SMITHSONIAN CONTRIBUTIONS TO KNOWLEDGE.
232

MEMOIR

ON THE

SECULAR VARIATIONS OF THE ELEMENTS

OF THE

ORBITS OF THE EIGHT PRINCIPAL PLANETS,

MERCURY, VENUS, THE EARTH, MARS, JUPITER, SATURN, URANUS, AND NEPTUNE;

WITH TABLES OF THE SAME;

TOGETHER WITH THE

OBLIQUITY OF THE ECLIPTIC, AND THE PRECESSION OF THE EQUINOXES
IN BOTH LONGITUDE AND RIGHT ASCENSION.

BY

JOHN N. STOCKWELL, M.A.

[ACCEPTED FOR PUBLICATION, DECEMBER, 1870.]

ADVERTISEMENT.

THIS memoir was presented to the Institution by Professor J. H. C. Coffin, Superintendent of the American Nautical Almanac, and was afterwards submitted to Prof. S. Newcomb of the Naval Observatory, both of whom recommended its adoption as one of the Smithsonian Contributions to Knowledge. The appropriation for printing at the time it was presented having been temporarily exhausted, a friend of science, who does not allow his name to be mentioned, furnished the means for its immediate publication.

JOSEPH HENRY,

Secretary.

SMITHSONIAN INSTITUTION,

May, 1872.

PREFACE.

THE computations of the secular inequalities, the results of which are given in the following Memoir, were commenced about ten years ago, and have been continued, during many interruptions, till the present time. In the spring of 1870 the calculations were so far advanced that the greater part of the Memoir was put in form for the press; but funds for printing it were not then available, and the computations were suffered to languish till late in autumn, when provision was made through the Smithsonian Institution for its publication. The work was then resumed, completed, and forwarded to the printer without delay; but an unexpected interval of leisure occurring during the process of publication, I availed myself of the opportunity thus presented, and prepared an additional chapter containing tabular values of the elements of the planetary orbits, together with the formulæ necessary for their convenient application. It is believed that this additional chapter will contribute largely to the usefulness of the work, and be found of great practical value in the researches of astronomers.

J. N. STOCKWELL.

CLEVELAND, May, 1872.

INTRODUCTION.

THE reciprocal gravitation of matter produces disturbances in the motions of the heavenly bodies, causing them to deviate from the elliptic paths which they would follow, if they were attracted only by the sun. The determination of the amount by which the actual place of a planet deviates from its true elliptic place at any time is called the problem of planetary perturbation. The analytical solution of this problem has disclosed to mathematicians the fact that the inequalities in the motions of the heavenly bodies are produced in two distinct ways. The *first* is a direct disturbance in the elliptic motion of the body; and the *second* is produced by reason of a variation of the *elements* of its elliptic motion. The elements of the elliptic motion of a planet are six in number, viz: the mean motion of the planet and its mean distance from the sun, the eccentricity and inclination of its orbit, and the longitude of the node and perihelion. The first two are invariable· the other four are subject to both periodic and secular variations.

The inequalities in the planetary motions which are produced by the direct action of the planets on each other, and depend for their amount only on their distances and mutual configurations, are called *periodic inequalities*, because they pass through a complete cycle of values in a comparatively short period of time; while those depending on the variation of the elements of the elliptic motion are produced with extreme slowness, require an immense number of ages for their full development, and are called *secular inequalities.* The general theory of all the planetary inequalities was completely developed by La Grange and La Place nearly a century ago; and the particular theory of each planet for the periodic inequalities was given by La Place in the *Mécanique Céleste.*

The determination of the periodic inequalities of the planets has hitherto received more attention from astronomers than has been bestowed upon the secular inequalities. This is owing in part to the immediate requirements of astronomy, and also in part to the less intricate nature of the problem. It is true that an approximate knowledge of the secular inequalities is necessary in the treatment of the periodic inequalities; but since the secular inequalities are produced with such extreme slowness, most astronomers have been content with the supposition that they are developed uniformly with the time. This supposition is sufficiently near the truth to be admissible in most astronomical investigations during the comparatively short period of time over which astronomical observations or human history extends; but since the values of these variations are derived from the equations of the differential variations of the elements at a particular epoch, it follows that they afford us no knowledge respecting the ultimate condition of the planetary system, or even a near approximation to its actual condition at a time only comparatively remote from the epoch of the elements on which they are founded. But aside from any considerations connected with the immediate needs of practical astronomy, the study of the secular inequalities is one of the most interesting and important departments of physical science, because their indefinite continuance in the same direction would ultimately seriously affect the stability of the planetary system. The demonstration that the secular inequalities of the planets are not indefinitely progressive, but may be expressed analytically by a series of terms depending on the *sines* and *cosines* of angles which increase uniformly with the time, is due to La Grange and La Place. It therefore follows that the secular inequalities are periodic, and differ from the ordinary periodic inequalities only in the length of time required to complete the cycle of their values. The amount by which the elements of any planet may ultimately deviate from their mean values can only be determined by the simultaneous integration of the differential equations of these elements, which is equivalent to the summation of all the infinitesimal variations arising from the disturbing forces of all the planets of the system during the lapse of an infinite period of time.

The simultaneous integration of the equations which determine the instantaneous variations of the elements of the orbits gives rise to a complete equation in which the unknown quantity is raised to a power denoted by the number of planets, whose mutual action is considered. La Grange first showed that if any of the roots of this equation were equal or imaginary, the finite expressions for the values of the elements would contain terms involving arcs of circles or exponential quantities, without the functions of *sine* and *cosine*, and as these terms would increase indefinitely with the time, they would finally render the orbits so very eccentrical that the stability of the planetary system would be destroyed. In order to determine whether the roots of the equation were all real and unequal, he substituted the approximate values of the elements and masses which were employed by astronomers at that time in the algebraic equations, and then by determining the roots he found them to be all real and unequal. It, therefore, followed, that for the particular values of the masses employed by La Grange, the equations which determine the secular variations contain neither arcs of a circle nor exponential quantities, without the signs of *sine* and *cosine ;* whence it follows that the elements of the orbits will perpetually oscillate about their mean values This investigation was valuable as a first attempt to fix the limits of the variations of the planetary elements; but, being based upon values of the masses which were, to a certain extent, gratuitously assumed, it was desirable that the important truths which it indicated should be established independently of any considerations of a hypothetic character. This magnificent generalization was effected by La Place. He proved that, whatever be the relative masses of the planets, the roots of the equations which determine the periods of the secular inequalities will all be real and unequal, provided the bodies of the system are subjected to this one condition, *that they all revolve round the sun in the same direction.* This condition being satisfied by all the members of the solar system, it follows that the orbits of the planets will never be very eccentrical or much inclined to each other by reason of their mutual attraction. The important truths in relation to the forms and positions of the planetary orbits are embodied in the two following theorems by the author of the *Mécanique Céleste:* I. *If the mass of each planet be multiplied by the product of the square of the eccentricity and square root of the mean distance, the sum of all these products will always retain the same magnitude.* II. *If the mass of each planet be multiplied by the product of the square of the inclination of the orbit and the square root of the mean distance, the sum of these products will always remain invariable.* Now, these quantities being computed for a given epoch, if their sum is found to be small, it follows from the preceding theorems that they will always remain so; consequently the eccentricities and inclinations cannot increase indefinitely, but will always be confined within narrow limits.

In order to calculate the limits of the variations of the elements with precision, it is necessary to know the correct values of the masses of all the planets. Unfortunately, this knowledge has not yet been attained. The masses of several of the planets are found to be considerably different from the values employed by La Grange in his investigations. Besides, he only took into account the action of the six principal planets which are within the orbit of Uranus. Consequently, his solution afforded only a first approximation to the limits of the secular variations of the elements.

The person who next undertook the computation of the secular inequalities was Pontécoulant, who, about the year 1834, published the third volume of his *Théorie Analytique du Système du Monde.* In this work he has given the results of his solution of this intricate problem. But the numerical values of the constants which he obtained are totally erroneous on account of his failure to employ a sufficient number of decimals in his computation. Our knowledge of the secular variations of the planetary orbits was, therefore, not increased by his researches.

In 1839 Le Verrier had completed his computation of the secular inequalities of the seven principal planets. This mathematician has given a new and accurate determination of the constants on which the amount of the secular inequalities depends; and has also given the coefficients for correcting the values of the constants for differential variations of the masses of the different planets. This investigation of Le Verrier's has been used as the groundwork of most of the subsequent corrections of the planetary elements and masses, and deservedly holds the first rank as authority concerning the secular variations of the planetary orbits. But Le Verrier's researches were far from being exhaustive, and he failed to notice some curious and interesting relations of a permanent character in the secular variations of the orbits of Jupiter, Saturn, and Uranus. Besides, the planet Neptune had not then been discovered; and the action of this planet considerably modifies the secular inequalities which would otherwise take place. We will now briefly glance at the results of our own labors on the subject.

On the first examination of the works of those authors who had investigated this problem, we perceived that the methods of reducing to numbers those analytical integrals which determine the secular variations of the elements were far from possessing that elegance and symmetry of form which usually characterize the formulas of astronomy. The first step, therefore, was to devise a system of algebraic equations, by means of which we should be enabled to obtain the values of the unknown quantities with the smallest amount of labor. It was soon found to be impracticable to deduce algebraic formulas for the constants, by the elimination of eight unknown quantities from as many linear symmetrical equations, of sufficient simplicity to be used in the deduction of exact results. It therefore became necessary to abandon the idea of a direct solution of the equations, and to seek for the best approximate method of obtaining rigorous values of the unknown quantities. This we have accomplished as completely as could be desired, and by means of the formulas which we have obtained, it is now possible to determine the secular variations of the planetary elements with less labor, perhaps, than would be necessary for the accurate determination of a comet's orbit. The methods and formulas are given in detail in the following Memoir.

After computing anew the numerical coefficients of the differential equations of the elements, we have substituted them in these equations, and have obtained, by means of successive approximations, the rigorous values of the constants corresponding to the assumed masses and elements. The details of the computation are given in the Memoir referred to, and it is unnecessary to speak of them here. We shall, therefore, only briefly allude to some of the conclusions to which our computations legitimately lead.

The object of our investigation has been the determination of the numerical values of the secular changes of the elements of the planetary orbits. These elements are four in number, viz: the eccentricities and inclinations of the orbits, and the longitudes of the nodes and perihelia. The questions that may legitimately arise in regard to the eccentricities and inclinations relate chiefly to their magnitude at any time; but we may also desire to know their rates of change at any time, and the limits within which they will perpetually oscillate. In regard to the nodes and perihelia, it is sometimes necessary to know their relative positions when referred to any plane and origin of coördinates; and also their mean motions, together with the amount by which their actual places can differ from their mean places. With respect to the magnitudes and positions of the elements, together with their rates of change, we may observe that our equations will give them for any required epoch, by merely substituting in the formulas the interval of time between the epoch required and that of the formulas, which is the beginning of the year 1850. A tabulation of the various planetary elements, of sufficient extent to supply the needs of the astronomer, is given at the close of the work. A similar tabulation of the elements of the earth's orbit of sufficient extent to be useful in geological investigations, does not come within the scope of our work; and we leave the computation of the elements for special epochs to those investigators who may need them in their researches. We shall here give the limits between which the eccentricities and inclinations will always oscillate, together with the mean motions of the perihelia and nodes on the fixed ecliptic of 1850; and shall also give the inclinations and nodes referred to the invariable plane of the planetary system.

For the planet Mercury, we find that the eccentricity is always included within the limits 0.1214943 and 0.2317185. The mean motion of its perihelion is $5''.463803$; and it performs a complete revolution in the heavens in 237,197 years. The maximum inclination of his orbit to the fixed ecliptic of 1850 is $10° 36' 20''$, and its minimum inclination is $3° 47' 8''$; while with respect to the invariable plane of the planetary system, the limits of the inclination are $9° 10' 41''$ and $4° 44' 27''$. The mean motion of the node of Mercury's orbit on the ecliptic of 1850, and on the invariable plane, is in both cases the same, and equal to $5''.126172$, making a complete revolution in the interval of 252,823 years. The amount by which the true place of the node can differ from its mean place on the ecliptic of 1850 is equal to $30° 8'$, while on the invariable plane this limit is only $18° 31'$.

For the planet Venus, we find that the eccentricity always oscillates between 0 and 0.0706329. Since the theoretical eccentricity of the orbit of Venus is a vanishing element, it follows that the perihelion of her orbit can have no mean motion, but may have any rate of motion, at different times, between nothing and infinity, both direct and retrograde. The position of her perihelion cannot therefore be determined within given limits at any very remote epoch by the assumption of any particular value for the mean motion, but it must be determined by direct computation from the finite formulas. The maximum inclination of her orbit to the ecliptic of 1850 is $4° 51'$, and to the invariable plane it is $3° 16'.3$; while the mean motion of her node on both planes is indeterminate, because the inferior limit of the inclination is in each case equal to nothing.

A knowledge of the elements of the earth's orbit is especially interesting and important on account of the recent attempts to establish a connection between geological phenomena and terrestrial temperatures, in so far as the latter is modified by the variable eccentricity of her orbit. The amount of light and heat received from the sun in the course of a year depends to an important extent on the eccentricity of the earth's orbit; but the distribution of the same over the surface of the earth depends on the relative position of the perihelion of the orbit with respect to the equinoxes, and on the obliquity of the ecliptic to the equator. These elements are subject to great and irregular variations; but their laws can now be determined with as much precision as the exigencies of science may require. We will now more carefully examine these elements, and the consequences to which their variations give rise.

As we have already computed the eccentricity of the earth's orbit at intervals of 10,000 years during a period of 2,000,000 years, by employing the constants which correspond to the assumed mass of the earth increased by its twentieth part, we here give the elements corresponding to this increased mass. We therefore find that the eccentricity of the earth's orbit will always be included within the limits of 0 and 0.0693888; and it consequently follows that the *mean* motion of the perihelion is indeterminate, although the actual motion and position at any time during a period of 2,000,000 years can be readily found by means of the tabular value of that element. The eccentricity of the orbit at any time can also be found by means of the same table.

The inclination of the apparent ecliptic to the fixed ecliptic of 1850 is always less than 4° 41′; while its inclination to the invariable plane of the planetary system always oscillates within the limits 0° 0′ and 3° 6′. It is also evident that the mean motion of the node of the apparent ecliptic on the fixed ecliptic of 1850, and also on the invariable plane, is wholly indeterminate.

The mean value of the precession of the equinoxes on the fixed ecliptic, and also on the apparent ecliptic, in a Julian year, is equal to 50″.438239; whence it follows, that the equinoxes perform a complete revolution in the heavens in the average interval of 25,694.8 years; but on account of the secular inequalities in their motion, the time of revolution is not always the same, but may differ from the mean time of revolution by 281.2 years. We also find that if the place of the equinox be computed for any time whatever, by using the mean value of precession, its place when thus determined can never differ from its true place to a greater extent than 3° 56′ 26″. The maximum and minimum values of precession in a Julian year are 52″.664080 and 48″.212398, respectively, and since the length of the tropical year depends on the annual precession, it follows that the maximum variation of the tropical year is equal to the mean time required for the earth to describe an arc which is equal to the maximum variation of precession. Now this latter quantity being 4″.451682, and the sidereal motion of the earth in a second of time being 0″.041067, it follows that the maximum variation of the tropical year is equal to 108.40 seconds of time. In like manner, if we take the difference between the present value of precession and the maximum and minimum values of the same quantity, we shall find that the tropical year may be shorter than at present by 59.13 seconds, and longer than at present by 49.27 seconds. We also find that the tropical year is now shorter than in the time of Hipparchus, by 11.30 seconds.

The obliquity of the equator to the apparent ecliptic, and also to the fixed ecliptic of 1850, has also been determined. The variations of this element follow a law similar to that which governs the variation of precession, although the maximum values of the inequalities are considerably smaller than those which affect this latter quantity. The mean value of the obliquity of both the apparent and fixed ecliptics to the equator is 23° 17′ 17″. The limits of the obliquity of the apparent ecliptic to the equator are 24° 35′ 58″ and 21° 58′ 36″; whence it follows that the greatest and least declinations of the sun at the solstices can never differ from each other to any greater extent than 2° 37′ 22″. And here we may mention a few, among the many happy, consequences which result from the spheroidal form of the earth. Were the earth a perfect sphere there would be no precession or change of obliquity arising from the attraction of the sun and moon; the equinoctial circle would form an invariable plane in the heavens, about which the solar orbit would revolve with an inclination varying to the extent of twelve degrees, and a motion equal to the planetary precession of the equinoctial points. The sun, when at the solstices, would, at some periods of time, attain the declination of 29° 17′ for many thousands of years; and again, at other periods, only to 17° 17′. The seasons would be subject to vicissitudes depending on the distance of the tropics from the equator, and the distribution of solar light and heat on the surface of the earth would be so modified as essentially to change the character of its vegetation, and the distribution of its animal life. But the spheroidal form of the earth so modifies the secular changes in the relative positions of the equator and ecliptic that the inequalities of precession and obliquity are reduced to less than one-quarter part of what they would otherwise be. The periods of the secular changes, which, in the case of a spherical earth, would require nearly two millions of years to pass through a complete cycle of values, are now reduced to periods which vary between 26,000 and 53,000 years. The secular motions which would take place in the case of a spherical earth are so modified by the actual condition of the terrestrial globe that changes in the position of the equinox and equator are now produced in a few centuries, which would otherwise require a period of many thousands of years. This consideration is of much importance in the investigation of the reputed antiquity and chronology of those ancient nations which attained proficiency in the science of astronomy, and the records of whose astronomical labors are the only remaining monument of a highly intellectual people, of whose existence every other trace has long since passed away. For it is evident that, if these changes were much slower than they are, a much longer time would be required in order to produce changes of sufficient magnitude to be detected by observation, and we should be unable to estimate the interval between the epochs of elements which differed by only a few thousand years, since they would manifestly be so nearly identical with our own that the value of legitimate conclusions would be greatly impaired by the unavoidable errors of the observations on which they were based.

The duration of the different seasons is also greatly modified by the eccentricity of the earth's orbit. At present the sun is north of the equator scarcely 186½ days, and south of the same circle about 178¾ days; thus making a difference of 7¾ days between the length of the summer and winter at present. But when the eccentricity of the orbit is nearly at its maximum, and its transverse axis also passes through the solstices, both of which conditions have, in past ages, been fulfilled, the summer, in one hemisphere, will have a period of 198¾ days, and a winter of only 166½ days, while, in the other hemispheres, these conditions will be reversed; the winter having a period of 198¾ days, and a summer of only 166½ days. The variations of the sun's distance from the earth in the course of a year, at such times, are also enormous, amounting to almost one-seventh part of its mean distance —a quantity scarcely less than 13,000,000 of miles!

Passing now to the consideration of the elements of the planet Mars, we find that the eccentricity of his orbit always oscillates within the limits 0.018475 and 0.139655; and the mean motion of his perihelion is 17″.784456. The maximum inclination of his orbit to the fixed ecliptic of 1850, and to the invariable plane of the planetary system, is 7° 28′ and 5° 56′ respectively. The minimum inclination to both planes being nothing, the mean motion of the node is indeterminate.

The secular variations of the orbits of Jupiter, Saturn, Uranus, and Neptune present some curious and interesting relations. These four planets compose a system by themselves, which is practically independent of the other planets of the system.

The maximum and minimum limits of the eccentricity of the orbits of these four planets are as follows:—

	Maximum eccentricity.	Minimum eccentricity.
Jupiter	0.0608274	0.0254928
Saturn	0.0843289	0.0123719
Uranus	0.0779652	0.0117576
Neptune	0.0145066	0.0055729

The maximum and minimum inclinations of their orbits to the invariable plane of the planetary system have the following values:—

	Maximum inclination.	Minimum inclination.
Jupiter	0° 28′ 56″	0° 14′ 23″
Saturn	1 0 39	0 47 16
Uranus	1 7 10	0 54 25
Neptune	0 47 21	0 33 43

The perihelia and nodes of their orbits have the following mean motions in a Julian year of 365¼ days:—

	Mean motion of perihelion.	Mean motion of node on the invariable plane.
Jupiter	+ 3″.716607	—25″.934567
Saturn	+ 22 .460848	— 25 .934567
Uranus	+ 3 .716607	— 2 .916082
Neptune	+ 0 .616685	— 0 .661666

But the most curious relation developed by this investigation pertains to the relative motions and positions of the perihelia and nodes of the three planets, Jupiter, Saturn, and Uranus. These relations are expressed by the two following theorems:—

I. *The mean motion of Jupiter's perihelion is exactly equal to the mean motion of the perihelion of Uranus, and the mean longitudes of these perihelia differ by exactly* 180°. II. *The mean motion of Jupiter's node on the invariable plane is exactly equal to that of Saturn, and the mean longitudes of these nodes differ by exactly* 180°.

We also perceive that the mean motion of Saturn's perihelion is very nearly six times that of Jupiter and Uranus, and this latter quantity is very nearly six times that of Neptune; or, more exactly, 985 times the mean motion of Jupiter's perihelion are equal to 163 times that of Saturn, and 440 times the mean motion of Neptune's perihelion are equal to 73 times that of Jupiter and Uranus. The perihelion of Saturn's orbit performs a complete revolution in the heavens in 57,700 years; the perihelia of Jupiter and Uranus in 348,700 years; while that of Neptune requires no less than 2,101,560 years to complete the circuit of the heavens. In like manner the nodes of Jupiter and Saturn, on the invariable plane, perform a complete revolution in 49,972 years; that of Uranus in 444,432 years; while the node of Neptune requires 1,958,692 years to traverse the circumference of the heavens. The motions of the nodes are retrograde, and those of the perihelia are direct; thus conforming to the same law of variation as that which obtains in the corresponding elements of the moon's motion.

We may here observe that the law which controls the motions and positions of the perihelia of the orbits of Jupiter and Uranus is of the utmost importance in relation to their mutual perturbations of Saturn's orbit. For, in the existing arrangement, the orbit of Saturn is affected only by the *difference* of the perturbations by Jupiter and Uranus; whereas, if the mean places of the perihelia of these two planets were the same, instead of differing by 180°, the orbit of Saturn would be affected by the sum of their disturbing forces. But notwithstanding this favoring condition, the elements of Saturn's orbit would be subject to very great perturbations from the superior action of Jupiter, were it not for the comparatively rapid motion of its perihelion; its equilibrium being maintained by the very act of perturbation. Indeed, the stability of Saturn's orbit depends to a very great extent upon the rapidly varying positions of its transverse axis. For, if the motions of the perihelia of Jupiter and Saturn were very nearly the same, the action of Jupiter on the eccentricity of Saturn's orbit would be at its maximum value during very long periods of time, and thereby produce great and permanent changes in the value of that element. But, in the existing conditions, the rapid motion of Saturn's orbit prevents such an accumulation of perturbation, and any increase of eccentricity is soon changed into a corresponding diminution. The same remark is also applicable to the perturbations of the forms of the orbits of Jupiter and Uranus by the disturbing action of Saturn; for the secular variations of Jupiter's orbit depend almost entirely upon the influence of Saturn, because the planet Neptune is too remote to produce much disturbance, and the mean disturbing influence of Uranus on the eccentricity of Jupiter's orbit is identically equal to nothing, by reason of the relation which always exists between the perihelia of their orbits. We may here observe that the eccentricity of the orbit of Saturn always increases, while that of Jupiter diminishes, and *vice versâ.*

The consequences which result from the mutual relations which always exist between the nodes of Jupiter and Saturn, on the invariable plane of the planetary system, are no less interesting or remarkable with respect to the *position* of the orbit of Uranus than those which result from the permanent relation between the perihelia of Jupiter and Uranus are with respect to the form of the orbit of Saturn. The mean disturbing force of Saturn on the inclination of the orbit of Uranus is about four times that of Jupiter; but as these two planets always act on the inclinations in opposite directions, it follows that the joint action of the two planets is equivalent to the action of a single planet at the distance of Saturn and having about three-fourths of his mass; so that the orbit of Uranus might attain a considerable inclination from the superior action of Saturn if allowed to accumulate during the lapse of an unlimited time, at its maximum rate of variation depending on the action of this planet. But such an accumulation of perturbation is rendered forever impossible by reason of the comparatively rapid motion of the nodes of Jupiter and Saturn, with respect to that of Uranus, on the invariable plane. By reason of this rapid motion, the secular changes of the inclination of the orbit of Uranus pass through a complete cycle of values in the period of 56,300 years. The corresponding cycle of perturbation in the eccentricity of Saturn's orbit is 69,140 years. It is the rapid motion of the orbit with respect to the forces in the one case, and the rapid motion of the forces with respect to the orbit in the other, that gives permanence of form and position to the orbits of Saturn and Uranus.

The mean angular distance between the perihelia of Jupiter and Uranus is exactly 180°; but the conditions of the variations of these elements are sufficiently elastic to allow of a considerable deviation on each side of their mean positions. The perihelion of Jupiter may differ from its mean place to the extent of 24° 10′, and that of Uranus to the extent of 47° 33′; and therefore the longitudes of the perihelia of these two planets can differ from 180° to the extent of 71° 43′. The nearest approach of the perihelia of these two planets is, therefore, 108° 17′.

In like manner the longitudes of the nodes of Jupiter and Saturn, on the invariable plane, can suffer considerable deviations from their mean positions. The actual position of Jupiter's node may differ from its mean place to the extent of 19° 38′; while that of Saturn may deviate from its mean place to the extent of 7° 7′. It therefore follows that their longitudes on the invariable plane can differ from 180° by only 26° 45′. Their nearest possible approach is 153° 15′, while their present distance apart is 166° 27′.

The inequalities in the eccentricity of Neptune's orbit are very small, and the two principal ones have periods of 613,900 years, and 418,060 years respectively. Strictly speaking, the periods of the secular inequalities of the eccentricities and perihelia are the same for all the planets; and the same remark is equally applicable to the nodes and inclinations. But the principal inequalities of the several planetary orbits are different, unless they are connected by some permanent relation, similar to that which exists between the perihelia of Jupiter and Uranus, or the nodes of Jupiter and Saturn. Thus the principal inequalities affecting the inclination of the orbits of Jupiter and Saturn have the same periods for each planet, and these periods are, for the two principal inequalities, 51,280 years, and 56,303 years. In like manner the principal inequalities in the eccentricities of Jupiter and Saturn depend on their mutual attraction, and have a period of 69,141 years. The secular inequalities of those orbits which have no vanishing elements are composed of terms, of very different orders of magnitude; and it frequently happens that two or three of these terms are greater than the sum of all the remaining ones. In such cases the variation of the corresponding element very approximately conforms to a much simpler law, and the maxima and minima repeat themselves according to definite and well-defined cycles. But with regard to the orbits of Venus, the Earth, and Mars, the rigorous expressions of the eccentricities and inclinations are composed of twenty-eight periodic terms, having coefficients of considerable magnitude; and this circumstance renders the law of their variations extremely intricate.

The method we have adopted for finding the coefficients of the corrections of the constants, depending on finite variations of the different planetary masses, consists in supposing that each planetary mass receives in succession a finite increment, and then finding the values of all the constants corresponding to this increased mass in the same manner as for the assumed masses. By this means we have a set of values corresponding to the assumed masses, and another set corresponding to a finite increment to each of the planetary masses. Then, taking the difference between the two sets of constants, and dividing by the increment, which produced it, we get the coefficient of the variation of the constants for any other finite increment of mass to the same planet; but, on account of the importance of the earth's mass, and the probable inaccuracy of its assumed value, we have prepared separate solutions corresponding to the several increments of $\frac{1}{20}$, $\frac{2}{20}$, and $\frac{3}{20}$ of its assumed mass; and a comparison of the values which the different solutions give for the superior limit of the eccentricity of the earth's orbit has suggested the inquiry whether there may not be some unknown physical relation between the masses and mean distances of the different planets. The same peculiarity in the elements of the orbit of *Venus* is also found to depend upon particular values of the mass of that planet. But without entering into details in regard to the peculiarity referred to, we here give the several values of the masses of these two planets which we have employed in our computations, and the corresponding values of the superior limit of the eccentricity of their orbits.

For Venus.		For the Earth.	
Mass m'	*Maximum* e'	Mass m''	*Maximum* e''.
m'_0	0.070633	m''_0	0.067735
$m'_0(1+\frac{1}{20})$	0.074872	$m''_0(1+\frac{1}{20})$	0.069389
$m'_0(1+\frac{2}{20})$	0.076075	$m''_0(1+\frac{2}{20})$	0.069649
$m'_0(1+\frac{3}{20})$	0.075029	$m''_0(1+\frac{3}{20})$	0.068089
$m'_0(1+\frac{4}{20})$	0.072098		

These numbers show that if the mass of Venus were to be increased, the superior limit of the eccentricity of her orbit would also increase until it had attained a maximum value, after which a further increase of her mass would diminish that limit; and the same remark is also applicable to the eccentricity of the earth's orbit.

The above numbers indicate that the superior limit of the eccentricity of the orbit of Venus is a maximum if the mass of that plant is equal to $m'_0\left(1+\frac{2.04}{20}\right)$; or, if $m'=\frac{1}{354490}$ of the sun's mass; and the superior limit of the eccentricity of the earth's orbit is a maximum if the earth's mass is equal to $m''_0\left(1+\frac{1.643}{20};\right)$ or, if $m''=\frac{1}{340700}$ of the sun's mass. But this value of the earth's mass corresponds to a solar parallax of 8″.730, a value closely approximating to the recent determinations of that element.

If, then, the mass of *Venus is* equal to $\frac{1}{354490}$, and the mass of the *earth is* equal to $\frac{1}{340700}$, it follows that the orbits of these two planets will ultimately become more eccentric from the mutual attraction of the other planets, than they would for any other values of their respective masses; and we may now inquire whether such coincidence between the superior limits of the eccentricities and the masses of these two planets has any physical significance, or is merely accidental.

Since the mean motions and mean distances of the planets are invariable, and independent of the eccentricities of the orbits, it would seem that there could be no connection between these elements, by means of which the stability of the system might be secured or impaired; but a more careful examination shows that, although the mean motions or times of revolution of the planets are invariable, their actual velocities, or the variation of their mean velocities, depends wholly on the eccentricities; and, were any of the planetary orbits to become extremely elliptical, the velocity of the planet would be subject to great variations of velocity,—moving with very great rapidity when in perihelion, and with extreme slowness when in the neighborhood of its aphelion; and it is evident that when the planet was in this latter position a small foreign force acting upon it might so change its velocity as to completely change the elements of its orbit, by causing it to fall upon the sun or fly off into remoter space. A system of bodies moving in very eccentrical orbits is, therefore, one of manifest instability; and if it can also be shown that a system of bodies moving in circular orbits is one of unstable equilibrium, it would seem that, between the two supposed conditions, a system might exist which should possess a greater degree of stability than either. The idea is thus suggested of the existence of a system of bodies in which the masses of the different bodies are so adjusted to their mean distances as to insure to the system a greater degree of permanence than would be possible by any other distribution of masses. The mathematical expression of a criterion for such distribution of masses has not yet been fully developed; and the preceding illustrations have been introduced here, more for the purpose of calling the attention of mathematicians and astronomers to this interesting problem than for any certain light we have yet been able to obtain in regard to its solution.

MEMOIR

ON

THE SECULAR VARIATIONS OF THE ELEMENTS OF THE ORBITS OF THE EIGHT PRINCIPAL PLANETS,

MERCURY VENUS THE EARTH MARS, JUPITER, SATURN, URANUS, AND NEPTUNE.

CHAPTER I.

ON THE SECULAR VARIATIONS OF THE ECCENTRICITIES AND PERIHELIA.

1. We shall assume as the basis of our computation the following differential equations, which determine the instantaneous variations of the eccentricities and places of the perihelia of the planetary orbits at any time. These equations are demonstrated by La Place, in Book II, Chapter VII, of the *Mécanique Céleste;* and by Pontécoulant, in Book II, Chapter VIII, of his *Théorie Analytique du Système du Monde*, and are as follows:—

$$\left.\begin{array}{l}
\dfrac{dh}{dt} = \Big\{(0,1)+(0,2)+(0,3)+\&c.\Big\}\, l - \boxed{0,1}\,l' - \boxed{0,2}\,l'' - \boxed{0,3}\,l''' - \&c.; \\[2ex]
\dfrac{dl}{dt} = -\Big\{(0,1)+(0,2)+(0,3)+\&c.\Big\}\, h + \boxed{0,1}\,h' + \boxed{0,2}\,h'' + \boxed{0,3}\,h''' + \&c.; \\[2ex]
\dfrac{dh'}{dt} = \Big\{(1,0)+(1,2)+(1,3)+\&c.\Big\}\, l' - \boxed{1,0}\,l - \boxed{1,2}\,l'' - \boxed{1,3}\,l''' - \&c.; \\[2ex]
\dfrac{dl'}{dt} = -\Big\{(1,0)+(1,2)+(1,3)+\&c.\Big\}\, h' + \boxed{1,0}\,h + \boxed{1,2}\,h'' + \boxed{1,3}\,h''' + \&c.; \\[1ex]
\quad \&c.
\end{array}\right\} \quad (A)$$

If we denote by e, e', e'', &c., ϖ, ϖ', ϖ'', &c., the eccentricities and longitudes of the perihelia of the orbits of *Mercury*, *Venus*, the *Earth*, &c., we shall have the following equations for the determination of these quantities:—

$$\left.\begin{array}{lll}
h = e \sin \varpi, & h' = e' \sin \varpi', & h'' = e'' \sin \varpi'', \ \&c., \\
l = e \cos \varpi, & l' = e' \cos \varpi', & l'' = e'' \cos \varpi'', \ \&c.
\end{array}\right\} \quad (1)$$

Whence we deduce

$$e^2 = h^2 + l^2, \quad e'^2 = h'^2 + l'^2, \quad e''^2 = h''^2 + l''^2, \ \&c.; \quad \tan \varpi = \frac{h}{l}, \quad \tan \varpi' = \frac{h'}{l'}, \ \&c. \quad (2)$$

If h, l, h', l', &c., are determined by the integrals of equations (A), for any time whatever, and substituted in equations (2), we shall obtain the corresponding values of e, e', &c., ϖ, ϖ', &c.

2. Now to find the integrals of equations (A), we shall suppose

$$\left.\begin{array}{lll} h=N\sin(gt+\beta), & h'=N'\sin(gt+\beta) & h''=N''\sin(gt+\beta), \text{ \&c.,} \\ l=N\cos(gt+\beta), & l'=N'\cos(gt+\beta) & l''=N''\cos(gt+\beta), \text{ \&c.} \end{array}\right\} (3)$$

If these values be substituted in equations (A), they will become,

$$\left.\begin{array}{l} Ng=\{(0,1)+(0,2)+(0,3)+\text{ \&c.}\}\,N-\boxed{0,1}N'-\boxed{0,2}N''-\boxed{0,3}N'''-\text{ \&c.} \\ N'g=\{(1,0)+(1,2)+(1,3)+\text{ \&c.}\}\,N'-\boxed{1,0}N-\boxed{1,2}N''-\boxed{1,3}N'''-\text{ \&c.} \\ N''g=\{(2,0)+(2,1)+(2,3)+\text{ \&c.}\}\,N''-\boxed{2,0}N-\boxed{2,1}N'-\boxed{2,3}N'''-\text{ \&c.} \\ \text{\&c.} \end{array}\right\} (B)$$

If we suppose the number of planets whose mutual action is considered, to be i, the number of these equations will be i; and by eliminating the constant quantities N, N', N'', &c., we shall obtain a final equation in g of the degree i.

3. The quantities $(0,1)$ and $(1,0)$, $\boxed{0,1}$, $\boxed{1,0}$; $(0,2)$ and $(2,0)$, $\boxed{0,2}$, $\boxed{2,0}$; $(1,2)$ and $(2,1)$; $\boxed{1,2}$ and $\boxed{2,1}$, &c., have some remarkable relations with each other, which not only facilitate their computation, but render the equations resulting from the elimination of N, N', N'', &c., much shorter and more commodious. The general expression for $(0,1)$ is

$$(0,1)=-\frac{3m'na^2a'(a,a')'}{4(a'^2-a^2)^2}. \qquad (4)$$

In this equation n and a denote the mean motion and mean distance of *Mercury*, m' denotes the mass of *Venus* and a' its mean distance from the sun. If we change n, a into n', a', and m', a' into m, a, respectively, $(0,1)$ will change it into $(1,0)$, and we shall have

$$(1,0)=-\frac{3mn'a'^2a(a',a)'}{4(a'^2-a^2)^2}. \qquad (5)$$

Now since $(a,a')'=(a',a)'$, equations (4) and (5) will give

$$(1,0)=(0,1)\frac{m}{na}\cdot\frac{n'a'}{m'}; \qquad (6)$$

we shall also have

$$\left.\begin{array}{l} (2,0)=(0,2)\dfrac{m}{na}\cdot\dfrac{n''a''}{m}; \\ (3,0)=(0,3)\dfrac{m}{na}\cdot\dfrac{n'''a'''}{m'''}. \\ \text{\&c.} \end{array}\right\} \qquad (7)$$

The same relations also hold with respect to the quantities $\boxed{0,1}$, $\boxed{1,0}$, $\boxed{0,2}$, $\boxed{2,0}$, &c., so that we shall also have

$$\left.\begin{array}{l} \boxed{1,0}=\boxed{0,1}\dfrac{m}{na}\cdot\dfrac{n'a'}{m'}; \\ \boxed{2,0}=\boxed{0,2}\dfrac{m}{na}\cdot\dfrac{n''a''}{m''}; \\ \boxed{3,0}=\boxed{0,3}\dfrac{m}{na}\cdot\dfrac{n'a'''}{m'''}. \\ \text{\&c.} \end{array}\right\} \qquad (8)$$

Therefore when we have computed the quantities $(0,1)$, $(0,2)$, $(0,3)$, &c., or the coefficient for an interior planet, depending on the action of an exterior planet, we shall obtain the corresponding coefficient for an exterior planet, depending on the action of an interior planet, by means of equations (6), (7), and (8).

