Next let $C = X + Y$.

Then $$AB(X+Y) = AB.(X+Y) = AB.X + AB.Y$$
$$= ABX + ABY \quad \text{.........(10).}$$

Lastly let $B = X + Y$.

Then $$A(X+Y)C = A.(X+Y)C = A.(XC+YC)$$
$$= A.XC + A.YC = AXC + AYC \quad \text{.........(11).}$$

Thus if E is any one of the factor matrices A, B, C, and if $E = X + Y$, the product ABC is the sum of the partial product obtained from ABC when E is replaced by X and the partial product obtained from ABC when E is replaced by Y; and this result is the theorem (D).

§ 48. Associative and distributive character of a product of any number of matrices.

1. *Proof of the associative property.*

We have seen that any two matrices A, B taken in this order give rise to a product matrix denoted by AB, and that any three matrices A, B, C taken in this order give rise to a definite product matrix ABC.

We will now consider the product of any four matrices A, B, C, D taken in this order, and *we will in the first place define* $ABCD$ *to mean* $A.BCD$, i.e. the product matrix obtained by multiplying the matrix A into the matrix BCD.

Since a product of three matrices is associative, we have

$$ABCD = A.BCD = A.(B.CD) = AB.CD$$
$$= (AB.C).D = ABC.D.$$

Having shown that

$$ABCD = A \,.\, BCD = AB \,.\, CD = ABC \,.\, D,$$

we have shown that we obtain the same product matrix $ABCD$ when we divide the four letters into *any two* smaller groups of adjacent letters. The smaller groups being themselves associative, it follows that we still obtain the same product matrix $ABCD$ when we divide the four letters *in any manner whatever* into groups composed of adjacent letters. Accordingly $ABCD$ is a definite product matrix which is always obtained however the four letters are associated, so long as their relative order remains unchanged. That is, *a product of four matrices is always associative.*

We will next consider the product of any five matrices A, B, C, D, E taken in this order, and *we will in the first place define $ABCDE$ to mean $A \,.\, BCDE$.*

Since products of three matrices and products of four matrices are associative, we have

$$\begin{aligned} ABCDE = A \,.\, BCDE &= A \,.\, (B \,.\, CDE) = AB \,.\, CDE \\ &= AB \,.\, (C \,.\, DE) = ABC \,.\, DE \\ &= (ABC \,.\, D) \,.\, E = ABCD \,.\, E. \end{aligned}$$

Having shown that

$$ABCDE = A \,.\, BCDE = AB \,.\, CDE = ABC \,.\, DE = ABCD \,.\, E,$$

we have shown that we obtain the same product matrix $ABCDE$ when we divide the five letters into *any two* smaller groups of adjacent letters. The smaller groups being themselves associative, it follows that we still obtain the same product matrix $ABCDE$ when we divide the five letters *in any manner whatever* into groups composed of adjacent letters. That is, *a product of any five matrices is always associative.*

In both cases the product is clearly in general not commutative.

Proceeding in this way, we obtain the following result:

A product of any number of matrices is always associative but is in general not commutative ....(A).

2. *Proof of the distributive character.*

Let $$X = ABC \dots DEF \dots KL$$

be a product of any number of matrices A, B, C, ... D, E, F, ... K, L.

First let $E = E_1 + E_2$, where E is any factor matrix of the product except the first and last, and E_1 and E_2 are matrices.

Let X_1 be the partial product obtained when we replace the factor matrix E in the complete product by E_1, and let X_2 be the partial product obtained when we replace E by E_2.

Let $$P = ABC \dots D, \quad Q = F \dots KL$$
be the product of all the factor matrices which precede E, and the product of all the factor matrices which follow E in the product X.

Then by the associative property of all matrix products and the distributive property of a product of two matrices, we have

$$\begin{aligned} X = PEQ = P \,.\, EQ &= P \,.\, (E_1 + E_2)\, Q \\ &= P \,.\, (E_1 Q + E_2 Q) = PE_1 Q + PE_2 Q \\ &= X_1 + X_2. \end{aligned}$$

Next let $A = A_1 + A_2$, and let X_1, X_2 be the partial products obtained when A is replaced by A_1, A_2 respectively in the complete product X.

Let $Q = BC \dots DEF \dots KL$ be the product of all the factor matrices which follow A in the product X.

Then $$X = AQ = (A_1 + A_2)\, Q = A_1 Q + A_2 Q = X_1 + X_2.$$

Finally let $L = L_1 + L_2$, and let X_1, X_2 be the partial products obtained when L is replaced by L_1, L_2 respectively in the complete product X.

Let $P = ABC \dots DEF \dots K$ be the product of all the factor matrices which precede L in the product X.

Then $$X = PL = P(L_1 + L_2) = PL_1 + PL_2 = X_1 + X_2.$$

We see then that if E is any factor matrix whatever of the product X, and if $E = E_1 + E_2$, then the product X is the sum of the two partial products obtained from X when E is replaced by E_1 and by E_2 respectively. This result is equivalent to the following statement:

A product of any number of matrices is always distributive ....(B).

§ 49. Active and passive rows in a product formed by a chain of matrix factors.

Let $[x]_m^n$ be the product matrix of any number of matrices

$$[a]_m^{\alpha'},\ [b]_\beta^{\beta'},\ \dots\ [l]_\lambda^n,$$

arranged in this order,

so that $$[a]_m^{\alpha'} [b]_\beta^{\beta'} [c]_\gamma^{\gamma'} [d]_\delta^{\delta'} [e]_\epsilon^{\epsilon'} \dots [l]_\lambda^n = [x]_m^n \quad \text{..................(1).}$$

The various matrices occurring in the chain on the left will be termed *factor matrices* or *matrix factors* of the product.

It will be observed that the number of horizontal rows in the product matrix is equal to the number of horizontal rows in the first factor matrix, and that the number of vertical rows in the product matrix is equal to the number of vertical rows in the last factor matrix.

For the product of the first two factors is a matrix of the form $[p]_m^{\beta'}$; the product of this matrix and the third factor is a matrix of the form $[q]_m^{\gamma'}$; and proceeding in this way the product of all the factors is a matrix of the form $[x]_m^n$.

The horizontal rows of the first factor matrix and the vertical rows of the last factor matrix will be called *active rows*. All other horizontal and vertical rows occurring in the factor matrices will be called *passive rows*.

The number of active or horizontal rows in the first factor matrix and the number of active or vertical rows in the last factor matrix will be called the *activities* of those respective matrices, and they will also be called the *activities* of the product. Thus the activity of the first factor matrix is m, the activity of the second factor matrix is n, and the activities of the product are m and n.

The smaller of the two activities m and n, which is the efficiency of the product matrix, will be called the *efficiency* of the product.

The number of vertical passive rows and the number of horizontal passive rows occurring in any factor matrix will each be called a *passivity* of that factor matrix and also a *passivity* of the product. The first factor matrix has one passivity α', the last factor matrix has one passivity λ. The intermediate factor matrix $[d]_\delta^{\delta'}$ has in general two passivities δ and δ', but if $\delta = \delta'$ it has only one passivity.

The ith horizontal or active row of the first factor matrix will be said to *correspond* to the ith horizontal row of the product matrix. So the ith vertical or active row of the last factor matrix will be said to *correspond* to the ith vertical row of the product matrix.

Again let $D = [d]_\delta^{\delta'}$, $E = [e]_\epsilon^{\epsilon'}$ be any two consecutive factor matrices in the chain. Then the ith vertical (passive) row in the matrix D on the left will be said to *correspond* to the ith horizontal (passive) row in the matrix E on the right, so long as these ith rows occur in both the matrices. If $\delta' > \epsilon$, there will be vertical rows in D whose corresponding horizontal rows in E are absent. Such rows will be called *redundant* final vertical passive rows of D. So if $\delta' < \epsilon$, there will be horizontal rows in E whose corresponding rows in D are absent. Such rows will be called redundant final horizontal passive rows of E.

Thus we have established a one-one correspondence between the active rows of the two extreme factor matrices and the rows of the product matrix. Also in every two adjacent factor matrices of the chain we have established a one-one correspondence between the non-redundant vertical passive rows in the matrix on the left and the non-redundant vertical passive rows in the matrix on the right.

In the case of any intermediate factor matrix the non-redundant horizontal passive rows correspond to the non-redundant vertical passive rows of the adjacent matrix on the left, and the non-redundant vertical passive rows correspond to the non-redundant horizontal passive rows of the adjacent matrix on the right.

The product is said to be of *standard form* or to be a *standard product* when in every pair of adjacent factor matrices the number of vertical rows in the matrix on the left is equal to the number of horizontal passive rows in the matrix on the left. In such a product there are no redundant final passive rows. This case occurs when

$$\alpha' = \beta,\ \beta' = \gamma,\ \gamma' = \delta,\ \ldots.$$

Thus a standard product has the form

$$[a]_m^\alpha [b]_\alpha^\beta [c]_\beta^\gamma [d]_\gamma^\delta [e]_\delta^\epsilon \ldots [l]_\lambda^n = [x]_m^n \quad\ldots\ldots\ldots\ldots(2).$$

The products which occur in practice are almost invariably of standard form, and in future chapters a product will always be understood to be a standard product. In the next article it will be shown that the consideration of products which are not of standard form is superfluous.

§ 50. Reduction of a product of any number of matrices to a product of standard form.

Let
$$[a]_m^{\alpha'} [b]_\beta^{\beta'} \ldots [d]_\delta^{\delta'} [e]_\epsilon^{\epsilon'} \ldots [l]_\lambda^n = [x]_m^n$$

be a product formed by any chain of matrix factors.

Let $[d]_\delta^{\delta'}$, $[e]_\epsilon^{\epsilon'}$ be any two successive factor matrices of the product and let

$$[d]_\delta^{\delta'} [e]_\epsilon^{\epsilon'} = [p]_\delta^{\epsilon'}.$$

In consequence of the associative property of the chain we can in it replace $[d]_\delta^{\delta'} [e]_\epsilon^{\epsilon'}$ by $[p]_\delta^{\epsilon'}$ without in any way altering the product matrix $[x]_m^n$.

If $\delta' > \epsilon$, we can increase ϵ to δ' by inserting in $[e]_\epsilon^{\epsilon'}$ additional final horizontal passive rows of 0's. We can also diminish δ' to ϵ by striking out the redundant final vertical passive rows in $[d]_\delta^{\delta'}$. By the properties of passive rows in a product of two matrices the matrix $[p]_\delta^{\epsilon'}$ and consequently the matrix $[x]_m^n$ is entirely unaffected by either of these operations.

If $\delta' < \epsilon$, we can increase δ' to ϵ by inserting in $[d]_\delta^{\delta'}$ additional final vertical passive rows of 0's. We can also diminish ϵ to δ' by striking out the redundant final horizontal passive rows in $[e]_\epsilon^{\epsilon'}$. By the properties of passive rows in a product of two matrices the matrix $[p]_\delta^{\epsilon'}$ and consequently the matrix $[x]_m^n$ is entirely unaffected by either of these operations.

Treating every pair of adjacent factor matrices in one of these ways, we finally reduce the product to the standard form.

The reduction is most conveniently effected by repetitions of the operation of striking out redundant final passive rows. The various factors of the reduced product have then their lowest orders and contain no redundant rows. The reduction can also be effected by repetitions of the operation of adding final passive rows of 0's, or again by a combination of both kinds of operations. In the following examples use is made of the abbreviated notations introduced in § 47.I.

Ex. i.
$$\begin{aligned}&[abcd]_{12}[\alpha\beta\gamma\delta]_{123}[pq]_{12345}[uvw]_{1234}\\ =&[abc]_{12}[\alpha\beta\gamma\delta]_{123}[pq]_{12345}[uvw]_{1234}\\ =&[abc]_{12}[\alpha\beta\gamma\delta]_{123}[pq]_{1234}[uvw]_{1234}\\ =&[abc]_{12}[\alpha\beta\gamma\delta]_{123}[pq]_{1234}[uvw]_{12}.\end{aligned}$$

Here we have at each step omitted redundant final passive rows.

Ex. ii.
$$\begin{aligned}&[abcd]_{12}[\alpha\beta\gamma\delta]_{123}[pq]_{12345}[uvw]_{1234}\\ =&[abcd]_{12}[\alpha\beta\gamma\delta]_{1230}[pq]_{12345}[uvw]_{1234}\\ =&[abcd]_{12}[\alpha\beta\gamma\delta 0]_{1230}[pq]_{12345}[uvw]_{1234}\\ =&[abcd]_{12}[\alpha\beta\gamma\delta 0]_{1230}[pq00]_{12345}[uvw]_{1234}.\end{aligned}$$

Here we have at each step inserted additional passive rows of 0's.

Ex. iii.
$$\begin{aligned}&[abc]_{12}[\alpha\beta\gamma\delta]_{1234}[pq]_{12345}[uvw]_{1234}\\ =&[abc]_{12}[\alpha\beta\gamma\delta]_{1234}[pq]_{12345}[uvw]_{12}\\ =&[abc]_{12}[\alpha\beta\gamma\delta 0]_{1234}[pq]_{12345}[uvw]_{12}\\ =&[abc]_{12}[\alpha\beta\gamma\delta 0]_{123}[pq]_{12345}[uvw]_{12}.\end{aligned}$$

Here we have omitted redundant final passive rows at the first and third steps, and we have added a final passive row of 0's at the second step.

§ 51. Expressions for the elements of a product matrix of any number of given matrices.

1. *Product of two matrices.*

If
$$[x]_m^n = [a]_m^{\alpha'} [b]_\beta^n$$
then
$$x_{ij} = \sum_u a_{iu} b_{uj} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
where u receives all integral values from 1 to the smaller of the two numbers α' and β.

This result is equivalent to the definition of a product of two matrices given in § 42.

2. *Product of three matrices.*

If
$$[x]_m^n = [a]_m^{\alpha'} [b]_\beta^{\beta'} [c]_\gamma^n,$$
then
$$x_{ij} = \sum_u \sum_v a_{iu} b_{uv} c_{vj} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$
where u ranges from 1 to the smaller of the two numbers α' and β,
and v ranges from 1 to the smaller of the two numbers β' and β,
the summations with respect to u and v being independent.

To prove this, let $[p]_\beta^n = [b]_\beta^{\beta'} [c]_\gamma^n$, so that $[x]_m^n = [a]_m^{\alpha'} [p]_\beta^n$.

Then by the previous case
$$x_{ij} = \sum_u a_{iu} p_{uj} = \sum_u a_{iu} (\sum_v b_{uv} c_{vj}) = \sum_u \sum_v a_{iu} b_{uv} c_{vj}.$$

3. *Product of four matrices.*

If
$$[x]_m^n = [a]_m^{\alpha'} [b]_\beta^{\beta'} [c]_\gamma^{\gamma'} [d]_\delta^n,$$
then
$$x_{ij} = \sum_u \sum_v \sum_w a_{iu} b_{uv} c_{vw} d_{wj} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$
where u ranges from 1 to the smaller of the two numbers α' and β,
v ranges from 1 to the smaller of the two numbers β' and γ,
and w ranges from 1 to the smaller of the two numbers γ' and δ.

To prove this, let $[p]_\beta^n = [b]_\beta^{\beta'} [c]_\gamma^{\gamma'} [d]_\delta^n$, so that $[x]_m^n = [a]_m^{\alpha} [p]_\beta^n$.

Then by the previous cases,
$$x_{ij} = \sum_u a_{iu} p_{uj} = \sum_u a_{iu} (\sum_v \sum_w b_{uv} c_{vw} d_{wj})$$
$$= \sum_u \sum_v \sum_w a_{iu} b_{uv} c_{vw} d_{wj}.$$

4. *Product of any number of matrices.*

The corresponding result for a product of any number of matrices will now be obvious.

For products of standard form the results are somewhat simpler.

Thus if
$$[x]_m^n = [a]_m^\alpha [b]_\alpha^\beta [c]_\beta^\gamma [d]_\gamma^n,$$
then
$$x_{ij} = \sum_u \sum_v \sum_w a_{iu} b_{uv} c_{vw} d_{wj},$$
where
u receives the values 1, 2, 3, ... α,
v receives the values 1, 2, 3, ... β,
w receives the values 1, 2, 3, ... γ.

§ 52. Properties of the passive rows in any product formed by a chain of matrix factors.

The properties enumerated in this and the following article are all deducible from the properties of a product of two matrices together with the associative property of any chain of matrix factors, in virtue of which any two adjacent factor matrices in the chain can be replaced by their product matrix. In a few cases the distributive property of the chain is also made use of.

The properties can also be deduced from the expressions given in § 51 for the elements of the product matrix.

1. *The product matrix is the sum of all the partial products which can be obtained from the chain by replacing any given pair of adjacent factor matrices by the matrices of a corresponding pair of their non-redundant passive rows, all other factor matrices being left unaltered.*

Let $AB \dots CDEF \dots L = X$ be the product of the matrices $A, B, \dots C, D, E, F, \dots L$ arranged in this order.

Let $P_1, P_2, P_3, \dots$ be the products of the matrices of pairs of corresponding passive rows in the product DE.

Let $X_1, X_2, X_3, \dots$ be the values assumed by X when we replace DE by $P_1, P_2, P_3 \dots$ in the chain.

By the properties of a product of two matrices
$$DE = P_1 + P_2 + P_3 + \dots.$$
Therefore
$$X = AB \dots C(P_1 + P_2 + P_3 + \dots) F \dots L.$$
By the distributive property of the chain this is equal to
$$AB \dots CP_1F \dots L + AB \dots CP_2F \dots L + AB \dots CP_3F \dots L + \dots.$$
That is,
$$X = X_1 + X_2 + X_3 + \dots .$$

Ex. i.
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} [l_1 \, m_1 \, n_1] \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} [l_2 \, m_2 \, n_2] \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{bmatrix}.$$

2. *Wherever the elements of any given passive row of any factor matrix occur in the elements of the product matrix, each of them is multiplied by an element of the corresponding passive row which occurs in the preceding or following matrix.*

This follows immediately from sub-article 1.

It can also be deduced from the general expressions for the elements of the product matrix given in § 51. As an illustration of this second method of proof, we will prove the theorem for the kth horizontal row of the factor matrix $[c]_{\gamma}^{\gamma'}$ in the product

$$[a]_m^{\alpha'} [b]_{\beta}^{\beta'} [c]_{\gamma}^{\gamma'} [d]_{\delta}^{n} = [x]_m^n.$$

We know that
$$x_{ij} = \sum_u \sum_v \sum_w a_{iu} b_{uv} c_{vw} d_{wj}.$$

If an element c_{kp} from the given row occurs in this expression for x_{ij}, it is multiplied by an element of the form b_{qk}, which is an element belonging to the kth vertical row of the matrix $[b]_{\beta}^{\beta'}$.

3. *The product matrix is unaffected by the insertion or omission of any number of final passive rows of 0's in any factor matrix.*

Let $AB \dots DE \dots L = X$ be the product.

Let D, E be any two adjacent factor matrices, and let $DE = P$.

In consequence of the associative property of the chain we can in it replace DE by P without altering the product matrix X in any way.

Now by the properties of a product of two matrices we can insert or remove any number of final vertical rows of 0's in D or any number of final horizontal rows of 0's in E without altering in any way the matrix P by which DE can be replaced. Consequently the product matrix X is the same after these changes have been made in D and E as it was before.

Ex. ii.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \epsilon_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \epsilon_2 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \epsilon_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \epsilon_2 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & 0 \\ l_2 & m_2 & 0 \\ l_3 & m_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{bmatrix},$$

or $$[abc]_{12}[\alpha\beta\gamma\delta\epsilon]_{12}[lm]_{123}[\lambda\mu\nu]_{123} = [abc]_{12}[\alpha\beta\gamma\delta\epsilon]_{12}[lm0]_{1230}[\lambda\mu\nu]_{123}.$$

4. *If any factor matrix contains redundant final passive rows, we can omit all or any number of them without in any way altering the product matrix.*

It follows that conversely we can insert arbitrary redundant final passive rows in any factor matrix without in any way altering the product matrix.

Using the same notation as in the last theorem, if there are redundant final vertical rows in D or redundant final horizontal rows in E, we can strike out some or all of these redundant rows without altering the matrix P which can replace DE in the chain. This follows from § 43.4. Consequently the product matrix X is unaffected when these changes are made in D and E.

Similarly if arbitrary redundant final vertical rows are inserted in D or if arbitrary redundant final horizontal rows are inserted in E, the matrix P and therefore the matrix X remains unaltered.

Ex. iii. $$[abc]_{12}[\alpha\beta\gamma\delta\epsilon]_{12}[lm]_{123}[\lambda\mu\nu]_{123}$$
$$= [abc]_{12}[\alpha\beta\gamma\delta\epsilon]_{12}[lm]_{123}[\lambda\mu\nu]_{12}$$
$$= [ab]_{12}[\alpha\beta\gamma]_{12}[lm]_{123}[\lambda\mu\nu]_{12}.$$

5. *If any non-redundant passive row in any factor matrix is a row of 0's, we can strike out that row and the corresponding passive row (which occurs in the preceding or following matrix) without in any way altering the product matrix.*

Let the product be $$AB \ldots DEF \ldots KL = X,$$

and let $$DE = P, \quad EF = Q.$$

If E is any factor matrix except the first and contains a horizontal passive row of 0's, we can strike out that row in E and the corresponding vertical passive row in D without altering P in any way. Therefore we can replace DE by the same matrix P in the chain both before and after this change, and consequently X remains unaffected by the change.

Again if E is any factor matrix except the last and contains a vertical passive row of 0's, we can strike out that row in E and the corresponding horizontal passive row in F without altering Q in any way. Therefore we can replace EF by the same matrix Q in the chain both before and after this change, and consequently X remains unaffected by the change.

Ex. iv.
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{bmatrix}$$
$$= \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{bmatrix}.$$

Ex. v. $[abc]_{12}\, [\alpha\beta 0\delta]_{123}\, [lmn]_{12345}\, [\lambda\mu\nu]_{12} = [abc]_{12}\, [\alpha\beta\delta]_{123}\, [lmn]_{1245}\, [\lambda\mu\nu]_{12}.$

6. *Conversely we can insert any additional non-redundant passive row of arbitrary elements in any position in any one of the factor matrices and an additional corresponding passive row of 0's in the corresponding position in the appropriate adjacent factor matrix without in any way altering the product matrix.*

The examples just given serve also to illustrate this converse theorem.

7. *The product matrix is unaltered when the sign of every element in two corresponding passive rows is changed. It is also unaltered when every element of one passive row is multiplied by k and every element of the corresponding passive row is divided by k.*

This is an immediate consequence of sub-article 2; also of sub-article 1.

Ex. vi.
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{bmatrix}$$
$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1, & -\beta_1, & \gamma_1, & \delta_1 \\ \alpha_2, & -\beta_2, & \gamma_2, & \delta_2 \\ \alpha_3, & -\beta_3, & \gamma_3, & \delta_3 \end{bmatrix} \begin{bmatrix} l_1, & m_1 \\ -l_2, & -m_2 \\ l_3, & m_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{bmatrix}.$$

8. *The product matrix is unaltered when the two sets of corresponding passive rows occurring in two adjacent factor matrices (one set vertical and the other set horizontal) are re-arranged in any manner, provided that the re-arrangement is exactly the same for both sets.*

This is an immediate consequence of sub-article 1.

Ex. vii. $[abc]_{12}\, [\alpha\beta\gamma]_{123}\, [lm]_{123}\, [\lambda\mu]_{12} = [abc]_{123}\, [\gamma\alpha\beta]_{123}\, [lm]_{312}\, [\lambda\mu]_{12}.$

Ex. viii. $[abcde]_{12}\, [\alpha\beta\gamma]_{123}\, [lm]_{123}\, [\lambda\mu]_{123} = [cbade]_{12}\, [\alpha\beta\gamma]_{321}\, [lm]_{123}\, [\lambda\mu]_{123}.$

9. *If we strike out any number of pairs of corresponding non-redundant passive rows in two given adjacent factor matrices of the chain and leave the other factor matrices unaltered, we obtain a partial product. Any sum of such partial products in which every pair of corresponding non-redundant passive rows of the two given adjacent factor matrices occurs once and only once is equal to the product matrix of the chain.*

This is at once seen when each partial product is expanded in the manner described in sub-article 1.

Ex. ix. $[abc]_{12}[a\beta\gamma\delta]_{1234}[lm]_{1234}[\lambda\mu]_{123}$

$$=[abc]_{12}[a\delta]_{1234}[lm]_{14}[\lambda\mu]_{123}+[abc]_{12}[\beta\gamma]_{1234}[lm]_{23}[\lambda\mu]_{123}.$$

Ex. x. $[abc]_{12}[a\beta\gamma\delta]_{1234}[lm]_{123}[\lambda\mu]_{123}$

$$=[abc]_{12}[a\delta]_{1234}[lm]_{1}[\lambda\mu]_{123}+[abc]_{12}[\beta\gamma]_{1234}[lm]_{23}[\lambda\mu]_{123}.$$

§ 53. Properties of the active rows in any product formed by a chain of matrix factors.

1. *The elements of any given row in the product matrix are homogeneous linear functions of the elements of the corresponding active row in the chain.*

The letters denoting matrices, let the product be

$$ABC \dots KL = X.$$

Let $P = ABC \dots K$, $Q = BC \dots KL$, be the product of all the matrices except the last, and the product of all the matrices except the first respectively, so that

$$X = AQ \quad \text{and} \quad X = PL.$$

Using the properties of a product of two matrices, it follows from the equation $X = AQ$ that the elements of the ith horizontal row of X are homogeneous linear functions of the elements of the ith horizontal or active row of the first factor matrix A.

Similarly from the equation $X = PL$, it follows that the elements of the jth vertical row of X are homogeneous linear functions of the elements of the jth vertical or active row of the last factor matrix L.

2. *The elements belonging to any given active row of the chain occur in the product matrix only in the elements of the corresponding row.*

Using the same notation as before, we see from the equation $AQ = X$ that the elements of the ith horizontal row of A occur only in those elements of X which belong to its ith horizontal row. So from the equation $PL = X$ we see that the elements of the jth vertical row of L occur only in those elements of X which belong to its jth vertical row.

3. *If any active row in either of the two extreme factor matrices is a row of 0's, the corresponding row in the product matrix is a row of 0's.*

This follows immediately from sub-article 1.

Ex. i. The product matrix of

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} a_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & 0 \\ \lambda_2 & \mu_2 & 0 \\ \lambda_3 & \mu_3 & 0 \end{bmatrix}$$

has the form
$$\begin{bmatrix} x_1 & y_1 & 0 \\ 0 & 0 & 0 \\ x_3 & y_3 & 0 \end{bmatrix}.$$

4. *We may strike out any active rows in one or both of the extreme factor matrices, provided that we strike out the corresponding rows in the product matrix.*

As in sub-article 1 let

$$X = ABC \dots KL = AQ = PL.$$

First let A' be obtained from A by striking out certain horizontal rows, and let X' be obtained from X by striking out the corresponding horizontal rows. Then from the equation $AQ = X$ it follows that $A'Q = X'$,

i.e. $$A'BC \dots KL = X'.$$

Secondly let L' be obtained from L by striking out certain vertical rows, and let X' be obtained from X by striking out the corresponding vertical rows. Then from the equation $PL = X$, it follows that $PL' = X'$,

i.e. $$ABC \dots KL' = X'.$$

Ex. ii. If

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} a_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \\ a_4 & \beta_4 & \gamma_4 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix},$$

then
$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} a_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \\ a_4 & \beta_4 & \gamma_4 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{bmatrix} \begin{bmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{bmatrix} = \begin{bmatrix} y_1 & z_1 \\ y_3 & z_3 \end{bmatrix}.$$

Ex. iii. If $$[abcde]_{123}[a\beta\gamma]_{123}[lm]_{12}[pqrs]_{123} = [xyzw]_{123},$$

then $$[abcde]_{13}[a\beta\gamma]_{123}[lm]_{12}[pqs]_{123} = [xyw]_{13}.$$

Ex. iv. Let $$[a]_m^r [b]_\beta^{\beta'} [c]_\gamma^{\gamma'} \dots [k]_\kappa^{\kappa'} [l]_s^n = [x]_m^n.$$

If we strike out all the horizontal rows of the first factor matrix and the product matrix except the ith, and all the vertical rows of the last factor matrix and the product matrix except the jth, and then equate the determinants of both sides in the resulting matrix equation, we obtain

$$x_{ij} = \det [a_{i1} a_{i2} \dots a_{ir}] [b]_\beta^{\beta'} [c]_\gamma^{\gamma'} \dots [k]_\kappa^{\kappa'} \begin{bmatrix} l_{1j} \\ l_{2j} \\ \vdots \\ l_{sj} \end{bmatrix}.$$

Thus the above theorem provides us with convenient expressions for the elements of the product matrix.

Ex. v. A product formed by any chain of matrix factors can always be expressed in the form

$$AMB = [a]_m^r M [b]_s^n,$$

where $A = [a]_m^r$ and $B = [b]_s^n$ are the first and last factor matrices and M is the product of all the intervening matrix factors. If we denote the matrix of the ith active row of A by a_i and the matrix of the jth active row of B by b_j, we have by Ex. iv

$$AMB = [a]_m^r [M] [b]_s^n = \begin{bmatrix} \det a_1 M b_1, & \det a_1 M b_2, & \dots & \det a_1 M b_n \\ \det a_2 M b_1, & \det a_2 M b_2, & \dots & \det a_2 M b_n \\ \dots & \dots & \dots & \dots \\ \det a_m M b_1, & \det a_m M b_2, & \dots & \det a_m M b_n \end{bmatrix}.$$

The special case in which there are only two factors in the chain is contained in formula (1) of § 42. It can be included in the above by regarding M as a unit matrix.

If the chain is of standard form, as is most usual, M will be of the form $[c]_r^s$.

5. *We can replace any of the active rows in one or both of the extreme factor matrices by rows of 0's, provided that we replace the corresponding rows in the product matrix by rows of 0's.*

Since this theorem is known to be true for a product of two matrices, we can use the same method of proof as in sub-article 4.

Ex. vi. If $$[abc]_{123} [\alpha\beta]_{123} [lm]_{12} [pqrs]_{12} = [xyzw]_{123},$$
then $$[abc]_{123} [\alpha\beta]_{123} [lm]_{12} [0q0s]_{12} = [0y0w]_{123}.$$

Ex. vii. If $$[abcde]_{123} [\alpha\beta\gamma]_{123} [lm]_{12} [pqrs]_{123} = [xyzw]_{123},$$
then $$[abcde]_{103} [\alpha\beta\gamma]_{123} [lm]_{12} [0q0s]_{123} = [0y0w]_{103},$$
where in $[abcde]_{103}$ the second horizontal row is a row of 0's.

6. *We can insert additional active rows of 0's anywhere in one or both of the extreme factor matrices, provided that we also insert similarly placed corresponding rows of 0's in the product matrix.*

This theorem is the converse of the preceding theorem.

7. *We can re-arrange the active rows in one or both of the extreme factor matrices in any manner we please, provided that the corresponding rows in the product matrix are subjected to exactly the same re-arrangements.*

Let $$X = ABC \dots KL = AQ = PL.$$

If similar re-arrangements of the corresponding horizontal rows of A and X convert them into A' and X', then from the equation $AQ = X$ we deduce that $A'Q = X'$ or

$$A'BC \dots KL = X'.$$

Again if similar re-arrangements of the corresponding vertical rows of L and X convert them into L' and X', then from the equation $PL = X$ we deduce that $PL' = X'$ or

$$ABC \dots KL' = X'.$$

Ex. viii. If

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \\ \lambda_4 & \mu_4 & \nu_4 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix},$$

then

$$\begin{bmatrix} a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \nu_1 & \mu_1 & \lambda_1 \\ \nu_2 & \mu_2 & \lambda_2 \\ \nu_3 & \mu_3 & \lambda_3 \\ \nu_4 & \mu_4 & \lambda_4 \end{bmatrix} = \begin{bmatrix} z_3 & y_3 & x_3 \\ z_1 & y_1 & x_1 \\ z_2 & y_2 & x_2 \end{bmatrix}.$$

We can prove this by prefixing and postfixing the matrices $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ on both sides of the first equation.

8. *If $ABC \dots KL = \Omega$ is a product formed by any chain of matrix factors we can insert anywhere in the first factor matrix A an arbitrary additional active row whose matrix is x, provided that we insert in the same situation in the product matrix Ω the corresponding horizontal row whose matrix is $xBC \dots KL$. Also we can insert anywhere in the last factor matrix L an arbitrary additional active row whose matrix is y, provided that we insert in the same situation in Ω the corresponding vertical row whose matrix is $ABC \dots Ky$.*

Let $ABC \dots K = P$, $BC \dots KL = Q$, so that $\Omega = PL = AQ$.

Let the insertion in A of the additional horizontal row whose matrix is x convert A into A', and let the insertion in Ω of the corresponding horizontal row whose matrix is $xBC \dots KL$ or xQ convert Ω into Ω'. Then from the equation $AQ = \Omega$ it follows by § 44.8 that $A'Q = \Omega'$, or

$$A'BC \dots KL = \Omega'.$$

Again let the insertion in L of the additional vertical row whose matrix is y convert L into L', and let the insertion in Ω of the corresponding vertical row whose matrix is $ABC \dots Ky$ or Py convert Ω into Ω'. Then from the equation $PL = \Omega$ it follows by § 44.8 that $PL' = \Omega'$, or

$$ABC \dots KL' = \Omega'.$$

Ex. ix. If

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix},$$

and

$$[xyz] \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{bmatrix} = [\xi\eta\zeta],$$

then

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ x & y & z \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ \xi & \eta & \zeta \\ p_3 & q_3 & r_3 \end{bmatrix}.$$

Ex. x. If

$$[abcde]_{12}[\alpha\beta\gamma]_{123}[lm]_{123}[\lambda\mu]_{123} = [pq]_{12},$$

and

$$[abcde]_{12}[\alpha\beta\gamma]_{123}[lm]_{123}[xyz]_{123} = [\xi\eta\zeta]_{12},$$

then

$$[abcde]_{12}[\alpha\beta\gamma]_{123}[lm]_{123}[x\lambda y\mu z]_{123} = [\xi p\eta q\zeta]_{12}.$$

9. *We can multiply all the elements of any active row of the chain by the scalar number k, provided that we also multiply all the elements of the corresponding row in the product matrix by k.*

This is an immediate consequence of sub-articles 1 and 2.

Ex. xi. If

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix},$$

then

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} \lambda_1, & k\mu_1, & \nu_1 \\ \lambda_2, & k\mu_2, & \nu_2 \\ \lambda_3, & k\mu_3, & \nu_3 \end{bmatrix} = \begin{bmatrix} p_1, & kq_1, & r_1 \\ p_2, & kq_2, & r_2 \\ p_3, & kq_3, & r_3 \end{bmatrix}.$$

10. *We can add to the elements of any active row in either of the extreme factor matrices the corresponding elements of any other active row of the same factor matrix each multiplied by the same scalar quantity k, provided that we deal similarly with the corresponding rows in the product matrix.*

Let $$X = ABC \dots KL = PL = AQ.$$

Let A become A' when to each element of the ith horizontal row of A there is added k times the corresponding element of the jth horizontal row of A, and let the product matrix X become X' when to each element of the ith horizontal row of X there is added k times the corresponding element of the jth horizontal row of X. Then from the equation $AQ = X$ it follows by § 44.10 that

$$A'Q = X' \quad \text{or} \quad A'BC \ldots KL = X'.$$

Similarly if L and X become respectively L' and X' when to each element of the ith vertical row there is added k times the corresponding element of the jth vertical row, it follows from the equation $PL = X$ that $PL' = X'$, or

$$ABC \ldots KL' = X'.$$

Ex. xii. If

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 & \pi_1 \\ \lambda_2 & \mu_2 & \nu_2 & \pi_2 \\ \lambda_3 & \mu_3 & \nu_3 & \pi_3 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix},$$

then

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \lambda_1, & \mu_1+k\pi_1, & \nu_1, & \pi_1 \\ \lambda_2, & \mu_2+k\pi_2, & \nu_2, & \pi_2 \\ \lambda_3, & \mu_3+k\pi_3, & \nu_3, & \pi_3 \end{bmatrix} = \begin{bmatrix} x_1, & y_1+kw_1, & z_1, & w_1 \\ x_2, & y_2+kw_2, & z_2, & w_2 \\ x_3, & y_3+kw_3, & z_3, & w_3 \end{bmatrix}.$$

We can prove this by postfixing the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & k & 0 & 1 \end{bmatrix}$ on both sides of the first equation.

§ 54. Other properties of the product matrix of a chain of matrix factors.

1. *Equivalence of conventionally equal matrices.*

If any factor matrix is replaced by another conventionally equal to it, the product matrix is either unaltered or is replaced by another matrix conventionally equal to it.

For we know that the product matrix is unaltered when final passive rows of 0's are added to or removed from any of the factor matrices; and we also know that the addition or removal of a final active row of 0's in either extreme factor matrix causes the addition or removal of a corresponding row of 0's in the product matrix.

Conversely if $X = ABC \dots KL$ is the product matrix of any chain of matrix factors, and if X' is any matrix conventionally equal to X, then X' can be expressed identically in the form $X' = A'BC \dots KL'$, where A' and L' are conventionally equal to A and L respectively.

It follows from the first result that in multiplying together any number of matrices we can consistently regard conventionally equal matrices as equivalent.

2. *Conjugate of a product of any number of matrices.*

Theorem. *The conjugate of the product of any number of matrices taken in a given order is identical with the product of the conjugates of the factor matrices taken in the reverse order.*

We have shown in § 45.4 that this theorem is true for a product of two matrices. Hence to prove that the theorem is true generally, it will be sufficient to show that if it is true for a product of n matrices, then it is also true for a product of $n + 1$ matrices.

Suppose that the theorem is true for any product whatever of n matrices.

Let $A_1, A_2, \dots A_n, A_{n+1}$ be any $n + 1$ matrices, and let X be their product matrix when they are taken in this order, so that

$$X = A_1 A_2 \dots A_n A_{n+1}.$$

Let $A_1', A_2', \dots A_n', A'_{n+1}, X'$ be the respective conjugate matrices of $A_1, A_2, \dots A_n, A_{n+1}, X$. Further let

$$P = A_1 A_2 \dots A_n$$

be the product matrix of the first n factor matrices, and let P' be the conjugate of P. We then have $X = PA_{n+1}$, and from this it follows by § 45.4 that $X' = A'_{n+1} P'$. But by supposition

$$P' = A_n' A'_{n-1} \dots A_2' A_1'.$$

Thus
$$X' = A'_{n+1} P' = A'_{n+1} A_n' \dots A_2' A_1'.$$

Thus if the theorem is true for all products of n matrices, it is also true for all products of $n + 1$ matrices. It follows that it is true generally.

Ex. i. If
$$[x]^n_m = [a]^\alpha_m [b]^\beta_n [c]^\gamma_\nu [d]^n_w,$$

then
$$\overline{\underline{x}}^m_n = \overline{\underline{d}}^w_n \, \overline{\underline{c}}^\nu_\gamma \, \overline{\underline{b}}^u_\beta \, \overline{\underline{a}}^m_\alpha .$$

Ex. ii. If

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix},$$

then
$$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

3. *Multiplication of a product by a scalar quantity.*

Since the elements of the product matrix are homogeneous linear functions of the elements of each factor matrix, the product matrix is multiplied by the scalar quantity k when all the elements of any one factor matrix are multiplied by k.

Ex. iii. If
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix},$$

then
$$k\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} kx_1 & ky_1 \\ kx_2 & ky_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} k\alpha_1 & k\beta_1 & k\gamma_1 \\ k\alpha_2 & k\beta_2 & k\gamma_2 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} kl_1 & km_1 \\ kl_2 & km_2 \\ kl_3 & km_3 \end{bmatrix}.$$

More generally, the product matrix is multiplied by the scalar quantity k when all the elements of one of every pair of corresponding passive rows in two adjacent factor matrices are multiplied by k. This follows from § 52.1 or § 52.2.

Ex. iv.
$$k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & k\beta_1 & \gamma_1 \\ \alpha_2 & k\beta_2 & \gamma_2 \\ \alpha_3 & k\beta_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} kl_1 & km_1 & kn_1 & kp_1 \\ l_2 & m_2 & n_2 & p_2 \\ kl_3 & km_3 & kn_3 & kp_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{bmatrix}.$$

§ 55. Special cases of a product formed by a chain of matrix factors.

1. *One factor a zero matrix.*

If any factor in a chain of standard or general form is a zero matrix, the product matrix is a zero matrix. For each element of the product matrix is a homogeneous linear function of the elements of the zero matrix, and is therefore equal to zero.

2. *One factor a unit matrix.*

If any factor matrix in a product of *standard form* is a unit matrix, it can be omitted. For if P is either of the adjacent factor matrices, we can replace the product of the unit matrix and P by P simply.

A standard product of any number of unit matrices is itself a unit matrix. A general product of any number of unit matrices is conventionally equal to a unit matrix.

Ex. i.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}.$$

Here the product of the first two factors is identical with the first factor. Also the product of the second and third factors is identical with the third factor.

3. *One factor a scalar matrix.*

If any factor matrix in a product of *standard form* is a scalar matrix with argument k, it can be struck out and replaced by the scalar multiplier k operating on the product of the remaining factor matrices or on any one of the remaining factor matrices. It can also be replaced by a scalar matrix with argument k occupying any other position in the chain, the standard form of the chain being preserved.

To prove this it is sufficient to observe that the product of the scalar matrix and either adjacent factor matrix is that factor matrix multiplied by the scalar quantity k.

Any number of scalar matrices with arguments $k_1, k_2, \ldots k_n$ occurring in a product of standard form can be replaced by the scalar multiplier $k_1k_2 \ldots k_n$ if there are other factor matrices in the product.

A standard product of scalar matrices with arguments $k_1, k_2, \ldots k_n$ is a scalar matrix with argument $k_1k_2 \ldots k_n$.

Ex. ii. $$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} a_1 & \beta_1 \\ a_2 & \beta_2 \\ a_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} ka_1 & k\beta_1 \\ ka_2 & k\beta_2 \\ ka_3 & k\beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix} \begin{bmatrix} a_1 & \beta_1 \\ a_2 & \beta_2 \\ a_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}$$

$$= k \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & \beta_1 \\ a_2 & \beta_2 \\ a_3 & \beta_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}.$$

NOTE. *The n! products of n given matrices.*

With n given factor matrices we can form n! chains by arranging them in all possible ways, and we can form n! products, one corresponding to each chain. These n! products are in general all different even when the given matrices are all square and have all the same order.

For example the values of the 6 products

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

are respectively

$$\begin{bmatrix} 11 & 15 \\ 20 & 27 \end{bmatrix}, \quad \begin{bmatrix} 13 & 8 \\ 41 & 25 \end{bmatrix}, \quad \begin{bmatrix} 37 & 8 \\ 5 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 23 & 6 \\ 35 & 9 \end{bmatrix}, \quad \begin{bmatrix} 5 & 6 \\ 23 & 27 \end{bmatrix}, \quad \begin{bmatrix} 31 & 17 \\ 2 & 1 \end{bmatrix}.$$

These are all different.

CHAPTER VII.

DETERMINOID OF A PRODUCT FORMED BY A CHAIN OF MATRIX FACTORS.

[The determinoid of a product of two given matrices is first considered in §§ 56 and 57. Then in §§ 58 and 59 the value of the determinoid of a product of any number of given matrices is found. If the efficiency of the product is η, the determinoid of the product is in general equal to a sum of products of minor determinants of order η formed from the factor matrices in a prescribed manner. Certain special cases are considered in §§ 60 and 62; and in § 63 the determinoid of any product is reduced to the determinoid of a product in which one of the activities is 1. Finally the preceding results are generalised in § 64, where exactly corresponding results are obtained for the algebraical sum of the affected minor determinants of any given order k of a product of given matrices.]

§ 56. Determinoid of a product of two given matrices.

In this chapter and all future chapters a product of any number of matrices will always be expressed in the standard form without redundant passive rows.

Let then $A=[a]_m^r$, $B=[b]_r^n$ be any two given matrices such that the number of vertical rows in A is equal to the number of horizontal rows in B, and let $X=[x]_m^n$ be their product matrix when they are taken in the above order, so that $X = AB$, or more fully

$$[x]_m^n = [a]_m^r\,[b]_r^n.$$

The number r is the passivity of the product, and the numbers m and n are the two activities of the product. The smaller of the two numbers m and n (or either of them if they are equal) will be denoted by η. Then η is the efficiency of the product. We shall proceed to investigate the value of $(x)_m^n$ or det X. The final results which will be obtained can be stated as follows:

Theorem I. *If the passivity of the product AB is less than its efficiency, then*

$$det\ \boldsymbol{AB} = 0.$$

Theorem II. *If the passivity of the product AB is not less than its efficiency, reduce the product AB to the product $A'B'$ of two square matrices of order η by striking out corresponding passive rows in both the factor matrices and active rows in the larger factor matrix; let Δ, Δ' be the determinants of the square matrices A', B', and let ω, ω' be their affects in the matrices A, B from which they are derived; then*

$$det\ AB = \Sigma(-1)^{\omega+\omega'}\Delta\Delta'.$$

Thus det AB can be expressed as an algebraical sum of the products of all possible pairs of determinants of order η derived from A and B respectively in the manner described.

General Formula equivalent to Theorem II.

Let the product $[x]_m^n = [a]_m^r [b]_r^n$ have efficiency η, where $\eta \nless r$. Then

$$(x)_m^n = \Sigma(-1)^{\omega+\omega'}(a_{mr})_\eta^\eta (b_{rn})_\eta^\eta,$$

where $[m_1 m_2 \ldots m_\eta]$, $[r_1 r_2 \ldots r_\eta]$, $[n_1 n_2 \ldots n_\eta]$ are minors of order η of $[1\ 2 \ldots m]$, $[1\ 2 \ldots r]$, $[1\ 2 \ldots n]$ respectively;

$$\omega = \textit{affect of } (a_{mr})_\eta^\eta \textit{ in } [a]_m^r, \qquad \omega' = \textit{affect of } (b_{rn})_\eta^\eta \textit{ in } [b]_r^n,$$

or $$\omega = \textit{affect of } [m]_\eta \textit{ in } [1\ 2 \ldots m], \qquad \omega' = \textit{affect of } [n]_\eta \textit{ in } [1\ 2 \ldots n];$$

and the summation extends over a complete set of distinct values of each of the minor sequences $[m]_\eta$, $[r]_\eta$, $[n]_\eta$.

It will be convenient to divide the investigation into several cases.

CASE I. $m = n = r = \eta$.

In this case A, B and X are all square matrices of order m, and their determinoids are all determinants of order m.

By the definition of the product of two determinants of the same order, we have

$$(a)_m^m (b)_m^m = (x)_m^m.$$

Therefore

$$\det X = (x)_m^m = (a)_m^m (b)_m^m = \det A \times \det B.$$

This result is equivalent to the following theorem:

Theorem A. *If A and B are both square matrices, then*

$$det\ AB = det\ A \times det\ B.$$

CASE II *a*. $\eta = n$, $r < \eta$.

In this case $\qquad m \nless n, \quad r < m, \quad r < n.$

Expanding the determinoid $(x)^n_m$ in terms of its simple minor determinants of order n, we have

$$\det X = (x)^n_m = \sum_p (-1)^\omega (x_{p1})^n_n = \sum_p (-1)^\omega \det [x_{p1}]^n_n \quad \ldots\ldots\ldots(1),$$

where $[p_1\, p_2 \ldots p_n]$ is any corranged minor of order n of $[1\ 2 \ldots m]$, and ω is the affect of $[p_1\, p_2 \ldots p_n]$ in $[1\ 2 \ldots m]$.

Now from the equation $[x]^n_m = [a]^r_m\, [b]^n_r$ it follows by the properties of active rows (see § 53.4), that

$$[x_{p1}]^n_n = [a_{p1}]^r_n\, [b]^n_r.$$

By the properties of passive rows (see § 52.3), we can insert $n-r$ additional final passive rows of 0's in each of the last factor matrices. When this is done, we obtain

$$[x_{p1}]^n_n = \begin{bmatrix} a_{p_1 1} & a_{p_1 2} & \ldots & a_{p_1 r} & 0 \ldots 0 \\ a_{p_2 1} & a_{p_2 2} & \ldots & a_{p_2 r} & 0 \ldots 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{p_n 1} & a_{p_n 2} & \ldots & a_{p_n r} & 0 \ldots 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{r1} & b_{r2} & \ldots & b_{rn} \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \quad \ldots\ldots(2).$$

This is an equation of the form considered in Case I. Applying Theorem A and noticing that the determinants of the square factor matrices on the right in (2) are both zero, we see that

$$\det [x_{p1}]^n_n = 0.$$

Hence from (1) it follows that in this case

$$\det X = \det AB = 0.$$

CASE II b. $\eta = m,\ r < \eta$.

In this case $\quad n \nless m, \quad r < m, \quad r < n.$

Expanding the determinant $(x)^n_m$ in terms of its simple minor determinants of order m, we have

$$\det X = (x)^n_m = \sum_q (-1)^\omega (x_{1q})^m_m = \sum_q (-1)^\omega \det [x_{1q}]^m_m \quad \ldots\ldots\ldots(3),$$

where $[q_1 q_2 \ldots q_m]$ is any corranged minor of order m of $[1\ 2 \ldots n]$, and ω is the affect of $[q_1 q_2 \ldots q_m]$ in $[1\ 2 \ldots n]$.

From the equation $[x]^n_m = [a]^r_m\, [b]^n_r$ it follows by the properties of active rows that

$$[x_{1q}]^m_m = [a]^r_m\, [b_{1q}]^m_r.$$

By the properties of passive rows we can insert $m-r$ additional final passive rows of 0's in each of the factor matrices on the right. They then become square matrices of order m whose determinants vanish, and applying Theorem A we see that

$$\det\,[x_{1q}]_m^m = 0.$$

Substituting this value in (3) we see that in this case

$$\det X = \det AB = 0.$$

The results obtained in Cases II a and II b can be summarised as follows:

Theorem B. *If the passivity of the product AB is less than its efficiency, then*

$$\det AB = 0.$$

Ex. i.

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \epsilon_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \epsilon_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 & \epsilon_3 \end{bmatrix} = 0,$$

or

$$\det\,[abc]_{1234}\,[\alpha\beta\gamma\delta\epsilon]_{123} = 0.$$

Here the passivity is 3 and the efficiency is 4.

Ex. ii. $\det\,[abc]_{12345}\,[\alpha\beta\gamma\delta\epsilon]_{123} = 0.$

Ex. iii. $\det \begin{bmatrix} a \\ b \\ c \end{bmatrix} [\alpha\beta\gamma\delta] = 0.$

CASE III a. $\eta = n,\ r = \eta.$

In this case $m \not< n, \quad r \not> m, \quad r = n.$

As in Case II a,

$$\det X = (x)_m^n = \sum_p (-1)^\omega \det\,[x_{p1}]_n^n = \sum_p (-1)^\omega \det\,[a_{p1}]_n^r\,[b]_r^n,$$

where $[p_1 p_2 \ldots p_n]$ is a corranged minor of $[1\,2 \ldots m]$ having affect ω.

Since $r = n$, we have by Case I

$$\det\,[a_{p1}]_r^n\,[b]_r^n = \det\,[a_{p1}]_n^n\,[b]_n^n = (a_{p1})_n^n\,(b)_n^n.$$

Observing that $(b)_n^n = \det B$ and is constant, and using § 30, we have

$$\det X = \sum_p (-1)^\omega (a_{p1})_n^n (b)_n^n = \left\{\sum_p (-1)^\omega (a_{p1})_n^n\right\} (b)_n^n$$

$$= (a)_m^n\,(b)_n^n = \det A \times \det B.$$

CASE III b. $\eta = m,\ r = \eta$.

In this case $n \nless m, \quad r = m, \quad r \ngtr n$.

As in Case II b,

$$\det X = (x)^n_m = \sum_q (-1)^\omega \det [x_{1q}]^m_m = \sum_q (-1)^\omega \det [a]^r_m [b_{1q}]^m_r,$$

where $[q_1 q_2 \ldots q_m]$ is a corranged minor of $[1\,2 \ldots n]$ having affect ω.

Since $r = m$, we have by Case I

$$\det [a]^r_m [b_{1q}]^m_r = \det [a]^m_m [b_{1q}]^m_m = (a)^m_m (b_{1q})^m_m.$$

Observing that $(a)^m_m = \det A$ and is constant, and using § 30, we have

$$\det X = \sum_q (-1)^\omega (a)^m_m (b_{1q})^m_m = (a)^m_m \sum_q (-1)^\omega (b_{1q})^m_m$$
$$= (a)^m_m (b)^n_m = \det A \times \det B.$$

The results obtained in Cases I, IIIa and IIIb can be summarised as follows:

Theorem C. *If the smaller of the two matrices A and B is a square matrix (or if the passivity of the product is equal to η), then*

$$\det AB = \det A \times \det B.$$

Ex. iv. $$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix},$$

or $$\det [abc]_{123} [\alpha\beta\gamma]_{123} = (abc)_{123} (\alpha\beta\gamma)_{123}.$$

Ex. v. $$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix},$$

or $$\det [abc]_{1234} [\alpha\beta\gamma]_{123} = (abc)_{1234} (\alpha\beta\gamma)_{123}.$$

Ex. vi. $$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{bmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{vmatrix},$$

or $$\det [ab]_{12} [\alpha\beta\gamma\delta]_{12} = (ab)_{12} (\alpha\beta\gamma\delta)_{12}.$$

CASE IV a. $\eta = n,\ r > \eta$.

In this case $m \nless n, \quad r > n$.

Expanding $(x)^n_m$ in terms of its simple minor determinants and making use of the properties of active rows, we have, as in Case IIa

$$\det X = (x)^n_m = \sum_p (-1)^\omega (x_{p1})^n_n = \sum_p (-1)^\omega \det [a_{p1}]^r_n [b]^n_r \quad \ldots\ldots(4),$$

where $[p_1 p_2 \ldots p_n]$ is a corranged minor of $[1\,2 \ldots m]$ having affect ω.

Also by the definition of a product of two matrices

$$(x_{p1})^n_n = \det [a_{p1}]^r_n [b]^n_r$$

$$= \begin{vmatrix} \sum_u a_{p_1 u} b_{u1}, & \sum_u a_{p_1 u} b_{u2}, & \ldots & \sum_u a_{p_1 u} b_{un} \\ \sum_u a_{p_2 u} b_{u1}, & \sum_u a_{p_2 u} b_{u2}, & \ldots & \sum_u a_{p_2 u} b_{un} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_u a_{p_n u} b_{u1}, & \sum_u a_{p_n u} b_{u2}, & \ldots & \sum_u a_{p_n u} b_{un} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots(5),$$

where u receives all integral values from 1 to r.

Expressing the last determinant as a sum of partial determinants we have

$$(x_{p1})^n_n = \sum \begin{vmatrix} a_{p_1 u_1} b_{u_1 1}, & a_{p_1 u_2} b_{u_2 2}, & \ldots & a_{p_1 u_n} b_{u_n n} \\ a_{p_2 u_1} b_{u_1 1}, & a_{p_2 u_2} b_{u_2 2}, & \ldots & a_{p_2 u_n} b_{u_n n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p_n u_1} b_{u_1 1}, & a_{p_n u_2} b_{u_2 2}, & \ldots & a_{p_n u_n} b_{u_n n} \end{vmatrix} = \sum U,$$

where each one of the suffixes $u_1, u_2, \ldots u_n$ receives independently of the others all the values $1, 2, \ldots r$.

If any two of the suffixes $u_1, u_2, \ldots u_n$ are equal, the determinant U vanishes. Hence the last summation may be considered to extend over every possible *arrangement* of n numbers $u_1, u_2, \ldots u_n$ all different selected from the numbers $1, 2, \ldots r$.

Now if $[q_1 q_2 \ldots q_n]$ is any one *corranged* minor of order n of $[1\,2 \ldots r]$, the sum of all the determinants U obtained by choosing for $u_1, u_2, \ldots u_n$ all possible arrangements of the numbers $q_1, q_2, \ldots q_n$ is

$$\begin{vmatrix} \sum_v a_{p_1 v} b_{v1}, & \sum_v a_{p_1 v} b_{v2}, & \ldots & \sum_v a_{p_1 v} b_{vn} \\ \sum_v a_{p_2 v} b_{v1}, & \sum_v a_{p_2 v} b_{v2}, & \ldots & \sum_v a_{p_2 v} b_{vn} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_v a_{p_n v} b_{v1}, & \sum_v a_{p_n v} b_{v2}, & \ldots & \sum_v a_{p_n v} b_{vn} \end{vmatrix} = V$$

where v receives all the values $q_1, q_2, \ldots q_n$.

But by the law of multiplication for determinants

$$V = (a_{pq})^n_n (b_{q1})^n_n.$$

Selecting the corranged minor $[q_1 q_2 \dots q_n]$ of $[1\ 2 \dots m]$ in all possible ways, we have

$$(x_{p1})_n^n = \sum_q V = \sum_q (a_{pq})_n^n (b_{q1})_n^n.$$

Substituting this value in (4), we obtain

$$\det X = \sum_p (-1)^\omega (x_{p1})_n^n = \sum_p \sum_q (-1)^\omega (a_{pq})_n^n (b_{q1})_n^n \quad \dots\dots\dots\dots(6),$$

where the summations with respect to $[p_1 p_2 \dots p_n]$ and $[q_1 q_2 \dots q_n]$ are independent.

Let σ be the affect of $[q_1 q_2 \dots q_n]$ in $[1\ 2 \dots r]$, so that $\omega + \sigma$ is the affect of $(a_{pq})_n^n$ in $[a]_m^r$ or A, and σ is the affect of $(b_{q1})_n^n$ in $[b]_r^n$ or B. Then equation (6) and the equations obtained from it by performing the summations with respect to $[p_1 p_2 \dots p_n]$ only and $[q_1 q_2 \dots q_n]$ only are respectively:

$$\det X = \sum_p \sum_q (-1)^{\omega+\sigma} (a_{pq})_n^n (-1)^\sigma (b_{q1})_n^n \quad \dots\dots\dots\dots(7),$$

$$\det X = \sum_q (a_{1q})_m^n (b_{q1})_n^n \quad \dots\dots\dots\dots(8),$$

$$\det X = \sum_p (-1)^\omega \det [a_{p1}]_n^r [b]_r^n \dots\dots\dots\dots(9).$$

Here $[p_1 p_2 \dots p_n]$ is any corranged minor of order n of $[1\ 2 \dots m]$,

$[q_1 q_2 \dots q_n]$ is any corranged minor of order n of $[1\ 2 \dots r]$,

ω is the affect of $[p_1 p_2 \dots p_n]$ in $[1\ 2 \dots m]$,

σ is the affect of $[q_1 q_2 \dots q_n]$ in $[1\ 2 \dots r]$,

and $\sum_p$, $\sum_q$ mean summations for all such values of $[p_1 p_2 \dots p_n]$, $[q_1 q_2 \dots q_n]$ respectively.

To deduce (8) from (7) we notice that when $[q_1 q_2 \dots q_n]$ remains constant, $(b_{q1})_n^n$ remains constant, whilst by § 30

$$\sum_p (-1)^\omega (a_{pq})_n^n = (a_{1q})_m^n.$$

Equation (9) simply gives the value of det X obtained in (4).

An inspection of equations (7), (8), (9) shows that they can be obtained as follows:

To obtain (8) we may strike out pairs of corresponding passive rows in the product AB till it is reduced to a product $A'B'$ in which B' is a square matrix of order η. When this is done in all possible ways, we have (using Theorem C) as an equation equivalent to (8)

$$\det AB = \Sigma \det A'B' = \Sigma \det A' \times \det B' \quad \dots\dots\dots\dots(8').$$

To obtain (9) we may strike out active rows of A in the product AB till it is reduced to a product $A'B$ in which A' and B are inversely similar matrices of common activity or efficiency η, and denote by ω the affect of A' in A. When this is done in all possible ways, we have as an equation equivalent to (9)

$$\det AB = \Sigma (-1)^{\omega} \det A'B \text{..........................} (9').$$

To obtain (7) we may strike out pairs of corresponding passive rows of A and B and active rows of A in the product AB till it is reduced to a product $A'B'$ in which A' and B' are both square matrices of order η, and denote the affects of A' and B' in A and B by ω and ω' respectively. When this process is carried out in all possible ways, we have as an equation equivalent to (7)

$$\det AB = \Sigma (-1)^{\omega+\omega'} A'B' = \Sigma \{(-1)^{\omega} \det A' . (-1)^{\omega'} \det B'\} \text{...} (7').$$

Equations (7), (8) and (9) are clearly also true in Cases I and IIIa.

CASE IV b. $\eta = m,\ r > \eta$.

In this case $n \nless m, \quad r > m.$

Expanding $(x)^n_m$ in terms of its simple minor determinants and using the properties of active rows, we have as in Case IIb

$$\det X = (x)^n_m = \sum_q (-1)^{\omega} (x_{1q})^m_m = \sum_q (-1)^{\omega} \det [a]^r_m [b_{1q}]^m_r,$$

where $[q_1 q_2 \ldots q_m]$ is a corranged minor of $[1\,2 \ldots n]$ having affect ω.

Evaluating $\det [a]^r_m [b_{1q}]^m_r$ by a method similar to that used in Case IVa, we have

$$\det X = \sum_q \sum_p (-1)^{\sigma} (a_{1p})^m_m (-1)^{\omega+\sigma} (b_{pq})^m_m \text{............} (10),$$

$$\det X = \sum_p (a_{1p})^m_m (b_{p1})^m_m \text{.....................................} (11),$$

$$\det X = \sum_q (-1)^{\omega} \det [a]^r_m [b_{1q}]^m_r \text{........................} (12),$$

where $[q_1 q_2 \ldots q_m]$ is a corranged minor of $[1\,2 \ldots n]$,

$[p_1 p_2 \ldots p_m]$ is a corranged minor of $[1\,2 \ldots r]$,

ω is the affect of $[q_1 q_2 \ldots q_m]$ in $[1\,2 \ldots n]$,

σ is the affect of $[p_1 p_2 \ldots p_m]$ in $[1\,2 \ldots r]$,

and $\sum_p, \sum_q$ denote summations with respect to $[p_1 p_2 \ldots p_m]$ and $[q_1 q_2 \ldots q_m]$.

Equations (10), (11) and (12) are clearly also true in Cases I and IIIb.

The results obtained in Cases IV*a* and IV*b* and given by equations (7), (8), (9), (10), (11), (12) can be summarised as follows:

Theorem D. *If the passivity of the product AB is not less than its efficiency, then $\det AB$ can be evaluated by any one of the following rules:*

RULE I. *Strike out pairs of corresponding passive rows in both factor matrices so as to form an auxiliary product $A'B'$ whose passivity is equal to the efficiency η. Then $\det AB$ is equal to the sum of the determinoids of all such auxiliary products. Since the smaller factor matrix is by this process reduced to a square matrix of order η, the determinoid of each auxiliary product is by Theorem C equal to the product of the determinoids of its factors. Thus*

$$\det AB = \Sigma \det A'B' = \Sigma \{\det A' . \det B'\}.$$

RULE II. *Strike out active rows in the larger factor matrix of the product AB so as to form an auxiliary product $A'B'$ of two inversely similar matrices of common activity and efficiency η. Give to the determinoid of the auxiliary product the sign determined by the affect ω of the reduced matrix in the matrix from which it has been derived. Then $\det AB$ is equal to the algebraical sum of the determinoids of all such auxiliary products. Thus*

$$\det AB = \Sigma (-1)^{\omega} \det A'B'.$$

RULE III. *Strike out corresponding passive rows in both factor matrices and active rows in the larger factor matrix of the product AB so as to reduce it to an auxiliary product $A'B'$ in which both factors are square matrices of order η. Give to the determinoid of the auxiliary product the sign determined by the sum of the affects ω, ω' of A', B' in A, B respectively. Then $\det AB$ is equal to the algebraical sum of the determinoids of all such auxiliary products. By Theorem A the determinoid of each auxiliary product is equal to the product of the determinants of its factors. Accordingly*

$$\det AB = \Sigma (-1)^{\omega+\omega'} \det A'B' = \Sigma \{(-1)^{\omega} \det A' . (-1)^{\omega'} \det B'\}.$$

The above rules may be made applicable to the case in which the passivity of the product is less than its efficiency if we understand that wherever there is no row to strike out, a row of 0's is to be inserted. With this understanding they can be applied to all products whatever of two matrices.

The final result given by Rule III is usually more conveniently obtained by using the result given by Rule I or that given by Rule II as an intermediate result.

In equations (7), (9), (10), (12) the minor sequences $[p]_r, [q]_r$ can clearly be deranged in any manner in each term. The summation then extends over a complete set of values of $[p]_r$ which are such that no two are simply derangements of one another and over a complete set of values of $[q]_r$ which are such that no two are simply derangements of one another. Consequently the general result given by Rule III of Theorem D can be expressed in the form given in the general formula following Theorem II.

Ex. vii. If $X=[abc]_{12}[\alpha\beta\gamma]_{123}=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$,

then by Rule I we have

$$\det X=\det\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}\begin{bmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}+\det\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$$
$$+\det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix}$$
$$=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\cdot\begin{vmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}+\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\cdot\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}+\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\cdot\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix}$$
$$=(bc)_{12}(\alpha\beta\gamma)_{23}+(ac)_{12}(\alpha\beta\gamma)_{13}+(ab)_{12}(\alpha\beta\gamma)_{12}.$$

If we expand the determinoids occurring in the last expression by § 30, we obtain

$$(\alpha\beta\gamma)_{pq}=(\alpha\beta)_{pq}-(\alpha\gamma)_{pq}+(\beta\gamma)_{pq}.$$

where $[pq]=[2\,3]$ or $[1\,3]$ or $[1\,2]$, and therefore

$$\begin{aligned}\det X= & \ (bc)_{12}(\alpha\beta)_{23}-(bc)_{12}(\alpha\gamma)_{23}+(bc)_{12}(\beta\gamma)_{23}\\ & +(ac)_{12}(\alpha\beta)_{13}-(ac)_{12}(\alpha\gamma)_{13}+(ac)_{12}(\beta\gamma)_{13}\\ & +(ab)_{12}(\alpha\beta)_{12}-(ab)_{12}(\alpha\gamma)_{12}+(ab)_{12}(\beta\gamma)_{12}.\end{aligned}$$

This is the result which Rule III gives.

Ex. viii. If $X=[abc]_{1234}[\alpha\beta]_{123}=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$,

then by Rule II we have

$$\det X=\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}-\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$$
$$+\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_4 & b_4 & c_4 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}+\det\begin{bmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$$
$$-\det\begin{bmatrix} a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}+\det\begin{bmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$$
$$\begin{aligned}= & \det[abc]_{12}[\alpha\beta]_{123}-\det[abc]_{13}[\alpha\beta]_{123}+\det[abc]_{14}[\alpha\beta]_{123}\\ & +\det[abc]_{23}[\alpha\beta]_{123}-\det[abc]_{24}[\alpha\beta]_{123}+\det[abc]_{34}[\alpha\beta]_{123}.\end{aligned}$$

If we now apply Rule I to each term, we have such results as

$$\det [abc]_{12} [a\beta]_{123} = (ab)_{12} (a\beta)_{12} + (ac)_{12} (a\beta)_{13} + (bc)_{12} (a\beta)_{23},$$

$$\det [abc]_{13} [a\beta]_{123} = (ab)_{13} (a\beta)_{12} + (ac)_{12} (a\beta)_{13} + (bc)_{13} (a\beta)_{23}.$$

Substituting these results, we have

$$\begin{aligned} \det X = \; & (a\beta)_{12} \{(ab)_{12} - (ab)_{13} + (ab)_{14} + (ab)_{23} - (ab)_{24} + (ab)_{34}\} \\ & + (a\beta)_{13} \{(ac)_{12} - (ac)_{13} + (ac)_{14} + (ac)_{23} - (ac)_{24} + (ac)_{34}\} \\ & + (a\beta)_{23} \{(bc)_{12} - (bc)_{13} + (bc)_{14} + (bc)_{23} - (bc)_{24} + (bc)_{34}\}. \end{aligned}$$

This is the result which Rule III gives.

§ 57. Determinoid of a product of two matrices in certain special cases.

1. *One factor a unit matrix.*

If one of the factor matrices is a unit matrix, it can be omitted.

For if in the product AB of two matrices the first factor matrix A is a unit matrix, then $AB = B$, and therefore $\det AB = \det B$; and if the second factor matrix B is a unit matrix, then $AB = A$, and therefore $\det AB = \det A$.

Ex. i. $$\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

or $$\det [1]_3^3 [ab]_{123} = (ab)_{123}.$$

Ex. ii. $$\det [abc]_{1234} [1]_3^3 = \det [abc]_{1234} = (abc)_{1234}.$$

2. *One factor a scalar matrix.*

If one of the factor matrices in the product AB is a scalar matrix with argument h, the determinoid of the product is the determinoid of the other factor matrix multiplied by h^η, where η is the efficiency of the product.

For the scalar matrix can be removed if each element of the other factor is multiplied by h. Thus if A is the scalar matrix,

$$\det AB = \det hB = h^\eta \det B;$$

and if B is the scalar matrix,

$$\det AB = \det hA = h^\eta \det A.$$

$$Ex. \text{ iii. } \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{bmatrix} = \det \begin{bmatrix} ha_1 & hb_1 & hc_1 \\ ha_2 & hb_2 & hc_2 \end{bmatrix} = h^2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

3. *Product of two inversely similar matrices.*

Let $[a]_m^r [b]_r^m$ be a product of two inversely similar matrices, or a product in which the two activities are equal. When $r > m$ let $\alpha_1, \alpha_2, \ldots \alpha_\mu$ be the simple minor determinants of $[a]_m^r$ formed by the omission of passive or vertical rows, and let $\beta_1, \beta_2, \ldots \beta_\mu$ be the corresponding simple minor determinants of $[b]_r^m$. Applying the results of § 56 to a product of this special kind we obtain the following theorem.

Theorem. (1) *If $r < m$, then*

$$det\ [a]_m^r [b]_r^m = 0.$$

(2) *If $r < m$, then*

$$det\ [a]_m^r [b]_r^m = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_\mu\beta_\mu,$$

i.e.

$$det\ [a]_m^r [b]_r^m = det\,[\alpha_1\, \alpha_2 \ldots \alpha_\mu] \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_\mu \end{bmatrix}.$$

In the latter formulae the minor determinants $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ may be corranged or deranged: also they may be affected or unaffected.

$$Ex. \text{ iv. } \det \begin{bmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} l_2 & l_3 \\ m_2 & m_3 \\ n_2 & n_3 \end{bmatrix} = |ab|_{23}\,|lm|_{23} + |ac|_{23}\,|ln|_{23} + |bc|_{23}\,|mn|_{23}$$

$$= \det\,[A_1 B_1 C_1] \begin{bmatrix} L_1 \\ M_1 \\ N_1 \end{bmatrix},$$

where A_1, B_1, C_1 and L_1, M_1, N_1 are the co-factors of a_1, b_1, c_1 and l_1, m_1, n_1 in

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} \quad \text{and} \quad \begin{matrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{matrix} \quad \text{respectively.}$$

§ 58. Determinoid of a product of any number of given matrices.

Let [illegible]

[illegible]

Theorem I. [illegible]

[illegible]

Theorem II. *In any given pair of adjacent factor matrices of the chain strike out pairs of corresponding passive rows until the number of corresponding passive rows in each of those matrices is reduced to η, and let X be thereby reduced to X'. If this is done in all possible ways, det X is the sum of all such terms as det X'.*

In this way the determinoid of the original product X can be expressed as a sum of determinoids of auxiliary products obtained from X by striking out pairs of corresponding passive rows in any selected pair of adjacent factor matrices.

To prove this theorem, let $D=[d]_\gamma^\delta$ and $E=[e]_\delta^\epsilon$ be the two selected adjacent factor matrices. When the product DE is reduced to a product of passivity η by striking out pairs of corresponding passive rows in its two factor matrices, let D and E become respectively $D'=[d_{1u}]_\gamma^\eta$, $E'=[e_{u1}]_\eta^\epsilon$, so that DE becomes

$$D'E'=[d_{1u}]_\gamma^\eta\,[e_{u1}]_\eta^\epsilon,$$

where $[u_1\, u_2 \ldots u_\eta]$ is any corranged minor of order η of $[1\,2 \ldots \delta]$.

Further let

$$AB \ldots D=[p]_m^\delta=P, \qquad E \ldots ST=[q]_\delta^n=Q,$$

so that by the properties of active rows (see § 53.4)

$$AB \ldots D'=[p_{1u}]_m^\eta=P', \qquad E' \ldots ST=[q_{u1}]_\eta^n=Q'.$$

Then $\quad X=AB \ldots DE \ldots ST=PQ, \qquad X'=AB \ldots D'E' \ldots ST=P'Q'.$

Now from the equation $X=PQ$ it follows by Rule I of § 56 that

$$\det X=\sum_u \det\,[p_{1u}]_m^\eta\,[q_{u1}]_\eta^n=\sum_u \det P'Q'=\sum_u \det X',$$

and this establishes the theorem.

Ex. ii.
$$\begin{aligned}&\det\,[abcd]_{12}[\alpha\beta\gamma]_{1234}[lmnp]_{123}[\lambda\mu\nu]_{1234}\\ =\;&\det\,[abcd]_{12}[\alpha\beta]_{1234}\;[lmnp]_{12}\;[\lambda\mu\nu]_{1234}\\ &+\det\,[abcd]_{12}[\alpha\gamma]_{1234}\;[lmnp]_{13}\;[\lambda\mu\nu]_{1234}\\ &+\det\,[abcd]_{12}[\beta\gamma]_{1234}\;[lmnp]_{23}\;[\lambda\mu\nu]_{1234}.\end{aligned}$$

Here the efficiency is 2 and we have reduced the two middle factor matrices. The theorem can be again applied to each term of the last sum so as to reduce the first two matrices or the last two matrices.

Theorem III. *In that extreme factor matrix of the chain which has the larger activity strike out active rows so as to reduce its activity to η, and let X be thereby reduced to X'. If this is done in all possible ways, det X is the sum of all such terms as $(-1)^\omega$ det X', where ω is the affect of the reduced extreme matrix in the original extreme matrix.*

In this way the determinoid of the original product X can be expressed as an algebraical sum of determinoids of products obtained from X by striking out active rows in the extreme factor matrix with the larger activity.

First suppose that $m > n$, so that the first factor matrix A has a larger activity than the last factor matrix T, and $\eta = n$.

Let $A' = [a_{u1}]_\eta^\alpha$ be any matrix formed from A by striking out $m-n$ horizontal rows, so that

$[u_1 u_2 \dots u_\eta]$ is any corranged minor of order η of $[1\,2 \dots m]$,

and let ω be the affect of A' in A or of $[u_1 u_2 \dots u_\eta]$ in $[1\,2 \dots m]$.

Further let $$B \dots DE \dots ST = [q]_\alpha^n = Q.$$

Then $X = AB \dots DE \dots ST = AQ, \quad X' = A'B \dots DE \dots ST = A'Q.$

From the equation $X = AQ$, it follows by Rule II of § 56 that

$$\det X = \sum_u (-1)^\omega \det [a_{u1}]_\eta^\alpha [q]_\alpha^n = \sum_u (-1)^\omega A'Q = \sum_u (-1)^\omega \det X'.$$

Next suppose that $n > m$, so that T has a larger activity than A, and $\eta = m$.

