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Measuring Portfolio Skewness

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Abstract

This research describes the proper specification and measurement of portfolio skewness and presents empirical results about the behavior of skewness parameters as portfolio size is altered. Since the analysis is mathematical in nature, the concepts developed in the paper can be applied to any security population.

MEASURING PORTFOLIO SKEWNESS

Traditional portfolio theory using the mean and variance of security return distributions is appropriate if investor utility functions are quadratic or if return distributions are characterized completely by their first two moments. Because of the restrictive assumptions required for mean-variance analysis, Arditti [2], Jean [20], Kraus and Litzenberger [22] and others [10, 21, 23, 30] have extended portfolio analysis to include the third moment of the expected return distribution. Recent papers by Simkowitz and Beedles [30], Kane [21], and Conine and Tamarkin [10] demonstrate that consideration of return distribution skewness can affect the measurement of expected utility and explain seemingly contradictory investor behavior such as antidiversification.[1]

The purpose of this paper is to develop the proper mathematical measurements for portfolio skewness and to examine the behavior of skewness statistics across different types of portfolios as portfolio size is varied. Our concern is with the proper estimation and measurement of portfolio skewness, not with the efficiency of any particular portfolio from an investment strategy or valuation perspective. Since the analysis is statistical in nature, the concepts can be applied to any security group.

Using several years of security data, skewness characteristics are calculated and compared for portfolios of common stocks, covered options and long option positions. The negatively skewed covered option returns permit an interesting contrast with the positively skewed stock and long option portfolios and illustrate that skewness and expected utility need not always deteriorate with diversification. Furthermore, because of the

extreme heterogeneity in security distribution moments within each port-
folio population, the data demonstrates the importance of considering the
variance in skewness and cross-sectional return deviations when estimating
portfolio skewness. Our results have important implications for research
[10, 21] using portfolio construction rules based on risk-skewness trade-
offs assuming portfolio parameter homogeneity.

In the following section the theoretical motivation for the analysis
of skewness is examined as well as analytical considerations regarding the
measurement of portfolio skewness. Part III describes the data base and
methodology employed, while Part IV presents the empirical results. Con-
clusions and implications are contained in the final section.

II. EXPECTED UTILITY AND THE MEASUREMENT OF PORTFOLIO SKEWNESS

Let R be a random variable representing return on investment and
U(R) a utility function quantifying the utility to an investor of the re-
turn, R. Let E(R) denote the expected value of the random variable R.
If the mathematical assumptions are satisfied, U may be expanded around
E(R) in a Taylor series and expectations taken of both sides to solve
for the expected investor utility of R:

$$E[U(R)] = U[E(R)] + \frac{U''[E(R)]}{2!} \sigma_R^2 + \frac{U'''[E(R)]}{3!} M_R^3 + \sum_{n=4}^{\infty} \frac{U^n[E(R)]}{n!} M_R^n \qquad (1)$$

where σ_R^2, M_R^3 and M_R^n represent the variance, skewness and higher moments
of R's probability distribution.

Sufficient conditions which require the inclusion of the third moment,
M_R^3, to evaluate expected utility include: a utility function of higher
order than quadratic, or the inadequacy of the mean and variance to describe

the distribution of returns. Research [2, 11, 21, 22, 28] has demonstrated that a preference (aversion) for positive (negative) skewness is consistent for rational investors having utility functions other than quadratic. Much research [2, 3, 4, 5, 6, 14, 15, 17, 18, 24] has examined the positive skewness typically found in stock returns. Regarding options, the empirical literature [19, 25, 26, 27, 32] consistently reports non-normal, skewed return distributions across different samples and various differencing intervals. Given the empirical presence and theoretical importance of skewness, an accurate measurement of portfolio skewness is crucial if the impact of alternative diversification policies upon investor utility is to be accurately assessed. Proper evaluation of portfolio skewness requires consideration of the following three skewness components.

A. The Expected Level of Time Series Skewness

Portfolio skewness traditionally is measured by the skewness in the time series return distribution. In a portfolio context, this skewness in return, M_n^3, on a portfolio of n securities is calculated as:

$$M_n^3 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} x_i x_j x_k M_{ijk} \qquad (2)$$

where x_i, x_j, and x_k represent the portfolio proportions invested in assets i, j, and k and M_{ijk} indicates the coskewness between the time series returns on i, j, and k.

Under a naive or random investment policy of equal investment in each security, equation (A5) gives the relationship between the expected level of skewness for a portfolio of n securities and the market level of skewness:[2,3]

$$E(M_n^3) = [\frac{1}{n^2(N-2)}] \left[[\frac{(N-1)(N-2)-(n-1)(n-2)}{(N-1)}] \overline{M}^3 + 3(n-1)(N-n)\overline{M}_{iij} \right]$$

$$+ [\frac{(n-1)(n-2)N^2}{n^2(N-1)(N-2)}] M_N^3 \qquad\qquad (A5)$$

where: $E(M_n^3)$ = expected skewness on a portfolio of n securities

n = number of securities in the portfolio

N = number of securities in the population

$\overline{M}^3$ = average skewness for a one security portfolio

$\overline{M}_{iij}$ = average curvilinear relationship for the population

M_N^3 = market (systematic) skewness for an equally-weighted portfolio of all N securities in the population

Examination of equation (A5) reveals that the effect on $E(M_n^3)$ of changing portfolio size is a priori indeterminate. Unlike portfolio return and variance, the parameters describing $E(M_n^3)$ can assume positive, negative or zero values; thus, $E(M_n^3)$ may increase, decrease or remain the same with diversification. In addition, equation (A5) indicates that expected skewness equals the average one security skewness, $\overline{M}^3$, when n = 1, then passes through the average curvilinear product, $\overline{M}_{iij}$, and eventually equals market portfolio skewness, M_N^3, when n = N.

Because coskewness structures differ across security groups, diversification will have different effects on $E(M_n^3)$ for alternative portfolios. Strategies including call options, for example, contain extreme leverage but low correlation of returns, whereas portfolios of stocks and covered options are more conservative with higher correlation in returns. Consequently, the relative magnitudes of systematic, M_N^3, and unsystematic, $(\overline{M}^3 - M_N^3)$, skewness will vary dramatically between these security groups and differences will exist in the effects of diversification upon the

absolute level of expected skewness. Furthermore, unlike portfolio return and risk, the proportionate change (percentage change in unsystematic skew) as n goes from one to N will also differ across security groups.[4]

Finally, it is important to realize that (A5) is a measure of only the _expected_ or _mean_ level of portfolio skewness at portfolio size n. For example, in a population containing 100 securities, 4950 (100 x 99/2) unique two security portfolios can be formed; (A5) is the cross-sectional average of the skewness found in these portfolios.

Previous research [5, 10, 21, 30] has focused only upon the expected level of time series skewness (A5) when evaluating the expected utility of alternative diversification policies. However, for investors holding portfolios smaller than the market, two additional components of skewness should be recognized. One, the variance in skewness, provides information about the dispersion of possible values of skewness at any portfolio size; the other, cross-sectional skewness, measures deviations of portfolio returns from the expected return of the market. Calculations for these skewness measures are described below.

B. The Variance in Skewness

If the time series skewness on a portfolio of n securities is the skewness measure that is of interest, then the investor should be concerned not only with its expected value (A5), but also with the dispersion of possible values it can assume; or, the variance in skewness. Analytically, the variance in skewness measure, equation (C32), allows the investor to infer how closely the skewness of any particular portfolio will compare to the expected level of skewness at a particular portfolio size. While previous analyses have neglected this statistic, ignoring

the variance in skewness requires the unrealistic assumption of portfolio skewness homogeneity--all portfolios of a given size have the same level of skewness.

$$E(M_{i,n}^3 - \overline{M}_n^3)^2 = (\tfrac{1}{n})^5 \left[(1 - \tfrac{n-1}{N-1}) E(M_i^3 - \overline{M}^3)^2 + 9(n-1) [1 - \tfrac{n(n-1)}{N(N-1)}] \times \right.$$

$$[\overline{M}_{iij}^2 + E(M_{iij}M_{ijj})] + 9(n-1)(n-2)[1 - \tfrac{n(n-1)(N-2)}{(n-2)N(N-1)}][E(M_{iij}M_{iik}) + 2E(M_{iij}M_{jjk})$$

$$+ E(M_{iij}M_{kkj})] + 9(n-1)(n-2)(n-3)[1 - \tfrac{n(n-1)(N-2)(N-3)}{(n-2)(n-3)N(N-1)}] E(M_{iij}M_{kkl})$$

$$+ 6(n-1)(n-2)[1 - \tfrac{n(n-1)(n-2)}{N(N-1)(N-2)}] \overline{M}_{ijk}^2 + 18(n-1)(n-2)(n-3)[1 - \tfrac{n(n-1)(n-2)(N-3)}{(n-3)N(N-1)(N-2)}] \times$$

$$E(M_{ijk}M_{ijl}) + 9(n-1)(n-2)(n-3)(n-4) [1 - \tfrac{n(n-1)(n-2)(N-3)(N-4)}{(n-3)(n-4)N(N-1)(N-2)}] \times$$

$$E(M_{ijk}M_{ilm}) + (n-1)(n-2)(n-3)(n-4)(n-5) [1 - \tfrac{n(n-1)(n-2)(N-3)(N-4)(N-5)}{(n-3)(n-4)(n-5)N(N-1)(N-2)}] \times$$

$$E(M_{ijk}M_{lmn}) + 6(n-1) [1 - \tfrac{n}{N}] [E(M_i^3 M_{iij}) + E(M_i^3 M_{ijj})] + 6(n-1)(n-2) \times$$

$$[1 - \tfrac{n(N-2)}{(n-2)N}] E(M_i^3 M_{jjk}) + 6(n-1)(n-2) [1 - \tfrac{n}{N}] E(M_i^3 M_{ijk}) + 2(n-1)(n-2)(n-3) \times$$

$$[1 - \tfrac{n(N-3)}{(n-3)N}] E(M_i^3 M_{jkl}) + 36(n-1)(n-2) [1 - \tfrac{n(n-1)}{N(N-1)}] E(M_{iij}M_{ijk}) + 18(n-1) \times$$

$$(n-2)(n-3) [1 - \tfrac{n(n-1)(N-3)}{(n-3)N(N-1)}] [E(M_{iij}M_{ikl}) + E(M_{iij}M_{jkl})] + 6(n-1)(n-2)(n-3) \times$$

$$\left. (n-4) [1 - \tfrac{n(n-1)(N-3)(N-4)}{(n-3)(n-4)N(N-1)}] E(M_{iij}M_{klm}) \right] \tag{C32}$$

where: $E(M_{i,n}^3 - \overline{M}_n^3)^2$ = variance in skewness, the variance associated with the distribution of skews of portfolios of size n

