A MECHANICAL TEXT-BOOK;

OR,

INTRODUCTION TO THE STUDY OF MECHANICS.

BY

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With Numerous Diagrams.

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PREFACE.

THIS book is designed as an Introduction to more abstruse works on Engineering and Mechanics, and in particular to those of the late Professor Rankine.

Its study demands only a previous acquaintance with the ordinary Rules of Arithmetic, and with the Elementary Algebraical Notation. A few pages have been devoted to the Differential and Integral Calculus, as these have been used in different parts of the book, their application having been in every instance explained.

Professor Rankine's *Manual of Applied Mechanics* has been taken as the model for this work, the only alteration being the treating of the Theory of *Motion* before that of *Force*, as more in harmony with modern practice, and as proposed by himself for the present purpose.

The general design of the work having been indicated, it only remains for me to explain briefly how my name has been connected with that of Professor Rankine on the Title-page, and also in what condition it was left at the time of his recent lamented death.

I was Professor Rankine's Assistant, and lectured for him during his illness, and it was whilst on a visit which his death suddenly terminated, that the arrangement was made which connected me with him in the task. My duty was simply to assist him in its preparation. On my mentioning to him that the amount of labour I should have to do hardly justified my name appearing with his as joint-author, he replied, that, owing to his state of health, more of the work might devolve upon me than I expected. The issue has proved the correctness of his surmise.

As to the state of the MS. at the time of his death, two hundred pages had been already completed, and the general scope and plan of the work decided upon. I need hardly say that his wishes have been implicitly carried out in every respect, so far as lay in my power. The work has been completed at the request of Professor Rankine's Executrix, and at that of the Publishers, at whose desire also I have undertaken the superintendence of New Editions of his other Scientific Manuals, some of which have already been submitted to the Public.

E. F. B.

GLASGOW, *October*, *1873*.

PREFACE TO SECOND EDITION.

THIS Second Edition has been carefully revised, and some additions have been made to the text.

E. F. B.

LONDON, *October*, 1874.

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MATHEMATICAL INTRODUCTION.

ARITHMETICAL RULES.

For convenience sake the following Arithmetical Rules are here given: they will be referred to hereafter in the designing of toothed gearing, under The Theory of Pure Mechanism, Part II.

Definition.—A *prime number* is one which is only divisible by the number 1.

1. **To find the Prime Factors of a Given Number.**—Try the prime numbers, 2, 3, 5, 7, 11, &c., as divisors in succession, until a prime number has been found to divide the given number without a remainder; then try whether and how many times over the quotient is again divisible by the same prime number, so as to obtain a quotient not divisible again by the same prime number; then try the division of that quotient by the next greater prime number; and so on until a quotient is obtained which is itself a prime number; that is, a number not divisible by any other number except 1. This final quotient and the series of divisors will be the prime factors of the given number. To test the accuracy of the process, multiply all the prime factors together; the product should be the given number.

2. **To find the Greatest Common Measure (otherwise called the greatest common divisor) of Two Numbers.**—Divide the greater number by the less, so as to obtain a quotient, and a remainder less than the divisor; divide the divisor by the remainder as a new divisor; that new divisor by the new remainder; and so on, until a remainder is obtained which divides the previous divisor without a remainder. That last remainder will be the required greatest common measure.

If the last remainder is 1, the two numbers are said to be "prime to each other."

Example.—Required, the greatest common measure of 1420 and 1808.

Divisor, 1420) 1808 (1, Quotient.
1420
Remainder, 388) 1420 (3, Quotient.
1164
Remainder, 256) 388 (1, Quotient.
256
Remainder, 132) 256 (1, Quotient.
132
Remainder, 124) 132 (1, Quotient.
124
Remainder, 8) 124 (15, Quotient.
120
Remainder, 4) 8 (2, Quotient.

The last remainder, 4, is the required greatest common measure.

Definition.—Ratio is the mutual relation of two quantities in respect of magnitude.

3. **To reduce the Ratio of Two Numbers to its Least Terms,** divide both numbers by their greatest common measure.

$$\text{For example, } \frac{1808 \div 4}{1420 \div 4} = \frac{452}{355}.$$

4. **To express the Ratio of Two Numbers in the form of a Continued Fraction.**—Let A be the lesser of the two numbers, and B the greater; and let a, b, c, d, &c., be the quotients obtained during the process of finding the greatest common measure of A and B. Then, in the equation

$$\frac{\mathrm{B}}{\mathrm{A}} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \&c.,}}}$$

the right-hand side is the continued fraction required.

To save space in printing, a continued fraction is often arranged as follows:—

$$a + \frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{d+}\ \&c.$$

The ratio of two incommensurable quantities is expressed by an endless continued fraction. For example, the ratio of the diagonal to the side of a square is expressed by $1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}$ &c., without end.

5. **To form a series of Approximations to a Given Ratio.**—Express

the ratio in the form of a continued fraction. Then write the quotients in their order; and in a line below them write $\frac{0}{1}$ to the left of the first quotient, and $\frac{1}{0}$ directly under the first quotient. Then calculate a series of fractions by the following rule:—Multiply the first quotient by the numerator of the fraction that is below it, and add the numerator of the fraction next to the left; the sum will be the numerator of a new fraction: multiply the first quotient by the denominator of the fraction that is below it, and add the denominator of the fraction that is next to the left; the sum will be the denominator of the new fraction; then write that new fraction under the second quotient, and treat the second quotient, the fraction below it, and the fraction next to the left, as before, to find a fraction which is to be written under the third quotient, and so on. For example:

Quotients,....a, b, c, d, &c.

$$\text{Fractions, } \frac{0}{1}, \frac{1}{0}, \frac{n}{m}, \frac{n'}{m'}, \frac{n''}{m''};$$

$$\frac{n}{m} = \frac{0+a}{1+0} = \frac{a}{1}; \quad \frac{n'}{m'} = \frac{1+bn}{0+bm}; \quad \frac{n''}{m''} = \frac{n+cn'}{m+cm'}; \text{ \&c.}$$

To take a particular case; let the given ratio be as before, $\frac{452}{355}$, then we have the following series:—

Quotients,..................		1	3	1	1	1	15	2	
Fractions,..............	$\frac{0}{1}$	$\frac{1}{0}$	$\frac{1}{1}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{9}{7}$	$\frac{14}{11}$	$\frac{219}{172}$	$\frac{452}{355}$
Less or greater than given ratio,.......	L	G	L	G	L	G	L	G	

The fractions in a series formed in the manner just described are called *converging fractions*, and they have the following properties:—*First*, each of them is in its least terms; *secondly*, the difference between any pair of consecutive converging fractions is equal to unity divided by the product of their denominators; for example, $\frac{9}{7} - \frac{5}{4} = \frac{36-35}{7 \times 4} = \frac{1}{28}$; $\frac{9}{7} - \frac{14}{11} = \frac{99-98}{7 \times 11} = \frac{1}{77}$; *thirdly*, they are alternately less and greater than the given ratio towards which they approximate, as indicated by the letters L and G in the example; and, *fourthly*, the difference between any one of them and the given ratio is less than the difference between that one and the next fraction of the series.

Fractions intermediate between the converging fractions may be found by means of the formula $\frac{hn+kn'}{hm+km'}$; where $\frac{n}{m}$ and $\frac{n'}{m'}$, are any two of the converging fractions, and h and k are any two whole numbers, positive or negative, that are prime to each other.

6. **Logarithms. Definitions.**—The *power* of a number is the product of itself multiplied a certain number of times. The *index* or *exponent* of the power is the small figure placed above the right-hand corner, which denotes the number of times the multiplication takes place. The *Logarithm* of a number to a given base is the index of the power to which the base must be raised to be equal to the given number. That number of which the indices of the powers are the logarithms, is called the *base* of the system. A suffix denotes the base of the logarithm; if $a^x = n$, x is the logarithm of the number n to the base a, or $\log_a n = x$.

Logarithms to the base 10 are called common logarithms.

7. The logarithm of 1 is 0.

8. The *common* logarithm of 10 is 1, and that of any power of 10 is the index of that power; in other words, it is equal to the number of noughts in the power; thus the common logarithm of 100 is 2; that of 1000, 3; and so on.

9. The common logarithm of ·1 is -1, and that of any power of ·1 is the index of that power with the negative sign; that is, it is equal to one more than the number of noughts between the decimal point and the figure 1, with the negative sign; for example, the common logarithm of ·01 is -2; that of ·001, -3; and so on.

10. The logarithms given in tables, are merely the fractional parts of the logarithms, correct to a certain number of places of decimals, without the integral parts or *indices;* which are supplied in each case according to the following rules:—

The index of the common logarithm of a number not less than 1 is one less than the number of integer places of figures in that number; that is to say, for numbers less than 10 and not less than 1, the index is 0; for numbers less than 100 and not less than 10, the index is 1; for numbers less than 1000, and not less than 100, the index is 2; and so on.

The index of the common logarithm of a decimal fraction less than 1 is *negative*, and is one more than the number of noughts between the decimal point and the significant figures; and the negative sign is usually written above instead of before the index; that is to say, for numbers less than 1 and not less than ·1, the index is $\bar{1}$; for numbers less than ·1 and not less than ·01, the index is $\bar{2}$; and so on.

The fractional part of a common logarithm is always positive, and depends solely upon the series of figures of which the number consists, and not upon the place of the decimal point amongst them.

Examples.

Number.	Logarithms.
377000	5·57634
37700	4·57634
3770	3·57634
377	2·57634
37·7	1·57634
3·77	0·57634
·377	$\bar{1}$·57634
·0377	$\bar{2}$·57634
·00377	$\bar{3}$·57634

and so on.

11. The logarithm of a product is the sum of the logarithms of its factors.

12. The logarithm of a power is equal to the logarithm of the root multiplied by the index of the power.

13. The logarithm of a quotient is found by subtracting the logarithm of the divisor from the logarithm of the dividend.

14. The logarithm of a root is found by dividing the logarithm of one of its powers by the index of that power.

Note.—In applying these principles to logarithms of numbers less than 1, it is to be observed that negative indices are to be subtracted instead of being added, and added instead of being subtracted.

15. To avoid the inconvenience which attends the use of negative indices to logarithms, it is a very common practice to put, instead of a negative index to the logarithm of a fraction, the *complement* (as it is called) of that index to 10; that is to say, 9 instead of $\bar{1}$, 8 instead of $\bar{2}$, 7 instead of $\bar{3}$, and so on. In such cases, it is always to be understood that each such complementary index has -10 combined with it; and to prevent mistakes, it is useful to prefix $-10+$ to it; for example,

Number.	Logarithm with Negative Index.	Logarithm with Complementary Index.
·377	$\bar{1}$·57634	$-10+9$·57634
·0377	$\bar{2}$·57634	$-10+8$·57634
·00377	$\bar{3}$·57634	$-10+7$·57634

16. To find the fractional part of the common logarithm of a number of five places of figures; take from the table the logarithm corresponding to the first three figures, and the difference between that logarithm and the next greater logarithm in the table; multiply that difference by the two remaining figures of the given number, and divide by 100; the quotient will be a correction, to be added to the logarithm already found.

Example.—Find the common logarithm of 37725.

Log. 377,	57634
Log. 378,	57749
Difference,	115
	× 25 ÷ 100
Correction,	29
Add log. 377,	57634
Log. 37725,	57663 *Answer.*

17. To find the natural number, or *antilogarithm*, corresponding to a common logarithm of five places of decimals, which is not in the table; find the next less, and the next greater logarithm in the table, and take their difference. Opposite the next less logarithm will be the first three figures of the antilogarithm. Subtract the next less logarithm from the given logarithm; annex two noughts to the remainder, and divide by the before-mentioned difference; the quotient will give two additional figures of the required antilogarithm. (The first of those figures may be a nought.)

Example.—Find the antilogarithm of the common logarithm ·57663.

Next less log. in table,	57634
Next greater,	57749
Difference,	115
Given logarithm,	57663
Subtract log. 377,	57634
Divide by difference,	115)2900
Two additional figures,	25

so that the answer is 37725.

Note.—The last two rules refer particularly to the tables in Rankine's *Useful Rules and Tables*, but are equally applicable to other tables. For instance, where the logarithm of a number of 5 figures is given in the tables; in these last two rules, for 3 read 5, and for 5 read 7.

TRIGONOMETRICAL RULES.

The following is a summary of the Principles and Chief Rules of Trigonometry:—

Definition.—Every expression which in any way contains a number, or depends for its value upon the value of the number, is said to be a *function* of that number, as $2x$, x^2, log. x, tan x are all functions of x.

18. **Trigonometrical Functions Defined.**—Suppose that A, B, C

stand for the three angles of a right-angled triangle, C being the right angle, and that a, b, c stand for the sides respectively opposite to those angles, c being the hypothenuse; then the various names of trigonometrical functions of the angle A have the following meanings:—

$$\sin A = \frac{a}{c}; \ \cos A = \frac{b}{c};$$

$$\text{versin } A = \frac{c-b}{c}; \ \text{coversin } A = \frac{c-a}{c};$$

$$\tan A = \frac{a}{b}; \ \text{cotan } A = \frac{b}{a};$$

$$\sec A = \frac{c}{b}; \ \text{cosec } A = \frac{c}{a}.$$

The *complement* of A means the angle B, such that A + B = a right angle; and the sine of each of those angles is the cosine of the other, and so of the other functions by pairs.

19. Relations amongst the Trigonometrical Functions of One Angle, A, and of its Supplement, 180° – A:—

$$\sin A = \sqrt{1 - \cos^2 A} = \frac{\tan A}{\sec A} = \frac{1}{\text{cosec } A};$$

$$\cos A = \sqrt{1 - \sin^2 A} = \frac{\text{cotan } A}{\text{cosec } A} = \frac{1}{\sec A};$$

$$\text{versin } A = 1 - \cos A;$$

$$\text{coversin } A = 1 - \sin A;$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{1}{\text{cotan } A} = \sin A \cdot \sec A = \sqrt{\sec^2 A - 1};$$

$$\text{cotan } A = \frac{\cos A}{\sin A} = \frac{1}{\tan A} = \cos A \cdot \text{cosec } A = \sqrt{\text{cosec}^2 A - 1};$$

$$\sec A = \frac{1}{\cos A} = \sqrt{1 + \tan^2 A};$$

$$\text{cosec } A = \frac{1}{\sin A} = \sqrt{1 + \text{cotan}^2 A}.$$

$$\sin (180° - A) = \sin A;$$
$$\cos (180° - A) = -\cos A;$$
$$\text{versin } (180° - A) = 1 + \cos A = 2 - \text{versin } A;$$
$$\text{coversin } (180° - A) = \text{coversin } A;$$
$$\tan (180° - A) = -\tan A;$$
$$\text{cotan } (180° - A) = -\text{cotan } A;$$
$$\sec (180° - A) = -\sec A;$$
$$\text{cosec } (180° - A) = \text{cosec } A.$$

20. The Circular Measure of an Angle.—If a right line as radius by revolution about a fixed point at its extremity as centre, traces out an angle from a fixed position, the angle may be measured by the ratio of the arc to the radius; this mode of measurement is called circular measure. The *unit* of circular measure is the angle whose arc is equal to the radius, that is, $360° \div 2\pi = (57° \, 17' \, 45'' = 206265'')$.

To compute sines, &c., approximately by series; reduce the angle to circular measure—that is, to radius-lengths and fractions of a radius-length let it be denoted by A. Then

$$\sin A = A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7} + \&c.$$

$$\cos A = 1 - \frac{A^2}{2} + \frac{A^4}{2.3.4} - \frac{A^6}{2.3.4.5.6} + \&c.$$

21. Trigonometrical Functions of Two Angles:—

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B;$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B;$$

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

22. Formulæ for the Solution of Plane Triangles.—Let A, B, C be the angles, and a, b, c the sides respectively opposite them.

I. *Relations amongst the Angles*—

$$A + B + C = 180°;$$

or if A and B are given,

$$C = 180° - A - B.$$

II. *When the Angles and One Side are given*, let a be the given side; then the other two sides are

$$b = a \cdot \frac{\sin B}{\sin A}; \quad c = a \cdot \frac{\sin C}{\sin A}.$$

III. *When Two Sides and the Included Angle are given*, let a, b be the given sides, C the given included angle; then

To find the third side. First Method:

$$c = \sqrt{(a^2 + b^2 - 2\,a\,b \cos C)};$$

Second Method: Make $\sin D = \frac{2\sqrt{a\,b}}{a+b} \cdot \cos \frac{C}{2}$; then

$$c = (a + b) \cos D.$$

Third Method: Make $\tan E = \frac{2\sqrt{a\,b}}{a-b} \cdot \sin \frac{C}{2}$; then

$$c = (a - b) \sec E.$$

To find the remaining Angles, A and B.

If the third side has been computed,

$$\sin A = \frac{a}{c} \cdot \sin C; \ \sin B = \frac{b}{c} \cdot \sin C.$$

If the third side has not been computed,

$$\tan \cdot \frac{A+B}{2} = \text{cotan} \frac{C}{2}; \ \tan \frac{A-B}{2} = \frac{a-b}{a+b} \text{cotan} \frac{C}{2};$$

$$A = \frac{A+B}{2} + \frac{A-B}{2}; \ B = \frac{A+B}{2} - \frac{A-B}{2}.$$

IV. *When the Three Sides are given, to find any one of the Angles,* such as C—

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,a\,b};$$

or otherwise, let

$$s = \frac{a+b+c}{2}; \text{ then}$$

$$\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{a\,b}}; \ \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{a\,b}};$$

$$\text{cotan} \frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}; \ \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}};$$

$$\sin C = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a\,b}.$$

Note.—In all trigonometrical problems, it is to be borne in mind, that small acute angles, and large obtuse angles, are most accurately determined by means of their *sines, tangents,* and *cosecants;* and angles approaching a right angle by their *cosines, cotangents,* and *secants.*

23. **To Solve a Right-angled Triangle.**—Let C denote the right angle; c the hypothenuse; A and B the two oblique angles; a and b the sides respectively opposite them.

Given, the right angle, another angle B, the hypothenuse c. Then

$$A = 90° - B; \ a = c \cdot \cos B; \ b = c \cdot \sin B.$$

Given, the right angle, another angle B, a side a,

$$A = 90° - B; \ b = a \cdot \tan B; \ c = a \cdot \sec B.$$

Given, the right angle, and the sides a, b,

$$\tan A = \frac{a}{b}; \ \tan B = \frac{b}{a}; \ c = \sqrt{a^2 + b^2}.$$

Given, the right angle, the hypothenuse c; a side a,

$$\sin A = \cos B = \frac{a}{c};\ b = \sqrt{c^2 - a^2}.$$

Given the three sides, a, b, c, which fulfilling the equation $c^2 = a^2 + b^2$, the triangle is known to be right-angled at C,

$$\sin A = \frac{a}{c};\ \sin B = \frac{b}{c}.$$

24. **To Express the Area of a Plane Triangle in terms of its Sides and Angles.**

Given, one side, c, and the angles.

$$\text{Area} = \frac{c^2}{2} \cdot \frac{\sin A \sin B}{\sin C}.$$

Given, two sides, b, c, and the included angle A.

$$\text{Area} = \frac{b\,c \cdot \sin A}{2}.$$

Given, the three sides a, b, c. Let $\frac{a+b+c}{2} = s$; then

$$\text{Area} = \sqrt{\left\{ s\,(s-a)\,(s-b)\,s - c) \right\}}.$$

RULES OF THE DIFFERENTIAL AND INTEGRAL CALCULUS.

25. *Definitions.*—A *function* has already been defined. When a function of one quantity is assumed equal to another quantity, both quantities are called *variables*, the one upon whose assumed value the other depends being called the *independent* variable, while the other, whose value depends upon it, is called the *dependent* variable. The expression $y = \phi\, x$ for instance denotes that the *dependent variable* y, depends for its value upon the *independent variable* x, or y is a *function* of x.

A quantity, x, may be assumed to be made up of an infinite number of infinitesimal parts, dx, this expression meaning simply one of the small infinitesimal differences of which x is made up, *i.e.*, $x = n \cdot dx$, where n is assumed to increase without limit, and dx to diminish without limit, this process of considering a quantity to be diminished without limit is called *differentiation.*

The quotient, if it has a limit formed by taking the difference of the function of a quantity, and the function of that quantity with a small increment, and dividing by the increment, is termed the *differential coefficient* of the function, with regard to the quantity $\frac{\phi\,(x+dx) - \phi\, x}{dx}$ is the differential coefficient of $\phi\, x$ with respect to x, this is generally written $\phi'\, x$; or otherwise the small increment or decrement of the dependent variable divided by that of the independent variable, the former being a function of the latter, is called the differential coefficient, thus $\frac{dy}{dx}$ is the differential coefficient of y with respect to x, it being always borne in mind that $\frac{dy}{dx}$ is *one* quantity, which cannot be divided into a numerator dy, and a denominator dx.

26. **Rules for finding differential coefficients,—**

If $y = C$ (a constant); $\frac{dy}{dx} = 0$.

The Differential Coefficient of the sum of functions is equal to the sum of the differential coefficients of the functions, or if $v = w + y + z$ where all of these quantities are functions of x, then $\frac{dv}{dx} = \frac{dw}{dx} + \frac{dy}{dx} + \frac{dz}{dx}$.

In the same way to find the differential coefficient of the difference, product, and quotient of functions of quantities.

If $v = y - z$, then $\frac{dv}{dx} = \frac{dy}{dx} - \frac{dz}{dx}$, where v, y, z, are functions of x.

If $v = w\,y\,z$, then $\frac{dv}{dx} = \frac{dw}{dx} \cdot y \cdot z + \frac{dy}{dx} \cdot w \cdot z + \frac{dz}{dx} \cdot w \cdot y$, where v, w, y, z, are functions of x.

If $v = \frac{y}{z}$, then $\frac{dv}{dx} = \frac{z \cdot \frac{dy}{dx} - y\frac{dz}{dx}}{z^2}$, where v, y, z, are functions of x.

If $\phi x = nx$, $\phi' x = n$; or otherwise let $\phi x = y = nx$, then $\frac{dy}{dx} = n$, thus if $\phi x = 7x$, $\phi' x = 7$.

If $\phi x = x^n$, $\phi' x = nx^{n-1}$, thus if $\phi x = x^7$, $\phi' x = 7x^6$.

If $\phi x = \log_a x$, $\phi' x = \frac{1}{x \log_\epsilon a}$, thus if $\phi x = \log_{10} x$, $\phi' x = \frac{1}{x \log_\epsilon 10} = \frac{1}{2{\cdot}30258x} = \frac{{\cdot}43429}{x}$

If $\phi x = \log_\epsilon x$, $\phi' x = \frac{1}{x}$.

If $\phi x = a^x$, $\phi' x = a^x \log_\epsilon a$.

If $\phi x = \epsilon^x$, $\phi' x = \epsilon^x$.

If $\phi x = \sin x$, $\phi' x = \cos x$.

If $\phi x = \cos x$, $\phi' x = -\sin x$.

If $\phi x = \tan x$, $\phi' x = \frac{1}{\cos^2 x}$.

Definition.—By sine $^{-1} x$ is meant the angle whose sine is x, thus if $y = \text{sine}^{-1} x$, $x = \text{sine } y$.

$$\text{If } \varphi x = \sin^{-1} x,\ \varphi' x = \frac{1}{\sqrt{(1-x^2)}}$$

$$\varphi x = \cos^{-1} x,\ \varphi' x = -\ \frac{1}{\sqrt{(1-x^2)}}$$

$$\varphi x = \tan^{-1} x,\ \varphi' x = \frac{1}{1+x^2}.$$

If $v = \phi\, x$, and $\psi\, v = \chi\, x$, then $\chi' x = \psi' v \cdot \phi' x$, or otherwise;

$$\text{If } v = \varphi\, x,\ y = \chi\, x,\ y = \psi\, v \text{ then } \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

If $\varphi\, x = \log.\ \sin x$, then $\varphi' x = \frac{1}{\sin x} \cdot \cos x$; here *log. sin x* is first differentiated with respect to *sin x*, and then *sin x* with respect to x.

Definition.—The differential coefficient of the differential coefficient is called the *second* differential coefficient; the differential coefficient of the second is called the *third;* and so on :—

Thus let $\varphi x = x^n$, then $\varphi' x = n x^{n-1}$, $\varphi'' x = n(n-1)x^{n-2}$, $\varphi''' x = n(n-1)(n-2)x^{n-3}$; where $\varphi'' x$ stands for the second differential coefficient, $\varphi''' x$ for the third differential coefficient. So let $\varphi x = \sin x$, then $\varphi' x = \cos x$, $\varphi'' x = -\sin x$, $\varphi''' x = -\cos x$. This process is called successive differentiation.

Another mode of representing this is the following :— $\frac{dy}{dx}$ is the differential coefficient of y with respect to x; the second differential coefficient $\frac{d\frac{dy}{dx}}{dx}$ is represented by $\frac{d^2 y}{dx^2}$, where $\frac{d^2 y}{dx^2}$ is a quantity which cannot be divided into a numerator $d^2 y$ and denominator $d x^2$, as already stated of the quantity $\frac{dy}{dx}$.

27. The following is an illustration of the application of the differential calculus to Geometry. In fig. 1, let O X, O Y be two axes of co-ordinates at right angles to each other. Let A B be a curve, whose equation is represented by $y = \varphi x$, *i.e.*, each point of the curve has ordinates and abscissæ bearing to each other the ratio represented by the equation, and all the points in the curve are known when their ordinates and abscissæ (co-ordinates) are known. Let O C $= a$ be a particular value of x, and A C $= b = \varphi a$ be a particular value of y;

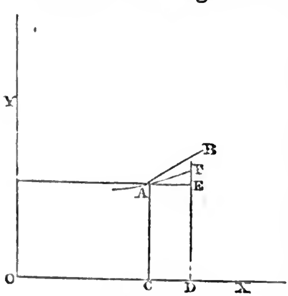


Fig. 1.

also let A E be a small increment of $x = \Delta x$, and E B a corresponding increment of $y = \Delta y$. The trigonometrical tangent of the angle E A B $= \frac{\Delta y}{\Delta x}$. Let there be a line, A T, lying between A B and A E, which makes with A E an angle, whose tangent is $\frac{d y}{d x} = \varphi' x$. Then as B approaches towards A, the line A B will ultimately be the line A T; that is, its limiting value as soon as A and B coincide, will be A T. Hence if x be an abscissa of a curve, and $\varphi x = y$ be an ordinate. the differential coefficient $\frac{d y}{d x} = \varphi' x$ is the trigonometrical tangent of the angle which the geometrical tangent of the curve makes with the axis of x, at the point where the abscissa is x.

The differential calculus will occasionally be applied in an elementary manner to portions of the following work, as, for example, in treating of the VARIED MOTION OF POINTS (Part I., Section 3).

28. The integral calculus is the inverse of the differential; it determines the whole magnitude of a quantity of which the differentials are given.

If a number of points be taken in a curve, and chords drawn joining the points, and also tangents drawn through the points intersecting each other, the sum of the one will be less than, and that of the other (intersected portions) will be greater than the length of the curve; if the chords, and tangents are increased in number, they will approximate to the length of the curve. The integral calculus is used for finding the exact length of the curve. A mechanical illustration is the computation of the space passed over by a point having varied motion.

29. **Approximate Computation of Integrals.**—The present article is intended to afford to those who have not made that branch of mathematics which treats of the process of *integration* a special study, some elementary information respecting it.

The meaning of the symbol of an integral, viz.:—

$$\int u \, d x,$$

is of the following kind:—

In fig. 2, let A C B D be a plane area, of which one boundary, A B, is a portion of an axis of abscissæ O X,—the opposite boundary, C D, a curve of any figure,—and the remaining boundaries A C, B D, ordinates perpendicular to O X, whose respective abscissæ, or distances from the origin O, are

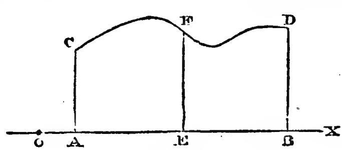


Fig. 2.

$$\overline{OA} = a; \ \overline{OB} = b.$$

Let $\overline{EF} = u$ be any ordinate whatsoever of the curve C D, and $\overline{OE} = x$ the corresponding abscissa. Then the integral denoted by the symbol,

$$\int_a^b u \, d\, x,$$

means, *the area of the figure* A C B D. The abscissæ a and b which are the least and greatest values of x, and which indicate the longitudinal extent of the area, are called the *limits of integration;* but when the extent of the area is otherwise indicated, the symbols of those limits are sometimes omitted.

When the relation between u and x is expressed by any ordinary algebraical equation, the value of the integral for a given pair of values of its limits can generally be found by means of formulæ which are contained in works on the Integral Calculus, or by means of mathematical tables.

Cases may arise, however, in which u cannot be so expressed in terms of x; and then approximate methods must be employed. Those approximate methods, of which two are here described, are founded upon the division of the area to be measured into bands by parallel and equi-distant ordinates, the approximate computation of the areas of those bands, and the adding of them together; and the more minute that division is, the more near is the result to the truth.

First Approximation.

Divide the area A C D B, as in fig. 3, into any convenient number of bands by parallel ordinates, whose uniform distance apart is $\Delta\, x$; so that if n be the number of bands, $n + 1$ will be the number of ordinates, and

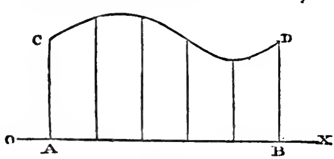


Fig. 3.

$$b - a = n \, \Delta \, x,$$

the length of the figure.

Let u', u'', denote the two ordinates which bound one of the bands; then the area of that band is

$$\frac{u' + u''}{2} \cdot \Delta\, x, \textit{ nearly;}$$

and consequently, adding together the approximate areas of all the bands,—denoting the extreme ordinates as follows,—

$$\overline{AC} = u_a; \ \overline{BD} = u_b;$$

and the intermediate ordinates by $u\,i$, we find for the approximate value of the integral (the symbol Σ denoting *sum*)—

$$\int_a^b u\,d\,x = \left(\frac{u_a}{2} + \frac{u_b}{2} + \Sigma \cdot u_i\right) \Delta\,x, \qquad (1.)$$

Second Approximation.

Divide the area A C D B, as in fig. 4, into an *even* number of bands, by parallel ordinates, whose uniform distance apart is $\Delta\,x$. The ordinates are marked alternately by plain lines and by dotted lines, so as to arrange the bands in pairs. Considering any one pair of bands, such as E F H G, and assuming that the curve F H is nearly a parabola, it appears from the properties of that curve, that the area of that pair of bands is

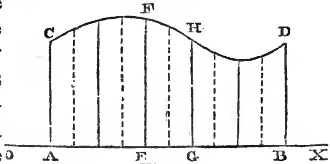


Fig. 4.

$$\frac{(u' + 4\,u'' + u''')\,\Delta\,x}{3}, \textit{ nearly};$$

in which u' and u''' denote the plain ordinates $\overline{E\,F}$ and $\overline{G\,H}$, and u'' the intermediate dotted ordinate; and consequently, adding together the approximate areas of all the pairs of bands, we find, for the approximate value of the integral—

$$\int_a^b u\,d\,x = \Big(u_a + u_b + 2\,\Sigma \cdot u_i \text{ (plain)}$$

$$+ \; 4\,\Sigma \cdot u_i \text{ (dotted)}\Big)\frac{\Delta\,x}{3}, \qquad (2.)$$

It is obvious, that if the values of the ordinates u required in these computations can be calculated, it is unnecessary to draw the figure to a scale, although a sketch of it may be useful to assist the memory.

When the symbol of integration is repeated, so as to make a *double integral,* such as

$$\int \int u \cdot d\,x\,d\,y,$$

or a *triple integral,* such as

$$\int \int \int u \cdot d\,x\,d\,y\,d\,z,$$

it is to be understood as follows:—

Let

$$v = \int u \cdot d\,x$$

be the value of this single integral for a given value of y. Construct a curve whose abscissæ are the various values of y within the prescribed limits, and its ordinates the corresponding values of v. Then the area of that curve is denoted by

$$\int v \cdot dy = \int \int u \cdot dx\, dy.$$

Next, let

$$t = \int v \cdot dy$$

be the value of this double integral for a given value of z. Construct a curve whose abscissæ are the various values of z within the prescribed limits, and its ordinates the corresponding values of t. Then the area of that curve is denoted by

$$\int t \cdot dz = \int \int v \cdot dy\, dz = \int \int \int u \cdot dx\, dy\, dz;$$

and so on for any number of successive integrations.

RULES FOR THE MENSURATION OF FIGURES AND FINDING OF CENTRES OF MAGNITUDE.

Section 1.—Areas of Plane Surfaces.

30. **Parallelogram.** *Rule A.*—Multiply the length of one of the sides by the perpendicular distance between that side and the opposite side.

Rule B.—Multiply together the lengths of two adjacent sides and the sine of the angle which they make with each other. (When the parallelogram is right-angled, that sine is $=1$.)

31. **Trapezoid** (or four-sided figure bounded by a pair of parallel straight lines, and a pair of straight lines not parallel). Multiply the half sum of the two parallel sides by the perpendicular distance between them.

32. **Triangle.** *Rule A.*—Multiply the length of any one of the sides by one-half of its perpendicular distance from the opposite angle.

Rule B.—Multiply one-half of the product of any two of the sides by the sine of the angle between them.

Rule C.—Multiply together the following four quantities: the half sum of the three sides, and the three remainders left after subtracting each of the three sides from that half sum; extract the square root of the quotient; that root will be the area required.

Note.—Any Polygon may be measured by dividing it into triangles, measuring those triangles, and adding their areas together.

33. **Parabolic Figures of the Third Degree.**—The parabolic

figures to which the following rules apply are of the following kind (see figs. 5 and 6.) One boundary is a straight line, A X, called the *base* or *axis;* two other boundaries are either points in that line, or straight lines at right angles to it, such as A B and X C, called *ordinates;* and the fourth boundary is a curve, B C, of the *parabolic class*, and of the *third degree;* that is, a curve whose *ordinate* (or perpendicular distance from the base A X) at any point is expressed by what is called an *algebraical function of the third degree* of the *abscissa* (or distance of that ordinate from a fixed point in the base). An algebraical function of the third degree of a quantity consists of terms not exceeding four in number, of which one may be constant, and the rest must be proportional to powers of that quantity not higher than the cube.

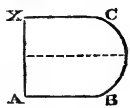


Fig. 5.

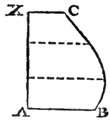


Fig. 6.

Rule A.—Divide the base, as in fig. 5, into two equal parts or intervals; measure the endmost ordinates, A B and X C, and the middle ordinate (which is dotted in the figure) at the point of divison; add together the endmost ordinates and *four times* the middle ordinate, and divide the sum by *six;* the quotient will be the *mean breadth* of the figure, which, being multiplied by the length of the base, A X, will give the area.

Rule B.—Divide the base, as in fig. 6, into three equal intervals; measure the endmost ordinates, A B and X C, and the two intermediate ordinates (which are dotted) at the points of division; add together the endmost ordinates and three times each of the intermediate ordinates; divide the sum by *eight;* the quotient will be the *mean breadth* of the figure, which, being multiplied by the length of the base, A X, will give the area.

In applying either of those rules to figures whose curved boundaries meet the base at one or both ends, the ordinate at each such point of meeting is to be made = 0.

34. **Any Plane Area.**—Draw an axis or base-line, A X, in a convenient position. The most convenient position is usually parallel to the greatest length of the area to be measured. Divide the length of the figure into a convenient number of equal intervals, and measure breadths in a direction perpendicular to the axis at the two ends of that length, and at the points of division, which breadths will, of course, be one more in number than the intervals. (For example, in fig. 7, the length of the figure is divided into ten equal intervals, and eleven breadths are measured at b_0, b_1, &c.) Then the following rules are exact, if the sides of the figures are bounded by straight lines, and by

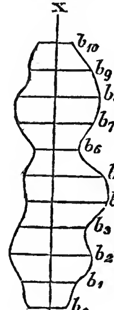


Fig. 7.

parabolic curves not exceeding the third degree, and are approximate for boundaries of any other figures.

Rule A.—(“*Simpson's First Rule,*” to be used when the number of intervals is even.)—Add together the two endmost breadths, *twice* every second intermediate breadth, and *four times* each of the remaining intermediate breadths; multiply the sum by the common interval between the breadths, and divide by 3; the result will be the area required.

For two intervals the multipliers for the breadths are 1, 4, 1 (as in Rule A of the preceding Article); for four intervals, 1, 4, 2, 4, 1; for six intervals, 1, 4, 2, 4, 2, 4, 1; and so on. These are called “Simpson's Multipliers.”*

Example.—Length, 120 feet, divided into six intervals of 20 feet each.

Breadths in Feet and Decimals.	Simpson's Multipliers.	Products.
17·28	1	17·28
16·40	4	65·60
14·08	2	28·16
10·80	4	43·20
7·04	2	14·08
3·28	4	13·12
0	1	0·00

Sum, 181·44

× Common interval, 20 feet.

÷ 3) 3628·8

Area required, 1209·6 square feet.

Rule B.—(“*Simpson's Second Rule*” to be used when the number of intervals is a multiple of 3.)—Add together the two endmost breadths, *twice* every third intermediate breadth, and *thrice* each of the remaining intermediate breadths; multiply the sum by the common interval between the breadths, and by 3; divide the product by 8; the result will be the area required.

“Simpson's Multipliers” in this case are, for three intervals, 1, 3, 3, 1; for six intervals, 1, 3, 3, 2, 3, 3, 1; for nine intervals, 1, 3, 3, 2, 3, 3, 2, 3, 3, 1; and so on.

Example.—Length, 120 feet, divided into six intervals of 20 feet each.

* This rule has been given in symbols at page 15.

Breadths in Feet and Decimals.	Simpson's Multipliers.	Products.
17·28	1	17·28
16·40	3	49·20
14·08	3	42·24
10.80	2	21·60
7·04	3	21·12
3·28	3	9·84
0	1	0·00
	Sum,	161·28
	× Common interval,	20 feet.
		3225·6
		× 3
	÷ 8)	9676·8
	Area required,	1209·6 square feet.

Remarks.—The preceding examples are taken from a parabolic figure of the third degree, for which both Simpson's Rules are exact; and the results of using them agree together precisely. For other figures, for which the rules are approximate only, the first rule is in general somewhat more accurate than the second, and is therefore to be used unless there is some special reason for preferring the second.

The probable extent of error in applying Simpson's First Rule to a given figure is, in most cases, nearly proportional to the fourth power of the length of an interval.

The errors are greatest where the boundaries of the figure are most curved, and where they are nearly perpendicular to the axis. In such positions of a figure the errors may be diminished by subdividing the axis into smaller intervals.

Rule C.—(“*Merrifield's Trapezoidal Rule,*” for calculating separately the areas of the parts into which a figure is subdivided by its equidistant ordinates or breadths.)—Write down the breadths in their order. Then take the *differences* of the successive breadths, distinguishing them into positive and negative, according as the breadths are increasing or diminishing, and write them opposite the intervals between the breadths. Then take the differences of those differences, or *second differences,* and write them opposite the intervals between the first differences, distinguishing them into positive and negative, according to the following principles:—

First Differences.	Second Difference.
Positive increasing, or Negative diminishing,	.........Positive.
Negative increasing, or Positive diminishing,	.........Negative.

In the column of second differences there will now be two blanks opposite the two endmost breadths; those blanks are to be filled up with numbers each forming an arithmetical progression with the two adjoining second differences, if these are unequal, or equal to them, if they are equal.

Divide each second difference by 12; this gives a *correction*, which is to be *subtracted* from the breadth opposite it if the second difference is *positive*, and *added* to that breadth if the second difference is *negative*.

Then to find the area of the division of the figure contained between a given pair of ordinates or breadths; *multiply the half sum of the corrected breadths by the interval between them.*

The area of the whole figure may be formed either by adding together the areas of all its divisions, or by adding together the halves of the endmost corrected breadths, and the whole of the intermediate breadths, and multiplying the sum by the common interval.

Example.—Length, 120 feet, divided into six intervals of 20 feet each.

Breadths in Feet and Decimals.	First Differences.	Second Differences.	Corrections.	Corrected Breadths. Feet.	Areas of Divisions. Sq. Feet.
17·28		(−1·92)	+0·16	17·44	
	−0·88				339·6
16·40		−1·44	+0·12	16·52	
	−2·32				306·8
14·08		−0·96	+0·08	14·16	
	−3·28				250·0
10·80		−0·48	+0·04	10·84	
	−3·76				178·8
7·04		0	0	7·04	
	−3·76				102·8
3·28		+0·48	−0·04	3·24	
	−3.28				31·6
0		(+0·96)	−0·08	−0·08	
			Total area, square feet,		1209·6

The second differences enclosed in parentheses at the top and bottom of the column are those filled in by making them form an arithmetical progression with the second differences adjoining them.

The last corrected breadth in the present example is negative, and is therefore subtracted instead of added in the ensuing computation.

Rule D.—(“*Common Trapezoidal Rule,*” to be used when a rough approximation is sufficient.) Add together the halves of the endmost breadths, and the whole of the intermediate breadths, and multiply the sum by the common interval.

Example.—The same as before.

	Feet.	
Half breadth at one end, 17·28 ÷ 2 =	8·64	
Intermediate breadths,	16·40	
	14·08	
	10·80	
	7·04	
	3·28	
Half breadth at the other end, . .	0	
	60·24	
× Common interval, ,	20	
Approximate area, ...	1204·8	square feet.
True area as before computed, . .	1209·6	
Error,	−4·8	square feet.

35. **Circle.**—The area of a circle is equal to its circumference multiplied by one-fourth of its diameter, and therefore to the square of the diameter multiplied by one-fourth of the ratio of the circumference to the diameter. The ratio of the area of a circle to the square of its diameter $\left(\text{which ratio is denoted by the symbol } \frac{\pi}{4}\right)$ is *incommensurable;* that is, not expressible exactly in figures; but it can be found approximately, to any required degree of precision. Its value has been computed to 250 places of decimals; but the following approximations are close enough for most purposes, scientific or practical:—

Approximate Values of $\frac{\pi}{4}$.	Errors in Fractions of the Circle, about
·7853981634 −	+ one-300,000,000,000th.
·785398 + ...	− one-5,000,000th.
·7854 −	+ one-400,000th.
$\frac{355}{4 \times 113}$ −	+ one-13,000,000th.
$\frac{11}{14}$ −	+ one-2,500th.

The *diameter of a circle equal in area to a given square* is very nearly 1·12838 × the side of the square. The following table gives examples of this:—

TABLE—MULTIPLIERS FOR CONVERTING

	Sides of Squares into Diameters of Equal Circles.	Diameters of Circles into sides of Equal Squares.	
1	1·12838	0·88623	1
2	2·25676	1·77245	2
3	3·38514	2·65868	3
4	4·51352	3·54491	4
5	5·64190	4·43113	5
6	6·77028	5·31736	6
7	7·89866	6·20359	7
8	9·02704	7·08981	8
9	10·15542	7·97604	9
10	11·28380	8·86227	10

36. The area of a **Circular Sector** (O A C B, fig. 8) is the same fraction of the whole circle that the angle A O B of the sector is of a whole revolution. In other words, multiply *half the square of the radius*, or *one-eighth of the square of the diameter*, by the circular measure (to radius unity) of the angle A O B; the product will be the area of the sector.

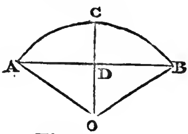


Fig. 8.

SECTION 2.—VOLUMES OF SOLID FIGURES.

37. **To Measure the Volume of any Solid.**—*Method I. By layers.*—Choose a straight axis in any convenient position. (The most convenient is usually parallel to the greatest length of the solid.) Divide the whole length of the solid, as marked on the axis, into a convenient number of equal intervals, and measure the sectional area of the solid upon a series of planes crossing the axis at right angles at the two ends and at the points of division. Then treat those areas as if they were the breadths of a plane figure, applying to them Rule A, B, or C of Article 34, page 17; and the result of the calculation will be the volume required. If Rule C is used, the volume will be obtained in separate layers.

Method II. By prisms or columns (" *Wooley's Rule*").—Assume a plane in a convenient position as a base, divide it into a network of equal rectangular divisions, and conceive the solid to be built of a set of rectangular prismatic columns, having those rectangular divisions for their sectional areas. Measure the thickness of the solid at the *centre* and at the *middle of each of the sides* of each of those rectangular columns; add together the doubles of all the thicknesses before-mentioned, which are in the interior of the solid, and the simple thicknesses which are at its boundaries; divide the sum by *six*, and multiply by the area of one rectangular division of the base.

SECTION 3.—LENGTHS OF CURVED LINES.

38. **To Calculate the Lengths of Circular Arcs.**—When the proportion of the arc to an entire circumference is given, the length of the arc, in terms of the radius, is to be calculated by multiplying that proportion by the well-known approximate value of the ratio of the circumference of a circle to its radius: viz., $\frac{\text{circumference}}{\text{radius}} = \frac{710}{113}$ nearly, $= 6{\cdot}283185$ nearly: the above ratio is commonly denoted by the symbol 2π; the reciprocal of the above ratio is very nearly $\frac{113}{710} = 0{\cdot}159155$ nearly; but it is often much more convenient in practice to proceed by drawing; and then the following rules are the most accurate yet known:—*

I. (Fig. 9). *To draw a straight line approximately equal to a given circular arc,* A B. Draw the straight chord B A; produce A to C, making A C $= \frac{1}{2}$ B A; about C, with the radius C B $= \frac{3}{2}$ B A, draw a circle; then draw the straight line A D, touching the given arc in A, and meeting the last-mentioned circle in D; A D will be the straight line required.

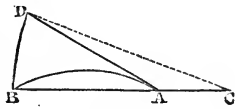


Fig. 9.

The error of this rule consists in the straight line being a little shorter than the arc: in fractions of the length of the arc, it is about $\frac{1}{1000}$ for an arc equal in length to its own radius; and it varies as the fourth power of the angle subtended by the arc; so that it may be diminished to any required extent by subdividing the arc to be measured by means of bisections. For example, in drawing a straight line approximately equal to an arc subtending 60°, the error is about $\frac{1}{900}$ of the length of the arc; divide the arc into two arcs, each subtending 30°; draw a straight line approximately equal to one of these, and double it; the error will be reduced to *one-sixteenth* of its former amount; that is, to about $\frac{1}{14400}$ of the length of the arc. The greatest angular extent of the arcs to which the rule is applied should be limited in each case according to the degree of precision required in the drawing.

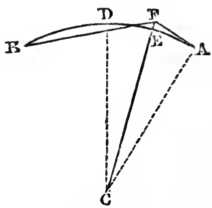


Fig. 10.

II. (Fig. 10). *To draw a straight line approximately equal to a given circular arc,* A B. (Another Method.) Let C be the centre of the arc. Bisect the arc A B in D, and the arc A D in E; draw the straight secant

* These rules are extracted from Papers read to the British Association in 1867, and published in the *Philosophical Magazine* for September and October of that year.

C E F, and the straight tangent A F, meeting each other in F; draw the straight line F B; then a straight line of the length A F + F B will be approximately equal in length to the arc A B.

The error of this rule, in fractions of the length of the arc, is just one-fourth of the error of Rule I., but in the contrary direction; and it varies as the fourth power of the angle subtended by the arc.

III. *To lay off upon a given circle an arc approximately equal in length to a given straight line.* In fig. 11, let A D be part of the circumference of the given circle, A one end of the required arc, and A B a straight line of the given length, drawn so as to touch the circle at the point A. In A B take A C = $\frac{1}{4}$ A B, and about C, with the radius C B = $\frac{3}{4}$ A B draw a circular arc B D, meeting the given circle in D. A D will be the arc required.

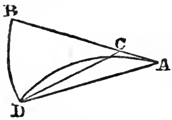


Fig. 11.

The error of this rule, in fractions of the given length, is the same as that of Rule I., and follows the same law.

IV. (Fig. 11.) *To draw a circular arc which shall be approximately equal in length to the straight line* A B, *shall with one of its ends touch that straight line at* A, *and shall subtend a given angle.* In A B take A C = $\frac{1}{4}$ A B; and about C, with the radius C B = $\frac{3}{4}$ A B, draw a circle, B D. Draw the straight line A D, making the angle B A D = one-half of the given angle, and meeting the circle B D in D. Then D will be the other end of the required arc, which may be drawn by well-known rules.

The error of this rule, in fractions of the given length, is the same with that of Rules I. and III., and follows the same law.

V. *To divide a circular arc, approximately, into any required number of equal parts.* By Rule I. or II., draw a straight line approximately equal in length to the given arc; divide that straight line into the required number of equal parts, and then lay off upon the given arc, by Rule III., an arc approximately equal in length to one of the parts of the straight line.

Rule V. becomes unnecessary when the number of parts is 2, 4, 8, or any other power of 2; for then the required division can be performed *exactly* by plane geometry.

VI. *To divide the whole circumference of a circle approximately into any required number of equal arcs.* When the required number of equal arcs is any one of the following numbers, the division can be made exactly by plane geometry, and the present rule is not needed :—any power of 2; 3; 3 × any power of 2; 5; 5 × any power of 2; 15; 15 × any power of 2.* In other cases

* It may be convenient here to state the methods of subdividing arcs and whole circles by plane geometry. (1.) *To bisect any circular arc* On the chord of the arc as a base, construct any convenient isosceles triangle, with the summit pointing away from the centre of the arc; a straight line from

proceed as follows:—Divide the circumference exactly, by plane geometry, into such a number of equal arcs as may be required, in order to give sufficient precision to the approximative part of the process. Let the number of equal arcs in that preliminary division be called n. Divide one of them, by means of Rule V., into the required number of equal parts; n times one of those parts will be one of the required equal arcs into which the whole circumference is to be divided.

Rules I., III., and V., are applicable to arcs of other curves besides the circle, provided the changes of curvature in such arcs are small and gradual.

39. **To Measure the Length of any Curve.**—Divide it into short arcs, and measure each of them by Rule I. of Article 38, page 23.

SECTION 4.—GEOMETRICAL CENTRES AND MOMENTS.

40. **Centre of Magnitude—General Principles.**—By the *magnitude* of a figure is to be understood its length, area, or volume, according as it is a line, a surface, or a solid.

The *centre of magnitude* of a figure is a point such that, if the figure be divided in any way into equal parts, the distance of the centre of magnitude of the whole figure from any given plane is the mean of the distances of the centres of magnitude of the several equal parts from that plane.

The *geometrical moment* of any figure relatively to a given plane is the product of its magnitude into the perpendicular distance of its centre from that plane.

I. *Symmetrical figure.*—If a plane divides a figure into two symmetrical halves, the centre of magnitude of the figure is in that plane; if the figure is symmetrically divided in the like manner by two planes, the centre of magnitude is in the line where those planes cut each other; if the figure is symmetrically divided by three planes, the centre of magnitude is their point of intersection; and if a figure has a *centre of figure* (for example, a circle, a sphere,

the centre of the arc to that summit will bisect the arc. (2.) *To mark the sixth part of the circumference of a circle.* Lay off a chord equal to the radius. (3.) *To mark the tenth part of the circumference of a circle.* In fig. 12, draw the straight line A B = the radius of the circle; and perpendicular to A B, draw B C = ½ A B. Join A C, and from it cut off C D = C B. A D will be the chord of one-tenth part of the circumference of the circle. (4.) *For the fifteenth part,* take the difference between one-sixth and one-tenth. It may be added that Gauss discovered a method of dividing the circumference of a circle by geometry exactly, when the number of equal parts is any prime number that is equal to 1 + a power of 2; such as $1+2^4=17$; $1+2^8=257$, &c.; but the method is too laborious for use in designing mechanism.

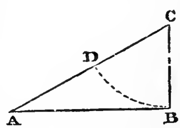


Fig. 12.

an ellipse, an ellipsoid, a parallelogram, &c.), that point is its centre of magnitude.

II. *Compound figure.*—To find the perpendicular distance from a given plane of the centre of a compound figure made up of parts whose centres are known. Multiply the magnitude of each part by the perpendicular distance of its centre from the given plane; distinguish the products (or *geometrical moments*) into positive or negative, according as the centres of the parts lie to one side or to the other of the plane; add together, separately, the positive moments and the negative moments: take the difference of the two sums, and call it positive or negative according as the positive or negative sum is the greater; this is the *resultant moment* of the compound figure relatively to the given plane; and its being positive or negative shews at which side of the plane the required centres lies. Divide the resultant moment by the magnitude of the compound figure; the quotient will be the distance required.

The centre of a figure in three dimensions is determined by finding its distances from three planes that are not parallel to each other. The best position for those planes is perpendicular to each other; for example, one horizontal, and the other two cutting each other at right angles in a vertical line. To determine the centre of a plane figure, its distances from two planes perpendicular to the plane of the figure are sufficient.

41. **Centre of a Plane Area.**—To find, approximately, the centre of any plane area.

Rule A.—Let the plane area be that represented in fig. 7 (of Article 34, page 17). Draw an axis, A X, in a convenient position, divide it into equal intervals, measure breadths at the ends and at the points of division, and calculate the area, as in Article 34.

Then multiply each breadth by its distance from one end of the axis (as A); consider the products as if they were the breadths of a new figure, and proceed by the rules of Article 34 to calculate the area of that new figure. The result of the operation will be the *geometrical moment* of the original figure relatively to a plane perpendicular to A X at the point A.

Divide the *moment* by the *area* of the original figure; the quotient will be the distance of the centre required from the plane perpendicular to A X at A.

Draw a second axis intersecting A X (the most convenient position being in general perpendicular to A X), and by a similar process find the distance of the centre from a plane perpendicular to the second axis at one of its ends; the centre will then be completely determined.

Rule B.—If convenient, the distance of the required centre from a plane cutting an axis at one of the intermediate points of division, instead of at one of its ends, may be computed as follows:—Take separately the moments of the two parts into which that plane divides the figure; the required centre will lie in the part which has the greater moment. Subtract the less moment from the greater; the remainder will be the *resultant moment* of the whole figure, which being divided by the whole area, the quotient will be the distance of the required centre from the plane of division.

Remark.—When the resultant moment is $=0$, the centre is in the plane of division.

Rule C.—To find the perpendicular distance of the centre from the axis A X. Multiply each breadth by the distance of the middle point of that breadth from the axis, and by the proper "Simpson's Multiplier," Article 34, page 18; distinguish the products into right-handed and left-handed, according as the middle points of the breadths lie to the right or left of the axis; take separately the sum of the right-handed products and the sum of the left-handed products; the required centre will lie to that side of the axis for which the sum is the greater; subtract the less sum from the greater, and multiply the remainder by $\frac{1}{3}$ of the common interval if Simpson's first rule is used, or by $\frac{3}{8}$ of the common interval if Simpson's second rule is used; the product will be the *resultant moment* relatively to the axis A X, which being divided by the area, the quotient will be the required distance of the centre from that axis.*

42. **Centre of a Volume.**—To find the perpendicular distance of the centre of magnitude of any solid figure from a plane perpendicular to a given axis at a given point, proceed as in Rule A of the preceding Article to find the moment relatively to the plane, substituting *sectional areas* for *breadths;* then divide the moment by the volume (as found by Article 37); the quotient will be the required distance.

To determine the centre completely, find its distances from three planes, no two of which are parallel. In general it is best that those planes should be perpendicular to each other.

43. **Centre of Magnitude of a Curved Line.**—*Rule A.—To find approximately the centre of magnitude of a very flat curved line.*—In fig. 13, let A D B be the arc. Draw the straight chord A B, which bisect in C; draw C D (the *deflection* of the arc) perpendicular to A B; from D lay off D E $=\frac{1}{3}$ C D; E will be very nearly the centre required.

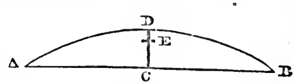


Fig. 13.

* The rules of this Article are expressed in symbols, as follows:—Let x and y be the perpendicular distances of any point in the plane area from two

This process is exact for a cycloidal arc whose chord, A B, is parallel to the base of the cycloid. For other curves it is approximate. For example, in the case of a circular arc, it gives D E too small; the error, for an arc subtending 60°, being about $\frac{1}{330}$ of the deflection, and its proportion to the deflection varying nearly as the square of the angular extent of the arc.

Rule B.—When the curved line is not very flat, divide it into very flat arcs; find their several centres of magnitude by Rule A, and measure their lengths; then treat the whole curve as a compound figure, agreeably to Rule II. of Article 40, page 26.

44. **Special Figures.**—I. *Triangle* (fig. 14).—From any two of the angles draw straight lines to the middle points of the opposite sides; these lines will cut each other in the centre required;—or otherwise,—from any one of the angles draw a straight line to the middle of the opposite side, and cut off one-third part from that line commencing at the side.

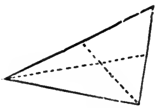


Fig. 14.

II. *Quadrilateral* (fig. 15).—Draw the two diagonals A C and B D, cutting each other in E. If the quadrilateral is a parallelogram, E will divide each diagonal into two equal parts, and will itself be the centre. If not, one or both of the diagonals will be divided into unequal parts by the point E. Let B D be a diagonal that is unequally divided. From D lay off D F in that diagonal = B E. Then the centre of the triangle F A C, found as in the preceding rule, will be the centre required.

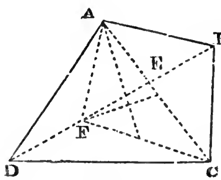


Fig. 15.

III. *Plane polygon.*—Divide it into triangles; find their centres, and measure their areas; then treat the polygon as a compound figure made up of the triangles, by Rule II. of Article 40, page 26.

IV. *Prism or cylinder with plane parallel ends.*—Find the centres of the ends; a straight line joining them will be the axis of the prism or cylinder, and the middle point of that line will be the centre required.

planes perpendicular to the area and to each other, and x_0 and y_0 the perpendicular distances of the centre of magnitude of the area from the same planes; then

$$x_0 = \frac{\iint x\,dx\,dy}{\iint dx\,dy}; \; y_0 = \frac{\iint y\,dx\,dy}{\iint dx\,dy}.$$

See Article 29, page 16.

V. *Tetrahedron, or triangular pyramid* (fig. 16).—Bisect any two opposite edges, as A D and B C, in E and F; join E F, and bisect it in G; this point will be the centre required.

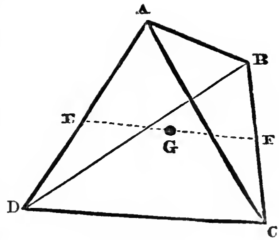


Fig. 16.

VI. *Any pyramid or cone with a plane base.*—Find the centre of the base, from which draw a straight line to the summit; this will be the axis of the pyramid or cone. From the axis cut off one-fourth of its length, beginning at the base; this will give the centre required.

VII. *Any polyhedron or plane-faced solid.*—Divide it into pyramids; find their centres and measure their volumes; then treat the whole solid as a compound figure by Rule II. of Article 22.

VIII. *Circular arc.*—In fig. 17, let A B be the arc, and C the the centre of the circle of which it is part. Bisect the arc in D, and join C D and A B. Multiply the radius C D by the chord A B, and divide by the length of the arc A D B; lay off the quotient C E upon C D; E will be the centre of magnitude of the arc.

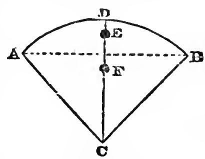


Fig. 17.

IX. *Circular sector*, C A D B, fig. 17.—Find C E as in the preceding rule, and make C F $= \frac{2}{3}$ C E; F will be the centre required.

X. *Sector of a flat ring.*—Let r be the external and r' the internal radius of the ring. Draw a circular arc of the same angular extent with the sector, and of the radius $\frac{2}{3} \cdot \frac{r^3 - r'^3}{r^2 - r'^2}$, and find its centre of magnitude by Rule VIII.

ELEMENTARY MECHANICAL NOTIONS.

Definition of General Terms and Division of the Subject.

45. **Mechanics** is the science of rest, motion, and force.

The *laws*, or *first principles* of mechanics, are the same for all bodies, celestial and terrestrial, natural and artificial.

The *methods of applying* the principles of mechanics to particular cases are more or less different, according to the circumstances of the case. Hence arise branches in the science of mechanics.

46. **Matter** (considered mechanically) is that which fills space.

47. **Bodies** are limited portions of matter. Bodies exist in three conditions—the solid, the liquid, and the gaseous. Solid bodies tend to preserve a definite size and shape. Liquid bodies tend to preserve a definite size only. Gaseous bodies tend to expand indefinitely. Bodies also exist in conditions intermediate between the solid and liquid, and possibly also between the liquid and the gaseous.

48. **A Material or Physical Volume** is the space occupied by a body or by a part of a body.

49. **A Material or Physical Surface** is the boundary of a body, or between two parts of a body.

50. **Line, Point, Physical Point, Measure of Length.**—In mechanics, as in geometry, a Line is the boundary of a surface, or between two parts of a surface; and a Point is the boundary of a line, or between two parts of a line; but the term "*Physical Point*" is sometimes used by mechanical writers to denote an *immeasurably small body*—a sense inconsistent with the strict meaning of the word "point;" but still not leading to error, so long as it is rightly understood.

In *measuring* the dimensions of bodies, the standard British unit of length is the *yard*, being the length at the temperature of 62° Fahrenheit, and at the mean atmospheric pressure, between the two ends of a certain bar which is kept in the office of the Exchequer, at Westminster.

In computations respecting motion and force, and in expressing the dimensions of large structures, the unit of length commonly employed in Britain is the *foot*, being one-third of the yard.

In expressing the dimensions of machinery, the unit of length commonly employed in Britain is the *inch*, being one-thirty-sixth part of the yard. Fractions of an inch are very commonly stated by mechanics and other artificers in halves, quarters, eighths, sixteenths, and thirty-second parts; but according to a resolution of the Institution of Mechanical Engineers, passed at the meeting held at Manchester in June, 1857, the practice has been introduced of expressing fractions of an inch in decimals.

The French unit of length is the mètre, being about $\frac{1}{40000000}$ of the earth's circumference, measured round the poles.

51. **Rest** is the relation between two points, when the straight line joining them does not change in length nor in direction.

A body is at rest relatively to a point, when every point in the body is at rest relatively to the first mentioned point.

52. **Motion** is the relation between two points when the straight line joining them changes in length, or in direction, or in both.

A body moves relatively to a point when any point in the body moves relatively to the first mentioned point.

53. **Fixed Point.**—When a single point is spoken of as having motion or rest, some other point, either actual or ideal, is always either expressed or understood, *relatively* to which the motion or rest of the first point takes place. Such a point is called a *fixed* point.

So far as the phenomena of motion alone indicate, the choice of a fixed point with which to compare the positions of other points appears to be arbitrary, and a matter of convenience alone; but when the laws of force, as affecting motion, come to be considered, it will be seen that there are reasons for calling certain points fixed, in preference to others.

In the mechanics of the solar system, the fixed point is what is called the *common centre of mass* of the bodies composing that system. In applied mechanics, the fixed point is either a point which is at rest relatively to the earth, or (if the structure or machine under consideration be movable from place to place on the earth), a point which is at rest relatively to the structure, or to the frame of the machine, as the case may be.

Points, lines, surfaces, and volumes, which are at rest relatively to a fixed point, are fixed.

54. **Cinematics.**—The comparision of motions with each other, without reference to their causes, is the subject of a branch of geometry called "*Cinematics.*"

55. **Force** is an action between two bodies, either causing or tending to cause change in their relative rest or motion.

The notion of force is first obtained directly by sensation; for the forces exerted by the voluntary muscles can be felt. The existence of forces other than muscular tension is inferred from their effects.

56. **Equilibrium** or **Balance** is the condition of two or more forces which are so opposed that their combined action on a body produces no change in its rest or motion.

The notion of balance is first obtained by sensation; for the forces exerted by voluntary muscles can be felt to balance sometimes each other, and sometimes external pressures.

57. **Dynamics—Statics and Kinetics.**—Forces may take effect, either by balancing other forces, or by producing change of motion. The former of those effects is the subject of *Statics;* the latter that of *Kinetics,* and the Science which treats of both is by modern practice entitled *Dynamics;* these, together with *Cinematics,* already defined, form the three great divisions of pure, abstract, or general mechanics.

58. **Structures and Machines.**—The works of human art to which the science of applied mechanics relates, are divided into two classes, according as the parts of which they consist are intended to rest or to move relatively to each other. In the former case they are called *Structures;* in the latter, *Machines.* Structures are subjects of Statics alone; Machines, when the motions of their parts are considered alone, are subjects of Cinematics; when the forces acting on and between their parts are also considered, machines are subjects of Dynamics.

PART I.

PRINCIPLES OF CINEMATICS, OR THE COMPARISON OF MOTIONS.

59. **Division of the Subject.**—The Science of Cinematics, and the fundamental notions of rest and motion to which it relates, having already been defined among the ELEMENTARY MECHANICAL NOTIONS, Articles 51, 52, 53, 54, it remains to be stated, that the principles of Cinematics, or the comparison of motions, will be divided and arranged in the present part of this treatise in the following manner :—

I.	Motions of Points.	
II.	„	Rigid Bodies or Systems.
III.	„	Pliable Bodies and Fluids.

CHAPTER I.

MOTIONS OF POINTS.

SECTION 1.—MOTIONS OF A PAIR OF POINTS.

60. **Fixed and Nearly Fixed Directions.**—From the definition of motion given in Article 52, it follows, that in order to determine the relative motion of a pair of points, which consists in the change of length and direction of the straight line joining them, that line must be compared, at the beginning and en[illegible] motion considered, with some fixed or standard length, [illegible] with at least two fixed directions. Standard lengths have already been considered in Article 50.

An *absolutely fixed direction* may be ascertained by means whose principles cannot be demonstrated until the subject of kinetics is considered. For the present it is sufficient to state, that when a solid body rotates free from the influence of any external force tending to change its rotation, there is an absolutely fixed direction called that of the *axis of angular momentum*, which bears certain relations to the successive positions of the body.

A *nearly fixed direction* is that of a straight line joining a pair of points in two bodies whose distance from each other is very great, such as the earth and a fixed star.

A *line fixed relatively to the earth* changes its absolute direction (unless parallel to the earth's axis) in a manner depending on the earth's rotation, and returns periodically to its original absolute direction at the end of each *sidereal day* of 86,164 seconds. This rate of change of direction is so slow compared with that which takes place in almost all pieces of mechanism to which cinematical and kinetic principles are applied, that in almost all questions of applied mechanics, directions fixed relatively to the earth may be treated as sufficiently nearly fixed for practical purposes.

When the motions of pieces of mechanism relatively to each other, or to the frame by which they are carried, are under consideration, directions fixed relatively to the frame, or to one of the pieces of the machine, may be considered provisionally as fixed for the purposes of the particular question.

POSTULATE.—Let it be granted that a line may represent a motion, where the term motion is employed to represent the path of motion, the direction and the velocity or length of motion in a unit of time. This is a self-evidently possible problem, for a line may be drawn to represent any path, in any direction to represent any direction of motion, and of any length to represent any length of motion, or velocity, limited always by the space within which motions can take place or lines be drawn.

61. Motion of a Pair of Points.—In fig. 18, let $A_1 B_1$ represent the relative situation of a pair of points at one instant, and $A_2 B_2$ the relative situation of the same pair of points at a later instant. Then the change of the straight line $\overline{AB}$ between those points, from the length and direction represented by $\overline{A_1 B_1}$ to the length and direction represented by $\overline{A_2 B_2}$, constitutes the *relative motion* of the pair of points A B, during the interval between the two instants of time considered.

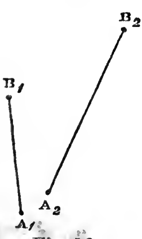


Fig. 18.

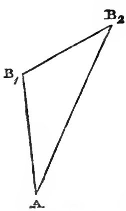


Fig. 19.

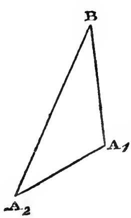


Fig. 20.

To represent that relative motion by one line, let there be drawn, from one point A, fig. 19, a pair of lines, $\overline{AB_1}$, $\overline{AB_2}$, equal and parallel to $\overline{A_1 B_1}$, $\overline{A_2 B_2}$, of fig. 18; then A represents one of the pair of points whose relative motion is under consideration, and B_1, B_2, represent the two successive positions of the other point B relatively to A; and the line $\overline{B_1 B_2}$ represents *the motion of* B *relatively to* A, which, for the purposes of the representation, is assumed to be fixed.

Or otherwise, as in fig. 20, from a single point B let there be drawn a pair of lines, $\overline{BA_1}$, $\overline{BA_2}$, equal and parallel to $\overline{A_1 B_1}$, $\overline{A_2 B_2}$, of fig. 18; then A_1 A_2, represent the two successive positions of A relatively to B; and the line $\overline{A_1 A_2}$, equal and parallel to $\overline{B_1 B_2}$ of fig. 19, but *pointing in the contrary direction*, represents *the motion of* A *relatively to* B.

62. **Fixed Point and Moving Point.**—In fig. 19, A is treated as the fixed point, and B as the moving point; and in fig. 20, B is treated as the fixed point, and A as the moving point; and these are simply two different methods of representing to the mind the same relation between the points A and B (see Article 53).

63. **Component and Resultant Motions.**—Let O be a point assumed as fixed, and A and B two successive positions of a second point relatively to O. In order to express mathematically the amount and direction of $\overline{AB}$, the motion of the second point relatively to O, that line may be compared with three *axes*, or lines in fixed directions, traversing the fixed point O, such as O X, O Y, O Z.

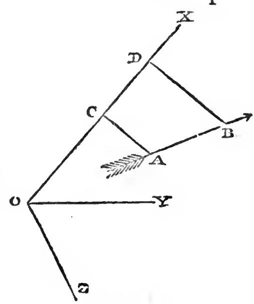


Fig. 21.

Through A and B draw straight lines A C, B D, parallel to the plane of O Y and O Z, and cutting the axis O X in C and D. Then $\overline{CD}$ is said to be the *component* of the motion of the second point relatively to O, *along*, or *in the direction of* the axis O X; and by a similar process are found the components of the motion $\overline{AB}$ along O Y and O Z. The entire motion $\overline{AB}$ is said to be the *resultant* of these components, and is evidently the diagonal of a parallelopiped of which the components are the sides.

The three axis are usually taken at right angles to each other; in which case A C and B D are perpendiculars let fall from A and B upon O X; and if α be the angle made by the direction of the motion $\overline{AB}$ with O X,

$$\overline{CD} = \overline{AB} \cdot \cos \alpha.$$

64. **The Measurement of Time** is effected by comparing the events, and especially the motions, which take place in intervals of time.

Equal times are the times occupied by the same body, or by equal and similar bodies, under precisely similar circumstances, in performing equal and similar motions. The *standard unit of time* is the period of the earth's rotation, or *sidereal day*, which has been proved by Laplace, from the records of celestial phenomena, not to have changed by so much as one *eight-millionth* part of its length in the course of the last two thousand years.

A subordinate unit is the *second*, being the time of one swing of a pendulum, so adjusted as to make 86,400 oscillations in 1·00273791 of a sidereal day; so that a sidereal day is 86164·09 seconds.

The length of a solar day is variable; but the *mean solar day*, being the exact mean of all its different lengths, is the period already mentioned of 1·00273791 of a sidereal day, or 86,400 seconds. The divisions of the mean solar day into 24 hours, of each hour into 60 minutes, and of each minute into 60 seconds, are familiar to all.

Fractions of a second are measured by the oscillations of small pendulums, or of springs, or by the rotations of bodies so contrived as to rotate through equal angles in equal times.

65. **Velocity** is the ratio of the number of units of length described by a point in its motion relatively to another point, to the number of units of time in the interval occupied in describing the length in question; and if that ratio is the same, whether it be computed for a longer or a shorter, an earlier or a later, part of the motion, the velocity is said to be UNIFORM. Velocity is expressed in *units of distance per unit of time*. For different purposes, there are employed various units of velocity, some of which, together with their proportions to each other, are given in the following table:—

Comparison of Different Measures of Velocity.

	Miles per hour.	Feet per second.	Feet per minute.	Feet per hour.
	1	= 1·4$\dot{6}$	= 88	= 5280
	0·68$\dot{1}\dot{8}$	= 1	= 60	= 3600
	0·011$\dot{3}\dot{6}$	= 0·01$\dot{6}$	= 1	= 60
	0·0001$\dot{8}\dot{9}\dot{3}$	= 0·0002$\dot{7}$	= 0·01$\dot{6}$	= 1
1 nautical mile per hour, or "knot,"......	= 1·1507	= 1·6877	= 101·262	= 6075·7$\dot{4}$

In treating of the general principles of mechanics, the *foot per second* is the unit of velocity commonly employed in Britain. The units of time being the same in all civilized countries, the proportions amongst their units of velocity are the same with those amongst their linear measures.

Component and resultant velocities are the velocities of component and resultant motions, and are related to each other in the same way with those motions, which have already been treated of in Article 63.

66. **Uniform Motion** consists in the combination of uniform velocity with uniform direction; that is, with motion along a straight line whose direction is fixed.

Section 2.—Uniform Motion of Several Points.

67. **Motion of Three Points.**—Theorem. *The relative motions of three points in a given interval of time are represented in direction and magnitude by the three sides of a triangle.* Let O, A, B, denote the three points. Any one of them may be taken as a fixed point; let O be so chosen; and let O X, O Y, O Z, fig. 22, be axes traversing it in fixed directions. Let A_1 and B_1 be the positions of A and B relatively to O at the beginning of the given interval of time, and A_2 and B_2 their positions at the end of that interval. Then $\overline{A_1 A_2}$ and $\overline{B_1 B_2}$ are the respective motions of A and B relatively to O. Complete the parallelogram $A_1 B_1 b A_2$; then because $\overline{A_2 b}$ is parallel and equal to $A_1 B_1$, b is the position which B would have at the end of the interval, *if it had no motion relatively to* A; but B_2 is the actual position of B at the end of the interval; therefore, $\overline{b B_2}$ is the motion of B relatively to A. Then in the triangle $B_1 b B_2$,

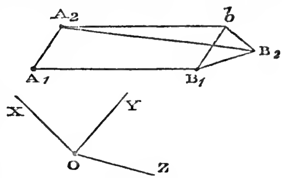


Fig. 22.

$\overline{B_1 b} = \overline{A_1 A_2}$ is the motion of A relatively to O,

$\overline{b B_2}$ is the motion of B relatively to A,

$\overline{B_1 B_2}$ is the motion of B relatively to O;

so that those three motions are represented by the three sides of a triangle.—Q. E. D.

This Theorem might be otherwise expressed by saying, *that if three moving points be considered in any order, the motion of the third relatively to the first is the resultant of the motion of the third relatively to the second, and of the motion of the second relatively to the first;* the word "*resultant*" being understood as already explained in Article 63.

68. **Motions of a Series of Points.**—Corollary. *If a series of points be considered in any order, and the motion of each point determined relatively to that which precedes it in the series, and if the relative motion of the last point and the first point be also determined, then will those motions be represented by the sides of a closed*

polygon. Let O be the first point, A, B, C, &c., successive points following it, M the last point but one, and N the last point; and, for brevity's sake, let the relative motion of two points, such as B and C, be denoted thus (B, C). Then by the Theorem of Article 67, (O, A), (A, B), and (O, B) are the three sides of a triangle; also (O, B), (B, C), and (O, C), are the three sides of a triangle; therefore (O, A), (A, B), (B, C), and (O, C), are the four sides of a quadrilateral; and by continuing the same process, it is shewn, that how great soever the number of points, (O, N), is the closing side of a polygon, of which (O, A), (A, B), (B, C), (C, D), &c., (M, N) are the other sides.—Q. E. D. In other words, *the motion of the last point relatively to the first is the resultant of the motions of each point of the series relatively to that preceding it.*

69. **The Parallelopiped of Motions.**—In fig. 23, let there be four points, O, A, B, C, of which one, O, is assumed as fixed, and is traversed by three axes in fixed directions, O X, O Y, O Z. In a given interval of time, let A have the motion $\overline{A_1 A_2}$ along or parallel to O X; let B have, in the same interval, the motion $\overline{b B_2}$ parallel to O Y, and relatively to A; then $\overline{B_1 B_2}$, the diagonal of the parallelogram whose sides are $\overline{B_1 b} = \overline{A_1 A_2}$ and $\overline{b B_2}$, is the motion of B relatively to O. Let C have, relatively to B, the motion $\overline{c C_2}$ parallel to O Z; then $\overline{C_1 C_2}$, the diagonal of the parallelopiped whose edges are $\overline{A_1 A_2}$, $\overline{b B_2}$, and $\overline{c C_2}$, is the motion of C relatively to O, being the resultant of the motions represented by those three edges. This is a *mechanical* explanation of the composition of motions, leading to results corresponding with the *geometrical* explanation of Article 63.

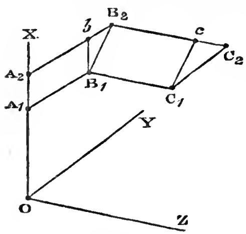


Fig. 23.

70. **Comparative Motion** is the relation which exists between the simultaneous motions of two points relatively to a third, which is assumed as fixed. The comparative motion of two points is expressed, in the most general case, by means of four quantities, viz. :—

(1.) The *velocity ratio*,* or the proportion which their velocities bear to each other, that is, the proportion borne to each other by the distances moved through by the two points in the same interval of time.

(2.) (3.) (4.) The *directional relation*,* which is the relation between the directions in which the two points are moving at the same instant, and which requires, for its complete expression, three

* These terms are adopted from Prof. Willis's work on Mechanism.

angles. Those three angles may be measured in different ways, and one of those ways is the following :—

(2.) The angle made by the directions of the compared motions with each other.

(3.) The angle made by a plane parallel to those two directions with a fixed plane.

(4.) The angle made by the intersection of those two planes with a fixed direction in the fixed plane.

Thus, the comparative motion of two points relatively to a third, is expressed by means of one of those groups of four elements which Sir William Rowan Hamilton has called "*quaternions.*" In most of the practical applications of cinematics, the motions to be compared are limited by conditions which render the comparision more simple than it is in the general case just described. In machines, for example, the motion of each point is limited to two directions, forward or backward in a fixed path; so that the comparative motion of two points is sufficiently expressed by means of the velocity ratio, together with a directional relation expressed by + or −, according as the motions at the instant in question are similar or contrary.