Let $T' = [t_{1r}]_\sigma^\eta$ be any matrix formed from T by striking out $n-m$ vertical rows, and let ω be the affect of T' in T. Further let

$$AB \dots DE \dots S = [p]_m^\sigma = P.$$

Then $X = AB \dots DE \dots ST = PT, \quad X' = AB \dots DE \dots ST' = PT'.$

From the equation $X = PT$, it follows by Rule II of § 56 that

$$\det X = \sum_r (-1)^\omega \det PT' = \sum_r (-1)^\omega \det X'.$$

Ex. iii. $\det [abcd]_{12} [\alpha\beta\gamma]_{1234} [lmnp]_{123} [\lambda\mu\nu]_{1234}$

$= \det [abcd]_{12} [\alpha\beta\gamma]_{1234} [lmnp]_{123} [\lambda\mu]_{1234}$

$- \det [abcd]_{12} [\alpha\beta\gamma]_{1234} [lmnp]_{123} [\lambda\nu]_{1234}$

$+ \det [abcd]_{12} [\alpha\beta\gamma]_{1234} [lmnp]_{123} [\mu\nu]_{1234}$.

Here the efficiency is 2 and we reduce the last factor matrix, which has the larger activity.

Each term in the last sum can be further expanded by Theorem II.

Theorem IV. *If the common passivity of any two consecutive factor matrices D, E of the chain is equal to the efficiency η, so that these two matrices contain exactly η pairs of corresponding passive rows, then* det X *is equal to* *the determinoid of the product of all factor matrices of the chain from the first up to D multiplied by the determinoid of the product of all factor matrices of the chain from E to the last, i.e.*

$$\det AB \ldots DE \ldots ST = (\det AB \ldots D) \times \det (E \ldots ST).$$

In this case $\delta = \eta$.

Let $AB \ldots D = [p]_m^\delta = [p]_m^\eta = P, \quad E \ldots ST = [q]_\delta^n = [q]_\eta^n = Q,$

so that $$X = [x]_m^n = [p]_m^\eta [q]_\eta^n = PQ.$$

Now PQ is a product of two matrices whose passivity and activity are both equal to η, for η is the smaller of the two numbers m and n. Consequently PQ is a product of two matrices in which the smaller factor matrix is a square matrix.

It follows by Theorem C of § 56 that

$$\det X = \det PQ = \det P \times \det Q = (\det AB \ldots D) \times (\det E \ldots ST).$$

Ex. iv. $$\det \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{bmatrix}$$

$$= \det \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{bmatrix} . \det \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{bmatrix}.$$

Ex. v. $$\det [abc]_{123} [\alpha\beta\gamma\delta]_{123} [lmn]_{1234} [\lambda\mu\nu]_{123}$$

$$= (abc)_{123} . \det [\alpha\beta\gamma\delta]_{123} [lmn]_{1234} [\lambda\mu\nu]_{123}$$

$$= (abc)_{123} . \det [\alpha\beta\gamma\delta]_{123} [lmn]_{1234} . (\lambda\mu\nu)_{123}.$$

Theorem V. *If any factor matrix H of the chain is a square matrix of order η, η being the efficiency of the chain, it can be replaced by its determinant regarded as a factor multiplying the determinoid of all the other factor matrices of the chain,*

i.e. $$\det X = \det X' . \det H.$$

where X' is obtained from the chain by striking out the factor matrix H.

If H is one of the inner factor matrices of the chain, the theorem can also be expressed in the form

$$\det X = \det P . \det H . \det Q = \det PQ . \det H$$

where P is the product of all the factor matrices which precede H in the chain, and Q is the product of all those which follow H in the chain.

If H is one of the extreme factor matrices, the theorem follows immediately from Theorem IV, for H is then one of a pair of adjacent factor matrices whose common passivity is η. This case is illustrated by Ex. v.

If H is one of the inner factor matrices of the chain, let

$$X = AB \dots DHE \dots ST$$

$$= [a]_m^\alpha [b]_\alpha^\beta \dots [d]_\gamma^\eta [h]_\eta^\eta [e]_\eta^\epsilon \dots [s]_\rho^\sigma [t]_\sigma^n = [x]_m^n.$$

Let $\qquad P = AB \dots D = [p]_m^\eta, \qquad Q = E \dots ST = [q]_\eta^n,$

so that

$$X = [p]_m^\eta [h]_\eta^\eta [q]_\eta^n = PHQ.$$

Then by Theorem IV

$$\det X = \det P \,.\, \det HQ, \quad \det HQ = \det H \,.\, \det Q, \quad \det PQ = \det P \,.\, \det Q.$$

Therefore

$$\det X = \det P \,.\, \det H \,.\, \det Q = (\det P \,.\, \det Q) \,.\, \det H$$

$$= \det PQ \,.\, \det H = \det X' \,.\, \det H.$$

Ex. vi. $\qquad \det [abc]_{12} [\alpha\beta]_{123} [lm]_{12} [\lambda\mu\nu]_{12}$

$$= \det [abc]_{12} [\alpha\beta]_{123} \,.\, \det [lm]_{12} \,.\, \det [\lambda\mu\nu]_{12}$$

$$= \det [abc]_{12} [\alpha\beta]_{123} \,.\, |\lambda\mu\nu|_{12} \,.\, |lm|_{12}$$

$$= \det [abc]_{12} [\alpha\beta]_{123} [\lambda\mu\nu]_{12} \,.\, |lm|_{12}.$$

Theorem VI. *Let all the factor matrices of the chain be reduced to square matrices of order η by striking out corresponding passive rows in adjacent factor matrices and active rows in the extreme factor matrix which has the greater activity. Let the product of the determinants of these square matrices be formed when each determinant has prefixed to it the sign determined by its affect in the matrix from which it has been derived. Then $\det X$ is the algebraical sum of all such products of derived determinants of order η.*

In forming each term in the expression for $\det X$ we may as an alternative give to the determinantal factor derived from the extreme factor matrix with the larger activity the sign determined by the affect of the sequence of the retained active rows in the sequence of the original active rows, and give to every other determinantal factor the positive sign.

To prove the theorem, let

$$X = [a]_m^\alpha [b]_\alpha^\beta [c]_\beta^\gamma \dots [s]_\rho^\sigma [t]_\sigma^n = ABC \dots ST.$$

By repeated applications of Theorem II, we obtain

$$\det X = \Sigma \det [a_{1\alpha}]^{\eta}_{m} [b_{\alpha\beta}]^{\eta}_{\eta} [c_{\beta\gamma}]^{\eta}_{\eta} \dots [s_{\rho\sigma}]^{\eta}_{\eta} [t_{\sigma 1}]^{n}_{\eta},$$

where $[\alpha_1\alpha_2 \dots \alpha_\eta]$ is any corranged minor of order η of $[1\,2 \dots \alpha]$,
$[\beta_1\beta_2 \dots \beta_\eta]$ is any corranged minor of order η of $[1\,2 \dots \beta]$,
$[\gamma_1\gamma_2 \dots \gamma_\eta]$ is any corranged minor of order η of $[1\,2 \dots \gamma]$,
....
$[\sigma_1\sigma_2 \dots \sigma_\eta]$ is any corranged minor of order η of $[1\,2 \dots \sigma]$,

and the summation extends over all possible values of these minor sequences. Since either m or n is equal to η, it follows by Theorems IV and V that

$$\det X = \Sigma \det [a_{1\alpha}]^{\eta}_{m} [t_{\sigma 1}]^{n}_{\eta} \,.\, (b_{\alpha\beta})^{\eta}_{\eta} (c_{\beta\gamma})^{\eta}_{\eta} \dots (s_{\rho\sigma})^{\eta}_{\eta}$$
$$= \Sigma (a_{1\alpha})^{\eta}_{m} (t_{\sigma 1})^{n}_{\eta} \,.\, (b_{\alpha\beta})^{\eta}_{\eta} (c_{\beta\gamma})^{\eta}_{\eta} \dots (s_{\rho\sigma})^{\eta}_{\eta}$$
$$= \Sigma (a_{1\alpha})^{\eta}_{m} (b_{\alpha\beta})^{\eta}_{\eta} (c_{\beta\gamma})^{\eta}_{\eta} \dots (s_{\rho\sigma})^{\eta}_{\eta} (t_{\sigma 1})^{n}_{\eta} \quad \dots\dots\dots\dots(1).$$

FIRST CASE. *Let* $\eta = n$, *so that* $m \nless \eta$ *and* $(t_{\sigma 1})^{n}_{\eta} = (t_{\sigma 1})^{\eta}_{\eta}$.

Then by § 30

$$(a_{1\alpha})^{\eta}_{m} = \Sigma (-1)^{\omega} (a_{m\alpha})^{\eta}_{\eta},$$

where $[m_1 m_2 \dots m_\eta]$ is any corranged minor of order η of $[1\,2 \dots m]$, and ω is the affect of $[m_1 m_2 \dots m_\eta]$ in $[1\,2 \dots m]$, which is also the vertical affect of $(a_{m\alpha})^{\eta}_{\eta}$ in $[a]^{a}_{m}$.

Substituting this value in (1), we have

$$\det X = \Sigma (-1)^{\omega} (a_{m\alpha})^{\eta}_{\eta} (b_{\alpha\beta})^{\eta}_{\eta} (c_{\beta\gamma})^{\eta}_{\eta} \dots (s_{\rho\sigma})^{\eta}_{\eta} (t_{\sigma 1})^{\eta}_{\eta} \quad \dots\dots(2),$$

where now the summation extends over all values of the minor sequences

$$[m_1 m_2 \dots m_\eta],\ [\alpha_1\alpha_2 \dots \alpha_\eta],\ [\beta_1\beta_2 \dots \beta_\eta], \dots [\sigma_1\sigma_2 \dots \sigma_\eta].$$

This proves the alternative form of the theorem for the present case.

Let ω_α = the affect of $[\alpha_1\alpha_2 \dots \alpha_\eta]$ in $[1\,2 \dots \alpha]$,
ω_β = the affect of $[\beta_1\beta_2 \dots \beta_\eta]$ in $[1\,2 \dots \beta]$,
....
ω_σ = the affect of $[\sigma_1\sigma_2 \dots \sigma_\eta]$ in $[1\,2 \dots \sigma]$.

Then from (2)

$$\det X = \Sigma (-1)^{\omega + \omega_\alpha} A' \,.\, (-1)^{\omega_\alpha + \omega_\beta} B' \,.\, (-1)^{\omega_\beta + \omega_\gamma} C' \dots (-1)^{\omega_\sigma} T' \quad \dots\dots\dots(3),$$

where $A' = (a_{m\alpha})^{\eta}_{\eta},\quad B' = (b_{\alpha\beta})^{\eta}_{\eta},\quad C' = (c_{\beta\gamma})^{\eta}_{\eta}, \dots T' = (t_{\sigma 1})^{\eta}_{\eta}$.

Since $\omega + \omega_\alpha$, $\omega_\alpha + \omega_\beta$, $\omega_\beta + \omega_\gamma$, ... ω_σ are respectively the affects of A' in A, B' in B, C' in C, ... T' in T, this proves the first form of the theorem for the present case.

SECOND CASE. *Let* $\eta = m$, *so that* $n \nless \eta$ *and* $(a_{1\alpha})^{\eta}_{m} = (a_{1\alpha})^{\eta}_{\eta}$.

Then $$(t_{\sigma 1})^{n}_{\eta} = \Sigma (-1)^{\omega'} (t_{\sigma n})^{\eta}_{\eta},$$

where $[n_1 n_2 \dots n_\eta]$ is any corranged minor of order η of $[1\,2 \dots n]$, and ω' is the affect of $[n_1\, n_2 \dots n_\eta]$ in $[1\,2 \dots n]$.

Substituting this value in (1), we have

$$\det X = \Sigma (-1)^{\omega'} (a_{1\alpha})^{\eta}_{\eta} (b_{\alpha\beta})^{\eta}_{\eta} (c_{\beta\gamma})^{\eta}_{\eta} \dots (s_{\rho\sigma})^{\eta}_{\eta} (t_{\sigma n})^{\eta}_{\eta} \quad \dots\dots\dots(4),$$

where now the summation extends over all values of the minor sequences

$$[\alpha_1\, \alpha_2 \dots \alpha_\eta],\quad [\beta_1\, \beta_2 \dots \beta_\eta],\quad \dots\quad [\sigma_1\, \sigma_2 \dots \sigma_\eta],\quad [n_1\, n_2 \dots n_\eta].$$

This proves the alternative form of the theorem for this second case.

Let $\omega_\alpha, \omega_\beta, \omega_\gamma, \dots \omega_\sigma$ be defined as in the first case. Then from (4)

$$\det X = \Sigma (-1)^{\omega_\alpha} A' . (-1)^{\omega_\alpha + \omega_\beta} B' . (-1)^{\omega_\beta + \omega_\gamma} C' \dots (-1)^{\omega_\sigma + \omega'} T' \dots (5),$$

where $$A' = (a_{1\alpha})^{\eta}_{\eta},\quad B' = (b_{\alpha\beta})^{\eta}_{\eta},\quad C' = (c_{\beta\gamma})^{\eta}_{\eta}, \dots\ T' = (t_{\sigma n})^{\eta}_{\eta}.$$

Equation (5) is equivalent to the first form of the theorem for this second case.

Both forms of the theorem have now been proved for all cases.

The values of the determinoid of any product

$$[x]^{n}_{m} = [a]^{\alpha}_{m} [b]^{\beta}_{\alpha} [c]^{\gamma}_{\beta} \dots [s]^{\sigma}_{\rho} [t]^{n}_{\sigma}$$

given by Theorem VI can conveniently be embodied in the two general formulae which follow:

Formula A. *The alternative form of Theorem VI is equivalent to the formula*

$$(x)^{n}_{m} = \Sigma (-1)^{\omega + \omega'} (a_{m\alpha})^{\eta}_{\eta} (b_{\alpha\beta})^{\eta}_{\eta} (c_{\beta\gamma})^{\eta}_{\eta} \dots (s_{\rho\sigma})^{\eta}_{\eta} (t_{\sigma n})^{\eta}_{\eta},$$

where $[m_1 m_2 \dots m_\eta]$, $[\alpha_1 \alpha_2 \dots \alpha_\eta]$, $[\beta_1 \beta_2 \dots \beta_\eta]$, ... $[n_1 n_2 \dots n_\eta]$ *are respectively corranged minors of order* η *of the sequences*

$$[1\,2 \dots m],\quad [1\,2 \dots \alpha],\quad [1\,2 \dots \beta],\quad \dots\quad [1\,2 \dots n],$$

and ω, ω' *are the affects of* $[m]_\eta$, $[n]_\eta$ *in* $[1\,2 \dots m], [1\,2 \dots n]$.

If $\eta = m$, we have

$$[m]_\eta = [1\,2 \dots m],\quad (a_{m\alpha})^{\eta}_{\eta} = (a_{1\alpha})^{\eta}_{\eta},\quad \text{and}\quad \omega = 0.$$

If $\eta = n$, we have

$$[n]_\eta = [1\,2 \dots n],\quad (t_{\sigma n})^{\eta}_{\eta} = (t_{\sigma 1})^{\eta}_{\eta},\quad \text{and}\quad \omega' = 0.$$

Thus in all cases one of the two quantities ω, ω' is zero.

Formula B. *The corresponding formula for the first form of Theorem VI is*

$$(x)^n_m = \Sigma (-1)^{\omega_a} (a_{m\alpha})^\eta_\eta \,.\, (-1)^{\omega_b} (b_{\alpha\beta})^\eta_\eta \,.\, (-1)^{\omega_c} (c_{\beta\gamma})^\eta_\eta \ldots (-1)^{\omega_t} (t_{\sigma n})^\eta_\eta,$$

where $\omega_a, \omega_b, \ldots \omega_t$ *are the affects of* $(a_{m\alpha})^\eta_\eta, (b_{\alpha\beta})^\eta_\eta, \ldots (t_{\sigma n})^\eta_\eta$ *in*

$$[a]^\alpha_m, \quad [b]^\beta_\alpha, \ \ldots \ [t]^n_\sigma.$$

Note. These formulae have been proved on the supposition that the minor sequences $[m]_\eta, [\alpha]_\eta, [\beta]_\eta, \ldots [n]_\eta$ are corranged. The summation then extends over all possible values of such minor sequences. But from § 52. 8 it follows that the minor sequences $[\alpha]_\eta, [\beta]_\eta, \ldots [\sigma]_\eta$ can be subjected to any derangements whatever in Formula A; and from Theorem V *a* of § 25 it follows that the minor sequences $[m]_\eta, [n]_\eta$ can be subjected to any derangements whatever in Formula A. Also from Theorem V *a* of § 25 it follows that in Formula B all the minor sequences can be subjected to any derangements whatever. Consequently in both formulae all the minor sequences can be subjected to any derangements whatever.

When the minor sequences are deranged, the summation with respect to $[\alpha]_\eta$ must extend over a complete set of values of that minor sequence, no two of which are simply derangements of one another, and similarly for the other minor sequences.

Ex. vii. Using Formula A, we have

$$\det [a]^4_3 [b]^3_4 [c]^2_3 = \Sigma (-1)^{\omega + \omega'} (a_{m\alpha})^2_2 (b_{\alpha\beta})^2_2 (c_{\beta n})^2_2$$
$$= \Sigma (-1)^{\omega} (a_{m\alpha})^2_2 (b_{\alpha\beta})^2_2 (c_{\beta 1})^2_2.$$

This is equivalent to

$$\det [a]^4_3 [b]^3_4 [c]^2_3 = \Sigma (-1)^\omega \begin{vmatrix} a_{pu} & a_{pv} \\ a_{qu} & a_{qv} \end{vmatrix} . \begin{vmatrix} b_{ux} & b_{uy} \\ b_{vx} & b_{vy} \end{vmatrix} . \begin{vmatrix} c_{x1} & c_{x2} \\ c_{y1} & c_{y2} \end{vmatrix},$$

where $[pq]$ is any corranged minor of order 2 of [1 2 3],
$[uv]$ is any corranged minor of order 2 of [1 2 3 4],
$[xy]$ is any corranged minor of order 2 of [1 2 3],
ω is the affect of $[pq]$ in [1 2 3].

Since there are 3 possible values of $[pq]$, 6 possible values of $[uv]$, and 3 possible values of $[xy]$, there are altogether 54 terms in the sum.

Ex. viii. $\det [a]^4_3 [b]^5_4 [c]^5_5 [d]^4_5 = \Sigma (-1)^{\omega'} (a_{1\alpha})^3_3 (b_{\alpha\beta})^3_3 (c_{\beta\gamma})^3_3 (d_{\gamma n})^3_3,$

where $[\alpha_1\alpha_2\alpha_3]$, $[\beta_1\beta_2\beta_3]$, $[\gamma_1\gamma_2\gamma_3]$, $[n_1n_2n_3]$ are respectively any corranged minors of order 3 of [1 2 3 4], [1 2 3 4 5], [1 2 3 4 5], [1 2 3 4], and ω is the affect of $[n_1n_2n_3]$ in [1 2 3 4].

The number of terms in the sum is $4 \times 10 \times 10 \times 4$ or 1600.

Ex. ix. If the two activities of the product are equal, then in Formula A, ω and ω' are both zero. In this case we may write

$$X = [x]^\eta_\eta = [a]^\alpha_\eta [b]^\beta_\alpha [c]^\gamma_\beta \ldots [s]^\sigma_\rho [t]^\eta_\sigma,$$
$$\det X = (x)^\eta_\eta = \Sigma (a_{1\alpha})^\eta_\eta (b_{\alpha\beta})^\eta_\eta (c_{\beta\gamma})^\eta_\eta \ldots (s_{\rho\sigma})^\eta_\eta (t_{\sigma 1})^\eta_\eta.$$

The minor sequences $[\alpha_1\alpha_2 \ldots \alpha_\eta]$, ... $[\sigma_1\sigma_2 \ldots \sigma_\eta]$ may be all either corranged or deranged, and $\det X$ is a *determinant* of order η.

Ex. x. If $$[x]_m^n = [a]_m^a [b]_a^\beta [c]_\beta^\gamma [d]_\gamma^n,$$

we have by the properties of active rows

$$[x_{ij}] = [a_{i1} a_{i2} \ldots a_{ia}] [b]_a^\beta [c]_\beta^\gamma \begin{bmatrix} d_{1j} \\ d_{2j} \\ \vdots \\ d_{\gamma j} \end{bmatrix}.$$

Therefore $$x_{ij} = \det [x_{ij}] = \Sigma a_{iu} b_{uv} c_{vw} d_{wj},$$

where u receives the values 1, 2, ... a; v receives the values 1, 2, ... β; w receives the values 1, 2, ... γ.

This is the expression for x_{ij} given by § 51.

§ 59. Progressive development of the determinoid of a product of any number of given matrices.

If $X = ABC \ldots ST$ is any standard product of given matrices whose efficiency is η, the full expansion of $\det X$ in terms of products of minor determinants of order η can be obtained directly by means of Theorem VI and one of Formulae A and B in § 58. This method has already been illustrated in Exs. vii and viii of § 58.

It is usually more convenient to develop the expansion systematically, step by step, by repeated applications of Theorems I—V of § 58. The most straightforward plan is to commence by reducing the extreme factor matrix which has the smaller activity to a square matrix of order η in all possible ways by omissions of corresponding passive rows in it and the next matrix; then to reduce the next matrix to a square matrix of order η in all possible ways by omissions of corresponding passive rows in it and the matrix next to it; and so to reduce each of the successive factor matrices in turn till the other extreme factor matrix is reached; and finally to reduce this extreme matrix to a square matrix of order η in all possible ways by omissions of active rows.

An alternative plan is to commence by reducing the extreme factor matrix which has the large activity to one whose activity is η in all possible ways by omissions of active rows. Then $\det X$ is expressed as the algebraical sum of a number of *determinants* of auxiliary products, each of which can be evaluated systematically as before starting from *either end* of the corresponding auxiliary chain.

Ex. i. Let $$X=[abcd]_{123}[\alpha\beta\gamma]_{1234}[lm]_{123}.$$

Replacing the last factor matrix by square matrices of order 2 and using Theorems II and V of § 58, we have

$$\begin{aligned}\det X = &\ \det[abcd]_{123}[\beta\gamma]_{1234}[lm]_{23}+\det[abcd]_{123}[\alpha\gamma]_{1234}[lm]_{13}\\ &+\det[abcd]_{123}[\alpha\beta]_{1234}[lm]_{12}\\ = &\ \det[abcd]_{123}[\beta\gamma]_{1234}.(lm)_{23}+\det[abcd]_{123}[\alpha\gamma]_{1234}.(lm)_{13}\\ &+\det[abcd]_{123}[\alpha\beta]_{1234}.(lm)_{12}.\end{aligned}$$

This is the first step in the process of expanding $\det X$. In it we have expressed $\det X$ as a sum of terms of the form $\det ABC=\det AB.\det C$, where C is a square matrix. The second step consists in now replacing the second matrix from the end in each term by square matrices of order 2. Using the same theorems as before, we have

$$\det X=S_1.(lm)_{23}+S_2.(lm)_{13}+S_3.(lm)_{12},$$

where
$$\begin{aligned}S = &\ \det[abcd]_{123}[\beta\gamma]_{1234}\\ = &\ \det[ab]_{123}[\beta\gamma]_{12}+\det[ac]_{123}[\beta\gamma]_{13}+\det[ad]_{123}[\beta\gamma]_{14}\\ &+\det[bc]_{123}[\beta\gamma]_{23}+\det[bd]_{123}[\beta\gamma]_{24}+\det[cd]_{123}[\beta\gamma]_{34}\\ = &\ (ab)_{123}(\beta\gamma)_{12}+(ac)_{123}(\beta\gamma)_{13}+(ad)_{123}(\beta\gamma)_{14}\\ &+(bc)_{123}(\beta\gamma)_{23}+(bd)_{123}(\beta\gamma)_{24}+(cd)_{123}(\beta\gamma)_{34},\\ S_2 = &\ (ab)_{123}(\alpha\gamma)_{12}+(ac)_{123}(\alpha\gamma)_{13}+(ad)_{123}(\alpha\gamma)_{14}\\ &+(bc)_{123}(\alpha\gamma)_{23}+(bd)_{123}(\alpha\gamma)_{24}+(cd)_{123}(\alpha\gamma)_{34},\\ S_3 = &\ (ab)_{123}(\alpha\beta)_{12}+(ac)_{123}(\alpha\beta)_{13}+(ad)_{123}(\alpha\beta)_{14}\\ &+(bc)_{123}(\alpha\beta)_{23}+(bd)_{123}(\alpha\beta)_{24}+(cd)_{123}(\alpha\beta)_{34}.\end{aligned}$$

In this second step we have expressed $\det X$ as a sum of 18 terms of the form $\det ABC=\det A.\det B.\det C$, where B and C are square matrices of order 2. The third step consists now in replacing the third matrix from the end in each term by square matrices of order 2. Since the third step is the last step, this is done by omissions of active rows. Using equations of the type

$$(ab)_{123}=(ab)_{12}-(ab)_{13}+(ab)_{23},$$

we have finally

$$\begin{aligned}\det X = &\ \{(ab)_{12}-(ab)_{13}+(ab)_{23}\}\{(\beta\gamma)_{12}(lm)_{23}+(\alpha\gamma)_{12}(lm)_{13}+(\alpha\beta)_{12}(lm)_{12}\}\\ &+\{(ac)_{12}-(ac)_{13}+(ac)_{23}\}\{(\beta\gamma)_{13}(lm)_{23}+(\alpha\gamma)_{13}(lm)_{13}+(\alpha\beta)_{13}(lm)_{12}\}\\ &+\{(ad)_{12}-(ad)_{13}+(ad)_{23}\}\{(\beta\gamma)_{14}(lm)_{23}+(\alpha\gamma)_{14}(lm)_{13}+(\alpha\beta)_{14}(lm)_{12}\}\\ &+\{(bc)_{12}-(bc)_{13}+(bc)_{23}\}\{(\beta\gamma)_{23}(lm)_{23}+(\alpha\gamma)_{23}(lm)_{13}+(\alpha\beta)_{23}(lm)_{12}\}\\ &+\{(bd)_{12}-(bd)_{13}+(bd)_{23}\}\{(\beta\gamma)_{24}(lm)_{23}+(\alpha\gamma)_{24}(lm)_{13}+(\alpha\beta)_{24}(lm)_{12}\}\\ &+\{(cd)_{12}-(cd)_{13}+(cd)_{23}\}\{(\beta\gamma)_{34}(lm)_{23}+(\alpha\gamma)_{34}(lm)_{13}+(\alpha\beta)_{34}(lm)_{12}\}.\end{aligned}$$

We have now expressed $\det X$ as the sum of 54 terms of the form

$$\pm\det ABC=\pm\det A.\det B.\det C,$$

where A, B, C are square matrices of order 2, and $\det A$, $\det B$, $\det C$ are determinants of order 2.

Ex. ii. Let $X=[abcd]_{123}[\alpha\beta\gamma]_{1234}[lm]_{123}$.

Replacing the first factor matrix by matrices with activity 2 and using Theorem III of § 58, we have

$$\det X=\det[abcd]_{12}[\alpha\beta\gamma]_{1234}[lm]_{123}-\det[abcd]_{13}[\alpha\beta\gamma]_{1234}[lm]_{123}$$
$$+\det[abcd]_{23}[\alpha\beta\gamma]_{1234}[lm]_{123}.$$

Thus $\det X$ is expressed as the sum of three *determinants*, each of which can be evaluated by the process pursued in Ex. i.

§ 60. Determinoid of a product in which one of the factors is a scalar matrix.

Let $X=ABC\ldots T=[a]_m^\alpha[b]_\alpha^\beta[c]_\beta^\gamma\ldots[t]_\sigma^n=[x]_m^n$ be any standard product of a number of given matrices in which the efficiency is η. Then in determining $\det X$ we can use the following results:

1. *Unit matrix.* If any factor matrix is a unit matrix, it can be simply omitted, and we have

$$\det X=\det X'$$

where X' is the product of the remaining factors of the chain.

For by § 55.2, $X=X'$; and since X and X' are similar matrices, it follows that $\det X=\det X'$.

2. *Scalar matrix.* If any factor matrix of the chain is a scalar matrix with argument h, it can be replaced by h^η multiplying the determinoid of the product X' of the remaining factor matrices, and we have

$$\det X=h^\eta\det X'.$$

For by § 55.3, $X=hX'$. Since X and X' are similar and X' has therefore efficiency η, it follows that $\det X=\det hX'=h^\eta\det X'$.

Clearly the scalar factor matrix can be removed to any new position in the chain provided that its order is so adjusted that the chain remains normal in form.

3. *Numerical factor.* If any factor matrix of the chain has the numerical or scalar factor h, we can show similarly that

$$\det X=h^\eta\det X'$$

where X' is the product which remains after the numerical factor h has been removed from the chain.

In fact the numerical factor h occurring anywhere in the chain can be replaced by a scalar matrix with argument h and *vice versâ*.

Ex. i. $\det [abc]_{12} [\alpha\beta\gamma\delta]_{123} [1]_1^1 [lm]_{1234} = \det [abc]_{12} [\alpha\beta\gamma\delta]_{123} [lm]_{1234}$.

Ex. ii.
$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \end{bmatrix} \begin{bmatrix} h & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{bmatrix} \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \\ l_3 & m_3 \\ l_4 & m_4 \end{bmatrix}$$
$$= \det h [abc]_{12} [\alpha\beta\gamma\delta]_{123} [lm]_{12} = h^2 \det [abc]_{12} [\alpha\beta\gamma\delta]_{123} [lm]_{12}.$$

§ 61. Expressions for the determinant of a product of any number of given matrices in which the two activities are equal.

We shall confine ourselves in the present article to products of four matrices, but the method adopted will be general in character and the results obtained can be at once extended to products of any number of matrices.

Let then
$$X = [a]_\eta^r [b]_r^s [c]_s^t [d]_t^\eta = ABCD = [x]_\eta^\eta$$
be a standard product of four factor matrices in which both extreme factor matrices have activity η, and no one of the passivities r, s, t is less than η. It will be observed that in this case $\det X$ is a *determinant*.

Referring to Formula A of § 58, we see that it has already been proved that
$$\det X = \Sigma\, (a_{1r})_\eta^\eta\, (b_{rs})_\eta^\eta\, (c_{st})_\eta^\eta\, (d_{t1})_\eta^\eta \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$
where $[r_1 r_2 \ldots r_\eta]$ is any minor of order η of $[1\ 2 \ldots r]$,
$[s_1 s_2 \ldots s_\eta]$ is any minor of order η of $[1\ 2 \ldots s]$,
$[t_1 t_2 \ldots t_\eta]$ is any minor of order η of $[1\ 2 \ldots t]$.

These minors may be either corranged or deranged, and if we consider that two minors of the same sequence are distinct when one is not simply a derangement of the other, the summation extends over all distinct values of these minor matrices.

The minor sequences $[r_1 r_2 \ldots r_\eta]$, $[s_1 s_2 \ldots s_\eta]$, $[t_1 t_2 \ldots t_\eta]$ have respectively ρ, σ and τ distinct values, where
$$\rho = \binom{r}{\eta}, \quad \sigma = \binom{s}{\eta}, \quad \tau = \binom{t}{\eta}.$$

Let ρ distinct values of the minor sequence $[r_1 r_2 \ldots r_\eta]$ arranged in any order be called the 1st, 2nd, ... uth, ... ρth values of that sequence.

Let σ distinct values of the minor sequence $[s_1 s_2 \dots s_\eta]$ arranged in any order be called the 1st, 2nd, ... vth, ... σth values of that sequence.

Let τ distinct values of the minor sequence $[t_1 t_2 \dots t_\eta]$ arranged in any order be called the 1st, 2nd, ... wth, ... τth values of that sequence.

When $[r_1 r_2 \dots r_\eta]$, $[s_1 s_2 \dots s_\eta]$, $[t_1 t_2 \dots t_\eta]$ have respectively their uth, vth and wth values, let $(a_{1r})_\eta^\eta$, $(b_{rs})_\eta^\eta$, $(c_{st})_\eta^\eta$, $(d_{t1})_\eta^\eta$ be denoted respectively by α_{1u}, β_{uv}, γ_{vw}, δ_{w1}.

Then

$$\det X = \sum_u \sum_v \sum_w \alpha_{1u} \beta_{uv} \gamma_{vw} \delta_{w1} \quad \dots\dots\dots\dots\dots\dots\dots (B),$$

where u, v, w are independent of one another, and

u receives all the values $1, 2, \dots \rho$,
v receives all the values $1, 2, \dots \sigma$,
w receives all the values $1, 2, \dots \tau$.

Now by § 51 we have

$$[\alpha]_1^\rho [\beta]_\rho^\sigma [\gamma]_\sigma^\tau [\delta]_\tau^1 = [e],$$

where

$$e = \sum_u \sum_v \sum_w \alpha_{1u} \beta_{uv} \gamma_{vw} \delta_{w1}.$$

Therefore

$$\det X = \det [\alpha]_1^\rho [\beta]_\rho^\sigma [\gamma]_\sigma^\tau [\delta]_\tau^1 = \det A'B'C'D' \quad \dots\dots\dots (C),$$

or

$$\det X = \det [\alpha_{11}\ \alpha_{12} \dots \alpha_{1\rho}] \begin{bmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1\sigma} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2\sigma} \\ \dots & \dots & \dots & \dots \\ \beta_{\rho 1} & \beta_{\rho 2} & \dots & \beta_{\rho\sigma} \end{bmatrix} \begin{bmatrix} \gamma_{11} & \gamma_{12} & \dots & \gamma_{1\tau} \\ \gamma_{21} & \gamma_{22} & \dots & \gamma_{2\tau} \\ \dots & \dots & \dots & \dots \\ \gamma_{\sigma 1} & \gamma_{\sigma 2} & \dots & \gamma_{\sigma\tau} \end{bmatrix} \begin{bmatrix} \delta_{11} \\ \delta_{21} \\ \vdots \\ \delta_{\tau 1} \end{bmatrix} \quad \dots\dots\dots (D).$$

We have thus expressed det $ABCD$ in the form det $A'B'C'D'$, where A', B', C', D' are matrices whose elements are the derived determinants of order η which can be formed from the matrices A, B, C, D respectively, and where each of the extreme factor matrices A', D' contains only one active row.

We may regard (D), or its equivalent (C), as a concise way of expressing the results of Theorem VI in § 58.

To explain more thoroughly the formation of the formula (D) we will introduce some further terms.

The minor matrix formed from A by retaining only the r_1th, r_2th, ... r_ηth vertical rows and arranging them in this order, and the minor matrix formed from B by retaining only the r_1th, r_2th, ... r_ηth horizontal rows and arranging them in this order will be called respectively the vertical minor of A and the horizontal minor of B corresponding to the sequence $[r_1 r_2 \dots r_\eta]$. Then A has ρ vertical minors and B has ρ horizontal minors corresponding one by one to the ρ values of $[r_1 r_2 \dots r_\eta]$.

Again the minor matrix formed from B by retaining only the s_1th, s_2th, ... s_ηth vertical rows and arranging them in this order, and the minor matrix formed from C by retaining only the s_1th, s_2th, ... s_ηth horizontal rows and arranging them in this order will be called respectively the vertical minor of B and the horizontal minor of C corresponding to the sequence $[s_1 s_2 \dots s_\eta]$. Then B has σ vertical minors and C has σ horizontal minors corresponding one by one to the σ values of $[s_1 s_2 \dots s_\eta]$.

Similarly C has τ vertical minors and D has τ horizontal minors corresponding one by one to the τ values of $[t_1 t_2 \dots t_\eta]$.

All these minors are inferior simple minor matrices of reduced order η.

Then α_{1u} is the determinant formed by the uth vertical minor of A, β_{uv} is the determinant formed by the intersection of the uth horizontal minor and the vth vertical minor of B, γ_{vw} is the determinant formed by the intersection of the vth horizontal minor and the wth vertical minor of C, and δ_{w1} is the determinant formed by the wth horizontal minor of D.

The elements of the uth horizontal row in B' are the simple minor determinants belonging to the uth horizontal minor of B, and the elements of the vth vertical row in B' are the simple minor determinants belonging to the vth vertical minor of B. Similarly the elements of the vth horizontal row in C' and the elements of the wth vertical row in C' are respectively the simple minor determinants belonging to the vth horizontal minor and the wth vertical minor of C.

The choice of the successive minor matrices of the factor matrices in forming equation (D) is not entirely arbitrary. For the successive horizontal minors of any one factor matrix must correspond to the successive vertical minors of the preceding factor matrix.

If

$R_1, R_2, \dots R_\rho$ are the 1st, 2nd, ... ρth values of the sequence $[r_1 r_2 \dots r_\eta]$,

$S_1, S_2, \dots S_\sigma$ are the 1st, 2nd, ... σth values of the sequence $[s_1 s_2 \dots s_\eta]$,

$T_1, T_2, \dots T_\tau$ are the 1st, 2nd, ... τth values of the sequence $[t_1 t_2 \dots t_\eta]$,

we may say that the vertical rows of A and the horizontal rows of B follow the scheme $(R_1, R_2, \dots R_\rho)$, the vertical rows of B and the horizontal rows of C follow the scheme $(S_1, S_2, \dots S_\sigma)$, and the vertical rows of C and the horizontal rows of D follow the scheme $(T_1, T_2, \dots T_\tau)$ as regards their orders of arrangement. The schemes $(R_1, R_2, \dots R_\rho)$, $(S_1, S_2, \dots S_\sigma)$, $(T_1, T_2, \dots T_\tau)$ can be chosen arbitrarily.

Since by Formula B of § 58 the formula (A) above remains true when every one of the determinantal factors has prefixed to it the sign determined by its affect in the matrix from which it is derived, the same is true of the formulae (B), (C) and (D). This can be also seen by applying the results contained in § 52.7 and § 53.9.

Ex. i. If

$$X = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ l_4 & m_4 & n_4 & p_4 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \\ \lambda_4 & \mu_4 & \nu_4 \end{bmatrix} = ABCD,$$

then $\det X = \det A'B'C'D'$, where

$$A' = \left[(abc)_{123},\ (abd)_{123},\ (acd)_{123},\ (bcd)_{123}\right],$$

$$B' = \begin{bmatrix} (\beta\gamma\delta)_{123} & (\alpha\gamma\delta)_{123} & (\alpha\beta\delta)_{123} & (\alpha\beta\gamma)_{123} \\ (\beta\gamma\delta)_{124} & (\alpha\gamma\delta)_{124} & (\alpha\beta\delta)_{124} & (\alpha\beta\gamma)_{124} \\ (\beta\gamma\delta)_{134} & (\alpha\gamma\delta)_{134} & (\alpha\beta\delta)_{134} & (\alpha\beta\gamma)_{134} \\ (\beta\gamma\delta)_{234} & (\alpha\gamma\delta)_{234} & (\alpha\beta\delta)_{234} & (\alpha\beta\gamma)_{234} \end{bmatrix},$$

$$C' = \begin{bmatrix} (lnp)_{234} & (lmp)_{234} & (lmn)_{234} & (mnp)_{234} \\ (lnp)_{134} & (lmp)_{134} & (lmn)_{134} & (mnp)_{134} \\ (lnp)_{124} & (lmp)_{124} & (lmn)_{124} & (mnp)_{124} \\ (lnp)_{123} & (lmp)_{123} & (lmn)_{123} & (mnp)_{123} \end{bmatrix}, \quad D' = \begin{bmatrix} (\lambda\mu\nu)_{134} \\ (\lambda\mu\nu)_{124} \\ (\lambda\mu\nu)_{123} \\ (\lambda\mu\nu)_{234} \end{bmatrix}.$$

Here the successive arrangements of the vertical rows of A, B, C, which may be chosen arbitrarily, follow the schemes

(1 2 3, 1 2 4, 1 3 4, 2 3 4), (2 3 4, 1 3 4, 1 2 4, 1 2 3), (1 3 4, 1 2 4, 1 2 3, 2 3 4)

respectively; also the minor determinants are unaffected.

Ex. ii. If $X = [lmnp]_{234} [abcd]_{1234} [a'b'c'd']_{1234} [l'm'n']_{1234}$

$$= \begin{bmatrix} l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ l_4 & m_4 & n_4 & p_4 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{bmatrix} a_1' & b_1' & c_1' & d_1' \\ a_2' & b_2' & c_2' & d_2' \\ a_3' & b_3' & c_3' & d_3' \\ a_4' & b_4' & c_4' & d_4' \end{bmatrix} \begin{bmatrix} l_1' & m_1' & n_1' \\ l_2' & m_2' & n_2' \\ l_3' & m_3' & n_3' \\ l_4' & m_4' & n_4' \end{bmatrix},$$

then $\det X = \det [LMNP]_1 [ABCD]_{1234} [A'B'C'D']_{1234} [P']_{1234}$

$$= \det [L_1 M_1 N_1 P_1] \begin{bmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{bmatrix} \begin{bmatrix} A_1' & B_1' & C_1' & D_1' \\ A_2' & B_2' & C_2' & D_2' \\ A_3' & B_3' & C_3' & D_3' \\ A_4' & B_4' & C_4' & D_4' \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \\ P_3' \\ P_4' \end{bmatrix},$$

where $[LMNP]_{1234}$, $[ABCD]_{1234}$, $[A'B'C'D']_{1234}$, $[L'M'N'P']_{1234}$ are the reciprocals of $[lmnp]_{1234}$, $[abcd]_{1234}$, $[a'b'c'd']_{1234}$, $[l'm'n'p']_{1234}$ respectively.

Here the successive arrangements of the vertical rows of the first three factor matrices follow the scheme (2 3 4, 1 3 4, 1 2 4, 1 2 3) in each case, and the minor determinants are affected, i.e. they have prefixed to them the signs determined by their affects in the matrices from which they are derived.

If the arrangements of the vertical rows of the first three factor matrices follow the schemes (2 3 4, 1 3 4, 1 2 4, 1 2 3), (1 3 4, 1 2 4, 1 2 3, 2 3 4), (1 2 4, 1 2 3, 2 3 4, 1 3 4) respectively, we obtain the formula

$$\det X = \det [LMNP]_1 [BCDA]_{1234} [C'D'A'B']_{2341} [P']_{3412}$$

$$= \det [L_1 M_1 N_1 P_1] \begin{bmatrix} B_1 & C_1 & D_1 & A_1 \\ B_2 & C_2 & D_2 & A_2 \\ B_3 & C_3 & D_3 & A_3 \\ B_4 & C_4 & D_4 & A_4 \end{bmatrix} \begin{bmatrix} C_2' & D_2' & A_2' & B_2' \\ C_3' & D_3' & A_3' & B_3' \\ C_4' & D_4' & A_4' & B_4' \\ C_1' & D_1' & A_1' & B_1' \end{bmatrix} \begin{bmatrix} P_3' \\ P_4' \\ P_1' \\ P_2' \end{bmatrix}.$$

§ 62. Determinant of a product of three matrices in which the two activities are equal.

We consider in this article a product of the form

$$X = [a]_\eta^r \, [b]_r^s \, [c]_s^\eta = ABC,$$

where the two activities and the efficiency are both equal to η. Such products are of very common occurrence. It will be assumed that neither r nor s is less than η, so that $\det X$ is not necessarily zero.

In this case the expansion of $\det X$ can be effected with peculiar ease, if we note that the expansion is a homogeneous linear function of the derived determinants of order η of each of the three factor matrices.

First Method. One method of proceeding is the following:

Select any corranged derived determinant α of order η belonging to the first factor matrix and any corranged derived determinant γ of order η belonging to the last factor matrix. Retain in the middle matrix only those rows which correspond to the rows of the first and last matrix retained in α and γ, and let β be the determinant of order η formed by these last retained rows. Then

$$\det X = \Sigma \alpha\beta\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

where all the terms are obtained by making all possible selections of α and γ.

We also have

$$\det X = \Sigma (-1)^{\omega_1} \alpha \,.\, (-1)^{\omega_2} \beta \,.\, (-1)^{\omega_3} \gamma \quad \ldots\ldots\ldots\text{(B)},$$

where ω_1, ω_2, ω_3 are the affects of α, β, γ in A, B, C.

Second Method. We may also proceed as follows:

Select any corranged derived determinant β of order η belonging to the middle matrix. Retain in the two extreme matrices only those passive rows which correspond to the rows of the middle matrix retained in β. Let α be the determinant formed by the rows retained in the first matrix, and let γ be the determinant formed by the rows retained in the last matrix. Then

$$\det X = \Sigma \alpha\beta\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{A}'),$$

where all the terms are obtained by selecting β in all possible ways. Defining ω_1, ω_2, ω_3 as before we also have

$$\det X = \Sigma (-1)^{\omega_1} \alpha \,.\, (-1)^{\omega_2} \beta \,.\, (-1)^{\omega_3} \gamma \quad \ldots\ldots\ldots(\text{B}').$$

When formula (B) or (B′) is used it is clearly quite immaterial in both cases whether the derived determinants α, β, γ are corranged or deranged.

Ex. i. $$\det [x \; y \; z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

If we use the first method, the determinoid consists of terms in

$$x^2,\quad xy,\quad xz,\quad yx,\quad y^2,\quad yz,\quad zx,\quad zy,\quad z^2$$

with co-factors $\quad a,\quad h,\quad g,\quad h,\quad b,\quad f,\quad g,\quad f,\quad c\quad$ respectively.

If we use the second method, the determinoid consists of terms in

$$a,\quad h,\quad g,\quad h,\quad b,\quad f,\quad g,\quad f,\quad c$$

with co-factors $\quad x^2,\quad xy,\quad xz,\quad yx,\quad y^2,\quad yz,\quad zx,\quad zy,\quad z^2\quad$ respectively.

The product of the three matrices is here a matrix with only one element, and we can obtain the above result immediately by direct multiplication. For we have

$$[x \; y \; z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x \; y \; z] \begin{bmatrix} ax + hy + gz \\ hx + by + fz \\ gx + fy + cz \end{bmatrix}$$

$$= [x(ax + hy + gz) + y(hx + by + fz) + z(gx + fy + cz)]$$

$$= [ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy] = [e],$$

and $$\det [e] = e = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

Ex. ii. $$\det [x \; y \; 1] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

Ex. iii. $$\det [x \; y \; z \; 1] \begin{bmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d.$$

Ex. iv. $$\det[l_1\ m_1\ n_1]\begin{bmatrix} a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix} l_2\\ m_2\\ n_2\end{bmatrix} = \det[l_2\ m_2\ n_2]\begin{bmatrix} a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix} l_1\\ m_1\\ n_1\end{bmatrix}$$

$$= al_1l_2+bm_1m_2+cn_1n_2+f(m_1n_2+m_2n_1)+g(n_1l_2+n_2l_1)+h(l_1m_2+l_2m_1).$$

The equality in the first line can be regarded as a consequence of the fact that the product matrix is in this case a matrix with only one element and is therefore necessarily self-conjugate.

Ex. v. $$\det[x\ y\ 1]\begin{bmatrix} a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix} x'\\ y'\\ 1\end{bmatrix} = \det[x'\ y'\ 1]\begin{bmatrix} a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix} x\\ y\\ 1\end{bmatrix}$$

$$=axx'+h(xy'+x'y)+byy'+g(x+x')+f(y+y')+c.$$

Ex. vi. $$\det\begin{bmatrix} l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{bmatrix}\begin{bmatrix} a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix} l_2 & l_3\\ m_2 & m_3\\ n_2 & n_3\end{bmatrix}$$

$$=AL_1^2+BM_1^2+CN_1^2+2FM_1N_1+2GN_1L_1+2HL_1M_1,$$

where $\begin{bmatrix} A & H & G\\ H & B & F\\ G & F & C\end{bmatrix}$, $\begin{bmatrix} L_1 & M_1 & N_1\\ L_2 & M_2 & N_2\\ L_3 & M_3 & N_3\end{bmatrix}$ are the reciprocals of $\begin{bmatrix} a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}$, $\begin{bmatrix} l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{bmatrix}$,

so that $$L_1=(mn)_{23},\quad M_1=-(ln)_{23}=(nl)_{23},\quad N_1=(lm)_{23}.$$

Here if we apply the second method and use equation (B′), we see that the determinoid consists of terms in

$$A,\quad H,\quad G,\quad H,\quad B,\quad F,\quad G,\quad F,\quad C,$$

with the co-factors

$$L_1^2,\quad L_1M_1,\quad L_1N_1,\quad M_1L_1,\quad M_1^2,\quad M_1N_1,\quad N_1L_1,\quad N_1M_1,\quad N_1^2.$$

If we apply the first method and use equation (A), it consists of terms in

$$L_1^2,\quad -L_1M_1,\quad L_1N_1,\quad -M_1L_1,\quad M_1^2,\quad -M_1N_1,\quad N_1L_1,\quad -N_1M_1,\quad N_1^2,$$

with the co-factors

$$A,\quad -H,\quad G,\quad -H,\quad B,\quad -F,\quad G,\quad -F,\quad C.$$

Ex. vii. The result of Ex. vi is obtained more easily from formula (D) of § 61. If we give to the minor determinants the signs determined by their affects, we obtain at once

$$\det\begin{bmatrix} l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{bmatrix}\begin{bmatrix} a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix} l_2 & l_3\\ m_2 & m_3\\ n_2 & n_3\end{bmatrix}$$

$$=\det\left[(mn)_{23},\ -(ln)_{23},\ (lm)_{23}\right]\begin{bmatrix} A & H & G\\ H & B & F\\ G & F & C\end{bmatrix}\begin{bmatrix} (mn)_{23}\\ -(ln)_{23}\\ (lm)_{23}\end{bmatrix}$$

$$= \det[L_1 M_1 N_1] \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix} \begin{bmatrix} L_1 \\ M_1 \\ N_1 \end{bmatrix}$$

$$= AL_1^2 + BM_1^2 + CN_1^2 + 2FM_1N_1 + 2GN_1L_1 + 2HL_1M_1.$$

If the minor determinants are unaffected, we obtain

$$\det \begin{bmatrix} l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} l_2 & l_3 \\ m_2 & m_3 \\ n_2 & n_3 \end{bmatrix}$$

$$= \det\left[(mn)_{23}, \quad (ln)_{23}, \quad (lm)_{23}\right] \begin{bmatrix} A, & -H, & G \\ -H, & B, & -F \\ G, & -F, & C \end{bmatrix} \begin{bmatrix} (mn)_{23} \\ (ln)_{23} \\ (lm)_{23} \end{bmatrix}$$

$$= \det[L_1, \ -M_1, \ N_1] \begin{bmatrix} A, & -H, & G \\ -H, & B, & -F \\ G, & -F, & C \end{bmatrix} \begin{bmatrix} L_1 \\ -M_1 \\ N_1 \end{bmatrix}.$$

If now we use § 52. 7 and change the signs of all elements in the second pair of corresponding passive rows in the first two matrices, and also the signs of all elements in the second pair of corresponding passive rows in the last two matrices, we obtain the previous result.

Ex. viii.
$$\det \begin{bmatrix} l_3 & m_3 & n_3 \\ l_1 & m_1 & n_1 \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix}$$

$$= \det \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} l_3 & l_1 \\ m_3 & m_1 \\ n_3 & n_1 \end{bmatrix}$$

$$= \det[L_2 M_2 N_2] \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix} \begin{bmatrix} L_3 \\ M_3 \\ N_3 \end{bmatrix} = \det[L_3 M_3 N_3] \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix} \begin{bmatrix} L_2 \\ M_2 \\ N_2 \end{bmatrix}$$

$$= AL_2L_3 + BM_2M_3 + CN_2N_3 + F(M_2N_3 + M_3N_2) + G(N_2L_3 + N_3L_2) + H(L_2M_3 + L_3M_2),$$

where the notation is the same as in Ex. vi, and therefore

$$L_2 = (mn)_{31}, \quad M_2 = (nl)_{31}, \quad N_2 = (lm)_{31}, \quad L_3 = (mn)_{12}, \quad M_3 = (nl)_{12}, \quad N_3 = (lm)_{12}.$$

Ex. ix. If
$$X = \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \\ p_1 & p_2 \end{bmatrix} \quad \text{.....................(1)},$$

then

$$\det X = \det[(\lambda\mu), (\lambda\nu), (\mu\nu)] \times \begin{bmatrix} (ab)_{12}, & (ac)_{12}, & (ad)_{12}, & (bc)_{12}, & (bd)_{12}, & (cd)_{12} \\ (ab)_{13}, & (ac)_{13}, & (ad)_{13}, & (bc)_{13}, & (bd)_{13}, & (cd)_{13} \\ (ab)_{23}, & (ac)_{23}, & (ad)_{23}, & (bc)_{23}, & (bd)_{23}, & (cd)_{23} \end{bmatrix} \begin{bmatrix} (lm) \\ (ln) \\ (lp) \\ (mn) \\ (mp) \\ (np) \end{bmatrix} \ldots (2),$$

where $(\lambda\mu)$ stands for $(\lambda\mu)_{12}$, (lm) for $(lm)_{12}$, and so on.

This follows at once from formula (D) of § 61. We can also obtain the result by the methods of the present article. For we know that $\det X$ is a homogeneous linear function both of $(\lambda\mu)$, $(\lambda\nu)$, $(\mu\nu)$ and of (lm), (ln), (lp), (mn), (mp), (np), and therefore we can write

$$\det X = \det[(\lambda\mu), (\lambda\nu), (\mu\nu)] \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{bmatrix} \begin{bmatrix} (lm) \\ (ln) \\ (lp) \\ (mn) \\ (mp) \\ (np) \end{bmatrix} \ldots\ldots\ldots (3),$$

where it remains to determine the middle matrix.

To determine e_{24}, we notice that in the expansion of (3) the only term in which e_{24} occurs is $(\lambda\nu)\, e_{24}\, (mn)$, and this is the only term in which both $(\lambda\nu)$ and (mn) occur as factors. Again in the expansion of $\det X$, the only term in which both $(\lambda\nu)$ and (mn) occur as factors is $(\lambda\nu)\,(bc)_{13}\,(mn)$. We have therefore

$$(\lambda\nu)\,(mn)\, e_{24} = (\lambda\nu)\,(mn)\,(bc)_{13}, \quad \text{or} \quad e_{24} = (bc)_{13}.$$

Determining all the elements of the matrix $[e]_3^6$ in this way, we obtain the result (2).

Ex. x. *Expansion of a bordered determinant.*

If
$$\Delta = \begin{vmatrix} z_0 & x_1 & x_2 & \ldots & x_m \\ y_1 & a_{11} & a_{12} & \ldots & a_{1m} \\ y_2 & a_{21} & a_{22} & \ldots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_m & a_{m1} & a_{m2} & \ldots & a_{mm} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1m} & y_1 \\ a_{21} & a_{22} & \ldots & a_{2m} & y_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mm} & y_m \\ x_1 & x_2 & \ldots & x_m & z_0 \end{vmatrix},$$

we can show that
$$\Delta = z_0\,(a)_m^m - \det[x_1 x_2 \ldots x_m]\, \overline{\underline{A}}_m^m \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

where $[A]_m^m$ *is the reciprocal of* $[a]_m^m$.

Taking Δ in the first form and expanding it in terms of the elements of the leading horizontal row, we have

$$\Delta = z_0 (a)^m_m + \sum_j (-1)^j x_j \begin{vmatrix} y_1 & a_{11} \ldots a_{1,j-1} & a_{1,j+1} \ldots a_{1m} \\ y_2 & a_{21} \ldots a_{2,j-1} & a_{2,j+1} \ldots a_{2m} \\ \cdots & \cdots & \cdots \\ y_m & a_{m1} \ldots a_{m,j-1} & a_{m,j+1} \ldots a_{mm} \end{vmatrix} = z_0 (a)^m_m + \sum_j (-1)^j x_j \Delta_j .$$

Now the co-factor of y_i in Δ_j is $(-1)^{j-1} A_{ij}$, and therefore

$$\Delta_j = (-1)^{j-1} \sum_i y_i A_{ij} .$$

We have therefore $\qquad \Delta = z_0 (a)^m_m - \sum\sum x_j y_i A_{ij} ,$

and this is equivalent to the result given above.

Ex. xi. Putting $z_0 = 0$ in Ex. x, we have

$$\begin{vmatrix} 0 & x_1 & x_2 \ldots x_m \\ y_1 & a_{11} & a_{12} \ldots a_{1m} \\ y_2 & a_{21} & a_{22} \ldots a_{2m} \\ \cdots & \cdots & \cdots \\ y_m & a_{m1} & a_{m2} \ldots a_{mm} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \ldots a_{1m} & y_1 \\ a_{21} & a_{22} \ldots a_{2m} & y_2 \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} \ldots a_{mm} & y_m \\ x_1 & x_2 \ldots x_m & 0 \end{vmatrix} = -\det [x_1 x_2 \ldots x_m] \overline{\underline{A}}^m_m \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} .$$

A generalisation of this result is given in § 116.

Ex. xii. *Expansion of a bordered determinoid.*

If

$$\Delta = \begin{vmatrix} z_0 & x_1 & x_2 \ldots x_n \\ y_1 & a_{11} & a_{12} \ldots a_{1n} \\ y_2 & a_{21} & a_{22} \ldots a_{2n} \\ \cdots & \cdots & \cdots \\ y_m & a_{m1} & a_{m2} \ldots a_{mn} \end{vmatrix} , \qquad \textit{where } m < n,$$

we can show that

$$\Delta = z_0 (a)^n_m - (-1)^m \begin{vmatrix} x_1 & x_2 \ldots x_n \\ a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \cdots & \cdots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{vmatrix} - \det [x_1 x_2 \ldots x_n] \overline{\underline{A}}^m_n \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} ,$$

where $[A]^n_m$ *is the reciprocal of* $[a]^n_m$.

When we expand Δ in terms of the elements of its leading long row, we obtain

$$\Delta = z_0 (a)^n_m + \sum_j (-1)^j x_j \begin{vmatrix} y_1 & a_{11} & a_{12} \ldots a_{1,j-1} & a_{1,j+1} \ldots a_{1n} \\ y_2 & a_{21} & a_{22} \ldots a_{2,j-1} & a_{2,j+1} \ldots a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_m & a_{m1} & a_{m2} \ldots a_{m,j-1} & a_{m,j+1} \ldots a_{mn} \end{vmatrix}$$

$$= z_0 (a)^n_m + \sum_j (-1)^j x_j \Delta_j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let Y_i be the co-factor of y_i in Δ_j, and let X_j be the co-factor of x_j in the determinoid formed from Δ by striking out the first vertical row. Then

$$Y_i = (-1)^{j-1} A_{ij} .$$

Expanding Δ_j in terms of the elements of its first vertical or short row, we obtain by § 29

$$\Delta_j = \sum_i y_i Y_i + \Delta_j' = (-1)^{j-1} \sum_i y_i A_{ij} + \Delta_j',$$

where

$$\Delta_j' = \begin{vmatrix} 0 & a_{11} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1n} \\ 0 & a_{21} & \dots & a_{2,j-1} & a_{2,j+1} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_{m1} & \dots & a_{m,j-1} & a_{m,j+1} & \dots & a_{mn} \end{vmatrix} = (-1)^m \begin{vmatrix} a_{11} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2,j-1} & a_{2,j+1} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{m,j-1} & a_{m,j+1} & \dots & a_{mn} \end{vmatrix}$$

$$= (-1)^{m+j-1} X_j.$$

Now substituting for Δ_j in (4), we obtain

$$\Delta = z_0 (a)_m^n + (-1)^m \sum x_j X_j - \sum\sum x_j y_i A_{ij},$$

and this is equivalent to the result given above.

Ex. xiii. *If*

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & y_1 \\ a_{21} & a_{22} & \dots & a_{2n} & y_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & y_m \\ x_1 & x_2 & \dots & x_n & z_0 \end{vmatrix}, \quad \textit{where } m < n,$$

then

$$\Delta = (-1)^{m+n} z_0 (a)_m^n + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \\ x_1 & x_2 & \dots & x_n \end{vmatrix} - (-1)^{m+n} \det [x_1 x_2 \dots x_n] \underline{\overline{A}}_n^m \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

where $[A]_m^n$ *is the reciprocal matrix of* $[a]_m^n$.

Ex. xiv. We will prove the identity

$$\{ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1\}\{ax_2^2 + by_2^2 + cz_2^2 + 2fy_2z_2 + 2gz_2x_2 + 2hx_2y_2\}$$
$$-\{ax_1x_2 + by_1y_2 + cz_1z_2 + f(y_1z_2 + y_2z_1) + g(z_1x_2 + z_2x_1) + h(x_1y_2 + x_2y_1)\}^2$$
$$= A(y_1z_2 - y_2z_1)^2 + B(z_1x_2 - z_2x_1)^2 + C(x_1y_2 - x_2y_1)^2$$
$$+ 2F(z_1x_2 - z_2x_1)(x_1y_2 - x_2y_1) + 2G(x_1y_2 - x_2y_1)(y_1z_2 - y_2z_1) + 2H(y_1z_2 - y_2z_1)(z_1x_2 - z_2x_1).$$

Let the matrix $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ be denoted by ϕ, and let $\begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix}$ be its reciprocal.

Then the expression on the left

$$= \det \begin{bmatrix} \det [x_1 y_1 z_1] \, \phi \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, & \det [x_1 y_1 z_1] \, \phi \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \\ \det [x_2 y_2 z_2] \, \phi \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, & \det [x_2 y_2 z_2] \, \phi \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \end{bmatrix}$$

$$= \det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} = \det [XYZ] \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$= AX^2 + BY^2 + CZ^2 + 2FYZ + 2GZX + 2HXY,$$

where $X = (y_1 z_2 - y_2 z_1), \quad Y = (z_1 x_2 - z_2 x_1), \quad Z = (x_1 y_2 - x_2 y_1).$

§ 63. Expressions for the determinoid of a product of any number of given matrices when the two activities are not equal.

Let $$X = [x]_m^n = [a]_m^r [b]_r^s [c]_s^t [d]_t^n = ABCD$$

be a product of four given matrices in which the efficiency is η and no one of the passivities r, s, t is less than η, and let the same notation be used as in § 64, k being equal to η.

Then by Formula A of § 58, we have

$$\det X = \Sigma (-1)^{\omega + \omega'} (a_{mr})_\eta^\eta (b_{rs})_\eta^\eta (c_{st})_\eta^\eta (d_{tn})_\eta^\eta \quad \ldots\ldots\ldots\ldots (A).$$

Here the summation extends over all distinct values of the minor sequences

$$[m_1 m_2 \ldots m_\eta], \quad [r_1 r_2 \ldots r_\eta], \quad [s_1 s_2 \ldots s_\eta], \quad [t_1 t_2 \ldots t_\eta], \quad [n_1 n_2 \ldots n_\eta],$$

and ω, ω' are the affects of $[m_1 m_2 \ldots m_\eta]$, $[n_1 n_2 \ldots n_\eta]$ in $[1\,2 \ldots m]$, $[1\,2 \ldots n]$ respectively.

Equation (A) is equivalent to

$$\det X = \Sigma (-1)^{\omega + \omega'} a_{pu} \beta_{uv} \gamma_{vw} \delta_{wq} \quad \ldots\ldots\ldots\ldots (B).$$

Here the summation extends over the values $1, 2, \ldots \mu$ of p, the values $1, 2, \ldots \rho$ of u, the values $1, 2, \ldots \sigma$ of v, the values $1, 2, \ldots \tau$ of w, and the values $1, 2, \ldots \nu$ of q; and ω, ω' are the vertical and horizontal affects of a_{pu}, δ_{wq} in A, D respectively.

In the present case

$$\mu = \binom{m}{\eta}, \quad \rho = \binom{r}{\eta}, \quad \sigma = \binom{s}{\eta}, \quad \tau = \binom{t}{\eta}, \quad \nu = \binom{n}{\eta}.$$

Since η is equal to either m or n, it follows that either p or q has the fixed value 1, and either μ or ν is equal to 1.

Equation (B) can be written in the form

$$\det X = \det [a'']_\mu^\rho [\beta]_\rho^\sigma [\gamma]_\sigma^\tau [\delta'']_\tau^\nu \quad \ldots\ldots\ldots\ldots (C),$$

where $$a''_{pu} = (-1)^{\omega + p - 1} a_{pu}, \qquad \delta''_{wq} = (-1)^{\omega' + q - 1} \delta_{wq}.$$

Here ω and $p-1$ are the vertical affects of a_{pu} in A and in $[a]_\mu^\rho$,

and ω' and $q-1$ are the horizontal affects of δ_{wq} in D and in $[\delta]_\tau^\nu$.

From (B) above, or from Formula B of § 58, we obtain

$$\det X = \Sigma\, a'_{pu}\beta'_{uv}\gamma'_{vw}\delta'_{wq} \quad \text{.................................(D)},$$

where $a'_{pu}, \beta'_{uv}, \gamma'_{vw}, \delta'_{wq}$ are the affected minor determinants corresponding to the unaffected minor determinants $a_{pu}, \beta_{uv}, \gamma_{vw}, \delta_{wq}$ respectively; and from this it follows that

$$\det X = \det [a'']_{\mu}^{\rho}\, [\beta']_{\rho}^{\sigma}\, [\gamma']_{\sigma}^{\tau}\, [\delta'']_{\tau}^{\nu} \quad \text{..............................(E)},$$

where now $\qquad a''_{pu} = (-1)^{p-1}\, a'_{pu}, \qquad \delta''_{wq} = (-1)^{q-1}\, \delta'_{wq}.$

Ex. i. $$\det \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \\ l_4 & m_4 & n_4 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \\ l_4 & m_4 & n_4 \end{bmatrix}$$

$$= \det \begin{bmatrix} P_1 \\ -P_2 \\ P_3 \\ -P_4 \end{bmatrix} [A_4\, B_4\, C_4\, D_4] \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix},$$

where $[LMNP]_{1234}$, $[ABCD]_{1234}$ are the reciprocals of $[lmnp]_{1234}$, $[abcd]_{1234}$ respectively.

This is obtained from formula (E), the active rows of the first and last factor matrices following the schemes (2 3 4, 1 3 4, 1 2 4, 1 2 3), (1 2 3) respectively, and the passive rows of the first and second pairs of adjacent factor matrices following the schemes (1 2 3), (2 3 4, 1 3 4, 1 2 4, 1 2 3).

Ex. ii. If $\qquad X = [lmn]_{1234} [abc]_{123} [\lambda\mu]_{123} = [x]_4^2,$

then

$$\det X = \det \begin{bmatrix} (mn)_{12}, & (nl)_{12}, & (lm)_{12} \\ (mn)_{13}, & (nl)_{13}, & (lm)_{13} \\ (mn)_{14}, & (nl)_{14}, & (lm)_{14} \\ -(mn)_{23}, & -(nl)_{23}, & -(lm)_{23} \\ -(mn)_{24}, & -(nl)_{24}, & -(lm)_{24} \\ -(mn)_{34}, & -(nl)_{34}, & -(lm)_{34} \end{bmatrix} \begin{bmatrix} (ab)_{23} & (ac)_{23} & (bc)_{23} \\ (ab)_{31} & (ac)_{31} & (bc)_{31} \\ (ab)_{12} & (ac)_{12} & (bc)_{12} \end{bmatrix} \begin{bmatrix} (\lambda\mu)_{12} \\ (\lambda\mu)_{13} \\ (\lambda\mu)_{23} \end{bmatrix}.$$

This is obtained from formula (C), the schemes for the active rows being (1 2, 1 3, 1 4, 2 3, 2 4, 3 4) and (1 2) respectively, and the schemes for the passive rows being (2 3, 3 1, 1 2) and (1 2, 1 3, 2 3).

§ 64. Algebraical sum of the affected minor determinants of order *k* of a product of given matrices.

Let the product be

$$[x]_m^n = [a]_m^{\alpha}\, [b]_{\alpha}^{\beta}\, [c]_{\beta}^{\gamma} \dots [s]_{\rho}^{\sigma}\, [t]_{\sigma}^{n} \quad \text{.................(1)},$$

or briefly $\qquad X = ABC \dots ST,$

and let S_k be the algebraical sum of all distinct determinants of order k derived from $[x]_m^n$, each determinant having the sign determined by its affect in $[x]_m^n$, and k being any number which does not exceed either m or n. Then the theorems and formulae given in §§ 56—58 can all be generalised by replacing det X by S and η by k. The generalised results can either be proved by the methods adopted in §§ 56—58 or deduced from the results obtained in those articles. The most fundamental results are those given in the following two theorems:

Theorem I. *If any one of the passivities* $\alpha, \beta, \gamma \ldots \rho, \sigma$ *is less than* k, *then*

$$S_k = 0.$$

Theorem II. *If no one of the passivities is less than* k, *then*

$$(1).\quad S_k = \Sigma(-1)^{\omega+\omega'}(a_{m\alpha})_k^k (b_{\alpha\beta})_k^k (c_{\beta\gamma})_k^k \ldots (s_{\rho\sigma})_k^k (t_{\sigma n})_k^k;$$

$$(2).\quad S_k = \Sigma(-1)^{\omega'}\quad (a_{1\alpha})_m^k (b_{\alpha\beta})_k^k (c_{\beta\gamma})_k^k \ldots (s_{\rho\sigma})_k^k (t_{\sigma n})_k^k;$$

$$(3).\quad S_k = \Sigma(-1)^{\omega}\quad (a_{m\alpha})_k^k (b_{\alpha\beta})_k^k (c_{\beta\gamma})_k^k \ldots (s_{\rho\sigma})_k^k (t_{\sigma 1})_k^n;$$

$$(4).\quad S_k = \Sigma\quad (a_{1\alpha})_m^k (b_{\alpha\beta})_k^k (c_{\beta\gamma})_k^k \ldots (s_{\rho\sigma})_k^k (t_{\sigma 1})_k^n;$$

where

$$[m_1 m_2 \ldots m_r],\ [\alpha_1 \alpha_2 \ldots \alpha_r],\ [\beta_1 \beta_2 \ldots \beta_r],\ \ldots\ [\sigma_1 \sigma_2 \ldots \sigma_r],\ [n_1 n_2 \ldots n_r]$$

are minors of the sequences

$$[1\ 2 \ldots m],\quad [1\ 2 \ldots \alpha],\quad [1\ 2 \ldots \beta],\quad \ldots\quad [1\ 2 \ldots \sigma],\quad [1\ 2 \ldots n]$$

respectively, and

$$\omega = \textit{affect of } [m_1 m_2 \ldots m_r] \textit{ in } [1\ 2 \ldots m],$$
$$\omega' = \textit{affect of } [n_1 n_2 \ldots n_r] \textit{ in } [1\ 2 \ldots n].$$

The summation extends over all distinct values of each of the minor sequences $[m]_r, [\alpha]_r, [\beta]_r, \ldots [\sigma]_r, [n]_r$.

The first form of S_k in Theorem II is a generalisation of Theorem II of § 56 and Theorem VI of § 58. It shows that S_k is equal to the algebraical sum of all products of determinants of order k which can be formed from the given product by striking out active rows in excess of r in number and corresponding passive rows in excess of r in number.

The second and third forms of S_k are generalisations of equations (9) and (12) in § 56. The fourth form of S_k is a generalisation of Theorem C of § 56.

The second, third and fourth forms can be deduced from the first form by performing one or both of the summations with respect to the minor sequences

$[m_1 m_2 \dots m_r]$, $[n_1 n_2 \dots n_r]$. Consequently in order to prove Theorem II completely, it is sufficient to prove the correctness of the first form of S_k.

Results corresponding to Theorems II—V of § 58 are easily deduced from Theorem II.

Proof of Theorem I. Suppose that at least one of the passivities $\alpha, \beta, \gamma, \dots \rho, \sigma$ *is less than* k, and let $(x_{mn})_k^k$ be any one of the derived determinants of $[x]_m^n$ of order k.

By the properties of active rows we deduce from equation (1) that

$$[x_{mn}]_k^k = [a_{m1}]_k^\alpha [b]_\alpha^\beta [c]_\beta^\gamma \dots [s]_\rho^\sigma [t_{1n}]_\sigma^k \text{ ....(2).}$$

Here the product on the right has efficiency k, and it follows by Theorem I of § 58 that

$$(x_{mn})_k^k = \det [x_{mn}]_k^k = 0.$$

Thus in this case every derived determinant of order k of $[x]_m^n$ vanishes, and therefore

$$S_k = 0.$$

Proof of Theorem II. Suppose that no one of the passivities $\alpha, \beta, \gamma, \dots \rho, \sigma$ *is less than* k, and let $(x_{mn})_k^k$ be any one of the derived determinants of $[x]_m^n$ of order k.

We obtain as before equation (2) in which the product on the right has efficiency k. Since no one of the passivities is less than k, it follows by Formula A of § 58 or by § 61 that

$$(x_{mn})_k^k = \Sigma (a_{m\alpha})_k^k (b_{\alpha\beta})_k^k (c_{\beta\gamma})_k^k \dots (s_{\rho\sigma})_k^k (t_{\sigma n})_k^k \text{ ....(3).}$$

Here $[m_1 m_2 \dots m_k]$, $[\alpha_1 \alpha_2 \dots \alpha_k]$, $[\beta_1 \beta_2 \dots \beta_k]$, $\dots [\sigma_1 \sigma_2 \dots \sigma_k]$, $[n_1 n_2 \dots n_k]$ are minor sequences of order k of the fundamental sequences

$$[1\ 2 \dots m],\quad [1\ 2 \dots \alpha],\quad [1\ 2 \dots \beta],\quad \dots [1\ 2 \dots \sigma],\quad [1\ 2 \dots n]$$

respectively; and the summation extends over all distinct values of each of the minor sequences $[\alpha]_k$, $[\beta]_k$, $\dots$ $[\sigma]_k$, the minor sequences $[m]_k$, $[n]_k$ remaining fixed.

Let ω, ω' be the affects of $[m]_k$, $[n]_k$ in $[1\ 2 \dots m]$, $[1\ 2 \dots n]$ respectively.

Then

$$(-1)^{\omega+\omega'} (x_{mn})_k^k = \Sigma (-1)^{\omega+\omega'} (a_{m\alpha})_k^k (b_{\alpha\beta})_k^k \dots (s_{\rho\sigma})_k^k (t_{\sigma n})_k^k \text{ ......(4),}$$

the summation being as before.

Now $\omega + \omega'$ is the affect of $(x_{mn})_k^k$ in $[x]_m^n$.

Therefore $$S_k = \Sigma (-1)^{\omega+\omega'} (x_{mn})_k^k \qquad \ldots\ldots(5),$$

the summation extending over all distinct values of each of the minor sequences $[m]_k$, $[n]_k$.

From (4) and (5) we obtain the first value of S_k given in Theorem II.

Thus Theorem II is proved.

Ex. i. Let $[x]_m^n = [a]_m^k [b]_k^n$, where $k \not< m$ and $k \not< n$.