$E(M_i^3 - \overline{M}^3)^2$ = variance associated with the distribution of skews of the individual securities (i = 1,...,N)

Equation (C32) reveals three aspects about the structure of portfolio skews that should be noted. First, the greatest uncertainty about the

skewness of a randomly selected portfolio occurs when n = 1, and declines
to zero when n = N. Second, the magnitude of the uncertainty at a given
size n is related to the risk characteristics of the individual securi-
ties. Deviations in returns become magnified in portfolio skewness
structures and stretch out the dispersion of portfolios' skews. Third,
the coskewness structure inherent in the security population affects
the rate at which skewness uncertainty is eliminated. Thus, the rate
at which (C32) approaches zero may vary considerably across security
populations.

Given the portfolio alternatives existing today, potential exists
for large levels of skewness uncertainties as well as enormous differences
across portfolio skewness uncertainty structures. For example, because
long positions in calls possess large deviations in returns, much larger
levels of skewness uncertainty would be expected in these portfolios rela-
tive to, say, covered option writing. On the other hand, the rate at
which uncertainty is eliminated will probably be faster for long option
positions, since there is less correlation in security returns.

Considering equations (A5) and (C32) together enables the investor
to assess the effect of portfolio size on expected portfolio skewness.
For portfolios in which $E(M_n^3)$ falls as portfolio size increases, as shown
in Figure 1, an investor might select a portfolio size at which the prob-
ability of portfolio skewness falling below some predetermined level is
less than, say 10%. For portfolios with increasing levels of $E(M_n^3)$,
shown in Figure 2, an additional incentive for diversification is provided.

Cross-Sectional Portfolio Skewness

Because the return on a particular portfolio may deviate not only from the expected return of the portfolio, but from the expected return of the market as well, cross-sectional portfolio skewness also should be considered by the investor. The term "cross-sectional" skewness refers to the skewness of portfolio expected returns about their average expected return on the market as depicted in Figure 3. For many portfolios the cross-sectional distribution of average returns is highly dispersed and extremely skewed. As diversification proceeds, the distribution of average returns collapses about its mean--the market return. If positive cross-sectional skewness is present, diversification can severely reduce the upside average return potential of the chosen portfolio.[5]

We believe that cross-sectional skewness is an important consideration in portfolio analysis and argue that the traditional manner in which portfolio skewness is measured, (A5), is incorrect since it ignores the cross-sectional average return skewness. A more complete measure of expected skewness is the skewness about the expected return on the market. This "total" skewness is composed of both the skewness in the return of the portfolio about the mean return on the portfolio (A5) and the skewness caused by the difference between the expected return on the portfolio and the expected return on the market. Equation (A5) can be adjusted for this extra element and the total expected skewness function is given as:
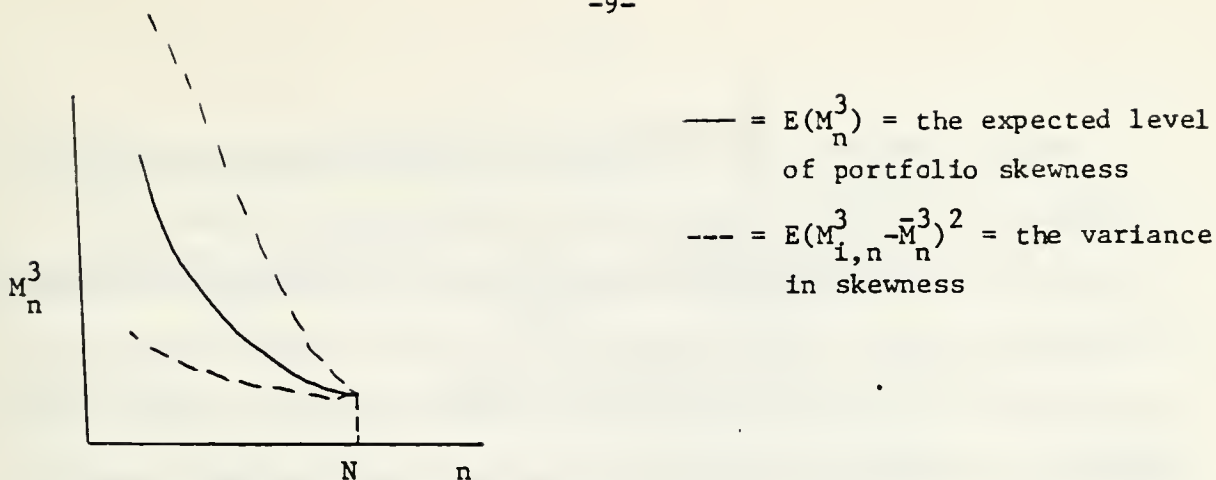
Figure 1: Diversification and its effects upon the
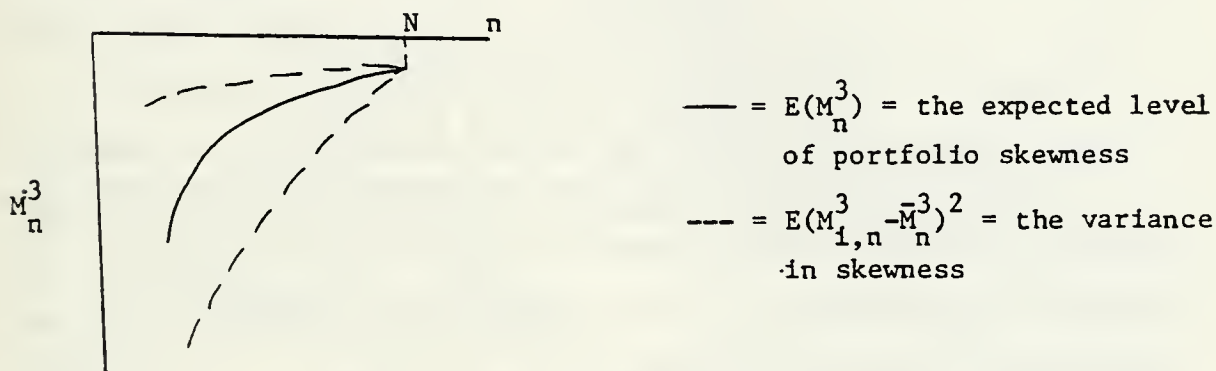variance about a declining mean level of portfolio skew

Figure 2: Diversification and its effects upon the
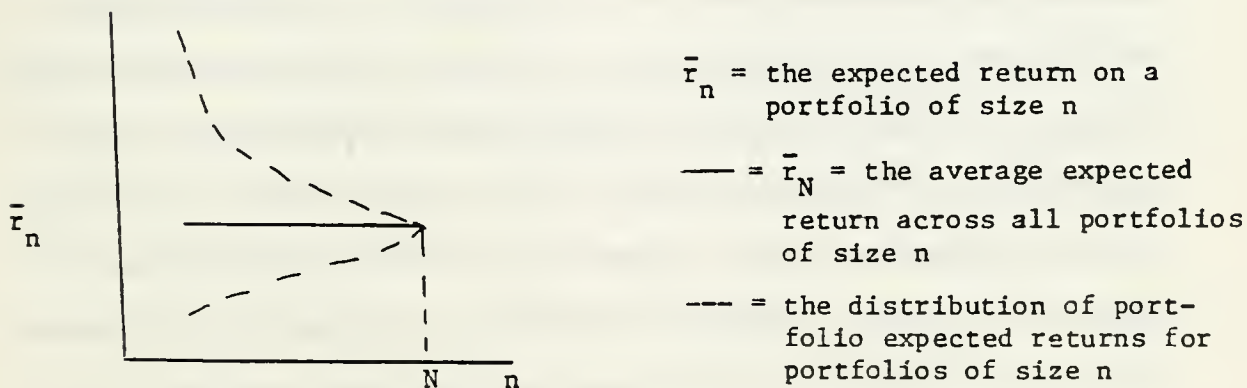variance about an increasing mean level of portfolio skew

Figure 3: Diversification and its effects upon the
distribution of portfolio expected returns

$$E(M_n^3) = [\frac{1}{n^2(N-2)}]\left[[\frac{(N-1)(N-2)-(n-1)(n-2)}{(N-1)}]\overline{M}^3 + 3(n-1)(N-n)\overline{M}_{iij}\right]$$

$$+ [\frac{(n-1)(n-2)N^2}{n^2(N-1)(N-2)}]M_N^3 + [\frac{1}{n^2(N-2)}]\left[[\frac{(N-1)(N-2)-(n-1)(n-2)}{(N-1)}]E(\overline{r}_i-\overline{r}_N)^3\right.$$

$$\left. + 3(n-1)(N-n) E[(\overline{r}_i-\overline{r}_N)^2(\overline{r}_j-\overline{r}_N)]\right] \qquad \text{(A5)} + \text{(D1)}$$

where: $E(\overline{r}_i-\overline{r}_N)^3$ = skewness of the individual securities' expected returns about the expected return on the population

$E[(\overline{r}_i-\overline{r}_N)^2(\overline{r}_j-\overline{r}_N)]$ = average return curvilinear relationship for the population

$\overline{r}_i$ = expected return on security i

$\overline{r}_N$ = expected return on the market

The following section illustrates the measurement of the three components of skewness described above for selected sample portfolios. Differences in the behavior of these statistics across security groups and for different levels of diversification are striking.