Section 3.—Varied Motion of Points.

71. Velocity and Direction of Varied Motion.—The motion of one point relatively to another may be varied, either by change of velocity, or by change of direction, or by both combined, which last case will now be considered, as being the most general.

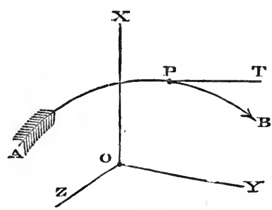


Fig. 24.

In fig. 24, let O represent a point assumed as fixed, O X, O Y, O Z, fixed directions, and A B part of the *path* or orbit traced by a second point in its varied motion relatively to O. At the instant when the second point reaches a given position, such as P, in its path, the *direction* of its motion is obviously that of $\overline{PT}$, a tangent to the path at P.

To find the velocity at the instant of passing P, let Δt denote an interval of time which includes that instant, and Δs the distance traced in that interval. Then

$$\frac{\Delta s}{\Delta t}$$

is an *approximation* to the velocity at the instant in question, which will approach continually nearer and nearer to the exact velocity as the interval Δt and the distance Δs are made shorter and shorter; and the *limit* towards which $\frac{\Delta s}{\Delta t}$ converges, as Δs and Δt are indefinitely diminished, and which is denoted by

$$v = \frac{d s}{d t}, \dots\dots\dots\dots (1.)$$

is the *exact* velocity at the instant of passing P. In the language of the differential calculus, the space is a function of the time and the velocity is the differential coefficient of the space with respect to the time, thus $s = \phi t$ and $\frac{d s}{d t} = \phi' t = v$. It will be seen hereafter that, the velocity (v) itself is a function of the time (t). This is the process called "*differentiation.*"

Should the velocity at each instant of time be known, then the distance $s_1 - s_0$, described during an interval of time $t_1 - t_0$, is found by *integration* (see Article 29), as follows:—

$$s_1 - s_0 = \int_{t_0}^{t_1} v \, d t \dots\dots\dots\dots (2.)$$

72. **Components of Varied Motion.**—All the propositions of the two preceding sections, respecting the composition and resolution of motions, are applicable to the velocities of varied motions at a given instant, each such velocity being represented by a line, such as $\overline{\text{PT}}$, in the direction of the tangent to the path of the point which moves with that velocity, at the instant in question. For example, if the axes O X, O Y, O Z, are at right angles to each other, and if the tangent $\overline{\text{PT}}$ makes with their directions respectively the angles α, β, γ then the three rectangular components of the velocity of the point parallel to those three axes are

$$v \cos \alpha; \; v \cos \beta; \; v \cos \gamma.$$

Let x, y, z, be the co-ordinates of any point, such as P, in the path A P B, as referred to the three given axes. If a point p be assumed indefinitely near to the point P, its co-ordinates will be $x + d x$, $y + d y$, $z + d z$, and if $d s$ have the already assumed value, $d x$, $d y$, $d z$, will be its projections on the three axes; that is, the lengths bounded by perpendiculars let fall from the extremities of $d s$ on the three respective axes. Then it is well known that

$$\cos \alpha = \frac{d x}{d s}; \; \cos \beta = \frac{d y}{d s}; \; \cos \gamma = \frac{d z}{d s};$$

and consequently the three components of the velocity $v \left(= \frac{d s}{d t} \right)$ are

$$v \cos \alpha = \frac{d x}{d t}; \; v \cos \beta = \frac{d y}{d t}; \; v \cos \gamma = \frac{d z}{d t}; \dots\dots (3.)$$

now by the Geometry of three dimensions

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

and hence these are related to their resultant by the equation

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = v^2 \ldots\ldots\ldots\ldots(4.)$$

73. **Uniformly-Varied Velocity.**—Let the velocity of a point either increase or diminish at an uniform rate; so that if t represents the time elapsed from a fixed instant when the velocity was v_0, the velocity at the end of that time shall be

$$v = v_0 + a\,t; \ldots\ldots\ldots\ldots(1.)$$

a being a constant quantity, which is the *rate of variation* of the velocity, and is called *acceleration* when positive, and *retardation* when negative. Then the *mean* velocity during the time t is

$$\frac{v_0 + v}{2} = \frac{v_0 + v_0 + a\,t}{2} = v_0 + \frac{a\,t}{2} \ldots\ldots\ldots\ldots(2.)$$

and the distance described is

$$s = v_0 t + \frac{a\,t^2}{2} \ldots\ldots\ldots\ldots(3.)$$

If there be no initial velocity, that is, if the body start from a state of rest, then $v = a\,t$ and $s = \frac{a\,t^2}{2}$, and these equations are illustrations of the use of the differential calculus; for first differentiate s with respect to t in the equation $s = \frac{a\,t^2}{2}$, and there is obtained $\frac{ds}{dt} (= v) = \frac{a \cdot 2\,t}{2} = a\,t$, which is the first equation, then differentiate $v = a\,t$, and there is obtained $\frac{dv}{dt} = a$. To find the velocity of a point, whose velocity is uniformly varied, at a given instant, and the rate of variation of that velocity, let the distances, Δs_1, Δs_2, described in two equal intervals of time, each equal to Δt, before and after the instant in question, be observed. Then the velocity at the instant between those intervals is

$$v = \frac{\Delta s_1 + \Delta s_2}{2\,\Delta\,t} \ldots\ldots\ldots\ldots(4.)$$

and its rate of variation is

$$a = \frac{\Delta v}{\Delta t} = \frac{\Delta s_2 - \Delta s_1}{(\Delta\,t)^2} \ldots\ldots\ldots\ldots(5.)$$

where the variation of velocity $=\frac{\Delta s_2 - \Delta s_1}{\Delta t}$, and the rate of variation being either acceleration or retardation, as the velocity of the point is being increased or diminished, is that quantity divided by Δt.

74. **Graphical Representation of Motions.**—Since in uniform motion the space is equal to the product of the velocity and time, and since in geometry a rectangular area is the product of a base line and perpendicular, an uniform motion may be represented by a rectangular area, as in fig. 25, where A B represents a certain number of *units* of time, and A C a certain number of units of velocity per *unit* of time. It will be noticed that in uniform motion, the velocity or number of units of velocity at each unit of time is the same, as at A, B, E. Varied motion and uniformly varied motion may also be graphically represented: in the first, the line C D will be a curve; and in the second, the line C D will form a constant angle with A B; hence in varied motion any ordinate, E F, depends upon the abscissa A E, and the mean velocity is the mean ordinate of a figure so formed, or is the quotient of the area (space) divided by the base (time), whereas in uniformly-varied motion, the space described depends upon the initial and final velocities alone, and not upon the intermediate velocities. Fig. 26 represents varied motion where the velocity at each point is represented by the ordinate at that point, and the mean velocity is equal to the area of the figure divided by the base A B. Fig. 27 represents uniformly-varied motion, and it is evident that, in order to estimate the area of the figure A B C D, that is, the space, it is only necessary to consider the initial and final velocities. In these figures, if the velocity be null at any point, there will be no ordinate at that point: if the direction of motion change, this will be represented by a change of sign of the ordinate or velocity.

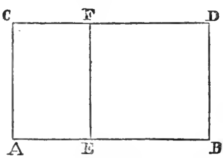


Fig. 25.

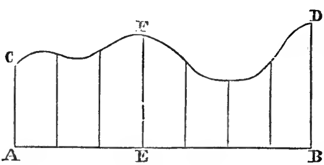


Fig. 26.

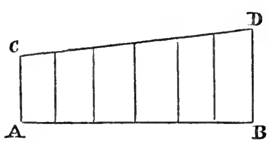


Fig. 27.

There is another method of graphically representing the motion of a point: in this the abscissæ represent the time, and the ordinates at each point the space passed over in the corresponding number of units of time, or the distance of the point from a certain datum point. In this case the space described in any number of units of time is equal to the difference of the lengths of the ordinates at the corresponding intervals, and the velocity is proportional to the quotient of the difference of the ordinates divided by the difference of the abscissæ.

75. **Varied Rate of Variation of Velocity.**—When the velocity of a point is neither constant nor uniformly-varied, its rate of variation may still be found by applying to the velocity the same operation of *differentiation*, which, in Article 73, was applied to the distance described in order to find the velocity. The result of this operation is expressed by the symbols,

$$a = \frac{d\,v}{d\,t} = \frac{d \cdot \frac{d\,s}{d\,t}}{d\,t} = \frac{d^2\,s}{d\,t^2};$$

and is the limit to which the quantity obtained by means of the formula 5 of Article 73 continually approximates, as the interval denoted by $\Delta\,t$ is indefinitely diminished. In the fraction $\frac{d \cdot \frac{d\,s}{d\,t}}{d\,t}$, $d\,s$ is the limit of the difference of either of the spaces Δs in equation (5), Article 73, and $d \cdot d\,s$, is the limit of the difference of that difference, viz., $\Delta s_2 - \Delta s_1$; that is, d in this fraction is represented by the minus sign $(-)$ in the other, and $d\,s$ by the limit of either of the quantities Δs_1, Δs_2. Here in the language of the differential calculus, the velocity (v) is a function of the time (t), and the acceleration (a) is the differential coefficient of the velocity with respect to the time, thus $v = \phi\,t$ and $a = \phi'\,t$, or $= \frac{d\,v}{d\,t}$. Also the velocity, v, being the differential coefficient of the space with respect to the time, see Article 71; the acceleration a is the 2nd differential coefficient of the space with respect to the time, or v being $\psi' t$, $a = \psi''\,t$.

76. **Combination of Uniform and Uniformly Accelerated Motion.**—Assume a pair of rectangular axes of co-ordinates. Let the uniform motion be represented by abscissæ along O X, and the uniformly accelerated motion by ordinates parallel to O Y; let O B $(= x) = v\,t$, represent the space described in the time t with the velocity v, and let O C $(= y) = \frac{a\,t^2}{2}$, represent the space de-

scribed with a uniform rate of acceleration, a, in the same time t, see Article 73, then $x^2 = v^2 t^2$ and $y = \frac{a t^2}{2}$, $\therefore$ $x^2 = y \frac{2 v^2}{a}$, where the square of any abscissa bears a constant ratio to the corresponding ordinate, and the path of the point is known by Conic Sections to be a Parabola. The same follows for any axes of co-ordinates; but if the direction of the uniformly accelerated motion be that of the uniform motion or directly opposed to it, the resultant direction will be the same as that of either motion, or will be that of the greater component.

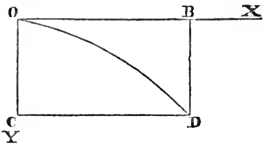


Fig. 28.

77. **Uniform Deviation** is the change of motion of a point which moves with uniform velocity in a circular path. The *rate* at which uniform deviation takes place is determined in the following manner:—

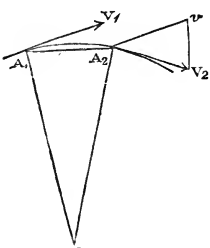


Fig. 29.

Let C, fig. 29, be the centre of the circular path described by a point A with an uniform velocity v, and let the radius $\overline{CA}$ be denoted by r. At the beginning and end of an interval of time Δt, let A_1 and A_2 be the positions of the moving point. Then

$$\text{the arc } A_1 A_2 = v \Delta t;$$

$$\text{the chord } A_1 A_2 = v \Delta t \cdot \frac{\text{chord}}{\text{arc}}$$

The velocities and directions at A_1 and A_2 are represented by the equal lines $\overline{A_1 V_1} = \overline{A_2 V_2} = v$, touching the circle at A_1 and A_2 respectively. From A_2 draw $\overline{A_2 v}$ equal and parallel to $\overline{A_1 V_1}$, and join $\overline{V_2 v}$. Then the velocity $\overline{A_2 V_2}$ may be considered as compounded of $\overline{A_2 v}$ and $\overline{v V_2}$; so that $\overline{v V_2}$ is the *deviation* of the motion during the interval Δt; and because the isosceles triangles $A_2 v V_2$, $C A_1 A_2$, are similar:—

$$\overline{v V_2} = \frac{\overline{A_2 V_2} \cdot \overline{A_1 A_2}}{\overline{C A}} = \frac{v^2 \cdot \Delta t}{r} \cdot \frac{\text{chord}}{\text{arc}},$$

deduced by substituting the value of $\overline{A_1 A_2}$ already found; and the *approximate rate* of that deviation being the deviation divided by the interval of time in which it occurs, is

$$\frac{v^2}{r} \cdot \frac{\text{chord}}{\text{arc}};$$

but the deviation does not take place by instantaneous changes of velocity, but by insensible degrees; so that the true rate of deviation is to be found by finding the limit to which the approximate rate continually approaches as the interval Δt is diminished indefinitely. Now the factor $\frac{v^2}{r}$ remains unaltered by that diminution; and the ratio of the chord to the arc approximates continually to equality; so that the limit in question, or *true rate of deviation*, is expressed by

$$\frac{v^2}{r} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

78. **Varying Deviation.**—When a point moves with a varying velocity, or in a curve not circular, or has both these variations of motion combined, the *rate of deviation* at a given instant is still represented by Equation 1 of Article 77, provided v be taken to denote the velocity, and r the radius of curvature of the path, of the point at the instant in question.

79. **The Resultant Rate of Variation** of the motion of a point is found by considering the rate of variation of velocity and the rate of deviation as represented by two lines, the former in the direction of a tangent to the path of the point, and the latter in the direction of the radius of curvature at the instant in question, and taking the diagonal of the rectangle of which those two lines are the sides, which has the following value:—

$$\sqrt{\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{r^2}} = \sqrt{\left\{\left(\frac{d^2 s}{dt^2}\right)^2 + \frac{1}{r^2}\left(\frac{ds}{dt}\right)^4\right\}} \ldots (1.)$$

the first term of the quantity under the first radical is the square of $\frac{dv}{dt}$ in Article 73, and the second the square of $\frac{v^2}{r}$, Equation (1), Article 77.

80. **The Rates of Variation of the Component Velocities** of a point parallel to three rectangular axes, are represented as follows:—

$$\frac{d^2 x}{dt^2};\ \frac{d^2 y}{dt^2};\ \frac{d^2 z}{dt^2};\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

and if a rectangular parallelopiped be constructed, of which the edges represent these quantities, its diagonal, whose length is

$$\sqrt{\left\{\left(\frac{d^2 x}{dt^2}\right)^2 + \left(\frac{d^2 y}{dt^2}\right)^2 + \left(\frac{d^2 z}{dt^2}\right)^2\right\}} \ldots\ldots\ldots\ldots (2.)$$

will represent the *resultant rate of variation*, already given in another form in Equation 1 of Article 79.

81. **The Comparison of the Varied Motions** of a pair of points

relatively to a third point assumed as fixed, is made by finding the ratio of their velocities, and the directional relation of the tangents of their paths at the same instant, in the manner already described in Article 70, as applied to uniform motions. It is evident that the comparative motions of a pair of points may be so regulated as to be constant, although the motion of each point is varied, provided the variations take place for both points at the same instant, and at rates proportional to their velocities.

CHAPTER II.

MOTIONS OF RIGID BODIES.

SECTION 1.—RIGID BODIES, AND THEIR TRANSLATION.

82. The term **Rigid Body** is to be understood to denote a body, or an assemblage of bodies, or a system of points, whose figure undergoes no alteration during the motion which is under consideration.

83. **Translation or Shifting** is the motion of a rigid body relatively to a fixed point, when the points of the rigid body have no motion relatively to each other; that is to say, when they all move with the same velocity and in the same direction at the same instant, so that no line in the rigid body changes its direction.

It is obvious that if three points in the rigid body, not in the same straight line, move in parallel directions with equal velocities at each instant, the body must have a motion of translation.

The paths of the different points of the body, provided they are all equal and similar, and at each instant parallel, may have any figure whatsoever.

SECTION 2.—SIMPLE ROTATION.

84. **Rotation or Turning** is the motion of a rigid body when lines in it change their direction. Any point in or rigidly attached to the body may be assumed as a fixed point to which to refer the motions of the other points. Such a point is called a *centre of rotation.*

85. **Axis of Rotation.**—THEOREM. *In every possible change of position of a rigid body, relatively to a fixed centre, there is a line traversing that centre whose direction is not changed.* In fig. 30, let O be the centre of rotation, and let A and B denote any two other points in the body, whose situations relatively to O are, before the turning, A_1, B_1, and after the turning, A_2, B_2. Join $\overline{A_1 A_2}$, $\overline{B_1 B_2}$, forming the isosceles triangles $O A_1 A_2$, $O B_1 B_2$. Bisect the bases of those triangles in C

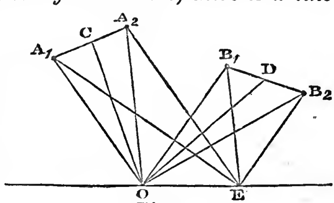


Fig. 30.

and D respectively, and through the points of bisection draw two planes perpendicular to the respective bases, intersecting each other in the straight line $\overline{OE}$, which must traverse O. Let E be any point in the line O E; then E A_1 A_2, and E B_1 B_2, are isosceles triangles; and E is at the same distance from O, A, and B, before and after the turning; therefore E is one and the same point in the body, whose place is unchanged by the turning; and this demonstration applies to every point in the straight line $\overline{OE}$; therefore that line is unchanged in direction.—Q. E. D.

In fig. 31, the same construction and reasoning being applied, the point E being supposed vertically above or below the point O, it is evident that the planes through O D, and O C intersect, and the axis will be represented by a straight line perpendicular to the plane of the paper through O and E.

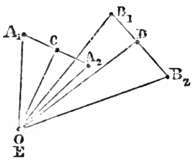


Fig. 31.

COROLLARY. It is evident that every line in the body, parallel to the axis, has its direction unchanged.

86. **The Plane of Rotation** is any plane perpendicular to the axis, such as any plane parallel to the plane of the paper, in fig. 31. The **Angle of Rotation**, or angular motion, is the angle made by the two directions, before and after the turning, of a line perpendicular to the axis, as $A_1 O A_2$, or $B_1 O B_2$, in fig. 31.

87. **The Angular Velocity** of a turning body is the ratio of the angle of rotation, expressed in terms of radius, to the number of units of time in the interval of time occupied by the angular motion. Speed of turning is sometimes expressed also by the number of turns or fractions of a turn in a given time. The relation between these two modes of expression is the following:—Let a be the angular velocity, as above defined, and T the turns in the same unit of time; then

$$T = \frac{a}{2\pi};$$
$$a = 2\pi T;$$
$$\left(2\pi = 6{\cdot}2831852 = \frac{710}{113}\right)^*$$

88. **Uniform Rotation** consists in uniformity of the angular

* The value of π may be easily remembered by taking the first three odd numbers twice each, and placing the six in a row, using the first three as the denominator, and the last three as the numerator of a fraction: we thus obtain 113 | 355 $= \frac{355}{113}$; this is a nearer approximation than 3·14159, and is generally much more easily employed in calculation.

velocity of the turning body, and constancy of the direction of its axis of rotation.

89. **Rotation common to all Parts of Body.**—Since the angular motion of rotation consists in the change of direction of a line in a plane of rotation, and since that change of direction is the same how short soever the line may be, it is evident that the condition of rotation, like that of translation, is common to every particle, how small soever, of the turning rigid body, and that the angular velocity of turning of each particle, how small soever, is the same with that of the entire body. This is otherwise evident by considering, that each part into which a rigid body can be divided turns completely about in the same time with every other part, and with the entire body.

90. **Right and Left-Handed Rotation.**—The *direction* of rotation round a given axis is distinguished in an arbitrary manner into *right-handed* and *left-handed.* One end of the axis is chosen, as that from which an observer is supposed to look along the direction of the axis towards the rotating body. Then if the body seems to the observer to turn in the same direction in which the sun seems to revolve to an observer north of the tropics, or in that in which the hands of a watch or clock revolve, the rotation is said to be *right-handed;* if in the contrary direction, *left-handed:* and it is usual to consider the angular velocity of right-handed rotation to be positive, and that of left-handed rotation to be negative; but this is a matter of convenience. It is obvious that the same rotation which seems right-handed when looked at from one end of the axis, seems left-handed when looked at from the other end.

91. **Relative Motion of a Pair of Points in a Rotating Body.**—Let O and A denote any two points in a rotating body; and considering O as fixed, let it be required to determine the motion of A relatively to an axis of rotation drawn through O. On that axis let fall a perpendicular from A; let r be the length of that perpendicular. Then the motion of A relatively to the axis traversing O is one of *revolution,* or *translation in a circular path of the radius r;* the centre of that circular path being at the point where the perpendicular from A meets the axis. If a be the angular velocity of the body, that is, the velocity of a point situate at the distance unity from the axis of rotation, then the *velocity* of A relatively to the axis traversing O is

$$v = a\,r; \qquad (1.)$$

and the *direction* of that velocity is at each instant perpendicular to the plane drawn through A and the axis. The *rate of deviation* of A in its motion relatively to the given axis is

$$\frac{v^2}{r} = a^2\,r; \qquad (2.)$$

in which the first expression is that already found in Article 77, and the second is deduced from the first by the aid of Equation 1 of this Article. It is evident that for a given rotation the motion of O relatively to an axis of rotation traversing A is exactly the same with that of A relatively to a parallel axis traversing O; for it depends solely on the angular velocity a, the perpendicular distance r of the moving point from the axis, and the direction of the axis; all which are the same in either case.

r is called the *radius-vector* of the moving point.

92. **Cylindrical Surface of Equal Velocities.**—If a cylindrical surface of circular cross section be described about an axis of rotation, all the points in that surface have equal velocities relatively to the axis, and the direction of motion of each point in the cylindrical surface relatively to the axis is a tangent to the surface in a plane perpendicular to the axis.

93. **Comparative Motions of Two Points relatively to an Axis.**—Let O, A, B, denote three points in a rotating rigid body; let O be considered as fixed, and let an axis of rotation be drawn through it. Then the *comparative* motions of A and B relatively to that axis are expressed as follows:—*The velocity-ratio is that of the radii-vectores of the points, and the directional relation consists in the angle between their directions of motion being the same with that between their radii-vectores.* Or symbolically: Let r_1, r_2, be the perpendicular distances of A and B from the axis traversing O, and v_1 and v_2 their velocities; then

$$\frac{v_2}{v_1}=\frac{r_2}{r_1};\ \text{and}\ \widehat{v_1\, v_2}=\widehat{r_1\, r_2}.$$

94. **Components of Velocity of a Point in a Rotating Body.**—The component parallel to an axis of rotation, of the velocity of a point in a rotating body relatively to that axis, is null. That velocity may be resolved into components in the plane of rotation. Thus let O, in fig. 32, represent an axis of rotation of a body whose plane of rotation is that of the figure; and let A be any point in the body whose radius-vector is $\overline{\mathrm{O\,A}}=r$. The velocity of that point being $v=a\,r$ (a representing the velocity of a point situated at the distance unity from the axis of rotation), let that velocity be represented by the line $\overline{\mathrm{A\,V}}$ perpendicular to O A. Let B A be any direction in the plane of rotation, along which it is desired to find the component of the velocity of A; and let $\angle$ V A U $= \theta$ be the angle made by that line with A V. From V let fall V U perpendicular

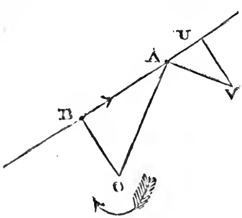


Fig. 32.

to B A; then $\overline{AU}$ represents the component in question; and denoting it by u,

$$u = v \cdot \cos\theta = a\,r \cdot \cos\theta \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

From O let fall O B perpendicular to B A. Then $\angle AOB = \angle VAU = \theta$; and the right-angled triangles O B A and A U V are similar; so that

$$\overline{AV} : \overline{AU} :: \overline{OA} : \overline{OB} = r \cos\theta \ldots\ldots\ldots\ldots(2.)$$

Now the *entire* velocity of B relatively to the axis O is

$$a\,r \cos\theta = u, \ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

so that *the component, along a given straight line in the plane of rotation, of the velocity of any point in that line, is equal to the velocity of the point where a perpendicular from the axis meets that line.*

Section 3.—Combined Rotations and Translations.

95. **Property of all Motions of Rigid Bodies.**—The foregoing proposition may be regarded as a particular case of the following, which is true of all motions of a rigid body.

The components, along a given straight line in a rigid body, of the velocities of the points in that line relatively to any point, whether in or attached to the body or otherwise, are all equal to each other; for otherwise, the distances between points in the given straight line must alter, which is inconsistent with the idea of rigidity.

96. **Helical Motion.**—Rotation is the only movement which a rigid body as a whole can have relatively to a point belonging to it or attached to it. But if the motion of the body be determined relatively to a point not attached to it, a translation may be combined with the rotation. When that translation takes place in the direction of the axis of rotation, the motion of the rigid body is said to be *helical*, or *screw-like*, because each point in the rigid body describes a helix or screw, or a part of a helix or screw.

Let v_1 denote the velocity of translation, parallel to the axis of rotation, which is common to all points of the body; this is called the *velocity of advance.* The advance during one complete turn of the rotating body is the *pitch* of each of the helical or screw-like paths described by its particles; that is, the distance, in a direction parallel to the axis, between one turn of each such helix and the next; and a being the angular velocity, so that $\frac{2\pi}{a}$ is the time of one turn (2π being the space traversed in one turn by a point at the distance unity from the axis), the value of the pitch (or the space passed over, which is equal to the product of the velocity and time) is

$$p = \frac{2\pi v_1}{a}; \text{ whence } v_1 = \frac{a\,p}{2\pi} \dots\dots\dots\dots\dots\dots (1.)$$

Let r, as before, be the radius-vector of any point in the body, and let

$$v_2 = a\,r \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2.)$$

denote its *velocity of revolution*, or velocity relatively to the axis, due to the rotation alone. Then the *resultant* velocity of that point is

$$v = \sqrt{v_1^2 + v_2^2} = a \cdot \sqrt{\left\{\frac{p^2}{4\pi^2} + r^2\right\}} \dots\dots\dots (3.)$$

The *inclination* of the helix described by that point to the *plane of rotation* is given by the equation

$$i = \text{arc} \cdot \tan \cdot \frac{v_1}{v_2} = \text{arc} \cdot \tan \cdot \frac{p}{2\pi r}; \dots\dots\dots\dots\dots (4.)$$

that is, an angle whose tangent is equal to v_1 divided by v_2, or to p divided by $2\pi r$, the tangent of that angle being the ratio of the pitch to the circumference of the circle described by the point relatively to the axis of rotation.

97. Problem.—**To find the Motion of a Rigid Body from the Motions of Three of its Points.**—Let A, B, C, fig. 33, be three points in a rigid body, and at a given instant let them have motions relatively to a point independent of the body, which motions are represented in velocity and direction by the three lines $\overline{AV_a}$, BV_b, CV_c. It is required to find the motion of the entire rigid body relatively to the same fixed point.

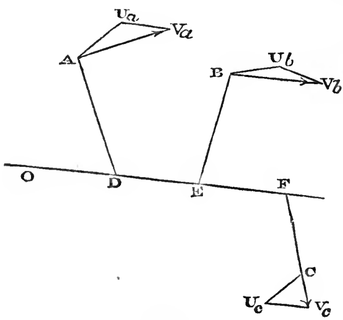


Fig. 33.

Through any point o, fig. 34, draw three lines oa, ob, oc, equal and parallel to the three lines $\overline{AV_a}$, $\overline{BV_b}$, CV_c. Through a, b, and c, draw a plane $a\,b\,c$, on which let fall a perpendicular $o\,n$ from o. Then $o\,n$ represents a component, which is common to the velocities of all the three points A, B, C, and must therefore be common to all the points in the body; that is, it is a *velocity of translation.*

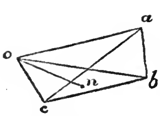


Fig. 34.

From the points V_a, V_b, V_c, draw lines $\overline{V_a U_a}$, $\overline{V_b U_b}$, $\overline{V_c U_c}$, equal and parallel to $o\,n$, but opposite in direction to it; and join

$\overline{A U_a}$, $\overline{B U_b}$, $\overline{C U_c}$, which will all be parallel to the same plane; that is, to the plane *a b c*. The last three lines will represent the component velocities which, along with the common velocity of translation parallel to *o n*, make up the resultant velocities of the three points. Through the point A draw a plane perpendicular to the component of its motion, which is parallel to *a b c*; that is, to $\overline{A U_a}$, and through B draw a plane perpendicular to $\overline{B U_b}$. These two planes will intersect each other in a line O D E, which will be parallel to *o n*. The perpendicular distances of that line from the points A B being unchanged by the motion, it represents one and the same line in or attached to the rigid body, and it is therefore the axis of rotation. A plane drawn through the third point, C, perpendicular to $\overline{C U_c}$, will cut the other two planes in the same axis: the three revolving component velocities

$$\overline{A U_a}, \overline{B U_b}, \overline{C U_c},$$

will be respectively proportional to the perpendicular distances, or *radii-vectores*, $\overline{A D}$, $\overline{B E}$, $\overline{C F}$,

of the three points from that axis; and the angular velocity will be equal to each of the three quotients made by dividing the revolving component velocities of the points by their respective radii-vectores. This rotation, combined with a translation parallel to the axis, with a velocity represented by *o n*, constitutes a *helical* or *screw-like* motion, being the required motion of the rigid body.—Q. E. I.

98. **Special Cases** of the preceding problem occur, in which either a more simple method of solution is sufficient, or the general method fails, and a special method has to be employed.

I. *When the motions of the points of the body are known to be all parallel to one plane*, it is sufficient to know the motions of two points, such as A, B, fig. 35. Let A O, B O, be two planes traversing A and B, and perpendicular to the respective directions of the simultaneous velocities of those points; if those planes cut each other, the entire motion is a rotation; the line of intersection of the planes O, being the axis of rotation, and the angular velocity, are found as in the last Article. If the two planes are parallel, the motion is a translation.

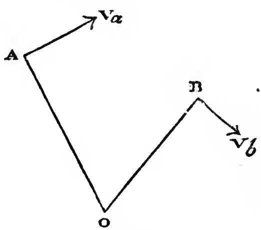


Fig. 35.

II. *If three points in the same plane have parallel motions oblique to the plane*, the motion is a translation.

III. *If three points in the same plane move perpendicularly to the plane*, as A B C, fig. 35 *a*, then, if their velocities are equal, the motion is a translation; and if their velocities are unequal, the motion is a rotation about the axis which is the intersection of the plane of the three points with the plane drawn through the extre-

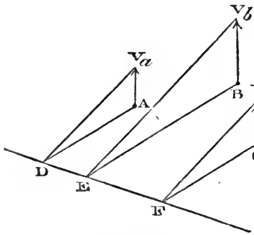


Fig. 35 *a.*

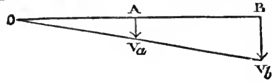


Fig. 35 *b.*

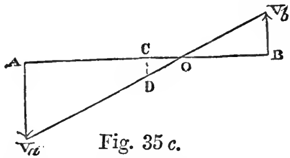


Fig. 35 *c.*

mities of the three lines which represent their velocities viz., through the points, V_a, V_b, V_c; the angular velocity being found as in Article 97.

If the plane of rotation is known, then the simultaneous velocities of two points, as A and B in figs. 35 *b* and 35 *c*, are sufficient to determine the axis O.

99. Rotation Combined with Translation in the Same Plane.—

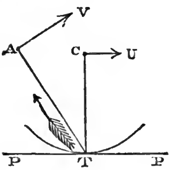


Fig. 36.

Let a body rotate about an axis C (fig. 36), fixed relatively to the body, with an angular velocity *a*, and at the same time let that axis have a motion of translation in a straight path perpendicular to the direction of the axis, with the velocity *u*, represented by the line $\overline{\mathrm{C\,U}}$. It is required to find the velocity and direction of motion of any point in the body. From the moving axis draw a straight line $\overline{\mathrm{C\,T}}$ perpendicular to that axis and to $\overline{\mathrm{C\,U}}$, and in that direction into which the rotation (as represented by the feathered arrow) tends to turn $\overline{\mathrm{C\,U}}$, and make

$$\overline{\mathrm{C\,T}} = \frac{u}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Then the point T has, in virtue of *translation along with* the axis C, a *forward* motion with the velocity *u*; and in virtue of *rotation about* that axis, it has a *backward* motion with the velocity

$$a \cdot \overline{\mathrm{C\,T}} = u,$$

equal and opposite to the former; and its resultant velocity is 0. Hence every point in the body, which comes in succession into the position T, situated at the distance $\frac{u}{a}$ from the axis C in the direction above described, is *at rest at the instant of its arriving at that position;* that is, it has just ceased to move in one direction, and is about to move in another direction; and this is true of every point which arrives at *a line traversing* T *parallel to* C. Consequently the resultant motion of the body, at any given instant, is the same as if it were rotating about the line which *at the instant in question* occupies the position T, parallel to C, at the distance $\frac{u}{a}$; and that line is called THE INSTANTANEOUS AXIS. To find the motion of any point A in the body at a given instant, let fall the perpendicular $\overline{\text{A T}}$ from that point on the instantaneous axis; then the motion of A is in the direction A V perpendicular to the plane of the instantaneous axis and of the *instantaneous radius-vector* $\overline{\text{A T}}$, and the velocity of that motion is

$$v = a \cdot \overline{\text{A T}} \qquad \qquad (2.)$$

100. **Rolling Cylinder; Trochoid.**—Every straight line parallel to the moving axis C, in a cylindrical surface described about C with the radius $\frac{u}{a}$, becomes in turn the instantaneous axis. Hence the motion of the body is the same with that produced by the rolling of such a cylindrical surface on a plane P T P parallel to C and to $\overline{\text{C U}}$, at the distance $\frac{u}{a}$.

The path described by any point in the body, such as A, which is not in the moving axis C, is a curve well known by the name of *trochoid.* The particular form of trochoid called the *cycloid,* is described by each of the points in the rolling cylindrical surface; being such a curve as is described by a nail in the tyre of a revolving wheel.

101. **Plane Rolling on Cylinder; Spiral Paths.**—Another mode of representing the combination of rotation with translation in the same plane as follows:—Let O, fig. 37, be an axis assumed as fixed, about which let the plane O C (containing the axis O) rotate (right-handedly, in the figure), with the angular velocity *a*. Let a rigid body have, *relatively to the rotating plane,* and in a direction perpendicular to it, a translation with the velocity *u*. In the plane O C, and at right angles to the axis O, take $\overline{\text{O T}} = \frac{u}{a}$, in such a direction that the velocity

$$u = a \cdot \overline{\text{O T}},$$

which the point T *in the rotating plane* has at a given instant, shall be in the contrary direction to the equal velocity of translation *u*, which the rigid body has relatively to the rotating

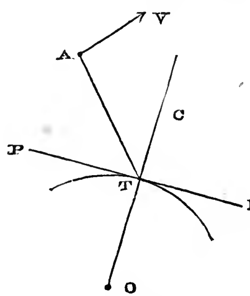


Fig. 37.

plane. Then each point *in the rigid body* which arrives at the position T, or at any position in a line traversing T parallel to the fixed axis O, is at rest *at the instant* of its occupying that position; therefore the line traversing T parallel to the fixed axis O is *the instantaneous axis;* the motion at a given instant of any point in the rigid body, such as A, is at right angles to the radius-vector A T drawn perpendicular to the instantaneous axis; and the velocity of that motion is given by the equation

$$v = a \cdot \overline{\mathrm{A\,T}}.$$

All the lines in the rigid body which successively occupy the position of instantaneous axis are situated in a plane of that body, P T P, perpendicular to O C; and all the positions of the instantaneous axis are situated in a cylinder described about O with the radius $\overline{\mathrm{O\,T}}$; so that the motion of the rigid body is such as is produced by the *rolling of the plane* P P *on the cylinder whose radius is* $\overline{\mathrm{O\,T}} = \frac{u}{a}$. Each point in the rigid body, such as A, describes a plane spiral about the fixed axis O. For each point in the *rolling plane,* P P, that spiral is the involute of the circle whose radius is $\overline{\mathrm{O\,T}}$. The simplest method of understanding the nature of this curve, is to wrap a cord round the perimeter of a cylinder, placed on a sheet of paper, to attach a tracing point to any point in the cord in juxtaposition with the cylinder, and then to unwrap the cord from the cylinder, keeping the cord always in the same plane parallel to the plane of the paper; the tracing point will trace the involute of a circle on the sheet of paper. For each point whose path of motion traverses the fixed axis O; that is, for each point in a plane of the rigid body traversing O parallel to P P, the spiral is Archimedean, having a radius-vector increasing by the length u for each angle a through which it rotates; this spiral is traced by a point moving uniformly from the centre along the radius, while the radius itself revolves.

102. **Combined Parallel Rotations.**—In figs. 38, 39, and 40, let O be an axis assumed as fixed, and O C a plane traversing that axis, and rotating about it with the angular velocity a. Let C be an axis in that plane, parallel to the fixed axis O; and about the moving axis C let a rigid body rotate with the angular velocity b *relatively to the plane* O C; and let the directions of the rotations a and b be distinguished by positive and negative signs. The body is said to have the rotations about the parallel axes O and C *combined* or *compounded,* and it is required to find the result of that combination of parallel rotations.

Fig. 38 respresents the case in which a and b are similar in direction; fig. 39, that in which a and b are in opposite directions, and b is the greater; and fig. 40, that in which a and b are in opposite directions, and a is the greater.

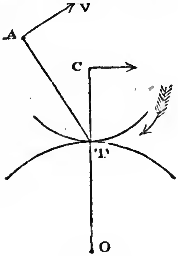


Fig. 38.

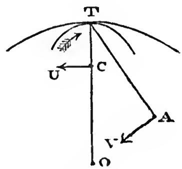


Fig. 39.

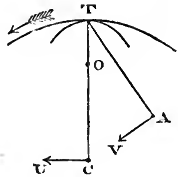


Fig. 40.

Let a common perpendicular O C to the fixed and moving axes be intersected in T by a straight line parallel to both those axes, in such a manner that the distances of T from the fixed and moving axes respectively shall be inversely proportional to the angular velocities of the component rotations about them, as is expressed by the following proportion:—

$$a : b : : \overline{\mathrm{CT}} : \overline{\mathrm{OT}} \quad \text{.........................(1.)}$$

When a and b are similar in direction, let T fall between O and C, as in fig. 38; when they are contrary, beyond, as in figs. 39 and 40. Then the velocity of the line T *of the plane* O C is $a \cdot \overline{\mathrm{OT}}$; and the velocity of the line T *of the rigid body, relatively to the plane* O C, is $b \cdot$ C T, equal in amount and contrary in direction to the former; therefore each line of the rigid body which arrives at the position T is at rest at the instant of its occupying that position, and is then *the instantaneous axis.* The *resultant angular velocity* is given by the equation

$$c = a + b; \quad \text{.........................(2.)}$$

regard being had to the directions or signs of a and b; that is to say, if we now take a and b to represent *arithmetical* magnitudes, and affix explicit signs to denote their directions, the direction of c will be the same with that of the greater; the case of fig. 38 will be represented by Equation 2, already given; and those of figs. 39 and 40 respectively by

$$c = b - a; \ c = a - b \quad \text{.................(2 A.)}$$

The relative proportions of a, b, and c, and of the distances between the fixed, moving, and instantaneous axes, are given by the equation

$$a : b : c : : \overline{\mathrm{CT}} : \overline{\mathrm{OT}} : \overline{\mathrm{OC}} \quad \text{.................(3.)}$$

The motion of any point, such as A, in the rigid body, is at each instant at right angles to the radius-vector $\overline{\mathrm{A\,T}}$ drawn from the point perpendicular to the instantaneous axis; and the velocity of that motion is

$$v = c \cdot \overline{\mathrm{A\,T}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

103. **Cylinder Rolling on Cylinder; Epitrochoids.**—All the lines in the rigid body which successively occupy the position of instantaneous axis are situated in a cylindrical surface described about C with the radius $\overline{\mathrm{C\,T}}$; and all the positions of the instantaneous axis are contained in a cylindrical surface described about O with the radius $\overline{\mathrm{O\,T}}$; therefore the resultant motion of the rigid body is that which is produced by rolling the former cylinder, attached to the body, on the latter cylinder, considered as fixed.

In fig. 38, a convex cylinder rolls on a convex cylinder; in fig. 39, a smaller convex cylinder rolls in a larger concave cylinder; in fig. 40, a larger concave cylinder rolls on a smaller convex cylinder.

Each point in the rolling rigid body traces, relatively to the fixed axis, a curve of the kind called *epitrochoids.* The epitrochoid traced by a point in the surface of the rolling cylinder is an *epicycloid.*

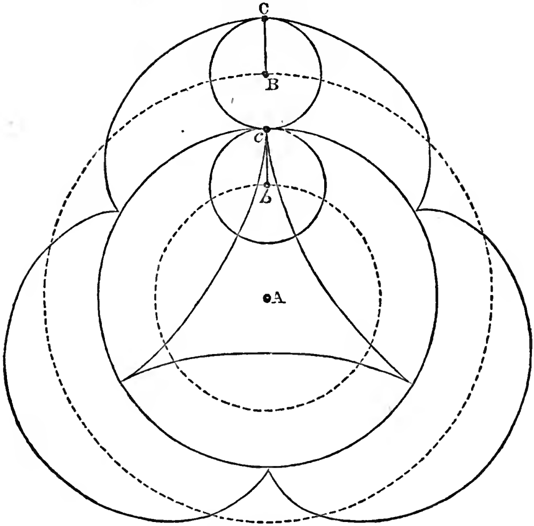
In certain cases, the epitrochoids become curves of a more simple class. For example, each point in the *moving axis* C traces a circle.

When a cylinder, as in fig. 39, rolls within a concave cylinder of *double its radius,* each point in the surface of the rolling cylinder moves backwards and forwards in a straight line, being a diameter of the fixed cylinder; each point in the axis of the rolling cylinder traces a circle of the same radius with that cylinder, and each other point in or attached to the rolling cylinder traces an ellipse of greater or less eccentricity, having its centre in the fixed axis O.

In the examples shewn in figs. 41, 42, and 43 the ratio of the rolling-circle to the base-circle* is $\frac{1}{3}$, so that the epitrochoids are three-lobed. Each figure shews an external and an internal epitrochoid, traced by rolling the rolling-circle outside and inside the base-circle respectively. The centres of the base-circles are marked A; those of the external rolling-circles, B; those of the internal rolling-circles, *b*; and the tracing points of the external and internal rolling-circles are marked C and *c* respectively.

In fig. 41 the tracing-points are in the circumferences of the rolling-circles; and the curves traced are epicycloids, distinguished by having *cusps* at the points where the tracing-point coincides with the base-circle. In fig. 42 the tracing points are inside the rolling-circles; and the curves traced are *prolate epitrochoids,* distinguished by their wave-like form. In fig. 43 the tracing-points

* The fixed circle is called a base-circle.



Fig, 41.

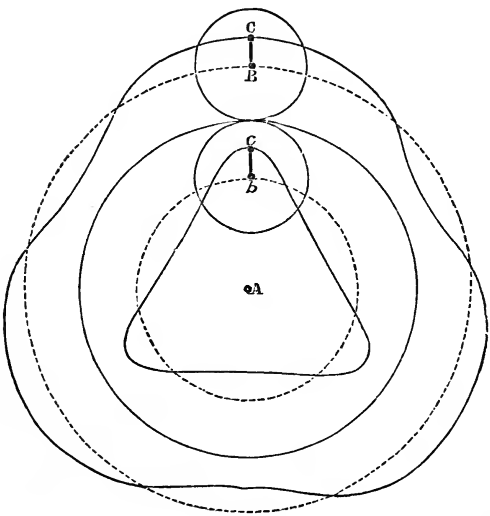


Fig. 42.

are outside the rolling-circles; and the curves traced are *curtate epitrochoids,* distinguished by their looped form.

An important property of curves traced by rolling is that at

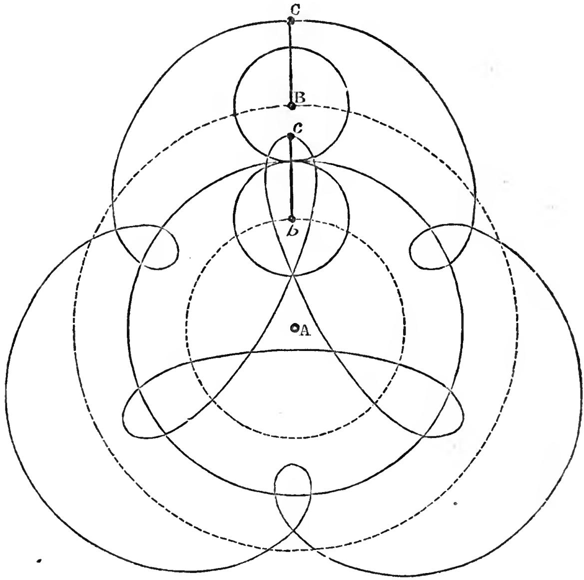


Fig. 43.

every instant the straight line joining the tracing-point and the pitch-point, or point of contact of the rolling-curve and base-curve, is normal to the traced curve at the tracing point.

The distance B C or *b c* may in each case be called the *tracing-arm.*

In mechanism for the tracing of epitrochoids (used chiefly in ornamental turning), the rolling and base-circles are the pitch-circles of a pair of spur-wheels, made with great accuracy.

Elliptic paths traced by rolling form a particular case of internal epitrochoids. In fig. 44 is represented a rolling-circle, which rolls inside a base-circle of exactly twice its radius. Then (considering a quarter of a revolution at a time), while the centre of the rolling-circle traces a quadrant, B *b*, of an equal circle about A, a point D in the circumference of the rolling-circle traces a straight line traversing A, and a point C, inside the rolling-circle, traces a quadrant, C *c*, of an ellipse whose semiaxes are A C = A B + B C, and A *c* = C D = A B − B C; also a point C′ outside the rolling-circle, but rigidly attached to it, traces a quadrant, C′ c′, of an ellipse whose semiaxes are A C′ = B C′ + A B, and A c′ = C′ D = B C′ − A B. The former may be called an *internal*, and the latter an *external*, ellipse. The proportions of the axes of either of them

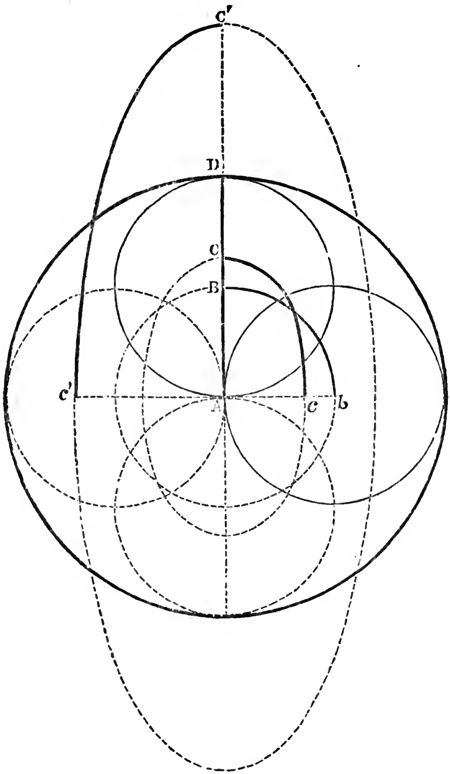


Fig. 44.

may be indefinitely varied by adjusting the position of the tracing-point; but in every internal ellipse the sum, and in every external ellipse the difference, of the semiaxes is equal to the diameter of the rolling-circle; that is, to the radius of the base-circle.

This is the principle of the mechanism commonly used for turning ellipses.

It is evident that by having a number of tracing-points carried by one rolling-circle, several ellipses differently proportioned and in different positions may be traced at the same time.

104. Equal and Opposite Parallel Rotations Combined.—Let a plane O C rotate with an angular velocity a about an axis O contained in the plane, and let a rigid body rotate about the axis C in that plane parallel to O, with an angular velocity $-a$, equal and opposite to that of the plane. Then the angular velocity of the rigid body is nothing; that is, its motion is one of *translation* only, all its points moving in equal circles of the radius $\overline{OC}$, with the velocity $a \cdot \overline{OC}$. This case is not capable of being represented by a rolling action.

105. Rotations about Intersecting Axes Combined.—In fig. 45, let O A be an axis assumed as fixed; and about it let the plane A O C rotate with the angular velocity a. Let O C be an axis in the rotating plane; and about that axis let a rigid body rotate with the angular velocity b relatively to the rotating plane.

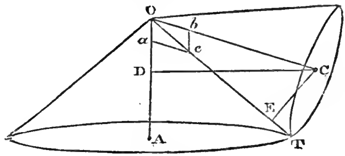


Fig. 45.

Because the point O in the rigid body is fixed, the instantaneous axis must traverse that point. The direction of that axis is determined, as before, by considering that each point which arrives at that line must have, in virtue of the rotation about O C, a velocity relatively to the rotating plane, equal and directly opposed to that which the coincident point of the rotating plane has. Hence it follows, that the ratio of the perpendicular distances of each point in the instantaneous axis from the fixed and moving axes respectively—that is, the ratio of the sines of the angles which the instantaneous axis makes with the fixed and moving axes—must be the reciprocal of the ratio of the component angular velocities about those axes; or symbolically, if O T be the instantaneous axis,

$$\sin AOT : \sin COT : : b : a \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

This determines the direction of the instantaneous axis, which may also be found by graphic construction as follows:—On O A take $\overline{Oa}$ proportional to a; and on O C take $\overline{Ob}$ proportional to b. Let those lines be taken in such directions, that to an observer looking from their extremities towards O, the component rotations seem both right-handed. Complete the parallelogram O b c a; the diagonal $\overline{Oc}$ will represent the direction of the instantaneous axis.

The resultant angular velocity about this instantaneous axis is found by considering, that if C be any point in the moving axis, the linear velocity of that point must be the same, whether computed from the angular velocity a of the rotating plane about the fixed axis O A, or from the resultant angular velocity c of the rigid body about the instantaneous axis. That is so say, let C D, C E, be perpendiculars from C upon O A, O T, respectively; then

$$a \cdot \overline{\mathrm{C\,D}} = c \cdot \overline{\mathrm{C\,E}};$$

but $\overline{\mathrm{C\,D}} : \overline{\mathrm{C\,E}} : : \sin \angle \mathrm{A\,O\,C} : \sin \angle \mathrm{C\,O\,T}$; and therefore

$$\sin \angle \mathrm{C\,O\,T} : \sin \angle \mathrm{A\,O\,C} : : a : c;$$

and, combining this proportion with that given in Equation 1, we obtain the following proportional equation:—

$$\left.\begin{array}{lcccccc} & \sin \angle \mathrm{C\,O\,T} & : & \sin \angle \mathrm{A\,O\,T} & : & \sin \angle \mathrm{A\,O\,C} \\ : : & a & : & b & : & c \\ : : & \overline{\mathrm{O}\,a} & : & \overline{\mathrm{O}\,b} & : & \overline{\mathrm{O}\,c} \end{array}\right\} \ldots\ldots(2.)$$

that is to say, *the angular velocities of the component and resultant rotations are each proportional to the sine of the angle between the axes of the other two;* and *the diagonal of the parallelogram* O b c a *represents both the direction of the instantaneous axis and the angular velocity about that axis.*

106. **Rolling Cones.**—All the lines which successively come into the position of instantaneous axis are situated in the surface of a cone described by the revolution of O T about O C; and all the positions of the instantaneous axis lie in the surface of a cone described by the revolution of O T about O A. Therefore the motion of the rigid body is such as would be produced by the rolling of the former of those cones upon the latter.

It is to be understood, that either of the cones may become a flat disc, or may be hollow, and touched internally by the other. For example, should ∠ A O T become a right angle, the fixed cone would become a flat disc; and should ∠ A O T become obtuse, that cone would be hollow, and would be touched internally by the rolling cone; and similar changes may be made in the rolling cone.

The path described by a point in or attached to the rolling cone is a *spherical epitrochoid;* but for the purposes of the present treatise, it is unnecessary to enter into details respecting the properties of that class of curves.

107. **Comparative Motions in Compound Rotations.**—The velocity ratio of two points in a rotating rigid body at any instant, is that of their perpendicular distances from its instantaneous axis; and the angle between the directions of motion of the two points is equal to that between the two planes which traverse the points and the instantaneous axis.

SECTION 4.—VARIED ROTATION.

108. **Variation of Angular Velocity** is measured like variation of linear velocity, by comparing the change which takes place in

the angular velocity of a rotating body, Δa, during a given interval of time, with the length of that interval, Δt, and the *rate of variation* is the value towards which the ratio of the change of angular velocity to the interval of time, $\frac{\Delta a}{\Delta t}$, converges, as the length of the interval is indefinitely diminished; being represented by $\frac{d a}{d t}$, and found by the operation of differentiation.

109. **Components of Varied Rotation.**—The most convenient way, in most cases, of expressing the mode of variation of a rotatory motion, is to resolve the angular velocity at each instant into three component angular velocities about three rectangular axes fixed in direction. The values of these components, at any instant shew at once the resultant angular velocity and the direction of the instantaneous axis. For example, let a_x, a_y, a_z, be the rectangular components of the angular velocity of a rigid body at a given instant,—

rotation about x from y towards z,
about y from z towards x,
and about z from x towards y,

being considered as positive; then

$$a = \sqrt{(a_x^2 + a_y^2 + a_z^2)} \quad \text{.....................(1.)}$$

is the resultant angular velocity, and

$$\cos \alpha = \frac{a_x}{a}; \; \cos \beta = \frac{a_y}{a}; \; \cos \gamma = \frac{a_z}{a}; \quad \text{..............(2.)}$$

are the cosines of the angles which the instantaneous axis makes with the axis of x, y, and z respectively.

CHAPTER III.

MOTIONS OF PLIABLE BODIES, AND OF FLUIDS.

110. **Division of the Subject.**—The subject of the present chapter will be considered under the following branches:—

I. The Motions of Flexible Cords.
II. The Motions of Fluids not altering in Volume.
III. The Motions of Fluids altering in Volume.

Section 1.—Motions of Flexible Cords.

111. **General Principles.**—As those relative motions of the points of a cord which may arise from its extensibility, belong to the subject of resistance to tension, which is a branch of that of strength and stiffness, the present section is confined to those motions of which a flexible cord is capable when the length, not merely of the whole cord, but of each part lying between two points fixed in the cord, is invariable, or sensibly invariable.

In order that the figure and motions of a flexible cord may be determined from cinematical considerations alone, independently of the magnitude and distribution of forces acting on the cord, its weight must be insensible compared with the tension on it, and it must everywhere be *tight;* and when that is the case, each part of the cord which is not straight is maintained in a curved figure by passing over a *convex* surface. The line in which a tight cord lies on a convex surface is the *shortest line* which it is possible to draw on that surface between each pair of points in the course of the cord. (It is a well-known principle of the geometry of curved surfaces, that the *osculating plane* or *tangential plane* at each point of such a line is perpendicular to the curved surface.)

Hence it appears, that the motions of a tight flexible cord of invariable length and insensible weight are regulated by the following principles:—

I. *The length between each pair of points in the cord is constant.*

II. *That length is the shortest line which can be drawn between its extremities over the surfaces by which the cord is guided.*

112. **Motions Classed.**—The motions of a cord are of two kinds—

I. Travelling of a cord along a track of invariable form; in which case the velocities of all points of the cord are equal.

II. Alteration of the figure of the track by the motion of the guiding surfaces.

Those two kinds of motion may be combined.

The most usual problems in practice respecting the motions of cords are those in which cords are the means of transmitting motion between two pieces in a train of mechanism. Such problems will be considered in Part II. of this treatise.

Next in point of frequency in practice is the problem to be considered in the ensuing Article.

113. **Cord Guided by Surfaces of Revolution.**—Let a cord in some portions of its course be straight, and in others guided by the surfaces of circular drums or pulleys, over each of which its track is a circular arc in a plane perpendicular to the axis of the guiding surface. Let r be the radius of any one of the guiding surfaces, i the angle of inclination which the two straight portions of the cord contiguous to that surface make with each other, expressed in length of arc to radius unity. Then the length of the portion of the cord which lies on that surface is $r\,i$; and if s be the length of any straight portion of the cord, the total length between two given points fixed in the cord may be expressed thus:—

$$\mathrm{L} = \Sigma \cdot s + \Sigma \cdot r\,i \qquad \text{(1.)}$$

Let c be the distance between the centres of a given adjacent pair of guiding surfaces, s the length of the straight portion of cord which lies between them, and r, r', their respective radii; then evidently

$$s = \sqrt{c^2 - (r \pm r')^2} \qquad \text{(2.)}$$

the $\left\{\begin{matrix}\text{sum}\\\text{difference}\end{matrix}\right\}$ of the radii being employed, according as the cord $\left\{\begin{matrix}\text{crosses}\\\text{does not cross}\end{matrix}\right\}$ the line of centres c.

The case most common in practice is that in which the *plies*, or straight parts of the cord, are all parallel to each other; so that $i = 180°$ in each case, while a certain number, n, of the guiding bodies or pulleys all move simultaneously in a direction parallel to the plies of the cord with the same velocity, u; where u represents the velocity of translation of the guiding surfaces, and v the longitudinal velocity of any point in the cord

$$v = 2\,n\,u \qquad \text{(3.)}$$

Section 2.—Motions of Fluids of Constant Density.

114. **Velocity and Flow.**—The density of a moving fluid mass may be either exactly invariable, from the constancy or the adjustment of its temperature and pressure, or sensibly invariable, from the smallness of the alterations of volume which the actual altera-

tions of pressure and temperature are capable of producing. The latter is the case in most problems of practical mechanics affecting liquids.

Conceive an ideal surface of any figure, and of the area A, to be situated within a fluid mass, the parts of which have motion relatively to that surface; and let u denote, as the case may be, the *uniform* velocity, or the *mean* value of the varying velocity, resolved in a direction perpendicular to A, with which the particles of the fluid pass A. Then

$$Q = u\,A \qquad (1.)$$

is the volume of fluid which passes from one side to the other of the surface A in an unit of time, and is called the *flow*, or *rate of flow*, through A.

When the particles of fluid move obliquely to A, let θ denote the angle which the direction of motion of any particle passing A makes with a normal to A, and v the velocity of that particle; then

$$u = v \cdot \cos \theta \qquad (2.)$$

115. **Principle of Continuity.**—Axiom. *When the motion of a fluid of constant density is considered relatively to an enclosed space of invariable volume which is always filled with the fluid, the flow into the space and the flow out of it, in any one given interval of time, must be equal*—a principle expressed symbolically by

$$\Sigma \cdot Q = 0 \qquad (3.)$$

The preceding self-evident principle regulates all the motions of fluids of constant density, when considered in a purely cinematical manner. The ensuing articles of this section contain its most usual applications.

116. **Flow in a Stream.**—A stream is a moving fluid mass, indefinitely extended in length, and limited transversely, and having a continuous longitudinal motion. At any given instant, let A, A′, be the areas of any two of its transverse sections, considered as fixed; u, u', the mean normal velocities through them; Q, Q′, the rates of flow through them; then in order that the principle of continuity may be fulfilled, those rates of flow must be equal; that is,

$u\,A = u'\,A' = Q = Q' =$ constant for all cross sections of the channel at the given instant; (1.)

consequently,

$$\frac{u'}{u} = \frac{A}{A'}; \qquad (2.)$$

or, *the normal velocities at a given instant at two fixed cross sections are inversely as the areas of these sections.*

117. **Pipes, Channels, Currents, and Jets.**—When a stream of

fluid completely fills a *pipe* or *tube*, the area of each cross section is given by the figure and dimensions of the pipe, and for similar forms of section varies as the square of the diameter. Hence the mean normal velocities of a stream flowing in a full pipe, at different cross sections of the pipe, are inversely as the squares of the diameters of those sections.

A *channel* partially encloses the stream flowing in it, leaving the upper surface free; and this description applies not only to channels commonly so called, but to pipes partially filled. In this case the area of a cross section of the stream depends not only on the figure and dimensions of the channel, but on the figure and elevation of the free upper surface of the stream.

A *current* is a stream bounded by other portions of fluid whose motions are different.

A *jet* is a stream whose surface is either free all round, or is touched by a solid body in a small portion of its extent only.

118. **Steady Motion** of a fluid relatively to a given space considered as fixed is that in which the velocity and direction of the motion of the fluid at each *fixed point* is uniform at every instant of the time under consideration; so that although the velocity and direction of the motion of a given particle of the fluid may vary while it is transferred from one point to another, that particle assumes, at each fixed point at which it arrives, a certain definite velocity and direction depending on the position of that point alone; which velocity and direction are successively assumed by each particle which successively arrives at the same fixed point.

The steady motion of a stream is expressed by the two conditions, that the area of each fixed cross section is constant, and that the flow through each cross section is constant, then the differential coefficient of a constant being equal to 0 (see Article 26, page 11),

$$\frac{d\,A}{d\,t}=0;\ \frac{d\,Q}{d\,t}=0 \qquad \text{(1.)}$$

If u represents the normal velocity of a fluid moving steadily, *at a given fixed point*,

$$\frac{d\,u}{d\,t}=0; \qquad \text{(2.)}$$

expresses the condition of steady motion.

119. **Motion of Pistons.**—Let a mass of fluid of invariable volume be enclosed in a vessel, two portions of the boundary of which (called *pistons*) are movable inwards and outwards, the rest of the boundary being fixed. Then, if motion be transmitted between the pistons by moving one inwards and the other outwards, it follows, from the invariability of the volume of the enclosed fluid, that the velocities of the two pistons at each instant

will be to each other in the inverse ratio of the areas of the respective projections of the pistons on planes normal to their directions of motion. This is the principle of the transmission of motion in the *hydraulic press* and *hydraulic crane.*

The *flow* produced by a piston whose velocity is u, and the area of whose projection on a plane perpendicular to the direction of its motion is A, is given, as in other cases, by the equation

$$Q = u\,A \qquad (1.)$$

Section 3.—Motions of Fluids of Varying Density.

120. **Flow of Volume and Flow of Mass.**—In the case of a fluid of varying density, the *volume,* which in an unit of time flows through a given area A, with a normal velocity u, is still represented, as for a fluid of constant density, by

$$Q = A\,u; \qquad (1.)$$

but the *absolute quantity,* or *mass* of fluid which so flows, bears no longer a constant proportion to that volume, but is proportional to the volume multiplied by the density. The density may be expressed, either in units of weight per unit of volume, or in arbitrary units suited to the particular case. Let ρ be the density; then the *flow of mass* may be thus expressed:—

$$\rho\,Q = \rho\,A\,u \qquad (2.)$$

121. **The Principle of Continuity,** as applied to fluids of varying density, takes the following form:—*the flow into or out of any fixed space of constant volume is that due to the variation of density alone.*

To express this symbolically, let there be a fixed space of the constant volume V, and in a given interval of time let the density of the fluid in it, which in the first place may be supposed uniform at each instant, change from ρ_1 to ρ_2. Then the mass of fluid which at the beginning of the interval occupied the volume V, occupies at the end of the interval the volume $\frac{V\rho_1}{\rho_2}$; and the difference of those volumes is the volume which flows through the surface bounding the space, *outward* if ρ_2 is less than ρ_1, *inward* if ρ_2 is greater than ρ_1. Let $t_2 - t_1$ be the length of the interval of time; then the rate of flow of volume is expressed as follows:—

$$Q = \frac{V\left(\frac{\rho_1}{\rho_2} - 1\right)}{t_2 - t_1} \qquad (1.)$$

PART II.

THEORY OF MECHANISM.

CHAPTER I.

DEFINITIONS AND GENERAL PRINCIPLES.

122. **Theory of Pure Mechanism Defined.**—*Machines* are bodies, or assemblages of bodies, which transmit and modify motion and force. The word "machine," in its widest sense, may be applied to every material substance and system, and to the material universe itself; but it is usually restricted to works of human art, and in that restricted sense it is employed in this treatise. A machine transmits and modifies motion when it is the means of making one motion cause another; as when the mechanism of a clock is the means of making the descent of the weight cause the rotation of the hands. A machine transmits and modifies force when it is the means of making a given kind of physical energy perform a given kind of work; as when the furnace, boiler, water, and mechanism of a marine steam engine are the means of making the energy of the chemical combination of fuel with oxygen perform the work of overcoming the resistance of water to the motion of a ship. The acts of transmitting and modifying motion, and of transmitting and modifying force, take place together, and are connected by a certain law; and until lately, they were always considered together in treatises on mechanics; but recently great advantage in point of clearness has been gained by first considering separately the act of transmitting and modifying motion. The principles which regulate this function of machines constitute a branch of Cinematics, called the *theory of pure mechanism.* The principles of the theory of pure mechanism having been first established and understood, those of the *theory of the work of machines,* which will form the subject of Part VI. of this work, which regulate the act of transmitting and modifying force, are much more readily demonstrated and apprehended than when the two departments of the theory of machines are mingled. The establishment of the theory of pure mechanism as an independent subject has been mainly ac-

complished by the labours of Professor Willis, whose nomenclature and methods are, to a great extent, followed in this treatise.

123. **The General Problem** of the theory of pure mechanism may be stated as follows:—*Given the mode of connection of two or more movable points or bodies with each other, and with certain fixed bodies; required the comparative motions of the movable points or bodies:* and conversely, *when the comparative motions of two or more movable points are given, to find their proper mode of connection.*

The term "comparative motion" is to be understood as in Articles 70, 81, 93, and 107. In those Articles, the comparative motions of points belonging to one body have already been considered. In order to constitute *mechanism,* two or more bodies must be so connected that their motions depend on each other through cinematical principles alone.

124. **Frame; Moving Pieces; Connectors; Bearings.**—The *frame* of a machine is a structure which supports the *moving pieces,* and regulates the path or kind of motion of most of them directly. In considering the movements of machines mathematically, the frame is considered as fixed, and the motions of the moving pieces are referred to it. The frame itself may have (as in the case of a ship or of a locomotive engine) a motion relatively to the earth, and in that case the motions of the moving pieces relatively to the earth are the resultants of their motions relatively to the frame, and of the motion of the frame relatively to the earth; but in all problems of pure mechanism, and in many problems of the work of machines, the motion of the frame relatively to the earth does not require to be considered.

The *moving pieces* may be distinguished into *primary* and *secondary;* the former being those which are directly carried by the frame, and the latter those which are carried by other moving pieces. The motion of a secondary moving piece relatively to the frame is the resultant of its motion relatively to the primary piece which carries it, and of the motion of that primary piece relatively to the frame.

Connectors are those secondary moving pieces, such as links, belts, cords, and chains, which transmit motion from one moving piece to another, when that transmission is not effected by immediate contact.

Bearings are the surfaces of contact of primary moving pieces with the frame, and of secondary moving pieces with the pieces which carry them. Bearings guide the motions of the pieces which they support, and their figures depend on the nature of those motions. The bearings of a piece which has a motion of translation in a straight line, must have plane or cylindrical surfaces, *exactly straight* in the direction of motion. The bearings of rotating pieces must have surfaces accurately turned to *figures of revolution,* such as cylinders, spheres, conoids, and flat discs. The bearing of a piece whose motion is helical, must be an *exact screw,* of a pitch equal to that of the helical motion (Article 96). Those parts of moving pieces which touch the bearings, should have surfaces accurately fitting those of the bearings. They may be distinguished into *slides,* for pieces which move in straight lines, *gudgeons, journals, bushes,* and *pivots,* for those which rotate, and *screws* for those which move helically.

125. **The Motions of Primary Moving Pieces** are limited by the fact, that in order that different portions of a pair of bearing surfaces may accurately fit each other during their relative motion, those surfaces must be either straight, circular, or helical; from which it follows, that the motions in question can be of three kinds only, viz. :—

I. *Straight translation,* or *shifting,* which is necessarily of limited extent, and which, if the motion of the machine is of indefinite duration, must be *reciprocating*; that is to say, must take place alternately in opposite directions. (See Part I., Chapter II., Section 1.)

II. *Simple rotation,* or *turning* about a fixed axis, which motion may be either continuous or reciprocating, being called in the latter case *oscillation.* (See Part I., Chapter II., Section 2.)

III. *Helical* or *screw-like motion,* to which the same remarks apply as to straight translation. (See Part I., Chapter II., Section 3, Article 96.)

126. **The Motions of Secondary Moving Pieces** relatively to the pieces which carry them, are limited by the same principles which apply to the motions of primary pieces relatively to the frame. But the motions of secondary moving pieces relatively to the frame may be any motions which can be compounded of straight translations and simple rotations according to the principles already explained in Part I., Chapter II., Section 3.

127. **An Elementary Combination** in mechanism consists of a pair of *primary moving pieces,* so connected that one transmits motion to the other.

The piece whose motion is the cause is called the *driver;* that whose motion is the effect, the *follower.* The *connection* between the driver and the follower may be—

I. By *rolling contact* of their surfaces, as in *toothless wheels.*

II. By *sliding contact* of their surfaces, as in *toothed wheels, screws, wedges, cams,* and *escapements.*

III. By *bands* or *wrapping connectors,* such as *belts, cords,* and *gearing-chains.*

IV. By *link-work,* such as *connecting rods, universal joints,* and *clicks.*

V. By *reduplication of cords*, as in the case of ropes and pulleys.

VI. By an *intervening fluid*, transmitting motion between two pistons.

The various cases of the transmission of motion from a driver to a follower are further classified, according as the relation between their *directions* of *motion* is constant or changeable, and according as the ratio of their velocities is constant or variable. This latter principle of classification was employed by Professor Willis, in the first edition of his *Principles of Mechanism*, as the foundation of a primary division of the subject of elementary combinations in mechanism into classes, which are subdivided according to the mode of connection of the pieces. In the present treatise, elementary combinations will be classed primarily according to the mode of connection; which is the classification employed by Professor Willis in the Edition of 1870.

128. **Line of Connection.**—In every class of elementary combinations, except those in which the connection is made by reduplication of cords, or by an intervening fluid, there is at each instant a certain straight line, called the *line of connection*, or *line of mutual action* of the driver and follower. In the case of rolling contact, this is any straight line whatsoever traversing the point of contact of the surfaces of the pieces; in the case of sliding contact, it is a line perpendicular to those surfaces at their point of contact; in the case of wrapping connectors, it is the centre line of that part of the connector by whose tension the motion is transmitted; in the case of link-work, it is the straight line passing through the points of attachment of the link to the driver and follower.

129. **Principle of Connection.**—The line of connection of the driver and follower at any instant being known, their comparative velocities are determined by the following principle:—*The respective linear velocities of a point in the driver, and a point in the follower, each situated anywhere in the line of connection, are to each other inversely as the cosines of the respective angles made by the paths of the points with the line of connection.* This principle might be otherwise stated as follows:—*The components, along the line of connection, of the velocities of any two points situated in that line, are equal.*

130. **Adjustments of Speed.**—The velocity-ratio of a driver and its follower is sometimes made capable of being changed at will, by means of apparatus for varying the position of their line of connection, as when a pair of rotating cones are embraced by a belt which can be shifted so as to connect portions of their surfaces of different diameters.

131. **A Train of Mechanism** consists of a series of moving pieces, each of which is follower to that which drives it, and driver to that which follows it.

132. **Agregate Combinations** in mechanism are those by which compound motions are given to secondary pieces.

CHAPTER II.

ON ELEMENTARY COMBINATIONS AND TRAINS OF MECHANISM.

Section 1.—Rolling Contact.

133. **Pitch Surfaces** are those surfaces of a pair of moving pieces, which touch each other when motion is communicated by rolling contact. The LINE OF CONTACT is that line which at each instant traverses all the pairs of points of the pair of pitch surfaces which are in contact.

134. **Smooth Wheels, Rollers, Smooth Racks.**—Of a pair of primary moving pieces in rolling contact, both may rotate, or one may rotate and the other have a motion of sliding, or straight translation. A rotating piece, in rolling contact, is called a *smooth wheel*, and sometimes a *roller;* a sliding piece may be called a *smooth rack.*

135. **General Conditions of Rolling Contact.**—The whole of the principles which regulate the motions of a pair of pieces in rolling contact follow from the single principle, *that each pair of points in the pitch surfaces, which are in contact at a given instant, must at that instant be moving in the same direction with the same velocity;* that this must be the case is evident from the rigidity of the bodies, for did the pair of points vary in velocity, it would follow that there was motion among the particles, or in a particle at least, of the body, which is contrary to the hypothesis of rigidity.

The direction of motion of a point in a rotating body being perpendicular to a plane passing through its axis, the condition, that each pair of points in contact with each other must move in the same direction leads to the following consequences:—

I. That when both pieces rotate, their axes, and all their points of contact, lie in the same plane.

II. That when one piece rotates and the other slides, the axis of the rotating piece, and all the points of contact, lie in a plane perpendicular to the direction of motion of the sliding piece.

The condition, that the velocities of each pair of points of contact must be equal, leads to the following consequences:—

III. That the angular velocities of a pair of wheels, in rolling contact, must be inversely as the perpendicular distances of any pair of points of contact from the respective axes.

IV. That the linear velocity of a smooth rack in rolling contact with a wheel, is equal to the product of the angular velocity of the wheel by the perpendicular distance from its axis to a pair of points of contact.

Respecting the line of contact, the above principles III. and IV. lead to the following conclusions:—

V. That for a pair of wheels with parallel axes, and for a wheel and rack, the line of contact is straight, and parallel to the axes or axis; and hence that the pitch surfaces are either plane or cylindrical (the term "cylindrical" including all surfaces generated by the motion of a straight line parallel to itself).

VI. That for a pair of wheels, with intersecting axes, the line of contact is also straight, and traverses the point of intersection of the axes; and hence that the rolling surfaces are conical, with a common apex (the term "conical" including all surfaces generated by the motion of a straight line which traverses a fixed point).

136. **Circular Cylindrical Wheels** are employed when an uniform velocity-ratio is to be communicated between parallel axes. Figs. 38, 39, and 40, of Article 102, may be taken to represent pairs of such wheels; C and O, in each figure, being the parallel axes of the wheels, and T a point in their line of contact. In fig. 38, both pitch surfaces are convex, the wheels are said to be in *outside gearing*, and their directions of rotation are contrary. In figs. 39 and 40, the pitch surface of the larger wheel is concave, and that of the smaller convex; they are said to be in *inside gearing*, and their directions of rotation are the same.

To represent the comparative motions of such pairs of wheels symbolically, let

$$\overline{\mathrm{O\,T}} = r_1,\ \overline{\mathrm{C\,T}} = r_2.$$

be their radii: let $\overline{\mathrm{O\,C}} = c$ be the *line of centres*, or perpendicular distance between the axes, so that for

$$\left.\begin{matrix}\text{outside}\\ \text{inside}\end{matrix}\right\}\ \text{gearing, } c = r_1 \pm r_2 \text{..............(1.)}$$

Let a_1, a_2, be the angular velocities of the wheels, and v the common linear velocity of their pitch surfaces; then

$$\left.\begin{matrix} v = a_1 r_1 = a_2 r_2; \\ c : r_1 : r_2 : : a_2 \pm a_1 : a_2 : a_1; \end{matrix}\right\} \text{............(2.)}$$

the sign $\pm$ applying to $\left\{\begin{matrix}\text{outside}\\ \text{inside}\end{matrix}\right\}$ gearing.

137. **A Straight Rack and Circular Wheel**, which are used when an uniform velocity-ratio is to be communicated between a sliding piece and a turning piece, may be represented by fig. 36 of Article 99, C being the axis of the wheel, P T P the plane surface of the rack, and T a point in their line of contact. Let r be the radius

of the wheel, a its angular velocity, and v the linear velocity of the rack; then

$$v = r\,a.$$

138. **Bevel Wheels**, whose pitch surfaces are frustra of regular cones are used to transmit an uniform angular velocity-ratio between a pair of axes which intersect each other. Fig. 45 of Article 105 will serve to illustrate this case; O A and O C being the pair of axes, intersecting each other in O, O T the line of contact, and the cones described by the revolution of O T about O A and O C respectively being the pitch surfaces, of which narrow zones or frustra are used in practice.

Let a_1, a_2, be the angular velocities about the two axes respectively; and let $i_1 = \angle \mathrm{A\,O\,T}$, $i_2 = \angle \mathrm{C\,O\,T}$, be the angles made by those axes respectively with the line of contact; then from the principle III. of Article 135 it follows, that the angular velocity-ratio is

$$\frac{a_2}{a_1} = \frac{\sin i_1}{\sin i_2}; \quad \ldots\ldots\ldots\ldots(1.)$$

Which equation serves to find the angular velocity-ratio when the axes and the line of contact are given.

Conversely, let the angle between the axes,

$$\angle \mathrm{A\,O\,C} = i_1 + i_2 = j,$$

be given, and also the ratio $\frac{a_2}{a_1}$; then the position of the line of contact is given by either of the two following equations:—

$$\left.\begin{aligned} \sin i_1 &= \frac{a_2 \sin j}{\sqrt{(a_1^2 + a_2^2 + 2\,a_1\,a_2 \cos j)}}; \\ \sin i_2 &= \frac{a_1 \sin j}{\sqrt{(a_1^2 + a_2^2 + 2\,a_1\,a_2 \cos j)}}; \end{aligned}\right\} \ldots\ldots\ldots\ldots(2.)$$

which are formed from equation (1) by substituting for i_1 its value $= (j - i_2)$, and for i_2 its value $= (j - i_1)$.

As this is the first instance of the use of Trigonometrical analysis, the method of formation of these equations will be explained:—

From Equation (1) it follows that—

$$\begin{aligned} \sin i_1 \cdot a_1 &= \sin i_2 \cdot a_2 \\ &= \sin (j - i_1) \cdot a_2 \\ &= \sin j \cdot \cos i_1 \cdot a_2 - \cos j \cdot \sin i_1 \cdot a_2 \\ &= \sin j \cdot \sqrt{(1 - \sin^2 i_1)} \cdot a_2 - \cos j \cdot \sin i_1 \cdot a_2. \end{aligned}$$

(See *Trigonometrical Rules,* Sections 19 and 21.)