Then if $(x_{pq})_k^k$ is any minor determinant of $[x]_m^n$ of order k, and if

$\omega =$ affect of $[p_1 p_2 \ldots p_k]$ in $[1\ 2 \ldots m]$, $\omega' =$ affect of $[q_1 q_2 \ldots q_k]$ in $[1\ 2 \ldots n]$,

we have $$[x_{pq}]_k^k = [a_{p1}]_k^k [b_{1q}]_k^k, \qquad (x_{pq})_k^k = (a_{p1})_k^k (b_{1q})_k^k.$$

Therefore $$S_k = \sum_p \sum_q (-1)^{\omega+\omega'} (a_{p1})_k^k (b_{1q})_k^k.$$

Since $$\sum_p (-1)^{\omega} (a_{p1})_k^k = (a)_m^k, \qquad \sum_q (-1)^{\omega'} (b_{1q})_k^k = (b)_k^n,$$

it follows that $$S_k = \sum_p (-1)^{\omega} (a_{p1})_k^k (b)_k^n = \Sigma (a)_m^k (b)_k^n \qquad \ldots\ldots(6).$$

The same result is obtained at once from the fourth value of S_k given by Theorem II.

Thus if the passivity k of a product of two matrices does not exceed the efficiency, then the algebraical sum of the affected minor determinants of order k of the product matrix is equal to the product of the determinoids of the factor matrices.

This result is a generalisation of Theorem C of § 56.

Ex. ii. If $$X = [abc]_{12345} [\alpha\beta\gamma\delta]_{123},$$

then $$S_3 = (abc)_{12345} (\alpha\beta\gamma\delta)_{123}.$$

CHAPTER VIII.

MATRICES OF MINOR DETERMINANTS.

[The complete matrices of the minor determinants of any given order of a fundamental matrix are first defined in § 65. Then in § 66 it is shown that any complete matrix of the minor determinants of order k of a product of given matrices is in general equal to any correspondingly formed product of complete matrices of the minor determinants of order k of the factor matrices, but that it vanishes when any passivity of the product is less than k. Finally in § 67 the values of the reciprocal and the conjugate reciprocal of any standard product of square matrices are found.]

§ 65. Matrices of the minor determinants of a fundamental matrix.

1. *Definition of a complete matrix of the minor determinants of given order of a given fundamental matrix.*

Let
$$X = [x]_m^n$$
be any given matrix and let k be any number which does not exceed either m or n, so that X has minor determinants of order k.

Let $[m_1\, m_2 \dots m_k]$ be any minor of order k of the sequence $[1\ 2 \dots m]$, and $[n_1\, n_2 \dots n_k]$ be any minor of order k of the sequence $[1\ 2 \dots n]$. Then the minor sequences $[m_1\, m_2 \dots m_k]$, $[n_1\, n_2 \dots n_k]$ have respectively μ and ν *distinct* values, where
$$\mu = \binom{m}{k}, \qquad \nu = \binom{n}{k}.$$

Here two minor sequences of a given fundamental sequence are regarded as distinct when one is not simply a derangement of the other.

Let any μ distinct values of the minor sequence $[m_1\, m_2 \dots m_k]$, arranged in any order, be called the 1st, 2nd, ... pth, ... μth values of that sequence.

Let any ν distinct values of the minor sequence $[n_1\, n_2 \dots n_k]$, arranged in any order, be called the 1st, 2nd, ... qth, ... νth values of that sequence.

Let the simple minor matrix of X formed by retaining only its m_1th, m_2th, ... m_kth horizontal rows and arranging them in this order be called the horizontal minor of X (of reduced order k) corresponding to the sequence $[m_1\, m_2 \ldots m_k]$.

Let the simple minor matrix of X formed by retaining only its n_1th, n_2th, ... n_kth vertical rows and arranging them in this order be called the vertical minor of X (of reduced order k) corresponding to the sequence $[n_1 n_2 \ldots n_k]$.

Then X has μ distinct horizontal minors (of reduced order k) corresponding respectively to the 1st, 2nd, ... pth, ... μth values of the sequence $[m_1 m_2 \ldots m_k]$; and it has ν distinct vertical minors (of reduced order k) corresponding respectively to the 1st, 2nd, ... qth, ... νth values of the sequence $[n_1\, n_2 \ldots n_k]$.

Let ξ_{pq} be the minor determinant of X of order k formed by the intersection of the pth horizontal minor and the qth vertical minor of X, so that $\xi_{pq} = (x_{mn})_k^k$, where $[m_1\, m_2 \ldots m_k]$ and $[n_1\, n_2 \ldots n_k]$ have respectively their pth and qth values.

Then the matrix $[\xi]_\mu^\nu$ *will be called a complete matrix of the (unaffected) minor determinants of order k of X.*

The determinants forming the elements of the pth horizontal row of $[\xi]_\mu^\nu$ are simple minor determinants of the pth horizontal minor of X; and the determinants forming the elements of the qth vertical row of $[\xi]_\mu^\nu$ are simple minor determinants of the qth vertical minor of X.

Every distinct minor determinant of order k of X occurs once and only once as an element in $[\xi]_\mu^\nu$.

2. *Schemes of formation for complete matrices of minor determinants.*

The elements of a given minor sequence $[m_1\, m_2 \ldots m_k]$ can be arranged in $k!$ different ways, and the elements of a given minor sequence $[n_1\, n_2 \ldots n_k]$ can be arranged in $k!$ different ways. Also any μ given distinct values of the minor sequence $[m_1\, m_2 \ldots m_k]$ can be arranged amongst themselves in $\mu!$ different ways, and any ν given distinct values of the minor sequence $[n_1\, n_2 \ldots n_k]$ can be arranged amongst themselves in $\nu!$ different ways. Accordingly the matrix $[\xi]_\mu^\nu$ defined above can be formed in $\mu!\ \nu!\ k!\ k!$ different ways.

To describe any particular one of these matrices, let the 1st, 2nd, ... pth, ... μth distinct values of the minor sequence $[m_1\, m_2 \ldots m_k]$ which are selected be denoted by $M_1, M_2, \ldots M_p, \ldots M_\mu$ respectively, and let the 1st, 2nd, ... qth, ... νth distinct values of the minor sequence $[n_1\, n_2 \ldots n_k]$ which are selected be denoted by $N_1, N_2, \ldots N_q, \ldots N_\nu$ respectively. Then we shall say that in the formation of $[\xi]_\mu^\nu$ the horizontal and vertical rows of $[x]_m^n$ follow the *schemes* $(M_1, M_2, \dots M_p, \dots M_\mu)$, $(N_1, N_2, \dots N_q, \dots N_\nu)$ respectively.

When the schemes $(M_1, M_2, \dots M_\mu)$, $(N_1, N_2, \dots N_\nu)$ are given, the matrix $[\xi]_\mu^\nu$ is completely determinate.

In the special case of a square fundamental matrix $[x]_m^m$, we have $n = m$, $\nu = \mu$, and it is possible for the sequences $N_1, N_2, \dots N_\mu$ to be identical with the sequences $M_1, M_2, \dots M_\mu$ respectively. When this is so, then in the formation of the matrix $[\xi]_\mu^\mu$ the horizontal and vertical rows of $[x]_m^m$ will be said to follow the *common scheme* $(M_1, M_2, \dots M_\mu)$.

Ex. i. Let $[\xi]_3^6$ be a matrix of the minor determinants of order 2 of $[x]_3^4$.

If
$$[\xi]_3^6 = \begin{bmatrix} \begin{pmatrix}43\\x\\12\end{pmatrix}, & \begin{pmatrix}42\\x\\12\end{pmatrix}, & \begin{pmatrix}41\\x\\12\end{pmatrix}, & \begin{pmatrix}23\\x\\12\end{pmatrix}, & \begin{pmatrix}31\\x\\12\end{pmatrix}, & \begin{pmatrix}12\\x\\12\end{pmatrix} \\ \begin{pmatrix}43\\x\\13\end{pmatrix}, & \begin{pmatrix}42\\x\\13\end{pmatrix}, & \begin{pmatrix}41\\x\\13\end{pmatrix}, & \begin{pmatrix}23\\x\\13\end{pmatrix}, & \begin{pmatrix}31\\x\\13\end{pmatrix}, & \begin{pmatrix}12\\x\\13\end{pmatrix} \\ \begin{pmatrix}43\\x\\23\end{pmatrix}, & \begin{pmatrix}42\\x\\23\end{pmatrix}, & \begin{pmatrix}41\\x\\23\end{pmatrix}, & \begin{pmatrix}23\\x\\23\end{pmatrix}, & \begin{pmatrix}31\\x\\23\end{pmatrix}, & \begin{pmatrix}12\\x\\23\end{pmatrix} \end{bmatrix},$$

then the horizontal and vertical rows of $[x]_3^4$ follow respectively the schemes

$$(1\,2,\ 1\,3,\ 2\,3) \quad\text{and}\quad (4\,3,\ 4\,2,\ 4\,1,\ 2\,3,\ 3\,1,\ 1\,2).$$

Here $M_1=[1\,2]$, $M_2=[1\,3]$, $M_3=[2\,3]$;
and $N_1=[4\,3]$, $N_2=[4\,2]$, $N_3=[4\,1]$, $N_4=[2\,3]$, $N_5=[3\,1]$, $N_6=[1\,2]$.

Ex. ii. If $[\xi]_4^4$ is that matrix of the minor determinants of order 3 of $[xyzw]_{1234}$ in which the horizontal and vertical rows follow the schemes

$$(1\,2\,3,\ 1\,2\,4,\ 4\,3\,2,\ 4\,3\,1) \quad\text{and}\quad (2\,4\,1,\ 2\,4\,3,\ 4\,3\,1,\ 2\,3\,1)$$

respectively, then

$$[\xi]_4^4 = \begin{bmatrix} (ywx)_{123}, & (ywz)_{123}, & (wzx)_{123}, & (yzx)_{123} \\ (ywx)_{124}, & (ywz)_{124}, & (wzx)_{124}, & (yzx)_{124} \\ (ywx)_{432}, & (ywz)_{432}, & (wzx)_{432}, & (yzx)_{432} \\ (ywx)_{431}, & (ywz)_{431}, & (wzx)_{431}, & (yzx)_{431} \end{bmatrix}.$$

Ex. iii. If $[\xi]_6^6$ is that matrix of the minor determinants of order 2 of $[xyzw]_{1234}$ in which the horizontal and vertical rows follow the common scheme

$$(1\,2,\ \ 1\,3,\ \ 1\,4,\ \ 2\,3,\ \ 2\,4,\ \ 3\,4),$$

then
$$[\xi]_6^6 = \begin{bmatrix} (xy)_{12}, & (xz)_{12}, & (xw)_{12}, & (yz)_{12}, & (yw)_{12}, & (zw)_{12} \\ (xy)_{13}, & (xz)_{13}, & (xw)_{13}, & (yz)_{13}, & (yw)_{13}, & (zw)_{13} \\ (xy)_{14}, & (xz)_{14}, & (xw)_{14}, & (yz)_{14}, & (yw)_{14}, & (zw)_{14} \\ (xy)_{23}, & (xz)_{23}, & (xw)_{23}, & (yz)_{23}, & (yw)_{23}, & (zw)_{23} \\ (xy)_{24}, & (xz)_{24}, & (xw)_{24}, & (yz)_{24}, & (yw)_{24}, & (zw)_{24} \\ (xy)_{34}, & (xz)_{34}, & (xw)_{34}, & (yz)_{34}, & (yw)_{34}, & (zw)_{34} \end{bmatrix}.$$

3. *Standard schemes. Standard matrices of minor determinants.*

Let
$$A = [a_1 \, a_2 \dots a_m]$$
be any fundamental sequence, and let the corranged minor sequences of order k of A be so arranged that

if a_p occurs before a_q in A, then all minors beginning with a_p are placed before all minors beginning with a_q;

if a_q occurs before a_r in A, then all minors beginning with $a_p a_q$ are placed before all minors beginning with $a_p a_r$;

if a_r occurs before a_s in A, then all minors beginning with $a_p a_q a_r$ are placed before all minors beginning with $a_p a_q a_s$;

and so on.

The corranged minor sequences of order k of A when so arranged will be said to form the *standard arrangement* of the distinct minor sequences of order k.

If $U = [u_1 u_2 \dots u_k]$, $V = [v_1 v_2 \dots v_k]$ are any two corranged minor sequences of order k of A, and if $[u_1 u_2 \dots u_{i-1}] = [v_1 v_2 \dots v_{i-1}]$ but $u_i \neq v_i$, then in the standard arrangement of the minor sequences of order k of A, the minor sequence U occurs before or after the minor sequence V according as u_i occurs before or after v_i in A.

When $M_1, M_2, \dots M_\mu$ is the standard arrangement of the distinct minor sequences of order k of $[1\,2 \dots m]$ and $N_1, N_2, \dots N_\nu$ is the standard arrangement of the distinct minor sequences of order k of $[1\,2 \dots n]$, then the schemes $(M_1, M_2, \dots M_\mu)$, $(N_1, N_2, \dots N_\nu)$ will be called *standard schemes*, and the corresponding matrix $[\xi]_\mu^\nu$ will be called the standard matrix of the minor determinants of order k of the fundamental matrix $[x]_m^n$.

In a standard matrix of the minor determinants of a square fundamental matrix the horizontal and vertical rows follow a common scheme.

Ex. iv. In the standard matrix $[\xi]_{10}^4$ of the minor determinants of order 3 of the matrix $[abcd]_{12345}$ the horizontal and vertical rows follow the schemes

$$(1\,2\,3,\ \ 1\,2\,4,\ \ 1\,2\,5,\ \ 1\,3\,4,\ \ 1\,3\,5,\ \ 1\,4\,5,\ \ 2\,3\,4,\ \ 2\,3\,5,\ \ 2\,4\,5,\ \ 3\,4\,5)$$

and $$(1\,2\,3,\ \ 1\,2\,4,\ \ 1\,3\,4,\ \ 2\,3\,4)$$ respectively.

Accordingly

$$[\xi]_{10}^4 = \begin{bmatrix} (abc)_{123}, & (abd)_{123}, & (acd)_{123}, & (bcd)_{123} \\ (abc)_{124}, & (abd)_{124}, & (acd)_{124}, & (bcd)_{124} \\ \cdots & \cdots & \cdots & \cdots \\ (abc)_{245}, & (abd)_{245}, & (acd)_{245}, & (bcd)_{245} \\ (abc)_{345}, & (abd)_{345}, & (acd)_{345}, & (bcd)_{345} \end{bmatrix}.$$

Ex. v. The matrix $[\xi]_3^6$ occurring in Ex. iii is the standard matrix of the minor determinants of order 2 of $[xyzw]_{1234}$.

4. *Complete matrices of affected minor determinants of given order.*

Let $[\xi]_\mu^\nu$ be defined as in sub-article 1, and let ω and ω' be the vertical and horizontal affects of the minor determinant ξ_{pq} in the fundamental matrix $[x]_m^n$, so that

$$\omega = \text{the affect of the } p\text{th value of } [m_1 m_2 \ldots m_k] \text{ in } [1\,2 \ldots m]$$
$$= \text{the affect of the } p\text{th horizontal minor matrix of } X \text{ in } X,$$
$$\omega' = \text{the affect of the } q\text{th value of } [n_1 n_2 \ldots n_k] \text{ in } [1\,2 \ldots n]$$
$$= \text{the affect of the } q\text{th vertical minor matrix of } X \text{ in } X;$$

and let

$$\xi'_{pq} = (-1)^{\omega+\omega'} \xi_{pq}.$$

Thus ξ'_{pq} is the minor determinant ξ_{pq} provided with the sign determined by its affect in X.

The matrix $[\xi']_\mu^\nu$ will be called a complete matrix of the affected minor determinants of order k of X.

Clearly $[\xi']_\mu^\nu = [\xi'']_\mu^\nu$, where $[\xi'']_\mu^\nu$ is some complete matrix of the *unaffected* minor determinants of order k of X which has different schemes of formation to $[\xi]_\mu^\nu$.

If $(M_1'', M_2'', \ldots M_p'', \ldots M_\mu'')$, $(N_1'', N_2'', \ldots N_q'', \ldots N_\nu'')$ are schemes of formation for $[\xi'']_\mu^\nu$, we choose M_p'' to be any derangement of the sequence M_p which has an even affect in $[1\,2 \ldots m]$ and N_q'' to be any derangement of the sequence N_q which has an even affect in $[1\,2 \ldots n]$; for an affected minor determinant can be regarded as a derangement of a minor determinant whose affect is even.

Ex. vi. Let $[\xi]_6^6$ be that matrix of the *affected* minor determinants of order 2 of $[abcd]_{123}$ in the formation of which the horizontal and vertical rows follow the standard schemes

$$(1\,2,\ 1\,3,\ 2\,3) \quad \text{and} \quad (1\,2,\ 1\,3,\ 1\,4,\ 2\,3,\ 2\,4,\ 3\,4);$$

in other words let $[\xi]_6^6$ be the standard matrix of the affected minor determinants of order 2 of $[abcd]_{123}$. Then

$$[\xi]_6^6 = \begin{bmatrix} +(ab)_{12}, & -(ac)_{12}, & +(ad)_{12}, & +(bc)_{12}, & -(bd)_{12}, & +(cd)_{12} \\ -(ab)_{13}, & +(ac)_{13}, & -(ad)_{13}, & -(bc)_{13}, & +(bd)_{13}, & -(cd)_{13} \\ +(ab)_{23}, & -(ac)_{23}, & +(ad)_{23}, & +(bc)_{23}, & -(bd)_{23}, & +(cd)_{23} \end{bmatrix}.$$

We also have

$$[\xi]_6^6 = \begin{bmatrix} (ab)_{12}, & (ca)_{12}, & (ad)_{12}, & (bc)_{12}, & (db)_{12}, & (cd)_{12} \\ (ab)_{31}, & (ca)_{31}, & (ad)_{31}, & (bc)_{31}, & (db)_{31}, & (cd)_{31} \\ (ab)_{23}, & (ca)_{23}, & (ad)_{23}, & (bc)_{23}, & (db)_{23}, & (cd)_{23} \end{bmatrix}.$$

Thus $[\xi]_6^6$ is that matrix of the *unaffected* minor determinants of order 2 in which the horizontal and vertical rows follow the schemes

$$(1\,2,\ 3\,1,\ 2\,3) \quad \text{and} \quad (1\,2,\ 3\,1,\ 1\,4,\ 2\,3,\ 4\,2,\ 3\,4)$$

respectively.

Ex. vii. If the minor sequences $M_1, M_2, \dots M_p, \dots M_\mu$ are so selected as to have all even affects in $[1\,2 \dots m]$, and if the minor sequences $N_1, N_2, \dots N_q, \dots N_\nu$ are so selected as to have all even affects in $[1\,2 \dots n]$; in other words if the μ successive distinct horizontal minors of $[x]_m^n$ have all even affects in $[x]_m^n$, and if the ν successive distinct vertical minors of $[x]_m^n$ have all even affects in $[x]_m^n$; then the matrix $[\xi]_\mu^\nu$ in which the horizontal and vertical rows of $[x]_m^n$ follow the schemes $(M_1, M_2, \dots M_\mu)$, $(N_1, N_2, \dots N_\nu)$ respectively can be regarded both as a matrix of the unaffected minor determinants of order k of X and as a matrix of the affected minor determinants of order k of X; for every element of $[\xi]_\mu^\nu$ is then a minor determinant of order k of X whose affect in X is even.

§ 66. Matrices of the minor determinants of a product formed by a chain of given matrix factors.

The result contained in formula (D) of § 61 is a particular case of a more general theorem which will be considered in the present article.

Let
$$X = ABC \dots ST$$
be a standard product of a number of given matrices in which the passivities are not subject to any restrictions. Let η be the efficiency of the product, i.e. the efficiency of the product matrix X, and let k be any number which is not greater than η, so that the product matrix X has derived determinants of order k. Then we will prove the following results:

Theorem I. *If any one of the passivities is less than k, then every complete matrix of the minor determinants (unaffected or affected) of order k of the product matrix X is a zero matrix.*

Theorem II. *If no one of the passivities is less than k, then every complete matrix of the minor determinants (unaffected or affected) of order k of the product matrix X is equal to the product of any set of correspondingly formed complete matrices of the minor determinants (unaffected or affected) of order k of the factor matrices $A, B, C, \dots S, T$.*

The nature of the correspondences alluded to in the enunciation of Theorem II will be explained in the proof.

We shall prove the above two theorems for products of four matrices, but it will be obvious that similar proofs can be given for products of any number of matrices.

Let then $X = ABCD$ ....(1),

or more fully $[x]_m^n = [a]_m^r [b]_r^s [c]_s^t [d]_t^n$ ....(2),

be a standard product of four given matrices A, B, C, D, the efficiency of the product being η; and let k be any number which is not greater than η.

Let $[m_1 m_2 \dots m_k]$ be any minor of order k of the sequence $[1\,2 \dots m]$, and let $[n_1 n_2 \dots n_k]$ be any minor of order k of the sequence $[1\,2 \dots n]$.

Then we can determine in many ways μ, ν distinct values of the minor sequences $[m_1 m_2 \dots m_k]$, $[n_1 n_2 \dots n_k]$ respectively, where

$$\mu = \binom{m}{k}, \qquad \nu = \binom{n}{k}.$$

Let $M_1, M_2, \dots M_p, \dots M_\mu$ be any one selection of μ distinct values of the minor sequence $[m_1 m_2 \dots m_k]$,

and let $N_1, N_2, \dots N_q, \dots N_\nu$ be any one selection of ν distinct values of the minor sequence $[n_1 n_2 \dots n_k]$.

Let $[\xi]_\mu^\nu$, $[\xi']_\mu^\nu$ be respectively the matrix of the unaffected minor determinants of order k of $[x]_m^n$ and the matrix of the affected minor determinants of order k of $[x]_m^n$ in which the horizontal and vertical rows of $[x]_m^n$ follow (in both cases) the schemes $(M_1, M_2, \dots M_\mu)$ and $(N_1, N_2, \dots N_\nu)$ respectively.

Then we will prove Theorems I and II by determining the values of the matrices $[\xi]_\mu^\nu$ and $[\xi']_\mu^\nu$.

Proof of Theorem I. Suppose that at least one of the passivities r, s, t is less than k.

Let the minor sequences $[m_1 m_2 \dots m_k]$ and $[n_1 n_2 \dots n_k]$ have respectively their pth and qth values, so that

$$\xi_{pq} = (x_{mn})_k^k.$$

From equation (2) it follows by the properties of active rows contained in § 53.4 and § 53.7 that

$$[x_{mn}]_k^k = [a_{m1}]_k^r [b]_r^s [c]_s^t [d_{1n}]_t^k \quad \text{.....................(3).}$$

The product on the right in (3) has one of its passivities less than the activity k, and therefore by Theorem I of § 58

$$(x_{mn})_k^k = \det [x_{mn}]_k^k = 0.$$

Thus $\xi_{pq} = 0$; and since $\xi'_{pq} = \pm \xi_{pq}$, we also have $\xi'_{pq} = 0$.

These results are true for all the values 1, 2, ... μ of p and all the values 1, 2, ... ν of q.

Consequently

$$[\xi]_\mu^\nu = 0, \qquad [\xi']_\mu^\nu = 0 \text{..........................(A).}$$

This establishes the truth of Theorem I.

Proof of Theorem II. Suppose that no one of the passivities r, s, t is less than k.

In this case every one of the factor matrices A, B, C, D has minor determinants of order k.

Let $[r_1 r_2 \dots r_k]$, $[s_1 s_2 \dots s_k]$, $[t_1 t_2 \dots t_k]$ be respectively any minor sequences of order k of the fundamental sequences $[1\ 2 \dots r]$, $[1\ 2 \dots s]$, $[1\ 2 \dots t]$.

Then we can determine in many ways ρ, σ, τ distinct values of the minor sequences $[r_1 r_2 \dots r_k]$, $[s_1 s_2 \dots s_k]$, $[t_1 t_2 \dots t_k]$ respectively, where

$$\rho = \binom{r}{k}, \qquad \sigma = \binom{s}{k}, \qquad \tau = \binom{t}{k}.$$

Let $R_1, R_2, \dots R_u, \dots R_\rho$ be any ρ distinct values of the minor sequence $[r_1 r_2 \dots r_k]$;

let $S_1, S_2, \dots S_v, \dots S_\sigma$ be any σ distinct values of the minor sequence $[s_1 s_2 \dots s_k]$;

and let $T_1, T_2, \dots T_w, \dots T_\tau$ be any τ distinct values of the minor sequence $[t_1 t_2 \dots t_k]$.

Let $[\alpha]_\mu^\rho$ be the matrix of the minor determinants of order k of $[a]_m^r$ in which the horizontal and vertical rows of $[a]_m^r$ follow the schemes

$(M_1, M_2, \dots M_p, \dots M_\mu)$ and $(R_1, R_2, \dots R_u, \dots R_\rho)$ respectively;

let $[\beta]_\rho^\sigma$ be the matrix of the minor determinants of order k of $[b]_r^s$ in which the horizontal and vertical rows of $[b]_r^s$ follow the schemes

$(R_1, R_2, \dots R_u, \dots R_\rho)$ and $(S_1, S_2, \dots S_v, \dots S_\sigma)$ respectively;

let $[\gamma]_\sigma^\tau$ be the matrix of the minor determinants of order k of $[c]_s^t$ in which the horizontal and vertical rows of $[c]_s^t$ follow the schemes

$(S_1, S_2, \dots S_v, \dots S_\sigma)$ and $(T_1, T_2, \dots T_w, \dots T_\tau)$ respectively;

and let $[\delta]_\tau^\nu$ be the matrix of the minor determinants of order k of $[d]_t^n$ in which the horizontal and vertical rows of $[d]_t^n$ follow the schemes

$$(T_1, T_2, \ldots T_w, \ldots T_\tau) \text{ and } (N_1, N_2, \ldots N_q, \ldots N_\nu) \text{ respectively.}$$

Then

$$\alpha_{pu} = (a_{mr})_k^k, \quad \beta_{uv} = (b_{rs})_k^k, \quad \gamma_{vw} = (c_{st})_k^k, \quad \delta_{wq} = (d_{tn})_k^k,$$

where $[m_1 m_2 \ldots m_k]$, $[r_1 r_2 \ldots r_k]$, $[s_1 s_2 \ldots s_k]$, $[t_1 t_2 \ldots t_k]$, $[n_1 n_2 \ldots n_k]$ have respectively their pth, uth, vth, wth, qth values.

The matrices $[\xi]_\mu^\nu$, $[\alpha]_\mu^\rho$, $[\beta]_\rho^\sigma$, $[\gamma]_\sigma^\tau$, $[\delta]_\tau^\nu$ *will be called a set of correspondingly formed complete matrices of the unaffected minor determinants of order k of X, A, B, C, D; and the product* $[\alpha]_\mu^\rho [\beta]_\rho^\sigma [\gamma]_\sigma^\tau [\delta]_\tau^\nu$ *will be called a product of the complete matrices of the unaffected minor determinants of order k of A, B, C, D formed in correspondence with* $[\xi]_\mu^\nu$.

The correspondences in the formation of the matrices of the minor determinants consist in the following facts:

(1) The horizontal or active rows of the first factor matrix $[a]_m^r$ and the horizontal rows of the product matrix $[x]_m^n$ follow a common scheme.

(2) The vertical or active rows of the last factor matrix $[d]_t^n$ and the vertical rows of the product matrix $[x]_m^n$ follow a common scheme.

(3) The two sets of mutually corresponding passive rows which occur in any two adjacent factor matrices follow a common scheme.

It should be noticed moreover that in the sense of § 49 the pth horizontal minor of X corresponds to the pth horizontal minor of A, the qth vertical minor of X corresponds to the qth vertical minor of D, and the uth, vth, wth vertical minors of A, B, C correspond respectively to the uth, vth, wth horizontal minors of B, C, D.

We will now determine the value of any element ξ_{pq} of the matrix $[\xi]_\mu^\nu$.

Let the minor sequences $[m_1 m_2 \ldots m_k]$, $[r_1 r_2 \ldots r_k]$, $[s_1 s_2 \ldots s_k]$, $[t_1 t_2 \ldots t_k]$, $[n_1 n_2 \ldots n_k]$ have respectively their pth, uth, vth, wth, qth values.

By the properties of active rows contained in § 53.4 and § 53.7 we deduce from the equation

$$[x]_m^n = [a]_m^r [b]_r^s [c]_s^t [d]_t^n$$

that

$$[x_{mn}]_k^k = [a_{m1}]_k^r [b]_r^s [c]_s^t [d_{1n}]_t^k \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Here the product on the right has both its activities equal to the efficiency k, and is a product of the kind considered in § 61.

Applying Formula A of § 58, we have

$$(x_{mn})_k^k = \det [x_{mn}]_k^k = \Sigma (a_{mr})_k^k (b_{rs})_k^k (c_{st})_k^k (d_{tn})_k^k \quad \text{.............(5)},$$

where the summation extends over all the ρ, σ, τ values of the minor sequences $[r_1 r_2 \dots r_k]$, $[s_1 s_2 \dots s_k]$, $[t_1 t_2 \dots t_k]$, whilst the two minor sequences $[m_1 m_2 \dots m_k]$, $[n_1 n_2 \dots n_k]$ remain fixed.

Equation (5) can be written in the form

$$\xi_{pq} = \sum_u \sum_v \sum_w \alpha_{pu} \beta_{uv} \gamma_{vw} \delta_{wq} \quad \text{.......................(6)},$$

where u, v, w are independent of one another; u receives the values 1, 2, ... ρ; v receives the values 1, 2, ... σ; w receives the values 1, 2, ... τ; and p and q remain fixed.

By § 51 all the equations of which (6) is typical are together equivalent to the matrix equation

$$[\xi]_\mu^\nu = [\alpha]_\mu^\rho [\beta]_\rho^\sigma [\gamma]_\sigma^\tau [\delta]_\tau^\nu \quad \text{.......................(B)}.$$

Having proved formula (B), we have proved Theorem II for matrices of unaffected minor determinants.

When the schemes $(M_1, M_2, \dots M_\mu)$, $(N_1, N_2, \dots N_\nu)$ are given, the matrix $[\xi]_\mu^\nu$ is completely determined, but the product on the right of (B) is still to a large extent arbitrary, since it is still open to us to select the schemes $(R_1, R_2, \dots R_\rho)$, $(S_1, S_2, \dots S_\sigma)$, $(T_1, T_2, \dots T_\tau)$ in any manner we please. Thus when $[\xi]_\mu^\nu$ is given, there are many products which can be placed on the right in (B); but by the properties of passive rows (see § 52.8) all these products are equal to one another.

To prove Theorem II for matrices of affected minor determinants, let

ω = the vertical affect of ξ_{pq} in X = the vertical affect of α_{pu} in A,

ω' = the horizontal affect of ξ_{pq} in X = the horizontal affect of δ_{wq} in D,

ω_1 = the horizontal affect of α_{pu} in A = the vertical affect of β_{uv} in B,

ω_2 = the horizontal affect of β_{uv} in B = the vertical affect of γ_{vw} in C,

ω_3 = the horizontal affect of γ_{vw} in C = the vertical affect of δ_{wq} in D.

Then by the definition of ξ'_{pq}, we have

$$\xi'_{pq} = (-1)^{\omega + \omega'} \xi_{pq}.$$

Let

$$\beta'_{uv} = (-1)^{\omega_1 + \omega_2} \beta_{uv}, \quad \gamma'_{vw} = (-1)^{\omega_2 + \omega_3} \gamma_{vw}, \quad \delta'_{wq} = (-1)^{\omega_3 + \omega} \delta_{wq}.$$

Then ξ'_{pq}, α'_{pu}, β'_{uv}, γ'_{vw}, δ'_{wq} are the determinants ξ_{pq}, α_{pu}, β_{uv}, γ_{vw}, δ_{wq}, each provided with the sign determined by its affect in the matrix from which it is derived.

From equation (6) it follows that

$$\xi'_{pq} = \sum_u \sum_v \sum_w \alpha'_{pu} \beta'_{uv} \gamma'_{vw} \delta'_{wq} \quad \text{.....................(7)},$$

and from this it follows by § 51 that

$$[\xi']^{\nu}_{\mu} = [\alpha']^{\rho}_{\mu} [\beta']^{\sigma}_{\rho} [\gamma']^{\tau}_{\sigma} [\delta']^{\nu}_{\tau} \quad \text{.....................(C)}.$$

The matrices occurring in formula (C) are the matrices of the affected minor determinants of order k of X, A, B, C, D formed by following the same schemes as before; and having proved formula (C), we have proved Theorem II for matrices of affected minor determinants.

NOTE 1. *Deduction of Theorem I from Theorem II.*

When one or more of the passivities is less than k, we can, using the properties of passive rows, replace the product on the right of (1) or (2) by one which has no passivity less than k by inserting pairs of corresponding additional final passive rows of 0's in the factor matrices; and we can then apply Theorem II. Since in this case one at least of the modified factor matrices will have all its minor determinants of order k equal to zero, we see in this way also that $[\xi]^{\nu}_{\mu} = 0$ and $[\xi']^{\nu}_{\mu} = 0$.

NOTE 2. *Standard equation for a matrix of the minor determinants of a given product.*

Equation (B) assumes many different forms according to the choice made of the schemes $(M_1, M_2, \ldots M_\mu)$, $(R_1, R_2, \ldots R_\rho)$, $(S_1, S_2, \ldots S_\sigma)$, $(T_1, T_2, \ldots T_\tau)$, $(N_1, N_2, \ldots N_\nu)$; but it gives in all cases one of the matrices of the minor determinants of order k of the product of the four given matrices. When in particular all the schemes are standard schemes, the equation will be called the *standard equation* for the matrix of the minor determinants of order k of the given product.

The standard equation always gives the standard matrix of the minor determinants; but there are of course other forms of the equation which also give the standard matrix.

Similar remarks may be made of equation (C).

NOTE 3. *Products not of standard form.*

It is clear that Theorems I and II remain true when the product of the given matrices is not a product of standard form.

Ex. i. Let
$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \\ l_4 & m_4 & n_4 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{bmatrix},$$

or
$$[xy]_{1234} = [lmn]_{1234} [abc]_{123} [\lambda\mu]_{123}.$$

Then

$$\begin{bmatrix} (xy)_{12} \\ (xy)_{13} \\ (xy)_{14} \\ (xy)_{23} \\ (xy)_{24} \\ (xy)_{34} \end{bmatrix} = \begin{bmatrix} (lm)_{12} & (ln)_{12} & (mn)_{12} \\ (lm)_{13} & (ln)_{13} & (mn)_{13} \\ (lm)_{14} & (ln)_{14} & (mn)_{14} \\ (lm)_{23} & (ln)_{23} & (mn)_{23} \\ (lm)_{24} & (ln)_{24} & (mn)_{24} \\ (lm)_{34} & (ln)_{34} & (mn)_{34} \end{bmatrix} \begin{bmatrix} (ab)_{12} & (ac)_{12} & (bc)_{12} \\ (ab)_{13} & (ac)_{13} & (bc)_{13} \\ (ab)_{23} & (ac)_{23} & (bc)_{23} \end{bmatrix} \begin{bmatrix} (\lambda\mu)_{12} \\ (\lambda\mu)_{13} \\ (\lambda\mu)_{23} \end{bmatrix}.$$

This is obtained from formula (B), the schemes for the horizontal and vertical rows of the three successive factor matrices being

(12, 13, 14, 23, 24, 34), (12, 13, 23), (12, 13, 23)

and (12, 13, 23), (12, 13, 23), (12) respectively.

Ex. ii. Let

$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{bmatrix},$$

where the product matrix must have the form given, since the product on the left is clearly self-conjugate.

Then $$\begin{bmatrix} A' & H' & G' \\ H' & B' & F' \\ G' & F' & C' \end{bmatrix} = \begin{bmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{bmatrix} \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix} \begin{bmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \\ N_1 & N_2 & N_3 \end{bmatrix},$$

where each matrix of large letters is the reciprocal of the corresponding matrix with small letters.

We obtain this result by equating the matrices of the affected minor determinants of order 2 of both sides, using the schemes

(23, 13, 12), (23, 13, 12), (23, 13, 12)

both for the horizontal rows and for the vertical rows of the three successive factor matrices.

We can also obtain it by equating the matrices of the unaffected minor determinants of order 2 using the schemes

(23, 31, 12), (23, 31, 12), (23, 31, 12).

Ex. iii. Let

$$\begin{bmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 & \pi_1 \\ \lambda_2 & \mu_2 & \nu_2 & \pi_2 \\ \lambda_3 & \mu_3 & \nu_3 & \pi_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \\ \alpha_4 & \beta_4 & \gamma_4 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}.$$

Then $$\begin{bmatrix} (yz)_{12} & (xz)_{12} & (xy)_{12} \\ (yz)_{13} & (xz)_{13} & (xy)_{13} \\ (yz)_{23} & (xz)_{23} & (xy)_{23} \end{bmatrix} = ABCD,$$

where
$$A = \begin{bmatrix} (pn)_{12} & (pm)_{12} & (nm)_{12} & (pl)_{12} & (nl)_{12} & (ml)_{12} \\ (pn)_{13} & (pm)_{13} & (nm)_{13} & (pl)_{13} & (nl)_{13} & (ml)_{13} \\ (pn)_{23} & (pm)_{23} & (nm)_{23} & (pl)_{23} & (nl)_{23} & (ml)_{23} \end{bmatrix},$$

$$B = \begin{bmatrix} (bc)_{43} & (ca)_{43} & (ab)_{43} \\ (bc)_{42} & (ca)_{42} & (ab)_{42} \\ (bc)_{32} & (ca)_{32} & (ab)_{32} \\ (bc)_{41} & (ca)_{41} & (ab)_{41} \\ (bc)_{31} & (ca)_{31} & (ab)_{31} \\ (bc)_{21} & (ca)_{21} & (ab)_{21} \end{bmatrix},$$

$$C = \begin{bmatrix} (\mu\nu)_{23} & (\nu\lambda)_{23} & (\lambda\mu)_{23} & (\lambda\pi)_{23} & (\mu\pi)_{23} & (\nu\pi)_{23} \\ (\mu\nu)_{31} & (\nu\lambda)_{31} & (\lambda\mu)_{31} & (\lambda\pi)_{31} & (\mu\pi)_{31} & (\nu\pi)_{31} \\ (\mu\nu)_{12} & (\nu\lambda)_{12} & (\lambda\mu)_{12} & (\lambda\pi)_{12} & (\mu\pi)_{12} & (\nu\pi)_{12} \end{bmatrix},$$

$$D = \begin{bmatrix} (\beta\gamma)_{23} & (\alpha\gamma)_{23} & (\alpha\beta)_{23} \\ (\beta\gamma)_{31} & (\alpha\gamma)_{31} & (\alpha\beta)_{31} \\ (\beta\gamma)_{12} & (\alpha\gamma)_{12} & (\alpha\beta)_{12} \\ (\beta\gamma)_{14} & (\alpha\gamma)_{14} & (\alpha\beta)_{14} \\ (\beta\gamma)_{24} & (\alpha\gamma)_{24} & (\alpha\beta)_{24} \\ (\beta\gamma)_{34} & (\alpha\gamma)_{34} & (\alpha\beta)_{34} \end{bmatrix}.$$

Here we have equated the matrices of the unaffected minor determinants of order 2, following the schemes

$$(12,\ 13,\ 23), \quad (23,\ 13,\ 12)$$

for the active rows of the first and last factor matrices respectively, and the schemes

$$(43,\ 42,\ 32,\ 41,\ 31,\ 21), \quad (23,\ 31,\ 12), \quad (23,\ 31,\ 12,\ 14,\ 24,\ 34)$$

for the passive rows of the first, second and third pairs of adjacent factor matrices.

Ex. iv. If $[xyz]_{1234} = [lmn]_{1234}[abc]_{123}[\lambda\mu\nu]_{123}$, the standard equation for the (unaffected) minor determinants of order 2 of the product matrix is

$$\begin{bmatrix} (xy)_{12} & (xz)_{12} & (yz)_{12} \\ (xy)_{13} & (xz)_{13} & (yz)_{13} \\ (xy)_{14} & (xz)_{14} & (yz)_{14} \\ (xy)_{23} & (xz)_{23} & (yz)_{23} \\ (xy)_{24} & (xz)_{24} & (yz)_{24} \\ (xy)_{34} & (xz)_{34} & (yz)_{34} \end{bmatrix} = \begin{bmatrix} (lm)_{12} & (ln)_{12} & (mn)_{12} \\ (lm)_{13} & (ln)_{13} & (mn)_{13} \\ (lm)_{14} & (ln)_{14} & (mn)_{14} \\ (lm)_{23} & (ln)_{23} & (mn)_{23} \\ (lm)_{24} & (ln)_{24} & (mn)_{24} \\ (lm)_{34} & (ln)_{34} & (mn)_{34} \end{bmatrix}$$

$$\times \begin{bmatrix} (ab)_{12} & (ac)_{12} & (bc)_{12} \\ (ab)_{13} & (ac)_{13} & (bc)_{13} \\ (ab)_{23} & (ac)_{23} & (bc)_{23} \end{bmatrix} \begin{bmatrix} (\lambda\mu)_{12} & (\lambda\nu)_{12} & (\mu\nu)_{12} \\ (\lambda\mu)_{13} & (\lambda\nu)_{13} & (\mu\nu)_{13} \\ (\lambda\mu)_{23} & (\lambda\nu)_{23} & (\mu\nu)_{23} \end{bmatrix}.$$

§ 67. Reciprocal and conjugate reciprocal of a standard product of square matrices.

In a standard product of square matrices, the matrices must all have the same order.

Let then $$[x]_m^m = [a]_m^m [b]_m^m [c]_m^m \dots [s]_m^m [t]_m^m \qquad \text{.....(1)}$$

be a product of any number of square matrices of the same order m, and let

$$[X]_m^m,\ [A]_m^m,\ [B]_m^m,\ [C]_m^m,\ \dots [S]_m^m,\ [T]_m^m$$

be the reciprocal matrices of

$$[x]_m^m,\ [a]_m^m,\ [b]_m^m,\ [c]_m^m,\ \dots [s]_m^m,\ [t]_m^m$$

respectively.

If in equation (1) we equate the matrices of the affected minor determinants of order $m-1$, the scheme for both the horizontal and the vertical minor matrices of reduced order $m-1$ being in every case

$$(2\,3\,4 \dots m,\ 1\,3\,4 \dots m,\ 1\,2\,4 \dots m,\ \dots 1\,2\,3 \dots \overline{m-1}),$$

where 1 is omitted in the first minor sequence, 2 in the second, 3 in the third, ... m in the mth, we obtain

$$[X]_m^m = [A]_m^m [B]_m^m [C]_m^m \dots [S]_m^m [T]_m^m \qquad \text{..............(2).}$$

If we equate the conjugates of both sides in equation (2), we obtain by § 54.2

$$\overline{\underline{X}}{}_m^m = \overline{\underline{T}}{}_m^m \overline{\underline{S}}{}_m^m \dots \overline{\underline{C}}{}_m^m \overline{\underline{B}}{}_m^m \overline{\underline{A}}{}_m^m \qquad \text{....................(3).}$$

Equations (2) and (3) lead to the following theorems:

Theorem I. *The reciprocal of a standard product of any number of square matrices taken in a given order is equal to the product of the reciprocals of the factor matrices taken in the same order.*

Theorem II. *The conjugate reciprocal of a standard product of any number of square matrices taken in a given order is equal to the product of the conjugate reciprocals of the factor matrices taken in the reverse order.*

NOTE. *Alternative Proof.*

We can prove Theorem I without making use of the general theorem of § 66.

Using the same notation as in the text, let $A'_{ij}, B'_{ij}, \dots X'_{ij}$ be the corranged minor determinants of $[a]_m^m, [b]_m^m, \dots [x]_m^m$ formed with the same rows as $A_{ij}, B_{ij}, \dots X_{ij}$ respectively, so that

$$A'_{ij} = (-1)^\omega A_{ij},\quad B'_{ij} = (-1)^\omega B_{ij},\ \dots\ X'_{ij} = (-1)^\omega X_{ij}.$$

where $$\omega = (i-1)+(j-1),$$
and therefore
$$[A'_{i1} A'_{i2} \dots A'_{im}] = (-1)^{i-1}[A_{i1} A_{i2} \dots A_{im}], \quad [B'_{1j} B'_{2j} \dots B'_{mj}] = (-1)^{j-1}[B_{1j} B_{2j} \dots B_{mj}].$$

First let $$[a]_m^m [b]_m^m = [x]_m^m.$$

Then by the properties of active rows

$$X'_{ij} = \det \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ \dots & \dots & \dots & \dots \\ a_{i-1,1} & a_{i-1,2} & \dots & a_{i-1,m} \\ a_{i+1,1} & a_{i+1,2} & \dots & a_{i+1,m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1,j-1} & b_{1,j+1} & \dots & b_{1m} \\ b_{21} & \dots & b_{2,j-1} & b_{2,j+1} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{m1} & \dots & b_{m,j-1} & b_{m,j+1} & \dots & b_{mm} \end{bmatrix}.$$

Here the corranged simple minor determinants of the two factor matrices on the right are $A'_{i1}, A'_{i2}, \dots A'_{im}$ and $B'_{1j}, B'_{2j}, \dots B'_{mj}$ respectively. Hence by § 57.3

$$X'_{ij} = \det [A'_{i1}\ A'_{i2} \dots A'_{im}] \begin{bmatrix} B'_{1j} \\ B'_{2j} \\ \vdots \\ B'_{mj} \end{bmatrix}, \quad \text{and therefore} \quad X_{ij} = \det [A_{i1}\ A_{i2} \dots A_{im}] \begin{bmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{mj} \end{bmatrix}.$$

It follows that $$[X]_m^m = [A]_m^m [B]_m^m.$$

Thus Theorem I is proved for a product with two factors.

Next let $$[a]_m^m [b]_m^m [c]_m^m = [x]_m^m.$$

Write $$[b]_m^m [c]_m^m = [p]_m^m, \quad \text{so that} \quad [a]_m^m [p]_m^m = [x]_m^m.$$

Then $$[X]_m^m = [A]_m^m [P]_m^m, \quad \text{and} \quad [P]_m^m = [B]_m^m [C]_m^m.$$

Therefore $$[X]_m^m = [A]_m^m [B]_m^m [C]_m^m.$$

Thus Theorem I is proved for a product with three factors.

Proceeding in this way we see that the theorem is true for a product with any number of factors.

Having proved Theorem I, we can deduce Theorem II as in the text.

Ex. i. *If a product of two square matrices of the same order is a non-zero scalar matrix, then the product is commutative.*

Let $$[a]_m^m [b]_m^m = k[1]_m^m, \quad \text{where } k \neq 0 \quad \dots\dots\dots(4).$$

Equating the determinants of both sides, we have

$$(a)_m^m (b)_m^m = k^m,$$

and from this we see that $$(a)_m^m \neq 0, \quad (b)_m^m \neq 0.$$

Let $$\Delta = (a)_m^m \neq 0.$$

Further let $[A]_m^m$, $[B]_m^m$ be the reciprocals of $[a]_m^m$, $[b]_m^m$.

Prefixing $\underline{\overline{A}}{}^m_m$ on both sides of (4), we obtain by Theorem B) of § 46.5 and by § 55.2

$$\Delta\,[b]^m_m = k\,\underline{\overline{A}}{}^m_m .$$

Now postfixing $[a]^m_m$ on both sides, we have

$$\Delta\,[b]^m_m\,[a]^m_m = k\,\Delta\,[1]^m_m .$$

Since $\Delta \neq 0$, we deduce that

$$[b]^m_m\,[a]^m_m = k\,[1]^m_m = [a]^m_m\,[b]^m_m .$$

Thus the factors of the product $[a]^m_m\,[b]^m_m$ can be commuted.

From this theorem we deduce Ex. xviii of § 46.

As an example we have

$$\begin{bmatrix} 2, & 4, & 6 \\ 4, & 6, & 2 \\ 8, & 2, & 2 \end{bmatrix}\begin{bmatrix} -2, & -1, & 7 \\ -2, & 11, & -5 \\ 10, & -7, & 1 \end{bmatrix} = \begin{bmatrix} 48, & 0, & 0 \\ 0, & 48, & 0 \\ 0, & 0, & 48 \end{bmatrix};$$

also

$$\begin{bmatrix} -2, & -1, & 7 \\ -2, & 11, & -5 \\ 10, & -7, & 1 \end{bmatrix}\begin{bmatrix} 2, & 4, & 6 \\ 4, & 6, & 2 \\ 8, & 2, & 2 \end{bmatrix} = \begin{bmatrix} 48, & 0, & 0 \\ 0, & 48, & 0 \\ 0, & 0, & 48 \end{bmatrix}.$$

Ex. ii. If a product of two square matrices of the same order is a zero matrix, then the product is not necessarily commutative.

As an example we have

$$\begin{bmatrix} 1, & 1, & 1 \\ 2, & 3, & 1 \\ 2, & 2, & 2 \end{bmatrix}\begin{bmatrix} 2, & -4, & 6 \\ -1, & 2, & -3 \\ -1, & 2, & -3 \end{bmatrix} = 0,$$

but

$$\begin{bmatrix} 2, & -4, & 6 \\ -1, & 2, & -3 \\ -1, & 2, & -3 \end{bmatrix}\begin{bmatrix} 1, & 1, & 1 \\ 2, & 3, & 1 \\ 2, & 2, & 2 \end{bmatrix} = \begin{bmatrix} 6, & 2, & 10 \\ -3, & -1, & -5 \\ -3, & -1, & -5 \end{bmatrix}.$$

If however one of the matrices is the conjugate reciprocal of the other, then the product is commutative.

As an example we have

$$\begin{bmatrix} 1, & 1, & 1 \\ 2, & 3, & 1 \\ 2, & 2, & 2 \end{bmatrix}\begin{bmatrix} 4, & 0, & -2 \\ -2, & 0, & 1 \\ -2, & 0, & 1 \end{bmatrix} = 0, \quad \text{and} \quad \begin{bmatrix} 4, & 0, & -2 \\ -2, & 0, & 1 \\ -2, & 0, & 1 \end{bmatrix}\begin{bmatrix} 1, & 1, & 1 \\ 2, & 3, & 1 \\ 2, & 2, & 2 \end{bmatrix} = 0.$$

Ex. iii. If
$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
then
$$\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

That is, if (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are the direction cosines of three mutually perpendicular straight lines, then (l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3) are also the direction cosines of three mutually perpendicular straight lines.

Ex. iv. If
$$\begin{bmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ l_4 & m_4 & n_4 & p_4 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ p_1 & p_2 & p_3 & p_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
then
$$\begin{bmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ p_1 & p_2 & p_3 & p_4 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ l_4 & m_4 & n_4 & p_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Ex. v. If $[l]_m^m \overline{\underline{l}}_m^m = [1]_m^m$, then $\overline{\underline{l}}_m^m [l]_m^m = [1]_m^m$.

CHAPTER IX.

RANK OF A MATRIX AND CONNECTIONS BETWEEN THE ROWS OF A MATRIX.

[In the first two articles, §§ 68 and 69, the rank of any matrix whose elements have all definite values is defined, and the meaning of a connection between its rows is explained. The next three articles, §§ 70—72, contain a number of theorems relating to the rank of such a matrix and the connections between its rows. In § 73 the rank of every complete matrix of the minor determinants of a given fundamental matrix is determined. Lastly in § 74 the preceding definitions and theorems are extended to matrices whose elements are rational integral functions of certain variables.]

§ 68. Rank of a matrix whose elements are constants.

Let $A = [a]_m^n$ be a matrix each of whose elements has a single definite value, and let s be any number which is not greater than the efficiency of the matrix. If all the derived determinants of order s of the matrix vanish, then all derived determinants of orders greater than s vanish; for every such determinant can be expanded in terms of derived determinants of order s.

The *rank* of the matrix A is the greatest order which a non-vanishing derived determinant can have. Thus the rank of the matrix A is r when A has at least one derived determinant of order r which does not vanish, whilst every derived determinant of order $r+1$ vanishes and therefore every derived determinant of order greater than r vanishes.

A matrix will be said to have rank 0 when all its elements vanish, i.e. when it is a zero matrix.

Clearly the rank of a matrix cannot exceed its efficiency. A matrix whose rank is less than its efficiency will be called a *degenerate matrix*. A matrix whose rank is equal to its efficiency will be called an *undegenerate matrix*. An undegenerate matrix has non-vanishing derived determinants of all orders not exceeding the efficiency; its determinoid may or may not be equal to zero. In a degenerate matrix of efficiency η, all derived determinants of order η vanish; also the determinoid of the matrix vanishes.

A *singular matrix* is one whose determinoid vanishes. A *non-singular matrix* is one whose determinoid does not vanish. An undegenerate matrix may be either non-singular or singular; and a singular matrix may be either undegenerate or degenerate. On the other hand a degenerate matrix is necessarily singular, and a non-singular matrix is necessarily undegenerate.

A determinoid will be said to have the same rank as its matrix.

Ex. i. Each of the matrices

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 11 & 9 & 25 \\ 3 & 4 & 1 & 0 \\ 5 & 7 & 2 & 1 \\ 4 & 11 & 7 & 17 \\ 2 & 5 & 3 & 7 \end{bmatrix}$$

has rank 2; for in each of them every derived determinant of order 3 vanishes but there are derived determinants of order 2 which do not vanish.

Ex. ii. *If A' is any derangement of a matrix A, then A' has the same rank as A.* For to every derived determinant Δ of order k of A there corresponds a derived determinant Δ' of order k of A' which is a derangement of Δ. Since $\Delta' = \pm\Delta$, it follows that Δ' does or does not vanish according as Δ does or does not vanish. Thus we can institute a one-one correspondence between the derived determinants of order k of A and A' such that to every vanishing or non-vanishing derived determinant of order k of one matrix there corresponds respectively a vanishing or non-vanishing derived determinant of order k of the other matrix.

Again if A' is formed from A by changing the signs of all the elements in some of the horizontal and vertical rows of A, it appears in the same way that A' has the same rank as A.

Ex. iii. The complete matrices of the minor determinants of given order k of any matrix A have all the same rank. For any one of them can be derived from any other by deranging its rows and by changing the signs of all the elements in some of its rows.

Ex. iv. The rank of a matrix A is r when a complete matrix of the minor determinants of order $r+1$ of A vanishes whilst a complete matrix of the minor determinants of order r of A does not vanish. The complete matrices of the minor determinants of all orders greater than r then vanish.

Ex. v. The rank of a matrix A is r when there are non-vanishing derived determinoids of A of efficiency r, but no non-vanishing derived determinoids of greater efficiency than r.

Ex. vi. A matrix and its conjugate have equal ranks.

Ex. vii. *If a product of two matrices is a non-zero scalar matrix, then both the factor matrices are undegenerate.*

Let $$[a]_m^n \, [b]_n^m = k\,[1]_m^m, \qquad \text{where } k \neq 0.$$

If $n < m$, the determinant of the product on the left is equal to zero, whilst the determinant of the matrix on the right is equal to k^m and is not zero.

Thus a product of two matrices cannot be a non-zero scalar matrix unless the active rows of the factor matrices are long rows.

Assuming that $n \nless m$, we obtain by equating the determinants of both sides

$$\Sigma (a_{1q})^m_m (b_{q1})^m_m = k^m \neq 0,$$

where $[q_1 q_2 \dots q_m]$ is any corranged minor of order m of $[1\,2 \dots n]$.

This result could not be true if all the determinants $(a_{1q})^m_m$ vanished, or if all the determinants $(b_{q1})^m_m$ vanished.

Hence both the factor matrices must have rank m and be undegenerate.

Ex. viii. *If two matrices are inverse to one another, then both of them are undegenerate.*

This is a particular case of Ex. vii. It follows that *a degenerate matrix has no inverse.*

Ex. ix. *If the determinoid* Δ *of a matrix does not vanish (i.e. if the matrix is non-singular), then the matrix itself and its reciprocal, its conjugate reciprocal and its inverse are all undegenerate.*

For with the matrix and its conjugate reciprocal we can form a product which is a non-zero scalar matrix with argument Δ. It follows from Ex. vii that the matrix and its conjugate reciprocal are undegenerate. The (principal) inverse matrix, being a non-zero scalar multiple of the conjugate reciprocal matrix, is also undegenerate. Further by Ex. viii every inverse matrix is undegenerate.

Ex. x. *If a square matrix is undegenerate (or non-singular), then its reciprocal, its conjugate reciprocal and its inverse are all undegenerate (or non-singular) square matrices.*

This is a particular case of Ex. ix.

Ex. xi. *If* $[A]^m_m$ *is the reciprocal of the square matrix* $[a]^m_m$, *and if* $(a)^m_m = \Delta$, *then the equation*

$$det\,[A]^m_m = (A)^m_m = \Delta^{m-1}$$

is true in all cases whether $[a]^m_m$ *is or is not degenerate.*

Equating the determinants of both sides in the equation

$$[a]^m_m \,\overline{\underline{A}}^m_m = \Delta\,[1]^m_m \quad\dots\dots\dots(1),$$

we have

$$\Delta\,(A)^m_m = \Delta^m \quad\dots\dots\dots(2).$$

Now equation (2), like equation (1), is an identity in the elements of $[a]^m_m$. Since Δ is a function of those same elements which does not vanish identically, it follows from (2) that

$$(A)^m_m = \Delta^{m-1} \quad\dots\dots\dots(3),$$

and equation (3) is also an *identity* in the elements of $[a]^m_m$.

We can deduce Ex. x from (3); for equation (3) shows that $(A)^m_m$ does or does not vanish according as Δ does or does not vanish.

Ex. xii. *If* $[A]_m^{m+1}$ *is the reciprocal of a matrix* $[a]_m^{m+1}$ *in which the number of short rows is greater by* 1 *than the number of long rows, and if*

$$(a)_m^{m+1} = det\,[a]_m^{m+1} = \Delta,$$

then
$$(A)_m^{m+1} = det\,[A]_m^{m+1} = (m+1)\,\Delta^{m-1}.$$

Let b_q, B_q be the corranged simple minor determinants of $[a]_m^{m+1}$, $[A]_m^{m+1}$ formed by striking out the qth vertical rows, this being true for all the values $1, 2, \dots (m+1)$ of q; and let

$$\omega = \text{affect of } b_q \text{ in } [a]_m^{m+1} = \text{affect of } B_q \text{ in } [A]_m^{m+1}.$$

Then

$$\begin{bmatrix} a_{11} & \dots & a_{1,q-1} & a_{1,q+1} & \dots & a_{1m} \\ a_{21} & \dots & a_{2,q-1} & a_{2,q+1} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{m,q-1} & a_{m,q+1} & \dots & a_{mm} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{m1} \\ \dots & \dots & \dots & \dots \\ A_{1,q-1} & A_{2,q-1} & \dots & A_{m,q-1} \\ A_{1,q+1} & A_{2,q+1} & \dots & A_{m,q+1} \\ \dots & \dots & \dots & \dots \\ A_{1m} & A_{2m} & \dots & A_{mm} \end{bmatrix}$$

$$= \begin{bmatrix} \Delta - a_{1q}A_{1q}, & -a_{1q}A_{2q}, & \dots & -a_{1q}A_{mq} \\ -a_{2q}A_{1q}, & \Delta - a_{2q}A_{2q}, & \dots & -a_{2q}A_{mq} \\ \dots & \dots & \dots & \dots \\ -a_{mq}A_{1q}, & -a_{mq}A_{2q}, & \dots & \Delta - a_{mq}A_{mq} \end{bmatrix}$$

$$= \begin{bmatrix} \Delta & 0 & \dots & 0 \\ 0 & \Delta & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta \end{bmatrix} - \begin{bmatrix} a_{1q} \\ a_{2q} \\ \vdots \\ a_{mq} \end{bmatrix} [A_{1q} \;\; A_{2q} \dots A_{mq}]$$

$$= \begin{bmatrix} \sqrt{\Delta} & 0 & \dots & 0 \\ 0 & \sqrt{\Delta} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{\Delta} \end{bmatrix} \begin{bmatrix} \sqrt{\Delta} & 0 & \dots & 0 \\ 0 & \sqrt{\Delta} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{\Delta} \end{bmatrix} + \begin{bmatrix} i\,a_{1q} \\ i\,a_{2q} \\ \vdots \\ i\,a_{mq} \end{bmatrix} [iA_{1q}, \; iA_{2q}, \dots iA_{mq}]$$

$$= \begin{bmatrix} ia_{1q} & \sqrt{\Delta} & 0 & \dots & 0 \\ ia_{2q} & 0 & \sqrt{\Delta} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ ia_{mq} & 0 & 0 & \dots & \sqrt{\Delta} \end{bmatrix} \begin{bmatrix} iA_{1q} & iA_{2q} & \dots & iA_{mq} \\ \sqrt{\Delta} & 0 & \dots & 0 \\ 0 & \sqrt{\Delta} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{\Delta} \end{bmatrix}.$$

Equating the determinants of both sides in this equation, we have by § 57.3

$$b_q B_q = \Delta^m - \Delta^{m-1}(a_{1q}A_{1q} + a_{2q}A_{2q} + \dots + a_{mq}A_{mq}).$$

Now by § 29 and § 30

$$\Delta = \det [a]_m^{m+1}$$

$$= a_{1q}A_{1q} + a_{2q}A_{2q} + \dots + a_{mq}A_{mq} + \begin{bmatrix} a_{11} & \dots & a_{1,q-1} & 0 & a_{1,q+1} & \dots & a_{1,m+1} \\ a_{21} & \dots & a_{2,q-1} & 0 & a_{2,q+1} & \dots & a_{2,m+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{m,q-1} & 0 & a_{m,q+1} & \dots & a_{m,m+1} \end{bmatrix}$$

$$= a_{1q}A_{1q} + a_{2q}A_{2q} + \dots + a_{mq}A_{mq} + (-1)^{\omega} b_q,$$

and therefore
$$a_{1q}A_{1q} + a_{2q}A_{2q} + \dots + a_{mq}A_{mq} = \Delta - (-1)^{\omega} b_q.$$

Thus $$b_q B_q = \Delta^m - \Delta^{m-1}(\Delta - (-1)^\omega b_q) = (-1)^\omega \Delta^{m-1} b_q.$$

Since this is an identity in the elements of $[a]_m^{m+1}$, and b_q is a function of those elements which does not vanish identically, it follows that

$$B_q = (-1)^\omega \Delta^{m-1}.$$

Consequently we have

$$(A)_m^{m+1} - \Sigma(-1)^\omega B_q = (m+1)\Delta^{m-1}.$$

We observe that all the affected simple minor determinants of $[A]_m^{m+1}$ have the common value Δ^{m-1}.

§ 69. Connections between the rows of a matrix whose elements are constants.

Let $A = [a]_m^n$ be a matrix of constant elements with m horizontal and n vertical rows.

There is said to be a *connection between* the horizontal rows or their matrices when there exist scalar multipliers $h_1, h_2, \ldots h_m$, not all zero, such that

$$h_1[a_{11}a_{12}\ldots a_{1n}] + h_2[a_{21}a_{22}\ldots a_{2n}] + \ldots + h_m[a_{m1}a_{m2}\ldots a_{mn}] = 0 \ldots\ldots (1).$$

The matrix equation (1) is equivalent to the m scalar equations

$$\begin{aligned} h_1a_{11} + h_2a_{21} + \ldots + h_ma_{m1} &= 0,\\ h_1a_{12} + h_2a_{22} + \ldots + h_ma_{m2} &= 0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\ldots\\ h_1a_{1m} + h_2a_{2m} + \ldots + h_ma_{mm} &= 0. \end{aligned}$$

By the properties of passive rows (see § 43.1), it can also be written in the form

$$[h_1\ h_2 \ldots h_m]\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ a_{m1} & a_{m2} & \ldots & a_{mn}\end{bmatrix} = 0 \quad \ldots\ldots\ldots\ldots \quad (2).$$

If h_i is not zero, the ith horizontal row (or its matrix) is said to be *connected with* the other horizontal rows (or their matrices). In this case the matrix of the ith horizontal row can be expressed as a homogeneous linear function of the matrices of the other horizontal rows.

When no such relation as (1) exists, the horizontal rows (or their matrices) are said to be *unconnected*.

Again there is said to be a *connection between* the vertical rows of A or their matrices when there exist constant scalar multipliers $k_1, k_2, \ldots k_n$, not all zero, such that

$$k_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + k_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + k_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots(3).$$

The matrix equation (3) is equivalent to the m scalar equations

$$\begin{aligned} k_1 a_{11} + k_2 a_{12} + \ldots + k_n a_{1n} &= 0, \\ k_1 a_{21} + k_2 a_{22} + \ldots + k_n a_{2n} &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ k_1 a_{m1} + k_2 a_{m2} + \ldots + k_n a_{mn} &= 0. \end{aligned}$$

It can also, by the properties of passive rows, be written in the form

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If k_i is not zero, the matrix of the ith vertical row can be expressed as a homogeneous linear function of the matrices of the other vertical rows, and in this case the ith vertical row (or its matrix) is said to be *connected with* the other vertical rows (or their matrices).

When no such relation as (3) exists, the vertical rows of A are said to be *unconnected*.

It should be observed that every row of the matrix is connected with itself, and therefore with all the parallel rows of the matrix including itself.

Ex. i. In the matrix $[a]_m^n$ we have

$$[a_{21}\, a_{22}\, a_{23} \ldots a_{2n}] = 0 \,.\, [a_{11}\, a_{12}\, a_{13} \ldots a_{1n}] + 1 \,.\, [a_{21}\, a_{22}\, a_{23} \ldots a_{2n}]$$
$$+ 0 \,.\, [a_{31}\, a_{32}\, a_{33} \ldots a_{3n}] + \ldots + 0 \,.\, [a_{m1}\, a_{m2}\, a_{m3} \ldots a_{mn}].$$

Thus the second horizontal row is connected with all the horizontal rows.

Ex. ii.
$$\begin{bmatrix} 2 & 11 & 9 & 25 \\ 3 & 4 & 1 & 0 \\ 5 & 7 & 2 & 1 \\ 4 & 11 & 7 & 17 \\ 2 & 5 & 3 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix} = 0,$$

or
$$0 \cdot \begin{bmatrix} 2 \\ 3 \\ 5 \\ 4 \\ 2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 11 \\ 4 \\ 7 \\ 11 \\ 5 \end{bmatrix} - 4 \cdot \begin{bmatrix} 9 \\ 1 \\ 2 \\ 7 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 25 \\ 0 \\ 1 \\ 17 \\ 7 \end{bmatrix} = 0.$$

Here there is a connection between the vertical rows of the large matrix. The 2nd, 3rd, and 4th vertical rows are each connected with the remaining vertical rows of the matrix.

§ 70. Theorems concerning connections between the rows of a matrix whose elements are constants.

The most important properties of connections are included in the following theorems.

Theorem I. *If there exists any connection between the horizontal rows or the vertical rows of a square matrix, the determinant of the matrix vanishes.*

Let the square matrix be $A = [a]_m^m$, and first suppose that there is a connection

$$h_1[a_{11}a_{12} \dots a_{1m}] + h_2[a_{21}a_{22} \dots a_{2m}] + \dots + h_m[a_{m1}a_{m2} \dots a_{mm}] = 0,$$

or
$$[h_1 h_2 \dots h_m][a]_m^m = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots (1),$$

between the horizontal rows of A, where $h_i \neq 0$.

By the properties of determinants we can transform the determinant $(a)_m^m$ into a determinant in which the ith horizontal row is a row of 0's by adding to the matrix of the ith horizontal row suitable multiples of the matrices of the other horizontal rows, namely by adding the matrices of the first, second, ... mth horizontal rows multiplied by $\frac{h_1}{h_i}, \frac{h_2}{h_i}, \dots \frac{h_m}{h_i}$ respectively. Accordingly $(a)_m^m = 0$. This result can also be obtained at once from the equation

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ h_1 & h_2 & \dots & h_m \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{im} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix};$$

for if in this equation we equate the determinants of both sides, we obtain

$$h_i (a)_m^m = 0, \quad \text{or} \quad (a)_m^m = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

Secondly suppose that there is a connection

$$k_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1}\end{bmatrix} + k_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2}\end{bmatrix} + \dots + k_m\begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{mm}\end{bmatrix} = 0, \quad \text{or} \quad [a]_m^m \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m\end{bmatrix} = 0 \quad \dots\dots\dots\dots\dots(3),$$

between the vertical rows of A, where $k_i \neq 0$. Then

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1i} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2i} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mi} & \dots & a_{mm}\end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & k_1 & \dots & 0 \\ 0 & 1 & \dots & k_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_m & \dots & 1\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & 0 & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & 0 & \dots & a_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & 0 & \dots & a_{mm}\end{bmatrix}.$$

Equating the determinants of both sides, we have

$$(a)_m^m k_i = 0, \quad \text{or} \quad (a)_m^m = 0 \quad \dots\dots\dots\dots\dots\dots(4).$$

Theorem II. *If Δ is any non-vanishing derived determinant of a matrix A, then the horizontal rows and also the vertical rows of A which occur in Δ are unconnected.*

Let $$A = [a]_m^n, \quad \Delta = (a_{pq})_r^r.$$

If there were a connection

$$[h_1 h_2 \dots h_r][a_{pl}]_r^n = 0$$

between the horizontal rows of A which are retained (or occur) in Δ, we should have by the properties of active rows

$$[h_1 h_2 \dots h_r][a_{pq}]_r^r = 0.$$

Thus there would exist a connection between the horizontal rows of Δ and therefore by Theorem I the determinant Δ would vanish, contrary to the hypothesis.

Again if there were any connection between the vertical rows of A which are retained in Δ we should have relations of the forms

$$[a_{1q}]_m^r \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r\end{bmatrix} = 0, \qquad [a_{pq}]_r^r \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r\end{bmatrix} = 0,$$

where $k_1, k_2, \dots k_r$ do not all vanish.

Thus there would exist a connection between the vertical rows of Δ and therefore Δ would vanish, contrary to the hypothesis.

Theorem III. *If r is the rank of a matrix A, and if Δ_r is one of its non-vanishing derived determinants of order r, then:*

(1) *the horizontal (and also the vertical) rows of A which occur in Δ_r are unconnected;*

(2) *all other horizontal (and similarly all other vertical) rows of A are connected with those occurring in Δ_r.*

The first part of this theorem is contained in Theorem II. It remains to prove the second part of the theorem.

Let $$A = [a]_m^n, \quad \text{and} \quad \Delta_r = (a_{pq})_r^r.$$

We will first assume that $r < m$, so that there are horizontal rows of A which do not occur in Δ_r. Let the uth horizontal row of A be one of these, and let

$$A_{r+1}^n = \begin{bmatrix} a_{p_1 1} & a_{p_1 2} \dots a_{p_1 n} \\ a_{p_2 1} & a_{p_2 2} \dots a_{p_2 n} \\ \dots & \dots \\ a_{p_r 1} & a_{p_r 2} \dots a_{p_r n} \\ a_{u1} & a_{u2} \dots a_{un} \end{bmatrix}, \quad A_{r+1}^r = \begin{bmatrix} a_{p_1 q_1} & a_{p_1 q_2} \dots a_{p_1 q_r} \\ a_{p_2 q_1} & a_{p_2 q_2} \dots a_{p_2 q_r} \\ \dots & \dots \\ a_{p_r q_1} & a_{p_r q_2} \dots a_{p_r q_r} \\ a_{uq_1} & a_{uq_2} \dots a_{uq_r} \end{bmatrix},$$

so that A_{r+1}^n is the matrix formed with the horizontal rows of A which occur in Δ_r and one additional horizontal row of A placed in the final position, and A_{r+1}^r is formed from A_{r+1}^n by retaining only those vertical rows which occur in Δ_r.

Further let $A_1, A_2, \dots A_r, A_{r+1}$ be the simple minor determinants formed from A_{r+1}^r by omitting its 1st, 2nd, ... rth, $(r+1)$th horizontal row respectively, each determinant having the sign determined by its affect in A_{r+1}^r.

Applying the prefactor $[A_1 \; A_2 \dots A_r \; A_{r+1}]$ to A_{r+1}^n, we have

$$[A_1 \; A_2 \dots A_r \; A_{r+1}] \begin{bmatrix} a_{p_1 1} & a_{p_1 2} \dots a_{p_1 n} \\ a_{p_2 1} & a_{p_2 2} \dots a_{p_2 n} \\ \dots & \dots \\ a_{p_r 1} & a_{p_r 2} \dots a_{p_r n} \\ a_{u1} & a_{u2} \dots a_{un} \end{bmatrix} = 0 \quad \dots\dots\dots\dots (5).$$

To see this we notice that the ith element of the product matrix on the left in (5)

$$= \det [A_1 \; A_2 \dots A_r \; A_{r+1}] \begin{bmatrix} a_{p_1 i} \\ a_{p_2 i} \\ \vdots \\ a_{p_r i} \\ a_{ui} \end{bmatrix} = \begin{vmatrix} a_{p_1 q_1} & a_{p_1 q_2} \dots a_{p_1 q_r} & a_{p_1 i} \\ a_{p_2 q_1} & a_{p_2 q_2} \dots a_{p_2 q_r} & a_{p_2 i} \\ \dots & \dots & \\ a_{p_r q_1} & a_{p_r q_2} \dots a_{p_r q_r} & a_{p_r i} \\ a_{uq_1} & a_{uq_2} \dots a_{uq_r} & a_{ui} \end{vmatrix} = \Delta_{r+1}.$$

the left-hand member of this last equation being the expansion of the determinant on the right in terms of the minor determinants of order r belonging to the first r vertical rows.

Now if i is one of the numbers $q_1, q_2, \dots q_r$, the determinant Δ_{r+1} has two identical vertical rows and therefore vanishes; and if i is not one of the numbers $q_1, q_2, \dots q_r$, then Δ_{r+1} is a derived determinant of A of order $r+1$ and vanishes because the rank of A is r. Thus the ith element of the product matrix on the left in (5) vanishes for all values of i, and consequently the product matrix is a zero matrix.

Since $A_{r+1} = \Delta_r \neq 0$, it follows from (1) that the uth horizontal row of A is connected with the horizontal rows of A which occur in Δ_r. Thus the second part of the theorem is proved for horizontal rows.

We will next assume that $r < n$, so that there are vertical rows of A which do not occur in Δ_r. Let the vth vertical row be one of these and let

$$A_m^{r+1} = \begin{bmatrix} a_{1q_1} & a_{1q_2} & \dots a_{1q_r} & a_{1v} \\ a_{2q_1} & a_{2q_2} & \dots a_{2q_r} & a_{2v} \\ \dots & \dots & \dots & \dots \\ a_{mq_1} & a_{mq_2} & \dots a_{mq_r} & a_{mv} \end{bmatrix}, \quad A_r^{r+1} = \begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} & \dots a_{p_1q_r} & a_{p_1v} \\ a_{p_2q_1} & a_{p_2q_2} & \dots a_{p_2q_r} & a_{p_2v} \\ \dots & \dots & \dots & \dots \\ a_{p_rq_1} & a_{p_rq_2} & \dots a_{p_rq_r} & a_{p_rv} \end{bmatrix},$$

so that A_m^{r+1} is the matrix formed with the vertical rows of A which occur in Δ_r and one additional vertical row of A placed in the final position, and A_r^{r+1} is formed from A_m^{r+1} by retaining only those horizontal rows which occur in Δ_r. Further let $A_1, A_2, \dots A_r, A_{r+1}$ be the simple minor determinants of A_r^{r+1} formed by omitting its 1st, 2nd, ... rth, $(r+1)$th vertical row, each determinant having the sign determined by its affect in A_r^{r+1}. Then by reasoning similar to that employed in the first case, we have

$$\begin{bmatrix} a_{1q_1} & a_{1q_2} & \dots a_{1q_r} & a_{1v} \\ a_{2q_1} & a_{2q_2} & \dots a_{2q_r} & a_{2v} \\ \dots & \dots & \dots & \dots \\ a_{mq_1} & a_{mq_2} & \dots a_{mq_r} & a_{mv} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \\ A_{r+1} \end{bmatrix} = 0 \quad \dots\dots\dots\dots(6),$$

for the ith element of the product matrix on the left is the determinant

$$\begin{vmatrix} a_{p_1q_1} & a_{p_1q_2} & \dots a_{p_1q_r} & a_{p_1v} \\ a_{p_2q_1} & a_{p_2q_2} & \dots a_{p_2q_r} & a_{p_2v} \\ \dots & \dots & \dots & \dots \\ a_{p_rq_1} & a_{p_rq_2} & \dots a_{p_rq_r} & a_{p_rv} \\ a_{iq_1} & a_{iq_2} & \dots a_{iq_r} & a_{iv} \end{vmatrix} = \Delta_{r+1},$$

which vanishes for all values of i from 1 to n.