Equations (6) and (7) may be written as follows:—

$$\left.\begin{array}{l} (1,0)\dfrac{m'}{n'a'}=(0,1)\dfrac{m}{na} \\ (2,0)\dfrac{m''}{n''a''}=(0,2)\dfrac{m}{na} \\ \text{\&c.} \end{array}\right\} \qquad (9)$$

We shall also similarly have

$$\left.\begin{array}{l} (2,1)\dfrac{m''}{n''a''}=(1,2)\dfrac{m'}{n'a'} \\ (3,1)\dfrac{m'''}{n'''a'''}=(1,3)\dfrac{m'}{n'a'} \\ (3,2)\dfrac{m'''}{n'''a'''}=(2,3)\dfrac{m''}{n''a''} \\ \text{\&c.} \end{array}\right\} \qquad (10)$$

We shall therefore have

$$\left.\begin{array}{l} (1,2)(2,0)(0,1)=(2,1)(1,0)(0,2), \\ (1,3)(3,2)(2,1)=(3,1)(1,2)(2,3), \\ (3,0)(0,2)(2,3)=(0,3)(3,2)(2,0), \\ (3,1)(1,0)(0,3)=(1,3)(3,0)(0,1), \\ \text{\&c.} \end{array}\right\}. \qquad (11)$$

We shall also have the following products of four factors

$$\left.\begin{array}{l} (0,1)(1,2)(2,3)(3,0)=(1,0)(0,3)(3,2)(2,1), \\ (0,1)(1,3)(3,2)(2,0)=(1,0)(0,2)(2,3)(3,1), \\ (0,2)(2,1)(1,3)(3,0)=(2,0)(0,3)(3,1)(1,2), \\ (1,2)(2,3)(3,4)(4,1)=(2,1)(1,4)(4,3)(3,2), \\ \text{\&c} \end{array}\right\}. \qquad (12)$$

And of five factors we have

$$\left.\begin{array}{l} (0,1)(1,2)(2,3)(3,4)(4,0)=(1,0)(0,4)(4,3)(3,2)(2,1), \\ (0,1)(1,2)(2,4)(4,3)(3,0)=(1,0)(0,3)(3,4)(4,2)(2,1), \\ (1,2)(2,3)(3,4)(4,5)(5,1)=(2,1)(1,5)(5,4)(4,3)(3,2), \\ (1,2)(2,4)(4,3)(3,5)(5,1)=(2,1)(1,5)(5,3)(3,4)(4,2), \\ \text{\&c.} \end{array}\right\}. \qquad (13)$$

And in like manner we may form the products of six, or any number of factors. Equations (11), (12), and (13) are very useful in reducing two terms to a single one. We may in like manner form the following equations:—

$$\left.\begin{array}{l} \boxed{0,1}\,\boxed{1,2}\,\boxed{2,0}=\boxed{1,0}\,\boxed{0,2}\,\boxed{2,1}, \\ \boxed{1,3}\,\boxed{3,2}\,\boxed{2,1}=\boxed{3,1}\,\boxed{1,2}\,\boxed{2,3}, \\ \boxed{2,3}\,\boxed{3,0}\,\boxed{0,2}=\boxed{3,2}\,\boxed{2,0}\,\boxed{0,3}, \\ \boxed{3,1}\,\boxed{1,0}\,\boxed{0,3}=\boxed{1,3}\,\boxed{3,0}\,\boxed{0,1}, \\ \text{\&c.} \end{array}\right\}. \qquad (14)$$

$$\left.\begin{array}{l}
\boxed{0,1}\,\boxed{1,2}\,\boxed{2,3}\,\boxed{3,0}=\boxed{1,0}\,\boxed{0,3}\,\boxed{3,2}\,\boxed{2,1}, \\
\boxed{0,1}\,\boxed{1,3}\,\boxed{3,2}\,\boxed{2,0}=\boxed{1,0}\,\boxed{0,2}\,\boxed{2,3}\,\boxed{3,1}, \\
\boxed{0,2}\,\boxed{2,1}\,\boxed{1,3}\,\boxed{3,0}=\boxed{2,0}\,\boxed{0,3}\,\boxed{3,1}\,\boxed{1,2}, \\
\boxed{1,2}\,\boxed{2,3}\,\boxed{3,4}\,\boxed{4,1}=\boxed{2,1}\,\boxed{1,4}\,\boxed{4,3}\,\boxed{3,2}, \\
\text{\&c.}
\end{array}\right\} . \qquad (15)$$

$$\left.\begin{array}{l}
\boxed{0,1}\,\boxed{1,2}\,\boxed{2,3}\,\boxed{3,4}\,\boxed{4,0}=\boxed{1,0}\,\boxed{0,4}\,\boxed{4,3}\,\boxed{3,2}\,\boxed{2,1}, \\
\boxed{0,1}\,\boxed{1,2}\,\boxed{2,4}\,\boxed{4,3}\,\boxed{3,0}=\boxed{1,0}\,\boxed{0,3}\,\boxed{3,4}\,\boxed{4,2}\,\boxed{2,1}, \\
\boxed{1,2}\,\boxed{2,3}\,\boxed{3,4}\,\boxed{4,5}\,\boxed{5,1}=\boxed{2,1}\,\boxed{1,5}\,\boxed{5,4}\,\boxed{4,3}\,\boxed{3,2}, \\
\boxed{1,2}\,\boxed{2,4}\,\boxed{4,3}\,\boxed{3,5}\,\boxed{5,1}=\boxed{2,1}\,\boxed{1,5}\,\boxed{5,3}\,\boxed{3,4}\,\boxed{4,2}, \\
\text{\&c.}
\end{array}\right\} \qquad (16)$$

4. The quantities $(0,1)$, $(0,2)$, $(0,3)$, &c., $(1,0)$, $(1,2)$, $(1,3)$, &c.; $\boxed{0,1}$, $\boxed{0,2}$, $\boxed{0,3}$, &c., $\boxed{1,0}$, $\boxed{1,2}$, $\boxed{1,3}$, &c.; depend on the masses and mean distances of the different planets. The analytical expressions of $(0,1)$ and $\boxed{0,1}$ are as follows: *Méc. Cél.* [1076], [1082], *Bowditch's Translation.*

$$(0,1)=-\frac{3m'na^2b_{-\frac{1}{2}}^{(1)}}{4(1-\alpha^2)^2}; \qquad (17)$$

$$\boxed{0,1}=-\frac{3m'na\{(1+\alpha^2)b_{-\frac{1}{2}}^{(1)}+\frac{1}{2}\alpha b_{-\frac{1}{2}}^{(0)}\}}{2(1-\alpha^2)^2}. \qquad (18)$$

In these equations n denotes the mean motion of the disturbed planet; m' the mass of the disturbing planet; α the ratio of the mean distances of the inner and outer planets, $b_{-\frac{1}{2}}^{(0)}$ and $b_{-\frac{1}{2}}^{(1)}$ depend entirely on α, and are given by equations [989] *Méc. Cél.* If we reduce the coefficients of the different powers of α to decimal numbers we shall have the following equations to determine $b_{-\frac{1}{2}}^{(0)}$ and $b_{-\frac{1}{2}}^{(1)}$.

$$\left.\begin{array}{rl}
\frac{1}{2}b_{-\frac{1}{2}}^{(0)}= & 1+0.25\alpha^2+0.015625\alpha^4+0.003906249\alpha^6+0.001525878\alpha^8 \\
& +0.0007476805\alpha^{10}+0.0004205703\alpha^{12}+0.0002596378\alpha^{14} \\
& +0.0001714015\alpha^{10}+0.0001190288\alpha^{18}+0.00008599836\alpha^{20} \\
& +0.00006414336\alpha^{22}+0.00004910978\alpha^{24}+0.00003843058\alpha^{20} \\
& +0.00003063663\alpha^{28}+0.00002481567\alpha^{30}+\text{\&c.}
\end{array}\right\} . \qquad (18)$$

$$\left.\begin{array}{rl}
b_{-\frac{1}{2}}^{(1)}= & -\alpha+0.125\alpha^3+0.015625\alpha^5+0.004882812\alpha^7+0.002136230\alpha^9 \\
& +0.001121521\alpha^{11}+0.0006608960\alpha^{13}+0.0004219114\alpha^{15} \\
& +0.0002856691\alpha^{17}+0.0002023490\alpha^{19}+0.0001485425\alpha^{21} \\
& +0.0001122509\alpha^{23}+0.00008688650\alpha^{25}+0.00006862602\alpha^{27} \\
& +0.00005514591\alpha^{29}+0.00004497838\alpha^{31}+\text{\&c.}
\end{array}\right\} . \qquad (19)$$

If we now multiply equation (19) by $1+\alpha^2$, and equation (18) by α, and put

$$\{(1+\alpha^2)b_{-\frac{1}{2}}^{(1)}+\tfrac{1}{2}\alpha b_{-\frac{1}{2}}^{(0)}\}\alpha=b^{(0)}, \qquad (20)$$

we shall have

$$\left.\begin{array}{rl}
b^{(0)}= & -0.625\alpha^4+0.15625\alpha^6+0.024414061\alpha^8+0.008544920\alpha^{10} \\
& +0.004005431\alpha^{12}+0.002202987\alpha^{14}+0.0013424452\alpha^{16} \\
& +0.0008789820\alpha^{18}+0.0006070469\alpha^{20}+0.0004368899\alpha^{22} \\
& +0.0003249368\alpha^{24}+0.0002482472\alpha^{26}+0.0001939431\alpha^{28} \\
& +0.00015440856\alpha^{30}+0.00012493996\alpha^{32}+\text{\&c.}
\end{array}\right\} . \qquad (21)$$

It will manifestly be unnecessary to compute $b^{(0)}_{-\frac{1}{2}}$ separately, since it is already included in the value of $b^{(0)}$. Therefore taking logarithmic coefficients of equations (19) and (21), we shall have the two following working formulæ for the computation of $b^{(0)}$ and $b^{(1)}_{-\frac{1}{2}}$.

$$\left.\begin{array}{l} b^{(0)} = -\alpha^4\{0.625-[9.1938200]\alpha^2-[8.3876400]\alpha^4-[7.9317080]\alpha^6 \\ \quad -[7.6026493]\alpha^8-[7.3430119]\alpha^{10}-[7.1278964]\alpha^{12} \\ \quad -[6.9439800]\alpha^{14}-[6.7832222]\alpha^{16}-[6.6403720]\alpha^{18} \\ \quad -[6.5117989]\alpha^{20}-[6.3948844]\alpha^{22}-[6.2876743]\alpha^{24} \\ \quad -[6.1886714]\alpha^{26}-[6.0967014]\alpha^{28}-\&c.\} \end{array}\right\} . \quad (22)$$

$$\left.\begin{array}{l} b^{(1)}_{-\frac{1}{2}} = -\alpha\{1-[9.0969100]\alpha^2-[8.1938200]\alpha^4-[7.6886700]\alpha^6 \\ \quad -[7.3296480]\alpha^8-[7.0498073]\alpha^{10}-[6.8201332]\alpha^{12} \\ \quad -[6.6252212]\alpha^{14}-[6.4558633]\alpha^{16}-[6.3061010]\alpha^{18} \\ \quad -[6.1718509]\alpha^{20}-[6.0501898]\alpha^{22}-[5.9389523]\alpha^{24} \\ \quad -[5.8364888]\alpha^{26}-[5.7415133]\alpha^{28}-[5.6530038]\alpha^{30}-\&c.\} \end{array}\right\} . \quad (23)$$

Then we shall have

$$(0,1) = -\frac{3m'n\alpha^2 b^{(1)}_{-\frac{1}{2}}}{4(1-\alpha^2)^2}, \qquad (24)$$

$$\boxed{0,1} = -\frac{3m'nb^{(0)}}{2(1-\alpha^2)^2}. \qquad (25)$$

5. To reduce the preceding formulæ to numbers we shall assume the following values of the invariable elements of the eight principal planets.

Invariable Elements of the Eight Principal Planets.

	Masses.	Mean motions in a Julian year.	Mean distances from the sun.
Mercury	$m = \frac{1+\mu}{4865751}$	$n = 5381016''.2$	$a = 0.3870987$
Venus	$m' = \frac{1+\mu'}{390000}$	$n' = 2106641\ .438$	$a' = 0.7233323$
The Earth . . .	$m'' = \frac{1+\mu''}{368689}$	$n'' = 1295977\ .440$	$a'' = 1.0000000$
Mars	$m''' = \frac{1+\mu'''}{2680637}$	$n''' = 689050\ .9023$	$a''' = 1.5236878$
Jupiter	$m^{IV} = \frac{1+\mu^{IV}}{1047.879}$	$n^{IV} = 109256\ .719$	$a^{IV} = 5.202798$
Saturn	$m^{V} = \frac{1+\mu^{V}}{3501.6}$	$n^{V} = 43996\ .127$	$a^{V} = 9.538852$
Uranus	$m^{VI} = \frac{1+\mu^{VI}}{24905}$	$n^{VI} = 15424\ .5094$	$a^{VI} = 19.183581$
Neptune	$m^{VII} = \frac{1+\mu^{VII}}{18780}$	$n^{VII} = 7873\ .993$	$a^{VII} = 30.03386$

From these quantities we shall obtain the following logarithms

m	log. 93.3128501;	a	log. 9.58782172;
m'	log. 94.4089354;	a'	log. 9.85933786;
m''	log. 94.4333398;	a''	log. 0.00000000;
m'''	log. 93.5717620;	a'''	log. 0.18289600;
m^{IV}	log. 96.9796889;	a^{IV}	log. 0.71623697;
m^{V}	log. 96.4557335;	a^{V}	log. 0.97949611;
m^{VI}	log. 95.6037135;	a^{VI}	log. 1.28292968;
m^{VII}	log. 95.7263044;	a^{VII}	log. 1.47761116;

n	log. 6.7308643;	$m \div na$	log. 86.9941641;
n'	log. 6.3235906;	$m' \div n'a'$	log. 88.2260070;
n''	log. 6.1125974;	$m'' \div n''a''$	log. 88.3207424;
n'''	log. 5.8382513;	$m''' \div n'''a'''$	log. 87.5506147;
n^{IV}	log. 5.0384481;	$m^{IV} \div n^{IV}a^{IV}$	log. 91.2250039;
n^{V}	log. 4.6434145;	$m^{V} \div n^{V}a^{V}$	log. 90.8328229;
n^{VI}	log. 4.1882114;	$m^{VI} \div n^{VI}a^{VI}$	log. 90.1325724;
n^{VII}	log. 3.8961950;	$m^{VII} \div n^{VII}a^{VII}$	log. 90.3524982.

The values of a, a', a'', &c., give the following values of α and $1-\alpha^2$.

For *Mercury* and	*Venus*	α=0.5351603	log. 9.72848386;	$1-\alpha^2$ log. 9.8534569;
"	*Earth*	α=0.3870987	log. 9.58782172;	$1-\alpha^2$ log. 9.9294980;
"	*Mars*	α=0.2540538	log. 9.40492572;	$1-\alpha^2$ log. 9.9710237;
"	*Jupiter*	α=0.07440202	log. 8.87158475;	$1-\alpha^2$ log. 9.9975892;
"	*Saturn*	α=0.04058127	log. 8.60832561;	$1-\alpha^2$ log. 9.9992842;
"	*Uranus*	α=0.02017864	log. 8.30489204;	$1-\alpha^2$ log. 9.9998231;
"	*Neptune*	α=0.01288874	log. 8.11021056;	$1-\alpha^2$ log. 9.9999279.
For *Venus* and	*Earth*	α=0.7233323	log. 9.85933786;	$1-\alpha^2$ log. 9.6783274;
"	*Mars*	α=0.4747247	log. 9.67644186;	$1-\alpha^2$ log. 9.8890979;
"	*Jupiter*	α=0.1390276	log. 9.14310089;	$1-\alpha^2$ log. 9.9915235;
"	*Saturn*	α=0.07583011	log. 8.87984175;	$1-\alpha^2$ log. 9.9974955;
"	*Uranus*	α=0.03770580	log. 8.57640818;	$1-\alpha^2$ log. 9.9993822;
"	*Neptune*	α=0.02408390	log. 8.38172670;	$1-\alpha^2$ log. 9.9997480.
For *Earth* and	*Mars*	α=0.6563025	log. 9.81710400;	$1-\alpha^2$ log. 9.7553161;
"	*Jupiter*	α=0.1922043	log. 9.28376303;	$1-\alpha^2$ log. 9.9836522;
"	*Saturn*	α=0.1048344	log. 9.02050389;	$1-\alpha^2$ log. 9.9952006;
"	*Uranus*	α=0.0521279	log. 8.71707032;	$1-\alpha^2$ log. 9.9988183;
"	*Neptune*	α=0.03329575	log. 8.52238884;	$1-\alpha^2$ log. 9.9995183.
For *Mars* and	*Jupiter*	α=0.2928593	log. 9.46665903;	$1-\alpha^2$ log. 9.9610571;
"	*Saturn*	α=0.1597349	log. 9.20339989;	$1-\alpha^2$ log. 9.9887751;
"	*Uranus*	α=0.07942665	log. 8.89996632;	$1-\alpha^2$ log. 9.9972515;
"	*Neptune*	α=0.05073232	log. 8.70528484;	$1-\alpha^2$ log. 9 9988808.

For *Jupiter* and *Saturn*	α=0.5454324	log. 9.73674086;	$1-\alpha^2$ log. 9.8466486;
" *Uranus*	α=0.2712110	log. 9.43330729;	$1-\alpha^2$ log. 9.9668195;
" *Neptune*	α=0.1732311	log. 9.23862581;	$1-\alpha^2$ log. 9.9867677.
For *Saturn* and *Uranus*	α=0.4972404	log. 9.69656643;	$1-\alpha^2$ log. 9.8766519;
" *Neptune*	α=0.3176033	log. 9.50188495;	$1-\alpha^2$ log. 9.9538216.
For *Uranus* and *Neptune*	α=0.6387318	log. 9.80531852;	$1-\alpha^2$ log. 9.7723377.

6. If we now substitute these several values of α inequations (22) and (23), we shall obtain the following values of the quantities $b^{(0)}$ and $b^{(1)}_{-\frac{1}{2}}$.

Mercury and	*Venus*	log. $b^{(0)} = 98.6758747n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.7120142n$;
"	*Earth*	log. $b^{(0)} = 98.1301668n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.5794468n$;
"	*Mars*	log. $b^{(0)} = 97.4084445n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.4013786n$;
"	*Jupiter*	log. $b^{(0)} = 95.2816171n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.8712839n$;
"	*Saturn*	log. $b^{(0)} = 94.2290035n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.6082362n$;
"	*Uranus*	log. $b^{(0)} = 93.0154041n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.3048699n$;
"	*Neptune*	log. $b^{(0)} = 92.2367042n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.1102016n$.
Venus and	*Earth*	log. $b^{(0)} = 99.1656339n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.8275394n$;
"	*Mars*	log. $b^{(0)} = 98.4754676n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.6636495n$;
"	*Jupiter*	log. $b^{(0)} = 96.3661735n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.1420477n$;
"	*Saturn*	log. $b^{(0)} = 65.3146217n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.8795292n$;
"	*Uranus*	log. $b^{(0)} = 94.1013583n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.5763310n$;
"	*Neptune*	log. $b^{(0)} = 93.3227239n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.3816952n$.
The *Earth* and	*Mars*	log. $b^{(0)} = 99.0105934n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.7915113n$;
"	*Jupiter*	log $b^{(0)} = 96.9268788n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.2817435n$;
"	*Saturn*	log. $b^{(0)} = 95.8766987n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.0199061n$;
"	*Uranus*	log. $b^{(0)} = 94.6638660n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.7169227n$;
"	*Neptune*	log. $b^{(0)} = 93.8853150n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.5223286n$.
Mars and	*Jupiter*	log. $b^{(0)} = 97.6529713n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.4619255n$;
"	*Saturn*	log. $b^{(0)} = 96.6066892n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.2020080n$;
"	*Uranus*	log. $b^{(0)} = 95.3950591n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.8996234n$:
"	*Neptune*	log. $b^{(0)} = 94.6167398n$;	log. $b^{(1)}_{-\frac{1}{2}} = 98.7051450n$.
Jupiter and	*Saturn*	log. $b^{(0)} = 98.7074558n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.7195915n$;
"	*Uranus*	log. $b^{(0)} = 97.5209527n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.4292578n$;
"	*Neptune*	log. $b^{(0)} = 96.7470972n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.2369875n$.
Saturn and	*Uranus*	log. $b^{(0)} = 98.5532200n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.6824668n$;
"	*Neptune*	log. $b^{(0)} = 97.7921436n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.4963019n$.
Uranus and	*Neptune*	log. $b^{(0)} = 98.9667121n$;	log. $b^{(1)}_{-\frac{1}{2}} = 99.7812070n$.

7. We must now substitute the values of $b_{-\frac{1}{2}}^{(1)}$, and the corresponding values of α, in equation (24); and change m', successively into m'', m''', m^{IV}, &c., and we shall obtain the values of $(0,1)$, $(0,2)$, $(0,3)$, &c., or the coefficient of the action of each of the planets on *Mercury*. The characters 0, 1, 2, 3, 4, 5, 6, 7, refer respectively to *Mercury*, *Venus*, the *Earth*, *Mars*, *Jupiter*, *Saturn*, *Uranus*, and *Neptune*. Then changing n into n', and m', into m'', m''', &c., we shall obtain the values of $(1,2)$, $(1,3)$, &c., or the action of each outer planet on *Venus*. And in like manner we shall obtain $(2,3)$, $(2,4)$, $(2,5)$, &c.; $(3,4)$, $(3,5)$, &c., or the action of each outer planet on each inner one. We shall then obtain $(1,0)$, $(2,0)$, $(2,1)$, &c., or the action of an inner planet on an outer one by means of the equations (6), (7), (10), &c.

In this manner we have obtained the following results:—

$$\begin{array}{lll}
(0,1) = (1+\mu'). & 2''.9986729 & \log.\ 0.4769291; \\
(0,2) = (1+\mu''). & 0\ .8617070 & \log.\ 9.9353596; \\
(0,3) = (1+\mu'''). & 0\ .0279815 & \log.\ 8.4468702; \\
(0,4) = (1+\mu^{IV}). & 1\ .6028375 & \log.\ 0.2048895; \\
(0,5) = (1+\mu^{V}). & 0\ .0772642 & \log.\ 8.8879781; \\
(0,6) = (1+\mu^{VI}). & 0\ .0013324 & \log.\ 7.1246469; \\
(0,7) = (1+\mu^{VII}). & 0\ .0004603 & \log.\ 6.6629969.
\end{array}$$

$$\begin{array}{lll}
(1,0) = (1+\mu). & 0''.1758273 & \log.\ 9.2450862; \\
(1,2) = (1+\mu''). & 6\ .6305873 & \log.\ 0.8215520; \\
(1,3) = (1+\mu'''). & 0\ .1020355 & \log.\ 9.0087513; \\
(1,4) = (1+\mu^{IV}). & 4\ .2028443 & \log.\ 0.6235433; \\
(1,5) = (1+\mu^{V}). & 0\ .1988873 & \log.\ 9.2986071; \\
(1,6) = (1+\mu^{VI}). & 0\ .0034100 & \log.\ 7.5327484; \\
(1,7) = (1+\mu^{VII}). & 0\ .0011765 & \log.\ 7.0706089.
\end{array}$$

$$\begin{array}{lll}
(2,0) = (1+\mu). & 0''.0406239 & \log.\ 8.6087813; \\
(2,1) = (1+\mu'). & 5\ .3310972 & \log.\ 0.7268166; \\
(2,3) = (1+\mu'''). & 0\ .2982001 & \log.\ 9.4745078; \\
(2,4) = (1+\mu^{IV}). & 7\ .0682646 & \log.\ 0.8493128; \\
(2,5) = (1+\mu^{V}). & 0\ .3265163 & \log.\ 9.5139049; \\
(2,6) = (1+\mu^{VI}). & 0\ .0055565 & \log.\ 7.7447989; \\
(2,7) = (1+\mu^{VII}). & 0\ .0019144 & \log.\ 7.2820328.
\end{array}$$

$$\begin{array}{lll}
(3,0) = (1+\mu). & 0''.0077700 & \log.\ 7.8904196; \\
(3,1) = (1+\mu'). & 0\ .4832186 & \log.\ 9.6841436; \\
(3,2) = (1+\mu''). & 1\ .7564488 & \log.\ 0.2446355; \\
(3,4) = (1+\mu^{IV}). & 14\ .6598964 & \log.\ 1.1661309; \\
(3,5) = (1+\mu^{V}). & 0\ .6313987 & \log.\ 9.8003037; \\
(3,6) = (1+\mu^{VI}). & 0\ .0105215 & \log.\ 8.0220791; \\
(3,7) = (1+\mu^{VII}). & 0\ .0036105 & \log.\ 7.5575701.
\end{array}$$

$(4,0)=(1+\mu\;).\ 0''.0000942.0$ log. 5.9740497;
$(4,1)=(1+\mu'\;).\ 0\ .0042125.6$ log. 7.6245464;
$(4,2)=(1+\mu''\;).\ 0\ .0088115.3$ log. 7.9450513;
$(4,3)=(1+\mu'''\;).\ 0\ .0031027.1$ log. 7.4917417;
$(4,5)=(1+\mu^{V}\;).\ 7\ .3963746.3$ log. 0.8690189;
$(4,6)=(1+\mu^{VI}).\ 0\ .0757628.5$ log. 8.8794563;
$(4,7)=(1+\mu^{VII}).\ 0\ .0240169.3$ log. 8.3805175.

$(5,0)=(1+\mu\;).\ 0''.0000112.0$ log. 5.0493193;
$(5,1)=(1+\mu'\;).\ 0\ .0004918.0$ log. 6.6917912;
$(5,2)=(1+\mu''\;).\ 0\ .0010042.1$ log. 7.0018244;
$(5,3)=(1+\mu'''\;).\ 0\ .0003296.8$ log. 6.5180955;
$(5,4)=(1+\mu^{IV}).18\ .2473541$ log. 1.2611999;
$(5,6)=(1+\mu^{VI}).\ 0\ .2782820.5$ log. 9.4444852;
$(5,7)=(1+\mu^{VII}).\ 0\ .0687398.9$ log. 8.8372088.

$(6,0)=(1+\mu\;).\ 0''.0000009.688$ log. 93.9862386;
$(6,1)=(1+\mu'\;).\ 0\ .0000422.85$ log. 95.6261830;
$(6,2)=(1+\mu''\;).\ 0\ .0000856.98$ log. 95.9329689;
$(6,3)=(1+\mu'''\;).\ 0\ .0000275.50$ log. 95.4401214;
$(6,4)=(1+\mu^{IV}).\ 0\ .9373198.2$ log. 99.9718878;
$(6,5)=(1+\mu^{V}\;).\ 1\ .3955188.2$ log. 0.1447357;
$(6,7)=(1+\mu^{VII}).\ 0\ .4332570.2$ log. 99.6367457.

$(7,0)=(1+\mu\;).\ 0''.0000002.0168$ log. 93.3046628;
$(7,1)=(1+\mu'\;).\ 0\ .0000087.926$ log. 94.9441177;
$(7,2)=(1+\mu''\;).\ 0\ .0000177.94$ log. 95.2502770;
$(7,3)=(1+\mu'''\;).\ 0\ .0000056.975$ log. 94.7556866;
$(7,4)=(1+\mu^{IV}).\ 0\ .1790701.5$ log. 99.2530232;
$(7,5)=(1+\mu^{V}\;).\ 0\ .2077464.0$ log. 99.3175335;
$(7,6)=(1+\mu^{VI}).\ 0\ .2611078.3$ log. 99.4168199.

In like manner formulas (25) and (8) will give the following values of $\boxed{0,1}$, $\boxed{1,0}$; $\boxed{0,2}$, $\boxed{2,0}$; $\boxed{1,2}$, $\boxed{2,1}$; &c.

$\boxed{0,1}=(1+\mu'\;).1''.926868$ log. 0.2848519;
$\boxed{0,2}=(1+\mu''\;).0.\ 4087579$ log. 99.6114662;
$\boxed{0,3}=(1+\mu'''\;).0.\ 008812816$ log. 97.9451147;
$\boxed{0,4}=(1+\mu^{IV}).0.\ 1489646$ log. 99.1730832;
$\boxed{0,5}=(1+\mu^{V}\;).0.\ 00391854$ log. 97.5931242;
$\boxed{0,6}=(1+\mu^{VI}).0.\ 0000336068$ log. 95.5264270;
$\boxed{0,7}=(1+\mu^{VII}).0.\ 00000741495$ log. 94.8701084.

$\boxed{1,0}=(1+\mu\;).0.''1129820$ log. 99.0530090;
$\boxed{1,2}=(1+\mu''\;).5.\ 520785$ log. 0.7420008;
$\boxed{1,3}=(1+\mu'''\;).0.\ 0587105$ log. 98.7687157;
$\boxed{1,4}=(1+\mu^{IV}).0.\ 7286137$ log. 99.8624973;
$\boxed{1,5}=(1+\mu^{V}\;).0.\ 0188385$ log. 98.2750461;
$\boxed{1,6}=(1+\mu^{VI}).0.\ 00016069$ log. 96.2059893;
$\boxed{1,7}=(1+\mu^{VII}).0.\ 0000354172$ log. 95.5492142.

$\boxed{2,0}=(1+\mu\ ).\ 0''.0192703$ log. 98.2848879;
$\boxed{2,1}=(1+\mu'\ ).\ 4\ .438798$ log. 0.6472654;
$\boxed{2,3}=(1+\mu'''\ ).\ 0\ .2293041$ log. 99.3604119;
$\boxed{2,4}=(1+\mu^{IV}).\ 1\ .690254$ log. 0.2279520;
$\boxed{2,5}=(1+\mu^{V}\ ).\ 0\ .0427287$ log. 98.6307197;
$\boxed{2,6}=(1+\mu^{VI}).\ 0\ .000361930$ log. 96.5586316;
$\boxed{2,7}=(1+\mu^{VII}).\ 0\ .0000796657$ log. 95.9012715.

$\boxed{3,0}=(1+\mu\ ).\ 0''.00244717$ log. 97.3886641;
$\boxed{3,1}=(1+\mu'\ ).\ 0\ .2780404$ log. 99.4441080;
$\boxed{3,2}=(1+\mu''\ ).\ 1\ .350640$ log. 0.1305396;
$\boxed{3,4}=(1+\mu^{IV}).\ 5\ .307483$ log. 0.7248886;
$\boxed{3,5}=(1+\mu^{V}\ ).\ 0\ .1256652$ log. 99.0992151;
$\boxed{3,6}=(1+\mu^{VI}).\ 0\ .001043788$ log. 97.0186122;
$\boxed{3,7}=(1+\mu^{VII}).\ 0\ .000228889$ log. 96.3596252.

$\boxed{4,0}=(1+\mu\ ).\ 0''.0000087547$ log. 94.9422434;
$\boxed{4,1}=(1+\mu'\ ).\ 0\ .00073030$ log. 96.8635004;
$\boxed{4,2}=(1+\mu''\ ).\ 0\ .00210713$ log. 97.3236905;
$\boxed{4,3}=(1+\mu'''\ ).\ 0\ .00112331$ log. 97.0504994;
$\boxed{4,5}=(1+\mu^{V}\ ).\ 4\ .835390$ log. 0.6844315;
$\boxed{4,6}=(1+\mu^{VI}).\ 0\ .0254429$ log. 98.4055666;
$\boxed{4,7}=(1+\mu^{VII}).\ 0\ .00518090$ log. 97.7144056.

$\boxed{5,0}=(1+\mu\ ).\ 0''.00000056815$ log. 93.7544654;
$\boxed{5,1}=(1+\mu'\ ).\ 0\ .0000465833$ log. 95.6682302;
$\boxed{5,2}=(1+\mu''\ ).\ 0\ .000131413$ log. 96.1186392;
$\boxed{5,3}=(1+\mu'''\ ).\ 0\ .0000656156$ log. 95.8170069;
$\boxed{5,4}=(1+\mu^{IV}).11\ .92923$ log. 1.0766125;
$\boxed{5,6}=(1+\mu^{VI}).\ 0.1671612$ log. 99.2231355;
$\boxed{5,7}=(1+\mu^{VII}).\ 0.0269346$ log. 98.4303106.

$\boxed{6,0}=(1+\mu\ ).\ 0''.000000024435$ log. 92.3880187;
$\boxed{6,1}=(1+\mu'\ ).\ 0\ .0000019926$ log. 94.2994239;
$\boxed{6,2}=(1+\mu''\ ).\ 0\ .00000558215$ log. 94.7468016;
$\boxed{6,3}=(1+\mu'''\ ).\ 0\ .0000027331$ log. 94.4366545;
$\boxed{6,4}=(1+\mu^{IV}).\ 0\ .3147736$ log. 99.4979981;
$\boxed{0,5}=(1+\mu^{V}\ ).\ 0\ .8382741$ log. 99.9233860;
$\boxed{6,7}=(1+\mu^{VII}).\ 0\ .3255696$ log. 99.5126438.

$\boxed{7,0}=(1+\mu\ ).\ 0''.000000003249$ log. 91.5117743;
$\boxed{7,1}=(1+\mu'\ ).\ 0\ .00000026468$ log. 93.4227230;
$\boxed{7,2}=(1+\mu''\ ).\ 0\ .000000740484$ log. 93.8695157;
$\boxed{7,3}=(1+\mu'''\ ).\ 0\ .000000361195$ log. 93.5577417;
$\boxed{7,4}=(1+\mu^{IV}).\ 0\ .0386288$ log. 98.5869113;
$\boxed{7,5}=(1+\mu^{V}\ ).\ 0\ .0814020$ log. 98.9106353;
$\boxed{7,6}=(1+\mu^{VI}).\ 0\ .1962086$ log. 99.2927180.

8. The values of $(0,1)$, $(0,2)$, &c., $\boxed{0,1}$, $\boxed{0,2}$, &c., $(1,0)$, $(1,2)$, &c., $\boxed{1,0}$, $\boxed{1,2}$, &c., being substituted in equations (B), supposing μ, μ', μ'', &c., to be equal to nothing, we shall have a series of numerical equations which are perfectly symmetrical in form, between g and the unknown quantities N, N', N'', &c. If we then eliminate N, N', N'' from these equations, we shall obtain a final equation in g, of a degree equal to the number of original equations. The construction of this final equation in g is the most delicate and intricate problem connected with the actual determination of the secular inequalities. Theoretically speaking, this equation may be formed by eliminating the quantities N, N', N'', &c., by the ordinary methods of elimination in algebra. But this method, though direct and simple in theory, leads to impracticable operations when we attempt to apply it to the formation of the equation of the eighth degree which is necessary in the simultaneous determination of the secular inequalities of the eight principal planets. The only actual merit of this method seems to be that of leaving the algebraic values of N',N'', N''', &c., in the successive eliminations, in very good form for computation, when the value of g has been determined; while its defects are twofold, as follows: *First*, it introduces foreign facts depending on g, which raise the final equation in g, to a very high degree; and *secondly*, it necessitates the employment of a very great number of decimals in the successive eliminations, in order to obtain a near approximation to the correct value of the final equation. The method of *determinants* enables us to form the final equation in g without actually performing the eliminations of the unknown quantities N, N', N'', &c. It also enables us to estimate, in advance, the exact amount of labor necessary for forming the final equation arising from any number of linear symmetrical equations. In the year 1860, we published a short paper on this interesting branch of analysis in GOULD'S *Astronomical Journal*, Vol. VI. From the explanation and formulæ there given, it follows that each of the given equations contains one binomial term of the form $g—a$, and that each term of the final equation contains one factor from each of the given equations; and also that the whole number of terms in the final equation is equal to the continued product of the numbers 1, 2, 3, 4, &c., to n inclusive; n denoting the number of given equations. In the present case n is equal to *eight;* there being one equation corresponding to each of the eight principal planets. The whole number of terms in the equation of the eighth degree is therefore equal to $1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 = 40320$. There are therefore 40320 distinct terms in the equation of the eighth degree, each of which contains eight factors which are either monomial or binomial. They are distributed in the following order:—

1	term	having	8	binomial	factors	producing	9	monomial	terms.
28	terms	"	6	"	"	"	196	"	"
112	"	"	5	"	"	"	672	"	"
630	"	"	4	"	"	"	3150	"	"
2464	"	"	3	"	"	"	9856	"	"
7420	"	"	2	"	"	"	22260	"	"
14832	"	"	1	"	"	"	29664	"	"
14833	"	without binomial factors				"	14833	"	"
Total 40320							80640		

The equation of the eighth degree when completely developed is therefore composed of 80640 distinct monomial terms, each of which contains eight factors. The actual formation of this equation could therefore with difficulty be brought within the compass of an ordinary lifetime; and we must, therefore, seek for a shorter and more expeditious method of attaining results which seem to necessarily involve such an immense expenditure of labor.