Squaring both sides, and transposing

$$\sin^2 j \cdot \sin^2 i_1 \cdot a_2^2 + (\sin i_1 \cdot a_1 + \cos j \cdot \sin i_1 \cdot a_2)^2 = \sin^2 j \cdot a_2^2$$

$$\sin^2 i_1 \quad a_2^2 - \cos^2 j \cdot \sin^2 i_1 \cdot a_2^2 + \sin^2 i_1 \cdot a_1^2 + \cos^2 j \cdot \sin^2 i_1 \cdot a_2^2 + 2 \sin i_1^2 \cdot a_1 \cdot \cos j \cdot a_2 = \sin^2 j \cdot a_2^2$$

$$\sin^2 i_1 \cdot a_2^2 + \sin^2 i_1 \cdot a_1^2 + 2 \sin i_1^2 \cdot a_1 \cdot \cos j \cdot a_2 = \sin^2 j \cdot a_2^2$$

$$\sin^2 i_1 = \frac{\sin^2 j \cdot a_2^2}{a_1^2 + a_2^2 + 2\, a_1 \cdot a_2 \cdot \cos j}$$

$$\therefore \sin \; i_1 = \frac{a_2 \cdot \sin j}{\sqrt{(a_1^2 + a_2^2 + 2\, a_1 \cdot a_2 \cdot \cos j)}}.$$

Graphically, the same problem is solved as follows :—On the two axes respectively, take lengths to represent the angular velocities of their respective wheels. Complete the parallelogram of which those lengths are the sides, and its diagonal will be the line of contact. As in the case of the rolling cones of Article 106, one of a pair of bevel wheels may be a flat disc, or a concave cone.

139. **Non-Circular Wheels** are used to transmit a variable velocity-ratio between a pair of parallel axes. In fig. 46, let C_1, C_2, represent the axes of such a pair of wheels; T_1, T_2, a pair of points which at a given instant touch each other in the line of contact (which line is parallel to the axes and in the same plane with them); and U_1, U_2, another pair of points, which touch each other at another instant of the motion; and let the four points, T_1, T_2, U_1, U_2, be in one plane perpendicular to the two axes, and to the line of contact. Then for every such set of four points, the two following equations must be fulfilled :—

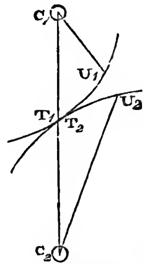


Fig. 46.

$$\overline{C_1 U_1} + \overline{C_2 U_2} = \overline{C_1 T_1} + \overline{C_2 T_2} = \overline{C_1 C_2}$$

$$\text{arc } T_1 U_1 = \text{arc } T_2 U_2$$

and those equations shew the geometrical relations which must exist between a pair of rotating surfaces in order that they may move in rolling contact round fixed axes.

Section 2.—Sliding Contact.

140. **Skew-Bevel Wheels** are employed to transmit an uniform

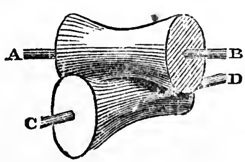


Fig. 47.

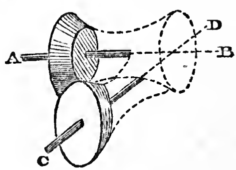


Fig. 48.

velocity-ratio between two axes which are neither parallel nor intersecting. The pitch surface of a skew-bevel wheel is a frustrum or zone of a *hyperboloid of revolution.* In fig. 47, a pair of large portions of such hyperboloids are shewn, rotating about axes A B, C D. In fig. 48 are shewn a pair of narrow zones of the same figures, such as are employed in practice.

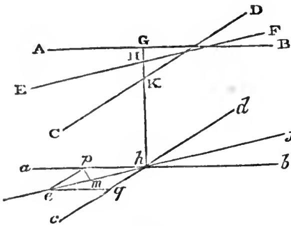


Fig. 49.

A hyperboloid of revolution is a surface resembling a sheaf or a dice box, generated by the rotation of a straight line round an axis from which it is at a constant distance, and to which it is inclined at a constant angle. If two such hyperboloids, equal or unequal, be placed in the closest possible contact, as in fig. 47, they will touch each other along one of the generating straight lines of each, which will form their line of contact, and will be inclined to the axes A B, C D, in opposite directions. The axes will neither be parallel, nor will they intersect each other.

The motion of two such hyperboloids, rotating in contact with each other, has sometimes been classed amongst cases of rolling contact; but that classification is not strictly correct; for although the component velocities of a pair of points of contact in a direction at right angles to the line of contact are equal, still, as the axes are neither parallel to each other nor to the line of contact, the velocities of a pair of points of contact have components along the line of contact, which are unequal, and their difference constitutes a *lateral sliding.*

The directions and positions of the axes being given, and the required angular velocity-ratio, $\frac{a_2}{a_1}$, it is required to find the *obliquities* of the generating line to the two axes, and its *radii vectores*, or least perpendicular distances from these axes.

In fig. 49, let A B, C D, be the two axes, and G K their common perpendicular.

On any plane normal to the common perpendicular G K h, draw $a\,b \parallel$ A B, $c\,d \parallel$ C D, in which take lengths in the following proportions:—

$$a_1 : a_2 : : \overline{h\,p} : \overline{h\,q};$$

complete the parallelogram $h\,p\,e\,q$, and draw its diagonal $e\,h\,f$; the line of contact E H F will be parallel to that diagonal.

From p let fall $p\,m$ perpendicular to $h\,e$. Then divide the common perpendicular G K in the ratio given by the proportional equation

$$\overline{h\,e} : \overline{e\,m} : \overline{m\,h} :: \overline{\mathrm{G\,K}} : \overline{\mathrm{G\,H}} : \overline{\mathrm{K\,H}};$$

then the two segments thus found will be the least distances of the line of contact from the axes.

The first pitch surface is generated by the rotation of the line E H F about the axis A B with the radius vector $\overline{\mathrm{G\,H}} = r_1$; the second, by the rotation of the same line about the axis C D with the radius vector $\overline{\mathrm{H\,K}} = r_2$.

To draw the hyperbola* which is the longitudinal section of a skew-bevel wheel whose generating line has a given radius vector and obliquity, let A G B, fig. 50, represent the axis, G H ⊥ A G B, the radius vector of the generating line, and let the straight line E G F make with the axis an angle equal to the obliquity of the generating line. H will be the vertex, and E G F one of the asymptotes,† of the required hyperbola. To find any number of points in that hyperbola, proceed as follows:—Draw X W Y parallel to G H, cutting G E in W, and make $\overline{\mathrm{X\,Y}} = \sqrt{(\overline{\mathrm{G\,H}}^2 + \overline{\mathrm{X\,W}}^2)}$. Then will Y be a point in the hyperbola.

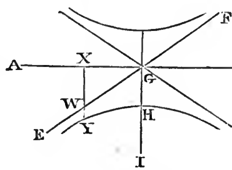


Fig. 50.

141. **Principle of Sliding Contact.**—The *line of action*, or *of connection*, in the case of sliding contact of two moving pieces, is the common perpendicular to their surfaces at the point where they touch; and the principle of their comparative motion is, that *the components, along that perpendicular, of the velocities of any two points traversed by it, are equal.*

Case 1. *Two shifting pieces,* in sliding contact, have linear velocities proportional to the secants of the angles which their directions of motion make with their line of action.

Case 2. *Two rotating pieces,* in sliding contact, have angular velocities inversely proportional to the perpendicular distances from their axes of rotation to their line of action, each multiplied by the sine of the angle which the line of action makes with the particular axis on which the perpendicular is let fall.

In fig. 51, let C_1, C_2, represent the axes of rotation of the two pieces; A_1, A_2, two portions of their respective surfaces; and T_1, T_2, a pair of points in those surfaces, which, at the instant under consideration, are in contact with each other. Let $P_1 P_2$ be the common perpendicular of the surfaces at the pair of points T_1, T_2;

* The *Hyperbola* is the curve traced out by a point which moves in such a manner that its distance from a given fixed point (I), continually bears the same ratio *greater than unity* to its distance from a given fixed line (A B).

† An *Asymptote* is a straight line whose distance from a curve diminishes as the curve extends away from the origin.

that is, the *line of action;* and let $\overline{C_1 P_1}$, $\overline{C_2 P_2}$, be the common perpendiculars of the line of action and of the two axes respectively. Then at the given instant, the components along the line $P_1 P_2$ of the velocities of the points P_1, P_2, are equal. Let i_1, i_2, be the angles which that line makes with the directions of the axes respectively. Let a_1, a_2, be the respective angular velocities of the moving pieces; then

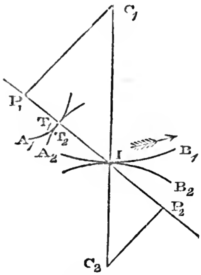


Fig. 51.

$$a_1 \cdot \overline{C_1 P_1} \cdot \sin i_1 = a_2 \cdot \overline{C_2 P_2} \cdot \sin i_2;$$

consequently,

$$\frac{a^2}{a_1} = \frac{\overline{C_1 P_1} \sin i_1}{\overline{C_2 P_2} \sin i_2}; \ldots\ldots\ldots\ldots(1.)$$

which is the principle stated above.

When the line of action is perpendicular in direction to both axes, then $\sin i_1 = \sin i_2 = 1$; and Equation 1 becomes

$$\frac{a_2}{a_1} = \frac{\overline{C_1 P_1}}{\overline{C_2 P_2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1\text{A}.)$$

When the axes are parallel, $i_1 = i_2$. Let I be the point where the line of action cuts the plane of the two axes; then the triangles $P_1 C_1 I$, $P_2 C_2 I$, are similar; so that Equation 1 A is equivalent to the following:—

$$\frac{a_2}{a_1} = \frac{\overline{I C_1}}{\overline{I C_2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1\text{B}.)$$

Case 3. *A rotating piece and a shifting piece,* in sliding contact, have their comparative motion regulated by the following principle:—Let $\overline{C P}$ denote the perpendicular distance from the axis of the rotating piece to the line of action; i the angle which the direction of the line of action makes with that axis; a the angular velocity of the rotating piece; v the linear velocity of the sliding piece; j the angle which its direction of motion makes with the line of action; then

$$v = a \cdot \overline{C P} \cdot \sin i \cdot \sec j \ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

When the line of action is perpendicular in direction to the axis of the rotating piece, $\sin i = 1$; and

$$v = a \cdot \overline{C P} \cdot \sec j = a \cdot \overline{I C}; \ldots\ldots\ldots\ldots\ldots\ldots(2\text{A}.)$$

where $\overline{I C}$ denotes the distance from the axis of the rotating piece to the point where the line of action cuts a perpendicular from that axis on the direction of motion of the shifting piece.

142. **Teeth of Wheels.**—The most usual method of communicating motion between a pair of wheels, or a wheel and a rack, and the only method which, by preventing the possibility of the rotation of one wheel unless accompanied by the other, insures the preservation of a given velocity-ratio exactly, is by means of the projections called *teeth.*

The *pitch surface* of a wheel is an ideal smooth surface, intermediate between the crests of the teeth and the bottoms of the spaces between them, which, by rolling contact with the pitch surface of another wheel, would communicate the same velocity-ratio that the teeth communicate by their sliding contact. In designing wheels, the forms of the ideal pitch surfaces are first determined, and from them are deduced the forms of the teeth.

Wheels with cylindrical pitch surfaces are called *spur wheels;* those with conical pitch surfaces, *bevel wheels;* and those with hyperboloidal pitch surfaces, *skew-bevel wheels.*

The *pitch line* of a wheel, or, in circular wheels, the *pitch circle,* is a transverse section of the pitch surface made by a surface perpendicular to it and to the axis; that is, in spur wheels, by a plane perpendicular to the axis; in bevel wheels, by a sphere described about the apex of the conical pitch surface; and in skew-bevel wheels, by any oblate spheroid generated by the rotation of an ellipse whose foci are the same with those of the hyperbola that generates the pitch surface.

The *pitch point* of a pair of wheels is the point of contact of their pitch lines; that is, the transverse section of the line of contact of the pitch surfaces.

Similar terms are applied to racks.

That part of the acting surface of a tooth which projects beyond the pitch surface is called the *face;* that which lies within the pitch surface, the *flank.*

The radius of the pitch circle of a circular wheel is called the *geometrical radius;* that of a circle touching the crests of the teeth is called the *real radius;* and the difference between those radii, the *addendum.*

143. **Pitch and Number of Teeth.**—The distance, measured along the pitch line, from the face of one tooth to the face of the next, is called the PITCH.

The pitch, and the number of teeth in circular wheels, are regulated by the following principles :—

I. In wheels which rotate continuously for one revolution or more, it is obviously necessary *that the pitch should be an aliquot part of the circumference.*

In wheels which reciprocate without performing a complete revolution, this condition is not necessary. Such wheels are called *sectors.*

II. In order that a pair of wheels, or a wheel and a rack, may work correctly together, it is in all cases essential *that the pitch should be the same in each.*

III. Hence, in any pair of circular wheels which work together, the numbers of teeth in a complete circumference are directly as the radii, and inversely as the angular velocities.

IV. Hence also, in any pair of circular wheels which rotate continuously for one revolution or more, the ratio of the numbers of teeth, and its reciprocal, the angular velocity-ratio, must be expressible in whole numbers.

V. Let n, N, be the respective numbers of teeth in a pair of wheels, N being the greater. Let t, T, be a pair of teeth in the smaller and larger wheel respectively, which at a particular instant work together. It is required to find, first, how many pairs of teeth must pass the line of contact of the pitch surfaces before t and T work together again (let this number be called a); secondly, with how many different teeth of the larger wheel the tooth t will work at different times (let this number be called b); and thirdly, with how many different teeth of the smaller wheel the tooth T will work at different times (let this be called c).

Case 1. If n is a divisor of N,

$$a = \mathrm{N}; \; b = \frac{\mathrm{N}}{n}; \; c = 1 \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Case 2. If the greatest common divisor of N and n be d a number less than n, so that $n = m\,d$, $\mathrm{N} = \mathrm{M}\,d$, then

$$a = m\,\mathrm{N} = \mathrm{M}\,n = \mathrm{M}\,m\,d; \; b = \mathrm{M}; \; c = m \ldots\ldots\ldots (2.)$$

Case 3. If N and n be prime to each other,

$$a = \mathrm{N}\,n; \; b = \mathrm{N}; \; c = n \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

It is considered desirable by millwrights, with a view to the preservation of the uniformity of shape of the teeth of a pair of wheels, that each given tooth in one wheel should work with as many different teeth in the other wheel as possible. They, therefore, study to make the numbers of teeth in each pair of wheels which work together such as to be either prime to each other, or to have their greatest common divisor as small as is possible consistently with the purposes of the machine.

VI. The *smallest* number of teeth which it is practicable to give to a pinion (that is, a small wheel), is regulated by the principle, that in order that the communication of motion from one wheel to another may be continuous, at least *one* pair of teeth should always be in action; and that in order to provide for the contingency of a tooth breaking, a *second* pair, at least, should be in action also. For reasons which will appear when the forms of teeth are considered, this principle gives the following as the least numbers of teeth which can be *usually* employed in pinions having teeth of the three classes of figures named below, whose properties will be explained in the sequel:—

I. Involute teeth,............................25.
II. Epicycloidal teeth,.........................12.
III. Cylindrical teeth, or *staves*,............... 6.

144. **Hunting Cog.**—When the ratio of the angular velocties of two wheels, being reduced to its least terms, is expressed by small numbers, less than those which can be given to wheels in practice, and it becomes necessary to employ multiples of those numbers by a common multiplier, which becomes a common divisor of the numbers of teeth in the wheels, millwrights and engine-makers avoid the evil of frequent contact between the same pairs of teeth, by giving one additional tooth, called a *hunting cog*, to the larger of the two wheels. This expedient causes the velocity-ratio to be not exactly but only approximately equal to that which was at first contemplated; and therefore it cannot be used where the exactness of certain velocity-ratios amongst the wheels is of importance as in clockwork.

145. **A Train of Wheelwork** consists of a series of axes, each having upon it two wheels, one of which is *driven* by a wheel on the preceding axis, while the other *drives* a wheel on the following axis. If the wheels are all in outside gearing, the direction of rotation of each axis is contrary to that of the adjoining axes. In some cases, a single wheel upon one axis answers the purpose both of receiving motion from a wheel on the preceding axis and giving motion to a wheel on the following axis. Such a wheel is called an *idle wheel:* it affects the direction of rotation only, and not the velocity-ratio.

Let the series of axes be distinguished by numbers 1, 2, 3, &c. . . . . m; let the numbers of teeth in the *driving wheels* be denoted by N's, each with the number of its axis affixed; thus, N_1, N_2, &c. . . . . N_{m-1}; and let the numbers of teeth in the *driven* or *following* wheels be denoted by n's, each with the number of its axis affixed; thus, n_2, n_3, &c. . . . . n_m. Then the ratio of the angular velocity a_m of the m^{th} axis to the angular velocity a_1 of the first axis is the product of the $m-1$ velocity-ratios of the successive elementary combinations, viz.:—

$$\frac{a_m}{a_1} = \frac{N_1 \cdot N_2 \cdot \&c. \ldots . N_{m-1}}{n_2 \cdot n_3 \cdot \&c. \ldots . n_m}; \qquad (1.)$$

that is to say, the velocity-ratio of the last and first axes is the ratio of the product of the numbers of teeth in the drivers to the product of the numbers of teeth in the followers; and it is obvious that so long as the same drivers and followers constitute the train, the *order* in which they succeed each other does not affect the resultant velocity-ratio.

Supposing all the wheels to be in outside gearing, then as each elementary combination reverses the direction of rotation, and as the number of elementary combinations, $m-1$, is one less than the number of axes, m, it is evident that if m is odd, the direction of rotation is preserved, and if even, reversed.

It is often a question of importance to determine the numbers of teeth in a train of wheels best suited for giving a determinate velocity-ratio to two axes. It was shewn by Young, that to do this with the *least total number of teeth*, the velocity-ratio of each elementary combination should approximate as nearly as possible 3·59. This would in many cases give too many axes; and as a useful practical rule it may be laid down, that from 3 to 6 ought to be the limit of the velocity-ratio of an elementary combination in wheelwork.

Let $\frac{B}{C}$ be the velocity-ratio required, reduced to its least terms, and let B be greater than C.

If $\frac{B}{C}$ is not greater than 6, and C lies between the prescribed minimum number of teeth (which may be called t), and its double $2t$, then one pair of wheels will answer the purpose, and B and C will themselves be the numbers required. Should B and C be inconveniently large, they are if possible to be resolved into factors, and those factors, or if they are too small multiples of them, used for the numbers of teeth. Should B or C, or both, be at once inconveniently large, and prime, then instead of the exact ratio $\frac{B}{C}$, some ratio approximating to that ratio, and capable of resolution into convenient factors, is to be found by the method of continued fractions. See MATHEMATICAL INTRODUCTION, page 2, Article 4.

Should $\frac{B}{C}$ be greater than 6, the best number of elementary combinations is found by dividing by 6 again and again till a quotient is obtained less than unity, when the number of divisions will be the required number of combinations, $m-1$.

Then, if possible, B and C themselves are to be resolved each into $m-1$ factors, which factors, or multiples of them, shall be not less than t, nor greater than $6t$; or if B and C contain inconveniently large prime factors, an approximate velocity-ratio, found by the method of continued fractions, is to be substituted for $\frac{B}{C}$, as before. When the prime factors of either B or C are fewer in number than $m-1$, the required number of factors is to be made up by inserting 1 as often as may be necessary. In multiplying factors that are too small to serve for numbers of teeth, prime numbers differing from those already amongst the factors are to be preferred as multipliers; and in general, where two or more factors require to be multiplied, different prime numbers should be used for the different factors.

So far as the resultant velocity-ratio is concerned, the *order* of the drivers N, and of the followers n, is immaterial; but to secure equable wear of the teeth, as explained in Article 143, Principle V., the wheels ought to be so arranged that for each elementary combination the greatest common divisor of N and n shall be either 1, or as small as possible; and if the preceding rules have been observed in the choice of multipliers, this will be insured by so placing each driving wheel that it shall work with a following wheel whose number of teeth does not contain any of the same multipliers; for the original numbers B and C contain no common factor except 1.

The following is an example of a case requiring the use of additional multipliers:—Let the required velocity-ratio, in its least terms, be

$$\frac{B}{C}=\frac{360}{7}.$$

To get a quotient less than 1, this ratio must be divided by 6 three times, therefore $m-1=3$. The prime factors of 360 are $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$; these may be combined so as to make three factors in various different ways; and the preference is to be given to that which makes these factors least unequal, viz., $5 \cdot 8 \cdot 9$. Hence, resolving numerator and denominator into three factors each, we have

$$\frac{B}{C}=\frac{5 \cdot 8 \cdot 9}{1 \cdot 1 \cdot 7}.$$

It is next necessary to multiply the factors of the numerator and denominator by a set of three multipliers. Suppose that the wheels to be used are of such a class that the smallest pinion has 12 teeth, then those multipliers must be such that none of their products by the existing factors shall be less than 12; and for reasons already given, it is advisable that they should be different prime numbers. Take the prime numbers, 2, 13, 17 (2 being taken to multiply 7); then the numbers of teeth in the followers will be

$$13 \times 1=13;\ 17 \times 1=17;\ 2 \times 7=14.$$

In distributing the multipliers amongst the factors of the numerator, let the smallest multiplier be combined with the largest factor, and so on; then we have

$$17 \times 5 = 85;\ 13 \times 8 = 104;\ 2 \times 9 = 18.$$

Finally, in combining the drivers with the followers, those numbers are to be combined which have no common factor; the result being the following train of wheels:—

$$\frac{85}{14} \cdot \frac{18}{13} \cdot \frac{104}{17} = \frac{360}{7}.$$

146. **Teeth of Spur-Wheels and Racks. General Principle.**—The figures of the teeth of wheels are regulated by the principle, *that the teeth of a pair of wheels shall give the same velocity-ratio by their sliding contact, which the ideal smooth pitch surfaces would give by their rolling contact.* Let B_1, B_2, in fig. 51, be parts of the pitch lines (that is, of cross sections of the pitch surfaces) of a pair of wheels with parallel axes, and I the pitch point (that is, a section of the line of contact). Then the angular velocities which would be given to the wheels by the rolling contact of those pitch lines are inversely as the segments $\overline{I\,C_1}$, $\overline{I\,C_2}$, of the line of centres; and this also is the proportion of the angular velocities given by a pair of surfaces in sliding contact whose line of action traverses the point I (Article 141, Case 2, Equation 1 B). Hence the condition of correct working for the teeth of wheels with parallel axes is, *that the line of action of the teeth shall at every instant traverse the line of contact of the pitch surfaces;* and the same condition obviously applies to a rack sliding in a direction perpendicular to that of the axis of the wheel with which it works.

147. **Teeth Described by Rolling Curves.**—From the principle of the preceding Article it follows, that at every instant, the position of the point of contact T_1 in the cross section of the acting surface of a tooth (such as the line $A_1\,T_1$ in fig. 51), and the corresponding position of the pitch point I in the pitch line $I\,B_1$ of the wheel to which that tooth belongs, are so related, that the line $I\,T_1$ which joins them is normal to the outline of the tooth $A_1\,T_1$ at the point T_1. Now, this is the relation which exists between the *tracing-point* T_1, *and the instantaneous axis or line of contact* I, in a rolling curve of such a figure, that being rolled upon the pitch surface B_1, its tracing-point T_1 traces the outline of the tooth. (As to rolling curves, see Articles 100, 101, 103, and 106).

In order that a pair of teeth may work correctly together, it is necessary and sufficient that the *instantaneous radii vectores* from the pitch point to the points of contact of the two teeth should coincide at each instant, as expressed by the equation

$$\overline{I\,T_1} = \overline{I\,T_2}; \qquad \text{(1.)}$$

and this condition is fulfilled *if the outlines of the two teeth be traced by the motion of the same tracing-point, in rolling the same rolling curve on the same side of the pitch surfaces of the respective wheels.*

The *flank* of a tooth is traced while the rolling curve rolls *inside* of the pitch line; the *face*, while it rolls *outside*. Hence it is evident that the *flanks* of the teeth of the driving wheel drive the *faces* of the teeth of the driven wheel; and that the *faces* of the teeth of the driving wheel drive the *flanks* of the teeth of the driven wheel. The former takes place while the point of contact of the teeth is *approaching* the pitch point, as in fig. 51, supposing the motion to be from P_1 towards P_2; the latter, after the point of contact has passed, and while it is *receding from*, the pitch point. The pitch point divides the path of the point of contact of the teeth into two parts, called the *path of approach* and the *path of recess;* and the lengths of those paths must be so adjusted, that two pairs of teeth at least shall be in action at each instant.

It is evidently necessary that the surfaces of contact of a pair of teeth should either be both convex, or that if one is convex and the other concave, the concave surface should have the flatter curvature.

148. **The Sliding of a Pair of Teeth on each Other,** that is, their relative motion in a direction perpendicular to their line of action, is found by supposing one of the wheels, such as 1, to be fixed, the line of centres $C_1 C_2$ to rotate backwards round C_1 with the angular velocity a_1, and the wheel 2 to rotate round C_2 as before with the angular velocity a_2 relatively to the line of centres $C_1 C_2$, so as to have the same motion as if its pitch surface *rolled* on the pitch surface of the first wheel. Thus the *relative* motion of the wheels is unchanged; but 1 is considered as fixed, and 2 has the resultant motion given by the principles of Article 102; that is, a rotation about the instantaneous axis I with the angular velocity $a_1 + a_2$. Hence the *velocity of sliding* is that due to this rotation about I, with the radius $\overline{\mathrm{I\,T}} = r$; that is to say, its value is

$$r\ (a_1 + a_2);\dots\dots\dots\dots\dots\dots(1.)$$

so that it is greater, the farther the point of contact is from the line of centres; and at the instant when that point, passing the line of centres, coincides with the *pitch point*, the velocity of sliding is null, and the action of the teeth is, for the instant, that of rolling contact.

The roots of the teeth slide towards each other during the approach, and from each other during the recess. To find the *amount* or *total distance* through which the sliding takes place, let t_1 be the time occupied by the approach, and t_2 that occupied by the recess; then the distance of sliding is

$$s = \int_0^{t_1} r(a_1 + a_2)\ d\,t + \int_0^{t_2} r(a_1 + a_2)\ d\,t;\dots\dots\dots(2.)$$

or in another form, if $d\,i$ denote an element of the change of angular position of one wheel relatively to the other, i_1 the amount of that change during the approach, and i_2 during the recess, then

$$(a_1 + a_2)\, d\, t = d\, i; \text{ and}$$

$$s = \int_0^{i_1} r\, d\, i + \int_0^{i_2} r\, d\, i \quad \text{.....................(3.)}$$

149. **The Arc of Contact on the Pitch Lines** is the length of that portion of the pitch lines which passes the pitch point during the action of one pair of teeth; and in order that two pairs of teeth at least may be in action at each instant, its length should be at least double of the pitch. It is divided into two parts, the arc of approach and the arc of recess. In order that the teeth may be of length sufficient to give the required duration of contact, the distance moved over by the point I upon the pitch line during the rolling of a rolling curve to describe the face and flank of a tooth, must be in all equal to the length of the required arc of contact. It is usual to make the arcs of approach and recess equal.

150. **The Length of a Tooth** may be divided into two parts, that of the face and that of the flank. For teeth in the driving wheel, the length of the flank depends on the arc of approach,—that of the face, on the arc of recess; for those in the following wheel, the length of the flank depends on the arc of recess,—that of the face, on the arc of approach.

151. **Involute Teeth for Circular Wheels.**—In fig. 52, let C_1, C_2, be the centres of two circular wheels, whose pitch circles are B_1, B_2. Through the pitch point I draw the intended *line of action* P_1, P_2, making the angle $CIP = \theta$ with the line of centres. From C_1, C_2, draw

$$\left.\begin{array}{l} \overline{C_1 P_1} = \overline{I C_1} \cdot \sin \theta, \\ \overline{C_2 P_2} = \overline{I C_2} \cdot \sin \theta, \end{array}\right\} \quad \text{.....................(1.)}$$

perpendicular to P_1 P_2, with which two perpendiculars as radii, describe circles (called *base circles*) D_1, D_2.

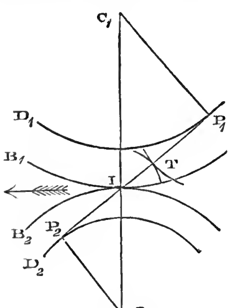


Fig. 52.

Suppose the base circles to be a pair of circular pulleys, connected by means of a cord whose course from pulley to pulley is P_1 I P_2. As the line of connection of those pulleys is the same with that of the proposed teeth, they will rotate with the required velocity-ratio. Now suppose a tracing-point T be fixed to the cord, so as to be carried along the path of contact P_1 I P_2. That point will trace, on a plane rotating along with the wheel 1, part of the involute of the base circle D_1, and on a plane rotating along with the wheel 2, part of the involute of the base circle D_2, and the two curves so traced will always touch each other in the required point of contact T, and will therefore fulfil the condition required by Article 146.

All involute teeth of the same pitch work smoothly together.

To find the length of the path of contact on either side of the pitch point I, it is to be observed that the distance between the fronts of two successive teeth, as measured along P_1 I P_2, is less than the pitch in the ratio sin θ : 1, for the former is proportional to $r \cdot sin\ \theta$, and the latter to $r \cdot \theta$, and consequently that if distances not less than the pitch $\times$ sin θ be marked off either way from I towards P_1 and P_2 respectively, as the extremities of the path of contact, and if the addendum circles be described through the points so found, there will always be at least two pairs of teeth in action at once. In practice, it is usual to make the path of contact somewhat longer, viz., about $2\frac{1}{4}$ times the pitch; and with this length of path and the value of θ which is usual in practice, viz., $75\frac{1}{2}°$, the addendum is about $\frac{3}{10}$ of the pitch.

The teeth of a *rack*, to work correctly with wheels having involute teeth, should have plane surfaces, perpendicular to the line of connection, and consequently making, with the direction of motion of the rack, angles equal to the before-mentioned angle θ.

152. **The Smallest Pinion with Involute Teeth** of a given pitch p, has its size fixed by the consideration that the path of contact of the flanks of the teeth, which must not be less than $p \cdot sin\ \theta$, cannot be greater than the distance along the line of action from the pitch point to the base circle, $\overline{\mathrm{I\,P}} = r \cdot \cos\theta$. Then $r = \dfrac{\mathrm{I\,P}}{\cos\theta}$ and substituting for I P its least possible value $p \cdot \sin\theta$, hence the *least radius* is

$$r = p \tan\theta; \qquad (1.)$$

which, for $\theta = 75\frac{1}{2}°$, gives for the radius $r = 3{\cdot}867\ p$, and for the circumference of the pitch circle, $p \times 3{\cdot}867 \times 2\ \pi = 24{\cdot}3\ p$; to which the next greater integer multiple of p is $25\ p$; and therefore *twenty-five*, as formerly stated, in Article 143, is the least number of *involute teeth* to be employed in a pinion.

153. **Epicycloidal Teeth.**—For tracing the figures of teeth, the most convenient rolling curve is the circle. The path of contact which a point in its circumference traces is identical with the circle itself; the flanks of the teeth are internal, and their faces external epicycloids, for wheels; and both flanks and faces are cycloids for a rack.

Wheels of the same pitch, with epicycloidal teeth traced by the same rolling circle, all work correctly with each other, whatsoever may be the numbers of their teeth; and they are said to belong to *the same set.*

For a pitch circle of twice the radius of the rolling or *describing* circle (as it is called), the internal epicycloid is a straight line, being in fact a diameter of the pitch circle; so that the flanks of the teeth for such a pitch circle are planes radiating from the axis. For a smaller pitch circle, the flanks would be convex, and *in-curved* or *under-cut*, which would be inconvenient; therefore the smallest wheel of a set should have its pitch circle of twice the radius of the describing circle, so that the flanks may be either straight or concave.

In fig. 53, let B be part of the pitch circle of a wheel, C C the line of centres, I the pitch-point, R the internal, and R′ the equal external describing circles, so placed as to touch the pitch circle and each other at I; let D I D′ be the path of contact, consisting of the path of approach D I, and the path of recess I D′. In order that there may always be at least two pairs of teeth in action, each of those arcs should be equal to the pitch.

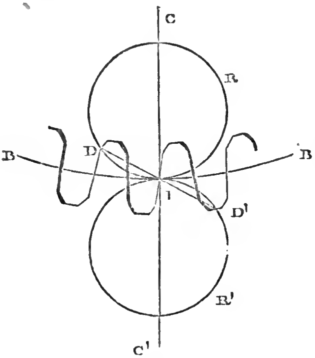


Fig. 53.

The angle θ, on passing the line of centres, is 90°; the least value of that angle is $\theta = \angle \text{C I D} = \angle \text{C' I D'}$.

It appears from experience that the least value of θ should be about 60°; therefore the arcs D I = I D′ should each be one-sixth of a circumference; therefore the circumference of the describing circle should be *six times the pitch.*

It follows that the smallest pinion of a set, in which pinion the flanks are straight, *should have twelve teeth,* as has been already stated in Article 143.

154. **Teeth of Wheel and Trundle.**—A *trundle*, as in fig. 54, has cylindrical pins called *staves* for teeth. The face of the teeth

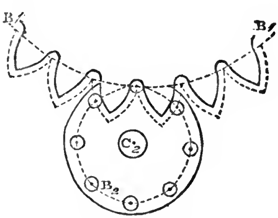


Fig. 54.

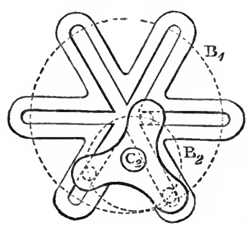


Fig. 55.

of a wheel suitable for driving it, in outside gearing, are described by first tracing external epicycloids by rolling the pitch circle B_2 of the trundle on the pitch circle B_1 of the driving wheel, with the centre of a stave for a tracing point, as shewn by the dotted lines, and then drawing curves parallel to and within the epicycloids, at a distance from them equal to the radius of a stave. Trundles having only six staves will work with large wheels.

To drive a trundle in *inside gearing*, the outlines of the teeth of the wheel should be curves parallel to internal epicycloids. A peculiar case of this is represented in fig. 55, where the radius of the pitch circle of the trundle is exactly one-half of that of the pitch circle of the wheel; the trundle has three equi-distant staves; and the internal epicycloids described by their centres while the pitch circle of the trundle is rolling within that of the wheel, are three straight lines, diameters of the wheel, making angles of 60° with each other. Hence the surfaces of the teeth of the wheel form three straight grooves intersecting each other at the centre, each being of a breadth equal to the diameter of a stave of the trundle.

155. **Dimensions of Teeth.**—Toothed wheels being in general intended to rotate either way, the *backs* of the teeth are made similar to the fronts. The *space* between two teeth, measured on the pitch circle, is made about one-fifth part wider than the thickness of the tooth on the pitch circle; that is to say,

$$\text{thickness of tooth} = \frac{5}{11} \text{ pitch},$$

$$\text{width of space} = \frac{6}{11} \text{ pitch}.$$

The difference of $\frac{1}{11}$ of the pitch is called the *back-lash.*

The clearance allowed between the points of teeth and the bottoms of the spaces between the teeth of the other wheel, is about one-tenth of the pitch.

The *thickness* of a tooth is fixed according to the principles of strength; and the *breadth* is so adjusted, that when multiplied by pitch, the product shall contain *one square inch* for each 160 lbs. of force transmitted by the teeth.

156. **The Teeth of a Bevel-Wheel** have acting surfaces of the conical kind, generated by the motion of a line traversing the apex of the conical pitch surface, while a point in it is carried round the traces of the teeth upon a spherical surface described about that apex.

The operations of describing the exact figures of the teeth of bevel-wheels, whether by involutes or by rolling curves, are in every respect analogous to those for describing the figures of the teeth of spur-wheels, except that in the case of bevel-wheels, all those operations are to be performed on the surface of a sphere described about the apex, instead of on a plane, substituting *poles* for *centres*, and *great circles* for *straight lines*.

In consideration of the practical difficulty, especially in the case of large wheels, of obtaining an accurate spherical surface, and of drawing upon it when obtained, the following *approximate* method, proposed originally by Tredgold, is generally used:—Let O, fig. 56, be the apex, and O C the axis of the pitch cone of a bevel-wheel; and let the largest pitch circle be that whose radius is $\overline{CB}$. Perpendicular to O B draw B A cutting the axis produced in A, let the outer rim of the pattern and of the wheel be made a portion of the surface of the cone whose apex is A and side A B. The narrow zone of that cone thus employed will approach sufficiently near to a zone of the sphere described about O with the radius O B, to be used in its stead. On a plane surface, with the radius A B, draw a circular arc B D; a sector of that circle will represent a portion of the surface of the cone A B C *developed*, or *spread out flat*. Describe the figures of teeth of the required pitch, suited to the pitch circle B D, as if it were that of a spur-wheel of the radius A B; those figures will be the required cross sections of the teeth of the bevel-wheel, made by the conical zone whose apex is A.

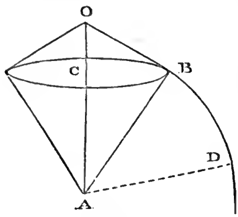


Fig. 56.

157. **The Teeth of Non-Circular Wheels** are described by rolling circles or other curves on the pitch surfaces, like the teeth of circular wheels; and when they are small compared with the wheels to which they belong, each tooth is nearly similar to the tooth of a circular wheel, having the same radius of curvature with the pitch surface of the actual wheel at the point where the tooth is situated.

158. A **Cam** or **Wiper** is a single tooth, either rotating continuously or oscillating, and driving a sliding or turning-piece, either constantly or at intervals. All the principles which have been stated in Article 141, as being applicable to sliding contact, are applicable to cams; but in designing cams, it is not usual to determine or take into consideration the form of the ideal pitch surface which would give the same comparative motion by rolling contact that the cam gives by sliding contact.

159. **Screws. Pitch.**—The figure of a screw is that of a convex or concave cylinder with one or more helical projections called *threads* winding round it. Convex and concave screws are distinguished technically by the respective names of *male* and *female*, or *external* and *internal*; a short internal screw is called a *nut*; and when a *screw* is not otherwise specified, *external* is understood.

The relation between the *advance* and the *rotation*, which compose the motion of a screw working in contact with a fixed nut or helical guide, has already been demonstrated in Article 96, Equation 1; and the same relation exists between the rotation of a screw about an axis fixed longitudinally relatively to the framework, and the advance of a nut in which that screw rotates, the nut being free to shift longitudinally, but not to turn. The advance of the nut in the latter case is in the direction opposite to that of the advance of the screw in the former case.

A screw is called *right-handed* or *left-handed*, according as its advance in a fixed nut is accompanied by right-handed or left-handed rotation, when viewed by an observer *from* whom the advance takes place. Fig. 57 represents a right-handed screw, and fig. 58 a left-handed screw.

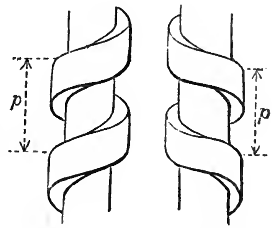


Fig. 57. Fig 58.

The *pitch* of a screw of one thread, and the *total pitch* of a screw of any number of threads, is the pitch of the helical motion of that screw, as explained in Article 96, and is the distance (marked p in figs. 57 and 58) measured parallel to the axis of the screw, between the corresponding points in two consecutive turns *of the same thread.*

In a screw of two or more threads, the distance measured parallel to the axis, between the corresponding points in *two adjacent threads*, may be called the *divided pitch.*

160. **Normal and Circular Pitch.**—When the pitch of a screw is not otherwise specified, it is always understood to be measured parallel to the axis. But it is sometimes convenient for particular purposes to measure it in other directions; and for that purpose a *cylindrical pitch surface* is to be conceived as described about the axis of the screw, intermediate between the crests of the threads and the bottoms of the grooves between them.

If a helix be now described upon the pitch cylinder, so as to cross each turn of each thread at right angles, the distance between two corresponding points on two successive turns of the same thread, measured along this *normal helix*, may be called the *normal pitch;* and when the screw has more than one thread, the normal pitch from thread to thread may be called the *normal divided pitch.*

The distance from thread to thread measured on a circle described on the pitch cylinder, and called the *pitch circle*, may be called the *circular pitch;* for a screw of one thread it is one circumference; for a screw of n threads

$$\frac{\text{one circumference}}{n}.$$

The following set of formulæ shew the relations amongst the different modes of measuring the pitch of a screw. The *pitch*, properly speaking, as originally defined, is distinguished as the *axial pitch*, and is the same for all parts of the same screw: the normal and circular pitch depend on the radius of the pitch cylinder.

Let r denote the radius of the pitch cylinder;

n, the number of threads;

i, the obliquity of the threads to the pitch circles, and of the normal helix to the axis;

$\left.\begin{matrix} P_a \\ \frac{P_a}{n} = p_a \end{matrix}\right\}$ the axial $\left\{\begin{matrix}\text{pitch;} \\ \text{divided pitch;}\end{matrix}\right.$

$\left.\begin{matrix} P_n \\ \frac{P_n}{n} = p_n \end{matrix}\right\}$ the normal $\left\{\begin{matrix}\text{pitch;} \\ \text{divided pitch;}\end{matrix}\right.$

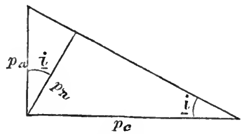


Fig. 59

p_c, the circular pitch;

Then

$$p_c = p_a \cdot \text{cotan}\, i = p_n \cdot \text{cosec}\, i = \frac{2\pi r}{n};$$

$$p_a = p_n \cdot \sec i = p_c \cdot \tan i = \frac{2\pi r \cdot \tan i}{n};$$

$$p_n = p_c \cdot \sin i = p_a \cdot \cos i = \frac{2\pi r \cdot \sin i}{n}.$$

Fig. 59 will make these formulæ clear, in which the several lines are lettered to represent the pitches: the hypotenuse of the larger triangle is the linear development on the plane of the paper of one coil of the screw which, it will be remarked, $= \sqrt{(p_a^2 + p_c^2)}$; p_n the normal pitch is normal to this: it is also evident from the figure that with a constant axial pitch, the normal and radial or circumferential pitch, as well as the angle of obliquity of the threads to the pitch cylinders, vary with the radii of those cylinders.

161. **Screw Gearing.**—A pair of convex screws, each rotating about its axis, are used as an elementary combination, to transmit motion by the sliding contact of their threads. Such screws are commonly called *endless screws*. At the point of contact of the screws, their threads must be parallel; and their line of connection is the common perpendicular to the acting surfaces of the threads at their point of contact. Hence the following principles:—

I. If the screws are both right-handed or both left-handed, the angle between the directions of their axes is the sum of their obliquities:—if one is right-handed and the other left-handed, that angle is the difference of their obliquities.

II. The normal pitch, for a screw of one thread, and the normal divided pitch, for a screw of more than one thread, must be the same in each screw.

III. The angular velocities of the screws are inversely as their number of threads.

162. **The Wheel and Screw** is an elementary combination of two screws, whose axes are at right angles to each other, both being right-handed or both left-handed. As the usual object of this combination is to produce a change of angular velocity in a ratio greater than can be obtained by any single pair of ordinary wheels, one of the screws is commonly wheel-like, being of large diameter and many-threaded, while the other is short and of few threads; and the angular velocities are inversely as the number of threads.

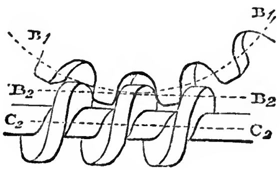


Fig. 60.

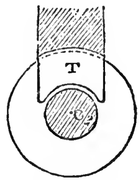


Fig. 61.

Fig. 60, represents a side view of this combination, and fig. 61 a cross section at right angles to the axis of the smaller screw. It has been shewn by Prof. Willis, that if each section of both screws be made by a plane perpendicular to the axis of the large screw or wheel, the outlines of the threads of the larger and smaller screw should be those of the teeth of a wheel and rack respectively: B_1 B_1, in fig. 60 for example, being the pitch circle of the wheel, and B_2 B_2 the pitch line of the rack.

The periphery and teeth of the wheel are usually hollowed to fit the screw, as shewn at T, fig. 61.

To make the teeth or threads of a pair of screws fit correctly and work smoothly, a hardened steel screw is made of the figure of the smaller screw, with its thread or threads notched so as to form a cutting tool; the larger screw, or wheel, is cast approximately of the required figure; the larger screw and the steel screw are fitted up in their proper relative position, and made to rotate in contact with each other by turning the steel screw, which cuts the threads of the larger screw to their true figure.

163. **The Relative Sliding of a Pair of Screws** at their point of contact is found thus:—Let r_1, r_2, be the radii of their pitch cylinders, and i_1, i_2, the obliquities of their threads to their pitch circles, one of which is to be considered as negative if the screws are contrary-handed. Let u be the common component of the velocities of a pair of points of contact along a line touching the pitch sur-

faces and perpendicular to the threads at the pitch point, and v the velocity of sliding of the threads over each other, where v may be considered to be made up of the algebraical sum of two quantities, v_1 and v_2, which act perpendicularly to u, and whose values are $v_1 = a_1 r_1 \cos i_1$, and $v_2 = a_2 r_2 \cos i_2$ the sum or difference being taken as the screws are similar or contrary-handed. Then

so that

$$\left.\begin{array}{l} u = a_1 r_1 \cdot \sin i_1 = a_2 r_2 \cdot \sin i_2; \\ a_1 = \dfrac{u}{r_1 \cdot \sin i_1}; \; a_2 = \dfrac{u}{r_2 \cdot \sin i_2}; \end{array}\right\} \dots\dots\dots\dots(1.)$$

and

$$v = a_1 r_1 \cdot \cos i_1 + a_2 r_2 \cdot \cos i_2 = u\,(\text{cotan}\, i_1 + \text{cotan}\, i_2)\dots\dots(2.)$$

When the screws are contrary-handed, the difference instead of the sum of the terms in Equation 2 is to be taken.

164. **Oldham's Coupling.**—A *coupling* is a mode of connecting a pair of shafts so that they shall rotate in the same direction, with the same mean angular velocity. If the axes of the shafts are in the same straight line, the coupling consists in so connecting their contiguous ends that they shall rotate as one piece; but if the axes are not in the same straight line, combinations of mechanism are required. A coupling for parallel shafts which acts by *sliding contact* was invented by Oldham, and is represented in fig. 62.

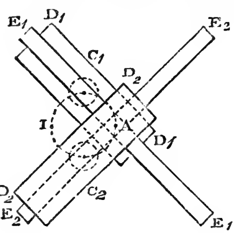


Fig. 62.

C_1, C_2, are the axes of the two parallel shafts; D_1, D_2, two cross-heads, facing each other, fixed on the ends of the two shafts respectively; E_1, E_1, a bar, sliding in a diametral groove in the face of D_1; E_2, E_2, a bar, sliding in a diametral groove in the face of D_2; those bars are fixed together at A, so as to form a rigid cross. The angular velocities of the two shafts and of the cross are all equal at every instant. The middle point of the cross, at A, revolves in the dotted circle described upon the line of centres C_1, C_2, as a diameter, twice for each turn of the shafts and cross; the instantaneous axis of rotation of the cross, at any instant, is at I, the point in the circle $C_1 C_2$, diametrically opposite to A.

Oldham's coupling may be used with advantage where the axes of the shafts are intended to be as nearly in the same straight line as is possible, but where there is some doubt as to the practicability or permanency of their exact continuity.

SECTION 3.—CONNECTION BY BANDS.

165. **Bands Classed.**—Bands, or wrapping connectors, for communicating motion between pulleys or drums rotating about fixed axes, or between rotating pulleys and drums and shifting pieces, may be thus classed:—

I. *Belts*, which are made of leather or of gutta percha, are flat and thin, and require nearly cylindrical pulleys. A belt tends to move towards that part of a pulley whose radius is greatest; pulleys for belts therefore, are slightly swelled in the middle, in order that the belt may remain on the pulley unless forcibly shifted. A belt when in motion is shifted off a pulley, or from one pulley on to another of equal size alongside of it, by pressing against that part of the belt which is moving *towards* the pulley.

II. *Cords*, made of catgut, hempen or other fibres, or wire, are nearly cylindrical in section, and require either drums with ledges, or grooved pulleys.

III. *Chains*, which are composed of links or bars jointed together, require pulleys or drums, grooved, notched, and toothed, so as to fit the links of the chains.

Bands for communicating continuous motion are *endless*.

Bands for communicating reciprocating motion have usually their ends made fast to the pulleys or drums which they connect, and which in this case may be sectors.

166. **Principle of Connection by Bands.**—The *line of connection* of a pair of pulleys or drums connected by means of a band, is the central line or axis of that part of the band whose tension transmits the motion. The principle of Article 129 being applied to this case, leads to the following consequences:—

I. *For a pair of rotating pieces*, let r_1, r_2, be the perpendiculars let fall from their axes on the centre line of the band, a_1, a_2, their angular velocities, and i_1, i_2, the angles which the centre line of the band makes with the two axes respectively. Then the longitudinal velocity of the band, that is, its component velocity in the direction of its own centre line, is

$$u = r_1 a_1 \sin i_1 = r_2 a_2 \sin i_2; \qquad (1.)$$

whence the angular velocity-ratio is

$$\frac{a_2}{a_1} = \frac{r_1 \sin i_1}{r_2 \sin i_2} \qquad (2.)$$

When the axes are parallel (which is almost always the case), $i_1 = i_2$, and

$$\frac{a_2}{a_1} = \frac{r_1}{r_2} \qquad (3.)$$

The same equation holds when both axes, whether parallel or not, are perpendicular in direction to that part of the band which transmits the motion; for then $\sin i_1 = \sin i_2 = 1$.

II. *For a rotating piece and a sliding piece*, let r be the perpendicular from the axis of the rotating piece on the centre line of the band, a the angular velocity, i the angle between the directions of the band and axis, u the longitudinal velocity of the band, j the angle between the direction of the centre line of the band and that of the motion of the sliding piece, and v the velocity of the sliding piece; then

$$u = r\,a \sin i = v \cos j; \qquad (4.)$$

for $r \sin i$ is the projection on the plane of motion of r, and u the longitudinal velocity of the band must necessarily be equal to $v \cos j$, the longitudinal velocity of the sliding piece owing to the rigidity of the band; and

$$v = \frac{r\,a \sin i}{\cos j} \qquad (5.)$$

When the centre line of the band is parallel to the direction of motion of the sliding piece, and perpendicular to the direction of the axis of the rotating piece, $\sin i\,(90°) = \cos j\,(0°) = 1$, and

$$v = u = r\,a. \qquad (6.)$$

167. **The Pitch Surface of a Pulley or Drum** is a surface to which the line of connection is always a tangent; that is to say, it is a surface parallel to the acting surface of the pulley or drum, and distant from it by half the thickness of the band.

168. **Circular Pulleys and Drums** are used to commuicate a

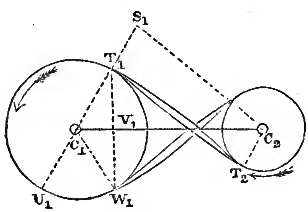


Fig. 63.

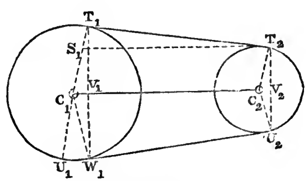


Fig. 64.

constant velocity-ratio. In each of them, the length denoted by r in the equations of Article 166 is constant, and is called the *effective radius*, being equal to the real radius of the pulley or drum added to half the thickness of the band.

A *crossed belt* connecting a pair of circular pulleys, as in fig. 63, reverses the direction of rotation; an *open belt*, as in fig. 64, preserves that direction.

169. **The length of an Endless Belt,** connecting a pair of pulleys whose effective radii are $\overline{C_1 T_1} = r_1$, $\overline{C_2 T_2} = r_2$, with parallel axes whose distance apart is $\overline{C_1 C_2} = c$, is given by formulæ founded on Equation 1 of Article 113, viz., $L = \Sigma \cdot s + \Sigma \cdot r i$. Each of the two equal straight parts of the belt is evidently of the length

$$\left.\begin{array}{l} s = T_1 T_2 = \sqrt{c^2 - (r_1 + r_2)^2} \text{ for a crossed belt;} \\ s = T_1 T_2 = \sqrt{c^2 - (r_1 - r_2)^2} \text{ for an open belt;} \end{array}\right\} \ldots\ldots\ldots(1.)$$

r_1 being the greater radius, and r_2 the less. Let i_1 be the arc to radius unity of the greater pulley, and i_2 that of the less pulley, with which the belt is in contact; then for a crossed belt

$$i_1 = i_2 = \left(\pi + 2 \text{ arc} \cdot \sin \frac{r_1 + r_2}{c}\right);$$

for the angle $V_1 C_1 W_1$ at the centre is double of the angle at the circumference $C_1 T_1 W_1$, and this is equal to the angle $S_1 C_2 C_1$ as they both differ from a right angle by the same angle $T_1 C_1 V_1$; and for an open belt, (2.)

$$i_1 = \left(\pi + 2 \text{ arc} \cdot \sin \frac{r_1 - r_2}{c}\right); \; i_2 = \left(\pi - 2 \text{ arc} \cdot \sin \frac{r_1 - r_2}{c}\right);$$

and the introduction of those values into Equation 1 of Article 113 gives the following results:—

For a crossed belt

$$L = 2\sqrt{c^2 - (r_1 + r_2)^2} + (r_1 + r_2) \cdot \left(\pi + 2 \text{ arc} \cdot \sin \cdot \frac{r_1 + r_2}{c}\right);$$

and if similar reasoning be applied, it may be shewn that for an open belt, (3.)

$$L = 2\sqrt{c^2 - (r_1 - r_2)^2} + \pi(r_1 + r_2) + 2(r_1 - r_2) \cdot \text{arc} \cdot \sin \cdot \frac{r_1 - r_2}{c}.$$

As the last of these equations would be troublesome to employ in a practical application to be mentioned in the next Article, an approximation to it, sufficiently close for practical purposes, is obtained by considering, that if $r_1 - r_2$ is small compared with c, $\sqrt{c^2 - (r_1 - r_2)^2} = c - \frac{(r_1 - r_2)^2}{2c}$ nearly, and arc $\cdot \sin \cdot \frac{r_1 - r_2}{c} = \frac{r_1 - r_2}{c}$ nearly; whence, for an open belt,

$$L \text{ nearly } = 2c + \pi(r_1 + r_2) + \frac{(r_1 - r_2)^2}{c} \ldots\ldots\ldots(3\text{A}.)$$

170. **Speed-Cones** (figs. 65, 66, 67, 68) are a contrivance for

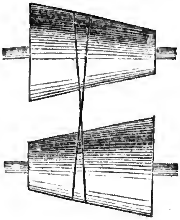


Fig. 65.

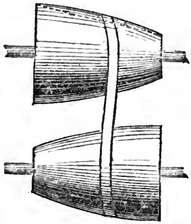


Fig. 66.

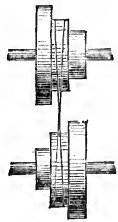


Fig. 67.

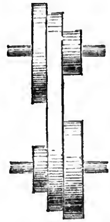


Fig. 68.

varying and adjusting the velocity-ratio communicated between a pair of parallel shafts by means of a belt, and may be either continuous cones or conoids, as in figs. 65, 66, whose velocity-ratio can be varied gradually while they are in motion by shifting the belt; or sets of pulleys whose radii vary by steps, as in figs. 67, 68, in which case the velocity-ratio can be changed by shifting the belt from one pair of pulleys to another.

In order that the belt may be equally tight in every possible position on a pair of speed-cones, the quantity L in the equations of Article 169 must be constant.

For a *crossed* belt, as in figs. 65 and 66, L depends solely on c and on $r_1 + r_2$. Now c is constant, because the axes are parallel, therefore the *sum of the radii* of the pitch circles connected in every position of the belt is to be constant. That condition is fulfilled by a pair of continuous cones generated by the revolution of two straight lines inclined opposite ways to their respective axes at equal angles, and by a set of pairs of pulleys in which the sum of the radii is the same for each pair.

For an *open* belt, the following practical rule is deduced from the approximate Equation 3A of Article 169:—

Let the speed-cones be equal and similar conoids, as in fig. 66, but with their large and small ends turned opposite ways. Let r_1 be the radius of the large end of each, r_2 that of the small end, r_0 that of the middle; and let y be the *sagitta*, measured perpendicular to the axis, of the arc by whose revolution each of the conoids is generated, or, in other words, the *bulging* of the conoids in the middle of their length; then

$$y = r_0 - \frac{r_1 + r_2}{2} = \frac{(r_1 - r_2)^2}{2\,\pi\,c} \dots\dots\dots\dots\dots\dots (1.)$$

where the second value is obtained from the first by considering that in Equation 3A, $2\,\pi\,r_0 = \pi\,(r_1 + r_2) + \frac{(r_1 - r_2)^2}{c}$; $2\,\pi = 6{\cdot}2832$; but 6 may be used in most practical cases without sensible error.

The radii at the middle and ends being thus determined, make the generating curve an arc either of a circle or of a parabola.

For a pair of stepped cones, as in fig. 68, let a series of *differences* of the radii, or values of $r_1 - r_2$, be assumed; then for each pair of pulleys, the sum of the radii is to be computed from the difference by the formula

$$r_1 + r_2 = 2\,r_0 - \frac{(r_1 - r_2)^2}{\pi c}; \qquad (2.)$$

$2\,r_0$ being that sum when the radii are equal.

Section 4.—Linkwork.

171. *Definitions.*—The pieces which are connected by linkwork, if they rotate or oscillate, are usually called *cranks, beams,* and *levers.* The *link* by which they are connected is a rigid bar, which may be straight or of any other figure; the straight figure being the most favourable to strength, is used when there is no special reason to the contrary. The link is known by various names under various circumstances, such as *coupling rod, connecting rod, crank rod, eccentric rod,* &c. It is attached to the pieces which it connects by two pins, about which it is free to turn. The effect of the link is to maintain the distance between the centres of those pins invariable; hence the line joining the centres of the pins is the *line of connection;* and those centres may be called the *connected points.* In a turning piece, the perpendicular let fall from its connected point upon its axis of rotation is the *arm* or *crank arm.*

172. **Principles of Connection.**—The whole of the equations already given in Article 166 for bands, are applicable to linkwork. The axes of rotation of a pair of turning pieces connected by a link are almost always parallel, and perpendicular to the line of connection; in which case the angular velocity-ratio at any instant is the reciprocal of the ratio of the common perpendiculars let fall from the line of connection upon the respective axes of rotation (Article 166, Equation 3).

173. **Dead Points.**—If at any instant the direction of one of the crank arms coincides with the line of connection, the common perpendicular of the line of connection and the axis of that crank arm vanishes, and the directional relation of the motions becomes indeterminate. The position of the connected point of the crank arm in question at such an instant is called a *dead point.* The velocity of the other connected point at such an instant is null, unless it also reaches a dead point at the same instant, so that the line of connection is in the plane of the two axes of rotation, in which case the velocity-ratio is indeterminate.

174. **Coupling of Parallel Axes.**—The only case in which an uni-

form angular velocity-ratio (being that of equality) is communicated by linkwork, is that in which two or more parallel shafts (such as those of the driving wheels of a locomotive engine) are made to rotate with constantly equal angular velocities, by having equal cranks, which are maintained parallel by a coupling rod of such a length that the line of connection is equal to the distance between the axes. The cranks pass their dead points simultaneously. To obviate the unsteadiness of motion which this tends to cause, the shafts are provided with a second set of cranks at right angles to the first, connected by means of a similar coupling rod, so that one set of cranks pass their dead points at the instant when the other set are farthest from theirs.

175. **The Comparative Motion of the Connected Points** in a piece of linkwork at a given instant is capable of determination by the method explained in Article 98; that is, by finding the instantaneous axis of the link; for the two connected points move in the same manner with two points in the link, considered as a rigid body.

If a connected point belongs to a turning piece, the direction of its motion at a given instant is perpendicular to the plane containing the axis and crank arm of the piece. If a connected point belongs to a shifting piece, the direction of its motion at any instant is given, and a plane can be drawn perpendicular to that direction.

The line of intersection of the planes perpendicular to the paths of the two connected points at a given instant, is the *instantaneous axis of the link* at that instant; and the *velocities of the connected points are directly as their distances from that axis.*

In drawing on a plane surface, the two planes perpendicular to the paths of the connected points are represented by two lines (being their sections by a plane normal to them), and the instantaneous axis by a point; and should the length of the two lines render it impracticable to produce them until they actually intersect, the velocity-ratio of the connected points may be found by the principle, that it is equal to the ratio of the segments which a line parallel to the line of connection cuts off from any two lines drawn from a given point, perpendicular respectively to the paths of the connected points.

Example I. *Two Rotating Pieces with Parallel Axes* (fig. 69)—Let C_1, C_2, be the parallel axes of the pieces; T_1, T_2, their connected points; $\overline{C_1 T_1}$, $\overline{C_2 T_2}$, their crank arms; $\overline{T_1 T_2}$, the link. At a given instant, let v_1 be the velocity of T_1; v_2 that of T_2.

To find the ratio of those velocities, produce $C_1 T_1$, $C_2 T_2$, till they intersect in K; K is the instantaneous axis of the link or connecting rod, and the velocity ratio is

$$v_1 : v :: \overline{K T_1} . \overline{K T_2} \qquad (1.)$$

Should K be inconveniently far off, draw any triangle with its sides respectively parallel to $C_1 T_1$, $C_2 T_2$, and $T_1 T_2$; the ratio of the two sides first mentioned will be the velocity-ratio required. For example, draw $C_2 A$ parallel to $C_1 T_1$, cutting $T_1 T_2$ in A, then

$$v_1 : v_2 : : \overline{C_2 A} : \overline{C_2 T_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

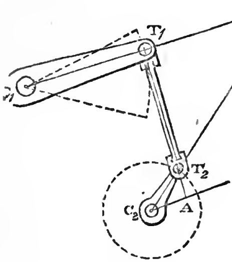


Fig. 69.

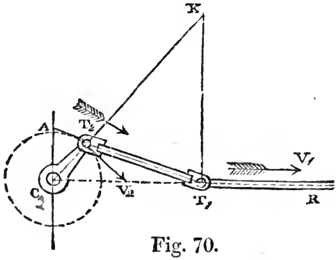


Fig. 70.

Example II. *Rotating piece and sliding piece* (fig. 70).—Let C_2 be the axis of a rotating piece, and T_1 R the straight line along which a sliding piece moves. Let T_1, T_2, be the connected points, $\overline{C_2 T_2}$ the crank arm of the rotating piece, and $\overline{T_1 T_2}$ the link or connecting rod. The point T_1, T_2, and the line T_1 R, are supposed to be in one plane, perpendicular to the axis C. Draw T_1 K perpendicular to T_1 R, intersecting $C_2 T_2$ in K; K is the instantaneous axis of the link; and

$$v_1 : v_2 : : K\,T_1 : K\,T_2$$

Or otherwise draw from a point C_2, $C_2 A$ perpendicular to T_1 R the direction of motion of the sliding piece, $C_2 T_2$ perpendicular to the direction of motion of the rotating piece, then the line $T_1 T_2$, or a line parallel thereto cuts off the segments $C_2 A$, $C_2 T_2$, or segments proportional thereto, and the velocity-ratio of the rotating piece to the sliding piece is as $C_2 T_2$ to $C_2 A$.

176. An **Eccentric** (fig. 71) being a circular disc keyed on a shaft, with whose axis its centre does not coincide, and used to give a reciprocating motion to a rod, is equivalent to a crank whose connected point is T, the centre of the eccentric disc, and whose crank arm is $\overline{CT}$, the distance of that point from the axis of the shaft, called the *eccentricity*.

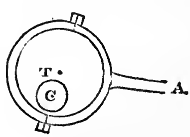


Fig. 71.

177. **The Length of Stroke** of a point in a reciprocating piece is the distance between the two ends of the path in which that point moves. When it is connected by a link with a point in a continuously rotating piece, the ends of the stroke of the reciprocating point correspond with the *dead points* of the continuously revolving piece (Article 173).

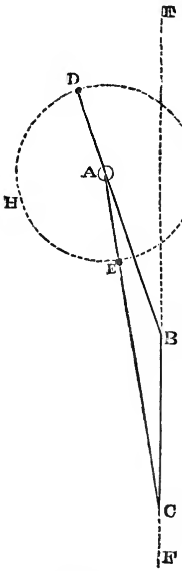


Fig. 72.

Let S = B C be the length of stroke of the reciprocating piece, L = E C = D B the length of the line of connection, and R = A E = A D the crank arm of the continuously turning piece. Then if the two ends of the stroke be in one straight line with the axis of the crank,

$$S = 2\,R; \quad\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

and if their ends be not in one straight line with that axis, then S, L − R, and L + R, are the three sides of a triangle, having the angle opposite S at that axis; so that if θ be the supplement of the arc between the dead points, D and E,

$$\left.\begin{array}{l} S^2 = (L-R)^2 + (L+R)^2 - 2\,(L-R) \\ \quad (L+R)\cos\theta; \\ S^2 = 2\,(L^2+R^2) - 2\,(L^2-R^2)\cos\theta; \\ \cos\theta = \dfrac{2\,L^2 + 2\,R^2 - S^2}{2(L^2-R^2)} \end{array}\right\}\ldots(2.)$$

178. **Hooke's Universal Joint** (fig. 73) is a contrivance for coupling shafts whose axes intersect each other in a point.

Let O be the point of intersection of the axes O C_1, O C_2, and i their angle of inclination to each other. The pair of shafts C_1, C_2, terminate in a pair of forks, F_1, F_2, in bearings at the extremities of which turn the gudgeons at the ends of the arms of a rectangular cross, having its centre at O. This cross is the link; the connected points are the centres of the bearings F_1, F_2. At each instant each of those points moves at right angles to the central plane of its shaft and fork, therefore the line of intersection of the central planes of the two forks, at any instant, is the instantaneous axis of the cross, and the *velocity-ratio* of the

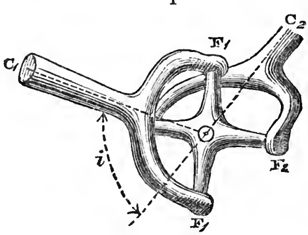


Fig. 73.

points F_1, F_2 (which, as the forks are equal, is also the *angular velocity-ratio* of the shafts), is equal to the ratio of the distances of those points from that instantaneous axis. The *mean* value of that velocity-ratio is that of equality; for each successive *quarter turn* is made by both shafts in the same time; but its actual value fluctuates between the limits,

$$\left.\begin{array}{l} \dfrac{a_2}{a_1} = \dfrac{r}{r \cdot \sin (90^\circ - i)} = \dfrac{1}{\cos i}, \text{ when } F_1 \text{ is in the plane of the axis;} \\ \dfrac{a_2}{a_1} = \cos i \text{ when } F_2 \text{ is in that plane.} \end{array}\right\} \ldots\ldots\ldots (1.)$$

179. The **Double Hooke's Joint** (fig. 74) is used to obviate the vibratory and unsteady motion caused by the fluctuation of the velocity-ratio indicated in the equation of Article 178. Between the two shafts to be connected, C_1, C_2, there is introduced a short intermediate shaft C_2, making equal angles with C_1 and C_3, connected with each of them by a Hooke's joint, and having both its own forks in the same plane.

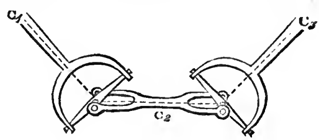


Fig. 74.

By this arrangement the angular velocities of the first and third shafts are equal to each other at every instant.

180. **A Click**, being a reciprocating bar, acting upon a rachet wheel or rack, which it pushes or pulls through a certain arc at each forward stroke, and leaves at rest at each backward stroke, is an example of intermittent linkwork. During the forward stroke, the action of the click is governed by the principles of linkwork; during the backward stroke, that action ceases. A *catch* or *pall*, turning on a fixed axis, prevents the ratchet wheel or rack from reversing its motion.

Section 5.—Reduplication of Cords.

181. *Definitions*—The combination of pieces connected by the several plies of a cord or rope consists of a pair of cases or frames called *blocks*, each containing one or more pulleys called *sheaves*. One of the blocks called the *fall-block*, B_1, is fixed; the other, or *running-block*, B_2, is movable to or from the fall-block, with which

it is connected by means of a rope of which one end is attached either to the fall-block or to the running-block, while the other end, T_1, called the *fall*, or *tackle-fall*, is free; while the intermediate portion of the rope passes alternately round the pulleys in the fall-block and running-block. The whole combination is called a *tackle* or *purchase*.

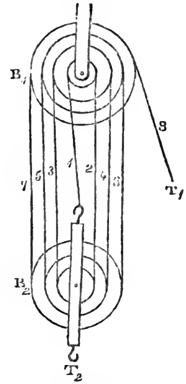


Fig. 75.

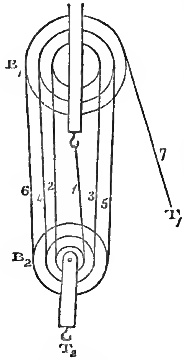


Fig. 75A.

182. The **Velocity-Ratio** chiefly considered in a tackle is that between the velocities of the running-block, u, and of the tackle-fall, v. That ratio is given by Equation 3 of Article 113 (which see), viz.:—

$$v = n\,u; \qquad (1.)$$

where n is the *number of plies* of rope by which the running-block is connected with the fall-block. Thus, in fig. 75 $n = 7$; and in fig. 75A $n = 6$.

182A. The **Velocity of any Ply** of the rope is found in the following manner:—

I. For a ply on the side of the fall-block next the tackle-fall, such as 2, 4, 6, fig. 75, and 3, 5, fig. 75A, it is to be considered what would be the velocity of that ply if it were itself the tackle-fall. Let that velocity be denoted by v', and let n' be the number of plies *between* the ply in question and the point of attachment by which the first ply (marked 1 in the figures) is fixed to one or other block. Then

$$v' = n'\,u \qquad (1.)$$

II. For a ply on the side of the fall-block farthest from the tackle-fall, the velocity is equal and contrary to that of the next succeeding ply, with which it is directly connected over one of the sheaves of the fall-block.

III. If the first ply, as in fig. 75A, is attached to the fall-block, its velocity is nothing; if to the running-block, its velocity is equal to that of the block.

183. **White's Tackle.**—The sheaves in a block are usually made all of the same diameter, and turn on a fixed pin; and they have, consequently, different angular velocities. But by making the diameter of each sheaf proportional to the velocity, *relatively to the*

block, of the ply of rope which it is to carry, the angular velocities of the sheaves in one block may be rendered equal, so that the sheaves may be made all in one piece, and may have journals turning in fixed bearings. This is called *White's Tackle*, from the inventor, and is represented in figs. 75 and 75A.

SECTION 6.—COMPARATIVE MOTION IN THE "MECHANICAL POWERS."

184. **Classification of the Mechanical Powers.**—"Mechanical Powers" is a name given to certain simple or elementary machines, all of which, with the single exception of the pulley, are more simple than even an elementary combination of a driver and follower; for, with that exception, a mechanical power consists essentially of only one primary moving piece; and the comparative motion taken into consideration is simply the velocity-ratio either of a pair of points in that piece, or of two components of the velocity of one point. There are two established classifications of the mechanical powers; an older classification, which enumerates six; and a newer classification, which ranges the six mechanical powers of the older system under three heads. The following table shews both these classifications:—

NEWER CLASSIFICATION	OLDER CLASSIFICATION.
THE LEVER,........................	The Lever, The Wheel and Axle.
THE INCLINED PLANE,...........	The Inclined Plane. The Wedge. The Screw.
THE PULLEY,........................	The Pulley.

In the present section the comparative motions in the mechanical powers are considered alone. The relations amongst the forces which act in those machines will be treated of in the kinetic division of this Treatise.

In the lever and the wheel and axle of the older classification, which are both comprehended under the lever of the newer classification, the primary moving piece turns about a fixed axis; and the comparative motion taken into consideration is the velocity-ratio of two points in that piece, which may be called respectively the *driving point* and the *following point*. The principle upon which that velocity-ratio depends has already been stated in Article 93, page 50—viz., that the velocity of each point is proportional to the radius of the circular path which it describes; that is, to its perpendicular distance from the axis of motion.

The distinction between the lever and the wheel and axle is this: that in the *lever*, the driving point, D, and the following point, F, are a pair of determinate points in the moving piece, as in figs. 76A to 76D; whereas in the *wheel* and *axle* they may be any pair of points which are situated respectively in a pair of cylindrical pitch-surfaces, D and F, described about the axis A, fig. 76.

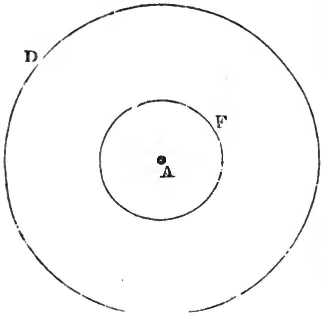


Fig. 76.

In each of these figures the plane of projection is normal to the axis, and A is the trace of the axis. In fig. 76, D and F are the traces of two cylindrical pitch-surfaces. In figs. 76A to 76D, D and F are the projections of the driving and following points respectively.

The axis of a lever is often called the *fulcrum*.

A lever is said to be *straight*, when the driving point, D, and following point, F, are in one plane traversing the axis A, as in figs. 76A, 76B, and 76C. In other cases the lever is said to be *bent*, as in fig. 76D.

Fig. 76A.

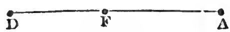


Fig 76B.

Fig. 76C.

The straight lever is said to be of one or other of three kinds, according to the following classification :—

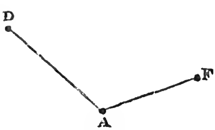


Fig. 76D.

In a *lever of the first kind*, fig. 76A, the driving and following points are at opposite sides of the fulcrum A.

In a *lever of the second kind*, fig. 76B, the driving and following points are at the same side of the fulcrum, and the driving point is the further from the fulcrum.

In a *lever of the third kind*, fig. 76C, the driving and following points are at the same side of the fulcrum, and the following point is the further from the fulcrum.

In the inclined plane, and in the wedge, the comparative motion considered is the velocity-ratio of the entire motion of a straight-sliding primary piece and one of the components of that motion; the principles of which velocity-ratio have been stated in Article 70, pages 38, 39.

In the inclined plane, fig. 76E, A A is the trace of a fixed plane; B, a block sliding on that plane in the direction B C; the plane of projection being perpendicular to the plane A A, and parallel to the direction of motion of B. B D is some direction oblique to B C. From any convenient point, C, in B C, let fall C D perpendicular to B D; then B D ÷ B C is the ratio of the component velocity in the direction B D to the entire velocity of B.

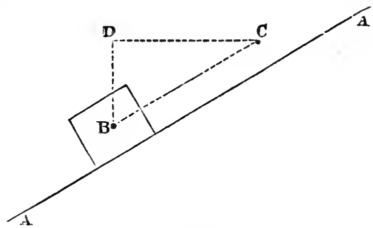
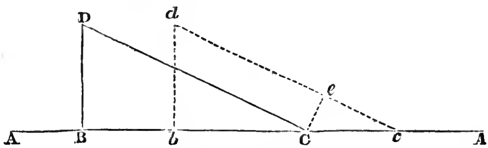


Fig. 76E.

In fig. 76F, A A is the trace of a fixed plane; B C D, the trace of a wedge which slides on that plane. While the wedge advances through the distance C *c*, its oblique face advances from the position C D to the position *c d*; and if C *e* be drawn normal to the plane C D, the ratio borne by the component velocity of the wedge



Fig, 76F.

in a direction normal to its oblique face to its entire velocity will be expressed by C *e* : C *c*.

In the screw the comparative motion considered is the ratio borne by the entire velocity of some point in, or rigidly connected with, the screw, to the velocity of advance of the screw.

The helical path of motion of a point in, or rigidly attached to, a screw may be developed (as has been already explained in Article 160, page 94) into a straight line: being the hypotenuse of a right angled triangle whose height is equal to the pitch of the screw, and its base to the circumference of a circle whose radius is the distance of the given point from the axis of the screw. Then if B D in fig. 76E be taken to represent the pitch of the screw, and D C, perpendicular to B D, the circumference of the circle described by the point in question about the axis, B C will be the development of one turn of the screw-line described by that point as it revolves and advances along with the screw; and B C ÷ B D will be the ratio of its entire velocity to the velocity of advance; just as in the case of a body sliding on an inclined plane, A A, parallel to B C. This shews why the screw is comprehended under the general head of the inclined plane, in the newer classification of the mechanical powers.

The term *pulley*, in treating of the mechanical powers, means any purchase or tackle of the class already described in Section 5 of this Chapter, pages 105 to 107.

Section 7.—Hydraulic Connection.

185. **The General Principle** of the communication of motion between two pistons by means of an intervening fluid of constant density has already been stated in Article 119, viz., that the velocities of the pistons are inversely as their areas, measured on planes normal to their directions of motion.

Should the density of the fluid vary, the problem is no longer one of pure mechanism; because in that case, besides the communication of motion from one piston to the other, there is an additional motion of one or other, or both pistons, due to the change of volume of the fluid.

186. **Valves** are used to regulate the communication of motion through a fluid, by opening and shutting passages through which the fluid flows; for example, a cylinder may be provided with valves which shall cause the fluid to flow in through one passage, and out through another. Of this use of valves, two cases may be distinguished.

I. *When the piston moves the fluid*, the valves may be what is called *self-acting;* that is, moved by the fluid. If there be two passages into the cylinder, one provided with a valve opening inwards, and the other with a valve opening outwards; then during the outward stroke of the piston the former valve is opened and the latter shut by the inward pressure of the fluid, which flows in through the former passage; and during the inward stroke of the piston, the former valve is shut and the latter opened by the outward pressure of the fluid, which flows out through the latter passage. This combination of cylinder, piston, and valves, constitutes a *pump*.

II. *When the fluid moves the piston*, the valves must be opened and shut by mechanism, or by hand. In this case the cylinder is a *working cylinder*.