Since $A_{r+1} = \Delta_r \neq 0$, it follows from (6) that the rth vertical row of A is connected with those vertical rows of A which occur in Δ_r.

Thus the second part of Theorem III is proved for vertical as well as for horizontal rows.

The following corollaries depend simply on the fact that the rank of a matrix is always either equal to or less than the number of long rows, and is less than the number of short rows when the matrix is not square.

COROLLARY 1. *If the long rows of a matrix A are unconnected, the rank of A must be equal to the number of long rows, i.e. must be equal to the efficiency of A.*

COROLLARY 2. *If there is any connection between the long rows of a matrix A, the rank of A must be less than the number of long rows, i.e. must be less than the efficiency of A.*

COROLLARY 3. *There are always connections between the short rows of a matrix which is not square.*

COROLLARY 4. *The number of rows in a set of unconnected short rows of a matrix A cannot exceed the efficiency of A.*

COROLLARY 5. *If a determinant vanishes, there are connections between its horizontal rows and also connections between its vertical rows. If a determinant does not vanish, then its horizontal rows and also its vertical rows are unconnected.*

Note 1. The proof of the theorem in the text depends on the fact that all determinants of the types Δ_{r+1} vanish. Observing the forms of Δ_{r+1}, it is clear that these determinants all vanish when the derived determinants of order $r+1$ of A which contain Δ_r as a minor all vanish. We have therefore the following theorem:

If Δ_r is a non-vanishing derived determinant of order r of a matrix A, and if all those derived determinants of order $r+1$ of A which contain Δ_r as a minor vanish, then every horizontal (or vertical) row of A is connected with those horizontal (or vertical) rows of A which occur in Δ_r.

Note 2. If we do not assume that Δ_r is *non-vanishing*, then under the same circumstances we can only conclude that there is some connection between the horizontal (or vertical) rows of A which occur in Δ_r and every other horizontal (or vertical) row of A.

Theorem IV. *If any s parallel rows of a matrix A are unconnected, they form a simple minor matrix of rank s having at least one non-vanishing derived determinant of order s.*

Let A_s be the matrix formed by the s rows. Then the s unconnected rows are necessarily long rows of A_s. Hence A_s has derived determinants of order s. These cannot all vanish, otherwise by Theorem III there would be connections between the s rows.

It follows that the rank of A_s is s. We could deduce this theorem from Corollary 1 of Theorem III.

It may be further observed that *if any s parallel rows of a matrix A are connected, they form a simple minor matrix A_s whose rank is less than s.* For if the s rows are short rows of A_s, the rank of A_s is necessarily less than s. And if the s rows are long rows of A_s, then by Theorem III the rank of A_s could only be s if the s rows were unconnected.

Theorem V. *If r is the rank of a matrix A, then:*

(1) *it is possible to select r unconnected rows of each kind in A;*

(2) *it is not possible to select more than r unconnected rows of either kind in A.*

The first part of this theorem is included in Theorem III.

The second part of the theorem follows from Theorem IV.

For if it were possible to select $r+1$ unconnected rows of either kind, the simple minor matrix of A formed by them would have a non-vanishing derived determinant of order $r+1$, and therefore A itself would have a non-vanishing derived determinant of order $r+1$. This however is impossible, since the rank of A is r.

Theorem VI. *If r is the rank of a matrix A, and if we select any r unconnected rows of A of either kind (horizontal or vertical), then every other row of A of the same kind is connected with the r selected rows.*

By Theorem IV the matrix of the r selected rows contains a non-vanishing derived determinant of order r. The theorem now follows from the second part of Theorem III when we take this determinant for Δ_r.

Theorem VII. *If all the horizontal (or vertical) rows of a matrix A are connected with r of them, the rank of A cannot exceed r. If those r rows are themselves unconnected, then the rank of A is equal to r; conversely if the rank of A is equal to r, then those r rows are unconnected.*

Let $A = [a]_m^n$, and first let all the horizontal rows of A be connected with the p_1th, p_2th, ... p_rth horizontal rows.

To prove the first part of the theorem in this case, we will show that if

$$\Delta = (a_{\alpha\beta})_{r+1}^{r+1}$$

is any derived determinant of A of order $r+1$, then $\Delta = 0$.

Let $$[a_{i1}\ a_{i2} \dots a_{in}] = [h_{i1}\ h_{i2} \dots h_{ir}]\,[a_{pj}]_r^n,$$

where i receives all the values $1, 2, \dots m$, be the connections of the various horizontal rows of A with the p_1th, p_2th, ... p_rth horizontal rows. These relations are together equivalent to

$$[a]_m^n = [h]_m^r \, [a_{p1}]_r^n .$$

By the properties of active rows it follows that

$$[a_{\alpha\beta}]_{r+1}^{r+1} = [h_{\alpha 1}]_{r+1}^{r} \, [a_{p\beta}]_r^{r+1} .$$

The product on the right has efficiency $r+1$ and passivity r. Since the passivity is less than the efficiency, the determinant of the product matrix is zero. Hence, equating the determinants of the two sides, we have

$$\Delta = 0.$$

Next let all the vertical rows of A be connected with the q_1th, q_2th, ... q_rth vertical rows. Then the single matrix equation which expresses all these connections has the form

$$[a]_m^n = [a_{1q}]_m^r \, [k]_r^n .$$

By the properties of active rows it follows that

$$[a_{\alpha\beta}]_{r+1}^{r+1} = [a_{\alpha q}]_{r+1}^{r} \, [k_{1\beta}]_r^{r+1} .$$

Equating the determinants of both sides, we have as before

$$\Delta = 0.$$

Thus in both cases every derived determinant of A of order $r+1$ vanishes, and therefore the rank of A cannot exceed r.

Thus the first part of Theorem VII is proved.

To prove the second part of the theorem we observe that if the r rows are themselves unconnected, the matrix formed by them has a non-vanishing derived determinant of order r, which is also a non-vanishing derived determinant of order r of the matrix A. Thus the rank of A cannot be less than r. But it has been shown that the rank of A cannot exceed r. Consequently the rank of A is equal to r.

Again if there were a connection between the r rows, then all the rows of A of the same kind would be connected with $r-1$ of them, and therefore by the first part of the theorem the rank of A could not exceed $r-1$. Hence if the rank of A is r, the r rows must be unconnected. Thus the third part of the theorem is proved.

Theorem VIII. *If a connection exists between any of the active rows of either extreme factor matrix in a product formed by a chain of matrix factors, then there is exactly the same connection between the corresponding rows of the product matrix.*

This follows from and is a generalisation of the properties of active rows considered in § 53.9 and § 53.10.

We can also prove it directly as follows.

Let $$[a]_m^\alpha [b]_\alpha^\beta [c]_\beta^\gamma \dots [s]_\rho^\sigma [t]_\sigma^n = [x]_m^n \quad \dots\dots\dots\dots\dots\dots(7).$$

First suppose that there is a connection

$$[h_1 \ h_2 \dots h_r] [a_{p1}]_r^\alpha = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(8)$$

between the p_1th, p_2th, ... p_rth horizontal rows of the first factor matrix $[a]_m^\alpha$.

By the properties of active rows (see § 53.4) it follows from (7) that

$$[a_{p1}]_r^\alpha [b]_\alpha^\beta [c]_\beta^\gamma \dots [s]_\rho^\sigma [t]_\sigma^n = [x_{p1}]_r^n.$$

Prefixing the matrix $[h_1 \ h_2 \dots h_r]$ on both sides of this last equation, and making use of (8) we obtain

$$[h_1 \ h_2 \dots h_r] [x_{p1}]_r^n = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(9).$$

Thus there is the same connection between the p_1th, p_2th, ... p_rth horizontal rows of $[x]_m^n$ as between the p_1th, p_2th, ... p_rth horizontal rows of $[a]_m^\alpha$.

Similarly the connection

$$[t_{1q}]_\sigma^r \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{bmatrix} = 0 \ \text{ leads to } \ [x_{1q}]_m^r \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{bmatrix} = 0.$$

That is, if there is any connection between the q_1th, q_2th, ... q_rth vertical rows of the last factor matrix $[t]_\sigma^n$, then there is the same connection between the q_1th, q_2th, ... q_rth vertical rows of the product matrix $[x]_m^n$.

It should be observed that the converse of the above theorem is not true. A connection between either set of rows in the product matrix does not necessarily lead to a connection between the corresponding active rows in an extreme factor matrix.

§ 71. Theorems concerning the rank of a matrix whose elements are constants.

Most of the following theorems serve to facilitate the determination of the rank of a given matrix.

Theorem I. *If Δ_r is a non-vanishing derived determinant of order r of a matrix A, and if all those derived determinants of A of order $r+1$ which contain Δ_r as a minor vanish, then all derived determinants of A of order $r+1$ vanish, and therefore the rank of A is r.*

If A has no derived determinants of order $r+1$, so that r is the efficiency of A, the rank of A is clearly equal to r.

We shall suppose that r is less than the efficiency of A, so that A has derived determinants of order $r+1$.

Let $$A = [a]_m^n, \qquad \Delta_r = (a_{pq})_r^r.$$

Then by Theorem III of § 70 the r horizontal rows of A which occur in Δ_r are unconnected. Further it has been shown in Note 1 of the same theorem that all other horizontal rows of A are connected with those r unconnected horizontal rows. It follows from Theorem VII of § 70 that the rank of A is r and that *all* derived determinants of A of order $r+1$ vanish.

The theorem can also be proved as follows:

Let

$$A_{r+1}^{n} = \begin{bmatrix} a_{p_1 1} & a_{p_1 2} & \cdots & a_{p_1 n} \\ a_{p_2 1} & a_{p_2 2} & \cdots & a_{p_2 n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p_r 1} & a_{p_r 2} & \cdots & a_{p_r n} \\ a_{u1} & a_{u2} & \cdots & a_{un} \end{bmatrix}, \qquad A_{r+1}^{r+1} = \begin{bmatrix} a_{p_1 q_1} & a_{p_1 q_2} & \cdots & a_{p_1 q_r} & a_{p_1 v} \\ a_{p_2 q_1} & a_{p_2 q_2} & \cdots & a_{p_2 q_r} & a_{p_2 v} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{p_r q_1} & a_{p_r q_2} & \cdots & a_{p_r q_r} & a_{p_r v} \\ a_{u1} & a_{u2} & \cdots & a_{ur} & a_{ur} \end{bmatrix},$$

where u is one of the suffixes $1, 2, \ldots m$ not included in $p_1, p_2, \ldots p_r$,
and v is one of the suffixes $1, 2, \ldots n$ not included in $q_1, q_2, \ldots q_r$.

By hypothesis det $A_{r+1}^{r+1} = 0$, i.e. A_{r+1}^{r+1} has no non-vanishing derived determinant of order $r+1$. Further A_{r+1}^{r+1} has a non-vanishing derived determinant $\Delta_r = (a_{pq})_r^r$ of order r. Therefore the rank of A_{r+1}^{r+1} is r and its last vertical row is connected with the first r vertical rows. It follows that all vertical rows of A_{r+1}^{n} are connected with those of its vertical rows which occur in Δ_r, and since $\Delta_r \neq 0$ those latter rows (r in number) are unconnected. Consequently by Theorem VII of § 70 A_{r+1}^{n} has rank r. By Theorem III of § 70 the first r horizontal rows of A_{r+1}^{n} are unconnected and the last horizontal row is connected with the first r horizontal rows. It follows that every horizontal row of A is connected with those r horizontal rows of A (themselves unconnected) which occur in Δ_r. Hence by Theorem VI of § 70, the rank of A is r.

Theorem II a. *If Δ_r is a non-vanishing derived determinant of order r of a matrix A, and if all those derived determinants of $\boldsymbol{A}$ of order $r+2$ which contain Δ_r as a minor vanish, then all the derived determinants of A of order $r+2$ vanish, and therefore the rank of A cannot exceed $r+1$.*

We will assume that A has derived determinants of order $r+2$.

If there is a derived determinant Δ_{r+1} of order $r+1$ containing Δ_r as a minor which does not vanish, then since all derived determinants of order $r+2$ which contain Δ_{r+1} as a minor vanish, it follows by Theorem I that the rank of A is $r+1$.

If on the other hand every derived determinant of order $r+1$ which contains Δ_r as a minor vanishes, then by Theorem I the rank of $\boldsymbol{A}$ is equal to r.

Thus the rank of $\boldsymbol{A}$ is either r or $r+1$, and cannot exceed $r+1$.

Theorem II b. *If Δ_r is a non-vanishing derived determinant of order r of a matrix A, and if all those derived determinants of A of order $r+s$ which contain Δ_r as a minor vanish, then all the derived determinants of A of order $r+s$ vanish, i.e. the rank of A cannot exceed $r+s-1$.*

It is here implicitly assumed that A has derived determinants of order $r+s$. If it has no derived determinants of that order, then the rank of A naturally cannot exceed $r+s-1$.

The above theorem has already been proved (in Theorem I and Theorem IIa) for the special cases in which $s=1$ and $s=2$.

To prove it in general, assume that it is true when $s=i$.

Let all the derived determinants of A of order $r+i+1$ which contain Δ_r as a minor vanish.

If A has a derived determinant Δ_{r+i} of order $r+i$ containing Δ_r as a minor which does not vanish, then since all derived determinants of A of order $r+i+1$ which contain Δ_{r+i} as a minor vanish, it follows from Theorem I that the rank of A is $r+i$.

On the other hand if every derived determinant of A of order $r+i$ containing Δ_r as a minor vanishes, then by the assumption the rank of A cannot exceed $r+i-1$.

We conclude that the rank of A cannot exceed $r+i$.

Thus if the theorem is true when $s=i$, it is true when $s=i+1$.

Since it has been proved to be true when $s=1$ and when $s=2$, it is true generally.

Ex. i. Let $\phi = \begin{bmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{bmatrix}$, and let $\begin{bmatrix} A & H & G & U \\ H & B & F & V \\ G & F & C & W \\ U & V & W & D \end{bmatrix}$ be the reciprocal of ϕ.

If $ab - h^2 \neq 0$, then necessary and sufficient conditions that the rank of ϕ shall be 2 are

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, \quad \begin{vmatrix} a & h & u \\ h & b & v \\ g & f & w \end{vmatrix} = 0, \quad \begin{vmatrix} a & h & g \\ h & b & f \\ u & v & w \end{vmatrix} = 0, \quad \begin{vmatrix} a & h & u \\ h & b & v \\ u & v & d \end{vmatrix} = 0.$$

These are equivalent to $\quad C=0, \quad D=0, \quad W=0$;

or
$$\begin{bmatrix} C & W \\ W & D \end{bmatrix} = 0.$$

Ex. ii. If $a \neq 0$, then necessary and sufficient conditions that the rank of ϕ shall not exceed 2 are

$$B=0, \quad C=0, \quad D=0, \quad F=0, \quad V=0, \quad W=0;$$

or
$$\begin{bmatrix} B & F & V \\ F & C & W \\ V & W & D \end{bmatrix} = 0.$$

Theorem III. *If $X = AB$ is a standard product of two matrices A and B, the rank of the product matrix X cannot exceed the rank of either of the factor matrices A and B.*

Let
$$A = [a]_m^s, \quad B = [b]_s^n, \quad X = [x]_m^n,$$
so that
$$[x]_m^n = [a]_m^s\,[b]_s^n.$$

First let r be the rank of A.

If $r = m$, the rank of X cannot exceed r, for it cannot exceed m.

If $r < m$, there is a connection between every $r+1$ horizontal rows of A, and therefore by Theorem VIII of § 70 there is a connection between every $r+1$ horizontal rows of X. Thus X cannot have $r+1$ unconnected horizontal rows, and therefore it cannot have a non-vanishing derived determinant of order $r+1$. It follows that the rank of X cannot exceed r.

Next let r be the rank of B.

If $r = n$, the rank of X cannot exceed r, for it cannot exceed n.

If $r < n$, there is a connection between every $r+1$ vertical rows of A, and therefore there is a connection between every $r+1$ vertical rows of X. Thus X cannot have $r+1$ unconnected vertical rows, and therefore it cannot have a non-vanishing derived determinant of order $r+1$. It follows that the rank of X cannot exceed r.

It may be observed that it is possible for the rank of X to be less than the ranks of both A and B.

Ex. iii.

$$\begin{bmatrix} 2, & 1, & 2, & 1 \\ 1, & 1, & 1, & 1 \end{bmatrix} \begin{bmatrix} 2, & -1, & 0 \\ 0, & 4, & -1 \\ -2, & 1, & 0 \\ 1, & -3, & 2 \end{bmatrix} = \begin{bmatrix} 1, & 1, & 1 \\ 1, & 1, & 1 \end{bmatrix}.$$

Here the ranks of the factor matrices are 2 and 3, and the rank of the product matrix is 1.

Ex. iv.

$$\begin{bmatrix} 2, & 1, & 2, & 1 \\ 1, & 1, & 1, & 1 \end{bmatrix} \begin{bmatrix} 1, & 2, & -3 \\ 2, & 3, & -1 \\ -1, & -2, & 3 \\ -2, & -3, & 1 \end{bmatrix} = 0.$$

Here the ranks of the factor matrices are 2 and 3, and the rank of the product matrix is 0.

Ex. v. *If* $[x]_m^n = [a]_m^s [b]_s^n$ *is a product of two matrices each of which has rank equal to the passivity s, then the product matrix* $[x]_m^n$ *also has rank s.*

By the theorem the rank of $[x]_m^n$ cannot exceed s.

Let $(a_{p1})_s^s$, $(b_{1q})_s^s$ be non-vanishing simple minor determinants of $[a]_m^s$, $[b]_s^n$. Then by the properties of active rows, we have

$$[x_{pq}]_s^s = [a_{p1}]_s^s \, [b_{1q}]_s^s, \quad \text{and therefore} \quad (x_{pq})_s^s = (a_{p1})_s^s \, (b_{1q})_s^s \neq 0.$$

Thus $[x]_m^n$ has a non-vanishing minor determinant of order s, and therefore its rank is s.

This result can also be obtained by equating the matrices of the minor determinants of order s on both sides of the given equation.

Ex. vi. *The product* $[x]_m^n = [a]_m^s [b]_s^n$ *in which the first and second factor matrices have ranks m and s respectively, has itself rank m.*

Let $[a]_s^s$ be a matrix of rank s formed by adding additional final horizontal rows to $[a]_m^s$, and let

$$[x]_s^n = [a]_s^s \, [b]_s^n.$$

Then by Ex. v the matrix $[x]_s^n$ has rank s, and all its horizontal rows are unconnected. Since $[x]_m^n$ is a minor of $[x]_s^n$, it follows that all the horizontal rows of $[x]_m^n$ are unconnected, or that $[x]_m^n$ has rank m.

Ex. vii. *The product* $[x]_m^n = [a]_m^s [b]_s^n$ *in which the first and second factor matrices have ranks s and n respectively, has itself rank n.*

A proof similar to that of Ex. vi can be given.

The results of the last three examples are summarised and proved in another way in Ex. iv of § 73.

Theorem IV. *If $X = ABC \dots ST$ is a standard product of any number of matrices, the rank of the product matrix X cannot exceed the rank of any one of the factor matrices.*

First let $X = ABC$ be a standard product of three matrices, and let $P = AB$, so that $X = PC$.

Then by Theorem III the rank of X cannot exceed the rank of either P or C, and the rank of P cannot exceed the rank of either A or B. Consequently the rank of X cannot exceed the rank of A or B or C.

Next let $X = ABCD$ be a standard product of four matrices, and let $P = ABC$, so that $X = PD$.

Then by Theorem III the rank of X cannot exceed the rank of either P or D. Also by the first case the rank of P cannot exceed the rank of A or B or C. Consequently the rank of X cannot exceed the rank of A or B or C or D.

Proceeding in this way, the theorem can be proved to be true for a standard product of any number of matrices.

Theorem V. *The rank of a matrix is unaltered when it is multiplied by or into any undegenerate square matrix, provided that the product so formed is a standard product.*

First let
$$[p]^m_m [a]^n_m = [x]^n_m \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(1),$$
where $[a]^n_m$ is any matrix whatever and $[p]^m_m$ is any square matrix of order and rank m.

We will show that the rank of $[x]^n_m$ is equal to the rank of $[a]^n_m$.

Let $\underline{\overline{P}}{}^m_m$ be the (principal) inverse matrix of $[p]^m_m$ as defined in § 46.6.

By Ex. x of § 68 the rank of $\underline{\overline{P}}{}^m_m$ is m.

Prefixing the matrix $\underline{\overline{P}}{}^m_m$ on both sides of (1), we obtain
$$\underline{\overline{P}}{}^m_m [x]^n_m = [a]^n_m \quad \dots\dots\dots\dots\dots\dots\dots\dots(2).$$

Applying Theorem III, we see from (1) that the rank of $[x]^n_m$ cannot be greater than the rank of $[a]^n_m$, and we see from (2) that the rank of $[a]^n_m$ cannot be greater than the rank of $[x]^n_m$, or that the rank of $[x]^n_m$ cannot be less than the rank of $[a]^n_m$. We conclude that the rank of $[x]^n_m$ is equal to the rank of $[a]^n_m$.

Next let $$[a]_m^n [q]_n^n = [x]_m^n \quad \text{.............................(3),}$$

where $[a]_m^n$ is any matrix whatever and $[q]_n^n$ is a square matrix whose order and rank are both n.

Let $\overline{\underline{Q}}{}_n^n$ be the inverse matrix of $[q]_n^n$; its rank is n.

Postfixing $\overline{\underline{Q}}{}_n^n$ on both sides of (3), we obtain

$$[x]_m^n \overline{\underline{Q}}{}_n^n = [a]_m^n \quad \text{.............................(4).}$$

From (3) we see that the rank of $[x]_m^n$ cannot be greater than the rank of $[a]_m^n$, and from (4) we see that the rank of $[x]_m^n$ cannot be less than the rank of $[a]_m^n$. We conclude that in this case also the rank of $[x]_m^n$ is equal to the rank of $[a]_m^n$. This completes the proof of Theorem V.

An immediate consequence of the theorem is that if

$$[p]_m^m [a]_m^n [q]_n^n = [x]_m^n,$$

where $[a]_m^n$ is any matrix whatever and $[p]_m^m$, $[q]_n^n$ are undegenerate square matrices of orders m and n respectively, then the rank of $[x]_m^n$ is equal to the rank of $[a]_m^n$.

The following corollaries are particular cases of the theorem.

COROLLARY 1. *If all the elements of any horizontal or vertical row of a matrix are multiplied by the same non-vanishing scalar quantity k, the rank of the matrix is unaltered.*

COROLLARY 2. *If to the elements of any horizontal or vertical row of a matrix we add the corresponding elements of any parallel row each multiplied by the same scalar quantity k, the rank of the matrix is unaltered.*

COROLLARY 3. *If the horizontal and vertical rows of a matrix are re-arranged amongst themselves in any manner, the rank of the matrix is unaltered.*

Ex. viii. $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ ka_2 & kb_2 & kc_2 & kd_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ has the same rank as $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$, when $k \neq 0$.

We deduce this from the theorem of the text by means of the equation

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ ka_2 & kb_2 & kc_2 & kd_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}.$$

Ex. ix. $\begin{bmatrix} a_1, & b_1+kd_1, & c_1, & d_1 \\ a_2, & b_2+kd_2, & c_2, & d_2 \\ a_3, & b_3+kd_3, & c_3, & d_3 \end{bmatrix}$ has the same rank as $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$.

This follows from the equation

$$\begin{bmatrix} a_1, & b_1+kd_1, & c_1, & d_1 \\ a_2, & b_2+kd_2, & c_2, & d_2 \\ a_3, & b_3+kd_3, & c_3, & d_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & k & 0 & 1 \end{bmatrix}.$$

Ex. x. $\begin{bmatrix} b_2 & d_2 & c_2 & a_2 \\ b_1 & d_1 & c_1 & a_1 \\ b_3 & d_3 & c_3 & a_3 \end{bmatrix}$ has the same rank as $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$.

This follows from the equation

$$\begin{bmatrix} b_2 & d_2 & c_2 & a_2 \\ b_1 & d_1 & c_1 & a_1 \\ b_3 & d_3 & c_3 & a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Ex. xi. *If in the product* $[x]_m^n = [a]_m^s [e]_s^s [b]_s^n$ *each of the factor matrices on the right has rank equal to the common passivity s, then the product matrix* $[x]_m^n$ *also has rank s.*

Writing $[a]_m^s [e]_s^s = [p]_m^s$, we have $[x]_m^n = [p]_m^s [b]_s^n$, where each factor matrix on the right has rank s. It follows from Ex. v that $[x]_m^n$ has rank s.

To prove the theorem more directly we first observe that the given equation shows that the rank of $[x]_m^n$ cannot exceed s. Then if $(a_{p1})_s^s$, $(b_{1q})_s^s$ are non-vanishing minor determinants of order s of $[a]_m^s$, $[b]_s^n$, we obtain from the given equation

$$[x_{pq}]_s^s = [a_{p1}]_s^s [e]_s^s [b_{1q}]_s^s, \quad (x_{pq})_s^s = (a_{p1})_s^s (e)_s^s (b_{1q})_s^s \neq 0.$$

Thus $[x]_m^n$, having a non-vanishing minor determinant of order s, must have rank s.

Ex. xii. *Equipotent matrices.*

Two similar matrices will be called *equipotent* when either, and therefore each, can be derived from the other by prefixing and postfixing undegenerate square matrices.

The necessary and sufficient condition that $[a]_m^n$, $[b]_m^n$ shall be equipotent can be expressed in any one of the forms

$$[p]_m^m [a]_m^n [q]_n^n = [h]_m^m [b]_m^n [k]_n^n,$$

$$[b]_m^n = [p]_m^m [a]_m^n [q]_n^n,$$

$$[a]_m^n = [h]_m^m [b]_m^n [k]_n^n,$$

where $[p]_m^m$, $[q]_n^n$, $[h]_m^m$ $[k]_n^n$ are undegenerate square matrices.

Two equipotent matrices have equal ranks.

Any matrix $[a]_m^n$ of rank r is equipotent with the matrix formed from the unit matrix $[1]_r^r$ by adding $m-r$ final horizontal rows of 0's and $n-r$ vertical rows of 0's.

Theorem VI. *If any row of a matrix A is connected with certain of the parallel rows, then the rank of A is equal to the rank of the matrix obtained from A by striking out that row.*

Since the rank of a matrix is unaffected by any re-arrangement of its rows, it will be sufficient to consider the case in which the row in question occupies the final position in the matrix.

Let the matrix be $[a]_{m+1}^{n+1}$ and let the final horizontal row be connected with the preceding horizontal rows, the equation expressing the connection being

$$[h_1\ h_2 \ldots h_m\ \ 1]\,[a]_{m+1}^{n+1} = 0.$$

We have then

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & 0 \\ h_1 & h_2 & \ldots & h_m & 1 \end{bmatrix} [a]_{m+1}^{n+1} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1,n+1} \\ a_{21} & a_{22} & \ldots & a_{2,n+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{m,n+1} \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$

It follows by Theorem V that the rank of $[a]_{m+1}^{n+1}$ is equal to the rank of the matrix on the right. Now the non-vanishing derived determinant of the highest order contained in that matrix is a derived determinant of the matrix $[a]_m^{n+1}$ formed by its first m horizontal rows. Consequently $[a]_{m+1}^{n+1}$ has the same rank as $[a]_m^{n+1}$, which is the matrix formed from $[a]_{m+1}^{n+1}$ by striking out its last horizontal row.

This establishes the theorem for horizontal rows. A similar proof can be given for vertical rows.

Ex. xiii. The matrices

$$\begin{bmatrix} 2 & 11 & 9 & 25 \\ 3 & 4 & 1 & 0 \\ 5 & 7 & 2 & 1 \\ 4 & 11 & 7 & 17 \\ 2 & 5 & 3 & 7 \end{bmatrix}, \quad \begin{bmatrix} 2 & 9 & 25 \\ 3 & 1 & 0 \\ 5 & 2 & 1 \\ 4 & 7 & 17 \\ 2 & 3 & 7 \end{bmatrix}, \quad \begin{bmatrix} 2 & 9 \\ 3 & 1 \\ 5 & 2 \\ 4 & 7 \\ 2 & 3 \end{bmatrix}$$

have all the same rank. For in the first matrix the second vertical row is obtained by adding the first and third vertical rows, and can be struck out. Again in the second matrix the third vertical row is obtained by multiplying the second vertical row by 3 and subtracting the first vertical row, and it can therefore be struck out.

The third matrix has clearly rank 2. Therefore the first matrix has rank 2.

Ex. xiv. *The augmented matrix* $[a, c]_m^{\rho, n}$ *has the same rank as* $[a]_m^\rho$ *when and only when each of the vertical rows of* $[c]_m^n$ *is connected with the vertical rows of* $[a]_m^\rho$.

Let $$[a]_m^\rho = A, \quad [c]_m^n = C, \quad [a, c]_m^{\rho, n} = A'.$$

First suppose that A and A' have common rank r. Then the matrix A has r unconnected vertical rows, and every vertical row of A' is connected with these. Consequently every vertical row of C is connected with them, and therefore every vertical row of C is connected with the vertical rows of A.

Next suppose that every vertical row of C is connected with the vertical rows of A. Then by Theorem VI the matrix A' has the same rank as A.

Ex. xv. *The augmented matrix* $\begin{bmatrix} b \\ c \end{bmatrix}_{\sigma, m}^n$ *has the same rank as* $[b]_\sigma^n$ *when and only when each of the horizontal rows of* $[c]_m^n$ *is connected with the horizontal rows of* $[b]_\sigma^n$.

Let $$[b]_\sigma^n = B, \quad [c]_m^n = C, \quad \begin{bmatrix} b \\ c \end{bmatrix}_{\sigma, m}^n = B'.$$

First suppose that B and B' have common rank s. Then the matrix B has s unconnected horizontal rows, and every horizontal row of B' is connected with these. Consequently every horizontal row of C is connected with them, and therefore every horizontal row of C is connected with the horizontal rows of A.

Next suppose that every horizontal row of C is connected with the horizontal rows of B. Then by Theorem VI the matrix B' has the same rank as B.

Theorem VII. *If a matrix A has rank r, and if we fix our attention on any particular s unconnected horizontal (or vertical) rows of A, where $s < r$, then we can always determine a set of r unconnected horizontal (or vertical) rows of A which includes these s particular rows.*

It will be sufficient to prove the theorem for horizontal rows.

Let Δ_s be a non-vanishing derived determinant of A belonging to the matrix formed by the s particular unconnected horizontal rows of A.

If all the derived determinants of A of order r which contain Δ_s as a minor vanished, then by Theorem II *b*, the rank of A could not exceed $r-1$, whereas A has by hypothesis rank r.

Hence there is at least one derived determinant Δ_r of A of order r containing Δ_s as a minor which does not vanish.

The r horizontal rows of A which occur in Δ_r (i.e. which contribute elements to Δ_r) are unconnected, and the s particular horizontal rows are all included among those r rows.

This establishes the theorem.

The theorem can also be easily deduced from Theorem VI.

In particular if we fix upon any one row (horizontal or vertical) of A which is not a row of 0's, then (the rank of A being r), we can always find $r-1$ parallel rows of A which together with that one row form a set of r unconnected rows of A.

NOTE 1. *Necessary and sufficient conditions that a given matrix shall have rank r.*

It appears from Theorem I that necessary and sufficient conditions for a matrix A being of rank r are:

(1) that A shall have a non-vanishing derived determinant Δ_r of order r;

(2) that every derived determinant of order $r+1$ of A which contains Δ_r as a minor shall vanish.

A more direct proof of this result is given in Ex. v of § 110.

NOTE 2. *Rank of a symmetrical or self-conjugate matrix.*

Special theorems relating to the rank of a symmetrical or self-conjugate square matrix are given in § 119.

§ 72. Rank of a product of two mutually conjugate matrices.

1. *Rank of a product of two mutually conjugate real matrices.*

Let
$$[a]_m^n \overline{\underline{a}}_n^m = [x]_m^m$$

be a product of two such matrices, each of rank r, and suppose that $(a_{pq})_r^r$ is a non-vanishing derived matrix of order r.

Then by the properties of active rows

$$[a_{p1}]_r^n \overline{\underline{a_{p1}}}_n^r = [x_{pp}]_r^r.$$

Equating the determinants of both sides, we have

$$(x_{pp})_r^r = \Delta_1^2 + \Delta_2^2 + \dots + \Delta_\nu^2 \quad \text{.................................(1)},$$

where $\Delta_1, \Delta_2, \dots \Delta_\nu$ are the simple minor determinants of the matrix $[a_{p1}]_r^n$, and are real quantities. The right-hand side of equation (1) can only vanish when $\Delta_1, \Delta_2, \dots \Delta_\nu$ all vanish. But one of these determinants is the non-vanishing determinant $(a_{pq})_r^r$, and therefore $(x_{pp})_r^r \neq 0$. Thus the product matrix $[x]_m^m$ has a non-vanishing minor determinant of order r, and since its rank cannot exceed the rank of $[a]_m^n$, it has rank r. We conclude that:

Any product of two mutually conjugate real matrices has the same rank as the factor matrices ....(A).

2. *Rank of a product of two mutually conjugate undegenerate matrices.*

An undegenerate matrix in which the sum of the squares of the simple minor determinants is equal to zero will be called an *extravagant matrix*. In an extravagant matrix the simple minor determinants are not all zero, but the sum of their squares vanishes. Clearly no real matrix, i.e. no matrix whose elements are all real, can be extravagant; also no matrix whose elements are all pure imaginary quantities can be extravagant.

Let $[a]_r^n$ be an undegenerate matrix of rank r, so that the horizontal rows are long rows and $n \nless r$. With this matrix and its conjugate matrix $\overline{\underline{a}}{}^{r}_{n}$ two products can be formed, viz. $\overline{\underline{a}}{}^{r}_{n}[a]_r^n$ and $[a]_r^n\overline{\underline{a}}{}^{r}_{n}$. In the first of these short rows are active; in the second long rows are active.

Consider first the product $\overline{\underline{a}}{}^{r}_{n}[a]_r^n = [x]_n^n$ *in which short rows are active.*

By Ex. v of § 71 the product matrix $[x]_n^n$ has rank r.

Consider next the product $[a]_r^n\overline{\underline{a}}{}^{r}_{n} = [y]_r^r$ *in which long rows are active.*

In this case $\det[y]_r^r$ is equal to the sum of the squares of the simple minor determinants of $[a]_r^n$. Hence $(y)_r^r$ does or does not vanish according as the sum of these squares does or does not vanish, i.e. according as the matrix $[a]_r^n$ is or is not extravagant.

We conclude that:

A product of two mutually conjugate undegenerate matrices of rank r has itself rank r except when the matrices are extravagant and long rows are active rows. In the exceptional case (which cannot occur when the matrices are real), the rank of the product is less than r ....(B).

Ex. From equation (1) we see that the product of two mutually conjugate matrices of rank r can only have rank less than r when every undegenerate active simple minor matrix of rank r is extravagant.

§ 73. Ranks of all complete matrices of the minor determinants of any given fundamental matrix.

When the rank of any matrix whatever is known, the rank of any complete matrix of its minor determinants of any given order is also known, as is shown by the theorems which follow.

Theorem I. *If a matrix is undegenerate, then every complete matrix of its minor determinants of order s is undegenerate for all possible values of s.*

Theorem II. *If* $[a]_m^n$ *is a matrix of rank r, and if* $[\mathbf{A}]_\mu^\nu$ *is a complete matrix of the minor determinants of order s of* $[a]_m^n$, *then:*

(1). *if* $s > r$, $[\mathbf{A}]_\mu^\nu$ *has rank* 0;

(2). *if* $s \ngtr r$, $[\mathbf{A}]_\mu^\nu$ *has rank* $\binom{r}{s}$.

Proof of Theorem I for a square matrix.

Let $[a]_m^m$ be any square matrix, and let $[\mathbf{A}]_\mu^\mu$ be any complete matrix of its minor determinants of order s, where $s \not> m$. Further let $[\mathbf{B}]_\mu^\mu$ be the matrix each of whose elements is the co-factor in $[a]_m^m$ of the corresponding element of $[\mathbf{A}]_\mu^\mu$. Then by Ex. ix of § 32, we have

$$[\mathbf{A}]_\mu^\mu \, \overline{\underline{\mathbf{B}}}_\mu^\mu = \Delta \, [1]_\mu^\mu \quad \text{..........................(1)},$$

where $$\Delta = (a)_m^m = \det [a]_m^m.$$

For the elements $\mathbf{A}_{i1}, \mathbf{A}_{i2}, \ldots \mathbf{A}_{i\mu}$ of the ith horizontal row of $[\mathbf{A}]_\mu^\mu$

and the elements $\mathbf{A}_{j1}, \mathbf{A}_{j2}, \ldots \mathbf{A}_{j\mu}$ of the jth horizontal row of $[\mathbf{A}]_\mu^\mu$

are corresponding simple minor determinants of two long-cut minor matrices U_i, U_j of $[a]_m^m$. Each of these long-cut minor matrices is composed of s horizontal rows of $[a]_m^m$ and has only its long rows deranged. If $j = i$, then U_j is the same as U_i; if $j \neq i$, then U_j is not composed of the same long rows of $[a]_m^m$ as U_i. Further $\mathbf{B}_{j1}, \mathbf{B}_{j2}, \ldots \mathbf{B}_{jm}$ are the co-factors of $\mathbf{A}_{j1}, \mathbf{A}_{j2}, \ldots \mathbf{A}_{jm}$ respectively. Therefore by Ex. ix of § 32, we have

$$\mathbf{A}_{i1}\mathbf{B}_{j1} + \mathbf{A}_{i2}\mathbf{B}_{j2} + \ldots + \mathbf{A}_{im}\mathbf{B}_{jm} = 0 \text{ or } \Delta \quad \text{..............(2)}$$

according as $j \neq i$ or $j = i$.

All such equations as (2) are together equivalent to (1).

By equating the determinants of both sides in (1), we obtain

$$(\mathbf{A})_\mu^\mu \; (\mathbf{B})_\mu^\mu = \Delta^\mu \quad \text{.. ....(3)}.$$

From (3) we conclude that when $\Delta \neq 0$, then $(\mathbf{A})_\mu^\mu \neq 0$; or that when $[a]_m^m$ is undegenerate, then $[\mathbf{A}]_\mu^\mu$ is undegenerate.

We have now proved that Theorem I is true for square matrices.

Ex. i. It is shown in § 113 that

$$(\mathbf{A})_\mu^\mu = \pm \Delta^\alpha, \quad (\mathbf{B})_\mu^\mu = \pm \Delta^\beta,$$

where $$\alpha = \binom{m-1}{s-1}, \quad \beta = \binom{m-1}{s} = \binom{m-1}{m-s-1},$$

the above equations being identities in the elements of $[a]_m^m$.

Proof of Theorem I for any matrix.

Let $[a]_m^n$ be any *undegenerate* matrix whatever, and let $[\mathbf{A}]_\mu^\nu$ be any complete matrix of its minor determinants of order s, so that

$$s \not> m, \quad s \not> n, \quad \mu = \binom{m}{s}, \quad \nu = \binom{n}{s}.$$

First suppose that long rows are horizontal, so that $m \not> n$, $\mu \not> \nu$.

Then $[a]_m^n$ has a non-vanishing simple minor determinant $(a_{1q})_m^m$ of order m, and therefore an *undegenerate* simple minor square matrix $[a_{1q}]_m^m$; and $[\mathbf{A}]_\mu^\nu$ has a simple minor matrix $[\mathbf{A}_{1v}]_\mu^\mu$ which is a matrix of the minor determinants of order s of $[a_{1q}]_m^m$. Since the square matrix $[a_{1q}]_m^m$ is undegenerate, therefore, as just proved, the matrix $[\mathbf{A}_{1v}]_\mu^\mu$ is undegenerate and $(\mathbf{A}_{1v})_\mu^\mu \neq 0$. Thus the matrix $[\mathbf{A}]_\mu^\nu$, having a non-vanishing simple minor determinant $(\mathbf{A}_{1v})_\mu^\mu$, is undegenerate and has rank μ.

Next suppose that long rows are vertical, so that $m \not< n$, $\mu \not< \nu$.

Then $[a]_m^n$ has a non-vanishing simple minor determinant $(a_{p1})_n^n$ and therefore an *undegenerate* simple minor square matrix $[a_{p1}]_n^n$; and $[\mathbf{A}]_\mu^\nu$ has a simple minor matrix $[\mathbf{A}_{u1}]_\nu^\nu$ which is a matrix of the minor determinants of order s of $[a_{p1}]_n^n$. Since the square matrix $[a_{p1}]_n^n$ is undegenerate, therefore, as just proved, the matrix $[\mathbf{A}_{u1}]_\nu^\nu$ is undegenerate and $(\mathbf{A}_{u1})_\nu^\nu \neq 0$. Thus the matrix $[\mathbf{A}]_\mu^\nu$, having a non-vanishing simple minor determinant $(\mathbf{A}_{u1})_\nu^\nu$, is undegenerate and has rank ν.

Proof of Theorem II.

The first part of the theorem is obvious, for if $s > r$, and if $[a]_m^n$ has rank r, then every minor determinant of order s of $[a]_m^n$ vanishes, and therefore every element of $[\mathbf{A}]_\mu^\nu$ vanishes.

To prove the second part of the theorem, let the matrix $[a]_m^n$ have rank r and let $(a_{pq})_r^r$ be one of its non-vanishing derived determinants of order r, where of course $r \not> m$ and $r \not> n$.

Then by Theorem III of § 70 every horizontal row of $[a]_m^n$ is connected with the horizontal rows of the minor matrix $[a_{p1}]_r^n$.

Therefore there exist quantities $h_{i1}, h_{i2}, \dots h_{ir}$ such that

$$[a_{i1} a_{i2} \dots a_{in}] = [h_{i1} h_{i2} \dots h_{ir}] \, [a_{p1}]_r^n,$$

and there exists a matrix $[h]_m^r$ such that

$$[a]_m^n = [h]_m^r \, [a_{p1}]_r^n \quad \dots\dots\dots\dots(4).$$

Again $[a_{p1}]_r^n$ is a matrix of rank r containing the non-vanishing simple minor determinant $(a_{pq})_r^r$, and by Theorem III of § 70 every one of its vertical rows is connected with the vertical rows of the minor matrix $[a_{pq}]_r^r$. Therefore there exist quantities $k_{1j}, k_{2j}, \dots k_{rj}$ and a matrix $[k]_r^n$ such that

$$\begin{bmatrix} a_{p_1 j} \\ a_{p_2 j} \\ \vdots \\ a_{p_r j} \end{bmatrix} = [a_{pq}]_r^r \begin{bmatrix} k_{1j} \\ k_{2j} \\ \vdots \\ k_{rj} \end{bmatrix}, \quad [a_{p1}]_r^n = [a_{pq}]_r^r \, [k]_r^n \quad \dots\dots\dots\dots(5).$$

From (4) and (5) we see that there exist matrices $[h]_m^r$, $[k]_r^n$ such that

$$[a]_m^n = [h]_m^r \, [a_{pq}]_r^r \, [k]_r^n \quad \dots\dots\dots\dots(6).$$

Since $[a]_m^n$ has rank r, it follows from Theorem IV of § 71 that none of the factor matrices on the right of (6) can have rank less than r. Also none of them can have rank greater than r. Accordingly the factor matrices on the right of (6) are all undegenerate and have all rank r.

When we equate corresponding matrices of the minor determinants of order s on both sides of (6), we obtain

$$[\mathbf{A}]_\mu^\nu = [H]_\mu^\rho \, [\alpha]_\rho^\rho \, [K]_\rho^\nu \quad \dots\dots\dots\dots(7)$$

where $[\mathbf{A}]_\mu^\nu$, $[H]_\mu^\rho$, $[\alpha]_\rho^\rho$, $[K]_\rho^\nu$ are complete matrices of the minor determinants of order s of $[a]_m^n$, $[h]_m^r$, $[a_{pq}]_r^r$, $[k]_r^n$ respectively, and

$$\mu = \binom{m}{s}, \quad \rho = \binom{r}{s}, \quad \nu = \binom{n}{s}.$$

Since $r \not> m$ and $r \not> n$, it follows that $\rho \not> \mu$ and $\rho \not> \nu$.

From the fact that the factor matrices on the right of (6) are all undegenerate, it follows by Theorem I that the factor matrices on the right of (7) are all undegenerate and have all rank ρ.

This being so, it follows by Ex. v or Ex. ix of § 71 that the product matrix on the right of (7) has rank ρ.

Accordingly $[\mathbf{A}]_\mu^\nu$ has rank ρ or $\binom{r}{s}$, and Theorem II is proved.

Let η be the efficiency of the matrix $[a]_m^n$, so that η is the smaller of the two numbers m and n. Then the theorem shows that the rank of any complete matrix $[\mathbf{A}]_\mu^\nu$ of the minor determinants of *given order* s of $[a]_m^n$ is dependent on the rank of $[a]_m^n$ in accordance with the following scheme:

Scheme I. Rank of $[\mathbf{A}]_\mu^\nu$ *for a given value of* s.

Rank of $[a]_m^n$	η	$\eta-1$	...	r	...	$s+1$	s	$<s$
Rank of $[\mathbf{A}]_\mu^\nu$	$\binom{\eta}{s}$	$\binom{\eta-1}{s}$	...	$\binom{r}{s}$	...	$s+1$	1	0

Again when the matrix $[a]_m^n$ has a *given rank* r, the rank of any complete matrix $[\mathbf{A}]_\mu^\nu$ of its minor determinants of order s is for the various possible values of s given by the following second scheme:

Scheme II. Rank of $[\mathbf{A}]_\mu^\nu$ *for a given value of* r.

Value of s	1	2	...	s	...	$r-1$	r	$>r$
Rank of $[\mathbf{A}]_\mu^\nu$	r	$\binom{r}{2}$	...	$\binom{r}{s}$	...	r	1	0

From the first scheme we see that we cannot determine $[a]_m^n$ so that $[\mathbf{A}]_\mu^\nu$ shall have an arbitrary rank, and we also deduce the following third theorem:

Theorem III. *The matrix* $[a]_m^n$ *has rank* r, *where* $r \not< s$, *when and only when the matrices of its minor determinants of order* s *have rank* $\binom{r}{s}$. *In particular the matrix* $[a]_m^n$ *has rank* r *when and only when the matrices of its minor determinants of order* r *have rank* 1.

Ex. ii. It has been shown in Ex. iii of § 68 that all complete matrices of the minor determinants of given order s of $[a]_m^n$ have the same rank. Thus when the rank of any one of them is known, the ranks of all of them are known.

Ex. iii. The actual values of the matrices $[h]_m^r$, $[k]_r^n$ occurring in equation (6) are determined in § 106, and are given in formula (B′) of that article.

Ex. iv. *If* $[a]_m^s\,[b]_s^n = [x]_m^n$ *is a product of two undegenerate matrices one of which has rank equal to the passivity* s, *then the product matrix has the greatest permissible rank, i.e. its rank is equal to the smaller of the ranks of the two factor matrices.*

Let the factor matrix with the smaller rank have rank r, and let the result of equating the matrices of the minor determinants of order r on the two sides of the above equation be

$$[A]_\mu^\sigma [B]_\sigma^\nu = [X]_\mu^\nu \quad \text{.....................................(8)},$$

where $\mu = \binom{m}{r}, \quad \nu = \binom{n}{r}, \quad \sigma = \binom{s}{r}.$

We have to show that in all the cases that are possible the matrix $[x]_m^n$ has rank r. This will be proved by showing that the matrix $[X]_\mu^\nu$ has rank 1.

CASE I. *Both factor matrices have rank s.*

In this case $r = s$, and equation (8) becomes

$$[A]_\mu^1 [B]_1^\nu = [X]_\mu^\nu.$$

Since $[A]_\mu^1$ has some non-vanishing element A_{i1} and $[B]_1^\nu$ has some non-vanishing element B_{1j}, it follows that $[X]_\mu^\nu$ has some non-vanishing element $X_{ij} = A_{i1} B_{1j}$.

Thus $[X]_\mu^\nu$ cannot have rank 0. Again since the factor matrices on the right of (8) have rank 1, it follows that the rank of $[X]_\mu^\nu$ cannot exceed 1. Accordingly $[X]_\mu^\nu$ has rank 1.

CASE II. $[a]_m^s$ *has rank s, and* $[b]_s^n$ *has rank n.*

In this case $r = n$; also $m \not< s$, and therefore $\mu \not< \sigma$.

Equation (8) becomes $\quad [A]_\mu^\sigma [B]_\sigma^1 = [X]_\mu^1.$

Since $[A]_\mu^\sigma$ is undegenerate and has rank σ, there cannot be any connection between its vertical rows, and therefore $[X]_\mu^1$ cannot have rank 0. Accordingly $[X]_\mu^1$ or $[X]_\mu^\nu$ has rank 1.

CASE III. $[a]_m^s$ *has rank m, and* $[b]_s^n$ *has rank s.*

In this case $r = m$; also $n \not< s$, and therefore $\nu \not< \sigma$.

Equation (8) becomes $\quad [A]_1^\sigma [B]_\sigma^\nu = [X]_1^\nu.$

Since $[B]_\sigma^\nu$ is undegenerate and has rank σ, there cannot be any connection between its horizontal rows, and therefore $[X]_1^\nu$ cannot have rank 0. Accordingly $[X]_1^\nu$ or $[X]_\mu^\nu$ has rank 1.

§ 74. Rank of a matrix and connections between its rows when its elements are rational integral functions of certain variables.

Let

$$\Phi(x) = \begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2n} \\ \dots & \dots & \dots & \dots \\ \phi_{m1} & \phi_{m2} & \dots & \phi_{mn} \end{bmatrix} = [\phi]_m^n$$

be a matrix in which the elements are rational integral functions of certain variables $x_1, x_2, \dots x_s$.

Such a matrix will sometimes be called a (rational integral) *functional matrix*.

We will define the *rank* of the matrix $\Phi(x)$ to be the greatest possible order of a derived determinant which does not vanish identically.

Thus if the rank of $\Phi(x)$ is r, there is at least one derived determinant of order r which does not vanish identically, but all derived determinants of orders higher than r, if such exist, vanish identically.

If $\Phi(x)$ becomes $\Phi(c)$ when definite values given by

$$[x_1 x_2 \dots x_s] = [c_1 c_2 \dots c_s]$$

are ascribed to the variables, then $\Phi(c)$ is a matrix whose elements are constants. The rank of $\Phi(c)$ cannot exceed the rank of $\Phi(x)$, but it may happen that the rank of $\Phi(c)$ is less than the rank of $\Phi(x)$.

The matrix $\Phi(x)$ is *degenerate* or *undegenerate* according as its rank is less than or equal to the efficiency of the matrix. It is *singular* or *non-singular* according as $\det \Phi(x)$ is or is not identically equal to zero. It is a *scalar matrix* when it has the form

$$\phi[1]_m^m = \begin{bmatrix} \phi & 0 & \dots & 0 \\ 0 & \phi & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \phi \end{bmatrix},$$

where ϕ is a rational integral function of the variables $x_1, x_2, \dots x_s$.

There is said to be a *connection between* the horizontal rows of the matrix $\Phi(x)$ when rational integral functions $\eta_1, \eta_2, \dots \eta_m$ of the variables $x_1, x_2, \dots x_s$ exist, one at least of which is not identically zero, such that

$$\eta_1[\phi_{11}\phi_{12}\dots\phi_{1n}] + \eta_2[\phi_{21}\phi_{22}\dots\phi_{2n}] + \dots + \eta_m[\phi_{m1}\phi_{m2}\dots\phi_{mn}] = 0 \quad \dots\dots\dots(1),$$

or

$$[\eta_1 \; \eta_2 \dots \eta_m]\begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2n} \\ \dots & \dots & \dots & \dots \\ \phi_{m1} & \phi_{m2} & \dots & \phi_{mn} \end{bmatrix} = 0,$$

for all values of the variables. If η_i does not vanish identically, the ith horizontal row is said to be *connected with* the remaining horizontal rows. In this case we have a relation of the form

$$[\phi_{i1}\phi_{i2}\dots\phi_{in}] = \chi_1[\phi_{11}\phi_{12}\dots\phi_{1n}] + \chi_2[\phi_{21}\phi_{22}\dots\phi_{2n}] + \dots + \chi_m[\phi_{m1}\phi_{m2}\dots\phi_{mn}]$$

$$= [\chi_1 \; \chi_2 \dots \chi_m]\begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2n} \\ \dots & \dots & \dots & \dots \\ \phi_{m1} & \phi_{m2} & \dots & \phi_{mn} \end{bmatrix},$$

where $\chi_1, \chi_2, \dots \chi_{i-1}, \chi_{i+1} \dots \chi_m$ are rational but not necessarily integral functions of $x_1, x_2, \dots x_s$, and the horizontal row $[\phi_{i1}\phi_{i2}\dots\phi_{in}]$ is omitted in the last matrix.

If no such relation as (1) exists, the horizontal rows of the matrix are said to be *unconnected*.

Similar definitions, as regards connections, apply to the vertical rows.

The theorems contained in the preceding articles of this chapter can be at once extended to functional matrices of the kind here considered.

NOTE. *Rank and connections of a matrix whose elements are any continuous functions of certain variables.*

Let the same notation be retained for the variables and the functions.

The matrix $[\phi]_m^n$ may be said to have rank r when it has a derived determinant of order r which does not vanish identically, whilst all derived determinants of order greater than r vanish identically, i.e. for all values of the variables.

There will be a connection between the horizontal rows of the matrix $[\phi]_m^n$ when there exists a relation of the form (1) in which $\eta_1, \eta_2, \ldots \eta_m$ are continuous functions of the variables and do not all vanish identically. In particular the ith horizontal row will be connected with the remaining horizontal rows when there exists a similar relation in which $\eta_i = 1$. The horizontal rows of the matrix will be unconnected when no such relation exists. Connections between the vertical rows are similarly defined.

The continuous functions being supposed capable of expansion by Taylor's Theorem, theorems corresponding to those given in the text can be established.

Ex. i. *If a product of two (rational integral) functional matrices is a non-zero scalar functional matrix, then both the factor matrices are undegenerate.*

Let $[\phi]_m^n [\psi]_n^m = \chi [1]_m^m$, where $\phi_{ij}, \psi_{ij}, \chi$ are rational integral functions of $x_1, x_2, \ldots x_s$ and χ does not vanish identically.

If $n < m$, we obtain by equating the determinants of both sides

$$\chi^m = 0$$

for all values of the variables. Since χ does not vanish identically this is impossible. Accordingly the product of the two matrices cannot be a scalar functional matrix when $n < m$.

Assuming then that $n \not< m$, we have by equating determinants

$$\Sigma (\phi_{1q})_m^m (\psi_{q1})_m^m = \chi^m,$$

where $[q_1 q_2 \ldots q_m]$ is any corranged minor of $[1\ 2 \ldots n]$ of order m.

Since χ^m does not vanish identically, it follows that there must be at least one of the determinants $(\phi_{1q})_m^m$ which does not vanish identically, and at least one of the determinants $(\psi_{q1})_m^m$ which does not vanish identically.

Hence both $[\phi]_m^m$ and $[\psi]_m^m$ must have rank m.

Ex. ii. *If $A = [a]_m^n$ is a matrix whose elements are functions of the variables $x_1, x_2, \ldots x_s$ and if Δ is any derived determinant of A which does not vanish identically, then the horizontal rows and also the vertical rows of A which occur in Δ are unconnected.*

The proof is exactly similar to the proof of Theorem II in § 70.

Let $\Delta=(a_{pq})_r^r$. If there were any connection between the horizontal rows of A, then there would exist an identical relation of the form

$$[u_1\, u_2 \dots u_r][a_{pq}]_r^r=0,$$

where $u_1, u_2, \dots u_r$ are certain rational integral functions of the variables $x_1, x_2, \dots x_s$ which do not all vanish identically. Such a relation could only exist if the determinant $(a_{pq})_r^r$ vanish identically, which by hypothesis is not the case.

Ex. iii. *If $A=[a]_m^n$ is a functional matrix of rank r whose elements are rational integral functions of the variables $x_1, x_2, \dots x_s$ and if Δ_r is a derived determinant of order r of A which does not vanish identically, then all horizontal rows (and similarly all vertical rows) of A are connected with those horizontal (or vertical) rows of A which occur in Δ_r.*

The proof is exactly similar to the proof of Theorem III in § 70.

If $\Delta_r=(a_{pq})_r^r$, we obtain the relation (5) of § 70.

Since $A_1, A_2, \dots A_r, A_{r+1}$ are now rational integral functions of $x_1, x_2, \dots x_s$, and A_{r+1} does not vanish identically, this relation shows that the uth horizontal row of A is connected with those horizontal rows of A which occur in Δ_r.

Ex. iv. *Equipotent functional matrices.*

Any two similar matrices $[\phi]_m^n$, $[\psi]_m^n$ whose elements are rational integral functions of certain variables $x_1, x_2, x_3 \dots$ are said to be equipotent when they are connected by a relation of any one of the equivalent forms

$$[\psi]_m^n=[a]_m^m[\phi]_m^n[b]_n^n \quad \dots\dots\dots\dots(1),$$

$$[\phi]_m^n=[c]_m^m[\psi]_m^n[d]_n^n \quad \dots\dots\dots\dots(2),$$

$$[a]_m^m[\phi]_m^n[b]_n^n=[c]_m^m[\psi]_m^n[d]_n^n \quad \dots\dots\dots\dots(3),$$

where $[a]_m^m$, $[b]_n^n$, $[c]_m^m$, $[d]_n^n$ are undegenerate square matrices *whose determinants have constant values*, but whose elements, subject to this condition, may be any rational integral functions of the variables $x_1, x_2, x_3, \dots$.

Clearly when a relation of any one of these forms is given, relations of the other forms can be deduced.

The following are some of the properties of such equipotent matrices:

(i) Two equipotent functional matrices have equal ranks.

(ii) If two functional matrices are equipotent, any rational integral function χ of the variables $x_1, x_2, x_3, \dots$ which is a common factor of all the elements of one matrix is also a common factor of all the elements of the other matrix. Consequently the highest common factor of all the elements of one matrix is also the highest common factor of the elements of the other matrix.

(iii) If two functional matrices are equipotent, any rational integral function χ which is a common factor of all the derived determinants of order r of one matrix is also a common factor of all the derived determinants of order r of the other matrix. Consequently the two sets of derived determinants of order r have the same highest common factor.

(iv) If $[\phi]_m^n$ is a functional matrix of rank r, it is equipotent with a matrix of the form

$$\begin{bmatrix} \chi_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \chi_2 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \chi_r & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix},$$

where $\chi_1, \chi_2, \dots \chi_r$ are rational integral functions of $x_1, x_2, x_3 \dots$. Further the functions $\chi_1, \chi_2, \dots \chi_r$ can be so arranged that each is a factor of the next following one.

The second property follows immediately from equations (1) and (2) which show that all the elements of either matrix are homogeneous linear functions of the elements of the other matrix. The third property follows similarly from the equations obtained by equating the matrices of the minor determinants of order r on both sides of (1) and (2).

CHAPTER X.

MATRIX EQUATIONS OF THE FIRST DEGREE.

[In the first three articles of this chapter, §§ 75—77, matrix equations of the first degree are defined, and equations of the specially simple form $X+A=B$ are considered. The next six articles, §§ 78—83, deal with the solution of equations of the forms $AX=C$, $XB=C$, $AXB=C$, where A, B, C are any known matrices. The special cases in which A and B are undegenerate square matrices are first considered in §§ 78—80; and the general cases are considered in §§ 81—83. The equation $AXB=C$ is the most general equation which is solved in this chapter. Finally §§ 84 and 85 give a summary of cases in which it is allowable to cancel a matrix factor which is common to both sides of a matrix equation.]

§ 75. Matrix equations of the first degree.

An equation in which the quantities equated are matrices is called a *matrix equation*. Each term in the equation may be either a single matrix or a product of several matrices. If the equation involves one and only one unknown matrix whose elements are to be determined so that the equation is satisfied, and if this unknown matrix does not occur more than once as a factor in any term of the equation, then the equation will be called a *matrix equation of the first degree*.

If the unknown matrix whose elements are to be determined is denoted by the letter X, then the most general matrix equation of the first degree can be put into the form

$$A_1XB_1 + A_2XB_2 + \dots + A_rXB_r = C \quad \dots\dots\dots\dots\dots\dots(1),$$

where $A_1, A_2, \dots A_r, B_1, B_2, \dots B_r$ and C are known matrices whose elements are given constants.

The equations considered in the present chapter are all reducible to the form

$$AXB = C \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2),$$

where A, B and C are known matrices whose elements are given constants.

Any value of the matrix X which satisfies the equation (1) or the equation (2) will be called a *solution* of that equation. A solution X from which every solution can be obtained by giving particular values to its elements will be called the *general solution* of the equation. A matrix X which satisfies the equation and has *all* its elements finite will be called a *finite solution* of the equation. If any one of the elements of X is infinite, then X will be called an *infinite solution* of the equation.

If there exists a matrix X which has all its elements finite and which satisfies the equation (1) or the equation (2), then the equation will be said to *admit of finite solution*, or sometimes (the word *finite* being understood) to *admit of solution*.

Equations of the special form $[a]_m^n\,[x]_n^1 = [c]_m^1$ or

$$[a]_m^n \,\lceil x \rfloor_n = \lceil c \rfloor_m \qquad \text{.................................(3)}$$

will be further considered in the next chapter. The solution of this special equation is equivalent to the solution of a system of linear equations in Algebra, and includes the solution of a linear vector equation in Vector Analysis.

§ 76. Solution of an equation of the form $X = A$.

Case I. *X and A are similar matrices.*

Let $X = [x]_m^n$, $A = [a]_m^n$, so that the equation is

$$[x]_m^n = [a]_m^n \qquad \text{.................................(1).}$$

In this case the matrix equality is identical and serves to determine the mn elements of X.

In fact equation (1) is equivalent to the mn scalar equations

$$x_{ij} = a_{ij} \qquad \text{..................................(2),}$$

where i receives the values 1, 2, ... m, and j receives the values 1, 2, ... n.

These mn equations determine the matrix X completely and uniquely.

Equation (1) is also equivalent to the m equations

$$[x_{i1}\ x_{i2}\ \dots\ x_{in}] = [a_{i1}\ a_{i2}\ \dots\ a_{in}] \qquad \text{....................(3),}$$

and to the n equations

$$\begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{mj} \end{bmatrix} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \qquad \text{.............................(4).}$$

CASE II. *X and A are dissimilar matrices.*

Let $X=[x]_m^n$, $A=[a]_p^q$, so that the equation is

$$[x]_m^n=[a]_p^q \quad \text{.............................}(5).$$

In this case the matrix equality is conventional.

Two elements of X and A which occupy the same positions in those two matrices, or which have the same horizontal affects and the same vertical affects in their respective matrices will be called corresponding elements of X and A. Thus if x_{ij} is an element occurring in X and a_{ij} an element occurring in A, then the two elements x_{ij}, a_{ij} are corresponding elements of X and A.

The equation is impossible unless those elements of A which have no corresponding elements in X are 0's. If every such element is 0, so that every row of A which has no corresponding row in X is a row of 0's, then the equation is reducible to the form considered in Case I by omissions and additions of final rows of 0's in A, and has a complete and unique solution.

Such equations have already been considered in § 39.

Ex. Consider the equation $$[x]_3^2=[a]_2^4,$$

or $$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}.$$

This equation has no solution unless $a_{13}=a_{14}=a_{23}=a_{24}=0$.

When these conditions are satisfied, it can be written

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ 0 & 0 \end{bmatrix}.$$

The equation is now reduced to the form considered in Case I, and it is satisfied when and only when

$$x_{11}=a_{11},\quad x_{12}=a_{12},\quad x_{21}=a_{21},\quad x_{22}=a_{22},\quad x_{31}=0,\quad x_{32}=0.$$

The necessary condition for a solution can be expressed in the form

$$\begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix}=0,$$

and the solution which exists when this condition is satisfied is equivalent to

$$[x_{31}\, x_{32}]=0, \qquad [x]_2^2=[a]_2^2.$$

§ 77. Solution of an equation of the form $X + A = B$.

If $X + A = B$, we have $X = B - A$, or $X = C$, where $C = B - A$, and is a known matrix. The equalities are of course conventional. Thus the equation can be reduced to one of the form considered in § 76.

It is a necessary condition for the existence of a solution that the matrix $B - A$ shall be conventionally equal to a matrix similar to X.

When this condition is satisfied, the equation has a unique solution.

Ex. i.
$$[x]_3^2 + [a]_2^3 = [b]_3^3.$$

This equation is equivalent to

$$[x]_3^2 = [b]_3^3 - [a]_2^3,$$

or

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} = \begin{bmatrix} b_{11}-a_{11}, & b_{12}-a_{12}, & b_{13}-a_{13} \\ b_{21}-a_{21}, & b_{22}-a_{22}, & b_{23}-a_{23} \\ b_{31} \quad , & b_{32} \quad , & b_{33} \end{bmatrix}.$$

It is only possible when $\quad b_{13}=a_{13}, \quad b_{23}=a_{23}, \quad b_{33}=0.$

When these conditions are satisfied, the equation is equivalent to

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} = \begin{bmatrix} b_{11}-a_{11}, & b_{12}-a_{12}, & 0 \\ b_{21}-a_{21}, & b_{22}-a_{22}, & 0 \\ b_{31} \quad , & b_{32} \quad , & 0 \end{bmatrix} = \begin{bmatrix} b_{11}-a_{11}, & b_{12}-a_{12} \\ b_{21}-a_{21}, & b_{22}-a_{22} \\ b_{31} \quad , & b_{32} \end{bmatrix},$$

and has the unique solution

$$x_{11}=b_{11}-a_{11}, \quad x_{21}=b_{21}-a_{21}, \quad x_{31}=b_{31},$$
$$x_{12}=b_{12}-a_{12}, \quad x_{22}=b_{22}-a_{22}, \quad x_{32}=b_{32}.$$

Ex. ii.
$$[x]_2^4 + [a]_2^1 = [b]_3^2.$$

This equation can be written $\quad [x]_2^4 = [b]_3^2 - [a]_2^1.$

The necessary condition for the possibility of a solution is $[b_{31}\, b_{32}]=0.$

When this condition is satisfied, the solution is given by

$$\begin{bmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{bmatrix} = 0, \qquad \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} b_{11}-a_{11}, & b_{12} \\ b_{21}-a_{21}, & b_{22} \end{bmatrix}.$$

§ 78. Solution of an equation of the form $AX = C$ when A is an undegenerate or non-singular square matrix.

Let $A = [a]_m^m$, $X = [x]_m^n$, $C = [c]_m^n$, so that the equation to be solved is

$$[a]_m^m [x]_m^n = [c]_m^n \quad \text{.............................(1)},$$

where $[a]_m^m$ has rank m.

We shall show that this equation has always a unique solution.

Let $\alpha = \det A = (a)_m^m$. Then by hypothesis $\alpha \neq 0$.

Let $[A]_m^m$ be the reciprocal and therefore $\overline{\underline{A}}{}_m^m$ the conjugate reciprocal of $[a]_m^m$.

Prefixing the matrix $\overline{\underline{A}}{}_m^m$ on both sides of equation (1), we obtain

$$\overline{\underline{A}}{}_m^m [a]_m^m [x]_m^n = \overline{\underline{A}}{}_m^m [c]_m^n = [\gamma]_m^n \quad \text{.....................(2)}$$

where $[\gamma]_m^n$ is a known matrix. Hence by § 46.5 we have

$$\begin{bmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \dots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \dots & \gamma_{2n} \\ \dots & \dots & \dots & \dots \\ \gamma_{m1} & \gamma_{m2} & \dots & \gamma_{mn} \end{bmatrix},$$

or

$$\alpha [x]_m^n = \overline{\underline{A}}{}_m^m [c]_m^n = [\gamma]_m^n \quad \text{.......................(A).}$$

This last equation determines the matrix X completely and uniquely, for it gives

$$\alpha x_{11} = \gamma_{11}, \quad \alpha x_{12} = \gamma_{12}, \ \dots \ \alpha x_{ij} = \gamma_{ij}, \ \dots \ \alpha x_{mn} = \gamma_{mn}.$$

If equation (1) has any solution, it must be that determined by equation (A). No other solution is possible.

Now prefix the matrix $[a]_m^m$ on both sides of (A). We then have

$$\alpha [a]_m^m [x]_m^n = [a]_m^m \overline{\underline{A}}{}_m^m [c]_m^n = \alpha [c]_m^n,$$

or

$$[a]_m^m [x]_m^n = [c]_m^n.$$

This follows from the fact that $[a]_m^m \overline{\underline{A}}{}_m^m$ is a scalar matrix with argument α.

Thus the value of X or $[x]_m^n$ given by (A) does actually satisfy (1), and therefore *equation* (1) *has a unique solution given by* (*A*).

Ex. i. Since

$$[\gamma]_m^n = \overline{\underline{A}}{}_m^m [c]_m^n,$$

we have

$$\gamma_{uv} = A_{1u} c_{1v} + A_{2u} c_{2v} + \dots + A_{mu} c_{mv}.$$

Thus γ_{uv} is the determinant obtained from a or $(a)_m^m$ by replacing the uth vertical row of $(a)_m^m$ by the vth vertical row of $[c]_m^n$.

Ex. ii. Since A is an undegenerate square matrix we see by Theorem V of § 71 that when the equation is satisfied, X must have the same rank as AX, i.e. the same rank as C.

Thus the solution $[x]^n_m$ *has the same rank as* $[c]^n_m$.

Ex. iii. If $m \ngtr n$, then equating the determinoids of both sides in formula (A), and making use of Ex. xi of § 68 and Theorem C of § 56, we obtain

$$a\,(x)^n_m = (c)^n_m.$$

Thus in this case the solution $[x]^n_m$ is or is not singular according as the matrix $[c]^n_m$ is or is not singular.

Ex. iv. If we denote the *inverse matrix* of $[a]^m_m$ by $\overline{\underline{A}}\,^m_m$, the solution of the equation $[a]^m_m\,[x]^n_m = [c]^n_m$ in the present case takes the form

$$[x]^n_m = \overline{\underline{A}}\,^m_m\,[c]^n_m \quad \text{.......................................(B).}$$

This is obtained from (A) by dividing both sides by a; or it can be obtained directly from (1) by prefixing $\overline{\underline{A}}\,^m_m$ on both sides, since now $\overline{\underline{A}}\,^m_m\,[a]^m_m = [1]^m_m$.

The inverse matrix $\overline{\underline{A}}\,^m_m$ is also undegenerate, having rank m.

If we denote the inverse matrix by A^{-1}, then from $AX = C$ we have

$$A^{-1}AX = A^{-1}C,$$

or

$$X = A^{-1}C \quad \text{.......................................(C).}$$

Ex. v. *Solution of the equation* $AX = 0$ *when* A *is an undegenerate square matrix.*

Putting $C = 0$ in the equation of the text, we see that the equation $AX = 0$ has in this case the unique solution

$$X = 0.$$

This also appears from Theorem V of § 71, which shows that X has the same rank as the zero matrix on the right, and must therefore itself be a zero matrix.

Ex. vi. We will solve the equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$

The conjugate reciprocal of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 2, & 1, & -7 \\ 0, & -5, & 5 \\ -4, & 3, & -1 \end{bmatrix}$.

Prefixing the latter matrix on both sides, we obtain

$$\begin{bmatrix} -10, & 0, & 0 \\ 0, & -10, & 0 \\ 0, & 0, & -10 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} = \begin{bmatrix} 2, & 1, & -7 \\ 0, & -5, & 5 \\ -4, & 3, & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix},$$

i.e.

$$-10 \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} = \begin{bmatrix} -17, & -11, & -1, & 5 \\ 5, & 5, & -5, & -5 \\ -1, & -3, & -3, & -5 \end{bmatrix},$$

or

$$\begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 17, & 11, & 1, & -5 \\ -5, & -5, & 5, & 5 \\ 1, & 3, & 3, & 5 \end{bmatrix}.$$

This solution gives $x_1 = \frac{17}{10}$, $y_1 = \frac{11}{10}$, $z_1 = \frac{1}{10}$, $w_1 = -\frac{5}{10}$, and so on.

Ex. vii. If we make use of Ex. i, we can write down the value of each individual element of X from a mere inspection of the equation. Thus in the case of the equation considered in Ex. vi, we have

$$z_2 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix}, \qquad w_2 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix},$$

where the second vertical row has been changed.

That is, we have $-10z_2 = -5$, $\quad -10w_2 = -5$,

or $z_2 = \frac{5}{10} = \frac{1}{2}$, $\quad w_2 = \frac{5}{10} = \frac{1}{2}$.

Ex. viii. To solve the equation

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} = \begin{bmatrix} a_1 & \beta_1 & \gamma_1 & \delta_1 \\ a_2 & \beta_2 & \gamma_2 & \delta_2 \\ a_3 & \beta_3 & \gamma_3 & \delta_3 \end{bmatrix},$$

in which $(abc)_{123} = \Delta \neq 0$, we prefix on both sides the matrix $\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$ which is the conjugate reciprocal of $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$.

We then obtain

$$\Delta \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} = \begin{bmatrix} (abc) & (\beta bc) & (\gamma bc) & (\delta bc) \\ (aac) & (a\beta c) & (a\gamma c) & (a\delta c) \\ (aba) & (ab\beta) & (ab\gamma) & (ab\delta) \end{bmatrix},$$

where (abc), (βbc), ... stand for $(abc)_{123}$, $(\beta bc)_{123}$, ...

§ 79. Solution of an equation of the form $XB = C$ when B is an undegenerate or non-singular square matrix.

Let $X = [x]_m^n$, $B = [b]_n^n$, $C = [c]_m^n$, so that the equation to be solved is

$$[x]_m^n [b]_n^n = [c]_m^n \quad \text{..............................(1)},$$

where $[b]_n^n$ has rank n.

We will show that this equation always has a unique solution.

Let $\beta = \det B = (b)_n^n$. Then by hypothesis $\beta \neq 0$.

Let $[B]_n^n$ be the reciprocal and therefore $\overline{\underline{B}}_n^n$ the conjugate reciprocal of $[b]_n^n$.