III. ILLUSTRATING THE CONCEPTS

The objectives of this section are to illustrate the differences in magnitude and behavior of the various components of portfolio skewness for different security populations. Five diverse portfolios were chosen as samples, primarily because of the differences between their return distributions. The five portfolios include common stocks, at-the-money and out-of-the-money covered option writing portfolios and at-the-money and out-of-the-money long option positions. After the data and methodology for calculating security returns are described, results are presented to illustrate the effects of diversification upon the elements of portfolio skewness.

The Data and Methodology

The sample chosen includes the 136 stocks having listed options available on December 31, 1975. Securities not having complete price data on the Compustat tapes over the period July 1, 1963 to December 31, 1978, were eliminated, resulting in 102 sample securities for analysis. Although the choice of this particular group introduces a selection bias in the study, these securities represent over one-third of the population of listed option securities; thus, these results may be inferred to the current universe of optionable stocks.

Since listed options were not available until 1973, six month premiums for the 102 stocks sample were generated for the 15 1/2 year sample period using the Black and Scholes pricing model, equation (3):

$$C = PN(D1) - Ke^{-rt}N(D2)$$

where:

$$D1 = [\ln(P/K) + (r + \frac{1}{2}V^2)t]/V\sqrt{t}$$
$$D2 = D1 - V\sqrt{t}$$

The beginning of period price P, was obtained from the Compustat tapes; time to maturity, t, was specified as 180 days; the daily equivalent of the six month commercial paper rate was proxied for the risk-free rate, r; and the variance rate, V^2, was estimated from the log of daily price changes obtained from the CRSP tapes for the six months prior to each option pricing date. The impact of dividends on the option premium was considered by reducing the stock price by the present value of dividends paid during the life of the option (see [8]). The above data were used to generate option premiums, C, across two exercise prices (K): at-the-money and 10 percent out-of-the-money.

Use of Black and Scholes beginning of period option premiums is be-
lieved necessary to generate a sample period of sufficient length and to
standardize stock price/exercise price ratios.[6] The similarity between
Black-Scholes model prices and actual premiums has been demonstrated
[7, 25] (in [25], the divergence between at-the-money model and market
premiums is reported to be about 0.1%).

Semiannual returns (gross of commissions) on each long option posi-
tion for the thirty-one six month holding periods were calculated by
dividing the beginning of period call value as determined by equation (5)
into the intrinsic value of the option at maturity. Intrinsic value is
the maximum of zero or the difference between stock price and striking
price at option maturity.

Semiannual returns on each covered writing position were calculated
by dividing the beginning stock price less the option premium received
into the sum of stock price at the end of the period plus dividends, less
the option's intrinsic value at maturity. Stock holding period returns
include price appreciation plus dividends. Commissions are ignored in
all transactions.

Return Distribution Statistics for Alternative Portfolios

Table I presents return distribution statistics for the five secur-
ity groups examined. Line 1 reveals that average returns increase (5.01%
to 21.00%) as one goes from writing strategies to call options with suc-
cessively higher exercise prices, while total risk as measured by the
average security variance, $\overline{\sigma}^2$, increases from 42.51 to 75,934.70. That
long positions in call options contain large amounts of systematic risk

TABLE I

Return Distribution Statistics for Alternative Samples[a]

	Symbol	Writing P/K=1.0	Writing P/K=.9	Stocks	Calls P/K=1.0	Calls P/K=.9
1. Sample mean return	$\bar{r}_N$	5.01%	5.41%	6.61%	17.60%	21.00%
2. Average security variance	$\bar{\sigma}^2$	130.51	207.97	696.90	31,869.90	75,934.70
3. Market variance	σ_N^2	42.51	74.80	277.90	8,735.90	14,156.90
4. Average security skewness	$\bar{M}^3$	-2423.13	-2945.90	29,001.52	18,311,019.90	138,872,696.00
5. Market skewness	M_N^3	-156.96	-147.08	5,354.31	875,729.92	2,239,765.17
6. Average curvilinear relationship	$\bar{M}_{11j}$	-600.52	-729.19	8,696.80	2,606,978.33	10,895,000.79
7. Skewness of mean portfolio returns	$E(\bar{r}_i-\bar{r}_N)^3$	1.28	1.81	46.54	14,698.36	168,882.34
8. Average mean return product	$E[(\bar{r}_i-\bar{r}_N)^2(\bar{r}_j-\bar{r}_N)]$	-.01	-.02	-.46	-145.53	-1,672.12
9. Variance in skewness	$E(M_i^3-\bar{M}^3)^2$	7,501,924.52	9,491,720.24	3,350,191,749.63	5,295,339,843.75 $\times 10^6$	6,354,172,509.58 $\times 10^8$

[a]Each sample consists of 102 securities and all statistics relate to six month differencing intervals over the period examined.

relative to stocks and covered writing portfolios is shown by the market variance, σ_N^2 (line 3).

Average security skewness $(\overline{M}^3)$ and systematic (market portfolio) skewness, (M_N^3), data presented in lines 4-5 exhibit a wide range of values and behavior. The data imply that for stocks and long option positions, diversification will reduce the positive skewness $(M_N^3 < \overline{M}_{iij} < \overline{M}^3$ where M_N^3, $\overline{M}_{iij}$ and $\overline{M}^3 > 0)$ whereas for option writing strategies, increasing portfolio size will lower (a benefit) the negative skewness $(M_N^3 > \overline{M}_{iij} > \overline{M}^3$ where M_N^3, $\overline{M}_{iij}$ and $\overline{M}^3 < 0).$[7]

Lines 7 thru 9 indicate the relative importance of the additional skewness component considerations. The variance in skewness (line 9) contains an enormous amount of uncertainty regarding the level of skewness for even the relatively low-risk option writing positions. The uncertainty becomes incredibly large for stocks and long option portfolios. In particular, the very large variance in skewness for out-of-the-money calls $(6,354,172,509.58 \times 10^8)$ is attributable to the dispersion in the underlying return distributions for these assets. While many options expired worthless, some showed returns of several thousand percent.

Deviations in returns also is documented in the cross-sectional average return skewness figures, $E(\overline{r}_i - \overline{r}_N)^3$. This skewness element is very large (162,882.34) for out-of-the-money calls and becomes smaller as one moves into stocks and covered writing. However, even the writing portfolios exhibit positive cross-sectional skewness, which largely is attributable to a few securities that generated high option premiums but exhibited an ex-post stability in prices. This resulted in large average returns for these positions, relative to other covered positions. Since

investors can diversify their holdings, it is instructive to examine the behavior of the various elements of portfolio skewness in response to changes in portfolio size.

Diversification and Changes in Portfolio Skewness

Using the summary skewness and coskewness data from Table I for each sample, equation (A5) allows the traditional time series skewness measures for any portfolio size to be calculated. Table II presents relationships between portfolio size and the time series skewness for the five portfolios examined. The results indicate a wide spectrum of portfolio size-skewness relationships. First, for the writing strategies, increasing portfolio size is beneficial in eliminating much of the negative skewness present in these security positions, with over 90% being potentially diversifiable. On the other hand, for stocks and call options, the diversification process has a damaging effect upon positive skewness. This is particularly evident for the out-of-the-money portfolio, where over 98% of the skewness is unsystematic and thus can be destroyed with diversification.

Second, the rate at which expected skewness approaches its market level varies considerably across the portfolios. For example, $E(M_n^3)$ changes least rapidly for the option writing strategies; thus, larger covered option portfolios are needed to replicate market skewness. On the other hand, slight changes in portfolio size in long option portfolios rapidly destroys positive skewness.

TABLE II

The Effects of Portfolio Size Upon the Time Series Element of Expected Skewness for Alternative Samples

Portfolio Size	Writing, P/K=1.0*	% Change in $\bar{M}^3-M_N^3$**	Writing, P/K=.9*	% Change in $\bar{M}^3-M_N^3$**	Stocks*	% Change in $\bar{M}^3-M_N^3$**	Calls, P/K=1.0*	% Change in $\bar{M}^3-M_N^3$**	Calls, P/K=.9*	% Change in $\bar{M}^3-M_N^3$**
1 ($\bar{M}^3$)	-2423.13	—	-2945.90	—	29,001.52	—	18,311,019.90	—	138,872,696.00	—
2	-1056.17	60.32	-1330.62	57.71	13,772.98	64.40	6,532,988.72	67.55	42,889,424.59	70.25
3	-701.45	75.97	-833.77	73.68	10,187.30	79.56	3,955,224.70	82.34	23,130,651.21	84.71
4	-543.02	82.96	-677.52	81.05	8,673.93	85.96	2,919,138.88	88.28	15,545,448.98	90.26
5	-454.02	86.89	-559.26	85.27	7,855.34	89.42	2,378,382.17	91.38	11,728,466.66	93.06
10	-289.64	94.15	-335.12	93.28	6,419.37	95.50	1,478,882.01	96.54	5,746,314.84	97.43
20	-214.26	97.47	-229.22	97.07	5,801.99	98.11	1,120,138.56	98.60	3,581,145.33	97.43
40	-178.28	99.06	-177.82	98.90	5,518.46	99.31	963,518.49	99.50	2,705,037.17	99.66
102	-156.96	100.00	-147.08	100.00	5,354.31	100.00	875,729.92	100.00	2,239,765.17	100.00
$1-(M_N^3/\bar{M}^3)$	93.52%		95.01%		81.54%		95.22%		98.39%	

*Calculated using equation (A5) and lines 4-6 of Table I.