187. **In the Hydraulic Press**, the rapid motion of a small piston in a pump causes the slow motion of a large piston in a working cylinder. The pump draws water from a reservoir, and forces it into the working cylinder: during the outward stroke of the pump piston, the piston of the working cylinder stands still; during the inward stroke of the pump piston, the piston of the working cylinder moves outward with a velocity as much less than that of the pump piston as its area is greater. When the piston of the

working cylinder has finished its outward stroke, which may be of any length, it is permitted to be moved inwards again by opening a valve by hand and allowing the water to escape.

188. **In the Hydraulic Hoist,** the slow inward motion of a large piston drives water from a large cylinder into a smaller cylinder, and causes a more rapid outward motion of the piston of the smaller cylinder. When the latter piston is to be moved inward, a valve between the two cylinders is closed, and the valve of an outlet from the smaller cylinder opened, by hand, so as to allow the water to escape from the smaller cylinder. The larger cylinder is filled and its piston moved outward, when required, by means of a pump, in a manner resembling the action of a hydraulic press.

Section 8.—Trains of Mechanism.

189. **Trains of Elementary Combinations** have been defined in Article 131, and illustrated in the case of wheelwork, in Article 145, and in the case of a double Hooke's joint, in Article 179. The general principle of their action is that the comparative motion of the first driver and last follower is expressed by a ratio, which is found by multiplying together the several velocity-ratios of the series of elementary combinations of which the train consists, each with the sign denoting the directional relation.

Two or more trains of mechanism may *converge* into one; as when the two pistons of a pair of steam engines, each through its own connecting rod, act upon one crank shaft. One train of mechanism may *diverge* into two or more; as when a single shaft, driven by a prime mover, carries several pulleys, each of which drives a different machine. The principles of comparative motion in such converging and diverging trains are the same as in simple trains.

CHAPTER III.

ON AGGREGATE COMBINATIONS.

190. **The General Principles** of aggregate combinations have already been given in Part I., Chapter II., Section 3. The problems to which those principles are to be applied may be divided into two classes.

I. Where a secondary moving piece is connected at three, or at two points, as the case may be, with three or with two other pieces whose motions are given; so that the problem is, *from the motions of three or of two points in the secondary piece, to find its motion as a whole, and the motion of any point in it.* The solution of this problem is given in Articles 97 and 98.

II. Where a secondary piece, C, is carried by another piece, B; and denoting the frame of the machine by A, there are given two out of the three motions of A, B, and C, relatively to each other, and the third is required. The motion of C relatively to A is the resultant of the motion of C relatively to B, and of B relatively to A; and the problem is solved by the methods already explained in Articles 99 to 107, inclusive.

Professor Willis distinguishes the effects of aggregate combinations into *aggregate velocities*, whether linear or angular, produced in secondary pieces by the combined action of different drivers, and *aggregate paths*, being the curves, such as cycloids and trochoids, epicycloids and epitrochoids, described by given points in such secondary pieces.

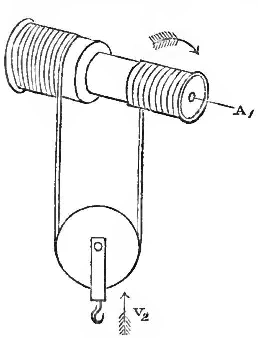


Fig. 77.

The following Articles give examples of two simple aggregate combinations.

191. **Differential Windlass.**—In fig. 77, the axis A_1 carries two barrels of different radii, r_1 being the greater, and r_2 the less. A running block containing a single pulley is hung by a rope which passes below the pulley, and has one end wound round the larger barrel, and the other wound the contrary way round the smaller barrel. When the two barrels rotate together with the common angular velocity a, the division of the rope which hangs from the larger barrel moves with the

velocity $a r_1$, and the division which hangs from the smaller barrel moves in the contrary direction with the velocity $-a r_2$ (whose direction is denoted by the negative sign). These are also the velocities of the two points at opposite extremities of a diameter of the pulley, where it is touched by the two vertical divisions of the rope. The velocity of the centre of the pulley is a mean between those two velocities; that is, their half-difference, because their signs are opposite; or denoting it by v,

$$v = \frac{a(r_1 - r_2)}{2} \text{..........................(1.)}$$

The *instantaneous axis* of the pulley may be found by the method of Article 98, as follows:—In fig. 35c, let A and B be the two ends of the horizontal diameter of the pulley, and let $\overline{A V_a} = a r_1$, and $\overline{B V_b} = a r_2$ represent their velocities; join $\overline{V_a V_b}$ cutting A B in O; this is the instantaneous axis. Now

$$\begin{aligned} & AO - OB = AC + CO - OB = BC + CO - OB = 2\,OC, \\ & AO + OB : AO - OB :: AV_a + BV_b : AV_a - BV_b, \\ & AB : 2\,OC :: a(r_1 + r_2) : a(r_1 - r_2); \end{aligned}$$

and hence the distance of the instantaneous axis from the centre or moving axis of the pulley is obviously

$$\overline{AB} \cdot \frac{r_1 - r_2}{2(r_1 + r_2)} \text{..........................(2.)}$$

The motion of the centre of the pulley is the same with that of a point in a rope wound on a barrel of the radius $\frac{r_1 - r_2}{2}$. The use of the contrivance is to obtain a slow motion of the pulley without using a small, and therefore a weak, barrel.

192. **Compound Screws.**—(Fig. 78). On the same axis let there be two screws $S_1 S_1$, and $S_2 S_2$, of the respective pitches

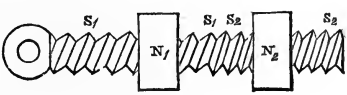


Fig. 78.

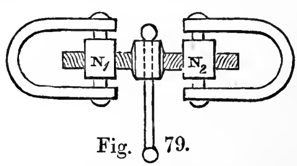


Fig. 79.

p_1 and p_2, p_1 being the greater, and let the screws in the first instance be both right-handed or both left-handed. Let N_1 and N_2 be two nuts, fitted on the two screws respectively. When the compound screw rotates with the angular velocity a, the nuts ap roach towards or recede from each other with the relative velocity

$$v = \frac{a\,(p_1 - p_2)}{2\,\pi};\qquad\qquad(1.)$$

being that due to a screw whose pitch is the *difference* of the two pitches of the compound screw. (See Article 96, Equation 1.) The object of this contrivance is to obtain the slow advance due to a fine pitch, together with the strength of large threads.

Fig. 79 represents a compound screw in which the two screws are contrary-handed, and the relative velocity of the nuts N_1 N_2 is that due to the *sum* of the two pitches; or as they are usually equal, to *double* the pitch of each screw. This combination is used in coupling railway-carriages.

PART III.

PRINCIPLES OF STATICS.

CHAPTER I.

SUMMARY OF GENERAL PRINCIPLES.

NATURE AND DIVISION OF THE SUBJECT.

THE present Chapter contains a summary of the Principles of Statics.

193. **Forces—Action and Re-action.**—Every force is an action exerted between a pair of bodies, tending to alter their condition as to relative rest and motion; it is exerted equally, and in contrary directions, upon each body of the pair. That is to say, if A and B be a pair of bodies acting mechanically on each other, the force exerted by A upon B is equal in magnitude and contrary in direction to the force exerted by B upon A. This principle is sometimes called *the equality of action and re-action.* It is analogous to that of relative motion, explained in Article 61, page 34.

194. **Forces, how Determined and Expressed.**—A force, as respects one of the two bodies between which it acts, is determined, or made known, when the following three things are known respecting it:—*first*, the *place*, or part of the body to which it is applied; *secondly*, the *direction* of its action; *thirdly*, its *magnitude.*

The PLACE of the application of a force to a body may be the whole of its volume, as in the case of gravity; or the surface at which two bodies touch each other, or the bounding surface between two parts of the same body, as in the case of pressure, tension, shearing stress, and friction.

Thus every force has its action distributed over a certain space, either a volume or a surface; and a force concentrated at a single point has no real existence. Nevertheless, in investigations respecting the action of a distributed force upon the position and movements, as a whole, of a rigid body, or of a body which without error may be treated as rigid, like the solid parts of a machine, fixed or moving, that force may be treated as if it were concentrated at a point or points, determined by suitable processes; and such is the use of those numerous propositions in statics which relate to forces concentrated at points; or *single forces*, as they are called.

The DIRECTION of a force is that of the motion which it tends to produce. A straight line drawn through the points of application of a single force, and along its direction, is the LINE OF ACTION of that force.

The MAGNITUDES of two forces are equal when, being applied to the same body in opposite directions along the same line of action, they balance each other.

The magnitude of a force is expressed arithmetically by stating in numbers its ratio to a certain *unit* or *standard* of force, which, for practical purposes, is usually the *weight* (or attraction towards the earth), at a certain latitude, and at a certain level, of a known mass of a certain material. Thus the British unit of force is the *standard pound avoirdupois;* which is the weight, in the latitude of London, of a certain piece of platinum kept in a public office.

For the sake of convenience, or of compliance with custom, other units of weight are occasionally employed in Britain, bearing certain ratios to the standard pound; such as—

The grain $= \frac{1}{7000}$ of a pound avoirdupois.
The troy pound = 5,760 grains = $0\cdot8\dot{2}28571\dot{4}$ pound avoirdupois.
The hundredweight = 112 pounds avoirdupois.
The ton = 2,240 pounds avoirdupois.

The French standard of weight is the *kilogramme*, which is the weight, in the latitude of Paris, of a certain piece of platinum kept in a public office. It was originally intended to be the weight of a cubic decimètre of pure water, measured at the temperature at which the density of water is greatest—viz., 4°·1 Cent., or 39°·4 Fahr., and under the pressure which supports a barometric column of 760 millimètres of mercury; but it is in reality a little heavier.

—— A kilogramme is 2·20462125 lbs. avoirdupois.
A pound avoirdupois is 0·4535926525 of a kilogramme.

For scientific purposes, forces are sometimes expressed in *Absolute Units.* The "Absolute Unit of Force" is a term used to denote the force which, acting on an unit of mass for an unit of time, produces an unit of velocity.

The unit of time employed is always a second.

The unity of velocity is in Britain one foot per second; in France one mètre per second.

The unit of mass is the mass of so much matter as weighs one unit of weight near the level of the sea, and in some definite latitude.

In Britain the latitude chosen is that of London; in France, that of Paris.

In Britain the unit of weight chosen is sometimes a grain, sometimes a pound avoirdupois; and it is equal to 32·187 of the corresponding absolute units of force. In France the unit of weight chosen is either a gramme or a kilogramme, and it is equal to 9·8087 of the corresponding absolute units of force. Each of those coefficients is denoted by the letter g.

195. **Measures of Force and Mass.**—If by the unit of force is understood the weight of a certain standard, such as the avoirdupois pound, then the mass of that standard is $1 \div g$; and the unit of mass is g times the mass of the standard; and this is the most convenient system for calculations connected with mechanical engineering, and is therefore followed in the present work.

But if we take for the unit of mass, the mass of the standard itself, then the unit of force is the *absolute unit;* and the weight of the standard in such units is expressed by g; for g is the velocity which a body's own weight, acting unbalanced, impresses on it in a second. This will be specially treated of in Part V. This is the system employed in many scientific writings, and in particular, in Thomson and Tait's *Natural Philosophy.* It has great advantages in a scientific point of view; but its use in calculations for practical purposes would be inconvenient, because of the prevailing custom of expressing forces in terms of the standard of weight.

196. **Representation of Forces by Lines.**—A single force may be represented in a drawing by a straight line; an extremity of the line indicating the point of application of the force,—the direction of the line, the direction of the force,—and the length of the line, the magnitude of the force, according to an arbitrary scale.

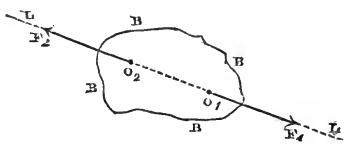


Fig. 80.

For example, in fig. 80, the fact that the body B B B B is acted upon at the point O_1 by a given force, may be expressed by drawing from O_1 a straight line $\overline{O_1 F_1}$ in the direction of the force, and of a length representing the magnitude of the force.

If the force represented by $\overline{O_1 F_1}$ is balanced by a force applied either at the same point, or at another point O_2 (which must be in the line of action L L of the force to be balanced), then the second force will be represented by a straight line $\overline{O_2 F_2}$, opposite in direction, and equal in length to $\overline{O_1 F_1}$, and lying in the same line of action L L.

If the body B B B B (fig. 81), be balanced by several forces acting in the same straight line LL, applied at points O_1 O_2, &c., and represented by lines $\overline{O_1 F_1}$, $\overline{O_2 F_2}$, &c.; then either direction in the line L L (such as the direction towards + L) is to be considered as positive, and the opposite direction (such as the direction towards – L) as negative; and if the sum of all the lines representing forces which point positively be equal to the sum of all those which point negatively, the algebraical sum of all the forces is nothing, and the body is balanced.

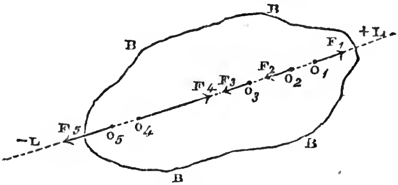


Fig. 81.

197. **Resultant and Component Forces—Their Magnitude.**—The Resultant of any combination of forces applied to one body is a single force capable of balancing that single force which balances the combined forces; that is to say, the resultant of the combined forces is equal and directly opposed to the force which balances the combined forces, and is *equivalent* to the combined forces so far as the balance of the body is concerned. The combined forces are called *components* of their resultant.

The resultant of a set of mutually balanced forces is nothing.

The *magnitudes* and *directions* of a resultant force and of its components are related to each other exactly in the same manner with the velocities and directions of resultant and component motions.

As to the *position* of the resultant, if the components act through one point, the resultant acts through that point also; but if the components do not act through one point, the position of the resultant is to be found by methods which will be stated further on.

198. **Equilibrium or Balance** is the condition of two or more forces which are so opposed that their combined action on a body produces no change in its rest or motion, and that each force merely *tends to* cause such change, without actually causing it.

In treatises on statics, the word *pressure* is often used to denote any balanced force; although in the popular sense that word is used to denote a force, of the nature of a thrust or push, distributed over a surface.

199. **Parallel Forces** are forces whose directions of motion are parallel, excepting couples and directly opposed forces.

200. **Couples.**—Two forces of equal magnitude applied to the same body in parallel and opposite directions, but not in the same

line of action (such as F, F, in fig. 82), constitute what is called a "*couple*."

The *arm* or *leverage* of a couple (L, fig. 82) is the perpendicular distance between the lines of action of the two equal forces.

The tendency of a couple is to turn the body to which it is applied in the *plane* of the couple—that is, the plane which contains the lines of action of the two forces. (The plane in which a body turns is any plane parallel to those planes in the body whose position is not altered by the turning). The turning of a body is said to be *right-handed* when it appears to a spectator to take place in the same direction with that of the hands of a watch, and *left-handed* when in the opposite direction; and couples are designated as right-handed or left-handed according to the direction of the turning which they tend to produce. The couple represented in fig. 82 appears right-handed to the reader.

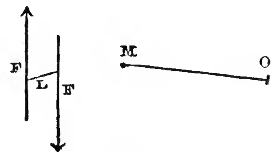


Fig. 82.

The *Moment* of a couple means the product of the magnitude of its force by the length of its arm (F L); and may be represented by the area of a rectangle whose sides are F and L. If the force be a certain number of pounds, and the arm a certain number of feet, the product of those two numbers is called the moment in *foot-pounds*, and similarly for other measures. The moment of a couple may also be represented by a single line on paper, by setting off upon its *axis* (that is, upon any line perpendicular to the plane of the couple) a length proportional to that moment (O M, fig. 82) in such a direction, that to an observer looking from O towards M the couple shall seem right-handed.

201. **The Centre of Parallel Forces** is the single point referred to in the following principle. The forces to which that principle is applied are in general either weights or pressures; and the point in question is then called the *Centre of Gravity* or the *Centre of Pressure*, as the case may be.

If there be given a system of points, and the mutual ratios of a system of parallel forces applied to those points, which forces have a single resultant, then there is one point, and one only, which is traversed by the line of action of the resultant of every system of parallel forces having the given mutual ratios and applied to the given system of points, whatsoever may be the absolute magnitudes of those forces and the angular position of their lines of action.

202. **Distributed Forces in General.**—In Article 194, page 115, it has already been explained, that the action of every real force is distributed throughout some volume, or over some surface. It is always possible, however, to find either a *single resultant*, or a

resultant couple, or a *combination of a single force with a couple*, to which a given distributed force is equivalent, so far as it affects the equilibrium of the body, or part of a body, to which it is applied.

In the application of Mechanics to Structures, the only force distributed throughout the volume of a body which it is necessary to consider, is its *weight*, or attraction towards the earth; and the bodies considered are in every instance so small as compared with the earth, that this attraction may, without appreciable error, be held to act in parallel directions at each point in each body. Moreover, the forces distributed over surfaces are either parallel at each point of their surfaces of application, or capable of being resolved into sets of parallel forces; hence, *parallel distributed forces* have alone to be considered; and every such force is statically equivalent either to a single resultant, or to a resultant couple.

The *intensity of a distributed force* is the ratio which the magnitude of that force, expressed in units of weight, bears to the space over which it is distributed, expressed in units of volume, or in units of surface, as the case may be. An *unit of intensity* is an unit of force distributed over an unit of volume or of surface, as the case may be; so that there are two kinds of units of intensity. For example, *one pound per cubic foot* is an unit of intensity for a force distributed throughout a volume, such as weight; and *one pound per square foot* is an unit of intensity for a force distributed over a surface, such as pressure or friction.

203. **Specific Gravity—Heaviness—Density—Bulkiness.**—I. *Specific Gravity* is the ratio of the weight of a given bulk of a given substance to the weight of the same bulk of pure water at a standard temperature. In Britain the standard temperature is 62° Fahr. = 16°·67 Cent. In France it is the temperature of the maximum density of water = 3°·94 Cent. = 39°·1 Fahr.

In rising from 39°·1 Fahr. to 62° Fahr., pure water expands in the ratio of 1·001118 to 1; but that difference is of no consequence in calculations of specific gravity for engineering purposes.

II. The *heaviness* of any substance is the weight of an unit of volume of it in units of weight. In British measures heaviness is most conveniently expressed in *lbs. avoirdupois to the cubic foot;* in French measures, in *kilogrammes to the cubic decimètre* (or to the litre). The values of the heaviness of water at 39°·1 Fahr., and at 62° Fahr., are respectively 62·425 and 62·355 lbs. to the cubic foot.

III. The *density* of a substance is either the number of units of *mass* in an unit of volume, in which case it is equal to the heaviness,—or the ratio of the mass of a given volume of the substance to the mass of an equal volume of water, in which case it is equal to the specific gravity. In its application to *gases*, the term

"Density" is often used to denote the ratio of the heaviness of a given gas to that of air, at the same temperature and pressure.

IV. The *bulkiness* of a substance is the number of units of volume which an unit of weight fills; and is the *reciprocal of the heaviness.* In British measures bulkiness is most conveniently expressed in *cubic feet to the lb. avoirdupois;* in French measures, in *cubic decimètres* (or in litres) *to the kilogramme.* Rise of temperature produces (with certain exceptions) increase of bulkiness. The linear expansion of a solid body is one-third of its expansion in bulk.

204. **The Centre of Gravity** of a body or of a system of bodies, is the point always traversed by the resultant of the weight of the body or system of bodies,—in other words, the *centre of parallel forces* for the weight of the body or system of bodies.

To *support* a body, that is, to balance its weight, the resultant of the supporting force must act through the centre of gravity.

When the centre of gravity of a *geometrical figure* is spoken of, it is to be understood to mean the point where the centre of gravity would be, if the figure were formed of a substance of uniform heaviness.

205. **The Centre of Pressure** in a plane surface is the point traversed by the resultant of a pressure that is exerted at that surface. When the intensity is uniform, the centre of pressure is at the *centre of magnitude* of the pressed surface.

206. **The Centre of Buoyancy** of a solid wholly or partly immersed in a liquid is the centre of gravity of the mass of liquid displaced. The resultant pressure of the liquid on the solid is equal to the weight of liquid displaced, and is exerted vertically upwards through the centre of buoyancy.

207. **The Intensity of Pressure** is expressed in units of weight on the unit of area; as pounds on the square inch, or kilogrammes on the square mètre; or by the height of a column of some fluid; or in *atmospheres,* the unit in this case being the average pressure of the atmosphere at the level of the sea.

CHAPTER II.

COMPOSITION, RESOLUTION, AND BALANCE OF FORCES.

SECTION 1.—FORCES ACTING THROUGH ONE POINT.

208. **Resultant of Forces Acting in One Straight Line.**—The resultant of any number of forces acting on one body in the same straight line of action, acts along that line, and is equal in magnitude to the sum of the component forces; it being understood, that when some of the component forces are opposed to the others, the word "*sum*" is to be taken in the algebraical sense; that is to say, that forces acting in the same direction are to be added to, and forces acting in opposite directions subtracted from each other.

When a system of forces acting along one straight line are balanced, the sum of the forces acting in one direction is equal to the sum of the forces acting in the opposite direction.

209. **Resultant and Balance of Inclined Forces—Parallelogram of Forces.**—The smallest number of inclined forces which can balance each other is three. Those three forces must act through one point, and in one plane. Their relation to each other depends on the following theorem, called the "PARALLELOGRAM OF FORCES," from which the whole science of statics may be deduced.

If two forces whose lines of action traverse one point be represented in direction and magnitude by the sides of a parallelogram, their resultant is represented by the diagonal.

For example, through the point O (fig. 83) let two forces act, represented in direction and magnitude by $\overline{OA}$ and $\overline{OB}$. The resultant or equivalent single force of those two forces is represented in direction and magnitude by the diagonal O C of the parallelogram O A C B. Its magnitude is given algebraically by the equation.

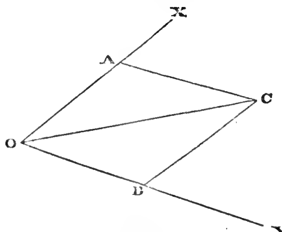


Fig. 83.

$$OC = \sqrt{\left\{ OA^2 + OB^2 + 2OA \cdot OB \cos AOB \right\}} \qquad (1.)$$

210. **Triangle of Forces.**—To balance the forces $\overline{OA}$ and $\overline{OB}$, a force is required equal and directly opposed to their resultant $\overline{OC}$. This may be expressed by saying, that *if the directions and*

magnitudes of three forces be represented by the three sides of a triangle, taken in the same order (such as $\overline{OA}$, $\overline{AC}$, $\overline{CO}$), *then those three forces, acting through one point, balance each other*, or in other words, that three forces in the same plane balance each other at one point, when each is proportional to the sine of the angle between the other two.

211. **Polygon of Forces.**—*If a number of forces acting through the same point be represented by lines equal and parallel to the sides of a closed polygon, taken in the same order, those forces balance each other.* To fix the ideas, let there be five forces acting through the point O (fig. 84), and represented in direction and magnitude by the lines F_1, F_2, F_3, F_4, F_5, which are equal and parallel to the sides of the closed polygon O A B C D O; viz. :—

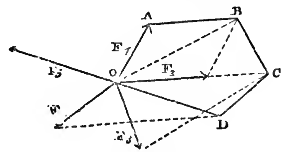


Fig. 84.

$$F_1 = \text{and} \parallel OA;\ F_2 = \text{and} \parallel AB;\ F_3 = \text{and} \parallel BC;$$
$$F_4 = \text{and} \parallel CD;\ F_5 = \text{and} \parallel DO.$$

Then, by the principle of the parallelogram of forces, the resultant of F_1 and F_2 is O B; the resultant of F_1, F_2, and F_3 is O C; the resultant of F_1, F_2, F_3, and F_4 is O D, equal and opposite to F_5, so that the final resultant is nothing.

The closed polygon may be either plane or "gauche"—that is, not in one plane.

212. **Principles of the Parallelopiped of Forces.**—The simplest gauche polygon is one of four sides. Let A O B C E F G H (fig. 85), be a parallelopiped whose diagonal is O H. Then any three successive edges so placed as to begin at O and end at H, form, together with the diagonal H O, a closed quadrilateral; consequently, if three forces F_1, F_2, F_3, acting through O, be represented by the three edges $\overline{AO}$, $\overline{OB}$, $\overline{OC}$, of a parallelopiped, the diagonal $\overline{OH}$ represents their resultant, and a fourth force F_4 equal and opposite to $\overline{OH}$ balances them.

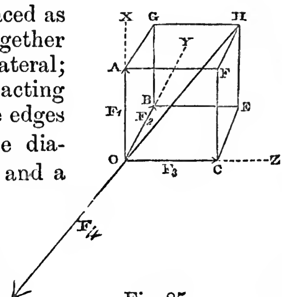


Fig. 85.

213. **Resolution of a Force into two Components.**—In order that a given single force may be resolvable into two components acting in given lines inclined to each other, it is necessary, *first*, that the lines of action of those components should intersect the line of action of the given force in one point; and *secondly*, that those three lines of action should be in one plane.

Returning then to fig. 83, let $\overline{OC}$ represent the given force, which it is required to resolve into two component forces, acting in the lines O X, O Y, which lie in one plane with O C, and intersect it in one point O.

Though C draw C A ∥ O Y, cutting O X in A, and C B ∥ O X, cutting O Y in B. Then will O A and O B represent the component forces required.

Two forces respectively equal to and directly opposed to $\overline{OA}$ and $\overline{OB}$ will balance $\overline{OC}$.

The magnitudes of the forces are in the following proportions:—

$$OC : OA : OB$$

$$:: \sin AOB : \sin BOC : \sin AOC \text{.............}(1.)$$

214. **Resolution of a Force into three Components.**—In order that a given single force may be resolvable into three components acting in given lines inclined to each other, it is necessary that the lines of action of the components should intersect the line of action of the given force in one point.

Returning to fig. 85, let $\overline{OH}$ represent the given force which it is required to resolve into three component forces, acting in the lines O X, O Y, O Z, which intersect O H in one point O.

Through H draw three planes parallel respectively to the planes Y O Z, Z O Y, X O Y, and cutting respectively O X in A, O Y in B, O Z in C. Then will $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, represent the component forces required.

Three forces respectively equal to, and directly opposed to $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$, will balance $\overline{OH}$.

215. **Resolution of a Force. Rectangular Components.**—The rectangular components of a force are those into which it is resolved when the directions of their lines of action are at right angles to each other.

For example, in fig. 85, suppose O X, O Y, O Z, to be three axes of co-ordinates at right angles to each other. Then $\overline{OH}$ is resolved into three rectangular components, A O, O B, O C, simply by letting fall from H perpendiculars on O X, O Y, O Z, cutting them at A, B, C, respectively.

Let the three rectangular components be denoted respectively by X, Y, Z, the resultant by R, and the angles which it makes with the components by α, β, γ, respectively; then the relations between the three rectangular components and their resultant are expressed by the following equations :—

$$X = R \cos \alpha; \; Y = R \cos \beta; \; Z = R \cos \gamma; \text{.........}(2.)$$

$$R^2 = X^2 + Y^2 + Z^2 \text{......................}(3.)$$

When the resultant is in the same plane with two of its components (as X and Y), the third component is null, and the Equations 2 and 3 take the following form:—

$$X = R \cos \alpha = R \sin \beta;\ Y = R \cos \beta = R \sin \alpha;\ Z = 0; \ldots (4.)$$

$$R^2 = X^2 + Y^2 \ldots\ldots\ldots\ldots (5.)$$

In using Equations 2, 3, 4, and 5, it is to be remembered that cosines of obtuse angles are negative.

216. **Resultant and Balance of any number of inclined Forces acting through one Point.**—To find this resultant by calculation, assume any three directions at right angles to each other as axes; resolve each force into three components (X, Y, Z) along those axes, and consider the components along a given axis which act in one direction as positive, and those which act in the opposite direction as negative; take the algebraical sums of the components along the three axes respectively ($\Sigma \cdot X$, $\Sigma \cdot Y$, $\Sigma \cdot Z$); these will be the *rectangular components of the resultant of all the forces;* and its magnitude and direction will be given by the following equations:—

$$R^2 = (\Sigma \cdot X)^2 + (\Sigma \cdot Y)^2 + (\Sigma \cdot Z)^2; \ldots\ldots\ldots (1.)$$

$$\cos \alpha = \frac{\Sigma \cdot X}{R};\ \cos \beta = \frac{\Sigma \cdot Y}{R};\ \cos \gamma = \frac{\Sigma \cdot Z}{R} \ldots\ldots (2.)$$

If the forces all act in one plane, two rectangular axes in that plane are sufficient, and the terms containing Z disappear from the equations.

If the forces balance each other, the components parallel to each axis balance each other independently; that is to say, the three following conditions are fulfilled:—

$$\Sigma \cdot X = 0;\ \Sigma \cdot Y = 0;\ \Sigma \cdot Z = 0. \ldots\ldots\ldots (3.)$$

If the forces all act in one plane, these *conditions of equilibrium* are reduced to two.

Section 2.—Resultant and Balance of Couples.

217. **Equivalent Couples.**—*If the moments of two couples acting in the same direction and in the same or parallel planes are equal, those couples are equivalent:* that is, their tendencies to turn the body to which they are applied are the same.

The following propositions are the chief consequences of the principle just stated:—

218. **Resultant of Couples.**—The resultant of any number of couples acting in the same or parallel planes is equivalent to a couple whose moment is the algebraical sum of the moments of the combined couples.

219. **Equilibrium of Couples with same Axis.**—Two opposite couples of equal moment in the same or parallel planes balance each other. Any number of couples in the same or parallel planes balance each other when the moments of the right-handed couples are together equal to the moments of the left-handed couples; in other words, when the resultant moment is nothing—a condition expressed algebraically by

$$\Sigma \cdot F L = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

220. **Parallelogram of Couples.**—If the two sides of a parallelogram represent the axes and moments of two couples acting on the same body in planes inclined to each other, the diagonal of the parallelogram will represent the axis and moment of the resultant couple, which is equivalent to those two.

In other words, three couples represented by the three sides of a triangle, taken in the same order, balance each other.

221. **Polygon of Couples.**—If any number of couples acting on the same body be represented by a series of lines joined end to end, and taken in the same order so as to form sides of a polygon, and if the polygon is closed, those couples balance each other.

These propositions are analogous to corresponding propositions relating to single forces; and couples, like single forces, can be resolved into components acting about two or three given axes.

222. **Resultant of a Couple and Single Force in Parallel Planes.**—Let M denote the moment of a couple applied to a body (fig. 86); and at a point O let a single force F be applied, in a plane parallel to that of the couple. For the given couple substitute an equivalent couple, consisting of a force $-F$ equal and directly opposed to F at O, and a force F acting through the point A, the arm $\overline{A\,O}$ perpendicular to F being $= \frac{M}{F}$, and parallel to the plane of the couple M. Then the forces at O balance each other, and F acting through A is the resultant of the single force F applied at O, and the couple M; that is to say, that if with a single force F there be combined a couple M whose plane is parallel to the force, the effect of that combination is to shift the line of action of the force parallel to itself through a distance $O\,A = \frac{M}{F}$;—to the left if M is right-handed—to the right if M is left-handed.

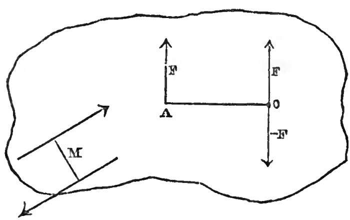


Fig. 86.

223. **Moment of Force with respect to an Axis.**—In fig. 87, let the straight line F represent a force. Let O X be any straight line perpendicular in direction to the line of action of the force, and not intersecting it, and let A B be the common perpendicular of those two lines. At B conceive a pair of equal and directly opposed forces to be applied in a line of action parallel to F, viz.:—$F' = F$, and $-F' = -F$. The supposed application of such a pair of balanced forces does not alter the statical condition of the body. Then the original single force F, applied in a line traversing A, is equivalent to the force F′ applied in a line traversing B, the point in O X which is nearest to A, combined with the couple composed of F and $-F'$, whose moment is $F \cdot AB$. This is called the *moment of the force* F *relatively to the axis* O X, and sometimes also, the *moment of the force* F *relatively to the plane* traversing O X, parallel to the line of action of the force.

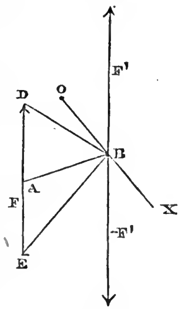


Fig. 87.

If from the point B there be drawn two straight lines B D and B E, to the extremities of the line F representing the force, the area of the triangle B D E being $= \frac{1}{2} F \cdot AB$, represents one-half of the moment of F relatively to O X.

Section 3.—Resultant and Balance of Parallel Forces.

224. **Magnitude of Resultant of Parallel Forces.**—A balanced system of parallel forces consists either of pairs of directly opposed equal forces, or of couples of equal forces, or of combinations of such pairs and couples.

Hence the following propositions as to the relations amongst the *magnitudes* of systems of parallel forces.

I. In a balanced system of parallel forces the sums of the forces acting in opposite directions are equal; in other words, the algebraical sum of the magnitudes of all the forces taken with their proper signs is nothing.

II. The magnitude of the resultant of any combination of parallel forces is the algebraical sum of the magnitudes of the forces.

The relations amongst the *positions* of the lines of action of balanced parallel forces remain to be shewn; and in this inquiry all pairs of directly opposed equal forces may be left out of consideration; for each such pair is independently balanced whatsoever its position may be; so that the question in each case is to be solved by means of the theory of couples.

The following is the simplest case:—

225. **Direction of Resultant of Parallel Forces—Principle of the Lever.**—*If three parallel forces applied to one body balance each other, they must be in one plane; the two extreme forces must act in the same direction; the middle force must act in the opposite direction; and the magnitude of each force must be proportional to the distance between the lines of action of the other two.* Let a body (fig. 88) be maintained in equilibrio by two opposite couples acting in the same plane, and of equal moments,

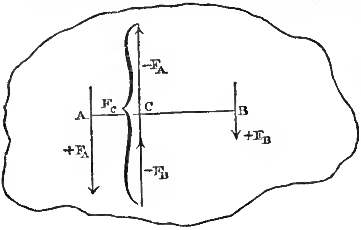


Fig. 88.

$$F_A L_A = F_B L_B,$$

and let those couples be so applied to the body that the lines of action of two of those forces, $-F_A - F_B$, which act in the same direction, shall coincide. Then those two forces are equivalent to the single middle force $F_C = -(F_A + F_B)$, equal and opposite to the sum of the extreme forces $+F_A$, $+F_B$, and in the same plane with them; and if the straight line A C B be drawn perpendicular to the lines of action of the forces, then

$$\overline{AC} = L_A; \; \overline{CB} = L_B; \; \overline{AB} = L_A + L_B;$$

and consequently

$$F_A : F_B : F_C :: \overline{CB} : \overline{AC} : \overline{AB}; \ldots\ldots\ldots\ldots (1.)$$

This proposition holds also when the straight line A C B crosses the lines of action of the three forces obliquely.

226. **To find the Resultant of Two Parallel Forces.**—The resultant is in the same plane with, and parallel to, the components. It is their sum or difference, according as they act in the same or contrary directions; and in the latter case its direction is that of the greater component. To find its line of action by construction, proceed as follows:—Fig. 89 representing the case in which the components act in the same direction, fig. 90 that in which they act in contrary directions. Let A D and B E be the components. Join A E and B D, cutting each other in F. In B D (produced in fig. 90) take B G = D F. Through G draw a line parallel to the components; this will be the line of action of the resultant. To find its magnitude by construction: parallel to A E, draw B C and D H, cutting the line of action of the resultant in C and H; C H will represent the resultant required; and a force equal and opposite to C H will balance A D and B E.

To find the line of action of the resultant by calculation; make either

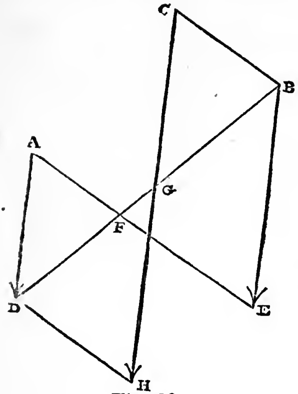


Fig. 89.

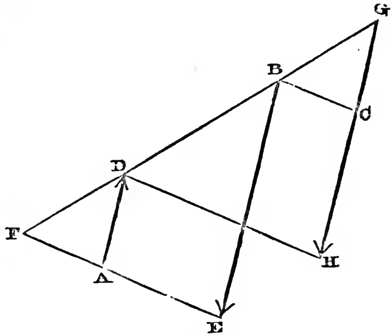


Fig. 90.

$$B\,G = \frac{A\,D \cdot D\,B}{C\,H}; \text{ or } D\,G = \frac{B\,E \cdot D\,B}{C\,H}.$$

When the two given parallel forces are opposite and equal, they form a couple, and have no single resultant.

227. **To find the Relative Proportions of Three Parallel Forces which Balance each other, Acting in One Plane: their Lines of Action being given.**—Across the three lines of action, in any convenient position, draw a straight line A C B, fig. 91, and measure the distances between the points where it cuts the lines of action. Then each force will be proportional to the distance between the lines of action of the other two. The direction of the middle force, C, is contrary to that of the other two forces, A and B.

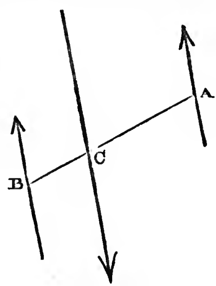


Fig. 91.

In symbols, let A, B, and C be the forces; then,

$$A + B + C = 0;\ A\,B : B\,C : C\,A :: C : A : B.$$

Each of the three forces is equal and opposite to the resultant of the other two; and each pair of forces are equal and opposite to the components of the third. Hence this rule serves to resolve a given force into two parallel components acting in given lines in the same plane.

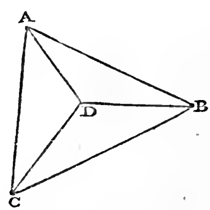


Fig. 92.

228. **To find the Relative Proportions of Four Parallel Forces which Balance each other, not Acting in One Plane: their Lines of Action being given.**—Conceive a plane to cross the lines of action in any convenient position; and in fig. 92 or fig. 93, let A, B, C, D repre-

sent the points where the four lines of action cut the plane. Draw the six straight lines joining those four points by pairs. Then the force which acts through each point will be proportional to the area of the triangle formed by the other three points.

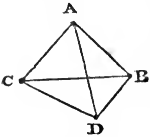


Fig. 93.

In fig. 92 the directions of the forces at A, B, and C are the same, and are contrary to that of the force at D. In fig. 93 the forces at A and D act in one direction, and those at B and C in the contrary direction.

In symbols, $A + B + C + D = 0;$

$$BCD : CDA : DAB : ABC$$
$$:: A : B : C : D.$$

Each of the four forces is equal and opposite to the resultant of the other three; and each set of three forces are equal and opposite to the components of the fourth. Hence the rule serves to resolve a force into three parallel components not acting in one plane.

229. **Moments of a Force with respect to a Pair of Rectangular Axes.**—In fig. 94, let F be any single force; O an arbitrarily-assumed point, called the "origin of co-ordinates;" $-X\,O+X$, $-Y\,O+Y$, a pair of axes traversing O, at right angles to each other and to the line of action of F. Let $A\,B = y$, be the common perpendicular of F and O X; let $A\,C = x$, be the common perpendicular of F and O Y. x and y are the "rectangular coordinates" of the line of action of F relatively to the axes $-X\,O+X$, $-Y\,O+Y$, respectively. According to the arrangement of the axes in the figure, x is to be considered as positive to the right, and negative to the left, of $-Y\,O+Y$; and y is to be considered as positive to the left, and negative to the right, of $-X\,O+X$; right and left referring to the spectator's right and left hand. In the particular case represented, x and y are both positive. Forces, in the figure, are considered as positive upwards, and negative downwards; and in the particular case represented, F is positive.

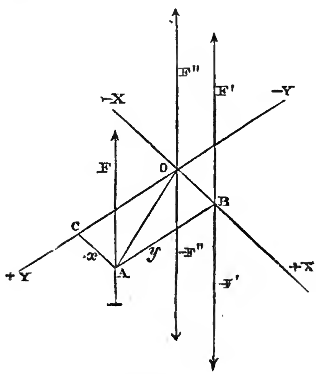


Fig. 94.

At B conceive a pair of equal and opposite forces, F′ and − F′, to be applied; F′ being equal and parallel to F, and in the same direction. Then, as in Article 223, F is equivalent to the single force

$F' = F$ applied at B, combined with the couple constituted by F and $-F'$ with the arm y, whose moment is $y\,F$; being positive in the case represented, because the couple is right-handed. Next, at the origin O, conceive a pair of equal and opposite forces, F'' and $-F''$, to be applied, F'' being equal and parallel to F and F', and in the same direction. Then the single force F' is equivalent to the single force $F'' = F' = F$ applied at O, combined with the couple constituted by F' and $-F''$ with the arm $OB = x$, whose moment is $-x\,F$; being negative in the case represented, because the couple is left-handed.

Hence, it appears finally, that a force F acting in a line whose co-ordinates with respect to a pair of rectangular axes perpendicular to that line are x and y, is equivalent to an equal and parallel force acting through the origin, combined with two couples whose moments are,

$y\,F$ relatively to the axis O X, and $-x\,F$ relatively to the axis O Y right-handed couples being considered positive; and + Y lying to the left of + X, as viewed by a spectator looking from + X towards O, with his head in the direction of positive forces.

230. **Balance of any System of Parallel Forces in one Plane.**— In order that any system of parallel forces whose lines of action are in one plane may balance each other, it is necessary and sufficient that the following conditions should be fulfilled:—

First—(As already stated) that the algebraical sum of the forces shall be nothing.

Secondly—That the algebraical sum of the moments of the forces relatively to any axis perpendicular to the plane in which they act shall be nothing,

two conditions which are expressed symbolically as follows:— Let F denote any one of the forces, considered as positive or negative, according to the direction in which it acts; let y be the perpendicular distance of the line of action of this force from an arbitrarily assumed axis O X, y also being considered as positive or negative, according to its direction; then,

$$\Sigma \cdot F = 0; \; \Sigma \cdot y\,F = 0.$$

In summing moments, right-handed couples are usually considered as positive, and left-handed couples as negative.

231. Let R denote the **Resultant of any System of Parallel Forces in one Plane,** and y_r, the distance of the line of action of that resultant from the assumed axis O X to which the positions of forces are referred; then,

$$R = \Sigma \cdot F;$$

$$y_r = \frac{\Sigma \cdot y\,F}{\Sigma \cdot F}.$$

In some cases the forces may have no single resultant, $\Sigma \cdot \mathrm{F}$ being $=0$; and then, unless the forces balance each other completely, their resultant is a couple of the moment $\Sigma \cdot y\,\mathrm{F}$.

232. **Balance of any System of Parallel Forces.**—In order that any system of parallel forces, whether in one plane or not, may balance each other, it is necessary and sufficient that the three following conditions shall be fulfilled:—

First—(As already stated) that the algebraical sum of the forces shall be nothing.

Secondly and Thirdly—That the algebraical sums of the moments of the forces, relatively to a pair of axes at right angles to each other, and to the lines of action of the forces, shall each be nothing, two conditions which are expressed symbolically as follows:—

Let O X and O Y denote the pair of axes; let F be the magnitude of any one of the forces; y its perpendicular distance from O X, and x its perpendicular distance from O Y; then,

$$\Sigma \cdot \mathrm{F}=0;\ \Sigma \cdot y\,\mathrm{F}=0;\ \Sigma \cdot x\,\mathrm{F}=0;$$

233. Let R denote the **Resultant of any System of Parallel Forces,** and x_r and y_r the distances of its line of action from two rectangular axes; then,

$$\mathrm{R}=\Sigma \cdot \mathrm{F};\ x_r=\frac{\Sigma \cdot x\,\mathrm{F}}{\Sigma \cdot \mathrm{F}};\ y_r=\frac{\Sigma \cdot y\,\mathrm{F}}{\Sigma \cdot \mathrm{F}}.$$

In some cases the forces may have no single resultant, $\Sigma \cdot \mathrm{F}$ being $=0$; and then, unless the forces balance each other completely, their resultant is a couple, whose axis, direction, and moment, are found as follows:—

Let $$\mathrm{M}_x=\Sigma \cdot y\,\mathrm{F};\ \mathrm{M}_y=-\Sigma \cdot x\,\mathrm{F};$$

be the moments of the pair of partial resultant couples about the axes O X and O Y respectively. From O, along those axes, set off two lines representing respectively M_x and M_y; that is to say, proportional to those moments in length, and pointing in the direction from which those couples must respectively be viewed in order that they may appear right-handed. Complete the rectangle whose sides are those lines; its diagonal will represent the axis, direction, and moment of the final resultant couple. Let M_r be the moment of this couple; then

$$\mathrm{M}_r=\sqrt{\left\{\mathrm{M}_x^2+\mathrm{M}_y^2\right\}}$$

and if θ be the angle which its axis makes with O X,

$$\cos\theta=\frac{\mathrm{M}_x}{\mathrm{M}_r}.$$

234. To find the Centre of Parallel Forces.—Let O in fig. 95 be any convenient point, taken as the origin of co-ordinates, and O X, O Y, O Z, three axes of co-ordinates at right angles to each other.

Let A be any one of the points to which the system of parallel forces in question is applied. From A draw x parallel to O X, and perpendicular to the plane Y Z, y parallel to O Y, and perpendicular to the plane Z X, and z parallel to O Z, and perpendicular to the plane X Y. ∴ x, y, and z are the rectangular co-ordinates of A, which, being known, the position of A is determined. Let F denote either the magnitude of the force applied at A, or any magnitude proportional to that magnitude. x, y, z, and F are supposed to be known for every point of the given system of points.

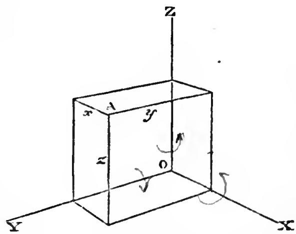


Fig. 95.

The position of the centre of parallel forces depends solely on the *proportionate* magnitudes of the parallel forces, not on their absolute magnitudes, nor on the *angular positions of their lines of actions;* so that for any system of parallel forces another may be substituted in any angular position : this is the statement of the principle of the centre of parallel forces given at Article 201, page 119. This is evident since, in considering the relations of parallel forces, they are not considered with reference to any particular plane, and hence these relations must hold for any plane.

First, conceive all the parallel forces to act in lines parallel to the plane Y Z. Then the distance of their resultant, and of the centre of parallel forces from that plane is

$$x_r = \frac{\Sigma \cdot x\,F}{\Sigma \cdot F} \text{..............................(1)}$$

Secondly, conceive all the parallel forces to act in lines parallel to the plane Z X. Then the distance of their resultant, and of the centre of parallel forces from that plane is

$$y_r = \frac{\Sigma \cdot y\,F}{\Sigma \cdot F} \text{..............................(2.)}$$

Thirdly, conceive all the parallel forces to act in lines parallel to the plane X Y. Then the distance of their resultant, and of the centre of parallel forces from that plane is

$$z_r = \frac{\Sigma \cdot z\,F}{\Sigma \cdot F} \text{..............................(3.)}$$

If the forces have no single resultant, so that $\Sigma \cdot F = 0$, there is no centre of parallel forces. This may be the case with pressures, but not with weights.

If the parallel forces applied to a system of points are all equal and in the same direction, it is obvious that the distance of the centre of parallel forces from any given plane is simply the mean of the distances of the points of the system from that plane.

SECTION 4.—OF ANY SYSTEM OF FORCES.

235. **Resultant and Balance of any System of Forces in One Plane.**—Let the plane be that of the axes O X and O Y in fig. 95; and in looking from Z towards O, let Y lie to the right of X, so that rotation from X towards Y shall be right-handed. Let x and y be the co-ordinates of the point of application of one of the forces, or of any point in its line of action, relatively to the assumed origin and axes. Resolve each force into two rectangular components X and Y, as in Article 215, page 125; then the rectangular components of the resultant are $\Sigma \cdot X$ and $\Sigma \cdot Y$; its magnitude is given by the equation

$$R^2 = (\Sigma \cdot X)^2 + (\Sigma \cdot Y)^2, \ldots\ldots (1.)$$

and the angle α_r which it makes with O X is found by the equations

$$\cos \alpha_r = \frac{\Sigma \cdot X}{R}; \ \sin \alpha_r = \frac{\Sigma \cdot Y}{R} \ldots\ldots (2.)$$

This angle is acute or obtuse according as $\Sigma \cdot X$ is positive or negative; and it lies to the right or left of O X according as $\Sigma \cdot Y$ is positive or negative.

The perpendicular distance from 0 of the line of action of any force is $x \sin \alpha - y \cos \alpha$, and hence the resultant moment of the system of forces about the axis O Z is

$$M = \Sigma (x Y - y X), \ldots\ldots (3.)$$

and is right or left-handed according as M is positive or negative.

The perpendicular distance of the resultant force R from O is

$$L = \frac{M}{R} \ldots\ldots (4.)$$

Let x_r and y_r be the co-ordinates of any point in the line of action of that resultant; then the equation of that line is*

$$x_r \Sigma \cdot Y - y_r \Sigma \cdot X = M. \ldots\ldots (5.)$$

* The method of obtaining this result by Co-ordinate Geometry is the

If $M=0$ the resultant acts through the origin O; if M has magnitude, and $R=0$ (in which case $\Sigma \cdot X=0$, $\Sigma \cdot Y=0$) the resultant is a couple. The conditions of equilibrium of the system of forces are

$$\Sigma \cdot X=0; \ \Sigma \cdot Y=0; \ M=0. \ \dots\dots\dots\dots\dots (6.)$$

236. **Resultant and Balance of any System of Forces.**—To find the resultant and the conditions of equilibrium of any system of forces acting through any system of points, the forces and points are to be referred to three rectangular axes of co-ordinates.

As before, let O in fig. 95, p. 133, denote the origin of co-ordinates, and O X, O Y, O Z, the three rectangular axes: and let them be arranged so that in looking from

$$\left.\begin{matrix} X \\ Y \\ Z \end{matrix}\right\} \text{towards O, rotation from} \left\{\begin{matrix} \text{Y towards Z} \\ \text{Z towards X} \\ \text{X towards Y} \end{matrix}\right\}$$

shall appear right-handed.

Let X, Y, Z, denote the rectangular components of any one of the forces; x, y, z, the co-ordinates of a point in its line of action.

Taking the algebraical sums of all the forces which act along the same axes, and of all the couples which act round the same axes,

following:—Let $OC=L$, $AB=R$, $\angle XAB=\alpha_r$; and let $EG=x_r$ and $OG=y_r$ be the co-ordinates of the point E. Then by Trigonometry $\sin \alpha_r=\sin OAC=\cos COA=\sin DOG=\cos DGO=\sin EGF$ and $-\cos \alpha_r=\cos OAC=\sin COA=\cos DOG$.

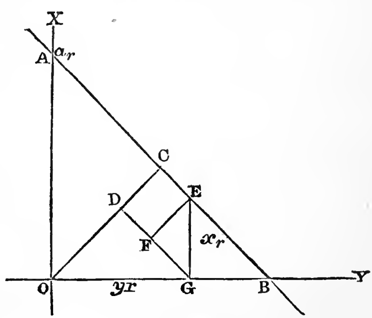


Fig. 96.

$$L=DC+OD=FE+OD$$
$$=EG\cdot\sin EGF+OG\cdot\cos DOG$$
$$=x_r\cdot\sin\alpha_r-y_r\cdot\cos\alpha_r$$

multiplying by R

$$L\cdot R=M=x_r\cdot R\cdot\sin\alpha_r-y_r\cdot R\cdot\cos\alpha_r$$
$$=x_r\cdot\Sigma Y-y_r\cdot\Sigma X$$

by substituting the values in Equation 2 *supra*.

the six following quantities are found, which compose the resultant of the given system of forces:—

Forces.

$$\Sigma \cdot X; \ \Sigma \cdot Y; \ \Sigma \cdot Z; \ \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Couples.

$$\left.\begin{array}{ll} \text{about } O X; & M_1 = \Sigma (y Z - z Y); \\ \quad " \quad O Y; & M_2 = \Sigma (z X - x Z); \\ \quad " \quad O Z; & M_3 = \Sigma (x Y - y X); \end{array}\right\} \ldots\ldots\ldots\ldots (2.)$$

found as already explained in Article 235.

The three forces are equivalent to a single force

$$R = \sqrt{\left\{ (\Sigma \cdot X)^2 + (\Sigma \cdot Y)^2 + (\Sigma \cdot Z)^2 \right\}} \ldots\ldots\ldots (3.)$$

acting through O in a line which makes with the axes the angles given by the equations

$$\cos \alpha = \frac{\Sigma \cdot X}{R}; \ \cos \beta = \frac{\Sigma \cdot Y}{R}; \ \cos \gamma = \frac{\Sigma \cdot Z}{R} \ldots\ldots\ldots (4.)$$

The three couples, M_1, M_2, M_3, are equivalent to one couple, whose magnitude is given by the equation

$$M = \sqrt{(M_1^2 + M_2^2 + M_3^2)}, \ \ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

and whose axis makes with the axes of co-ordinates the angles given by the equations

$$\cos \lambda = \frac{M_1}{M}; \ \cos \mu = \frac{M_2}{M}; \ \cos \nu = \frac{M_3}{M}, \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

in which $\left\{\begin{array}{c} \lambda \\ \mu \\ \nu \end{array}\right\}$ denote respectively the angles made by the axis of M with $\left\{\begin{array}{c} O X \\ O Y \\ O Z \end{array}\right\}$

The *conditions of equilibrium* of the system of forces may be expressed in either of the two following forms:—

$$\Sigma \cdot X = 0; \ \Sigma \cdot Y = 0; \ \Sigma \cdot Z = 0; \ M_1 = 0; \ M_2 = 0; \ M_3 = 0; \ (7.)$$

or

$$R = 0; \ M = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.)$$

When the system is not balanced, its resultant may fall under one or other of the following cases:—

Case I.—*When* $M = 0$, the resultant is the single force R acting through O.

CASE II.—*When the axis of* M *is at right angles to the direction of* R,—a case expressed by the following equation:—

$$\cos \alpha \cos \lambda + \cos \beta \cos \mu + \cos \gamma \cos \nu = 0; \ldots\ldots (9.)$$

(an equation of Co-ordinate Geometry)

the resultant of M and R is a single force equal and parallel to R, acting in a plane perpendicular to the axis of M, and at a perpendicular distance from O given by the equation

$$L = \frac{M}{R} \ldots\ldots\ldots\ldots\ldots\ldots (10.)$$

CASE III. *When* $R = 0$, there is no single resultant; and the only resultant is the couple M.

CASE IV. *When the axis of* M *is parallel to the line of action of* R, that is, when either

$$\lambda = \alpha; \ \mu = \beta; \ \nu = \gamma, \ldots\ldots\ldots\ldots (11.)$$

or

$$\lambda = -\alpha; \ \mu = -\beta; \ \nu = -\gamma; \ldots\ldots\ldots (12.)$$

there is no single resultant; and the system of forces is equivalent to the force R and the couple M, being incapable of being farther simplified.

CASE V.—*When the axis of* M *is oblique to the direction of* R, making with it the angle given by the equation

$$\cos \theta = \cos \lambda \cos \alpha + \cos \mu \cos \beta + \cos \nu \cos \gamma, \ldots (13.)$$

the couple M is to be resolved into two rectangular components, viz:—

M sin θ round an axis perpendicular to R, and in the plane containing the direction of R and of the axis of M;
M cos θ round an axis parallel to R. (14.)

The force R and the couple M sin θ are equivalent, as in Case II., to a single force equal and parallel to R, whose line of action is in a plane perpendicular to that containing R and axis of M, and whose perpendicular distance from O is

$$L = \frac{M \sin \theta}{R} \ldots\ldots\ldots\ldots\ldots\ldots (15.)$$

The couple M cos θ, whose axis is parallel to the line of action of R, is incapable of further combination.

Hence it appears finally, that every system of forces which is not self-balanced, is equivalent either, (A); to a single force, as in

Cases I. and II. (B); to a couple, as in Case III. (C); to a force, combined with a couple whose axis is parallel to the line of action of the force, as in Cases IV. and V. This can occur with inclined forces only; for the resultant of any number of parallel forces is either a single force or a couple.

237. **Parallel Projections or Transformations in Statics.**—If two figures be so related, that for each point in one there is a corresponding point in the other, and that to each pair of equal and parallel lines in the one, there corresponds a pair of equal and parallel lines in the other, those figures are said to be PARALLEL PROJECTIONS of each other.

The relations between such a pair of figures is expressed algebraically as follows:—Let any figure be referred to axes of co-ordinates, whether rectangular or oblique; let x, y, z, denote the co-ordinates of any point in it, which may be denoted by A: let a second figure be constructed from a second set of axes of co-ordinates, either agreeing with, or differing from, the first set as to rectangularity or obliquity; let x', y', z', be the co-ordinates in the second figure, of the point A′ which corresponds to any point A in the first figure, and let those co-ordinates be so related to the co-ordinates of A, that for each pair of corresponding points, A, A′, in the two figures, the three pairs of corresponding co-ordinates shall bear to each other three constant ratios, such as

$$\frac{x'}{x} = a; \ \frac{y'}{y} = b; \ \frac{z'}{z} = c;$$

then are those two figures parallel projections of each other.

For example, all circles and ellipses are parallel projections of each other; so are all spheres, spheroids, and ellipsoids; so are all triangles; so are all triangular pyramids; so are all cylinders; so are all cones.

The following are the geometrical properties of parallel projections which are of most importance in statics:—

I. A parallel projection of a system of three points, lying in one straight line and dividing it in a given proportion, is also a system of three points, lying in one straight line and dividing it in the same proportion.

II. A parallel projection of a system of parallel lines, whose lengths bear given ratios to each other, is also a system of parallel lines whose lengths bear the same ratios to each other.

III. A parallel projection of a closed polygon is a closed polygon.

IV. A parallel projection of a parallelogram is a parallelogram.

V. A parallel projection of a parallelopiped is a parallelopiped.

VI. A parallel projection of a pair of parallel plane surfaces, whose areas are in a given ratio, is also a pair of parallel plane surfaces, whose areas are in the same ratio.

VII. A parallel projection of a pair of volumes having a given ratio, is a pair of volumes having the same ratio.

The following are the mechanical properties of parallel projections in connection with the principles set forth in this section:—

VIII. If two systems of points be parallel projections of each other; and if to each of those systems there be applied a system of parallel forces bearing to each other the same system of ratios, then the *centres of parallel forces* for those two systems of points will be parallel projections of each other, mutually related in the same manner with the other pairs of corresponding points in the two systems.

IX. If a *balanced system of forces* acting through any system of points be represented by a system of lines, then will any parallel projection of that system of lines represent a balanced system of forces; and if any two systems of forces be represented by lines which are parallel projections of each other, the lines, or sets of lines, representing their *resultants*, are corresponding parallel projections of each other,—it being observed that *couples* are to be represented by pairs of lines, as pairs of opposite forces, or by areas, and not by single lines along their axes.

CHAPTER III.

DISTRIBUTED FORCES.

SECTION 1.—CENTRES OF GRAVITY.

238. **Centre of Gravity of a Symmetrical Homogeneous Body.**—If a body is *homogeneous*, or of equal specific gravity throughout, and so far *symmetrical* as to have a *centre of figure*; that is, a point within the body, which bisects every diameter of the body drawn through it, that point is also the centre of gravity of the body.

Amongst the bodies which answer this description, are the sphere, the ellipsoid, the circular cylinder, the elliptic cylinder, prisms whose bases have centres of figure, and parallelopipeds, whether right or oblique.

239. **The Common Centre of Gravity of a Set of Bodies** whose several centres of gravity are known, is the *centre of parallel forces* for the weights of the several bodies, each considered as acting through its centre of gravity. (See Article 234, p. 133.)

240. **Planes of Symmetry—Axes of Symmetry.**—If a homogeneous body be of a figure which is *symmetrical* on either side of a given plane, the centre of gravity is in that plane. If two or more such *planes of symmetry* intersect in one line, or *axis of symmetry*, the centre of gravity is in that axis. If three or more planes of symmetry intersect each other in a point, that point is the centre of gravity.

241. To find the **Centre of Gravity of a Homogeneous Body of any Figure,** assume three rectangular co-ordinate planes in any convenient position, as in fig. 95, p. 133.

To find the distance of the centre of gravity of the body from one of those planes (for example, that of Y Z), conceive the body to be divided into indefinitely thin plane layers parallel to that plane. Let s denote the area of any one of those layers, and $d\,x$ its thickness, so that $s\,d\,x$ is the volume of the layer, and

$$V = \int s\,d\,x,$$

the volume of the whole body, being the sum of the volumes of the layers. Let x be the perpendicular distance of the centre of the layer $s\,d\,x$ from the plane of Y Z. Then the perpendicular distance x_o of the centre of gravity of the body from that plane is given by the equation

$$x_o = \frac{\int x\,s\,d\,x}{V}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

Find, by a similar process, the distances y_o, z_o, of the centre of gravity from the other two co-ordinate planes, and its position will be completely determined.

If the centre of gravity is previously known to be in a particular plane, it is sufficient to find by the above process its distances from *two* planes perpendicular to that plane and to each other.

If the centre of gravity is previously known to be in a particular line, it is sufficient to find its distance from *one* plane, perpendicular to that line.

242. **If the Specific Gravity of the Body Varies,** let w be the mean heaviness of the layer $s\,d\,x$, so that

$$W = \int w\,s\,d\,x,$$

is the weight of the body. Then

$$x_o = \frac{\int x w s d x}{W}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

243. **Centre of Gravity found by Addition.**—When the figure of a body consists of parts, whose respective centres of gravity are known, the centre of gravity of the whole is to be found as in Article 239.

244. **Centre of Gravity found by Subtraction.**—When the figure of a homogeneous body, whose centre of gravity is sought, can be made by taking away a figure whose centre of gravity is known from a larger figure whose centre of gravity is known also, the following method may be used:—

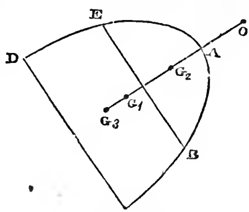


Fig. 97.

Let A C D be the larger figure, G_1 its known centre of gravity, W_1 its weight. Let A B E be the smaller figure, whose centre of gravity G_2 is known, W_2 its weight. Let E B C D be the figure whose centre of gravity G_3 is sought, made by taking away A B E from A C D, so that its weight is

$$W_3 = W_1 - W_2.$$

Join $G_1 G_2$; G_3 will be in the prolongation of that straight line beyond G_1. In the same straight line produced, take any point O as origin of co-ordinates. Make $\overline{O G_1} = x_1$; $\overline{O G_2} = x_2$, $\overline{O G_3}$ (the unknown quantity) $= x_3$.

Then

$$x_3 = \frac{x_1 W_1 - x_2 W_2}{W_1 - W_2} \text{..........................} (3.)$$

245. **Centre of Gravity Altered by Transposition.**—In fig. 98, let A B C D be a body of the weight W_0, whose centre of gravity G_0 is known. Let the figure of this body be altered, by transposing a part whose weight is W_1, from the position E C F to the position F D H, so that the new figure of the body is A B H E. Let G_1 be the original, and G_2 the new position of the centre of gravity of the transposed part. Then the centre of gravity of the whole body will be shifted to G_3, in a direction $G_0 G_3$ parallel to $G_2 G_1$, and through a distance given by the formula.

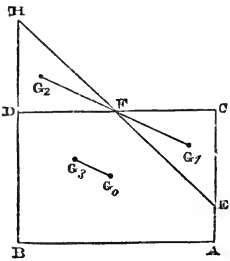


Fig. 98.

$$\overline{G_0 G_3} = \overline{G_1 G_2} \frac{W_1}{W_0} \text{..........................} (4.)$$

246. **Centre of Gravity found by Projection or Transformation.**—If the figures of two homogeneous bodies are parallel projections of each other, the centres of gravity of those two bodies are corresponding points in those parallel projections.

To express this symbolically,—as in Article 237, let x, y, z, be the co-ordinates, rectangular or oblique, of any point in the figure of the first body; x', y', z', those of the corresponding point in the second body; x_o, y_o, z_o, the co-ordinates of the centre of gravity of the first body; x'_o, y'_o, z'_o, those of the centre of gravity of the second body, then

$$\frac{x'_o}{x_o} = \frac{x'}{x}; \ \frac{y'_o}{y_o} = \frac{y'}{y}; \ \frac{z'_o}{z_o} = \frac{z'}{z}. \text{..................} (5.)$$

This theorem facilitates much the finding of the centres of gravity of figures which are parallel projections of more simple or more symmetrical figures.

For example, let it be supposed that the centre of gravity of a sector of a circular disc has been found (Case IX. Article 44), and let it be required to find the centre of gravity of a sector of an elliptic disc. In fig. 99, let A B′ A B′ be the ellipse, A O A = 2 a, and B′ O B′ = 2 b, its axes, and C′ O D′ the sector whose centre of gravity is required. About the centre of the ellipse, O, describe the circle, A B A B, whose radius is the semi-axis major a. Through C′ and D′ respectively draw E C′ C and F D′ D, parallel to O B, and cutting the circle in C and D respectively; the circular sector C O D is the parallel projection of the elliptic sector C′ O D′. Let G be the centre of gravity of the sector of the circular disc, its co-ordinates being

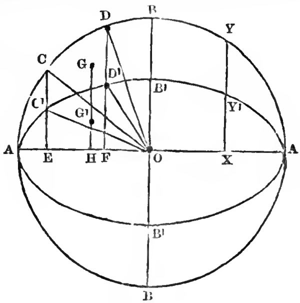


Fig. 99.

$$\overline{\mathrm{O\,H}} = x_o;\ \overline{\mathrm{H\,G}} = y_o.$$

Then the co-ordinates of the centre of gravity G′ of the sector of the elliptic disc are

$$\left.\begin{array}{l}\overline{\mathrm{O\,H}} = x'_o = x_o; \\ \overline{\mathrm{H\,G'}} = y'_o = \dfrac{b\,y_o}{a};\end{array}\right\} \text{.....................} (6.)$$

247. **Centre of Gravity found Experimentally.**—The centre of gravity of a body of moderate size may be found approximately by experiment, by hanging it up successively by a single cord in two different positions, and finding the single point in the body which in both positions is intersected by the axes of the cord.

Section 2.—Of Stress.

248. **Stress—its Intensity.**—The word Stress has been adopted as a general term to comprehend various forces which are exerted between contiguous bodies, or parts of bodies, and which are distributed over the surface of contact of the masses between which they act.

The Intensity of a stress is its amount in units of weight, divided by the extent of the surface over which it acts, in units of area.

The following table gives a comparison of various units in which the intensity of stress is expressed :—

	Pounds on the square foot.	Pounds on the square inch.
One pound on the square inch,....	144	1
One pound on the square foot,.....	1	$\frac{1}{144}$
One inch of mercury (that is, weight of a column of mercury at 32° Fahr., one inch high), ...	70·73	0·4912
One foot of water (at 39°·1 Fahr.),	62·425	0·4335
One inch of water (at 39°·1 Fahr.),	5·2021	0·036125
One foot of water (at 62° Fahr.),...	62·355	0·43302
One inch of water (at 62° Fahr.),...	5·19625	0·036085
One atmosphere, of 29·922 inches of mercury, or 760 millimètres,	2116·4	14·7
One foot of air, at 32° Fahr., and under the pressure of one atmosphere, ...	0·080728	0·0005606
One kilogramme on the square mètre, ...	0·20481	0·00142228
One kilogramme on the square millimètre, ...	204810	1422·28
One millimètre of mercury,........	2·7847	0·01934

249. **Classes of Stress.**—The various kinds of stress may be thus classed :—

I. *Thrust*, or *Pressure*, is the force which acts between two contiguous bodies, or parts of a body, when each pushes the other from itself.

II. *Pull*, or *Tension*, is the force which acts between two contiguous bodies, or parts of a body, when each draws the other towards itself.

Pressure and tension may be either *normal* or *oblique*, relatively to the surface at which they act.

III. *Shear*, or *Tangential Stress*, is the force which acts between two contiguous bodies, or parts of a body, when each draws the other sideways, in a direction parallel to their surface of contact.

In expressing a Thrust and a Pull in parallel directions algebraically, if one is treated as positive, the other must be treated as negative. The choice of the positive or negative sign for either is a matter of convenience.

The word "*Pressure*," although, strictly speaking, equivalent to "*thrust*," is sometimes applied to *stress* in general; and when this is the case, it is to be understood that thrust is treated as positive.

The following are the processes for finding the *magnitude of the resultant* of a stress distributed over a plane surface, and the *centre* *of stress;* that is, the point where the line of action of that resultant cuts the plane surface:—

250. **In Stress of Uniform Intensity,** the magnitude of the resultant is the product of that intensity and the area of the surface; and the centre of stress is at the centre of magnitude of the surface. Or in symbols, let S be the area of the surface, p the intensity of the stress, P its resultant, then—

$$P = p\,S.$$

251. **In Stress of Varying Intensity, but of One Sign,** there is all tension, or all pressure, or all shear in one direction.

In fig. 100, let A A be the given plane surface at which the stress acts; O X, O Y, two rectangular axes of co-ordinates in its plane; O Z, a third axis perpendicular to that plane. Conceive a solid to exist, bounded at one end by the given plane surface A A, laterally by a cylindrical or prismatic surface generated by the motion of a straight line parallel to O Z round the outline of A A, and at the other end by a surface B B, of such a figure, that its ordinate z at any point shall be proportional to the intensity of the stress at the point a of the surface A A from which that ordinate proceeds, as shewn by the equation

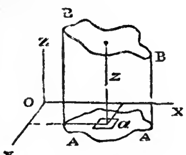


Fig. 100.

$$z = \frac{p}{w} \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

where p represents the intensity of the stress and w the heaviness, or weight per unit.

Conceive the surface A A to be divided into an indefinite number of small rectangular areas, each denoted by $d\,x\,d\,y$, and so small that the stress on each is sensibly uniform; the entire area being

$$S = \int\int d\,x\,d\,y.$$

The volume of the ideal solid will be

$$V = \int\int z \cdot d\,x\,d\,y \ldots\ldots\ldots\ldots(2.)$$

So that if it be conceived to consist of a material whose heaviness is $w = \frac{p}{z}$, the amount of the stress will be equal to the weight of the solid; that is to say,

$$P = \int\int p\,d\,x\,d\,y = w\,V \ldots\ldots\ldots\ldots(3.)$$

The *centre of stress* is the point on the surface A A perpendicularly opposite the centre of gravity of the ideal solid.

The simplest, and at the same time the commonest, case of this kind is where the stress is *uniformly-varying;* that is, where its intensity at a given point is simply proportional to the perpendicular distance of that point from a given straight line in the plane of the surface A A. To express this symbollically, take the straight line in question for the axis O Y; conceive the substance to be divided into bands by lines parallel to O Y; let y denote the length of one of these bands, and $d\,x$ its breadth, so that $y\,d\,x$ is its area, and $S = \int y\,d\,x$ the area of the whole surface. Let x be the perpendicular distance of the centre of a band from the *line of no stress* O Y, and let the intensity of the stress there be

$$p = a\,x; \quad \text{(4.)}$$

a being a constant coefficient; then the amount or resultant of the stress is

$$P = \int p\,y\,d\,x = a \int x\,y\,d\,x; \quad \text{(5.)}$$

and the perpendicular distance of the centre of stress from O Y is

$$x_0 = \frac{\int p\,x\,y\,d\,x}{\int p\,y\,d\,x} = a \frac{\int x^2\,y\,d\,x;}{P} \quad \text{(6.)}$$

252. **In Stress of Contrary Signs,** for example, pressure at one part of the surface and tension at another, the resultants and centres of stress of the pressure and tension are to be found separately. Those partial resultants are then to be treated as a pair of parallel forces acting through the two respective centres of stress; their final resultant will be equal to their difference, if any, acting through a point found as in Article 226, page 128.

If the total pressure and total tension are equal to each other, they have no single resultant and no single centre of stress: their resultant being a couple, whose moment is equal to the total stress of either kind multiplied by the perpendicular distance between the resultant of the pressure and the resultant of the tension.

SECTION 3.—PRINCIPLES OF HYDROSTATICS AND INTERNAL STRESS OF SOLIDS.

253. **Pressure and Balance of Fluids: Principles of Hydrostatics.**—*Fluid* is a term opposed to *solid*, and comprehending the liquid and gaseous conditions of bodies. The property common to the liquid and the gaseous conditions is that of *not tending to preserve a definite shape*, and the possession of this property by a body in perfection throughout all its parts, constitutes that body *a perfect fluid.*

A necessary consequence of that property is the following principle, which is the foundation of the whole science of hydrostatics:—

I. *In a perfect fluid, when still, the pressure exerted at a given point is normal to the surface on which it acts, and of equal intensity for all positions of that surface.*

The following are some of the most useful consequences of that principle:—

II. *A surface of equal pressure in a still fluid mass is everywhere perpendicular to the direction of gravity;* that is, horizontal throughout. In other words, the pressure at all points at the same level is of equal intensity.

III. *The intensity of the pressure at the lower of two points in a still fluid mass is greater than the intensity at the higher point, by an amount equal to the weight of a vertical column of the fluid whose height is the difference of elevation of the points, and base an unit of area.*

To express this symbolically, let p_0 denote the intensity of the pressure at the higher of two points in a fluid mass, and p_1 the intensity at a point whose vertical depth below the former point is x. Let w be the *mean heaviness* of the layer of fluid between those two points; then

$$p_1 = p_0 + w\,x \quad \text{.............................(1.)}$$

In a gas, such as air, w varies, being nearly proportional to p; but in a liquid, such as water, the variations of w are too small to be considered in practical cases.

For example, let the upper of the two points be the surface of a mass of water where it is exposed to the air; then p_0 is the atmospheric pressure; let the depth x of the second point below the surface be given in feet, and let the temperature be 39°·1; then

$$p_1 \text{ in lbs. on the square foot} = p_0 + 62{\cdot}425\,x \text{.........(2.)}$$

In many questions relating to engineering, the pressure of the atmosphere may be left out of consideration, as it acts with sensibly equal intensity on all sides of the bodies exposed to it, and so balances its own action. The pressure calculated, in such cases, is the *excess* of the pressure of the water above the atmospheric pressure, which may be thus expressed,—

$$p' = p_1 - p_0 = 62{\cdot}425\,x \text{ nearly}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

IV. The pressure of a liquid on a *floating* or *immersed body*, is equal to the weight of the volume of fluid displaced by that body; and the resultant of that pressure acts vertically upwards through the centre of gravity of that volume; which centre of gravity is called the "*centre of buoyancy.*"

V. The pressure of a liquid against *a plane surface immersed in it* is perpendicular to that surface in direction: its magnitude is equal to the weight of a volume of the liquid, found by multiplying the area of the surface by the depth to which its centre of gravity is immersed.

VI. The *centre of pressure* on such a surface, if the surface is horizontal, coincides with its centre of gravity; if the surface is vertical or sloping, the centre of pressure is always below the centre of gravity of the surface, and is found by considering that the pressure is an *uniformly-varying* stress, whose intensity at a given point varies as the distance of that point from the line where the given plane surface (produced if necessary) intersects the upper surface of the liquid.

To express the last two principles by symbols in the case in which the pressed surface is vertical or sloping, let the line where the plane of that surface cuts the upper surface of the liquid be taken as the axis O Y. Let θ denote the angle of inclination of the pressed surface to the horizon. Conceive that surface to be divided by parallel horizontal lines into an indefinite number of narrow bands. Let y be the length of any one of those bands, $d\,x$ its breadth, x the distance of its centre from O Y; then $y\,d\,x$ is its area, $x \sin\theta$ the depth at which it is immersed; and if w be the weight of unity of volume of the fluid, the intensity of the pressure on that band is

$$p = w\,x \sin\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

The whole area of the pressed surface, being the sum of the areas of all the bands, is $S = \int y\,d\,x$; the whole pressure upon it is

$$P = \int p\,y\,d\,x = w \sin\theta \int x\,y\,d\,x;\ldots\ldots\ldots\ldots\ldots(5.)$$

the mean intensity of the pressure is

$$\frac{P}{S} = \frac{\int p\,y\,d\,x}{\int y\,d\,x} = w \sin\theta \frac{\int x\,y\,d\,x}{\int y\,d\,x};\ldots\ldots\ldots\ldots\ldots(6.)$$

and the distance of the centre of pressure from O Y is

$$x_0 = \frac{\int x\,p\,y\,d\,x}{\mathrm{P}} = \frac{\int x^2\,y\,d\,x}{\int x\,y\,d\,x} \quad \text{.................(7.)}$$

For example, let the sloping pressed surface be rectangular, like a sluice, or the back of a reservoir-wall; and in the first instance, let it extend from the surface of a mass of water down to a distance x_1, measured along the slope, so that its lower edge is immersed to the depth $x_1 \sin\theta$. Then its centre of gravity is immersed to the depth $x_1 \sin\theta \div 2$, and the mean intensity of the pressure in lbs. on the square foot, is

$$\frac{\mathrm{P}}{\mathrm{S}} = \frac{62 \cdot 4\, x_1 \sin\theta}{2} \quad \text{.........................(8.)}$$

The breadth y is constant; so that the area of the surface is $\mathrm{S} = x_1\,y$; and the total pressure is

$$\mathrm{P} = \frac{62 \cdot 4\, x_1^2\, y \sin\theta}{2} \quad \text{......................(9.)}$$

The distance of the centre of pressure from the upper edge is

$$x_0 = \frac{2}{3}\; x_1 \quad \text{.........................(10.)}$$

Next, let the upper edge, instead of being at the surface of the water, be at the distance x_2 from it, so as to be immersed to the depth $x_2 \sin\theta$. Then the centre of gravity of the pressed surface is immersed to the depth $(x_1 + x_2) \sin\theta \div 2$, and the mean intensity of the pressure upon it, in lbs. on the square foot, is

$$\frac{\mathrm{P}}{\mathrm{S}} = \frac{62 \cdot 4\,(x_1 + x_2) \sin\theta}{2}; \quad \text{................(11.)}$$

the area of the surface is $(x_1 - x_2)\,y$, and the total pressure on it

$$\mathrm{P} = \frac{62 \cdot 4\,(x_1^2 - x_2^2)\,y \sin\theta}{2} \quad \text{.................(12.)}$$

The distance of the centre of pressure from the line O Y is

$$x_0 = \frac{2}{3} \cdot \frac{x_1^3 - x_2^3}{x_1^2 - x_2^2} \quad \text{......................(13.)}$$

254. **Compound Internal Stress of Solids.**—If a body be conceived to be divided into two parts by an ideal plane traversing it in any direction, the force exerted between those two parts at the plane of division is an *internal stress.*

According to the principles stated in the preceding article, the internal stress at a given point in a fluid is normal and of equal intensity for all positions of the ideal plane of division. In a solid body, on the other hand, the stress may be either normal, oblique, or shearing; and it may vary in direction and intensity, as the position of the ideal plane of division varies.