Postfixing the matrix $\overline{\underline{B}}_n^n$ on both sides of (1), we obtain

$$[x]_m^n [b]_n^n \overline{\underline{B}}_n^n = [c]_m^n \overline{\underline{B}}_n^n = [\gamma]_m^n \quad \text{.....................(2)},$$

where $[\gamma]_m^n$ is a known matrix. Hence by § 46.5 we have

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} \begin{bmatrix} \beta & 0 & \dots & 0 \\ 0 & \beta & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \dots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \dots & \gamma_{2n} \\ \dots & \dots & \dots & \dots \\ \gamma_{m1} & \gamma_{m2} & \dots & \gamma_{mn} \end{bmatrix},$$

or

$$\beta [x]_m^n = [c]_m^n \overline{\underline{B}}_n^n = [\gamma]_m^n \quad \text{........................(A)}.$$

This last equation determines the matrix X completely and uniquely, for it gives

$$\beta x_{11} = \gamma_{11}, \ \beta x_{12} = \gamma_{12}, \ \dots \ \beta x_{ij} = \gamma_{ij}, \ \dots \ \beta x_{mn} = \gamma_{mn}.$$

If equation (1) has any solution, it must be that determined by the equation (A). No other solution is possible.

Now postfix the matrix $[b]_n^n$ on both sides of (A). We then have

$$\beta [x]_m^n [b]_n^n = [c]_m^n \overline{\underline{B}}_n^n [b]_n^n = \beta [c]_m^n,$$

or

$$[x]_m^n [b]_n^n = [c]_m^n.$$

This follows from the fact that $\overline{\underline{B}}_n^n [b]_n^n$ is a scalar matrix with argument β.

Thus the value of X or $[x]_m^n$ given by (A) does actually satisfy (1), and therefore *equation* (1) *has a unique solution given by* (A).

Ex. i. Since

$$[\gamma]^n_m = [c]^n_m \overline{\underline{B}}^n_n,$$

we have

$$\gamma_{sr} = B_{r1} c_{s1} + B_{r2} c_{s2} + \ldots + B_{rn} c_{sn}.$$

Thus γ_{sr} is the determinant formed from β or $|b|^n_n$ by replacing the rth horizontal row of $[b]^n_n$ by the sth horizontal row of $[c]^n_m$.

Ex. ii. Since B is an undegenerate square matrix we see by Theorem V of § 71 that when the equation is satisfied, X must have the same rank as XB, i.e. the same rank as C.

Thus the solution $[x]^n_m$ has the same rank as $[c]^n_m$.

Ex. iii. If $n > m$, then equating the determinants of both sides in formula (A) and making use of Ex. xi of § 68 and Theorem C of § 56, we obtain

$$\beta |x|^n_m = |c|^n_m.$$

Thus in this case the solution $[x]^n_m$ is or is not singular according as the matrix $[c]^n_m$ is or is not singular.

Ex. iv. If we denote the *inverse matrix* of $[b]^n_n$ by $\overline{\underline{B}}^n_n$, the solution of

$$[x]^n_m [b]^n_n = [c]^n_m$$

in the present case takes the form

$$[x]^n_m = [c]^n_m \overline{\underline{B}}^n_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (B).$$

We obtain this by postfixing $\overline{\underline{B}}^n_n$ on both sides of (1).

If we denote the inverse matrix of B by B^{-1}, the solution of $XB = C$ in the present case takes the form

$$X = CB^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (C).$$

Ex. v. *Solution of the equation $XB = 0$, when B is an undegenerate square matrix.*

Putting $C = 0$ in the equation of the text, we see that the equation $XB = 0$ has the unique solution

$$X = 0.$$

This also appears from Theorem V of § 71, which shows that X must have the same rank as the zero matrix on the right of the equation, and must therefore itself be a zero matrix.

Ex. vi. The solution of the equation $[b]^n_n [x]^n_m = [c]^n_m$ can be deduced from the solution of the equation considered in § 78.

For equating the conjugates of both sides we have the equivalent equation

$$\overline{\underline{[x]}}^m_n \overline{\underline{[b]}}^n_n = \overline{\underline{[c]}}^m_n.$$

Prefixing $[B]_n^n$, the conjugate reciprocal of $[b]_n^n$, on both sides, we have

$$\beta \, \overline{\underline{x}}_n^{\,m} = [B]_n^n \, \overline{\underline{c}}_n^{\,m}.$$

If we now equate the conjugates of both sides, we obtain

$$\beta [x]_m^n = [c]_m^n \, \overline{\underline{B}}_n^{\,n}.$$

Ex. vii. We will solve the equation

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{bmatrix}.$$

The conjugate reciprocal of $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 4, & -2, & 1 \\ -2, & 4, & -2 \\ -8, & 4, & 1 \end{bmatrix}$.

Postfixing the latter matrix on both sides, we obtain

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} 4, & -2, & 1 \\ -2, & 4, & -2 \\ -8, & 4, & 1 \end{bmatrix},$$

i.e. $$6 \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} -18, & 12, & 3 \\ -20, & 22, & -5 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -18, & 12, & 3 \\ -20, & 22, & -5 \end{bmatrix}.$$

This solution gives

$$x_1 = -3, \qquad x_2 = -\frac{10}{3}, \qquad y_1 = 2, \qquad y_2 = \frac{11}{3}, \qquad z_1 = \frac{1}{2}, \qquad z_2 = -\frac{5}{6}.$$

Ex. viii. If we make use of Ex. i, we can write down the value of each individual element of X from a mere inspection of the equation.

Thus in the case of the equation considered in Ex. vii, we have

$$y_2 \begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 4 & 2 \\ 4 & 0 & 2 \end{vmatrix}, \qquad z_2 \begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 4 & 2 \end{vmatrix};$$

or $$6y_2 = 22, \quad 6z_2 = -5; \qquad \text{i.e. } y_2 = \frac{11}{3}, \quad z_2 = -\frac{5}{6}.$$

Ex. ix. To solve the equation

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{bmatrix},$$

in which $(abc)_{123} = \Delta \neq 0$, we postfix

$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$, the conjugate reciprocal of $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, on both sides.

We then obtain

$$\Delta \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} (\alpha bc) & (a\alpha c) & (ab\alpha) \\ (\beta bc) & (a\beta c) & (ab\beta) \\ (\gamma bc) & (a\gamma c) & (ab\gamma) \\ (\delta bc) & (a\delta c) & (ab\delta) \end{bmatrix},$$

where (αbc), $(a\alpha c)$, ... stand for $(\alpha bc)_{123}$, $(a\alpha c)_{123}$, ...

Ex. x. If $[a]_m^m$ is any undegenerate square matrix whose determinant and reciprocal are a and $[A]_m^m$, the solutions of the equations

$$[a]_m^m [x]_m^m = [1]_m^m, \qquad [x]_m^m [a]_m^m = [1]_m^m$$

are

$$[x]_m^m = \frac{1}{a} \overline{\underline{A}}\,{}_m^m, \qquad [x]_m^m = \frac{1}{a} \overline{\underline{A}}\,{}_m^m.$$

Thus an undegenerate square matrix has one and only one inverse matrix, viz. its principal inverse matrix as defined in § 16.6. *This matrix is both an inverse pre-factor and an inverse post-factor.*

Ex. xi. Let

$$[a]_m^m [b]_m^m = k[1]_m^m, \qquad \text{where } k \neq 0.$$

Let a, β be the determinants and $[A]_m^m$, $[B]_m^m$ the reciprocals of $[a]_m^m$, $[b]_m^m$. Then $a\beta = k^m \neq 0$.

Solving for $[a]_m^m$ and $[b]_m^m$ respectively, we have

$$[a]_m^m = \frac{k}{\beta} \overline{\underline{B}}\,{}_m^m, \qquad [b]_m^m = \frac{k}{a} \overline{\underline{A}}\,{}_m^m,$$

or

$$\overline{\underline{B}}\,{}_m^m = \frac{k^{m-1}}{a} [a]_m^m, \qquad \overline{\underline{A}}\,{}_m^m = \frac{k^{m-1}}{\beta} [b]_m^m.$$

Thus if a product of two square matrices of the same order is a non-zero scalar matrix, each of the two matrices is a scalar multiple of the conjugate reciprocal (or of the inverse) of the other.

§ 80. Solution of an equation of the form $AXB = C$ when A and B are undegenerate or non-singular square matrices.

Let $A = [a]_m^m$, $X = [x]_m^n$, $B = [b]_n^n$, $C = [c]_m^n$, so that the equation to be solved is

$$[a]_m^m [x]_m^n [b]_n^n = [c]_m^n \quad \text{..........................} (1),$$

where $[a]_m^m$ has rank m and $[b]_n^n$ has rank n.

Let $$\alpha = \det A = (a)_m^m, \quad \beta = \det B = (b)_n^n.$$

Then by hypothesis $$\alpha \neq 0 \quad \text{and} \quad \beta \neq 0.$$

Let $[A]_m^m$, $[B]_n^n$ be the reciprocals and therefore $\overline{\underline{A}}_m^m$, $\overline{\underline{B}}_n^n$ the conjugate reciprocals of $[a]_m^m$, $[b]_n^n$.

Prefixing the matrix $\overline{\underline{A}}_m^m$ and postfixing the matrix $\overline{\underline{B}}_n^n$ on both sides of (1), we obtain

$$\overline{\underline{A}}_m^m [a]_m^m [x]_m^n [b]_n^n \overline{\underline{B}}_n^n = \overline{\underline{A}}_m^m [c]_m^n \overline{\underline{B}}_n^n = [\gamma]_m^n \quad \text{...........(2)},$$

where $[\gamma]_m^n$ is a known matrix. Hence by § 46.5 we have

$$\begin{bmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} \begin{bmatrix} \beta & 0 & \dots & 0 \\ 0 & \beta & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \dots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \dots & \gamma_{2n} \\ \dots & \dots & \dots & \dots \\ \gamma_{m1} & \gamma_{m2} & \dots & \gamma_{mn} \end{bmatrix},$$

or

$$\alpha\beta [x]_m^n = \overline{\underline{A}}_m^m [c]_m^n \overline{\underline{B}}_n^n = [\gamma]_m^n \quad \text{.....................(A)}.$$

This last equation determines the matrix X completely and uniquely, for it gives

$$\alpha\beta x_{11} = \gamma_{11},\ \alpha\beta x_{12} = \gamma_{12},\ \dots\ \alpha\beta x_{ij} = \gamma_{ij},\ \dots\ \alpha\beta x_{mn} = \gamma_{mn}.$$

If equation (1) has any solution, it must be that determined by the equation (A). No other solution is possible.

If now we prefix the matrix $[a]_m^m$ and postfix the matrix $[b]_n^n$ on both sides of (A) and observe that $[a]_m^m \overline{\underline{A}}_m^m = \alpha [1]_m^m$, $\overline{\underline{B}}_n^n [b]_n^n = \beta [1]_n^n$, we obtain

$$\alpha\beta [a]_m^m [x]_m^n [b]_n^n = [a]_m^m \overline{\underline{A}}_m^m [c]_m^n \overline{\underline{B}}_n^n [b]_n^n = \alpha\beta [c]_m^n,$$

or

$$[a]_m^m [x]_m^n [b]_n^n = [c]_m^n.$$

This shows that the value of X or $[x]_m^n$ given by (A) is actually a solution of equation (1). *Thus equation* (1) *has a unique solution given by* (A).

Ex. i. Making use of Theorem V of § 71, we see that when the equation $AXB = C$ is satisfied the matrix X must have the same rank as the matrix C.

Thus the solution $[x]_m^n$ *has the same rank as* $[c]_m^n$.

In fact $[x]_m^n$ and $[c]_m^n$ are equipotent matrices.

Ex. ii. *Solution of the equation* $AXB = 0$, *when* A *and* B *are undegenerate square matrices.*

Putting $C=0$ in the equation of the text, we see that the equation $AXB=0$ has the unique solution

$$X=0.$$

This also appears from Theorem V of § 71, which shows that X must have the same value as the zero matrix on the right, and must therefore itself be a zero matrix.

Ex. iii. We will solve the equation

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{bmatrix}.$$

The conjugate reciprocals of

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{are} \quad \begin{bmatrix} 4, & -3 \\ -1, & 2 \end{bmatrix}, \quad \begin{bmatrix} 4, & -2, & 1 \\ -2, & 4, & -2 \\ -8, & 4, & 1 \end{bmatrix}.$$

Prefixing and postfixing these conjugate reciprocal matrices on both sides of the above equation, we have

$$\begin{bmatrix} 5, & 0 \\ 0, & 5 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4, & -3 \\ -1, & 2 \end{bmatrix} \begin{bmatrix} 2, & 1, & 3 \\ 1, & 4, & 2 \end{bmatrix} \begin{bmatrix} 4, & -2, & 1 \\ -2, & 4, & -2 \\ -8, & 4, & 1 \end{bmatrix},$$

i.e. $$30 \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} -12, & -18, & 27 \\ -22, & 32, & -13 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} -12, & -18, & 27 \\ -22, & 32, & -13 \end{bmatrix}.$$

Accordingly

$$x_1 = -\frac{2}{5}, \quad y_1 = -\frac{3}{5}, \quad z_1 = \frac{9}{10}, \quad x_2 = -\frac{11}{15}, \quad y_2 = \frac{16}{15}, \quad z_2 = -\frac{13}{30}.$$

Ex. iv. If $\overline{A}^m_m$, $\overline{B}^n_n$ are the *inverse matrices* of $[a]^m_m$, $[b]^n_n$, the solution of the equation $[a]^m_m [x]^n_m [b]^n_n = [c]^n_m$ in the present case is

$$[x]^n_m = \overline{A}^m_m [c]^n_m \overline{B}^n_n \quad \text{..................................} (B).$$

Ex. v. If the inverse matrices of A and B are denoted by A^{-1}, B^{-1}, the solution of the equation $AXB=C$ in the present case is

$$X=A^{-1}CB^{-1} \quad \text{..} (C).$$

Ex. vi. The equations $AX=C$, $XB=C$ are the particular cases of the equation $AXB=C$ obtained by taking B and A respectively to be unit matrices. Thus the solutions obtained in §§ 78 and 79 can be deduced from the solution obtained in the present article.

§ 81. Solution of any equation of the form $AX = C$.

1. *Augmented and unaugmented matrices of the equation.*

We shall assume that the equation to be solved is

$$[a]_m^\rho [x]_\rho^n = [c]_m^n \quad \text{.............................(1)},$$

so that $$A = [a]_m^\rho, \quad X = [x]_\rho^n, \quad C = [c]_m^n.$$

One special case of this equation, that in which A is an undegenerate square matrix, has already been considered in § 78.

The matrices

$$A = [a]_m^\rho, \qquad A' = [a, c]_m^{\rho, n},$$

or $$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1\rho} \\ a_{21} & a_{22} & \dots & a_{2\rho} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{m\rho} \end{bmatrix}, \quad A' = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1\rho} & c_{11} & c_{12} & \dots & c_{1n} \\ a_{21} & a_{22} & \dots & a_{2\rho} & c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{m\rho} & c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix},$$

will be called respectively the *unaugmented matrix* and the *augmented matrix* of the equation.

2. *Reduction of the equation to an irreducible equation of the same kind.*

An equation of the form $AX = C$ will be said to be *irreducible* when the horizontal rows of the augmented matrix A' are unconnected. We will now prove the following two theorems:

Theorem I. *If the ith horizontal row of the augmented matrix A' of the equation $[a]_m^\rho [x]_\rho^n = [c]_m^n$ is connected with certain other horizontal rows of A', then the given equation can be replaced by the equation obtained from it by striking out the ith horizontal rows of $[a]_m^\rho$ and $[c]_m^n$.*

Theorem II. *If the augmented matrix A' of the equation $[a]_m^\rho [x]_\rho^n = [c]_m^n$ has rank r, then by striking out $m-r$ corresponding horizontal rows of A and C we can reduce the equation to an irreducible equation of the form $[a]_r^\rho [x]_\rho^n = [c]_r^n$.*

To prove Theorem I it will be sufficient to prove it for the case $i = m$; for by the properties of active rows we can re-arrange the horizontal rows of A and C so that the ith horizontal row in each comes into the final position, and then the ith horizontal row of A' also comes into the final position in A'.

Suppose then that the mth horizontal row of A' is connected with the preceding $m-1$ horizontal rows by the relation

$$[a_{m1}\, a_{m2} \dots a_{m\rho}\, c_{m1}\, c_{m2} \dots c_{mn}] = [h_1\, h_2 \dots h_{m-1}]\, [a, c]_{m-1}^{\rho, n} \quad \text{......(2)}.$$

By the properties of active rows, this relation is equivalent to the two relations

$$[a_{m1}\,a_{m2}\,\ldots\,a_{m\rho}] = [h_1\,h_2\,\ldots\,h_{m-1}]\,[a]_{m-1}^{\rho} \quad\ldots\ldots\ldots\ldots(3),$$

$$[c_{m1}\,c_{m2}\,\ldots\,c_{mn}] = [h_1\,h_2\,\ldots\,h_{m-1}]\,[c]_{m-1}^{n} \quad\ldots\ldots\ldots\ldots(4).$$

To prove the theorem we have to show that the solutions of the equation

$$[a]_{m-1}^{\rho}\,[x]_{\rho}^{n} = [c]_{m-1}^{n} \quad\ldots\ldots\ldots\ldots(5)$$

are identical with the solutions of the equation

$$[a]_{m}^{\rho}\,[x]_{\rho}^{n} = [c]_{m}^{n} \quad\ldots\ldots\ldots\ldots(1).$$

Now if (1) is satisfied, it follows from the properties of active rows that (5) also is satisfied.

Again if (5) is satisfied, we have

$$[h_1\,h_2\,\ldots\,h_{m-1}]\,[a]_{m-1}^{\rho}\,[x]_{\rho}^{n} = [h_1\,h_2\,\ldots\,h_{m-1}]\,[c]_{m-1}^{n},$$

i.e., making use of (3) and (4), we have

$$[a_{m1}\,a_{m2}\,\ldots\,a_{m\rho}]\,[x]_{\rho}^{n} = [c_{m1}\,c_{m2}\,\ldots\,c_{mn}] \quad\ldots\ldots\ldots\ldots(6).$$

From (5) and (6) we deduce that

$$[a]_{m}^{\rho}\,[x]_{\rho}^{n} = [c]_{m}^{n}.$$

Thus equations (1) and (5) have identical solutions, and therefore in determining the solutions of (1), we can replace it by (4).

We have now proved Theorem I.

Next let A' have rank r and let the p_1th, p_2th, ... p_rth horizontal rows of A' be unconnected. Then all other horizontal rows of A' are connected with these. Therefore by Theorem I we can strike out all horizontal rows of A and C in the given equation (1) except the p_1th, p_2th, ... p_rth and so replace (1) by

$$[a_{p1}]_{r}^{\rho}\,[x]_{\rho}^{n} = [c_{p1}]_{r}^{n} \quad\ldots\ldots\ldots\ldots(7).$$

The horizontal rows of the augmented matrix

$$\begin{bmatrix} a_{p_11} & a_{p_12} & \ldots & a_{p_1\rho} & c_{p_11} & c_{p_12} & \ldots & c_{p_1n} \\ a_{p_21} & a_{p_22} & \ldots & a_{p_2\rho} & c_{p_21} & c_{p_22} & \ldots & c_{p_2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{p_r1} & a_{p_r2} & \ldots & a_{p_r\rho} & c_{p_r1} & c_{p_r2} & \ldots & c_{p_rn} \end{bmatrix} = [a_{p1},\, c_{p1}]_{r}^{\rho,\, n}$$

of equation (7) are unconnected; and equation (7) is therefore an *irreducible* equation of the *form* $[a]_{r}^{\rho}\,[x]_{\rho}^{n} = [c]_{r}^{n}$. Thus Theorem II is proved. The horizontal rows of A and C in equation (1) can always be so arranged that the first r horizontal rows of A' are unconnected. When this is done, the reduced equation actually is $[a]_r^\rho [x]_\rho^n = [c]_r^n$.

When we reduce the equation $AX = C$ by Theorem I, the rows struck out from A' are connected with the remaining rows of A', and therefore also the rows struck out from A are connected with the remaining rows of A. Consequently by Theorem VI of § 71 the omission of these rows produces no alteration in the ranks of A and A'. We have therefore the following theorem:

Theorem III. *When an equation of the form $AX = C$ is reduced by repeated applications of Theorem I, the augmented and unaugmented matrices of the reduced equation have respectively the same ranks as the augmented and unaugmented matrices of the original equation.*

We see then that in equation (7), $[a_{p1}]_r^\rho$ has the same rank as $[a]_m^\rho$. Similarly $[c_{p1}]_r^n$ has the same rank as $[c]_m^n$, and $[a_{p1}, c_{p1}]_r^{\rho, n}$ has the same rank as $[a, c]_m^{\rho, n}$.

Ex. i. In the equation

$$\begin{bmatrix} 1 & 3 & 4 \\ 1 & 1 & 2 \\ 5 & 3 & 8 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} = \begin{bmatrix} 10 & 1 & 25 & 9 \\ 4 & 1 & 11 & 7 \\ 14 & 5 & 41 & 33 \end{bmatrix},$$

the augmented matrix $$\begin{bmatrix} 1 & 3 & 4 & 10 & 1 & 25 & 9 \\ 1 & 1 & 2 & 4 & 1 & 11 & 7 \\ 5 & 3 & 8 & 14 & 5 & 41 & 33 \end{bmatrix}$$

has rank 2, for the sum of the matrices of the first and third horizontal rows is equal to six times the matrix of the second horizontal row.

We can strike out the second horizontal row and replace the equation by the irreducible equation

$$\begin{bmatrix} 1 & 3 & 4 \\ 5 & 3 & 8 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} = \begin{bmatrix} 10 & 1 & 25 & 9 \\ 14 & 5 & 41 & 33 \end{bmatrix}.$$

In this case we can strike out either the first or the second or the third horizontal row in both A and C.

3. *Necessary and sufficient condition for the existence of a finite solution of the equation.*

This condition is contained in the following theorem:

Theorem IV. *The equation $AX = C$ has a finite solution or admits of finite solution when and only when the augmented matrix A' has the same rank as the unaugmented matrix A.*

If there is a finite matrix $[x]_\rho^n$ satisfying the equation

$$[a]_m^\rho [x]_\rho^n = [c]_m^n \quad \text{.............................(1)},$$

then the form of this equation shows that each of the vertical rows of $[c]_m^n$ is connected with the vertical rows of $[a]_m^\rho$. Again if each of the vertical rows of $[c]_m^n$ is connected with the vertical rows of $[a]_m^\rho$, then by the definition of a connection there must exist a relation of the form (1) in which the matrix $[x]_\rho^n$ is finite, i.e. has all its elements finite.

Thus equation (1) *admits of finite solution when and only when each of the vertical rows of* $[c]_m^n$ *is connected with the vertical rows of* $[a]_m^\rho$ ....(8).

This result is equivalent to the above theorem, as has been shown in Ex. xiv of § 71.

Alternative proof of Theorem IV. We will commence by showing that the condition

$$\text{rank of } A' = \text{rank of } A \quad \text{.........................(9)}$$

is a *necessary* condition for the existence of a finite solution.

Suppose that there is a finite matrix $[x]_\rho^n$ satisfying equation (1).

Then if there is any connection

$$[h_1 h_2 \ldots h_m] [a]_m^\rho = 0 \quad \text{...........................(10)}$$

between the horizontal rows of A, we see by prefixing $[h_1 h_2 \ldots h_m]$ on both sides of (1) that

$$[h_1 h_2 \ldots h_m] [c]_m^n = 0 \quad \text{..........................(11)},$$

and from (10) and (11) it follows that

$$[h_1 h_2 \ldots h_m] [a, c]_m^{\rho, n} = 0 \quad \text{..........................(12)}.$$

This shows that to every connection between the horizontal rows of A there corresponds exactly the same connection between the horizontal rows of A'. Consequently A' cannot have more unconnected horizontal rows than A, and the rank of A' cannot exceed the rank of A.

Further the rank of A' cannot be less than the rank of A; for every non-vanishing derived determinant of A is also a non-vanishing derived determinant of A'.

It follows that when equation (1) admits of finite solution, A' must have the same rank as A.

Thus (9) is a *necessary* condition for the existence of a finite solution.

Now in the next sub-article a finite solution of equation (1) will actually be obtained whenever the condition (9) is satisfied.

Thus (9) is a *sufficient* as well as a necessary condition for the existence of a finite solution.

When equation (1) admits only of infinite solutions, we cannot deduce (11) from (10). In fact it will be shown in sub-article 7 that there may be infinite solutions when the condition (9) is not satisfied.

Ex. ii. In the equation of Ex. i the unaugmented and augmented matrices

$$\begin{bmatrix} 1 & 3 & 4 \\ 1 & 1 & 2 \\ 5 & 3 & 8 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 3 & 4 & 10 & 1 & 25 & 9 \\ 1 & 1 & 2 & 4 & 1 & 11 & 7 \\ 5 & 3 & 8 & 14 & 5 & 41 & 33 \end{bmatrix}$$

have common rank 2.

Consequently the equation admits of finite solution.

4. *General solution of the equation $AX = C$ when it admits of finite solution.*

As before let the given equation be

$$[a]_m^\rho [x]_\rho^n = [c]_m^n \quad \text{..............................(1).}$$

And now let A and A' have common rank r, so that equation (1) admits of finite solution. Then A and A' have a common non-vanishing derived determinant of order r. Let $(a_{pq})_r^r$ be any such determinant. We will consider first the general case in which there is no restriction on $(a_{pq})_r^r$, and then the special case in which $(a)_r^r$ the leading derived determinant of order r of $[a]_m^\rho$ does not vanish and $(a_{pq})_r^r$ is taken to be $(a)_r^r$.

CASE I. $(a_{pq})_r^r = \alpha \neq 0$.

By sub-article 2 the solutions of the given equation (1) are identical with the solutions of the irreducible equation

$$[a_{p1}]_r^\rho [x]_\rho^n = [c_{p1}]_r^n \quad \text{..........................(13),}$$

whose augmented and unaugmented matrices both have rank r.

By the properties of passive rows (see § 52.9), we can write (13) in the form

$$[a_{pq}]_r^r [x_{q1}]_r^n + [a_{p\tau}]_r^{\rho - r} [x_{\tau 1}]_{\rho - r}^n = [c_{p1}]_r^n,$$

or

$$[a_{pq}]_r^r [x_{q1}]_r^n = [c_{p1}]_r^n - [a_{p\tau}]_r^{\rho - r} [x_{\tau 1}]_{\rho - r}^n = [\xi]_r^n \quad \text{...........(14),}$$

where $[\tau_1 \tau_2 \dots \tau_{\rho-r}]$ is complementary to $[q_1 q_2 \dots q_r]$ in $[1\ 2 \dots \rho]$.

Equation (14) can be solved for the matrix $[x_{q1}]_r^n$ by § 78.

Let $(a_{pq})_r^r = \alpha$, and let $[A_{pq}]_r^r$ be the reciprocal matrix of $[a_{pq}]_r^r$. Then prefixing $\overline{\underline{A_{pq}}}_r^r$, the conjugate reciprocal of $[a_{pq}]_r^r$, on both sides of (14), we obtain by § 78 the following result:

First form of the general solution.

If A and A' have common rank r, and if $(a_{pq})_r^r = \alpha \neq 0$, then equation (1) is satisfied when and only when

$$\alpha \left[x_{q1}\right]_r^n = \boxed{A_{pq}}_r^r \left[c_{p1}\right]_r^n - \boxed{A_{pq}}_r^r \left[a_{p\tau}\right]_r^{\rho-r} \left[x_{\tau 1}\right]_{\rho-r}^n ,$$

or

$$\alpha \left[x_{q1}\right]_r^n = \left[\gamma\right]_r^n - \left[\alpha_{1\tau}\right]_r^{\rho-r} \left[x_{\tau 1}\right]_{\rho-r}^n ,$$

where γ_{ij} is the value of the determinant formed from the determinant $(a_{pq})_r^r$ when its ith vertical row is replaced by the jth vertical row of $\left[c_{p1}\right]_r^n$,

and α_{ij} is the value of the determinant formed from the determinant $(a_{pq})_r^r$ when its ith vertical row is replaced by the jth vertical row of $\left[a_{p1}\right]_r^\rho$......(A).

Formula (A) gives the general solution of equation (1). In this solution arbitrary values can be assigned to the $n(\rho - r)$ unknown elements which do not occur in $\left[x_{q1}\right]_r^n$, and the remaining rn unknown elements, viz. those which do occur in $\left[x_{q1}\right]_r^n$, are expressed as unique linear functions of these.

If $x_{q_i j}$ is any one of the elements of $\left[x_{q1}\right]_r^n$, we have

$$\alpha x_{q_i j} = \gamma_{ij} - \alpha_{i\tau_1} x_{\tau_1 j} - \alpha_{i\tau_2} x_{\tau_2 j} - \dots - \alpha_{i\tau_{\rho-r}} x_{\tau_{\rho-r} j} \quad \text{.........(15).}$$

We have now proved the following theorem:

Theorem V. *If the unaugmented matrix A and the augmented matrix A' of the equation $\left[a\right]_m^\rho \left[x\right]_\rho^n = \left[c\right]_m^n$ have common rank r, then the equation admits of finite solution, and the general solution expresses rn of the ρn unknown elements as linear (in general non-homogeneous) functions of the remaining $(\rho - r)\, n$ unknown elements to which arbitrary values may be assigned.*

Since the general solution contains $(\rho - r)\, n$ arbitrary elements, which may be real or imaginary, the total number of solutions is $\infty^{2(\rho-r)n}$. In the special case in which $r = \rho$, no arbitrary elements occur in the general solution, and there is a single unique finite solution. In fact this special case is equivalent to that considered in § 78.

NOTE 1. *The simplest particular solution in Case I.*

This is obtained by assigning zero values to all the (arbitrary) elements of $\left[x_{\tau 1}\right]_{\rho-r}^n$ and is given by

$$\alpha \left[x_{q1}\right]_r^n = \boxed{A_{pq}}_r^r \left[c_{p1}\right]_r^n = \left[\gamma\right]_r^n , \quad \left[x_{\tau 1}\right]_{\rho-r}^n = 0.$$

This particular solution can be obtained by solving the equation

$$\left[a_{pq}\right]_r^r \left[x_{q1}\right]_r^n = \left[c_{p1}\right]_r^n ,$$

and it has the same rank as $\left[c_{p1}\right]_r^n$.

NOTE 2. *Alternative method of solution in Case I.*

The general solution can also be obtained directly from the reduced equation (13) by prefixing the conjugate reciprocal matrix $\overline{\underline{A_{pq}}}{}_r^r$ on both sides before transferring terms to the right-hand side. This method is often more convenient in practice.

Writing
$$\overline{\underline{A_{pq}}}{}_r^r \, [a_{p1}]_r^\rho = [a]_r^\rho,$$

so that
$$[a_{1q}]_r^r = \overline{\underline{A_{pq}}}{}_r^r \, [a_{pq}]_r^r = a\,[1]_r^r \quad \text{.............................(16)},$$

the general solution is given by
$$[a]_r^\rho \, [x]_\rho^n = [\gamma]_r^n \quad \text{.......................................(A')}.$$

The result (16) shows that formula (A′) can by the properties of passive rows be transformed into the second equation in (A). It also shows that if we equate each of the rn elements of the product matrix on the left in (A′) to the corresponding element of the matrix on the right, we obtain rn scalar equations each of which determines one of the unknown elements occurring in $[x_{q1}]_r^n$. In fact these scalar equations are the same as the rn scalar equations given by the second matrix equation in (A).

CASE II. $(a)_r^r = \alpha \neq 0$.

We can always before proceeding to the solution of the given equation re-arrange (if necessary) the horizontal rows of A and C and the passive rows of A and X in such a manner as to form an equivalent equation in which $(a)_r^r$, the leading derived determinant of order r of A, does not vanish. Thus every case can always be reduced to this special case.

In the present case we can replace the given equation (1) by the irreducible equation
$$[a]_r^\rho \, [x]_\rho^n = [c]_r^n \quad \text{........................(17)},$$

and this by the properties of passive rows can be written in the form
$$[a]_r^r \, [x]_r^n = [\xi]_r^n \quad \text{............................(18)},$$

where now
$$[\xi]_r^n = [c]_r^n - \begin{bmatrix} a_{1,r+1} & a_{1,r+2} & \cdots & a_{1\rho} \\ a_{2,r+1} & a_{2,r+2} & \cdots & a_{2\rho} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r,r+1} & a_{r,r+2} & \cdots & a_{r\rho} \end{bmatrix} \begin{bmatrix} x_{r+1,1} & x_{r+1,2} & \cdots & x_{r+1,n} \\ x_{r+2,1} & x_{r+2,2} & \cdots & x_{r+2,n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{\rho 1} & x_{\rho 2} & \cdots & x_{\rho n} \end{bmatrix} \quad \text{......(19)}.$$

Writing $(a)_r^r = \alpha$, and denoting the reciprocal matrix of $[a]_r^r$ by $[A]_r^r$, we can solve equation (18) for $[x]_r^n$ by prefixing $\overline{\underline{A}}{}_r^r$ on both sides as in § 78; and we then obtain the following result:

Second form of the general solution.

If A and A' have common rank r, and if $(a)_r^r = \alpha \neq 0$, equation (1) is satisfied when and only when

$$\alpha [x]_r^n = \overline{\underline{A}}_r^r [\xi]_r^n,$$

or $$\alpha [x]_r^n = [\gamma]_r^n - \begin{bmatrix} \alpha_{1,r+1} & \alpha_{1,r+2} & \cdots & \alpha_{1\rho} \\ \alpha_{2,r+1} & \alpha_{2,r+2} & \cdots & \alpha_{2\rho} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{r,r+1} & \alpha_{r,r+2} & \cdots & \alpha_{r\rho} \end{bmatrix} \begin{bmatrix} x_{r+1,1} & x_{r+1,2} & \cdots & x_{r+1,n} \\ x_{r+2,1} & x_{r+2,2} & \cdots & x_{r+2,n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{\rho 1} & x_{\rho 2} & \cdots & x_{\rho n} \end{bmatrix},$$

where γ_{ij} is the determinant formed from $(a)_r^r$ when its ith vertical row is replaced by the jth vertical row of $[c]_r^n$, and α_{ij} is the determinant formed from $(a)_r^r$ when its ith vertical row is replaced by the jth vertical row of $[a]_r^\rho$, so that in particular $[\alpha]_r^r = \alpha [1]_r^r$ ....(B).

Formula (B) gives the general solution of equation (1) in this special case. The solution expresses the rn elements of $[x]_r^n$ as linear functions of the remaining $(\rho - r)n$ unknown elements to which arbitrary values may be assigned.

If x_{ij} is any one of the elements of $[x]_r^n$, we have

$$\alpha x_{ij} = \gamma_{ij} - \alpha_{i,r+1} x_{r+1,j} - \alpha_{i,r+2} x_{r+2,j} - \ldots - \alpha_{i\rho} x_{\rho j} \quad \text{........(20).}$$

Formula (B) can of course be deduced from formula (A) by putting

$$[p_1 p_2 \ldots p_r] = [q_1 q_2 \ldots q_r] = [1\ 2 \ldots r], \quad [\tau_1 \tau_2 \ldots \tau_{\rho-r}] = [(r+1), (r+2), \ldots \rho].$$

NOTE 3. *The simplest particular solution in Case II.*

This is obtained by assigning zero values to all the arbitrary elements and is given by

$$a [x]_\rho^n = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{r1} & \gamma_{r2} & \cdots & \gamma_{rn} \\ 0 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$

This particular solution can be determined by solving the equation

$$[a]_r^r [x]_r^n = [c]_r^n,$$

and it has the same rank as $[c]_r^n$.

NOTE 4. *Alternative method of solution in Case II.*

The general solution can also be obtained in this case directly from the reduced equation (17) by prefixing the matrix $\overline{\underline{A}}_r^r$ on both sides.

Writing $$\overline{\underline{A}}_r^r\,[a]_r^\rho = [a]_r^\rho \qquad\qquad \text{(21)},$$

so that $$[a]_r^r = \overline{\underline{A}}_r^r\,[a]_r^r = a\,[1]_r^r,$$

the general solution is given by

$$[a]_r^\rho\,[x]_\rho^n = [\gamma]_r^n,$$

where γ_{ij}, a_{ij} have the same meanings as in (B).

Since $[a]_r^r = a\,[1]_r^r$, the last equation is the same as

$$\begin{bmatrix} a & 0 \dots 0 & a_{1,\,r+1} & a_{1,\,r+2} \dots a_{1\rho} \\ 0 & a \dots 0 & a_{2,\,r+1} & a_{2,\,r+2} \dots a_{2\rho} \\ \dots & \dots & \dots & \dots \\ 0 & 0 \dots a & a_{r,\,r+1} & a_{r,\,r+2} \dots a_{r\rho} \end{bmatrix} [x]_\rho^n = [\gamma]_r^n \qquad\qquad \text{(B}'\text{)}.$$

By the properties of passive rows formula (B′) is equivalent to the second equation in (A).

If we equate each of the rn elements of the product matrix on the left in (B′) to the corresponding element of the matrix on the right, we obtain rn scalar equations each of which determines one of the unknown elements occurring in $[x]_r^n$.

Ex. iii. We will solve the equation

$$\begin{bmatrix} 1 & 3 & 4 & 3 & 2 \\ 1 & 1 & 2 & 5 & 2 \\ 2 & 3 & 0 & 1 & 3 \\ 1 & 0 & 1 & 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \end{bmatrix} = \begin{bmatrix} 26 & 16 & 24 \\ 28 & 18 & 12 \\ 22 & 21 & 9 \\ 29 & 19 & 6 \end{bmatrix}.$$

It is easily seen that the augmented and unaugmented matrices have common rank 3. In fact if the 1st, 2nd, 3rd and 4th horizontal rows of the augmented matrix are denoted by a_1, a_2, a_3, a_4 we have the relation $a_1 = 3a_2 - 2a_4$, showing that not more than three of the horizontal rows are unconnected.

First solution. Using the non-vanishing minor determinant $\begin{vmatrix} 1 & 3 & 4 \\ 1 & 1 & 2 \\ 2 & 3 & 0 \end{vmatrix}$ of the unaugmented matrix, we can replace the given equation by the irreducible equation

$$\begin{bmatrix} 1 & 3 & 4 & 3 & 2 \\ 1 & 1 & 2 & 5 & 2 \\ 2 & 3 & 0 & 1 & 3 \end{bmatrix} [x\;y\;z]_{12345} = \begin{bmatrix} 26 & 16 & 24 \\ 28 & 18 & 12 \\ 22 & 21 & 9 \end{bmatrix},$$

or $$\begin{bmatrix} 1 & 3 & 4 \\ 1 & 1 & 2 \\ 2 & 3 & 0 \end{bmatrix} [x\;y\;z]_{123} = \begin{bmatrix} 26 & 16 & 24 \\ 28 & 18 & 12 \\ 22 & 21 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 5 & 2 \\ 1 & 3 \end{bmatrix} [x\;y\;z]_{45}.$$

Prefixing $\begin{bmatrix} -6, & 12, & 2 \\ 4, & -8, & 2 \\ 1, & 3, & 2 \end{bmatrix}$, the conjugate reciprocal of $\begin{bmatrix} 1 & 3 & 4 \\ 1 & 1 & 2 \\ 2 & 3 & 0 \end{bmatrix}$, we obtain the general solution in the form

$$10\,[x\ y\ z]_{123} = \begin{bmatrix} 224, & 162, & 18 \\ -76, & -38, & 18 \\ 66, & 28, & 42 \end{bmatrix} - \begin{bmatrix} 44, & 18 \\ -26, & -2 \\ 16, & 2 \end{bmatrix} [x\ y\ z]_{45},$$

or
$$5\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 112, & 81, & 9 \\ -38, & -19, & 9 \\ 33, & 14, & 21 \end{bmatrix} - \begin{bmatrix} 22, & 9 \\ -13, & -1 \\ 8, & 1 \end{bmatrix} \begin{bmatrix} x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \end{bmatrix},$$

or
$$\begin{bmatrix} 5x_1-112, & 5y_1-81, & 5z_1-9 \\ 5x_2+38, & 5y_2+19, & 5z_2-9 \\ 5x_3-33, & 5y_3-14, & 5z_3-21 \end{bmatrix} = \begin{bmatrix} -22x_4-9x_5, & -22y_4-9y_5, & -22z_4-9z_5 \\ 13x_4+x_5, & 13y_4+y_5, & 13z_4+z_5 \\ -8x_4-x_5, & -8y_4-y_5, & -8z_4-z_5 \end{bmatrix},$$

where $x_4, y_4, z_4, x_5, y_5, z_5$ are arbitrary.

Second solution. If we use the non-vanishing determinant $\begin{vmatrix} 1 & 5 & 2 \\ 3 & 1 & 3 \\ 0 & 6 & 2 \end{vmatrix}$, we can replace the given equation by the irreducible equation

$$\begin{bmatrix} 1 & 1 & 2 & 5 & 2 \\ 2 & 3 & 0 & 1 & 3 \\ 1 & 0 & 1 & 6 & 2 \end{bmatrix} [x\ y\ z]_{12345} = \begin{bmatrix} 28 & 18 & 12 \\ 22 & 21 & 9 \\ 29 & 19 & 6 \end{bmatrix},$$

or
$$\begin{bmatrix} 1 & 5 & 2 \\ 3 & 1 & 3 \\ 0 & 6 & 2 \end{bmatrix} [x\ y\ z]_{245} = \begin{bmatrix} 28 & 18 & 12 \\ 22 & 21 & 9 \\ 29 & 19 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} [x\ y\ z]_{13}.$$

Prefixing $\begin{bmatrix} -16, & 2, & 13 \\ -6, & 2, & 3 \\ 18, & -6, & -14 \end{bmatrix}$, the conjugate reciprocal of $\begin{bmatrix} 1 & 5 & 2 \\ 3 & 1 & 3 \\ 0 & 6 & 2 \end{bmatrix}$,

we obtain the general solution in the form

$$10\,[x\ y\ z]_{245} = \begin{bmatrix} 27, & -1, & 96 \\ 37, & 9, & 36 \\ 34, & 68, & -78 \end{bmatrix} + \begin{bmatrix} 1, & -19 \\ 1, & -9 \\ -8, & 22 \end{bmatrix} [x\ y\ z]_{13},$$

where $[xyz]_{13}$ is arbitrary.

Ex. iv. We will solve the irreducible equation

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{bmatrix},$$

in which $(ab)_{12} = \Delta \neq 0$.

The equation can be written

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{bmatrix} - \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}.$$

Prefixing $\begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}$, the conjugate reciprocal of $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, on both sides, we obtain

$$\Delta \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} (pb), & (qb), & (rb) \\ (ap), & (aq), & (ar) \end{bmatrix} - \begin{bmatrix} (cb), & (db) \\ (ap), & (aq) \end{bmatrix} \begin{bmatrix} x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix},$$

where $(pb), (qb), \ldots$ stand for $(pb)_{12}, (qb)_{12}, \ldots$.

Ex. v. The equation $$[a]_m^n \, [x]_n^m = [1]_m^m$$

admits of finite solution when and only when $[a,\ 1]_m^{n,m}$ has the same rank as $[a]_m^n$, i.e. when and only when $[a]_m^n$ has rank m.

If $n>m$ and $[a]_m^n$ is undegenerate, the general solution contains $(n-m)n$ arbitrary elements.

If $n=m$ and $[a]_m^n$ is undegenerate, the equation has a unique solution which is finite.

In other cases the equation has no finite solution.

Thus the matrix $[a]_m^n$ has an inverse post-factor when and only when its rank is m. This condition being satisfied, it has an infinite number of inverse post-factors when $n>m$, and a single unique inverse post-factor when $n=m$.

Ex. vi. The general solution of the equation of Ex. iii can be obtained as follows:

First solution. Reducing the equation to

$$\begin{bmatrix} 1 & 3 & 4 & 3 & 2 \\ 1 & 1 & 2 & 5 & 2 \\ 2 & 3 & 0 & 1 & 3 \end{bmatrix} [x\ y\ z]_{12345} = \begin{bmatrix} 26 & 16 & 24 \\ 28 & 18 & 12 \\ 22 & 21 & 9 \end{bmatrix},$$

and prefixing the matrix $\begin{bmatrix} -6, & 12, & 2 \\ 4, & -8, & 2 \\ 1, & 3, & -2 \end{bmatrix}$, we obtain

$$\begin{bmatrix} 10, & 0, & 0, & 44, & 18 \\ 0, & 10, & 0, & -26, & -2 \\ 0, & 0, & 10, & 16, & 2 \end{bmatrix} [x\ y\ z]_{12345} = \begin{bmatrix} 224, & 162, & 18 \\ -76, & -38, & 18 \\ 66, & 28, & 42 \end{bmatrix},$$

or $$\begin{bmatrix} 5, & 0, & 0, & 22, & 9 \\ 0, & 5, & 0, & -13, & -1 \\ 0, & 0, & 5, & 8, & 1 \end{bmatrix} [x\ y\ z]_{12345} = \begin{bmatrix} 112, & 81, & 9 \\ -38, & -19, & 9 \\ 33, & 14, & 21 \end{bmatrix}.$$

This gives $5x_1 + 22x_4 + 9x_5 = 112, \quad 5y_1 + 22y_4 + 9y_5 = 81,$ etc.,

or $5x_1 = 112 - 22x_4 - 9x_5, \quad 5y_1 = 81 - 22y_4 - 9y_5,$ etc.

as in Ex. iii.

Second solution. Reducing the equation to

$$\begin{bmatrix} 1 & 1 & 2 & 5 & 2 \\ 2 & 3 & 0 & 1 & 3 \\ 1 & 0 & 1 & 6 & 2 \end{bmatrix} [x\ y\ z]_{12345} = \begin{bmatrix} 28 & 18 & 12 \\ 22 & 21 & 9 \\ 29 & 19 & 6 \end{bmatrix},$$

and prefixing the matrix $\begin{bmatrix} -16, & 2, & 13 \\ -6, & 2, & 3 \\ 18, & -6, & -14 \end{bmatrix}$, we obtain

$$\begin{bmatrix} 1, & -10, & -19, & 0, & 0 \\ 1, & 0, & -9, & -10, & 0 \\ -8, & 0, & 22, & 0, & -10 \end{bmatrix} [x\ y\ z]_{12345} = \begin{bmatrix} -27, & 1, & -96 \\ -37, & -9, & -36 \\ -34, & -68, & 78 \end{bmatrix}.$$

This is equivalent to the second solution in Ex. iii.

Ex. vii. To solve the equation $[abcd]_{12}[xyz]_{1234} = [pqr]_{12}$ of Ex. iv, we prefix $\begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}$ on both sides, and so obtain

$$\begin{bmatrix} \Delta & 0 & (cb) & (db) \\ 0 & \Delta & (ac) & (ad) \end{bmatrix} [x\ y\ z]_{1234} = \begin{bmatrix} (pb) & (qb) & (rb) \\ (ap) & (aq) & (ar) \end{bmatrix},$$

or

$$\Delta\,[x\ y\ z]_{12} = \begin{bmatrix} (pb) & (qb) & (rb) \\ (ap) & (aq) & (ar) \end{bmatrix} - \begin{bmatrix} (cb) & (db) \\ (ac) & (ad) \end{bmatrix} [x\ y\ z]_{34}.$$

5. *General solution of an irreducible equation of the form $AX = C$ which admits of finite solution.*

This is the only case which it is necessary to consider, since an equation which is not irreducible can always be replaced by an irreducible equation.

If

$$[a]_r^\rho\,[x]_\rho^n = [c]_r^n \qquad \text{.........................(22)}$$

is an irreducible equation in which $(a_{1q})_r^r \neq 0$, then to obtain the general solution we use the properties of passive rows to write (22) in the form

$$[a_{1q}]_r^r\,[x_{q1}]_r^n = [\xi]_r^n \qquad \text{.........................(23),}$$

and then solve (23) for $[x_{q1}]_r^n$ as in § 78 by prefixing the conjugate reciprocal of $[a_{1q}]_r^r$ on both sides.

As an alternative we may obtain the solution for $[x_{q1}]_r^n$ directly from equation (22) by prefixing the conjugate reciprocal of $[a_{1q}]_r^r$ on both sides.

The matrix $[x_{q1}]_r^n$ for which we solve is derived from $[x]_\rho^n$ by retaining only those passive rows which correspond to the passive rows of $[a]_r^\rho$ occurring in $[a_{1q}]_r^r$.

6. *Solution of the special equation* $AX = 0$.

If the matrix C vanishes the given equation (1) becomes

$$[a]_m^\rho \, [x]_\rho^n = 0 \quad \text{..............................(24)}.$$

In this case the augmented matrix A' is the same as the unaugmented matrix A. Hence the equation always admits of finite solution. In fact $X = 0$ or $[x]_\rho^n = 0$ is always a solution.

First let $[a]_m^\rho$ have rank r and let $(a)_r^r = \alpha \neq 0$.

Then the equation can be replaced by the irreducible equation

$$[a]_r^\rho \, [x]_\rho^n = 0,$$

or

$$[a]_r^r \, [x]_r^n = -\begin{bmatrix} a_{1,r+1} & a_{1,r+2} & \cdots & a_{1\rho} \\ a_{2,r+1} & a_{2,r+2} & \cdots & a_{2\rho} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r,r+1} & a_{r,r+2} & \cdots & a_{r\rho} \end{bmatrix} \begin{bmatrix} x_{r+1,1} & x_{r+1,2} & \cdots & x_{r+1,n} \\ x_{r+2,1} & x_{r+2,2} & \cdots & x_{r+2,n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{\rho 1} & x_{\rho 2} & \cdots & x_{\rho n} \end{bmatrix}.$$

Prefixing $\overline{\underline{A}}{}_r^r$ on both sides, we obtain as the general solution

$$\alpha [x]_r^n = -\begin{bmatrix} \alpha_{1,r+1} & \alpha_{1,r+2} & \cdots & \alpha_{1\rho} \\ \alpha_{2,r+1} & \alpha_{2,r+2} & \cdots & \alpha_{2\rho} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{r,r+1} & \alpha_{r,r+2} & \cdots & \alpha_{r\rho} \end{bmatrix} \begin{bmatrix} x_{r+1,1} & x_{r+1,2} & \cdots & x_{r+1,n} \\ x_{r+2,1} & x_{r+2,2} & \cdots & x_{r+2,n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{\rho 1} & x_{\rho 2} & \cdots & x_{\rho n} \end{bmatrix} \quad \text{...(C)}.$$

This determines the rn unknown elements occurring in $[x]_r^n$ as *homogeneous* linear functions of the remaining $(\rho - r)n$ unknown elements to which arbitrary values may be assigned. The quantities $\alpha_{v,r+v}$ have the same values as in formula (B).

Since

$$\alpha \begin{bmatrix} x_{r+1,1} & x_{r+1,2} & \cdots & x_{r+1,n} \\ x_{r+2,1} & x_{r+2,2} & \cdots & x_{r+2,n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{\rho 1} & x_{\rho 2} & \cdots & x_{\rho n} \end{bmatrix} = \begin{bmatrix} \alpha & 0 & \cdots & 0 \\ 0 & \alpha & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha \end{bmatrix} \begin{bmatrix} x_{r+1,1} & x_{r+1,2} & \cdots & x_{r+1,n} \\ x_{r+2,1} & x_{r+2,2} & \cdots & x_{r+2,n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{\rho 1} & x_{\rho 2} & \cdots & x_{\rho n} \end{bmatrix},$$

the general solution can be expressed in the form

$$\alpha [x]_\rho^n = \begin{bmatrix} \alpha_{1,r+1} & \alpha_{1,r+2} & \cdots & \alpha_{1\rho} \\ \alpha_{2,r+1} & \alpha_{2,r+2} & \cdots & \alpha_{2\rho} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{r,r+1} & \alpha_{r,r+2} & \cdots & \alpha_{r\rho} \\ -\alpha & 0 & \cdots & 0 \\ 0 & -\alpha & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\alpha \end{bmatrix} \begin{bmatrix} x_{r+1,1} & x_{r+1,2} & \cdots & x_{r+1,n} \\ x_{r+2,1} & x_{r+2,2} & \cdots & x_{r+2,n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{\rho 1} & x_{\rho 2} & \cdots & x_{\rho n} \end{bmatrix} \quad \text{......(D)}.$$

If $[x]_\rho^n$ is any one of the solutions of (24), its first r horizontal rows are by (C) connected with the last $\rho - r$ horizontal rows.

Hence by Theorem VI of § 71 the rank of $[x]_\rho^n$ is equal to the rank of

$$\begin{bmatrix} x_{r+1,1} & x_{r+1,2} & \dots & x_{r+1,n} \\ x_{r+2,1} & x_{r+2,2} & \dots & x_{r+2,n} \\ \dots & \dots & \dots & \dots \\ x_{\rho 1} & x_{\rho 2} & \dots & x_{\rho n} \end{bmatrix}.$$

But this last matrix, being entirely arbitrary, can have any rank from 0 to the smaller of the two numbers $\rho - r$ and n. Since there is no loss of generality in assuming that $(a)_r^r \neq 0$, we deduce the following theorem:

Theorem VI. *The equation* $[a]_m^\rho [x]_\rho^n = 0$, *where* $[a]_m^\rho$ *has rank* r, *has finite solutions of all ranks from* 0 *to the smaller of the two numbers* $\rho - r$ *and* n.

If $r = \rho$, then $X = 0$ is the only solution of equation (24). If $r < \rho$, then there are other solutions. We deduce the following theorem:

Theorem VII. *The equation* $AX = 0$ *leads to* $X = 0$ *as a necessary consequence when and only when the rank of* A *is equal to the passivity of the product* AX. *It has non-zero solutions when and only when the rank of* A *is less than the passivity of the product* AX.

Next let $[a]_m^\rho$ have rank r and let $(a_{pq})_r^r = \alpha \neq 0$.

Then the equation can be replaced by the irreducible equation

$$[a_{p1}]_r^\rho [x]_\rho^n = 0,$$

or

$$[a_{pq}]_r^r [x_{q1}]_r^n = -[a_{p\tau}]_r^{\rho - r} [x_{\tau 1}]_{\rho - r}^n,$$

where $[\tau_1 \tau_2 \dots \tau_{\rho - r}]$ is the complement of $[q_1 q_2 \dots q_r]$ in $[1\ 2 \dots \rho]$.

Prefixing $\overline{\underline{A_{pq}}}_r^r$ on both sides, we obtain as the general solution of (24)

$$\alpha [x_{q1}]_r^n = -[\alpha]_r^{\rho - r} [x_{\tau 1}]_{\rho - r}^n \quad \text{.......................(E)},$$

where the quantities α_{uv} have the same values as in formula (A).

This determines each of the rn unknown elements occurring in $[x_{q1}]_r^n$ as *homogeneous* linear functions of the remaining $(\rho - r)n$ unknown elements, to which arbitrary values may be assigned.

Ex. viii. The general case can be reduced to this special case. For the equation

$$[a]_m^\rho [x]_\rho^n + [c]_m^n = 0$$

can be written in either of the forms

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1\rho} & c_{11} & c_{12} & \dots & c_{1n} \\ a_{21} & a_{22} & \dots & a_{2\rho} & c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{m\rho} & c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{\rho 1} & x_{\rho 2} & \dots & x_{\rho n} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} = 0,$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1\rho} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2\rho} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{m\rho} & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{\rho 1} & x_{\rho 2} & \dots & x_{\rho n} \\ c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix} = 0.$$

Ex. ix. *If there are r unconnected connections between the horizontal rows of the matrix $[a]_m^n$, the matrix cannot have more than $m-r$ unconnected horizontal rows.*

This is another way of expressing the result given by Theorem VI.

For if $[h]_r^m [a]_m^n = 0$, where the horizontal rows of $[h]_r^m$ are unconnected, or where $[h]_r^m$ has rank r, the rank of $[a]_m^n$ cannot exceed $m-r$.

7. *Infinite solutions of the equation $AX = C$, where $C \neq 0$.*

If $[x]_\rho^n$ is a matrix some of whose elements are infinite, it can be expressed in the form $[x]_\rho^n = k[X]_\rho^n$, where $[X]_\rho^n$ is a matrix whose elements are all finite but not all zero and k is some infinite scalar quantity. The infinite matrix $k[X]_\rho^n$ will be a solution of the equation $[a]_m^\rho [x]_\rho^n = [c]_m^n$ if and only if $[a]_m^\rho [X]_\rho^n = \frac{1}{k}[c]_m^n = 0$.

Conversely if $[X]_\rho^n$ is a finite solution of equation $[a]_m^\rho [x]_\rho^n = 0$ whose elements are not all zero, then $k[X]_\rho^n$, where k is a scalar quantity, will be a solution of the equation $[a]_m^\rho [x]_\rho^n = [c]_m^n$ if and only if $[a]_m^\rho [X]_\rho^n = \frac{1}{k}[c]_m^n = 0$, i.e. if and only if k is infinite.

We have thus proved the following theorem:

Theorem VIII. *The infinite solutions of the equation* $[a]_m^\rho [x]_\rho^n = [c]_m^n$ *are given by* $[x]_\rho^n = k[X]_\rho^n$ *where* k *is an infinite scalar quantity and* $[X]_\rho^n$ *is a finite non-zero solution of the equation* $[a]_m^\rho [x]_\rho^n = 0$.

We know from Theorem V that the equation $[a]_m^\rho [x]_\rho^n = 0$ has non-zero solutions when and only when the rank of $[a]_m^\rho$ is less than ρ. Accordingly we have the following further theorem:

Theorem IX. *The equation* $[a]_m^\rho [x]_\rho^n = [c]_m^n$ *has infinite solutions when and only when the rank of the unaugmented matrix* $[a]_m^\rho$ *is less than the passivity* ρ.

We now see how to determine all possible solutions, both finite and infinite, of the equation $[a]_m^\rho [x]_\rho^n = [c]_m^n$ or $AX = C$.

If we denote the ranks of A and A' by r and r' respectively, we have the following results:

If $r = r' < \rho$, the equation has both finite and infinite solutions.

If $r = r' = \rho$, the equation has a unique finite solution.

If $r < \rho$ and $r' > r$, the equation has only infinite solutions.

If $r = \rho$ and $r' > r$, the equation has no solution.

§ 82. Solution of any equation of the form $XB = C$.

1. *Augmented and unaugmented matrices of the equation.*

We shall assume that the equation to be solved is

$$[x]_m^\sigma [b]_\sigma^n = [c]_m^n \quad \text{..............................(1)},$$

so that $X = [x]_m^\sigma, \quad B = [b]_\sigma^n, \quad C = [c]_m^n.$

One special case of this equation, that in which B is an undegenerate square matrix, has already been considered in § 79.

If we equate the conjugates of both sides of (1), we obtain the equivalent equation

$$\overline{\underline{b}}{}_n^\sigma \, \overline{\underline{x}}{}_\sigma^m = \overline{\underline{c}}{}_n^m \quad \text{..............................(2)}.$$

Equation (2) has the form considered in § 81, and the solutions of (1) are the conjugates of the solutions of (2). Hence the properties and solutions of (1) can be deduced from the properties and solutions of (2) which can be obtained from the results of § 81.

We can however treat equation (1) independently by methods similar to those used in § 81. The vertical rows of B and C will now play the same parts as the horizontal rows of A and C in § 81, and we shall have to postfix matrices instead of prefixing them.

The matrices

$$B = [b]_\sigma^n = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{\sigma 1} & b_{\sigma 2} & \dots & b_{\sigma n} \end{bmatrix}, \quad B' = \begin{bmatrix} b \\ c \end{bmatrix}_{\sigma, m}^n = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{\sigma 1} & b_{\sigma 2} & \dots & b_{\sigma n} \\ c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

will be called respectively the *unaugmented matrix* and the *augmented matrix* of the equation (1).

2. *Reduction of the equation to an irreducible equation of the same kind.*

An equation of the form $XB = C$ will be said to be *irreducible* when the vertical rows of the augmented matrix B' are unconnected. As regards the reduction of the equation we have the following theorems:

Theorem I. *If the ith vertical row of the augmented matrix B' of the equation $[x]_m^\sigma [b]_\sigma^n = [c]_m^n$ is connected with certain other vertical rows of B', then the given equation can be replaced by the equation obtained from it by striking out the ith vertical rows of $[b]_\sigma^n$ and $[c]_m^n$.*

Theorem II. *If the augmented matrix B' of the equation $[x]_m^\sigma [b]_\sigma^n = [c]_m^n$ has rank s, then by striking out $n-s$ corresponding vertical rows of B and C we can reduce the equation to an irreducible equation of the form $[x]_m^\sigma [b]_\sigma^s = [c]_m^s$.*

Theorem III. *When an equation of the form $XB = C$ is reduced by repeated applications of Theorem I, the augmented and unaugmented matrices of the reduced equation have respectively the same ranks as the augmented and unaugmented matrices of the original equation.*

The proofs of these theorems are similar to the proofs of the corresponding theorems in § 81.

3. *Necessary and sufficient condition for the existence of a finite solution of the equation.*

This condition is contained in the following theorem:

Theorem IV. *The equation $XB = C$ has a finite solution or finite solutions when and only when the augmented matrix B' has the same rank as the unaugmented matrix B.*

The proof is similar to the proof of Theorem IV in § 81.

4. *General solution of the equation $XB = C$ when it admits of finite solution.*

Let the augmented matrix B' and the unaugmented matrix B of the given equation

$$[x]_m^\sigma [b]_\sigma^n = [c]_m^n \quad \dots\dots\dots\dots\dots\dots\dots\dots(1)$$

have common rank s, so that the equation (1) admits of finite solution.

Then B and B' have a common non-vanishing derived determinant of order s. Let $(b_{pq})_s^s$ be such a determinant. We will consider first the general case in which there is no restriction on $(b_{pq})_s^s$, and then the special case in which $(b)_s^s$ the leading derived determinant of order s of $[b]_\sigma^n$ does not vanish and $(b_{pq})_s^s$ is taken to be $(b)_s^s$.

Case I. $(b_{pq})_s^s = \beta \neq 0$.

By sub-article 2 the solutions of the given equation (1) are identical with the solutions of the irreducible equation

$$[x]_m^\sigma [b_{1q}]_\sigma^s = [c_{1q}]_m^s \dots\dots\dots\dots\dots\dots\dots\dots(3),$$

whose augmented and unaugmented matrices both have rank s.

By the properties of passive rows (see § 52.9), we can write (3) in the form

$$[x_{1p}]_m^s [b_{pq}]_s^s + [x_{1\tau}]_m^{\sigma-s} [b_{\tau q}]_{\sigma-s}^s = [c_{1q}]_m^s,$$

or
$$[x_{1p}]_m^s [b_{pq}]_s^s = [c_{1q}]_m^s - [x_{1\tau}]_m^{\sigma-s} [b_{\tau q}]_{\sigma-s}^s = [\xi]_m^s \quad \dots\dots\dots(4),$$

where $[\tau_1 \tau_2 \dots \tau_{\sigma-s}]$ is complementary to $[p_1 p_2 \dots p_s]$ in $[1\, 2 \dots \sigma]$.

Equation (4) can be solved for the matrix $[x_{1p}]_m^s$ by § 79.

Let $(b_{pq})_s^s = \beta$, and let $[B_{pq}]_s^s$ be the reciprocal matrix of $[b_{pq}]_s^s$. Then postfixing $\left\lfloor \overline{B_{pq}} \right\rfloor_s^s$, the conjugate reciprocal of $[b_{pq}]_s^s$, on both sides of (4), we obtain by § 79 the following result:

First form of the general solution.

If B and B' have common rank s, and if $(b_{pq})_s^s = \beta \neq 0$, then equation (1) *is satisfied when and only when*

$$\beta\,[x_{1p}]_m^s = [c_{1q}]_m^s \,\overline{\underline{B_{pq}}}_s^s - [x_{1\tau}]_m^{\sigma-s}\,[b_{\tau q}]_{\sigma-s}^s\,\overline{\underline{B_{pq}}}_s^s,$$

or
$$\beta\,[x_{1p}]_m^s = [\gamma]_m^s - [x_{1\tau}]_m^{\sigma-s}\,[\beta_{\tau 1}]_{\sigma-s}^s,$$

where γ_{ij} is the determinant formed from $(b_{pq})_s^s$ when its jth horizontal row is replaced by the ith horizontal row of $[c_{1q}]_m^s$, and β_{ij} is the determinant formed from $(b_{pq})_s^s$ when its jth horizontal row is replaced by the ith horizontal row of $[b_{1q}]_\sigma^s$, so that in particular $[\beta_{p1}]_s^s = \beta\,[1]_s^s$ ... .(A).

Formula (A) gives the general solution of equation (1). In this solution arbitrary values can be assigned to the $m(\sigma - s)$ unknown elements which do not occur in $[x_{1p}]_m^s$, and the remaining ms unknown elements, viz. those which do occur in $[x_{1p}]_m^s$, are expressed as unique linear functions of these.

If x_{ip_j} is any one of the elements of $[x_{1p}]_m^s$, we have

$$\beta x_{ip_j} = \gamma_{ij} - x_{i\tau_1}\beta_{\tau_1 j} - x_{i\tau_2}\beta_{\tau_2 j} - \dots - x_{i\tau_{\sigma-s}}\beta_{\tau_{\sigma-s} j} \quad \text{........(5).}$$

We have now proved the following theorem:

Theorem V. *If the unaugmented matrix B and the augmented matrix B' of the equation $[x]_m^\sigma\,[b]_\sigma^n = [c]_m^n$ have common rank s, then the equation admits of finite solution, and the general solution expresses ms of the $m\sigma$ unknown elements as linear (in general non-homogeneous) functions of the remaining $m(\sigma - s)$ unknown elements to which arbitrary values may be assigned.*

Since the general solution contains $m(\sigma - s)$ arbitrary elements, which may be real or imaginary, the total number of solutions is $\infty^{2m(\sigma-s)}$. In the special case in which $s = \sigma$, no arbitrary elements occur in the general solution, and there is a single unique finite solution. In fact this special case is equivalent to that considered in § 79.

NOTE 1. *The simplest particular solution in Case I.*

This is obtained by assigning zero values to all the (arbitrary) elements of $[x_{1\tau}]_m^{\sigma-s}$ and is given by

$$\beta\,[x_{1p}]_m^s = [c_{1q}]_m^s\,\overline{\underline{B_{pq}}}_s^s = [\gamma]_m^s, \qquad [x_{1\tau}]_m^{\sigma-s} = 0.$$

This particular solution can be obtained by solving the equation

$$[x_{1p}]_m^s\,[b_{pq}]_s^s = [c_{1q}]_m^s,$$

and it has the same rank as $[c_{1q}]_m^s$.

Note 2. *Alternative method of solution in Case I.*

The general solution can also be obtained directly from the reduced equation (3) by postfixing the conjugate reciprocal matrix $\overline{\underline{B_{pq}}}{}_s^s$ on both sides before transferring terms to the right-hand side.

Writing $$[b_{1q}]_\sigma^s \, \overline{\underline{B_{pq}}}{}_s^s = [\beta]_\sigma^s,$$

so that $$[\beta_{p1}]_s^s = [b_{pq}]_s^s \, \overline{\underline{B_{pq}}}{}_s^s = \beta\,[1]_s^s \quad \text{..............................(6)},$$

the general solution is given by

$$[x]_m^\sigma \, [\beta]_\sigma^s = [\gamma]_m^s \quad \text{.......................................(A')},$$

where γ_{ij} and β_{ij} have the same meanings as in (A).

Using (6) formula (A′) can by the properties of passive rows be transformed into the second equation in (A). Moreover when we equate each of the ms elements of the product matrix on the left in (A′) to the corresponding element of the matrix on the right, we obtain ms scalar equations each of which determines one of the unknown elements occurring in $[x_{1p}]_m^s$. In fact these scalar equations are the same as the ms scalar equations given by the second matrix equation in (A).

Case II. $(b)_s^s = \beta \neq 0$.

In this case (to which every other case can be reduced) we can replace the given equation (1) by the irreducible equation

$$[x]_m^\sigma \, [b]_\sigma^s = [c]_m^s \quad \text{..............................(7)},$$

and this by the properties of passive rows can be written in the form

$$[x]_m^s \, [b]_s^s = [\xi]_m^s \quad \text{..............................(8)},$$

where now

$$[\xi]_m^s = [c]_m^s - \begin{bmatrix} x_{1,s+1} & x_{1,s+2} & \cdots & x_{1\sigma} \\ x_{2,s+1} & x_{2,s+2} & \cdots & x_{2\sigma} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m,s+1} & x_{m,s+2} & \cdots & x_{m\sigma} \end{bmatrix} \begin{bmatrix} b_{s+1,1} & b_{s+1,2} & \cdots & b_{s+1,s} \\ b_{s+2,1} & b_{s+2,2} & \cdots & b_{s+2,s} \\ \cdots & \cdots & \cdots & \cdots \\ b_{\sigma 1} & b_{\sigma 2} & \cdots & b_{\sigma s} \end{bmatrix} \quad \text{..................(9)}.$$

Writing $(b)_s^s = \beta$, and denoting the reciprocal of $[b]_s^s$ by $[B]_s^s$, we can solve equation (8) for $[x]_m^s$ by postfixing $\overline{\underline{B}}{}_s^s$ on both sides as in § 79, and we then obtain the following result:

Second form of the general solution.

If B and B' have common rank s, and if $(b)_s^s = \beta \neq 0$, then equation (1) is satisfied when and only when

$$\beta\,[x]_m^s = [\xi]_m^s \, \overline{\underline{B}}{}_s^s.$$

$$\text{or} \qquad \beta\,[x]_m^s = [\gamma]_m^s - \begin{bmatrix} x_{1,s+1} & x_{1,s+2} & \dots & x_{1\sigma} \\ x_{2,s+1} & x_{2,s+2} & \dots & x_{2\sigma} \\ \dots & \dots & \dots & \dots \\ x_{m,s+1} & x_{m,s+2} & \dots & x_{m\sigma} \end{bmatrix} \begin{bmatrix} \beta_{s+1,1} & \beta_{s+1,2} & \dots & \beta_{s+1,s} \\ \beta_{s+2,1} & \beta_{s+2,2} & \dots & \beta_{s+2,s} \\ \dots & \dots & \dots & \dots \\ \beta_{\sigma 1} & \beta_{\sigma 2} & \dots & \beta_{\sigma s} \end{bmatrix},$$

where γ_{ij} *is the determinant formed from* $(b)_s^s$ *when its jth horizontal row is replaced by the ith horizontal row of* $[c]_m^s$, *and* β_{ij} *is the determinant formed from* $(b)_s^s$ *when its jth horizontal row is replaced by the ith horizontal row of* $[b]_\sigma^s$, *so that in particular* $(\beta)_s^s = \beta\,[1]_s^s$ ....(B).

Formula (B) gives the general solution of equation (1) in this special case. The solution expresses the ms elements of $[x]_m^s$ as linear functions of the remaining $m(\sigma - s)$ unknown elements to which arbitrary values may be assigned.

If x_{ij} is any one of the elements of $[x]_m^s$, we have

$$\beta x_{ij} = \gamma_{ij} - x_{i,s+1}\,\beta_{s+1,j} - x_{i,s+2}\,\beta_{s+2,j} - \dots - x_{i\sigma}\,\beta_{\sigma j} \quad \text{.........(10).}$$

Formula (B) can of course be deduced from formula (A) by putting

$$[p_1\,p_2 \dots p_r] = [q_1\,q_2 \dots q_r] = [1\;2 \dots r], \quad [\tau_1\,\tau_2 \dots \tau_{\sigma-s}] = [(s+1),\ (s+2),\ \dots \sigma].$$

NOTE 3. *The simplest particular solution in Case II.*

This is obtained by assigning zero values to all the arbitrary elements, and is given by

$$\beta\,[x]_m^\sigma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \dots & \gamma_{1s} & 0 & \dots & 0 \\ \gamma_{21} & \gamma_{22} & \dots & \gamma_{2s} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{m1} & \gamma_{m2} & \dots & \gamma_{ms} & 0 & \dots & 0 \end{bmatrix}.$$

This particular solution can be determined by solving the equation

$$[x]_m^s\,[b]_s^s = [c]_m^s,$$

and it has the same rank as $[c]_m^s$.

NOTE 4. *Alternative method of solution in Case II.*

The general solution can also be obtained in this case directly from the reduced equation (7) by postfixing the conjugate reciprocal matrix $\overline{\underline{B}}_s^s$ on both sides.

Writing $$[b]_\sigma^s\,\overline{\underline{B}}_s^s = [\beta]_\sigma^s,$$

so that $$[\beta]_s^s = [b]_s^s\,\overline{\underline{B}}_s^s = \beta\,[1]_s^s \quad \text{..................................(11),}$$

the general solution is given by

$$[x]_m^\sigma\,[\beta]_\sigma^s = [\gamma]_m^s \quad \text{...(B′),}$$

where γ_{ij}, β_{ij} have the same meanings as in (B).

Since $[\beta]_s^s = \beta\,[1]_s^s$, formula (B′) is equivalent to the second equation in (B). If we equate each of the ms elements of the product matrix on the left in (B′) to the corresponding element of the matrix on the right, we obtain ms scalar equations each of which determines one of the unknown elements occurring in $[x]_m^s$. In fact these scalar equations are the same as the ms scalar equations given by the second matrix equation in (B).

5. *General solution of an irreducible equation of the form $XB = C$ which admits of finite solution.*

This is the only case which it is necessary to consider, since an equation which is not irreducible can always be replaced by an irreducible equation.

If
$$[x]_m^\sigma\,[b]_\sigma^s = [c]_m^s \quad \text{.............................}(12)$$
is an irreducible equation in which $(b_{p1})_s^s \neq 0$, then to obtain the general solution we use the properties of passive rows to write (12) in the form
$$[x_{1p}]_m^s\,[b_{p1}]_s^s = [\xi]_m^s \quad \text{.............................}(13),$$
and then solve (13) for $[x_{1p}]_m^s$ as in § 79 by postfixing the conjugate reciprocal of $[b_{p1}]_s^s$ on both sides.

As an alternative we may obtain the solution for $[x_{1p}]_m^s$ directly from (12) by prefixing the conjugate reciprocal of $[b_{p1}]_s^s$ on both sides.

The matrix $[x_{1p}]_m^s$ for which we solve is derived from $[x]_m^\sigma$ by retaining only those passive rows which correspond to the passive rows of $[b]_\sigma^s$ occurring in $[b_{p1}]_s^s$.

Ex. i. We will solve the equation

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 3 & 0 \\ 4 & 2 & 0 & 1 \\ 3 & 5 & 1 & 6 \\ 2 & 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 26 & 28 & 22 & 29 \\ 16 & 18 & 21 & 19 \\ 24 & 12 & 9 & 6 \end{bmatrix}.$$

As in Ex. iii of § 81 the augmented and unaugmented matrices have common rank 3.