** $= \dfrac{\bar{M}^3-E(M_Q^3)}{\bar{M}^3-M_N^3}$, any sample's % may differ slightly due to rounding.

Diversification and the Uncertainty About Skewness

In Table III, the relationships between portfolio size and the var-
iance in skewness (equation (C32)) are presented. The results demonstrate
that diversification quickly reduces uncertainty about portfolio skewness.
For example, by the three security level about 97% of the variance in
skewness has been eliminated for the writing strategies and about 99%
has been removed for stocks and long option positions.

Due to the differences in magnitudes of the numbers, the results
presented in Tables II-III illustrate that the traditional method of eval-
uating diversification and portfolio risk in terms of the percent of
diversifiable risk eliminated can be misleading when applied to portfolio
skewness measures. First, much smaller portfolios are required to achieve
similar absolute levels of expected skewness. For example, a forty secur-
ity out-of-the-money option portfolio contains a greater expected skewness
(2,705,037.17) and variance in skewness (3349.61 x 10^8) than a ten
security at-the-money option portfolio (1,478,822.01 and 332,036.55 x 10^6).
Second, even though 99.9% of the variance in skewness has been eliminated
at the five security level for out-of-the-money options, the magnitude
of this uncertainty is 4,253,340.92 x 10^8. This is over 36 million times
the mean level of skewness (4,253,340.92 x 10^8 vs. 11,728,466.66). On
the other hand, the uncertainty about the level of skewness for at-the-
money writing at the five security level is about 62 times the mean level
of skewness (28,444.72 vs. -454.02). Furthermore, these levels of un-
certainty are still quite large for large portfolios. Note that the
level of skewness uncertainty for a forty security at-the-money writing
portfolio is 3971.33 which is about 22 times the mean level of skewness

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TABLE III

The Effects of Portfolio Size Upon the Variance in Skewness for Alternative Samples

Portfolio Size	Writing P/K=1.0*	% Change**	Writing, P/K=.9*	% Change**	Stocks*	% Change**	Calls P/K=1.0* (x 10^6)	% Change**	Calls P/K=.9* (x 10^8)	% Change**
1	7,501,924.52	—	9,491,720.24	—	3,350,191,749.63	—	5,295,339,843.75	—	6,354,172,509.58	—
2	718,607.07	90.42	1,839,819.33	80.62	208,491,903.52	93.78	205,951,542.67	96.11	234,636,492.17	96.31
3	156,017.54	97.92	504,077.40	94.69	38,817,422.08	98.84	31,744,956.21	99.40	35,491,793.59	99.44
4	67,534.65	99.10	248,911.98	97.38	13,970,484.58	99.58	9,820,963.64	99.81	10,687,179.40	99.83
5	28,444.72	99.62	136,376.97	98.56	6,966,548.94	99.79	4,019,097.83	99.92	4,253,340.92	99.93
10	12,166.62	99.84	76,682.56	99.19	1,230,611.22	99.96	332,036.55	99.99	308,950.08	99.99
20	8,193.09	99.89	47,064.58	99.50	290,508.16	99.99	40,632.02	99.99	29,523.33	99.99
40	3,971.33	99.95	21,444.49	99.77	66,899.54	99.99	6,982.29	99.99	3,349.61	99.99
102	0	100.00	0	100.00	0	100.00	0	100.00	0	100.00

*Calculated using equation (C32)

** $= \dfrac{E(M_1^3 - \bar{M}^3)^2 - E(M_{1,n}^3 - \bar{M}_n^3)^2}{E(M_1^3 - \bar{M}^3)^2}$, any sample's % may differ slightly due to rounding

(-178.28). For out-of-the-money options, this same comparison yields a factor of over 123,000 (3,349.61 x 10^8 vs. 2,705,037.17).

These results reveal that the level of skewness uncertainty can differ dramatically across alternative stock/option portfolio positions. Also the data indicate that the variance in skewness can be quite large even for large portfolios of "relatively" low risk assets. Thus, a dilemma exists for investors seeking to establish an appropriate portfolio size, given a preference (aversion) for positive (negative) skewness. For the option writing portfolios, diversification is beneficial since it reduces the negative skewness present in these positions. However, since the level changes more slowly (see Table II) than for stocks or long options, more securities are required to achieve the same benefits of diversification. Furthermore, the uncertainty about the level of skewness is quite large, even for large portfolios. These factors may motivate option writers to hold relatively larger portfolios than would be selected if only the expected level of time series skewness was considered.

On the other hand, investors holding long positions in stocks and options face a different and more complex tradeoff. Increasing portfolio size reduces positive skewness (an undesirable effect), while at the same time reducing the uncertainty about skewness (a positive consideration). Given the extreme variance in skewness present in the sample portfolios, it appears that some investors may be motivated to increase portfolio size so that a more precise estimate of portfolio skewness could be obtained.

Cross-Sectional Portfolio Skewness

Values for the cross-sectional component of expected skewness (equation (D1)) at varying portfolio sizes are presented in Table IV. As indicated by the first column under each group, the values fall rapidly for all portfolios as portfolio size is increased. The importance of this element can be gauged through an analysis of the effects that diversification has upon the maximum potential return at each portfolio size. This is shown in the second column under each strategy in Table IV. For example, over the 15 1/2 year sample period, for out-of-the-money options, there is the opportunity to earn a return of 264.94% at the one security level. At the ten security level, the largest portfolio return is 120.56%, about one-half the size one amount. The effect that increasing portfolio size has upon the upside return potential becomes less dramatic as one moves from option buying to stocks to option writing. Recognition of this component of skewness may result in a willingness on the part of the investor to trade some of the reduction in return uncertainty to preserve some upside return potential, especially in the case of option buying.

IV. CONCLUSION

This paper has presented three components of portfolio skewness and illustrated the magnitude and diversity of these statistics across five portfolio groups. The results indicate that investors who hold portfolios smaller than the market should consider not only time series skewness but also the variance in skewness and cross sectional skewness in returns when determining an appropriate portfolio size.

TABLE IV

The Effects of Portfolio Size Upon the Cross-Sectional Component of Expected Skewness
for Alternative Samples

Portfolio Size	Writing P/K=1.0*		Writing P/K=.9*		Stocks*		Calls P/K=1.0*		Calls P/K=.9*	
	Average Return Skewness*	Maximum $\bar{r}_n$**	Average Return Skewness*	Maximum $\bar{r}_n$**	Average Return Skewness*	Maximum $\bar{r}_n$**	Average Return Skewness*	Maximum $\bar{r}_n$**	Average Return Skewness*	Maximum $\bar{r}_n$**
1	1.28	8.72%	1.81	9.96%	46.54	18.48%	14,698,36	111.39%	168,822.34	264.94%
2	.31	8.70	.44	9.65	11.29	18.48	3,565.44	102.34	40,966.29	200.11
3	.13	8.63	.19	9.54	4.87	17.30	1,536.78	98.43	17,657.21	177.58
4	.07	8.60	.10	9.45	2.65	16.40	837.88	93.03	9,626.97	165.65
5	.05	8.57	.06	9.34	1.64	15.84	519.48	89.23	5,968.60	155.23
10	.01	8.20	.01	8.98	.35	14.14	109.79	75.49	1,261.36	120.56
20	.00	7.57	.00	8.44	.06	12.16	18.50	59.34	212.49	89.63
40	.00	6.63	.00	7.44	.00	10.12	1.24	44.80	14.24	64.67
102	0	5.01, $\bar{r}_N$	0	5.41, $\bar{r}_N$	0	6.61, $\bar{r}_N$	0	17.60, $\bar{r}_N$	0	21.00, $\bar{r}_N$

*Calculated using equation (01) and lines 7 and 8 of Table I.

** = maximum return possible for an equally-weighted n security portfolio.

Investors in positively skewed portfolios such as common stocks or long options may be motivated to hold portfolios smaller than the market to preserve the greatest amount of positive skew. However, the extreme uncertainty about expected skewness in these portfolios is an encouragement for some diversification. Conversely, option writing portfolios possess negative time series skewness which can be reduced through diversification. Because greater certainty about the skewness estimate is possible with larger portfolios, investors in these securities should be motivated to hold the market portfolio of covered call positions. Results presented above are consistent with observed investor behavior of antidiversification for investors in stock and options [31].

Footnotes

[1] In this paper the term "skewness" will refer to a distribution's third moment. Many authors use the term "skewness" to denote the third moment divided by the cube of the standard deviation.

[2] The assumption of an equal weighting scheme is consistent with the literature which examines the effect of portfolio size on portfolio distribution parameters (for example, see [12, 30]). An equal or random investment policy is optimal when one is unable to predict future return distribution parameters. Knowledge of the future structure of returns implies that security weights (investment proportions) can be adjusted to improve the parameter structures of portfolios.

[3] The derivation of all results are presented in the accompanying Appendixes.