9. For this purpose we shall resume equations (B) of § 2, and shall suppose

$$\left.\begin{array}{l}
\boxed{0,0}=g-\{(0,1)+(0,2)+(0,3)+\text{\&c.}\}; \\
\boxed{1,1}=g-\{(1,0)+(1,2)+(1,3)+\text{\&c.}\}; \\
\boxed{2,2}=g-\{(2,0)+(2,1)+(2,3)+\text{\&c.}\}; \\
\boxed{3,3}=g-\{(3,0)+(3,1)+(3,2)+\text{\&c.}\}; \\
\boxed{4,4}=g-\{(4,0)+(4,1)+(4,2)+\text{\&c.}\}; \\
\boxed{5,5}=g-\{(5,0)+(5,1)+(5,2)+\text{\&c.}\}; \\
\boxed{6,6}=g-\{(6,0)+(6,1)+(6,2)+\text{\&c.}\}; \\
\boxed{7,7}=g-\{(7,0)+(7,1)+(7,2)+\text{\&c.}\};
\end{array}\right\} . \quad (26)$$

By this means equations (B) will be reduced to the following:—

$$\left.\begin{array}{l}
\boxed{0,0}N+\boxed{0,1}N'+\boxed{0,2}N''+\boxed{0,3}N'''+\boxed{0,4}N^{IV}+\boxed{0,5}N^{V}+\boxed{0,6}N^{VI}+\boxed{0,7}N^{VII}=0, \\
\boxed{1,1}N'+\boxed{1,0}N+\boxed{1,2}N''+\boxed{1,3}N'''+\boxed{1,4}N^{IV}+\boxed{1,5}N^{V}+\boxed{1,6}N^{VI}+\boxed{1,7}N^{VII}=0, \\
\boxed{2,2}N''+\boxed{2,0}N+\boxed{2,1}N'+\boxed{2,3}N'''+\boxed{2,4}N^{IV}+\boxed{2,5}N^{V}+\boxed{2,6}N^{VI}+\boxed{2,7}N^{VII}=0, \\
\boxed{3,3}N'''+\boxed{3,0}N+\boxed{3,1}N'+\boxed{3,2}N''+\boxed{3,4}N^{IV}+\boxed{3,5}N^{V}+\boxed{3,6}N^{VI}+\boxed{3,7}N^{VII}=0, \\
\boxed{4,4}N^{IV}+\boxed{4,0}N+\boxed{4,1}N'+\boxed{4,2}N''+\boxed{4,3}N'''+\boxed{4,5}N^{V}+\boxed{4,6}N^{VI}+\boxed{4,7}N^{VII}=0, \\
\boxed{5,5}N^{V}+\boxed{5,0}N+\boxed{5,1}N'+\boxed{5,2}N''+\boxed{5,3}N'''+\boxed{5,4}N^{IV}+\boxed{5,6}N^{VI}+\boxed{5,7}N^{VII}=0, \\
\boxed{6,6}N^{VI}+\boxed{6,0}N+\boxed{6,1}N'+\boxed{6,2}N''+\boxed{6,3}N'''+\boxed{6,4}N^{IV}+\boxed{6,5}N^{V}+\boxed{6,7}N^{VII}=0, \\
\boxed{7,7}N^{VII}+\boxed{7,0}N+\boxed{7,1}N'+\boxed{7,2}N''+\boxed{7,3}N'''+\boxed{7,4}N^{IV}+\boxed{7,5}N^{V}+\boxed{7,6}N^{VI}=0,
\end{array}\right\} . \quad (B')$$

Now since the coefficients of N^{IV}, N^{V}, N^{VI}, and N^{VII} of the first four of the equations (B′) are independent of g, and also the coefficients of N, N', N'' and N''', in the last four of equations (B′), we may divide them into two distinct groups, and determine the values of g, N', N'', N''', &c., by successive approximations. We shall therefore suppose

$$\left.\begin{array}{l}
b\ =\boxed{0,4}N^{IV}+\boxed{0,5}N^{V}+\boxed{0,6}N^{VI}+\boxed{0,7}N^{VII}, \\
b'\ =\boxed{1,4}N^{IV}+\boxed{1,5}N^{V}+\boxed{1,6}N^{VI}+\boxed{1,7}N^{VII}, \\
b''=\boxed{2,4}N^{IV}+\boxed{2,5}N^{V}+\boxed{2,6}N^{VI}+\boxed{2,7}N^{VII}, \\
b'''=\boxed{3,4}N^{IV}+\boxed{3,5}N^{V}+\boxed{3,6}N^{VI}+\boxed{3,7}N^{VII},
\end{array}\right\} . \quad (27)$$

$$\left.\begin{array}{l}
b_1=\boxed{4,0}N+\boxed{4,1}N'+\boxed{4,2}N''+\boxed{4,3}N''', \\
b_2=\boxed{5,0}N+\boxed{5,1}N'+\boxed{5,2}N''+\boxed{5,3}N''', \\
b_3=\boxed{6,0}N+\boxed{6,1}N'+\boxed{6,2}N''+\boxed{6,3}N''', \\
b_4=\boxed{7,0}N+\boxed{7,1}N'+\boxed{7,2}N''+\boxed{7,3}N'''.
\end{array}\right\} . \quad (28)$$

Substituting these quantities in equations (B′), they will become

$$\left.\begin{array}{l}
\boxed{0,0}N+\boxed{0,1}N'\ \boxed{0,2}N''+\boxed{0,3}N'''+b\ =0, \\
\boxed{1,1}N'+\boxed{1,0}N\ \boxed{1,2}N''+\boxed{1,3}N'''+b'\ =0, \\
\boxed{2,2}N''+\boxed{2,0}N\ \boxed{2,1}N'+\boxed{2,3}N'''+b''=0, \\
\boxed{3,3}N'''+\boxed{3,0}N\ \boxed{3,1}N'+\boxed{3,2}N''+b'''=0;
\end{array}\right\} . \quad (B'')$$

$$\left.\begin{array}{l}
\boxed{4,4}N^{IV}+\boxed{4,5}N^{V}\boxed{4,6}N^{VI}+\boxed{4,7}N^{VII}+b_1=0,\\
\boxed{5,5}N^{V}+\boxed{5,4}N^{IV}\boxed{5,6}N^{VI}+\boxed{5,7}N^{VII}+b_2=0,\\
\boxed{6,6}N^{VI}+\boxed{6,4}N^{IV}\boxed{6,5}N^{V}+\boxed{6,7}N^{VII}+b_3=0,\\
\boxed{7,7}N^{VII}+\boxed{7,4}N^{IV}\boxed{7,5}N^{V}+\boxed{7,6}N^{VI}+b_4=0.
\end{array}\right\} \quad (B''')$$

If we now suppose b, b', b'', &c. to be equal to nothing, and eliminate N', N'', and N''' from equations (B''), and N^{V}, N^{VI}, and N^{VII} from equations (B'''), the resulting equations will be divisible by N and N^{IV} respectively; and we shall have

$$\left.\begin{array}{l}
\boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}-\boxed{3,2}\boxed{2,3}\boxed{0,0}\boxed{1,1}-\boxed{3,1}\boxed{1,3}\boxed{0,0}\boxed{2,2}-\boxed{3,0}\boxed{0,3}\boxed{1,1}\boxed{2,2}\\
-\boxed{2,1}\boxed{1,2}\boxed{0,0}\boxed{3,3}-\boxed{2,0}\boxed{0,2}\boxed{1,1}\boxed{3,3}-\boxed{1,0}\boxed{0,1}\boxed{2,2}\boxed{3,3}+\{\boxed{3,2}\boxed{2,1}\boxed{1,3}\\
+\boxed{3,1}\boxed{1,2}\boxed{2,3}\}\boxed{0,0}+\{\boxed{3,0}\boxed{0,2}\boxed{2,3}+\boxed{3,2}\boxed{2,0}\boxed{0,3}\}\boxed{1,1}+\{\boxed{3,1}\boxed{1,0}\boxed{0,3}\\
+\boxed{3,0}\boxed{0,1}\boxed{1,3}\}\boxed{2,2}+\{\boxed{2,0}\boxed{0,1}\boxed{1,2}+\boxed{2,1}\boxed{1,0}\boxed{0,2}\}\boxed{3,3}\\
+\boxed{3,2}\boxed{2,3}\boxed{1,0}\boxed{0,1}+\boxed{3,1}\boxed{1,3}\boxed{2,0}\boxed{0,2}+\boxed{3,0}\boxed{0,3}\boxed{2,1}\boxed{1,2}\\
-\{\boxed{3,0}\boxed{0,2}\boxed{2,1}\boxed{1,3}+\boxed{3,1}\boxed{1,2}\boxed{2,0}\boxed{0,3}\}-\{\boxed{3,1}\boxed{1,0}\boxed{0,2}\boxed{2,3}\\
+\boxed{3,2}\boxed{2,0}\boxed{0,1}\boxed{1,3}\}-\{\boxed{3,0}\boxed{0,1}\boxed{1,2}\boxed{2,3}+\boxed{3,2}\boxed{2,1}\boxed{1,0}\boxed{0,3}\}
\end{array}\right\} \begin{array}{c}=0;\\(29)\end{array}$$

$$\left.\begin{array}{l}
\boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}-\boxed{7,6}\boxed{6,7}\boxed{4,4}\boxed{5,5}-\boxed{7,5}\boxed{5,7}\boxed{4,4}\boxed{6,6}-\boxed{7,4}\boxed{4,7}\boxed{5,5}\boxed{6,6}\\
-\boxed{6,5}\boxed{5,6}\boxed{4,4}\boxed{7,7}-\boxed{6,4}\boxed{4,6}\boxed{5,5}\boxed{7,7}-\boxed{5,4}\boxed{4,5}\boxed{6,6}\boxed{7,7}+\{\boxed{7,6}\boxed{6,5}\boxed{5,7}\\
+\boxed{7,5}\boxed{5,6}\boxed{6,7}\}\boxed{4,4}+\{\boxed{7,4}\boxed{4,6}\boxed{6,7}+\boxed{7,6}\boxed{6,4}\boxed{4,7}\}\boxed{5,5}\\
+\{\boxed{7,5}\boxed{5,4}\boxed{4,7}+\boxed{7,4}\boxed{4,5}\boxed{5,7}\}\boxed{6,6}+\{\boxed{6,4}\boxed{4,5}\boxed{5,6}+\boxed{6,5}\boxed{5,4}\boxed{4,6}\}\boxed{7,7}\\
+\boxed{7,5}\boxed{5,7}\boxed{6,4}\boxed{4,6}+\boxed{7,4}\boxed{4,7}\boxed{6,5}\boxed{5,6}+\boxed{7,6}\boxed{6,7}\boxed{5,4}\boxed{4,5}\\
-\{\boxed{7,4}\boxed{4,6}\boxed{6,5}\boxed{5,7}+\boxed{7,5}\boxed{5,6}\boxed{6,4}\boxed{4,7}\}-\{\boxed{7,5}\boxed{5,4}\boxed{4,6}\boxed{6,7}\\
+\boxed{7,6}\boxed{6,4}\boxed{4,5}\boxed{5,7}\}-\{\boxed{7,4}\boxed{4,5}\boxed{5,6}\boxed{6,7}+\boxed{7,6}\boxed{6,5}\boxed{5,4}\boxed{4,7}\}
\end{array}\right\} \begin{array}{c}=0.\\(30)\end{array}$$

Each of these equations is evidently of the fourth degree in g, and consequently has four roots, which may be found by any of the ordinary methods of finding the roots of numerical equations. These roots will be only approximate, because we have neglected b, b', b'', &c., in the determination of equations (29) and (30). If we substitute the approximate roots derived from equation (29) in any three of equations (B''), we can find by elimination the values of N', N'', and N''' in terms of N, which remains indeterminate. When N', N'', and N''' have been thus determined we must substitute their values in equations (28), and we shall obtain the values b_1, b_2, b_3, and b_4 in terms of N; and these quantities are then to be substituted in equations (B'''), together with the corresponding value of g; and we shall then obtain the values of N^{IV}, N^{V}, N^{VI}, and N^{VII}, in terms N. But instead of performing this operation separately for each of the roots, in the manner described above, it is better to deduce a system of algebraic equations, not only for the purpose of facilitating the numerical calculations, but also for the purpose of devising checks to the accuracy of the different parts of the computations.

10. If we now assume the following quantities

$$\left.\begin{array}{l}
A=\boxed{0,0}\boxed{2,2}-\dfrac{\boxed{1,0}\boxed{0,3}}{\boxed{1,3}}\boxed{2,2}-\dfrac{\boxed{1,2}\boxed{2,3}}{\boxed{1,3}}\boxed{0,0}+\dfrac{\boxed{1,0}\boxed{0,2}\boxed{2,3}}{\boxed{1,3}}+\dfrac{\boxed{1,2}\boxed{2,0}\boxed{0,3}}{\boxed{1,3}}\\
\qquad\qquad\qquad\qquad -\boxed{2,0}\boxed{0,2};
\end{array}\right\}(31)$$

$$\left.\begin{array}{l}
A'=\boxed{0,0}\boxed{3,3}-\dfrac{\boxed{1,3}\boxed{3,2}}{\boxed{1,2}}\boxed{0,0}-\dfrac{\boxed{1,0}\boxed{0,2}}{\boxed{1,2}}\boxed{3,3}+\dfrac{\boxed{1,0}\boxed{0,3}\boxed{3,2}}{\boxed{1,2}}+\dfrac{\boxed{1,3}\boxed{3,0}\boxed{0,2}}{\boxed{1,2}}\\
\qquad\qquad\qquad\qquad -\boxed{3,0}\boxed{0,3};
\end{array}\right\}(32)$$

$$\left.\begin{aligned}A''=\boxed{0,0}\boxed{1,1}-\frac{\boxed{2,1}\boxed{1,3}}{\boxed{2,3}}\boxed{0,0}-\frac{\boxed{2,0}\boxed{0,3}}{\boxed{2,3}}\boxed{1,1}+\frac{\boxed{2,0}\boxed{0,1}\boxed{1,3}}{\boxed{2,3}}+\frac{\boxed{2,1}\boxed{1,0}\boxed{0,3}}{\boxed{2,3}}\\-\boxed{1,0}\boxed{0,1};\end{aligned}\right\}\tag{33}$$

$$\left.\begin{aligned}D=\boxed{1,1}\boxed{2,2}-\frac{\boxed{0,2}\boxed{2,3}}{\boxed{0,3}}\boxed{1,1}-\frac{\boxed{0,1}\boxed{1,3}}{\boxed{0,3}}\boxed{2,2}+\frac{\boxed{0,1}\boxed{1,2}\boxed{2,3}}{\boxed{0,3}}+\frac{\boxed{0,2}\boxed{2,1}\boxed{1,3}}{\boxed{0,3}}\\-\boxed{2,1}\boxed{1,2};\end{aligned}\right\}\tag{34}$$

$$\left.\begin{aligned}D'=\boxed{1,1}\boxed{3,3}-\frac{\boxed{0,3}\boxed{3,2}}{\boxed{0,2}}\boxed{1,1}-\frac{\boxed{0,1}\boxed{1,2}}{\boxed{0,2}}\boxed{3,3}+\frac{\boxed{0,1}\boxed{1,3}\boxed{3,2}}{\boxed{0,2}}+\frac{\boxed{0,3}\boxed{3,1}\boxed{1,2}}{\boxed{0,2}}\\-\boxed{3,1}\boxed{1,3};\end{aligned}\right\}\tag{35}$$

$$\left.\begin{aligned}D''=\boxed{2,2}\boxed{3,3}-\frac{\boxed{0,2}\boxed{2,1}}{\boxed{0,1}}\boxed{3,3}-\frac{\boxed{0,3}\boxed{3,1}}{\boxed{0,1}}\boxed{2,2}+\frac{\boxed{0,3}\boxed{3,2}\boxed{2,1}}{\boxed{0,1}}+\frac{\boxed{0,2}\boxed{2,3}\boxed{3,1}}{\boxed{0,1}}\\-\boxed{3,2}\boxed{2,3};\end{aligned}\right\}\tag{36}$$

$$\left.\begin{aligned}A_1=\boxed{4,4}\boxed{6,6}-\frac{\boxed{5,6}\boxed{6,7}}{\boxed{5,7}}\boxed{4,4}-\frac{\boxed{5,4}\boxed{4,7}}{\boxed{5,7}}\boxed{6,6}+\frac{\boxed{5,4}\boxed{4,6}\boxed{6,7}}{\boxed{5,7}}+\frac{\boxed{5,6}\boxed{6,4}\boxed{4,7}}{\boxed{5,7}}\\-\boxed{6,4}\boxed{4,6};\end{aligned}\right\}\tag{37}$$

$$\left.\begin{aligned}A_2=\boxed{4,4}\boxed{7,7}-\frac{\boxed{5,7}\boxed{7,6}}{\boxed{5,6}}\boxed{4,4}-\frac{\boxed{5,4}\boxed{4,6}}{\boxed{5,6}}\boxed{7,7}+\frac{\boxed{5,4}\boxed{4,7}\boxed{7,6}}{\boxed{5,6}}+\frac{\boxed{5,7}\boxed{7,4}\boxed{4,6}}{\boxed{5,6}}\\-\boxed{7,4}\boxed{4,7};\end{aligned}\right\}\tag{38}$$

$$\left.\begin{aligned}A_3=\boxed{4,4}\boxed{5,5}-\frac{\boxed{6,5}\boxed{5,7}}{\boxed{6,7}}\boxed{4,4}-\frac{\boxed{6,4}\boxed{4,7}}{\boxed{6,7}}\boxed{5,5}+\frac{\boxed{6,4}\boxed{4,5}\boxed{5,7}}{\boxed{6,7}}+\frac{\boxed{6,5}\boxed{5,4}\boxed{4,7}}{\boxed{6,7}}\\-\boxed{5,4}\boxed{4,5};\end{aligned}\right\}\tag{39}$$

$$\left.\begin{aligned}D_1=\boxed{5,5}\boxed{6,6}-\frac{\boxed{4,6}\boxed{6,7}}{\boxed{4,7}}\boxed{5,5}-\frac{\boxed{4,5}\boxed{5,7}}{\boxed{4,7}}\boxed{6,6}+\frac{\boxed{4,5}\boxed{5,6}\boxed{6,7}}{\boxed{4,7}}+\frac{\boxed{4,6}\boxed{6,5}\boxed{5,7}}{\boxed{4,7}}\\-\boxed{6,5}\boxed{5,6};\end{aligned}\right\}\tag{40}$$

$$\left.\begin{aligned}D_2=\boxed{5,5}\boxed{7,7}-\frac{\boxed{4,7}\boxed{7,6}}{\boxed{4,6}}\boxed{5,5}-\frac{\boxed{4,5}\boxed{5,6}}{\boxed{4,6}}\boxed{7,7}+\frac{\boxed{4,5}\boxed{5,7}\boxed{7,6}}{\boxed{4,6}}+\frac{\boxed{4,7}\boxed{7,5}\boxed{5,6}}{\boxed{4,6}}\\-\boxed{7,5}\boxed{5,7};\end{aligned}\right\}\tag{41}$$

$$\left.\begin{aligned}D_3=\boxed{6,6}\boxed{7,7}-\frac{\boxed{4,7}\boxed{7,5}}{\boxed{4,5}}\boxed{6,6}-\frac{\boxed{4,6}\boxed{6,5}}{\boxed{4,5}}\boxed{7,7}+\frac{\boxed{4,7}\boxed{7,6}\boxed{6,5}}{\boxed{4,5}}+\frac{\boxed{4,6}\boxed{6,7}\boxed{7,5}}{\boxed{4,5}}\\-\boxed{7,6}\boxed{6,7};\end{aligned}\right\}\tag{42}$$

$$B=\left\{\boxed{2,2}-\frac{\boxed{1,2}\boxed{2,3}}{\boxed{1,3}}\right\}b;\quad B'=\left\{\boxed{3,3}-\frac{\boxed{1,3}\boxed{3,2}}{\boxed{1,2}}\right\}b;\quad B''=\left\{\boxed{1,1}-\frac{\boxed{2,1}\boxed{1,3}}{\boxed{2,3}}\right\}b;\tag{43}$$

$$\left.\begin{aligned}C&=\left\{\frac{\boxed{0,2}\boxed{2,3}}{\boxed{0,3}}-\boxed{2,2}\right\}\frac{\boxed{0,3}}{\boxed{1,3}}b'; & C'&=\left\{\frac{\boxed{0,3}\boxed{3,2}}{\boxed{0,2}}-\boxed{3,3}\right\}\frac{\boxed{0,2}}{\boxed{1,2}}b';\\ C''&=\left\{\frac{\boxed{0,3}\boxed{2,1}}{\boxed{2,3}}-\boxed{0,1}\right\}b'; & C'''&=\left\{\frac{\boxed{0,2}\boxed{3,1}}{\boxed{3,2}}-\boxed{0,1}\right\}b';\end{aligned}\right\}\tag{44}$$

$$\left.\begin{aligned}E&=\left\{\frac{\boxed{0,3}\boxed{1,2}}{\boxed{1,3}}-\boxed{0,2}\right\}b''; & E'&=\left\{\frac{\boxed{0,1}\boxed{1,3}}{\boxed{0,3}}-\boxed{1,1}\right\}\frac{\boxed{0,3}}{\boxed{2,3}}b'';\\ E''&=\left\{\frac{\boxed{0,3}\boxed{3,1}}{\boxed{0,1}}-\boxed{3,3}\right\}\frac{\boxed{0,1}}{\boxed{2,1}}b''; & E'''&=\left\{\frac{\boxed{0,1}\boxed{3,2}}{\boxed{3,1}}-\boxed{0,2}\right\}b'';\end{aligned}\right\}\tag{45}$$

$$\left.\begin{array}{ll} F=\left\{\dfrac{\boxed{0,2}\,\boxed{1,3}}{\boxed{1,2}}-\boxed{0,3}\right\}b'''; & F'=\left\{\dfrac{\boxed{0,1}\,\boxed{2,3}}{\boxed{2,1}}-\boxed{0,3}\right\}b'''; \\ F''=\left\{\dfrac{\boxed{0,2}\,\boxed{2,1}}{\boxed{0,1}}-\boxed{2,2}\right\}\dfrac{\boxed{0,1}}{\boxed{3,1}}b'''; & F'''=\left\{\dfrac{\boxed{0,1}\,\boxed{1,2}}{\boxed{0,2}}-\boxed{1,1}\right\}\dfrac{\boxed{0,2}}{\boxed{3,2}}b'''; \end{array}\right\} \quad (46)$$

$$B_1=\left\{\boxed{6,6}-\frac{\boxed{5,6}\,\boxed{6,7}}{\boxed{5,7}}\right\}b_1;\ B_2=\left\{\boxed{7,7}-\frac{\boxed{5,7}\,\boxed{7,6}}{\boxed{5,6}}\right\}b_1;\ B_3-\left\{\boxed{5,5}-\frac{\boxed{6,5}\,\boxed{5,7}}{\boxed{6,7}}\right\}b_1; \quad (47)$$

$$\left.\begin{array}{ll} C_1=\left\{\dfrac{\boxed{4,6}\,\boxed{6,7}}{\boxed{4,7}}-\boxed{6,6}\right\}\dfrac{\boxed{4,7}}{\boxed{5,7}}b_2; & C_2=\left\{\dfrac{\boxed{4,7}\,\boxed{7,6}}{\boxed{4,6}}-\boxed{7,7}\right\}\dfrac{\boxed{4,6}}{\boxed{5,6}}b_2; \\ C_3=\left\{\dfrac{\boxed{4,7}\,\boxed{6,5}}{\boxed{6,7}}-\boxed{4,5}\right\}b_2; & C_4=\left\{\dfrac{\boxed{4,6}\,\boxed{7,5}}{\boxed{7,6}}-\boxed{4,5}\right\}b_2; \end{array}\right\} \quad (48)$$

$$\left.\begin{array}{ll} E_1=\left\{\dfrac{\boxed{4,7}\,\boxed{5,6}}{\boxed{5,7}}-\boxed{4,6}\right\}b_3; & E_2=\left\{\dfrac{\boxed{4,5}\,\boxed{5,7}}{\boxed{4,7}}-\boxed{5,5}\right\}\dfrac{\boxed{4,7}}{\boxed{6,7}}b_3; \\ E_3=\left\{\dfrac{\boxed{4,7}\,\boxed{7,5}}{\boxed{4,5}}-\boxed{7,7}\right\}\dfrac{\boxed{4,5}}{\boxed{6,5}}b_3; & E_4=\left\{\dfrac{\boxed{4,5}\,\boxed{7,6}}{\boxed{7,5}}-\boxed{4,6}\right\}b_3; \end{array}\right\} \quad (49)$$

$$\left.\begin{array}{ll} F_1=\left\{\dfrac{\boxed{4,6}\,\boxed{5,7}}{\boxed{5,6}}-\boxed{4,7}\right\}b_4; & F_2=\left\{\dfrac{\boxed{4,5}\,\boxed{6,7}}{\boxed{6,5}}-\boxed{4,7}\right\}b_4; \\ F_3=\left\{\dfrac{\boxed{4,6}\,\boxed{6,5}}{\boxed{4,5}}-\boxed{6,6}\right\}\dfrac{\boxed{4,5}}{\boxed{7,5}}b_4; & F_4=\left\{\dfrac{\boxed{4,5}\,\boxed{5,6}}{\boxed{4,6}}-\boxed{5,5}\right\}\dfrac{\boxed{4,6}}{\boxed{7,6}}b_4; \end{array}\right\} \quad (50)$$

$$\left.\begin{array}{lll} f=B+C+E; & f'=B'+C'+F; & f''=B''+C''+E'; \\ f'''=B'+E''+F'; & f^{IV}=B+E'''+F''; & f^{V}=B''+C'''+F'''; \end{array}\right\} \quad (51)$$

$$\left.\begin{array}{lll} f_1=B_1+C_1+E_1; & f_2=B_2+C_2+F_1; & f_3=B_3+C_3+E_2; \\ f_4=B_2+E_3+F_2; & f_5=B_1+E_4+F_2; & f_4=B_3+C_4+F_4; \end{array}\right\} \quad (52)$$

we shall have the following system of equations for the rigorous determination of g, N', N'', N''', &c.—observing that the terms of equations (29) and (30), which are inclosed by braces, are reduced to single terms by means of equations (14) and (15).

$$\left.\begin{array}{l} \boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}-\boxed{3,2}\boxed{2,3}\boxed{0,0}\boxed{1,1}-\boxed{3,1}\boxed{1,3}\boxed{0,0}\boxed{2,2} \\ -\boxed{3,0}\boxed{0,3}\boxed{1,1}\boxed{2,2}-\boxed{2,1}\boxed{1,2}\boxed{0,0}\boxed{3,3}-\boxed{2,0}\boxed{0,2}\boxed{1,1}\boxed{3,3} \\ -\boxed{1,0}\boxed{0,1}\boxed{2,2}\boxed{3,3}+2\boxed{3,2}\boxed{2,1}\boxed{1,3}\boxed{0,0}+2\boxed{3,0}\boxed{0,2}\boxed{2,3}\boxed{1,1} \\ +2\boxed{3,1}\boxed{1,0}\boxed{0,3}\boxed{2,2}+2\boxed{2,0}\boxed{0,1}\boxed{1,2}\boxed{3,3}+\boxed{3,2}\boxed{2,3}\boxed{1,0}\boxed{0,1} \\ +\boxed{3,1}\boxed{1,3}\boxed{2,0}\boxed{0,2}+\boxed{3,0}\boxed{0,3}\boxed{2,1}\boxed{1,2}-2\boxed{3,0}\boxed{0,2}\boxed{2,1}\boxed{1,3} \\ -2\boxed{3,1}\boxed{1,0}\boxed{0,2}\boxed{2,3}-2\boxed{3,0}\boxed{0,1}\boxed{1,2}\boxed{2,3} \end{array}\right\}=(\chi,\chi_1,\chi_2,\chi_3); \quad (53)$$

$$\left.\begin{array}{l} \boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}-\boxed{7,6}\boxed{6,7}\boxed{4,4}\boxed{5,5}-\boxed{7,5}\boxed{5,7}\boxed{4,4}\boxed{6,6} \\ -\boxed{7,4}\boxed{4,7}\boxed{5,5}\boxed{6,6}-\boxed{6,5}\boxed{5,6}\boxed{4,4}\boxed{7,7}-\boxed{6,4}\boxed{4,6}\boxed{5,5}\boxed{7,7} \\ -\boxed{5,4}\boxed{4,5}\boxed{6,6}\boxed{7,7}+2\boxed{7,6}\boxed{6,5}\boxed{5,7}\boxed{4,4}+2\boxed{7,4}\boxed{4,6}\boxed{6,7}\boxed{5,5} \\ +2\boxed{7,5}\boxed{5,4}\boxed{4,7}\boxed{6,6}+2\boxed{6,4}\boxed{4,5}\boxed{5,6}\boxed{7,7}+\boxed{7,5}\boxed{5,7}\boxed{6,4}\boxed{4,6} \\ +\boxed{7,4}\boxed{4,7}\boxed{6,5}\boxed{5,6}+\boxed{7,6}\boxed{6,7}\boxed{5,4}\boxed{4,5}-2\boxed{7,4}\boxed{4,6}\boxed{6,5}\boxed{5,7} \\ -2\boxed{7,5}\boxed{5,4}\boxed{4,6}\boxed{6,7}-2\boxed{7,4}\boxed{4,5}\boxed{5,6}\boxed{6,7} \end{array}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7); \quad (54)$$

$$\left.\begin{aligned}
&\left\{\frac{\boxed{0,2}}{\boxed{1,2}}D'f-\frac{\boxed{0,3}}{\boxed{1,3}}Df'\right\}\div\left\{\frac{\boxed{0,3}}{\boxed{1,3}}-\frac{\boxed{0,2}}{\boxed{1,2}}\right\}N\\
=&\left\{\frac{\boxed{0,1}}{\boxed{2,1}}D''f''-\frac{\boxed{0,3}}{\boxed{2,3}}Df'''\right\}\div\left\{\frac{\boxed{0,3}}{\boxed{2,3}}-\frac{\boxed{0,1}}{\boxed{2,1}}\right\}N\\
=&\left\{\frac{\boxed{0,2}}{\boxed{3,2}}D'f^{IV}-\frac{\boxed{0,1}}{\boxed{3,1}}D''f^{V}\right\}\div\left\{\frac{\boxed{0,1}}{\boxed{3,1}}-\frac{\boxed{0,2}}{\boxed{3,2}}\right\}N
\end{aligned}\right\}=(\chi,\chi_1,\chi_2,\chi_3);\quad(55)$$

$$\left.\begin{aligned}
&\left\{\frac{\boxed{4,6}}{\boxed{5,6}}D_2f_1-\frac{\boxed{4,7}}{\boxed{5,7}}D_1f_2\right\}\div\left\{\frac{\boxed{4,7}}{\boxed{5,7}}-\frac{\boxed{4,6}}{\boxed{5,6}}\right\}N^{IV}\\
=&\left\{\frac{\boxed{4,5}}{\boxed{6,5}}D_3f_3-\frac{\boxed{4,7}}{\boxed{6,7}}D_1f_4\right\}\div\left\{\frac{\boxed{4,7}}{\boxed{6,7}}-\frac{\boxed{4,5}}{\boxed{6,5}}\right\}N^{IV}\\
=&\left\{\frac{\boxed{4,6}}{\boxed{7,6}}D_2f_5-\frac{\boxed{4,5}}{\boxed{7,5}}D_3f_6\right\}\div\left\{\frac{\boxed{4,5}}{\boxed{7,5}}-\frac{\boxed{4,6}}{\boxed{7,6}}\right\}N^{IV}
\end{aligned}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7);\quad(56)$$

$$N'=\frac{AN+f}{\{\boxed{0,3}\div\boxed{1,3}\}D};\qquad N'=\frac{A'N+f'}{\{\boxed{0,2}\div\boxed{1,2}\}D'};\quad(57)$$

$$N''=\frac{A''N+f''}{\{\boxed{0,3}\div\boxed{2,3}\}D};\qquad N''=\frac{A'N+f'''}{\{\boxed{0,1}\div\boxed{2,1}\}D''};\quad(58)$$

$$N'''=\frac{AN+f^{IV}}{\{\boxed{0,1}\div\boxed{3,1}\}D''};\qquad N'''=\frac{A''N+f^{V}}{\{\boxed{0,2}\div\boxed{3,2}\}D'};\quad(59)$$

$$\left.\begin{aligned}
N=\frac{\boxed{1,2}\,\boxed{0,3}\,Df'-\boxed{1,3}\,\boxed{0,2}\,D'f''}{\boxed{1,3}\,\boxed{0,2}\,AD'-\boxed{1,2}\,\boxed{0,3}\,A'D}&=\frac{\boxed{2,1}\,\boxed{0,3}\,Df'''-\boxed{2,3}\,\boxed{0,1}\,D''f''}{\boxed{2,3}\,\boxed{0,1}\,A''D''-\boxed{2,1}\,\boxed{0,3}\,A'D}\\
&=\frac{\boxed{3,2}\,\boxed{0,1}\,D''f^{V}-\boxed{3,1}\,\boxed{0,2}\,D'f^{IV}}{\boxed{3,1}\,\boxed{0,2}\,AD'-\boxed{3,2}\,\boxed{0,1}\,A''D''}
\end{aligned}\right\};\quad(60)$$

$$N^{V}=\frac{A_1N^{IV}+f_1}{\{\boxed{4,7}\div\boxed{5,7}\}D_1};\qquad N^{V}=\frac{A_2N^{IV}+f_2}{\{\boxed{4,6}\div\boxed{5,6}\}D_2};\quad(61)$$

$$N^{VI}=\frac{A_3N^{IV}+f_3}{\{\boxed{4,7}\div\boxed{6,7}\}D_1};\qquad N^{VI}=\frac{A_2N^{IV}+f_4}{\{\boxed{4,5}\div\boxed{6,5}\}D_3};\quad(62)$$

$$N^{VII}=\frac{A_1N^{IV}+f_5}{\{\boxed{4,5}\div\boxed{7,5}\}D_3};\qquad N^{VII}=\frac{A_3N^{IV}+f_6}{\{\boxed{4,6}\div\boxed{7,6}\}D_2};\quad(63)$$

$$\left.\begin{aligned}
N^{IV}=\frac{\boxed{4,7}\,\boxed{5,6}\,D_1f_2-\boxed{4,6}\,\boxed{5,7}\,D_2f_1}{\boxed{4,6}\,\boxed{5,7}\,A_1D_2-\boxed{4,7}\,\boxed{5,6}\,A_2D_1}&=\frac{\boxed{4,7}\,\boxed{6,5}\,D_1f_4-\boxed{4,5}\,\boxed{6,7}\,D_3f_3}{\boxed{4,5}\,\boxed{6,7}\,A_3D_3-\boxed{4,7}\,\boxed{6,5}\,A_2D_1}\\
&=\frac{\boxed{4,5}\,\boxed{7,6}\,D_3f_6-\boxed{4,6}\,\boxed{7,5}\,D_2f_5}{\boxed{4,6}\,\boxed{7,5}\,A_1D_2-\boxed{4,5}\,\boxed{7,6}\,A_3D_3}
\end{aligned}\right\}.\quad(64)$$

11. Equations (53) to (64) are entirely rigorous. They are, moreover, under a very simple and convenient form for the computation of N', N'', N''', &c.; and, as there are duplicate and independent formulæ for all these quantities, any error that may accidentally creep into the computation of one formula is at once detected by computation of the other. They have also this additional advantage: sometimes one of the formulæ for N, N', N'', &c., gives value for these quantities of the form

$N=\frac{a-a'}{a''}$, in which a is very nearly equal to a', and the computation of these quantities cannot be readily effected by logarithms with sufficient precision to give their *difference* correct to more than three or four significant figures; and, in all these cases, the other formula for the same quantity gives a value which is free from this source of error.

The computation of the successive approximations to the values of the required quantities is then arranged as follows:—

We first find the roots of equation (53), on the supposition that χ is equal to nothing. We shall designate these roots by g, g_1, g_2, and g_3, and the corresponding values of the second member by χ, χ_1, χ_2, and χ_3. The roots of equation (54) are also designated by g_4, g_5, g_6, and g_7, and the corresponding values of the second member by χ_4, χ_5, χ_6, and χ_7. When g, g_1, g_2, &c. have been determined, we must transform equation (53) into others whose roots shall be smaller by the values g, g_1, g_2, and g_3. Then if we denote the corrections to be applied to g, g_1, &c., in order to obtain their correct values by δg, δg_1, δg_2, and δg_3, we shall have a system of equations for the determination of δg, δg_1, δg_2, and δg_3, of the following form:—

$$\left.\begin{array}{l} \delta g^4+a\ \delta g^3+b\ \delta g^2+c\ \delta g=\chi\ ; \\ \delta g_1^4+a'\ \delta g_1^3+b'\ \delta g_1^2+c'\ \delta g_1=\chi_1; \\ \delta g_2^4+a''\ \delta g_2^3+b''\ \delta g_2^2+c''\ \delta g_2=\chi_2; \\ \delta g_3^4+a'''\delta g_3^3+b'''\delta g_3^2+c'''\delta g_3=\chi_3; \end{array}\right\} . \quad (65)$$

$$\delta g+\delta g_1+\delta g_2+\delta g_3+\delta g_4+\delta g_5+\delta g_6+\delta g_7=0. \qquad (66) \quad \text{[Equation of condition.]}$$

The equations for the determination of δg_4, δg_5, δg_6, and δg_7 are entirely similar. Then, having determined the approximate values of g, g_1, g_2, &c., we must substitute them in succession in equations (31—42) inclusive, and we shall obtain the values of A, A', A'', &c., D, D', D'', &c., which are to be substituted in equations (57—59), and we shall obtain the approximate values of N', N'', and N'''. These quantities are then to be substituted in equations (28), and we shall get the values of b_1, b_2, b_3, and b_4; which quantities, together with the value of g, are to be substituted in equations (47—50) inclusive, and we obtain B_1, B_2, B_3, C_1, C_2, &c., &c. Then equation (52) will give f_1, f_2, &c., which are to be substituted in equations (61—64), and we shall obtain N^{IV}, N^{V}, N^{VI}, and N^{VII}. Equations (27) will then give b, b', b'', and b''', which are to be substituted along with g in equations (43—46). Then equations (51) will give f, f', f'', &c., which being substituted in equations (54) will give the value of χ, on which the value of δg depends in the first of equations (65). When δg has been determined we must add it to the approximate value of g, and repeat the whole computation with the corrected value of g, and by this means we shall obtain the correct values of g, N', N'', N''', &c. In like manner we shall obtain g_1, N_1', N_1'', N_1''', &c.; g_2, N_2', N_2'', &c.

12. We shall now reduce the preceding formulæ to numbers, and illustrate by a numerical example the extreme simplicity of the formulæ (which appear so unwieldy in their algebraic form), and the comparative facility with which the required quantities can be obtained.