If we make use of the non-vanishing minor determinant $\begin{vmatrix} 1 & 3 & 0 \\ 5 & 1 & 6 \\ 2 & 3 & 2 \end{vmatrix}$ of the unaugmented matrix, we can replace the given equation by the irreducible equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{12345} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 0 & 1 \\ 5 & 1 & 6 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 28 & 22 & 29 \\ 18 & 21 & 19 \\ 12 & 9 & 6 \end{bmatrix} \quad \text{.........................(14).}$$

By the properties of passive rows this is equivalent to

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{245} \begin{bmatrix} 1 & 3 & 0 \\ 5 & 1 & 6 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 28 & 22 & 29 \\ 18 & 21 & 19 \\ 12 & 9 & 6 \end{bmatrix} - \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{13} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

Postfixing

$$\begin{bmatrix} -16, & -6, & 18 \\ 2, & 2, & -6 \\ 13, & 3, & -14 \end{bmatrix}, \text{ the conjugate reciprocal of } \begin{bmatrix} 1 & 3 & 0 \\ 5 & 1 & 6 \\ 2 & 3 & 2 \end{bmatrix}, \text{ on both sides,}$$

we obtain as the general solution

$$10 \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{245} = \begin{bmatrix} 27, & 37, & 34 \\ -1, & 9, & 68 \\ 96, & 36, & -78 \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{13} \begin{bmatrix} 1, & 1, & -8 \\ -19, & -9, & 22 \end{bmatrix} \quad \text{.........(15),}$$

where the elements of $[xyz]_{13}$ are arbitrary.

Ex. ii. In solving the equation of Ex. i we may postfix the matrix

$$\begin{bmatrix} -16, & -6, & 18 \\ 2, & 2, & -6 \\ 13, & 3, & -14 \end{bmatrix}$$

on both sides of (14). We then obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{12345} \begin{bmatrix} 1, & 1, & -8 \\ -10, & 0, & 0 \\ -19, & -9, & 22 \\ 0, & -10, & 0 \\ 0, & 0, & -10 \end{bmatrix} = \begin{bmatrix} -27, & -37, & -34 \\ 1, & -9, & -68 \\ -96, & -36, & 78 \end{bmatrix},$$

which by the properties of passive rows is equivalent to (15).

Ex. iii. We will solve the irreducible equation

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \\ r_1 & r_2 \end{bmatrix},$$

in which $(ab)_{12} = \Delta \neq 0$.

The equation can be written

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \\ r_1 & r_2 \end{bmatrix} - \begin{bmatrix} x_3 & x_4 \\ y_3 & y_4 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}.$$

Postfixing $\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$, the conjugate reciprocal of $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, on both sides, we obtain

$$\Delta \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} = \begin{bmatrix} (pb) & (ap) \\ (qb) & (aq) \\ (rb) & (ar) \end{bmatrix} - \begin{bmatrix} x_3 & x_4 \\ y_3 & y_4 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} (cb) & (ac) \\ (db) & (ad) \end{bmatrix},$$

where $(pb), (ap), \ldots$ stand for $(pb)_{12}, (ap)_{12}, \ldots$.

Ex. iv. If we postfix $\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$ on both sides of the equation of Ex. iii in its original form, we obtain

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \\ (cb) & (ac) \\ (db) & (ad) \end{bmatrix} = \begin{bmatrix} (pb) & (ap) \\ (qb) & (aq) \\ (rb) & (ar) \end{bmatrix},$$

which is equivalent to the previous result.

Ex. v. The equation $[x]_n^m [a]_m^n = [1]_n^n$

admits of finite solution when and only when $\begin{bmatrix} a \\ 1 \end{bmatrix}_{m,n}^n$ has the same rank as $[a]_m^n$, i.e. when and only when $[a]_m^n$ has rank n.

If $m > n$ and $[a]_m^n$ is undegenerate, the general solution contains $n(m-n)$ arbitrary elements.

If $m = n$ and $[a]_m^n$ is undegenerate, the equation has a unique solution which is finite.

In other cases the equation has no finite solution.

Thus the matrix $[a]_m^n$ has an inverse pre-factor when and only when its rank is n. This condition being satisfied, it has an infinite number of inverse pre-factors when $m > n$, and a single unique inverse pre-factor when $m = n$.

6. *Solution of the special equation $XB = 0$.*

If the matrix C vanishes the given equation (1) becomes

$$[x]_m^\sigma [b]_\sigma^n = 0 \quad \text{..............................(16).}$$

In this case the augmented matrix B' is the same as the unaugmented matrix B. Hence the equation always admits of finite solution.

In fact $X = 0$, or $[x]_m^\sigma = 0$ is always a solution.

The general solution is obtained from (A) or (B) by putting $C = 0$ or $[\gamma]_m^n = 0$.

Thus if $\beta = (b_{pq})_s^s \neq 0$, we have from (A)

$$\beta\,[x_{1p}]_m^s = -[x_{1\tau}]_m^{\sigma-s}\,[\beta]_{\sigma-s}^s \quad\text{.......................(C)},$$

where $[\tau_1\,\tau_2 \ldots \tau_{\sigma-s}]$ is the complement of $[p_1\,p_2 \ldots p_s]$ in $[1\,2 \ldots \sigma]$; and if $\beta = (b)_s^s \neq 0$, we have from (B)

$$\beta\,[x]_m^s = -\begin{bmatrix} x_{1,s+1} & x_{1,s+2} & \ldots & x_{1\sigma} \\ x_{2,s+1} & x_{2,s+2} & \ldots & x_{2\sigma} \\ \ldots & \ldots & \ldots & \ldots \\ x_{m,s+1} & x_{m,s+2} & \ldots & x_{m\sigma} \end{bmatrix} \begin{bmatrix} \beta_{s+1,1} & \beta_{s+1,2} & \ldots & \beta_{s+1,s} \\ \beta_{s+2,1} & \beta_{s+2,2} & \ldots & \beta_{s+2,s} \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{\sigma 1} & \beta_{\sigma 2} & \ldots & \beta_{\sigma s} \end{bmatrix}$$

.....................(D).

In formulae (C) and (D) arbitrary values may be assigned to those elements of $[x]_m^\sigma$ which occur on the right.

We can replace (D) by

$$\beta\,[x]_m^\sigma = -\begin{bmatrix} x_{1,s+1} & x_{1,s+2} & \ldots & x_{1\sigma} \\ x_{2,s+1} & x_{2,s+2} & \ldots & x_{2\sigma} \\ \ldots & \ldots & \ldots & \ldots \\ x_{m,s+1} & x_{m,s+2} & \ldots & x_{m\sigma} \end{bmatrix} \begin{bmatrix} \beta_{s+1,1} & \beta_{s+1,2} & \ldots & \beta_{s+1,s} & -\beta & 0 & \ldots & 0 \\ \beta_{s+2,1} & \beta_{s+2,2} & \ldots & \beta_{s+2,s} & 0 & -\beta & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta_{\sigma 1} & \beta_{\sigma 2} & \ldots & \beta_{\sigma s} & 0 & 0 & \ldots & -\beta \end{bmatrix}$$

..................(E).

The following theorems can be proved in the same way as the corresponding theorems in sub-article 6 of § 81.

Theorem VI. *The equation* $[x]_m^\sigma\,[b]_\sigma^n = 0$, *where* $[b]_\sigma^n$ *has rank s, has finite solutions of all ranks from* 0 *to the smaller of the two numbers m and* $\sigma - s$.

Theorem VII. *The equation* $XB = 0$ *leads to* $X = 0$ *as a necessary consequence when and only when the rank of B is equal to the passivity of the product XB. It has non-zero solutions when and only when the rank of B is less than the passivity of the product XB.*

Ex. vi. The general case can be reduced to this special case. For the equation

$$[x]_m^\sigma\,[b]_\sigma^n + [c]_m^n = 0$$

can be written in the form

$$\begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1\sigma} & 1 & 0 & \ldots & 0 \\ x_{21} & x_{22} & \ldots & x_{2\sigma} & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{m1} & x_{m2} & \ldots & x_{m\sigma} & 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{\sigma 1} & b_{\sigma 2} & \ldots & b_{\sigma n} \\ c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ c_{m1} & c_{m2} & \ldots & c_{mn} \end{bmatrix} = 0.$$

Ex. vii. *If there are s unconnected connections between the vertical rows of the matrix* $[x]_m^n$, *the matrix cannot have more than* $n-s$ *unconnected vertical rows.*

This can be proved in the same way as Ex. ix of § 81, and is merely another way of expressing the result of Theorem VI.

7. *Infinite solutions of the equation* $XB = C$, *when* $C \neq 0$.

Concerning the infinite solutions we have the following two theorems, the proofs of which are similar to the proofs of the corresponding theorems in § 81.

Theorem VIII. *The infinite solutions of the equation* $[x]_m^\sigma [b]_\sigma^n = [c]_m^n$ *are given by* $[x]_m^\sigma = k[X]_m^\sigma$ *where k is an infinite scalar quantity and* $[X]_m^\sigma$ *is a finite non-zero solution of the equation* $[x]_m^\sigma [b]_\sigma^n = 0$.

Theorem IX. *The equation* $[x]_m^\sigma [b]_\sigma^n = [c]_m^n$ *has infinite solutions when and only when the rank of the unaugmented matrix* $[b]_\sigma^n$ *is less than the passivity* σ.

If we denote the ranks of B and B' by s and s' respectively, we have the following results:

If $s = s' < \sigma$, the equation has both finite and infinite solutions.

If $s = s' = \sigma$, the equation has a unique finite solution.

If $s < \sigma$ and $s' > s$, the equation has only infinite solutions.

If $s = \sigma$ and $s' > s$, the equation has no solution.

§ 83. Solution of any equation of the form $AXB = C$.

1. *Augmented and unaugmented matrices of the equation.*

We shall assume that the equation to be solved is

$$[a]_m^\rho [x]_\rho^\sigma [b]_\sigma^n = [c]_m^n \quad \text{..........................(1)},$$

so that $\quad A = [a]_m^\rho, \quad X = [x]_\rho^\sigma, \quad B = [b]_\sigma^n, \quad C = [c]_m^n.$

One special case of this equation, that in which A and B are both undegenerate square matrices, has been already considered in § 80. A second special case, that in which B is a unit matrix, has been considered in § 81. A third special case, that in which A is a unit matrix, has been considered in § 82.

The matrices

$$A = [a]_m^\rho, \quad B = [b]_\sigma^n$$

will be called the *unaugmented matrices* of the equation.

The matrices

$$A' = [a, c]_m^{\rho, n}, \quad B' = \begin{bmatrix} b \\ c \end{bmatrix}_{\sigma, m}^{n},$$

$$\text{or} \quad A' = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1\rho} & c_{11} & c_{12} & \dots & c_{1n} \\ a_{21} & a_{22} & \dots & a_{2\rho} & c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{m\rho} & c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}, \qquad B' = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{\sigma 1} & b_{\sigma 2} & \dots & b_{\sigma n} \\ c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix},$$

will be called the corresponding *augmented matrices* of the equation.

2. *Reduction of the equation to an irreducible equation of the same kind.*

An equation of the form $AXB = C$ will be said to be irreducible when the horizontal rows of the augmented matrix A' are unconnected and also the vertical rows of the augmented matrix B' are unconnected.

We will now prove the following theorems:

Theorem I. *If the ith horizontal row of the augmented matrix A' of the equation $AXB = C$ is connected with certain other horizontal rows of A', then the given equation can be replaced by the equation obtained from it by striking out the ith horizontal rows of A and C.*

Theorem II. *If the jth vertical row of the augmented matrix B' of the equation $AXB = C$ is connected with certain other vertical rows of B', then the given equation can be replaced by the equation obtained from it by striking out the jth vertical rows of B and C.*

Theorem III. *If the augmented matrices A' and B' of the equation $[a]_m^\rho [x]_\rho^\sigma [b]_\sigma^n = [c]_m^n$ or $AXB = C$ have ranks r and s respectively, then by striking out $m - r$ corresponding horizontal rows of A and C and $n - s$ corresponding vertical rows of B and C, the equation can be reduced to an irreducible equation of the form $[a]_r^\rho [x]_\rho^\sigma [b]_\sigma^s = [c]_r^s$.*

Proof of Theorem I. Let the horizontal rows of A' other than the ith be the p_1th, p_2th, ... p_{m-1}th. If the ith horizontal row of A' is connected with certain of the other horizontal rows of A', there must exist a relation of the form

$$[a_{i1}\, a_{i2} \ldots a_{i\rho}\, c_{i1}\, c_{i2} \ldots c_{in}] = [h_1\, h_2 \ldots h_{m-1}]\, [a_{p1},\, c_{p1}]_{m-1}^{\rho,\, n}.$$

By the properties of active rows it follows that

$$[a_{i1}\, a_{i2} \ldots a_{i\rho}] = [h_1\, h_2 \ldots h_{m-1}]\, [a_{p1}]_{m-1}^{\rho} \qquad \text{.................(2)},$$

$$[c_{i1}\, c_{i2} \ldots c_{in}] = [h_1\, h_2 \ldots h_{m-1}]\, [c_{p1}]_{m-1}^{n} \qquad \text{.................(3)}.$$

To prove the theorem we have to show that the solutions of the given equation

$$[a]_m^{\rho}\, [x]_{\rho}^{\sigma}\, [b]_{\sigma}^{n} = [c]_m^{n} \qquad \text{..........................(1)}$$

are identical with the solutions of the equation

$$[a_{p1}]_{m-1}^{\rho}\, [x]_{\rho}^{\sigma}\, [b]_{\sigma}^{n} = [c_{p1}]_{m-1}^{n} \qquad \text{.......................(4)}.$$

Now if (1) is satisfied, it follows by the properties of active rows that (4) also is satisfied.

Again if (4) is satisfied, we have

$$[h_1\, h_2 \ldots h_{m-1}]\, [a_{p1}]_{m-1}^{\rho}\, [x]_{\rho}^{\sigma}\, [b]_{\sigma}^{n} = [h_1\, h_2 \ldots h_{m-1}]\, [c_{p1}]_{m-1}^{n},$$

i.e., making use of (2) and (3), we have

$$[a_{i1}\, a_{i2} \ldots a_{i\rho}]\, [x]_{\rho}^{\sigma}\, [b]_{\sigma}^{n} = [c_{i1}\, c_{i2} \ldots c_{in}] \qquad \text{.................(5)}.$$

But (4) and (5) are together equivalent to (1).

Hence if (4) is satisfied, (1) also is satisfied. Thus (1) and (4) have identical solutions.

By Theorem VI of § 71, the augmented and unaugmented matrices of the reduced equation (4) have the same ranks as the corresponding augmented and unaugmented matrices of the original equation (1).

Proof of Theorem II. Let the vertical rows of B' other than the jth be the q_1th, q_2th, ... q_{n-1}th. If the jth vertical row of B' is connected with certain of the other vertical rows of B', there must exist a relation of the form

$$\begin{bmatrix} b_{1j} \\ \vdots \\ b_{\sigma j} \\ c_{1j} \\ \vdots \\ c_{mj} \end{bmatrix} = \begin{bmatrix} b_{1q} \\ c_{1q} \end{bmatrix}_{\sigma,\, m}^{n-1} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-1} \end{bmatrix},$$

which by the properties of active rows is equivalent to the two relations

$$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{\sigma j} \end{bmatrix} = [b_{1q}]_{\sigma}^{n-1} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-1} \end{bmatrix}, \quad \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix} = [c_{1q}]_{m}^{n-1} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-1} \end{bmatrix} \quad \text{.........(6).}$$

To prove the theorem we have to show that the solutions of the given equation

$$[a]_m^{\rho} [x]_{\rho}^{\sigma} [b]_{\sigma}^{n} = [c]_m^{n} \quad \text{..........................(1)}$$

are identical with the solutions of the equation

$$[a]_m^{\rho} [x]_{\rho}^{\sigma} [b_{1q}]_{\sigma}^{n-1} = [c_{1q}]_m^{n-1} \quad \text{.......................(7).}$$

Now if (1) is satisfied, it follows by the properties of active rows that (7) also is satisfied.

Again if (7) is satisfied, then if we postfix the matrix $\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-1} \end{bmatrix}$ on both sides of the equation and make use of (6), we obtain

$$[a]_m^{\rho} [x]_{\rho}^{\sigma} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{\sigma j} \end{bmatrix} = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix} \quad \text{..............................(8).}$$

Since (7) and (8) are together equivalent to (1), we see that if (7) is satisfied, then (1) also is satisfied. Thus (1) and (7) have identical solutions.

By Theorem VI of § 71, the augmented and unaugmented matrices of the reduced equation (7) have the same ranks as the corresponding augmented and unaugmented matrices of the original equation (1).

Proof of Theorem III. Let A', B' have ranks r and s respectively. Let the p_1th, p_2th, ... p_rth horizontal rows of A' be unconnected and let the q_1th, q_2th, ... q_sth vertical rows of B' be unconnected. Then all horizontal rows of A' are connected with its p_1th, p_2th, ... p_rth horizontal rows, and therefore by Theorem I we can strike out all the horizontal rows of A and C other than the p_1th, p_2th, ... p_rth and so replace the given equation (1) by

$$[a_{p1}]_r^{\rho} [x]_{\rho}^{\sigma} [b]_{\sigma}^{n} = [c_{p1}]_r^{n} \quad \text{..........................(9).}$$

Now there are the same connections between the vertical rows of the augmented matrix $B'' = \begin{bmatrix} b_{11} \\ c_{p1} \end{bmatrix}_{\sigma,\, r}^{n}$ of equation (9) as between the vertical rows of the augmented matrix B' of equation (1), and the matrix B'' has the same rank s as B'. Hence all vertical rows of B'' are connected with its q_1th, q_2th, ... q_sth vertical rows, and these are themselves unconnected. By Theorem II we can now strike out all the vertical rows of $[b]_\sigma^n$, $[c_{p1}]_r^n$ other than the q_1th, q_2th, ... q_sth and so replace (9) by

$$[a_{p1}]_r^\rho \, [x]_\rho^\sigma \, [b_{1q}]_\sigma^s = [c_{pq}]_r^s \quad \text{.......................(10).}$$

In this last equation the augmented and unaugmented matrices have the same ranks as in the original equation (1). Thus, using the notation of § 2.8, the augmented matrices $[a_{p1}, c_{pq}]_r^{\rho, s}$ and $\begin{bmatrix} b_{1q} \\ c_{pq} \end{bmatrix}_{\sigma, r}^s$ of equation (10) have ranks r and s respectively, and the equation is therefore irreducible. Thus we have reduced the given equation to an *irreducible* equation of the *form*

$$[a]_r^\rho \, [x]_\rho^\sigma \, [b]_\sigma^s = [c]_r^s \quad \text{..........................(11).}$$

If the first r horizontal rows of A' are unconnected and the first s vertical rows of B' are unconnected, then the reduced equation actually is (11).

In the course of the above proofs we have established the following theorem, which is a deduction from Theorem VI of § 71.

Theorem IV. *When an equation of the form $AXB = C$ is reduced by repeated applications of Theorems I and II, the ranks of the augmented and unaugmented matrices of the reduced equation are the same as the ranks of the corresponding augmented and unaugmented matrices of the original equation.*

Also the matrix C has the same rank in the reduced equation as in the original equation.

3. *Necessary and sufficient conditions for the existence of a finite solution of the equation $AXB = C$.*

These conditions are contained in the following theorem:

Theorem V. *The equation $AXB = C$ has a finite solution or finite solutions when and only when the augmented matrix A' has the same rank as the unaugmented matrix A and also the augmented matrix B' has the same rank as the unaugmented matrix B.*

According to this theorem necessary and sufficient conditions for the existence of a finite solution are

$$\text{rank of } A' = \text{rank of } A, \qquad \text{rank of } B' = \text{rank of } B \text{.........(12).}$$

Proof of the necessity of these conditions. Let $[x]_\rho^\sigma$ be a finite matrix satisfying the given equation

$$[a]_m^\rho \, [x]_\rho^\sigma \, [b]_\sigma^n = [c]_m^n \quad \text{..........................(1).}$$

Writing $[x]_\rho^\sigma [b]_\sigma^n = [k]_\rho^n$, we have $[c]_m^n = [a]_m^\rho [k]_\rho^n$.

This shows that every vertical row of $[c]_m^n$ is connected with the vertical rows of $[a]_m^\rho$, or by Theorem VI of § 71 that the matrix $[a, c]_m^{\rho, n}$ has the same rank as the matrix $[a]_m^\rho$.

Writing $[a]_m^\rho [x]_\rho^\sigma = [h]_m^\sigma$, we have $[c]_m^n = [h]_m^\sigma [b]_\sigma^n$.

This shows that every horizontal row of $[c]_m^n$ is connected with the horizontal rows of $[b]_\sigma^n$, or by Theorem VI of § 71 that the matrix $\begin{bmatrix} b \\ c \end{bmatrix}_{\sigma, m}^n$ has the same rank as the matrix $[b]_\sigma^n$.

Thus if the given equation (1) has a finite solution, the conditions (12) must be satisfied.

Alternative proof of Theorem V. Let the equation (1) have a finite solution $[x]_\rho^\sigma$. First let A have rank r. We will show that A' also has rank r. Clearly the rank of A' cannot be less than r. It is therefore sufficient to show that the rank of A' is not greater than r.

Let $[a_{p1}, c_{p1}]_{r+1}^{\rho, n}$ be the matrix formed by any $r+1$ horizontal rows of A'. Since there is a connection between every $r+1$ horizontal rows of A, there must exist a relation of the form

$$[h_1 h_2 \dots h_{r+1}] [a_{p1}]_{r+1}^\rho = 0 \quad \text{.........................(13)},$$

where $h_1, h_2, \dots h_{r+1}$ are not all zero.

Now from (1) we deduce by the properties of active rows that

$$[a_{p1}]_{r+1}^\rho [x]_\rho^\sigma [b]_\sigma^n = [c_{p1}]_{r+1}^n .$$

Prefixing $[h_1 h_2 \dots h_{r+1}]$ on both sides and making use of (13), we obtain

$$[h_1 h_2 \dots h_{r+1}] [c_{p1}]_{r+1}^n = 0 \quad \text{.........................(14)},$$

where (13) and (14) are together equivalent to

$$[h_1 h_2 \dots h_{r+1}] [a_{p1}, c_{p1}]_{r+1}^{\rho, n} = 0 \quad \text{.......................(15)}.$$

It follows that there is a connection between every $r+1$ horizontal rows of A', and that the rank of A' cannot be greater than r. Accordingly A' has the same rank r as A.

Next if B has rank s, we can show in a similar way that B' also has rank s.

Proof of the sufficiency of these conditions.

In equation (1) let A and A' have common rank r, and let B and B' have common rank s. Further let $(a_{pu})_r^r \neq 0$, $(b_{vq})_s^s \neq 0$, these being minor determinants of orders r, s of A, B respectively. Then by sub-article 2, we can replace the given equation (1) by the irreducible equation

$$[a_{p1}]_r^{\rho}\,[x]_{\rho}^{\sigma}\,[b_{1q}]_{\sigma}^{s}=[c]_r^{s}\quad\ldots\ldots\ldots\ldots(16).$$

Since by Theorem IV the unaugmented and augmented matrices of (16) have the same ranks as the corresponding unaugmented and augmented matrices of (1), the matrices $[a_{p1}]_r^{\rho}$, $[b_{1q}]_{\sigma}^{s}$ must have ranks r and s respectively, and therefore $\rho \nless r$, $\sigma \nless s$.

When all elements of $[x]_{\rho}^{\sigma}$ except those occurring in the minor matrix $[x_{uv}]_r^{s}$ are put equal to zero, equation (16) becomes

$$[a_{pu}]_r^{r}\,[x_{uv}]_r^{s}\,[b_{vq}]_s^{s}=[c]_r^{s}\quad\ldots\ldots\ldots\ldots(17).$$

Now by § 80, equation (17) admits of a finite solution for $[x_{uv}]_r^{s}$.

Accordingly we can determine a finite solution of (16) in which all elements of $[x]_{\rho}^{\sigma}$ except those occurring in the minor matrix $[x_{uv}]_r^{s}$ have zero values, and the finite matrix $[x]_{\rho}^{\sigma}$ thus determined is a finite solution of the given equation (1).

Thus when the conditions (12) are satisfied, the given equation (1) admits of finite solution.

4. *General solution of the equation $AXB=C$ when it admits of finite solution.*

As before let the given equation be

$$[a]_m^{\rho}\,[x]_{\rho}^{\sigma}\,[b]_{\sigma}^{n}=[c]_m^{n}\quad\ldots\ldots\ldots\ldots(1).$$

Let A and A' have common rank r, and let B and B' have common rank s, so that equation (1) admits of finite solution.

Let $(a_{pu})_r^{r}$ be any non-vanishing derived determinant of order r of A, and let $(b_{vq})_s^{s}$ be any non-vanishing derived determinant of order s of B. We will consider first the general case in which there are no restrictions on $(a_{pu})_r^{r}$, $(b_{vq})_s^{s}$, and afterwards the special case in which $(a)_r^{r}$, $(b)_s^{s}$, the leading derived determinants of orders r, s of A, B, do not vanish and $(a_{pu})_r^{r}$, $(b_{vq})_s^{s}$ are taken to be $(a)_r^{r}$, $(b)_s^{s}$ respectively.

CASE I. $(a_{pu})_r^r = \alpha \neq 0, \quad (b_{vq})_s^s = \beta \neq 0.$

By Theorems I—IV the given equation (1) can be replaced by the irreducible equation

$$[a_{p1}]_r^\rho \, [x]_\rho^\sigma \, [b_{1q}]_\sigma^s = [c_{pq}]_r^s \quad \text{.......................(18)},$$

where $\rho \nless r$ and $\sigma \nless s$.

Now let $[X]_\rho^\sigma$ be the matrix formed from $[x]_\rho^\sigma$ by putting all the elements belonging to the minor matrix $[x_{uv}]_r^s$ equal to zero and leaving all the other elements unaltered, and let $[X']_\rho^\sigma$ be the matrix formed from $[x]_\rho^\sigma$ by leaving all the elements belonging to the minor matrix $[x_{uv}]_r^s$ unaltered and putting all the other elements equal to zero.

Then by the definition of addition

$$[x]_\rho^\sigma = [X]_\rho^\sigma + [X']_\rho^\sigma ;$$

and by the distributive law

$$[a_{p1}]_r^\rho \, [x]_\rho^\sigma \, [b_{1q}]_\sigma^s = [a_{p1}]_r^\rho \, [X']_\rho^\sigma \, [b_{1q}]_\sigma^s + [a_{p1}]_r^\rho \, [X]_\rho^\sigma \, [b_{1q}]_\sigma^s .$$

Since all the horizontal rows of $[X']_\rho^\sigma$ except the u_1th, u_2th, ... u_rth rows are rows of 0's, and all the vertical rows of $[X']_\rho^\sigma$ except the v_1th, v_2th, ... v_sth rows are rows of 0's, it follows by the properties of passive rows (see § 52.5) that

$$[a_{p1}]_r^\rho \, [x]_\rho^\sigma \, [b_{1q}]_\sigma^s = [a_{pu}]_r^r \, [x_{uv}]_r^s \, [b_{vq}]_s^s + [a_{p1}]_r^\rho \, [X]_\rho^\sigma \, [b_{1q}]_\sigma^s \quad \text{......(19)}.$$

The result here obtained is clearly an immediate consequence of the fact that the elements of the product matrix $[a_{p1}]_r^\rho \, [x]_\rho^\sigma \, [b_{1q}]_\sigma^s$ are homogeneous linear functions of the elements of the middle factor matrix.

Making use of (19) we can write (18) in the form

$$[a_{pu}]_r^r \, [x_{uv}]_r^s \, [b_{vq}]_s^s = [c_{pq}]_r^s - [a_{p1}]_r^\rho \, [X]_\rho^\sigma \, [b_{1q}]_\sigma^s = [\xi]_r^s \quad \text{......(20)},$$

where $[\xi]_r^s$ involves only those elements of $[x]_\rho^\sigma$ which do not occur in the minor $[x_{uv}]_r^s$.

Equation (20) can be solved for the matrix $[x_{uv}]_r^s$ as in § 80.

Let $(a_{pu})_r^r = \alpha$, $(b_{vq})_s^s = \beta$; and let $[A_{pu}]_r^r$, $[B_{vq}]_s^s$ be the reciprocal matrices of $[a_{pu}]_r^r$, $[b_{vq}]_s^s$ respectively. Then prefixing and postfixing $\overline{\underline{A_{pu}}}_r^r$, $\overline{\underline{B_{vq}}}_s^s$, the conjugate reciprocals of $[a_{pu}]_r^r$, $[b_{vq}]_s^s$, on both sides of (20), we obtain the following result:

First form of the general solution.

If A and A' have common rank r, and B and B' have common rank s, and if $(a_{pu})_r^r = \alpha \neq 0$ and $(b_{vq})_s^s = \beta \neq 0$, then equation (1) is satisfied when and only when

$$\alpha\beta\,[x_{uv}]_r^s = \overline{\underline{A_{pu}}}_r^r\,[c_{pq}]_r^s\,\overline{\underline{B_{vq}}}_s^s - \overline{\underline{A_{pu}}}_r^r\,[a_{p1}]_r^\rho\,[X]_\rho^\sigma\,[b_{1q}]_\sigma^s\,\overline{\underline{B_{vq}}}_s^s,$$

or

$$\alpha\beta\,[x_{uv}]_r^s = [\gamma]_r^s - [\alpha]_r^\rho\,[X]_\rho^\sigma\,[\beta]_\sigma^s,$$

where α_{ij} is the value of the determinant formed when the ith vertical row of $(a_{pu})_r^r$ is replaced by the jth vertical row of $[a_{p1}]_r^\rho$; β_{ij} is the value of the determinant formed when the jth horizontal row of $(b_{vq})_s^s$ is replaced by the ith horizontal row of $[b_{1q}]_\sigma^s$; and $[X]_\rho^\sigma$ is the matrix formed from $[x]_\rho^\sigma$ by putting $[x_{uv}]_r^s = 0$ ....(A).

It may be observed that the values of the minor matrices $[\alpha_{1u}]_r^r$, $[\beta_{v1}]_s^s$ of $[\alpha]_r^\rho$, $[\beta]_\sigma^s$ are given by

$$[\alpha_{1u}]_r^r = \alpha\,[1]_r^r, \qquad [\beta_{v1}]_s^s = \beta\,[1]_s^s \quad \text{................(21);}$$

i.e. if j has one of the values $u_1, u_2, \ldots u_r$, then $\alpha_{ij} = 0$ or α according as $i \neq j$ or $i = j$; and if i has one of the values $v_1, v_2, \ldots v_s$, then $\beta_{ij} = 0$ or β according as $j \neq i$ or $j = i$.

Formula (A) gives the general solution of equation (1). In this solution arbitrary values can be assigned to the $\rho\sigma - rs$ unknown elements which do not occur in $[x_{uv}]_r^s$, and the remaining rs unknown elements, viz. those which do occur in $[x_{uv}]_r^s$, are expressed uniquely in terms of those $\rho\sigma - rs$ arbitrary elements.

Since the elements of $[\xi]_r^s$ in (20) are *linear* functions of the $\rho\sigma - rs$ unknown elements which do not occur in $[x_{uv}]_r^s$, the same is true of the elements of $[x_{uv}]_r^s$ determined by the solution of (20).

We have now proved the following theorem:

Theorem VI. *If A and A' have common rank r, and B and B' have common rank s, then the equation $[a]_m^\rho\,[x]_\rho^\sigma\,[b]_\sigma^n = [c]_m^n$ admits of finite solution, and the general solution of the equation expresses rs of the $\rho\sigma$ unknown elements* *as linear (in general non-homogeneous) functions of the remaining* $\rho\sigma - rs$ *unknown elements to which arbitrary values may be assigned.*

Since the general solution contains $\rho\sigma - rs$ arbitrary elements, which may be real or imaginary, the total number of solutions is $\infty^{2(\rho\sigma - rs)}$. In the special case in which $r = \rho$ and $s = \sigma$, no arbitrary elements occur in the general solution, and there is a single unique finite solution. In fact this special case is equivalent to that considered in § 80.

NOTE 1. *The simplest particular solution in Case I.*

This is obtained by assigning zero values to all the elements of $[X]_\rho^\sigma$ and is given by

$$\alpha\beta\,[x_{uv}]_r^s = \boxed{A_{pu}}_r^r\,[c_{pq}]_r^s\,\boxed{B_{vq}}_s^s = [\gamma]_r^s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(22),$$

all other elements of $[x]_\rho^\sigma$ having zero values.

This particular solution can be determined by solving the equation

$$[a_{pu}]_r^r\,[x_{uv}]_r^s\,[b_{vq}]_s^s = [c_{pq}]_r^s,$$

and it has the same rank as $[c_{pq}]_r^s$.

NOTE 2. *Alternative proof of* (19).

Let $[\lambda_1\,\lambda_2 \ldots \lambda_{\rho-r}]$ be complementary to $[u_1\,u_2 \ldots u_r]$ in $[1\,2 \ldots \rho]$,
and $[\mu_1\,\mu_2 \ldots \mu_{\sigma-s}]$ be complementary to $[v_1\,v_2 \ldots v_s]$ in $[1\,2 \ldots \sigma]$.

Then by the properties of passive rows (see § 52.8 and § 52.9) we have

$$\begin{aligned}[a_{p1}]_r^\rho\,[x]_\rho^\sigma\,[b_{1q}]_\sigma^s &= [a_{pu}]_r^r\,[x_{u1}]_r^\sigma\,[b_{1q}]_\sigma^s + [a_{p\lambda}]_r^{\rho-r}\,[x_{\lambda 1}]_{\rho-r}^\sigma\,[b_{1q}]_\sigma^s,\\ &= [a_{pu}]_r^r\,[x_{uv}]_r^s\,[b_{vq}]_s^s + [a_{pu}]_r^r\,[x_{u\mu}]_r^{\sigma-s}\,[b_{\mu q}]_{\sigma-s}^s\\ &\quad + [a_{p\lambda}]_r^{\rho-r}\,[x_{\lambda v}]_{\rho-r}^s\,[b_{vq}]_s^s + [a_{p\lambda}]_r^{\rho-r}\,[x_{\lambda\mu}]_{\rho-r}^{\sigma-s}\,[b_{\mu q}]_{\sigma-s}^s\end{aligned} \quad \ldots\ldots\ldots(23).$$

Putting $[x_{uv}]_r^s = 0$ in the identity (23), we obtain as a particular case

$$\begin{aligned}[a_{p1}]_r^\rho\,[X]_\rho^\sigma\,[b_{1q}]_\sigma^s &= [a_{pu}]_r^r\,[x_{u\mu}]_r^{\sigma-s}\,[b_{\mu q}]_{\sigma-s}^s + [a_{p\lambda}]_r^{\rho-r}\,[x_{\lambda v}]_{\rho-r}^s\,[b_{vq}]_s^s\\ &\quad + [a_{p\lambda}]_r^{\rho-r}\,[x_{\lambda\mu}]_{\rho-r}^{\sigma-s}\,[b_{\mu q}]_{\sigma-s}^s\end{aligned} \quad \ldots\ldots\ldots(24).$$

From (23) and (24) we see that

$$[a_{p1}]_r^\rho\,[x]_\rho^\sigma\,[b_{1q}]_\sigma^s = [a_{pu}]_r^r\,[x_{uv}]_r^s\,[b_{vq}]_s^s + [a_{p1}]_r^\rho\,[X]_\rho^\sigma\,[b_{1q}]_\sigma^s \quad \ldots\ldots\ldots(19).$$

Equation (23) can be deduced from the general identity

$$[a,\ b]_m^{p,\,q}\begin{bmatrix}\alpha, & \beta\\ \gamma, & \delta\end{bmatrix}_{p,\,q}^{r,\,s}\begin{bmatrix}c\\ d\end{bmatrix}_{r,\,s}^{n}$$

$$= [a]_m^p\,[\alpha]_p^r\,[c]_r^n + [a]_m^p\,[\beta]_p^s\,[d]_s^n + [b]_m^q\,[\gamma]_q^r\,[c]_r^n + [b]_m^q\,[\delta]_q^s\,[d]_s^n \quad \ldots\ldots(25),$$

which can be proved in the same way. For by § 52.8, we have

$$[a_{pi}]_r^\rho \; [x]_\rho^\sigma \; [b_{iq}]_\sigma^s = [a_{pu}, \; a_{p\lambda}]_r^{r,\,\rho-r} \begin{bmatrix} x_{uv}, & x_{u\mu} \\ x_{\lambda v}, & x_{\lambda\mu} \end{bmatrix}_{r,\,\rho-r}^{s,\,\sigma-s} \begin{bmatrix} b_{vq} \\ b_{\mu q} \end{bmatrix}_{s,\,\sigma-s}^{s}.$$

Note 3. *Alternative method of solution in Case I.*

We can also obtain the general solution directly from the reduced equation (18) by prefixing and postfixing $\overline{\underline{A_{pu}}}_r^r$, $\overline{\underline{B_{vq}}}_s^s$ on both sides. This gives

$$[a]_r^\rho \; [x]_\rho^\sigma \; [\beta]_\sigma^s = [\gamma]_r^s \quad \text{.................................(A')},$$

where a_{ij}, β_{ij}, γ_{ij} have the same meanings as in (A).

If in (A′) we put $[x]_\rho^\sigma = [X]_\rho^\sigma + [X']_\rho^\sigma$ and make use of (21), we obtain the second equation in (A).

By applying the identity (22) in (A′) or the identity (23) in (A) we obtain more expanded forms of the solution.

Note 4. *Extension of the abbreviated notation for augmented matrices.*

In Note 2 we have used an obvious extension of the notation for augmented matrices given in § 2.4.

When we write

$$X = \begin{bmatrix} a, & b \\ c, & d \end{bmatrix}_{m,\,\mu}^{n,\,\nu}, \qquad \det X = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}_{m,\,\mu}^{n,\,\nu},$$

we understand X to be the matrix formed by placing the horizontal rows of the second of the two matrices

$$[a, \; b]_m^{n,\,\nu}, \qquad [c, \; d]_\mu^{n,\,\nu}$$

immediately below the horizontal rows of the first without altering their relative order, or by placing the vertical rows of the second of the two matrices

$$\begin{bmatrix} a \\ c \end{bmatrix}_{m,\,\mu}^{n}, \qquad \begin{bmatrix} b \\ d \end{bmatrix}_{m,\,\mu}^{\nu}$$

immediately to the right of the vertical rows of the first without altering their relative order.

Thus

$$X = \begin{bmatrix} a_{11} \dots a_{1n} & b_{11} \dots b_{1\nu} \\ \dots\dots\dots & \dots\dots\dots \\ a_{m1} \dots a_{mn} & b_{m1} \dots b_{m\nu} \\ c_{11} \dots c_{1n} & d_{11} \dots d_{1\nu} \\ \dots\dots\dots & \dots\dots\dots \\ c_{\mu 1} \dots c_{\mu n} & d_{\mu 1} \dots d_{\mu\nu} \end{bmatrix}.$$

More generally when we write

$$Y = \begin{bmatrix} a_{xy}, & b_{pq} \\ c_{uv}, & d_{\xi\eta} \end{bmatrix}_{m,\,\mu}^{n,\,\nu}, \qquad \det Y = \begin{pmatrix} a_{xy}, & b_{pq} \\ c_{uv}, & d_{\xi\eta} \end{pmatrix}_{m,\,\mu}^{n,\,\nu},$$

we understand Y to be the matrix formed by placing the horizontal rows of the second of the two matrices

$$[a_{xy},\ b_{pq}]_m^{n,\ \nu}, \qquad [c_{uv},\ d_{\xi\eta}]_\mu^{n,\ \nu}$$

immediately below the horizontal rows of the first without altering their relative order.

The conjugates of the matrices X and Y will be denoted by X' and Y', where

$$X' = \overline{\underline{\begin{matrix} a, & c \\ b, & d \end{matrix}}}_{n,\ \nu}^{m,\ \mu}, \qquad Y' = \overline{\underline{\begin{matrix} a_{xy}, & c_{uv} \\ b_{pq}, & d_{\xi\eta} \end{matrix}}}_{n,\ \nu}^{m,\ \mu}.$$

CASE II. $(a)_r^r = \alpha \neq 0,\ (b)_s^s = \beta \neq 0.$

Any case whatever can be reduced to this special case by suitable re-arrangements of corresponding horizontal rows of A and C, corresponding vertical rows of B and C, corresponding passive rows of A and X, and corresponding passive rows of X and B.

The given equation

$$[a]_m^\rho\ [x]_\rho^\sigma\ [b]_\sigma^n = [c]_m^n \qquad \text{.........................(1)}$$

can in this case be replaced by the irreducible equation

$$[a]_r^\rho\ [x]_\rho^\sigma\ [b]_\sigma^s = [c]_r^s \qquad \text{.......................(26),}$$

in which $\rho \nless r$ and $\sigma \nless s$.

Passing from the general case to this special case by putting

$$[p_1\, p_2 \dots p_r] = [u_1\, u_2 \dots u_r] = [1\ 2 \dots r],$$

$$[q_1\, q_2 \dots q_s] = [v_1\, v_2 \dots v_s] = [1\ 2 \dots s],$$

$$[X]_\rho^\sigma = \begin{bmatrix} 0 & \dots & 0 & x_{1,\,s+1} & \dots & x_{1\sigma} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & x_{r,\,s+1} & \dots & x_{r\sigma} \\ x_{r+1,\,1} & \dots & x_{r+1,\,s} & x_{r+1,\,s+1} & \dots & x_{r+1,\,\sigma} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{\rho 1} & \dots & x_{\rho s} & x_{\rho,\,s+1} & \dots & x_{\rho\sigma} \end{bmatrix} \qquad \text{............(27),}$$

we see that equation (26) can be expressed in the form

$$[a]_r^r\ [x]_r^s\ [b]_s^s = [c]_r^s - [a]_r^\rho\ [X]_\rho^\sigma\ [b]_\sigma^s = [\xi]_r^s \qquad \text{...........(28).}$$

Writing $(a)_r^r = \alpha,\ (b)_s^s = \beta$; denoting the reciprocals of $[a]_r^r$, $[b]_s^s$ by $[A]_r^r$, $[B]_s^s$; and solving equation (28) for $[x]_r^s$ as in § 80 by prefixing and postfixing the matrices $\overline{\underline{A}}_r^r$, $\overline{\underline{B}}_s^s$; we obtain the following result:

Second form of the general solution.

If A and A' have common rank r and B and B' have common rank s, and if $(a)_r^r = \alpha \neq 0$ and $(b)_s^s = \beta \neq 0$, then equation (1) is satisfied when and only when

$$\alpha\beta\,[x]_r^s = \overline{\underline{A}}_r^r\,[c]_r^s\,\overline{\underline{B}}_s^s - \overline{\underline{A}}_r^r\,[a]_r^\rho\,[X]_\rho^\sigma\,[b]_\sigma^s\,\overline{\underline{B}}_s^s,$$

or

$$\alpha\beta\,[x]_r^s = [\gamma]_r^s - [\alpha]_r^\rho\,[X]_\rho^\sigma\,[\beta]_\sigma^s,$$

where α_{ij} is the value of the determinant formed when the ith vertical row of $(a)_r^r$ is replaced by the jth vertical row of $[a]_r^\rho$; β_{ij} is the value of the determinant formed when the jth horizontal row of $(b)_s^s$ is replaced by the ith horizontal row of $[b]_\sigma^s$; and $[X]_\rho^\sigma$ is given by (27) and is formed from $[x]_\rho^\sigma$ by putting $[x]_r^s = 0$ ....(B).

Formula (B) gives the general solution of equation (1) in this special case. The solution expresses the rs elements of $[x]_r^s$ as linear functions of the $\rho\sigma - rs$ remaining unknown elements to which arbitrary values may be assigned.

Note 5. *The simplest particular solution in Case II.*

This is obtained by assigning zero values to all the arbitrary elements and is given by

$$[x]_\rho^\sigma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1s} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{r1} & \cdots & \gamma_{rs} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \gamma, & 0 \\ 0, & 0 \end{bmatrix}_{r,\ \rho - r}^{s,\ \sigma - s} \qquad \text{..................(29).}$$

This particular solution can be obtained by solving the equation

$$[a]_r^r\,[x]_r^s\,[b]_s^s = [c]_r^s,$$

and it has the same rank as $[c]_r^s$.

Note 6. *Forms of $[a]_r^\rho$, $[\beta]_\sigma^s$ in Case II.*

It will be observed that the matrices $[a]_r^\rho$, $[\beta]_\sigma^s$ have the forms

$$[a]_r^\rho = \begin{bmatrix} a & 0 & \cdots & 0 & a_{1,r+1} & \cdots & a_{1\rho} \\ 0 & a & \cdots & 0 & a_{2,r+1} & \cdots & a_{2\rho} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a & a_{r,r+1} & \cdots & a_{r\rho} \end{bmatrix}, \qquad [\beta]_\sigma^s = \begin{bmatrix} \beta & 0 & \cdots & 0 \\ 0 & \beta & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \beta \\ \beta_{s+1,1} & \beta_{s+1,2} & \cdots & \beta_{s+1,s} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{\sigma 1} & \beta_{\sigma 2} & \cdots & \beta_{\sigma s} \end{bmatrix} \quad \text{....(30)},$$

so that $$[a]_r^r = a\,[1]_r^r, \qquad [\beta]_s^s = \beta\,[1]_s^s.$$

NOTE 7. *Alternative method of solution in Case II.*

The general solution can also be obtained directly from the reduced equation (26) by prefixing and postfixing $\overline{\underline{A}}_r^r$, $\overline{\underline{B}}_s^s$ respectively. We then obtain

$$[a]_r^\rho\, [x]_\rho^\sigma\, [\beta]_\sigma^s = \overline{\underline{A}}_r^r\, [c]_r^s\, \overline{\underline{B}}_s^s = [\gamma]_r^s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{B}'),$$

where a_{ij}, β_{ij}, γ_{ij} have the same values as in (B). Proceeding as in Note 3, we can deduce from (B′) the formula contained in (B).

Ex. i. We will solve the equation

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 17 & 11 \\ 1 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 2 & 4 & 1 \\ 6 & 2 & 7 & 1 \\ 4 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 4 & 9 & 6 & 7 \\ 1 & 4 & 3 & 2 \end{bmatrix} \quad \ldots\ldots\ldots\ldots(31).$$

Here the horizontal rows a_1, a_2, a_3 of A' are connected by the relation $a_2 = a_1 + 2a_3$, and the vertical rows b_1, b_2, b_3, b_4 of B' are connected by the relation $b_3 = b_1 + b_2 - b_4$. Also A and B have non-vanishing derived determinants of orders 2 and 3 respectively. Consequently A and A' have common rank 2, and B and B' have common rank 3.

First Method. Basing the solution on the non-vanishing minor determinants $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$, $\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{vmatrix}$ of A and B, we can replace the given equation by the irreducible equation

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 6 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots(32).$$

Using the formula (20), we see that equation (32) can be written

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & w_1 \\ x_3 & y_3 & w_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \\ 1 & 8 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & z_1 & 0 \\ x_2 & y_2 & z_2 & w_2 \\ 0 & 0 & z_3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 6 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}$$
$$\ldots\ldots\ldots\ldots(33).$$

The conjugate reciprocals of

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{are} \quad \begin{bmatrix} 4, & -3 \\ -1, & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 4, & -2, & 1 \\ -2, & 4, & -2 \\ -8, & 4, & 1 \end{bmatrix}.$$

Prefixing the first of these and postfixing the second on both sides of (33) we obtain as the general solution of (31)

$$30\begin{bmatrix} x_1 & y_1 & w_1 \\ x_3 & y_3 & w_3 \end{bmatrix} = \begin{bmatrix} -12, & -18, & 27 \\ -22, & 32, & -13 \end{bmatrix} - \begin{bmatrix} 5, & -20, & 0 \\ 0, & 15, & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & z_1 & 0 \\ x_2 & y_2 & z_2 & w_2 \\ 0 & 0 & z_3 & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 12 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$
$$\ldots\ldots\ldots\ldots(34),$$

where x_2, y_2, z_2, w_2, z_1, z_3 are arbitrary.

Second Method. Instead of using formula (20) we may proceed as follows. By the properties of passive rows

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 8 & 4 \end{bmatrix} [x\ y\ z\ w]_{123} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 6 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} [x\ y\ z\ w]_{13} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 6 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 8 \end{bmatrix} [x\ y\ z\ w]_{2} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 6 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} [x\ y\ w]_{13} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix} [6\ 2\ 1]$$

$$+ \begin{bmatrix} 1 \\ 8 \end{bmatrix} [x\ y\ z\ w]_{2} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 6 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}.$$

Hence equation (24) can be written

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} [x\ y\ w]_{13} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix} [6\ 2\ 1] - \begin{bmatrix} 1 \\ 8 \end{bmatrix} [x\ y\ z\ w]_{2} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 6 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}.$$

Prefixing and postfixing the same matrices as before, we have

$$30 \begin{bmatrix} x_1 & y_1 & w_1 \\ x_3 & y_3 & w_3 \end{bmatrix}$$

$$= \begin{bmatrix} -12, & -18, & 27 \\ -22, & 32, & -13 \end{bmatrix} - 5 \begin{bmatrix} z_1 \\ z_3 \end{bmatrix} [12,\ 0,\ 3] - \begin{bmatrix} -20 \\ 15 \end{bmatrix} [x_2\ y_2\ z_2\ w_2] \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 12 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix}.$$

This is clearly equivalent to (31).

Third Method. If we prefix and postfix the conjugate reciprocal matrices on both sides of equation (32), we obtain

$$\begin{bmatrix} 5, & -20, & 0 \\ 0, & 15, & 5 \end{bmatrix} [x\ y\ z\ w]_{123} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 12 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 12, & -18, & 27 \\ -22, & 32, & 13 \end{bmatrix}.$$

Writing $$[x\ y\ z\ w]_{123} = \begin{bmatrix} x_1 & y_1 & 0 & w_1 \\ 0 & 0 & 0 & 0 \\ x_3 & y_3 & 0 & w_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & z_1 & 0 \\ x_2 & y_2 & z_2 & w_2 \\ 0 & 0 & z_3 & 0 \end{bmatrix},$$

this is at once reducible to (34).

Ex. ii. In solving the reduced equation it is usually better to write it in such a form that the leading derived determinants do not vanish. The reduced equation (32) is already so arranged. To solve it we may replace it by

$$\begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \\ 1 & 8 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & w_1 \\ 0 & 0 & 0 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 6 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}.$$

We now prefix $\begin{bmatrix} 8, & -1 \\ -1, & 2 \end{bmatrix}$ and postfix $\begin{bmatrix} 0, & -1, & 1 \\ 3, & 2, & -2 \\ -6, & 2, & 1 \end{bmatrix}$ on both sides.

We thus obtain the general solution of (32) or (31) in the form

$$45 \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} -120, & 37, & 29 \\ 15, & 16, & -13 \end{bmatrix} - \begin{bmatrix} 15 & 0 & 20 \\ 0 & 15 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & w_1 \\ 0 & 0 & 0 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} \begin{bmatrix} 3, & 0, & 0 \\ 0, & 3, & 0 \\ 0, & 0, & 3 \\ -12, & 0, & 6 \end{bmatrix},$$

where x_3, y_3, z_3, w_1, w_2, w_3 are arbitrary.

If we expand the last product by the identity (25) of Note 1 we obtain

$$45 \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} -120, & 37, & 29 \\ 15, & 16, & -13 \end{bmatrix} - 15 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} [-12,\ 0,\ 6] - 3 \begin{bmatrix} 20 \\ 5 \end{bmatrix} [x_3\ y_3\ z_3]$$
$$+ \begin{bmatrix} 20 \\ 5 \end{bmatrix} [-12,\ 0,\ 6]\ w_3.$$

5. *General solution of an irreducible equation of the form* $AXB = C$ *which admits of finite solution.*

This is the only case which it is necessary to consider, since an equation which is not irreducible can always be replaced by an irreducible equation.

If $$[a]_r^{\rho}\ [x]_{\rho}^{\sigma}\ [b]_{\sigma}^{s} = [c]_r^{s} \quad \text{..........................(35)}$$

is an irreducible equation in which $(a_{1u})_r^r \neq 0$, $(b_{v1})_s^s \neq 0$, then to obtain the general solution we use the properties of passive rows to write (35) in the form

$$[a_{1u}]_r^r\ [x_{uv}]_r^s\ [b_{v1}]_s^s = [\xi]_r^s \quad \text{.......................(36),}$$

and then solve (36) for $[x_{uv}]_r^s$ as in § 80 by prefixing the conjugate reciprocal of $[a_{1u}]_r^r$ and postfixing the conjugate reciprocal of $[b_{v1}]_s^s$ on both sides.

It has been shown in sub-article 4 that

$$[\xi]_r^s = [c]_r^s - [a]_r^\rho \, [X]_\rho^\sigma \, [b]_\sigma^s,$$

where $[X]_\rho^\sigma$ is obtained from $[x]_\rho^\sigma$ by putting $[x_{uv}]_r^s = 0$.

As an alternative we may obtain the solution for $[x_{uv}]_r^s$ directly from equation (35) by prefixing the conjugate reciprocal of $[a_{1u}]_r^r$ and postfixing the conjugate reciprocal of $[b_{v1}]_s^s$ on both sides.

The matrix $[x_{uv}]_r^s$ for which we solve is derived from $[x]_\rho^\sigma$ by retaining only those rows of $[x]_\rho^\sigma$ which correspond to the passive rows of $[a]_r^\rho$ occurring in $[a_{1u}]_r^r$ and to the passive rows of $[b]_\sigma^s$ occurring in $[b_{v1}]_s^s$.

Ex. iii. To solve the equation

$$[abc]_{12} [xyzw]_{123} [pqr]_{1234} = [\alpha\beta\gamma]_{12}$$

for $[xyzw]_{12}$ when $(bc)_{12} \neq 0$ and $(pqr)_{134} \neq 0$, we write it in the form

$$[bc]_{12} [xzw]_{23} [pqr]_{134} = [\alpha\beta\gamma]_{12} - [abc]_{12} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ 0 & y_2 & 0 & 0 \\ 0 & y_3 & 0 & 0 \end{bmatrix} [pqr]_{1234}.$$

Denoting the reciprocals of $[bc]_{12}$, $[pqr]_{134}$ by $[BC]_{12}$, $[PQR]_{134}$, and prefixing and postfixing the conjugates of these respectively, we obtain as the general solution

$$(bc)_{12} (pqr)_{134} \begin{bmatrix} x_2 & z_2 & w_2 \\ x_3 & z_3 & w_3 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ C_1 & C_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} P_1 & P_3 & P_4 \\ Q_1 & Q_3 & Q_4 \\ R_1 & R_3 & R_4 \end{bmatrix}$$

$$- \begin{bmatrix} (ac)_{12}, & (bc)_{12}, & 0 \\ (ba)_{12}, & 0, & (bc)_{12} \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \\ 0 & y_2 & 0 & 0 \\ 0 & y_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} (pqr)_{134}, & 0, & 0 \\ (pqr)_{234}, & (pqr)_{124}, & (pqr)_{132} \\ 0, & (pqr)_{134}, & 0 \\ 0, & 0, & (pqr)_{134} \end{bmatrix},$$

where $x_1, y_1, z_1, w_1, y_2, y_3$ are arbitrary.

Here the last product can be replaced by

$$y_1 \begin{bmatrix} (ac)_{12} \\ (ba)_{12} \end{bmatrix} [(pqr)_{234}, (pqr)_{124}, (pqr)_{132}]$$

$$+ (bc)_{12} \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} [(pqr)_{234}, (pqr)_{124}, (pqr)_{132}] + (pqr)_{134} \begin{bmatrix} (ac)_{12} \\ (ba)_{12} \end{bmatrix} [x_1 \; z_1 \; w_1].$$

6. *Solution of the special equation* $AXB = 0$.

When the matrix C vanishes, the given equation (1) becomes

$$[a]_m^\rho \, [x]_\rho^\sigma \, [b]_\sigma^n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37).$$

In this case A' is the same matrix as A and B' is the same matrix as B. Hence the equation always admits of finite solution. In fact $X = 0$ or $[x]_\rho^\sigma = 0$ is always a solution.

If A and B have ranks r and s respectively, and if $(a)_r^r = \alpha \neq 0$, $(b)_s^s = \beta \neq 0$, we first replace (37) by the irreducible equation

$$[a]_r^\rho \, [x]_\rho^\sigma \, [b]_\sigma^s = 0,$$

or

$$[a]_r^r \, [x]_r^s \, [b]_s^s = -[a]_r^\rho \, [X]_\rho^\sigma \, [b]_\sigma^s,$$

where

$$[X]_\rho^\sigma = \begin{bmatrix} 0 & \ldots & 0 & x_{1,s+1} & \ldots & x_{1\sigma} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & x_{r,s+1} & \ldots & x_{r\sigma} \\ x_{r+1,1} & \ldots & x_{r+1,s} & x_{r+1,s+1} & \ldots & x_{r+1,\sigma} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{\rho 1} & \ldots & x_{\rho s} & x_{\rho,s+1} & \ldots & x_{\rho\sigma} \end{bmatrix} \quad \ldots\ldots\ \ldots\ldots(38).$$

Then prefixing $\overline{\underline{A}}_r^r$ and postfixing $\overline{\underline{B}}_s^s$ we obtain the general solution in the form

$$\alpha\beta \, [x]_r^s = -\overline{\underline{A}}_r^r \, [a]_r^\rho \, [X]_\rho^\sigma \, [b]_\sigma^s \, \overline{\underline{B}}_s^s = -[\alpha]_r^\rho \, [X]_\rho^\sigma \, [\beta]_\sigma^s \quad \ldots\ldots(C).$$

Here α_{ij}, β_{ij} have the same values as in formula (B).

This solution determines each of the rs elements of $[x]_r^s$ as *homogeneous* linear functions of the remaining $\rho\sigma - rs$ elements to which arbitrary values may be assigned.

If $r = \rho$ and $s = \sigma$ there are no arbitrary elements in the solution. In this case $X = 0$ is the only solution. In all other cases there are other solutions besides $X = 0$.

Since there is no loss of generality in assuming that $(a)_r^r \neq 0$ and $(b)_s^s \neq 0$, we have proved the following theorem:

Theorem VII. *The equation $AXB = 0$ leads to $X = 0$ as a necessary consequence when and only when the rank of each of the extreme factor matrices* **A** *and B is equal to its passivity. The equation has non-zero solutions when and only when the rank of one of the extreme factor matrices is less than its passivity.*

More generally if A and B have ranks r and s respectively and if

$$(a_{pu})^r_r = \alpha \neq 0, \qquad (b_{vq})^s_s = \beta \neq 0,$$

we first replace (37) by the irreducible equation

$$[a_{p1}]^\rho_r \, [x]^\sigma_\rho \, [b_{1q}]^s_\sigma = 0 \quad \text{..................... ...(39)},$$

or

$$[a_{pu}]^r_r \, [x_{uv}]^s_r \, [b_{vq}]^s_s = -\, [a_{p1}]^\rho_r \, [X]^\sigma_\rho \, [b_{1q}]^s_\sigma \quad \text{..............(40)},$$

where $[X]^\sigma_\rho$ is obtained from $[x]^\sigma_\rho$ by putting $[x_{uv}]^s_r = 0$.

Then prefixing $\boxed{A_{pu}}^r_r$ and postfixing $\boxed{B_{vq}}^s_s$, we obtain the general solution in the form

$$\alpha\beta \, [x_{uv}]^s_r = -\, \boxed{A_{pu}}^r_r \, [a_{p1}]^\rho_r \, [X]^\sigma_\rho \, [b_{1q}]^s_\sigma \, \boxed{B_{vq}}^s_s = -\, [\alpha]^\rho_r \, [X]^\sigma_\rho \, [\beta]^s_\sigma$$

$$\text{......................(D)},$$

where α_{ij}, β_{ij} have the same values as in formula (A).

This solution determines each of the rs elements of $[x_{uv}]^s_r$ as *homogeneous* linear functions of the other $\rho\sigma - rs$ unknown elements to which arbitrary values may be assigned.

Ex. iv. The equation $[a]^\rho_m \, [x]^\sigma_\rho \, [b]^n_\sigma = 0$ is satisfied when $[a]^\rho_m \, [x]^\sigma_\rho = 0$ and when $[x]^\sigma_\rho \, [b]^n_\sigma = 0$. Hence by § 81.6 and § 82.6 if r_1 is the smaller of the two numbers σ, $\rho - r$, and if s_1 is the smaller of the two numbers ρ, $\sigma - s$, the equation certainly has solutions of all ranks from 0 to the greater of the two numbers r_1, s_1.

Ex. v. *If a determinant* $(a)^m_m$ *of order* m *contains a zero minor matrix* $[a_{uv}]^q_p$ *of orders* p *and* q, *then if* $p+q>m$ *the determinant necessarily vanishes, but if* $p+q=m$ *or* $<m$, *the other elements of the determinant can be so chosen that the determinant does not vanish.*

The first part of this theorem can be proved as follows. The determinant can be expanded in terms of the simple minor determinants of the matrix $[a_{1v}]^q_m$.

This matrix contains p horizontal rows of 0's. The number of the remaining horizontal rows is $m-p$, which is less than q. Consequently every simple minor determinant of order q contains one or more rows of 0's, and vanishes. It follows that the determinant $(a)^m_m$ vanishes.

The second part of the theorem can be proved by observing that the non-vanishing determinant of the unit matrix $[1]^m_m$ has zero minor matrices the sum of whose orders is equal to m or to any number less than m.

Ex. vi. *If* R *is the least of the three numbers* ρ, σ, $(\rho - r)+(\sigma - s)$, *the matrix* $[X]^\sigma_\rho$ *can have any rank from* 0 *to* R, *but cannot have rank greater than* R.

First suppose that $(\rho - r)+(\sigma - s)$ is the least of the three numbers. Then $[X]^\sigma_\rho$ has a derived determinant of order $(\rho - r)+(\sigma - s)$ which contains a zero minor matrix with

$\sigma - s$ horizontal rows of 0's and $\rho - r$ vertical rows of 0's and has all its other elements arbitrary. By Ex. v the arbitrary elements can be so chosen that this determinant does not vanish. Every derived determinant of greater order $(\rho - r) + (\sigma - s) + u$ contains a zero minor matrix with $\sigma - s + u$ horizontal rows of 0's and $\rho - r + u$ vertical rows of 0's, and therefore vanishes by Ex. v.

Next suppose that ρ is the least of the three numbers, so that $\sigma \nless r + s$. Then $[X]_\rho^\sigma$ has a derived determinant of order ρ which contains a zero minor matrix with r horizontal rows of 0's and $\rho - (\sigma - s)$ vertical rows of 0's and has all its other elements arbitrary. By Ex. v the arbitrary elements can be so chosen that this determinant does not vanish.

The case in which σ is the least of the three numbers can be treated similarly.

Thus $[X]_\rho^\sigma$ can always have rank as great as R, but can never have rank greater than R.

Further when it has any given rank we can always reduce its rank to any lower number by making certain rows to consist of zero elements only. The above theorem is therefore true.

Ex. vii. *The equation* $[a]_m^\rho [x]_\rho^\sigma [b]_\sigma^n = [c]_m^n$ *can certainly have solutions of rank as great as* R. For the arbitrary elements in $[X]_\rho^\sigma$ can be so chosen that $[X]_\rho^\sigma$ has a non-vanishing derived determinant of order R whose matrix is a derangement of a unit matrix, and the value of this determinant is unaltered when any zero minor matrix is replaced by a matrix of arbitrary elements.

By Theorem IV of § 71 *the equation can have no solution whose rank is less than the rank of* $[c]_m^n$.

7. *Infinite solutions of the equation* $AXB = C$, *where* $C \neq 0$.

Any infinite matrix $k[X]_\rho^\sigma$, where k is an infinite scalar quantity and $[X]_\rho^\sigma$ is a finite non-zero matrix, satisfies the given equation (1) when and only when $[a]_m^\rho [X]_\rho^\sigma [b]_\sigma^n = 0$. Conversely if $[X]_\rho^\sigma$ is any finite non-zero matrix such that $[a]_m^\rho [X]_\rho^\sigma [b]_\sigma^n = 0$, then $k[X]_\rho^\sigma$ satisfies the given equation when and only when k is infinite. We have therefore the following theorem:

Theorem VIII. *The infinite solutions of the equation* $[a]_m^\rho [x]_\rho^\sigma [b]_\sigma^n = [c]_m^n$ *are given by* $[x]_\rho^\sigma = k[X]_\rho^\sigma$, *where* k *is an infinite scalar quantity and* $[X]_\rho^\sigma$ *is a finite non-zero solution of the equation* $[a]_m^\rho [x]_\rho^\sigma [b]_\sigma^n = 0$.

We know by Theorem VII that the equation $[a]_m^\rho [x]_\rho^\sigma [b]_\sigma^n = 0$ has finite non-zero solutions when and only when either the rank of $[a]_m^\rho$ is less than ρ or the rank of $[b]_\sigma^n$ is less than σ. Accordingly we have the following theorem:

Theorem IX. *The equation $AXB = C$ has infinite solutions when and only when one of the unaugmented matrices A, B has rank less than its passivity in the product AXB.*

Denoting the ranks of A, B, A', B' by r, s, r', s', we can classify the solutions of the equation $[a]^{\rho}_{m}[x]^{\sigma}_{\rho}[b]^{n}_{\sigma} = [c]^{n}_{m}$ as follows:

CASE I. $r = r'$, $s = s'$. There are finite solutions.

(1) If $r = r' = \rho$, and $s = s' = \sigma$, there is a unique finite solution.

(2) If $r < \rho$ or $s < \sigma$, there are both finite and infinite solutions.

CASE II. $r < r'$ or $s < s'$. There are no finite solutions.

(1) If $r = \rho$, $s = \sigma$, there is no solution of any kind.

(2) If $r < \rho$ or $s < \sigma$, there are infinite solutions.

§ 84. Cancellation of matrix factors in a matrix equation.

The first three of the following results have been proved in §§ 81—83; and the remaining results are immediately deducible from them. All products are supposed to be in standard form.

(1) *The equation $AX = 0$ leads to $X = 0$ as a necessary consequence when and only when the rank of A is equal to its passivity in the product AX.*

(2) *The equation $XB = 0$ leads to $X = 0$ as a necessary consequence when and only when the rank of B is equal to its passivity in the product XB.*

(3) *The equation $AXB = 0$ leads to $X = 0$ as a necessary consequence when and only when the ranks of A and B are equal to their passivities in the product AXB.*

(4) *The equation $AX = AY$ leads to $X = Y$ as a necessary consequence when and only when the rank of A is equal to its passivity in the products AX, AY.*

This follows from (1) when we write the equation in the form

$$A(X - Y) = 0.$$

(5) *The equation $XB = YB$ leads to $X = Y$ as a necessary consequence when and only when the rank of B is equal to its passivity in the products XB, YB.*

This follows from (2) when we write the equation in the form

$$(X - Y)B = 0.$$

(6) *The equation $AXB = AYB$ leads to $X = Y$ as a necessary consequence when and only when the ranks of A and B are equal to their passivities in the products AXB, AYB.*

This follows from (3) when we write the equation in the form

$$A(X - Y)B = 0.$$

Ex. If $[abc]_{1234}$ has rank 3, and if

$$[abc]_{1234}[xy]_{123} = [abc]_{1234}[pqrs]_{123},$$

then

$$[xy]_{123} = [pqrs]_{123},$$

i.e.

$$[rs]_{123} = 0, \quad \text{and} \quad [xy]_{123} = [pq]_{123}.$$

§ 85. Cancellation of matrix factors in a matrix identity.

An equation which involves certain constants and certain variables and is true for all values of the variables is called an *identity*.

In the following matrix identities variable elements will be denoted by the letters x, y, z with suffixes, and constant elements by earlier letters of the alphabet with suffixes. Similarly X, Y, Z will denote matrices whose elements are independent variables, and A, B, C, ... will denote matrices whose elements are constants.

(1) *Any identity of the form $AX = C$ leads to $A = 0$, $C = 0$ as necessary consequences. In particular the identity $AX = 0$ leads to $A = 0$.*

If the equation $[a]_m^\rho [x]_\rho^n = [c]_m^n$ is identically true, then the equation

$$a_{i1}x_{1j} + a_{i2}x_{2j} + \ldots + a_{i\rho}x_{\rho j} = c_{ij}$$

is identically true. We have therefore $a_{i1} = a_{i2} = \ldots = a_{i\rho} = 0$, and $c_{ij} = 0$. Similarly every element of $[a]_m^\rho$ and every element of $[c]_m^n$ must vanish. Accordingly the given matrix equation can only be identically true when $[a]_m^\rho = 0$ and $[c]_m^n = 0$.

Corollary. *Any identity of the form $AX = BX + C$ leads to $A = B$ and $C = 0$. In particular the identity $AX = BX$ leads to $A = B$.*

This is deduced from the above theorem by writing the identity in the form

$$(A - B)X = C.$$

Ex. i. The identity $[a]_m^n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$ leads to $[a]_m^n = 0$.

Ex. ii. The identity $[a]_m^{n+1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = 0$ leads to $[a]_m^{n+1} = 0$.

For it is equivalent to

$$[a]_m^n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} a_{1,\,n+1} \\ a_{2,\,n+1} \\ \vdots \\ a_{m,\,n+1} \end{bmatrix} = 0.$$

Ex. iii. The identity $[a]_m^{n+1} \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1r} \\ x_{21} & x_{22} & \dots & x_{2r} \\ \vdots & & & \\ x_{n1} & x_{n2} & \dots & x_{nr} \\ 1 & 1 & \dots & 1 \end{bmatrix} = 0$ leads to $[a]_m^{n+1} = 0$.

Using the properties of active rows we can deduce this from Ex. ii.

(2) *Any identity of the form $XB = C$ leads to $B = 0$, $C = 0$ as necessary consequences. In particular the identity $XB = 0$ leads to $B = 0$.*

The proof is similar to the proof of (1).

Corollary. *Any identity of the form $XA = XB + C$ leads to $A = B$, $C = 0$. In particular the identity $XA = XB$ leads to $A = B$.*

(3) *Any identity of the form $XAY = B$ leads to $A = 0$, $B = 0$ as necessary consequences. In particular the identity $XAY = 0$ leads to $A = 0$.*

For from (1) it follows that $B = 0$ and that $XA = 0$ for all values of X. From the second of these results it follows by (2) that $A = 0$.

Corollary. *Any identity of the form $XAY = XBY + C$ leads to $A = B$, $C = 0$. In particular the identity $XAY = XBY$ leads to $A = B$.*

To prove this we write the identity in the form $X(A - B)Y = C$, and this leads to $A - B = 0$, $C = 0$.

Ex. iv. The identity

$$[x_1 \; x_2 \dots x_m]\,[a]_m^n \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [x_1 \; x_2 \dots x_m]\,[b]_m^n \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

leads to

$$[a]_m^n = [b]_m^n.$$

(4) *The identity $[a]_m^{r+s} \begin{bmatrix} x \\ c \end{bmatrix}_{r,s}^n = 0$, in which $[c]_s^n$ is known to have rank s, leads to $[a]_m^{r+s} = 0$.*

The identity $[x, c]_m^{r,s}\,[b]_{r+s}^n = 0$, in which $[c]_m^s$ is known to have rank s, leads to $[b]_{r+s}^n = 0$.

The identity $[x, a]_m^{p,q}[c]_{p+q}^{r+s} \begin{bmatrix} y \\ b \end{bmatrix}_{r,s}^n = 0$, in which $[a]_m^q$, $[b]_s^n$ are known to have ranks q and s respectively, leads to $[c]_{p+q}^{r+s} = 0$.

To prove the first result we will consider the identity $[a, b]_m^{r,s} \begin{bmatrix} x \\ c \end{bmatrix}_{r,s}^n = 0$. This can be written $[a]_m^r [x]_r^n + [b]_m^s [c]_s^n = 0$.

By (1) it therefore leads to $[a]_m^r = 0$, $[b]_m^s [c]_s^n = 0$.

If $[c]_s^n$ is known to have rank s, the last result shows that $[b]_m^s = 0$. Since $[a]_m^r = 0$ and $[b]_m^s = 0$, we have $[a, b]_m^{r,s} = 0$. This proves the first result.

The second result can be proved similarly by considering the identity

$$[x, c]_m^{r,s} \begin{bmatrix} a \\ b \end{bmatrix}_{r+s}^n = 0.$$

From the third identity we deduce by the first result that $[x, a]_m^{p,q} [c]_{p+q}^{r+s} = 0$ for all values of $[x]_m^p$, and from this it follows that $[c]_{p+q}^{r+s} = 0$.

From these three results we deduce the following three corollaries.

The identity $[a]_m^{r+s} \begin{bmatrix} x \\ c \end{bmatrix}_{r,s}^n = [b]_m^{r+s} \begin{bmatrix} x \\ c \end{bmatrix}_{r,s}^n$, *in which* $[c]_s^n$ *has rank* s, *leads to* $[a]_m^{r+s} = [b]_m^{r+s}$.

The identity $[x, c]_m^{r,s} [a]_{r+s}^n = [x, c]_m^{r,s} [b]_{r+s}^n$, *in which* $[c]_m^s$ *has rank* s, *leads to* $[a]_{r+s}^n = [b]_{r+s}^n$.

The identity $[x, a]_m^{p,q} [\alpha]_{p+q}^{r+s} \begin{bmatrix} y \\ b \end{bmatrix}_{r,s}^n = [x, a]_m^{p,q} [\beta]_{p+q}^{r+s} \begin{bmatrix} y \\ b \end{bmatrix}_{r,s}^n$, *in which* $[a]_m^q$, $[b]_s^n$ *have ranks* q, s, *leads to* $[\alpha]_{p+q}^{r+s} = [\beta]_{p+q}^{r+s}$.

Ex. v. Each of the identities

$$[x_1\ x_2 \dots x_m\ 1]\ [a]_{m+1}^{n+1} = 0,$$

$$[a]_{m+1}^{n+1} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ 1 \end{bmatrix} = 0, \qquad [x_1\ x_2 \dots x_m\ 1]\ [a]_{m+1}^{n+1} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ 1 \end{bmatrix} = 0,$$

leads to $[a]_{m+1}^{n+1} = 0$.

Ex. vi The identity $\begin{bmatrix} x_1 & y_1 & z_1 & 1 & 0 & 0 \\ x_2 & y_2 & z_2 & 0 & 1 & 0 \\ x_3 & y_3 & z_3 & 0 & 0 & 1 \end{bmatrix} [a]_6^4 = 0$ leads to $[a]_6^4 = 0$.

Ex. vii. The identity $[x_1\ x_2\ x_3\ 1]\ [a]_4^3 \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} = 0$ leads to $[a]_4^3 = 0$.

(5) *Any identity of the form* $[x]_m^\rho [a]_\rho^\sigma [x]_\sigma^n = [c]_m^n$ *leads to* $[a]_\rho^\sigma = 0$, $[c]_m^n = 0$ *as necessary consequences. In particular the identity* $[x]_m^\rho [a]_\rho^\sigma [x]_\sigma^n = 0$ *leads to* $[a]_\rho^\sigma = 0$.

By equating corresponding elements on both sides we obtain identities of the type

$$\sum_u \sum_v x_{iu} a_{uv} x_{vj} = c_{ij}.$$

Since there are no two terms on the left which contain the same product $x_{iu} x_{vj}$, it follows that $a_{uv} = 0$, $c_{ij} = 0$.