[4] Since the expected level of portfolio return is constant (and equals $\bar{r}_N$, the average market return) for all n, the percentage change in $E(r_n)$ equals zero for all n for all security groups. From Elton and Gruber [12, p. 419], the portfolio expected risk equivalent of equation (5A) is:

$$E(\sigma_n^2) = \frac{1}{n} \bar{\sigma}^2 \left(1 - \frac{n-1}{N-1}\right) + \left(\frac{N}{N-1}\right)\left(\frac{n-1}{n}\right) \sigma_N^2,$$

where $\bar{\sigma}^2$ is the average one security variance and σ_N^2 is the market portfolio risk. The percentage of unsystematic risk $(\bar{\sigma}^2 - \sigma_N^2)$ which has

been eliminated at portfolio size n is given by the ratio:
$[\bar{\sigma}^2 - E(\sigma_n^2)]/(\bar{\sigma}^2 - \sigma_N^2)$ which equals $\frac{N(n-1)}{n(N-1)}$, and thus is the same for all
security groups and independent of the levels of the population parameters
$\bar{\sigma}^2$ and σ_N^2. On the other hand, the percentage change in unsystematic
skew is given by $[\bar{M}^3 - E(M_n^3)]/(\bar{M}^3 - M_N^3)$ and will be different for different
populations because of the influence of the parameter $\bar{M}_{iij}$.

[5]For example, assume there are only three possible n security
portfolios with return characteristics as shown below, and the investor
randomly selects one of these portfolios. Considering only the time
series skewness of each portfolio would imply that additional diversifi-
cation will have no further effect on portfolio skewness. However, if
cross-sectional skewness is examined it becomes evident that consider-
able skewness is still present in the possible portfolio returns. Even
though the average expected return is 30% regardless of the portfolio
combination selected, the upside return potential (average return skew-
ness) differs significantly across the possible portfolio combinations.
By selecting only one portfolio, there is the opportunity to earn 70%,
whereas combining all three portfolios produces a maximum potential
return of only 30%. Diversifying destroys the average return skewness;
consequently, the investor may be willing to trade some of the reduction
in return uncertainty to preserve the upside average return potential
and thus may be motivated to hold a portfolio smaller than the market.

Portfolio	Average Return $(\bar{r}_n)$	Time Series Risk (σ_n^2)	Time Series Skewness (M_n^3)
A	70%	0	0
B	10	0	0
C	10	0	0
Market	30%	0	0

[6]While it would be informative to use actual premiums, we believe
that deficiencies in the historical data base could provide misleading
results. These data problems include:

a) A short time period for analysis. The CBOE began trading
   listed options in 1973 on only sixteen securities.

b) Nonavailability of listed contracts for desired stock price/
   exercise price ratios. It has not been until the last few
   years that sufficient varieties of stock price/exercise price
   ratios have been available on most securities.

Further research can incorporate actual premiums once the listed option
market becomes more complete and the historical data base has been gener-
ated. The objective of our analysis was to select a sample of reasonable
size and sufficient duration so as to provide meaningful measures of
portfolio skewness.

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[7]Because the stock skewness statistic is highly sensitive to the differencing interval used [14], caution should be exercised when comparing the results of our stock sample data with previous studies of security skewness. Simkowitz and Beedles [30] used a 549 common stock sample and observed skewness to change from positive to negative values with increasing portfolio size. Their analysis was based on monthly, rather than six-month, returns. Merton, Scholes and Gladstein [25] used six-month returns and report mean, variance and positive skewness values similar to our results. Studies of common stock portfolios have reported both positive and negative skewness measures for various market periods [3, 5, 6, 15, 17, 19, 24, 32].

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Appendix A

The Expected Level of Portfolio Skewness

Equation (A5) is developed by decomposing equation (4) into its unsystematic and market components. The skewness of any equally-weighted $(x_i = \frac{1}{n})$ portfolio containing n securities is:

$$M_n^3 = \sum_{i=1}^{n} (\frac{1}{n})^3 M_i^3 + \sum_{i=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} (\frac{1}{n})^3 M_{iij} + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \sum_{k=1}^{n} (\frac{1}{n})^3 M_{ijk} \qquad (A1)$$

where $M_i^3 = E(r_i - \bar{r}_i)^3$, $M_{iij} = E[(r_i - \bar{r}_i)^2 (r_j - \bar{r}_j)]$ and $M_{ijk} = E[(r_i - \bar{r}_i) \times (r_j - \bar{r}_j)(r_k - \bar{r}_k)]$. In (A1) there are n terms like M_i^3, $3n(n-1)$ terms like M_{iij} and $n(n-1)(n-2)$ terms like M_{ijk} for a total of n^3 terms. Taking expected values:

$$E(M_n^3) = n(\frac{1}{n})^3 \bar{M}^3 + 3n(n-1)(\frac{1}{n})^3 \bar{M}_{iij} + n(n-1)(n-2)(\frac{1}{n})^3 \bar{M}_{ijk}$$

$$= (\frac{1}{n})^2 \bar{M}^3 + [\frac{3(n-1)}{n^2}] \bar{M}_{iij} + [\frac{(n-1)(n-2)}{n^2}] \bar{M}_{ijk} \qquad (A2)$$

where $\bar{M}^3$ is the average one security skewness for all securities in the population, $\bar{M}_{iij}$ is the average curvilinear product for the population and $\bar{M}_{ijk}$ is the population triplicate product average.

The relationship between $E(M_n^3)$ and market skewness can be developed by recognizing that the market portfolio skewness, M_N^3, of an equally weighted portfolio of all N securities in the population is given by:

$$M_N^3 = (\frac{1}{N})^2 \bar{M}^3 + [\frac{3(N-1)}{N^2}] \bar{M}_{iij} + [\frac{(N-1)(N-2)}{N^2}] \bar{M}_{ijk} \qquad (A3)$$

or: $$\bar{M}_{ijk} = [\frac{N^2}{(N-1)(N-2)}] \left[M_N^3 - (\frac{1}{N})^2 \bar{M}^3 - [\frac{3(N-1)}{N^2}] \bar{M}_{iij} \right] \qquad (A4)$$

Substituting (A4) into (A2) and rearranging produces equation (A5) in the text:

$$E(M_n^3) = [\frac{1}{N^2(N-2)}] \left[[\frac{(N-1)(N-2) - (n-1)(n-2)}{(N-1)}] \bar{M}^3 + 3(n-1)(N-n) \bar{M}_{iij} \right]$$

$$+ [\frac{(n-1)(n-2)N^2}{n^2(N-1)(N-2)}] M_N^3 \tag{A5}$$

Appendix B

Useful Formulas

In deriving the variability about portfolio skewness in Appendix C, it is useful to know the $E(\frac{1}{n} \sum_{i=1}^{n} Z_i)^2$, where Z represents any variable of interest. From Elton and Gruber [12, p. 427]:

$$E(\frac{1}{n} \sum_{i=1}^{n} Z_i)^2 = \frac{1}{n} (1 - \frac{n-1}{N-1}) \sigma_Z^2 + (\bar{Z})^2 \tag{B1}$$

where $\bar{Z}$ and σ_Z^2 represent the mean and variance about Z.

In Appendix D, we need to know the $E(\frac{1}{n} \sum_{i=1}^{n} Z_i)^3$. Using (A1) and (A2), we see that:

$$E(\frac{1}{n} \sum_{i=1}^{n} Z_i)^3 = (\frac{1}{n})^3 E(\sum_{i=1}^{n} Z_i^3 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} Z_i^2 Z_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\substack{k=1 \\ i \neq j = k}}^{n} Z_i Z_j Z_k)$$

$$= (\frac{1}{n})^2 \bar{Z}^3 + [\frac{3(n-1)}{n^2}] E(Z_i^2 Z_j) + [\frac{(n-1)(n-2)}{n^2}] E(Z_i Z_j Z_k) \tag{B2}$$

But, for a population of N securities:

$$\bar{Z} = \frac{1}{N} \sum_{i=1}^{N} Z_i$$

$$(\bar{Z})^3 = (\frac{1}{N} \sum_{i=1}^{N})^3$$

$$= (\frac{1}{N})^2 \bar{Z}^3 + [\frac{3(N-1)}{N^2}] E(Z_i^2 Z_j) + [\frac{(N-1)(N-2)}{N^2}] E(Z_i Z_j Z_k)$$

or: $E(Z_i Z_j Z_k) = [\frac{N^2}{(N-1)(N-2)}][(\bar{Z})^3 - (\frac{1}{N})^2 \bar{Z}^3 - [\frac{3(N-1)}{N^2}] E(Z_i^2 Z_j)]$ (B3)

Substituting (B3) into (B2):

$$E(\frac{1}{n} \sum_{i=1}^{n} Z_i)^3 = [\frac{1}{N^2(N-2)}] \left[[\frac{(N-1)(N-2)-(n-1)(n-2)}{(N-1)}] \bar{Z}^3 + 3(n-1)(N-n)E(Z_i^2 Z_j) \right]$$

$$+ [\frac{(n-1)(n-2)N^2}{n^2(N-1)(N-2)}] (\bar{Z})^3$$ (B4)

But: $M_Z^3 = E(Z-\bar{Z})^3 = \bar{Z}^3 - 3\bar{Z}^2\bar{Z} + 2(\bar{Z})^3,$

Or: $\bar{Z}^3 = M_Z^3 + 3\bar{Z}^2\bar{Z} - 2(\bar{Z})^3$ (B5)

Substituting (B5) into (B4):

$$E(\frac{1}{n} \sum_{i=1}^{n} Z_i)^3 = [\frac{1}{N^2(N-2)}] \left[[\frac{(N-1)(N-2)-(n-1)(n-2)}{(N-1)}] [M_Z^3 + 3\bar{Z}^2\bar{Z} - 2(\bar{Z})^3] \right.$$

$$\left. + 3(n-1)(N-n)E(Z_i^2 Z_j) \right] + [\frac{(n-1)(n-2)N^2}{n^2(N-1)(N-2)}] (\bar{Z})^3$$ (B6)