Substituting the values of $(0,1)$, $(0,2)$, $(0,3)$, &c., given in § 7, in equations (26), we shall get the following values of $\boxed{0,0}$, $\boxed{1,1}$, $\boxed{2,2}$, &c.:—

$$\left.\begin{array}{ll} \boxed{0,0}=g-\ \ 5''.5702558; & \boxed{4,4}=g-\ \ 7''.5123754; \\ \boxed{1,1}=g-11\ .3147682; & \boxed{5,5}=g-18\ .5962129; \\ \boxed{2,2}=g-13\ .0721730; & \boxed{6,6}=g-\ \ 2\ .7662522; \\ \boxed{3,3}=g-17\ .5528645; & \boxed{7,7}=g-\ \ 0\ .6479569. \end{array}\right\}. \qquad (67)$$

These give

$$\boxed{0,0}\boxed{1,1}=g^2-16.8850240.g+\ 63.02615319170556; \qquad (68)$$
$$\boxed{0,0}\boxed{2,2}=g^2-18.6424288.g+\ 72.81534747185340; \qquad (69)$$
$$\boxed{0,0}\boxed{3,3}=g^2-23.1231203.g+\ 97.77394528773910; \qquad (70)$$
$$\boxed{1,1}\boxed{2,2}=g^2-24.3869412.g+147.90860736529860; \qquad (71)$$
$$\boxed{1,1}\boxed{3,3}=g^2-28.8676327.g+198.60659306350890; \qquad (72)$$
$$\boxed{2,2}\boxed{3,3}=g^2-30.6250375.g+229.45408138955850; \qquad (73)$$

$$\boxed{4,4}\boxed{5,5}=g^2-26.1085883.g+139.70173232312266; \qquad (74)$$
$$\boxed{4,4}\boxed{6,6}=g^2-10.2786276.g+\ 20.78112497747588; \qquad (75)$$
$$\boxed{4,4}\boxed{7,7}=g^2-\ \ 8.1603323.g+\ \ 4.86769547582026; \qquad (76)$$
$$\boxed{5,5}\boxed{6,6}=g^2-21.3624651.g+\ 51.44181484629338; \qquad (77)$$
$$\boxed{5,5}\boxed{7,7}=g^2-19.2441698.g+\ 12.04954446242401; \qquad (78)$$
$$\boxed{6,6}\boxed{7,7}=g^2-\ \ 3.4142091.g+\ \ 1.79241220013018; \qquad (79)$$

$$\left.\begin{array}{l} \boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=g^4-47.5100615.g^3+809.584727769664.g^2 \\ \qquad -5804.515976137376.g+14461.60808412039 \end{array}\right\}; \qquad (80)$$

$$\left.\begin{array}{l} \boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=g^4-29.5227974.g^3+230.634324285266.g^2 \\ \qquad -523.768277980466.g+250.403089395286 \end{array}\right\}. \qquad (81)$$

13. We shall now give the computation of equations (31—50), and also of (53) and (54) in full, because we can then readily correct them for any assumed changes in the adopted masses; and observe at the same time that the coefficients in equations (55—64) are independent of the masses of the planets, and consequently are not affected by their supposed variation.

Computation of A.

$$\begin{array}{rrl} \boxed{0,0}\boxed{2,2} = & g^2-18.642428800.g+ & 72.815347472 \\ -\boxed{0,0}\boxed{1,2}\boxed{2,3}\div\boxed{1,3} = & -21.562395123.g+ & 120.10805649 \\ -\boxed{2,2}\boxed{1,0}\boxed{0,3}\div\boxed{1,3} = & -\ \ 0.016959303.g+ & 0.22169494 \\ +\boxed{1,0}\boxed{0,2}\boxed{2,3}\div\boxed{1,3} = & + & 0.1803729465 \\ +\boxed{1,2}\boxed{2,0}\boxed{0,3}\div\boxed{1,3} = & + & 0.01596936 \\ -\boxed{2,0}\boxed{0,2} \qquad = & - & 0.00787688 \\ \text{Sum of terms} \quad A = & g^2-40.22178322.g+ & 193.3335643 \end{array}$$

Computation of A′.

$$
\begin{array}{lrl}
 & \boxed{0,0}\boxed{3,3} & =g^2-23.123120300.g+97.77394529 \\
- & \boxed{0,0}\boxed{1,3}\boxed{3,2}\div\boxed{1,2} & = -\ 0.014363307.g+\ 0.0800073 \\
- & \boxed{3,3}\boxed{1,0}\boxed{0,2}\div\boxed{1,2} & = -\ 0.008365164.g+\ 0.1468326 \\
+ & \boxed{1,0}\boxed{0,3}\boxed{3,2}\div\boxed{1,2} & = \qquad\qquad\qquad\quad +\ 0.0002436 \\
+ & \boxed{1,3}\boxed{3,0}\boxed{0,2}\div\boxed{1,2} & = \qquad\qquad\qquad\quad +\ 0.0000106 \\
- & \boxed{3,0}\boxed{0,3} & = \qquad\qquad\qquad\quad -\ 0.0000216 \\
\text{Sum of terms} & A' & = g^2-23.14584877.g+98.0010178
\end{array}
$$

Computation of A″.

$$
\begin{array}{lrl}
 & \boxed{0,0}\boxed{1,1} & =g^2-16.885024000.g+63.02615319 \\
- & \boxed{0,0}\boxed{2,1}\boxed{1,3}\div\boxed{2,3} & = -\ 1.136499372.g+\ 6.33059222 \\
- & \boxed{1,1}\boxed{2,0}\boxed{0,3}\div\boxed{2,3} & = -\ 0.000740612.g+\ 0.00837985 \\
+ & \boxed{2,0}\boxed{0,1}\boxed{1,3}\div\boxed{2,3} & = \qquad\qquad\qquad\quad +\ 0.00950700 \\
+ & \boxed{2,1}\boxed{1,0}\boxed{0,3}\div\boxed{2,3} & = \qquad\qquad\qquad\quad +\ 0.01927424 \\
- & \boxed{1,0}\boxed{0,1} & = \qquad\qquad\qquad\quad -\ 0.21770124 \\
\text{Sum of terms} & A'' & =g^2-18.022263984.g+69.17620526
\end{array}
$$

Computation of D.

$$
\begin{array}{lrl}
 & \boxed{1,1}\boxed{2,2} & =g^2-24.386941200.g+147.908607365 \\
- & \boxed{1,1}\boxed{0,2}\boxed{2,3}\div\boxed{0,3} & = -10.635634399g.+120.339738 \\
- & \boxed{2,2}\boxed{0,1}\boxed{1,3}\div\boxed{0,3} & = -12.836685468.g+167.803373 \\
+ & \boxed{0,1}\boxed{1,2}\boxed{2,3}\div\boxed{0,3} & = \qquad\qquad\qquad\quad +276.789684 \\
+ & \boxed{0,2}\boxed{2,1}\boxed{1,3}\div\boxed{0,3} & = \qquad\qquad\qquad\quad +\ 12.087392 \\
- & \boxed{2,1}\boxed{1,2} & = \qquad\qquad\qquad\quad -\ 24.5056485 \\
\text{Sum of terms} & D & =g^2-47.859261067.g+700.423146
\end{array}
$$

Computation of D′.

$$
\begin{array}{lrl}
 & \boxed{1,1}\boxed{3,3} & =g^2-28.867632700.g+198.606593 \\
- & \boxed{1,1}\boxed{0,3}\boxed{3,2}\div\boxed{0,2} & = -\ 0.029119780.g+\ 0.329484 \\
- & \boxed{3,3}\boxed{0,1}\boxed{1,2}\div\boxed{0,2} & = -26.024746029.g+456.808841 \\
+ & \boxed{0,1}\boxed{1,3}\boxed{3,2}\div\boxed{0,2} & = \qquad\qquad\qquad\quad +\ 0.373801 \\
+ & \boxed{0,3}\boxed{3,1}\boxed{1,2}\div\boxed{0,2} & = \qquad\qquad\qquad\quad +\ 0.033095 \\
- & \boxed{3,1}\boxed{1,3} & = \qquad\qquad\qquad\quad -\ 0.016324 \\
\text{Sum of terms} & D' & =g^2-\ 54.92149851.g+656.135490
\end{array}
$$

Computation of D″.

$$
\begin{array}{lrl}
 & \boxed{2,2}\boxed{3,3} & =g^2-30.625037500.g+229.4540814 \\
- & \boxed{2,2}\boxed{0,3}\boxed{3,1}\div\boxed{0,1} & = -\ 0.001271660.g+\ 0.0166233 \\
- & \boxed{3,3}\boxed{0,2}\boxed{2,1}\div\boxed{0,1} & = -\ 0.941628729.g+\ 16.5282815 \\
+ & \boxed{0,3}\boxed{3,2}\boxed{2,1}\div\boxed{0,1} & = \qquad\qquad\qquad\quad +\ 0.0274200 \\
+ & \boxed{0,2}\boxed{2,3}\boxed{3,1}\div\boxed{0,1} & = \qquad\qquad\qquad\quad +\ 0.0135249 \\
- & \boxed{3,2}\boxed{2,3} & = \qquad\qquad\qquad\quad -\ 0.3097073 \\
\text{Sum of terms} & D'' & = g^2-31.56793789.g+245.7302238
\end{array}
$$

Computation of A_1.

$$\begin{array}{rl}
\boxed{4,4}\,\boxed{6,6} & = g^2 - 10.278627600.g + 20.78112498 \\
-\boxed{4,4}\,\boxed{5,6}\,\boxed{6,7} \div \boxed{5,7} & = -\ 2.020545804.g + 15.17909859 \\
-\boxed{6,6}\,\boxed{5,4}\,\boxed{4,7} \div \boxed{5,7} & = -\ 2.294602698.g + \ 6.34744976 \\
+\boxed{5,4}\,\boxed{4,6}\,\boxed{6,7} \div \boxed{5,7} & = \qquad\qquad\qquad + \ 3.66870084 \\
+\boxed{5,6}\,\boxed{6,4}\,\boxed{4,7} \div \boxed{5,7} & = \qquad\qquad\qquad + \ 0.01012112 \\
-\boxed{6,4}\,\boxed{4,6} & = \qquad\qquad\qquad - \ 0.00800875 \\
\text{Sum of terms} & A_1 = g^2 - 14.593776102.g + 45.97848654
\end{array}$$

Computation of A_2.

$$\begin{array}{rl}
\boxed{4,4}\,\boxed{7,7} & = g^2 - 8.160332300.g + 4.86769548 \\
-\boxed{4,4}\,\boxed{5,7}\,\boxed{7,6} \div \boxed{5,6} & = -0.031614994.g + 0.23750370 \\
-\boxed{7,7}\,\boxed{5,4}\,\boxed{4,6} \div \boxed{5,6} & = -1.815697936.g + 1.17649401 \\
+\boxed{5,4}\,\boxed{4,7}\,\boxed{7,6} \div \boxed{5,6} & = \qquad\qquad\qquad + 0.07254385 \\
+\boxed{5,7}\,\boxed{7,4}\,\boxed{4,6} \div \boxed{5,6} & = \qquad\qquad\qquad + 0.00015836 \\
-\boxed{4,7}\,\boxed{7,4} & = \qquad\qquad\qquad - 0.00020013 \\
\text{Sum of terms} & A_2 = g^2 - 10.00764523.g + 6.35419527
\end{array}$$

Computation of A_3.

$$\begin{array}{rl}
\boxed{4,4}\,\boxed{5,5} & = g^2 - 26.108588300.g + 139.70173232 \\
-\boxed{4,4}\,\boxed{6,5}\,\boxed{5,7} \div \boxed{6,7} & = -\ 0.069351012.g + \ 0.52099084 \\
-\boxed{5,5}\,\boxed{6,4}\,\boxed{4,7} \div \boxed{6,7} & = -\ 0.005009102.g + \ 0.09315032 \\
+\boxed{6,4}\,\boxed{4,5}\,\boxed{5,7} \div \boxed{6,7} & = \qquad\qquad\qquad + \ 0.12592049 \\
+\boxed{6,5}\,\boxed{5,4}\,\boxed{4,7} \div \boxed{6,7} & = \qquad\qquad\qquad + \ 0.15913300 \\
-\boxed{5,4}\,\boxed{4,5} & = \qquad\qquad\qquad - \ 57.68249007 \\
\text{Sum of terms} & A_3 = g^2 - 26.182948414.g + \ 82.91843690
\end{array}$$

Computation of D_1.

$$\begin{array}{rl}
\boxed{5,5}\,\boxed{6,6} & = g^2 - 21.362465100.g + \ 51.44181485 \\
-\boxed{5,5}\,\boxed{4,6}\,\boxed{6,7} \div \boxed{4,7} & = -\ 1.598839244.g + \ 29.732355 \\
-\boxed{6,6}\,\boxed{4,5}\,\boxed{5,7} \div \boxed{4,7} & = -25.138334450.g + \ 69.538973 \\
+\boxed{4,5}\,\boxed{5,6}\,\boxed{6,7} \div \boxed{4,7} & = \qquad\qquad\qquad + \ 50.793156 \\
+\boxed{4,6}\,\boxed{6,5}\,\boxed{5,7} \div \boxed{4,7} & = \qquad\qquad\qquad + \ 0.110881 \\
-\boxed{6,5}\,\boxed{5,6} & = \qquad\qquad\qquad - \ 0.140127 \\
\text{Sum of terms} & D_1 = g^2 - 48.09963879.g + 201.477053
\end{array}$$

Computation of D_2.

$$\begin{array}{rl}
\boxed{5,5}\,\boxed{7,7} & = g^2 - 19.244169800.g + 12.04954446 \\
-\boxed{5,5}\,\boxed{4,7}\,\boxed{7,6} \div \boxed{4,6} & = -\ 0.039953699.g + \ 0.7429875 \\
-\boxed{7,7}\,\boxed{4,5}\,\boxed{5,6} \div \boxed{4,6} & = -31.768769968.g + 20.5847937 \\
+\boxed{4,5}\,\boxed{5,7}\,\boxed{7,6} \div \boxed{4,6} & = \qquad\qquad\qquad + \ 1.0043694 \\
+\boxed{4,7}\,\boxed{7,5}\,\boxed{5,6} \div \boxed{4,6} & = \qquad\qquad\qquad + \ 0.0027708 \\
-\boxed{7,5}\,\boxed{5,7} & = \qquad\qquad\qquad - \ 0.0021925 \\
\text{Sum of terms} & D_2 = g^2 - 51,052893467.g + 34.3822734
\end{array}$$

Computation of D_3.

$$\boxed{6,6}\boxed{7,7}=g^2-3.4142091000.g+1.7924122001$$
$$-\boxed{6,6}\boxed{4,7}\boxed{7,5}\div\boxed{4,5}=-0.0000872187.g+0.0002412688$$
$$-\boxed{7,7}\boxed{4,6}\boxed{6,5}\div\boxed{4,5}=-0.0044108379.g+0.0028580329$$
$$+\boxed{4,7}\boxed{7,6}\boxed{6,5}\div\boxed{4,5}=+0.0001762293$$
$$+\boxed{4,6}\boxed{6,7}\boxed{7,5}\div\boxed{4,5}=+0.0001394486$$
$$-\boxed{7,6}\boxed{6,7}=-0.0638795429$$
Sum of terms $D_3=g^2-3.4187071566.g+1.7319476368$

Computation of B, B', and B''.

$$\boxed{2,2}=g-13''.0721730$$
$$-\boxed{1,2}\boxed{2,3}\div\boxed{1,3}=-21\,.5623951$$
Sum $=B\div b=g-34\,.6345681;$

$$\boxed{3,3}=g-17''.552864500$$
$$-\boxed{1,3}\boxed{3,2}\div\boxed{1,2}=-\ 0\,.014363307$$
Sum $=B'\div b=g-17\,.567227807;$

$$\boxed{1,1}=g-11''.31476820$$
$$-\boxed{2,1}\boxed{1,3}\div\boxed{2,3}=-\ 1\,.13649937$$
Sum $=B''\div b=g-12\,.45126757.$

Computation of C, C', C'', and C'''.

$$-\boxed{2,2}=13''.0721730-g$$
$$+\boxed{0,2}\boxed{2,3}\div\boxed{0,3}=10\,.6356344$$
Sum $=23\,.7078074-g$

log. $\{\boxed{0,3}\div\boxed{1,3}\}=[9\,.176390].$

Therefore $C'=\{23''.7078074-g\}\times[9\,.1763990]b'.$

$$-\boxed{3,3}=17\,.55286450-g$$
$$+\boxed{0,3}\boxed{3,2}\div\boxed{0,2}=0\,.02911978$$
Sum $=17\,.58198428-g$

log. $\{\boxed{0,2}\div\boxed{1,2}\}=[8\,.8694654]$

Therefore $C'=\{17''.58198428-g\}\times[8\,.8694654]b'.$

$$+\boxed{0,3}\boxed{2,1}\div\boxed{2,3}=+\ 0.170596$$
$$-\boxed{0,1}=-\ 1.926868$$
Sum $=C''\div b'=-\ 1.756272$

Therefore $C''=-[0.2445917]b'.$

$$+\boxed{0,2}\boxed{3,1}\div\boxed{3,2}=+\ 0.084146$$
$$-\boxed{0,1}=-\ 1.926868$$
Sum $=C'''\div b'=-\ 1.842722$

Therefore $C'''=-[0.2654598]b'.$

Computation of B_1, B_2, and B_3.

$$\boxed{6,6}=g-\ 2''.7662522$$
$$-\boxed{5,6}\boxed{6,7}\div\boxed{5,7}=-\ 2\,.0205458$$
Sum $=B_1\div b_1=g-\ 4\,.7867980;$

$$\boxed{7,7}=g-\ 0''.64795690$$
$$-\boxed{5,7}\boxed{7,6}\div\boxed{5,6}=-\ 0\,.0316149937$$
Sum $=B_2\div b_1=g-\ 0\,.6795718937;$

$$\boxed{5,5}=g-18\,.5962129$$
$$-\boxed{6,5}\boxed{5,7}\div\boxed{6,7}=-\ 0\,.069351012$$
Sum $=B_3\div b_1=g-18\,.665563912.$

Computation of C_1, C_2, C_3, and C_4.

$$-\boxed{6,6}=2''.7662522\ -g$$
$$+\boxed{4,6}\boxed{6,7}\div\boxed{4,7}=1\,.598839244$$
Sum $=4\,.365091444-g$

log. $\{\boxed{4,7}\div\boxed{5,7}\}=[9\,.2840950]$

Therefore $C_1=\{4''.365091444-g\}\times[9\,.2840950]b_2.$

$$-\boxed{7,7}=0\,.6479569\ -g$$
$$+\boxed{4,7}\boxed{7,6}\div\boxed{4,6}=0\,.0399536996$$
Sum $=0\,.6879105996-g$

log. $\{\boxed{4,6}\div\boxed{5,6}\}=[9\,.1824311]$

Therefore $C_2=\{0''.6879105996-g\}\times[9\,.1824311]b_2.$

$$\boxed{4,7}\boxed{6,5}\div\boxed{6,7}=+\ 0.01333975$$
$$-\boxed{4,5}=-\ 4.835390$$
Sum $=C_3\div b_2=-\ 4.822050$

Therefore $C_3=-[0.6832317]b_2.$

$$\boxed{4,6}\boxed{7,5}-\boxed{7,6}=+\ 0.01055562$$
$$-\boxed{4,5}=-\ 4.835390$$
Sum $=C_4\div b_2=-\ 4.8248344$

Therefore $C_4=-[0.6834824]b_2.$

Computation of E, E', E'', and E'''.

$+\boxed{0,3}\boxed{1,2}\div\boxed{1,3}=+\ 0.8287048$
$-\boxed{0,2} \qquad =-\ 0.4087579$
Sum $=E\div b''=+\ 0.4199469$
Therefore $E=+[9.6231944]b''$.

$-\boxed{1,1}=+\ 11''.3147682-g$
$+\boxed{0,1}\boxed{1,3}\div\boxed{0,3}=+\ 12\ .8366855$
Sum $=+\ 24\ .1514537-g$
log. $\{\boxed{0,3}\div\boxed{2,3}\}=\ [\ 8\ .5847028]$
Therefore $E'=\ \{24''.1514537-g\}\times$
$[\ 8\ .5847028]b''$.

$-\boxed{3,3}=+\ 17''.5528645-g$
$+\boxed{0,3}\boxed{3,1}\div\boxed{0,1}=+\ \ 0\ .00127166$
Sum $=+\ 17\ .55413616-g$
log. $\{\boxed{0,1}\div\boxed{2,1}\}=\ [9\ .6375865]$
Therefore $E''=\ \{17''.55413616-g\}\times$
$[9\ .6375865]b''$.

$\boxed{0,1}\boxed{3,2}\div\boxed{3,1}=+\ 9.360164$
$-\boxed{0,2}=-\ 0.408758$
Sum $=E'''\div b''=+\ 8.951406$
Therefore $E'''=+[0.9518913]b''$.

Computation of E_1, E_2, E_3, and E_4.

$+\boxed{4,7}\boxed{5,6}\div\boxed{5,7}=+\ 0.03215367$
$-\boxed{4,6} \qquad =-\ 0.02544290$
Sum $=E_1\div b_3=+\ 0.00671077$
Therefore $E_1=\ [7.8267723]b_3$.

$-\boxed{5,5}=+18''.5962129-g$
$+\boxed{4,5}\boxed{5,7}\div\boxed{4,7}=+25\ .1383344$
Sum $=+43\ .7345473-g$
log. $\{\boxed{4,7}\div\boxed{6,7}\}=\ [8\ .2017618]$
Therefore $E_2=\ \{43\ .7345473-g\}\times$
$[8\ .2017618]b_3$.

$-\boxed{7,7}=+0\ .6479569-g$
$+\boxed{4,7}\boxed{7,5}\div\boxed{4,5}=+0\ .000047219$
Sum $=+0\ .648004119-g$
log. $\{\boxed{4,5}\div\boxed{6,5}\}=\ [0\ .7610455]$
Therefore $E_3=\ \{0''.648004119-g\}\times$
$[0\ .7610455]b_3$.

$\boxed{4,5}\boxed{7,6}\div\boxed{7,5}=+11.655051$
$-\boxed{4,6} \qquad =-\ 0.025443$
Sum $=E_4\div b_3=+11.629608$
Therefore $E_4=\ [1.0655652]b_3$.

Computation of F, F', F'', and F'''.

$\boxed{0,2}\boxed{1,3}\div\boxed{1,2}=+\ 0.004346915$
$-\boxed{0,3} \qquad =-\ 0.008812816$
Sum $=F\div b'''=-\ 0.004465901$
Therefore $F=-[7.6499091]b'''$.

$\boxed{0,1}\boxed{2,3}\div\boxed{2,1}=+\ 0.09954018$
$-\boxed{0,3} \qquad =-\ 0.00881282$
Sum $=F'\div b'''=+\ 0.09072736$
Therefore $F'=+[8.9577383]b'''$.

$-\boxed{2,2}=+13''.07217300-g$
$+\boxed{0,2}\boxed{2,1}\div\boxed{0,1}=+\ 0\ .94162873$
Sum $=+14\ .01380173-g$
log. $\{\boxed{0,1}\div\boxed{3,1}\}=\ [0\ .8407439]$
Therefore $F''=\ \{14''.01380173-g\}\times$
$[0\ .8407439]b'''$.

$-\boxed{1,1}=+11''.3147682-g$
$+\boxed{0,1}\boxed{1,2}\div\boxed{0,2}=+26\ .0247460$
Sum $=+37\ .3395142-g$
log. $\{\boxed{0,2}\div\boxed{3,2}\}=\ [9\ .4809266]$
Therefore $F'''=\ \{37''.3395142-g\}\times$
$[9\ .4809266]b'''$.

Computation of F_1, F_2, F_3, and F_4.

$\boxed{4,6}\boxed{5,7}\div\boxed{5,6}=+\ 0.004099602$
$-\boxed{4,7} \qquad =-\ 0.005180905$
Sum $=F_1\div b_4=-\ 0.001081303$
Therefore $F_1=-[7.0339474]b_4$.

$\boxed{4,5}\boxed{6,7}\div\boxed{6,5}=+\ 1.8779726$
$-\boxed{4,7} \qquad =-\ 0.0051809$
Sum $=F_2\div b_4=+\ 1.8727917$
Therefore $F_2=+[0.2724895]b_4$.

$-\boxed{6,6}=+2''.7662522-g$
$+\boxed{4,6}\boxed{6,5}\div\boxed{4,5}=+0\ .004410838$
Sum $=+2\ .770663038-g$
log. $\{\boxed{4,5}\div\boxed{7,5}\}=\ [1\ .7737962]$
Therefore $F_3=\ \{2''.770663038-g\}\times$
$[1\ .7737962]b_4$.

$-\boxed{5,5}=+18\ .5962129-g$
$+\boxed{4,5}\boxed{5,6}\div\boxed{4,6}=+31\ .76876997$
Sum $=+50\ .364982867-g$
log. $\{\boxed{4,6}\div\boxed{7,6}\}=\ [\ 9\ .11284867]$
Therefore $F_4=\ \{50''.3649829-g\}\times$
$[\ 9\ .1128486]b_4$.

Computation of the Equations of the Fourth Degree.

$$-\boxed{2,1}\boxed{1,2}\boxed{0,0}\boxed{3,3}=-24.50564851.g^2+566.64705848.g-2396.01393568$$
$$-\boxed{3,2}\boxed{2,3}\boxed{0,0}\boxed{1,1}=-0.30970734.g^2+5.22941589.g-19.51966234$$
$$-\boxed{1,0}\boxed{0,1}\boxed{2,2}\boxed{3,3}=-0.21770124.g^2+6.66710860.g-49.95243773$$
$$-\boxed{3,1}\boxed{1,3}\boxed{0,0}\boxed{2,2}=-0.01632389.g^2+0.30431699.g-1.18862983$$
$$-\boxed{2,0}\boxed{0,2}\boxed{1,1}\boxed{3,3}=-0.00787688.g^2+0.22738681.g-1.56439984$$
$$-\boxed{3,0}\boxed{0,3}\boxed{1,1}\boxed{2,2}=-0.00002157.g^2+0.00052594.g-0.00318986$$
$$+2\boxed{3,2}\boxed{2,1}\boxed{1,3}\boxed{0,0}=+0.70396439.g-3.92126177$$
$$+2\boxed{2,0}\boxed{0,1}\boxed{1,2}\boxed{3,3}=+0.40998748.g-7.19645470$$
$$+2\boxed{3,1}\boxed{1,0}\boxed{0,3}\boxed{2,2}=+0.00055368.g-0.00723785$$
$$+2\boxed{3,0}\boxed{0,2}\boxed{2,3}\boxed{1,1}=+0.00045875.g-0.00519060$$
$$+\boxed{1,0}\boxed{0,1}\boxed{3,2}\boxed{2.3}=+0.06742367$$
$$+\boxed{2,1}\boxed{1,2}\boxed{3,0}\boxed{0,3}=+0.00052850$$
$$+\boxed{2,0}\boxed{0,2}\boxed{3,1}\boxed{1,3}=+0.00012858$$
$$-2\boxed{3,0}\boxed{0,1}\boxed{1,2}\boxed{2,3}=-0.01193875$$
$$-2\boxed{3,1}\boxed{1,0}\boxed{0,2}\boxed{2,3}=-0.00588878$$
$$-2\boxed{3,0}\boxed{0,2}\boxed{2,1}\boxed{1,3}=-0.00052136$$

Sum of preceding terms $=-25.05727943.g^2+580.19077701.g+2479.32266834$

Add $\boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=$

$$g^4-47.5100615.g^3+809.58472777.g^2-5804.51597614.g-14461.60808412$$

Sum is the value of equation (53).

Whence we get
$$\left.\begin{array}{l} g^4-47.5100615.g^3+784.52744834.g^2 \\ -5224.32519913.g+11982.28541576 \end{array}\right\}=(\chi,\chi_1,\chi_2,\chi_3). \quad (82)$$

In like manner,

$$-\boxed{5,4}\boxed{4,5}\boxed{6,6}\boxed{7,7}=-57.682490072.g^2+196.940082515.g-103.390798939$$
$$-\boxed{6,5}\boxed{5,6}\boxed{4,4}\boxed{7,7}=-0.140126895.g^2+1.143482030.g-0.682095055$$
$$-\boxed{7,6}\boxed{6,7}\boxed{4,4}\boxed{5,5}=-0.063879543.g^2+1.667804687.g-0.924082807$$
$$-\boxed{6,4}\boxed{4,6}\boxed{5,5}\boxed{7,7}=-0.008008749.g^2+0.154121732.g-0.096501781$$
$$-\boxed{7,5}\boxed{5,7}\boxed{4,4}\boxed{6,6}=-0.002192532.g^2+0.022536218.g-0.045563277$$
$$-\boxed{7,4}\boxed{4,7}\boxed{5,5}\boxed{6,6}=-0.000200132.g^2+0.004275316.g-0.010295162$$
$$+2\boxed{6,4}\boxed{4,5}\boxed{5,6}\boxed{7,7}=+0.508856230.g-0.329716905$$
$$+2\boxed{7,5}\boxed{5,4}\boxed{4,7}\boxed{6,6}=+0.010061979.g-0.027833971$$
$$+2\boxed{7,6}\boxed{6,5}\boxed{5,7}\boxed{4,4}=+0.008860222.g-0.066561313$$
$$+2\boxed{7,4}\boxed{4,6}\boxed{6,7}\boxed{5,5}=+0.000639958.g-0.011900801$$
$$+\boxed{5,4}\boxed{4,5}\boxed{7,6}\boxed{6,7}=+3.684731101$$
$$+\boxed{6,5}\boxed{5,6}\boxed{7,4}\boxed{4,7}=+0.000028044$$
$$+\boxed{6,4}\boxed{4,6}\boxed{7,5}\boxed{5,7}=+0.000017559$$
$$-2\boxed{7,4}\boxed{4,5}\boxed{5,6}\boxed{6,7}=-0.020330689$$
$$-2\boxed{7,5}\boxed{5,4}\boxed{4,6}\boxed{6,7}=-0.016087486$$
$$-2\boxed{7,4}\boxed{4,6}\boxed{6,5}\boxed{5,7}=-0.000044382$$

Sum of preceding terms $=-57.896897923.g^2+200.460720887.g-109.937035864$

Add $\boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=$

$$g^4-29.5227974.g^3+230.634324285.g^2-523.768277980.g+250.403089395$$

Sum is value of equation (54).

Whence we get
$$\left.\begin{array}{l} g^4-29.5227974.g^3+172.737426362.g^2 \\ -323.307557093.g+140.466053531 \end{array}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7). \quad (83)$$

Equations (55–64) reduced to numbers are as follows, in which the numbers inclosed in brackets are logarithms.

$$N' = [0.8236010]\frac{AN+f}{D} = [1.1305346]\frac{A'N+f'}{D'}; \quad (84)$$

$$N'' = [1.4152972]\frac{A''N+f''}{D} = [0.3624135]\frac{A'N+f'''}{D'}; \quad (85)$$

$$N''' = [9.1592561]\frac{AN+f^{IV}}{D''} = [0.5190734]\frac{A''N+f^{V}}{D'}; \quad (86)$$

$$N = \frac{[9.1763990]Df' - [8.8694654]D'f}{[8.8694654]AD' - [9.1763990]A'D}; \quad (87)$$

$$N = \frac{[8.5847028]Df''' - [9.6375865]D'f''}{[9.6375865]A''D' - [8.5847028]A'D}; \quad (88)$$

$$N = \frac{[0.8407439]D'f^{V} - [9.4809266]D'f^{IV}}{[9.4809266]AD' - [0.8407439]A''D''}; \quad (89)$$

$$N^{V} = [0.7159050]\frac{A_1N^{IV}+f_1}{D_1} = [0.8175689]\frac{A_2N^{IV}+f_2}{D_2}; \quad (90)$$

$$N^{VI} = [1.7982382]\frac{A_3N^{IV}+f_3}{D_1} = [9.2389545]\frac{A_2N^{IV}+f_4}{D_3}; \quad (91)$$

$$N^{VII} = [8.2262038]\frac{A_1N^{IV}+f_5}{D_3} = [0.8871514]\frac{A_3N^{IV}+f_6}{D_2}; \quad (92)$$

$$N^{IV} = \frac{[9.2840950]D_1f_2 - [9.1824311]D_2f_1}{[9.1824311]A_1D_2 - [9.2840950]A_2D_1}; \quad (93)$$

$$N^{IV} = \frac{[8.2017618]D_1f_4 - [0.7610455]D_3f_3}{[0.7610455]A_3D_3 - [8.2017618]A_2D_1}; \quad (94)$$

$$N^{IV} = \frac{[1.7737962]D_3f_6 - [9.1128486]D_2f_5}{[9.1128486]A_1D_2 - [1.7737962]A_3D_3}; \quad (95)$$

$$\left.\begin{aligned} &\left\{[9.9882719]D'f - [0.2952055]Df'\right\}\frac{1}{N} \\ =&\left\{[8.9873765]Df''' - [0.0402602]D''f''\right\}\frac{1}{N} \\ =&\left\{[8.6595749]D'f^{IV} - [0.0193922]D''f^{V}\right\}\frac{1}{N} \end{aligned}\right\} = (\chi, \chi_1, \chi_2, \chi_3); \quad (96)$$

$$\left.\begin{aligned} &\left\{[0.5787943]D_2f_1 - [0.6804582]D_1f_2\right\}\frac{1}{N^{IV}} \\ =&\left\{[7.4419161]D_1f_4 - [0.0011998]D_3f_3\right\}\frac{1}{N^{IV}} \\ =&\left\{[7.3400015]D_2f_5 - [0.0009491]D_3f_6\right\}\frac{1}{N^{IV}} \end{aligned}\right\} = (\chi_4, \chi_5, \chi_6, \chi_7); \quad (97)$$

We shall here repeat and number the equations which we have computed, for convenience of future reference. By this means we shall obtain the following

Fundamental Equations for the Adopted Masses.

$$A = g^2 - 40.22178322\ .g + 193.3335643; \quad (98)$$
$$A' = g^2 - 23.14584877\ .g + 98.0010178; \quad (99)$$
$$A'' = g^2 - 18.02226398\ .g + 69.17620526; \quad (100)$$
$$A_1 = g^2 - 14.593776102\ .g + 45.97848654; \quad (101)$$
$$A_2 = g^2 - 10.00764523\ .g + 6.35419527; \quad (102)$$
$$A_3 = g^2 - 26.182948414\ .g + 82.91843690; \quad (103)$$

$$D = g^2 - 47.859261067\ .g + 700.423146; \quad (104)$$
$$D' = g^2 - 54.92149851\ .g + 656.135490; \quad (105)$$
$$D'' = g^2 - 31.56793789\ .g + 245.7302238; \quad (106)$$
$$D_1 = g^2 - 48.09963879\ .g + 201.477053; \quad (107)$$
$$D_2 = g^2 - 51.052893467\ .g + 34.3822734; \quad (108)$$
$$D_3 = g^2 - 3.4187071566.g + 1.7319476368; \quad (109)$$

$$\left.\begin{array}{l} B = \{g - 34.6345681\}b;\ B' = \{g - 17.567227807\}b; \\ \qquad\qquad\qquad\qquad B'' = \{g - 12.4512675772\}b; \end{array}\right\} \quad (110)$$

$$\left.\begin{array}{l} C = \{23.7078074 - g\}[9.1763990]b'; \\ C' = \{17.58198428 - g\}[8.8694654]b'; \\ C'' = -[0.2445917]b'; \\ C''' = -[0.2654598]b'; \end{array}\right\} \quad (111)$$

$$\left.\begin{array}{l} E = +[9.6231944]b'' \\ E' = \{24.1514537 - g\}[8.5847028]b''; \\ E'' = \{17.55413616 - g\}[9.6375865]b''; \\ E''' = +[0.9518913]b''; \end{array}\right\} \quad (112)$$

$$\left.\begin{array}{l} F = -[7.6499091]b'''; \\ F' = +[8.9577383]b'''; \\ F'' = \{14.01380173 - g\}[0.8407439]b'''; \\ F''' = \{37.3395142 - g\}[9.4809266]b'''; \end{array}\right\} \quad (113)$$

$$\left.\begin{array}{l} B_1 = \{g - 4.7867980\}b_1;\ B_2 = \{g - 0.6795718937\}b_1; \\ \qquad\qquad\qquad\qquad B_3 = \{g - 18.665563912\}b_1; \end{array}\right\} \quad (114)$$

$$\left.\begin{array}{l} C_1 = \{4.365091444 - g\}[9.2840950]b_2; \\ C_2 = \{0.6879105996 - g\}[9.1824311]b_2; \\ C_3 = -[0.6882317]b_2; \\ C_4 = -[0.6834824]b_2; \end{array}\right\} \quad (115)$$

$$\left.\begin{array}{l} E_1 = +[7.8267723]b_3; \\ E_2 = \{43.7345473 - g\}[8.2017618]b_3; \\ E_3 = \{0.648004119 - g\}[0.7610455]b_3; \\ E_4 = +[1.0655652]b_3; \end{array}\right\} \quad (116)$$

$$\left.\begin{array}{l} F_1 = -[7.0339474]b_4; \\ F_2 = +[0.2724895]b_4; \\ F_3 = \{2.770663038 - g\}[1.7737962]b_4; \\ F_4 = \{50.3649829 - g\}[9.1128486]b_4; \end{array}\right\} \quad (117)$$

$$\left.\begin{aligned}
b&=\{0.1489647\ldots[9.1730832]\}N^{\text{iv}}+[7.5931242]N^{\text{v}}+[95.5264270]N^{\text{vi}}\\
&\qquad+[94.8701084]N^{\text{vii}}\\
b'&=\{0.7286137\ldots[9.8624973]\}N^{\text{iv}}+[8.2750461]N^{\text{v}}+[96.2059893]N^{\text{vi}}\\
&\qquad+[95.5492142]N^{\text{vii}}\\
b''&=\{1.690254\ \ldots[0.2279520]\}N^{\text{iv}}+[8.6307197]N^{\text{v}}+[96.5586316]N^{\text{vi}}\\
&\qquad+[95.9012715]N^{\text{vii}}\\
b'''&=\{5.307482\ \ldots[0.7248886]\}N^{\text{iv}}+[9.0992151]N^{\text{v}}+[97.0186122]N^{\text{vi}}\\
&\qquad+[96.3596252]N^{\text{vii}}
\end{aligned}\right\}\quad(118)$$

$$\left.\begin{aligned}
b_1&=\{0.000008754742\ \ldots[94.9422434]\}N+[96.8635004]N'\\
&\qquad+[97.3236905]N''+[97.0504994]N''';\\
b_2&=\{0.0000005681531\ \ldots[93.7544654]\}N+[95.6682302]N'\\
&\qquad+[96.1186392]N''+[95.8170069]N''';\\
b_3&=\{0.00000002443536\ \ldots[92.3880187]\}N+[94.2994239]N'\\
&\qquad+[94.7468016]N''+[94.4366545]N''';\\
b_4&=\{0.000000003249184\ldots[91.5117743]\}N+[93.4227230]N'\\
&\qquad+[93.8695157]N''+[93.5577417]N''';
\end{aligned}\right\}\quad(119)$$

We have given the natural and logarithmic coefficients of N and N^{iv}, in the values of b, b', b'', &c., because the values of the other quantities are determined in functions of these, and they will therefore be wanted.