(6) *If* $[a]_m^m$ *is a self-conjugate square matrix of order m, the identity*

$$[x_1\ x_2 \ldots x_m]\ [a]_m^m \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = [c] \qquad \textit{leads to} \qquad [a]_m^m = 0,\quad c = 0.$$

Here the product on the left is equal to the matrix $[c]$, where $c = \Sigma a_{ii} x_i^2 + 2\Sigma a_{ij} x_i x_j$, the first summation extending over all values of i from 1 to m, and the second summation over all corranged minor sequences $[ij]$ of order 2 of $[1\ 2 \ldots m]$. Thus the given identity is equivalent to the identity

$$\Sigma a_{ii} x_i^2 + 2\Sigma a_{ij} x_i x_j = c.$$

This can only be true when

$$a_{ii} = 0, \qquad a_{ij} = 0, \qquad c = 0.$$

Ex. viii. The identity

$$[x\ y\ z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \text{leads to} \qquad \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = 0.$$

Ex. ix. If $[a]_m^m$ is a self-conjugate square matrix, the identity

$$[x]_p^m\ [a]_m^m\ \overline{x}_m^p = 0 \quad \text{leads to} \quad [a]_m^m = 0.$$

(7) *If* $[a]_{m+1}^{m+1}$ *is a self-conjugate square matrix of order* $m+1$, *the identity*

$$[x_1\ x_2 \ldots x_m\ \ 1]\ [a]_{m+1}^{m+1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ 1 \end{bmatrix} = 0 \qquad \textit{leads to} \qquad [a]_{m+1}^{m+1} = 0.$$

The above identity is equivalent to the identity

$$\Sigma a_{ii} x_i^2 + 2\Sigma a_{ij} x_i x_j + 2\Sigma a_{i,m+1} x_i + a_{m+1,m+1} = 0,$$

the first and third summations extending over all values of i from 1 to m, and the second summation over all corranged minor sequences $[ij]$ of order 2 of $[1\ 2 \ldots m]$. This can only be true when every element of the matrix $[a]_{m+1}^{m+1}$ vanishes.

Ex. x. The identity

$$[x\ \ y\ \ z\ \ 1]\begin{bmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0 \quad \text{leads to} \quad \begin{bmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{bmatrix} = 0.$$

Ex. xi. The identity

$$[x_1\ x_2 \ldots x_m\ \ 1]\ [a]_{m+1}^{m+1}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ 1 \end{bmatrix} = 0 \quad \text{leads to} \quad [a]_{m+1}^{m+1} = 0.$$

This is a particular case of (4).

Ex. xii. *If* $[a]_{m+r}^{m+r}$ *is a self-conjugate square matrix, the identity*

$$\begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1r} & 1 & 0 & \ldots & 0 \\ x_{21} & x_{22} & \ldots & x_{2r} & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \ldots & x_{mr} & 0 & 0 & \ldots & 1 \end{bmatrix}[a]_{m+r}^{m+r}\begin{bmatrix} x_{11} & x_{21} & \ldots & x_{m1} \\ x_{12} & x_{22} & \ldots & x_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ x_{1r} & x_{2r} & \ldots & x_{mr} \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} = 0 \quad \textit{leads to} \quad [a]_{m+r}^{m+r} = 0.$$

For if we denote the product matrix on the left by $[c]_m^m$, then by the theorem (7) and by Ex. xi the identity obtained by equating c_{ij} to zero shows that

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1r} & a_{1,\,r+j} \\ a_{21} & a_{22} & \ldots & a_{2r} & a_{2,\,r+j} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r2} & \ldots & a_{rr} & a_{r,\,r+j} \\ a_{r+i,\,1} & a_{r+i,\,2} & \ldots & a_{r+i,\,r} & a_{r+i,\,r+j} \end{bmatrix} = 0.$$

Since this is true for all values of i from 1 to m and all values of j from 1 to m, it follows that every element of the matrix $[a]_{m+r}^{m+r}$ is equal to zero.

Ex. xiii. *If* $[c]_{m+r}^{m+r}$ *is a self-conjugate matrix and* $[a]_r^r$ *has rank* r, *the identity*

$$[x,a]_r^{m,\,r}\ [c]_{m+r}^{m+r}\begin{bmatrix} x \\ a \end{bmatrix}_{m,\,r}^{r} = 0 \quad \textit{leads to} \quad [c]_{m+r}^{m+r} = 0.$$

For if we prefix the inverse of $[a]_r^r$ and postfix its conjugate on both sides, we obtain an identity of the form considered in Ex. xii.

(8) *If* $[c]_{n+r}^{n+r}$ *is a self-conjugate square matrix and* $[a]_m^r$ *has rank* r, *the identity*

$$[x,a]_m^{n,\,r}\ [c]_{n+r}^{n+r}\begin{bmatrix} x \\ a \end{bmatrix}_{n,\,r}^{m} = 0 \quad \textit{leads to} \quad [c]_{n+r}^{n+r} = 0.$$

For by striking out active rows we can deduce an identity of the form considered in Ex. xiii. The following corollary can be immediately deduced from this theorem.

If $[c]_{n+r}^{n+r}$ *is a self-conjugate square matrix and* $[u]_m^r$ *has rank* r, *the identity*

$$[x, u]_m^{n,r}\,[a]_{n+r}^{n+r}\begin{bmatrix} x \\ u \end{bmatrix}_{n,r}^{m} = [x, u]_m^{n,r}\,[\beta]_{n+r}^{n+r}\begin{bmatrix} x \\ u \end{bmatrix}_{n,r}^{m} \quad \textit{leads to} \quad [a]_{n+r}^{n+r} = [\beta]_{n+r}^{n+r}.$$

Ex. xiv. The identity

$$[x\ y\ z\ 1]\begin{bmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = [x\ y\ z\ 1]\begin{bmatrix} a' & h' & g' & u' \\ h' & b' & f' & v' \\ g' & f' & c' & w' \\ u' & v' & w' & d' \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

leads to

$$\begin{bmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{bmatrix} = \begin{bmatrix} a' & h' & g' & u' \\ h' & b' & f' & v' \\ g' & f' & c' & w' \\ u' & v' & w' & d' \end{bmatrix}.$$

This can be seen directly since the identity is equivalent to

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d$$
$$= a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy + 2u'x + 2v'y + 2w'z + d'.$$

CHAPTER XI.

SOLUTION OF ANY SYSTEM OF LINEAR ALGEBRAIC EQUATIONS.

[Connections between linear functions and between linear equations are first considered in § 86. In §§ 87—89 any system of linear algebraic equations is replaced by a single equivalent matrix equation, and the general solution of the system is then obtained by the methods of Chapter x. In §§ 90—93 complete sets of unconnected solutions of any system of linear algebraic equations are defined, and by means of them a symmetrical form is given to the general solution. In the two following articles, §§ 94 and 95, properties of mutually orthogonal solutions are considered; in particular it is shown that a system of linear equations whose matrix is real has always a complete set of unconnected mutually orthogonal real solutions. In § 96 another interpretation is given to connections between linear equations. Finally § 97 deals with the conditions for the existence of functional dependences between algebraic functions.]

§ 86. Connections between linear functions and between linear equations.

Let

$$\left.\begin{array}{l} f_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + a_{1,n+1} = 0 \\ f_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + a_{2,n+1} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f_m = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + a_{m,n+1} = 0 \end{array}\right\} \quad \dots\dots\dots\dots(1)$$

be a system of m linear equations in the n variables $x_1, x_2, \dots x_n$; so that $f_1, f_2, \dots f_m$ are a system of m linear functions of the same variables.

There is said to be a *connection between* the m linear functions $f_1, f_2, \dots f_m$ and also a *connection between* the m linear equations $f_1 = 0, f_2 = 0, \dots f_m = 0$ when there exist *constants* $h_1, h_2, \dots h_m$, not all zero, such that

$$h_1 f_1 + h_2 f_2 + \dots + h_m f_m = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(2)$$

identically, i.e. when there is a connection between the horizontal rows of the matrix

$$[a]_m^{n+1} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n+1} \\ a_{21} & a_{22} & \dots & a_{2,n+1} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{m,n+1} \end{bmatrix}$$

of the form

$$[h_1 h_2 \dots h_m][a]_m^{n+1} = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(3).$$

If there exists a relation of the form (2) or (3) in which $h_i \neq 0$, then the ith function f_i is said to be *connected with* the other functions, and also the ith equation $f_i = 0$ is said to be *connected with* the other equations. This case occurs when the ith horizontal row of the matrix $[a]_m^{n+1}$ is connected with the other horizontal rows.

If no such relation as (2) or (3) exists, then the m functions $f_1, f_2, \ldots f_m$ are said to be *unconnected*, and also the m equations $f_1 = 0, f_2 = 0, \ldots f_m = 0$ are said to be *unconnected*. This case occurs when the horizontal rows of the matrix $[a]_m^{n+1}$ are unconnected.

A set of values of $x_1, x_2, \ldots x_n$ which are all finite and which satisfy all the equations (1) constitute a *finite solution* of the system of equations, or a finite common solution of all the m equations (1). When at least one such set of values exists, the system of equations will be said to *admit of finite solution*. In this case it will sometimes be said for the sake of brevity that the m equations themselves admit of finite solution, this being understood to mean that they have a finite *common* solution.

Note. It will be shown in § 96 that if the system of equations (1) admits of finite solution, then there are connections between the m equations when and only when some of the equations are redundant, being necessary consequences of the rest. In determining the common solutions of the m equations these redundant equations can be omitted.

§ 87. Solution of any system of linear equations. First treatment.

The m scalar equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_m \end{array}\right\} \qquad \ldots\ldots\ldots\ldots(1)$$

are together equivalent to the single matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}, \text{ or } [a]_m^n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \qquad \ldots\ldots\ldots\ldots(2).$$

An equivalent form of this matrix equation is

$$[x_1 x_2 \ldots x_n]\,\overline{\underline{a}}_n^m = [c_1 c_2 \ldots c_m] \qquad \ldots\ldots\ldots\ldots(3).$$

Abbreviated modes of writing (2) and (3) are

$$[a]_m^n \,\overline{\underline{x}}_n = \overline{\underline{c}}_m, \qquad [x]_n\,\overline{\underline{a}}_n^m = [c]_m \qquad \ldots\ldots\ldots\ldots(4).$$

Since there is a one-one correspondence between the solutions of the system of equations (1) and the solutions of the matrix equation (2), any matrix $\underset{n}{\overline{x}}$ satisfying the equation (2) may be called a solution of the system of equations (1), and the most general value of $\underset{n}{\overline{x}}$ satisfying (2) may be called the general solution of the system of equations (1). Further the system of equations (1) admits of finite solution when and only when the matrix equation (2) admits of finite solution.

The matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad A' = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & c_1 \\ a_{21} & a_{22} & \dots & a_{2n} & c_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & c_m \end{bmatrix},$$

which are respectively the unaugmented and the augmented matrices of the matrix equation (2) will also be called the *unaugmented* and *augmented matrices* of the system of equations (1).

Regarding (2) as a special equation of the form $AX = C$, we can deduce its solution from the results of § 81. Applying the theorems of § 81 to this particular case we have in the first instance the following results:

(i) *If any one of the m equations (1) is connected with the remaining equations, it is redundant and can be omitted when we are determining the common solutions of the m equations. Accordingly if A' has rank r, so that r and not more than r of the equations are unconnected, we can replace the system of equations (1) by a system composed of r of their number which are unconnected. The solutions of the reduced system are identical with the solutions of the original system; and the matrices A, A', C have the same ranks for the reduced system as for the original system.*

(ii) *The system of equations (1) admits of finite solution when and only when the rank of A' is equal to the rank of A.*

(iii) *The system of equations (1) has infinite solutions when and only when the rank of A is less than n.*

(iv) *The system of equations (1) has solutions finite or infinite when and only when the rank of A' does not exceed n, i.e. when the number of unconnected equations does not exceed the number of variables. If the rank of A' exceeds n, i.e. is equal to $n+1$, then there are no solutions, neither finite solutions nor infinite solutions.*

(v) *When A and A' have common rank r, so that the system of equations admits of finite solution, the general solution expresses r of the unknowns $x_1, x_2, \dots x_n$ as linear functions of the remaining unknowns to which* *arbitrary values may be assigned. In the special case in which $r = n$, so that the number of unconnected equations is equal to the number of variables, there is a unique finite solution and no infinite solution.*

(vi) *The infinite solutions are given by*

$$[x_1\, x_2 \ldots x_n] = k\,[X_1\, X_2 \ldots X_n],$$

where k is an infinite scalar quantity, and $X_1, X_2, \ldots X_n$ are finite quantities, not all zero, which satisfy the system of homogeneous equations obtained from (1) by putting $c_1 = c_2 = \ldots = c_m = 0$.

To prove (iv) we observe that the rank of A' must always either be equal to the rank of A or exceed the rank of A by 1. If A' has rank $n+1$, then A has rank n, and therefore there are no finite solutions and no infinite solutions. On the other hand when the rank of $A' = r < n+1$, then if the rank of $A = r$, there are certainly finite solutions, and if the rank of $A = r - 1 < n$, there are certainly infinite solutions.

When there are only infinite solutions, we obtain them by solving a system of homogeneous equations as in § 89.

We will proceed to find the general solution when finite solutions exist.

CASE I. *Let A and A' have common rank r, and let $(a)_r^r = \alpha \neq 0$.*

Every case in which A and A' have common rank r can be reduced to this, for the equations (1) and the variables occurring in them can in every case be so arranged that $(a)_r^r$ is one of the non-vanishing derived determinants of order r of $[a]_m^n$.

The system of equations (1) can now be replaced by the system of r unconnected equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = c_r \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5);$$

or by the irreducible matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{r1} & a_{r2} & \ldots & a_{rn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

in which $$n \nless r.$$

By the properties of passive rows equation (6) can be written

$$[a]_r^r \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} - \begin{bmatrix} a_{1,r+1} & a_{1,r+2} & \cdots & a_{1n} \\ a_{2,r+1} & a_{2,r+2} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r,r+1} & a_{r,r+2} & \cdots & a_{rn} \end{bmatrix} \begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_r \end{bmatrix} \quad \ldots\ldots(7),$$

where $\xi_i = c_i - a_{i,r+1} x_{r+1} - a_{i,r+2} x_{r+2} - \ldots - a_{in} x_n$.

We now obtain the general solution by prefixing $\overline{\underline{A}}{}_r^r$, the conjugate reciprocal of $[a]_r^r$, on both sides of (7); for by § 84 the equation obtained when this is done is equivalent to (7). The result is as follows:

First form of the general solution.

The general solution of the system of equations (1) *is given in Case I by*

$$\alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = \overline{\underline{A}}{}_r^r \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} - \overline{\underline{A}}{}_r^r \begin{bmatrix} a_{1,r+1} & a_{1,r+2} & \cdots & a_{1n} \\ a_{2,r+1} & a_{2,r+2} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r,r+1} & a_{r,r+2} & \cdots & a_{rn} \end{bmatrix} \begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix},$$

or

$$\alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_r \end{bmatrix} - \begin{bmatrix} \alpha_{1,r+1} & \alpha_{1,r+2} & \cdots & \alpha_{1n} \\ \alpha_{2,r+1} & \alpha_{2,r+2} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{r,r+1} & \alpha_{r,r+2} & \cdots & \alpha_{rn} \end{bmatrix} \begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix},$$

where γ_i *is the value of the determinant formed when the ith vertical row of* $(a)_r^r$ *is replaced by the vertical row of* $\overline{\underline{c}}_r$, *and* α_{ij} *is the value of the determinant formed when the ith vertical row of* $(a)_r^r$ *is replaced by the jth vertical row of* $[a]_r^n$ ....(A).

This general solution expresses the r unknowns $x_1, x_2, \ldots x_r$ as unique linear functions of the remaining $n-r$ unknowns to which arbitrary values can be assigned.

If x_i is any one of the elements $x_1, x_2, \ldots x_r$, we have

$$\alpha x_i = \gamma_i - \alpha_{i,r+1} x_{r+1} - \alpha_{i,r+2} x_{r+2} - \ldots - \alpha_{in} x_n \quad \ldots\ldots\ldots\ldots(8).$$

Ex. i. *The simplest particular solution in Case I.*

This is obtained by assigning zero values to all the arbitrary elements and is given by

$$a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = \overline{\underline{A}}{}_r^r \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_r \end{bmatrix}, \qquad [x_{r+1} \; x_{r+2} \ldots x_n] = 0 \quad \ldots\ldots\ldots\ldots(9).$$

This particular solution corresponds to the unique solution of the equation

$$[a]_r^r \, \overline{\underline{x}}_r = \overline{\underline{c}}_r .$$

Ex. ii. *Alternative method of solution in Case I.*

We can also obtain the general solution directly from the reduced equation (6) by prefixing the matrix $\overline{\underline{A}}^r_r$ on both sides. This gives

$$[a]^n_r \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_r \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a & 0 \ldots 0 & a_{1,r+1} & a_{1,r+2} \ldots a_{1n} \\ 0 & a \ldots 0 & a_{2,r+1} & a_{2,r+2} \ldots a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 \ldots a & a_{r,r+1} & a_{r,r+2} \ldots a_{rn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_r \end{bmatrix} \quad \text{.........(A')},$$

where γ_i, a_{ij} have the same meanings as in (A).

It will be observed that since

$$[a]^n_r = \overline{\underline{A}}^r_r \, [a]^n_r, \qquad \text{therefore} \qquad [a]^r_r = \overline{\underline{A}}^r_r \, [a]^r_r = a\,[1]^r_r \quad \text{...........(10)}.$$

By the properties of passive rows formula (A′) is equivalent to the second equation in (A).

If in formula (A′) we equate each of the r elements of the product matrix on the left to the corresponding element of the matrix on the right, we obtain r equations each of which determines one of the unknowns $x_1, x_2, \ldots x_r$ in terms of the remaining unknowns.

Case II. *Let A and A' have common rank r, and let $(a_{pq})^r_r = \alpha \neq 0$.*

This is the most general case possible when the given system of equations admits of finite solution.

Using the notation of § 2.7, the given system of equations (1) is equivalent to the matrix equation

$$[a]^n_m \, \overline{\underline{x}}_n = \overline{\underline{c}}_m \,.$$

This again can be replaced by the irreducible equation

$$[a_{p1}]^n_r \, \overline{\underline{x}}_n = \overline{\underline{c_p}}_r \quad \text{..........................(11)},$$

where $n \nless r$; moreover by the properties of passive rows (11) can be written in the form

$$[a_{pq}]^r_r \, \overline{\underline{x_q}}_r = \overline{\underline{c_p}}_r - [a_{p\tau}]^{n-r}_r \, \overline{\underline{x_\tau}}_{n-r} \quad \text{..................(12)},$$

where $[\tau_1\, \tau_2 \ldots \tau_{n-r}]$ is complementary to $[q_1\, q_2 \ldots q_r]$ in $[1\ 2 \ldots n]$.

As in § 81 we obtain the general solution of (12) by prefixing $\overline{\underline{A_{pq}}}^r_r$, the conjugate reciprocal of $[a_{pq}]^r_r$, on both sides. The result is as follows:

Second form of the general solution.

The general solution of the system of equations (1) *is given in Case II by*

$$\alpha \, \overline{\underline{x_q}}_r = \overline{\underline{A_{pq}}}^r_r \, \overline{\underline{c_p}}_r - \overline{\underline{A_{pq}}}^r_r \, [a_{p\tau}]^{n-r}_r \, \overline{\underline{x_\tau}}_{n-r} \,,$$

or $$\alpha\, \overline{\underline{x_q}}_r = \overline{\underline{\gamma}}_r - [\alpha_{1\tau}]_r^{n-r}\, \overline{\underline{x_\tau}}_{n-r},$$

where γ_i is the value of the determinant formed when the ith vertical row of $(a_{pq})_r^r$ is replaced by the vertical row of $\overline{\underline{c_p}}_r$, and α_{ij} is the value of the determinant formed when the ith vertical row of $(a_{pq})_r^r$ is replaced by the jth vertical row of $[a_{p1}]_r^n$ ....(B).

This general solution expresses the r unknowns $x_{q_1}, x_{q_2}, \ldots x_{q_r}$ as unique linear functions of the remaining $n-r$ unknowns $x_{\tau_1}, x_{\tau_2}, \ldots x_{\tau_{n-r}}$ to which arbitrary values may be assigned.

If x_{q_i} is any one of the elements $x_{q_1}, x_{q_2}, \ldots x_{q_r}$, we have

$$\alpha x_{q_i} = \gamma_i - \alpha_{i\tau_1} x_{\tau_1} - \alpha_{i\tau_2} x_{\tau_2} - \ldots - \alpha_{i\tau_{n-r}} x_{\tau_{n-r}} \quad \ldots\ldots\ldots\ldots(13).$$

Ex. iii. *The simplest particular solution in Case II.*

This is obtained by assigning zero values to all the arbitrary elements and is given by

$$a\, \overline{\underline{x_q}}_r = \overline{\underline{A_{pq}}}_r^r\, \overline{\underline{c_p}}_r = \overline{\underline{\gamma}}_r, \quad \overline{\underline{x_\tau}}_{n-r} = 0 \quad \ldots\ldots\ldots\ldots(14),$$

or $$a\,[x_q]_r = [c_p]_r\, [A_{pq}]_r^r = [\gamma]_r, \quad [x_\tau]_{n-r} = 0.$$

This particular solution corresponds to the unique solution of the equation

$$[a_{pq}]_r^r\, \overline{\underline{x_q}}_r = \overline{\underline{c_p}}_r.$$

Ex. iv. *Alternative method of solution in Case II.*

We can also obtain the general solution directly from the reduced equation (11) by prefixing the matrix $\overline{\underline{A_{pq}}}_r^r$ on both sides. This gives

$$[a]_r^n\, \overline{\underline{x}}_n = \overline{\underline{\gamma}}_r \quad \ldots\ldots\ldots\ldots(B'),$$

where γ_i, a_{ij} have the same meanings as in (B).

Since $[a]_r^n = \overline{\underline{A_{pq}}}_r^r\, [a_{p1}]_r^n$, we have by the properties of active rows

$$[a_{1q}]_r^r = \overline{\underline{A_{pq}}}_r^r\, [a_{pq}]_r^r = a\,[1]_r^r \quad \ldots\ldots\ldots\ldots(15);$$

i.e. $a_{iq_j} = 0$ or a according as $j \neq i$ or $j = i$.

Thus if in (B′) we equate each of the r elements of the product matrix on the left to the corresponding element of the matrix on the right, we obtain r equations each of which expresses one of the quantities $x_{q_1}, x_{q_2}, \ldots x_{q_r}$ in terms of the remaining $n-r$ unknowns.

In fact when we use (15) we can by the properties of passive rows transform (B′) into the second equation in (A).

In solving any system of linear equations whatever, we commence by replacing the system by an equivalent system of unconnected equations. Hence it would be sufficient to consider the solution of any system of *unconnected* linear equations.

If $$[a]_r^n \, \overline{\underline{x}}_n = \overline{\underline{c}}_r \qquad \text{..............................(16)}$$

is the irreducible matrix equation equivalent to any system of unconnected linear equations, the procedure followed in finding the general solution of (16) can be briefly described as follows.

If $(a)_r^r \neq 0$, we use the properties of passive rows to re-write (16) in the form

$$[a]_r^r \, \overline{\underline{x}}_r = \overline{\underline{\xi}}_r \qquad \text{..............................(17)},$$

and then solve (17) for $\overline{\underline{x}}_r$ as in § 78 by prefixing the conjugate reciprocal matrix $\overline{\underline{A}}_r^r$. As an alternative we can obtain the general solution by prefixing $\overline{\underline{A}}_r^r$ in equation (16).

If $(a_{1q})_r^r \neq 0$, we use the properties of passive rows to re-write (16) in the form

$$[a_{1q}]_r^r \, \overline{\underline{x_q}}_r = \overline{\underline{\xi}}_r \qquad \text{..........................(18)},$$

and then solve (18) for $\overline{\underline{x_q}}_r$ as in § 78 by prefixing the conjugate reciprocal matrix $\overline{\underline{A_{1q}}}_r^r$. As an alternative we can obtain the general solution by prefixing $\overline{\underline{A_{1q}}}_r^r$ in equation (16).

In the second or general case the unknown elements for which we solve are those whose coefficients in the given equations are contained in the non-vanishing determinant $(a_{1q})_r^r$.

Ex. v. In solving the system of equations (1), we may use the second form of the equivalent matrix equation, viz.

$$[x_1 x_2 \ldots x_n] \, \overline{\underline{a}}_n^m = [c_1 c_2 \ldots c_m], \qquad \text{or} \qquad [x]_n \, \overline{\underline{a}}_n^m = [c]_m \qquad \text{............(3)}.$$

If $(a)_r^r \neq 0$, we can replace (3) by the irreducible equation

$$[x_1 x_2 \ldots x_n] \, \overline{\underline{a}}_n^r = [c_1 c_2 \ldots c_r], \qquad \text{or} \qquad [x]_n \, \overline{\underline{a}}_n^r = [c]_r,$$

and can further write the last equation in the form

$$[x_1 x_2 \dots x_r] \overline{\underline{a}}_r^r = [\xi_1 \xi_2 \dots \xi_r], \qquad \text{or} \qquad [x]_r \overline{\underline{a}}_r^r = [\xi]_r.$$

To solve, we *postfix* $[A]_r^r$ in either of the last two equations.

If $(a_{pq})_r^r \neq 0$, we can replace (3) by the irreducible equation

$$[x_1 x_2 \dots x_n] \overline{\underline{a_{p1}}}_n^r = [e_{p_1} e_{p_2} \dots e_{p_r}], \qquad \text{or} \qquad [x]_n \overline{\underline{a_{p1}}}_n^r = [e_p]_r,$$

and can further write the last equation in the form

$$[x_{q_1} x_{q_2} \dots x_{q_r}] \overline{\underline{a_{pq}}}_r^r = [\xi_1 \xi_2 \dots \xi_r], \qquad \text{or} \qquad [x_q]_r \overline{\underline{a_{pq}}}_r^r = [\xi]_r.$$

To solve, we *postfix* $[A_{pq}]_r^r$ in either of the last two equations.

Ex. vi. To solve the system of four unconnected scalar equations

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1w=e_1 \\ a_2x+b_2y+c_2z+d_2w=e_2 \\ a_3x+b_3y+c_3z+d_3w=e_3 \\ a_4x+b_4y+c_4z+d_4w=e_4 \end{aligned}\right\} \qquad \text{or} \qquad \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix},$$

where $(abcd)_{1234}=(abcd)\neq 0$, we denote the reciprocal matrix of $[abcd]_{1234}$ by $[ABCD]_{1234}$, and have

$$\begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \\ D_1 & D_2 & D_3 & D_4 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \\ D_1 & D_2 & D_3 & D_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix},$$

i.e. we have

$$\begin{bmatrix} (abcd) & 0 & 0 & 0 \\ 0 & (abcd) & 0 & 0 \\ 0 & 0 & (abcd) & 0 \\ 0 & 0 & 0 & (abcd) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} (ebcd) \\ (aecd) \\ (abed) \\ (abce) \end{bmatrix}, \quad \text{or} \quad (abcd) \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} (ebcd) \\ (aecd) \\ (abed) \\ (abce) \end{bmatrix}.$$

Thus the general solution is given by

$$(abcd)\,x=(ebcd), \qquad (abcd)\,y=(aecd), \qquad (abcd)\,z=(abed), \qquad (abcd)\,w=(abce).$$

In this example the number of unconnected equations is equal to the number of variables. Accordingly there is a unique finite solution and no infinite solution.

Ex. vii. To solve the system of two unconnected scalar equations

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1w=e_1 \\ a_2x+b_2y+c_2z+d_2w=e_2 \end{aligned}\right\} \qquad \text{or} \qquad \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

in which $(bd)_{12} = (bd) \neq 0$, we prefix $\begin{bmatrix} B_1 & B_2 \\ D_1 & D_2 \end{bmatrix}$, the conjugate reciprocal matrix of $\begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix}$, and obtain

$$\begin{bmatrix} B_1 & B_2 \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

or

$$\begin{bmatrix} (ad) & (bd) & (cd) & 0 \\ (ba) & 0 & (bc) & (bd) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} (ed) \\ (be) \end{bmatrix}.$$

Thus the general solution is given by

$$(bd)\,y = (ed) - (ad)\,x - (cd)\,z, \qquad (bd)\,w = (be) - (ba)\,x - (bc)\,z,$$

where x and z are arbitrary.

We may also write the given matrix equation in the form

$$\begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix},$$

and prefix the conjugate reciprocal of $[bd]_{12}$. We then obtain

$$\begin{bmatrix} B_1 & B_2 \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \begin{bmatrix} B_1 & B_2 \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix},$$

or

$$(bd) \begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} (ed) \\ (be) \end{bmatrix} - \begin{bmatrix} (ad) & (cd) \\ (ba) & (bc) \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix},$$

which is equivalent to the previous result.

§ 88. Solution of any system of linear equations. Second treatment.

In general investigations a system of m linear equations in n variables $x_1, x_2, \ldots x_n$ is more conveniently exhibited in the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + a_{1,n+1} &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + a_{2,n+1} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + a_{m,n+1} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1).$$

This system of equations is equivalent to either of the matrix equations

$$[a]_m^{n+1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = 0, \qquad [x_1 x_2 \ldots x_n\ 1]\,\overline{\underline{a}}_{n+1}^{\,m} = 0 \quad \ldots\ldots\ldots(2).$$

In order that we may be able to write these matrix equations still more briefly *we will introduce the convention that*

$$x_{n+1} = 1 \qquad \text{.................................(3).}$$

The equations (2) can then be written in the forms

$$[a]_m^{n+1} \lceil x \rfloor_{n+1} = 0, \qquad [x]_{n+1} \lceil a \rfloor_{n+1}^{m} = 0 \qquad \text{................(4).}$$

We will now regard the matrices $[a]_m^n$, $[a]_m^{n+1}$ as being respectively the *unaugmented matrix* and the *augmented matrix* of the system of equations (1) or of the matrix equations (2). The theorems of § 81 and the results (i)—(vi) of § 87 remain true. For $[a]_m^n$ is the unaugmented matrix A as defined in § 81, and $[a]_m^{n+1}$ has the same rank as the augmented matrix A' defined in § 81, since it is obtained from A' by changing the sign of every element in the last vertical row.

The system of equations (1) admits of finite solution when and only when the rank of $[a]_m^{n+1}$ is equal to the rank of $[a]_m^n$; it admits of infinite solutions when and only when the rank of $[a]_m^n$ is less than n; it admits of solutions of some kind, finite or infinite, when and only when the rank of $[a]_m^{n+1}$ does not exceed n, i.e. when the number of unconnected equations does not exceed the number of variables.

We here give as before the general solution of the system of linear equations (1) when it admits of finite solution. All infinite solutions, whether there are finite solutions or not, can be obtained as in § 89.

CASE I. *Let* $[a]_m^n$ *and* $[a]_m^{n+1}$ *have common rank* r, *and let* $(a)_r^r = \alpha \neq 0$.

The system of equations (1) can be replaced by the irreducible matrix equation of which equivalent forms are

$$[a]_r^{n+1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = 0, \qquad [a]_r^r \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = - \begin{bmatrix} a_{1,r+1} & a_{1,r+2} & \cdots & a_{1,n+1} \\ a_{2,r+1} & a_{2,r+2} & \cdots & a_{2,n+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r,r+1} & a_{r,r+2} & \cdots & a_{r,n+1} \end{bmatrix} \begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \\ 1 \end{bmatrix}$$

..............(5).

Let $[A]_r^r$ be the reciprocal matrix of $[a]_r^r$, and let

$$\lceil A \rfloor_r^r [a]_r^{n+1} = [\alpha]_r^{n+1}, \text{ so that } [\alpha]_r^r = \lceil A \rfloor_r^r [a]_r^r = \alpha [1]_r^r \qquad \text{......(6).}$$

Then prefixing the matrix $\overline{\underline{A}}{}^{r}_{r}$ in equation (5) in either of its forms in order to obtain the general solution, we have the following result:

First form of the general solution.

The general solution of the system of equations (1) *is given in Case I by either of the formulae*

$$[\alpha]^{n+1}_{r}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & \dots & 0 & \alpha_{1,r+1} & \alpha_{1,r+2} & \dots & \alpha_{1,n+1} \\ 0 & \alpha & \dots & 0 & \alpha_{2,r+1} & \alpha_{2,r+2} & \dots & \alpha_{2,n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha & \alpha_{r,r+1} & \alpha_{r,r+2} & \dots & \alpha_{r,n+1} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = 0 \quad \dots(\text{A}'),$$

$$\alpha\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = -\begin{bmatrix} \alpha_{1,r+1} & \alpha_{1,r+2} & \dots & \alpha_{1,n+1} \\ \alpha_{2,r+1} & \alpha_{2,r+2} & \dots & \alpha_{2,n+1} \\ \dots & \dots & \dots & \dots \\ \alpha_{r,r+1} & \alpha_{r,r+2} & \dots & \alpha_{r,n+1} \end{bmatrix}\begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \\ 1 \end{bmatrix} \quad \dots\dots\dots\dots\dots(\text{A}),$$

where α_{ij} is the determinant formed by replacing the ith vertical row of $(a)^r_r$ by the jth vertical row of $[a]^{n+1}_r$.

Formula (A) expresses each of the r variables $x_1, x_2, \dots x_r$ as a unique linear function of the remaining $n-r$ variables $x_{r+1}, x_{r+2}, \dots x_n$. Formula (A′), being equivalent to the same r scalar equations as formula (A), also does this.

Case II. *Let $[a]^n_m$ and $[a]^{n+1}_m$ have common rank r, and let $(a_{pq})^r_r = \alpha \neq 0$, where $(a_{pq})^r_r$ is a derived determinant of order r of $[a]^n_m$.*

In this case let $[\tau_1 \tau_2 \dots \tau_{n+1-r}]$ be a minor sequence of $[1\ 2 \dots (n+1)]$ complementary to the minor sequence $[q_1 q_2 \dots q_r]$.

It is always open to us to suppose that $\tau_{n+1-r} = n+1$. In fact when the complementary minor sequence is corranged, this is necessarily the case.

The system of equations (1) can now be replaced by the irreducible matrix equation of which equivalent forms are

$$[a_{pi}]^{n+1}_{r}\,\overline{\underline{x}}_{n+1} = 0, \qquad [a_{pq}]^{r}_{r}\,\overline{\underline{x_q}}_{r} = -[a_{p\tau}]^{n+1-r}_{r}\,\overline{\underline{x_\tau}}_{n+1-r} \quad \dots\dots(7).$$

Let $[A_{pq}]^r_r$ be the reciprocal matrix of $[a_{pq}]^r_r$, and let

$$\overline{\underline{A_{pq}}}{}^{r}_{r}\,[a_{pi}]^{n+1}_{r} = [\alpha]^{n+1}_{r}, \qquad \text{so that } [\alpha_{iq}]^{r}_{r} = \overline{\underline{A_{pq}}}{}^{r}_{r}\,[a_{pq}]^{r}_{r} = \alpha\,[1]^{r}_{r} \quad \dots(8).$$

Then prefixing the matrix $\underset{r}{\overset{r}{\lceil A_{pq} \rfloor}}$ in either of the equations (7) we obtain equations which by § 84 are equivalent to them, and which give the general solution; and we have the following result:

Second form of the general solution.

The general solution of the system of equations (1) *is given in Case II by either of the formulae*

$$[\alpha]_r^{n+1} \underset{n+1}{\lceil x \rfloor} = 0 \quad \text{.............................(B}'\text{)},$$

$$\alpha \underset{r}{\lceil x_q \rfloor} = -[\alpha_{1\tau}]_r^{n+1-r} \underset{n+1-r}{\lceil x_\tau \rfloor} \quad \text{.....................(B)},$$

where α_{ij} *is the determinant formed by replacing the ith vertical row of* $(a_{pq})_r^r$ *by the jth vertical row of* $[a_{p1}]_r^{n+1}$.

Formula (B) expresses the r variables $x_{q_1}, x_{q_2}, \ldots x_{q_r}$ as unique linear functions of the remaining $n-r$ variables. Formula (B′), being equivalent to the same r scalar equations as formula (B), also does this.

We can deduce Case I from Case II by putting

$$[p_1 p_2 \ldots p_r] = [q_1 q_2 \ldots q_r] = [1\ 2 \ldots r].$$

The case in which (1) is a system of *unconnected* equations, which is the only case which it is really necessary to consider, can be deduced from Cases I and II by putting $m = r$, $[p_1 p_2 \ldots p_r] = [1\ 2 \ldots r]$.

Ex. i. *System of n unconnected linear equations in n variables.*

Let $$[a]_n^{n+1} \underset{n+1}{\lceil x \rfloor} = 0, \qquad \text{where} \qquad x_{n+1} = 1 \quad \text{........................(9)},$$

be a system of n unconnected linear equations in the n variables $x_1, x_2, \ldots x_n$.

The system admits of finite solution when and only when $[a]_n^n$ has rank n. Supposing this to be so, let $(a)_n^n = a \neq 0$, and let $[A]_n^n$ be the reciprocal matrix of $[a]_n^n$; also let

$$\underset{n}{\overset{n}{\lceil A \rfloor}} [a]_n^{n+1} = [a]_n^{n+1}, \qquad \text{so that} \qquad [a]_n^n = \underset{n}{\overset{n}{\lceil A \rfloor}} [a]_n^n = a\,[1]_n^n.$$

Then the system of equations and its general solution can be exhibited in the forms

$$[a]_n^n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = - \begin{bmatrix} a_{1,\,n+1} \\ a_{2,\,n+1} \\ \vdots \\ a_{n,\,n+1} \end{bmatrix}, \qquad a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = - \begin{bmatrix} a_{1,\,n+1} \\ a_{2,\,n+1} \\ \vdots \\ a_{n,\,n+1} \end{bmatrix},$$

where the second equation is obtained from the first by prefixing $\underset{n}{\overset{n}{\lceil A \rfloor}}$.

Thus there is one and only one solution, viz. that given by

$$\frac{x_1}{a_{1,\,n+1}} = \frac{x_2}{a_{2,\,n+1}} = \dots = \frac{x_n}{a_{n,\,n+1}} = \frac{1}{a} \quad\dots\dots(10).$$

This unique solution is finite, since $a \neq 0$.

Now let $a_1, a_2, \dots a_n, a_{n+1}$ be the *affected* simple minor determinants of the augmented matrix $[a]_n^{n+1}$ formed by striking out its 1st, 2nd, ... nth, $(n+1)$th vertical rows respectively, so that $a_1, a_2, \dots a_n, a_{n+1}$ have all even affects in $[a]_n^{n+1}$.

Let i be any one of the numbers 1, 2, ... n. Then $a_{i,\,n+1}$ is the determinant which is formed when the ith vertical row of $(a)_n^n$ is replaced by the last vertical row of $[a]_n^{n+1}$, and is therefore a derangement of a_i. The affect of $a_{i,\,n+1}$ in $[a]_n^{n+1}$ is the affect of $[1, 2, \dots (i-1), (n+1), (i+1), \dots n]$ in $[1\ 2 \dots (n+1)]$, which is equal to $n-(i-1)+(n-i)$ or $2(n-i)+1$, and is odd. Since $a_{i,\,n+1}$ is a derangement of a_i, and since $a_{i,\,n+1}$, a_i have respectively odd and even affects in $[a]_n^{n+1}$, it follows from Theorem V a or Theorem II b of § 25 that $a_{i,\,n+1} = -a_i$. Further it is clear that $a = a_{n+1}$. Accordingly equations (10) can be replaced by

$$\frac{x_1}{a_1} = \frac{x_2}{a_2} = \dots = \frac{x_n}{a_n} = \frac{1}{a_{n+1}} \quad\dots\dots(11),$$

or by

$$[x_1 x_2 \dots x_n\ 1] = \rho\,[a_1 a_2 \dots a_n a_{n+1}] \quad\dots\dots(C),$$

where ρ is an unspecified scalar quantity, and is necessarily equal to $\dfrac{1}{a_{n+1}}$.

We have thus proved the following theorem:

Theorem. *If (1) is any system of unconnected equations in which the number of equations is equal to the number of variables, and if the system admits of finite solution, i.e. if $[a]_n^n$ has rank n, then the system has a unique solution given by the formula*

$$[x_1 x_2 \dots x_n\ 1] = \rho\,[a_1 a_2 \dots a_n a_{n+1}],$$

where ρ is a scalar quantity. In other words, $x_1, x_2, \dots x_n, 1$ are proportional to the affected simple minor determinants of the augmented matrix $[a]_n^{n+1}$ formed by omitting its 1st, 2nd, ... nth, $(n+1)$th vertical rows ......(D).

In the limit when $a = a_{n+1} = 0$ and $[a]_n^{n+1}$ has rank n, the solution is infinite, and by Ex. i of § 89, it is still given by formula (C), ρ being infinite.

Theorem (D) can be at once deduced from the obvious equation

$$[a]_n^{n+1}\ \overline{\underline{a}}_{n+1} = 0.$$

The ith element of the product matrix on the left is by § 27 the determinant formed by adding the additional horizontal row $[a_{i1}\, a_{i2} \dots a_{i,\,n+1}]$ to $[a]_n^{n+1}$ in the leading position, and this determinant has two identical horizontal rows and therefore vanishes. The above equation shows that $a_{n+1}\ \overline{\underline{x}}_{n+1} = \overline{\underline{a}}_{n+1}$ is a solution of (9), and this must therefore be the unique solution of (9).

Ex. ii. If for the matrix equation (2) we use the form

$$[x_1 x_2 \dots x_n\, 1]\ \overline{\underline{a}}{}^{n}_{n+1} = 0,$$

it can be solved as in § 82.

If $(a)^r_r \neq 0$, we can reduce it to $[x_1 x_2 \dots x_n\, 1]\ \overline{\underline{a}}{}^{r}_{n+1} = 0$, and solve by *postfixing* $[A]^r_r$.

If $(a_{pq})^r_r \neq 0$, we can reduce it to $[x_1 x_2 \dots x_n\, 1]\ \overline{\underline{a_{p1}}}{}^{r}_{n+1} = 0$, and solve by *post-fixing* $[A_{pq}]^r_r$.

Ex. iii. We will solve the system of three unconnected scalar equations

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ p \\ q \\ 1 \end{bmatrix} = 0 \quad \dots\dots\dots\dots(12),$$

in which x, y, z, p, q are to be determined, in the two cases $(abc) \neq 0$, $(bce) \neq 0$.

To do this we write it in the forms

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = - \begin{bmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix} \quad \dots\dots\dots\dots(13),$$

$$\begin{bmatrix} b_1 & c_1 & e_1 \\ b_2 & c_2 & e_2 \\ b_3 & c_3 & e_3 \end{bmatrix} \begin{bmatrix} y \\ z \\ q \end{bmatrix} = - \begin{bmatrix} a_1 & d_1 & f_1 \\ a_2 & d_2 & f_2 \\ a_3 & d_3 & f_3 \end{bmatrix} \begin{bmatrix} x \\ p \\ 1 \end{bmatrix} \quad \dots\dots\dots\dots(14).$$

Let the conjugate reciprocals of

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \begin{bmatrix} b_1 & c_1 & e_1 \\ b_2 & c_2 & e_2 \\ b_3 & c_3 & e_3 \end{bmatrix} \quad \text{be} \quad \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}, \quad \begin{bmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ E_1 & E_2 & E_3 \end{bmatrix}$$

respectively in the two cases, the two notations being independent of one another.

Prefixing the first conjugate reciprocal in equation (13) in the first case, and the second conjugate reciprocal in equation (14) in the second case, we obtain

$$(abc) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = - \begin{bmatrix} (dbc) & (ebc) & (fbc) \\ (adc) & (aec) & (afc) \\ (abd) & (abe) & (abf) \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix},$$

$$(bce) \begin{bmatrix} y \\ z \\ q \end{bmatrix} = - \begin{bmatrix} (ace) & (dce) & (fce) \\ (bae) & (bde) & (bfe) \\ (bca) & (bcd) & (bcf) \end{bmatrix} \begin{bmatrix} x \\ p \\ 1 \end{bmatrix},$$

in the two cases respectively.

Thus when $(abc) \neq 0$, the general solution is given by

$$(abc)\,x = -(dbc)\,p - (ebc)\,q - (fbc),$$
$$(abc)\,y = -(adc)\,p - (aec)\,q - (afc),$$
$$(abc)\,z = -(abd)\,p - (abe)\,q - (abf),$$

where p and q are arbitrary;

and when $(bce) \neq 0$, the general solution is given by

$$(bce)\,y = -(ace)\,x - (dce)\,p - (fce),$$
$$(bce)\,z = -(bae)\,x - (bde)\,p - (bfe),$$
$$(bce)\,q = -(bca)\,x - (bcd)\,p - (bcf),$$

where x and p are arbitrary.

These solutions can also be obtained by prefixing the conjugate reciprocal matrices directly to equation (12). Thus in the second case if we prefix the second conjugate reciprocal in equation (12), we obtain

$$\begin{bmatrix} (ace) & (bce) & 0 & (dce) & 0 & (fce) \\ (bae) & 0 & (bce) & (bde) & 0 & (bfe) \\ (bca) & 0 & 0 & (bcd) & (bce) & (bcf) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ p \\ q \\ 1 \end{bmatrix} = 0.$$

This last equation is equivalent to three scalar equations which give the same general solution as that obtained before in the second case.

Ex. iv. We will solve the system of equations

$$2x - 3y + 9z + w - 3 = 0,$$
$$3x + y - 5z + 3w - 5 = 0,$$
$$2x + 6y - 8z + w + 7 = 0,$$
$$x + 2y + 6z - w + 9 = 0.$$

Here we may strike out the third equation and write the remaining equations the form

$$\begin{bmatrix} 2, & -3, & 9 \\ 3, & 1, & -5 \\ 1, & 2, & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1, & 3 \\ -3, & 5 \\ 1, & -9 \end{bmatrix} \begin{bmatrix} w \\ 1 \end{bmatrix}.$$

Prefixing $\begin{bmatrix} 16, & 36, & 6 \\ -23, & 3, & 37 \\ 5, & -7, & 11 \end{bmatrix}$, the conjugate reciprocal of $\begin{bmatrix} 2, & -3, & 9 \\ 3, & 1, & -5 \\ 1, & 2, & 6 \end{bmatrix}$,

we obtain
$$146 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -118, & 174 \\ 51, & -387 \\ 27, & -119 \end{bmatrix} \begin{bmatrix} w \\ 1 \end{bmatrix}.$$

Thus the general solution is given by

$$146x = -118w + 174, \qquad 146y = 51w - 387, \qquad 146z = 27w - 119,$$

where w is arbitrary.

§ 89. Solution of any system of homogeneous linear equations.

The system of m homogeneous linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right\} \ldots\ldots\ldots\ldots(1)$$

is equivalent to the single homogeneous matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0, \text{ or } [a]_m^n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \ldots\ldots\ldots(2).$$

The latter equation can also be written in the form

$$[x_1 x_2 \ldots x_n] \overline{\underline{a}}_n^m = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and in the still briefer forms

$$[a]_m^n \overline{\underline{x}}_n = 0, \quad [x]_n \overline{\underline{a}}_n^m = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Regarding the matrix equation (2) as a special equation of the form $AX = 0$, its general solution is contained in § 81.6. It can also be derived from the general solution in § 87 by putting $c_1 = c_2 = \ldots = c_m = 0$, or from the general solution in § 88 by putting $a_{1,n+1} = a_{2,n+1} = \ldots = a_{m,n+1} = 0$.

The augmented matrix is now identical with the unaugmented matrix $[a]_m^n$, and this may be called simply the matrix of the system.

From the theorems of § 81 or § 88 we have the following results:

(i) *By repeatedly omitting equations which are connected with the remaining equations, we can reduce the given system of equations to a system of r unconnected homogeneous linear equations, where r is the rank of $[a]_m^n$. The matrix of the reduced system of equations has the same rank as the matrix of the original system.*

(ii) *The system of homogeneous linear equations (1) always admits of finite solution. In fact $[x_1 x_2 \ldots x_n] = 0$ is always a solution.*

(iii) *If r is the rank of $[a]_m^n$, the matrix of the system, the general solution expresses r of the variables $x_1, x_2, \ldots x_n$ as homogeneous linear functions of the remaining $n - r$ variables to which arbitrary values may be assigned. In the special case in which $r = n$, so that the number of unconnected* *equations is equal to the number of variables, the system has only the zero solution* $[x_1 x_2 \dots x_n] = 0$.

(iv) *The system has non-zero solutions when and only when the rank of its matrix* ${[a]}^n_m$ *is less than* n, *i.e. when and only when the number of unconnected equations is less than the number of variables.*

(v) *The system has infinite solutions when and only when it has non-zero solutions.*

We will proceed to find the general solution of the system of equations (1) in all cases.

CASE I. *Let* ${[a]}^n_m$ *have rank* r, *and let* ${(a)}^r_r = \alpha \neq 0$.

Clearly any case whatever can be reduced to this.

The system of equations (1) can in this case be replaced by the system of r unconnected homogeneous linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}x_1 + a_{r2}x_2 + \dots + a_{rn}x_n = 0 \end{array}\right\} \quad \dots\dots\dots\dots\dots\dots(5),$$

which is equivalent to the irreducible matrix equation

$${[a]}^n_r \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0, \quad \text{or} \quad {[a]}^r_r \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = -\begin{bmatrix} a_{1,r+1} & a_{1,r+2} & \dots & a_{1n} \\ a_{2,r+1} & a_{2,r+2} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{r,r+1} & a_{r,r+2} & \dots & a_{rn} \end{bmatrix} \begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} \quad \dots(6),$$

where $n \nless r$.

Let ${[A]}^r_r$ be the reciprocal matrix of ${[a]}^r_r$, and let

$$\overline{\underline{A}}^r_r \, {[a]}^n_r = {[\alpha]}^n_r, \quad \text{so that} \quad {[\alpha]}^r_r = \overline{\underline{A}}^r_r \, {[a]}^r_r = \alpha \, {[1]}^r_r \quad \dots\dots\dots(7).$$

Then prefixing the matrix $\overline{\underline{A}}^r_r$ in either of the equations (6) we obtain equations which by § 84 are equivalent to them, and which give the general solution; and we have the following result:

First form of the general solution.

The general solution of the system of equations (1) *is given by either of the formulae*

$${[\alpha]}^n_r \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha & 0 & \dots & 0 & \alpha_{1,r+1} & \alpha_{1,r+2} & \dots & \alpha_{1n} \\ 0 & \alpha & \dots & 0 & \alpha_{2,r+1} & \alpha_{2,r+2} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha & \alpha_{r,r+1} & \alpha_{r,r+2} & \dots & \alpha_{rn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \quad \dots(\mathrm{A}'),$$

$$\alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = - \begin{bmatrix} \alpha_{1,r+1} & \alpha_{1,r+2} & \dots & \alpha_{1n} \\ \alpha_{2,r+1} & \alpha_{2,r+2} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{r,r+1} & \alpha_{r,r+2} & \dots & \alpha_{rn} \end{bmatrix} \begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} \text{..............(A)},$$

where α_{ij} *is the determinant formed by replacing the ith vertical row of* $(a)_r^r$ *by the jth vertical row of* $[a]_r^n$.

Formula (A) expresses each of the r variables $x_1, x_2, \dots x_r$ as a unique homogeneous linear function of the remaining $n-r$ variables $x_{r+1}, x_{r+2}, \dots x_n$. Formula (A′), being equivalent to the same r scalar equations as formula (A), serves the same purpose.

CASE II. *Let* $[a]_m^n$ *have rank* r, *and let* $(a_{pq})_r^r = \alpha \neq 0$.

This is the most general case possible.

In this case let $[\tau_1 \tau_2 \dots \tau_{n-r}]$ be a minor sequence of $[1\ 2 \dots n]$ complementary to the minor sequence $[q_1 q_2 \dots q_r]$. Then the system of equations (1) can be replaced by the irreducible matrix equation of which equivalent forms are

$$[a_{p1}]_r^n \overline{\underline{x}}_n = 0, \qquad [a_{pq}]_r^r \overline{\underline{x_q}}_r = -[a_{p\tau}]_r^{n-r} \overline{\underline{x_\tau}}_{n-r} \text{ ....(8)}.$$

Let $[A_{pq}]_r^r$ be the reciprocal matrix of $[a_{pq}]_r^r$, and let

$$\overline{\underline{A_{pq}}}_r^r [a_{p1}]_r^n = [\alpha]_r^n, \quad \text{so that } [\alpha_{1q}]_r^r = \overline{\underline{A_{pq}}}_r^r [a_{pq}]_r^r = \alpha [1]_r^r \text{ ......(9)}.$$

Then prefixing the matrix $\overline{\underline{A_{pq}}}_r^r$ on both sides of either of the equations (8) we obtain equations which by § 84 are equivalent to them, and which give the general solution; and we have the following result:

Second form of the general solution.

The general solution of the system of equations (1) *is given in Case II by either of the formulae*

$$[\alpha]_r^n \overline{\underline{x}}_n = 0 \text{ ....(B′)},$$

$$\alpha \overline{\underline{x_q}}_r = -[\alpha_{1\tau}]_r^{n-r} \overline{\underline{x_\tau}}_{n-r} \text{ ....(B)},$$

where α_{ij} *is the determinant formed by replacing the ith vertical row of* $(a_{pq})_r^r$ *by the jth vertical row of* $[a_{p1}]_r^n$.

Formula (B) expresses the r variables $x_{q_1}, x_{q_2}, \dots x_{q_r}$ as unique homogeneous linear functions of the remaining $n-r$ variables $x_{\tau_1}, x_{\tau_2}, \dots x_{\tau_{n-r}}$. Formula (B′), being equivalent to the same r scalar equations as formula (B), serves the same purpose.

We can deduce Case I from Case II by putting

$$[p_1 p_2 \dots p_r] = [q_1 q_2 \dots q_r] = [1\ 2 \dots r].$$

The case in which (1) is a system of *unconnected* homogeneous linear equations is the only case which it is really necessary to consider. The general solution in this case can be deduced from that found in Cases I and II above by putting $m = r$ and $[p_1 p_2 \dots p_r] = [1\ 2 \dots r]$.

Ex. i. *System of $n-1$ unconnected homogeneous linear equations in n variables.*

Let $$[a]_{n-1}^{n} \, \overline{\underline{x}}_{n} = 0 \qquad \text{.......(10)}$$

be a system of $n-1$ unconnected homogeneous linear equations in the n variables $x_1, x_2, \dots x_n$.

Then $[a]_{n-1}^{n}$ has rank $n-1$, and we may without loss of generality suppose that the variables and the corresponding vertical rows of $[a]_{n-1}^{n}$ are so arranged that

$$(a)_{n-1}^{n-1} = a \neq 0.$$

Let $[A]_{n-1}^{n-1}$ be the reciprocal matrix of $[a]_{n-1}^{n-1}$, and let

$$\overline{\underline{A}}_{n-1}^{n-1} [a]_{n-1}^{n} = [\alpha]_{n-1}^{n}, \qquad \text{so that} \qquad [\alpha]_{n-1}^{n-1} = a\,[1]_{n-1}^{n-1}.$$

Then by first re-writing (10) and then prefixing the matrix $\overline{\underline{A}}_{n-1}^{n-1}$, we have

$$[a]_{n-1}^{n-1}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = -x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{n-1,n} \end{bmatrix}, \qquad a\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = -x_n \begin{bmatrix} \alpha_{1n} \\ \alpha_{2n} \\ \vdots \\ \alpha_{n-1,n} \end{bmatrix}.$$

Thus the general solution is given by

$$a\,[x_1 x_2 \dots x_{n-1}] = -x_n\,[\alpha_{1n} \alpha_{2n} \dots \alpha_{n-1,n}],$$

or $$\frac{x_1}{\alpha_{1n}} = \frac{x_2}{\alpha_{2n}} = \dots = \frac{x_{n-1}}{\alpha_{n-1,n}} = -\frac{x_n}{a} \qquad \text{.......(11)},$$

where x_n is arbitrary.

Now let $a_1, a_2, \dots a_n$ be the *affected* simple minor determinants of $[a]_{n-1}^{n}$ formed by striking out its 1st, 2nd, ... nth vertical rows respectively, so that $a_1, a_2, \dots a_n$ have all even affects in $[a]_{n-1}^{n}$.

Let i be any one of the numbers $1, 2, \ldots n-1$. Then a_{in} is formed from $(a)_{n-1}^{n-1}$ by replacing its ith vertical row by the last vertical row of $[a]_{n-1}^{n}$, and is therefore a derangement of a_i. The affect of a_{in} in $[a]_{n-1}^{n}$ is the affect of $[1, 2, \ldots (i-1), n, (i+1), \ldots (n-1)]$ in $[1\ 2 \ldots n]$, which is equal to $(n-1)-(i-1)+(n-1-i)$ or $2(n-i)-1$ and is odd. Since a_{in} is a derangement of a_i, and since a_{in}, a_i have respectively odd and even affects in $[a]_{n-1}^{n}$, it follows from Theorem Va or Theorem 11b of § 25 that $a_{in} = -a_i$. Further it is clear that $a = a_n$.

Accordingly equations (11) can be replaced by

$$\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_{n-1}}{a_{n-1}} = \frac{x_n}{a_n},$$

or by $$[x_1\, x_2 \ldots x_n] = \rho\, [a_1\, a_2 \ldots a_n] \quad \text{.................................(C)},$$

where ρ is an arbitrary scalar quantity.

Formula (C) remains true when the elements on both sides are subjected to similar re-arrangements, and is therefore true independently of the hypothesis that $(a)_{n-1}^{n-1}$ is a non-vanishing determinant. It gives the general solution of equation (10) in all cases.

We have thus proved the following theorem:

Theorem. *A system of unconnected homogeneous linear equations in which the number of equations is less by 1 than the number of variables determines uniquely the ratios of the variables to one another. Each variable is proportional to the corresponding affected simple minor determinant of the matrix of the system* ....(D).

Theorem (D) can also be deduced from the obvious equation

$$[a]_{n-1}^{n}\ \overline{\underline{a}}_{n} = 0.$$

Equation (11) shows that the ratios of $x_1, x_2, \ldots x_n$ to one another are uniquely determined, and the above equation shows that $\overline{\underline{x}}_{n} = \overline{\underline{a}}_{n}$ is a particular solution of (10). It follows that the general solution of (10) is given by (C).

Ex. ii. If for the matrix equation (2) we use the form

$$[x_1 x_2 \ldots x_n]\ \overline{\underline{a}}_{n}^{m} = 0, \qquad \text{or} \qquad [x]_n\ \overline{\underline{a}}_{n}^{m} = 0,$$

it can be solved as in § 82.

If $(a)_r^r \neq 0$, we can reduce it to $[x_1 x_2 \ldots x_n]\ \overline{\underline{a}}_{n}^{r} = 0$, and solve by *postfixing* $[A]_r^r$.

If $(a_{pq})_r^r \neq 0$, we can reduce it to $[x_1 x_2 \ldots x_n]\ \overline{\underline{a_{p1}}}_{n}^{r} = 0$, and solve by *postfixing* $[A_{pq}]_r^r$.

Ex. iii. The general solution of the system of three unconnected equations

$$a_1 x + b_1 y + c_1 z + d_1 \xi + e_1 \eta + f_1 \zeta = 0,$$
$$a_2 x + b_2 y + c_2 z + d_2 \xi + e_2 \eta + f_2 \zeta = 0,$$
$$a_3 x + b_3 y + c_3 z + d_3 \xi + e_3 \eta + f_3 \zeta = 0,$$

in which x, y, z, ξ, η, ζ are to be determined, and $(abc)\neq 0$, is given by

$$\begin{aligned}(abc)\,x &= -(dbc)\,\xi-(ebc)\,\eta-(fbc)\,\zeta,\\ (abc)\,y &= -(adc)\,\xi-(aec)\,\eta-(afc)\,\zeta,\\ (abc)\,z &= -(abd)\,\xi-(abe)\,\eta-(abf)\,\zeta,\end{aligned}$$

where ξ, η and ζ are arbitrary.

This is obtained as in Ex. iii of § 88 by replacing $[xyz\!:\!pq1]$ by $[xyz\!:\!\xi\eta\zeta]$.

Ex. iv. We will solve the system of four equations

$$\begin{aligned}2x-3y+9z+\ \ w-3&=0,\\ 3x+\ \ y-5z+3w-5&=0,\\ 2x+6y-8z+\ \ w+2&=0,\\ x+2y+6z-\ \ w+7&=0.\end{aligned}$$

Here the augmented matrix has rank 4 and the unaugmented matrix has rank 3. Therefore the system has infinite solutions but no finite solution.

To determine the infinite solutions we have to solve the system of homogeneous equations obtained from the above by putting the constant terms equal to zero. This system is equivalent to the homogeneous matrix equation

$$\begin{bmatrix}2, & -3, & 9, & 1\\ 3, & 1, & -5, & 3\\ 1, & 2, & 6, & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}=0, \qquad\text{or}\qquad \begin{bmatrix}2, & -3, & 9\\ 3, & 1, & -5\\ 1, & 2, & 6\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1\\ -3\\ 1\end{bmatrix}w.$$

Prefixing $\begin{bmatrix}16, & 36, & 6\\ -23, & 3, & 37\\ 5, & -7, & 11\end{bmatrix}$, the conjugate reciprocal of $\begin{bmatrix}2, & -3, & 9\\ 3, & 1, & -5\\ 1, & 2, & 6\end{bmatrix}$,

we obtain
$$146\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-118\\ 51\\ 27\end{bmatrix}w,$$

or
$$-\frac{x}{118}=\frac{y}{51}=\frac{z}{27}=\frac{w}{146},$$
where w is arbitrary.

Thus the general solution of the corresponding system of homogeneous equations is

$$[x\,y\,z\,w]=\rho\,[-118,\ 51,\ 27,\ 146],$$

where ρ is arbitrary.

The infinite solutions of the original system of equations are obtained from this by giving infinite values to ρ.

Ex. v. The general solution of the system of homogeneous equations

$$\begin{aligned}2x_1-3x_2+9x_3+\ \ x_4-3x_5&=0,\\ 3x_1+\ \ x_2-5x_3+3x_4-5x_5&=0,\\ 2x_1+6x_2-8x_3+\ \ x_4+7x_5&=0,\\ x_1+2x_2+6x_3-\ \ x_4+9x_5&=0,\end{aligned}$$

is obtained as in Ex. iv of § 88 by putting $[x_1\, x_2\, x_3\, x_4\, x_5]$ in place of $[x\ y\ z\ w\ 1]$. It is given by

$$146\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -118, & 174 \\ 51, & -387 \\ 27, & -119 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix},$$

where x_4 and x_5 are arbitrary.

§ 90. Unconnected solutions of a system of homogeneous linear equations.

Let the system contain exactly r unconnected equations and be equivalent to the irreducible matrix equation

$$[a]_r^n \overline{\underline{x}}_n = [a]_r^n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \quad \text{.......................(1)},$$

in which $[a]_r^n$ has rank r.

By a solution of equation (1) we shall mean a value of the second factor matrix on the left-hand side which satisfies the equation. A non-zero solution is one in which $x_1, x_2, \ldots x_n$ are not all zero. We shall proceed to prove the following theorem:

Theorem. *The equation* (1) *in which the rank of* $[a]_r^n$ *is* r *has always* $n-r$ *and never more than* $n-r$ *unconnected finite non-zero solutions of the form* $[x_1\, x_2 \ldots x_n] = [\alpha_{s1}\, \alpha_{s2} \ldots \alpha_{sn}]$ ....(A).

We know by § 81 or § 89 that the equation can be solved and finite solutions be obtained when some $n-r$ of the quantities $x_1, x_2, \ldots x_n$ have arbitrarily assigned finite values. It may be supposed without loss of generality that $(a)_r^r \neq 0$. Then the equation can be solved when the values of $x_{r+1}, x_{r+2}, \ldots x_n$ are arbitrary.

If $r = n$, the equation has no non-zero solution.

If $r < n$, then since $x_{r+1}, x_{r+2}, \ldots x_n$ are arbitrary, we can determine $n-r$ different finite non-zero solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1n} \end{bmatrix}, \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2n} \end{bmatrix}, \ldots \begin{bmatrix} \alpha_{n-r,1} \\ \alpha_{n-r,2} \\ \vdots \\ \alpha_{n-r,n} \end{bmatrix}$$

by putting

$$[x_{r+1}\, x_{r+2} \dots x_n] = [1\, 0 \dots 0],\ [0\, 1 \dots 0],\ \dots\ [0\, 0 \dots 1]$$

in turn. These $n-r$ solutions are unconnected, for the matrix $\underset{n}{\overset{n-r}{\alpha}}$ formed by them has a non-vanishing derived determinant

$$\begin{vmatrix} \alpha_{1,r+1} & \alpha_{2,r+1} & \dots & \alpha_{n-r,r+1} \\ \alpha_{1,r+2} & \alpha_{2,r+2} & \dots & \alpha_{n-r,r+2} \\ \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{n-r,n} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{vmatrix}$$

of order $n-r$.

Thus the equation has always $n-r$ unconnected finite non-zero solutions.

Now let $[x_1\, x_2 \dots x_n] = [\alpha_{s1}\, \alpha_{s2} \dots \alpha_{sn}]$, where s receives the values $1, 2, \dots (n+1-r)$ be any $n+1-r$ finite solutions of the equation, so that

$$[a]_r^n \underset{n}{\overset{n+1-r}{\alpha}} = [a]_r^n \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n+1-r,1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n+1-r,2} \\ \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{n+1-r,n} \end{bmatrix} = 0 \quad \dots\dots\dots\dots(2).$$

Equation (2) shows that there are r unconnected connections between the horizontal rows of the matrix $\underset{n}{\overset{n+1-r}{\alpha}}$. Hence by Ex. ix of § 81 that matrix cannot have more than $n-r$ unconnected horizontal rows and its rank cannot exceed $n-r$. It follows that there must be some connection between its vertical rows, i.e. some connection must exist between every $n+1-r$ finite solutions of equation (1). Thus equation (1) cannot have more than $n-r$ unconnected finite solutions.

That the rank of $\underset{n}{\overset{n+1-r}{\alpha}}$ cannot exceed $n-r$ can also be seen by prefixing the conjugate reciprocal of $[a]_r^r$ on both sides of equation (2), as is done in the case of the corresponding equation in § 92.I.

It appears from the above reasoning and from formulae A and B of § 89 that *if the matrix* $[a]_r^n$ *is real*, i.e. if all its elements are real, *then we can always determine $n-r$ real unconnected finite solutions of equation* (1).

Ex. i. If $\underset{n}{\overset{n-r}{a}}$ is the matrix whose vertical rows are any $n-r$ unconnected finite non-zero solutions of (1), then clearly no vertical row of that matrix is a row of 0's.

If the ith horizontal row of $\underset{n}{\overset{n-r}{a}}$ is a row of 0's, then in the equation

$$[a]_r^n \underset{n}{\overset{n-r}{a}} = 0$$

we can strike out the ith pair of passive rows. The resulting equation shows that the rank of the matrix formed from $[a]_r^n$ by striking out the ith vertical row cannot exceed $r-1$; or that there is a connection between the horizontal rows of that matrix, whereas there is no connection between the horizontal rows of $[a]_r^n$ itself. It follows that by prefixing a suitable matrix $[h_1 h_2 \dots h_r]$ in equation (1), we can deduce the equation $x_i=0$. *Thus the ith horizontal row of* $\overline{\underline{a}}_n^{n-r}$ *can only be a row of 0's when the given system of equations is equivalent to a system of equations, one of which is $x_i=0$.*

Ex. ii. When s unconnected solutions of equation (1) of the form

$$[x_1 x_2 \dots x_n] = [a_{s1} a_{s2} \dots a_{sn}]$$

have been found, we determine an $(s+1)$th solution unconnected with them by putting

$$\begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_{r+s} \end{bmatrix} = \begin{bmatrix} a_{1,\,r+1} & a_{2,\,r+1} & \dots & a_{s,\,r+1} \\ a_{1,\,r+2} & a_{2,\,r+2} & \dots & a_{s,\,r+2} \\ \dots & \dots & \dots & \dots \\ a_{1,\,r+s} & a_{2,\,r+s} & \dots & a_{s,\,r+s} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_s \end{bmatrix},$$

$$x_{r+s+1} \neq k_1 a_{1,\,r+s+1} + k_2 a_{2,\,r+s+1} + \dots + k_s a_{s,\,r+s+1},$$

and then solving the given equation. This can be done so long as

$$r+s+1 \not> n, \qquad \text{or} \qquad s+1 \not> n-r.$$

By successive applications of this process all possible complete sets of unconnected solutions can be found.

Ex. iii. If s is the greatest possible number of unconnected solutions which the equation $[a]_m^n \overline{\underline{x}}_n = 0$ can have, and if r is the rank of $[a]_m^n$, then

$$r + s = n.$$

Ex. iv. *If the matrix $[a]_r^n$ of equation* (1) *is non-extravagant (i.e. if the sum of the squares of its simple minor determinants does not vanish), and if* $[x_1 x_2 \dots x_n] = [a_{11} a_{12} \dots a_{1n}]$, $[a_{21} a_{22} \dots a_{2n}]$, ... $[a_{s1} a_{s2} \dots a_{sn}]$ *are any s unconnected solutions of the equation, then the matrix* $\begin{bmatrix} a \\ a \end{bmatrix}_{r,\,s}^n$ *has rank $r+s$, and all its horizontal rows are unconnected.*

To prove this, suppose that there exists a connection of the form

$$[h_1 h_2 \dots h_r \;\; k_1 k_2 \dots k_s] \begin{bmatrix} a \\ a \end{bmatrix}_{r,\,s}^n = 0 \quad \dots\dots\dots\dots\dots(3),$$

or

$$[h_1 h_2 \dots h_r]\, [a]_r^n + [k_1 k_2 \dots k_s]\, [a]_s^n = 0,$$

or

$$\overline{\underline{a}}_n^r \; \overline{\underline{h}}_r = - \overline{\underline{a}}_n^s \; \overline{\underline{k}}_s \quad \dots\dots\dots\dots\dots(4).$$

From the given equation we have $[a]_r^n \overline{\underline{a}}_n^s = 0$, and therefore $[a]_r^n \overline{\underline{a}}_n^s \overline{\underline{k}}_s = 0$. From (4) it follows that

$$[a]_r^n \overline{\underline{a}}_n^r \overline{\underline{h}}_r = 0 \quad \dots\dots\dots\dots\dots(5).$$

Now if $[a]_r^n$ is non-extravagant, we know by § 72. 2 that the product matrix $[a]_r^n \overline{\underline{a}}_n^r$ has rank r, and that no connection of the form (5) can exist. Consequently $h_1 = h_2 = \ldots = h_r = 0$. Equation (4) then reduces to $\overline{\underline{a}}_n^s \overline{\underline{k}}_s = 0$. But by hypothesis the vertical rows of $\overline{\underline{a}}_n^s$ are unconnected, and no connection of this latter form can exist. Consequently $k_1 = k_2 = \ldots = k_s = 0$. Thus when $[a]_r^n$ is non-extravagant, no connection of the form (3) can exist.

Ex. v. *When* $[a]_r^n$ *is non-extravagant (in particular when it is real), the equation* (1) *has no solution connected with the active rows of* $[a]_r^n$.

This is a particular case of Ex. iv.

Ex. vi. *When* $[a]_r^n$ *is extravagant, the irreducible equation* (1) *has at least one finite non-zero solution connected with the active rows of* $[a]_r^n$.

For when $[a]_r^n$ is extravagant, there exists a relation of the form

$$[a]_r^n \overline{\underline{a}}_n^r \overline{\underline{h}}_r = 0.$$

Then $$\overline{\underline{x}}_n = \overline{\underline{a}}_n^r \overline{\underline{h}}_r, \qquad \text{or} \qquad [x_1 x_2 \ldots x_n] = [h_1 h_2 \ldots h_r]\,[a]_r^n,$$

is a finite non-zero solution of equation (1), and it is connected with the horizontal rows of $[a]_r^n$.

Ex. vii. *The irreducible equation* (1) *has or has not solutions connected with the active rows of* $[a]_r^n$ *according as the matrix* $[a]_r^n$ *is or is not extravagant.*

This statement summarises the results of Exs. v and vi.

An irreducible equation whose matrix is extravagant may be called an *extravagant equation.* An extravagant equation has solutions connected with its active rows; a non-extravagant equation cannot have such solutions.

Ex. viii. *If* $[a]_r^n \overline{\underline{a}}_n^r$ *has rank s, the irreducible equation* (1) *has* $r-s$ *and not more than* $r-s$ *unconnected solutions connected with the active rows of* $[a]_r^n$.

If $[a]_r^n \overline{\underline{a}}_n^r$ has rank s, then by Theorem VI of § 81 there exists a matrix $[k]_r^{r-s}$ of rank $r-s$ such that

$$[a]_r^n \overline{\underline{a}}_n^r [k]_r^{r-s} = 0.$$

Thus the vertical rows of the product matrix

$$\overline{\underline{a}}_n^r [k]_r^{r-s} = \overline{\underline{y}}_n^{r-s}$$

are solutions of equation (1), and they are unconnected because by Ex. vii of § 71 or

Ex. iv of § 73 the product matrix $\underline{\overline{y}}{}_n^{r-s}$ has rank $r-s$. Accordingly equation (1) has $r-s$ unconnected solutions connected with the active rows of the factor matrix $[a]_r^n$.

Again suppose that equation (1) has ρ unconnected solutions connected with the active rows of $[a]_r^n$, and let

$$\underline{\overline{z}}{}_n^{\rho} = \underline{\overline{a}}{}_n^{r}\,[h]_r^{\rho}$$

be the matrix formed by them. Then we have

$$[a]_r^n\,\underline{\overline{a}}{}_n^{r}\,[h]_r^{\rho} = 0,$$

where $[h]_r^{\rho}$ has rank ρ. By Ex. vii of § 82, the rank of $[a]_r^n\,\underline{\overline{a}}{}_n^{r}$ cannot exceed $r-\rho$. Hence if $[a]_r^n\,\underline{\overline{a}}{}_n^{r}$ has rank s, it follows that $s \not> r-\rho$, or $\rho \not> r-s$.

Thus equation (1) cannot have more than $r-s$ unconnected solutions connected with the active rows of $[a]_r^n$.

Ex. ix. *All solutions of the irreducible equation* (1) *are connected with the active rows of the factor matrix* $[a]_r^n$ *when and only when the rank of* $[a]_r^n\,\underline{\overline{a}}{}_n^{r}$ *is equal to* $2r-n$.

This case can only occur when $r \not< \frac{1}{2}n$.

Ex. x. If $[a]_r^n\,\underline{\overline{a}}{}_n^{r}$ has rank 0, i.e. if the active rows of $[a]_r^n$ are all mutually orthogonal and extravagant, then equation (1) has r and not more than r unconnected solutions connected with the active rows of $[a]_r^n$. In this case all the r active rows of $[a]_r^n$ are themselves solutions of equation (1).

Ex. xi. *A solution of the irreducible equation* (1) *which is connected with the active rows of the matrix* $[a]_r^n$ *is necessarily extravagant.*

Let $[x_1 x_2 \ldots x_n] = [h_1 h_2 \ldots h_r]\,[a]_r^n$ be such a solution.