Appendix C

The Variability in Portfolio Skewness

If we define $\sigma^2(M_n^3)$ as the dispersion of portfolio skews at a given portfolio size n, then using the result from (A1) and (A2):

$$\sigma^2(M_n^3) = E(M_{i,n}^3 - \bar{M}_n^3)^2 = E\left[\{\sum_{i=1}^{n} (\frac{1}{n})^3 M_i^3 - (\frac{1}{n}) \bar{M}^3\} + [\sum_{i=1}^{n} \sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{i=1}^{n} (\frac{1}{n})^3 M_{iij} \right.$$

$$\left. - [\frac{3n(n-1)}{n^3}] \bar{M}_{iij}] + [\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \sum_{k=1}^{n} (\frac{1}{n})^3 M_{ijk} - [\frac{n(n-1)(n-2)}{n^3}] \bar{M}_{ijk}] \right]^2$$ (C1)

Expression (C1) contains three squared terms and twelve cross-product terms. Analyzing the first squared term:

$$E[\sum_{i=1}^{n} (\frac{1}{n})^3 M_i^3 - (\frac{1}{n})^2 \bar{M}^3]^2 = (\frac{1}{n})^4 E[\frac{1}{n} \sum_{i=1}^{n} (M_i^3 - \bar{M}^3)]^2$$

Using (B1), we note that $Z_i = M_i^3 - \bar{M}^3$ which means that $E(Z_i) = \bar{Z} = 0$

Thus: $\quad E[\sum_{i=1}^{n} (\frac{1}{n})^3 M_i^3 - (\frac{1}{n}) \bar{M}^3]^2 = (\frac{1}{n})^5 (1 - \frac{n-1}{N-1}) E(M_i^3 - \bar{M}^3)^2 \qquad$ (C2)

For the second squared term in (C1):

$$E[\sum_{i=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} (\frac{1}{n})^3 M_{iij} - [\frac{3n(n-1)}{n^3}] \bar{M}_{iij}]^2 = (\frac{1}{n})^6 E[\sum_{i=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} M_{iij}]^2$$

$$- [\frac{6n(n-1)}{n^6}] \bar{M}_{iij} E(\sum_{i=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} M_{iij}) + [\frac{3n(n-1)}{n^3}]^2 (\bar{M}_{iij})^2$$

$$= (\frac{1}{n})^6 E[\sum_{i=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} M_{iij}]^2 - [\frac{3n(n-1)}{n^3}]^2 (\bar{M}_{iij})^2 \qquad (C3)$$

But, through decomposition we find that:

$$E[\sum_{i=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} M_{iij}]^2 = 9n(n-1) [\bar{M}_{iij}^2 + E(M_{iij} M_{ijj})] + 9n(n-1)(n-2) \times$$

$$[E(M_{iij} M_{iik}) + 2 E(M_{iij} M_{jjk}) + E(M_{iij} M_{kkj})] + 9n(n-1)(n-2)(n-3) E(M_{iij} M_{kkl})$$

$$(C4)$$

Similarly: $(\bar{M}_{iij})^2 = \left[\dfrac{\displaystyle\sum_{i=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}M_{iij}}{3N(N-1)}\right]^2 = \left[\dfrac{1}{9N^2(N-1)^2}\right]\left[9N(N-1)[\bar{M}_{iij}^2\right.$

$+ E(M_{iij}M_{ijj})] + 9N(N-1)(N-2)[E(M_{iij}M_{iik}) + 2E(M_{iij}M_{jjk}) + E(M_{iij}M_{kkj})]$

$+ 9N(N-1)(N-2)(N-3)\left. E(M_{iij}M_{kkl})\right]$ (C5)

Substituting (C4) and (C5) into (C3) and rearranging yields (C6):

$E\left[\displaystyle\sum_{\substack{i=1}}^{n}\sum_{\substack{i=1}}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}(\tfrac{1}{n})^3 M_{iij} - [\dfrac{3n(n-1)}{n^3}]\bar{M}_{iij}\right]^2 = (\tfrac{1}{n})^5\left[9(n-1)[1 - \dfrac{n(n-1)}{N(N-1)}] \times \right.$

$[\bar{M}_{iij}^2 + E(M_{iij}M_{ijj})] + 9(n-1)(n-2)[1 - \dfrac{n(n-1)(N-2)}{(n-2)N(N-1)}][E(M_{iij}M_{iik})$

$+ 2E(M_{iij}M_{jjk}) + E(M_{iij}M_{kkj})] + 9(n-1)(n-2)(n-3)[1 - \dfrac{n(n-1)(N-2)(N-3)}{(n-2)(n-3)N(N-1)}] \times$

$E(M_{iij}M_{kkl})\Big]$ (C6)

For the third squared term in (C1):

$E\left[\displaystyle\sum_{\substack{i=1}}^{n}\sum_{\substack{j=1}}^{n}\sum_{\substack{k=1\\i\neq j\neq k}}^{n}(\tfrac{1}{n})^3 M_{ijk} - [\dfrac{n(n-1)(n-2)}{n^3}]\bar{M}_{ijk}\right]^2 = (\tfrac{1}{n})^6 E\left[\displaystyle\sum_{\substack{i=1}}^{n}\sum_{\substack{j=1}}^{n}\sum_{\substack{k=1\\i\neq j\neq k}}^{n} M_{ijk}\right]^2$

$- [\dfrac{2n(n-1)(n-2)}{n^6}]\bar{M}_{ijk}\, E\left[\displaystyle\sum_{\substack{i=1}}^{n}\sum_{\substack{j=1}}^{n}\sum_{\substack{k=1\\i\neq j\neq k}}^{n} M_{ijk}\right] + [\dfrac{n(n-1)(n-2)}{n^6}]^2(\bar{M}_{ijk})^2$

$= (\tfrac{1}{n})^6 E\left[\displaystyle\sum_{\substack{i=1}}^{n}\sum_{\substack{j=1}}^{n}\sum_{\substack{k=1\\i\neq j\neq k}}^{n} M_{ijk}\right]^2 - [\dfrac{n(n-1)(n-2)}{n^3}]^2 (\bar{M}_{ijk})^2$ (C7)

Expanding the first bracketed term in (C7):

$$E[\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} M_{ijk}]^2 = 6n(n-1)(n-2)\,\bar{M}_{ijk}^2 + 18n(n-1)(n-2)(n-3) \times$$

$$E(M_{ijk} M_{ijl}) + 9n(n-1)(n-2)(n-3)(n-4)\,E(M_{ijk} M_{ilm}) + n(n-1)(n-2)(n-3)(n-4) \times$$

$$(n-5)\,E(M_{ijk} M_{lmn}) \tag{C8}$$

Also:

$$(\bar{M}_{ijk})^2 = \left[\frac{\sum_{\substack{i=1 \ j=1 \ k=1 \\ i\neq j \neq k}}^{n}\;\;\;^{n}\;\;\;^{n} M_{ijk}}{N(N-1)(N-2)}\right]^2 = [\frac{1}{N^2(N-1)^2(N-2)^2}]\left[6N(N-1)(N-2)\,\bar{M}_{ijk}^2\right.$$

$$+ 18N(N-1)(N-2)(N-3)\,E(M_{ijk} M_{ijl}) + 9N(N-1)(N-2)(N-3)(N-4)\,E(M_{ijk}M_{ilm})$$

$$\left.+ N(N-1)(N-2)(N-3)(N-4)(N-5)\,E(M_{ijk} M_{lmn})\right] \tag{C9}$$

Substituting (C8) and (C9) into (C7) and rearranging produces (C10):

$$E[\sum_{\substack{i=1 \ j=1 \ k=1 \\ i\neq j \neq k}}^{n}\;\;^{n}\;\;^{n} (\frac{1}{n})^3 M_{ijk} - [\frac{n(n-1)(n-2)}{n^3}]\,\bar{M}_{ijk}]^2 = (\frac{1}{n})^5 \left[6(n-1)(n-2) \times\right.$$

$$[1 - \frac{n(n-1)(n-2)}{N(N-1)(N-2)}]\,\bar{M}_{ijk}^2 + 18(n-1)(n-2)(n-3)\,[1 - \frac{n(n-1)(n-2)(N-3)}{(n-3)N(N-1)(N-2)}]\,E(M_{ijk} M_{ijl})$$

$$+ 9(n-1)(n-2)(n-3)(n-4)\,[1 - \frac{n(n-1)(n-2)(N-3)(N-4)}{(n-3)(n-4)N(N-1)(N-2)}]\,E(M_{ijk} M_{ilm})$$

$$\left.+ (n-1)(n-2)(n-3)(n-4)(n-5)\,[1 - \frac{n(n-1)(n-2)(N-3)(N-4)(N-5)}{(n-3)(n-4)(n-5)N(N-1)(N-2)}]\,E(M_{ijk} M_{lmn})\right]$$

$$\tag{C10}$$

Now consider the twelve cross-product terms in (C1). They are:

$$2E[\sum_{i=1}^{n} (\frac{1}{n})^3 M_i^3 \quad \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} (\frac{1}{n})^3 M_{iij}] \tag{C11}$$

$$-2[\frac{3n(n-1)}{n^3}] \bar{M}_{iij} \; E[\sum_{j=1}^{n} (\frac{1}{n})^3 M_i^3] = [\frac{-6(n-1)}{n^4}] \bar{M}^3 \bar{M}_{iij} \tag{C12}$$

$$-2(\frac{1}{n})^2 \bar{M}^3 \; E[\sum_{i=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} (\frac{1}{n})^3 M_{iij}] = [\frac{-6(n-1)}{n^4}] \bar{M}^3 \bar{M}_{iij} \tag{C13}$$

$$2(\frac{1}{n})^2 \bar{M}^3 [\frac{3n(n-1)}{n^3}] \bar{M}_{iij} = [\frac{6(n-1)}{n^4}] \bar{M}^3 \bar{M}_{iij} \tag{C14}$$