14. If we now suppose the second members of equations (82) and (83) to be equal to nothing, we shall obtain the following values of g, g_1, g_2, &c.:—

$$\begin{aligned}
g&=\ 5''.46370645; & g_4&=\ 0''.61668516;\\
g_1&=\ 7\ .24769852; & g_5&=\ 2\ .72772365;\\
g_2&=17\ .01424590; & g_6&=\ 3\ .71780374;\\
g_3&=17\ .78441063; & g_7&=22\ .46058485.
\end{aligned}$$

We must now transform equation (82) in four others whose roots shall be respectively less by g, g_1, g_2, and g_3. Putting δg for the root of the first transformed equation, we shall have the following equation to determine δg.

$$\delta g^4-25.6552\delta g^3+184.8969\delta g^2-253.8812\delta g+\chi=0.$$

But since δg, δg_1, &c. are very small quantities, we may neglect δg^3, δg^4, in these transformed equations, and we shall then get by dividing by the coefficients of δg

$$\delta g=\frac{\chi}{253.8812}+\frac{184.8969}{253.8812}.\delta g^2.$$

We may first neglect the last term of this equation, and we shall obtain a first approximation to the value of δg, with which we can compute the last term of the equation. If we perform the same process with the other roots, we shall obtain the following equations for determining δg, δg_1, δg_2, &c.

$$\delta g=+[7.59537\ ]\chi+[9.8623\ ]\delta g^2\ ; \qquad (120)$$
$$\delta g_1=-[7.73616\ ]\chi_1-[9.5602\ ]\delta g_1^2; \qquad (121)$$
$$\delta g_2=+[8.061074]\chi_2+[0.04511]\delta g_2^2; \qquad (122)$$
$$\delta g_3=-[8.00000\ ]\chi_3-[0.16856]\delta g_3^2; \qquad (123)$$
$$\delta g_4=+[7.84466\ ]\chi_4+[9.92529]\delta g_4^2; \qquad (124)$$
$$\delta g_5=-[8.38464\ ]\chi_5+[9.76863]\delta g_5^2; \qquad (125)$$
$$\delta g_6=+[8.23997\ ]\chi_6-[0.10692]\delta g_6^2; \qquad (126)$$
$$\delta g_7=-[6.09265\ ]\chi_7-[9.17554]\delta g_7^2. \qquad (127)$$

15. We shall now give the computation of N', N'', &c., for the root g. Using $g=5''.46370645$, we get $g^2=29.8520882$. Substituting these in equations (98–109), we get the following values of A, A', &c., D, D', &c.

$A=+\ 3.425636$	log. 0.5347412;	$D=+468.78628$	log. 2.6709749;
$A'=+\ 1.390983$	" 0.1433218;	$D'=+385.91263$	" 2.5864890;
$A''=+\ 0.5599336$	" 9.7481365;	$D''=+103.10436$	" 2.0132764;
$A_1=-\ 3.9055338$	" $0.5916804n$;	$D_1=-\ 31.473166$	" $1.4979404n$;
$A_2=-18.472552$	" $1.2665269n$;	$D_2=-214.70366$	" $2.3318394n$;
$A_3=-30.285419$	" $1.4812336n$;	$D_3=+\ 12.899760$	" 1.1105816.

Substituting A, A', A'', D, D', D'', in equations (84–86), and neglecting f, f', &c., we shall get the following computation of N', N'', and N'''.

Computation of N' & N''.		*Computation of N' & N'''.*		*Computation of N'' & N'''.*	
constant	log. 0.8236010	constant	log. 1.1305346	constant	log. 0.3624135
A	" 0.5347412	A'	" 0.1433218	A'	" 0.1433218
$1\div D$	" 7.3290251	$1\div D'$	" 7.4135110	$1\div D''$	" 7.9867236
A''	" 9.7481365	A''	" 9.7481365	A	" 0.5347412
constant	" 1.4152972	constant	" 0.5190734	constant	" 9.1592561
N'	" 8.6873673	N'	" 8.6873674	N''	" 8.4924589
N''	" 8.4924588	N'''	" 7.6807209	N'''	" 7.6807209

The computation of these quantities is thus seen to be correct, since independent formulæ give the same values. We may now use these values of N', N'', and N''', in equations (119), and we shall obtain the following values of b_1, b_2, b_3, and b_4.

$b_1=+0.0001151786N$	log. 96.0613718;
$b_2=+0.000007234609N$	" 94.8594150;
$b_3=+0.0000003080273N$	" 93.4885892;
$b_4=+0.00000004087918N$	" 92.6115022;

Equations (114–117) will now give,

$B_1=+0.00007796538N$	log. 95.8919018;
$B_2=+0.0005510300N$	" 96.7411752;
$B_3=-0.001520571N$	" 97.1820068n;
$C_1=-0.00000152882N$	" 94.1843555n;
$C_2=-0.00000525886N$	" 94.7208918n;
$C_3=-0.0000348856N$	" 95.5426467n;
$C_4=-0.0000349058N$	" 95.5428974n;
$E_1=+0.000000002067N$	" 91.3153615;
$E_2=+0.000000187594N$	" 93.2732190;
$E_3=-0.00000855646N$	" 94.9322943n;
$E_4=+0.00000358228N$	" 94.5541593;
$F_1=-0.0000000000442N$	" 89.6454496n;
$F_2=+0.000000076558N$	" 92.8839917;
$F_3=-0.00000653946N$	" 94.8155417n;
$F_4=+0.000000238018N$	" 93.3766095.

Substituting these quantities in equations (52) we get,

$$\begin{aligned}
f_1&=+0.00007643863N \quad \text{log. } 95.8833129;\\
f_2&=+0.0005457711N \quad \text{"} \quad 96.7370106;\\
f_3&=-0.001555269N \quad \text{"} \quad 97.1918055n;\\
f_4&=+0.0005425501N \quad \text{"} \quad 96.7344399;\\
f_5&=+0.00007500820N \quad \text{"} \quad 95.8751088;\\
f_6&=-0.001555239N \quad \text{"} \quad 97.1917972n.
\end{aligned}$$

With these values of f_1, f_2, &c., and A_1, A_2, A_3, D_1, D_2, and D_3, either of the equations (93–95) will give

$$N^{IV}=-0.00005102365N \quad \text{log. } 95.7077716n.$$

Therefore we shall easily find

$$A_1N^{IV}=+0.0001992746N, \quad A_2N^{IV}=+0.0009425372N, \quad A_3N^{IV}=+0.001545273N.$$

Then

$$\begin{aligned}
&A_1N^{IV}+f_1=+0.0002757132N; &&A_2N^{IV}+f_2=+0.0014883083N;\\
& &&A_3N^{IV}+f_3=-0.000009996N;\\
&A_1N^{IV}+f_5=+0.0002742828N; &&A_2N^{IV}+f_4=+0.0014850873N;\\
& &&A_3N^{IV}+f_6=-0.000009966N.
\end{aligned}$$

Equations (90–92) will now give the following:—

Computation of N^V and N^{VI}.

constant	log.	0.7159050
$A_1N^{IV}+f_1$	"	96.4404576
$1\div D_1$	"	98.5020596n
$A_3N^{IV}+f_3$	"	94.9998262n
constant	"	1.7982382
N^V	"	95.6584222n
N^{VI}	"	95.3001240

Computation of N^V and N^{VII}.

constant	log.	0.8175689
$A_2N^{IV}+f_2$	"	97.1726929
$1\div D_2$	"	97.6681606n
$A_3N^{IV}+f_6$	"	94.9985209n
constant	"	0.8871514
N^V	"	95.6584224n
N^{VII}	"	93.5538329

Computation of N^{VI} and N^{VII}.

constant	log.	99.2389545
$A_2N^{IV}+f_4$	"	97.1717519
$1\div D_3$	"	98.8894184
$A'N^{IV}+f_5$	"	96.4381985
constant	"	98.2262038
N^{VI}	"	95.3001248
N^{VII}	"	93.5538207

The small differences in the different values of N^{VI} and N^{VII}, in this example, are owing to the circumstance that A_3N^{IV} is nearly equal to f_3 and f_6, and has a contrary sign, which renders $A_3N^{IV}+f_3$ and $A_3N^{IV}+f_6$ very small quantities. We must therefore reject these values and use those depending on f_4 and f_5.

Having found the values of N^{IV}, N^{V}, N^{VI}, and N^{VII}, we must now substitute them in equations (118), and we shall get the following values of b, b', b'', and b'''.

$$
\begin{array}{lll}
b &= -0.000007745108N & \text{log. } 94.8890274n; \\
b' &= -0.00003787069N & \text{“ } 95.5783033n; \\
b'' &= -0.00008781746N & \text{“ } 95.9435809n; \\
b''' &= -0.0002754383N & \text{“ } 96.4400243n.
\end{array}
$$

Substituting these quantities, together with g, in equations (110–113), we get

$$
\begin{array}{lll}
B &= +0.0002259314N & \text{log. } 96.3539766; \\
B' &= +0.00009374307N & \text{“ } 95.9719391; \\
B'' &= +0.00005411940N & \text{“ } 95.7333530; \\
C &= -0.0001037110N & \text{“ } 96.0158248n; \\
C' &= -0.00003397893N & \text{“ } 95.5312097n; \\
C'' &= +0.00006651123N & \text{“ } 95.8228950; \\
C''' &= +0.00006978516N & \text{“ } 95.8437631; \\
E &= -0.0000368787N & \text{“ } 95.5667753n; \\
E' &= -0.00006307260N & \text{“ } 95.7998407n; \\
E'' &= -0.0004609025N & \text{“ } 96.6636091n; \\
E''' &= -0.0007860900N & \text{“ } 96.8954722n; \\
F &= +0.00000123008N & \text{“ } 94.0899334; \\
F' &= -0.00002498980N & \text{“ } 95.3977626n; \\
F'' &= -0.01632072N & \text{“ } 98.2127394n; \\
F''' &= -0.002657125N & \text{“ } 97.4244119n.
\end{array}
$$

These quantities being substituted in equations (51) we shall obtain

$$
\begin{array}{lll}
f &= +0.0000853417N & \text{log. } 95.9311615; \\
f' &= +0.00006099422N & \text{“ } 95.7852886; \\
f'' &= +0.00005755803N & \text{“ } 95.7601059; \\
f''' &= -0.0003921492N & \text{“ } 96.5934513n; \\
f^{IV} &= -0.01688088N & \text{“ } 98.2273951n; \\
f^{V} &= -0.002533220N & \text{“ } 97.4036729n.
\end{array}
$$

Now substituting f, f', &c., D, D', D'', in equations (96), each one of them will give

$$\chi = +0.0243677.$$

This value of χ is to be substituted in equation (120), and we shall find

$$\delta g = +0''.00009598,$$

and consequently

$$g = 5''.4638024.$$

We have thus obtained a first approximation to the values of the required quantities. In order to get a closer approximation we must repeat the whole computation by using the corrected value of g, and the values of f, f', f'', &c., already found. But instead of computing new values of A, A', A'', &c., D, D', D'', &c., by the formulæ (98–109), it is much more convenient, as well as less laborious, to compute corrections to these quantities depending on δg by formulæ similar to the following:—

$$
\left.
\begin{array}{l}
\delta A = 2g\delta g + \delta g^2 - \delta g 40.22178 \ldots; \\
\delta A' = 2g\delta g + \delta g^2 - \delta g 23.14584 \ldots;
\end{array}
\right\} \quad (128)
$$

Then we shall have

$$\left.\begin{array}{l}\text{Corrected } A = A + \delta A\,; \\ \text{“} \quad A' = A' + \delta A'; \\ \text{\&c.}\end{array}\right\} \qquad (129)$$

In this manner we shall obtain for the root g,

$$\begin{array}{lll}
g = & 5''.4638027\,; & \\
N' & = +0\,.04864325N & \log.\ 98.6870226\,; \\
N'' & = +0\,.03104381N & \text{“}\ \ 98.4919751\,; \\
N''' & = +0\,.004766696N & \text{“}\ \ 97.6782174\,; \\
N^{IV} & = -0\,.00005096246N & \text{“}\ \ 95.7072504n\,; \\
N^{V} & = -0\,.00004548871N & \text{“}\ \ 95.6579036n\,; \\
N^{VI} & = +0\,.00001992532N & \text{“}\ \ 95.2994053\,; \\
N^{VII} & = +0\,.000000357473N & \text{“}\ \ 93.5532432.
\end{array}$$

By performing similar calculations with reference to g_1, g_2, g_3, &c., we shall obtain the following values:—

$$\begin{array}{lll}
g_1 = & 7''.2484269\,; & \\
N_1' & = -0\,.7493120N_1 & \log.\ \ 9.8746627n\,; \\
N_1'' & = -0\,.5714185N_1 & \text{“}\ \ 9.7569543n\,; \\
N_1''' & = -0\,.0946700N_1 & \text{“}\ \ 8.9762124n\,; \\
N_1^{IV} & = +0\,.0003959309N_1 & \text{“}\ \ 96.5976194\,; \\
N_1^{V} & = +0\,.0004045142N_1 & \text{“}\ \ 96.6069338\,; \\
N_1^{VI} & = -0\,.0001020591N_1 & \text{“}\ \ 96.0088516n\,; \\
N_1^{VII} & = -0\,.000004173243N_1 & \text{“}\ \ 94.6204737n.
\end{array}$$

$$\begin{array}{lll}
g_2 = & 17''.0143734\,; & \\
N_2' & = -\ 7\,.644764N_2 & \log.\ \ 0.8833640n\,; \\
N_2'' & = +\ 7\,.708429N_2 & \text{“}\ \ 0.8869658\,; \\
N_2''' & = +15\,.38340N_2 & \text{“}\ \ 1.1870524\,; \\
N_2^{IV} & = -\ 0\,.0007220048N_2 & \text{“}\ \ 96.8585401n\,; \\
N_2^{V} & = -\ 0\,.004362647N_2 & \text{“}\ \ 97.6397511n\,; \\
N_2^{VI} & = +\ 0\,.000267212N_2 & \text{“}\ \ 96.4269521\,; \\
N_2^{VII} & = +\ 0\,.00001963376N_2 & \text{“}\ \ 95.2930035.
\end{array}$$

$$\begin{array}{lll}
g_3 = & 17''.7844562\,; & \\
N_3' & = -\ 8\,.277302N_3 & \log.\ \ 0.9178888n\,; \\
N_3'' & = +10\,.207100N_3 & \text{“}\ \ 1.0089024\,; \\
N_3''' & = -49\,.61991N_3 & \text{“}\ \ 1.6956560n\,; \\
N_3^{IV} & = +\ 0\,.0006685073N_3 & \text{“}\ \ 96.8251062\,; \\
N_3^{V} & = +\ 0\,.006909513N_3 & \text{“}\ \ 97.8394474\,; \\
N_3^{VI} & = -\ 0\,.0003927175N_3 & \text{“}\ \ 96.5940804n\,; \\
N_3^{VII} & = -\ 0\,.00002909962N_3 & \text{“}\ \ 95.4638874n.
\end{array}$$

$g_4 =$	$0''.6166849;$	
N_4	$=+\ 0.1213144 N_4^{IV}$	log. 9.0839135;
N_4'	$=+\ 0.1843578 N_4^{IV}$	" 9.2656615;
N_4''	$=+\ 0.2135417 N_4^{IV}$	" 9.3294826;
N_4'''	$=+\ 0.3454671 N_4^{IV}$	" 9.5384067;
N_4^{V}	$=+\ 1.127803 N_4^{IV}$	" 0.0522322;
N_4^{VI}	$=+\ 24.49974 N_4^{IV}$	" 1.3891617;
N_4^{VII}	$=+157.8892 N_4^{IV}$	" 2.1983503.
$g_5 =$	$2''.7276592;$	
N_5	$=+\ 0.2925180 N_5^{IV}$	log. 9.4661526;
N_5'	$=+\ 0.2866292 N_5^{IV}$	" 9.4573205;
N_5''	$=+\ 0.3000735 N_5^{IV}$	" 9.4772276;
N_5'''	$=+\ 0.3995364 N_5^{IV}$	" 9.6015563;
N_5^{V}	$=+\ 0.9103640 N_5^{IV}$	" 9.9592151;
N_5^{VI}	$=+15.29769 N_5^{IV}$	" 1.1846258;
N_5^{VII}	$=-\ 1.497460 N_5^{IV}$	" 0.1753554n.
$g_6 =$	$3''.7166075;$	
N_6	$=+0.5675117 N_6^{IV}$	log. 9.7539748;
N_6'	$=+0.3847378 N_6^{IV}$	" 9.5851649;
N_6''	$=+0.3786204 N_6^{IV}$	" 9.5782040;
N_6'''	$=+0.4354816 N_6^{IV}$	" 9.6389698;
N_6^{V}	$=+0.7901060 N_6^{IV}$	" 9.8976854;
N_6^{VI}	$=-1.0394174 N_6^{IV}$	" 0.0167908n;
N_6^{VII}	$=+0.03291265 N_6^{IV}$	" 8.5173628.
$g_7 =$	$22''.4608479;$	
N_7	$=-\ 0.006236689 N_7^{IV}$	log. 97.7949541n;
N_7'	$=+\ 0.02030250 N_7^{IV}$	" 98.3075494;
N_7''	$=-\ 0.1520658 N_7^{IV}$	" 99.1820317n;
N_7'''	$=-\ 0.9615594 N_7^{IV}$	" 99.9829761n;
N_7^{V}	$=-\ 3.091803 N_7^{IV}$	" 0.4902118n;
N_7^{VI}	$=+\ 0.1154716 N_7^{IV}$	" 99.0624752;
N_7^{VII}	$=+\ 0.00872852 N_7^{IV}$	" 97.9409405.

16. We have thus determined all the roots appertaining to the equation of the eighth degree, together with the ratios of the constant quantities N', N'', &c., to N; N_1', N_1'', &c., to N_1; and the ratios of the similar quantities relative to each of the other roots. The complete integrals of the equations (A) will therefore be the sums of all the corresponding terms depending on g, g_1, g_2, &c., and we shall therefore have

$$\left.\begin{array}{l}
h=N\sin(gt+\beta)+N_1\sin(g_1t+\beta_1)+N_2\sin(g_2t+\beta_2)+\&c.,\\
h'=N'\sin(gt+\beta)+N_1'\sin(g_1t+\beta_1)+N_2'\sin(g_2t+\beta_2)+\&c.,\\
h''=N''\sin(gt+\beta)+N_1''\sin(g_1t+\beta_1)+N_2''\sin(g_2t+\beta_2)+\&c.,\\
\&c.;\\
l=N\cos(gt+\beta)+N_1\cos(g_1t+\beta_1)+N_2\cos(g_2t+\beta_2)+\&c.,\\
l'=N'\cos(gt+\beta)+N_1'\cos(g_1t+\beta_1)+N_2'\cos(g_2t+\beta_2)+\&c.,\\
l''=N''\cos(gt+\beta)+N_1''\cos(g_1t+\beta_1)+N_2''\cos(g_2t+\beta_2)+\&c.,\\
\&c.
\end{array}\right\}\quad(C)$$

β, β_1, β_2, &c., being arbitrary constant quantities. These equations contain twice as many constant quantities as there are roots g, g_1, g_2, &c., of the differential equations (A). These constant quantities are indeterminate by analysis; but they may be deduced from the values of h, h', h'', &c., l, l', l'', &c., at any given epoch. If we suppose $t=0$, in equations (C), and also suppose the first members to be known quantities, the preceding equations will give the values of these constant quantities by direct elimination. But in order to determine them more conveniently, we shall resume the differential equations (A), and we shall multiply the first, third, fifth, &c., by $Nm\div na$, $N'm'\div n'a'$, $N''m''\div n''a''$, &c., and they will become

$$\left.\begin{array}{l}
N\dfrac{m}{na}\dfrac{dh}{dt}=N\dfrac{m}{na}\left\{(0,1)+(0,2)+(0,3)+\&c.,\right\}l\\
\qquad -N\dfrac{m}{na}\left\{\boxed{0,1}l'+\boxed{0,2}l''+\boxed{0,3}l'''+\&c.,\right\}\\
N'\dfrac{m'}{n'a'}\dfrac{dh'}{dt}=N'\dfrac{m'}{n'a'}\left\{(1,0)+(1,2)+(1,3)+\&c.,\right\}l'\\
\qquad -N'\dfrac{m'}{n'a'}\left\{\boxed{1,0}l+\boxed{1,2}l''+\boxed{1,3}l'''+\&c.,\right\}\\
N''\dfrac{m''}{n''a''}\dfrac{dh''}{dt}=N''\dfrac{m''}{n''a''}\left\{(2,0)+(2,1)+(2,3)+\&c.,\right\}l''\\
\qquad -N''\dfrac{m''}{n''a''}\left\{\boxed{2,0}l+\boxed{2,1}l'+\boxed{2,3}l'''+\&c.,\right\}\\
\qquad \&c.
\end{array}\right\}\quad(130)$$

If we add these equations, and in their sum substitute the values of $N\{(0,1)+(0,2)+(0,3)+\&c.\}$, $N'\{(1,0)+(1,2)+(1,3)+\&c.\}$, &c., deduced from equations (B), we shall obtain

$$\left.\begin{array}{l}
N\dfrac{m}{na}\dfrac{dh}{dt}+N'\dfrac{m'}{n'a'}\dfrac{dh'}{dt}+N''\dfrac{m''}{n''a''}\dfrac{dh}{dt}+\&c.\\
=g\left\{Nl\dfrac{m}{na}+N'l'\dfrac{m'}{n'a'}+N''l''\dfrac{m''}{n''a''}+\&c.\right\}\\
+N\left\{l'\left(\dfrac{m'}{n'a'}\boxed{1,0}-\dfrac{m}{na}\boxed{0,1}\right)+l''\left(\dfrac{m''}{n''a''}\boxed{2,0}-\dfrac{m}{na}\boxed{0,2}\right)+\&c.\right\}\\
+N'\left\{l\left(\dfrac{m}{na}\boxed{0,1}-\dfrac{m'}{n'a'}\boxed{1,0}\right)+l''\left(\dfrac{m''}{n''a''}\boxed{2,1}-\dfrac{m'}{n'a'}\boxed{1,2}\right)\right.\\
\qquad\left.+l'''\left(\dfrac{m'''}{n'''a'''}\boxed{3,1}-\dfrac{m'}{n'a'}\boxed{1,3}\right)+\&c.\right\}
\end{array}\right\}\quad(131)$$

But each of the factors of l, l', l'', &c., in the terms of the second member of this equation, which are independent of g, becomes nothing by means of equations (9); and we shall therefore have

$$\left.\begin{aligned} &N\frac{m}{na}\frac{dh}{dt}+N'\frac{m'}{n'a'}\frac{dh'}{dt}+N''\frac{m''}{n''a''}\frac{dh''}{dt}+\text{\&c.}\\ &=g\left\{Nl\frac{m}{na}+N'l'\frac{m'}{n'a'}+N''l''\frac{m''}{n''a''}+N'''l'''\frac{m'''}{n'''a'''}+\text{\&c.}\right.\end{aligned}\right\} \qquad (132)$$

If we substitute in this equation the preceding values of h, h', h'', &c., l, l', l'', &c., we shall have, by comparing the coefficients of the same cosines,

$$\left.\begin{aligned} &NN_1\frac{m}{na}+N'N_1'\frac{m'}{n'a'}+N''N_1''\frac{m''}{n''a''}+N'''N_1'''\frac{m'''}{n'''a'''}+\text{\&c.}=0;\\ &NN_2\frac{m}{na}+N'N_2'\frac{m'}{n'a'}+N''N_2''\frac{m''}{n''a''}+N'''N_2'''\frac{m'''}{n'''a'''}+\text{\&c.}=0;\\ &NN_3\frac{m}{na}+N'N_3'\frac{m'}{n'a'}+N''N_3''\frac{m''}{n''a''}+N'''N_3'''\frac{m'''}{n'''a'''}+\text{\&c.}=0;\\ &\text{\&c.}\end{aligned}\right\} \qquad (133)$$

If we now multiply the preceding values of h, h', h'', &c., respectively $N\frac{m}{na}$, $N'\frac{m'}{n'a'}$, $N''\frac{m''}{n''a''}$, we shall have by means of equations (133),

$$\left.\begin{aligned} &N\frac{m}{na}h+N'\frac{m'}{n'a'}h'+N''\frac{m''}{n''a''}h''+N'''\frac{m'''}{n'''a'''}h'''+\text{\&c.}\\ &=\left\{N^2\frac{m}{na}+N'^2\frac{m'}{n'a'}+N''^2\frac{m''}{n''a''}+N'''^2\frac{m'''}{n'''a'''}+\text{\&c.}\right\}\sin(gt+\beta)\end{aligned}\right\} \qquad (134)$$

In like manner the preceding values of l, l', l'', &c., will give,

$$\left.\begin{aligned} &N\frac{m}{na}l+N'\frac{m'}{n'a'}l'+N''\frac{m''}{n''a''}l''+N'''\frac{m'''}{n'''a'''}l'''+\text{\&c.}\\ &=\left\{N^2\frac{m}{na}+N'^2\frac{m'}{n'a'}+N''^2\frac{m''}{n''a''}+N'''^2\frac{m'''}{n'''a'''}+\text{\&c.}\right\}\cos(gt+\beta)\end{aligned}\right\} \qquad (135)$$

If the origin of the time t, be fixed at the epoch for which the values of h, l, h', l', &c., are supposed to be known; the two preceding equations will give,

$$\tan\beta=\frac{N\frac{m}{na}h+N'\frac{m'}{n'a'}h'+N''\frac{m''}{n''a''}h''+N'''\frac{m'''}{n'''a'''}h'''+\text{\&c.}}{N\frac{m}{na}l+N'\frac{m'}{n'a'}l'+N''\frac{m''}{n''a''}l''+N'''\frac{m'''}{n'''a'''}l'''+\text{\&c.}} \qquad (136)$$

Since N', N'', N''', &c. are given in terms of N, this indeterminate quantity will disappear from the expression of $\tan\beta$. Having found β, we may substitute it in either of equations (134) or (135), and we shall obtain the value of N; and consequently N', N'', N''', &c. will be known. We shall thus have the system of constant quantities corresponding to the root g. Changing in the preceding equations the root g successively into g_1, g_2, g_3, &c., we shall obtain the values of the constant quantities, corresponding to each of these roots.

If we now put the first members of equations (134) and (135) equal to x and y respectively and the coefficients of sin or cos $(gt+\beta)$ in the same equations equal to z, we shall have,

$$x=z\sin(gt+\beta); \qquad y=z\cos(gt+\beta). \qquad (137)$$

Whence

$$\tan(gt+\beta)=x\div y. \qquad (138)$$

17. In order to find the values of x and y, it is necessary to know the values of h and l. We shall therefore suppose that at the epoch of 1850, the eccentricities and places of the perihelia of the eight principal planets have the following values,

Mercury,	$e\ =0.2056179$	$\log. e\ =9.3130610;$	$\varpi\ = 75°\ 7'\ 0''.0;$
Venus,	$e'\ =0.00684184$	$\log. e'\ =7.8351730;$	$\varpi'\ =129\ 28\ 51.7;$
The Earth,	$e''\ =0.01677120$	$\log. e''\ =8.2245642;$	$\varpi''\ =100\ 21\ 41.0;$
Mars,	$e'''\ =0.0931324$	$\log. e'''\ =8.9691008;$	$\varpi'''\ =333\ 17\ 47.8;$
Jupiter,	$e^{\text{IV}}=0.0482388$	$\log. e^{\text{IV}}=8.6833965;$	$\varpi^{\text{IV}}=\ 11\ 54\ 53.1;$
Saturn,	$e^{\text{V}}\ =0.0559956$	$\log. e^{\text{V}}\ =8.7481539;$	$\varpi^{\text{V}}\ =\ 90\ \ 6\ 12.0;$
Uranus,	$e^{\text{VI}}=0.0462149$	$\log. e^{\text{VI}}=8.6647821;$	$\varpi^{\text{VI}}=170\ 34\ 17.6;$
Neptune,	$e^{\text{VII}}=0.00917396$	$\log. e^{\text{VII}}=7.9625568;$	$\varpi^{\text{VII}}=\ 50\ 16\ 39.1.$

Now since $h=e\sin\varpi$, $l=e\cos\varpi$, $h'=e'\sin\varpi'$, $l'=e'\cos\varpi'$, &c., we shall obtain the following values

$h\ =+0.198720,$	$\log. h\ =9.2989408;$
$h'\ =+0.0052808,$	$\log. h'\ =7.7226976;$
$h''\ =+0.0164977,$	$\log. h''\ =8.2174237;$
$h'''\ =-0.0418510,$	$\log. h'''\ =8.6217066n;$
$h^{\text{IV}}=+0.0099592,$	$\log. h^{\text{IV}}=7.9982241;$
$h^{\text{V}}\ =+0.0559955,$	$\log. h^{\text{V}}\ =8.7481532;$
$h^{\text{VI}}=+0.0075707,$	$\log. h^{\text{VI}}=7.8791377;$
$h^{\text{VII}}=+0.0070561,$	$\log. h^{\text{VII}}=7.8485672;$
$l\ =+0.052813,$	$\log. l\ =8.7227434;$
$l'\ =-0.0043502,$	$\log. l'\ =7.6385092n;$
$l''\ =-0.0030164,$	$\log. l''\ =7.4794898n;$
$l'''\ =+0.083199,$	$\log. l'''\ =8.9201199;$
$l^{\text{IV}}=+0.0471996,$	$\log. l^{\text{IV}}=8.6739377;$
$l^{\text{V}}\ =-0.00010988,$	$\log. l^{\text{V}}\ =6.0042715n;$
$l^{\text{VI}}=-0.0455906,$	$\log. l^{\text{VI}}=8.6588752n;$
$l^{\text{VII}}=+0.0058628,$	$\log. l^{\text{VII}}=7.7681049.$

We have already given the logarithms of $m\div na$, $m'\div n'a'$, $m''\div n''a''$, &c., in § 5; and if we add them successively to log. h and log. l, log. h' and log. l', &c., we shall obtain the following values of the logarithms of the constants, for the given epoch, which enter into the values of x and y.

$\log. h\frac{m}{na}=86.2924049$; $\log. l\frac{m}{na}=85.7169075$;

$\log. h'\frac{m'}{n'a'}=85.9487046$; $\log. l'\frac{m'}{n'a'}=85.8645162n$;

$\log. h''\frac{m''}{n''a''}=86.5381661$; $\log. l''\frac{m''}{n''a''}=85.8002322n$;

$\log. h'''\frac{m'''}{n'''a'''}=86.1723213n$; $\log. l'''\frac{m'''}{n'''a'''}=86.4707346$;

$\log. h^{IV}\frac{m^{IV}}{n^{IV}a^{IV}}=89.2232280$; $\log. l^{IV}\frac{m^{IV}}{n^{IV}a^{IV}}=89.8989416$;

$\log. h^{V}\frac{m^{V}}{n^{V}a^{V}}=89.5809761$; $\log. l^{V}\frac{m^{V}}{n^{V}a^{V}}=86.8370944n$;

$\log. h^{VI}\frac{m^{VI}}{n^{VI}a^{VI}}=88.0117101$; $\log. l^{VI}\frac{m^{VI}}{n^{VI}a^{VI}}=88.7914476n$;

$\log. h^{VII}\frac{m^{VII}}{n^{VII}a^{VII}}=88.2010654$; $\log. l^{VII}\frac{m^{VII}}{n^{VII}a^{VII}}=88.1206031$.

18. These quantities are now to be substituted in equations (136) and (137), in connection with the values of N', N'', N''', &c., corresponding to the different roots.

For the root $g=5''.4638027$, we find

$$x=+\frac{1847539}{10^{20}}N,\quad y=+\frac{64178.9}{10^{20}}N,\quad z=+\frac{10467760}{10^{23}}N^2.$$

Whence $\beta=88°\ 0'\ 37''.8$, and $\log. N=9.2470063$.

Therefore for root g, we have the following values,

$N\ =+0.1766064$, $N^{IV}=-0.0000090002$,
$N'\ =+0.0085906$, $N^{V}\ =-0.0000080335$,
$N''\ =+0.0054825$, $N^{VI}=+0.0000035168$,
$N'''=+0.00084182$, $N^{VII}=+0.000000063131$.

In like manner we shall find for the root $g_1=7''.2484269$,

$$x_1=+\frac{1654860.4}{10^{20}}N_1,\quad y_1=+\frac{4347600.6}{10^{20}}N_1,\quad z_1=+\frac{173037154}{10^{20}}N^2.$$

Whence $\beta_1=20°\ 50'\ 19''.4$, and $\log. N_1=8.4294913$.

Therefore for the root g_1 we have

$N_1\ =+0.02688384$, $N_1^{IV}=+0.000010644$,
$N_1'\ =-0.02014438$, $N_1^{V}\ =+0.000010875$,
$N_1''\ =-0.01536192$, $N_1^{VI}=-0.000002744$,
$N_1'''=-0.00254509$, $N_1^{VII}=-0.0000001122$.

For the root $g_2=17''.0143734$, we find,

$$x_2=-\frac{18893046}{10^{20}}N_2,\quad y_2=+\frac{40873423}{10^{20}}N_2,\quad z_2=+\frac{3.068840}{10^{10}}N_2^2.$$

$\beta_2=335°\ 11'\ 31''.4$, $\log. N_2=7.1665150$.

$N_2\ =+0.001467287$, $N_2^{IV}=-0.000001059$,
$N_2'\ =-0.01121706$, $N_2^{V}\ =-0.000006401$,
$N_2''=+0.01131047$, $N_2^{VI}=+0.000000392$,
$N_2'''=+0.02257186$, $N_2^{VII}=+0.0000000288$.

For the root $g_3=17''.7844562$, we find,

$$x_3=+\frac{0.01310358}{10^{10}}N_3,\ y_3=-\frac{0.014106133}{10^{10}}N_3,\ z_3=+\frac{12.083015}{10^{10}}N_3^2$$

$$\beta_3=137^\circ\ 6'\ 36''.5, \qquad \log.\ N_3=7.2023283.$$

$$\begin{array}{ll} N_3\ =+0.001593413, & N_3^{IV}=+0.0000010652, \\ N_3'\ =-0.01318916, & N_3^{V}\ =+0.0000110097, \\ N_3''=+0.01626412, & N_3^{VI}=-0.0000006258, \\ N_3'''=-0.0790650, & N_3^{VII}=-0.00000004636. \end{array}$$

For the root $g_4=0''.6166849$, we find,

$$x_4=+\frac{3.3572052}{10^{10}}N_4^{IV},\ y_4=+\frac{1.360283}{10^{10}}N_4^{IV},\ z_4=+\frac{56970.44}{10^{10}}N_4^{IV2}.$$

$$\beta_4=67^\circ\ 56'\ 34''.9, \qquad \log.\ N_4^{IV}=95.8033371.$$

$$\begin{array}{ll} N_4\ =+0.000007714, & N_4^{IV}=+0.000063582, \\ N_4'\ =+0.000011722, & N_4^{V}\ =+0.000071708, \\ N_4''=+0.000013577, & N_4^{VI}=+0.00155775, \\ N_4'''=+0.000021946, & N_4^{VII}=+0.01003893. \end{array}$$

For the root $g_5=2''.7276592$, we find,

$$x_5=+\frac{0.6475797}{10^{10}}N_5^{IV},\ y_5=-\frac{0.1743025}{10^{10}}N_5^{IV},\ z_5=+\frac{345.0396}{10^{10}}N_5^{IV2}.$$

$$\beta_5=105^\circ\ 3'\ 52''.9, \qquad \log.\ N_5^{IV}=97.2886122.$$

$$\begin{array}{ll} N_5\ =+0.000568545, & N_5^{IV}=+0.001943624, \\ N_5'\ =+0.000557100, & N_5^{V}\ =+0.00176940, \\ N_5''=+0.000583230, & N_5^{VI}=+0.02973295, \\ N_5'''=+0.000776548, & N_5^{VII}=-0.002910500. \end{array}$$

For the root $g_6=3''.7166075$, we find,

$$x_6=+\frac{0.4583186}{10^{10}}N_6^{IV},\ y_6=+\frac{0.8566965}{10^{10}}N_6^{IV},\ z_6=+\frac{22.51127}{10^{10}}N_6^{IV2}.$$

$$\beta_6=28^\circ\ 8'\ 45''.8, \qquad \log.\ N_6^{IV}=98.6350825.$$

$$\begin{array}{ll} N_6\ =+0.02449386, & N_6^{IV}=+0.04316011, \\ N_6'\ =+0.01660532, & N_6^{V}\ =+0.03410106, \\ N_6''=+0.01634130, & N_6^{VI}=-0.04486144, \\ N_6'''=+0.01879544, & N_6^{VII}=+0.001420513. \end{array}$$

For the root $g_7=22''.4608479$, we find,

$$x_7=-\frac{1.0095026}{10^{10}}N_7^{IV},\ y_7=+\frac{0.7872144}{10^{10}}N_7^{IV},\ z_7=+\frac{81.86052}{10^{10}}N_7^{IV2}.$$

$$\beta_7=307^\circ\ 56'\ 50''.1, \qquad \log.\ N_7^{IV}=98.1941887.$$

$$\begin{array}{ll} N_7\ =-0.00009753, & N_7^{IV}=+0.01563827, \\ N_7'\ =+0.00031750, & N_7^{V}\ =-0.04835044, \\ N_7''=-0.00237805, & N_7^{VI}=+0.001805776, \\ N_7'''=-0.01503712, & N_7^{VII}=+0.000136499. \end{array}$$

If these values be substituted in equations (C), we shall have the complete values of h, h', h'', &c., l, l', l'', &c., from which we can obtain the eccentricity and place of the perihelia by means of the formulas,

$$\tan\varpi = h \div l; \quad e = \sqrt{h^2 + l^2} = h \operatorname{cosec}\varpi = l \sec\varpi. \tag{139}$$

19. If, in equations (C), we put, instead of h and l, their values, $e\sin\varpi$, and $e\cos\varpi$, we shall get

$$e\sin\varpi = N\sin(gt+\beta) + N_1\sin g_1t+\beta_1) + N_2\sin(g_2t+\beta_2) + \text{\&c.} \tag{140}$$
$$e\cos\varpi = N\cos(gt+\beta) + N'\cos g_1t+\beta_1) + N_2\cos(g_2t+\beta_2) + \text{\&c.} \tag{141}$$

Multiplying equation (140) by $\cos(gt+\beta)$, and (141) by $\sin(gt+\beta)$, the difference of their products will give

$$e\sin(\varpi - gt - \beta) = N_1\sin\{(g_1-g)t+\beta_1-\beta\} + N_2\sin\{(g_2-g)t+\beta_2-\beta\} + \text{\&c.} \tag{142}$$

If we multiply equation (140) by $\sin(gt+\beta)$, and (141) by $\cos(gt+\beta)$ the sum of their products will give

$$\left.\begin{aligned} &e\cos(\varpi - gt - \beta) = \\ &\quad N + N_1\cos\{(g_1-g)t+\beta_1-\beta\} + N_2\cos\{(g_2-g)t+\beta_2-\beta\} + \text{\&c.} \end{aligned}\right\} \tag{143}$$

Dividing equation (142) by (143), we shall eliminate e, and get,

$$\tan(\varpi - gt - \beta) = \frac{N_1\sin\{(g_1-g)t+\beta_1-\beta\} + N_2\sin\{(g_2-g)t+\beta_2-\beta\} + \text{\&c.}}{N + N_1\cos\{(g_1-g)t+\beta_1-\beta\} + N_2\cos\{(g_2-g)t+\beta_2-\beta\} + \text{\&c.}} \tag{144}$$

When the sum $N_1+N_2+N_3+$&c., of the coefficients of the cosines of the denominator, all taken positively, is less than N, $\tan(\varpi - gt - \beta)$ cannot become infinite; the angle $\varpi - gt - \beta$ cannot therefore become equal to a right angle; consequently, the mean motion of the perihelion will be in that case equal to gt. It is also easy to see that when all the cosines of the denominator of $\tan(\varpi - gt - \beta)$ become equal to ± 1, N_1, N_2, N_3, &c., being supposed positive, the denominator will be either a *maximum* or a *minimum*, and the numerator will be equal to nothing; $\tan(\varpi - gt - \beta)$ will then be equal to nothing; $\varpi - gt - \beta$ will therefore become equal to nothing, and equation (143) will become,

$$e = N \pm \{N_1 + N_2 + N_3 + \text{\&c.}\}; \tag{145}$$

Consequently, the maximum and minimum values of e will be

$$\left.\begin{aligned} &\textit{maximum}\ e = N + N_1 + N_2 + N_3 + \text{\&c.}; \\ \text{and } &\textit{minimum}\ e = N - \{N_1 + N_2 + N_3 + \text{\&c.}\} \end{aligned}\right\} \tag{146}$$

We shall now substitute the numbers which we have already computed, in these equations, for the purpose of determining the maximum and minimum values of the eccentricities, and the mean motions of the perihelia.