Substituting in (1), we have

$$[a]_r^n\,\underline{\overline{a}}{}_n^{r}\,\underline{\overline{h}}{}_r = 0.$$

Prefixing $[h]_r$, we have

$$[h]_r\,[a]_r^n\,\underline{\overline{a}}{}_n^{r}\,\underline{\overline{h}}{}_r = 0,$$

i.e. $$[x]_n\,\underline{\overline{x}}{}_n = 0, \qquad \text{or} \qquad x_1^2 + x_2^2 + \ldots + x_n^2 = 0.$$

Thus the solution $[x_1 x_2 \ldots x_n]$ is extravagant.

From Ex. vi we see that *an extravagant equation has always at least one extravagant non-zero solution.*

An equation can also have extravagant solutions which are not connected with the active rows of its matrix.

Ex. xii. When the ith active row of the matrix $[a]_r^n$ is itself a solution of the given equation (1), then both the matrix itself and its ith active row are extravagant.

Ex. xiii. Both non-extravagant and extravagant equations have in general both non-extravagant and extravagant solutions.

Ex. xiv. The equation $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$ is extravagant. In accordance with Exs. vi and vii it has the solution $[x\ y\ z\ w] = [0\ 0\ 1\ i]$, which is connected with the last two active rows of the matrix of the equation and is extravagant.

Ex. xv. The equation $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$ has a solution $[x\ y\ z\ w] = [0\ 0\ 1\ i]$ formed by the last active row of the matrix of the equation. In accordance with Ex. xii the matrix itself and its last active row are both extravagant.

Ex. xvi. The equation $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$ is extravagant. In accordance with Exs. vii, viii and xi it has the extravagant solution $[x\ y\ z\ w] = [0\ 0\ 1\ i]$ connected with the active rows of its matrix and the non-extravagant solution $[x\ y\ z\ w] = [1, -1, 0, 0]$ which is not connected with the active rows of the matrix. These two solutions form a complete set of mutually unconnected solutions.

Ex. xvii. The equation $\begin{bmatrix} 2 & 1 & 1 & i \\ 1 & 2 & 1 & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is non-extravagant. In accordance with Ex. xiii it has the extravagant solution $[x\ y\ z\ w] = [0\ 0\ 1\ i]$ (connected with the active rows) and the non-extravagant solution $[x\ y\ z\ w] = [1, 1, -3, 0]$ (not connected with the active rows). These two solutions form a complete set of mutually unconnected solutions.

Ex. xviii. There exist equations all of whose solutions are extravagant. Two such equations are

$$\begin{bmatrix} \sqrt{2} & 0 & -i & i \\ 0 & \sqrt{2} & i & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0, \qquad \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3}i & \frac{2}{3}i \\ 0 & 1 & 0 & \frac{2}{3}i & \frac{1}{3}i \\ 0 & 0 & 1 & \frac{2}{3}i & -\frac{2}{3}i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ p \\ q \end{bmatrix} = 0.$$

The general solution of the first of these equations is

$$\sqrt{2}\,[x\ y\ z\ w] = [i\,(z-w),\ -i\,(z+w),\ \sqrt{2}\,z,\ \sqrt{2}\,w],$$

where z and w are arbitrary, and this general solution is extravagant. In this case the general solution is connected with the active rows, and therefore all solutions are connected with them.

The general solution of the second equation is

$$[x\ y\ z\ p\ q] = [-\tfrac{1}{3}i\,(p+2q),\ -\tfrac{1}{3}i\,(2p+q),\ \tfrac{2}{3}i\,(p-q),\ p,\ q],$$

where p and q are arbitrary. We obtain a particular solution connected with the active rows when $p=q$. No other particular solution is connected with the active rows.

Such equations are considered more generally in Exs. iii and iv of § 94.

§ 91. Symmetrical form of the general solution of a system of homogeneous linear equations.

Let the system of equations be equivalent to the irreducible matrix equation

$$[a]_r^n \,\overline{\underline{x}}_n = [a]_r^n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The general solution of this equation has been obtained in § 89 in a form which is not symmetrical in the variables. The object of the present article is to find a symmetrical form for the general solution.

Let

$$\begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1n} \end{bmatrix},\ \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2n} \end{bmatrix},\ \ldots \begin{bmatrix} \alpha_{n-r,1} \\ \alpha_{n-r,2} \\ \vdots \\ \alpha_{n-r,n} \end{bmatrix},\ \text{and} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

be respectively any $n-r$ unconnected particular finite solutions and any finite solution whatever of (1). Then between these $n+1-r$ solutions there exists some connection; and since the first $n-r$ solutions are unconnected, the connection can be expressed in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \ldots & \alpha_{n-r,1} \\ \alpha_{12} & \alpha_{22} & \ldots & \alpha_{n-r,2} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{1n} & \alpha_{2n} & \ldots & \alpha_{n-r,n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-r} \end{bmatrix} = \overline{\underline{\alpha}}_n^{\,n-r}\,\overline{\underline{\lambda}}_{n-r} \quad \ldots\ldots\ldots(A).$$

That is, it is possible to find finite quantities $\lambda_1, \lambda_2, \ldots \lambda_{n-r}$ such that (A) is true. Thus every finite solution of the given equation (1) can be expressed in the form (A) by ascribing suitable *finite* values to the parameters

$\lambda_1, \lambda_2, \ldots \lambda_{n-r}$; and this mode of expressing the solution is *unique*, otherwise there would be a connection between the vertical rows of $\underline{\overline{\alpha}}{}_n^{n-r}$. Conversely whatever finite values $\lambda_1, \lambda_2, \ldots \lambda_{n-r}$ may have, the value of $\underline{\overline{x}}{}_n$ given by (A) satisfies equation (1) and constitutes a finite solution of the equation. Since every infinite solution is obtained from some finite non-zero solution by multiplying it by an infinite scalar quantity, formula (A) also gives the infinite solutions when the restriction that the parameters $\lambda_1, \lambda_2, \ldots \lambda_{n-r}$ are to be all finite is removed. Hence we have the following result:

Theorem. *The general solution of the irreducible equation* (1) *can be represented by the formula* (*A*) *in which the vertical rows of* $\underline{\overline{\alpha}}{}_n^{n-r}$ *are any* $n-r$ *unconnected particular finite solutions of the equation, and* $\lambda_1, \lambda_2, \ldots \lambda_{n-r}$ *are arbitrary parameters. We obtain all finite solutions by giving to the parameters values which are all finite. We obtain infinite solutions when one at least of the parameters is infinite* ....(B).

Ex. i. We can write equation (1) in the form

$$[x_1 x_2 \ldots x_n]\, \underline{\overline{a}}{}_n^{r} = 0 \qquad \text{.......................................(2)},$$

and the general solution (A) in the form

$$[x_1 x_2 \ldots x_n] = [\lambda_1 \lambda_2 \ldots \lambda_{n-r}]\; [a]_{n-r}^{n} \qquad \text{..........................(3)},$$

where the horizontal rows of $[a]_{n-r}^{n}$ are unconnected particular solutions of (2) of the form

$$[x_1 x_2 \ldots x_n] = [a_{s1}\, a_{s2} \ldots a_{sn}].$$

Ex. ii. If some of the parameters in formula (A) are infinite, it can be expressed in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \rho\, \underline{\overline{a}}{}_n^{n-r} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n-r} \end{bmatrix},$$

where ρ is infinite and $k_1, k_2, \ldots k_{n-r}$ are finite.

This is by definition an infinite solution.

Ex. iii. If $\underline{\overline{a}}{}_n^{m}$, where $m \not< n-r$, is a matrix of rank $n-r$ whose vertical rows are finite solutions of equation (1), then the general solution of (1) can be expressed in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{m1} \\ a_{12} & a_{22} & \ldots & a_{m2} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n} & a_{2n} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} = \underline{\overline{a}}{}_n^{m}\; \underline{\overline{\lambda}}{}_m,$$

where $\lambda_1, \lambda_2, \ldots \lambda_m$ are arbitrary parameters.

For the matrix $\underset{n}{\overset{m}{a}}$ contains $n-r$ vertical rows which are unconnected finite non-zero solutions of (1).

Ex. iv. Particular solutions of the equation

$$ax+by+cz=0$$

are

$$[x\ y\ z]=[0,\ c,\ -b],\ [-c,\ 0,\ a],\ [b,\ -a,\ 0].$$

If $a\neq 0$, the general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -c, & b \\ 0, & -a \\ a, & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}.$$

In all cases the general solution is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0, & -c, & b \\ c, & 0, & -a \\ -b, & a, & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix}.$$

It is here supposed that a, b, c are not all zero.

Ex. v. The general solution of the equation

$$(b_1c_2-b_2c_1)x+(c_1a_2-c_2a_1)y+(a_1b_2-a_2b_1)z=0,$$

whose coefficients do not all vanish, is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \mu \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix},$$

where λ and μ are arbitrary.

Ex. vi. Particular solutions of the unconnected equations

$$(bc)x+(ca)y+(ab)z=0,$$
$$(bd)x+(da)y+(ab)w=0,$$

are

$$[x\ y\ z\ w]=[a_1\ b_1\ c_1\ d_1],\ [a_2\ b_2\ c_2\ d_2].$$

The general solution is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + \mu \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}.$$

§ 92. Unconnected solutions of any system of linear equations.

1. *Unconnected augmented solutions.*

Let the system contain exactly r unconnected equations admitting of finite solutions, and let it be equivalent to the irreducible matrix equation

$$[a]_r^{n+1} \underset{n+1}{\underline{\overline{x}}} = [a]_r^{n+1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = 0 \quad \text{.....................(1)},$$

where $[a]_r^n$ has rank r, and where it is to be understood that $x_{n+1} = 1$.

If $x_1, x_2, \dots x_n$ are values of the variables which satisfy this equation, we shall call the matrix $[x_1 x_2 \dots x_n 1]$ or its conjugate an *augmented solution* and the matrix $[x_1 x_2 \dots x_n]$ or its conjugate an *unaugmented solution.* A solution will generally mean an augmented solution. We will proceed to prove the following theorem:

Theorem I. *The equation* (1), *in which the rank of* $[a]_r^n$ *is* r, *has always* $n+1-r$ *unconnected finite augmented solutions of the form*

$$[x_1 x_2 \dots x_n 1] = [\alpha_{s1} \alpha_{s2} \dots \alpha_{sn} 1],$$

and can never have more than $n+1-r$ *unconnected finite solutions of this form.*

We know by § 87 that the equation can be solved and finite solutions be obtained when some $n-r$ of the quantities $x_1, x_2, \dots x_n$ are arbitrary. It may be supposed without loss of generality that $(a)_r^r \neq 0$. Then the equation can be solved when $x_{r+1}, x_{r+2}, \dots x_n$ are arbitrary.

If $r = n$, the equation has only one finite solution.

If $r < n$, then since $x_{r+1}, x_{r+2}, \dots x_n$ are arbitrary, we can determine $n+1-r$ different finite solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1n} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2n} \\ 1 \end{bmatrix}, \quad \dots \quad \begin{bmatrix} \alpha_{n+1-r,1} \\ \alpha_{n+1-r,2} \\ \vdots \\ \alpha_{n+1-r,n} \\ 1 \end{bmatrix}$$

by putting $[x_{r+1} x_{r+2} \dots x_n] = [1\ 0 \dots 0], [0\ 1 \dots 0], \dots [0\ 0 \dots 1], [0\ 0 \dots 0]$ in turn. These $n+1-r$ solutions are unconnected, for in the matrix

$$\underset{n+1}{\overset{n+1-r}{\alpha}} = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n+1-r,1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n+1-r,2} \\ \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{n+1-r,n} \\ 1 & 1 & \dots & 1 \end{bmatrix} \quad \dots\dots\dots\dots(2)$$

formed by them, the determinant formed by the last $n+1-r$ horizontal rows is

$$\begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix},$$

and is a non-vanishing derived determinant of order $n+1-r$.

Thus the equation (1) *has always* $n+1-r$ *unconnected finite solutions.*

In using the abbreviated notation on the left for the matrix (2), we must understand that

$$\alpha_{s,n+1} = 1 \text{ for all values of } s \quad \dots\dots\dots\dots(3).$$

Now let $[x_1 x_2 \dots x_n 1] = [\alpha_{s1} \alpha_{s2} \dots \alpha_{sn} 1]$, where s receives the values $1, 2, \dots (n+2-r)$, be any $n+2-r$ finite solutions of equation (1), so that

$$[a]_r^{n+1} \underset{n+1}{\overset{n+2-r}{\alpha}} = 0 \quad \dots\dots\dots\dots(4).$$

Prefixing the conjugate reciprocal of $[a]_r^r$ on both sides of the last equation, we obtain the result

$$\begin{bmatrix} \Delta & 0 & \dots & 0 & \beta_{1,r+1} & \dots & \beta_{1,n+1} \\ 0 & \Delta & \dots & 0 & \beta_{2,r+1} & \dots & \beta_{2,n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta & \beta_{r,r+1} & \dots & \beta_{r,n+1} \end{bmatrix} \underset{n+1}{\overset{n+2-r}{\alpha}} = 0 \quad \dots\dots\dots\dots(5),$$

where $\Delta = (a)_r^r \neq 0$, and β_{ij} is obtained from $(a)_r^r$ by replacing its ith vertical row by the jth vertical row of $[a]_r^{n+1}$.

Equation (5) is equivalent to r equations each connecting one of the first r horizontal rows of $\underset{n+1}{\overset{n+2-r}{\alpha}}$ with the last $n+1-r$ horizontal rows, and shows that the rank of that matrix cannot exceed $n+1-r$. Using Ex. ix of § 81, the same conclusion can be drawn immediately from equation (4), which shows that there are r unconnected connections between the horizontal rows of the matrix $\underset{n+1}{\overset{n+2-r}{\alpha}}$, and that the rank of the matrix cannot therefore exceed $n+1-r$.

It follows that there must be some connection between the $n+2-r$ vertical rows of the matrix $\overbrace{\underbrace{\alpha}_{n+1}}^{n+2-r}$, i.e. there must be some connection between every $n+2-r$ finite augmented solutions of equation (1).

Thus the equation (1) *cannot have more than* $n+1-r$ *unconnected finite augmented solutions.*

2. *Unconnected unaugmented solutions.*

Let

$$[x_1 x_2 \dots x_n] = [\alpha_{11}\,\alpha_{12} \dots \alpha_{1n}],\ [\alpha_{21}\,\alpha_{22} \dots \alpha_{2n}],\ \dots\ [\alpha_{s1}\,\alpha_{s2} \dots \alpha_{sn}] \quad \dots\dots(6)$$

be any s *unconnected* unaugmented solutions of equation (1). Then clearly

$$[x_1 x_2 \dots x_n\, 1] = [\alpha_{11}\,\alpha_{12} \dots \alpha_{1n}\, 1],\ [\alpha_{21}\,\alpha_{22} \dots \alpha_{2n}\, 1],\ \dots\ [\alpha_{s1}\,\alpha_{s2} \dots \alpha_{sn}\, 1] \dots(7)$$

are s unconnected augmented solutions of the equation; for if the former matrices are unconnected, so also are the latter matrices.

Again let (6) represent any s *connected* unaugmented solutions of equation (1). Then there exists a relation of the form

$$[h_1 h_2 \dots h_s]\,[\alpha]_s^n = 0 \quad \dots\dots\dots\dots(8).$$

Since the matrices (7) are solutions of the equation (1), we have

$$[a]_r^{n+1}\ \overbrace{\underbrace{\alpha}_{n+1}}^{s} = 0, \text{ or } [\alpha]_s^{n+1}\ \overbrace{\underbrace{a}_{n+1}}^{r} = 0 \quad \dots\dots\dots\dots(9).$$

Prefixing the matrix $[h_1 h_2 \dots h_s]$ on both sides of (9) in its second form, and making use of (8) and (3), we obtain

$$(h_1 + h_2 + \dots + h_s)\,[a_{1,n+1}\, a_{2,n+1} \dots a_{r,n+1}] = 0 \quad \dots\dots\dots(10).$$

The vanishing of the first factor in (10) means that the solutions (7) are connected. The vanishing of the second factor means that $a_{1,n+1}$, $a_{2,n+1}$, ... $a_{r,n+1}$ are all zero, or that equation (1) represents a system of homogeneous equations. When the system of equations is non-homogeneous, the first factor in (10) must vanish, and the solutions (7) are connected.

Thus when the given equations are not all homogeneous, the solutions (7) are or are not connected according as the solutions (6) are or are not connected; and therefore if the solutions (7) are unconnected, then also the solutions (6) are unconnected. Since there exist $n+1-r$ unconnected solutions of the form (7), there also exist $n+1-r$ unconnected solutions of the form (6).

When however the given equations are all homogeneous, and the solutions (7) are unconnected, it does not necessarily follow that the solutions (6) are unconnected. In fact there are $n+1-r$ unconnected solutions of the form (7), but by § 90 there are not more than $n-r$ unconnected solutions of the form (6). We have now established the following theorem:

Theorem II. *When the equation* (1), *in which the rank of* $[a]_r^n$ *is* r, *represents a system of non-homogeneous equations (i.e. when* $a_{1,n+1}, a_{2,n+1}, \ldots a_{r,n+1}$ *are not all zero), the equation has always* $n+1-r$ *unconnected finite solutions of the form* $[x_1 x_2 \ldots x_n] = [\alpha_{s1} \alpha_{s2} \ldots \alpha_{sn}]$, *and can never have more than* $n+1-r$ *unconnected finite solutions of this form.*

NOTE. *Alternative proof of Theorem II.*

Theorem II can be proved directly in the same way as Theorem I. We first observe that $[x_1 x_1 \ldots x_n] = [0\ 0 \ldots 0]$ satisfies equation (1) only when $[a_{1,n+1}\ a_{2,n+1} \ldots a_{r,n+1}]=0$. Hence it cannot be a solution when the given equations are non-homogeneous. Let

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix}, \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{bmatrix}, \ldots \begin{bmatrix} a_{n+1-r,1} \\ a_{n+1-r,2} \\ \vdots \\ a_{n+1-r,n} \end{bmatrix}$$

be the unaugmented solutions of (1) obtained when we put

$$[x_{r+1}\ x_{r+2} \ldots x_n] = [1\ 0 \ldots 0],\ [0\ 1 \ldots 0],\ \ldots\ [0\ 0 \ldots 1],\ [0\ 0 \ldots 0].$$

Then, supposing that the given equations are not all homogeneous, the first r elements of the last solution cannot all vanish.

Let $a_{n+1-r,i} = a \neq 0$, where $i \not> r$. Then in the matrix $\overline{a}_n^{n+1-r}$ formed by the above $n+1-r$ solutions, the ith horizontal row and the last $n-r$ horizontal rows form the determinant

$$\begin{vmatrix} a_{i1} & a_{i2} & \ldots & a_{i,n+1} & a \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & 0 \end{vmatrix}.$$

Since this is a non-vanishing derived determinant of order $n+1-r$, the matrix $[a]_n^{n+1-r}$ has rank $n+1-r$, and the above $n+1-r$ solutions are unconnected.

Thus the given equation has $n+1-r$ unconnected unaugmented solutions.

Again, since there is a connection between every $n+2-r$ augmented solutions, there is a connection between every $n+2-r$ unaugmented solutions. Accordingly the equation cannot have more than $n+1-r$ unconnected unaugmented solutions.

3. *Unconnected infinite solutions.*

We will now discard the restriction that the equation (1) shall admit of finite solution. In all cases, whether it does or does not admit of finite solution, every infinite solution corresponds to a definite finite non-zero solution of the homogeneous equation corresponding to (1), and any number of infinite solutions may be considered to have exactly the same connections as the corresponding finite solutions of that homogeneous equation.

Let the rank of $[a]_r^n$ be s. If $s=n$, there are no infinite solutions. If $s<n$, there are exactly $n-s$ unconnected infinite solutions.

Ex. i. When $s+1$ unconnected solutions of equation (1) of the form

$$[x_1\ x_2 \dots x_n\ 1] = [a_{s1}\ a_{s2} \dots a_{sn}\ 1]$$

have been found, we determine an $(s+2)$th solution unconnected with these by putting

$$\begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \vdots \\ x_{r+s} \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1,r+1} & a_{2,r+1} & \dots & a_{s+1,r+1} \\ a_{1,r+2} & a_{2,r+2} & \dots & a_{s+1,r+2} \\ \dots & \dots & \dots & \dots \\ a_{1,r+s} & a_{2,r+s} & \dots & a_{s+1,r+s} \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{s+1} \end{bmatrix},$$

$$x_{r+s+1} = k_1 a_{1,r+s+1} + k_2 a_{2,r+s+1} + \dots + k_{s+1} a_{s+1,r+s+1},$$

where $k_1, k_2, \dots k_{s+1}$ are any $s+1$ scalar quantities such that $k_1+k_2+\dots+k_{s+1}=1$, and then solving the given equation.

This can be done so long as $r+s+1 \not> n$, or $s+2 \not> n-r+1$.

By successive applications of this process all possible complete sets of unconnected solutions can be found.

Ex. ii. *If the augmented matrix* $[a]_r^{n+1}$ *of the irreducible equation* (1) *is non-extravagant and if* $[x_1\ x_2 \dots x_n\ 1] = [a_{11}\ a_{12} \dots a_{1n}\ 1]$, $[a_{21}\ a_{22} \dots a_{2n}\ 1]$, $\dots$ $[a_{s1}\ a_{s2} \dots a_{sn}\ 1]$ *are any s unconnected augmented solutions of the equation, then the matrix* $\begin{bmatrix} a \\ a \end{bmatrix}_{r,s}^{n+1}$ *has rank* $r+s$, *and all its horizontal rows are unconnected.*

This can be deduced from Ex. iv of § 90, or proved in the same manner.

Ex. iii. *When* $[a]_r^{n+1}$ *is non-extravagant (in particular when it is real), the irreducible equation* (1) *has no augmented solution connected with the active rows of* $[a]_r^{n+1}$.

This is a particular case of Ex. ii.

Ex. iv. When $[a]_r^{n+1}$ is extravagant, we cannot conclude that the irreducible equation (1) has an augmented solution connected with the active rows of $[a]_r^{n+1}$ unless we know that $[a]_r^n$ is non-extravagant.

§ 93. Symmetrical form of the general solution of any system of linear equations.

1. *Augmented solutions.*

Let the system of equations admit of finite solution and be equivalent to the irreducible matrix equation

$$[a]_r^{n+1} \overline{x}_{n+1} = [a]_r^{n+1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = 0 \quad \dots\dots\dots\dots\dots\dots(1),$$

in which $[a]_r^n$ has rank r, and $x_{n+1} = 1$.

Let $\begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1n} \\ 1 \end{bmatrix}$, $\begin{bmatrix} \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2n} \\ 1 \end{bmatrix}$, ... $\begin{bmatrix} \alpha_{n+1-r,1} \\ \alpha_{n+1-r,2} \\ \vdots \\ \alpha_{n+1-r,n} \\ 1 \end{bmatrix}$, and $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix}$

be respectively any $n+1-r$ particular unconnected finite solutions of the equation, and any finite solution whatever. Then between these $n+2-r$ solutions there exists some connection, and since the first $n+1-r$ solutions are unconnected, the connection can be put into the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n+1-r,1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n+1-r,2} \\ \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{n+1-r,n} \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n+1-r} \end{bmatrix} \quad\text{.................(A)},$$

or

$$\overline{\underline{x}}_{n+1} = \overline{\underline{\alpha}}_{n+1}^{n+1-r} \ \overline{\underline{\lambda}}_{n+1-r},$$

where $\qquad \alpha_{s,\,n+1} = 1$ for all values of s ....(2).

That is, it is possible to find finite quantities $\lambda_1, \lambda_2, \dots \lambda_{n+1-r}$ such that (A) is true, and these quantities have unity for their sum. Thus every finite augmented solution of the equation (1) can be expressed in the form (A) by ascribing suitable finite values to the parameters $\lambda_1, \lambda_2, \dots \lambda_{n+1-r}$; and this mode of expressing the solution is unique, otherwise there would be a connection between the vertical rows of $\overline{\underline{\alpha}}_{n+1}^{n+1-r}$.

Conversely whatever finite values $\lambda_1, \lambda_2, \dots \lambda_{n+1-r}$ may have, subject to the condition

$$\lambda_1 + \lambda_2 + \dots + \lambda_{n+1-r} = 1 \quad\text{.......................(3)},$$

the value of $\overline{\underline{x}}_{n+1}$ given by (A) satisfies equation (1) and constitutes a finite solution of the equation. Hence we have the following result:

Theorem I. *Formula (A), in which the vertical rows of* $\overline{\underline{\alpha}}_{n+1}^{n+1-r}$ *are any $n+1-r$ particular finite unconnected solutions of equation* (1), *and* $\lambda_1, \lambda_2, \dots \lambda_{n+1-r}$ *are arbitrary finite parameters subject to the condition* (3), *is a general expression for all finite augmented solutions of equation* (1).

In formula (A) put

$$\lambda_{n+1-r} = 1 - \lambda_1 - \lambda_2 - \dots - \lambda_{n-r} \quad\text{....................(4)},$$

$$[\alpha_{n+r-1,1},\ \alpha_{n+r-1,2},\ \dots\ \alpha_{n+r-1,n}\ 1] = [\alpha_1\ \alpha_2 \dots \alpha_n\ 1] \text{...........(5)}.$$

Then expanding the product on the right of (A) by the property of passive rows contained in § 43.1, we see that (A) is equivalent to

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_{11}-\alpha_1, & \alpha_{21}-\alpha_1, & \dots & \alpha_{n-r,1}-\alpha_1, & \alpha_1 \\ \alpha_{12}-\alpha_2, & \alpha_{22}-\alpha_2, & \dots & \alpha_{n-r,2}-\alpha_2, & \alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{1n}-\alpha_n, & \alpha_{2n}-\alpha_n, & \dots & \alpha_{n-r,n}-\alpha_n, & \alpha_n \\ 0, & 0, & \dots & 0, & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-r} \\ 1 \end{bmatrix} \quad \dots\dots(6).$$

If we strike out the last horizontal row and the last vertical row of the large matrix occurring in equation (6), the vertical rows of the matrix which remains are clearly unconnected finite solutions of the homogeneous equation

$$[a]_r^n \, \underline{\overline{x}}_n = [a]_r^n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \quad \dots\dots\dots\dots\dots\dots(7),$$

which we will call the *homogeneous equation corresponding to the given equation* (1).

Now writing

$$[\alpha_{s1}-\alpha_1, \ \alpha_{s2}-\alpha_2, \ \dots \ \alpha_{sn}-\alpha_n] = [l_{s1} \ l_{s2} \ \dots \ l_{sn}] \dots\dots\dots\dots(8),$$

we can replace (5) by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{21} & \dots & l_{n-r,1} & \alpha_1 \\ l_{12} & l_{22} & \dots & l_{n-r,2} & \alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ l_{1n} & l_{2n} & \dots & l_{n-r,n} & \alpha_n \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-r} \\ 1 \end{bmatrix} \quad \dots\dots\dots\dots(B),$$

where now $\lambda_1, \lambda_2, \dots \lambda_{n-r}$ are entirely arbitrary; $\underline{\overline{\alpha}}_{n+1}$ is any particular finite augmented solution of equation (1); and the vertical rows of $\underline{\overline{l}}_n^{n-r}$ are any $n-r$ unconnected finite solutions of equation (7).

Every finite augmented solution of the given equation (1) can be expressed in the form (B) by ascribing suitable finite values to the parameters $\lambda_1, \lambda_2, \dots \lambda_{n-r}$, and there is only one mode of so expressing it. Conversely whatever finite values the parameters $\lambda_1, \lambda_2, \dots \lambda_{n-r}$ may have, the value of $\underline{\overline{x}}_{n+1}$ given by (B) clearly satisfies equation (1) and constitutes a finite solution of it. Thus we have the following second result:

Theorem II. *Formula (B), in which the vertical rows of* $\underline{\overline{l}}_n^{n-r}$ *are any particular* $n-r$ *unconnected finite solutions of the homogeneous equation* *corresponding to* (1); $[x_1 x_2 \dots x_n 1] = [\alpha_1 \alpha_2 \dots \alpha_n 1]$ *is any particular finite augmented solution of equation* (1) *itself; and* $\lambda_1, \lambda_2, \dots \lambda_{n-r}$ *are arbitrary finite parameters; is a general expression for all finite augmented solutions of equation* (1).

2. *Unaugmented solutions.*

Formula (A) is equivalent to

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n+1-r,1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n+1-r,2} \\ \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{n+1-r,n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n+1-r} \end{bmatrix} = \underset{n}{\overset{n+1-r}{\alpha}} \; \underset{n+1-r}{\lambda} \quad \dots\dots(C),$$

where $\lambda_1, \lambda_2, \dots \lambda_{n+1-r}$ are arbitrary finite parameters subject to the condition

$$\lambda_1 + \lambda_2 + \dots + \lambda_{n+1-r} = 1.$$

If the given equation (1) is non-homogeneous, the vertical rows of $\underset{n}{\overset{n+1-r}{\alpha}}$ are finite unconnected unaugmented solutions of it, and (C) is a general expression for all finite unaugmented solutions.

If the given equation (1) is homogeneous, there is a connection between the vertical rows of $\underset{n}{\overset{n+1-r}{\alpha}}$, and we replace (C) by formula (A) of § 91.

We can always write $[\lambda_1 \lambda_2 \dots \lambda_{n+1-r}] = \frac{1}{\rho} [k_1 k_2 \dots k_{n+1-r}]$ where ρ is finite and not zero, and determine ρ so that condition (3) is satisfied. Thus we can replace (C) by

$$(k_1 + k_2 + \dots + k_{n+1-r}) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n+1-r,1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n+1-r,2} \\ \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{n+1-r,n} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n+1-r} \end{bmatrix} \dots\dots(D),$$

where now $k_1, k_2, \dots k_{n+1-r}$ are arbitrary finite quantities whose sum is not zero.

We have now the following theorem:

Theorem III. *If the given equation* (1) *is non-homogeneous, (i.e. if* $a_{1,n+1}, a_{2,n+1}, \dots a_{r,n+1}$ *are not all zero), then (C) and (D) are general expressions for all finite unaugmented solutions of the equation. In these formulae the vertical rows of* $\underset{n}{\overset{n+1-r}{\alpha}}$ *are any particular* $n+1-r$ *unconnected finite unaugmented solutions of equation* (1); $\lambda_1, \lambda_2, \dots \lambda_{n+1-r}$ *are arbitrary finite parameters whose sum is unity; and* $k_1, k_2, \dots k_{n+1-r}$ *are arbitrary finite parameters whose sum is not zero.*

Again in formula (B) the last of the equivalent $n+1$ scalar equations is superfluous, and we can replace the formula by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} l_{11} & l_{21} & \dots & l_{n-r,1} & \alpha_1 \\ l_{12} & l_{22} & \dots & l_{n-r,2} & \alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ l_{1n} & l_{2n} & \dots & l_{n-r,n} & \alpha_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-r} \\ 1 \end{bmatrix} \quad \dots\dots\dots\dots\dots(\text{E}),$$

or

$$\begin{bmatrix} x_1-\alpha_1 \\ x_2-\alpha_2 \\ \vdots \\ x_n-\alpha_n \end{bmatrix} = \begin{bmatrix} l_{11} & l_{21} & \dots & l_{n-r,1} \\ l_{12} & l_{22} & \dots & l_{n-r,2} \\ \dots & \dots & \dots & \dots \\ l_{1n} & l_{2n} & \dots & l_{n-r,n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-r} \end{bmatrix} = \overline{\underline{l}}^{\,n-r}_{\,n} \; \overline{\underline{\lambda}}_{\,n-r} \quad \dots\dots(\text{F}).$$

If the given equation (1) is non-homogeneous, the vertical rows of the large matrix in (E) are unconnected. If the given equation (1) is homogeneous, the last vertical row is connected with the preceding vertical rows, and can be taken to be a row of 0's. In this case formula (E) can be reduced to formula (A) of § 91.

We now have the following theorem:

Theorem IV. *If the given equation (1) is non-homogeneous, then formula (E) is a general expression for all finite unaugmented solutions of the equation. In this formula the first $n-r$ vertical rows of the large matrix are unconnected finite unaugmented solutions of the homogeneous equation corresponding to (1); the last vertical row is a finite unaugmented solution of equation (1) itself; and $\lambda_1, \lambda_2, \dots \lambda_{n-r}$ are arbitrary finite parameters.*

Instead of deducing (C) from (A), we can prove it in the same way that (A) was proved. Also we can deduce (E) from (C) in the same way that (B) was deduced from (A).

3. *Infinite solutions*

We assume still that $[a]^n_r$ has rank r. Since an infinite solution of (1) is an infinite scalar multiple of a finite solution of the corresponding homogeneous equation, the general formula for infinite solutions of (1) is

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \rho \begin{bmatrix} l_{11} & l_{21} & \dots & l_{n-r,1} \\ l_{12} & l_{22} & \dots & l_{n-r,2} \\ \dots & \dots & \dots & \dots \\ l_{1n} & l_{2n} & \dots & l_{n-r,n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-r} \end{bmatrix} = \rho\, \overline{\underline{l}}^{\,n-r}_{\,n} \; \overline{\underline{\lambda}}_{\,n-r} \quad \dots\dots\dots(\text{G}),$$

where the vertical rows of $\overline{\underline{l}}^{\,n-r}_{\,n}$ are unconnected finite solutions of the homogeneous equation corresponding to (1); $\lambda_1, \lambda_2, \dots \lambda_{n-r}$ are arbitrary finite parameters; and ρ is an infinite scalar number.

These infinite solutions can be included in the preceding formulae. Thus (A) and (C) give infinite solutions when one and therefore at least two of the parameters are infinite; (B) and (E) give infinite solutions when any one of the parameters is infinite; (D) gives infinite solutions when the parameters are finite and such that $k_1 + k_2 + \dots + k_{n-r} = 0$.

Ex. i. The general solution of (1) is the sum of the general solution of the homogeneous equation corresponding to (1) and any one particular solution of the equation (1) itself.

Ex. ii. The equation $\begin{bmatrix} 0, & -n, & m \\ n, & 0, & -l \\ -m, & l, & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ \beta \\ \gamma \end{bmatrix}$

admits of finite solution when and only when $la+m\beta+n\gamma=0$.

When this condition is satisfied, it has rank 2, and the general solution is

$$(l^2+m^2+n^2)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \rho \begin{bmatrix} l \\ m \\ n \end{bmatrix} + \begin{bmatrix} \beta n - \gamma m \\ \gamma l - an \\ am - \beta l \end{bmatrix},$$

where ρ is arbitrary.

Here the first term on the right corresponds to the general solution of the corresponding homogeneous equation, and the second term on the right corresponds to a particular solution of the complete equation.

The general solution can be written in the form

$$(l^2+m^2+n^2)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \rho \begin{bmatrix} l \\ m \\ n \end{bmatrix} + \begin{bmatrix} 0, & -\gamma, & \beta \\ \gamma, & 0, & -a \\ -\beta, & a, & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} \rho, & -\gamma, & \beta \\ \gamma, & \rho, & -a \\ -\beta, & a, & \rho \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix},$$

where ρ is arbitrary.

§ 94. Mutually orthogonal solutions of a system of homogeneous linear equations.

Let the system of equations be equivalent to the irreducible matrix equation

$$[a]_r^n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0, \quad \text{or} \quad [a]_r^n \,\underset{n}{\overline{x}} = 0 \quad \dots\dots\dots\dots\dots\dots\dots (1).$$

A solution $[x_1\, x_2 \dots x_n] = [\alpha_1\, \alpha_2 \dots \alpha_n]$ of this equation which is such that $\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 = 1$ will be called a *unit solution*.

If $\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 = 0$, the solution is called in accordance with § 72.2 an *extravagant solution*; if $\alpha_1, \alpha_2, \dots \alpha_n$ are all zero, it is a *zero solution*.

Clearly no real non-zero solution can be extravagant. Only imaginary solutions can be extravagant.

With every non-extravagant solution $[x_1 x_2 \dots x_n] = [\alpha_1 \alpha_2 \dots \alpha_n]$ we can associate two *corresponding unit solutions* by dividing the matrix on the right by ρ, where $\rho^2 = \alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2$.

Two solutions $[x_1 x_2 \dots x_n] = [\alpha_{i1} \alpha_{i2} \dots \alpha_{in}]$, $[\alpha_{j1} \alpha_{j2} \dots \alpha_{jn}]$ are said to be *orthogonal* to one another when

$$[\alpha_{i1}\, \alpha_{i2} \dots \alpha_{in}] \begin{bmatrix} \beta_{j1} \\ \beta_{j2} \\ \vdots \\ \beta_{jn} \end{bmatrix} = 0, \text{ or } \alpha_{i1}\beta_{j1} + \alpha_{i2}\beta_{j2} + \dots + \alpha_{in}\beta_{jn} = 0.$$

Any number of solutions are *mutually orthogonal* when every pair of them are orthogonal. An extravagant solution is orthogonal to itself or self-orthogonal. Conversely every self-orthogonal solution is extravagant.

The necessary and sufficient condition that the s solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1n} \end{bmatrix}, \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2n} \end{bmatrix}, \dots \begin{bmatrix} \alpha_{s1} \\ \alpha_{s2} \\ \vdots \\ \alpha_{sn} \end{bmatrix}$$

shall be mutually orthogonal and non-extravagant can be expressed in the form

$$[\alpha]_s^n \,\overline{\underline{\alpha}}_n^s = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{s1} & \alpha_{s2} & \dots & \alpha_{sn} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{s1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{s2} \\ \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{sn} \end{bmatrix} = \begin{bmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_s \end{bmatrix} \quad \dots\dots\dots(2),$$

where $k_1 \neq 0$, $k_2 \neq 0$, ... $k_s \neq 0$. In fact $k_i^2 = \alpha_{i1}^2 + \alpha_{i2}^2 + \dots + \alpha_{in}^2$.

Similarly the necessary and sufficient condition that the s solutions shall be mutually orthogonal unit solutions is

$$[\alpha]_s^n \,\overline{\underline{\alpha}}_n^s = [1]_s^s \quad \dots\dots\dots(3).$$

Clearly from every set of s mutually orthogonal non-extravagant solutions we can deduce a set of s mutually orthogonal unit solutions, and there are or are not s mutually orthogonal unit solutions according as there are or are not s mutually orthogonal non-extravagant solutions.

If there were any connection $[p_1 p_2 \dots p_s][\alpha]_s^n = 0$ between the horizontal rows of $[\alpha]_s^n$ in equation (2), we should have

$$[p_1 p_2 \dots p_s] \begin{bmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_s \end{bmatrix} = 0,$$

i.e. $[p_1 k_1, p_2 k_2, \dots p_s k_s] = 0$, or $p_1 = p_2 = \dots = p_s = 0$. Hence no such connection can exist, and therefore the rank of $[\alpha]_s^n$ cannot be less than s. In fact the matrix on the right in (2) has rank s, and its rank cannot exceed the rank of either factor matrix on the left. Thus again the rank of $[\alpha]_s^n$ cannot be less than s. We have then the following result:

If the given equation (1) *has s mutually orthogonal non-extravagant solutions, they must be unconnected. In particular any s mutually orthogonal real solutions must be unconnected, and any s mutually orthogonal unit solutions must be unconnected* ....(A).

Since equation (1) cannot have more than $n - r$ unconnected solutions, we deduce the following second result:

The given irreducible equation (1) *cannot have more than $n - r$ mutually orthogonal non-extravagant solutions. In particular it cannot have more than $n - r$ mutually orthogonal real solutions, and it cannot have more than $n - r$ mutually orthogonal unit solutions* ....(B).

Understanding a real matrix to be one whose elements are all real, we proceed to establish a third theorem:

Theorem. *When the undegenerate matrix $[a]_r^n$ is real, a complete set of $n - r$ real (unconnected) mutually orthogonal unit solutions of the given equation* (1) *can always be determined in an infinite number of ways, r being less than n*...(C).

To prove this we first note that the given equation (1) can be solved when some $n - r$ of the quantities $x_1, x_2, \dots x_n$ are arbitrary, and real solutions be obtained. We can therefore find real unit solutions. Let $[x_1 x_2 \dots x_n] = [\alpha_{11} \alpha_{12} \dots \alpha_{1n}]$ be any one real unit solution. To find a second solution orthogonal with it, we have to solve the equation

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{r1} & a_{r2} & \dots & a_{rn} \\ \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0, \quad \text{or} \quad \begin{bmatrix} a \\ \alpha \end{bmatrix}_{r,1}^n \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0.$$

If $r+1<n$, the rank of the matrix of this equation cannot exceed $r+1$ (in fact by Ex. iv of § 90 its rank is equal to $r+1$), and we can certainly find solutions in which $n-r-1$ of the variables $x_1, x_2, \ldots x_n$ are arbitrary. Since every solution is a homogeneous function of the arbitrary variables and has real coefficients, we can certainly find a real unit solution

$$[x_1\ x_2\ \ldots\ x_n]=[\alpha_{21}\ \alpha_{22}\ \ldots\ \alpha_{2n}],$$

and this solution and the previous one are two mutually orthogonal real unit solutions of the equation (1).

In general suppose that s mutually orthogonal real unit solutions

$$[x_1\ x_2\ \ldots\ x_n]=[\alpha_{11}\ \alpha_{12}\ \ldots\ \alpha_{1n}],\ [\alpha_{21}\ \alpha_{22}\ \ldots\ \alpha_{2n}],\ \ldots\ [\alpha_{s1}\ \alpha_{s2}\ \ldots\ \alpha_{sn}]$$

of equation (1) have been found. Then to find an $(s+1)$th solution orthogonal with all these, we have to solve the equation

$$\begin{bmatrix} a \\ \alpha \end{bmatrix}_{r,\,s}^{n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0.$$

The rank of the matrix of this equation is certainly not greater than $r+s$ (in fact by Ex. iv of § 90 it is equal to $r+s$); hence if $r+s<n$, we can certainly find solutions of the last equation in which some $n-r-s$ of the variables $x_1, x_2, \ldots x_n$ are arbitrary. We can therefore certainly find a real unit solution $[x_1\ x_2\ \ldots\ x_n]=[\alpha_{s+1,1}\ \alpha_{s+1,2}\ \ldots\ \alpha_{s+1,n}]$, and this solution and the previous ones form a system of $s+1$ mutually orthogonal real unit solutions of the equation (1).

Thus the number of mutually orthogonal real unit solutions can always be increased from s to $s+1$ so long as $s<n-r$ or $s+1<n+1-r$. That is, we can always determine a set of $n-r$ real mutually orthogonal unit solutions, and these are necessarily unconnected. We have thus proved theorem (C).

NOTE. *Results corresponding to* (C) *when* $[a]_r^n$ *is not real.*

When the matrix $[a]_r^n$ is not necessarily real, we can follow a similar process, determining at each stage a *non-extravagant* solution, so long as this is possible. Since in general this is possible throughout, *we can in general determine a complete set of* $n-r$ *mutually orthogonal non-extravagant solutions, and therefore a complete set of* $n-r$ *mutually orthogonal unit solutions* (which are not necessarily real). If however at any stage of the process we arrive at an equation $\begin{bmatrix} a \\ a \end{bmatrix}_{r,\,i}^{n} \overline{x}_n = 0$ which has only extravagant solutions, then the above argument fails. In fact when $[a]_r^n$ is not real there are exceptional cases (for example whenever $[a]_r^n$ is extravagant) in which it is not possible to determine a complete set of $n-r$ mutually orthogonal unit solutions.

Ex. i. Let $[x\ y\ z\ w]=[\mu, -\mu, \lambda, i\lambda]$, $[\mu', -\mu', \lambda', i\lambda']$ be any two solutions of the extravagant equation in Ex. xvi of § 90. They are mutually orthogonal only if $\mu\mu'=0$. If $\mu=0$, the first solution is extravagant; if $\mu'=0$, the second solution is extravagant. Thus there do not exist two mutually orthogonal non-extravagant solutions.

Ex. ii. We will find a complete set of mutually orthogonal real unit solutions of the equations

$$2x_1-3x_2+9x_3+x_4-3x_5=0,$$
$$3x_1+x_2-5x_3+3x_4-5x_5=0;$$

or

$$\begin{bmatrix} 2, & -3, & 9, & 1, & -3 \\ 3, & 1, & -5, & 3, & -5 \end{bmatrix} \overline{\underline{x}}_5 = 0 \quad \ldots\ldots(4).$$

Equation (4) can be solved when x_3, x_4, x_5 are arbitrary.

Putting $x_4=0$, $x_5=0$, $x_3\neq 0$, we obtain the real solution

$$[x_1\ x_2\ x_3\ x_4\ x_5]=[6,\ 37,\ 11,\ 0,\ 0]=[a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15}] \ldots\ldots(a).$$

We next find a real non-zero solution of the equation

$$\begin{bmatrix} 2, & -3, & 9, & 1, & -3 \\ 3, & 1, & -5, & 3, & -5 \\ 6, & 37, & 11, & 0, & 0 \end{bmatrix} \overline{\underline{x}}_5 = 0 \quad \ldots\ldots(5).$$

Equation (5) can be solved when x_2 and x_3 are arbitrary. Putting $x_3=0$, $x_2\neq 0$, we obtain the real solution

$$[x_1\ x_2\ x_3\ x_4\ x_5]=[148,\ -24,\ 0,\ 145,\ 171]=[a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25}] \ldots\ldots(\beta).$$

We then find a real non-zero solution of the equation

$$\begin{bmatrix} 2, & -3, & 9, & 1, & -3 \\ 3, & 1, & -5, & 3, & -5 \\ 6, & 37, & 11, & 0, & 0 \\ 148, & -24, & 0, & 145, & 171 \end{bmatrix} \overline{\underline{x}}_5 = 0 \quad \ldots\ldots(6).$$

Equation (6) can be solved when x_1 is arbitrary. The conjugate reciprocal of the matrix

$$\begin{bmatrix} -3, & 9, & 1, & -3 \\ 1, & -5, & 3, & -5 \\ 37, & 11, & 0, & 0 \\ -24, & 0, & 145, & 171 \end{bmatrix} \text{ is } \begin{bmatrix} -13618, & 666, & 14172, & -44 \\ 45806, & -22422, & 4416, & 148 \\ 32196, & 63378, & 2466, & 2418 \\ -29212, & -52806, & -102, & 1294 \end{bmatrix}.$$

Prefixing this matrix in equation (6), we obtain the real solution

$$[x_1\ x_2\ x_3\ x_4\ x_5]=[-286470,\ 35641,\ 36373,\ 313593,\ -12971] \ldots\ldots(\gamma).$$

The solutions (a), (β), (γ) form a complete set of three mutually orthogonal real solutions. In fact

$$\begin{bmatrix} 6, & 37, & 11, & 0, & 0 \\ 148, & -24, & 0, & 145, & 171 \\ -286470, & 35641, & 36373, & 313593, & -12971 \end{bmatrix} \begin{bmatrix} 6, & 148, & -286470 \\ 37, & -24, & 35641 \\ 11, & 0, & 36373 \\ 0, & 145, & 313593 \\ 0, & 171, & -12971 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix},$$

where $k_1=1526, \quad k_2=62746, \quad k_3=183166953400.$

Dividing the solutions (a), (β), (γ) by $\sqrt{k_1}$, $\sqrt{k_2}$, $\sqrt{k_3}$, we obtain three mutually orthogonal real unit solutions.

Ex. iii. *If all the solutions of the irreducible equation* (1) *are extravagant, then every two solutions are mutually orthogonal. Conversely if every two solutions are mutually orthogonal, then all solutions are extravagant.*

There will be no loss of generality in assuming that $(a)_r^r \neq 0$. Then using the same notation as in formula (A) of § 89, the general solution of (1) is given by

$$a\,[x_1\, x_2 \dots x_n] = [-\Sigma\, a_{1u}\, x_u,\ -\Sigma\, a_{2u}\, x_u,\ \dots\ -\Sigma a_{ru}\, x_u,\ a x_{r+1},\ a x_{r+2}, \dots\ a x_n],$$

where in the summations u receives the values $r+1$, $r+2, \dots n$, and where $x_{r+1}, x_{r+2}, \dots x_n$ are arbitrary.

Thus all solutions will be extravagant if and only if

$$(\Sigma\, a_{1u}\, x_u)^2 + (\Sigma\, a_{2u}\, x_u)^2 + \dots + (\Sigma\, a_{ru}\, x_u)^2 + a^2\left(x_{r+1}^2 + x_{r+2}^2 + \dots + x_n^2\right) = 0 \quad \dots(7)$$

identically, i.e. for all values of $x_{r+1}, x_{r+2}, \dots x_n$.

Also every two solutions will be mutually orthogonal if and only if

$$(\Sigma\, a_{1u}\, x_u)\ (\Sigma\, a_{1u}\, x_u') + (\Sigma\, a_{2u}\, x_u)\ (\Sigma\, a_{2u}\, x_u') + \dots + (\Sigma\, a_{ru}\, x_u)\ (\Sigma\, a_{ru}\, x_u')$$
$$+ a^2\,(x_{r+1}\, x'_{r+1} + x_{r+2}\, x'_{r+2} + \dots + x_n\, x_n') = 0 \quad \dots\dots(8)$$

identically, i.e. for all values of $x_{r+1}, x_{r+2}, \dots x_n, x'_{r+1}, x'_{r+2}, \dots x_n'$.

Now the coefficients of the various terms in (7) are the same as the coefficients of the various terms in (8). Accordingly the coefficients of the expression on the left in (7) all vanish when and only when the coefficients of the expression on the left in (8) all vanish. Thus each of the conditions (7) and (8) is satisfied when and only when the other condition is satisfied. This proves the above theorem.

Equating the coefficients to zero, we see that the necessary and sufficient conditions that all solutions shall be extravagant or that every two solutions shall be mutually orthogonal can be expressed in the form

$$\begin{bmatrix} a_{1,r+1} & a_{2,r+1} \dots & a_{r,r+1} \\ a_{1,r+2} & a_{2,r+2} \dots & a_{r,r+2} \\ \dots & \dots & \dots \\ a_{1n} & a_{2n} \dots & a_{rn} \end{bmatrix} \begin{bmatrix} a_{1,r+1} & a_{1,r+2} \dots & a_{1n} \\ a_{2,r+1} & a_{2,r+2} \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{r,r+1} & a_{r,r+2} \dots & a_{rn} \end{bmatrix} + a^2\,[1]_{n-r}^{n-r} = 0 \quad \dots\dots(9).$$

We can also obtain (9) as follows. Utilising the properties of passive rows we can express the conditions (7) and (8) respectively in the forms

$$[x_1\ x_2 \dots x_n]\ \phi \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0, \qquad [x_1\ x_2 \dots x_n]\ \phi \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = 0,$$

where ϕ is the complete matrix occurring on the left in (9).

By § 85 each of these conditions is equivalent to $\phi = 0$, i.e. to (9).

Ex. iv. Every equation which has only extravagant solutions can be reduced to the form

$$\begin{bmatrix} 1 & 0 \dots 0 & a_{1,r+1} & a_{1,r+2} \dots a_{1n} \\ 0 & 1 \dots 0 & a_{2,r+1} & a_{2,r+2} \dots a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 \dots 1 & a_{r,r+1} & a_{r,r+2} \dots a_{rn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0,$$

where $$\begin{bmatrix} a_{1,r+1} & a_{2,r+1} & \dots & a_{r,r+1} \\ a_{1,r+2} & a_{2,r+2} & \dots & a_{r,r+2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{rn} \end{bmatrix} \begin{bmatrix} a_{1,r+1} & a_{1,r+2} & \dots & a_{1n} \\ a_{2,r+1} & a_{2,r+2} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{r,r+1} & a_{r,r+2} & \dots & a_{rn} \end{bmatrix} = -[1]_{n-r}^{n-r}.$$

Illustrations are given in Ex. xviii of § 90.

Ex. v. *If the real irreducible equation* $[a]_r^n \overline{x}_n = 0$ *admits of finite solutions in which* $x_n \neq 0$, *then we can in an infinite number of ways find a real solution*

$$[x_1 \ x_2 \dots x_n] = [a_{r+1,1} \ a_{r+1,2} \dots a_{r+1,n}]$$

in which $x_n \neq 0$ *such that the real equation* $[a]_{r+1}^n \overline{x}_n = 0$ *is also irreducible and also admits of solutions in which* $x_n \neq 0$, *provided only that* $r+1 < n$.

The irreducible equation $[a]_r^n \overline{x}_n = 0$ admits of solutions in which $x_n \neq 0$ when and only when it admits of solutions in which $x_n = 1$. By § 88 this is the case when and only when $[a]_r^n$ has the same rank as $[a]_r^{n-1}$, i.e. when and only when $[a]_r^{n-1}$ has rank r.

We have therefore to show that if $r+1<n$ and if $[a]_r^{n-1}$ has rank r, then $[a]_{r+1}^{n-1}$ has rank $r+1$. There is no loss of generality in assuming that $(a)_r^r \neq 0$. Accordingly we will prove the theorem by showing that if $r+1<n$ and $(a)_r^r \neq 0$, then $(a)_{r+1}^{r+1} \neq 0$.

If $(a)_r^r \neq 0$, we can solve the equation $[a]_r^n \overline{x}_n = 0$ when $x_{r+1}, x_{r+2}, \dots x_n$ are arbitrary. Let $[x_1 \ x_2 \dots x_n] = [a_{r+1,1} \ a_{r+1,2} \dots a_{r+1,n}]$ be the solution obtained when $[x_{r+1} \ x_{r+2} \dots x_n] = [x \ 0 \dots 0 \ y]$, so that $[a_{r+1,r+1} \ a_{r+1,r+2} \dots a_{r+1,n-1} \ a_{r+1,n}] = [x \ 0 \dots 0 \ y]$.

Since the solution is homogeneous in $x_{r+1}, x_{r+2}, \dots x_n$, as shown in formula (A) of § 89, we can write

$$a_{r+1,1} = p_1 x + q_1 y, \quad a_{r+1,2} = p_2 x + q_2 y, \dots a_{r+1,r} = p_r x + q_r y,$$

where $p_1, p_2, \dots q_1, q_2, \dots$ do not contain x and y and are real.

We can also write

$$(a)_{r+1}^{r+1} = Ux + Vy,$$

where U and V do not contain x or y and are real.

Putting $x=1$, $y=0$, we see that $[x_1 \ x_2 \dots x_r \ x_{r+1} \ x_{r+2} \dots x_n] = [p_1 \ p_2 \dots p_r \ 1 \ 0 \dots 0]$ is a solution of the equation $[a]_r^n \overline{x}_n = 0$, and that therefore $[x_1 \ x_2 \dots x_r \ x_{r+1}] = [p_1 \ p_2 \dots p_r \ 1]$ is a solution of the real equation $[a]_r^{r+1} \overline{x}_{r+1} = 0$. It follows by Ex. iv of § 90 that the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} & a_{1,r+1} \\ a_{21} & a_{22} & \dots & a_{2r} & a_{2,r+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{r1} & a_{r2} & \dots & a_{rr} & a_{r,r+1} \\ p_1 & p_2 & \dots & p_r & 1 \end{bmatrix}$$

has rank $r+1$. Since the determinant of this matrix is U, it follows that $U \neq 0$. Hence

$Ux + Vy$ vanishes only when $x = -\frac{V}{U}y$. Accordingly we can assign real values to x and y such that

$$a_{r+1,\,n} = y \neq 0 \quad \text{and} \quad (a)_{r+1}^{r+1} = Ux + Vy \neq 0.$$

This establishes the theorem.

Ex. vi. *If in the irreducible equation* $[a]_r^n \overline{\underline{\,x\,}}_n = 0$, $[a]_r^n$ *is real and* $[a]_r^{n-1}$ *has rank* r, *then we can in an infinite number of ways determine a complete set of* $n-r$ *(unconnected) mutually orthogonal real solutions in each of which* $x_n \neq 0$; *consequently we can in an infinite number of ways determine a complete set of* $n-r$ *(unconnected) mutually orthogonal real unit solutions, in each of which* $x_n \neq 0$.

Since the given equation is irreducible, r is of necessity less than n. We first determine a real solution

$$[x_1\, x_2 \ldots x_n] = [a_{r+1,\,1}\; a_{r+1,\,2} \ldots a_{r+1,\,n}]$$

of the equation $[a]_r^n \overline{\underline{\,x\,}}_n = 0$ such that $a_{r+1,\,n} \neq 0$ and $[a]_{r+1}^{n-1}$ has rank $r+1$. We then determine a real solution

$$[x_1\, x_2 \ldots x_n] = [a_{r+2,\,1}\; a_{r+2,\,2} \ldots a_{r+2,\,n}]$$

of the equation $[a]_{r+1}^n \overline{\underline{\,x\,}}_n = 0$ such that $a_{r+2,\,n} \neq 0$ and $[a]_{r+2}^{n-1}$ has rank $r+2$. Proceeding in this way, we finally determine a real solution

$$[x_1\, x_2 \ldots x_n] = [a_{n1}\; a_{n2} \ldots a_{nn}]$$

of the equation $[a]_{n-1}^n \overline{\underline{\,x\,}}_n = 0$ such that $a_{nn} \neq 0$.

We have then determined $n-r$ mutually orthogonal real solutions of the original equation in each of which $x_n \neq 0$.

We can deduce a set of $n-r$ mutually orthogonal real solutions in each of which $x_n = 1$.

§ 95. Mutually orthogonal solutions of any system of linear equations.

Let the system of equations be equivalent as in § 88 to the irreducible equation

$$[x_1\, x_2 \ldots x_n\; 1]\, \overline{\underline{\,a\,}}_{n+1}^{\,r} = 0$$

or

$$[a]_r^{n+1} \overline{\underline{\,x\,}}_{n+1} = 0, \text{ where } x_{n+1} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then we have the following theorem:

Theorem. *If the irreducible equation* (1) *admits of finite solution, and if the matrix* $[a]_r^{n+1}$ *is real, then we can in an infinite number of ways determine a complete set of* $n+1-r$ *(unconnected) mutually orthogonal real augmented solutions; also, when the equation is non-homogeneous, we can in an infinite number of ways determine a complete set of* $n+1-r$ *(unconnected) mutually orthogonal real unaugmented non-zero solutions* ....(A).

We here consider that two augmented solutions $[x_1 x_2 \dots x_n 1]$, $[y_1 y_2 \dots y_n 1]$ are mutually orthogonal when $x_1 y_1 + x_2 y_2 + \dots + x_n y_n + 1 = 0$, and that two unaugmented solutions $[x_1 x_2 \dots x_n]$, $[y_1 y_2 \dots y_n]$ are mutually orthogonal when $x_1 y_1 + x_2 y_2 + \dots + x_n y_n = 0$. The truth of the theorem for augmented matrices follows immediately from Ex. vi of § 94.

To prove it for unaugmented matrices, we make use of Ex. v of § 94 and find in turn real non-zero solutions

$$[x_1 x_2 \dots x_n] = [a_{r+1,1}\ a_{r+1,2} \dots a_{r+1,n}],\ [a_{r+2,1}\ a_{r+2,2} \dots a_{r+2,n}],$$
$$[a_{r+3,1}\ a_{r+3,2} \dots a_{r+3,n}],\ \dots\ [a_{n+1,1}\ a_{n+1,2} \dots a_{n+1,n}]$$

of the equations

$$[a]_r^{n+1} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = 0, \qquad \begin{bmatrix} a_{11} & \dots & a_{1n} & a_{1,n+1} \\ a_{21} & \dots & a_{2n} & a_{2,n+1} \\ \dots & \dots & \dots & \dots \\ a_{r1} & \dots & a_{rn} & a_{r,n+1} \\ a_{r+1,1} & \dots & a_{r+1,n} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = 0,$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} & a_{1,n+1} \\ \dots & \dots & \dots & \dots \\ a_{r1} & \dots & a_{rn} & a_{r,n+1} \\ a_{r+1,1} & \dots & a_{r+1,n} & 0 \\ a_{r+2,1} & \dots & a_{r+2,n} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{bmatrix} = 0, \dots$$

such that $[a]_r^n$, $[a]_{r+1}^n$, $[a]_{r+2}^n$, ... $[a]_n^n$ have ranks r, $r+1$, $r+2$, ... n. We then have $n-r+1$ unaugmented real non-zero solutions which are mutually orthogonal.

Ex. In the case of the equation

$$[1\ 0\ 2\ 1] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0,$$

a complete set of three mutually orthogonal augmented solutions is

$$[x\ y\ z\ 1] = [-1,\ 1,\ 0,\ 1],\ [-1,\ -2,\ 0,\ 1],\ [1,\ 0,\ -1,\ 1],$$

and a complete set of three mutually orthogonal unaugmented solutions is

$$[x\ y\ z] = [-1,\ 1,\ 0],\ [-1,\ -1,\ 0],\ [0,\ 0,\ -\tfrac{1}{2}].$$

§ 96. Theorems relating to the connections between linear equations.

We will commence this article by proving the following fundamental theorem:

Theorem I. *If*

$$\left.\begin{aligned} f_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + a_{1,n+1} = 0 \\ f_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + a_{2,n+1} = 0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f_m &= a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + a_{m,n+1} = 0 \end{aligned}\right\} \quad \dots\dots\dots\dots(1)$$

is a system of m unconnected linear equations which admits of finite solution, then all solutions of this system of equations will satisfy the additional equation

$$y = b_1 x_1 + b_2 x_2 + \dots + b_n x_n + b_{n+1} = 0 \quad \dots\dots\dots(2)$$

if and only if equation (2) is connected with the equations (1) according to the definition of connection given in § 86, i.e. if and only if the matrix $[b_1\ b_2 \dots b_n\ b_{n+1}]$ *is connected with the horizontal rows of the matrix* $[a]_m^{n+1}$.

Since the system of *unconnected* equations (1) admits of finite solution, we know that the matrices $[a]_m^n$ and $[a]_m^{n+1}$ both have rank m. We will assume, as we may without loss of generality, that the equations are so written that

$$(a)_m^m = a \neq 0.$$

Then by formula (A) of § 88 the general solution of the system of equations (1) is given by

$$a\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = -\overline{\underline{A}}_m^m \begin{bmatrix} a_{1,m+1} & a_{1,m+2} & \dots & a_{1,n+1} \\ a_{2,m+1} & a_{2,m+2} & \dots & a_{2,n+1} \\ \dots & \dots & \dots & \dots \\ a_{m,m+1} & a_{m,m+2} & \dots & a_{m,n+1} \end{bmatrix} \begin{bmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \\ 1 \end{bmatrix} \quad \dots\dots\dots(3),$$

where $[A]_m^m$ is the reciprocal matrix of $[a]_m^m$, and $x_{m+1}, x_{m+2}, \dots x_n$ are arbitrary.

Also equation (2) can be written in the form

$$[b_1\ b_2 \dots b_m]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = -[b_{m+1}\ b_{m+2} \dots b_{n+1}]\begin{bmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \\ 1 \end{bmatrix} \quad \dots\dots\dots\dots(4).$$

Substituting from (3) in (4), we see that all solutions of (1) will satisfy (2) if and only if

$$[b_1\ b_2 \dots b_m]\overline{\underline{A}}_m^m \begin{bmatrix} a_{1,m+1} & a_{1,m+2} & \dots & a_{1,n+1} \\ a_{2,m+1} & a_{2,m+2} & \dots & a_{2,n+1} \\ \dots & \dots & \dots & \dots \\ a_{m,m+1} & a_{m,m+2} & \dots & a_{m,n+1} \end{bmatrix} \begin{bmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \\ 1 \end{bmatrix}$$

$$= a\,[b_{m+1}\ b_{m+2} \dots b_{n+1}]\begin{bmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \\ 1 \end{bmatrix}$$

for all values of $x_{m+1}, x_{m+2}, \dots x_n$.

Consequently, by § 85, all solutions of (1) will satisfy (2) if and only if

$$[b_1\ b_2 \dots b_m]\overline{\underline{A}}_m^m \begin{bmatrix} a_{1,m+1} & a_{1,m+2} & \dots & a_{1,n+1} \\ a_{2,m+1} & a_{2,m+2} & \dots & a_{2,n+1} \\ \dots & \dots & \dots & \dots \\ a_{m,m+1} & a_{m,m+2} & \dots & a_{m,n+1} \end{bmatrix} - a\,[b_{m+1}\ b_{m+2} \dots b_{n+1}] = 0 \dots(5).$$

Now, making use of Ex. x of § 62, equation (5) is

$$[B_{m+1}\ B_{m+2} \dots B_{n+1}] = 0,$$

where

$$B_{m+u} = \det\,[b_1\ b_2 \dots b_m]\ \overline{A}^{\,m}_{\,m} \begin{bmatrix} a_{1,\,m+u} \\ a_{2,\,m+u} \\ \vdots \\ a_{m,\,m+u} \end{bmatrix} - (a)^m_m\, b_{m+u}$$

$$= - \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1m} & a_{1,\,m+u} \\ a_{21} & a_{22} & \dots & a_{2m} & a_{2,\,m+u} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} & a_{m,\,m+u} \\ b_1 & b_2 & \dots & b_m & b_{m+u} \end{vmatrix} \qquad \dots\dots(6).$$

By Theorem 1 or Note 1 of § 71 the vanishing of the $n+1-m$ determinants B_{m+u} is a necessary and sufficient condition that the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,\,n} & a_{1,\,n+1} \\ a_{21} & a_{22} & \dots & a_{2,\,n} & a_{2,\,n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & a_{m,\,n+1} \\ b_1 & b_2 & \dots & b_n & b_{n+1} \end{bmatrix}$$

shall have rank m. Further since the first m horizontal rows of this matrix are unconnected, the matrix has rank m when and only when its last horizontal row is connected with its first m horizontal rows, i.e. when and only when the matrix $[b_1\ b_2 \dots b_{n+1}]$ is connected with the horizontal rows of the matrix $[a]^{n+1}_m$.

Thus we have proved that all solutions of (1) satisfy (2) when and only when the equation (2) is connected with the equations (1).

We easily deduce the two theorems which follow.

Theorem II. *If* (1) *is any system whatever of linear equations (not necessarily unconnected) which admits of finite solution, and if* (2) *is any other linear equation in the same variables, then all solutions of* (1) *will satisfy* (2) *if and only if equation* (2) *is connected with the equations* (1).

For if the first r of equations (1) are unconnected and the rest are connected with these, the solutions of the system (1) are identical with the common solutions of the first r of equations (1), and these solutions all satisfy (2) if and only if (2) is connected with the first r of equations (1). Moreover (2) is connected with the first r of equations (1) when and only when it is connected with the equations (1).

The last conclusion is obvious from the theory of connections between matrices, but can also be obtained by observing that g is a homogeneous linear function of $f_1, f_2, \dots f_r$ when and only when it is a homogeneous linear function of $f_1, f_2, \dots f_m$.

Theorem III. *If* (1) *is any system of linear equations admitting of finite solution, then some of these equations will be superfluous, being satisfied by all solutions of the remaining equations, if and only if there exist connections between the equations.*

For any one of the equations is superfluous if and only if it is connected with the rest of the equations.

From the above theorems we see that we could give the following alternative definitions of connection in the case of a system of linear equations admitting of finite solution:

(1) There are said to exist connections between a system of linear equations admitting of finite solution when some of the equations are satisfied by all common solutions of the rest, i.e. when some of the equations are necessary consequences of the rest, and are therefore superfluous.

(2) Any one of the equations is connected with the rest when it is satisfied by all common solutions of the rest, i.e. when it is a necessary consequence of the rest, and is therefore superfluous.

(3) The equations are unconnected when none of them are satisfied by all common solutions of the rest, i.e. when none of them are necessary consequences of the rest or none of them are superfluous.

We shall retain the definitions of connection given in § 86, and regard the above properties as consequences of those definitions.

§ 97. Functional dependences between a number of rational integral functions of several independent variables.

Let the variables be $x_1, x_2, \ldots x_n$ and the functions $\phi_1, \phi_2, \ldots \phi_m$.

Also let ϕ_{ij} be defined for the values $1, 2, \ldots m$ of i and the values $1, 2, \ldots n$ of j by the equation

$$\frac{\partial \phi_i}{\partial x_j} = \phi_{ij}.$$

Then we will prove the following theorem.

Theorem. *There is a functional dependence of the form* $F(\phi_1, \phi_2, \ldots \phi_m) = 0$ *between the rational integral functions* $\phi_1, \phi_2, \ldots \phi_m$ *if and only if the rank of the matrix*

$$\Phi = [\phi]_m^n = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{m1} & \phi_{m2} & \cdots & \phi_{mn} \end{bmatrix}$$

is less than m. If the rank of Φ *is r, where* $r < m$, *then r of the functions are functionally independent and the remaining* $m - r$ *functions are functionally dependent on these* ...(A).

First suppose that there exists a relation of the form

$$F(\phi_1, \phi_2, \ldots \phi_m) = 0 \quad \ldots\ldots(1),$$

which is true for all values of the variables.

Let the variables receive any independent infinitesimal increments $\delta x_1, \delta x_2, \ldots \delta x_n$, and let $\delta\phi_1, \delta\phi_2, \ldots \delta\phi_m$ be the corresponding increments in the functions. Then

$$\frac{\partial F}{\partial \phi_1}\delta\phi_1 + \frac{\partial F}{\partial \phi_2}\delta\phi_2 + \ldots + \frac{\partial F}{\partial \phi_m}\delta\phi_m = 0 \quad \ldots\ldots(2),$$

where $\frac{\partial F}{\partial \phi_1}, \frac{\partial F}{\partial \phi_2}, \ldots \frac{\partial F}{\partial \phi_m}$ are functions of $x_1, x_2, \ldots x_n$ which are not all identically equal to zero. Assuming that these functions can be expanded by Taylor's Theorem within a certain domain, it follows that within a certain domain there exists a relation of the form

$$\eta_1\,\delta\phi_1 + \eta_2\,\delta\phi_2 + \ldots + \eta_m\,\delta\phi_m = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where $\eta_1, \eta_2, \ldots \eta_m$ are rational integral functions of $x_1, x_2, \ldots x_n$ which do not all vanish identically. Regarding $\delta\phi_1, \delta\phi_2, \ldots \delta\phi_m$ as rational integral functions of $x_1, x_2, \ldots x_n$ containing arbitrary parameters $\delta x_1, \delta x_2, \ldots \delta x_n$, we have for the values $1, 2, \ldots m$ of i

$$\delta\phi_i = \phi_{i1}\,\delta x_1 + \phi_{i2}\,\delta x_2 + \ldots + \phi_{in}\,\delta x_n,$$

i.e. we have

$$[\delta\phi_1\ \delta\phi_2 \ldots \delta\phi_m] = [\delta x_1\ \delta x_2 \ldots \delta x_n]\ \left\lceil \phi \right\rfloor_n^m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (3) and (4), and then from Ex. i of § 85 it follows that

$$[\delta x_1\ \delta x_2 \ldots \delta x_n]\ \left\lceil \phi \right\rfloor_n^m \left\lceil \eta \right\rfloor_m = 0, \qquad \left\lceil \phi \right\rfloor_n^m \left\lceil \eta \right\rfloor_m = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The last result shows that everywhere within a certain domain there exists a connection between the vertical rows of $\left\lceil \phi \right\rfloor_n^m$ or between the horizontal rows of Φ. Consequently all minor determinants of Φ of order m vanish throughout that domain; and since they are integral functions, it follows that they vanish identically, and that the rank of Φ is less than m.

Thus a functional dependence of the form (1) *can only exist when the rank of* $[\phi]_m^n$ *is less than* m. *In particular if* $[\phi]_m^n$ *has rank* m, *the functions* $\phi_1, \phi_2, \ldots \phi_m$ *are functionally independent* $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$(6).

These conclusions can also be obtained as follows. Let $[\phi]_m^n$ have rank m. If we give any definite set of values to the variables $x_1, x_2, \ldots x_n$ and arbitrary values to $\delta\phi_1, \delta\phi_2, \ldots \delta\phi_m$, we can regard (4) as an equation for determining $\delta x_1, \delta x_2, \ldots \delta x_n$ in terms of $\delta\phi_1, \delta\phi_2, \ldots \delta\phi_m$. Since m cannot be greater than n, the augmented matrix of the equation has the same rank as the unaugmented matrix, except possibly for certain special values of the variables, and therefore the equation admits of finite solutions, or in this case of infinitesimal solutions. This shows that we can give such values to $\delta x_1, \delta x_2, \ldots \delta x_n$ that $\delta\phi_1, \delta\phi_2, \ldots \delta\phi_m$ assume any prescribed small values. Accordingly $\phi_1, \phi_2, \ldots \phi_m$ can vary independently of one another.

Thus if $[\phi]_m^n$ *has rank* m, *the functions* $\phi_1, \phi_2, \ldots \phi_m$ *can vary independently of one another and are functionally independent. Consequently a functional dependence of the form* (1) *can only exist when the rank of* $[\phi]_m^n$ *is less than* m $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$(7).

Now let Φ have rank r, where $r<m$, and suppose that $(\phi)_r^r$ is one of those derived determinants of order r of Φ which do not vanish identically. Then $[\phi]_r^n$ has rank r, and therefore by (6) or (7) the functions $\phi_1, \phi_2, \ldots \phi_r$ are functionally independent. By the properties of active rows, we deduce from (4) that

$$[\delta\phi_1\ \delta\phi_2 \ldots \delta\phi_r\ \delta\phi_u] = [\delta x_1\ \delta x_2\ \ldots\ \delta x_n] \begin{bmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{r1} & \phi_{u1} \\ \phi_{12} & \phi_{22} & \cdots & \phi_{r2} & \phi_{u2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_{1n} & \phi_{2n} & \cdots & \phi_{rn} & \phi_{un} \end{bmatrix} \quad \ldots\ldots\ldots(8).$$

We know that the last vertical row in the large matrix is connected with the first r vertical rows. Therefore there exists a relation of the form

$$\eta\,\delta\phi_u = \eta_1\,\delta\phi_1 + \eta_2\,\delta\phi_2 + \ldots + \eta_r\,\delta\phi_r,$$

where $\eta, \eta_1, \eta_2, \ldots \eta_r$ are rational integral functions of $x_1, x_2, \ldots x_n$, and η does not vanish identically; or a relation of the form

$$\delta\phi_u = \psi_1\,\delta\phi_1 + \psi_2\,\delta\phi_2 + \ldots + \psi_r\,\delta\phi_r \quad \ldots\ldots(9),$$

where $\psi_1, \psi_2, \ldots \psi_r$ are rational functions of $x_1, x_2, \ldots x_n$.

Since $\phi_1, \phi_2, \ldots \phi_r$ can be regarded as independent variables, the expression on the right is a perfect differential, and there exists a relation of the form

$$\phi_u = f(\phi_1, \phi_2, \ldots \phi_r) \quad \ldots\ldots(10).$$

Thus when the rank of the matrix $[\phi]_m^n$ *is r, r of the functions* $\phi_1, \phi_2, \ldots \phi_m$ *are functionally independent, and each of the remaining functions is functionally dependent on these r functions* ......(11).

We have now completely established the theorem (A).

NOTE. *Functional dependences between any m continuous functions of n independent variables.*

We will retain the same notation as before for the variables and the functions.

Then if the functions ϕ_{ij} and their first derivatives are continuous and capable of expansion by Taylor's Theorem, it can be shown that the theorem (A) is still true. In fact the proof of (7) is the same as before, and from (8) we deduce a relation of the form (9) where $\psi_1, \psi_2, \ldots \psi_r$ are continuous functions of the variables.

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The references are to pages. For *determinoid* and *determinant* the abbreviations *detoid* and *detant* are used.

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