255. **Conjugate Stresses—Principal Stresses.**—If two planes traverse a point in a body, and the direction of the stress on the first plane is parallel to the second plane, then the direction of the stress on the second plane is parallel to the first plane. Such a pair of stresses are said to be *conjugate;* and if they are both normal to their planes of application (and consequently perpendicular to each other) they are called *principal stresses.* Three conjugate stresses, or three principal stresses, may act through one point; but in the present treatise it is sufficient to consider two.

Fig. 101 represents a pair of conjugate oblique tensions acting in the direction X X and Y Y through a prismatic particle A B C D.

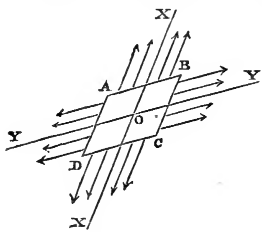


Fig. 101.

The rectangular directions in which principal stresses—that is, direct pulls and thrusts—act, through a given point in a solid, are called *axes of stress.*

In a fluid, the stress at a given point being of equal intensity in all directions, every direction has the property of an axis of stress. A solid *may* be in the same condition with a fluid as to stress; but it may also have the principal stresses at a given point of different intensities. In a mass of loose grains, the ratio of those intensities has a limit depending on friction:—in a firm continuous solid, the principal stresses at a point may bear any ratio to each other, and may be either of the same or of opposite kinds.

256. **The Shearing Stress,** on two planes traversing a point in a solid at right angles to each other, is of equal intensity.

257. **A Pair of Equal and Opposite Principal Stresses**; that is, a pull and a thrust of equal intensity acting through a particle of a solid in directions at right angles to each other, are equivalent to a pair of shearing stresses of the same intensity on a pair of planes at right angles to each other, and making angles of 45° with the first pair of planes.

258. **Combination of any Two Principal Stresses.**

Problem.—A pair of principal stresses of any intensities, and of the same or opposite kinds, being given, it is required to find the direction and intensity of the stress on a plane in any position at right angles to the plane parallel to which the two principal stresses act.

Let O X and O Y (figs. 102 and 103) be the directions of the two principal stresses; O X being the direction of the greater stress.

Let p_1 be the intensity of the greater stress;
and p_2 that of the less.

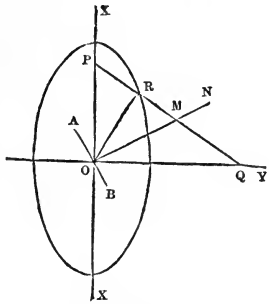


Fig. 102.

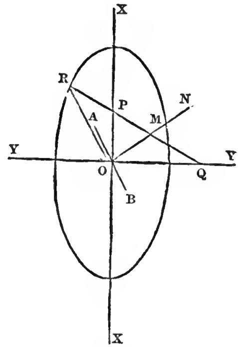


Fig. 103.

The kind of stress to which each of these belongs, pull or thrust, is to be distinguished by means of the algebraical signs. If a pull is considered as positive, a thrust is to be considered as negative, and *vice versâ*. It is in general convenient to consider that kind of stress as positive to which the greater principal stress belongs. Fig. 102 represents the case in which p_1 and p_2 are of the same kind; fig. 103 the case in which they are of opposite kinds. In all the following equations, the sign of p_2 is held to be *implied* in that symbol; that is to say, when p_2 is of the contrary kind to p_1, the sign applied to its arithmetical value, in computing by means of the equations, is to be reversed.

Let A B be the plane on which it is required to ascertain the direction and intensity of the stress, and O N a normal to that plane, making with the axis of greatest stress the angle

$$\angle \text{X O N} = \widehat{xn}.$$

On O N take $\overline{\text{O M}} = \frac{p_1 + p_2}{2}$; this will represent a normal stress on A B of the same kind with the greater principal stress, and of an intensity which is a mean between the intensities of the two principal stresses.

Through M draw P M Q, making with the axes of stress the same angles which O N makes, but in the opposite direction; that is to say, take $\overline{MP} = \overline{MQ} = \overline{MO}$. On the line thus found set off from M towards the axis of greatest stress, $\overline{MR} = \frac{p_1 - p_2}{2}$.

Join $\overline{OR}$. Then will that line represent the direction and intensity of the stress on A B.

In fig. 102, p_1 and p_2 are represented as being of the same kind; and $\overline{MR}$ is consequently less than $\overline{OM}$, so that $\overline{OR}$ falls on the same side of O X with O N; that is to say, $\widehat{nr} < \widehat{xn}$. In fig. 103, p_1 and p_2 are of opposite kinds, $\overline{MR}$ is greater than $\overline{OM}$, and O R falls on the opposite side of O X to O M; that is to say, $\widehat{nr} > \widehat{xn}$.

The locus of the point M is a circle of the radius $\frac{p_1 + p_2}{2}$, and that of the point R, an ellipse whose semi-axes are p_1 and p_2, and which may be called the Ellipse of Stress, because its semi-diameter in any direction represents the intensity of the stress in that direction.

259. **Deviation of Principal Stresses by a Shearing Stress.**—Problem. Let p_x and p_y denote the original intensities of a pair of principal stresses acting at right angles to each other through one particle of a solid. Suppose that with these there is combined a shearing stress of the intensity q, acting in the same plane with the original pulls or thrusts; it is required to find the new intensities and new directions of the principal stresses.

To assist the conception of this problem, the original stresses referred to are represented in fig. 104, as acting through a particle of the form of a square prism. The principal stresses, both original and new, are represented as tensions, although any or all of them might be pressures. In the formulæ annexed, tensions are considered positive, pressures negative; angles lying to the right of A A are considered as positive, to the left as negative; and a shearing stress is considered as positive or negative according as it tends to make the upper right-hand and lower left-hand corner of the square particle acute or obtuse.

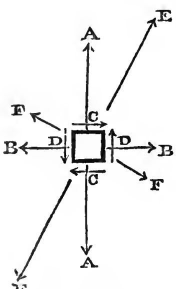


Fig. 104.

The arrows A A represent the greater original tension p_x; the arrows B B, the less original tension p_y; C, C, D, D, represent the positive shear of the intensity q, as acting at the four faces of the particle. The combination of this shear with the original tensions is equivalent to a new pair of principal tensions, oblique to the original pair. The greater new principal tension, p_1, is represented by the arrows E, E; it deviates to the right of p_x through an angle which will be denoted by θ. The less new principal tension p_1 is represented by the arrows F, F; it deviates through the same angle to the right of p_y.

Then the intensities of the new principal stresses are given by the equations,

$$\left.\begin{aligned} p_1 &= \frac{p_x + p_y}{2} + \sqrt{\left\{\frac{(p_x - p_y)^2}{4} + q^2\right\}}; \\ p_2 &= \frac{p_x + p_y}{2} - \sqrt{\left\{\frac{(p_x - p_y)^2}{4} + q^2\right\}}; \end{aligned}\right\} \text{..........(3.)}$$

and the double of the angle of deviation by either of the following,

$$\tan 2\theta = \frac{2q}{p_x - p_y}; \text{ or } \operatorname{cotan} 2\theta = \frac{p_x - p_y}{2q} \text{............(4.)}$$

The greatest value of θ is 45°, when $p_x = p_y$.

The new principal stresses are to be conceived as acting normally on the faces of a new square prism.

260. **Parallel Projection of Distributed Forces.**—In applying the principles of parallel projection to distributed forces, it is to be borne in mind that those principles, as stated in Article 237, are applicable to lines representing the *amounts* or *resultants* of distributed forces, and *not their intensities.* The relations amongst the intensities of a system of distributed forces, whose resultants have been obtained by the method of projection, are to be arrived at by a subsequent process of dividing each projected resultant by the projected space over which it is distributed.

261. **Friction** is that force which acts between two bodies at their surface of contact so as to resist their sliding on each other, and which depends on the force with which the bodies are pressed together. It is a kind of shearing stress. The following law respecting the friction of solid bodies has been ascertained by experiment:—

The friction which a given pair of solid bodies, with their surfaces in a given condition, are capable of exerting, is simply proportional to the force with which they are pressed together.

If a body be acted upon by a force tending to make it slide on another, then so long as that force does not exceed the amount fixed by this law, the friction will be equal and opposite to it, and will balance it.

There is a limit to the exactness of the above law, when the pressure becomes so intense as to crush or indent the parts of the bodies at and near their surface of contact. At and beyond that limit the friction increases more rapidly than the pressure; but that limit ought never to be attained in any structure. For some substances, especially those whose surfaces are sensibly indented by a moderate pressure, such as timber, the friction between a pair of surfaces which have remained for some time at rest relatively to each other, is somewhat greater than that between the same pair of surfaces when sliding on each other. That excess, however, of the *friction of rest* over the *friction of motion*, is instantly destroyed by a slight vibration; so that the *friction of motion* is alone to be taken into account, as contributing to the stability of a structure.

The friction between a pair of surfaces is calculated by multiplying the force with which they are directly pressed together, by a factor called the *coefficient of friction*, which has a special value depending on the nature of the materials and the state of the surfaces. Let F denote the friction between a pair of surfaces; N, the force, in a direction perpendicular to the surfaces, with which they are pressed together; and f the coefficient of friction; then

$$F = f N \qquad (1.)$$

The coefficient of friction of a given pair of surfaces is the *tangent* of an angle called the *angle of repose*, being the greatest angle which an oblique pressure between the surfaces can make with a perpendicular to them, without making them slide on aceh other.

Let P denote the amount of an oblique pressure between two plane surfaces, inclined to their common normal at the angle of repose φ; then

$$F = f N = N \tan \varphi = P \sin \varphi = \frac{f P}{\sqrt{1 + f^2}} \qquad (2.)$$

The angle of repose is the steepest inclination of a plane to the horizon, at which a block of a given substance will remain balanced on it without sliding down.

The *intensity* of the friction between two surfaces bears the same proportion to the intensity of the pressure that the whole friction bears to the whole pressure.

The following is a table of the angle of repose φ, the coefficient of friction $f = \tan \varphi$, and its reciprocal $1 : f$, for various materials—condensed from the tables of General Morin, and other sources, and arranged in a few comprehensive classes. The values of those constants which are given in the table have reference to the *friction of motion*.

Surfaces.	φ	f	$\frac{1}{f}$
Dry masonry and brickwork, ...	31° to 35°	0·6 to 0·7	1·67 to 1·43
Masonry and brickwork with wet mortar, ...	25½°	0·47	2·1
Masonry and brickwork, with slightly damp mortar, ...	36½°	0·74	1·35
Wood on stone, ...	22°	about 0·4	2·5
Iron on stone, ...	35° to 16⅔°	0·7 to 0·3	1·43 to 3·33
Masonry on dry clay, ...	27°	0·51	1·96
,, on moist clay, ...	18¼°	0·33	3
Earth on earth, ...	14° to 45°	0·25 to 1·0	4 to 1
,, ,, dry sand, clay, and mixed earth, ...	21° to 37°	0·38 to 0·75	2·63 to 1·33
Earth on earth, damp clay, ...	45°	1·0	1
,, ,, wet clay, ...	17°	0·31	3·23
,, ,, shingle and gravel,	35° to 48°	0·7 to 1·11	1·43 to 0·9
Wood on wood, dry, ...	14° to 26½°	·25 to ·5	4 to 2
,, ,, soaped, ...	11⅓° to 2°	·2 to ·04	5 to 25
Metals on oak, dry, ...	26½° to 31°	·5 to ·6	2 to 1·67
,, ,, wet, ...	13½° to 14½°	·24 to ·26	4·17 to 3·85
,, ,, soapy, ...	11½°	·2	5
Metals on elm, dry, ...	11½° to 14	.2 to ·25	5 to 4
Bronze on lignum vitæ, constantly wet, ...	3° ?	·05 ?	20 ?
Hemp on oak, dry, ...	28°	·53	1·89
,, ,, wet, ...	18½°	·33	3
Leather on oak, ...	15° to 19½°	·27 to ·38	3·7 to 2·86
Leather on metals, dry, ...	29½°	·56	1·79
,, ,, wet, ...	20°	·36	2·78
,, ,, greasy, ...	13°	·23	4·35
,, ,, oily, ...	8½°	·15	6·67
Metals on metals, dry, ...	8½° to 11½°	·15 to ·2	6·67 to 5
,, ,, wet and clean, ..	16½°	·3	3·33
,, ,, damp and slimy,	8°	·14	7·14
Smooth surfaces, occasionally greased	4° to 4½°	·07 to ·08	14·3 to 12·5
Smooth surfaces, continually greased, ...	3°	·05	20
Smoothest and best greased surfaces,	1¾° to 2°	·03 to ·036	33·3 to 27·6

PART IV.

THEORY OF STRUCTURES.

CHAPTER I.

SUMMARY OF PRINCIPLES OF STABILITY AND STRENGTH.

Section 1.—Of Structures in General.

262. **A Structure** consists of portions of solid materials, put together so as to preserve a definite form and arrangement of parts, and to withstand external forces tending to disturb such form and arrangement. As the parts of a structure are intended to remain at rest relatively to each other, the forces which act on the whole structure, and on each of its parts, should be *balanced*, so that the mechanical principles on which the permanence and efficiency of structures depend for the most part belong to Statics, or the science of balanced forces.

The *materials* of a structure may be more or less stiff, like stone, timber, and metals, or loose, like earth.

In the present chapter are given a summary of mechanical principles applicable to structures.

263. **Pieces — Joints — Supports — Foundations.**—A structure consists of two or more solid bodies, called its *pieces*, which touch each other and are connected at portions of their surfaces, called *joints*. This statement may appear to be applicable to structures of stiff materials only; but, nevertheless, it comprehends masses of earth also, if they are considered as consisting of a very great number of very small pieces, touching each other at innumerable joints.

Although the pieces of a structure are fixed relatively to each other, the structure as a whole may be either fixed or movable relatively to the earth.

A fixed structure is supported on a part of the solid material of the earth, called the *foundation* of the structure; the pressures by which the structure is supported, being the resistances of the various parts of the foundation, may be more or less oblique.

A movable structure may be supported, as a ship, by floating in water, or as a carriage, by resting on the solid ground through wheels. When such a structure is actually in motion, it partakes to a certain extent of the properties of a machine; and the determination of the forces by which it is supported requires the consideration of kinetic as well as of statical principles; but when it is not in actual motion, though capable of being moved, the pressures which support it are determined by the principles of statics; and it is obvious that they have their resultant equal and directly opposed to the weight of the structure.

264. **The Conditions of Equilibrium of a Structure** are the three following:—

I. *That the forces exerted on the whole structure by external bodies shall balance each other.*—The forces to be considered under this head are—(1.) the *Attraction of the Earth*—that is, the *weight* of the structure; (2.) the *External Load*, arising from the pressures exerted against the structure by bodies not forming part of it nor of its foundation; (these two kinds of forces constitute the *gross* or *total load*); (3.) the *Supporting Pressures*, or resistance of the foundation. Those three classes of forces will be spoken of together as the *External Forces.*

II. *That the forces exerted on each piece of the structure shall balance each other.*—These consist of—(1.) the *Weight* of the piece, and (2.) the *External Load* on it, making together the *Gross Load;* and (3.) the *Resistances*, or forces exerted at the joints, between the piece under consideration and the pieces in contact with it.

III. *That the forces exerted on each of the parts into which each piece of the structure can be conceived to be divided shall balance each other.*—Suppose an ideal surface to divide any part of any one of the pieces of the structure from the remainder of the piece; the forces which act on the part so considered are—(1.) its weight, and (2.) (if it is at the external surface of the piece) the external force applied to it, if any, making together its *gross load;* (3.) the *stress*, or force, exerted at the ideal surface of division, between the part in question and the other parts of the piece.

265. **Stability, Strength, and Stiffness.**—It is necessary to the permanence of a structure, that the three foregoing conditions of equilibrium should be fulfilled, not only under one amount and one mode of distribution of load, but under all the variations of the load as to amount and mode of distribution which can occur in the use of the structure.

Stability consists in the fulfilment of the *first* and *second* conditions of equilibrium of a structure under all variations of the load within given limits. A structure which is deficient in stability gives way by the displacement of its pieces from their proper positions.

When a structure, or one of its parts, is *flexible*, like the chain of a suspension bridge, or in any other way free to move, its stability consists in a tendency to recover its original figure and position after having been disturbed.

Strength consists in the fulfilment of the *third* condition of equilibrium of a structure for all loads not exceeding prescribed limits; that is to say, the greatest internal stress produced in any part of any piece of the structure, by the prescribed greatest load, must be such as the material can bear, not merely without immediate breaking, but without such injury to its texture as might endanger its breaking in the course of time.

A piece of a structure may be rendered unfit for its purpose, not merely by being broken, but by being stretched, compressed, bent, twisted, or otherwise strained out of its proper shape. It is necessary, therefore, that each piece of a structure should be of such dimensions that its alteration of figure under the greatest load applied to it shall not exceed given limits. This property is called *stiffness*, and is so connected with strength that it is necessary to consider them together.

Section 2.—Balance and Stability of Frames, Chains, and Blocks.

266. **A Frame** is a structure composed of bars, rods, links, or cords, attached together or supported by *joints*, such as occur in carpentry, in frames of metal bars, and in structures of ropes and chains, fixing the ends of two or more pieces together, but offering little or no resistance to change in the relative angular positions of those pieces. In a joint of this class, the *centre of resistance*, or point through which the resultant of the resistance to displacement of the pieces connected at the joint acts, is at or near the middle of the joint, and does not admit of any variation of position consistently with security.

The *line of resistance* of a frame is a line traversing the *centres of resistance* of the joints, and is in general a polygon, having its angles at these centres.

267. **A Single Bar** in a frame may act as a Tie, a Strut, or a Beam.

I. A *tie* has equal and directly opposite forces applied to its two ends, acting outwards, or *from* each other. The bar is in a state of *tension*, and the stress exerted between any two divisions of it is a *pull*, equal and opposite to the applied forces. A *rope* or *chain* will answer the purpose of a tie.

The equilibrium of a movable tie is stable; for if its angular position be deviated, the forces applied to its ends, which originally were directly opposed, now constitute a *couple* tending to restore the tie to its original position.

II. A *strut* has equal and directly opposite forces applied to its two ends, acting inwards, or *towards* each other. The bar is in a state of compression, and the stress exerted between any two divisions of it is a *thrust* equal and opposite to the applied forces. It is obvious that a flexible body will *not* answer the purpose of a strut.

The equilibrium of a movable strut is unstable; for if its angular position be deviated, the forces applied to its ends, which originally were directly opposed, now constitute a couple tending to make it deviate still farther from its original position.

In order that a strut may have stability, its ends must be prevented from deviating laterally. Pieces connected with the ends of a strut for this purpose are called *stays.*

III. A *beam* is a bar supported at two points, and loaded in a direction perpendicular or oblique to its length.

CASE I.—Let the supporting pressures be parallel to each other and to the direction of the load; and let the load act *between* the points of support, as in fig. 105; where P represents the resultant of the gross load, including the weight of the beam itself; L, the point where the line of action of that resultant intersects the axis of the beam; R_1, R_2, the two supporting pressures or resistances of the props parallel to, and in the same plane with P, and acting through the points S_1, S_2, in the axis of the beam.

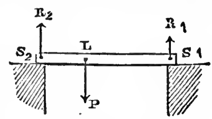


Fig. 105.

Then, according to the principle of the lever, Article 225, page 128, each of those three forces is proportional to the distance between the lines of action of the other two; and the load is equal to the sum of the two supporting pressures; that is to say,

$$P : R_1 : R_2 :: \overline{S_1 S_2} : \overline{L S_2} : \overline{L S_1}; \ldots\ldots\ldots\ldots (1.)$$

$$\text{and } P = R_1 + R_2 \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

CASE II.—Let the load act *beyond* the points of support, as in fig. 106, which represents a cantilever or projecting beam, held up by a wall or other prop at S_1, held down by a notch in a mass of masonry or otherwise at S_2, and loaded so that P is the resultant of the load, including the weight of the beam. Then the proportional Equation 1. remains exactly as before; but the load is equal to the difference of the supporting pressures; that is to say,

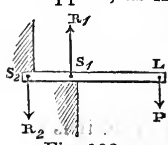


Fig. 106.

$$P = R_1 - R_2.$$

In these examples the beam is represented as horizontal; but the same principles would hold if it were inclined.

Case III.—Let the directions of the supporting forces R_1, R_2, be now inclined to that of the resultant of the load, P, as in fig. 107. This case is that of the equilibrium of three forces treated of in Article 209, page 122, and consequently the following principles apply to it:—

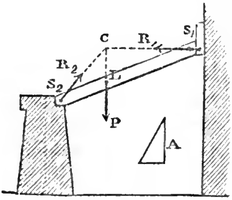


Fig. 107.

The lines of action of the supporting forces and of the resultant of the load must be in one plane.

They must intersect in one point (C, fig. 107).

Those three forces must be proportional to the three sides of a triangle A, respectively parallel to their directions.

Problem.—Given, the resultant of the load in magnitude and position, P, the line of action of one of the supporting forces, R_1, and the centre of resistance of the other, S_2; required, the line of action of the second supporting force, and the magnitudes of both.

Produce the line of action of R_1, till it cuts the line of action of P at the point C; join C S_2; this will be the line of action of R_2; construct a triangle A with its sides respectively parallel to those three lines of action; the ratios of the sides of that triangle will give the ratios of the forces.

To express this algebraically, let i_1, i_2, be the angles made by the lines of action of the supporting forces with that of the resultant of the load; then

$$P : R_1 : R_2 : : \sin (i_1 + i_2) : \sin i_2 : \sin i_1. \quad \text{.........(4.)}$$

The same piece in a frame may act at once as a beam and tie, or as a beam and strut; or it may act alternately as a strut and as a tie, as the action of the load varies.

The load tends to break a tie by tearing it asunder, a strut by crushing it, and a beam by breaking it across. The power of materials to resist those tendencies will be considered in a later section.

268. **Distributed Loads.**—Before applying the principles of the present section to frames in which the load, whether external or arising from the weight of the bars, is distributed over their length, it is necessary to reduce that distributed load to an equivalent load, or series of loads, applied at the centres of resistance. The steps in this process are as follows:—

I. Find the resultant load on each single bar.

II. Resolve that load, as in Article 267, Equation 1, page 159, into two parallel components acting through the centres of resistance at the two ends of the bar.

III. At each centre of resistance where two bars meet, combine the component loads due to the loads on the two bars into one resultant, which is to be considered as the total load acting through that centre of resistance.

IV. When a centre of resistance is also a point of support, the component load acting through it, as found by step II. of the process, is to be left out of consideration until the supporting force required by the system of loads at the other joints has been determined; with this supporting force is to be compounded a force equal and opposite to the component load acting directly through the point of support, and the resultant will be the total supporting force.

In the following Articles of this section, all the frames will be supposed to be loaded only at those centres of resistance which are *not* points of support; and, therefore, in those cases in which components of the load act directly through the points of support also, forces equal and opposite to such components must be combined with the supporting forces as determined in the following Articles, in order to complete the solution.

269. **Frames of Two Bars.**—Figures 108, 109, and 110, represent cases in which a frame of two bars, jointed to each at the point L, is loaded at that point with a given force, P, and is sup-

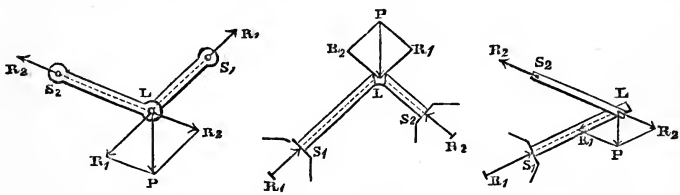


Fig. 108. Fig. 109. Fig. 110.

ported by the connection of the bars at their farther extremities, S_1, S_2, with fixed bodies. It is required to find the stress on each bar, and the supporting forces at S_1 and S_2.

Resolve the load P (as in Article 213, page 123) into two components, R_1, R_2, acting along the respective lines of resistance of the two bars. Those components are the loads borne by the two bars respectively; to which loads the supporting forces at S_1, S_2, are equal and directly opposed.

The symbolical expression of this solution is as follows:—Let i_1, i_2, be the respective angles made by the lines of resistance of the bars with the line of action of the load; then

$$P : R_1 : R_2 : : \sin (i_1 + i_2) : \sin i_2 : \sin i_1.$$

The inward or outward direction of the forces acting along each bar indicates that the stress is a thrust or a pull, and the bar a strut or a tie, as the case may be. Fig. 108 represents the case of two ties; fig. 109 that of two struts (such as a pair of rafters abutting against two walls); fig. 110 of a strut, L S_1, and a tie, L S_2 (such as the jib and the tie-rod of a crane).

A frame of two bars is *stable* as regards deviations in the plane of its lines of resistance.

With respect to *lateral* deviations of angular position, in a direction perpendicular to that plane, a frame of two ties is stable; so also is a frame consisting of a strut and a tie, when the direction of the load inclines *from* the line S_1 S_2, joining the points of support.

A frame consisting of a strut and a tie, when the direction of the load inclines *towards* the line S_1 S_2, and a frame of two struts in all cases, are unstable laterally, unless provided with lateral stays.

These principles are true of *any pair of adjacent bars whose farther centres of resistance are fixed;* whether forming a frame by themselves, or a part of a more complex frame.

270. **Triangular Frames.**—Let fig. 111 represent a frame, consisting of three bars, A, B, C, connected at the three joints 1, 2, 3,—viz., C and A at 1, A and B at 2, B and C at 3. Let a load P_1 be applied at the joint 1 in any given direction; let supporting forces, P_2, P_3, be applied at the joints 2, 3; the lines of action of those two forces must be in the same plane with that of P_1, and must either be parallel to it or intersect it in one point. The latter case is taken first, because its solution comprehends that of the former.

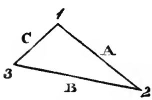


Fig. 111.

The three external forces balance each other, and are therefore proportional to the three sides of a triangle respectively parallel to their directions. In fig. 112, let A B C be such a triangle, in which

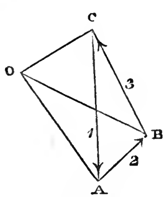


Fig. 112.

$\overline{C\ A}$ represents P_1,
$\overline{A\ B}$,, P_2,
$\overline{B\ C}$,, P_3,

Draw C O parallel to the bar C, and A O parallel to the bar A, meeting in the point O, and join B O, which will be parallel to B.

The lengths of the three lines radiating from O will represent the stresses on the bars to which they are respectively parallel.

When the three external forces are parallel to each other, the triangle of forces A B C of fig. 112, becomes a straight line C A, as in fig. 113, divided into two segments by the point B. Let straight lines radiate from O to A, B, C, respectively parallel to the bars of the frame; then if the load C A be applied at 1 (fig. 111), A B applied at 2, and B C applied at 3, are the supporting forces required to balance it; and the radiating lines O A, O B, O C, represent the stresses on the bars A, B, C, respectively, as before.

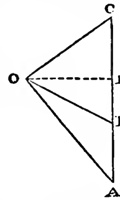


Fig. 113.

From O let fall O H perpendicular to C A, the common direction of the external forces. Then that line will represent a component of the stress, which is of equal amount in each bar. When C A, as is usually the case, is vertical, O H is horizontal; and the force represented by it is called the "*horizontal thrust*" of the frame. *Horizontal Stress or Resistance* would be a more precise term; because the force in question is a pull in some parts of the frame, and a thrust in others.

In fig. 111, A and C are *struts*, and B a *tie*. If the frame were exactly inverted, all the forces would bear the same proportions to each other; but A and C would be *ties*, and B a strut.

The trigonometrical expression of the relations amongst the forces acting in a triangular frame, under parallel forces, is as follows:—

Let a, b, c, denote the respective angles of inclination of the bars A, B, C, to the line O H (that is, in general, to a horizontal line); viz., the angles A O H, B O H, C O H of fig. 113, then

$$\text{Horizontal Stress O H} = \frac{\text{load C A}}{\tan c \pm \tan a} \text{.........(1.)}$$

$$\begin{matrix}\text{Supporting} \\ \text{Forces}\end{matrix} \left\{ \begin{matrix} \text{A B} = \text{O H} \cdot (\tan a \mp \tan b); \\ \text{B C} = \text{O H} \cdot (\tan b \pm \tan c); \end{matrix} \right\} \text{......(2.)}$$

$$\text{The sign} \left\{ \begin{matrix} + \\ - \end{matrix} \right\} \begin{matrix}\text{is to be used when the two} \\ \text{inclinations are in}\end{matrix} \left\{ \begin{matrix}\text{opposite directions,} \\ \text{the same direction.}\end{matrix} \right.$$

$$\text{Stresses} \left\{ \begin{matrix} \text{O A} = \text{O H} \cdot \sec a \\ \text{O B} = \text{O H} \cdot \sec b \\ \text{O C} = \text{O H} \cdot \sec c \end{matrix} \right\} \text{...............(3.)}$$

271. **Polygonal Frame.**—In fig. 114, let A, B, C, D, E, be the lines of resistance of the bars of a frame connected together at the joints, whose centres of resistance are, 1 between A and B, 2 between B and C, 3 between C and D, 4 between D and E, and 5 between E and A. In the

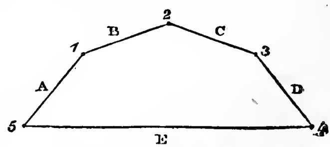


Fig. 114.

figure, the frame consists of five bars; but the principle is applicable to any number. From a point O, in fig. 115, (which may be called the *Diagram of Forces*), draw radiating lines O A, O B, O C, O D, O E, parallel respectively to the lines of resistance of the bars; and on those radiating lines take any lengths whatsoever, to represent the stresses on the several bars, which may have any magnitudes within the limits of strength of the material. Join the points thus found by straight lines, so as to form a closed polygon A B C D E A; then the sides of the polygon will represent a system of forces, which, being applied to the joints of the frame, will balance each other; each such force being applied to the joint between the bars whose lines of resistance are parallel to the pair of radiating lines that enclose the side of the polygon of forces representing the force in question.

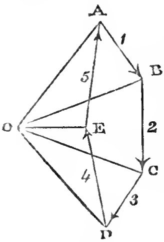
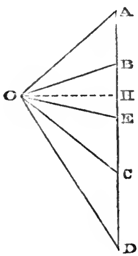


Fig. 115.

When the external forces are parallel to each other, the polygon of forces of fig. 115 becomes a straight line A D, as in fig. 116, divided into segments by the radiating lines; and each segment represents the external force which acts at the joint of the bars whose lines of resistance are parallel to the radiating lines that bound the segment. Moreover, the segment of the line A D which is intercepted between the radiating lines parallel to the lines of resistance of *any two bars whether contiguous or not*, represents the resultant of the external forces which act at points *between the bars*.



Fig, 116.

Thus, A D represents the total load, consisting of the three portions A B, B C, C D, applied at 1, 2, 3, respectively. D A represents the total supporting force, equal and opposite to the load, consisting of the two portions D E, E A, applied at 4 and 5 respectively. A C represents the resultant of the load applied between the bars A and C; and similarly for any other pair of bars.

From O draw O H perpendicular to A D; then that line represents a component of the stress, whose amount is the same in each bar of the frame. When the load, as is usually the case, is vertical, that component is called the "*horizontal thrust*" of the frame, and, as in Article 270, might more correctly be called *horizontal stress* or *resistance*, seeing that it is a pull in some of the bars and a thrust in others.

The trigonometrical expression of those principles is as follows: —

Let the force O H be denoted simply by H.

Let i, i', denote the inclinations to O H of the lines of resistance of *any two bars*, contiguous or not.

Let R, R′, be the respective stresses which act along those bars.

Let P be the resultant of the external forces acting through the joint or joints between those two bars.

Then

$$P = H\,(\tan i \pm \tan i');\ldots\ldots(1.)$$

$$R = H \cdot \sec i;\ R' = H \cdot \sec i' \ldots\ldots(2.)$$

The $\left\{\begin{matrix}\text{sum}\\ \text{difference}\end{matrix}\right\}$ of the tangents of the inclinations is to be used, according as the inclinations are $\left\{\begin{matrix}\text{opposite}\\ \text{similar.}\end{matrix}\right\}$

272. **Open Polygonal Frame.**—When the frame, instead of being closed, as in fig. 114, is converted into an open frame, by the omission of one bar, such as E, the corresponding modification is made in the diagram of inclined forces, fig. 115, by omitting the lines O E, D E, E A, and in the diagram of parallel forces, fig. 116, by omitting the line O E. Then, in both diagrams, D O and O A represent the *supporting forces* respectively, equal and directly opposed to the stresses along the extreme bars of the frame, D and A, which must be exerted by the supports (called in this case *abutments*), at the points 4 and 5, against the ends of those bars, in order to maintain the equilibrium.

In the case of parallel loads, the following formulæ give the horizontal stress and supporting pressures.

Let i_d and i_a denote the angles of inclination of the bars D and A respectively.

Let $R_d = O\,D$ and $R_a = O\,A$ be the stresses along them.

Let $\Sigma \cdot P = A\,D$ denote the total load on the frame; then,

$$H = \frac{\Sigma \cdot P}{\tan i_d + \tan i_a};\ldots\ldots(1.)$$

$$R_d = H \cdot \sec i_d;\ R_a = H \cdot \sec i_a \ldots\ldots(2.)$$

273. **Polygonal Frame—Stability.**—The stability or instability of a polygonal frame depends on the principles stated in Article 267, page 159, viz., that if a bar be free to change its angular position, then if it is a tie it is stable, and if a strut, unstable; and that a strut may be rendered stable by fixing its ends.

For example, in the frame of fig. 114, E is a tie, and stable; A, B, C, and D, are struts, free to change their angular position, and therefore unstable.

But these struts may be rendered stable in the plane of the frame by means of stays; for example, let two stay-bars connect the joints 1 with 4, and 3 with 5; then the points 1, 2, and 3, are all fixed, so that none of the struts can change their angular positions. The same effect might be produced by two stay-bars connecting the joint 2 with 5 and 4.

The frame, as a whole, is unstable, as being liable to overturn laterally, unless provided with lateral stays, connecting its joints with fixed points.

Now, suppose the frame to be exactly inverted, the loads at 1, 2, and 3, and the supporting forces at 4 and 5, being the same as before. Then E becomes a strut; but it is stable, because its ends are fixed in position; and A, B, C, and D becomes ties, and are stable without being stayed.

An open polygon consisting of ties, such as is formed by A, B, C, and D, when inverted, is called by mathematicians, a *funicular polygon*, because it may be made of ropes.

It is to be observed, that the stability of an *unstayed* polygon of ties is of the kind which admits of *oscillation* to and fro about the position of equilibrium. That oscillation may be injurious in practice, and stays may be required to prevent it.

274. **Bracing of Frames.**—A *brace* is a stay-bar on which there is a permanent stress. If the distribution of the loads on the joints of a polygonal frame, though consistent with its equilibrium as a whole, be not consistent with the equilibrium of each bar, then, in the diagram of forces, when converging lines respectively parallel to the lines of resistance are drawn from the angles of the polygon of external forces, those converging lines, instead of meeting in one point, will be found to have gaps between them. The lines necessary to fill up those gaps will indicate the forces to be supplied by means of the resistance of braces.*

The resistance of a brace introduces a pair of equal and opposite forces, acting along the line of resistance of the brace, upon the pair of joints which it connects. It therefore does not alter the *resultant* of the forces applied to that pair of joints in amount nor in position, but only the *distribution* of the components of that resultant on the pair of joints.

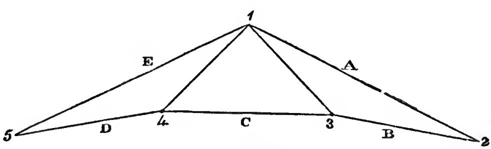


Fig. 117.

To exemplify the use of braces, and the mode of determining the stresses on them, let fig. 117 represent a frame such as frequently

* This method of treating *braced* frames contains an improvement suggested by Prof. Clerk Maxwell in 1867.

occurs in iron roofs, consisting of two struts or rafters, A and E, and three tie-bars, B, C, and D, forming a polygon of five sides, jointed at 1, 2, 3, 4, 5, loaded vertically at 1, and supported by the vertical resistance of a pair of walls at 2 and 5. The joints 3 and 4 having no loads applied to them, are connected with 1 by the braces 1 4 and 1 3.

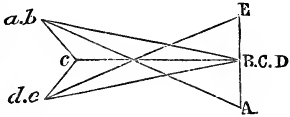


Fig. 118.

To make the diagram of forces (fig. 118), draw the vertical line E, A, as in Article 271, to represent the direction of the load and of the supporting forces.

The two segments of that line, A B and D E, are to be taken to represent the supporting forces at 2 and 5; and the whole line E A will represent the load at 1. From the ends, and from the point of division of the *scale of external forces*, E A, draw straight lines parallel respectively to the lines of resistance of the frame, each line being drawn from the point in E A that is marked with the corresponding letter. Then A *a* and B *b*, meeting at *a*, *b*, will represent the stresses along A and B respectively; and E *e* and D *d*, meeting in D *e*, will represent the stresses along D and E respectively; but those four lines, instead of meeting each other and C *c* parallel to C in one point, leave *gaps*, which are to be filled up by drawing straight lines parallel to the *braces:* that is to say, from *a*, *b*, to *c*, parallel to 1 3; and from *d*, *e*, to *c* parallel to 4 1. Then those straight lines will represent the stresses along the braces to which they are respectively parallel; and C *c* will represent the tension along C. To each joint in the frame, fig. 117, there corresponds, in fig. 118, a triangle, or other closed polygon, having its sides respectively parallel, and therefore proportional, to the forces that act at that joint. For example,

Joints,	1,	2,	3,	4,	5,
Polygons,	E A *a c e* E;	A B *b* A;	B *c b* B;	D *d c* D;	D E *e*D.

The order of the letters indicates the directions in which the forces act relatively to the joints.

Another method of treating simple cases of bracing is illustrated by fig. 119. A and B are two struts, forming the two halves of

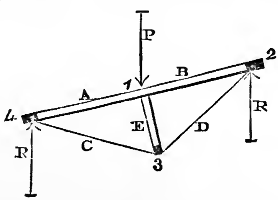


Fig. 119.

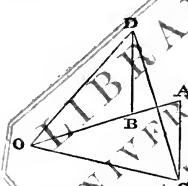


Fig. 120.

one straight bar; C and D are two equal tie-rods; E, a strut brace. A vertical rod P is applied at the joint 1, between A and B; two vertical supporting pressures, each denoted by $R = P \div 2$, act at the joints 4 and 2. The joint 3 has no external load.

Fig. 120 is the diagram of forces, constructed as follows:— Through a point O draw O B A parallel to A and B, O C parallel to C, and O D parallel to D. Make O D = O C; join C D; this line will be parallel to the brace E, and perpendicular to O A.

Through D and C draw vertical lines D B, C A; these, being equal to each other, are to be taken to represent the two supporting pressures R; and their sum D B + A C will represent the load P. The equal tensions on C and D will be represented by O C and O D, and the thrusts along A, B, and E, by O A, O B, and C D.

The polygon of external forces in this case is the crossed quadrilateral A C D B, in which C A and B D represent (as already stated) the supporting pressures, and D C and A B the components of the load P respectively parallel and perpendicular to the brace E. When A and B are horizontal, and E vertical, A B in fig. 120 vanishes, and B D and C A coincide with the two halves of C D.

275. **Rigidity of a Truss.**—The word *truss* is applied in carpentry to a triangular frame, and to a polygonal frame to which rigidity is given by staying and bracing, so that its figure shall be incapable of alteration by turning of the bars about their joints. If each joint were like a hinge, incapable of offering any resistance to alteration of the relative angular position of the bars connected by it, it would be necessary, in order to fulfil the condition of rigidity, that every polygonal frame should be divided by the lines of resistance of stays and braces into triangles and other polygons, so arranged that every polygon of four or more sides should be surrounded by triangles on all but two sides and the included angle at farthest: for every unstayed polygon of four sides or more, with flexible joints, is flexible, unless all the angles except one be fixed by being connected with triangles.

Sometimes, however, a certain amount of stiffness in the joints of a frame, and sometimes the resistance of its bars to bending, is relied upon to give rigidity to the frame, when the load upon it is subject to small variations only in its mode of distribution. For example, in the truss of fig. 121, the tie-beam A A is made in one piece, or in two or more pieces so connected together as to act like one piece; and part of its weight is suspended from the joints C, C, by the rods C B, C B. These rods also serve to make the resistance of the tie-beam A A to being bent act so as to prevent the

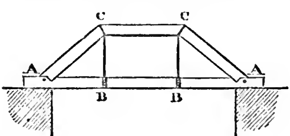


Fig. 121.

struts A C, C C, C A, from deviating from their proper angular positions, by turning on the joints A, C, C, A. If A B, B B, and B A, were three distinct pieces, with flexible joints at B B, it is evident that the frame might be disfigured by distortion of the quadrangle B C C B.

The object of stiffening a truss by braces is to enable it to sustain loads variously distributed; for were the load always distributed in one way, a frame might be designed of a figure exactly suited to that load, so that there should be no need of bracing.

The variations of load produce variations of stress on all the pieces of the frame, but especially on the braces; and each piece must be suited to withstand the greatest stress to which it is liable.

Some pieces, and especially braces, may have to act sometimes as struts and sometimes as ties, according to the mode of distribution of the load.

276. **Secondary and Compound Trussing.**—A *secondary truss* is a truss which is supported by another truss.

When a load is distributed over a great number of centres of resistance, it may be advantageous, instead of connecting all those centres by one polygonal frame, to sustain them by means of several small trusses, which are supported by larger trusses, and so on, the whole structure of secondary trusses resting finally on one large truss, which may be called the *primary truss*. In such a combination the same piece may often form part of different trusses; and then the stress upon it is to be determined according to the following principle:—

When the same bar forms at the same time part of two or more different frames, the stress on it is the resultant of the several stresses to which it is subject by reason of its position in the several frames.

In a *Compound Truss*, several frames, without being distinguishable into primary and secondary, are combined and connected in such a manner that certain pieces are common to two or more of them, and require to have their stresses determined by the principle above stated.

Example.—Fig. 122, represents a kind of secondary trussing common in the framework of iron roofs.

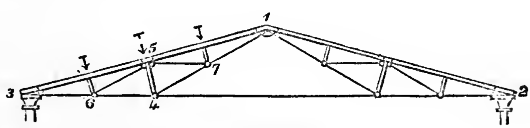


Fig. 122.

The entire frame is supported by pillars at 2 and 3, each of which sustains in all, half the weight.

1 2 3 is the *primary truss*, consisting of two rafters 1 3, 1 2, and a tie-rod 2 3.

The weight of a division of the roof is distributed over the rafters.

The middle point of each rafter is supported by a *secondary truss;* one of those is marked 1 4 3; it consists of a strut, 1 3 (the rafter itself), two ties 4 1, 4 3, and a strut-brace, 5 4, for transmitting the load, applied at 5, to the point where the ties meet.

Each of the two larger secondary trusses just described supports two *smaller secondary trusses* of similar form and construction to itself; two of those are marked 1 7 5, 5 6 3; and the subdivision of the load might be carried still farther.

In determining the stresses on the pieces of this structure, it is indifferent, so far as mathematical accuracy is concerned, whether we commence with the primary truss or with the secondary trusses; but by commencing with the primary truss, the process is rendered more simple.

(1.) *Primary Truss* 1 2 3. Let W denote the weight of the roof; then $\frac{1}{2}$ W is distributed over each rafter, the resultants acting through the middle points of the rafters. Divide each of those resultants into two equal and parallel components, each equal to $\frac{1}{4}$ W, acting through the ends of the rafter; then $\frac{1}{4}$ W is to be considered as directly supported at 3, $\frac{1}{4}$ W at 2, and $\frac{1}{4}$ W + $\frac{1}{4}$ W = $\frac{1}{2}$ W at 1; therefore the load at the joint 1 is

$$P = \tfrac{1}{2} W.$$

Let i be the inclination of the rafters to the horizon; then by the equations of Article 270.

$$H = \frac{\frac{1}{2} W}{2 \tan i} = \frac{W}{4 \tan i}; \qquad (1.)$$

This is the pull upon the horizontal tie-rod of the primary truss, 2 3; and the thrust on each of the rafters 1 3, 1 2, is given by the equation

$$R = H \sec i = \frac{W \operatorname{cosec} i}{4} \qquad (2.)$$

(2.) *Secondary Truss* 1 4 3 5. The rafter 1 3 has the load $\frac{1}{2}$ W dsstributed over it; and reasoning as before, we are to leave two quarters of this out of the calculation, as being directly supported at 1 and 3, and to consider one-half, or $\frac{1}{4}$ W, as being the vertical load at the point 5. The truss is to be considered as consisting of a polygon of four pieces, 5 1, 1 4, 4 3, 3 5, two of which happen to be in the same straight line, and of the strut-brace, 5 4, which exerts obliquely upwards against 5, and obliquely downwards against 4, a thrust equal to the component perpendicular to the rafter of the load $\frac{1}{4}$ W; which thrust is given by the equation

$$R_{54} = \tfrac{1}{4} W \cos i \text{......................(3.)}$$

Then we easily obtain the following values of the stresses on the rafter and ties, in which each stress is distinguished by having affixed to the letter R the numbers denoting the two joints between which it acts.

$$\left.\begin{array}{ll} \text{Pulls on ties} & \left\{ R_{43} = R_{41} = \dfrac{R_{54}}{2 \sin i} = \dfrac{1}{8} W \operatorname{cotan} i; \right. \\ \text{Thrusts on rafter} & \left\{ \begin{array}{l} R_{35} = \dfrac{R_{54}}{2 \tan i} + \dfrac{1}{8} W \sin i = \dfrac{1}{8} W \operatorname{cosec} i, \\ R_{51} = \dfrac{R_{54}}{2 \tan i} - \dfrac{1}{8} W \sin i = \dfrac{1}{8} W (\operatorname{cosec} i - 2 \sin i); \end{array} \right. \end{array} \right\} (4.)$$

The difference between the thrusts on the two divisions of the rafter,

$$R_{35} - R_{51} = \tfrac{1}{4} W \sin i,$$

is the component *along the rafter* of the load at the point 5.

(3.) *Smaller Secondary Trusses*, 1 7 5, 5 6 3.—These trusses are similar in every respect to the larger secondary trusses, except that the load on each point is one-half, and consequently each of the stresses is reduced to one-half of the corresponding stress in the Equations 3 and 4.

(4.) *Resultant Stresses.* The pull on the middle division of the great tie-rod 2 3 is simply that due to the primary truss, 1 2 3. The pull on the tie 4 7 is simply that due to the secondary truss 1 4 3. The pulls on the ties 5 7, 5 6, are simply those due to the smaller secondary trusses, 1 5 7, 5 6 3. But agreeably to the Theorem stated at the commencement of this article, the pull on the tie 1 7 is the sum of those due to the larger secondary truss 1 4 3, and the smaller secondary truss 1 7 5. The pull on 6 4 is the sum of those due to the primary truss 1 2 3, and to the larger secondary truss 1 4 3. The pull on 6 3 is the sum of those due to the primary truss 1 2 3, to the larger secondary truss 1 4 3, and to the smaller secondary truss 5 6 3. The thrust on each of the four divisions of the rafter 1 3, is the sum of three thrusts, due respectively to the primary truss, the larger secondary truss, and one or other of the smaller secondary trusses.

277. **Resistance of a Frame at a Section.**—The labour of calculating the stress on the bars of a frame may sometimes be abridged by the application of the following principle:—

If a frame be acted upon by any system of external forces, and if that frame be conceived to be completely divided into two parts by an ideal surface, the stresses along the bars which are intersected by that *surface, balance the external forces which act on each of the two parts of the frame.*

In most cases which occur in practice, the lines of resistance of the bars, and the lines of action of the external forces, are all in one vertical plane, and the external forces are vertical. In such cases the most convenient position for an assumed plane of section is vertical, and perpendicular to the plane of the frame. Take the vertical line of intersection of these two planes for an axis of co-ordinates,—say for the axis of y, and any convenient point in it for the origin O; let the axis of x be horizontal, and in the plane of the frame, and the axis of z horizontal, and in the plane of section.

The external forces applied to the part of the frame at one side of the plane of section (either may be chosen), being combined, as in Article 235, page 134, give three data—viz., the total force along $x = \Sigma \cdot \mathrm{X}$; the total force along $y = \Sigma \cdot \mathrm{Y}$; and the moment of the couple acting round $z = \mathrm{M}$; and the bars which are cut by the plane of section must exert resistances capable of balancing those two forces and that couple. If not more than three bars are cut by the plane of section, there are not more than three unknown quantities, and three relations between them and given quantities, so that the problem is determinate; if more than three bars are cut by the plane of section, the problem is or may be indeterminate.

The formulæ to which this reasoning leads are as follows:—Let x be positive in a direction from the plane of section towards the part of the structure which is considered in determining $\Sigma \cdot \mathrm{X}, \Sigma \cdot \mathrm{Y}$, and M; let $+y$ be measured upwards; let angles measured from Ox towards $+y$, that is, upwards, be positive; and let the lines of resistance of the three bars cut by the plane of section make the angles i_1, i_2, i_3, with x. Let n_1, n_2, n_3, be the perpendicular distances of those three lines of resistance from O, distances lying

$$\left\{ \begin{matrix} \text{upwards} \\ \text{downwards} \end{matrix} \right\} \text{from O}\,x \text{ being considered as} \left\{ \begin{matrix} \text{positive} \\ \text{negative.} \end{matrix} \right\}$$

Let R_1, R_2, R_3, be the resistances, or total stresses, along the three bars, pulls being positive, and thrusts negative. Then we have the following three equations:—

$$\left. \begin{aligned} \Sigma \cdot \mathrm{X} &= \mathrm{R}_1 \cos i_1 + \mathrm{R}_2 \cos i_2 + \mathrm{R}_3 \cos i_3; \\ \Sigma \cdot \mathrm{Y} &= \mathrm{R}_1 \sin i_1 + \mathrm{R}_2 \sin i_2 + \mathrm{R}_3 \sin i_3; \\ -\mathrm{M} &= \mathrm{R}_1 n_1 + \mathrm{R}_2 n_2 + \mathrm{R}_3 n_3; \end{aligned} \right\} \ldots\ldots\ldots (1.)$$

from which the three quantities sought, R_1, R_2, R_3 can be found.

Speaking with reference to the given plane of section, $\Sigma \cdot \mathrm{X}$ may be called the *normal stress*, $\Sigma \cdot \mathrm{Y}$, the *shearing stress*, and M, the *moment of flexure*, or *bending stress;* for it tends to bend the frame at the section under consideration. M is to be considered as $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ according as it tends to make the frame become concave $\left\{\begin{matrix}\text{upwards}\\ \text{downwards.}\end{matrix}\right\}$

The following is one of the simplest examples of the solution of a problem by the *method of polygons*, and the *method of sections.* Fig. 121 represents a truss of a form very common in carpentry (already referred to in Article 275), and consisting of three struts, A C, C C, C A, a tie-beam A A, and two suspension-rods, C B, C B, which serve to suspend part of the weight of the tie-beam from the joints C C, and also to stiffen the truss in the manner mentioned in Article 275.

Let i denote the equal and opposite inclinations of the rafters A C, C A, to the horizontal tie-beam A A; and leaving out of consideration the portions of the load directly supported at A, A, let P, P, denote equal vertical loads applied at C, C, and - P, - P, equal upward vertical supporting forces applied at A, A, by the resistance of the props. Let H denote the pull on the tie-beam, R the thrust on each of the sloping rafters, and T the thrust on the horizontal strut C C.

Proceeding by the *method of polygons*, as in Article 271, we find at once,

$$\left.\begin{aligned} H = -T &= P \text{ cotan } i; \\ R &= -P \text{ cosec } i. \end{aligned}\right\} \quad \text{......................(2.)}$$

(Thrusts being considered as negative.)

To solve the same question by the *method of sections*, suppose a vertical section to be made by a plane traversing the centre of the right hand joint C; take that centre for the origin of co-ordinates; let x be positive towards the right, and y positive downwards; let x_1, y_1, be the co-ordinates of the centre of resistance at the right hand point of support A. When the plane of section traverses the centre of resistance of a joint, we are at liberty to suppose either of the two bars which meet at that joint on opposite sides of the plane of section to be cut by it at an insensible distance from the joint.

First, consider the plane of section as cutting C A. The forces and couple acting on the part of the frame to the right of the section are

$$F_x = 0; \; F_y = -P$$
$$M = -Px_1.$$

Then, observing that for the strut A C, $n = 0$, and that for the tie A A, $n = y_1$, we have, by the equations 1 of this Article

$$R \cos i + H = F_x = 0;$$
$$R \sin i = -P;$$
$$H y_1 = -M = +P x_1;$$

whence we obtain, from the last equation,

$$H = \frac{P x_1}{y_1} = P \text{ cotan } i$$

from the first, or from the second

$$R = -\frac{H}{\cos i} = -P \text{ cosec } i \qquad \ldots\ldots\ldots\ldots(3.)$$

Next, conceive the section to cut C C at an insensible distance to the left of C. Then the equal and opposite applied forces + P at C, and - P at A, have to be taken into account; so that

$$F_x = 0; \; F_y = 0; \; M = -P x_1;$$

from the first of which equations we obtain

$$H + T = F_x = 0, \text{ and}$$
$$T = -H = -P \text{ cotan } i \ldots\ldots\ldots\ldots(4.)$$

In the example just given, the method of sections is tedious and complex as compared with the method of polygons, and is introduced for the sake of illustration only.

278. **Balance of a Chain or Cord.**—A loaded chain may be looked upon as a polygonal frame whose pieces and joints are so numerous that its figure may without sensible error be treated as a continuous curve. The following are the principles respecting the equilibrium of loaded chains and cords which are of most importance in practice.

I. *Balance of a Chain in general.*—Let D A C, in fig. 123, represent a flexible cord or chain supported at the points C and D, and loaded by forces in any direction, constant or varying, distributed over its whole length with constant or varying intensity.

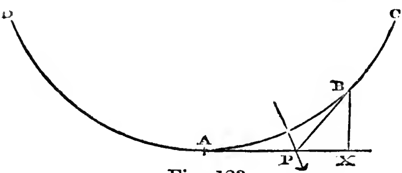


Fig. 123.

Let A and B be any two points in this chain; from those points draw tangents to the chain, A P and B P, meeting in P. The load acting on the chain between the points A and B is balanced by the pulls along the chain at those two points respectively; those pulls must respectively act along the tangents A P, B P; hence the resultant of the load between A and B acts through the point of intersection of the tangents at A and B; and that load, and the tensions on the chain at A and B, are respectively proportional to the sides of a triangle parallel to their directions.

II. *Chain under Vertical Load.—Curve of Equilibrium.*—If the direction of the load be everywhere parallel and vertical, draw a vertical straight line, C D, fig. 124, to represent the total load, and from its ends draw C O and D O, parallel to two tangents at the points of support of the chain, and meeting in O; those lines will represent the tensions on the chain at its points of support.

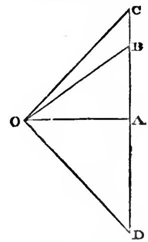


Fig. 124.

Let A, in fig. 123, be the lowest point of the chain. In fig. 124, draw the horizontal line O A; this will represent the horizontal component of the tension of the chain at every point, and if O B be parallel to a tangent to the chain at B (fig. 123), A B will represent the portion of the load supported between A and B, and O B the tension at B.

To express this algebraically, let

H = O A = *horizontal tension* along the chain at A;

R = O B = pull along the chain at B;

P = A B = load on the chain between A and B;

$i = \angle$ X P B (fig. 123) = $\angle$ A O B fig. 124) = inclination of chain at B;

then,

$$P = H \tan i;\ R = \sqrt{(P^2 + H^2)} = H \sec i \dots\dots\dots\dots (1.)$$

To deduce from these formulæ an equation by which the form of the curve assumed by the chain can be determined when the distribution of the load is known, let that curve be referred to rectangular, horizontal, and vertical co-ordinates, measured from the lowest point A, fig. 123, the co-ordinates of B being, A X $= x$, X B $= y$, then $\tan i = \frac{dy}{dx} = \frac{P}{H}$, a differential equation, which enables the form assumed by the cord (or "curve of equilibrium") to be determined when the distribution of the load is known.

279. **Stability of Blocks.**—The conditions of stability of a single block supported upon another body at a plane joint may be thus summed up:—

In fig. 125, let A A represent the upper block, B B part of the supporting body, e E the joint, C its centre of pressure, P C the resultant of the whole pressure distributed over the joint, N C, T C, its components perpendicular and parallel to the joints respectively. Then the conditions of stability are the following:—

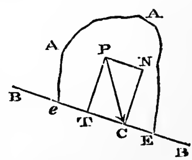


Fig. 125.

I. *In order that the block may not slide, the obliquity of the* *pressure must not exceed the angle of repose* (Article 261, page 154), that is to say,

$$\angle \text{P C N} \leqq \phi \text{ .... (1.)}$$

II. *In order that the block may be in no danger of overturning, the ratio which the deviation of the centre of pressure from the centre of figure of the joint bears to the length of the diameter of the joint traversing those two centres, must not exceed a certain fraction.* The value of that fraction varies, according to circumstances, from one-eighth to three-eighths.

The first of these conditions is called that of *stability of friction,* the second, that of *stability of position.*

In a structure composed of a series of blocks, or of a series of courses so bonded that each may be considered as one block, which blocks or courses press against each other at plane joints, the two conditions of stability must be fulfilled at each joint.

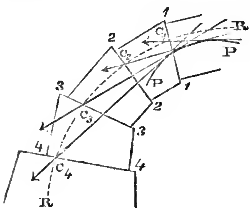


Fig. 126.

Let fig. 126 represent part of such a structure, 1, 1, 2, 2, 3, 3, 4, 4, being some of its plane joints.

Suppose the centre of pressure C_1 of the joint 1, 1, to be known, and also the amount and direction of the pressure, as indicated by the arrow traversing C_1. With that pressure combine the weight of the block 1, 2, 2, 1, together with any other external force which may act on that block; the resultant will be the total pressure to be resisted at the joint 2, 2, which will be given in magnitude, direction, and position, and will intersect that joint in the centre of pressure C_2. By continuing this process there are found the centres of pressure C_3, C_4, &c., of any number of successive joints, and the directions and magnitudes of the resultant pressures acting at those joints.

The magnitude and position of the resultant pressure at any joint whatsoever, and consequently the centre of pressure at that joint, may also be found simply by taking the resultant of all the forces which act on one of the parts into which that joint divides the structure.

The centres of pressure at the joints are sometimes called *centres of resistance.* A line traversing all those centres of resistance, such as the dotted line R R, in fig. 126, has received from Mr. Moseley the name of the "*line of resistance;*" and that author has also shewn how in many cases the equation which expresses the form of that line may be determined, and applied to the solution of useful problems.

The straight lines representing the resultant pressures may be all parallel, or may all lie in the same straight line, or may all intersect in one point. The more common case, however, is that in which those straight lines intersect each other in a series of points, so as to form a polygon. A curve, such as P P, in fig. 126 touching all the sides of that polygon, is called by Mr. Moseley the "*line of pressures.*"

The properties which the line of resistance and line of pressures must have, in order that the conditions of stability may be fulfilled, are, as already stated, the following:—

To insure stability of position, *the line of resistance must not deviate from the centre of figure of any joint by more than a certain fraction of the diameter of the joint, measured in the direction of deviation.*

To insure stability of friction, *the normal to each joint must not make an angle greater than the angle of repose with a tangent to the line of pressures drawn through the centre of resistance of that joint.*

Conceive a line to pass through all the limiting positions of the centre of resistance of the joint, so as to enclose a space beyond which that centre must not be found.

The product of the weight of the structure into the horizontal distance of a point in this line from a vertical line traversing the centre of gravity of the structure is the MOMENT OF STABILITY *of the structure, when the applied thrust acts in a vertical plane parallel to that horizontal distance, and tends to overturn the structure in the direction of the given point in the line limiting the position of the centre of resistance;* for that, according to Article 222, is the moment of the couple, which, being combined with a single force equal to the weight of the structure, transfers the line of action of that force parallel to itself through a distance equal to the given horizontal distance of the centre of resistance from the centre of gravity of the structure. The applied couple usually consists of the thrust of a frame, or an arch, or the pressure of a fluid, or of a mass of earth, against the structure, together with the equal, opposite, and parallel, but not directly opposed, resistance of the joint to that lateral force.

To express this symbolically, let t be the length of the diameter of the joint where it is cut by the vertical plane traversing the centre of gravity of the structure and parallel to the applied thrust; let j be the inclination of that diameter to the horizon; let $q\,t$ be the distance of the given limiting centre of resistance from the middle point of that diameter, and $q'\,t$ the distance from the same middle point to the point where the diameter is cut by the vertical line through the centre of gravity of the structure, and let W be the weight of the structure. Then the moment of stability is

$$W\,(q \pm q')\,t \cos j; \qquad (1.)$$

the sign $\left\{ {+ \atop -} \right\}$ being used according as the centre of resistance, and the vertical line through the centre of gravity, lie towards $\left\{ \begin{array}{l} \text{opposite sides} \\ \text{the same side} \end{array} \right\}$ of the middle of the diameter.

Let h denote the height of the structure above the middle of the plane joint which is its base, b the breadth of that joint in a direction perpendicular or conjugate to the diameter t, and w the weight of an unit of volume of the material. Then we shall have

$$W = n \cdot w\, h\, b\, t \text{.......................} (2.)$$

where n is a *numerical factor* depending on the *figure* of the structure, and on the angles which the dimensions, h, b, t, make with each other; that is, the angles of obliquity of the co-ordinates to which the figure of the structure is referred. Introducing this value of the weight of the structure into the formula 1, we find the following value for the moment of stability:—

$$n\,(q \pm q') \cos j \cdot w \cdot h\, b\, t^2 \text{..................} (3.)$$

This quantity is divided by points into three factors, viz. :—

(1.) $n\,(q \pm q') \cos j$, a *numerical factor*, depending on the *figure* of the structure, the *obliquities* of its co-ordinates, and the *direction* in which the applied force tends to overturn it.

(2.) w, the specific gravity of the material.

(3.) $h\, b\, t^2$, a geometrical factor, depending on the dimensions of the structure.

Now the first factor is the same in all structures having figures of the same class, with co-ordinates of equal obliquity, and exposed to similarly applied external forces; that is say, to all structures whose figures, together with the lines of action of the applied forces, are *parallel projections of each other, with co-ordinates of equal obliquity;* hence for any set of structures which fulfil that condition, the moments of stability are proportional to—

I. The specific gravity of the material;

II. The height;

III. The breadth;

IV. The *square* of the thickness; that is, of the dimension of the base which is parallel to the vertical plane of the applied force.

280. **Transformation of Blockwork Structures.**—If a structure composed of blocks have stability of position when acted on by forces represented by a given system of lines, then will a structure whose figure is a parallel projection of the original structure have stability of position when acted on by forces represented by the corresponding parallel projection of the original system of lines; also, the centres of pressure in the new structure will be the corresponding projections of the centres of pressure in the original structure.

The question, whether the new structure obtained by transformation will possess *stability of friction* is an independent problem.

CHAPTER II.

PRINCIPLES AND RULES RELATING TO STRENGTH AND STIFFNESS.

281. **The Object of this Chapter** is to give a summary of the principles, and of the general rules of calculation, which are applicable to problems of strength and stiffness, whatsoever the particular material may be.

SECTION I.—OF STRENGTH AND STIFFNESS IN GENERAL.

282. **Load, Stress, Strain, Strength.**—The *load*, or combination of external forces, which is applied to any piece, moving or fixed, in a structure or machine, produces *stress* amongst the particles of that piece, being the combination of forces which they exert in resisting the tendency of the load to disfigure and break the piece, accompanied by *strain*, or alteration of the volumes and figures of the whole piece, and of each of its particles.

If the load is continually increased, it at length produces either *fracture* or (if the material is very tough and ductile) such a disfigurement as is practically equivalent to fracture, by rendering the piece useless.

The *Ultimate Strength* of a body is the load required to produce fracture in some specified way. The *Proof Strength* is the load required to produce the greatest strain of a specific kind consistent with safety; that is, with the retention of the strength of the material unimpaired. A load exceeding the proof strength of the body, although it may not produce instant fracture, produces fracture eventually by long-continued application and frequent repetition.

The *Working Load* on each piece of a machine is made less than the ultimate strength, and less than the proof strength, in certain ratios determined partly by experiment and partly by practical experience, in order to provide for unforeseen contingencies.

Each solid has as many different kinds of strength as there are different ways in which it can be strained or broken, as shewn in the following classification:—

	Strain.	Fracture.
Elementary	Extension	Tearing.
	Compression	Crushing.
Compound	Distortion	Shearing.
	Twisting	Wrenching.
	Bending	Breaking across

283. **Coefficients or Moduli of Strength** are quantities expressing the *intensity* of the stress under which a piece of a given material gives way when strained in a given manner; such intensity being expressed in units of weight for each unit of sectional area of the layer of particles at which the body first begins to yield. In Britain, the ordinary unit of intensity employed in expressing the strength of materials is the *pound avoirdupois on the square inch.*

Coefficients of strength are of as many different kinds as there are different ways of breaking a body. Their use will be explained in the sequel.

Coefficients of strength, when of the same kind, may still vary according to the direction in which the stress is applied to the body. Thus the tenacity, or resistance to tearing, of most kinds of wood is much greater against tension exerted along than across the grain.

284. **Factors of Safety.**—A factor of safety, in the ordinary sense, is the ratio in which the load that is just sufficient to overcome instantly the strength of a piece of material is greater than the greatest safe ordinary working load.

The proper value for the factor of safety depends on the nature of the material; it also depends upon how the load is applied. The load upon any piece in a structure or in a machine is distinguished into *dead load* and *live load.* A *dead load* is a load which is put on by imperceptible degrees, and which remains steady; such as the weight of a structure, or of the fixed framing in a machine. A *live load* is one that is or may be put on suddenly, or accompanied with vibration; like a swift train travelling over a railway bridge; or like most of the forces exerted by and upon the moving pieces in a machine.

It can be shewn that in most cases which occur in practice a live load produces, or is liable to produce, *twice,* or very nearly twice, the effect, in the shape of stress and strain, which an equal dead load would produce. The *mean* intensity of the stress produced by a suddenly applied load is no greater than that produced by the same load acting steadily; but in the case of the suddenly applied load, the stress begins by being insensible, increases to double its mean intensity, and then goes through a series of fluctuations, alternately below and above the mean, accompanied by vibration of the strained body. Hence the ordinary practice is to make the factor of safety for a live load *double* of the factor of safety for a dead load.

A distinction is to be drawn between *real* and *apparent* factors of safety. A real factor of safety is the ratio in which the ultimate or breaking stress is greater than the real working stress at the time when the straining action of the load is greatest. The apparent factor of safety has to be made greater than the real factor of safety in those cases in which the calculation of strength is based, not upon the greatest straining action of the load, but upon a mean straining action, which is exceeded by the greatest straining action in a certain proportion. In such cases the apparent factor of safety is the product obtained by multiplying the real factor of safety by the ratio in which the greatest straining action exceeds the mean.

Another class of cases in which the apparent exceeds the real factor of safety is when there are additional straining actions besides that due to the transmission of motive power, and when those additional actions, instead of being taken into account in detail, are allowed for in a rough way by means of an increase of the factor of safety. A third class of cases is when there is a possibility of an increased load coming by accident to act upon the piece under consideration. For example, a steam engine may drive two lines of shafting, exerting half its power on each; one may suddenly break down, or be thrown out of gear, and the engine may for a short time exert its whole power on the other.

The following table shews the ordinary values of real factors of safety:—

	Real Factors of Safety. Dead Load.	Live Load
Perfect materials and workmanship,....	2	4
Ordinary materials and workmanship—		
Metals,....................................	3	6
Wood, Hempen Ropes,.................	from 3 to 5	10
Masonry and Brickwork,..............	4	8

The following are examples of apparent factors of safety:—

Real Factor of Safety, 6	Ratio in which Greatest Effort exceeds Mean Effort, nearly.	Apparent Factor of Safety.
Steam engines acting against a constant resistance—		
Single engine,............................	1·6	9·6
Pair of engines driving cranks at right angles,..................................	1·1	6·6
Three engines driving equiangular cranks,..................................	1·05	6·3
Ordinary cases of varying effort and resistance,..............................	2·0	12·0
Lines of shafting in millwork; apparent factor of safety for twisting stress due to motive power, to cover allowances for bending actions, accidental extra load, &c.,........................		from 18 to 36

Almost all the experiments hitherto made on the strength of materials give coefficients or moduli of *ultimate strength;* that is, coefficients expressing the intensity of the stress exerted by the most severely strained particles of the material just before it gives way. In calculations for the purpose of designing framework or machinery to bear a given working load, there are two ways of using the factor of safety,—one is, to multiply the working load by the factor of safety, so as to determine the breaking load, and use this load in the calculation, along with the modulus of ultimate strength: the other is, to divide the modulus of ultimate strength by the factor of safety, and thus to find a modulus or coefficient of *working stress,* which is to be used in the calculation, along with the *working load.* It is obvious that the two methods are mathematically equivalent, and must lead to the same result; but the latter is on the whole the more convenient in designing machines.

285. The **Proof** or **Testing** by experiment of the strength of a piece of material is conducted in two different ways, according to the object in view.

I. If the piece is to be *afterwards used,* the testing load must be so limited that there shall be no possibility of its impairing the strength of the piece; that is, it must not exceed the *proof strength,* being from one-third to one-half of the ultimate strength. About double or treble of the working load is in general sufficient. Care should be taken to avoid vibrations and shocks when the testing load approaches near to the proof strength.

II. If the piece is to be *sacrificed* for the sake of ascertaining the strength of the material, the load is to be increased by degrees until the piece breaks, care being taken, especially when the breaking point is approached, to increase the load by small quantities at a time, so as to get a sufficiently precise result.

The *proof strength* requires much more time and trouble for its determination than the ultimate strength. One mode of approximating to the proof strength of a piece is to apply a moderate load and remove it, apply the same load again and remove it, two or three times in succession, observing at each time of application of the load the *strain* or alteration of figure of the piece when loaded, by stretching, compression, bending, distortion, or twisting, as the case may be. If that alteration does *not sensibly increase* by repeated applications of the same load, the load is within the limit of proof strength. The effects of a greater and a greater load being successively tested in the same way, a load will at length be reached whose successive applications produce increasing disfigurements of the piece; and this load will be greater than the proof strength, which will lie between the last load and the last load but one in the series of experiments.

It was formerly supposed that the production of a *set*—that is, a disfigurement which continues after the removal of the load—was a test of the proof strength being exceeded; but Mr. Hodgkinson shewed that supposition to be erroneous, by proving that in most materials a set is produced by almost any load, how small soever.

The strength of bars and beams to resist breaking across, and of axles to resist twisting, can be tested by the application of known weights either directly or through a lever.

To test the tenacity of rods, chains, and ropes, and the resistance of pillars to crushing, more powerful and complex mechanism is required. The apparatus most commonly employed is the hydraulic press. In computing the stress which it produces, no reliance ought to be placed on the load on the safety valve, or on a weight hung to the pump handle, as indicating the intensity of the pressure, which should be ascertained by means of a pressure gauge. This remark applies also to the proving of boilers by water pressure. From experiments by Messrs. Hick and Lüthy it appears that, in calculating the stress produced on a bar by means of a hydraulic press, the friction of the collar may be allowed for by deducting a force equivalent to the pressure of the water upon an area of a length equal to the circumference of the collar, and one-eightieth of an inch broad.

For the exact determination of general laws, although the load may be applied at one end of the piece to be tested by means of a hydraulic press, it ought to be resisted and measured at the other end by means of a combination of levers.

286. **Stiffness or Rigidity, Pliability, their Moduli or Coefficients.** —Rigidity or stiffness is the property which a solid body possesses of resisting forces tending to change its figure. It may be expressed as a quantity, called a *modulus or coefficient of stiffness*, by taking the ratio of the intensity of a given stress of a given kind to the strain, or alteration of figure, with which that stress is accompanied—that strain being expressed as a quantity by dividing the alteration of some dimension of the body by the original length of that dimension. In most materials which are used in machinery, the moduli of stiffness, though not exactly constant, are nearly constant for stresses not exceeding the proof strength.

The reciprocal of a modulus of stiffness may be called a "*modulus of pliability;*" that is to say,

$$\text{Modulus of Stiffness} = \frac{\text{Intensity of Stress}}{\text{Strain}};$$

$$\text{Modulus of Pliability} = \frac{\text{Strain}}{\text{Intensity of Stress}}.$$

287. **The Elasticity of a Solid** consists of stiffness, or resistance to change of figure, combined with the power of recovering the

original figure when the straining force is withdrawn. If that recovery is complete and immediate, the body is *perfectly elastic;* if there is a *set,* or permanent change of figure, after the removal of the straining force, the body is *imperfectly elastic.* The elasticity of no solid substance is absolutely perfect, but that of many substances is nearly perfect when the stress does not exceed the proof strength, and may be made sensibly perfect by restricting the stress within small enough limits.

Moduli or *Coefficients of Elasticity* are the values of moduli of stiffness when the stress is so limited that the value of each of those moduli is sensibly constant, and the elasticity of the body sensibly perfect.

288. **Resilience or Spring** is the quantity of *mechanical work** required to produce the proof stress on a given piece of material, and is equal to the product of the *proof strain,* or alteration of figure, into the mean load which acts during the production of that strain; that is to say, in general, very nearly one-half of the proof load.

289. **Heights or Lengths of Moduli of Stiffness and Strength.**—The term *height* or *length,* as applied to a modulus or coefficient of strength or of stiffness, means the length of an imaginary vertical column of the material to which the modulus belongs, whose weight would cause a pressure on its base equal in intensity to the stress expressed by the given modulus. Hence

Height of a modulus in feet

$$= \frac{\text{Modulus in lbs. on the square foot}}{\text{Heaviness of material in lbs. to the cubic foot}};$$

$$= \frac{\text{Modulus in lbs. on the square inch}}{\text{Weight of 12 cubic inches of the material}};$$

Height of a modulus in inches

$$= \frac{\text{Modulus in lbs. on the square inch}}{\text{Heaviness of material in lbs. to the cubic inch}};$$

Height of a modulus in mètres

$$= \frac{\text{Modulus in kilogrammes on the square mètre}}{\text{Heaviness of material in kilogrammes to the cubic mètre}}.$$

Section 2.—Of Resistance to Direct Tension.

290. **Strength, Stiffness, and Resilience of a Tie.**—The word *tie* is here used to denote any piece in framing or in mechanism, such

* *Mechanical Work,* which will be fully treated of in Part VI., may be defined as the product of a *force* into the *space* through which it acts.

as a rod, bar, band, cord, or chain, which is under the action of a pair of equal and opposite longitudinal forces tending to stretch it, and to tear it asunder. The common magnitude of those two forces is the load; and it is equal to the product of the sectional area of the piece into the intensity of the tensile stress. The values of that intensity, corresponding to the immediate breaking load, the proof load, and the working load, are called respectively the moduli or coefficients of *ultimate tenacity*, of *proof tension*, and of *working tension.*

In symbols, let P be the load, S the sectional area, and p the intensity of the tensile stress; then

$$\mathrm{P} = p\,\mathrm{S}. \qquad (1.)$$

If the sectional area varies at different points, the *least* area is to be taken into account in calculations of strength.

The elongation of a tie produced by any load, P, not exceeding the proof load, is found as follows, provided the sectional area is uniform:—

Let x denote the original length of the tie, Δx the elongation, and $\alpha = \frac{\Delta x}{x}$ the extension; that is, the *proportion* which that elongation bears to the original length of the bar, being the numerical measure of the strain.

Let E denote the *modulus of direct elasticity*, or resistance to stretching. Then

$$\alpha = \frac{p}{\mathrm{E}};\ \Delta x = \alpha\,x = \frac{p}{\mathrm{E}}\,x. \qquad (2.)$$

Let f' denote the *proof tension* of the material, so that $f'\,\mathrm{S}$ is the proof load of the tie; then the *proof extension* is $f' \div \mathrm{E}$.

The Resilience or Spring of the tie, or the work done in stretching it to the limit of proof strain, is computed as follows. The length, as before, being x, the elongation of the tie produced by the proof load is $f'\,x \div \mathrm{E}$. The force which acts through this space has for its least value 0, for its greatest value $\mathrm{P} = f'\,\mathrm{S}$, and for its mean value $f'\,\mathrm{S} \div 2$; so that the work done in stretching the tie to the proof strain, that is, its *resilience* or *spring*, is

$$\frac{f'\,\mathrm{S}}{2} \cdot \frac{f'\,x}{\mathrm{E}} = \frac{f'^2}{\mathrm{E}} \cdot \frac{\mathrm{S}\,x}{2}. \qquad (3.)$$

The coefficient $f'^2 \div \mathrm{E}$, by which one-half of the volume of the tie is multiplied in the above formula, is called the MODULUS OF RESILIENCE.