$$2E[\sum_{i=1}^{n} (\frac{1}{n})^3 M_i^3 \quad \sum_{\substack{i=1 \\ i \neq j \neq k}}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (\frac{1}{n})^3 M_{ijk}] \tag{C15}$$

$$-2[\frac{n(n-1)(n-2)}{n^3}] \bar{M}_{ijk} \; E[\sum_{i=1}^{n} (\frac{1}{n})^3 M_i^3] = [\frac{-2(n-1)(n-2)}{n^4}] \bar{M}^3 \bar{M}_{ijk} \tag{C16}$$

$$-2(\frac{1}{n})^2 \bar{M}^3 \; E[\sum_{\substack{i=1 \\ i \neq j \neq k}}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (\frac{1}{n})^3 M_{ijk}] = [\frac{-2(n-1)(n-2)}{n^4}] \bar{M}^3 \bar{M}_{ijk} \tag{C17}$$

$$2(\frac{1}{n})^2 \bar{M}^3 [\frac{n(n-1)(n-2)}{n^3}] \bar{M}_{ijk} = [\frac{2(n-1)(n-2)}{n^4}] \bar{M}^3 \bar{M}_{ijk} \tag{C18}$$

$$2E[\sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} (\frac{1}{n})^3 M_{iij} \quad \sum_{\substack{i=1 \\ i \neq j \neq k}}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (\frac{1}{n})^3 M_{ijk}] \tag{C19}$$

$$-2[\frac{n(n-1)(n-2)}{n^3}] \bar{M}_{ijk} \; E[\sum_{i=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} (\frac{1}{n})^3 M_{iij}] = [\frac{-6(n-1)^2(n-2)}{n^4}] \bar{M}_{iij} \bar{M}_{ijk} \tag{C20}$$

$$-2\left[\frac{3n(n-1)}{n^3}\right] \bar{M}_{iij} \, E\left[\sum_{\substack{i=1 \\ i \neq j \neq k}}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{1}{n}\right)^3 M_{ijk}\right] = \left[\frac{-6(n-1)^2(n-2)}{n^4}\right] \bar{M}_{iij} \, \bar{M}_{ijk}$$

$$2\left[\frac{3n(n-1)}{n^3}\right] \bar{M}_{iij} \left[\frac{n(n-1)(n-2)}{n^3}\right] \bar{M}_{ijk} = \left[\frac{6(n-1)^2(n-2)}{n^4}\right] \bar{M}_{iij} \, \bar{M}_{ijk} \qquad \text{(C22)}$$

Note that (C13) cancels (C14), (C17) cancels (C18) and (C21) cancels (C22). This leaves (C11) and (C12), (C15) and (C16), and (C19) and (C20).

Concerning (C11) and (C12), (C11) can be decomposed as:

$$\text{(C11)} = 2\left(\frac{1}{n}\right)^6 \left[3n(n-1)[E(M_i^3 M_{iij}) + E(M_i^3 M_{iij})] + 3n(n-1)(n-2) \, E(M_i^3 M_{jjk})\right]$$

As for (C12), examine the expression: $\sum_{i=1}^{N} \frac{M_i^3}{N} \sum_{i=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} \frac{M_{iij}}{3N(N-1)}$

In this expression there are $3N(N-1)$ terms like $(M_i^3 M_{iij})$, $3n(n-1)$ terms like $(M_i^3 M_{iij})$ and $3n(n-1)(n-2)$ terms of the form $(M_i^3 M_{jjk})$ for a total of $3N^2(N-1)$ terms. Thus:

$$\bar{M}^3 \bar{M}_{iij} = \left[\frac{1}{3N^2(N-1)}\right] \left[3N(N-1)[E(M_i^3 M_{iij}) + E(M_i^3 M_{iij})] + 3N(N-1)(N-2) \, E(M_i^3 M_{jjk})\right]$$

$$\text{(C24)}$$

Substituting (C23) for (C11) and (C24) for (C12) and then adding (C23) and (C24) and rearranging produces (C25):

$$\text{(C23)} + \text{(C24)} = \left(\frac{1}{n}\right)^5 \left[6(n-1)[1 - \frac{n}{N}][E(M_i^3 M_{iij}) + E(M_i^3 M_{iij})] + 6(n-1)(n-2) \times \right.$$

$$\left. [1 - \frac{n(N-2)}{(n-2)N}] \, E(M_i^3 M_{jjk}) \right] \qquad \text{(C25)}$$

Working with (C15) and (C16), (C15) can be expanded as:

$$\text{(C15)} = 2\left(\frac{1}{n}\right)^6 \left[3n(n-1)(n-2) \, E(M_i^3 M_{ijk}) + n(n-1)(n-2)(n-3) \, E(M_i^3 M_{jkl})\right] \qquad \text{(C26)}$$

As for (C16), consider the expression: $\sum\limits_{i=1}^{N} \dfrac{M_i^3}{N} \sum\limits_{i=1}^{N} \sum\limits_{\substack{j=1 \\ i\neq j \neq k}}^{N} \sum\limits_{k=1}^{N} \dfrac{M_{ijk}}{N(N-1)(N-2)}$,

where there are $3N(N-1)(N-2)$ terms like $(M_i^3 \, M_{ijk})$ and $N(N-1)(N-2)(N-3)$ terms

of the form $(M_i^3 \, M_{jkl})$ for a total of $N^2(N-1)(N-2)$ terms. Hence:

$$\bar{M}^3 \, \bar{M}_{ijk} = [\dfrac{1}{N^2(N-1)(N-2)}] \left[3N(N-1)(N-2) \, E(M_i^3 \, M_{ijk}) + N(N-1)(N-2)(N-3)E(M_i^3 \, M_{jkl}) \right]$$

(C27)

Substituting (C26) for (C15) and (C27) for (C16) and then adding (C26)

and (C27) and rearranging produces (C28):

$$(C26) + (C27) = (\tfrac{1}{n})^5 \left[6(n-1)(n-2) \, [1 - \tfrac{n}{N}] \, E(M_i^3 \, M_{ijk}) + 2(n-1)(n-2)(n-3) \times \right.$$

$$\left. [1 - \tfrac{n(N-3)}{(n-3)N}] \, E(M_i^3 \, M_{jkl}) \right]$$

(C28)

Finally, working with (C19) and (C20), we find that (C19) can be

expressed as:

$$(C19) = 2(\tfrac{1}{n})^6 \left[18n(n-1)(n-2) \, E(M_{iij} \, M_{ijk}) + 9n(n-1)(n-2)(n-3) \, [E(M_{iij} \, M_{ikl}) \right.$$

$$\left. + E(M_{iij} \, M_{jkl})] + 3n(n-1)(n-2)(n-3)(n-4) \, E(M_{iij} \, M_{klm}) \right]$$

(C29)

As for (C20), examine: $\sum\limits_{i=1}^{N} \sum\limits_{\substack{i=1 \\ i\neq j}}^{N} \sum\limits_{j=1}^{N} \dfrac{M_{iij}}{3N(N-1)} \sum\limits_{i=1}^{N} \sum\limits_{\substack{j=1 \\ i\neq j \neq k}}^{N} \sum\limits_{k=1}^{N} \dfrac{M_{ijk}}{N(N-1)(N-2)}$,

which has $18N(N-1)(N-2)$ terms of the form $(M_{iij} \, M_{ijk})$, $9N(N-1)(N-2)(N-3)$

terms like $(M_{iij} \, M_{ikl})$ and $3N(N-1)(N-2)(N-3)(N-4)$ terms like $(M_{iij} \, M_{klm})$

for a total of $3N^2(N-1)^2(N-2)$ terms. Therefore:

$$\bar{M}_{iij} \, \bar{M}_{ijk} = [\dfrac{1}{3N^2(N-1)^2(N-2)}] \left[18N(N-1)(N-2) \, E(M_{iij} \, M_{ijk}) + 9N(N-1)(N-2)(N-3) \times \right.$$

$$\left. E(M_{iij} \, M_{ikl}) + 3N(N-1)(N-2)(N-3)(N-4) \, E(M_{iij} \, M_{klm}) \right]$$

(C30)

Substituting (C29) for (C19) and (C30) for (C20) and then adding (C29) and (C30) and rearranging produces (C31):

$$(C29) + (C30) = (\tfrac{1}{n})^5 \left[36(n-1)(n-2)[1 - \tfrac{n(n-1)}{N(N-1)}] \, E(M_{iij} M_{ijk}) + 18(n-1)(n-2) \; \times \right.$$

$$(n-3) \, [1 - \tfrac{n(n-1)(N-3)}{(n-3)N(N-1)}] \, [E(M_{iij} M_{ikl}) + E(M_{iij} M_{jkl})]$$

$$\left. + \, 6(n-1)(n-2)(n-3)(n-4) \, [1 - \tfrac{n(n-1)(N-3)(N-4)}{(n-3)(n-4)N(N-1)}] \, E(M_{iij} M_{klm}) \right] \qquad (C31)$$