20. For the planet *Mercury*, we have,

Maximum $e = N + N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 = 0.2317185$. One-half of this is 0.1158593, which being less than N, it follows that N exceeds the sum of all the remaining terms; consequently, the mean annual motion of Mercury's perihelion is equal to g or $5''.4638027$, and performs a complete revolution in 237197 years; and the minimum value of the eccentricity is 0.1214943.

For the planet *Venus* we have,

Maximum $e'=N'+N_1'+N_2'+$&c.$=0.0706329$. One-half of this is 0.0353165. As this number exceeds any *one* of the coefficients, N', N_1', N_2', &c., it follows that the perihelion of the orbit of Venus *has no mean motion*, and that the minimum value of its eccentricity is zero.

For the *Earth* we have,

Maximum $e''=N''+N_1''+N_2''+$&c.$=0.0677352$. One-half of this is 0.0338676. As this number exceeds any *one* of the coefficients N'', N_1'', N_2'', &c., it follows that the perihelion of the Earth's orbit *has no mean motion*, and that the minimum value of the eccentricity is zero.

For the planet *Mars* we have,

Maximum $e'''=N'''+N_1'''+N_2'''+$&c.$=0.1396547$. One-half of this is 0.0698274. As this number is less than N_3''', it follows that the perihelion of the orbit of Mars has a mean annual motion equal to g_3 or 17″.7844562; and that the minimum eccentricity of his orbit is equal to 0.0184753. We shall here observe that a small variation in the assumed mass of the Earth would produce a considerable variation in the limits of eccentricity and mean motion of the perihelion.

For the planet *Jupiter* we have,

Maximum $e^{IV}=N^{IV}+N_1^{IV}+N_2^{IV}+$&c.$=0.0608274$. One-half of this is 0.0304137. As this number is less than N_6^{IV}, it follows that the perihelion of the orbit of Jupiter has a mean annual motion equal to g_6 or 3″.7166075; and that the minimum value of the eccentricity is equal to 0.0254928.

For the planet *Saturn* we have,

Maximum $e^{V}=N^{V}+N_1^{V}+N_2^{V}+$&c.$=0.0843289$. One-half of this is equal to 0.0421644. As this number is less than N_7^{V}, it follows that the perihelion of Saturn's orbit has a mean annual motion equal to g_7 or 22″.4608479; and that the minimum value of the eccentricity is equal to 0.0123719.

For the planet *Uranus* we have,

Maximum $e^{VI}=N^{VI}+N_1^{VI}+N_2^{VI}+$&c.$=0.0779652$. One-half of this is 0.0389826. As this number is less than N_6^{VI}, it follows that the perihelion of the orbit of Uranus has a mean annual motion equal to g_6 or 3″.7166075; and that the minimum value of the eccentricity is equal to 0.0117576.

For the planet *Neptune* we have,

Maximum $e^{VII}=N^{VII}+N_1^{VII}+N_2^{VII}+$&c.$=0.0145066$. One-half of this is 0.0072533. As this number is less than N_4^{VII}, it follows that the perihelion of Neptune's orbit has a mean annual motion equal to g_4 or 0″.6166849; and that the minimum value of the eccentricity is equal to 0.0055712.

21. We see, by the preceding article, that the mean motions of the perihelia of *Jupiter* and *Uranus* are exactly equal. It follows from this circumstance that the

mean relative positions of their perihelia will always be the same; and we shall now inquire what their mean relative positions are. For this purpose we shall resume equation (144), and substitute in it the values corresponding to these two planets. By this means we shall get the following equations,

$$\left.\begin{array}{l}\tan(\varpi^{IV}-g_6t-\beta_6)= \\ \qquad \dfrac{N^{IV}\sin\{(g-g_6)t+\beta-\beta_6\}+N_1^{IV}\sin\{(g_1-g_6)t+\beta_1-\beta_6\}+\&c.}{N_6^{IV}+N^{IV}\cos\{(g-g_6)t+\beta-\beta_6\}+N_1^{IV}\cos\{(g_1-g_6)t+\beta_1-\beta_6\}+\&c.},\end{array}\right\} \quad (147)$$

$$\left.\begin{array}{l}\tan(\varpi^{VI}-g_6t-\beta_6)= \\ \qquad \dfrac{N^{VI}\sin\{(g-g_6)t+\beta-\beta_6\}+N_1^{VI}\sin\{(g_1-g_6)t+\beta_1-\beta_6\}+\&c.}{N_6^{VI}+N^{VI}\cos\{(g-g_6)t+\beta-\beta_6\}+N_1^{VI}\cos\{(g_1-g_6)t+\beta_1-\beta_6\}+\&c.}.\end{array}\right\} \quad (148)$$

Now, since the mean values of the numerators of these equations are each equal to nothing, and the signs of the denominators depend wholly on N_6^{IV} and N_6^{VI}; it follows that ϖ^{IV} will always be equal to ϖ^{VI}, if N_6^{IV} has the same sign as N_6^{VI}; and ϖ^{IV} will always differ from ϖ^{VI} by two right angles if N_6^{IV} and N_6^{VI} have different signs. According to the numbers which we have calculated, N_6^{IV} and N_6^{VI} have different signs; consequently, the mean longitudes of the perihelia of the orbits of Jupiter and Uranus differ by a semicircumference.

For the purpose of determining the maximum values of $\tan(\varpi^{IV}-g_6t-\beta_6)$ and $\tan(\varpi^{VI}-g_6t-\beta_6)$, we may suppose them to be of the following form,

$$\tan(\varpi^{IV}-g_6t-\beta_6)=\frac{\{N^{IV}+N_1^{IV}+N_2^{IV}+N_3^{IV}+N_4^{IV}+N_6^{IV}+N_7^{IV}\}\sin\alpha}{N_6^{IV}+\{N^{IV}+N_1^{IV}+N_2^{IV}+N_3^{IV}+N_4^{IV}+N_5^{IV}+N_7^{IV}\}\cos\alpha}; \quad (149)$$

$$\tan(\varpi^{VI}-g_6t-\beta_6)=\frac{\{N^{VI}+N_1^{VI}+N_2^{VI}+N_3^{VI}+N_4^{VI}+N_5^{VI}+N_7^{VI}\}\sin\alpha}{N_6^{VI}+\{N^{VI}+N_1^{VI}+N_2^{VI}+N_3^{VI}+N_4^{VI}+N_5^{VI}+N_7^{VI}\}\cos\alpha}; \quad (150)$$

the coefficients of $\cos\alpha$ being supposed positive. These equations evidently attain their maximum values when $\cos\alpha$ is equal to the quotient of its coefficient divided by the constant term of the denominator, taken negatively. If we reduce them to numbers, they will become

$$\tan(\varpi^{IV}-g_6t-\beta_6)=\frac{0.0176673\sin\alpha}{0.0431601+0.0176673\cos\alpha}; \quad (151)$$

$$\tan(\varpi^{VI}-g_6t-\beta_6)=\frac{0.0331038\sin\alpha}{-0.0448614+0.0331038\cos\alpha}. \quad (152)$$

The first of these evidently attains a maximum value when $\alpha=\pm 114°\ 10'$; and the second one when $\alpha=\pm 42°\ 27'$. Consequently, we shall find

Maximum value of $(\varpi^{IV}-g_6t-\beta_6)=\pm 24°\ 10'$;

and " " " $(\varpi^{VI}-g_6t-\beta\ )=180°\pm 47°\ 33'$.

The nearest approach of the perihelia of these two planets will therefore be 108° 17'.

The mean annual motions of the perihelia of the four planets *Jupiter*, *Saturn*, *Uranus*, and *Neptune* being

Jupiter and *Uranus* . . . . 3″.7166075;
Saturn . . . . 22 .4608479;
Neptune. . . . . 0 .6166849;

it follows that the mean motion of *Saturn's* perihelion is very nearly six times that of *Jupiter* and *Uranus;* and this latter quantity is very nearly six times that of *Neptune;* or, more exactly, 985 times the mean motion of *Jupiter's* perihelion is equal to 163 times that of *Saturn;* and 440 times the mean motion of *Neptune's* perihelion is equal to 73 times that of *Jupiter* and *Uranus.* It also follows that the perihelion of *Saturn* performs a complete revolution in the heavens in 57700 years; that of *Jupiter* and *Uranus* in 348700 years; while that of *Neptune* requires no less than 2101560 years to perform the circuit of the heavens.

22. Having determined the numerical values of all the constants which enter into the integrals of the differential equations (A), corresponding to the assumed masses, it now remains to determine the changes which would be introduced into these constants, on the supposition that the masses were changed from m to $m(1+\mu)$, m' to $m'(1+\mu')$, &c., or, on the supposition that the masses of all the planets were multiplied by a factor $(1+\alpha)$, α being any supposed correction of the mass of the planet, and greater than -1. If we suppose α, or μ, μ', μ'', &c. to be equal to -1, it is evident the whole system of differential equations would vanish. The effect of changing all the masses, in the ratio of 1 to $1+\alpha$, on the equation of the eighth degree, would be equivalent to multiplying the coefficients of the different powers of g by $(1+\alpha)^{8-n}$, n denoting the exponent of g in the given term. Consequently, the coefficient of g^8 would remain unaltered; that of g^7 would be multiplied by $1+\alpha$; that of g^6 would be multiplied by $(1+\alpha)^2$, &c.; while the term of the equation which is independent of g would be multiplied by $(1+\alpha)^8$. It is evident that these changes are such as would be produced by multiplying each of the roots of the equation by $1+\alpha$; consequently, we shall have the following theorem:—

If the masses of all the planets be simultaneously increased in the ratio of 1 *to* $1+\alpha$, *all the roots of the equation in* g *will be increased in the same ratio.*

It is also evident that, if the masses of all the planets be multiplied by $1+\alpha$, the values of A, A', A'', &c., D, D', D'', &c. will all be multiplied by $(1+\alpha)^2$; and, as they are all multiplied by the same quantity, it is manifest that the ratios of the quantities N, N', N'', &c., will remain unaltered. And since the ratios of N, N', N'', &c., remain unaltered, it is evident that $\tan\beta$ will be unchanged, and consequently the values of N, N', N'', &c. will not be changed. It therefore follows that, *if the masses of all the planets be simultaneously increased in any given ratio, the magnitudes of the secular inequalities will remain unchanged.*

To illustrate, we shall observe that if the masses of all the planets be supposed to be doubled, the intensity of the disturbing forces would be doubled; but, according to the preceding theorem, the roots g, g_1, g_2, &c. would be doubled; and consequently the disturbing forces would operate in the same direction during only one-half of the time; and since a double force acting during one-half the time produces the same effect as a single force acting during a double interval, it follows that the magnitude of the resulting inequalities will remain unchanged.

23. The rigorous determination of the separate effects of the corrections of the masses μ, μ', μ'', &c. on the values of the constants which determine the secular variations of the elements, when the masses simultaneously vary, is a much more difficult and intricate problem than that of the determination of the secular inequalities themselves. For, if we employ the masses in indeterminate forms, $m(1+\mu)$, $m'(1+\mu')$, $m''(1+\mu'')$, &c., instead of m, m', m'', &c., it is evident that the solutions of the differential equations would contain terms depending on μ, μ', μ'', &c., and on all the powers and products of these quantities up to μ^7, μ'^7, μ''^7, inclusive, in addition to the terms already calculated. And if we neglect the powers and products of μ, μ', μ'', &c., above the first, it is evident that our solution would be very imperfect, unless μ, μ', μ'', &c. were very small quantities. Unfortunately, this is not the case, for the masses of some of the planets are still very imperfectly known; and consequently the terms depending on the powers and products of μ, μ', μ'', &c. ought not to be neglected. There seems to be only one practicable method of determining the effects of the corrections of the masses on the values of the constants which we have already determined. And this method consists in supposing the mass of each planet, in succession, to be increased by a finite quantity, μ_0, and then determine anew all the constants in the same manner as with the assumed masses. If we then subtract the values of the constants which depend on the assumed masses, from the values of the constants which result from the corrected mass of the planet, we shall obtain the coefficient of the correction depending on μ, by dividing the difference of the constants by μ_0, or the finite variation of the mass of the planet. In this way we get the whole variation resulting from the assumed variation of mass, or, in other words, we retain the terms depending on all the *powers* μ_0, μ_0^2, μ_0^3, &c., and neglect only the terms depending on the products μ, μ', μ'', &c., when they simultaneously vary. As this is the method which we have adopted, we shall here give the resulting fundamental equations, together with the values of the constants determined by their solution.

24. We shall now suppose the mass of *Mercury* to be increased to two and one-half times its assumed value. In this case μ_0 will be equal to $1\frac{1}{2}$, and $m=\frac{1+\mu_0}{4865751}=\frac{2.5}{4865751}=\frac{1}{1946300.4}$. This value of the mass of *Mercury* is very nearly the same as that employed by astronomers, during the early part of the present century, and is doubtless considerably larger than the actual value. But as the perturbations produced by this planet are very small, a considerable variation of its mass will produce only a small variation in the values of the fundamental equations. We shall now compute the effect which this change in the mass of *Mercury* produces in the fundamental equations.

If we now put $1+\mu=2.5$ in the values of $(1,0)$, $(2,0)$, $(3,0)$, &c., $\boxed{1,0}$, $\boxed{2,0}$, $\boxed{3,0}$, &c., and substitute the resulting numbers in equations (26), we shall get the following values of $\boxed{0,0}$, $\boxed{1,1}$, $\boxed{2,2}$, &c.

$$\left.\begin{array}{ll}
\boxed{0,0}=g-\ \ 5''.5702558; & \boxed{4,4}=g-\ \ 7''.5125167;\\
\boxed{1,1}=g-11\ .5785090; & \boxed{5,5}=g-18\ .5962297;\\
\boxed{2,2}=g-13\ .1331088; & \boxed{6,6}=g-\ \ 2\ .7662536;\\
\boxed{3,3}=g-17\ .5645194; & \boxed{7,7}=g-\ \ 0\ .6479572.
\end{array}\right\}\quad(153)$$

These values give

$$\boxed{0,0}\boxed{1,1}=g^2-17.1487648.g+\ \ 64.4952569126;\qquad(154)$$
$$\boxed{0,0}\boxed{2,2}=g^2-18.7033646.g+\ \ 73.1547754652;\qquad(155)$$
$$\boxed{0,0}\boxed{3,3}=g^2-23.1347752.g+\ \ 97.83886606206;\qquad(156)$$
$$\boxed{1,1}\boxed{2,2}=g^2-24.7116178.g+152.06181843878;\qquad(157)$$
$$\boxed{1,1}\boxed{3,3}=g^2-29.1430284.g+203.37094595357;\qquad(158)$$
$$\boxed{2,2}\boxed{3,3}=g^2-30.6976282.g+230.67674429991;\qquad(159)$$

$$\boxed{4,4}\boxed{5,5}=g^2-26.1087464.g+139.70448617829;\qquad(160)$$
$$\boxed{4,4}\boxed{6,6}=g^2-10.2787703.g+\ \ 20.78152636644;\qquad(161)$$
$$\boxed{4,4}\boxed{7,7}=g^2-\ \ 8.1604739.g+\ \ \ 4.86778928589;\qquad(162)$$
$$\boxed{5,5}\boxed{6,6}=g^2-21.3624833.g+\ \ 51.44188735405;\qquad(163)$$
$$\boxed{5,5}\boxed{7,7}=g^2-19.2441869.g+\ \ 12.04956092697;\qquad(164)$$
$$\boxed{6,6}\boxed{7,7}=g^2-\ \ 3.4142108.g+\ \ \ 1.792413937146.\qquad(165)$$

$$\left.\begin{array}{l}
\boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=g^4-47.8463930.g^3+\ \ 821.59840712.g^2\\
\qquad\qquad-5935.67265019.g+14877.555887385
\end{array}\right\}(166)$$

$$\left.\begin{array}{l}
\boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=g^4-29.5229572.g^3+230.63766404.g^2\\
\qquad\qquad-523.77824644.g+250.40826811
\end{array}\right\}(167)$$

The difference between these values and the similar quantities depending on the assumed masses being denoted by Δ, we shall find the following values,

$$\left.\begin{array}{ll}
\Delta\boxed{0,0}=\ \ 0. & \Delta\boxed{4,4}=-0.0001413,\\
\Delta\boxed{1,1}=-0.2637408, & \Delta\boxed{5,5}=-0.0000168,\\
\Delta\boxed{2,2}=-0.0609358, & \Delta\boxed{6,6}=-0.0000014,\\
\Delta\boxed{3,3}=-0.0116549, & \Delta\boxed{7,7}=-0.0000003.
\end{array}\right\}\quad(168)$$

$$\left.\begin{array}{l}
\Delta\boxed{0,0}\boxed{1,1}=-0.2637408.g+1.4691037;\\
\Delta\boxed{0,0}\boxed{3,3}=-0.0116549.g+0.06492077;\\
\Delta\boxed{1,1}\boxed{3,3}=-0.2753957.g+4.7643529;\\
\Delta\boxed{0,0}\boxed{2,2}=-0.0609358.g+0.33942800;\\
\Delta\boxed{1,1}\boxed{2,2}=-0.3246766.g+4.1532110\ ;\\
\Delta\boxed{2,2}\boxed{3,3}=-0.0725907.g+1.2226629\ ;\\
\Delta\boxed{4,4}\boxed{5,5}=-0.0001581.g+0.00275386;\\
\Delta\boxed{4,4}\boxed{7,7}=-0.0001416.g+0.00009381;\\
\Delta\boxed{5,5}\boxed{7,7}=-0.0000171.g+0.00001647;\\
\Delta\boxed{4,4}\boxed{6,6}=-0.0001427.g+0.00040139;\\
\Delta\boxed{5,5}\boxed{6,6}=-0.0000182.g+0.00007250;\\
\Delta\boxed{6,6}\boxed{7,7}=-0.0000017.g+0.000017370.
\end{array}\right\}\quad(169)$$

The following calculation of ΔA must serve as an example for the computation of all similar quantities.

$$\begin{array}{rl}
\Delta\boxed{0,0}\boxed{2,2} = & -\ 0.06093580.g + \ 0.33942800 \\
-\ \boxed{2,2}\boxed{1,0}\boxed{0,3}\,\mu \div \boxed{1,3} = & 0.02543896.g + \ 0.33254241 \\
-\Delta\boxed{2,2}\boxed{1,0}\boxed{0,3}(1+\mu)\div\boxed{1,3} = & + \ 0.00258360 \\
\text{Sum of last three terms of } A\times\mu = & + \ 0.28269800 \\
\text{Sum of terms} = \quad \Delta A = & -\ 0.08637476.g + \ 0.95725201 \\
\text{Computed} \quad A = & g^2 - 40.22178322.g + 193.3335643 \\
\text{Corrected} \quad A = & g^2 - 40.30815798.g + 194.2908163.
\end{array}$$

If we proceed in like manner with A', A'', &c., we shall obtain the following system of fundamental equations for $\mu = +\frac{3}{2}$.

Fundamental Equations for $\mu = \frac{3}{2}$; *or for the case where* $m = \frac{1}{1946300.4}$.

$$A = g^2 - 40.30815798.g + 194.2908163; \qquad (170)$$
$$A' = g^2 - 23.17005142.g + 98.2867801; \qquad (171)$$
$$A'' = g^2 - 18.28711570.g + 70.37498708; \qquad (172)$$
$$A_1 = g^2 - 14.59391880.g + 45.9791766; \qquad (173)$$
$$A_2 = g^2 - 10.00778683.g + 6.35429409; \qquad (174)$$
$$A_3 = g^2 - 26.183106514.g + 82.92120064; \qquad (175)$$

$$D = g^2 - 48.18393767.g + 708.16361; \qquad (176)$$
$$D' = g^2 - 55.19689421.g + 661.210840; \qquad (177)$$
$$D'' = g^2 - 31.64052859.g + 246.9639388; \qquad (178)$$
$$D_1 = g^2 - 48.09965699.g + 201.477187; \qquad (179)$$
$$D_2 = g^2 - 51.05291057.g + 34.3823000; \qquad (180)$$
$$D_3 = g^2 - 3.4187088566.g + 1.7319493752. \qquad (181)$$

$$\left.\begin{array}{l} B = \{g - 34.6955039\}b; \quad B' = \{g - 17''.578882777\}b; \\ \phantom{B = \{g - 34.6955039\}b; \quad} B'' = \{g - 12\,.71500846\}b; \end{array}\right\} \qquad (182)$$

$$\left.\begin{array}{l} C = \{23.7687432 - g\}[9.1763990]b'; \\ C' = \{17.59363918 - g\}[8.8694654]b'; \\ C'' = -[0.2445917]b'; \\ C''' = -[0.2654598]b'. \end{array}\right\} \qquad (183)$$

$$\left.\begin{array}{l} E = +[9.6231944]b''; \\ E' = \{24.4151946 - g\}[8.5847028]b''; \\ E'' = \{17.56579106 - g\}[9.6375865]b''; \\ E''' = +[0.9518913]b''. \end{array}\right\} \qquad (184)$$

$$\left.\begin{array}{l} F = -[7.6499091]b'''; \\ F' = +[8.9577383]b'''; \\ F'' = \{14.07473753 - g\}[0.8407439]b'''; \\ F''' = \{37.6032551 - g\}[9.4809266]b'''. \end{array}\right\} \qquad (185)$$

$$\left.\begin{array}{l} B_1 = \{g - 4.7867994\}b_1; \quad B_2 = \{g - 0.6795721937\}b_1; \\ \phantom{B_1 = \{g - 4.7867994\}b_1; \quad} B_3 = \{g - 18.665580712\}b_1; \end{array}\right\} \qquad (186)$$

$$\left.\begin{aligned}
C_1 &= \{4.365092844 \;-g\}[9.2840950]b_2;\\
C_2 &= \{0.6879108996-g\}[9.1824311]b_2;\\
C_3 &= -[0.6832317]b_2;\\
C_4 &= -[0.6834824]b_2;
\end{aligned}\right\} \quad (186)$$

$$\left.\begin{aligned}
E_1 &= +[7.8267723]b_3;\\
E_2 &= \{43.7345641 \;-g\}[8.2017618]b_3;\\
E_3 &= \{\;0.648004419-g\}[0.7610455]b_3;\\
E_4 &= +[1.0655652]b_3;
\end{aligned}\right\} \quad (187)$$

$$\left.\begin{aligned}
F_1 &= -[7.0339474]b_4;\\
F_2 &= +[0.2724895]b_4;\\
F_3 &= \{\;2.770664438-g\}[1.7737962]b_4;\\
F_4 &= \{50.3649997 \;-g\}[9.1128486]b_4;
\end{aligned}\right\} \quad (188)$$

$$\left.\begin{aligned}
g^4-47.8463930.g^3+796.20272817.g^2-5344.10960488.g\\
+12307.3911771
\end{aligned}\right\} =(\chi, \chi_1, \chi_2, \chi_2); \quad (189)$$

$$\left.\begin{aligned}
g^4-29.5229572.g^3+172.74076612.g^2-\;323.31739720.g\\
+140.47094066
\end{aligned}\right\} =(\chi_4, \chi_5, \chi_6, \chi_7). \quad (190)$$

And lastly, the values of b_1, b_2, b_3, and b_4 become,

$$\left.\begin{aligned}
b_1 &= \{0.00002188686 \ldots\ldots [95.3401834]\{N+[96.8635004]N'\\
&\qquad +[97.3236905]\;N''+[97.0504994]N''';\\
b_2 &= \{0.000001420383 \ldots\ldots [94.1524054]\{N+[95.6682302]N'\\
&\qquad +[96.1186392]\;N''+[95.8170069]N''';\\
b_3 &= \{0.0000000610884 \ldots [92.7859587]\{N+[94.2994239]N'\\
&\qquad +[94.7468016]\;N''+[94.4366545]N''';\\
b_4 &= \{0.00000000812296 \ldots [91.9097143]\{N+[93.4227230]N'\\
&\qquad +[93.8695157]\;N''+[93.5577417]N'''.
\end{aligned}\right\} \quad (191)$$

The values of b, b', b'', and b''' remain the same as in equations (118).

Equations (189) and (190) give, when the second members are put equal to 0,

$$\left.\begin{aligned}
g &= 5''.34449540; & g_4 &= 0''.616685510;\\
g_1 &= 7\,.53805074; & g_5 &= 2\,.72773208;\\
g_2 &= 17\,.12785195; & g_6 &= 3\,.71791243;\\
g_3 &= 17\,.83599491; & g_7 &= 22\,.46062783.
\end{aligned}\right\} \quad (192)$$

The solutions of equations (170–191) will now give the following values, remembering that the coefficients of equations (84–97) remain unchanged. For the root g, we get,

$$g=5''.3446763;$$

N'	$=+0.10318901N$	log. 99.0136340;
N''	$=+0.0652640N$	" 98.8146736;
N'''	$=+0.01001010N$	" 98.0004383;
N^{IV}	$=-0.0001170589N$	" 96.0684043n;
N^{V}	$=-0.0001036091N$	" 96.0153978n;
N^{VI}	$=+0.0000476249N$	" 95.6778341;
N^{VII}	$=+0.000000750320N$	" 93.8752464.

For the root g_1, we get the following values,

$g_1 = 7''.5386995;$		
$N_1' = -0.8746970 N_1$	log.	$99.9418576n;$
$N_1'' = -0.6900382 N_1$	"	$99.8388736n;$
$N_1''' = -0.1163716 N_1$	"	$99.0658468n;$
$N_1^{IV} = +0.0004334367 N_1$	"	$96.6369257;$
$N_1^{V} = +0.0004535340 N_1$	"	$96.6566098;$
$N_1^{VI} = -0.0001067081 N_1$	"	$96.0281973n;$
$N_1^{VII} = -0.000004636406 N_1$	"	$94.6661815n.$

For the root g_2, we get the following values,

$g_2 = 17''.1279859;$		
$N_2' = -\ 7.663670 N_2$	log.	$0.8844368n;$
$N_2'' = +\ 7.458903 N_2$	"	$0.8726750;$
$N_2''' = +18.20022 N_2$	"	$1.2600767;$
$N_2^{IV} = -\ 0.0007514405 N_2$	"	$96.8758946n;$
$N_2^{V} = -\ 0.004832818 N_2$	"	$97.6842004n;$
$N_2^{VI} = +\ 0.0002927626 N_2$	"	$96.4665156;$
$N_2^{VII} = +\ 0.00002153572 N_2$	"	$95.3331593.$

For the root g_3, we get the following values,

$g_3 = 17''.8360300;$		
$N_3' = -\ 8.244660 N_3$	log.	$0.9161728n;$
$N_3'' = +\ 9.718342 N_3$	"	$0.9875922;$
$N_3''' = -39.93574 N_3$	"	$1.6013617n;$
$N_3^{IV} = +\ 0.0004835988 N_3$	"	$96.6844852;$
$N_3^{V} = +\ 0.005252281 N_3$	"	$97.7203480;$
$N_3^{VI} = -\ 0.0002970608 N_3$	"	$96.4728453n;$
$N_3^{VII} = -\ 0.00002202336 N_3$	"	$95.3428835n.$

For the root g_4, we get the following values,

$g_4 = 0''.6166852;$		
$N_4 = +\ 0.1197280 N_4^{IV}$	log.	$9.0781956;$
$N_4' = +\ 0.1807174 N_4^{IV}$	"	$9.2570000;$
$N_4'' = +\ 0.2114780 N_4^{IV}$	"	$9.3252650;$
$N_4''' = +\ 0.3450307 N_4^{IV}$	"	$9.5378578;$
$N_4^{V} = +\ 1.127821 N_4^{IV}$	"	$0.0522403;$
$N_4^{VI} = +\ 24.50074 N_4^{IV}$	"	$1.3891794;$
$N_4^{VII} = +157.8951 N_4^{IV}$	"	$2.1983687.$

For the root g_5, we get the following values,

$g_5 = 2''.7276680$;

$N_5 = + 0.2886746 N_5^{IV}$	log. 9.4604086;
$N_5' = + 0.2816212 N_5^{IV}$	" 9.4496654;
$N_5'' = + 0.2969611 N_5^{IV}$	" 9.4726996;
$N_5''' = + 0.3989168 N_5^{IV}$	" 9.6008823;
$N_5^{V} = + 0.9103832 N_5^{IV}$	" 9.9592242;
$N_5^{VI} = +15.29957 N_5^{IV}$	" 1.1846793;
$N_5^{VII} = - 1.497633 N_5^{IV}$	" 0.1754053*n*.

For the root g_6, we get the following values,

$g_6 = 3''.7167141$;

$N_6 = +0.5652238 N_6^{IV}$	log. 9.7522204;
$N_6' = +0.3828501 N_6^{IV}$	" 9.5830288;
$N_6'' = +0.3770050 N_6^{IV}$	" 9.5763472;
$N_6''' = +0.4350723 N_6^{IV}$	" 9.6385614;
$N_6^{V} = +0.7901118 N_6^{IV}$	" 9.8976886;
$N_6^{VI} = -1.039307 N_6^{IV}$	" 0.0167438*n*;
$N_6^{VII} = +0.03290417 N_6^{IV}$	" 8.5172509.

For the root g_7, we get the following values,

$g_7 = 22''.4608918$;

$N_7 = -0.006358233 N_7^{IV}$	log. 7.8033365*n*;
$N_7' = +0.02171584 N_7^{IV}$	" 8.3367766;
$N_7'' = -0.1536641 N_7^{IV}$	" 9.1865723*n*;
$N_7''' = -0.9634742 N_7^{IV}$	" 9.9838401*n*;
$N_7^{V} = -3.091784 N_7^{IV}$	" 0.4902090*n*;
$N_7^{VI} = +0.1154710 N_7^{IV}$	" 9.0624732;
$N_7^{VII} = +0.00872848 N_7^{IV}$	" 7.9409386.

25. We must now substitute the numbers we have computed, in the last article, in equations (134) and (135); making use of the logarithms of $\frac{m'}{n'a'}h'$, $\frac{m}{n'a'}l'$, $\frac{m''}{n''a''}$, &c., which were used for the adopted masses, but using the following values of log. $\frac{m}{na}$, log. $\frac{m}{na}h$, log. $\frac{m}{na}l$.

$$\text{log.}\ \frac{m}{na} = 87.3921041;\ \text{log.}\ \frac{m}{na}h = 86.6903449;\ \text{log.}\ \frac{m}{na}l = 86.1148475.$$

For the root $g = 5''.344676$, we get,

$$x = +\frac{4618325}{10^{20}}N;\ y = +\frac{259345}{10^{20}}N;\ z = +\frac{27356380}{10^{20}}N^2.$$

Whence $\beta = 86^\circ\ 47'\ 9''.2$; and log. $N = 9.2281141$.

$N = +0.1690885$;	$N^{IV} = -0.0000198$;
$N' = +0.0174481$;	$N^{V} = -0.0000175$;
$N'' = +0.0110355$;	$N^{VI} = +0.00000805$;
$N''' = +0.0016926$;	$N^{VII} = +0.000000127$.

For the root $g_1=7''.538699$, we get,

$$x_1=+\frac{4456102}{10^{20}}N_1;\quad y_1=+\frac{5531411}{10^{20}}N_1;\quad z_1=+\frac{253588600}{10^{20}}N_1^2;$$

Whence $\beta_1=38^\circ\ 13'\ 16''.2$; log. $N_1=8.4434891$.

$N_1\ =+0.0277645$; $N_1^{IV}=+0.00001203$;
$N_1'\ =-0.0242855$; $N_1^{V}\ =+0.00001259$;
$N_1''=-0.0191586$; $N_1^{VI}=-0.000002963$;
$N_1'''=-0.0032310$; $N_1^{VII}=-0.0000001287$.

For the root $g_2=17''.127986$, we get,

$$x_2=-\frac{22856900}{10^{20}}N_2;\quad y_2=+\frac{49895050}{10^{20}}N_2;\quad z_2=+\frac{3.332267}{10^{10}}N_2^2;$$

Whence $\beta_2=335^\circ\ 23'\ 35''.1$; log. $N_2=7.2167768$.

$N_2\ =+0.0016473$; $N_2^{IV}=-0.00000124$;
$N_2'\ =-0.0126245$; $N_2^{V}\ =-0.00000796$;
$N_2''=+0.0122872$; $N_2^{VI}=+0.000000482$;
$N_2'''=+0.0299815$; $N_2^{VII}=+0.0000000355$.

For the root $g_3=17''.836030$, we get,

$$x_3=+\frac{111304510}{10^{20}}N_3;\quad y_3=-\frac{112878600}{10^{20}}N_3;\quad z_3=+\frac{8.789905}{10^{10}}N_3^2.$$

Whence $\beta_3=135^\circ\ 24'\ 8''.2$; log. $N_3=7.2561145$.

$N_3\ =+0.0018035$; $N_3^{IV}=+0.000000872$;
$N_3'\ =-0.0148692$; $N_3^{V}\ =+0.00000947$;
$N_3''=+0.0175270$; $N_3^{VI}=-0.000000536$;
$N_3'''=-0.0720238$; $N_3^{VII}=-0.0000000397$.

For the root $g_4=0''.6166852$, we get,

$$x_4=+\frac{3.358365}{10^{10}}N_4^{IV};\quad y_4=+\frac{1.360318}{10^{10}}N_4^{IV};\quad z_4=+\frac{56975.27}{10^{10}}N_4^{IV2}.$$

Whence $\beta_4=67^\circ\ 56'\ 57''.9$; log. $N_4^{IV}=95.8034310$.

$N_4\ =+0.00000761$; $N_4^{IV}=+0.00006359$;
$N_4'\ =+0.00001098$; $N_4^{V}\ =+0.00007172$;
$N_4''=+0.00001345$; $N_4^{VI}=+0.00155816$;
$N_4'''=+0.00002194$; $N_4^{VII}=+0.0100415$.

For the root $g_5=2''.727688$, we get,

$$x_5=+\frac{0.6475863}{10^{10}}N_5^{IV};\quad y_5=-\frac{0.1743984}{10^{10}}N_5^{IV};\quad z_5=+\frac{345.0960}{10^{10}}N_5^{IV2}.$$

Whence $\beta_5=105^\circ\ 4'\ 20''.9$; log. $N_5^{IV}=7.2885615$.

$N_5\ =+0.00056101$; $N_5^{IV}=+0.0019434$;
$N_5'\ =+0.00054730$; $N_5^{V}\ =+0.0017692$;
$N_5''=+0.00057711$; $N_5^{VI}=+0.0297331$;
$N_5'''=+0.00077525$; $N_5^{VII}=-0.0029105$.

For the root $g_6=3''.716714$, we get,

$$x_6=+\frac{0.4584871}{10^{10}}N_6^{IV};\ y_6=+\frac{0.8567341}{10^{10}}N_6^{IV};\ z_6=+\frac{22.51143}{10^{10}}N_6^{IV2}.$$

Whence $\beta_6=28^\circ\ 9'\ 13''.5$; log. $N_6^{IV}=8.6351298$.

$$\begin{array}{ll} N_6\ =+0.0243977; & N_6^{IV}=+0.0431648; \\ N_6'\ =+0.0165257; & N_6^{V}\ =+0.0341050; \\ N_6''=+0.0162732; & N_6^{VI}=-0.0448615; \\ N_6'''=+0.0187798; & N_6^{VII}=+0.0014203. \end{array}$$

For the root $g_7=22''.460892$, we get,

$$x_7=-\frac{1.009497}{10^{10}}N_7^{IV};\ y_7=+\frac{0.7872134}{10^{10}}N_7^{IV};\ z_7=+\frac{81.35929}{10^{10}}N_7^{IV2}.$$

Whence $\beta_7=307^\circ\ 56'\ 50''.6$; log. $N_7^{IV}=8.1941936$.

$$\begin{array}{ll} N_7\ =-0.0000994; & N_7^{IV}=+0.0156384; \\ N_7'\ =+0.0003396; & N_7^{V}\ =-0.0483507; \\ N_7''=-0.0024030; & N_7^{VI}=+0.0018058; \\ N_7'''=-0.0150672; & N_7^{VII}=+0.00013650. \end{array}$$

The difference between the values here given, and the values depending on the adopted masses, manifestly measures the increment arising from the supposition that $\mu=\frac{3}{2}$; and two-thirds of this difference is the coefficient of μ, in the expression for the values of the constants corresponding to any other value of μ.