A *sudden pull* of $f'\,\mathrm{S} \div 2$, or one-half of the proof load, being applied to the bar, will produce the entire proof strain of $f' \div \mathrm{E}$, which is produced by the *gradual* application of the proof load itself; for the work performed by the action of the constant force

$f'S \div 2$, through a given space, is the same with the work performed by the action, through the same space, of a force increasing at an uniform rate from 0 up to $f'S$. Hence a tie, to resist with safety the sudden application of a given pull, requires to have twice the strength that is necessary to resist the gradual application and steady action of the same pull. This is an illustration of the principle, that the factor of safety for a live load is twice that for a dead load.

291. **Thin Cylindrical and Spherical Shells.**—Let r denote the radius of a thin hollow cylinder, such as the shell of a high-pressure boiler;

t, the thickness of the shell;

f, the ultimate tenacity of the material, in pounds per square inch;

p, the intensity of the pressure, in pounds per square inch, required to burst the shell. This ought to be taken at SIX TIMES the effective working pressure—*effective pressure* meaning the excess of the pressure from within above the pressure from without, which last is usually the atmospheric pressure, of 14·7 lbs. on the square inch or thereabouts.

Then

$$p = \frac{f\,t}{r}; \quad \text{(1.)}$$

and the proper proportion of thickness to radius is given by the formula,—

$$\frac{t}{r} = \frac{p}{f}. \quad \text{(2.)}$$

Thin spherical shells are *twice as strong* as cylindrical shells of the same radius and thickness.

The tenacity of good wrought-iron boiler-plates is about 50,000 lbs.

SECTION 3.—OF RESISTANCE TO DISTORTION AND SHEARING.

292. **Distortion and Shearing Stress in General.**—In framework and mechanism many cases occur in which the principal pieces, such as plates, links, bars, or beams, being themselves subjected to tension, pressure, twisting, or bending, are connected with each other at their joints by rivets, bolts, pins, keys, or screws, which are under the action of a shearing force, tending to make them give way by the sliding of one part over another.

Every shearing stress is equivalent to a pair of direct stresses of the same intensity, one tensile and the other compressive, exerted in directions making angles of 45° with the shearing stress. Hence it follows that a body may give way to a shearing stress either by actual shearing, at a plane parallel to the direction of the shearing force, or by tearing, in a direction making an angle of 45° with that force. The manner of breaking depends on the structure of the material, hard and brittle materials giving way by tension, and soft and tough materials by shearing.

When a shearing force does not exceed the limit within which moduli of stiffness are sensibly constant, it produces distortion of the body on which it acts. Let q denote the intensity of shearing stress applied to the four lateral faces of an originally square prismatic particle, so as to distort it; and let ν be the *distortion*, expressed by the *tangent of the difference between each of the distorted angles of the prism and a right angle;* then

$$\frac{q}{\nu} = \mathrm{C}, \qquad (1.)$$

is the *modulus of transverse elasticity*, or *resistance to distortion.*

One mode of expressing the distortion of an originally square prism is as follows:—Let α denote the proportionate elongation of one of the diagonals of its end, and $-\alpha$ the proportionate shortening of the other; then the distortion is

$$\nu = 2\,\alpha.$$

The ratio $\frac{\mathrm{C}}{\mathrm{E}}$ of the modulus of transverse elasticity to the modulus of direct elasticity defined in Article 287, page 184, has different values for different materials, ranging from 0 to $\frac{1}{2}$. For wrought-iron and steel it is about $\frac{1}{3}$.

SECTION 4.—OF RESISTANCE TO TWISTING AND WRENCHING.

293. **Twisting or Torsion in General.**—Torsion is the condition of strain into which a cylindrical or prismatic body is put when a pair of couples of equal and opposite moment, tending to make it rotate about its axis in contrary directions, are applied to its two ends. Such is the condition of shafts which transmit motive power. The moment is called the *twisting moment*, and at each cross-section of the bar it is resisted by an equal and opposite moment of stress. Each particle of the shaft is in a state of distortion, and exerts shearing stress.

In British measures, twisting moments are expressed in *inch-lbs.*

294. **Strength of a Cylindrical Shaft.**—A cylindrical shaft, A B,

fig. 127, being subjected to the twisting moment of a pair of equal and opposite couples applied to the cross-sections, A and B, it is required to find the condition of stress and strain at any intermediate cross-section, such as S, and also the angular displacement of any cross-section relatively to any other.

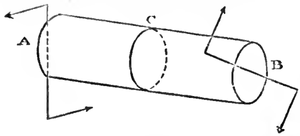


Fig. 127.

From the uniformity of the figure of the bar, and the uniformity of the twisting moment, it is evident that the condition of stress and strain of all cross-sections is the same; also, because of the circular figure of each cross-section, the condition of stress and strain of all particles at the same distance from the axis of the cylinder must be alike.

Suppose a circular layer to be included between the cross-section S, and another cross-section at the longitudinal distance $d\,x$ from it. The twisting moment causes one of those cross-sections to rotate relatively to the other, about the axis of the cylinder, through an angle which may be denoted by $d\,\theta$. Then if there be two points at the same distance, r, from the axis of the cylinder, one in the one cross-section and the other in the other, which points were originally in one straight line parallel to the axis of the cylinder, the twisting moment shifts one of those points laterally, relatively to the other, through the distance $r\,d\,\theta$. Consequently, the part of the layer which lies between those points is in a condition of *distortion*, in a plane perpendicular to the radius r; and the distortion is expressed by the ratio

$$\nu = r \cdot \frac{d\,\theta}{d\,x}; \qquad (1.)$$

which varies *proportionally to the distance from the axis.* There is therefore a *shearing stress* at each point of the cross-section, whose direction is perpendicular to the radius drawn from the axis to that point, and whose intensity is *proportional to that radius,* being represented by

$$q = \mathrm{C}\nu = \mathrm{C}\,r \cdot \frac{d\,\theta}{d\,x}. \qquad (2.)$$

The STRENGTH of the shaft is determined in the following manner:—Let q_1 be the limit of the shearing stress to which the material is to be exposed, being the *ultimate* resistance to wrenching if it is to be broken, the *proof* resistance if it is to be tested, and the *working* resistance if the working moment of torsion is to be determined. Let r_1 be the external radius of the axle. Then

q_1 is the value of q at the distance r_1 from the axis; and at any other distance, r, the intensity of the shearing stress is

$$q = \frac{q_1 r}{r_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

Conceive the cross-section to be divided into narrow concentric rings, each of the breadth $d\,r$. Let r be the *mean radius* of one of these rings. Then its area is $2\,\pi\,r\,d\,r$; the intensity of the shearing stress on it is that given by Equation 3, and the leverage of that stress relatively to the axis of the cylinder is r; consequently the moment of the shearing stress of the ring in question, being the product of the three quantities,

$$\frac{q_1 r}{r_1},\ r, \text{ and } 2\,\pi\,r\,d\,r \quad \text{is} \quad \frac{2\,\pi\,q_1}{r_1} \cdot r^3\,d$$

which being integrated for all the rings from the centre to the circumference of the cross-section, gives for the moment of torsion, and of resistance to torsion,

$$\mathrm{M} = \frac{\pi}{2}\,q_1\,r_1^3 = \frac{\pi}{16}\,q_1\,h_1^3; \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

if $h = 2\,r_1$ be the diameter of the shaft,

$$\left(\frac{\pi}{2} = 1{\cdot}5708\,;\ \frac{\pi}{16} = 0{\cdot}196 \text{ nearly}\right).$$

If the axle is *hollow*, h_0 being the diameter of the hollow, the moment of torsion becomes

$$\mathrm{M} = \frac{\pi}{16} \cdot q_1 \frac{h_1^4 - h_0^4}{h_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

The following formulæ serve to calculate the diameters of shafts when the twisting moment and stress are given; solid shafts:—

$$h_1 = \left(\frac{5{\cdot}1\ \mathrm{M}}{q_1}\right)^{\frac{1}{3}}; \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

hollow shafts—

$$h_1 = \left\{\frac{5{\cdot}1\ \mathrm{M}}{q_1\left(1 - \frac{h_0^4}{h_1^4}\right)}\right\}^{\frac{1}{3}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

SECTION 5.—OF RESISTANCE TO BENDING AND CROSS-BREAKING.

295. **Resistance to Bending in General.**—In explaining the principles of the resistance which bodies oppose to bending and cross-breaking, it is convenient to use the word *beam* as a general term

to denote the body under consideration; but those principles are applicable, not only to beams for supporting weights, but to levers, cross-heads, cross-tails, shafts, journals, cranks, and all pieces in machinery or framework to which forces are applied tending to bend them and to break them across; that is to say, forces transverse to the axis of the piece.

Conceive a beam which is acted upon by a combination of parallel transverse forces that balance each other, to be divided into two parts by an imaginary transverse section; and consider separately the conditions of equilibrium of one of those parts. The external transverse forces which act on that part, and constitute the load on it, do not necessarily balance each other. Their resultant may be found by the rule of Article 233, page 132. That resultant is called the *Shearing Load* at the cross-section under consideration, and it is balanced by the *Shearing Stress* exerted by the particles which that cross-section traverses. The resultant moment of the same set of forces, relatively to the same cross-section, may be found by the same rule; it is called the *Bending Moment* at that cross-section, and it is balanced (if the beam is strong enough) by the *Moment of Stress* exerted by the particles which the cross-section traverses, called also the *Moment of Resistance*. That moment of stress is due wholly to longitudinal stress, and it is exerted in the following way:—The bending of the beam causes the originally straight layers of particles to become curved; those near the concave side of the beam become shortened; those near the convex side, lengthened; the shortened layers exert longitudinal thrust; the lengthened layers, longitudinal tension; the resultant thrust and the resultant tension are equal and opposite, and compose a couple, whose moment is the moment of stress, equal and opposite to the bending moment.

In the solution of problems respecting the transverse strength of beams, it is necessary to determine the shearing load and bending moment produced by the transverse external forces at different cross-sections, and especially at those cross-sections at which they act most intensely, and the relations between the dimensions and figure of a cross-section of the beam, and the moment of stress which that cross-section is capable of exerting, so that each cross-section, and especially that at which the bending moment is greatest, may have sufficient strength.

296. **Calculation of Shearing Loads and Bending Moments.**—In the formulæ which follow, the shearing load at a given cross-section will be denoted by F, and the bending moment by M. In British measures it is most convenient to express the bending moment in *inch-lbs.*, because of the transverse dimensions of pieces in machines being expressed in inches.

The mathematical process for finding F and M at any given cross-section of a beam, though always the same in principle, may be varied considerably in detail. The following is on the whole the most convenient way of conducting it:—

Fig. 128 represents a beam *supported* at both ends, and loaded between them. Fig. 129 represents a *bracket;* that is, a beam *supported* and *fixed* at one end, and loaded on a projecting portion. P, Q, represent in each case the supporting forces; in fig. 128, W_1,

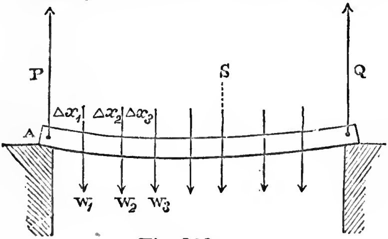


Fig. 128.

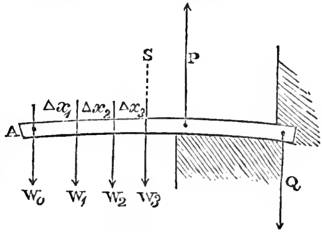


Fig. 129.

W_2, W_3, &c., represent portions of the load; in fig. 129, W_0 represents the endmost portion of the load, and W_1, W_2, W_3, other portions; in both figures, Δx_1, Δx_2, Δx_3, &c., denote the lengths of the intervals into which the lines of action of the portions of the load divide the longitudinal axis of the beam. The forces marked W may be the weights of parts of the beam itself, or of bodies carried by it; or they may be forces exerted by moving pieces in a machine on each other; or, in short, they may be any external transverse forces. If the body called the beam is a shaft, P and Q will be the bearing pressures.

The figures represent the load as applied at detached points; but when it is continuously distributed, the length of any indefinitely short portion of the beam may be denoted by $d\,x$, the intensity of the load upon it *per unit of length* by w, and the amount of the load upon it by $w\,d\,x$.

The process to be gone through will then consist of the following steps:—

STEP I. *To find the Supporting Forces or Bearing Pressures,* P *and* Q.—Assume any convenient point in the longitudinal axis as origin of co-ordinates, and find the distance x_0 of the resultant of the load from it, by the rule of Article 233, page 132; that is to say,

$$\left.\begin{array}{l} x_0 = \dfrac{\Sigma \cdot x\,W}{\Sigma \cdot W}; \text{ or} \\[2ex] {}_0 = \dfrac{\int x\,w\,d\,x}{\int w\,d\,x}. \end{array}\right\} \dots\dots\dots\dots(2.)$$

Then, by the rule of Article 227, page 129, find the two supporting forces or bearing pressures, P and Q; that is to say, let R be the resultant load, and P R and R Q its distances from the points of support; and make

$$\left.\begin{array}{l} PQ : PR : QR \\ :: R : Q : P. \end{array}\right\} \qquad (3.)$$

STEP II. *To find the shearing loads at a series of sections.*—In what position soever the origin of co-ordinates may have been during the previous step, assume it now, in a beam supported at both ends, to be at one of the points of support (as A, fig. 128), and in a bracket to be at the loaded point farthest from the fixed end (as A, fig. 129). Consider P as positive and W as negative.

Then the shearing load in any given interval of the length of the beam is the resultant of all the forces acting on the beam from the origin to that interval; so that it has the series of values,

In Fig. 128.

$F_{01} = P$;
$F_{12} = P - W_1$;
$F_{23} = P - W_1 - W_2$;
$F_{34} = P - W_1 - W_2 - W_3$; &c.;
and generally,

$$F = P - \Sigma \cdot W; \qquad (4.)$$

In Fig. 129.

$-F_{01} = W_0$;
$-F_{12} = W_0 + W_1$;
$-F_{23} = W_0 + W_1 + W_2$;
$-F_{34} = W_0 + W_1 + W_2 + W_3$, &c.;
and generally,

$$-F = \Sigma \cdot W; \qquad (5.)$$

so that the shearing loads which act in a series of intervals of the length of the beam can be computed by successive subtractions or successive additions, as the case may be.

For a continuously distributed load, these equations become respectively,

In a beam supported at both ends, $F = P - \int_0^{x'} w\,dx$; ...(6.)

In a bracket, $-F = \int_0^{x'} w\,dx$; ...(7.)

in which expressions, x' denotes the distance from the origin, A, to the plane of section under consideration.

The positive and negative signs distinguish the two contrary directions of the distortion which the shearing load tends to produce.

The Greatest Shearing Load acts in a beam supported at both ends, close to one or other of the points of support, and its value is either P or Q. In a bracket, the greatest shearing load on the projecting part acts close to the outer point of support, and its value is equal to the entire load.

In a beam supported at both ends the **Shearing Load vanishes**, or changes from positive to negative at some intermediate section, whose position may be found from Equation 4 or Equation 6, by making $F = 0$. At the second point of support, $F = -Q$.

STEP III. *To find the bending moments at a series of sections.*—At the origin A there is no bending moment. Multiply the length of each of the intervals Δx of the longitudinal axis of the beam by the shearing load F, which acts throughout that interval; the first of the products so obtained is the bending moment at the inner end of the first interval; and by adding to it the other products successively, there are obtained the bending moments at the inner ends of the other intervals in succession.*

That is to say,—bending moment

at the origin A; $M_0 = 0$;
at the line of action of W_1; $M_1 = F_{01} \cdot \Delta x_1$;
" " " W_2; $M_2 = F_{01} \cdot \Delta x_1 + F_{12} \Delta x_2$;
" " " W_3; $M_3 = F_{01} \cdot \Delta x_1 + F_{12} \Delta x_2 + F_{23} \cdot \Delta x_3$;
&c. &c.

and generally, $M = \Sigma \cdot F \Delta x$............(8.)

If the divisions Δx are of equal length, this becomes

$$M = \Delta x \cdot \Sigma F; \qquad (9.)$$

and for a continuously distributed load,

$$M = \int_0^x F\, d x. \qquad (10.)$$

The three preceding Equations 8, 9, and 10, are applicable to beams whether supported at both ends or fixed at one end. By substituting for F in Equation 10 its values as given by Equations 6 and 7 respectively, we obtain the following results:—

For a beam supported at both ends,

$$M = P_1 x' - \int_0^{x'} \int_0^{x} w\, d x^2$$

$$= P_1 x' - \int_0^{x'} (x' - x)\, w\, d x; \qquad (11.)$$

For a beam fixed at one end,

$$- M = \int_0^{x'} \int_0^{x} w\, d x^2 = \int_0^{x'} (x' - x)\, w\, d x; \qquad (12.)$$

in the latter of which equations the symbols $-M$ denotes that the bending moment acts downwards.

* This process is substantially the same with that employed by Mr. Herbert Latham, in his work *On Iron Bridges,* to compute the stress in a half-lattice girder.

The Greatest Bending Moment acts, in a bracket, at the outer point of support; and in a beam supported at both ends, at the section where the shearing load vanishes; found, as already stated in Step II., from the Equation $F = 0$.

When the transverse forces applied to a beam supported at both ends are symmetrically distributed relatively to its middle section, the Greatest Bending Moment acts at that section; and it is sometimes convenient to assume a point in that section as the origin of co-ordinates.

Step IV. *To deduce the shearing load and bending moment in one beam from those in another beam similarly supported and loaded.*—This is done by the aid of the following principle:—

When beams differing in length and in the amounts of the loads upon them are similarly supported, and have their loads similarly distributed, the shearing loads at corresponding sections in them vary as the total loads, and the bending moments as the products of the loads and lengths.

This principle may be expressed by symbols in either of the two following ways:—

First, Let l, l', denote the lengths of two beams, similarly supported; let W, W′, denote their total loads, similarly distributed; let F, F′, be the shearing forces, and M, M′, the bending moments, at sections similarly situated in the two beams; then

$$W : W' :: F : F'; \qquad (13.)$$

$$lW : l'W' :: M : M'. \qquad (14.)$$

Secondly, Let k and m be two numerical factors, depending on the way in which a beam is supported, the mode of distribution of its load, and the position of the cross-section under consideration; then

$$F = k\,W; \qquad (15.)$$

$$M = m\,W\,l. \qquad (16.)$$

The length between the points of support of a beam supported at the ends, as in fig. 128, is often called the *span.*

297. Examples.—In the following formulæ, which are examples of the application of the principles of the preceding Article to the cases which occur most frequently in practice, W denotes the total load;

w, when the load is distributed, the load per unit of length of the beam;

c, in brackets, the length of the free part of the bracket;

c, in beams either loaded or supported at both ends, the *half span,* between the extreme points of load or support and the middle;

M, the greatest bending moment.

I. Bracket fixed at on end and loaded at the other, ... } $M = c\,W$......(1.)

II. Bracket fixed at one end and uniformerly loaded, ... } $M = \frac{c\,W}{2} = \frac{w\,c^2}{2}$....(2.)

III. Beam supported at both ends and loaded at an intermediate point, whose distance from the middle of the span is x, ... } $M = \frac{(c^2 - x^2)\,W}{2\,c}$....(3.)

IV. Beam supported at both ends and loaded in the middle, ... } $M = \frac{c\,W}{2}$........(4.)

V. Beam supported at both ends and uniformly loaded, ... } $M = \frac{c\,W}{4} = \frac{w\,c^2}{2}$.....(5.)

In Example III. the greatest force exerted is $\frac{c \pm x}{2\,c} W$, and the leverage with which it acts is $c \mp x$; and Examples IV. and V. follow from it by making $x = o$.

VI. If a beam has equal and opposite couples applied to its two ends; for example, if the beam in fig. 130 has the couple of equal and opposite forces P_1 applied at A and B, and the couple of equal and opposite forces P_2 at C and D, and if the opposite moments $P_1 \cdot AB = P_2 \cdot CD = M$ are equal, then each of the endmost divisions, A B and C D, is in the condition of a bracket fixed at one end and loaded at the other (Example 1.); and the middle division B C is acted upon by the *uniform bending moment* M, and by no shearing load.

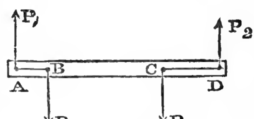


Fig. 130.

VII. Let a beam of the half span c be loaded with an uniformly distributed load of w units of weight per unit of span; and at a point whose distance from the middle of the span is a, let there be applied an additional load W. It is required to find x, the distance from the middle of the span at which the greatest bending moment is exerted, and M, that greatest moment.

Make

$$\frac{W}{2\,c\,w} = m;$$

then the solutions are as follows:—

CASE 1.—When $\frac{a}{c} =$ or $> \frac{m}{1+m}$; $x = m\,(c - a)$; and

$$M = \frac{w c^2}{2}\left(1 + m - \frac{m a}{c}\right) \text{ ....(6.)}$$

CASE 2.—When $\frac{a}{c}$ = or $\angle \frac{m}{1+m}$; $x = a$; and

$$M = \frac{w c^2}{2}(1 + 2m)\left(1 - \frac{a^2}{c^2}\right) \text{..................(7.)}$$

In the following case both sets of formulæ give the same result; when $\frac{a}{c} = \frac{m}{1+m}$; $x = a = m(c - a)$; and

$$M = \frac{w c^2}{2}\left(\frac{1 + 2m}{1 + m}\right)^2. \text{ ....(8.)}$$

298. **Bending Moments produced by Longitudinal and Oblique Forces.**—When a bar is acted upon at a given cross-section by any external force, whose line of action, whether transverse, oblique, or parallel to the axis of the bar, does not traverse the centre of magnitude of that cross-section, that force exerts a moment upon that cross-section equal to the product of the force into the perpendicular distance of its line of action from the centre of the cross-section, and that moment is to be balanced by the moment of longitudinal stress at the cross-section.

The external force may be resolved into a longitudinal and a transverse component. The longitudinal component is balanced by an uniform longitudinal tension or pressure, as the case may be, exerted at the cross-section, and combined with the stress which resists the bending moment; and the transverse component is resisted by shearing stress.

299. **Moment of Stress—Transverse Strength.**—The bending moment at each cross-section of a beam bends the beam so as to make any originally plane longitudinal layer of the beam perpendicular to the plane in which the load acts, become concave in the direction towards which the moment acts, and convex in the opposite direction. Thus, fig. 131 represents a side view of a short portion of a bent beam; C C′ is a layer, originally plane, which is now bent so as to become concave at one side and convex at the other.

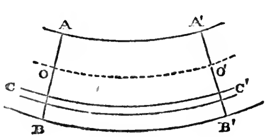


Fig. 131.

The layers at and near the concave side of the beam, A A′, are shortened, and the layers near the convex side, B B′ lengthened, by the bending action of the load. There is one intermediate surface, O O′, which is neither lengthened nor shortened; it is called the "*neutral surface.*" The particles at that surface are not necessarily, however, in a state devoid of strain; for, in common with the other particles of the beam, they are compressed and extended in a pair of diagonal directions, making angles of 45° with the neutral surface, by the shearing action of the load, when such action exists.

The condition of the particles of a beam, produced by the combined bending and shearing actions of the load, is illustrated by fig. 132, which represents a vertical longitudinal section of a rectangular beam, supported at the ends, and loaded at intermediate points. It is covered with a network consisting of two sets of curves cutting each other at right angles. The curves convex upwards are *lines of direct thrust;* those convex downwards are *lines of direct tension.* A pair of tangents to the pair of curves which traverse any particle are the *axes of stress* of that particle. The *neutral surface* is cut by both sets of curves at angles of 45°. At that vertical section of the beam where the shearing load vanishes, and the bending moment is greatest, both sets of curves become parallel to the neutral surface.

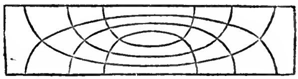


Fig. 132.

When a beam breaks under the bending action of its load, it gives way, either by the crushing of the compressed side, A A′, or by the tearing of the stretched side, B B′.

In fig. 133, A represents a beam of a granular material, like cast iron, giving way by the crushing of the compressed side, out of which a sort of wedge is forced. B represents a beam giving way by the tearing asunder of the stretched side.

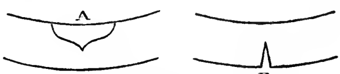


Fig. 133.

The *resistance* of a beam to bending and cross-breaking at any given cross-section is the moment of a couple, consisting of the thrust along the longitudinally-compressed layers, and the equal and opposite tension along the longitudinally-stretched layers.

It has been found by experiment, that in most cases which occur in practice, the longitudinal stress of the layers of a beam may, without material error, be assumed to be *uniformly varying*, its intensity being simply proportional to the distance of the layer from the neutral surface.

Let fig. 134 represent a cross-section of a beam (such as that represented in fig. 131), A the compressed side, B the extended side, C any layer, and O O the *neutral axis* of the section, being the line in which it is cut by the neutral surface. Let p denote the intensity of the stress along the layer C, and y the distance of that layer from the neutral axis. Because the stress is uniformly varying, $p \div y$ is a constant quantity. Let that constant be denoted for the present by a.

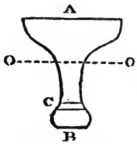


Fig. 134.

Let z be the breadth of the layer C, and $d\,y$ its thickness;

Then the amount of stress along it is

$$p\,z\,d\,y = a\,y\,z\,d\,y;$$

the amount of the stress along all the layers at the given cross-section is

$$a \int y\,z\,d\,y;$$

and this amount must be nothing,—in other words, the total thrust and total tension at the cross-section must be equal,—because the forces applied to the beam are wholly transverse; from which it follows that

$$\int y\,z\,d\,y = 0, \qquad \text{(1.)}$$

and the *neutral axis traverses the centre of magnitude of the cross-section.* This principle enables the neutral axis to be found by the aid of the methods explained in Section 1, Chapter III., Part III.

To find the greatest value of the constant $p \div y$ consistent with the strength of the beam at the given cross-section, let y_a be the distance of the compressed side, and y_b that of the extended side from the neutral axis; f_a the greatest thrust, and f_b the greatest tension which the material can bear in the form of a beam; compute $f_a \div y_a$, and $f_b \div y_b$, and adopt the *less* of those two quantities as the value of $p \div y$, which may now be denoted by $f \div y_1$; f being f_a or f_b, and y_1 being y_a or y_b, according as the beam is liable to give way by crushing or by tearing.

For the best economy of material, the two quotients ought to be equal; that is to say,

$$\frac{f}{y_1} = \frac{f_a}{y_a} = \frac{f_b}{y_b} = \frac{f_a + f_b}{h}; \qquad \text{(1 A.)}$$

and this gives what is called a *cross-section of equal strength.*

The moment relatively to the neutral axis, of the stress exerted along any given layer of the cross-section, is

$$y\,p\,z\,d\,y = \frac{f}{y_1} y^2\,z\,d\,y;$$

and the sum of all such moments, being the MOMENT OF STRESS, or MOMENT OF RESISTANCE of the given cross-section of the beam to breaking across, is given by the formula,

$$M = \int p\, y\, z\, d\, y = \frac{f}{y_1}\int y^2\, z\, d\, y; \ldots\ldots\ldots\ldots\ldots (2.)$$

or making $\int y^2\, z\, d\, y = I$,

$$M = \frac{f\,I}{y_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2\text{ A.})$$

When the *breaking* load is in question, the coefficient f is what is called the MODULUS OF RUPTURE of the material.

When the *proof load* or *working load* is in question, the coefficient f is the modulus of rupture divided by a suitable *factor of safety*, which, for the working stress in parts of machinery that are made of metal, is usually 6, and for the parts made of wood, 10. Thus, the *working modulus* f is usually 9,000 lbs. on the square inch for wrought iron, 4,500 for cast iron, and from 1,000 to 1,200 for wood.

The factor denoted by I in the preceding equation is what is called the "*geometrical moment of inertia*" of the cross-section of the beam. For sections whose figures are similar, or are parallel projections of each other, the moments of inertia are to each other as the breadths, and as the cubes of the depths of the sections, and the values of y_1 are as the depths. If, therefore, b be the breadth and h the depth of the rectangle circumscribing the cross-section of a given beam at the point where the moment of stress is greatest, we may put

$$I = n'\, b\, h^3, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

$$y_1 = m'\, h, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

n and m' being numerical factors depending on the form of section, and making $n' \div m' = n$, the moment of resistance may be thus expressed,—

$$M = n\, f\, b\, h^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

Hence it appears that the *resistances of similar cross-sections to cross-breaking are as their breadths and as the squares of their depths.*

The relation between the load and the dimensions of a beam is found by equating the value of the greatest bending moment in terms of the load and span of the beam, as given in Article 296, Equations 10, 11, 12, 16, to the value of the moment of resistance of the beam, at the cross-section where that greatest bending moment acts, as given in Equation 5 of this Article.

The depth h is usually fixed by considerations of stiffness, and then the unknown quantity is the breadth, b. Sometimes, as when the cross-section is circular or square, we have $b = h$; and then we have h^3, instead of $b\,h^2$ in Equation 5, which is solved so as to give h by extraction of the cube root. The following are the formulæ for these calculations:—

$$b = \frac{\text{M}}{n\,f\,h^2};\ldots\ldots\ldots\ldots(6.)$$

and when $h = b$,

$$h = \left(\frac{\text{M}}{n\,f}\right)^{\frac{1}{3}},\ldots\ldots\ldots\ldots(6\text{ A.})$$

Examples of the Numerical Factors in Equations 3, 4, 5 and 6.

Form of Cross-Sections.	$n' = \frac{\text{I}}{b\,h^3}$.	$m' = \frac{y_1}{h}$	$n = \frac{\text{M}}{f\,b\,h^2}$.
I. Rectangle $b\,h$, (including square)	$\frac{1}{12}$	$\frac{1}{2}$	$\frac{1}{6}$
II. Ellipse— Vertical axis h, Horizontal axis b, (including circle)	$\frac{\pi}{64} = \frac{1}{20{\cdot}4} = 0{\cdot}0491$	$\frac{1}{2}$	$\frac{\pi}{32} = \frac{1}{10{\cdot}2} = 0{\cdot}0982$
III. Hollow rectangle, $b\,h - b'\,h'$; also I-formed section, where b' is the sum of the breadths of the lateral hollows,	$\frac{1}{12}\left(1 - \frac{b'\,h'^3}{b\,h^3}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1 - \frac{b'\,h'^3}{b\,h^3}\right)$
IV. Hollow square— $h^2 - h'^2$,	$\frac{1}{12}\left(1 - \frac{h'^4}{h^4}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1 - \frac{h'^4}{h^4}\right)$
V. Hollow ellipse,	$\frac{\pi}{64}\left(1 - \frac{b'\,h'^3}{b\,h^3}\right)$	$\frac{1}{2}$	$\frac{\pi}{32}\left(1 - \frac{b'\,h'^3}{b\,h^3}\right)$
VI. Hollow circle,	$\frac{\pi}{64}\left(1 - \frac{h'^4}{h^4}\right)$	$\frac{1}{2}$	$\frac{\pi}{32}\left(1 - \frac{h'^4}{h^4}\right)$
VII. Isosceles triangle; base b, height h; y_1 measured from summit,	$\frac{1}{36}$	$\frac{2}{3}$	$\frac{1}{24}$

300. **Allowance for Weight of Beam — Limiting Length of Beam.**—When a beam is of great span, its own weight may bear a proportion to the load which it has to carry, sufficiently great to require to be taken into account in determining the dimensions of the beam. The following is the process to be performed for that purpose, when the load is uniformly distributed, and the

beam of uniform cross-section. Let W' be the external working load, s_1 its factor of safety, s_2 a factor of safety suited to a steady load, like the weight of the beam.

Let b' denote the breadth of any part of the beam, as computed by considering the *external breaking load alone*, $s_1 W'$. Compute the weight of the beam from that *provisional* breadth, and let it be denoted by B'. Then $\frac{s_1 W'}{s_1 W' - s_2 B'}$ is the proportion in which the *gross* breaking load exceeds the external part of that load. Consequently, if for the *provisional* breadth b' there be substituted the *exact* breadth,

$$b = \frac{b' s_1 W'}{s_1 W' - s_2 B'}, \quad \text{.......................(1.)}$$

the beam will now be strong enough to bear both the proposed external load W', and its own weight, which will now be

$$B = \frac{B' s_1 W'}{s_1 W' - s_2 B'}; \quad \text{.......................(2.)}$$

and the true gross breaking load will be

$$W = s_1 W' + s_2 B = \frac{s_1^2 W'^2}{s_1 W' - s_2 B'}. \quad \text{...........(3.)}$$

As the factor of safety for a steady load is in general one-half of that for a moving load, s_1 may be made $= 2 s_2$; in which case the preceding formulæ become

$$b = \frac{2 b' W'}{2 W' - B'}; \quad \text{.......................(4.)}$$

$$B = \frac{2 B' W'}{2 W' - B'}; \quad \text{.......................(5.)}$$

$$W = \frac{2 s_1 W'^2}{2 W' - B'}, \quad \text{.......................(6.)}$$

In all these formulæ, both the external load and the weight of the beam are treated as if uniformly distributed—a supposition which is sometimes exact, and always sufficiently near the truth for the purposes of the present Article.

The gross load of beams of similar figures and proportions, varying as the breadth and square of the depth directly, and inversely as the length, is proportional to the square of a given linear dimension. The weights of such beams are proportional to the cubes of corresponding linear dimensions. Hence the weight increases at a faster rate than the gross load; and for each particular figure of a beam of a given material and proportion of its dimensions, there must be a certain size at which the beam will bear its own weight only, without any additional load.

To reduce this to calculation, let the uniformly distributed gross breaking load of a beam of a given figure be expressed as follows:—

$$W = s_1 W' + s_2 B = \frac{M}{m\,l} = \frac{8\,n\,f\,h\,A}{l};\qquad(7.)$$

the value of m for an uniformly distributed load and rectangular cross-section being $\frac{1}{8}$; and $n\,f\,h\,A$ being $= n\,f\,b\,h^2$, Equation 5, Article 299; l, h and A being the length, depth, and sectional area of the beam, f the modulus of rupture, and n a factor depending on the form of cross-section. The weight of the beam will be expressed by

$$B = k\,w'\,l\,A;\qquad(8.)$$

w' being the weight of an unit of volume of the material, and k a factor depending on the figure of the beam. Then the ratio of the weight of the beam multiplied by its proper factor of safety to the gross breaking load is

$$\frac{s_2 B}{W} = \frac{s_2\,k\,w'\,l^2}{8\,n\,f\,h};\qquad(9.)$$

which increases in the simple ratio of the length, if the proportion $l \times h$ is fixed. When this is the case, the length L of a beam, whose weight (treated as uniformly distributed) is its working load, is given by the condition $s_2 B = W$; that is,

$$L = \frac{8\,n\,f\,h}{s_2\,k\,w'\,l};\qquad(10.)$$

This *limiting length* having once been determined for a given class of beams, may be used to compute the ratios of the gross breaking load, weight of the beam, and external working load to each other, for a beam of the given class, and of any smaller length, l, according to the following proportional equation:—

$$L : \frac{l}{s_2} : \frac{L - l}{s_1} :: W : B : W';\qquad(11.)$$

Section 6.—Of Resistance to Thrust or Pressure.

301. Resistance to Compression and Direct Crushing.—Resistance to *longitudinal compression*, when the proof stress is not exceeded, is sensibly equal to the resistance to stretching, and is expressed by the same modulus of elasticity, denoted by E. When that limit is exceeded, it becomes irregular.

The present Article has reference to direct and simple crushing only, and is limited to those cases in which the pillars, blocks, struts, or rods along which the thrust acts are not so long in proportion to their diameter as to have a sensible tendency to give way by bending sideways. Those cases comprehend—

Stone and brick pillars and blocks of ordinary proportions;

Pillars, rods, and struts of cast iron, in which the length is not more than five times the diameter, approximately;

Pillars, rods, and struts of wrought iron, in which the length is not more than ten times the diameter, approximately;

Pillars, rods, and struts of dry timber, in which the length is not more than about five times the diameter.

In such cases the rules for the strength of ties (Article 290) are approximately applicable, substituting *thrust* for *tension*, and using the proper modulus of resistance to direct crushing instead of the tenacity.

Blocks whose lengths are less than about once-and-a-half their diameters offer greater resistance to crushing than that given by the rules; but in what proportion is uncertain.

The modulus of resistance to direct crushing often differs considerably from the tenacity. The nature and amount of those differences depend mainly on the modes in which the crushing takes place. These may be classed as follows:—

I. *Crushing by splitting* (fig. 135) into a number of nearly prismatic fragments, separated by smooth surfaces whose general direction is nearly parallel to the direction of the load, is characteristic of very hard homogeneous substances, in which the resistance to direct crushing is greater than the tenacity; being in many examples about double.

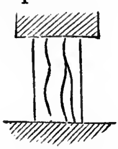


Fig. 135.

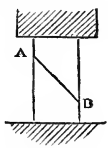


Fig. 136.

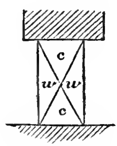


Fig. 137.

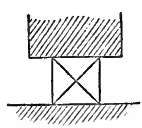


Fig. 138.

II. *Crushing by shearing or sliding* of portions of the block along oblique surfaces of separation is characteristic of substances of a granular texture, like cast iron, and most kinds of stone and brick. Sometimes the sliding takes place at a single plane surface, like A B in fig. 136; sometimes two cones or pyramids are formed, like *c*, *c* in fig, 137, which are forced towards each other, and split or drive outwards a number of wedges surrounding them, like *w*, *w*, in the same figure. Sometimes the block splits into four wedges, as in fig. 138. In substances which are crushed by shearing, the resistance to crushing is always much greater than the tenacity; for example, in cast iron it is from four times to six times.

III. *Crushing by bulging*, or lateral swelling and spreading of the block which is crushed, is characteristic of ductile and tough materials, such as wrought iron. Owing to the gradual manner in which materials of this nature give way to a crushing load, it is difficult to determine their resistance to that load exactly. That resistance is in general less, and sometimes considerably less, than the tenacity. In wrought iron, the resistance to the direct crushing of pillars or struts of moderate length, as nearly as it can be ascertained, is from $\frac{2}{3}$ to $\frac{4}{5}$ of the tenacity.

IV. *Crushing by buckling or crippling* is characteristic of fibrous subtances, such as wood, under the action of a thrust along the fibres. It consists in a lateral bending and wrinkling of the fibres, sometimes accompanied by a splitting of them asunder.

V. *Crushing by cross-breaking* is the mode of fracture of columns and struts in which the length greatly exceeds the diameter, under the breaking load they yield sideways, and are broken across like beams under a transverse load.

PART V.

PRINCIPLES OF KINETICS.

CHAPTER I.

SUMMARY OF GENERAL PRINCIPLES.

NATURE AND DIVISION OF THE SUBJECT.

THE present Chapter contains a summary of the Principles of Kinetics.

302. **Effort; Resistance; Lateral Force.**—Let F denote a force applied to a moving point, and θ the angle made by the direction of that force with the direction of the motion of the point. Then, by the principles of Article 215, the force F may be resolved into two rectangular components, one along, and the other across, the direction of motion of the point, viz:—

The *direct* force, $F \cos \theta$.
The *lateral* force, $F \sin \theta$.

A direct force is further distinguished, according as its acts *with* or *against* the motion of the point (that is, according as θ is acute or obtuse), by the name of *effort*, or of *resistance*, as the case may be. Hence, each force applied to a moving point may be thus decomposed:—

Effort, $P = F \cos \theta$, if θ is acute;
Resistance, $R = F \cos (\pi - \theta)$ if θ is obtuse;
Lateral Force, $Q = F \sin \theta$.

303. **The Conditions of Uniform Motion** of a pair of points are, that the forces applied to each of them shall balance each other; that is to say, *that the lateral forces applied to each point shall balance each other, and that the efforts applied to each point shall balance the resistances.*

The direction of a force being, as stated in Article 194, that of the motion which it tends to produce, it is evident that the balance of lateral forces is the condition of *uniformity of direction* of motion, that is, of motion in a straight line; and that the balance of efforts and resistances is the condition of *uniformity of velocity*.

304. **Work** consists in moving against resistance. The work is said to be *performed*, and the resistance *overcome*. Work is measured by the product of the resistance into the distance through which its point of application is moved. The *unit of work* commonly used in Britain is a resistance of one pound overcome through a distance of one foot, and is called a *foot-pound*.

305. **Energy** means *capacity for performing work*. The *energy of an effort*, or *potential energy*, is measured by the product of the effort into the distance through which its point of application is *capable* of being moved. The unit of energy is the same with the unit of work.

When the point of application of an effort *has been moved* through a given distance, energy is said to have been *exerted* to an amount expressed by the product of the effort into the distance through which its point of application has been moved.

306. **The Conservation of Energy**, in the case of uniform motion, means the fact, that *the energy exerted is equal to the work performed.*

307. **The Principle of Virtual Velocities** is the name given to the application of the principle of the conservation of energy to the determination of the conditions of equilibrium amongst the forces externally applied to any connected system of points.

308. **The Mass, or Inertia,** of a body, is a quantity proportional to the unbalanced force which is required in order to produce a given definite change in the motion of the body in a given interval of time.

It is known that the weight of a body, that is, the attraction between it and the earth, at a fixed locality on the earth's surface, acting unbalanced on the body for a fixed interval of time (*e. g.*, for a second), produces a change in the body's motion, which is the same for all bodies whatsoever. Hence it follows, that the *masses of all bodies are proportional to their weights at a given locality on the earth's surface.*

This fact has been learned by experiment; but it can also be shewn that it is necessary to the permanent existence of the universe; for if the gravity of all bodies whatsoever were not proportional to their respective masses, it would not produce similar and equal changes of motion in all bodies which arrive at similar positions with respect to other bodies, and the different parts which make up stars and systems would not accompany each other in their motions, never departing beyond certain limits, but would be dispersed and reduced to chaos. Neither an imponderable body, nor a body whose gravity, as compared with its mass, differs in the slightest conceivable degree from that of other bodies, can belong to the system of the universe.*

* See the Rev. Dr. Whewell's demonstration "that all matter gravitates."

309. **The Centre of Mass** of a body is its centre of gravity, found in the manner explained in Part III., Chapter III., Section 1.

310. **The Momentum** of a body means, the product of its mass into its velocity relatively to some point assumed as fixed. The momentum of a body, like its velocity, can be resolved into components, rectangular or otherwise, in the manner already explained for motions in Part I., Chapter I.

311. **The Resultant Momentum** of a system of bodies is the resultant of their separate momenta, compounded as if they were motions or statical couples.

312. **Variations and Deviations of Momentum** are the products of the mass of a body into the rates of variation of its velocity and deviation of its direction, found as explained in Part I., Chapter I., Section 3.

313. **Impulse** is the product of an unbalanced force into the *time* during which it acts unbalanced, and can be resolved and compounded exactly like force. If F be a force, and $d\,t$ an interval of time during which it acts unbalanced, $\mathrm{F}\,d\,t$ is the impulse exerted by the force during that time. The impulse of an unbalanced force in an unit of time is the magnitude of the force itself.

314. **Impulse, Accelerating, Retarding, Deflecting.**—Corresponding to the resolution of a force applied to a moving body into effort or resistance, as the case may be, and lateral force as explained in Article 302, there is a resolution of impulse into accelerating or retarding impulse, which acts with or against the body's motion, and deflecting impulse, which acts across the direction of the body's motion. Thus, if θ, as before, be the angle which the unbalanced force F makes with the body's path during an indefinitely short interval, $d\,t$.

$\mathrm{P}\,d\,t = \mathrm{F}\cos\theta \cdot d\,t$ is accelerating impulse if θ is acute;
$\mathrm{R}\,d\,t = \mathrm{F}\cos(\pi - \theta) \cdot d\,t$ is retarding impulse if θ is obtuse;
$\mathrm{Q}\,d\,t = \mathrm{F}\sin\theta \cdot d\,t$ is deflecting impulse.

315. **A Deviating Force** is one which acts unbalanced in a direction perpendicular to that of a body's motion, and changes that direction without changing the velocity of the body.

316. **Centrifugal Force** is the force with which a revolving body reacts on the body that guides it, and is equal and opposite to the deviating force with which the guiding body acts on the revolving body.

In fact, as has been stated in Article 193, every force is an action between two bodies; and *deviating force* and *centrifugal force* are but two different names for the same force, applied to it according as its action on the revolving body or on the guiding body is under consideration at the time.

317. **The Actual Energy** of a moving body relatively to a fixed

point is the product of the *mass* of the body into *one-half* of the *square of its velocity,* that is to say, it is represented by

$$\frac{m\,v^2}{2} = \frac{\mathrm{W}\,v^2}{2\,g} = \mathrm{W}\,h$$

The product $m\,v^2$, the double of the actual energy of a body, was formerly called its *vis-viva.* Actual energy, being the product of a *weight* into a *height,* is expressed, like potential energy and work, in *foot-pounds* (Articles 304, 305.)

318. **Energy Stored and Restored.**—A body alternately accelerated and retarded, so as to be brought back to its original speed, performs work by means of its retardation exactly equal in amount to the potential energy exerted in producing its acceleration; and that amount of energy may be considered as *stored* during the acceleration, and *restored* during the retardation.

319. **The Transformation of Energy** is a term applied to such processes as the expenditure of potential energy in the production of an equal amount of actual energy, and *vice versa.*

320. **Periodical Motion.**—If a body moves in such a manner that it periodically returns to its original velocity, then at the end of each period, the entire variation of its actual energy is nothing; and in each such period the whole potential energy exerted is equal to the whole work performed, exactly as in the case of a body moving uniformly (Article 306.)

321. **A Reciprocating Force** is a force which acts alternately as an effort and as an equal and opposite resistance, according to the direction of motion of the body. The work which a body performs in moving against a reciprocating force is employed in increasing its own potential energy, and is not lost by the body.

322. **Collision** is a pressure of inappreciably short duration between two bodies.

323. **The Moment of Inertia** of an indefinitely small body, or "physical point," relatively to a given axis, is the product of the mass of the body, or of some quantity proportional to the mass, such as the weight, into the square of its perpendicular distance from the axis.

324. **The Radius of Gyration** of a body about a given axis is that length whose square is the *mean of all the squares* of the distances of the indefinitely small equal particles of the body from the axis, and is found by dividing the moment of inertia by the mass.

325. **The Centre of Percussion** of a body, for a given axis, is a point so situated, that if part of the mass of the body were concentrated at that point, and the remainder at the point directly opposite in the given axis, the statical moment of the weight so distributed, and its moment of inertia about the given axis, would

be the same as those of the actual body in every position of the body.

326. The subjects to which the principles of kinetics relate will be classed in the following manner:—

I. Uniform Motion.
II. Varied Translation of Points and Rigid Bodies.
III. Rotations of Rigid Bodies.
IV. Motions of Fluids.

CHAPTER II.

ON UNIFORM MOTION UNDER BALANCED FORCES.

327. **First Law of Motion.**—*A body under the action of no force, or of balanced forces, is either at rest, or moves uniformly.* (Uniform motion has been defined in Article 66.)

Such is the first law of motion as usually stated; but in that statement is implied something more than the literal meaning of the words; for it is understood, that the *rest* or *motion of the body* to which the law refers, is its rest or motion *relatively to another body which is also under the action of no force or of balanced forces.* Unless this implied condition be fulfilled, the law is not true. Therefore the complete and explicit statement of the first law of motion is as follows:—

If a pair of bodies be each under the action of no force, or of balanced forces, the motion of each of those bodies relatively to the other is either none or uniform.

The first law of motion has been learned by experience and observation: not directly, for the circumstances supposed in it never occur; but indirectly, from the fact that its consequences, when it is taken in conjunction with other laws, are in accordance with all the phenomena of the motions of bodies.

The first law of motion may be regarded as a consequence of the definitions of *force* and of *balance* (Articles 55, 56); at the same time it is to be observed, that the framing of those definitions has been guided by experimental knowledge.

CHAPTER III.

ON THE VARIED TRANSLATION OF POINTS AND RIGID BODIES.

SECTION 1.—LAW OF VARIED TRANSLATION.

328. **Second Law of Motion.**—*Change of momentum is proportional to the impulse producing it.* In this statement, as in that of the first law of motion, Article 327, it is implied that the motion of the moving body under consideration is referred to a fixed point or body whose motion is uniform. In questions of applied mechanics, the motion of any part of the earth's surface may be treated as uniform without sensible error in practice. The units of mass and of force may be so adapted to each other as to make *change of momentum equal to the impulse producing it.* (See Articles 330, 331.)

329. **General Equations of Dynamics.**—To express the second law of motion algebraically, two methods may be followed: the first method being to resolve the change of momentum into direct variation and deviation, and the impulse into direct and deflecting impulse; and the second method being to resolve both the change of momentum and the impulse into components parallel to three rectangular axes.

First method. m being the mass of the body, v its velocity, and r the radius of curvature of its path, it follows from Articles 73 and 75 that the *rate of direct variation* of its momentum is

$$m \frac{d v}{d t} = m \cdot \frac{d^2 s}{d t^2};$$

and from Articles 77 and 78, that the rate of deviation of its momentum is

$$m \frac{v^2}{r}.$$

Equating these respectively to the direct and lateral impulse per unit of time, exerted by an unbalanced force F, making an angle θ with the direction of the body's motion, we find the two following equations (see Article 314):—

$$\text{P or } -\text{R} = \text{F} \cos \theta = m \cdot \frac{d v}{d t} = m \frac{d^2 s}{d t^2}; \quad \text{(1.)}$$

$$\text{Q} = \text{F} \sin \theta = \frac{m v^2}{r}. \quad \text{(2.)}$$

The radius of curvature r is in the direction of the deviating force Q.

Second method. As in Article 80, let the velocity of the body be resolved into three rectangular components, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$; so that the three component rates of variation of its momentum are

$$m\frac{d^2 x}{dt^2},\ m\frac{d^2 y}{dt^2},\ m\frac{d^2 z}{dt^2}.$$

Also let the unbalanced force F, making the angles α, β, γ, with the axes of co-ordinates, and its impulse per unit of time, be resolved into three components, F_x, F_y, F_z. Then we obtain

$$\left.\begin{aligned} F_x &= F \cos \alpha = m \cdot \frac{d^2 x}{dt^2}; \\ F_y &= F \cos \beta = m \frac{d^2 y}{dt^2}; \\ F_z &= F \cos \gamma = m \frac{d^2 z}{dt^2}; \end{aligned}\right\} \dots\dots\dots\dots(3.)$$

three equations, which are substantially identical with the Equations 1 and 2.

330. **Mass in Terms of Weight.**—A body's own weight, acting unbalanced on the body, produces velocity towards the earth, increasing at a rate per second denoted by the symbol g, whose numerical value is as follows:—Let λ denote the latitude of the place, h its elevation above the mean level of the sea,

$$g_1 = 32{\cdot}1695 \text{ feet, or } 9{\cdot}8051 \text{ mètres, per second;}$$

being the value of g for $\lambda = 45°$ and $h = 0$, and

$$R = 20900000 \text{ feet, or } 6370000 \text{ mètres, nearly,}$$

being the earth's mean radius; then

$$g = g_1 \cdot (1 - 0{\cdot}00284 \cos 2\lambda) \cdot \left(1 - \frac{2h}{R}\right) \dots\dots\dots(1.)$$

For latitudes exceeding 45°, it is to be borne in mind that cos 2 λ is negative, and the terms containing it as a factor have their signs reversed.

For practical purposes connected with ordinary machines, it is sufficiently accurate to assume

$$g = 32{\cdot}2 \text{ feet, or } 9{\cdot}81 \text{ mètres, per second nearly} \dots\dots(2.)$$

If, then, a body of the weight W be acted upon by an unbalanced force F, the change of velocity in the direction of F produced in a second will be

$$\frac{\mathrm{F}}{m} = \frac{\mathrm{F}\,g}{\mathrm{W}};$$

whence

$$m = \frac{\mathrm{W}}{g} \quad \text{.........................(3.)}$$

is the expression for the *mass* of a body in terms of its weight, suited to make a change of momentum *equal* to the impulse producing it. m being absolutely constant for the same body, g and W vary in the same proportion at different elevations and in different latitudes.

331. **An Absolute Unit of Force** is the force which, acting during an unit of time on an arbitrary unit of mass, produces an unit of velocity. In Britain, the unit of time being a second (as it is elsewhere), and the unit of velocity one foot per second, the unit of mass employed is the mass whose weight in vacuo at London and at the level of the sea is a standard avoirdupois pound.

The *weight* of an unit of mass, in any given locality, has for its value, in absolute units of force, the coefficient g. When the *unit of weight* is employed as the unit of force, instead of the *absolute unit*, the corresponding unit of mass becomes g times the unit just mentioned: that is to say, in British measures, the mass of 32·2 lbs.; or in French measures, the mass of 9·81 kilogrammes.

332. **The Motion of a Falling Body**, under the unbalanced action of its own weight, a sensibly uniform force, is a case of the uniformly varied velocity described in Article 73. In the equations of that Article, for the rate of variation of velocity a, is to be substituted the coefficient g, mentioned in the last Article. Then if v_0 be the velocity of the body at the beginning of an interval of time t, its velocity at the end of that time is

$$v = v_0 + g\,t, \quad \text{.........................(1.)}$$

the mean velocity during that time is

$$\frac{v_0 + v}{2} = v_0 + \frac{g\,t}{2}, \quad \text{.........................(2.)}$$

and the vertical height fallen through is

$$h = v_0\,t + \frac{g\,t^2}{2} \quad \text{.........................(3.)}$$

The preceding equations give the final velocity of the body, and the height fallen through, each in terms of the initial velocity and the time. To obtain the height in terms of the initial and final velocities, or *vice versa*, Equation 2 is to be multiplied by $v - v_0 = g\,t$, the acceleration, and compared with Equation 3; giving the following results :—

$$\left.\begin{array}{c}\frac{v^2 - v_0^2}{2} = v_0\, g\, t + \frac{g^2\, t^2}{2} = g\, h\,; \\ h = \frac{v^2 - v_0^2}{2\, g}.\end{array}\right\} \ldots\ldots\ldots\ldots\ldots(4.)$$

When the body falls from a state of rest, v_0 is to be made $=0$; so that the following equations are obtained :—

$$v = g\, t\,;\ h = \frac{g\, t^2}{2} = \frac{v^2}{2\, g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

The height h in the last equation is called *the height or fall due to the velocity* v; and that velocity is called *the velocity due to the height or fall* h.

Should the body be at first projected vertically upwards, the initial velocity v_0 is to be made negative. To find the height to which it will rise before reversing its motion and beginning to fall, v is to be made $=0$ in the last of the Equations 4; then

$$h = -\frac{v_0^2}{2\, g}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.)$$

being a rise equal to the fall due to the initial velocity v_0.

333. **An Unresisted Projectile**, or a projectile to whose motion there is no sensible resistance, has a motion compounded of the vertical motion of a falling body, and of the horizontal motion due to the horizontal component of its velocity of projection. In fig. 139, let O represent the point from which the projectile is originally projected in the direction O A, making the angle X O A $=\theta$ with a horizontal line O X in the same vertical plane with O A. Let horizontal distances parallel to O X be denoted by x, and vertical ordinates parallel to O Z by z, positive upwards, and negative downwards. In the equations of vertical motion, the symbol h of the equations of Article 332 is to be replaced by $-z$, because of h and z being measured in opposite directions.

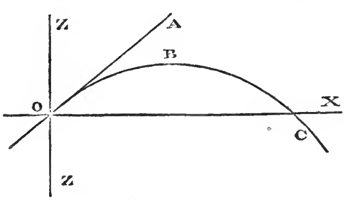


Fig. 139.

Let v_0 be the velocity of projection. Then at the instant of projection, the components of that velocity are,

$$\text{horizontal, } \frac{d\, x}{d\, t} = v_0 \cos\theta;\ \text{vertical, } \frac{d\, z}{d\, t} = v_0 \sin\theta;$$

and after the lapse of a given time t, those components have become

$$\left.\begin{aligned} \frac{d\,x}{d\,t} &= v_0 \cos\theta = \text{constant}; \\ \frac{d\,z}{d\,t} &= v_0 \sin\theta - g\,t. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1.)$$

Hence the co-ordinates of the body at the end of the time t are

$$\left.\begin{aligned} &\text{horizontal,} \quad x = v_0 \cos\theta \cdot t; \\ &\text{vertical,} \quad z = v_0 \sin\theta \cdot t - \frac{g\,t^2}{2}; \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(2.)$$

the Equations 2 being those of which the differential coefficients are Equations 1, and because $t = \frac{x}{v_0 \cos\theta}$, those co-ordinates are thus related,

$$z = x \cdot \tan\theta - \frac{g}{2 v_0^2 \cos^2\theta} \cdot x^2; \ldots\ldots\ldots\ldots(3.)$$

an equation which shews the path O B C of the projectile to be a parabola with a vertical axis, touching O A in O.

The total velocity of the projectile at a given instant, being the resultant of the components given by Equation 1, has for the value of its square (remembering that $\sin^2\theta + \cos^2\theta = 1$),

$$v^2 = \frac{dx^2}{dt^2} + \frac{dz^2}{dt^2} = v_0^2 - 2 v_0 \sin\theta \cdot g\,t + g^2\,t^2 = v_0^2 - 2\,g\,z; \ldots..(4.)$$

from the last form of which is obtained the equation

$$z = \frac{v_0^2 - v^2}{2\,g}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

which, being compared with Equation 4 of Article 332, shews that *the relation between the variation of vertical elevation, and the variation of the square of the resultant velocity, is the same, whether the velocity is in a vertical, inclined, or horizontal direction.*

The resistance of the air prevents any actual projectile near the earth's surface from moving exactly as an unresisted projectile. The approximation of the motion of an actual projectile to that of an unresisted projectile is the closer, the slower is the motion, and the heavier the body, because of the resistance of the air increasing with the velocity, and because of its proportion to the body's weight being dependent upon that of the body's surface to its weight.

334. **An Uniform Effort or Resistance**, unbalanced, causes the velocity of a body to vary according to the law expressed by this equation,

$$\frac{d\,v}{d\,t} = f\,g; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

where f is the constant ratio which the unbalanced force bears to the weight of the moving body, positive or negative according to the direction of the force; so that by substituting $f\,g$ for g in the equations of Article 332, those equations are transformed into the equations of motion of the body in question, h being taken to represent the distance traversed by it in a positive direction.

In the apparatus known by the name of its inventor, Attwood, for illustrating the effect of uniform moving forces, this principle is applied in order to produce motions following the same law with those of falling bodies. Two weights, P and R, of which P is the greater, are hung to the opposite ends of a cord passing over a finely constructed pulley. Considering the masses of the cord and pulley to be insensible, the weight of the mass to be moved is P + R, and the moving force P − R, being less than the weight in the ratio,

$$f = \frac{\mathrm{P} - \mathrm{R}}{\mathrm{P} + \mathrm{R}}.$$

consequently the two weights move according to the same law with a falling body, but more slowly in the ratio of f to 1.

335. **Deviating Force of a Single Body.**—It is part of the first law of motion, that if a body moves under no force, or balanced forces, it moves in a straight line.

It is one consequence of the second law of motion, that in order that a body may move in a curved path, it must be continually acted upon by an unbalanced force at right angles to the direction of its motion, the direction of the force being that towards which the path of the body is curved, and its magnitude bearing the same ratio to the weight of the body that the height due to the body's velocity bears to half the radius of curvature of its path.

This principle is expressed symbolically as follows:—

Half radius of curvature.		Height due to velocity.		Body's weight.		Deviating force.
$\frac{r}{2}$	:	$\frac{v^2}{2\,g}$	: :	W	:	$\mathrm{Q} = \frac{\mathrm{W} v^2}{g\,r}$......(1.)

or otherwise that the acceleration produced by gravity, bears the same ratio to the rate of deviation, that the weight bears to the magnitude of the deviating force, which may be symbolically expressed $g : \frac{v^2}{r} :: \mathrm{W} : \mathrm{Q} = \frac{\mathrm{W}\, v^2}{g\,r}$.

In the case of projectiles, just described, and of the heavenly bodies, deviating force is supplied by that component of the mutual attraction of two masses which acts perpendicular to the direction of their relative motion. In machines, deviating force is supplied by the strength or rigidity of some body, which *guides* the deviating mass, making it move in a curve.

A pair of free bodies attracting each other have both deviated motions, the attraction of each guiding the other; and their deviations of momentum are equal in equal times; that is, their deviations of motion are inversely as their masses.

In a machine, each revolving body tends to press or draw the body which guides it away from its position, in a direction from the centre of curvature of the path of the revolving body; and that tendency is resisted by the strength and stiffness of the guiding body, and of the frame with which it is connected.

336. **A Revolving Simple Pendulum** consists of a small mass A, suspended from a point C by a rod or cord C A of insensibly small weight as compared with the mass A, and revolving in a circle about a vertical axis C B. The tension of the rod is the resultant of the weight of the mass A, acting vertically, and of its centrifugal force, acting horizontally; and therefore the rod will assume such an inclination that

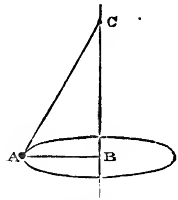


Fig. 140.

$$\frac{\text{height } \overline{\mathrm{B\,C}}}{\text{radius } \overline{\mathrm{A\,B}}} = \frac{\text{weight}}{\text{centrifugal force}} = \frac{g\,r}{v^2} \dots\dots\dots\dots (1.)$$

where $r = \mathrm{A\,B}$. Let n be the *number of turns per second* of the pendulum; then

$$v = 2\,\pi\,n\,r;$$

and therefore, making $\overline{\mathrm{B\,C}} = h$,

$$h = \frac{g\,r^2}{v^2} = \frac{g}{4\,\pi^2\,n^2}$$

$$= (\text{in the latitude of London})\ \frac{0{\cdot}8154 \text{ foot}}{n^2} = \frac{9{\cdot}7848 \text{ inches}}{n^2} \dots\dots (2.)$$

When the speed of revolution varies, the inclination of the pendulum varies so as to adjust the height to the varying speed.

337. **Deviating Force in Terms of Angular Velocity.**—If the radius of curvature of the path of a revolving body be regarded as a sort of *arm* of constant or variable length at the end of which the body is carried, the angular velocity of that arm is given by the expression,

$$a = \frac{v}{r} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.)$$

Let $a\,r$ be substituted for v in the value of deviating force of Article 335, and that value becomes

$$\mathrm{Q} = \frac{\mathrm{W}\,a^2\,r}{g} \dots\dots\dots\dots\dots\dots\dots\dots\dots (2.)$$

In the case of a body revolving with uniform velocity in a circle, like the bob A of the revolving pendulum of Article 336, $a = 2\pi n$, where n is the number of revolutions per second, so that

$$Q = \frac{4\pi^2 W n^2 r}{g}; \quad \text{(3.)}$$

from which equation the height of a revolving pendulum might be deduced with the same result as in the last Article.

338. **A Simple Oscillating Pendulum** consists of an indefinitely small weight A, fig. 141, hung by a cord or rod of insensible weight A C from a point C, and swinging in a vertical plane to and fro on either side of a central point D vertically below C. The path of the weight or *bob* is a circular arc, A D E.

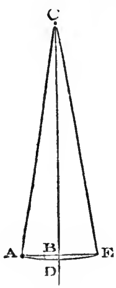


Fig. 141.

The weight W of the bob, acting vertically, may be resolved at any instant into two components, viz.:—

$$W \cdot \cos \angle DCA = W \cdot \frac{\overline{BC}}{\overline{CA}},$$

acting along C A, and balanced by the tension of the rod or cord, and

$$W \cdot \sin \angle DCA = W \cdot \frac{\overline{AB}}{\overline{CA}},$$

acting in the direction of a tangent to the arc, towards D, and unbalanced. The motion of A depends on the latter force.

When the arc A D E is small compared with the length of the pendulum A C, it very nearly coincides with the chord A B E; and the horizontal distance A B, to which the moving force is proportional, is very nearly equal to the distance of the bob from D, the central point of its oscillations. Then if the length of the pendulum, $\overline{CA}$, be denoted by l, we have approximately, for small arcs of oscillation,

$$\left.\begin{aligned} \frac{1}{n} &= 2\pi\sqrt{\frac{l}{g}}; \text{ and} \\ l &= \frac{g}{4\pi^2 n^2}; \end{aligned}\right\} \quad \text{(1.)}$$

and the following statement shews the connection between a simple oscillating and revolving pendulum, viz., *that the length of a simple oscillating pendulum, making a given number of small double oscillations in a second, is sensibly equal to the height of a revolving pendulum, making the same number of revolutions in a second.*

SECTION 2.—VARIED TRANSLATION OF A SYSTEM OF BODIES.

339. **Conservation of Momentum.**—THEOREM. *The mutual actions of a system of bodies cannot change their resultant momentum.* (Resultant momentum has been defined in Article 311.) Every force is a pair of equal and opposite actions between a pair of bodies; in any given interval of time it constitutes a pair of equal and opposite impulses on those bodies, and produces equal and opposite momenta. Therefore the momenta produced in a system of bodies by their mutual actions neutralize each other, and have no resultant, and cannot change the resultant momentum of the system.

340. **Motion of Centre of Gravity.**—COROLLARY. *The variations of the motion of the centre of gravity of a system of bodies are wholly produced by forces exerted by bodies external to the system;* for the motion of the centre of gravity is that which, being multiplied by the total mass of the system, gives the resultant momentum, and this can be varied by external forces only.

It follows that in all dynamical questions in which the mutual actions of a certain system of bodies are alone considered, the centre of gravity of that system of bodies may be correctly treated as a point whose motion is none or uniform; because its motion cannot be changed by the forces under consideration.

341. **The Angular Momentum,** relatively to a fixed point, of a body having a motion of translation, is the product of the momentum of the body into the perpendicular distance of the fixed point from the line of direction of the motion of the body's centre of gravity at the instant in question. Let m be the mass of the body, v its velocity, l the length of the before-mentioned perpendicular; then

$$m\,v\,l = \frac{W\,v\,l}{g}$$

is the angular momentum relatively to the given point.

Angular momenta are compounded and resolved like forces, each angular momentum being represented by a line whose length is proportional to the magnitude of the angular momentum, and whose direction is perpendicular to the plane of the motion of the body and of the fixed point, and such, that when the motion of the body is viewed from the extremity of the line, the radius vector of the body seems to have right-handed rotation. The direction of such a line is called the *axis* of the angular momentum which it represents. The *resultant angular momentum* of a system of bodies is the resultant of all their angular momenta relatively to their common centre of gravity; and the axis of that resultant angular momentum is called the *axis of angular momentum* of the system. The term *angular momentum* was introduced by Mr. Hayward.

342. **Angular Impulse** is the product of the moment of a couple of forces (Article 200) into the time during which it acts. Let F be the force of a couple, l its leverage, and $d\,t$ the time during which it acts, then

$$\mathrm{F}\,l\,d\,t$$

is the angular impulse. Angular impulses are compounded and resolved like the moments of couples.

343. **Relations of Angular Impulse and Angular Momentum.**—THEOREM. *The variation, in a given time, of the angular momentum of a body, is equal to the angular impulse producing that variation, and has the same axis.* This is a consequence which is deduced from the second law of motion in the following manner:—Conceive an unbalanced force F to be applied to a body m, and an equal, opposite, and parallel force, to a fixed point, during the interval $d\,t$; and let l be the perpendicular distance from the fixed point to the line of action of the first force. Then the couple in question exerts the angular impulse

$$\mathrm{F}\,l\,d\,t.$$

At the same time, the body m acquires a variation of momentum in the direction of the force applied to it, of the amount

$$m\,d\,v = \mathrm{F}\,d\,t;$$

so that relatively to the fixed point, the variation of the body's angular momentum is

$$m\,l\,d\,v = \mathrm{F}\,l\,d\,t;$$

being equal to the angular impulse, and having the same axis.—Q. E. D.

344. **Conservation of Angular Momentum.**—THEOREM. *The resultant angular momentum of a system of bodies cannot be changed in magnitude, nor in the direction of its axis, by the mutual actions of the bodies.*

Considering the common centre of gravity of the system of bodies as a fixed point, conceive that for each force with which one of the bodies of the system is urged in virtue of the combined action of all the other bodies upon it, there is an equal, opposite, and parallel force applied to the common centre of gravity, so as to form a couple. The forces with which the bodies act on each other are equal and opposite in pairs, and their resultant is nothing; therefore, the resultant of the ideal forces conceived to act at the common centre of gravity is nothing, and the supposition of these forces does not effect the equilibrium or motion of the system. Also, the resultant of all the couples thus formed is nothing; therefore, the resultant of their angular impulses is nothing; therefore, the resultant of the several variations of angular momentum produced by those angular impulses is nothing; therefore, the resultant angular momentum of the system is invariable in amount and in the direction of its axis.—Q. E. D.

345. **Collision.**—The most useful problem in cases of collision is, when two bodies whose masses are given move before the collision in one straight line with given velocities, and it is required to find their velocities after the collision. The two bodies form a system whose resultant momentum and internal energy are each unaltered by the collision; but a certain fraction of the internal energy disappears as visible motion, and appears as vibration and heat. If the bodies are equal, similar, and perfectly elastic, that fraction is nothing.

Let m_1, m_2, be the masses of the two bodies, and u_1, u_2, their velocities before the collision, whose directions should be indicated by their signs. Then the velocity of their common centre of gravity is

$$u_0 = \frac{u_1 m_1 + u_2 m_2}{m_1 + m_2}; \qquad (1.)$$

and this is not altered by the collision.

CHAPTER IV.

ROTATIONS OF RIGID BODIES.

346. **The Motion of a Rigid Body**, or of a body which sensibly preserves the same figure, has already been shewn in Part I., Chapter II., to be always capable of being resolved at each instant into a translation and a rotation; and by the aid of the principles explained in Section 3 of that chapter, the component rotation can always be conceived to take place about an axis traversing the centre of gravity of the body, and to be combined, if necessary, with a translation of the whole body in a curved or straight path along with its centre of gravity. The variations of the *momentum* of the translation, whether in amount or in direction, are due to the resultant force acting through the centre of gravity of the body, and are exactly the same with those of the momentum of the entire mass if it were concentrated at that centre; the variations of the *angular momentum* of the rotation are due to the resultant couple which is combined with that resultant force. The variations of *actual energy* are due to both causes.

When the translation of the centre of gravity of a rotating body, and its rotation about an axis traversing that centre, are known, the motion of every point in the body is determined by cinematical principles, which have been explained in Part I., Chapter II., Section 3.

SECTION 1.—ON MOMENTS OF INERTIA, RADII OF GYRATION, AND CENTRES OF PERCUSSION.

347. **The Moment of Inertia** of an indefinitely small body, or "physical point," relatively to a given axis, is the product of the mass of the body, or of some quantity proportional to the mass, such as the weight, into the square of its perpendicular distance from the axis: thus in the following equation:—

$$\frac{\mathrm{I}}{g} = m\, r^2 = \frac{\mathrm{W}\, r^2}{g}, \qquad \text{(1.)}$$

r is the perpendicular distance of the mass m, whose weight is W, from a given axis; and the moment of inertia, according to the unit employed, is either I, or $\mathrm{I} \div g$; the former, when the unit is the moment of inertia of an unit of *weight* at the end of an arm whose length is unity; and the latter, when the unit is the moment of inertia of an unit of *mass* at the end of the same arm. The former is the more convenient unit, and will be employed in this treatise.

By an extension of the term "moment of inertia," it is applied to the product of any quantity, such as a volume, or an area, into the square of the distance of the point to which that quantity relates from a given axis; but in the remainder of this treatise the term will be used in its strict sense, and according to the unit of measure already specified; that is, in British measures, moment of inertia will be expressed by the product of a certain number of *pounds avoirdupois* into the square of a certain number of *feet*.

The geometrical relations amongst moments of inertia, to which the present section refers, are independent of the unit of measure.

348. The **Moment of Inertia of a System of Physical Points**, relatively to a given axis, is the sum of the moments of inertia of the several points; that is,

$$I = \Sigma \cdot W r^2 \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

349. The **Moment of Inertia of a Rigid Body** is the sum of the moments of inertia of all its parts, and is found by integration; that is, by conceiving the body to be divided into small parts of regular figure, multiplying the mass of each of those parts into the square of the distance of its centre of gravity from the axis, adding the products together, and finding the value towards which their sum converges when the size of the small parts is indefinitely diminished. For example, let the body be conceived to be built up of rectangular molecules, whose dimensions are $d x$, $d y$, and $d z$, the volume of each $d x \, d y \, d z$, and the mass of unity of volume w. Then

$$I = \int\int\int r^2 w \cdot d x \, d y \, d z \ldots\ldots\ldots\ldots (1.)$$

Hence follows the general principle that propositions relative to the geometrical relations amongst the moments of inertia of systems of points are made applicable to continuous bodies by substituting integration for ordinary summation; that is, for example, by putting $\int\int\int$ for Σ, and $w \cdot d x \, d y \, d z$ for W.

350. **The Radius of Gyration** of a body about a given axis is that length whose square is the *mean of all the squares* of the distances of the indefinitely small equal particles of the body from the axis, and is found by dividing the moment of inertia by the mass, thus,

$$\varrho^2 = \frac{I}{\Sigma \cdot W} = \frac{\Sigma \cdot W r^2}{\Sigma W} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

When symbols of integration are used, this becomes

$$\varrho^2 = \frac{\int\int\int r^2\, w \cdot d\,x\,d\,y\,d\,z}{\int\int\int w \cdot d\,x\,d\,y\,d\,z} \text{..................(2.)}$$

351. **Components of Moment of Inertia.**—Let the positions of the particles of a body be referred to three rectangular axes, one of which, O X, is that about which the moment of inertia is to be taken. Then the square of the radius vector of any particle is

$$r^2 = y^2 + z^2;$$

so that the moment of inertia round the axis of x is

$$I_x = \Sigma \cdot W\, y^2 + \Sigma \cdot W\, z^2; \text{.....................(1.)}$$

that is to say, *the moment of inertia of a body round a given axis may be found by adding together the sum of the products of the masses of the particles, each multiplied by the square of each of its distances from a pair of planes cutting each other at right angles in the given axis.*

In the same manner it may be shewn that the moments of inertia of the same body round the other two axes are given by the equations

$$I_y = \Sigma \cdot W\, z^2 + \Sigma \cdot W\, x^2; \; I_z = \Sigma \cdot W\, x^2 + \Sigma \cdot W\, y^2 \text{......(2.)}$$

352. **Moments of Inertia Round Parallel Axes Compared.**—THEOREM. *The moment of inertia of a body about any given axis is equal to its moment of inertia about an axis traversing its centre of gravity parallel to the given axis, added to the moment of inertia about the given axis due to the whole mass of the body concentrated at its centre of gravity.*

This theorem may be expressed as follows:—Let I_0 be the moment of inertia of a body about an axis traversing its centre of gravity in any given direction, and I the moment of inertia of the same body about an axis parallel to the former at the perpendicular distance r_0; then

$$I = r_0^2 \cdot \Sigma\, W + I_0 \text{........................(1.)}$$

COROLLARY I. The radius of gyration (ϱ) of a body about any axis is equal to the hypotenuse of a right-angled triangle, of which the two sides are respectively equal to the radius of gyration of the body about an axis traversing the centre of gravity parallel to the given axis (ρ_0), and to the perpendicular distance between these axes (r_0). That is to say,

$$\rho^2 = r_0^2 + \rho_0^2 \quad \text{.............................(2.)}$$

Corollary II. The moment of inertia of a body about an axis traversing its centre of gravity in a given direction, is less than the moment of inertia of the same body about any other axis parallel to the first.

Corollary III. The moments of inertia of a body about all axes parallel to each other, which lie at equal distances from its centre of gravity, are equal.

353. **Combined Moments of Inertia.**—Theorem. *The combined moment of inertia of a rigidly connected system of bodies about a given axis, is equal to the combined moment of inertia which the system would have about the given axis, if each body were concentrated at its own centre of gravity, added to the sum of the several moments of inertia of the bodies, about axes traversing their respective centres of gravity, parallel to the given axis.*

Let W now denote the mass of *one of the bodies*, I_0 its moment of inertia about an axis traversing its own centre of gravity parallel to the given common axis, and r_0 the distance of its centre of gravity from that common axis. Then the moment of inertia of that body about the common axis, according to Article 352, Equation 1, is

$$I = W\, r_0^2 + I_0.$$

Consequently, the combined moment of inertia of the system of bodies is

$$\Sigma I = \Sigma \cdot W\, r_0^2 + \Sigma I_0; \quad \text{.....................(1).}$$

—Q. E. D.

354. **Examples of Moments of Inertia and Radii of Gyration** of homogeneous bodies of some of the more simple and ordinary figures, are given in the following tables. In each case, the axis is supposed to traverse the *centre of gravity* of the body; for the principles of Article 352 enable any other case to be easily solved. The axes are also supposed, in each case, to be *axes of symmetry* of the figure of the body.

The column headed W gives the mass of the body; that headed I_0 gives the moment of inertia; that headed ρ_0^2, the *square* of the radius of gyration. The mass of an unit of volume is in each case denoted by *w*.