Thus, the variance in portfolio skewness at portfolio size n is the sum of (C2), (C6), (C10), (C25), (C28) and (C31):

$$E(M_{i,n}^3 - \bar{M}_n^3)^2 = (\tfrac{1}{n})^5 \left[(1 - \tfrac{n-1}{N-1}) \, E(M_i^3 - \bar{M}^3)^2 + 9(n-1) \, [1 - \tfrac{n(n-1)}{N(N-1)}] \; \times \right.$$

$$[\bar{M}_{iij}^2 + E(M_{iij} M_{ijj})] + 9(n-1)(n-2) \, [1 - \tfrac{n(n-1)(N-2)}{(n-2)N(N-1)}] \, [E(M_{iij} M_{iik}) + 2E(M_{iij} M_{jj\ell})$$

$$+ \, E(M_{iij} M_{kkj})] + 9(n-1)(n-2)(n-3) \, [1 - \tfrac{n(n-1)(N-2)(N-3)}{(n-2)(n-3)N(N-1)}] \, E(M_{iij} M_{kkl})$$

$$+ \, 6(n-1)(n-2) \, [1 - \tfrac{n(n-1)(n-2)}{N(N-1)(N-2)}] \, \bar{M}_{ijk}^2 + 18(n-1)(n-2)(n-3) \, [1 - \tfrac{n(n-1)(n-2)(N-3)}{(n-3)N(N-1)(N-2)}] \; \times$$

$$E(M_{ijk} M_{ijl}) + 9(n-1)(n-2)(n-3)(n-4) \, [1 - \tfrac{n(n-1)(n-2)(N-3)(N-4)}{(n-3)(n-4)N(N-1)(N-2)}] \; \times$$

$$E(M_{ijk} M_{ilm}) + (n-1)(n-2)(n-3)(n-4)(n-5) \, [1 - \tfrac{n(n-1)(n-2)(N-3)(N-4)(N-5)}{(n-3)(n-4)(n-5)N(N-1)(N-2)}] \; \times$$

$$E(M_{ijk} M_{lmn}) + 6(n-1) \, [1 - \tfrac{n}{N}] \, [E(M_i^3 M_{iij}) + E(M_i^3 M_{ijj})] + 6(n-1)(n-2) \; \times$$

$$[1 - \tfrac{n(N-2)}{(n-2)N}] \, E(M_i^3 M_{jjk}) + 6(n-1)(n-2) \, [1 - \tfrac{n}{N}] \, E(M_i^3 M_{ijk}) + 2(n-1)(n-2)(n-3) \; \times$$

$$[1 - \tfrac{n(N-3)}{(n-3)N}] \, E(M_i^3 M_{jkl}) + 36(n-1)(n-2) \, [1 - \tfrac{n(n-1)}{N(N-1)}] \, E(M_{iij} M_{ijk}) + 18(n-1) \; \times$$

$$(n-2)(n-3) \, [1 - \tfrac{n(n-1)(N-3)}{(n-3)N(N-1)}] \, [E(M_{iij} M_{ikl}) + E(M_{iij} M_{jkl})] + 6(n-1)(n-2)(n-3) \; \times$$

$$(n-4) \, [1 - \tfrac{n(n-1)(N-3)(N-4)}{(n-3)(n-4)N(N-1)}] \, E(M_{iij} M_{klm}) \Bigg] \qquad (C32)$$

Appendix D

Another Element of Portfolio Skewness

Traditionally, portfolio skewness has been measured by the skewness

in the portfolio return distribution over time (Appendix A). However, a

more appropriate measure of portfolio skewness is total skewness which

measures the skewness of a portfolio's return about the market's expected

return and includes both the skewness in a portfolio mean return (equation

(A5)) as well as the skewness of the portfolio's average return about the

mean return on the market, $E(\bar{r}_n - \bar{r}_N)^3$. Note that: $E(\bar{r}_n - \bar{r}_N)^3 = E[\frac{1}{n} \sum_{i=1}^{n} (\bar{r}_i - \bar{r}_N)].^3$

Using (B6), we see that $Z_i = (\bar{r}_i - \bar{r}_N)$ which means that $E(Z_i) = \bar{Z} = 0$.

Thus, $3\bar{Z}^2\bar{Z} = -2(\bar{Z})^3 = (\bar{Z})^3 = 0$ and $M_Z^3 = E(\bar{r}_i - \bar{r}_N)^3$ and

$E(Z_i^2 Z_j) = E[(\bar{r}_i - \bar{r}_N)^2 (\bar{r}_j - \bar{r}_N)]$. Hence:

$$E(\bar{r}_n - \bar{r}_N)^3 = [\frac{1}{N^2(N-2)}] \left[[\frac{(N-1)(N-2)-(n-1)(n-2)}{(N-1)}] E(\bar{r}_i - \bar{r}_N)^3 + 3(n-1)(N-n) \times \right.$$

$$\left. E[(\bar{r}_i - \bar{r}_N)^2 (\bar{r}_j - \bar{r}_N)] \right] \tag{D1}$$

Substituting (C29) for (C19) and (C30) for (C20) and then adding (C29) and (C30) and rearranging produces (C31):

$$(C29) + (C30) = (\frac{1}{n})^5 \left[36(n-1)(n-2)[1 - \frac{n(n-1)}{N(N-1)}] E(M_{iij} M_{ijk}) + 18(n-1)(n-2) \times \right.$$

$$(n-3) [1 - \frac{n(n-1)(N-3)}{(n-3)N(N-1)}] [E(M_{iij} M_{ikl}) + E(M_{iij} M_{jkl})]$$

$$\left. + 6(n-1)(n-2)(n-3)(n-4) [1 - \frac{n(n-1)(N-3)(N-4)}{(n-3)(n-4)N(N-1)}] E(M_{iij} M_{klm}) \right] \quad (C31)$$

Thus, the variance in portfolio skewness at portfolio size n is the sum of (C2), (C6), (C10), (C25), (C28) and (C31):

$$E(M_{i,n}^3 - \bar{M}_n^3)^2 = (\frac{1}{n})^5 \left[(1 - \frac{n-1}{N-1}) E(M_i^3 - \bar{M}^3)^2 + 9(n-1) [1 - \frac{n(n-1)}{N(N-1)}] \times \right.$$

$$[\bar{M}_{iij}^2 + E(M_{iij} M_{ijj})] + 9(n-1)(n-2) [1 - \frac{n(n-1)(N-2)}{(n-2)N(N-1)}] [E(M_{iij} M_{iik}) + 2E(M_{iij} M_{jjk})$$

$$+ E(M_{iij} M_{kkj})] + 9(n-1)(n-2)(n-3) [1 - \frac{n(n-1)(N-2)(N-3)}{(n-2)(n-3)N(N-1)}] E(M_{iij} M_{kkl})$$

$$+ 6(n-1)(n-2) [1 - \frac{n(n-1)(n-2)}{N(N-1)(N-2)}] \bar{M}_{ijk}^2 + 18(n-1)(n-2)(n-3) [1 - \frac{n(n-1)(n-2)(N-3)}{(n-3)N(N-1)(N-2)}] \times$$

$$E(M_{ijk} M_{ijl}) + 9(n-1)(n-2)(n-3)(n-4) [1 - \frac{n(n-1)(n-2)(N-3)(N-4)}{(n-3)(n-4)N(N-1)(N-2)}] \times$$

$$E(M_{ijk} M_{ilm}) + (n-1)(n-2)(n-3)(n-4)(n-5) [1 - \frac{n(n-1)(n-2)(N-3)(N-4)(N-5)}{(n-3)(n-4)(n-5)N(N-1)(N-2)}] \times$$

$$E(M_{ijk} M_{lmn}) + 6(n-1) [1 - \frac{n}{N}] [E(M_i^3 M_{iij}) + E(M_i^3 M_{ijj})] + 6(n-1)(n-2) \times$$

$$[1 - \frac{n(N-2)}{(n-2)N}] E(M_i^3 M_{jjk}) + 6(n-1)(n-2) [1 - \frac{n}{N}] E(M_i^3 M_{ijk}) + 2(n-1)(n-2)(n-3) \times$$

$$[1 - \frac{n(N-3)}{(n-3)N}] E(M_i^3 M_{jkl}) + 36(n-1)(n-2) [1 - \frac{n(n-1)}{N(N-1)}] E(M_{iij} M_{ijk}) + 18(n-1) \times$$

$$(n-2)(n-3) [1 - \frac{n(n-1)(N-3)}{(n-3)N(N-1)}] [E(M_{iij} M_{ikl}) + E(M_{iij} M_{jkl})] + 6(n-1)(n-2)(n-3) \times$$

$$\left. (n-4) [1 - \frac{n(n-1)(N-3)(N-4)}{(n-3)(n-4)N(N-1)}] E(M_{iij} M_{klm}) \right] \quad (C32)$$

Appendix D

Another Element of Portfolio Skewness

Traditionally, portfolio skewness has been measured by the skewness in the portfolio return distribution over time (Appendix A). However, a more appropriate measure of portfolio skewness is total skewness which measures the skewness of a portfolio's return about the market's expected return and includes both the skewness in a portfolio mean return (equation (A5)) as well as the skewness of the portfolio's average return about the mean return on the market, $E(\bar{r}_n - \bar{r}_N)^3$. Note that: $E(\bar{r}_n - \bar{r}_N)^3 = E[\frac{1}{n} \sum\limits_{i=1}^{n} (\bar{r}_i - \bar{r}_N)]^3$. Using (B6), we see that $Z_i = (\bar{r}_i - \bar{r}_N)$ which means that $E(Z_i) = \bar{Z} = 0$. Thus, $3\bar{Z}^2\bar{Z} = -2(\bar{Z})^3 = (\bar{Z})^3 = 0$ and $M_Z^3 = E(\bar{r}_i - \bar{r}_N)^3$ and $E(Z_i^2 Z_j) = E[(\bar{r}_i - \bar{r}_N)^2 (\bar{r}_j - \bar{r}_N)]$. Hence:

$$E(\bar{r}_n - \bar{r}_N)^3 = [\frac{1}{N^2(N-2)}] \left[[\frac{(N-1)(N-2)-(n-1)(n-2)}{(N-1)}] E(\bar{r}_i - \bar{r}_N)^3 + 3(n-1)(N-n) \times \right.$$

$$\left. E[(\bar{r}_i - \bar{r}_N)^2(\bar{r}_j - \bar{r}_N)] \right] \tag{D1}$$