26. We shall now suppose that $\mu'=+\frac{1}{20}$, and the mass of *Venus* will become $m'=(1+\frac{1}{20})\div 390000=1\div 371428.6$. Using this mass, we shall find,

$$\left.\begin{array}{ll} \Delta\boxed{0,0}=-0'.1499336; & \Delta\boxed{4,4}=-0''.0002106; \\ \Delta\boxed{1,1}=\ 0. \qquad\quad ; & \Delta\boxed{5,5}=-0\ .0000246; \\ \Delta\boxed{2,2}=-0\ .2665548; & \Delta\boxed{6,6}=-0\ .0000021; \\ \Delta\boxed{3,3}=-0\ .0241609; & \Delta\boxed{7,7}=-0\ .0000004. \end{array}\right\}\ (193)$$

Whence,

$$\left.\begin{array}{ll} \boxed{0,0}=g-\ 5''.7201894; & \boxed{4,4}=g-\ 7''.5125860; \\ \boxed{1,1}=g-11\ .3147682; & \boxed{5,5}=g-18\ .5962375; \\ \boxed{2,2}=g-13\ .3387278; & \boxed{6,6}=g-\ 2\ .7662543; \\ \boxed{3,3}=g-17\ .5770254; & \boxed{7,7}=g-\ 0\ .6479573. \end{array}\right\}\ (194)$$

These quantities give,

$$\boxed{0,0}\boxed{1,1}=g^2-17.0349576.g+\ 64.7226171211; \qquad (195)$$
$$\boxed{0,0}\boxed{2,2}=g^2-19.0589172.g+\ 76.3000493710; \qquad (196)$$
$$\boxed{0,0}\boxed{3,3}=g^2-23.2972148.g+100.5439143766; \qquad (197)$$
$$\boxed{1,1}\boxed{2,2}=g^2-24.6534960.g+150.9246131399; \qquad (198)$$
$$\boxed{1,1}\boxed{3,3}=g^2-28.8917936.g+198.8799680465; \qquad (199)$$
$$\boxed{2,2}\boxed{3,3}=g^2-30.9157532.g+234.4551573443; \qquad (200)$$

$$\boxed{4,4}\boxed{5,5}=g^2-26.1088235.g+139.7058334952; \qquad (201)$$
$$\boxed{4,4}\boxed{6,6}=g^2-10.2788403.g+\ 20.7817233266; \qquad (202)$$
$$\boxed{4,4}\boxed{7,7}=g^2-\ 8.1605433.g+\ \ 4.8678349406; \qquad (203)$$
$$\boxed{5,5}\boxed{6,6}=g^2-21.3624918.g+\ 51.4419219482; \qquad (204)$$
$$\boxed{5,5}\boxed{7,7}=g^2-19.2441948.g+\ 12.0495678407; \qquad (205)$$
$$\boxed{6,6}\boxed{7,7}=g^2-\ 3.4142116.g+\ \ 1.79241466734. \qquad (206)$$

$$\left.\begin{array}{l}\boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=g^4-\ 47.9507108.g^3+\ 825.82631940.g^2 \\ \qquad\qquad -5994.8821218.g+15174.5513808\end{array}\right\}; \qquad (207)$$

$$\left.\begin{array}{l}\boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=g^4-\ 29.5230351.g^3+230.63929622.g^2 \\ \qquad\qquad -523.78311550.g+250.41078507\end{array}\right\}. \qquad (208)$$

We shall therefore obtain the following

Fundamental Equations for $\mu'=+\frac{1}{20}$; *or for* $m'=\frac{1}{371428.6}$.

$$A\ =g^2-40.63827162.g\ +200.0557143; \qquad (209)$$
$$A'=g^2-23.31994327.g\ +100.7733425; \qquad (210)$$
$$A''=g^2-18.22902235.g\ +\ 71.3586722; \qquad (211)$$
$$A_1=g^2-14.5939888.g\ +\ 45.9795152; \qquad (212)$$
$$A_2=g^2-10.00785623.g\ +\ \ 6.35434212; \qquad (213)$$
$$A_3=g^2-26.183183614.g\ +\ 82.92255281; \qquad (214)$$

$$D\ =g^2-48.76765014.g\ +728.640657; \qquad (215)$$
$$D'=g^2-56.24689671.g\ +679.929057; \qquad (216)$$
$$D''=g^2-31.8586536.g\ +250.7543893; \qquad (217)$$
$$D_1=g^2-48.09966549.g\ +201.477252; \qquad (218)$$
$$D_2=g^2-51.052918467.g\ +\ 34.3823105; \qquad (219)$$
$$D_3=g^2-\ 3.4187096566.g+\ \ 1.731950106. \qquad (220)$$

$$\left.\begin{array}{ll}B=\{g-34.9011230\}b; & B'=\{g-17.591388707\}b; \\ & B''=\{g-12.508092541\}b;\end{array}\right\} \qquad (221)$$

$$\left.\begin{array}{l}C\ =\ \{23.9743622\ -g\}[9.1763990]b'; \\ C'=\ \{17.60614518-g\}[8.8694654]b; \\ C''=-[0.2657810]b'; \\ C'''=-[0.2866491]b';\end{array}\right\} \qquad (222)$$

$$\left.\begin{array}{l}E\ =+[9.6231944]b''; \\ E'=\ \{24.7932880\ -g\}[8.5847028]b''; \\ E''=\ \{17.57829706-g\}[9.6375865]b''; \\ E'''=+[0.9518913]b'';\end{array}\right\} \qquad (223)$$

$$\left.\begin{array}{l}F\ =-[7.6499091]b'''; \\ F'\ =+[8.9577383]b'''; \\ F''=\ \{14.28035653-g\}[0.8407439]b'''; \\ F'''=\ \{38.6407515\ -g\}[9.4809266]b''';\end{array}\right\} \qquad (224)$$

$$\left.\begin{array}{ll} B_1=\{g-4.7868001\}b_1; & B_2=\{g-\ 0.6795722937\}b_1; \\ & B_3=\{g-18.665588512\}b_1; \end{array}\right\} \quad (225)$$

$$\left.\begin{array}{l} C_1=\ \{4.365093544\ \ -g\}[9.2840950]b_2; \\ C_2=\ \{0.68791009996-g\}[9.1824311]b_2; \\ C_3=-[0.6832317]b_2; \\ C_4=-[0.6834824]b_2; \end{array}\right\} \quad (226)$$

$$\left.\begin{array}{l} E_1=+[7.8267723]b_3; \\ E_2=\ \{43.7345719\ \ \ -g\}[8.2017618]b_3; \\ E_3=\ \{\ 0.648004519-g\}[0.7610455]b_3; \\ E_4=+[1.0655652]b_3; \end{array}\right\} \quad (227)$$

$$\left.\begin{array}{l} F_1=-[7.0339474]b_4; \\ F_2=+[0.2724895]b_4; \\ F_3=\ \{\ 2.770665138-g\}[1.7737962]b_4; \\ F_4=\ \{50.36500746\ \ -g\}[9.1128486]b_4; \end{array}\right\} \quad (228)$$

$$\left.\begin{array}{r} g^4-47.9507108.g^3+799.53205638.g^2-5381.3548732.g \\ +12499.1914947 \end{array}\right\}=(\chi,\chi_1,\chi_2,\chi_3); \quad (229)$$

$$\left.\begin{array}{r} g^4-29.5230351.g^3+172.74239830.g^2-323.32220515.g \\ +140.47332174 \end{array}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7). \quad (230)$$

The values of b, b', b'', and b''' are given by equations (118); and the values of b_1, b_2, b_3, and b_4 are given by equations (119), by simply adding log. $(1+\frac{1}{20})=$ [0.0211893] to the coefficients of N'.

Putting equations (229) and (230), equal to nothing, they will give,

$$\left.\begin{array}{ll} g=\ 5''.59773937; & g_4=\ 0''.61668564; \\ g_1=\ 7\ .25215980; & g_5=\ 2\ .72773622; \\ g_2=17\ .20072233; & g_6=\ 3\ .71796565; \\ g_3=17\ .90008930; & g_7=22\ .46064759. \end{array}\right\} \quad (231)$$

The equations just computed will give the following values:—

For the root g, we get,

$g=5''.5978504;$

$N'\ =+0.05341742N$	log. 98.7276830;
$N''\ =+0.03480002N$	" 98.5415794;
$N'''\ =+0.00540574N$	" 97.7328551;
$N^{IV}=-0.00005306593N$	" 95.7248158n;
$N^{V}\ =-0.00004782067N$	" 95.6796157n;
$N^{VI}=+0.00001988778N$	" 95.2985863;
$N^{VII}=+0.0000004029593N$	" 93.6052612.

For the root g_1, we get,

$g_1 = 7''.2528745;$

$N_1' = -0.6557708 N_1$	log.	$99.8167520n;$
$N_1'' = -0.5021007 N_1$	"	$99.7007908n;$
$N_1''' = -0.08380783 N_1$	"	$98.9232841n;$
$N_1^{IV} = +0.0003520860 N_1$	"	$96.5466488;$
$N_1^{V} = +0.0003598465 N_1$	"	$96.5561173;$
$N_1^{VI} = -0.00009068952 N_1$	"	$95.9575571n;$
$N_1^{VII} = -0.000003712050 N_1$	"	$94.5696138n.$

For the root g_2, we get,

$g_2 = 17''.2008675;$

$N_2' = -\ 7.2870812 N_2$	log.	$0.8625536n;$
$N_2'' = +\ 7.5219298 N_2$	"	$0.8763293;$
$N_2''' = +21.345970 N_2$	"	$1.3293159;$
$N_2^{IV} = -\ 0.0008132064 N_2$	"	$96.9102008n;$
$N_2^{V} = -\ 0.005455322 N_2$	"	$97.7368204n;$
$N_2^{VI} = +\ 0.0003281043 N_2$	"	$96.5160119;$
$N_2^{VII} = +\ 0.00002415603 N_2$	"	$95.3830256.$

For the root g_3, we get,

$g_3 = 17''.9001137;$

$N_3' = -\ 7.82904 N_3$	log.	$0.8937084n;$
$N_3'' = +\ 9.67313 N_3$	"	$0.9855672;$
$N_3''' = -33.37815 N_3$	"	$1.5234623n;$
$N_3^{IV} = +\ 0.000347642 N_3$	"	$96.5411317;$
$N_3^{V} = +\ 0.004032944 N_3$	"	$97.6056222;$
$N_3^{VI} = -\ 0.000226715 N_3$	"	$96.3554809n;$
$N_3^{VII} = -\ 0.00001681933 N_3$	"	$95.2258087n.$

For the root g_4, we get,

$g_4 = 0''.61668534;$

$N_4 = +\ 0.1207828 N_4^{IV}$	log.	$9.0820049;$
$N_4' = +\ 0.1835444 N_4^{IV}$	"	$9.2637411;$
$N_4'' = +\ 0.2119770 N_4^{IV}$	"	$9.3262888;$
$N_4''' = +\ 0.3449880 N_4^{IV}$	"	$9.5378038;$
$N_4^{V} = +\ 1.127829 N_4^{IV}$	"	$0.0522431;$
$N_4^{VI} = +\ 24.50116 N_4^{IV}$	"	$1.3891867;$
$N_4^{VII} = +157.8978 N_4^{IV}$	"	$2.1983761.$

For the root g_5, we get,

$g_5 = 2''.7276716;$

$N_5 = +\ 0.2857445 N_5^{IV}$	log.	$9.4559779;$
$N_5' = +\ 0.2850941 N_5^{IV}$	"	$9.4549882;$
$N_5'' = +\ 0.2978305 N_5^{IV}$	"	$9.4739691;$
$N_5''' = +\ 0.3989201 N_5^{IV}$	"	$9.6008859;$
$N_5^{V} = +\ 0.9103906 N_5^{IV}$	"	$9.9592278;$
$N_5^{VI} = +15.30032 N_5^{IV}$	"	$1.1847004;$
$N_5^{VII} = -\ 1.497701 N_5^{IV}$	"	$0.1754251n.$

For the root g_6, we get,

$$
\begin{array}{lll}
g_6=3''.7167656; & & \\
N_6 \;=+0.5404720N_6^{IV} & \text{log. } 9.7327733; \\
N_6' \;=+0.3822413N_6^{IV} & \text{“ } 9.5823376; \\
N_6'' \;=+0.3758324N_6^{IV} & \text{“ } 9.5748787; \\
N_6''' =+0.4347748N_6^{IV} & \text{“ } 9.6382644; \\
N_6^{V} =+0.7901155N_6^{IV} & \text{“ } 9.8976906; \\
N_6^{VI} =-1.039253N_6^{IV} & \text{“ } 0.0167214n; \\
N_6^{VII}=+0.03290012N_6^{IV} & \text{“ } 8.5171974.
\end{array}
$$

For the root g_7, we get,

$$
\begin{array}{lll}
g_7=22''.4609133; & & \\
N_7 \;=-0.006641048N_7^{IV} & \text{log. } 7.8222366n; \\
N_7' \;=+0.02354937N_7^{IV} & \text{“ } 8.3719794; \\
N_7'' \;=-0.1585801N_7^{IV} & \text{“ } 9.2002488n; \\
N_7''' =-0.9647544N_7^{IV} & \text{“ } 9.9844168n; \\
N_7^{V} =-3.091771N_7^{IV} & \text{“ } 0.4902072n; \\
N_7^{VI} =+0.1154698N_7^{IV} & \text{“ } 9.0624686; \\
N_7^{VII}=+0.008728388N_7^{IV} & \text{“ } 7.9409340.
\end{array}
$$

27. These numbers are now to be substituted in equations (134) and (135). We must also add the logarithm of $(1+\mu')$ to the logarithms of $\frac{m'}{n'a'}$, $\frac{m'}{n'a'}h'$, and $\frac{m'}{n'a'}l'$, which were used for the adopted masses; and by this means they will become,

$$\text{log. } \frac{m'}{n'a'}=88.2471963; \text{ log. } \frac{m'}{n'a'}h' =85.9698939; \text{ log. } \frac{m'}{n'a'}\,l'= 85.8857055n.$$

We shall therefore get for the root $g=5''.5978504$;

$$x=+\frac{1853795}{10^{20}}N, \; y=+\frac{41624.4}{10^{20}}N, \; z=\frac{10625807}{10^{20}}N^2.$$

Whence $\beta=88°\ 42'\ 49''.4$; log. $N=9.2418092$.

$$
\begin{array}{ll}
N \;=+0.1745056; & N^{IV}=-0.0000092603; \\
N' \;=+0.0093216; & N^{V} \;=-0.000008345; \\
N'' \;=+0.0060728; & N^{VI}=+0.000003470; \\
N''' =+0.00094333; & N^{VII}=+0.0000000703.
\end{array}
$$

For the root $g_1=7''.252874$, we get,

$$x_1=+\frac{1689761}{10^{20}}N_1, \; y_1=+\frac{3937391}{10^{20}}N_1, \; z_1=\frac{138888320}{10^{20}}N_1^2.$$

Whence $\beta_1=23°\ 23'\ 37''.0$; log. $N_1=98.4892507$.

$$
\begin{array}{ll}
N_1 \;=+0.0308497; & N_1^{IV}=+0.000010862; \\
N_1' \;=-0.0202303; & N_1^{V} \;=+0.000011101; \\
N_1'' \;=-0.0154897; & N_1^{VI}=-0.0000027974; \\
N_1''' =-0.00258544; & N_1^{VII}=-0.00000011452.
\end{array}
$$

For the root $g_2=17''.200867$, we get,

$$x_2=-\frac{32718590}{10^{20}}N_2,\ y_2=+\frac{57870410}{10^{20}}N_2,\ z_2=\frac{3.742550}{10^{10}}N_2^2.$$

Whence $\beta_2=330°\ 31'\ 1''.9$; log. $N_2=97.2495186$.

$N_2\ =+0.00177621$;	$N_2^{IV}=-0.0000014445$;
$N_2'=-0.0129441$;	$N_2^{V}\ =-0.0000096903$;
$N_2''=+0.0133612$;	$N_2^{VI}=+0.00000058281$;
$N_2'''=+0.0379170$;	$N_2^{VII}=+0.00000004291$.

For the root $g_3=17''.900114$, we get,

$$x_3=+\frac{93612050}{10^{20}}N_3,\ y_3=-\frac{95375820}{10^{20}}N_3,\ z_3=\frac{7.000929}{10^{10}}N_3^2.$$

Whence $\beta_3=135°\ 32'\ 5''.0$; log. $N_3=97.2807821$.

$N_3\ =+0.0019089$;	$N_3^{IV}=+0.00000066361$;
$N_3'=-0.0149448$;	$N_3^{V}\ =+0.00000769847$;
$N_3''=+0.0184650$;	$N_3^{VI}=-0.00000043278$;
$N_3'''=-0.0637154$;	$N_3^{VII}=-0.00000003210$.

For the root $g_4=0''.6166853$, we get,

$$x_4=+\frac{3.357379}{10^{10}}N_4^{IV},\ y_4=+\frac{1.360319}{10^{10}}N_4^{IV},\ z_4=\frac{56977.20}{10^{10}}N_4^{IV2}.$$

Whence $\beta_4=67°\ 56'\ 36''.7$; log. $N_4^{IV}=95.8033066$.

$N_4\ =+0.000007679$;	$N_4^{IV}=+0.000063578$;
$N_4'=+0.000011669$;	$N_4^{V}\ =+0.000071705$;
$N_4''=+0.000013477$;	$N_4^{VI}=+0.00155773$;
$N_4'''=+0.000021934$;	$N_4^{VII}=+0.01003882$.

For the root $g_5=2''.727672$, we get,

$$x_5=+\frac{0.6476120}{10^{10}}N_5^{IV},\ y_5=-\frac{0.1744696}{10^{10}}N_5^{IV},\ z_5=\frac{345.1504}{10^{10}}N_5^{IV2}.$$

Whence $\beta_5=105°\ 4'\ 40''.0$; log. $N_5^{IV}=97.2885211$.

$N_5\ =+0.00055526$;	$N_5^{IV}=+0.00194322$;
$N_5'=+0.00055400$;	$N_5^{V}\ =+0.00176909$;
$N_5''=+0.00057875$;	$N_5^{VI}=+0.0297318$;
$N_5'''=+0.00077519$;	$N_5^{VII}=-0.00291036$.

For the root $g_6=3''.716766$, we get,

$$x_6=+\frac{0.4583190}{10^{10}}N_6^{IV},\ y_6=+\frac{0.8566834}{10^{10}}N_6^{IV},\ z_6=\frac{22.51091}{10^{10}}N_6^{IV2}.$$

Whence $\beta_6=28°\ 8'\ 47''.2$; log. $N_6^{IV}=98.6350843$.

$N_6\ =+0.0233269$;	$N_6^{IV}=+0.0431603$;
$N_6'=+0.0164976$;	$N_6^{V}\ =+0.0341016$;
$N_6''=+0.0162167$;	$N_6^{VI}=-0.0448545$;
$N_6'''=+0.0187650$;	$N_6^{VII}=+0.0014200$.

For the root $g_7=22''.460913$, we get,

$$x_7=-\frac{1.009492}{10^{10}}N_7^{IV},\ y_7=+\frac{0.7872137}{10^{10}}N_7^{IV},\ z_7=\frac{81.858790}{10^{10}}N_7^{IV2}.$$

Whence $\beta_7=307°\ 56'\ 51''.1$; log. $N_7^{IV}=98.1941950$.

$N_7\ =-0.0001039$;	$N_7^{IV}=+0.0156385$;
$N_7'=+0.00036828$;	$N_7^{V}\ =-0.0483506$;
$N_7''=-0.00247996$;	$N_7^{VI}=+0.0018058$;
$N_7'''=-0.0150873$;	$N_7^{VII}=+0.0001365$.

28. We shall now suppose $\mu''=+\frac{1}{20}$; or, the mass of Earth to be $m''=(1+\frac{1}{20})\div 368689=1\div 351132.4$. Using this mass, we shall find,

$$\left.\begin{array}{ll} \Delta\boxed{0,0}=-0''.0430854; & \Delta\boxed{4,4}=-0''.0004406; \\ \Delta\boxed{1,1}=-0\ .3315294; & \Delta\boxed{5,5}=-0\ .0000502; \\ \Delta\boxed{2,2}=\ \ 0\ . \qquad\quad ; & \Delta\boxed{6,6}=-0\ .0000043; \\ \Delta\boxed{3,3}=-0\ .0878224; & \Delta\boxed{7,7}=-0\ .0000009. \end{array}\right\} \quad (232)$$

Whence we get,

$$\left.\begin{array}{ll} \boxed{0,0}=g-\ \ 5''.6133412; & \boxed{4,4}=g-\ \ 7''.5128160; \\ \boxed{1,1}=g-11\ .6462976; & \boxed{5,5}=g-18\ .5962631; \\ \boxed{2,2}=g-13\ .0721730; & \boxed{6,6}=g-\ \ 2\ .7662565; \\ \boxed{3,3}=g-17\ .6406869; & \boxed{7,7}=g-\ \ 0\ .6479578. \end{array}\right\} \quad (233)$$

These quantities give the following equations,

$$\left.\begin{array}{l} \boxed{0,0}\boxed{1,1}=g^2-17.2596388.g+\ \ 65.3746421455; \\ \boxed{0,0}\boxed{2,2}=g^2-18.6855142.g+\ \ 73.3785672744; \\ \boxed{0,0}\boxed{3,3}=g^2-23.2540281.g+\ \ 99.0231945721; \\ \boxed{1,1}\boxed{2,2}=g^2-24.7184706.g+152.2424170367; \\ \boxed{1,1}\boxed{3,3}=g^2-29.2869845.g+205.4486895058; \\ \boxed{2,2}\boxed{3,3}=g^2-30.7128599.g+230.6021109956; \end{array}\right\} \quad (234)$$

$$\left.\begin{array}{l} \boxed{4,4}\boxed{5,5}=g^2-26.1090791.g+139.7103029579; \\ \boxed{4,4}\boxed{6,6}=g^2-10.2790725.g+\ \ 20.7823760933; \\ \boxed{4,4}\boxed{7,7}=g^2-\ \ 8.1607738.g+\ \ \ \ 4.8679877272; \\ \boxed{5,5}\boxed{6,6}=g^2-21.3625196.g+\ \ 51.4420336761; \\ \boxed{5,5}\boxed{7,7}=g^2-19.2442209.g+\ \ 12.0495937275; \\ \boxed{6,6}\boxed{7,7}=g^2-\ \ 3.4142143.g+\ \ \ \ 1.79241747598. \end{array}\right\} \quad (235)$$

$$\left.\begin{array}{l} \boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=g^4-\ \ 47.9724987.g^3\ +\ \ 826.06962154.g^2 \\ \qquad\qquad -5987.951367523.g+15075.53048434 \end{array}\right\} \quad (236)$$

$$\left.\begin{array}{l} \boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=g^4-\ \ 29.5232934.g^3\ +230.64471166.g^2 \\ \qquad\qquad -523.79928387.g\ +250.41918860 \end{array}\right\} \quad (237)$$

We shall therefore obtain the following

Fundamental Equations for $\mu''=+\frac{1}{20}$; *or, for* $m''=1\div351132.4$.

$$\left.\begin{aligned}
A&=g^2-41.34298838.g+200.8870856;\\
A'&=g^2-23.27675657.g+99.2516206;\\
A''&=g^2-18.39687878.g+71.57390627;\\
A_1&=g^2-14.594221002.g+45.9806378;\\
A_2&=g^2-10.00808673.g+6.35450308;\\
A_3&=g^2-26.183439214.g+82.92703835;
\end{aligned}\right\}: \qquad (238)$$

$$\left.\begin{aligned}
D&=g^2-48.72257219.g+727.694841;\\
D'&=g^2-55.34085031.g+665.272796;\\
D''&=g^2-31.70284173.g+247.7780603;\\
D_1&=g^2-48.09969329.g+201.477460;\\
D_2&=g^2-51.052944567.g+34.3823533;\\
D_3&=g^2-3.4187123566.g+1.7319529170.
\end{aligned}\right\} \qquad (239)$$

$$B=\{g-35.7126879\}b;\quad \left.\begin{aligned}B'&=\{g-17.655050207\}b;\\ B''&=\{g-12.782796972\}b;\end{aligned}\right\} \qquad (240)$$

$$\left.\begin{aligned}
C&=\{24.2395891-g\}[9.1763990]b';\\
C'&=\{17.66980668-g\}[8.8694654]b';\\
C''&=-[0.2445917]b';\\
C'''&=-[0.2654598]b'.
\end{aligned}\right\} \qquad (241)$$

$$\left.\begin{aligned}
E&=+[9.6443837]b'';\\
E'&=\{24.4829831-g\}[8.5847028]b'';\\
E''&=\{17.64195856-g\}[9.6375865]b'';\\
E'''&=+[0.9730806]b''.
\end{aligned}\right\} \qquad (242)$$

$$\left.\begin{aligned}
F&=-[7.6499091]b''';\\
F'&=+[8.9577383]b''';\\
F''&=\{14.06088317-g\}[0.8407439]b''';\\
F'''&=\{37.6710436-g\}[9.4809266]b'''.
\end{aligned}\right\} \qquad (243)$$

$$B_1=\{g-4.7868023\}b_1;\quad \left.\begin{aligned}B_2&=\{g-0.6795727937\}b_1;\\ B_3&=\{g-18.665614112\}b_1;\end{aligned}\right\} \qquad (244)$$

$$\left.\begin{aligned}
C_1&=\{4.365095744-g\}[9.2840950]b_2;\\
C_2&=\{0.6879114996-g\}[9.1824311]b_2;\\
C_3&=-[0.6832317]b_2;\\
C_4&=-[0.6834824]b_2;
\end{aligned}\right\} \qquad (245)$$

$$\left.\begin{aligned}
E_1&=+[7.8267723]b_3;\\
E_2&=\{43.7345975-g\}[8.2017618]b_3;\\
E_3&=\{0.648005019-g\}[0.7610455]b_3;\\
E_4&=+[1.0655652]b_3;
\end{aligned}\right\} \qquad (246)$$

$$\left.\begin{aligned}
F_1&=-[7.0339474]b_4;\\
F_2&=+[0.2724895]b_4;\\
F_3&=\{2.770667338-g\}[1.7737962]b_4;\\
F_4&=\{50.3650331-g\}[9.1128486]b_4;
\end{aligned}\right\} \qquad (247)$$

$$\left.\begin{array}{r} g^4-47.9724987.g^3+799.77118048.g^2-5375.58619284.g \\ +12441.50566776 \end{array}\right\}=(\chi,\chi_1,\chi_2,\chi_2); \qquad (248)$$

$$\left.\begin{array}{r} g^4-29.5232934.g^3+172.74781374.g^2-\ 323.33816841.g \\ +140.48125235 \end{array}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7). \qquad (249)$$

The values of b, b', b'', and b''' are given by equations (118); and the values of b_1, b_2, b_3, and b_4 are given by equations (119), by simply adding log. $(1+\frac{1}{20})=$ [0.0211893], to the coefficients of N''.

Putting equations (248) and (249), equal to nothing, they will give,

$$\left.\begin{array}{ll} g=\ 5''.5094502; & g_4=\ 0''.61668624; \\ g_1=\ 7\ .3146241; & g_5=\ 2\ .7277498; \\ g_2=17\ .2173969; & g_6=\ 3\ .7181422; \\ g_3=17\ .9310275; & g_7=22\ .4607151. \end{array}\right\} \qquad (250)$$

The equations just computed, will now give the following values:

For the root g, we get,

$$\begin{array}{lll} g=5''.5095453; & & \\ N'\ =+0.04709515N & \text{log.} & 98.6729762; \\ N''\ =+0.03032428N & \text{“} & 98.4817906; \\ N'''\ =+0.004803897N & \text{“} & 97.6815864; \\ N^{IV}=-0.0000499083N & \text{“} & 95.6981728n; \\ N^{V}\ =-0.00004469284N & \text{“} & 95.6502379n; \\ N^{VI}=+0.00001922807N & \text{“} & 95.2839357; \\ N^{VII}=+0.0000003604268N & \text{“} & 93.5568171. \end{array}$$

For the root g_1, we get,

$$\begin{array}{lll} g_1=7.3153801; & & \\ N_1'\ =-0.7533911N_1 & \text{log.} & 9.8770206n; \\ N_1''\ =-0.5814061N_1 & \text{“} & 9.7644796n; \\ N_1'''\ =-0.0996916N_1 & \text{“} & 8.9986586n; \\ N_1^{IV}=+0.0004072400N_1 & \text{“} & 6.6098504; \\ N_1^{V}\ =+0.0004183444N_1 & \text{“} & 6.6215340; \\ N_1^{VI}=-0.0001038256N_1 & \text{“} & 6.0163044n; \\ N_1^{VII}=-0.000004308711N_1 & \text{“} & 4.6343474n. \end{array}$$

For the root g_2, we get,

$$\begin{array}{lll} g_2=17''.2175318; & & \\ N_2'\ =-\ 7.713321N_2 & \text{log.} & 0.8872414n; \\ N_2''\ =+\ 7.200735N_2 & \text{“} & 0.8573768; \\ N_3'''\ =+19.05947N_2 & \text{“} & 1.2801108; \\ N_3^{IV}=-\ 0.0007473015N_2 & \text{“} & 96.8734959n; \\ N_2^{V}\ =-\ 0.005061101N_2 & \text{“} & 97.7042450n; \\ N_2^{VI}=+\ 0.0003038889N_2 & \text{“} & 96.4827148; \\ N_2^{VII}=+\ 0.00002237723N_2 & \text{“} & 95.3498063. \end{array}$$

For the root g_3, we get,

$g_3 = 17''.9310572$;

$N_3' = -\ 8.306614 N_3$	log. $0.9194240n$;
$N_3'' = +\ 9.36934 N_3$	" 0.9717089;
$N_3''' = -37.82400 N_3$	" $1.5777676n$;
$N_3^{IV} = +\ 0.000407767 N_3$	" 96.6104120;
$N_3^{V} = +\ 0.00487549 N_3$	" 97.6880184;
$N_3^{VI} = -\ 0.0002732457 N_3$	" $96.4365534n$;
$N_3^{VII} = -\ 0.00002027655 N_3$	" $95.3069941n$.

For the root g_4, we get,

$g_4 = 0''.6166859$;

$N_4 = +\ 0.1210308 N_4^{IV}$	log. 9.0828960;
$N_4' = +\ 0.1840970 N_4^{IV}$	" 9.2650466;
$N_4'' = +\ 0.2134317 N_4^{IV}$	" 9.3292589;
$N_4''' = +\ 0.3445194 N_4^{IV}$	" 9.5372136;
$N_4^{V} = +\ 1.127857 N_4^{IV}$	" 0.0522542;
$N_4^{VI} = +\ 24.50277 N_4^{IV}$	" 1.3892152;
$N_4^{VII} = +157.9085 N_4^{IV}$	" 2.1984055.

For the root g_5, we get,

$g_5 = 2''.7276841$;

$N_5 = +\ 0.2888086 N_5^{IV}$	log. 9.4606101;
$N_5' = +\ 0.2846311 N_5^{IV}$	" 9.4542824;
$N_5'' = +\ 0.2991853 N_5^{IV}$	" 9.4759404;
$N_5''' = +\ 0.3984217 N_5^{IV}$	" 9.6003430;
$N_5^{V} = +\ 0.9104170 N_5^{IV}$	" 9.9592404;
$N_5^{VI} = +15.30294 N_5^{IV}$	" 1.1847750;
$N_5^{VII} = -\ 1.497941 N_5^{IV}$	" $0.1754948n$.

For the root g_6, we get,

$g_6 = 3''.7169230$;

$N_6 = +0.5533846 N_6^{IV}$	log. 9.7430272;
$N_6' = +0.3799388 N_6^{IV}$	" 9.5797136;
$N_6'' = +0.3762970 N_6^{IV}$	" 9.5755308;
$N_6''' = +0.4342462 N_6^{IV}$	" 9.6377360;
$N_6^{V} = +0.790124 N_6^{IV}$	" 9.8976952;
$N_6^{VI} = -1.039090 N_6^{VI}$	" $0.0166530n$;
$N_6^{VII} = +0.0328877 N_6^{IV}$	" 8.5170337.

For the root g_7, we get,

$g_7 = 22''.4609846$;

$N_7 = -0.006671589 N_7^{IV}$	log. $7.8242293n$;
$N_7' = +0.02618465 N_7^{IV}$	" 8.4180469;
$N_7'' = -0.1544778 N_7^{IV}$	" $9.1888659n$;
$N_7''' = -0.9765513 N_7^{IV}$	" $9.9896951n$;
$N_7^{V} = -3.091724 N_7^{IV}$	" $0.4902020n$;
$N_7^{VI} = +0.1154676 N_7^{IV}$	" 9.0624601;
$N_7^{VII} = +0.008728216 N_7^{IV}$	" 7.9409255.

29. These numbers are now to be substituted in equations (134) and (135). We must also add the logarithm of $(1+\mu'')$ to those of $\frac{m''}{n''a''}$, $\frac{m''}{n''a''}h''$, and $\frac{m''}{n''a''}l''$, in order to obtain the numbers which are to be used in this computation.

For the root $g=5''.5095453$, we get,

$$x=+\frac{1853601}{10^{20}}N;\quad y=+\frac{73698.3}{10^{20}}N;\quad z=\frac{10443191}{10^{20}}.$$

Whence $\beta=87°\ 43'\ 23''.3$; log. $N=9.2495260$.

$$\begin{aligned} N &=+0.1776340; & N^{IV} &=-0.000008865; \\ N' &=+0.0083657; & N^{V} &=-0.000007939; \\ N'' &=+0.0053866; & N^{VI} &=+0.000003416; \\ N''' &=+0.00085332; & N^{VII} &=+0.0000000640. \end{aligned}$$

For the root $g_1=7''.315380$, we get,

$$x_1=+\frac{1595251}{10^{20}}N_1;\quad y_1=+\frac{4450991}{10^{20}}N_1;\quad z_1=+\frac{180052670}{10^{20}}N_1^2.$$

Whence $\beta_1=19°\ 43'\ 4''.3$; log. $N_1=98.4192990$.

$$\begin{aligned} N_1 &=+0.0262603; & N_1^{IV} &=+0.000010694; \\ N_1' &=-0.0197843; & N_1^{V} &=+0.000010986; \\ N_1'' &=-0.0152679; & N_1^{VI} &=-0.0000027265; \\ N_1''' &=-0.00261793; & N_1^{VII} &=-0.00000011315. \end{aligned}$$

For the root $g_2=17''.217532$, we get,

$$x_2=-\frac{27629730}{10^{20}}N_2;\quad y_2=+\frac{51666060}{10^{20}}N_2;\quad z_2=\frac{3.432446}{10^{10}}N_2^2.$$

Whence $\beta_2=331°\ 51'\ 47''.6$; log. $N_2=97.2322182$.

$$\begin{aligned} N_2 &=+0.00170694; & N_2^{IV} &=-0.0000012756; \\ N_2' &=-0.0131662; & N_2^{V} &=-0.000008639; \\ N_2'' &=+0.0122912; & N_2^{VI} &=+0.0000005187; \\ N_2''' &=+0.0325334; & N_2^{VII} &=+0.0000000382. \end{aligned}$$

For the root $g_3=17''.931057$, we get,

$$x_3=+\frac{104020250}{10^{20}}N_3;\quad y_3=-\frac{108060520}{10^{20}}N_3;\quad z_3=\frac{8.174639}{10^{10}}N_3^2.$$

Whence $\beta_3=136°\ 5'\ 29''.0$; log. $N_3=97.2635963$.

$$\begin{aligned} N_3 &=+0.0018348; & N_3^{IV} &=+0.0000007482; \\ N_3' &=-0.0152412; & N_3^{V} &=+0.000008946; \\ N_3'' &=+0.0171912; & N_3^{VI} &=-0.0000005014; \\ N_3''' &=-0.0694007 & N_3^{VII} &=-0.0000000372. \end{aligned}$$

For the root $g_4=0''.6166859$, we get,

$$x_4=+\frac{3.357579}{10^{10}}N_4^{IV};\ y_4=+\frac{1.360360}{10^{10}}N_4^{IV};\ z_4=\frac{56981.24}{10^{10}}N_4^{IV2}.$$

Whence $\beta_4=67°\ 56'\ 38''.9$; log. $N_4^{IV}=95.8033000$.

$N_4\ =+0.000007695$;	$N_4^{IV}=+0.000063577$;
$N_4'\ =+0.00001170$;	$N_4^{V}\ =+0.000071706$;
$N_4''=+0.000013569$;	$N_4^{VI}=+0.00155781$;
$N_4'''=+0.000021904$;	$N_4^{VII}=+0.01003935$.

For the root $g_5=2''.727684$, we get,

$$x_5=+\frac{0.6476505}{10^{10}}N_5^{IV};\ y_5=-\frac{0.1746351}{10^{0}}N_5^{IV};\ z_5=\frac{345.2619}{10^{10}}N_5^{IV2}.$$

Whence $\beta_5=105°\ 5'\ 26''.0$; log. $N_5^{IV}=97.2884328$.

$N_5\ =+0.00056110$;	$N_5^{IV}=+0.00194282$;
$N_5'\ =+0.00055299$;	$N_5^{V}\ =+0.00176878$;
$N_5''=+0.00058126$;	$N_5^{VI}=+0.02973089$;
$N_5'''=+0.00077406$;	$N_5^{VII}=-0.00291023$.

For the root $g_6=3''.716923$, we get,

$$x_6=\frac{0.4583310}{10^{10}}N_6^{IV};\ y_6=+\frac{0.8566739}{10^{10}}N_6^{IV};\ z_6=+\frac{22.51056}{10^{10}}N_6^{IV2}.$$

Whence $\beta_6=28°\ 8'\ 50''.4$; log. $N_6^{IV}=98.6350898$.

$N_6\ =+0.0238845$;	$N_6^{IV}=+0.04316083$;
$N_6'\ =+0.0163985$;	$N_6^{V}\ =+0.03410240$;
$N_6''=+0.0162413$;	$N_6^{VI}=-0.04484797$;
$N_0'''=+0.0187424$;	$N_6^{VII}=+0.00141946$.

For the root $g_7,=22''.460985$, we get,

$$x_7=-\frac{1.009476}{10^{10}}N_7^{IV};\ y_7=+\frac{0.7872105}{10^{10}}N_7^{IV};\ z_7=+\frac{81.85731}{10^{10}}N_7^{IV2}.$$

Whence $\beta_7=307°\ 56'\ 52''.2$; log. $N_7^{IV}=98.1941978$.