Body.	Axis.	W	I_0	ϱ_0^2
I. Sphere of radius r,........	Diameter	$\frac{4\pi wr^3}{3}$	$\frac{8\pi wr^5}{15}$	$\frac{2r^2}{5}$
II. Spheroid of revolution—polar semi-axis a, equatorial radius r,...........	Polar axis	$\frac{4\pi war^2}{3}$	$\frac{8\pi war^4}{15}$	$\frac{2r^2}{5}$
III. Ellipsoid — semi-axes, a, b, c,......................	Axis, $2a$	$\frac{4\pi wabc}{3}$	$\frac{4\pi wabc(b^2+c^2)}{15}$	$\frac{b^2+c^2}{5}$
IV. Spherical shell—external radius r, internal r',....	Diameter	$\frac{4\pi w(r^3-r'^3)}{3}$	$\frac{8\pi w(r^5-r'^5)}{15}$	$\frac{2(r^5-r'^5)}{5(r^3-r'^3)}$
V. Spherical shell, insensibly thin — radius r, thickness dr,....................	Diameter	$4\pi wr^2dr$	$\frac{8\pi wr^4dr}{3}$	$\frac{2r^2}{3}$
VI. Circular cylinder—length $2a$, radius r,.............	Longitudinal axis, $2a$	$2\pi war^2$	πwar^4	$\frac{r^2}{2}$
VII. Elliptic cylinder—length $2a$, transverse semi-axes b, c,......................	Longitudinal axis, $2a$	$2\pi wabc$	$\frac{\pi wabc(b^2+c^2)}{2}$	$\frac{b^2+c^2}{4}$
VIII. Hollow circular cylinder—length $2a$, external radius r, internal r', ...	Longitudinal axis, $2a$	$2\pi wa(r^2-r'^2)$	$\pi wa(r^4-r'^4)$	$\frac{r^2+r'^2}{2}$
IX. Hollow circular cylinder, insensibly thin — length $2a$, radius r, thickness dr,	Longitudinal axis, $2a$	$4\pi wardr$	$4\pi war^3dr$	r^2
X. Circular cylinder—length $2a$, radius r, ...	Transverse diameter	$2\pi war^2$	$\frac{\pi war^2(3r^2+4a^2)}{6}$	$\frac{r^2}{4}+\frac{a^2}{3}$
XI. Elliptic cylinder—length $2a$, transverse semi-axes b, c,......................	Transverse axis, $2b$	$2\pi wabc$	$\frac{\pi wabc(3c^2+4a^2)}{6}$	$\frac{c^2}{4}+\frac{a^2}{3}$
XII. Hollow circular cylinder—length $2a$, external radius r, internal r', ...	Transverse diameter	$2\pi wa(r^2-r'^2)$	$\frac{\pi wa}{6}\left\{3(r^4-r'^4)+4a^2(r^2-r'^2)\right\}$	$\frac{r^2+r'^2}{4}+\frac{a^2}{3}$
XIII. Hollow circular cylinder, insensibly thin — radius r, thickness dr,..........	Transverse diameter	$4\pi wardr$	$\pi wa(2r^3+\frac{4}{3}a^2r)dr$	$\frac{r^2}{2}+\frac{a^2}{3}$
XIV. Rectangular prism — dimensions $2a$, $2b$, $2c$,.....	Axis, $2a$	$8wabc$	$\frac{8wabc(b^2+c^2)}{3}$	$\frac{b^2+c^2}{3}$
XV. Rhombic prism — length $2a$, diagonals $2b$, $2c$,....	Axis, $2a$	$4wabc$	$\frac{2wabc(b^2+c^2)}{3}$	$\frac{b^2+c^2}{6}$
XVI. Rhombic prism, as above,	Diagonal, $2b$	$4wabc$	$\frac{2wabc(c^2+2a^2)}{3}$	$\frac{c^2}{6}+\frac{a^2}{3}$

355. **The Centre of Percussion** of a body, for a given axis, is a point so situated, that if part of the mass of the body were concentrated at that point, and the remainder at the point directly opposite in the given axis, the statical moment of the weight so distributed (Article 223), and its moment of inertia about the given axis, would be the same as those of the actual body in every position of the body.

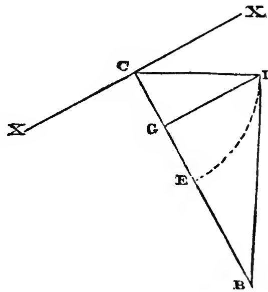


Fig. 142.

In fig. 142 let X X be the given axis, and G the centre of gravity of the body. It is evident, in the first place, that the centre of percussion must be somewhere in the perpendicular C G B let fall from the centre of gravity on the given axis. Secondly, in order that the statical moment of the whole mass, concentrated partly at C, and partly at the centre of percussion B (still unknown), may be the same with that of the actual body, the centre of gravity must be unaltered by that concentration of mass; that is to say, the masses concentrated at B and C must be inversely as the distances of those points from G. Hence denoting the weights of those masses by the letters B and C respectively, and the weight of the whole body by W, we have the proportion

$$W : C : B :: \overline{BC} : \overline{GB} : \overline{GC} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Lastly, in order that the moment of inertia of the mass as supposed to be concentrated at B and C, about the axis X X, may be the same with that of the actual body, we must have

$$B \cdot \overline{BC}^2 = W\varrho^2 = W\ (\varrho_0^2 + r_0^2) \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

where $r_0 = \overline{GC}$, and ϱ_0 is the radius of gyration of the body about an axis parallel to X X and traversing G; and substituting for B its value from Equation 1, viz., $B = Wr_0 \div \overline{BC}$, we find, for the distance of the centre of percussion from the axis,

$$\overline{BC} = \frac{\varrho^2}{r_0} = \frac{\varrho_0^2}{r_0} + r_0; \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

and for its distance from the centre of gravity,

$$\overline{GB} = \overline{BC} - r_0 = \frac{\varrho_0^2}{r_0}. \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

The last equation may also be expressed in the form

$$\overline{GB} \cdot \overline{GC} = \varrho_0^2; \qquad (5.)$$

which preserves the same value when $\overline{GB}$ and $\overline{GC}$ are interchanged; thus shewing, that if a new axis parallel to the original axis X X be made to traverse the original centre of percussion, the new centre of percussion is the point C in the original axis.

The proportion in which the mass of the body is to be considered as distributed between B and C takes the following form, when each of the last three terms of the proportion 1 is multiplied by $r_0 = \overline{GC}$:—

$$W : C : B :: \varrho_0^2 + r_0^2 : \varrho_0^2 : r_0^2 \qquad (6.)$$

The preceding solution is represented by the following geometrical construction:—Draw $\overline{GD}$ perpendicular to $\overline{CG}$ and $= \varrho_0$; join $\overline{CD}$, perpendicular to which draw $\overline{DB}$ cutting $\overline{CG}$ produced in B; this point is the centre of percussion.

Also, $\overline{CD} = \varrho$, the radius of gyration about X X; and D B is the radius of gyration about an axis traversing B parallel to X X.

If $\overline{CE}$ be taken $= \overline{CD}$, E is sometimes called the **Centre of Gyration** of the body for the axis X X.

Section 2.—On Uniform Rotation.

356. **The Momentum** of a body rotating about its centre of gravity is nothing, according to the principle of Article 344. As every motion of a rigid body can be resolved into a translation, and a rotation about its centre of gravity, the rotation will be supposed to take place about the centre of gravity of the body throughout this section.

357. **The Angular Momentum** is found in the following manner:—Let x denote the axis of rotation, and y and z any two axes fixed in the body, perpendicular to it and to each other. Let a be the angular velocity of rotation. Then the velocity of any particle W, whose radius vector is $r = \sqrt{y^2 + z^2}$, is

$$a\, r = a \sqrt{y^2 + z^2},$$

and the angular momentum of that particle, *relatively to the axis of rotation*, is

$$\frac{W a r^2}{g} = \frac{W a}{g} (z^2 + y^2);$$

being the *product of its moment of inertia into its angular velocity*, divided by g, because of the weights of the particles having been used in computing the moment of inertia.

358. **The Actual Energy of Rotation** of a body rotating about its centre of gravity, being the sum of the masses of its particles, each multiplied into one-half of the square of its velocity, is found as follows:—a being the angular velocity of rotation, the linear velocity of any particle whose distance from the axis of rotation is r, is

$$v = a\, r;$$

and the actual energy of that particle, its weight being W, is

$$\frac{\mathrm{W}\, v^2}{2\, g} = \frac{\mathrm{W}\, a^2\, r^2}{2\, g}; \qquad (1.)$$

being the *moment of inertia* of the particle multiplied by $\frac{a^2}{2\, g}$. Hence for the whole body the actual energy of rotation is

$$\mathrm{E} = \frac{a^2\, \mathrm{I}}{2\, g}; \qquad (2.)$$

that is to say, *actual energy bears the same relation to angular velocity and moment of inertia that it does to linear velocity and weight.*

CHAPTER V.

MOTIONS OF FLUIDS.

359. **Division of the Subject.**—The mode of division that will be employed in this chapter in treating of the motions of fluids is founded on the distinction between motions not sensibly affected by friction, and those which are so affected. The motions of fluids not sensibly affected by friction, and therefore governed by pressure and weight only, take place according to laws which are exactly known; so that any difficulty which exists in tracing their consequences, in particular cases, arises from mathematical intricacy alone. The laws of the friction of fluids, on the other hand, are only known approximately and empirically; and the mode of operation of that force amongst the particles of a fluid is not yet thoroughly understood; so that the solution of a particular problem has often to be deduced, not from first principles representing the condensed results of all experience, but from experiments of a special class, suited to the problem under consideration.

The following is the division of the subject of this chapter:—

I. Motions of Liquids under Gravity and Pressure alone.

II. Motions of Liquids affected by Friction.

Section 1.—Motions of Liquids without Friction.

360. **Dynamic Head.**—Let p denote the intensity of the pressure of the liquid at a given point and ϱ the *weight of an unit of volume;* then the quotient $\frac{p}{\varrho}$ is what is called the *height, or head, due to the pressure;* that is, the height of a column of the liquid, of the uniform specific gravity ϱ, whose weight per unit of base would be equal to the pressure p. Now, let a vertical ordinate z be measured *positively downwards* from a datum horizontal plane, ϱz is the weight of a column of liquid per unit of base extending down from that plane to a particle under consideration; $p - \varrho z$ is the difference between the intensity of the actual pressure at that particle and the pressure due to its depth below the datum horizontal plane; and

$$\frac{p}{\varrho} - z = h \quad \text{..............................(1.)}$$

is the *height* or *head due to that difference of intensity,* being what will be termed the *dynamic head.* When z is measured *positively* *upwards* from a datum horizontal plane, its sign is to be changed; so that the expression for the dynamic head in that case becomes

$$\frac{p}{\varrho} + z = h \text{..............................} (2.)$$

361. **Law of Dynamic Head for Steady Motion.**—This principle may be stated thus:—*In steady motion, the sum of the height due to the velocity of a particle and of its dynamic head is constant,* or symbolically

$$\frac{V^2}{2g} + h = \text{constant.}$$

This equation applies to the particles which successively occupy the same fixed point, as well as to each individual particle.

362. **The Total Energy** of a particle of a moving liquid without friction is expressed by multiplying the expression in the previous equation by the weight of the particle W, thus:—

$$\frac{W V^2}{2g} + W h;$$

in which $\frac{W V^2}{2g}$ is the *actual energy* of the particle, and $W h$ is its *potential energy;* because, from the last Article it appears, that by the diminution of $W h$, $\frac{W V^2}{2g}$ may be increased by an equal amount, and *vice versa;* so that *the dynamic head of a particle is its potential energy per unit of weight.* In the case of *steady motion,* the total energy of each particle is constant; and the total energy of each of the equal particles which successively occupy the same position is the same.

363. **The Free Surface** of a moving liquid mass, being that which is in contact with the air only, is characterized by the pressure being uniform all over it, and equal to that of the atmosphere. Let p_1 be the atmospheric pressure, z_1 the vertical ordinate, measured *positively upwards* from a given horizontal plane, of any point in the free surface of the liquid, and h_1 the dynamic head at the same point; then it appears from Article 360, Equation 2, that for that surface,

$$h_1 - z_1 = \frac{p_1}{\varrho} = \text{constant} \text{....................} (1.)$$

364. **A Surface of Equal Pressure** is characterized by an analogous equation,

$$h - z = \frac{p}{\varrho} = \text{constant}; \text{....................} (1.)$$

and all surfaces of equal pressure fulfil the differential equation,

$$d\,h = d\,z; \qquad \text{(2.)}$$

for the differential coefficient of a constant being equal to 0 $d\,h - d\,z = 0$ Equation 1, and $\therefore d\,h = d\,z$ which, for *steady motion*, becomes

$$d\,z = d\,h = -\,d \cdot \frac{V^2}{2\,g}; \qquad \text{(3.)}$$

found by differentiating the equation of Article 361, expressing that the variations of actual energy are those due to the variations of level simply.

365. **Motion in Plane Layers** is a state which is either exactly or approximately realized in many ordinary cases of liquid motion;

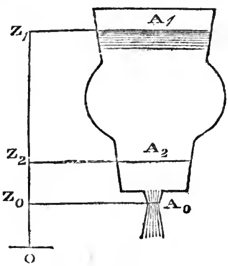


Fig. 143.

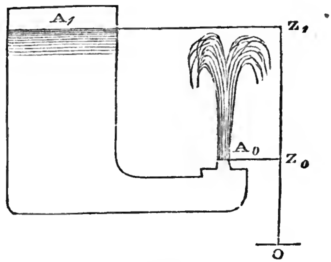


Fig. 144.

and the assumption of which is often used as a first approximation to the solution of various questions in hydraulics. It consists in the motions of all the particles in one plane being parallel to each other, perpendicular to the plane, and equal in velocity. It is illustrated by the three figures 143, 144, and 145, each of which represents a reservoir containing liquid up to the elevation $\overline{O\,Z_1} = z_1$ above a given datum, and discharging the liquid from an orifice A_0 at the smaller elevation $\overline{O\,Z_0} = z_0$. The liquid moves exactly or nearly in plane layers at the upper surfuce A_1 and at the orifice A_0. Let these symbols denote the areas of the upper surface and of the issuing stream respectively.

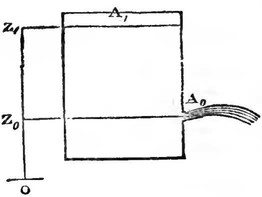


Fig. 145.

Let Q denote the rate of flow per second, v_1 the velocity of descent of the liquid at the upper surface, v_0 its velocity of outflow from the orifice; then, according to Article 116, the equation of continuity is

$$\left.\begin{array}{l} v_1 A_1 = v_0 A_0 = Q; \\ \text{or } v_1 = \dfrac{Q}{A_1};\ v_0 = \dfrac{Q}{A_0}. \end{array}\right\} \dots\dots\dots\dots (1.)$$

The pressures at the upper surface and at the orifice respectively are each equal to the atmospheric pressure; hence the difference of dynamic head is simply the difference of elevation; that is to say,

$$h_1 - h_0 = z_1 - z_0;$$

therefore, according to Article 361 and Article 364, Equations 2 and 3,

$$\frac{v_0^2 - v_1^2}{2g} = \frac{v_0^2}{2g}\left(1 - \frac{A_0^2}{A_1^2}\right) = z_1 - z_0 \dots\dots\dots\dots (2.)$$

This gives for the velocity of outflow,

$$v_0 = \sqrt{\left\{\frac{2g(z_1 - z_0)}{1 - \dfrac{A_0^2}{A_1^2}}\right\}}; \dots\dots\dots\dots (3.)$$

from which can be computed the rate of flow or discharge by means of Equation 1.

366. **The Contracted Vein** is the name given to a portion of a jet of fluid at a short distance from an orifice in a plate, which is smaller in diameter and in area than the orifice, owing to a spontaneous contraction which the jet undergoes after leaving the orifice.

The area of the narrowed part of the contracted vein is in every case to be considered as the *virtual* or *effective outlet*, and used for A_0 in the equations of the last Article.

The ratio of the area of the contracted vein, or effective orifice, to that of the actual orifice, is called the *coefficient of contraction*. For sharp edged orifices in thin plates, it has different values for different figures and proportions of the orifice, ranging from about 0·58 to 0·7, and being on an average about $\frac{5}{8}$. It diminishes somewhat for great pressures, and for dynamic heads of six feet and upwards may be taken at about 0·6. The most elaborate table of those coefficients is that of Poncelet and Lesbros.

For orifices with edges that are not sharp and thin, the discharge is modified sensibly by friction.

SECTION 2.—MOTIONS OF LIQUIDS WITH FRICTION.

367. **General Laws of Fluid Friction.**—It is known by experiment, that between a fluid, and a solid surface over which it glides,

there is exerted a resistance to their relative motion which is proportional to their surface of contact, and to the density of the fluid, and is approximately proportional to the square of the velocity of the relative motion; that is, the resistance is approximately proportional *to the weight of a prism of the fluid, whose base is the surface of contact, and its height the height due to the relative velocity.*

Let S be the surface of contact, v the velocity, ϱ the weight of an unit of volume of the fluid, and f a factor called the *coefficient of friction;* then

$$R = f \varrho S \frac{v^2}{2g}, \quad \text{.........................(1.)}$$

is the amount of the friction at the surface S.

The coefficient f is not absolutely constant at different velocities. The mode of calculation employed in practice, where the velocity is one of the unknown quantities to be determined, is to find an approximate value of the velocity from the *mean* value of f; then to compute the value of f corresponding to that approximate velocity, and use it to compute the velocity more exactly.

The following are some of the values of the coefficients of friction, according to different authorities, for streams of WATER, gliding over various surfaces; v being the *mean velocity of the stream,* in feet per second:—

Iron pipes (Darcy). Let d = diameter of pipe in feet; then,

$$f = 0{\cdot}0043 \left(1 + \frac{1}{9\,d}\right) + \frac{0{\cdot}001}{v}\left(1 + \frac{1}{18\,d}\right);$$

or for velocities that are not very small,

$$f = 0{\cdot}005\left(1 + \frac{1}{12\,d}\right).$$

Iron pipes, value of f for first approximation, 0·0064

Beds of rivers (Weisbach),...... $f = a + \frac{b}{v}$; $a = 0{\cdot}0074$.
$b = 0{\cdot}00023$ foot.

Beds of rivers, value of f for first approximation,......................... } 0·0076.

A collection of numerous formulæ for fluid friction, proposed by different authors, together with tables of the results of the best formulæ, is contained in Mr. Neville's work on hydraulics. The formulæ of many authors, though differing in appearance, are founded on the same, or nearly the same, experimental data, being chiefly those of Du Buat, with additions by subsequent inquirers; and their practical results do not materially differ. The two formulæ given above, on the authority of Darcy, for iron pipes, are based on his experiments as recorded in his treatise *du Mouvement de l'Eau dans les Tuyaux.*

368. **Internal Fluid Friction.**—Although the particles of fluids have no transverse elasticity—that is, no tendency to recover a certain figure after having been distorted—it is certain that they resist being made to slide over each other, and that there is a lateral communication of motion amongst them; that is, that there is a tendency of particles which move side by side in parallel lines to assume the same velocity. The laws of this lateral communication of motion, or internal friction of fluids, are not known exactly; but its effects are known thus far:—that the energy due to differences of velocity, which it causes to disappear, is replaced by heat in the proportion of one thermal unit of Fahrenheit's scale for 772 foot pounds of energy, and that it causes the friction of a stream against its channel to take effect, not merely in retarding the film of fluid which is immediately in contact with the sides of the channel, but in retarding the whole stream, so as to reduce its motion to one approximating to a motion in plane layers perpendicular to the axis of the channel (Article 365).

369. **Friction in an Uniform Stream.**—It is this last fact which renders possible the existence of an open stream of uniform section, velocity, and declivity. In hydraulic calculations respecting the resistance of this, or any other stream, the value given to the velocity is its mean value throughout a given cross-section of the stream A,

$$v = \frac{Q}{A} \dots\dots\dots\dots (1.)$$

The greatest velocity in each cross-section of a stream takes place at the point most distant from the rubbing surface of the channel. Its ratio to the mean velocity is given by the following empirical formula of Prony, where V is the greatest velocity in feet per second:—

$$\frac{v}{V} = \frac{7 \cdot 71 + V}{10 \cdot 25 + V} \dots\dots\dots\dots (2.)$$

In an uniform stream, the dynamic head which would otherwise have been expended in producing increase of actual energy, is wholly expended in overcoming friction. Consider a portion of the stream whose length is l, and fall z. The loss of head is equal to the fall of the surface of the stream, according to Article 363; and the expenditure of potential energy in a second is accordingly

$$z \varrho Q = z \varrho v A.$$

Equating this to the work performed in a second in overcoming friction, viz., v R, Equation 1, Article 367, we find

$$z \varrho v A = f \varrho S \frac{v^3}{2g};$$

or dividing by common factors, and by the area of section A, we find for the value of the fall in terms of the velocity

$$z = f \cdot \frac{S}{A} \cdot \frac{v^2}{2g} \dots\dots (3.)$$

Let s be what is called the *wetted perimeter* of the cross-section of the stream; that is, the cross-section of the rubbing surface of the stream and channel; then

$$S = l s;$$

and dividing both sides of Equation 3 by l, we find for the relation between the rate of declivity and the velocity,

$$\sin i = \frac{z}{l} = f \frac{s}{A} \cdot \frac{v^2}{2g} \dots\dots (4.)$$

$\frac{A}{s}$ is what is called the "HYDRAULIC MEAN DEPTH" of the stream; and as the friction is inversely proportional to it, it is evident that the figure of cross-section of channel which gives the least friction is that whose hydraulic mean depth is greatest, viz., a semicircle. When the stability of the material limits the side-slope of the channel to a certain angle, Mr. Neville has shewn that the figure of least friction consists of a pair of straight side-slopes of the given inclination connected at the bottom by an arc of a circle whose radius is the depth of liquid in the middle of the channel; or, if a flat bottom be necessary, by a horizontal line touching that arc. For such a channel, the hydraulic mean depth is half of the depth of liquid in the middle of the channel.

370. **Varying Stream.**—In a stream whose area of cross-section varies, and in which, consequently, the mean velocity varies at different cross-sections, the loss of dynamic head is the sum of that expended in overcoming friction, and of that expended in producing increased velocity, when the velocity increases, or the difference of those two quantities when the velocity diminishes, which difference may be positive or negative, and may represent either a loss or a gain of head. The following method of representing this principle symbolically is the most convenient for practical purposes. In fig. 146, let the origin of co-ordinates be taken at a point O *completely below* the part of the stream to be considered; let horizontal abscissæ x be measured *against* the direction of flow, and vertical ordinates to the surface of the stream, z, upwards. Consider any indefinitely short portion of the stream whose

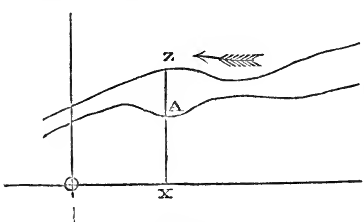


Fig. 146.

horizontal length is $d\,x$; in practice this may almost always be considered as equal to the actual length. The fall in that portion of the stream is $d\,z$, and the acceleration $-d\,v$, because of v being opposite to x. Then modifying the expression for the loss of head due to friction in Equation 3 of Article 369 to meet the present case, and adding the loss of head due to acceleration, we find

$$d\,z = f \cdot \frac{s\,d\,x}{\text{A}} \cdot \frac{v^2}{2\,g} - \frac{v\,d\,v}{g} \text{..................(1.)}$$

In applying this differential equation to the solution of any particular problem, for v is to be put Q ÷ A, and for A and s are to be put their values in terms of x and z. Thus is obtained a differential equation between z and x, and the constant quantity Q, the flow per second. If Q is known, then it is sufficient to know the value of z for one particular value of x, in order to be able to determine the integral equation between z and x If Q is unknown, the values of z for two particular values of x, or of z and $\frac{d\,z}{d\,x}$ (the declivity), for one particular value of x, are required for the solution, which comprehends the determination of the value of O.

371. **The Friction in a Pipe Running Full** produces loss of dynamic head according to the same law with the friction in a channel, except that the dynamic head is now the sum of the elevation of the pipe above a given level, and of the height due to the pressure within it. The differential equation which expresses this is as follows:—Let $d\,l$ be the length of an indefinitely short portion of a pipe measured *in* the direction of flow, s its internal circumference, A its area of section, z its elevation above a given level, p the pressure within it, h the dynamic head. Then the loss of head is

$$-d\,h = -d\,z - \frac{d\,p}{\varrho} = \frac{v\,d\,v}{g} + f \cdot \frac{s\,d\,l}{\text{A}} \cdot \frac{v^2}{2\,g} \text{...........(1.)}$$

The ratio $\frac{d\,h}{d\,l}$ is called the *virtual* or *hydraulic declivity*, being the rate of declivity of an open channel of the same flow, area, and hydraulic mean depth. This may differ to any extent from the *actual declivity* of the pipe, $\frac{d\,z}{d\,l}$.

When the pipe is of uniform section, $d\,v = 0$, and the first term of the right-hand side of Equation 1 vanishes.

When the section of the pipe varies, s and A are given functions of l. If Q is given, $v = \text{Q} \div \text{A}$ is also a given function of l; and to solve the equation completely, there is only required in addition the value of h for one particular value of l. If Q is unknown, the values of h for two particular values of l, or of h and $\frac{d\,h}{d\,l}$ for one particular value of l, are required for the solution, which comprehends the determination of Q.

372. **Resistance of Mouthpieces.**—A mouthpiece is the part of a channel or pipe immediately adjoining a reservoir. The internal friction of the fluid on entering a mouthpiece causes a loss of head equal to the height due to the velocity multiplied by a constant depending on the figure of the mouthpiece, whose values for certain figures have been found empirically; that is to say, let $-\Delta\,h$ be the loss of head; then

$$-\Delta\,h = \frac{f'\,v^2}{2\,g}, \quad \text{.........................(1.)}$$

f' being a constant.

For the mouthpiece of a cylindrical pipe, issuing from the flat side of a reservoir, and making the angle i with a normal to the side of the reservoir, according to Weisbach,

$$f' = 0{\cdot}505 + 0{\cdot}303 \sin i + 0{\cdot}226 \sin^2 i \quad \text{............(2.)}$$

373. **The Resistance of Curves and Knees** in pipes causes a loss of head equal to the height due to the velocity multiplied by a coefficient, whose values, according to Weisbach, are given by the following formulæ:—For *curves*, let i be the arc to radius unity, r the radius of curvature of the centre line of the pipe, and d its diameter.

Then for a circular pipe,

$$f'' = \frac{i}{\pi}\left\{0{\cdot}131 + 1{\cdot}847\left(\frac{d}{2\,r}\right)^{\frac{7}{2}}\right\};$$

and for a rectangular pipe,

$$f'' = \frac{i}{\pi}\left\{0{\cdot}124 + 3{\cdot}104\left(\frac{d}{2\,r}\right)^{\frac{7}{2}}\right\}; \quad \text{..............(1.)}$$

for *knees*, or sudden bends, let i be the angle made by the two portions of the pipe at either side of the knee with each other; then

$$f'' = 0{\cdot}9457 \sin^2 \frac{i}{2} + 2{\cdot}047 \sin^4 \frac{i}{2} \quad \text{...................(2.)}$$

374. **A Sudden Enlargement** of the channel in which a stream of liquid flows, causes a sudden diminution of the mean velocity in the same proportion as that in which the area of section is in-

creased. Thus, let v_1 be the velocity in the narrower portion of the channel, and let m be the number expressing the ratio in which the channel is suddenly enlarged: the velocity in the enlarged part is $\frac{v_1}{m}$. Now it appears from experiment, that the actual energy due to the velocity of the narrow stream *relatively* to the wide stream, that is, to the difference $v_1\left(1-\frac{1}{m}\right)$, is expended in overcoming the internal fluid friction of eddies, and so producing heat; so that there is a *loss of total head*, represented by

$$\frac{v_1^2}{2g}\left(1-\frac{1}{m}\right)^2 \dots\dots\dots\dots (1.)$$

375. **The General Problem** of the flow of a stream with friction is thus expressed:—Let $h_1 + \frac{v_1^2}{2g}$, and $h_2 + \frac{v_2^2}{2g}$, be the total heads at the beginning and end of the stream respectively; then the loss of total head is represented by

$$h_1 - h_2 + \frac{v_1^2 - v_2^2}{2g} = \Sigma \cdot \mathrm{F}\,\frac{v^2}{2g} \dots\dots\dots\dots (1.)$$

where the right-hand side of the equation represents the sum of all the losses of head due to the friction in various parts of the channel.

PART VI.

THEORY OF MACHINES.

CHAPTER I.

DEFINITIONS AND GENERAL PRINCIPLES.

376. **Nature and Division of the Subject.**—In the present Part of this work, machines are to be considered not merely as modifying motion, but also as modifying force, and transmitting energy from one body to another. The theory of machines consists chiefly in the application of the principles of dynamics to trains of mechanism; and therefore much of the present Part of this treatise will consist of references back to Parts II. and V.

There are two fundamentally different ways of considering a machine, each of which must be employed in succession, in order to obtain a complete knowledge of its working.

I. In the first place is considered the action of the machine during a certain period of time, with a view to the determination of its EFFICIENCY; that is, the ratio which the *useful* part of its work bears to the whole expenditure of energy. The motion of every ordinary machine is either uniform or periodical; and therefore the principle of the equality of energy and work is fulfilled, either constantly, or periodically at the end of each period or cycle of changes in the motion of the machine.

II. In the second place is to be considered the action of the machine during intervals of time less than its period or cycle, if its motion is periodic, in order to determine the law of the periodic changes in the motions of the pieces of which the machine consists, and of the periodic or reciprocating forces by which such changes are produced.

377. **A Prime Mover** is an engine, or combination of moving pieces, which serves to transfer energy from those bodies which naturally develop it, to those by means of which it is to be employed, and to transform energy from the various forms in which

it may occur, such as chemical affinity, heat, or electricity, into the form of mechanical energy, or energy of force and motion. The mechanism of a prime mover comprehends all those parts by means of which it regulates its own operations.

The *useful work* of a prime mover is the energy which it transmits to any machine driven by it; and its *efficiency* is the ratio of that useful work to the whole energy received by it from a natural source of energy.

The *effect* or *available power* of a prime mover is its useful work in some given unit of time, such as a second, a minute, an hour, or a day.

378. **The Regulator** of a prime mover is some piece of apparatus by which the rate at which it receives energy from the source of energy can be varied.

379. **A Governor** is a self-acting adjusting apparatus, usually consisting of a pair of rotating pendulums, whose angle of deviation from their axis depends upon the speed.

380. **Fluctuations of Speed** in a machine are caused by the alternate excess of the energy received above the work performed, and of the work performed above the energy received, which produce an alternate increase and diminution of actual energy.

381. **A Fly-Wheel** is a wheel with a heavy rim, whose great moment of inertia reduces the coefficient of fluctuation of speed to a certain fixed amount.

382. **A Brake** is employed to stop a machine in a shorter time than can be done by simply suspending the effort of the prime mover.

383. **Useful and Lost Work.**—The whole work performed by a machine is distinguished into *useful work*, being that performed in producing the effect for which the machine is designed, and *lost work* being that performed in producing other effects.

384. **Useful and Prejudicial Resistance** are overcome in performing useful work and lost work respectively.

385. **The Efficiency** of a machine is a fraction expressing the ratio of the useful work to the whole work performed, which is equal to the energy expended. The limit to the efficiency of a machine is *unity*, denoting the efficiency of a perfect machine in which no work is lost. The object of improvements in machines is to bring their efficiency as near to unity as possible.

386. **Power and Effect; Horse Power.**—The *power* of a machine is the energy exerted, and the *effect*, the useful work performed, in some interval of time of definite length.

The unit of power called conventionally a *horse power*, is 550 foot pounds per second, or 33,000 foot pounds per minute, or 1,980,000 foot pounds per hour. The effect is equal to the power multiplied by the efficiency.

387. **Driving Point; Train; Working Point.**—The driving point is that through which the resultant effort of the prime mover acts. The train is the series of pieces which transmit motion and force from the driving point to the working point, through which acts the resultant of the resistance of the useful work.

388. **Points of Resistance** are points in the train of mechanism through which the resultants of prejudicial resistances act.

389. **Efficiencies of Pieces of a Train.**—The useful work of an intermediate piece in a train of mechanism consists in driving the piece which follows it, and is less than the energy exerted upon it by the amount of the work lost in overcoming its own friction. Hence the efficiency of such an intermediate piece is the ratio of the work performed by it in driving the following piece, to the energy exerted on it by the preceding piece; and it is evident that *the efficiency of a machine is the product of the efficiencies of the series of moving pieces which transmit energy from the driving point to the working point.* The same principles apply to a train of *successive machines*, each driving that which follows it.

CHAPTER II.

OF THE PERFORMANCE OF WORK BY MACHINES.

SECTION 1.—OF WORK.

390. **The Action of a Machine** is to produce motion against Resistance. For example, if the machine is one for lifting solid bodies, such as a crane, or fluid bodies, such as a pump, its action is to produce upward motion of the lifted body against the resistance arising from gravity; that is, against its own weight: if the machine is one for propulsion, such as a locomotive engine, its action is to produce horizontal or inclined motion of a load against the resistance arising from friction, or from friction and gravity combined: if it is one for shaping materials, such as a planing machine, its action is to produce relative motion of the tool and of the piece of material shaped by it, against the resistance which that material offers to having part of its surface removed; and so of other machines.

391. **Work.**—The action of a machine is measured, or expressed as a definite quantity, by multiplying the motion which it produces into the resistance, or force directly opposed to that motion, which it overcomes; the product resulting from that multiplication being called WORK.

In Britain, the distances moved through by pieces of mechanism are usually expressed in feet; the resistances overcome, in pounds avoirdupois; and quantities of work, found by multiplying distances in feet by resistances in pounds, are said to consist of so many *foot-pounds.* Thus the work done in lifting a weight of one pound, through a height of one foot, is *one foot-pound;* the work done in lifting a weight of twenty pounds, through a height of one hundred feet, is $20 \times 100 = 2{,}000$ foot-pounds.

In France, distances are expressed in mètres, resistances overcome in kilogrammes, and quantities of work in what are called *kilogrammètres,* one kilogrammètre being the work performed in lifting a weight of one killogramme through a height of one mètre.

392. **The Rate of Work** of a machine means, the quantity of work which it performs in some given interval of time, such as a second, a minute, or an hour. It may be expressed in units of work (such as foot-pounds) per second, per minute, or per hour, as the case

may be; but there is a peculiar unit of power appropriated to its expression, called a HORSE-POWER, which is, in Britain,

550 foot-pounds per second,
or 33,000 foot-pounds per minute,
or 1,980,000 foot-pounds per hour.

In France, the term FORCE DE CHEVAL is applied to the following rate of work:—

		Foot-pounds.
75 kilogrammètres per second	=	$542\frac{1}{2}$
or 4,500 kilogrammètres per minute	=	32,549
or 270,000 kilogrammètres per hour	=	1,952,932

being about one-seventieth part less than the British horse-power.

393. **Velocity.**—If the *velocity of the motion* which a machine causes to be performed against a given resistance be given, then the product of that velocity into the resistance obviously gives the rate of work, or effective power. If the velocity is given in feet per second, and the resistance in pounds, then their product is the rate of work in foot-pounds per second, and so of minutes, or hours, or other units of time.

It is usually most convenient, for purposes of calculation, to express the velocities of the parts of machines either in feet per second or in feet per minute. For certain kinetic calculations the second is the more convenient unit of time · in stating the performance of machines for practical purposes, the minute is the unit most commonly employed.

394. **Work in Terms of Angular Motion.**—When a resisting force opposes the motion of a part of a machine which moves round a fixed axis, such as a wheel, an axis, or a crank, the product of the amount of that resistance into its *leverage* (that is, the perpendicular distance of the line along which it acts from the fixed axis) is called the *moment*, or *statical moment*, of the resistance. If the resistance is expressed in pounds, and its leverage in feet, then its moment is expressed in terms of a measure which may be called a *foot-pound*, but which, nevertheless, is a quantity of an entirely different kind from a foot-pound of work.

Suppose now that the body to whose motion the resistance is opposed turns through any number of revolutions, or parts of a revolution; and let T denote the angle through which it turns, expressed in revolutions, and parts of a revolution, also, let

$$2\pi = 6{\cdot}2832 = \frac{710}{113}$$

denote, as is customary, the ratio of the circumference of a circle to its radius. Then the distance through which the given resistance is overcome is expressed by

$$\text{the leverage} \times 2\pi \times \mathrm{T};$$

that is, by the product of the circumference of a circle whose radius is the leverage, into the number of turns and fractions of a turn made by the rotating body.

The distance thus found being multiplied by the resistance overcome, gives the work performed; that is to say,

The work performed
=the resistance × the leverage × 2 π × T:

But the product of the resistance into the leverage is what is called the *moment* of the resistance, and the product $2\pi\mathrm{T}$ is called the *angular motion* of the rotating body; consequently,

The work performed
= the moment of the resistance × the angular motion.

The mode of computing the work indicated by this last equation is often more convenient than the direct mode already explained in Article 391.

The angular motion $2\pi\mathrm{T}$ of a body during some definite unit of time, as a second or a minute, is called its *angular velocity;* that is to say, *angular velocity is the product of the turns and fractions of a turn made in an unit of time into the ratio of the circumference of a circle to its radius.* Hence it appears that

The rate of work
= the moment of the resistance × the angular velocity.

395. **Work in Terms of Pressure and Volume.**—If the resistance overcome be a pressure uniformly distributed over an area, as when a piston drives a fluid before it, then the amount of that resistance is equal to the intensity of the pressure, expressed in units of force on each unit of area (for example, in pounds on the square inch, or pounds on the square foot) multiplied by the area of the surface at which the pressure acts, if that area is perpendicular to the direction of the motion; or, if not, then by the projection of that area on a plane perpendicular to the direction of motion. In practice, when the *area of a piston* is spoken of, it is always understood to mean the projection above mentioned.

Now, when a plane area is multiplied into the distance through which it moves in a direction perpendicular to itself, if its motion is straight, or into the distance through which its centre of gravity moves, if its motion is curved, the product is the *volume of the space traversed* by the piston.

Hence the work performed by a piston in driving a fluid before it, or by a fluid in driving a piston before it, may be expressed in either of the following ways:—

Resistance × *distance traversed*
= *intensity of pressure* × *area* × *distance traversed;*
= *intensity of pressure* × *volume traversed.*

In order to compute the work in foot-pounds, if the pressure is stated in pounds on the square foot, the area should be stated in square feet, and the volume in cubic feet; if the pressure is stated in pounds on the square inch, the area should be stated in square inches, and the volume in units, each of which is a prism of one foot in length, and one square inch in area; that is, of $\frac{1}{144}$ of a cubic foot in volume.

396. **Algebraical Expressions for Work.**—To express the results of the preceding articles in algebraical symbols, let

s denote the distance in feet through which a resistance is overcome in a given time;

R, the amount of the resistance overcome in pounds.

Also, supposing the resistance to be overcome by a piece which turns about an axis, let

T be the number of turns and fractions of a turn made in the given time, and $i = 2\pi T = 6{\cdot}2832\ T$ the angular motion in the given time; and let

l be the leverage of the resistance; that is, the perpendicular distance of the line along which it acts from the axis of motion; so that $s = i\,l$, and $R\,l$ is the statical moment of the resistance. Supposing the resistance to be a pressure, exerted between a piston and a fluid, let A be the area or projected area of a piston, and p the intensity of the pressure in pounds per unit of area.

Then the following expressions all give quantities of work in the given time in foot-pounds:—

$$R\,s;\ i\,R\,l;\ p\,A\,s;\ i\,p\,A\,l.$$

The last of these expressions is applicable to a piston turning on an axis, for which l denotes the distance from the axis to the centre of gravity of the area A.

397. **Work against an Oblique Force.**—The resistance directly due to a force which acts against a moving body in a direction oblique to that in which the body moves, is found by resolving that force into two components, one at right angles to the direction of motion, which may be called a *lateral force,* and which must be balanced by an equal and opposite lateral force, unless it takes effect by altering the direction of the body's motion, and the other component directly opposed to the body's motion, which is the

resistance required. That resolution is effected by means of the well-known principle of the parallelogram of forces as follows:—

In fig. 147, let A represent the point at which a resistance is overcome, A B the direction in which that point is moving, and let $\overline{\mathrm{AF}}$ be a line whose direction and length represent the direction and magnitude of a force obliquely opposed to the motion of A.

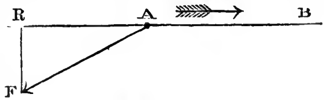


Fig. 147.

From F upon B A produced, let fall the perpendicular $\overline{\mathrm{FR}}$; the length of that perpendicular will represent the magnitude of the lateral component of the oblique force, and the length $\overline{\mathrm{AR}}$ will represent the direct component or resistance.

The work done against an oblique resisting force may also be calculated by resolving the motion into a direct component in the line of action of the force, and a transverse component, and multiplying the whole force by the direct component of the motion.

398. **Summation of Quantities of Work.**—In every machine, resistances are overcome during the same interval of time, by different moving pieces, and at different points in the same moving piece; and the whole work performed during the given interval is found by adding together the several products of the resistances into the respective distances through which they are simultaneously overcome. It is convenient, in algebraical symbols, to denote the result of that summation by the symbol—

$$\Sigma \cdot \mathrm{R}\, s; \qquad (1.)$$

in which Σ denotes the operation of taking the sum of a set of quantities of the kind denoted by the symbols to which it is prefixed.

When the resistances are overcome by pieces turning upon axes, the above sum may be expressed in the form—

$$\Sigma \cdot i\, \mathrm{R}\, l; \qquad (2.)$$

and so of other modes of expressing quantities of work.

The following are particular cases of the summation of quantities of work performed at different points :—

I. In a *shifting piece*, or one which has the kind of movement called *translation* only, the velocities of every point at a given instant are equal and parallel; hence, in a given interval of time, the motions of all the points are equal; and the work performed is to be found by multiplying the *sum of the resistances* into the motion as a common factor; an operation expressed algebraically thus—

$$s\, \Sigma\, \mathrm{R}; \qquad (3.)$$

II. For a *turning piece*, the angular motions of all the points during a given interval of time are equal; and the work performed is to be found by multiplying the *sum of the moments of the resistances relatively to the axis* into the angular motion as a common factor—an operation expressed algebraically thus—

$$i \Sigma \cdot \mathrm{R}\, l; \qquad (4.)$$

The sum denoted by $\Sigma \cdot \mathrm{R}\, l$ is the *total moment of resistance* of the piece in question.

III. In every *train of mechanism*, the *proportions* amongst the motions performed during a given interval of time by the several moving pieces, can be determined from the mode of connection of those pieces, independently of the absolute magnitudes of those motions, by the aid of the *Theory of Pure Mechanism*, Part II. This enables a calculation to be performed which is called *reducing the resistances to the driving point;* that is to say, determining the resistances, which, if they acted directly at the point where the motive power is applied to the machine, would require the same quantity of work to overcome them with the actual resistances.

Suppose, for example, that by the principles of pure mechanism it is found, that a certain point in a machine, where a resistance R is to be overcome, moves with a velocity bearing the ratio $n : 1$ to the velocity of the driving point. Then the work performed in overcoming that resistance will be the same as if a resistance n R were overcome directly at the driving point. If a similar calculation be made for each point in the machine where resistance is overcome, and the results added together, as the following symbol denotes :—

$$\Sigma \cdot n\, \mathrm{R}, \qquad (5.)$$

that sum is the *equivalent resistance at the driving point;* and if in a given interval of time the driving point moves through the distance s, then the work performed in that time is—

$$s\, \Sigma \cdot n\, \mathrm{R}. \qquad (6.)$$

The process above described is often applied to the steam engine, by reducing all the resistances overcome to equivalent resistances acting directly against the motion of the piston.

A similar method may be applied to the moments of resistances overcome by rotating pieces, so as to reduce them to *equivalent moments at the driving axle.* Thus, let a resistance R, with the leverage l, be overcome by a piece whose angular velocity of rotation bears the ratio $n : 1$ to that of the driving axle. Then the equivalent moment of resistance at the driving axle is $n\, \mathrm{R}\, l$; and if a similar calculation be made for each rotating piece in the machine which overcomes resistance, and the results added together, the sum—

$$\Sigma \cdot n\, \mathrm{R}\, l \qquad (7.)$$

is the total *equivalent moment of resistance* at the driving axle; and if in a given interval of time the driving axle turns through the arc i to radius unity, the work performed in that time is—

$$i \Sigma \cdot n R l. \qquad (8.)$$

IV. *Centre of gravity.*—The work performed in lifting a body is *the product of the weight of the body into the height through which its centre of gravity is lifted.*

If a machine lifts the centres of gravity of several bodies at once to heights either the same or different, the whole quantity of work performed in so doing is the sum of the several products of the weights and heights; but that quantity can also be computed by *multiplying the sum of all the weights into the height through which their common centre of gravity is lifted.*

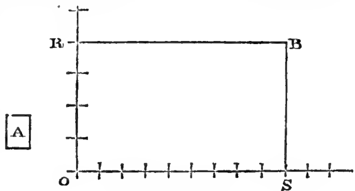


Fig. 148.

399. Representation of Work by an Area.—As a quantity of work is the product of two quantities, a force and a motion, it may be represented by the area of a plane figure, which is the product of two dimensions. Let the base of the rectangle A, fig. 148, represent *one foot* of motion, and its height *one pound* of resistance; then will its area represent one foot-pound of work.

In the larger rectangle, let the base $\overline{OS}$ represent a certain motion s, on the same scale with the base of the unit-area A; and let the height $\overline{OR}$ represent a certain resistance R, on the same scale with the height of the unit-area A; then will the number of times that the rectangle $\overline{OS} \cdot \overline{OR}$ contains the unit-rectangle A, express the number of foot-pounds in the quantity of work R s, which is performed in overcoming the resistance R through the distance s.

400. Work against Varying Resistance.—In fig. 149, let distances, as before, be represented by lengths measured along the base line O X of the figure; and let the magnitudes of the resistance overcome at each instant be represented by the lengths of ordinates drawn perpendicular to O X, and parallel to O Y:—For example, when the working body has moved through the distance represented by $\overline{OS}$, let the resistance be represented by the ordinate $\overline{SR}$.

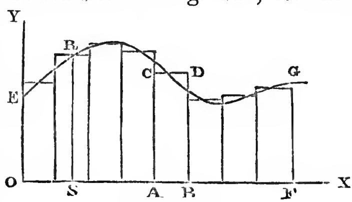


Fig. 149.

If the resistance were constant, the summits of those ordinates would lie in a straight line parallel to O X, like R B in fig. 148; but if the resistance varies continuously as the motion goes on, the summits of the ordinates will lie in a line, straight or curved, such as that marked E R G, fig. 149, which is not parallel to O X.

The values of the resistance at each instant being represented by the ordinates of a given line E R G, let it now be required to determine the work performed against that resistance during a motion represented by $\overline{\mathrm{O\,F}} = s$.

Suppose the area O E G F to be divided into bands by a series of parallel ordinates, such as A C and B D, and between the upper ends of those ordinates let a series of short lines, such as C D, be drawn parallel to O X, so as to form a stepped or serrated outline, consisting of lines parallel to O X and O Y alternately, and *approximating* to the given continuous line E G.

Now conceive the resistance, instead of varying continuously, to remain constant during each of the series of divisions into which the motion is divided by the parallel ordinates, and to change abruptly at the instants between those divisions, being represented for each division by the height of the rectangle which stands on that division: for example, during the division of the motion represented by A B, let the resistance be represented by A C, and so for other divisions.

Then the work performed during the division of the motion represented by $\overline{\mathrm{A\,B}}$, on the supposition of alternate constancy and abrupt variation of the resistance, is represented by the rectangle $\overline{\mathrm{A\,B}} \cdot \overline{\mathrm{A\,C}}$; and the whole work performed, on the same supposition during the whole motion $\overline{\mathrm{O\,F}}$, is represented by the sum of all the rectangles lying between the parallel ordinates; and inasmuch as the supposed mode of variation of the resistance represented by the stepped outline of those rectangles is an approximation to the real mode of variation represented by the continuous line E G, and is a closer approximation the closer and the more numerous the parallel ordinates are, so the sum of the rectangles is an approximation to the exact representation of the work performed against the continuously varying resistance, and is a closer approximation the closer and more numerous the ordinates are, and by making the ordinates numerous and close enough, can be made to differ from the exact representation by an amount less than any given difference.

But the sum of those rectangles is also an approximation to the area O E G F, bounded above by the continuous line E G, and is a closer approximation the closer and the more numerous the ordinates are, and by making the ordinates numerous and close enough, can be made to differ from the area O E G F by an amount less than any given difference.

Therefore the area O E G F, *bounded by the straight line* O F, *which* *represents the motion, by the line* E G, *whose ordinates represent the values of the resistance, and by the two ordinates* O E *and* F G, *represents exactly the work performed.* (See Article 34, page 17).

The MEAN RESISTANCE during the motion is found by dividing the area O E G F by the motion $\overline{\text{O F}}$.

401. **Useful Work and Lost Work.**—The useful work of a machine is that which is performed in effecting the purpose for which the machine is designed. The lost work is that which is performed in producing effects foreign to that purpose. The resistances overcome in performing those two kinds of work are called respectively *useful resistance* and *prejudicial resistance.*

The useful work and the lost work of a machine together make up its *total* or *gross work.*

In a pumping engine, for example, the useful work in a given time is the product of the weight of water lifted in that time into the height to which it is lifted : the lost work is that performed in overcoming the friction of the water in the pumps and pipes, the friction of the plungers, pistons, valves, and mechanism, and the resistance of the air pump and other parts of the engine.

For example, the useful work of a marine steam engine in a given time is the product of the resistance opposed by the water to the motion of the ship, into the distance through which she moves: the lost work is that performed in overcoming the resistance of the water to the motion of the propeller through it, the friction of the mechanism, and the other resistances of the engine, and in raising the temperature of the condensation water, of the gases which escape by the chimney, and of adjoining bodies.

There are some cases, such as those of muscular power and of windmills, in which the useful work of a prime mover can be determined, but not the lost work.

402. **The Work Performed against Friction** in a given time, between a pair of rubbing surfaces, is the product of that friction into the distance through which one surface slides over the other.

When the motion of one surface relatively to the other consists in rotation about an axis, the work performed may also be calculated by multiplying the relative *angular motion* of the surfaces to radius unity into the *moment of friction;* that is, the product of the friction into its leverage, which is the mean distance of the rubbing surfaces from the axis.

For a cylindrical journal, the leverage of the friction is simply the radius of the journal.

For a *flat pivot,* the leverage is two-thirds of the radius of the pivot.

For a *collar,* let r and r' be the inner and outer radii; then the leverage of the friction is

$$\frac{2}{3} \cdot \frac{r^3 - r'^3}{r^2 - r'^2} \qquad \text{(1.)}$$

In the *cup and ball* pivot, the end of the shaft, and the step on which it presses, present two recesses facing each other, into which are fitted two shallow cups of steel or hard bronze. Between the concave spherical surfaces of those cups is placed a steel ball, being either a complete sphere, or a lens having convex surfaces of a somewhat less radius than the concave surfaces of the cups. The moment of friction of this pivot is at first almost inappreciable, from the extreme smallness of the radius of the circles of contact of the ball and cups; but as they wear, that radius and the moment of friction increase.

By the rolling of two surfaces over each other without sliding, a resistance is caused, which is called sometimes "rolling friction," but more correctly *rolling resistance*. It is of the nature of a *couple* resisting rotation; its *moment* is found by multiplying the normal pressure between the rolling surfaces by an *arm* whose length depends on the nature of the rolling surfaces; and the work lost in an unit of time in overcoming it is the product of its moment by the *angular velocity* of the rolling surfaces relatively to each other. The following are approximate values of the arm in *decimals of a foot*:—

Oak upon oak,.........................	0·006	(Coulomb).
Lignum-vitæ on oak,...................	0·004	,,
Cast-iron on cast-iron,...............	0·002	(Tredgold).

The work lost in friction produces HEAT in the proportion of one British thermal unit, being so much heat as raises the temperature of a pound of water 1° of Fahr., for every 772 foot-pounds of lost work.

The heat produced by friction, when moderate in amount, is useful in softening and liquefying unguents; but when excessive it is prejudicial by decomposing the unguents, and sometimes even by softening the metal of the bearings, and raising their temperature so high as to set fire to neighbouring combustible matters.

Excessive heating is prevented by a constant and copious supply of a good unguent. When the velocity of rubbing is about four or five feet per second, the elevation of temperature is found to be, with good fatty and soapy unguents, 40° to 50° Fahr., with good mineral unguents, about 30°. The effect of friction upon the efficiency of machines will be considered at the end of this Part.

403. **Work of Acceleration.**—In order that the velocity of a body's motion may be changed, it must be acted upon by some other body with a force in the direction of the change of velocity, which force is proportional directly to the change of velocity, and to the mass of the body acted upon, and inversely to the time occupied in producing the change. If the change is an acceleration or increase of velocity, let the first body be called the *driven body*, and the second

the *driving body.* Then the force must act upon the driven body in the direction of its motion. Every force being a pair of equal and opposite actions between a pair of bodies, the same force which accelerates the driven body is a *resistance* as respects the driving body.

For example, during the commencement of the stroke of the piston of a steam engine, the velocity of the piston and of its rod is accelerated; and that acceleration is produced by a certain part of the pressure between the steam and the piston, being the excess of that pressure above the whole resistance which the piston has to overcome. The piston and its rod constitute the driven body; the steam is the driving body; and the same part of the pressure which accelerates the piston, acts as a *resistance* to the motion of the steam, in addition to the resistance which would have to be overcome if the velocity of the piston were uniform.

The resistance due to acceleration is computed in the following manner:—It is known by experiment, that if a body near the earth's surface is accelerated by the attraction of the earth,—that is, by its own weight, or by a force equal to its own weight, its velocity goes on continually increasing very nearly at the rate of 32·2 *feet per second of additional velocity, for each second during which the force acts.* This quantity varies in different latitudes, and at different elevations, but the value just given is near enough to the truth for purposes of mechanical engineering. For brevity's sake, it is usually denoted by the symbol g; so that, if at a given instant the velocity of a body is v_1 feet per second, and if its own weight, or an equal force, acts freely on it in the direction of its motion for t seconds, its velocity at the end of that time will have increased to

$$v_2 = v_1 + g\,t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

If the acceleration be at any different rate per second, *the force necessary to produce that acceleration, being the resistance on the driving body due to the acceleration of the driven body, bears the same proportion to the driven body's weight which the actual rate of acceleration bears to the rate of acceleration produced by gravity acting freely.* (In mètres per second, $g = 9·81$ nearly.)

To express this by symbols, let the weight of the driven body be denoted by W. Let its velocity at a given instant be v_1 feet per second; and let that velocity increase at an uniform rate, so that at an instant t seconds later, it is v_2 feet per second.

Let f denote the rate of acceleration; then

$$f = \frac{v_2 - v_1}{t}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

and the force R necessary to produce it will be given by the proportion,

$$g : f :: W : R;$$

that is to say,

$$R = \frac{f W}{g} = \frac{W (v_2 - v_1)}{g t} \quad \text{.....................(3.)}$$

The factor $\frac{W}{g}$, in the above expression, is called the MASS of the driven body; and being the same for the same body, in what place soever it may be, is held to represent the *quantity of matter* in the body. (See Article 195, page 117.)

The product $\frac{W v}{g}$ of the mass of a body into its velocity at any instant, is called its MOMENTUM; so that the resistance due to a given acceleration is equal to *the increase of momentum divided by the time which that increase occupies.*

If the product of a force by which a body is accelerated, equal and opposite to the resistance due to acceleration, into the time during which it acts, be called IMPULSE, the same principle may be otherwise stated by saying, that the *increase of momentum is equal to the impulse by which it is caused.*

If the rate of acceleration is not constant, but variable, the force R varies along with it. In this case, the value, at a given instant of the rate of acceleration, is represented by $f = \frac{d v}{d t}$, and the corresponding value of the force is

$$R = \frac{f W}{g} = \frac{W}{g} \cdot \frac{d v}{d t} \quad \text{.....................(4.)}$$

The WORK PERFORMED in accelerating a body is the product of the resistance due to the rate of acceleration into the distance moved through by the driven body while the acceleration is going on. The resistance is equal to the mass of the body, multiplied by the increase of velocity, and divided by the time which that increase occupies. The distance moved through is the product of the mean velocity into the same time. Therefore, the work performed is equal to the mass of the body multiplied by the increase of the velocity, and by the mean velocity; that is, *to the mass of the body, multiplied by the increase of the half-square of its velocity.*

To express this by symbols, in the case of an uniform rate of acceleration, let s denote the distance moved through by the driven body during the acceleration; then

$$s = \frac{v_2 + v_1}{2} t; \quad \text{..............................(5.)}$$

which being multiplied by Equation 3, gives for the work of acceleration,

$$R\,s = \frac{W}{g} \cdot \frac{v_2 - v_1}{t} \cdot \frac{v_2 + v_1}{2} \cdot t = \frac{W}{g} \cdot \frac{v_2^2 - v_1^2}{2} \ldots\ldots\ldots\ldots (6.)$$

In the case of a variable rate of acceleration, let v denote the mean velocity, and $d\,s$ the distance moved through, in an interval of time $d\,t$ so short that the increase of velocity $d\,v$ is indefinitely small compared with the mean velocity. Then

$$d\,s = v\,d\,t; \ldots\ldots\ldots\ldots (7.)$$

which being multiplied by Equation 4, gives for the work of acceleration during the interval $d\,t$,

$$R\,d\,s = \frac{W}{g} \cdot \frac{d\,v}{d\,t} \cdot v\,d\,t$$

$$= \frac{W}{g} \cdot v\,d\,v; \ldots\ldots\ldots\ldots (8.)$$

and the *integration* of this expression (see Article 29) gives for the work of acceleration during a finite interval,

$$\int R\,d\,s = \frac{W}{g} \int v\,d\,v = \frac{W}{g} \cdot \frac{v_2^2 - v_1^2}{2} \ldots\ldots\ldots\ldots (9.)$$

being the same with the result already arrived at in Equation 6.

From Equation 9 it appears that *the work performed in producing a given acceleration depends on the initial and final velocities,* v_1 *and* v_2, *and not on the intermediate changes of velocity.*

If a body falls freely under the action of gravity from a state of rest through a height h, so that its initial velocity is 0, and its final velocity v, the work of acceleration performed by the earth on the body is simply the product $W\,h$ of the weight of the body into the height of fall. Comparing this with Equation 6, we find—

$$h = \frac{v^2}{2\,g} \ldots\ldots\ldots\ldots (10.)$$

This quantity is called the *height*, or *fall, due to the velocity* v; and from Equations 6 and 9 it appears that *the work performed in producing a given acceleration is the same with that performed in lifting the driven body through the difference of the heights due to its initial and final velocities.*

If work of acceleration is performed by a prime mover upon bodies which neither form part of the prime mover itself, nor of the machines which it is intended to drive, that work is lost; as when a marine engine performs work of acceleration on the water that is struck by the propeller.

Work of acceleration performed on the moving pieces of the prime mover itself, or of the machinery driven by it, is not necessarily lost, as will afterwards appear. (Article 413.)

404. **Summation of Work of Acceleration.**—If several pieces of a machine have their velocities increased at the same time, the work performed in accelerating them is the sum of the several quantities of work due to the acceleration of the respective pieces; a result expressed in symbols by

$$\Sigma \left\{ \frac{W}{g} \cdot \frac{v_2^2 - v_1^2}{2} \right\} \quad \text{.........................(1.)}$$

The process of finding that sum is facilitated and abridged in certain cases by special methods.

I. *Accelerated Rotation.*—Let a denote the angular velocity of a solid body rotating about a fixed axis;—that is, as explained in Article 87, the velocity of a point in the body whose radius-vector, or distance from the axis, is unity.

Then the velocity of a particle whose distance from the axis is r is

$$v = a\, r; \quad \text{..............................(2.)}$$

and if in a given interval of time the angular velocity is accelerated from the value a_1, to the value a_2, the increase of the velocity of the particle in question is

$$v_2 - v_1 = r\,(a_2 - a_1) \quad \text{.......................(3.)}$$

Let w denote the weight, and $\frac{w}{g}$ the mass of the particle in question. Then the work performed in accelerating it, being equal to the product of its mass into the increase of the half-square of its velocity, is also equal to *the product of its mass into the square of its radius-vector, and into the increase of the half-square of the angular velocity;* that is to say, in symbols,

$$\frac{w}{g} \cdot \frac{v_2^2 - v_1^2}{2} = \frac{w\, r^2}{g} \cdot \frac{a_2^2 - a_1^2}{2} \quad \text{.....................(4.)}$$

To find the work of acceleration for the whole body, it is to be conceived to be divided into small particles, whose velocities at any given instant, and also their accelerations, are proportional to their distances from the axis; then the work of acceleration is to be found for each particle, and the results added together. But in the sum so obtained, the increase of the half-square of the angular velocity is a common factor, having the same value for each particle of the body; and the rate of acceleration produced by gravity, $g = 32{\cdot}2$ is a common divisor. It is therefore sufficient *to add together the* *products of the weight of each particle* (w) *into the square of its radius-vector* (r^2), *and to multiply the sum so obtained* ($\Sigma \cdot w r^2$) by *the increase of the half-square of the angular velocity* $\left(\frac{1}{2}(a_2^2 - a_1^2)\right)$, *and divide by the rate of acceleration due to gravity* (g). The result, viz.:—

$$\Sigma \left\{ \frac{w}{g} \cdot \frac{v_2^2 - v_1^2}{2} \right\} = \frac{a_2^2 - a_1^2}{2g} \cdot \Sigma w r^2 \text{.............(5.)}$$

is the work of acceleration sought. In fact, the sum $\Sigma w r^2$ is *the weight of a body, which, if concentrated at the distance unity from the axis of rotation, would require the same work to produce a given increase of angular velocity which the actual body requires.*

405. **Reduced Inertia.**—If in a certain machine, a moving piece whose weight is W has a velocity always bearing the ratio $n : 1$ to the velocity of the driving point, it is evident that when the driving point undergoes a given acceleration, the work performed in producing the corresponding acceleration in the piece in question is the same with that which would have been required if a weight n^2 W had been concentrated at the driving point, the work performed in producing the acceleration depending on the *square* of the velocity.

If a similar calculation be performed for each moving piece in the machine, and the results added together, the sum

$$\Sigma \cdot n^2 W \text{............................(1.)}$$

gives the weight which, being concentrated at the driving point, would require the same work for a given acceleration of the driving point that the actual machine requires; so that if v_1 is the initial, and v_2 the final velocity of the driving point, the work of acceleration of the whole machine is

$$\frac{v_2^2 - v_1^2}{2g} \cdot \Sigma \cdot n^2 W \text{........................(2.)}$$

This operation may be called *the reduction of the inertia to the driving point.* Mr. Moseley, by whom it was first introduced into the theory of machines, calls the expression (1.) the "*coefficient of steadiness.*"

In finding the reduced inertia of a machine, the mass of each rotating piece is to be treated as if concentrated at a distance from its axis equal to its radius of gyration ϱ; so that if v represents the velocity of the driving point at any instant, and a the corresponding angular velocity of the rotating piece in question, we are to make

$$n^2 = \frac{a^2 \varrho^2}{v^2} \text{..............................(3.)}$$

in performing the calculation expressed by the formula (1.)

406. **Summary of Various Kinds of Work.**—In order to present at one view the symbolical expression of the various modes of performing work described in the preceding articles, let it be supposed that in a certain interval of time dt the driving point of a machine moves through the distance ds; that during the same time its centre of gravity is elevated through the height dh; that resistances, any one of which is represented by R, are overcome at points, the respective ratios of whose velocities to that of the driving point are denoted by n; that the weight of any piece of the mechanism is W, and that n' denotes the ratio of its velocity (or if it rotates, the ratio of the velocity of the end of its radius of gyration) to the velocity of the driving point; and that the driving point, whose mean velocity is $v = \frac{ds}{dt}$, undergoes the acceleration dv. Then the *whole work performed* during the interval in question is

$$dh \cdot \Sigma \mathrm{W} + ds \cdot \Sigma n \mathrm{R} + \frac{v\, dv}{g} \cdot \Sigma n'^2 \mathrm{W} \qquad (1.)$$

The *mean total resistance, reduced to the driving point,* may be computed by dividing the above expression by the motion of the driving point $ds = v\,dt$, giving the following result:—

$$\frac{dh}{ds} \cdot \Sigma \mathrm{W} + \Sigma n \mathrm{R} + \frac{dv}{g\,dt} \cdot \Sigma n'^2 \mathrm{W} \qquad (2.)$$

Section 2.—Of Energy, Power, and Efficiency.

407. **Condition of Uniform Speed.**—According to the first law of motion, in order that a body may move uniformly, the forces applied to it, if any, must balance each other; and the same principle holds for a machine consisting of any number of bodies.

When the *direction* of a body's motion varies, but not the *velocity*, the lateral force required to produce the change of direction depends on the principles set forth in Article 335; but the condition of balance still holds for the forces which act *along* the direction of the body's motion, that is, for the *efforts* and *resistances;* so that, whether for a single body or for a machine, the condition of *uniform velocity* is, that the *efforts shall balance the resistances.*

In a machine, this condition must be fulfilled for each of the single moving pieces of which it consists.

It also follows, from the principles of statics, that in any body, system, or machine, that condition is fulfilled when *the sum of the products of the efforts into the velocities of their respective points of action is equal to the sum of the products of the resistances into the velocities of the points where they are overcome.*

Thus, let v be the velocity of a *driving point*, or point where an effort P is applied; v' the velocity of a *working point*, or point where a resistance R is overcome; the condition of uniform velocity for any body, system, or machine is

$$\Sigma \cdot P v = \Sigma \cdot R v' \text{.........................(1.)}$$

If there be only one driving point, or if the velocities of all the driving points be alike, then P being the total effort, the single product $P v$ may be put in in place of the sum $\Sigma \cdot P v$; reducing the above equation to

$$P v = \Sigma \cdot R v' \text{...........................(2.)}$$

Referring now to Article 398, let the machine be one in which the *comparative* or *proportionate* velocities of all the points at a given instant are known independently of their absolute velocities, from the construction of the machine; so that, for example, the velocity of the point where the resistance R is overcome bears to that of the driving point the ratio

$$\frac{v'}{v} = n;$$

then the condition of uniform speed may be thus expressed:—

$$P = \Sigma \cdot n R; \text{...........................(3.)}$$

that is, *the total effort is equal to the sum of the resistances reduced to the driving point.*

408 **Energy—Potential Energy.**—*Energy* means *capacity for performing work*, and is expressed, like work, by the product of a force into a space.

The energy of an effort, sometimes called "*potential energy*" (to distinguish it from another form of energy to be referred to in Article 414), is the *product of the effort into the distance through which it is capable of acting.* Thus, if a weight of 100 pounds be placed at an elevation of 20 feet above the ground, or above the lowest plane to which the circumstances of the case admit of its descending, that weight is said to possess potential energy to the amount of $100 \times 20 = 2{,}000$ *foot-pounds;* which means, that in descending from its actual elevation to the lowest point of its course, the weight is *capable of performing work* to that amount.

To take another example, let there be a reservoir containing 10,000,000 gallons of water, in such a position that the centre of gravity of the mass of water in the reservoir is 100 feet above the lowest point to which it can be made to descend while overcoming resistance. Then as a gallon of water weighs 10 lbs., the weight of the store of water is 100,000,000 lbs., which being multiplied by the height through which that weight is capable of acting, 100 feet, gives 10,000,000,000 foot-pounds for the potential energy of the weight of the store of water.

409. **Equality of Energy Exerted and Work Performed, or the Conservation of Energy.**—When an effort actually does drive its point of application through a certain distance, energy to the amount of the product of the effort into that distance is said to be *exerted;* and the potential energy, or energy which remains *capable of being exerted,* is to that amount diminished.

When the energy is exerted in driving a machine at an uniform speed, it is *equal to the work performed.*

To express this algebraically, let t denote the time during which the energy is exerted, v the velocity of a driving point at which an effort P is applied, s the distance through which it is driven, v' the velocity of any working point at which a resistance R is overcome, s' the distance through which it is driven; then

$$s = v\,t;\; s' = v'\,t;$$

and multiplying Equation 1 of Article 407 by the time t, we obtain the following equation:—

$$\Sigma \cdot \mathrm{P}\,v\,t = \Sigma \cdot \mathrm{R}\,v'\,t = \Sigma \cdot \mathrm{P}\,s = \Sigma \cdot \mathrm{R}\,s';\ldots\ldots\ldots\ldots(1.)$$

which expresses the equality of energy exerted, and work performed, for constant efforts and resistances.

When the efforts and resistances vary, it is sufficient to refer to Articles 400 and 29, to shew that the same principle is expressed as follows:—

$$\Sigma \int \mathrm{P}\,d\,s = \Sigma \int \mathrm{R}\,d\,s';\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

where the symbol $\int$ expresses the operation of finding the work performed against a varying resistance, or the energy exerted by a varying effort, as the case may be; and the symbol Σ expresses the operation of adding together the quantities of energy exerted, or work performed, as the case may be, at different points of the machine.

410. **Various Factors of Energy.**—A quantity of energy, like a quantity of work, may be computed by multiplying either a force into a distance, or a statical moment into an angular motion, or the intensity of a pressure into a volume. These processes have already been explained in detail in Articles 394 and 395, pages 244 to 246.

411. **The Energy Exerted in Producing Acceleration** is equal to the work of acceleration, whose amount has been investigated in Articles 403 and 404, pages 252 to 257.

412. **The Accelerating Effort** by which a given increase of velocity in a given mass is produced, and which is exerted by the *driving body* against the *driven body,* is equal and opposite to the resistance due to acceleration which the driven body exerts against the driving body, and whose amount has been given in Articles

403 and 404. Referring, therefore, to Equations 4 and 8 of Article 403, we find the two following expressions, the first of which gives the accelerating effort required to produce a given acceleration $d\,v$ in a body whose weight is W, when the *time* $d\,t$ in which that acceleration is to be produced is given, and the second, the same accelerating effort, when the *distance* $d\,s = v\,d\,t$ in which the acceleration is to be produced is given:—

$$P = \frac{W}{g} \cdot \frac{d\,v}{d\,t} \qquad (1.)$$

$$= \frac{W}{g} \cdot \frac{v\,d\,v}{d\,s} = \frac{W}{g} \cdot \frac{d\,(v^2)}{2\,d\,s} \qquad (2.)$$

Referring next to Article 404, page 257, we find, from Equation 5, that the work of acceleration corresponding to an increase $d\,a$ in the angular velocity of a rotating body whose moment of inertia is I, is

$$\frac{I \cdot d\,(a^2)}{2\,g} = \frac{I\,a\,d\,a}{g}.$$

Let $d\,t$ be the *time*, and $d\,i = a\,d\,t$ the *angular motion* in which that acceleration is to be produced; let P be the accelerating effort, and l its *leverage*, or the perpendicular distance of its line of action from the axis; then, according as the time $d\,t$, or the angle $d\,i$, is given, we have the two following expressions for the *accelerating couple*:—

$$P\,l = \frac{I}{g} \cdot \frac{d\,a}{d\,t} \qquad (3.)$$

$$= \frac{I}{g} \cdot \frac{a\,d\,a}{d\,i} = \frac{I}{g} \cdot \frac{d\,(a^2)}{2\,d\,i} \qquad (4.)$$

Lastly, referring to Article 405, page 257, Equation 2, we find, that if a train of mechanism consists of various parts, and if W be the weight of any one of those parts, whose velocity v' bears to that of the driving point v the ratio $\frac{v'}{v} = n$, then the accelerating effort which must be applied to the driving point, in order that, during the interval $d\,t$, in which the driving point moves through the distance $d\,s = v\,d\,t$, that point may undergo the acceleration $d\,v$, and each weight W the corresponding acceleration $n\,d\,v$, is given by one or other of the two formulæ—

$$P = \frac{\Sigma\,n^2\,W}{g} \cdot \frac{d\,v}{d\,t} \qquad (5.)$$

$$= \frac{\Sigma\,n^2\,W}{g} \cdot \frac{v\,d\,v}{d\,s} = \frac{\Sigma\,n^2\,W}{g} \cdot \frac{d\,(v^2)}{2\,d\,s} \qquad (6.)$$

both of which are derived from the equation $\mathrm{P}\, d\, s = \mathrm{P}\, v \cdot d\, t = \frac{v \cdot d\, v}{g} \cdot \Sigma\, n^2\, w$.

413. **Work During Retardation—Energy Stored and Restored.**—In order to cause a given retardation, or diminution of the velocity of a given body, in a given time, or while it traverses a given distance, resistance must be opposed to its motion equal to the effort which would be required to produce in the same time, or in the same distance, an acceleration equal to the retardation.

A moving body, therefore, while being retarded, *overcomes resistance* and *performs work;* and that work is equal to the energy exerted in producing an acceleration of the same body equal to the retardation.

It is for this reason that it has been stated, in Article 403, that the work performed in accelerating the speed of the moving pieces of a machine is not necessarily lost; for those moving pieces, by returning to their original speed, are capable of performing an equal amount of work in overcoming resistance; so that the performance of such work is not prevented, but only deferred. Hence energy exerted in acceleration is said to be *stored;* and when by a subsequent and equal retardation an equal amount of work is performed, that energy is said to be *restored.*

The algebraical expressions for the relations between a retarding resistance, and the retardation which it produces in a given body by acting during a given time or through a given space, are obtained from the equations of Article 412 simply by putting R, the symbol for a resistance, instead of P, the symbol for an effort, and $-d\, v$, the symbol for a retardation, instead of $d\, v$, the symbol for an acceleration.

414. **The Actual Energy** of a moving body is the work which it is capable of performing against a retarding resistance before being brought to rest, and is equal to the energy which must be exerted on the body to bring it from a state of rest to its actual velocity. The value of that quantity is the *product of the weight of the body into the height from which it must fall to acquire its actual velocity;* that is to say,

$$\frac{\mathrm{W}\, v^2}{2\, g} \qquad \text{(1.)}$$

The total actual energy of a system of bodies, each moving with its own velocity, is denoted by

$$\frac{\Sigma \cdot \mathrm{W}\, v^2}{2\, g}; \qquad \text{(2.)}$$

and when those bodies are the pieces of a machine, whose velocities bear definite ratios (any one of which is denoted by n) to the velocity of the driving point v, their total actual energy is

$$\frac{v^2}{2g} \cdot \Sigma n^2 W, \quad \text{(3.)}$$

being *the product of the reduced inertia* (or coefficient of steadiness, as Mr. Moseley calls it) *into the height due to the velocity of the driving point.*

The actual energy of a rotating body whose angular velocity is a, and moment of inertia $\Sigma W r^2 = I$, is

$$\frac{a^2 I}{2g}; \quad \text{(4.)}$$

that is, *the product of the moment of inertia into the height due to the velocity,* a, *of a point, whose distance from the axis of rotation is unity.*

When a given amount of energy is alternately stored and restored by alternate increase and diminution in the speed of a machine, the actual energy of the machine is alternately increased and diminished by that amount.

Actual energy, like motion, is *relative* only. That is to say, in computing the actual energy of a body, which is the capacity it possesses of performing work upon certain other bodies by *reason of its motion,* it is the motion *relatively to those other bodies* that is to be taken into account.

For example, if it be wished to determine how many turns a wheel of a locomotive engine, rotating with a given velocity, would make, before being stopped *by the friction of its bearings only,* supposing it lifted out of contact with the rails,—the actual energy of that wheel is to be taken *relatively to the frame of the engine* to which those bearings are fixed, and is simply the actual energy due to the rotation. But if the wheel be supposed to be detached from the engine, and it is inquired *how high it will ascend up a perfectly smooth inclined plane before being stopped by the attraction of the earth,* then its actual energy is to be taken *relatively to the earth;* that is to say, to the energy of rotation already mentioned, is to be added the energy due to the *translation* or forward motion of the wheel along with its axis.

415. **A Reciprocating Force** is a force which acts alternately as an effort and as an equal and opposite resistance, according to the direction of motion of the body. Such a force is the weight of a moving piece whose centre of gravity alternately rises and falls; and such is the elasticity of a perfectly elastic body. The work which a body performs in moving against a reciprocating force is employed in increasing its own potential energy, and is not lost by

the body; so that by the motion of a body alternately against and with a reciprocating force, energy is *stored* and *restored*, as well as by alternate acceleration and retardation.

Let Σ W denote the weight of the whole of the moving pieces of any machine, and h a height through which the common centre of gravity of them all is alternately raised and lowered. Then the quantity of energy—

$$h \Sigma \mathrm{W},$$

is stored while the centre of gravity is rising, and restored while it is falling.

These principles are illustrated by the action of the plungers of a single-acting pumping steam engine. The weight of those plungers acts as a resistance while they are being lifted by the pressure of the steam on the piston; and the same weight acts as effort when the plungers descend and drive before them the water with which the pump barrels have been filled. Thus the energy exerted by the steam on the piston is stored during the up-stroke of the plungers; and during their down-stroke the same amount of energy is restored, and employed in performing the work of raising water and overcoming its friction.

416. **Periodical Motion.**—If a body moves in such a manner that it periodically returns to its original velocity, then at the end of each period, the entire variation of its actual energy is nothing; and if, during any part of the period of motion, energy has been stored by acceleration of the body, the same quantity of energy exactly must have been during another part of the period restored by retardation of the body.

If the body also returns in the course of the same period to the same position relatively to all bodies which exert reciprocating forces on it—for example, if it returns periodically to the same elevation relatively to the earth's surface—any quantity of energy which has been stored during one part of the period by moving against reciprocating forces must have been exactly restored during another part of the period.

Hence *at the end of each period, the equality of energy and work, and the balance of* mean effort *and* mean resistance, *holds with respect to the driving effort and the resistances, exactly as if the speed were uniform and the reciprocating forces null;* and all the equations of Articles 407 and 409 are applicable to periodic motion, provided that in the equations of Article 407, and Equation 1 of Article 409, P, R, and v are held to denote the *mean values* of the efforts, resistances, and velocities,—that s and s' are held to denote spaces moved through in one or more *entire periods,*—and that in Equation 2 of Article 409, the integrations denoted by $\int$ be held to extend to one or more *entire periods.*

These principles are illustrated by the steam engine. The velocities of its moving parts are continually varying, and those of some of them, such as the piston, are periodically reversed in direction. But at the end of each period, called a *revolution*, or *double-stroke*, every part returns to its original position and velocity; so that the *equality of energy and work*, and the *equality of the mean effort to the mean resistance reduced to the driving point*,—that is, the equality of the mean effective pressure of the steam on the piston to the mean total resistance reduced to the piston—hold for one or any whole number of *complete revolutions*, exactly as for uniform speed.

It thus appears that (as stated at the commencement of this Part) there are two fundamentally different ways of considering a periodically moving machine, each of which must be employed in succession, in order to obtain a complete knowledge of its working.

"I. In the first place is considered the action of the machine during one or more whole periods, with a view to the determination of the relation between the mean resistances and mean efforts, and of the EFFICIENCY; that is the ratio which the *useful* part of its work bears to the whole expenditure of energy. The motion of every ordinary machine is either uniform or periodical.

"II. In the second place is to be considered the action of the machine during intervals of time less than its period, in order to determine the law of the periodic changes in the motions of the pieces of which the machine consists, and of the periodic or reciprocating forces by which such changes are produced."

417. **Starting and Stopping.**—The *starting* of a machine consists in setting it in motion from a state of rest, and bringing it up to its proper mean velocity. This operation requires the exertion, besides the energy required to overcome the mean resistance, of an additional quantity of energy equal to the actual energy of the machine when moving with its mean velocity, as found according to the principles of Article 414, page 262.

If, in order to *stop* a machine, the effort of the prime mover is simply suspended, the machine will continue to go until work has been performed in overcoming resistances equal to the actual energy due to the speed of the machine at the time of suspending the effort of the prime mover.

In order to diminish the time required by this operation, the resistance may be increased by means of the friction of a *brake.* Brakes will be further described in the sequel.

418. **The Efficiency** of a machine is a fraction expressing the ratio of the useful work to the whole work, which is equal to the energy expended. The COUNTER-EFFICIENCY is the reciprocal of the efficiency, and is the ratio in which the energy expended is greater than the useful work. The object of improvements in

machines is to bring their efficiency and counter-efficiency as near to unity as possible.

As to useful and lost work, see Article 401. The algebraical expression of the efficiency of a machine having uniform or periodical motion, is obtained by introducing the distinction between useful and lost work into the equations of the conservation of energy, Article 409. Thus, let P denote the mean effort at the driving point; s, the space described by it in a given interval of time, being a whole number of periods of revolutions; R_1, the mean useful resistance; s_1, the space through which it is overcome in the same interval; R_2, any one of the wasteful resistances; s_2, the space through which it is overcome; then

$$P\,s = R\,s_1 + \Sigma \cdot R_2\,s_2; \quad \text{(1.)}$$

and the efficiency of the machine is expressed by

$$\frac{R_1\,s_1}{P\,s} = \frac{R_1\,s_1}{R_1\,s_1 + \Sigma \cdot R_2\,s_2} \quad \text{(2.)}$$

In many cases the lost work of a machine, $R_2\,s_2$, consists of a constant part, and of a part bearing to the useful work a proportion depending in some definite manner on the sizes, figures, arrangement, and connection of the pieces of the train, on which also depends the constant part of the lost work. In such cases the whole energy expended and the efficiency of the machine are expressed by the equations

$$\left.\begin{aligned} P\,s &= (1 + A)\,R_1\,s_1 + B; \\ \frac{R_1\,s_1}{P\,s} &= \frac{1}{1 + A + \dfrac{B}{R_1\,s_1}} \end{aligned}\right\} \quad \text{(3.)}$$

and the first of these is the mathematical expression of what Mr. Moseley calls the "modulus" of a machine.

The useful work of an intermediate piece in a train of mechanism consists in driving the piece which follows it, and is less than the energy exerted upon it by the amount of the work lost in overcoming its own friction. Hence the efficiency of such an intermediate piece is the ratio of the work performed by it in driving the following piece, to the energy exerted on it by the preceding piece; and it is evident that *the efficiency of a machine is the product of the efficiencies of the series of moving pieces which transmit energy from the driving point to the working point.* The same principle applies to a train of *successive machines*, each driving that which follows it; and to counter-efficiency as well as to efficiency.

419. **Power and Effect—Horse Power.**—The *power* of a machine

is the energy exerted, and the *effect*, the useful work performed, in some interval of time of definite length, such as a second, a minute, an hour, or a day.

The unit of power called conventionally a *horse-power*, is 550 foot-pounds per second, or 33,000 foot-pounds per minute, or 1,980,000 foot-pounds per hour. The effect is equal to the power multiplied by the efficiency; and the power is equal to the effect multiplied by the counter-efficiency. The loss of power is the difference between the effect and the power. As to the French "Force de Cheval," see Article 392, page 244. It is equal to 0·9863 of a British horse-power; and a British horse-power is 1·0139 of a French force de cheval.

420. **General Equation.**—The following general equation presents at one view the principles of the action of machines, whether moving uniformly, periodically, or otherwise :—

$$\int \mathrm{P}\, d\, s = \Sigma \int \mathrm{R}\, d\, s' \pm h\, \Sigma\, \mathrm{W} + \Sigma \cdot \frac{\mathrm{W}\,(v_2^2 - v_1^2)}{2\, g};$$

where W is the weight of any moving piece of the machine;

h, when positive, the elevation, and when negative, the depression, which the common centre of gravity of all the moving pieces undergoes in the interval of time under consideration; v_1 the velocity at the beginning, and v_2 the velocity at the end, of the interval in question, with which a given particle of the machine of the weight W is moving;

g, the acceleration which gravity causes in a second, or 32·2 feet per second, or 9·81 mètres per second.

$\int \mathrm{R}\, d\, s'$, the work performed in overcoming any resistance during the interval in question;

$\int \mathrm{P}\, d\, s$, the energy exerted during the interval in question.

The second and third terms of the right-hand side, when positive, are *energy stored;* when negative, *energy restored.*

The principle represented by the equation is expressed in words as follows :—

The energy exerted, added to the energy restored, is equal to the energy stored added to the work performed.

421. **The Principle of Virtual Velocities,** when applied to the uniform motion of a machine, is expressed by Equation 3 of Article 407, already given in page 259; or in words as follows :—*The effort is equal to the sum of the resistances reduced to the driving point;* that is, each multiplied by the ratio of the velocity of its working point to the velocity of the driving point. The same principle, when applied to reciprocating forces and to re-actions due to varying speed, as well as to passive resistances, is expressed by

means of a modified form of the general equation of Article 420, obtained in the following manner:—Let n denote either the ratio borne at a given instant by the velocity of a given working point, where the resistance R is overcome, to the velocity of the driving point, or the mean value of that ratio during a given interval of time; let n'' denote the corresponding ratio for the vertical ascent or descent (according as it is positive or negative) of a moving piece whose weight is W; let n' denote the corresponding ratio for the mean velocity of a mass whose weight is W, undergoing acceleration or retardation, and $\frac{d\,v'}{d\,t}$ either the rate of acceleration of that mass, if the calculation relates to an instant, or the mean value of that rate, if to a finite interval of time. Then the effort at the instant, or the mean effort during the given interval, as the case may be, is given by the following equation:—

$$\mathrm{P} = \Sigma \cdot n\,\mathrm{R} + \Sigma \cdot n''\,\mathrm{W} + \Sigma \cdot \frac{n'\,\mathrm{W}\,d\,v'}{g\,d\,t}.$$

If the ratio n', which the velocity of the mass W bears to that of the driving point, is constant, we may put $\frac{d\,v'}{d\,t} = \frac{n'\,d\,v}{d\,t}$, where $\frac{d\,v}{d\,t}$ denotes the rate of acceleration of the driving point; and then the third term of the foregoing expression becomes $\frac{d\,v}{g\,d\,t}\,\Sigma \cdot n'^2\,\mathrm{W}$, as in formula 2 of Article 406, page 258.

422. **Forces in the Mechanical Powers, Neglecting Friction—Purchase.**—The mechanical powers, considered as means of modifying motion only, have been considered in Section 6, Part II., pages 107 to 110. When friction is neglected, any one of the mechanical powers may be regarded as *an uniformly-moving simple machine, in which one effort balances one resistance;* and in which, consequently, according to the principle of virtual velocities, or of the equality of energy exerted and work done, *the effort and resistance are to each other inversely as the velocities along their lines of action of the points where they are applied.*

In the older writings on mechanics, the effort is called the *power*, and the resistance the *weight;* but it is desirable to avoid the use of the word "power" in this sense, because of its being very commonly used in a different sense—viz., the rate at which energy is exerted by a prime mover; and the substitution of "resistance" for "weight" is made in order to express the fact, that the principle just stated applies to the overcoming of all sorts of resistance, and not to the lifting of weights only.

The weight of the moving piece itself in a mechanical power may either be wholly supported at the bearing, if the piece is balanced; or if not, it is to be regarded as divided into two parallel components, one supported directly at the bearing, and the other being included in the effort or in the resistance, as the case may be.

The relation between the effort and the resistance in any mechanical power may be deduced from the principles of statics; viz.:—In the case of the LEVER (including the *wheel* and *axle*), from the balance of couples of equal and opposite moments; in the case of the INCLINED PLANE (including the *wedge* and the *screw*), from the parallelogram of forces; and in the case of the pulley, from the composition of parallel forces. The principle of virtual velocities, however, is more convenient in calculation.

The *total load* in a mechanical power is the resultant of the effort, the resistance, the lateral components of the forces acting at the driving and working points, and the weight directly carried at the bearings; and it is equal and directly opposed to the re-action of the bearings or supports of the machine.

By the *purchase* of a mechanical power is to be understood the ratio borne by the resistance to the effort, which is equal to the ratio borne by the velocity of the driving point to that of the working point. This term has already been employed in connection with the pulley.

The following are the results of the principle of virtual velocities, as applied to determine the purchase in the several mechanical powers:—

I. LEVER.—The effort and resistance are to each other in the inverse ratio of the perpendicular distances of their lines of action from the axis of rotation or fulcrum; so that the *purchase* is the ratio which the perpendicular distance of the effort from the axis bears to the perpendicular distance of the resistance from the axis.

Under the head of the lever may be comprehended all turning or rocking primary pieces in mechanism which are connected with their drivers and followers by linkwork.

II. WHEEL AND AXLE.—The purchase is the same as in the case of the lever; and the perpendicular distances of the lines of action of the effort and of the resistance from the axis are the radii of the pitch-circles of the wheel and of the axle respectively.

Under the head of the wheel and axle may be comprehended all turning or rocking primary pieces in mechanism which are connected with their drivers and followers by means of rolling contact, of teeth, or of bands. By the "wheel" is to be understood the pitch-cylinder of that part of the piece which is driven; and by the "axle," the pitch-cylinder of that part of the piece which drives.

III. INCLINED PLANE, and IV. WEDGE.—Here the purchase, or ratio of the resistance to the effort, is the ratio borne by the whole velocity of the sliding body (represented by B C in fig. 76E, and C c in fig. 76F, page 109) to that component of the velocity (represented by B D in fig. 76E, and C e in fig. 76F, page 109) which is directly opposed to the resistance: it being understood that the effort is exerted in the direction of motion of the sliding body.

The term *inclined plane* may be used when the resistance to the motion of a body that slides along a guiding surface consists of its own weight, or of a force applied to a point in it by means of a link; and the term *wedge*, when that resistance consists of a pressure applied to a plane surface of the moving body, oblique to its direction of motion.

V. SCREW. Let the resistance (R) to the motion of a screw be a force acting along its axis, and directly opposed to its advance; and let the effort (P) which drives the screw be applied to a point rigidly attached to the screw, and at the distance r from the axis, and be exerted in the direction of motion of that point. Then, while the screw makes one revolution, the working point advances against the resistance through a distance equal to the pitch (p); and at the same time the driving point moves in its helical path through the distance $\sqrt{(4\pi^2 r^2 + p^2)}$; therefore the purchase of the screw, neglecting friction, is expressed as follows:—

$$\frac{R}{P} = \frac{\sqrt{4\pi^2 r^2 + p^2}}{p}$$

$$= \frac{\text{length of one coil of path of driving point}}{\text{pitch}}.$$

VI. PULLEY.—In the pulley without friction, the purchase is the ratio borne by the resistance which opposes the advance of the running block to the effort exerted on the hauling part of the rope; and it is expressed by the number of plies of rope by which the running block is connected with the fixed block.

VII. The HYDRAULIC PRESS, when friction is neglected, may be included amongst the mechanical powers, agreeably to the definition of them given at the beginning of this Article. By the resistance is to be understood the force which opposes the outward motion of the press-plunger; and by the effort, the force which drives inward the pump-plunger. The intensity of the pressure exerted between each of the two plungers and the fluid is the same; therefore the amount of the pressure exerted between each plunger and the fluid is proportional to the area of that plunger; so that the purchase of the hydraulic press is expressed as follows:—

$$\frac{R}{P} = \frac{\text{transverse area of press-plunger}}{\text{transverse area of pump-plunger}};$$

and this is the reciprocal of the ratio of the velocities of those plungers, as already shewn in Article 185, page 110.

The purchase of a train of mechanical powers is the product of the purchases of the several elementary parts of that train.

The object of producing a purchase expressed by a number greater than unity is, to enable a resistance to be overcome by means of an effort smaller than itself, but acting through a greater distance; and the use of such a purchase is found chiefly in machines driven by muscular power, because of the effort being limited in amount.

SECTION 3.—OF DYNAMOMETERS.

423. **Dynamometers** are instruments for measuring and recording the energy exerted and work performed by machines. They may be classed as follows:—

I. Instruments which merely *indicate the force* exerted between a driving body and a driven body, leaving the *distance* through which that force is exerted to be observed independently.

II. Instruments which *record* at once the *force, motion,* and *work* of a machine, by drawing a line, straight or curved, as the case may be, whose abscissæ represent the distances moved through, its ordinates the resistances overcome, and its area the work performed (as in fig. 149, page 249).

A dynamometer of this class consists essentially of two principle parts: a spring whose deflection indicates the force exerted between a driving body and a driven body; and a band of paper, or a card, moving at right angles to the direction of deflection of the spring with a velocity bearing a known constant proportion to the velocity with which the resistance is overcome. The spring carries a pen or pencil, which marks on the paper or card the required line. The Steam Engine Indicator is an example of this class of instruments.

III. Instruments called *Integrating Dynamometers,* which record the work performed, but not the resistance and motion separately.

424. **Steam Engine Indicator.**—This instrument was invented by Watt, and has been improved by other inventors, especially M'Naught and Richards. Its object is to record, by means of a diagram, the intensity of the pressure exerted by steam against one of the faces of a piston at each point of the piston's motion, and so to afford the means of computing, according to the principles of Articles 395 and 400, first, the energy exerted by the steam in driving the piston during the forward stroke; secondly, the work lost by the piston in expelling the steam from the cylinder during the return stroke; and thirdly, the difference of those quantities,

which is the *available* or *effective* energy exerted by the steam on the piston, and which, being multiplied by the number of strokes per minute and divided by 33,000 foot-pounds, gives the INDICATED HORSE-POWER.

The indicator in a common form is represented by fig. 150. A B is a cylindrical case. Its lower end, A, contains a smaller cylinder, fitted with a piston, which cylinder, by means of the screwed nozzle at its lower end, can be fixed in any convenient position on a tube communicating with that end of the engine-cylinder where the work of the steam is determined. The communication between the engine-cylinder and the indicator-cylinder can be opened and shut at will by means of the cock K. When it is open, the intensity of the pressure of the steam on the engine-piston and on the indicator-piston is the same, or nearly the same.

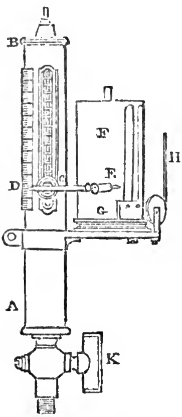


Fig. 150.

The upper end, B, of the cylindrical case contains a spiral spring, one end of which is attached to the piston, or to its rod, and the other to the top of the casing. The indicator-piston is pressed from below by the steam, and from above by the atmosphere. When the pressure of the steam is equal to that of the atmosphere, the spring retains its unstrained length, and the piston its original position. When the pressure of the steam exceeds that of the atmosphere, the piston is driven outwards, and the spring compressed; when the pressure of the steam is less than that of the atmosphere, the piston is driven inwards, and the spring extended. The compression or extension of the spring indicates the difference, upward or downward, between the pressure of the steam and that of the atmosphere.

A short arm, C, projecting from the indicator piston-rod carries at one side a pointer, D, which shews the pressure on a scale whose zero denotes the *pressure of the atmosphere*, and which is graduated into pounds on the square inch both upwards and downwards from that zero. At the other side the short arm has a longer arm jointed to it, carrying a pencil, E.

F is a brass drum, which rotates backward and forward about a vertical axis, and which, when about to be used, is covered with a piece of paper called a "card." It is alternately pulled round in one direction by the cord H, which wraps on the pulley G, and pulled back to its original position by a spring contained within itself. The cord H is to be connected with the mechanism of the steam engine in any convenient manner which shall ensure that the velocity of rotation of the drum shall at every instant bear a constant ratio to that of the steam engine piston: the back and forward motion of the surface of the drum representing that of the steam engine piston on a reduced scale. This having been done, and before opening the cock K, the pencil is to be placed in contact with the drum during a few strokes, when it will mark on the card a line which, when the card is afterwards spread out flat, becomes a straight line. This line, whose position indicates the pressure of the atmosphere, is called the *atmospheric line*. In fig. 151 it is represented by A A.

The cock K is opened, and the pencil, moving up and down with the variations of the pressure of the steam, traces on the card during each complete or double stroke a curve such as B C D E B. The ordinates drawn to that curve from any point in the atmospheric line, such as $\overline{HK}$ and $\overline{HG}$, indicate the differences between the pressure of the steam and the atmospheric pressure at the corresponding point of the motion of the piston. The ordinates of the part B C D E represent the pressures of the steam during the forward stroke, when it is driving the piston, those of the part E B represent the pressures of the steam when the piston is expelling it from the cylinder.

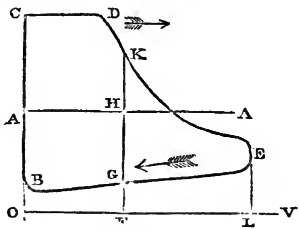


Fig. 151.

To found exact investigations on the indicator-diagrams of steam engines, the atmospheric pressure at the time of the experiment ought to be ascertained by means of a barometer; but this is generally omitted; in which case the atmospheric pressure may be assumed at its mean value, being 14·7 lbs. on the square inch, or 2116·3 lbs. on the square foot, at and near the level of the sea.

Let $\overline{AO} = \overline{HF}$ be ordinates representing the pressure of the atmosphere. Then O F V parallel to A A is the *absolute* or *true* zero line of the diagram, corresponding to *no pressure;* and ordinates drawn to the curve from that line represent the absolute intensities of the pressure of steam. Let O B and L E be ordinates touching the ends of the diagram; then

$\overline{OL}$ represents the *volume* traversed by the piston at each single stroke ($= s$ A, where s is the length of the stroke and A the area of the piston);

The area O B C D E L O represents the energy exerted by the steam on the piston during the forward stroke;

The area O B E L O represents the work lost in expelling the steam during the return stroke;

The area B C D E B, being the difference of the above areas, represents the *effective work* of the steam on the piston during the complete stroke.

Those areas can be found by the rules of Article 34, page 17; and the common trapezoidal rule, D, page 21, is in general sufficiently accurate. The number of intervals is usually ten, and of ordinates eleven.

The *mean forward pressure*, the *mean back pressure*, and the *mean effective pressure*, are found by dividing those three areas respectively by the volume s A, which is represented by $\overline{\text{O L}}$.

Those mean pressures, however, can be found by a direct process, without first measuring the areas, viz.:—having multiplied each ordinate, or breadth, of the area under consideration by the proper multiplier, divide the sum of the products by the sum of the multipliers, which process, when the common trapezoidal rule is used, takes the following form: add together the halves of the endmost ordinates, and the whole of the other ordinates, and divide by the number of intervals. That is, let b_0 be the first, b_n the last, and b_1, b_2, &c., the intermediate breadths; then let n be the number of intervals, and b_m the mean breadth; then

$$b_m = \frac{1}{n}\left(\frac{b_0 + b_n}{2} + b_1 + b_2 + \text{\&c.}\right); \quad \ldots\ldots\ldots\ldots\ldots(1.)$$

and this represents the mean forward pressure, mean back pressure, or mean effective pressure, as the case may be. Let p_e be the mean effective pressure; then the effective energy exerted by the steam on the piston during each double stroke is the product of the mean effective pressure, the area of the piston, and the length of stroke, or

$$p_e \text{A}\, s; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

and if N be the number of double strokes in a minute, the *indicated power in foot-pounds per minute*, in a single-acting engine, is

$$p_e \text{A N}\, s; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

from which the *indicated horse-power* is found by dividing by 33,000.

In a *double-acting engine* the steam acts alternately on either side of the piston; and to measure the power accurately, two indicators should be used at the same time, communicating respectively with the two ends of the cylinder. Thus a pair of diagrams will be obtained, one representing the action of the steam on each face of the piston. The mean effective pressure is to be found as above for each diagram separately, and then, if the areas of the two faces of the piston are sensibly equal, *the mean of those two results* is to be taken as the *general mean effective pressure;* which being multiplied by the area of the piston, the length of stroke, and *twice* the number of double strokes or revolutions in a minute, gives the indicated power per minute; that is to say, if p'' denotes the general mean effective pressure, the indicated power per minute is

$$p'' \mathrm{A} \cdot 2 \mathrm{N} s; \quad \text{(4.)}$$

If the two faces of the piston are sensibly of unequal areas (as in "trunk engines"), the indicated power is to be computed separately for each face, and the results added together.

If there are two or more cylinders, the quantities of power indicated by their respective diagrams are to be added together.

The reactions of the moving parts of the indicator, combined with the elasticity of the spring, cause oscillations of its piston. In order that the errors thus produced in the indicated pressures at particular instants may be as small as possible, and may neutralize each other's effects on the whole indicated power, the moving masses ought to be as small as practicable, and the spring as stiff as is consistent with shewing the pressures on a visible scale. In Richard's indicator this is effected by the help of a train of very light linkwork, which causes the pencil to shew the movements of the spring on a magnified scale.

The *friction* of the moving parts of the indicator tends on the whole to make the indicated power and indicated mean effective pressure less than the truth, but to what extent is uncertain.

Every indicator should have the accuracy of the graduation of its scale of pressures frequently tested by comparison with a standard pressure gauge.

The indicator may obviously be used for measuring the energy exerted by any fluid, whether liquid or gaseous, in driving a piston; or the work performed by a pump, in lifting, propelling, or compressing any fluid.

CHAPTER III.

OF REGULATING APPARATUS.

425. **Regulating Apparatus Classed—Brake—Fly—Governor.**—The effect of all regulating apparatus is to control the speed of machinery. A regulating instrument may act simply by consuming energy, so as to prevent acceleration, or produce retardation, or stop the machine if required; it is then called a *brake;* or it may act by storing surplus energy at one time, and giving it out at another time when energy is deficient: in this case it is called a *fly;* or it may act by adjusting the power of the prime mover to the work to be done, when it is called a *governor.* The use of a brake involves waste of power. A fly and a governor, on the other hand, promote economy of power and economy of strength.

Section 1.—Of Brakes.

426. **Brakes Defined and Classed.**—The contrivances here comprehended under the general title of *Brakes* are those by means of which friction, whether exerted amongst solid or fluid particles, is purposely opposed to the motion of a machine, in order either to stop it, to retard it, or to employ superfluous energy during uniform motion. The use of a brake involves waste of energy, which is in itself an evil, and is not to be incurred unless it is necessary to convenience or safety.

Brakes may be classed as follows:—

I. *Block-brakes,* in which one solid body is simply pressed against another, on which it rubs.

II. *Flexible brakes,* which embrace the periphery of a drum or pulley.

III. *Pump-brakes,* in which the resistance employed is the friction amongst the particles of a fluid forced through a narrow passage.

IV. *Fan-brakes,* in which the resistance employed is that of a fluid to a fan rotating in it.

427. **Action of Brakes in General.**—The work disposed of by a brake in a given time is the product of the resistance which it produces into the distance through which that resistance is overcome in a given time.

To *stop* a machine, the brake must employ work to the amount of the whole actual energy of the machine, as already stated in Article 417. To *retard* a machine, the brake must employ work to an amount equal to the difference between the actual energies of the machine at the greater and less velocities respectively.

To *dispose of surplus energy*, the brake must employ work equal to that energy; that is, the resistance caused by the brake must balance the surplus effort to which the surplus energy is due; so that if n is the ratio which the velocity of rubbing of the brake bears to the velocity of the driving point, P, the *surplus effort* at the driving point, and R the resistance of the brake, we ought to have—

$$R = \frac{P}{n}. \qquad \qquad (1.)$$

It is obviously better, when practicable, to store surplus energy, or to prevent its exertion, than to dispose of it by means of a brake.

When the action of a brake composed of solid material is long-continued, a stream of water must be supplied to the rubbing surfaces, to abstract the heat that is produced by the friction, according to the law stated in Article 402, page 252.

428. **Block-Brakes.**—When the motion of a machine is to be controlled by pressing a block of solid material against the rim of a rotating drum, it is advisable, inasmuch as it is easier to renew the rubbing surface of the block than that of the drum, that the drum should be of the harder, and the block of the softer material—the drum, for example, being of iron, and the block of wood. The best kinds of wood for this purpose are those which have considerable strength to resist crushing, such as elm, oak, and beech. The wood forms a facing to a frame of iron, and can be renewed when worn.

When the brake is pressed against the rotating drum, the direction of the pressure between them is obliquely opposed to the motion of the drum, so as to make an angle with the radius of the drum equal to the *angle of repose* of the rubbing surfaces (denoted by φ; see page 154). The component of that oblique pressure in the direction of a tangent to the rim of the drum is the friction (R); the component perpendicular to the rim of the drum is the normal pressure (N) required in order to produce that friction, and is given by the equation

$$N = \frac{R}{f}; \qquad \qquad (1.)$$

f being the coefficient of friction, and the proper value of R being determined by the principles stated in Article 427.

It is in general desirable that the brake should be capable of effecting its purpose when pressed against the drum by means of the strength of one man, pulling or pushing a handle with one hand or one foot. As the required normal pressure N is usually considerably greater than the force which one man can exert, a lever, or screw, or a train of levers, screws, or other convenient mechanism, must be interposed between the brake block and the handle, so that when the block is moved towards the drum, the handle shall move at least through a distance as many times greater than the distance by which the block *directly* approaches the drum, as the required normal pressure is greater than the force which the man can exert.

Although a man may be able occasionally to exert with one hand a force of 100 lbs., or 150 lbs., for a short time, it is desirable that, in working a brake, he should not be required to exert a force greater than he can keep up for a considerable time, and exert repeatedly in the course of a day, without fatigue—that is to say, about 20 lbs. or 25 lbs.

429. **The Brakes of Carriages** are usually of the class just described, and are applied either to the wheels themselves or to drums rotating along with the wheels. Their effect is to stop or to retard the rotation of the wheels, and make them slip, instead of rolling on the road or railway. The resistance to the motion of a carriage which is caused by its brake may be less, but cannot be greater, than the friction of the stopped or retarded wheels on the road or rails under the load which rests on those wheels. The distance which a carriage or train of carriages will run on a level line during the action of the brakes before stopping, is found by dividing the actual energy of the moving mass before the brakes are applied, by the sum of the ordinary resistance and of the additional resistance caused by the brakes; in other words, that distance is as many times greater than the height due to the speed as the weight of the moving mass is greater than the total resistance.

The *skid*, or *slipper-drag*, being placed under a wheel of a carriage, causes a resistance due to the friction of the skid upon the road or rail under the load that rests on the wheel.

Section 2.—Of Fly-Wheels.

430. **Periodical Fluctuations of Speed** in a machine are caused by the alternate excess and deficiency of the energy exerted above the work performed in overcoming resisting forces, which produce an alternate increase and diminution of actual energy, according to the law explained in Article 413, page 262.

To determine the greatest fluctuation of speed in a machine moving periodically, take A B C, in fig. 152, to represent the motion of the driving point during one period; let the effort P of the prime mover at each instant be represented by the ordinate of the curve D G E I F; and let the sum of the resistances, reduced to the driving point as in Article 398, at each instant, be denoted by R, and represented by the ordinate of the curve D H E K F, which cuts the former curve at the ordinates A D, B E, C F. Then the integral,

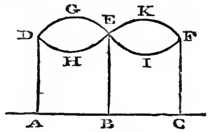


Fig. 152.

$$\int (P - R)\, d\, s,$$

being taken for any part of the motion, gives the excess or deficiency of energy, according as it is positive or negative. For the entire period A B C, this integral is nothing. For A B, it denotes an *excess of energy received*, represented by the area D G E H; and for B C, an equal *excess of work performed*, represented by the equal area E K F I. Let those equal quantities be each represented by Δ E. Then the actual energy of the machine attains a maximum value at B, and a minimum value at A and C, and Δ E is the difference of those values.

Now let v_0 be the mean velocity, v_1 the greatest velocity, v_2 the least velocity of the driving point, and $\Sigma \cdot n^2$ W the *reduced inertia* of the machine (see Article 405, page 257); then

$$\frac{v_1^2 - v_2^2}{2\,g} \cdot \Sigma \cdot n^2\, W = \Delta\, E; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

which, being divided by the *mean actual energy*,

$$\frac{v_0^2}{2\,g} \cdot \Sigma\, n^2 W = E_0,$$

gives

$$\frac{v_1^2 - v_2^2}{v_0^2} = \frac{\Delta\, E}{E_0}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

and observing that $v_0 = (v_1 + v_2) \div 2$, we find

$$\frac{v_1 - v_2}{v_0} = \frac{\Delta \cdot E}{2\, E_0} = \frac{g\, \Delta\, E}{v_0^2\, \Sigma \cdot n^2\, W}; \ldots\ldots\ldots\ldots\ldots (3.)$$

a ratio which may be called the *coefficient of fluctuation of speed* or of *unsteadiness*.

The ratio of the periodical excess and deficiency of energy Δ E to the whole energy exerted in one period or revolution, $\int \mathrm{P}\, d\, s$, has been determined by General Morin for steam engines under various circumstances, and found to be from $\frac{1}{10}$ to $\frac{1}{4}$ for single-cylinder engines. For a pair of engines driving the same shaft, with cranks at right angles to each other, the value of this ratio is about one-fourth, and for three engines with cranks at 120°, one-twelfth of its value for single-cylinder engines.

The following table of the ratio, $\Delta \mathrm{E} \div \int \mathrm{P}\, d\, s$, for *one revolution* of steam engines of different kinds is extracted and condensed from General Morin's works:—

Non-Expansive Engines.

$\frac{\text{Length of connecting rod}}{\text{Length of crank}} =$	8	6	5	4
$\Delta \mathrm{E} \div \int \mathrm{P}\, d\, s =$	·105	·118	·125	·132

Expansive Condensing Engines.

Connecting rod = crank × 5.

Fraction of Stroke at which steam is cut off,	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
$\Delta \mathrm{E} \div \int \mathrm{P}\, d\, s =$	·163	·173	·178	·184	·189	·191

Expansive Non-Condensing Engines.

Steam cut off at	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
$\Delta \mathrm{E} \div \int \mathrm{P}\, d\, s =$	·160	·186	·209	·232

For double-cylinder expansive engines, the value of the ratio $\Delta \mathrm{E} \div \int \mathrm{P}\, d\, s$ may be taken as equal to that for single-cylinder non-expansive engines.

For *tools working at intervals*, such as punching, slotting, and plate-cutting machines, coining presses, &c., Δ E is nearly equal to the whole work performed at each operation.

431. **Fly-Wheels.**—A fly-wheel is a wheel with a heavy rim, whose great moment of inertia being comprehended in the

reduced moment of inertia of a machine, reduces the coefficient of fluctuation of speed to a certain fixed amount, being about $\frac{1}{32}$ for ordinary machinery, and $\frac{1}{50}$ or $\frac{1}{60}$ for machinery for fine purposes.

Let $\frac{1}{m}$ be the intended value of the coefficient or fluctuation of speed, and Δ E, as before, the fluctuation of energy. If this is to be provided for by the moment of inertia, I, of the fly-wheel alone, let a_0 be its mean angular velocity; then Equation 3 of Article 430 is equivalent to the following :—

$$\frac{1}{m} = \frac{g \Delta \mathrm{E}}{a_0^2 \mathrm{I}}; \qquad (1.)$$

$$\mathrm{I} = \frac{m g \Delta \mathrm{E}}{a_0^2}; \qquad (2.)$$

the second of which equations gives the requisite moment of inertia of the fly-wheel.

The fluctuation of energy may arise either from variations in the effort exerted by the prime mover, or from variations in the resistance, or from both those causes combined. When but one fly-wheel is used, it should be placed in as direct connexion as possible with that part of the mechanism where the greatest amount of the fluctuation originates; but when it originates at two or more points, it is best to have a fly-wheel in connection with each of those points.

For example, let there be a steam engine which drives a shaft that traverses a workshop, having at intervals upon it pulleys for driving various machine-tools. The steam engine should have a fly-wheel of its own, as near as practicable to its crank, adapted to that value of Δ E which is due to the fluctuations of the effort applied to the crank-pin above and below the mean value of that effort, and which may be computed by the aid of General Morin's tables, quoted in Article 430; and each machine tool should also have a fly-wheel, adapted to a value of Δ E equal to the whole work performed by the tool at one operation.

As the rim of a fly-wheel is usually heavy in comparison with the arms, it is often sufficiently accurate for practical purposes to take the moment of inertia as simply equal to the weight of the rim multiplied by the square of the mean between its outside and inside radii—a calculation which may be expressed thus :—

$$\mathrm{I} = \mathrm{W} r^2; \qquad (3.)$$

whence the weight of the rim is given by the formula—

$$\mathrm{W} = \frac{m g \Delta \mathrm{E}}{a_0^2 r^2} = \frac{m g \Delta \mathrm{E}}{v'^2}, \qquad (4.)$$

if v' be the velocity of the rim of the fly-wheel.

In millwork the ordinary values of the product mg, the unit of time being the second, lie between 1,000 and 2,000 feet, or approximately between 300 and 600 mètres. In pumping-machinery it is sometimes only about 300 feet, or 90 mètres.

The rim of the fly-wheel of a factory steam engine is very often provided with teeth, or with a belt, in order that it may directly drive the machinery of the factory.

Section 3.—Of Governors.

432. **The Regulator** of a prime mover is some piece of apparatus by which the rate at which it receives energy from the source of energy can be varied; such as the sluice or valve which adjusts the size of the orifice for supplying water to a water-wheel, the apparatus for varying the surface exposed to the wind by windmill sails, the throttle-valve which adjusts the opening of the steam pipe of a steam engine, the damper which controls the supply of air to its furnace, and the expansion gear which regulates the volume of steam admitted into the cylinder at each stroke of the piston.

In prime movers whose speed and power have to be frequently and rapidly varied at will, such as locomotives and winding engines for mines, the regulator is adjusted by hand. In other cases the regulator is adjusted by means of a self-acting instrument driven by the prime mover to be regulated, and called a Governor.

The special construction of the different kinds of regulators is a subject for a treatise on prime movers. In the present treatise it is sufficient to state that in every governor there is a moving piece which acts on the regulator through a suitable train of mechanism, and which is itself made to move in one direction or in another according as the prime mover is moving too fast or too slow.

The object of a governor, properly so called, is to preserve a certain uniform speed, either exactly or approximately; and such is always the case in millwork. There are other cases, as in marine steam engines, where it may be considered sufficient to prevent sudden variations of speed, without preserving an uniform speed; and in those cases an apparatus may be used possessing only in part the properties of a governor: this may be called a *fly-governor*, to distinguish it from a governor proper.

Governors proper may be distinguished into *position-governors, disengagement-governors*, and *differential governors:* a position-governor being one in which the moving piece that acts on the regulator assumes positions depending on the speed of motion, as in the common steam engine governor, which consists of a pair of revolving pendulums acting directly on a train of mechanism which adjusts the throttle-valve: a disengaging-governor being one which, when the speed deviates above or below its proper value, throws the regulator into gear with one or other of two trains of mechanism which move it in contrary directions so as to diminish or increase the speed, as the case may require, as in water-mill governors; and a differential-governor being one which, by means of an aggregate combination, moves the regulator in one direction or in another with a speed proportional to the difference between the actual speed and the proper speed of the engine.

In almost all governors the action depends on the centrifugal force exerted by two or more masses which revolve round an axis. By another classification, different from that which has already been described, governors may be distinguished into *gravity-governors*, in which gravity is the force that opposes the centrifugal force; and *balanced-governors*, in which the actions of gravity on the various moving parts of the governor are mutually balanced, and the centrifugal force is opposed by the elasticity of a spring.

Governors may be further distinguished into those which are truly isochronous—that is to say, which remain without action on the regulator at one speed only; and those which are nearly isochronous—that is to say, which admit of some variation of the permanent or steady speed when the resistance overcome by the engine varies; and lastly, governors may be distinguished into those which are specially adapted to one speed, and those which can be adjusted at will to different speeds.

433. **Pendulum-Governors.**—A pendulum-governor is the simplest kind of gravity-governor. It has a vertical spindle, driven by the engine to be regulated; and from that spindle there hang, at opposite sides, a pair of revolving pendulums, which, by the positions that they assume at different speeds, act on the regulator.

The relation between the height of a simple revolving pendulum and the number of turns which it makes per second has already been stated in Article 336; but for the sake of convenience it may here be repeated:—Let h denote the height or *altitude* of the pendulum ($=$ O H in fig. 153), and T the number of turns per second; then

$$h = \frac{g}{4\pi^2 T^2} = \frac{\cdot 815 \text{ foot}}{T^2} = \frac{9\cdot 78 \text{ inches}}{T^2} = \frac{0\cdot 248 \text{ mètre}}{T^2} \ldots\ldots (1.)$$

If the rods of the revolving pendulums are jointed, as in fig. 154, not to a point in the vertical axis, but to a pair of points, such as C, c, in arms projecting from that axis, the height is to be measured to the point O, where the lines of tension of the rods cut the axis.

In most cases which occur in practice, the balls are so heavy, as compared with the rods, that the height may be measured without sensible error from the level of the centres of the balls to the point O, where the lines of suspension cut the axis. This amounts to neglecting the effects both of the weight and of the centrifugal force of the rods.

The ordinary steam engine governor invented by Watt, which is represented in fig. 153, is a position-governor, and acts on the

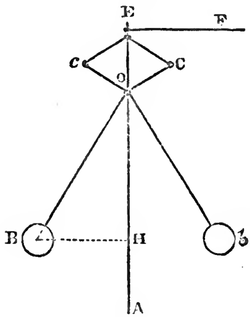


Fig. 153.

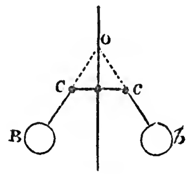


Fig. 154.

regulator by means of the variation of its altitude, through a train of levers and linkwork. That train may be very much varied in detail. In the example shewn in the figure, the lever O C forms one piece with the ball-rod O B, and the lever O c with the ball-rod O b; so that when the speed falls too low, the balls B, b, by approaching the spindle, cause the point E to rise; and when the speed rises too high, the balls, by receding from the spindle, cause the point E to fall. At the point E there is a collar, held in the forked end of the lever E F, which communicates motion to the regulator.

The ordinary pendulum-governor is not truly isochronous; for when, in order to adapt the opening of the regulator to different loads, it rotates with its revolving pendulums at different angles to the vertical axis, the altitude h assumes different values, corresponding to different speeds.

As in Article 431, let the utmost extent of fluctuation of the speed of the engine between its highest and lowest limits be the fraction $\frac{1}{m}$ of the mean speed; let h be the altitude of the governor corresponding to the mean speed; and let k be the utmost extent of variation of the altitude between its smaller limit, when the regulator is shut, and its greater limit, when the regulator is full open. Then we have the following proportion:—

$$1 : \left(1 + \frac{1}{2m}\right)^2 - \left(1 - \frac{1}{2m}\right)^2 :: h : k;$$

and consequently

$$\frac{k}{h} = \frac{2}{m} \qquad \text{(3.)}$$

434. **Loaded Pendulum-Governor.**—From the balls of the common governor, whose collective weight is (say) A, let there be hung by a pair of links of lengths equal to the ball-rods, a load B, capable of sliding up and down the spindle, and having its centre of gravity in the axis of rotation. Then the centrifugal force is that due to A alone; and the effect of gravity is that due to A + 2 B; for when the ball-rods shift their position, the load B moves through twice the vertical distance that the balls move through, and is therefore equivalent, to a double load, 2 B, acting directly on the balls. Consequently the altitude for a given speed is greater than that of a simple revolving pendulum, in the ratio $1 + \frac{2\,B}{A}$; a given *absolute* variation of altitude in moving the regulator produces a proportionate variation of speed smaller than in the common governor, in the ratio $\frac{A}{A + 2\,B}$; and the governor is said to be *more sensitive* than a common governor, in the ratio of A : A + 2 B. Such is the construction of Porter's governor.

The links by which the load B is hung may be attached, not to the balls themselves, but to any convenient pair of points in the ball-rods; the links, and the parts of the ball-rods to which they are jointed, always forming a rhombus, or equilateral parallelogram. Let q be the ratio borne by each of the sides of that rhombus to the length on the ball-rods from the centre of a ball to the point where the line of suspension cuts the axis; then in the preceding expressions 2 q B is to be substituted for 2 B.

In the one case 2 B, and in the other 2 q B, is the weight, applied directly at A, which would be *statically equivalent* to the load B, applied where it is.

CHAPTER IV.

OF THE EFFICIENCY AND COUNTER-EFFICIENCY OF PIECES, COMBINATIONS, AND TRAINS IN MECHANISM.

435. **Nature and Division of the Subject.**—The terms *Efficiency* and *Counter-efficiency* have already been explained in Article 418, page 265; and the laws of friction, the most important of the wasteful resistances which cause the efficiency of a machine to be less than unity, have been stated in Articles 261 and 402, pages 153 and 251. In the present chapter are to be set forth the effects of wasteful resistance, and especially of friction, on the efficiency and counter-efficiency of single pieces, and of combinations and trains of pieces, in mechanism. In practical calculations the *counter-efficiency* is in general the quantity best adapted for use; because the useful work to be done in an unit of time, or *effective power*, is in general given; and from that quantity, by multiplying it by the counter-efficiency, of the machine—that is, by the continued product of the counter-efficiencies of all the successive pieces and combinations by means of which motion is communicated from the driving-point to the useful working-point—is to be deduced the value of the expenditure of energy in an unit of time, or *total power*, required to drive the machine. In symbols, let U be the useful work to be done per second; c, c', c'', &c., the counter-efficiencies of the several parts of the train; T, the total energy to be expended per second; then

$$T = c \cdot c' \cdot c'' \cdot \&c. \, .. U \qquad (1.)$$

When the mean effort required at the driving-point can conveniently be computed by reducing each resistance to the driving-point, and adding together the reduced resistances (as in Article 407, page 253, and Article 421, page 267), the ratio in which the actual effort required at the driving-point is greater than what the required effort would be, in the absence of wasteful resistance, is expressed by the continued product of the counter-efficiencies of the parts of the train, as follows: let P_0 be the effort required, in the absence of wasteful resistance; P, the actual effort required; then

$$P = c \cdot c' \cdot c'' \cdot \&c. \dots P_0; \qquad (2.)$$

and in determining the efficiency or the counter-efficiency of a single piece, the most convenient method of proceeding often consists in comparing together the efforts required to drive that piece, with and without friction, and thus finding the ratios

$$\frac{P_0}{P} = \text{efficiency}; \ \frac{P}{P_0} = \text{counter-efficiency} \ldots\ldots\ldots (3.)$$

In the ensuing sections of this chapter, the efficiency of single primary pieces is first treated of, and then that of the various modes of connexion employed in elementary combinations.

SECTION 1.—EFFICIENCY AND COUNTER-EFFICIENCY OF PRIMARY PIECES.

436. **Efficiency of Primary Pieces in General.**—A primary piece in mechanism, moving with an uniform velocity, is balanced under the action of four forces, viz.:—

I. The re-action of the piece which it drives: this may be called the *Useful Resistance*, and denoted by R;

II. The *weight* of the piece itself: this may be denoted by W;

III. The *effort* by which the piece is driven: this may be denoted by P; and its values with and without friction by P_0 and P_1 respectively;

IV. The resultant pressure at the bearings, or *bearing-pressure*, which may be denoted by Q; and which of course is equal and directly opposed to the resultant of the first three forces.

In the absence of friction, the bearing-pressure would be normal to the bearing surface. The effect of friction is, that the line of action of the bearing-pressure becomes oblique to the bearing-surface, making with the normal to that surface the angle of repose (φ), whose tangent ($f = \tan \varphi$) is the coefficient of friction (see Article 261, page 154); and the amount of the friction is expressed by Q sin φ, or very nearly by f Q, when the coefficient of friction is small.

In the class of problems to which this chapter relates, the first two forces—that is, the useful resistance R, and the weight W—are given in magnitude, position, and direction; and in most cases it is convenient to find their resultant, in magnitude, position, and direction, by the rules of statics: that is to say, if the line of action of R is vertical, by Article 226, page 128; and if inclined, by the rules given or referred to in Article 209, page 122. In what follows, the resultant of the useful resistance and weight will be called the *given force*, and denoted by R′.

The third force—that is, the effort required in order to drive the piece at an uniform speed—is given in position and direction; for its line of action is the line of connection of the piece under consideration with the piece that drives it. The magnitude of the effort is one of the quantities to be found.

The fourth force—that is, the bearing-pressure—has to be found in position, direction, and magnitude. The general principles according to which it is determined are the following:—

First, That if the lines of action of the given force and the effort are parallel to each other, the line of action of the resultant bearing-pressure must be parallel to them both; and that if they are inclined to each other, the line of action of the resultant bearing-pressure must traverse their point of intersection.

Secondly, That at the centre of pressure, where the line of action of the resultant bearing-pressure cuts the bearing surfaces, it makes an angle with the common normal of those surfaces equal to their angle of repose, and in such a direction that its tangential component (being the friction) is directly opposed to the relative sliding motion of that pair of surfaces over each other.

Thirdly, That the given force, the effort, and the bearing-pressure, form a system of three forces that balance each other; and are therefore proportional to the three sides of a triangle parallel respectively to their directions.

437. **Efficiency of a Straight-sliding Piece.**—In fig. 155, let A A be a straight guiding-surface, upon which there slides, in the direction marked by the feathered arrow, the moving piece B. Let C D represent the *given force*, being the resultant of the useful resistance and of the weight of the piece B. (The figure shews the motion of B as horizontal; but it may be in any direction.) Let C J be the line of action of the effort by which the piece B is driven.

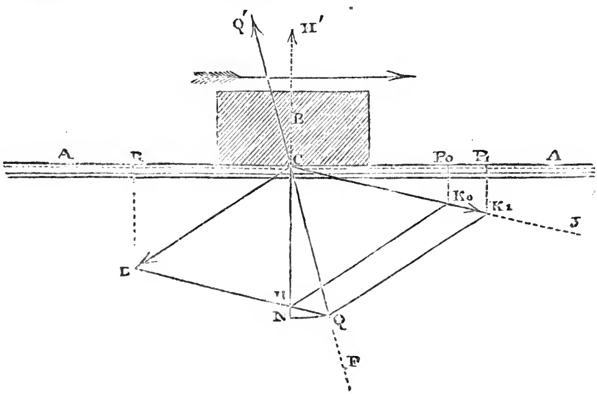


Fig. 155.

Draw C N perpendicular to A A; and C F making the angle N C F = the angle of repose. Through D, parallel to C J, draw the straight line D H Q, cutting C N in H, and C F in Q; and through H and Q, and parallel to D C, draw H K_0 and Q K_1, cutting C J in K_0 and K_1 respectively. Produce H C to H′, and Q C to Q′, making C H′ = H C, and C Q′ = Q C.

Then, in the absence of friction, C H′ will represent the resultant bearing-pressure exerted upon B by A A; and C K_0 = D H will represent the force in the given direction C J required to drive B at an uniform speed; and when friction is taken into account, C Q′ will represent the resultant bearing-pressure, and C K_1 the actual driving force required; and we shall have

$$\text{the efficiency} = \frac{C K_0}{C K_1}; \text{ and the counter-efficiency} = \frac{C K_1}{C K_0}$$

If from D, K_0, and K_1 there be let fall upon A A the perpendiculars D R, K_0 P_0, and K_1 P_1, C R will represent the direct resistance to the advance of B; C P_0, the direct effort in the absence of friction; and C P_1, the direct effort taking friction into account; so that the distance P_0 P_1 will represent the friction itself; which is also represented by Q N perpendicular to C N.

To express these results by symbols, let C D = R′ (the given force); let the acute angle A C D be denoted by α, and the acute angle A C J by β; and let φ denote the angle of repose N C Q.

Then, in the triangle C D H, we have $\angle$ D C H $= \frac{\pi}{2} - \alpha$, and C H D $= \frac{\pi}{2} - \beta$; and in the triangle C Q D, we have $\angle$ D C Q $= \frac{\pi}{2} - \alpha + \varphi$, and $\angle$ C Q D $= \frac{\pi}{2} - \beta - \varphi$; consequently

$$D H = R' \frac{\cos \alpha}{\cos \beta}; \; D Q = R' \cdot \frac{\cos (\alpha - \varphi)}{\cos (\beta + \varphi)};$$

whence it follows that the efficiency and counter-efficiency are given by the following equations:—

$$\text{Efficiency} = \frac{P_0}{P_1} = \frac{D H}{D Q} = \frac{\cos \alpha \cdot \cos (\beta + \varphi)}{\cos \beta \cdot \cos (\alpha - \varphi)} = \frac{1 - f \tan \beta}{1 + f \tan \alpha} \quad (1.)$$

$$\text{Counter-efficiency} = \frac{P_1}{P_0} = \frac{1 + f \tan \alpha}{1 - f \tan \beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

It is to be remarked, that the efficiency diminishes to nothing when cotan $\beta = f$; that is to say, when β is the complement of the angle of repose, φ. In other words, if the oblique effort is applied in the direction C Q, no force, how great soever, will be sufficient to keep the piece B in motion.

438. **Efficiency of an Axle.**—In fig. 156, let the circle A A A represent the trace of the bearing-surface of an axle on a plane perpendicular to its axis of rotation, O—in other words, the trans-

verse section of that surface. Let the arrow near the letter N represent the direction of rotation. Let C D be the given force; that is, as before, the resultant of the weight of the whole piece that rotates with the axle, and of the useful resistance or re-action exerted on that piece by the piece which it drives; C J, the line of action of the effort by which the rotating piece is driven.

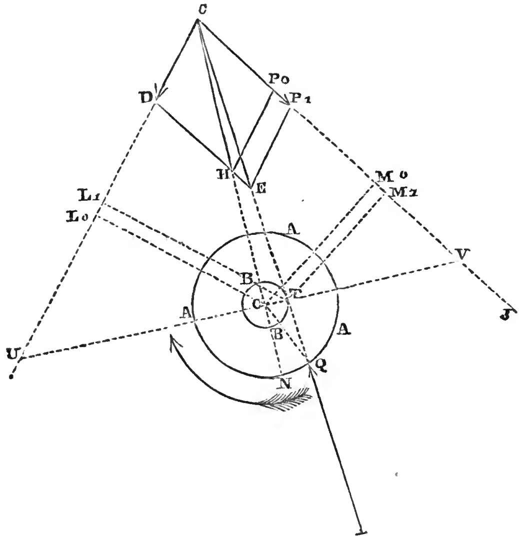


Fig. 156.

Let r denote the radius of the bearing-surface.

About O describe the small circle B B, with a radius = $r \sin \varphi = f r$, very nearly. Draw the line of action, C T Q, of the resultant bearing-pressure, touching the small circle at that side which will make the bearing-pressure resist the rotation. In the case in which C D and C J intersect each other in a point, C, as shewn in the figure, C T Q will traverse that point also; and in the case in which the lines of action of the given force and the effort are parallel to each other, C T Q will be parallel to both. The centre of bearing-pressure is at Q; and O Q T $= \varphi$, the angle of repose.

In the former case the efficiency may be found by parallelograms of forces, as follows:—Draw the straight line C O N; this would be the line of action of the resultant bearing-pressure in the absence of friction, and N would be the centre of bearing-pressure. Through D, parallel to C J, draw D H E, cutting C O N in H, and C T Q in E. Through H and E, parallel to D C, draw H P_0

and E P_1. Then, in the absence of friction, H C would represent the bearing-pressure, and C P_0 = D H the effort; the actual bearing-pressure is represented by E C, and the actual effort by C P_1 = D E. Hence the efficiency and counter-efficiency are as follows:—

$$\frac{P_0}{P_1} = \frac{D\,H}{D\,E}; \quad \frac{P_1}{P_0} = \frac{D\,E}{D\,H} \quad \ldots\ldots\ldots\ldots (1.)$$

Another method, applicable whether the forces are inclined or parallel, is as follows:—From the axis of rotation O, let fall O L_0 and O M_0 perpendicular respectively to the lines of action of the given force and of the effort. Then, by the balance of moments, the effort in the absence of friction is

$$P_0 = R' \cdot \frac{O\,L_0}{O\,M_0}.$$

From a convenient point in the actual line of action, C Q, of the bearing-pressure (such, for example, as T, where it touches the small circle B B), let fall T L_1 and T M_1 perpendicular respectively to the same pair of lines of action; then the actual effort will be

$$P_1 = R' \cdot \frac{T\,L_1}{T\,M_1}.$$

Hence the efficiency and the counter-efficiency have the following value:—

$$\left.\begin{aligned} \frac{P_0}{P_1} &= \frac{O\,L_0 \cdot T\,M_1}{O\,M_0 \cdot T\,L_1}; \\ \frac{P_1}{P_0} &= \frac{O\,M_0 \cdot T\,L_1}{O\,L_0 \cdot T\,M_1}; \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2.)$$

The same results are expressed, to a degree of approximation sufficient for practical purposes, by the following trigonometrical formulæ:—Let O $L_0 = l$; O $M_0 = m$; $\angle$ C O $L_0 = \alpha$; $\angle$ C O $M_0 = \beta$. Then we have, very nearly,

$$\frac{P_0}{P_1} = \frac{l}{m} \cdot \frac{m - f\,r \sin\beta}{l + f\,r \sin\alpha} = \frac{1 - \dfrac{f\,r}{m} \cdot \sin\beta}{1 + \dfrac{f\,r}{l} \cdot \sin\alpha} \quad \ldots\ldots (3.)$$

In making use of the preceding formula, it is to be observed that the *contrary algebraical signs* of sin α and sin β apply to those cases in which the two angles α and β lie at contrary sides of O C. In the cases in which those angles lie at the same side of O C, their algebraical signs are the same; and in the formula they are to be made *both positive* or *both negative*, according as β is *less* or *greater* than α; so that the efficiency may be always expressed by a fraction less than unity. That is to say,

$$\text{If } \beta > \alpha;\ \frac{P_0}{P_1} = \frac{1 - \frac{fr}{m}\sin\beta}{1 - \frac{fr}{l}\sin\alpha};\text{...........(3 A.)}$$

$$\text{If } \beta < \alpha;\ \frac{P_0}{P_1} = \frac{1 + \frac{fr}{m}\sin\beta}{1 + \frac{fr}{l}\sin\alpha};\text{...........(3 B.)}$$

When the lines of action intersect, let O C be denoted by c; then $l = c \cos \alpha$, and $m = c \cos \beta$; and consequently the three preceding equations take the following form;—

$$\beta \text{ and } \alpha \text{ of contrary signs; } \frac{P_0}{P_1} = \frac{1 - \frac{fr}{c}\tan\beta}{1 + \frac{fr}{c}\tan\alpha};\text{.................(4.)}$$

β and α of the same sign;

$$\beta > \alpha;\ \frac{P_0}{P_1} = \frac{1 - \frac{fr}{c}\tan\beta}{1 - \frac{fr}{c}\tan\alpha};\text{...........(4 A.)}$$

$$\beta < \alpha;\ \frac{P_0}{P_1} = \frac{1 + \frac{fr}{c}\tan\beta}{1 + \frac{fr}{c}\tan\alpha};\text{...........(4 B.)}$$

When the lines of action of the forces are parallel, we have sin β and sin $\alpha = +1$ or -1, as the case may be; and the formulæ take the following shape:—

When l and m lie at contrary sides of O, the piece is a "lever of the first kind;" and

$$\frac{P_0}{P_1} = \frac{1 - \frac{fr}{m}}{1 + \frac{fr}{l}}\text{.........................(5.)}$$

When l and m lie at the same side of O;

If $m > l$, the piece is a "lever of the second kind;" and

$$\frac{P_0}{P_1} = \frac{1 - \frac{fr}{m}}{1 - \frac{fr}{l}} \qquad (5A.)$$

If $m < l$, the piece is a "lever of the third kind;" and

$$\frac{P_0}{P_1} = \frac{1 + \frac{fr}{m}}{1 + \frac{fr}{l}} \qquad (5B.)$$

(As to levers of the first, second, and third kinds, see Article 184, page 108.)

The following method is applicable whether the forces are inclined or parallel; in the former case it is approximate, in the latter exact. Through O, perpendicular to O C, draw U O V, cutting the lines of action of the given force and of the effort in U and V respectively. The point where this transverse line cuts the small circle B B coincides exactly with T when the forces are parallel, and is very near T when they are inclined; and in either case the letter T will be used to denote that point. Then

$$\frac{P_0}{P_2} = \frac{\mathrm{O\,U}}{\mathrm{O\,V}} \cdot \frac{\mathrm{T\,V}}{\mathrm{T\,U}} \qquad (6.)$$

It is evident that with a given radius and a given coefficient of friction, the efficiency of an axle is the greater the more nearly the effort and the given force are brought into direct opposition to each other, and also the more distant their lines of action are from the axis of rotation.

439. **Efficiency of a Screw.**—The efficiency of a screw acting as a primary piece is nearly the same with that of a block sliding on a straight guide, which represents the development of a helix situated midway between the outer and inner edges of the screw-thread; the block being acted upon by forces making the same angles with the straight guide that the actual forces do with that helix. As to the development of a helix, see Article 160, page 94; and as to the efficiency of a piece sliding along a straight guide, see Article 437, page 288.

Section 2.—Efficiency and Counter-efficiency of Modes of Connection in Mechanism.

440. **Efficiency of Modes of Connection in General.**—In an elementary combination consisting of two pieces, a driver and a

follower, there is always some work lost in overcoming wasteful resistance occasioned by the mode of connection; the result being that the work done by the driver at its working-point is greater than the work done upon the follower at its driving-point, in a proportion which is *the counter-efficiency of the connection;* and the reciprocal of that proportion is *the efficiency of the connection.* In calculating the efficiency or the counter-efficiency of a train of mechanism, therefore, the factors to be multiplied together comprise not only the efficiencies, or the counter-efficiencies, of the several primary pieces considered separately, but also those of the several modes of connection by which they communicate motion to each other.

441. **Efficiency of Rolling Contact.**—The work lost when one primary piece drives another by rolling contact is expended in overcoming the *rolling resistance* of the pitch-surfaces, a kind of resistance whose mode of action has been explained in Article 402, page 251; and the value of that work in units of work per second is given by the expression $a\,b\,N$; in which N is the normal pressure exerted by the pitch-surfaces on each other; b, a constant arm, of a length depending on the nature of the surfaces (for example 0·002 of a foot = 0·6 millimètre for cast iron on cast iron, see page 252); and a the relative angular velocity of the surfaces.

The useful work per second is expressed by $u\,f\,N$, in which f is the coefficient of friction of the surfaces, and u the common velocity of the pitch lines. Hence the *counter-efficiency* is

$$c = 1 + \frac{a\,b}{u\,f} \qquad \text{(1.)}$$

Let p_1 and p_2 be the lengths of two perpendiculars let fall from the two axes of rotation on the common tangent of the two pitch-lines; if the pieces are circular wheels, those perpendiculars will be the radii. Then the absolute angular velocities of the pieces are respectively $\frac{u}{p_1}$ and $-\frac{u}{p_2}$; and their relative angular velocity is therefore

$$a = u\left(\frac{1}{p_1} + \frac{1}{p_2}\right);$$

which value being substituted in Equation 1, gives for the counter-efficiency the following value:—

$$c = 1 + \frac{1}{f}\left(\frac{b}{p_1} + \frac{b}{p_2}\right) \qquad \text{(2.)}$$

It is assumed that the normal pressure is not greater than is necessary in order to give sufficient friction to communicate the motion.

It is evident, from the smallness of b, that the lost work in this case must be almost always a very small fraction of the whole.

442. **Efficiency of Sliding Contact in General.**—In fig. 157, let T be the point of contact of a pair of moving pieces connected by sliding contact. Let the plane of the figure be that containing the directions of motion of the two particles which touch each other at the point T; and let T V be the velocity of the driving-particle, and T W the velocity of the following particle; whence V W will represent the velocity of sliding, and T U, perpendicular to V W, the common component of the velocities of the two particles along their line of connection R T P. C T C, parallel to V W, and perpendicular to R T P, is a common tangent to the two acting surfaces at the point T; the arrow A represents the direction in which the driver slides relatively to the follower; and the arrow B, the direction in which the follower slides relatively to the driver.

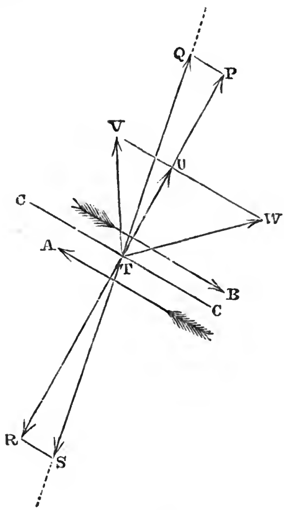


Fig. 157.

Along the line of connection, that is, normal to the acting surfaces at T, lay off T P to represent the effort exerted by the driver on the follower, and T R (= − T P) to represent the equal and opposite useful resistance exerted by the follower against the driver. Draw S T Q, making with R T P an angle equal to the angle of repose of the rubbing surfaces, (see Article 261, page 154), and inclined in the proper direction to represent forces opposing the sliding motion; draw P Q and R S parallel to C C. Then T Q will represent the resultant pressure exerted by the driver on the follower, and T S (= − T Q), the equal and opposite resultant pressure exerted by the follower against the driver, and P Q = − R S will represent the friction which is overcome, through the distance V W, in each second; while the useful resistance, T R, is overcome through the distance T U. Hence the useful work per second is T U · T R; the lost work is V W · R S; and the counter-efficiency is

$$c = 1 + \frac{\mathrm{V\,W} \cdot \mathrm{R\,S}}{\mathrm{T\,U} \cdot \mathrm{T\,R}} \qquad (1.)$$

Let the angle U T V $= \alpha$, the angle U T W $= \beta$, and let f be the coefficient of friction. Then we have—

$$\frac{\mathrm{V\,W}}{\mathrm{T\,U}} = \tan \alpha + \tan \beta\,; \ \frac{\mathrm{R\,S}}{\mathrm{T\,R}} = f\,;$$

and consequently

$$c = 1 + f\,(\tan \alpha + \tan \beta) \text{..................... (2.)}$$

443. **Efficiency of Teeth.**—It has already been shewn, in Article 148, page 87, that the relative velocity of sliding of a pair of teeth in outside gearing is expressed at a given instant by

$$(a_1 + a_2)\ t;$$

where t denotes the distance at that instant of the point of contact from the pitch-point. (In inside gearing the angular velocity of the greater wheel is to be taken with the negative sign.)

The distance t is continually varying from a maximum at the beginning and end of the contact, to nothing at the instant of passing the pitch-point. Its *mean value* may be assumed, with sufficient accuracy for practical purposes, to be sensibly equal to *one-half* of its greatest value; and in the formulæ which follow, the symbol t stands for that mean value.

Let P be the mutual pressure exerted by the teeth; f, the coefficient of friction; then the work lost per second through the friction of the teeth is

$$(a_1 + a_2)\ t\,f\,\mathrm{P}.$$

Let u be the common velocity of the two pitch-circles; θ, the *mean obliquity* of the line of connection to the common tangent of the pitch-circles; then $u \cos \theta$ is the mean value of the common component of the velocities of the acting surfaces of the teeth along the line of connection; and the useful work done per second is expressed by

$$\mathrm{P}\,u \cos \theta.$$

so that the counter-efficiency is

$$c = 1 + \frac{(a_1 + a_2)\,t\,f}{u \cos \theta} \text{........................ (1.)}$$

Let r_1 and r_2 be the radii of the two pitch circles; then we have

$$a_1 = \frac{u}{r_1};\ a_2 = \frac{u}{r_2};$$

and consequently

$$c = 1 + f\,t \sec \theta \left\{ \frac{1}{r_1} + \frac{1}{r_2} \right\} \text{.................. (2.)}$$

If two pairs of teeth at least are to be in action at each instant (as in the case of involute teeth, and of some epicycloidal teeth), and if the pitch be denoted by p, we have t see $\theta = \frac{p}{2}$; and therefore

$$c = 1 + \frac{fp}{2}\left\{\frac{1}{r_1} + \frac{1}{r_2}\right\} = 1 + \pi f\left\{\frac{1}{n_1} + \frac{1}{n_2}\right\}; \ldots\ldots (3.)$$

where n_1 and n_2 are the number of teeth in the two wheels.

In many examples of epicycloidal teeth, especially where small pinions are used, the duration of the contact is only $\frac{2}{3}$ or $\frac{3}{4}$ of that assumed in Equation 3; and the work lost in friction is less in the same proportion.

444. **Efficiency of Bands.**—A band, such as a leather belt or a hempen rope, which is not perfectly elastic, requires the expenditure of a certain quantity of work—first to bend it to the curvature of a pulley, and then to straighten it again; and the quantity of work so lost has been found by experiment to be nearly the same as would be required in order to overcome an additional resistance, varying directly as the sectional area of the band, directly as its tension, and inversely as the radius of the pulley. In the following formulæ for leather belts, the stiffness is given as estimated by Reuleaux (*Constructionslehre für Maschinenbau*, § 307).

Let T be the mean tension of the belt; S, its sectional area; r, the radius of the pulley; b, a constant divisor determined by experiment; R′, the resistance due to stiffness; then

$$R' = \frac{S\,T}{b\,r} \ldots\ldots\ldots\ldots (1.)$$

b (for leather) = 3·4 inch = 87 millimètres.

To apply this to an endless belt connecting a pair of pulleys of the respective radii r_1 and r_2, let T_1 and T_2 be the tensions of the two sides of the belt. Then the useful resistance is $T_1 - T_2$, the mean tension is $\frac{T_1 + T_2}{2}$; and the additional resistance due to stiffness is

$$\frac{T_1 + T_2}{2} \cdot \frac{S}{b}\left\{\frac{1}{r_1} + \frac{1}{r_2}\right\};$$

consequently the counter-efficiency is

$$\left.\begin{aligned} c &= 1 + \frac{T_1 + T_2}{2(T_1 - T_2)} \cdot \frac{S}{b}\left\{\frac{1}{r_1} + \frac{1}{r_2}\right\}; \\ &= 1 + \frac{N + 1}{2(N - 1)} \cdot \frac{S}{b}\left\{\frac{1}{r_1} + \frac{1}{r_2}\right\}; \end{aligned}\right\} \ldots\ldots (2.)$$

N denoting $\frac{T_1}{T_2}$. The sectional area, S, of a leather belt is given by the formula

$$S = \frac{T_1}{p}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

where p denotes the safe working tension of leather belts, in units of weight per unit of area; its value being, according to Morin,

0·2 kilogramme on the square millimètre, or
285 lbs. on the square inch.

The ordinary thickness of the leather of which belts are made is about 0·16 of an inch, or 4 millimètres; and from this and from the area the breadth may be calculated. A double belt is of double thickness, and gives the same area with half the breadth of a single belt.

When a band runs at a high velocity, the *centrifugal tension*, or tension produced by centrifugal force, must be added to the tension required for producing friction on the pulleys, in order to find the total tension at either side of the band, with a view to determining its sectional area and its stiffness. The centrifugal tension is given by the following expression :—

$$\frac{w\,S\,v^2}{g}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

in which w is the heaviness (being, for leather belts, nearly equal to that of water); S, the sectional area; v, the velocity; and g, gravity (= 32·2 feet, or 9·81 mètres per second).

When centrifugal force is taken into consideration, the following formula is to be used for calculating the sectional area; T_1 being the tension at the driving-side of the belt, *exclusive of centrifugal tension* :—

$$S = \frac{T_1}{p - \frac{w\,v^2}{g}}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

and the following formula for the counter-efficiency :—

$$c = 1 + \frac{T_1 + T_2 + \frac{2\,w\,v^2}{g}}{2\,(T_1 - T_2)} \cdot \frac{S}{b} \cdot \left\{ \frac{1}{r_1} + \frac{1}{r_2} \right\} \ldots\ldots\ldots(6.)$$

For calculating the efficiency of hempen ropes used as bands, it is unnecessary in such questions as that of the present article to use a more complex formula than that of Eytelwein—viz.,

$$R' = \frac{D^2\,T}{b'\,r}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7.)$$

where D is the diameter of the rope, and $b' = 54$ millimètres = 2·125 inches.

In all the formulæ, $\frac{D^2}{b'}$ is to be substituted for $\frac{S}{b}$. The proper value of D^2 is given by the formula

$$D^2 = \frac{T_1}{p'}; \qquad \qquad (8.)$$

where $p' = 1{,}000$ for measures in inches and lbs.; and
$p' = 0·7$ for measures in millimètres and killogrammes.

445. **Efficiency of Linkwork.**—In fig. 158, let $C_1 T_1$, $C_2 T_2$ be two levers, turning about parallel axes at C_1 and C_2, and connected with each other by the link $T_1 T_2$; T_1 and T_2 being the connected points.

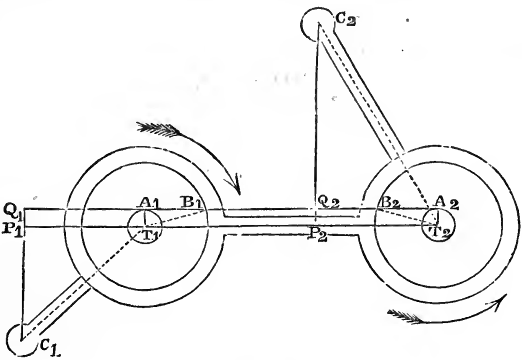


Fig. 158.

The pins, which are connected with each other by means of the link, are exaggerated in diameter, for the sake of distinctness. Let $C_1 T_1$ be the driver, and $C_2 T_2$ the follower, the motion being as shewn by the arrows. From the axes let fall the perpendiculars $C_1 P_1$, $C_2 P_2$, upon the line of connection. Then the angular velocities of the driver and follower are inversely as those perpendiculars; and, in the absence of friction, the driving moment of the first lever and the working moment of the second are directly as those perpendiculars; the driving pressure being exerted along the line of connection $T_1 T_2$. Let M_2 be the working moment; and let M_0 be the driving moment in the absence of friction; then we have

$$M_0 = \frac{M_2 \cdot C_1 P_1}{C_2 P_2}.$$

To allow for the friction of the pins, multiply the radius of each pin by the sine of the angle of repose; that is, very nearly by the coefficient of friction; and with the small radii thus computed, $T_1 A_1$ and $T_2 A_2$, draw small circles about the connected points. Then draw a straight line, $Q_1 A_1 B_1 Q_2 A_2 B_2$, touching both the small circles, and in such a position as to represent the line of action of a force that resists the motion of both pins in the eyes of the link. This will be the line of action of the resultant force exerted through the link. Let fall upon it the perpendiculars $C_1 Q_1$, $C_2 Q_2$; these will be proportional to the actual driving moment and working moment respectively; that is to say, let M_1 be the driving moment, including friction; then

$$M_1 = \frac{M_2 \cdot C_1 Q_1}{C_2 Q_2}.$$

Comparing this with the value of the driving moment without friction, we find for the counter-efficiency

$$c = \frac{M_1}{M_0} = \frac{C_1 Q_1 \cdot C_2 P_2}{C_2 Q_2 \cdot C_1 P_1}; \quad \text{(1.)}$$

and for the efficiency

$$\frac{1}{c} = \frac{M_0}{M_1} = \frac{C_2 Q_2 \cdot C_1 P_1}{C_1 Q_1 \cdot C_2 P_2}. \quad \text{(2.)}$$

446. **Efficiency of Blocks and Tackle.**—(See Articles 181, 182, pages 105 and 106.)—In a tackle composed of a fixed and a running block containing sheaves connected together by means of a rope, let the number of plies of rope by which the blocks are connected with each other be n. This is also the collective number of sheaves in the two blocks taken together, and is the number expressing the *purchase*, when friction is neglected.

Let c denote the counter-efficiency of a single sheave, as depending on its friction on the pin, according to the principles of Article 373, page 290. Let c' denote the counter-efficiency of the rope, when passing over a single sheave, determined by the principles of Article 444, the tension being taken as nearly equal to $\frac{R}{n}$; where R is the useful load, or resistance opposed to the motion of the running block. $R \div n$ is also the effort to be exerted on the hauling part of the rope, in the absence of friction. Then the counter-efficiency of the tackle will be expressed approximately by

$$(c c')^n; \quad \text{(1.)}$$

so that the actual or effective purchase, instead of being expressed by n, will be expressed by

$$n (c c')^{-n}. \quad \text{(2.)}$$

447. Efficiency of Connection by means of a Fluid.—When motion is communicated from one piston to another by means of an intervening mass of fluid, as described in Articles 185 to 188, pages 110 and 111, the efficiencies and counter-efficiencies of the two pistons have in the first place to be taken into account; that is to say, with ordinary workmanship and packing, the efficiency of each piston may be taken at 0·9 nearly; while with a carefully made cupped leather collar the counter-efficiency of a plunger may be taken at the following value:—

$$1 - \frac{4\,b}{d}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

in which d is the diameter of the plunger; and b a constant, whose value is from 0·01 to 0·015 of an inch, or from 0·25 to 0·38 of a millimètre. For if c be the circumference of the plunger, and p the effective pressure of the liquid, the whole amount of the pressure on the plunger is $\frac{p\,c\,d}{4}$; and the pressure required to overcome the friction is $p\,c\,b$.

The efficiency and counter-efficiency of the intervening mass of fluid remain to be considered; and if that fluid is a liquid, and may therefore be regarded as sensibly incompressible, these quantities depend on the work which is lost in overcoming the resistance of the passage which the liquid has to traverse.

To prevent unnecessary loss of work, that passage should be as wide as possible, and as nearly as possible of uniform transverse section; and it should be free from sudden enlargements and contractions, and from sharp bends, all necessary enlargements and contractions which may be required being made by means of gradually tapering conoidal parts of the passage, and all bends by means of gentle curves. When those conditions are fulfilled, let Q be the volume of liquid which is forced through the passage in a second; S, the sectional area of the passage; then,

$$v = \frac{Q}{S}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

is the velocity of the stream of fluid. Let b denote the wetted border or circumference of the passage; then,

$$m = \frac{S}{b}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

is what is called the *hydraulic mean depth* of the passage. In a cylindrical pipe, $m = \frac{1}{4}$ diameter. Let l be the length of the passage, and w the heaviness of the liquid. Then the loss of pressure in overcoming the friction of the passage is

$$p' = \frac{f\,l}{m} \cdot \frac{w\,v^2}{2\,g}; \qquad (4.)$$

in which g denotes gravity, and f a coefficient of friction whose value, for water in cylindrical cast-iron pipes, according to the experiments of Darcy, is

$$f = 0{\cdot}005 \left(1 + \frac{1}{12\,d}\right);^* \qquad (5.)$$

d being the diameter of the pipe in feet.

Let p be the pressure on the driven or following piston; then the pressure on the driving piston is $p + p'$; and the *counter-efficiency of the fluid* is

$$1 + \frac{p'}{p}; \qquad (6.)$$

which, being multiplied by the product of the counter-efficiencies of the two pistons, gives the *counter-efficiency of the intervening liquid.*

When the intervening fluid is *air*, there is a loss of work through friction of the passage, depending on principles similar to those of the friction of liquids; and there is a further loss through the escape by conduction of the heat produced by the compression of the air.

The friction which has to be overcome by the air, and which causes a certain loss of pressure between the compressing pumps and the working machinery, consists of two parts, one occasioned by the resistance of the valves, and the other by the friction along the internal surface of pipes.

To overcome the resistance of valves, about *five per cent.* of the effective pressure may be allowed.

The friction in the pipes depends on their length and diameter, and on the velocity of the current of air through them. It is nearly proportional to the square of the velocity of the air.

A velocity of about *forty feet per second* for the air in its compressed state has been found to answer in practice. The diameter of pipe required in order to give that velocity can easily be computed, when the dimensions of the cylinders of the machinery to be driven, and the number of strokes per minute, are given.

When the diameter of a pipe is so adjusted that the velocity of the air is 40 feet per second, the pressure expended in overcoming its friction may be estimated at *one per cent. of the total or absolute*

* When the diameter is expressed in millimètres, for $\frac{1}{12\,d}$ substitute $\frac{25{\cdot}4}{d}$.

pressure of the air, for every five hundred diameters of the pipe that its length contains.

Although the abstraction from the air of the heat produced by the compression involves a certain sacrifice of motive power (say from 30 to 35 per cent.) still the effects of the heated air are so inconvenient in practice, that it is desirable to cool it to a certain extent during or immediately after the compression. This may be effected by injecting water in the form of spray into the compressing pumps; and for that purpose a small forcing pump of about $\frac{1}{100}$th of the capacity of the compressing pumps has been found to answer in practice. The air may thus be cooled down to about 104° Fahr. or 40° Cent.

The factor in the counter-efficiency due to the loss of heat expresses the ratio in which the volume of air as discharged from the compressing pump at a high temperature is greater than the volume of the same air when it reaches the working machinery at a reduced temperature; which ratio may be calculated approximately by taking *two-sevenths of the logarithm of the absolute working pressure of the compressed air in atmospheres, and finding the corresponding natural number.* That is to say, let p_0 denote one atmosphere (= at the level of the sea 14·7 lbs. on the square inch, or 10,333 kilogrammes on the square mètre); let p_1 be the absolute working pressure of the air, so that $p_1 - p_0$ is the effective pressure; then the counter-efficiency due to the escape of heat is,

$$c = \left(\frac{p_1}{p_0}\right)^{\frac{2}{7}} \quad \text{.........................(7.)}$$

From examples of the practical working of compressed air, when used to transmit motive power to long distances, it appears that in order to provide for leakage and various other imperfections in working, the capacity of the compressing pumps should be very nearly double of the net volume of uncompressed air required; and it has also been found necessary, in working the compressing pumps, to provide from three to four times the power of the machinery driven by the compressed air.

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THE END.

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