$N_7\ =-0.0001043$;	$N_7^{IV}=+0.01563860$;
$N_7'\ =+0.0004095$;	$N_7^{V}\ =-0.04835037$;
$N_7''=-0.0024158$;	$N_7^{VI}=+0.00180575$;
$N_7'''=-0.0152719$;	$N_7^{VII}=+0.000136497$

30. The assumed mass of the earth was obtained on the hypothesis that the mean equatorial parallax of the sun is only $8''.50$. Recent investigations indicate that the sun's parallax is greater by at least one-thirtieth part. Consequently the assumed mass of the earth is too small by its *one-ninth* or *one-tenth* part; and on account of the importance of this element in most astronomical theories, we shall here give another determination of the constants corresponding to an increment of $\frac{1}{10}$ to the earth's mass. This new determination will be useful as an indication to

what extent the variation of the constants is proportional to the variation of the masses; and will also enable us to interpolate the constants for intermediate values of the earth's mass.

If we now suppose $\mu''=+\frac{1}{10}$, we shall find $m''=\frac{1.1}{368689}=\frac{1}{335172}$. And by using this value of the mass we shall find,

$$\left.\begin{array}{ll}\Delta\boxed{0,0}=-0''.0861707, & \Delta\boxed{4,4}=-0''.0008812,\\ \Delta\boxed{1,1}=-0\ .6630587, & \Delta\boxed{5,5}=-0\ .0001004,\\ \Delta\boxed{2,2}=\ \ 0\ . \qquad\quad , & \Delta\boxed{6,6}=-0\ .0000086,\\ \Delta\boxed{3,3}=-0\ .1756449, & \Delta\boxed{7,7}=-0\ .0000018,\end{array}\right\} \quad (251)$$

Whence we get,

$$\left.\begin{array}{ll}\boxed{0,0}=g-\ \ 5''.6564265; & \boxed{4,4}=g-\ \ 7''.5132566;\\ \boxed{1,1}=g-11\ .9778269; & \boxed{5,5}=g-18\ .5963133;\\ \boxed{2,2}=g-13\ .0721730; & \boxed{6,6}=g-\ \ 2\ .7662608;\\ \boxed{3,3}=g-17\ .7285094; & \boxed{7,7}=g-\ \ 0\ .6479587.\end{array}\right\} \quad (252)$$

These quantities will give the following equations,

$$\left.\begin{array}{l}\boxed{0,0}\boxed{1,1}=g^2-17.6342534.g+\ \ 67.75169748957285;\\ \boxed{0,0}\boxed{2,2}=g^2-18.7285995.g+\ \ 73.94178576978450;\\ \boxed{0,0}\boxed{3,3}=g^2-23.3849359.g+100.28001037565910;\\ \boxed{1,1}\boxed{2,2}=g^2-25.0499999.g+156.57622540085370;\\ \boxed{1,1}\boxed{3,3}=g^2-29.7063363.g+212.34901678822286;\\ \boxed{2,2}\boxed{3,3}=g^2-30.8006824.g+231.75014190892620.\end{array}\right\} \quad (253)$$

$$\left.\begin{array}{l}\boxed{4,4}\boxed{5,5}=g^2-26.1095699.g+139.71887363689278;\\ \boxed{4,4}\boxed{6,6}=g^2-10.2795174.g+\ \ 20.78362721292128;\\ \boxed{4,4}\boxed{7,7}=g^2-\ \ 8.1612153.g+\ \ \ \ 4.86827997930242;\\ \boxed{5,5}\boxed{6,6}=g^2-21.3625741.g+\ \ 51.44225250630864;\\ \boxed{5,5}\boxed{7,7}=g^2-19.2442720.g+\ \ 12.04964299066071;\\ \boxed{6,6}\boxed{7,7}=g^2-\ \ 3.4142195.g+\ \ \ \ 1.79242275182896.\end{array}\right\} \quad (254)$$

$$\left.\begin{array}{r}\boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=g^4-48.4349358.g^3+842.64887773.g^2\\ -6173.539244345.g+15701.465507779\end{array}\right\} \quad (255)$$

$$\left.\begin{array}{r}\boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=g^4-29.5237894.g^3+230.655099078.g^2\\ -523.830290018.g+250.4352879666\end{array}\right\} \quad (256)$$

We shall therefore obtain the following

Fundamental Equations for $\mu''=+\frac{1}{10}$; *or for* $m''=1\div335172$.

$$\left.\begin{array}{l}A\ =g^2-42.46419343.g\ +208.5335061;\\ A'=g^2-23.40766437.g\ +100.5097899;\\ A''=g^2-18.77149338.g\ +\ \ 74.00017359;\\ A_1=g^2-14.594665902.g+\ \ 45.9827890;\\ A_2=g^2-10.00852823.g\ +\ \ \ \ 6.35481085;\\ A_3=g^2-26.183930014.g+\ \ 82.93563982;\end{array}\right\} \quad (257)$$

$$\left.\begin{aligned}D&=g^2-49.585883207.g+755.319136;\\D'&=g^2-55.76020211.g+674.468336;\\D''&=g^2-31.83774566.g+249.8341677;\\D_1&=g^2-48.09974779.g+201.477866;\\D_2&=g^2-51.052995667.g+34.3824324;\\D_3&=g^2-3.4187175566.g+1.7319581971.\end{aligned}\right\}\tag{258}$$

$$\left.\begin{aligned}B=\{g-36''.7908076\}b;\quad B'&=\{g-17''.742872707\}b;\\B''&=\{g-13\;.114326272\}b;\end{aligned}\right\}\tag{259}$$

$$\left.\begin{aligned}C&=\{24.7713708-g\}[9.1763990]b';\\C'&=\{17.75762918-g\}[8.8694654]b';\\C''&=-[0.2445917]b';\\C'''&=-[0.2654598]b';\end{aligned}\right\}\tag{260}$$

$$\left.\begin{aligned}E&=+[9.6645871]b'';\\E'&=\{24.8145124-g\}[8.5847028]b'';\\E''&=\{17.72978106-g\}[9.6375865]b'';\\E'''&=+[0.9932840]b'';\end{aligned}\right\}\tag{261}$$

$$\left.\begin{aligned}F&=-[7.6499091]b''';\\F'&=+[8.9577383]b''';\\F''&=\{14.10796460-g\}[0.8407439]b''';\\F'''&=\{38.0025729-g\}[9.4809266]b'''.\end{aligned}\right\}\tag{262}$$

$$\left.\begin{aligned}B_1=\{g-4.78680657\}b_1;\quad B_2&=\{g-0.6795736737\}b_1;\\B_3&=\{g-18.665664332\}b_1;\end{aligned}\right\}\tag{263}$$

$$\left.\begin{aligned}C_1&=\{4.365100014-g\}[9.2840950]b_2;\\C_2&=\{0.6879123796-g\}[9.1824311]b_2;\\C_3&=-[0.6832317]b_2;\\C_4&=-[0.6834824]b_2;\end{aligned}\right\}\tag{264}$$

$$\left.\begin{aligned}E_1&=+[7.8267723]b_3;\\E_2&=\{43.7346477-g\}[8.2017618]b_3;\\E_3&=\{0.648005897-g\}[0.7610455]b_3;\\E_4&=+[1.0655652]b_3;\end{aligned}\right\}\tag{265}$$

$$\left.\begin{aligned}F_1&=-[7.0339474]b_4;\\F_2&=+[0.2724895]b_4;\\F_3&=\{2.770671608-g\}[1.7737962]b_4;\\F_4&=\{50.3650833-g\}[9.1128486]b_4;\end{aligned}\right\}\tag{266}$$

$$\left.\begin{aligned}&g^4-48.4349358.g^3+815.10927504.g^2\\&-5528.66691090.g+12909.37803021\end{aligned}\right\}(=\chi,\chi_1,\chi_2,\chi_3);\tag{267}$$

$$\left.\begin{array}{r} g^4 - 29.5237894.g^3 + 172.758201154.g^2 \\ -323.368779974.g + 140.4964513342 \end{array}\right\} (=\chi_4, \chi_5, \chi_6, \chi_7). \qquad (268)$$

The values of b, b', b'', and b''' are given by equations (118); and the values of b_1, b_2, b_3, and b_4 are given by equations (119), by simply adding log. $(1+\frac{1}{10})=$ 0.0413927 to the coefficient of N''.

Putting equations (267) and (268) equal to nothing, they will give,

$$\left.\begin{array}{ll} g = 5''.554906079, & g_4 = 0''.616687321, \\ g_1 = 7\,.378294649, & g_5 = 2\,.727775991, \\ g_2 = 17\,.403271601, & g_6 = 3\,.718480731, \\ g_3 = 18\,.098463471, & g_7 = 22\,.460845357. \end{array}\right\} \quad (268^1)$$

Equations (257–268), in connection with equations (84–97), will now give the following values:—

For the root g, we get,

$$g = 5''.5550002,$$

$N' = +0.04569346N$	log. 98.6598541;
$N'' = +0.02968146N$	" 98.4724852;
$N''' = +0.00482527N$	" 97.6853181;
$N^{IV} = -0.00004897096N$	" 95.6899385n;
$N^{V} = -0.00004399572N$	" 95.6434103n;
$N^{VI} = +0.00001859827N$	" 95.2694725;
$N^{VII} = +0.0000003632732N$	" 93.5602333.

For the root g_1, we get,

$$g_1 = 7''.3790776,$$

$N_1' = -0.755884N_1$	log. 99.8784552n;
$N_1'' = -0.590047N_1$	" 99.7708867n;
$N_1''' = -0.1045547N_1$	" 99.0193438n;
$N_1^{IV} = +0.0004179936N_1$	" 96.6211696;
$N_1^{V} = +0.0004316384N_1$	" 96.6351200;
$N_1^{VI} = -0.0001054817N_1$	" 96.0231771n;
$N_1^{VII} = -0.000004437763N_1$	" 94.6471640n.

For the root g_2, we get,

$$g_2 = 17''.4034121,$$

$N_2' = -\ 7.766638N_2$	log. 0.8902330n;
$N_2'' = +\ 6.689090N_2$	" 0.8253671;
$N_2''' = +23.91929N_2$	" 1.3787482;
$N_2^{IV} = -\ 0.000785956N_2$	" 96.8953983n;
$N_2^{V} = -\ 0.005986225N_2$	" 97.7771530n;
$N_2^{VI} = +\ 0.0003529385N_2$	" 96.5476990;
$N_2^{VII} = +\ 0.00002604318N_2$	" 95.4156940.

For the root g_3, we get,

$$g_3=18''.0984790,$$

$N_3' =- 8.350533N_3$	log. $0.9217142n$;
$N_3'' =+ 8.673780N_3$	" 0.9382084;
$N_3''' =-28.56700N_3$	" $1.4558646n$;
$N_3^{IV} =+ 0.0002228091N_3$	" 96.3479330;
$N_3^{V} =+ 0.003251222N_3$	" 97.5120466;
$N_3^{VI} =- 0.000179346N_3$	" $96.2536923n$;
$N_3^{VII} =- 0.00001332988N_3$	" $95.1248263n$.

For the root g_4, we get,

$$g_4=0''.6166870,$$

$N_4 =+ 0.1207555N_4^{IV}$	log. 9.0819069;
$N_4' =+ 0.1838474N_4^{IV}$	" 9.2644574;
$N_4' =+ 0.2133264N_4^{IV}$	" 9.3290446;
$N_4''' =+ 0.3435830N_4^{IV}$	" 9.5360316;
$N_4^{V} =+ 1.127906N_4^{IV}$	" 0.0522730;
$N_4^{VI} =+ 24.50543N_4^{IV}$	" 1.3892623;
$N_4^{VII} =+157.9263N_4^{IV}$	" 2.1984543.

For the root g_5, we get,

$$g_5=2''.7277089;$$

$N_5 =+ 0.285290N_5^{IV}$	log. 9.4552860;
$N_5' =+ 0.2827626N_5^{IV}$	" 9.4514220;
$N_5'' =+ 0.2983537N_5^{IV}$	" 9.4747313;
$N_5''' =+ 0.397320N_5^{IV}$	" 9.5991402;
$N_5^{V} =+ 0.910471N_5^{IV}$	" 9.9592660;
$N_5^{VI} =+15.30821N_5^{IV}$	" 1.1849243;
$N_5^{VII} =- 1.498422N_5^{IV}$	" $0.1756342n$.

For the root g_6, we get,

$$g_6=3''.7172386;$$

$N_6 =+0.5402483N_6^{IV}$	log. 9.7325934;
$N_6' =+0.3755153N_6^{IV}$	" 9.5746276;
$N_6'' =+0.3741537N_6^{IV}$	" 9.5730500;
$N_6''' =+0.4330305N_6^{IV}$	" 9.6365185;
$N_6^{V} =+0.7901418N_6^{IV}$	" 9.8977050;
$N_6^{VI} =-1.038754N_6^{IV}$	" $0.0165127n$;
$N_6^{VII} =+0.03286241N_6^{IV}$	" 8.5166994.

For the root g_7, we get,

$$g_7=22''.4611216;$$

$N_7 =-0.007172393N_7^{IV}$	log. $97.8556641n$;
$N_7' =+0.0327495N_7^{IV}$	" 98.5152046;
$N_7'' =-0.1572018N_7^{IV}$	" $99.1964576n$;
$N_7''' =-0.9919734N_7^{IV}$	" $99.9965000n$;
$N_7^{V} =-3.091662N_7^{IV}$	" $0.4901920n$;
$N_7^{VI} =+0.1154641N_7^{IV}$	" 99.0624468;
$N_7^{VII} =+0.00872795N_7^{IV}$	" 97.9409123.

31. These numbers are now to be substituted in equations (134) and (135). We must also add the logarithm of $(1+\mu'')$ or 0.0413927 to those of $\frac{m''}{n''a''}$, $\frac{m''}{n''a''}h''$, and $\frac{m''}{n''a''}l''$, in order to obtain the numbers to be used in this computation.

For the root $g=5''.5550002$, we get,

$$x=+\frac{1859252}{10^{20}};\ y=+\frac{982149.1}{10^{20}};\ z=-\frac{10422045}{10^{20}}.$$

Whence $\beta=87°\ 28'\ 12''.8$, log. $N=9.2518088$.

$$\begin{array}{ll} N\ =+0.1785701; & N^{IV}=-0.000008745; \\ N'\ =+0.0081595; & N^{V}\ =-0.000007856; \\ N''\ =+0.0053002; & N^{VI}=+0.000003321; \\ N'''\ =+0.00086522; & N^{VII}=+0.0000000649. \end{array}$$

For the root $g_1=7''.3790776$, we get,

$$x_1=+\frac{1535521}{10^{20}};\ y_1=+\frac{4548921}{10^{20}};\ z_1=-\frac{186590940}{10^{20}}.$$

Whence $\beta_1=18°\ 39'\ 8''.9$; log. $N_1=98.4104497$.

$$\begin{array}{ll} N_1\ =+0.0257306; & N_1^{IV}=+0.000010755; \\ N_1'=-0.0194493; & N_1^{V}\ =+0.000011106; \\ N_1''=-0.0151823; & N_1^{VI}=-0.000002714; \\ N_1'''=-0.00269026; & N_1^{VII}=-0.0000001142. \end{array}$$

For the root $g_2=17''.4034121$, we get,

$$x_2=-\frac{39188050}{10^{20}};\ y_2=+\frac{65870040}{10^{20}};\ z_2=\frac{4.079210}{10^{10}}.$$

Whence $\beta_2=329°\ 15'\ 1''.2$; log. $N_2=97.2739118$.

$$\begin{array}{ll} N_2\ =+0.0018789; & N_2^{IV}=-0.000001477; \\ N_2'=-0.0145930; & N_2^{V}\ =-0.000011248; \\ N_2''=+0.0125684; & N_2^{VI}=+0.0000006631; \\ N_2'''=+0.0449428; & N_2^{VII}=+0.00000004893. \end{array}$$

For the root $g_3=18''.0984790$, we get,

$$x_3=+\frac{82704460}{10^{20}};\ y_3=-\frac{81987110}{10^{20}};\ z_3=\frac{5.806087}{10^{10}}.$$

Whence $\beta_3=134°\ 45'\ 1''.6$; log. $N_3=97.3022770$.

$$\begin{array}{ll} N_3\ =+0.0020058; & N_3^{IV}=+0.0000004469; \\ N_3'=-0.0167491; & N_3^{V}\ =+0.0000065211; \\ N_3''=+0.0173974; & N_3^{VI}=-0.0000003597; \\ N_3'''=-0.0572983; & N_3^{VII}=-0.00000002674. \end{array}$$

For the root $g_4=0''.6166870$, we get,

$$x_4=+\frac{3.357912}{10^{10}};\quad y_4=+\frac{1.360959}{10^{10}};\quad z_4=\frac{56997.71}{10^{10}}.$$

Whence $\beta_4=67°\ 56'\ 42''.3$; log. $N_4^{IV}=95.8032384$.

$$\begin{aligned} N_4 &=+0.000007676; & N_4^{IV} &=+0.000063565;\\ N_4' &=+0.000011687; & N_4^{V} &=+0.000071695;\\ N_4'' &=+0.000013561; & N_4^{VI} &=+0.00155764;\\ N_4''' &=+0.000021841; & N_4^{VII} &=+0.01003850. \end{aligned}$$

For the root $g_5=2''.7277089$, we get,

$$x_5=+\frac{0.6477216}{10^{10}};\quad y_5=-\frac{0.1749684}{10^{10}};\quad z_5=\frac{345.4842}{10^{10}}.$$

Whence $\beta_5=105°\ 6'\ 59''.3$; log. $N_5^{IV}=97.2882538$.

$$\begin{aligned} N_5 &=+0.00055404; & N_5^{IV} &=+0.0019420;\\ N_5' &=+0.00054913; & N_5^{V} &=+0.0017682;\\ N_5'' &=+0.00057941; & N_5^{VI} &=+0.0297289;\\ N_5''' &=+0.00077160; & N_5^{VII} &=-0.00290997. \end{aligned}$$

For the root $g_6=3''.7172386$, we get,

$$x_6=+\frac{0.4583439}{10^{10}};\quad y_6=+\frac{0.8566511}{10^{10}};\quad z_6=\frac{22.50983}{10^{10}}.$$

Whence $\beta_6=28°\ 8'\ 55''.1$; log. $N_6^{IV}=98.6350978$.

$$\begin{aligned} N_6 &=+0.0233180; & N_6^{IV} &=+0.0431616;\\ N_6' &=+0.0162078; & N_6^{V} &=+0.0341038;\\ N_6'' &=+0.0161491; & N_6^{VI} &=-0.0448343;\\ N_6''' &=+0.0186903; & N_6^{VII} &=+0.00141840. \end{aligned}$$

For the root $g_7=22''.4611216$, we get,

$$x_7=-\frac{1.009450}{10^{10}};\quad y_7=+\frac{0.7872062}{10^{10}};\quad z_7=\frac{81.85450}{10^{10}}.$$

Whence $\beta_7=307°\ 56'\ 54''.3$; log. $N_7^{IV}=98.1942049$.

$$\begin{aligned} N_7 &=-0.00011217; & N_7^{IV} &=+0.0156388;\\ N_7' &=+0.00051216; & N_7^{V} &=-0.0483500;\\ N_7'' &=-0.0024585; & N_7^{VI} &=+0.0018057;\\ N_7''' &=-0.0155133; & N_7^{VII} &=+0.00013650. \end{aligned}$$

32. We shall now suppose $\mu'''=+1$; and the mass of *Mars* will become,

$$m'''=\frac{1+1}{2680637}=\frac{1}{1340318.5}.$$

Using this mass, we shall find,

$$\left.\begin{aligned} \Delta\boxed{0,0} &=-0''.0279815; & \Delta\boxed{4,4} &=-0''.0031027;\\ \Delta\boxed{1,1} &=-0\ .1020355; & \Delta\boxed{5,5} &=-0\ .0003297;\\ \Delta\boxed{2,2} &=-0\ .2982001; & \Delta\boxed{6,6} &=-0\ .0000275;\\ \Delta\boxed{3,3} &=\ \ 0\ . \qquad ; & \Delta\boxed{7,7} &=-0\ .0000057. \end{aligned}\right\}\quad(269)$$

Whence we get,

$$\left.\begin{array}{ll}
\boxed{0,0}=g-\ 5''.5982373; & \boxed{4,4}=g-\ 7''.5154781; \\
\boxed{1,1}=g-11\ .4168037; & \boxed{5,5}=g-18\ .5965426; \\
\boxed{2,2}=g-13\ .3703731; & \boxed{6,6}=g-\ 2\ .7662797; \\
\boxed{3,3}=g-17\ .5528645; & \boxed{7,7}=g-\ 0\ .6479626.
\end{array}\right\}\ (270)$$

From these quantities we get the following equations,

$$\left.\begin{array}{l}
\boxed{0,0}\boxed{1,1}=g^2-17.0150410.g+\ 63.9139763201; \\
\boxed{0,0}\boxed{2,2}=g^2-18.9686104.g+\ 74.8505214033; \\
\boxed{0,0}\boxed{3,3}=g^2-28.1511018.g+\ 98.2651007657; \\
\boxed{1,1}\boxed{2,2}=g^2-24.7871768.g+152.6469250785; \\
\boxed{1,1}\boxed{3,3}=g^2-28.9696682.g+200.3976083692; \\
\boxed{2,2}\boxed{3,3}=g^2-30.9232376.g+234.6883473387;
\end{array}\right\}\ (271)$$

$$\left.\begin{array}{l}
\boxed{4,4}\boxed{5,5}=g^2-26.1120207.g+139.7619086460; \\
\boxed{4,4}\boxed{6,6}=g^2-10.2817578.g+\ 20.7899145038; \\
\boxed{4,4}\boxed{7,7}=g^2-\ 8.1634407.g+\ \ 4.8697487299; \\
\boxed{5,5}\boxed{6,6}=g^2-21.3628223.g+\ 51.4432382846; \\
\boxed{5,5}\boxed{7,7}=g^2-19.2445052.g+\ 12.0498640941; \\
\boxed{6,6}\boxed{7,7}=g^2-\ 3.4142423.g+\ \ 1.79244578674.
\end{array}\right\}\ (272)$$

$$\left.\begin{array}{r}
\boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=g^4-47.9382786.g^3+824.7624793.g^2 \\
-5969.6589279.g+14999.865474416;
\end{array}\right\}\ (273)$$

$$\left.\begin{array}{r}
\boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=g^4-29.5262630.g^3+230.707120045.g^2 \\
-523.98540191.g+250.515644299.
\end{array}\right\}\ (274)$$

We shall therefore obtain the following

Fundamental Equations for $\mu'''=+1$; *or, for* $m'''=1\div1340318.5$.

$$\left.\begin{array}{lll}
A\ =g^2-40.54796482.g & +195.9771436; \\
A'=g^2-23.18819358.g & +\ 98.5732170; \\
A''=g^2-18.15228098.g & +\ 70.09590592; \\
A_1=g^2-14.596906302.g & +\ 45.9936083; \\
A_2=g^2-10.01075363.g & +\ \ 6.35635695; \\
A_3=g^2-26.186380814.g & +\ 82.97883004;
\end{array}\right\}\ (275)$$

$$\left.\begin{array}{lll}
D\ =g^2-48.259496667.g & +710.074578; \\
D'=g^2-55.05265379.g & +658.652503; \\
D''=g^2-31.86740965.g & +250.7131091; \\
D_1=g^2-48.09999599.g & +201.479694; \\
D_2=g^2-51.053228867.g & +\ 34.3827873; \\
D_3=g^2-\ 3.4187403566.g & +\ \ 1.7319812509.
\end{array}\right\}\ (276)$$

$$\left.\begin{array}{l} B=\{g-34.9327682\}b;\quad B'=\{g-17.581591114\}b'; \\ \qquad\qquad B''=\{g-12.553303072\}b; \end{array}\right\} \quad (277)$$

$$\left.\begin{array}{l} C=\{24.0060075\ -g\}[9.1763990]b'; \\ C'=\{17.61110406-g\}[8.8694654]b'; \\ C''=-[0.2445917]b'; \\ C'''=-[0.2654598]b'; \end{array}\right\} \quad (278)$$

$$\left.\begin{array}{l} E=+[9.6231944]b''; \\ E'=\{24.2534892\ -g\}[8.5847028]b''; \\ E''=\{17.55540782-g\}[9.6375865]b''; \\ E'''=+[0.9518913]b''; \end{array}\right\} \quad (279)$$

$$\left.\begin{array}{l} F=-[7.9509391]b'''; \\ F'=+[9.2587683]b'''; \\ F''=\{14.31200183-g\}[0.8407439]b'''; \\ F'''=\{37.4415497\ -g\}[9.4809266]b'''; \end{array}\right\} \quad (280)$$

$$\left.\begin{array}{l} B_1=\{g-4.7868255\}b_1;\quad B_2=\{g-0.6795775937\}b_1; \\ \qquad\qquad B_3=\{g-18.665893612\}b_1; \end{array}\right\} \quad (281)$$

$$\left.\begin{array}{l} C_1=\{4.365118944\ -g\}[9.2840950]b_2; \\ C_2=\{0.6879162996-g\}[9.1824311]b_2; \\ C_3=-[0.6832317]b_2; \\ C_4=-[0.6834824]b_2; \end{array}\right\} \quad (282)$$

$$\left.\begin{array}{l} E_1=+[7.8267723]b_3; \\ E_2=\{43.7348770\ \ -g\}[8.2017618]b_3; \\ E_3=\{\ 0.648009819-g\}[0.7610455]b_3; \\ E_4=+[1.0655652]b_3; \end{array}\right\} \quad (283)$$

$$\left.\begin{array}{l} F_1=-[7.0339474]b_4; \\ F_2=+[0.2724895]b_4; \\ F_3=\{\ 2.770690538-g\}[1.7737962]b_4; \\ F_4=\{50.365313\ \ \ -g\}[9.1128486]b_4; \end{array}\right\} \quad (284)$$

$$\left.\begin{array}{l} g^4-47.9382786.g^3\ +799.3791471.g^2 \\ -5382.3863166.g+12482.1014329 \end{array}\right\}=(\chi,\chi_1,\chi_2,\chi_3); \quad (285)$$

$$\left.\begin{array}{l} g^4-29.5262630.g^3\ +172.81022212.g^2 \\ -323.52210151.g+140.57248634 \end{array}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7). \quad (286)$$

The values of b, b', b'', and b''' are given by equations (118); and the values of b_1, b_2, b_3, and b_4 are given by equations (119), by simply adding [0.3010300] to the coefficients of N'''.

Putting equations (285) and (286), equal to nothing, they will give,

$$\left.\begin{array}{ll} g = 5''.4982776; & g_4 = 0''.616692122; \\ g_1 = 7\ .4032880; & g_5 = 2\ .7279047; \\ g_2 = 17\ .0209128; & g_6 = 3\ .7201830; \\ g_3 = 18\ .0158003; & g_7 = 22\ .4614832. \end{array}\right\} \quad (287)$$

The equations just computed will now give the following values:—

For the root g, we get,

$$g = 5''.4983682;$$

$N' = +0.04574900N$	log.	98.6603816;
$N'' = +0.02849177N$	"	98.4547195;
$N''' = +0.004428614N$	"	97.6462679;
$N^{IV} = -0.00004876604N$	"	95.6881175n;
$N^{V} = -0.00004363627N$	"	95.6398476n;
$N^{VI} = +0.00001885617N$	"	95.2754534;
$N^{VII} = +0.0000003497564N$	"	93.5437656.

For the root g_1, we get,

$$g_1 = 7''.4040670;$$

$N_1' = -0.8078317N_1$	log.	9.9073209n;
$N_1'' = -0.6055137N_1$	"	9.7821240n;
$N_1''' = -0.10224573N_1$	"	9.0096452n;
$N_1^{IV} = +0.0004271000N_1$	"	6.6305296;
$N_1^{V} = +0.0004419796N_1$	"	6.6454022;
$N_1^{VI} = -0.0001073512N_1$	"	6.0308069n;
$N_1^{VII} = -0.00000454067N_1$	"	4.6571202n.

For the root g_2, we get,

$$g_2 = 17''.0211649;$$

$N_2' = -\ 7.637284N_2$	log.	0.8829390n;
$N_2'' = +\ 7.417000N_2$	"	0.8702282;
$N_2''' = +14.83900N_2$	"	1.1714047;
$N_2^{IV} = -\ 0.001116683N_2$	"	97.0479297n;
$N_2^{V} = -\ 0.006781928N_2$	"	97.8313532n;
$N_2^{VI} = +\ 0.0004152514N_2$	"	96.6183110;
$N_2^{VII} = +\ 0.00003050897N_2$	"	95.4844276.

For the root g_3, we get,

$$g_3 = 18''.0158631;$$

$N_3' = -\ 8.466438N_3$	log.	0.9277008n;
$N_3'' = +10.65257N_3$	"	1.0274544;
$N_3''' = -26.00480N_3$	"	1.4150536n;
$N_3^{IV} = +\ 0.0005716405N_3$	"	6.7571230;
$N_3^{V} = +\ 0.007478025N_3$	"	7.8737870;
$N_3^{VI} = -\ 0.0004156815N_3$	"	6.6187606n;
$N_3^{VII} = -\ 0,00003086801N_3$	"	5.4895086n.

For the root g_4, we get,

$g_4=0''.61669170$;

$N_4 =+ 0.1217372 N_4^{IV}$	log.	9.0854233;
$N_4' =+ 0.1852979 N_4^{IV}$	"	9.2678704;
$N_4'' =+ 0.2150961 N_4^{IV}$	"	9.3326326;
$N_4''' =+ 0.345612 N_4^{IV}$	"	9.5385887;
$N_4^{V} =+ 1.128136 N_4^{IV}$	"	0.0523616;
$N_4^{VI} =+ 24.51803 N_4^{IV}$	"	1.3894856;
$N_4^{VII} =+158.0099 N_4^{IV}$	"	2.1986843.

For the root g_5, we get,

$g_5=2''.7278184$;

$N_5 =+ 0.2903833 N_5^{IV}$	log.	9.4629717;
$N_5' =+ 0.2858690 N_5^{IV}$	"	9.4561671;
$N_5'' =+ 0.2999592 N_5^{IV}$	"	9.4770622;
$N_5''' =+ 0.3995210 N_5^{IV}$	"	9.6015396;
$N_5^{V} =+ 0.9107056 N_5^{IV}$	"	9.9593780;
$N_5^{VI} =+15.33141 N_5^{IV}$	"	1.1855820;
$N_5^{VII} =- 1.500545 N_5^{IV}$	"	0.1762490*n*.

For the root g_6, we get,

$g_6=3''.7186222$;

$N_6 =+0.5567669 N_6^{IV}$	log.	9.7456734;
$N_6' =+0.3805960 N_6^{IV}$	"	9.5804642;
$N_6'' =+0.3754068 N_6^{IV}$	"	9.5745022;
$N_6''' =+0.4351473 N_6^{IV}$	"	9.6386363;
$N_6^{V} =+0.7902211 N_6^{IV}$	"	9.8977486;
$N_6^{VI} =-1.0372986 N_6^{IV}$	"	0.0159038*n*;
$N_6^{VII} =+0.03275252 N_6^{IV}$	"	8.5152447.

For the root g_7, we get,

$g_7=22''.4619460$;

$N_7 =-0.00561240 N_7^{IV}$	log.	7.7491486*n*;
$N_7' =+0.01444352 N_7^{IV}$	"	8.1596730;
$N_7'' =-0.1296412 N_7^{IV}$	"	9.1127430*n*;
$N_7''' =-0.967188 N_7^{IV}$	"	9.9855109*n*;
$N_7^{V} =-3.091169 N_7^{IV}$	"	0.4901228*n*;
$N_7^{VI} =+0.1154386 N_7^{IV}$	"	9.0623510;
$N_7^{VII} =+0.008726036 N_7^{IV}$	"	7.9408170.

33. These numbers are now to be substituted in equations (134) and (135). We must also add the logarithm of $(2=1+\mu''')$ to those of $\frac{m'''}{n'''a'''}$, $\frac{m'''}{n'''a'''}h'''$, and $\frac{m'''}{n'''a'''}l'''$, in order to obtain the numbers which are to be used in this computation.

For the root $g=5''.4983682$, we get,

$$x=+\frac{1840691}{10^{20}}N;\ y=+\frac{98053.1}{10^{20}}N;\ z=+\frac{10390430}{10^{20}}N^2.$$

Whence $\beta=86^\circ\ 57'\ 2''.7$; log. $N=9.2489632$.

$N\ =+0.1774036$; $N^{IV}=-0.000008651$;
$N'\ =+0.00811605$; $N^{V}\ =-0.000007741$;
$N''\ =+0.0050546$; $N^{VI}=+0.0000033452$;
$N'''\ =+0.00078565$; $N^{VII}=+0.00000006205$.

For the root $g_1=7''.404067$, we get,

$$x_1=+\frac{1842718}{10^{20}}N_1;\ y_1=+\frac{4337257}{10^{20}}N_1;\ z_1=\frac{197199700}{10^{20}}N_1^{\,2}.$$

Whence $\beta_1=23^\circ\ 1'\ 6''.9$; log. $N_1=8.3783427$.

$N_1\ =+0.0238970$; $N_1^{\,IV}=+0.000010206$;
$N_1'\ =-0.0193047$; $N_1^{\,V}\ =+0.000010562$;
$N_1''\ =-0.0144699$; $N_1^{\,VI}=-0.000002565$;
$N_1'''\ =-0.0024433$; $N_1^{\,VII}=-0.0000001085$.

For the root $g_2=17''.021165$, we get,

$$x_2=-\frac{51010670}{10^{20}}N_2,\ y_2=+\frac{80108830}{10^{20}}N_2,\ z_2=\frac{3.698921}{10^{10}}N_2^2.$$

Whence $\beta_2=327^\circ\ 30'\ 44''.5$; log. $N_2=97.4095166$.

$N_2\ =+0.00256754$; $N_2^{IV}=-0.0000028671$;
$N_2'=-0.0196090$; $N_2^{V}\ =-0.000017413$;
$N_2''=+0.0190434$; $N_2^{VI}=+0.0000010662$;
$N_2'''=+0.0380997$; $N_2^{VII}=+0.00000007833$.

For the root $g_3=18''.015863$, we get,

$$x_3=+\frac{137961060}{10^{20}}N_3;\ y_3=-\frac{149026100}{10^{20}}N_3;\ z_3=\frac{8.388122}{10^{10}}N_3^2.$$

Whence $\beta_3=137^\circ\ 12'\ 28''.8$; log. $N_3=97.3840054$.

$N_3\ =+0.00242106$; $N_3^{IV}=+0.0000013840$;
$N_3'=-0.0204978$; $N_3^{V}\ =+0.000018105$;
$N_3''=+0.0257905$; $N_3^{VI}=-0.0000010064$;
$N_3'''=-0.0629592$; $N_3^{VII}=-0.00000007473$.

For the root $g_4=0''.6166917$, we get,

$$x_4=+\frac{3.359379}{10^{10}}N_4^{IV};\ y_4=+\frac{1.360856}{10^{10}}N_4^{IV};\ z_4=\frac{57058.06}{10^{10}}N_4^{IV2}.$$

Whence $\beta_4=67^\circ\ 56'\ 51''.2$; log. $N_4^{IV}=95.8029370$.

$N_4\ =+0.000007733$; $N_4^{IV}=+0.000063524$;
$N_4'=+0.000011770$; $N_4^{V}\ =+0.000071663$;
$N_4''=+0.000013664$; $N_4^{VI}=+0.00155748$;
$N_4'''=+0.000021955$; $N_4^{VII}=+0.01003740$.

For the root $g_5=2''.727818$, we get,

$$x_5=+\frac{0.6479474}{10^{10}}N_5^{IV};\ y_5=-\frac{.1763111}{10^{10}}N_5^{IV};\ z_5=\frac{346.4665}{10^{10}}N_5^{IV2}.$$

Whence $\beta_5=105°\ 13\ 19''.4$; log. $N_5^{IV}=97.2873891$.

$N_5\ =+0.000562809$;	$N_5^{IV}=+0.00193816$;
$N_5'=+0.000554059$;	$N_5^{V}\ =+0.00176509$;
$N_5''=+0.000581368$;	$N_5^{VI}=+0.0297147$;
$N_5'''=+0.000774334$;	$N_5^{VII}=-0.00290829$.

For the root $g_6=3''.718622$, we get,

$$x_6=+\frac{0.4583136}{10^{10}}N_6^{IV},\ y_6=+\frac{0.8566921}{10^{10}}N_6^{IV},\ z_6=\frac{22.50708}{10^{10}}N_6^{IV2}.$$

Whence $\beta_6=28°\ 8'\ 45''.3$; log. $N_6^{IV}=98.6351607$.

$N_6\ =+0.0240344$;	$N_6^{IV}=+0.0431679$;
$N_6'=+0.0164295$;	$N_6^{V}\ =+0.0341122$;
$N_6''=+0.0162055$;	$N_6^{VI}=-0.0447780$;
$N_6'''=+0.0187844$;	$N_6^{VII}=+0.0014139$.

For the root $g_7=22''.461946$, we get,

$$x_7=-\frac{1.009108}{10^{10}}N_7^{IV};\ y_7=+\frac{0.7869276}{10^{10}}N_7^{IV};\ z_7=\frac{81.83667}{10^{10}}N_7^{IV2}.$$

Whence $\beta_7=307°\ 56'\ 52''.8$; log. $N_7^{IV}=98.1941499$.

$N_7\ =-0.0000878$;	$N_7^{IV}=+0.0156369$;
$N_7'=+0.0002258$;	$N_7^{V}\ =-0.0483362$;
$N_7''=-0.0020272$;	$N_7^{VI}=+0.0018051$;
$N_7'''=-0.0151238$;	$N_7^{VII}=+0.0001364$.

34. We shall now suppose $\mu^{IV}=+\frac{1}{100}$; and the mass of *Jupiter* will become, $m^{IV}=\frac{1+\mu^{IV}}{1047.879}=\frac{1.01}{1047.879}=\frac{1}{1037.504}$. Using this value of *Jupiter's* mass, we shall find,

$$\left.\begin{array}{ll}\Delta\boxed{0,0}=-0''.0160284; & \Delta\boxed{4,4}=\ 0''. \quad ; \\ \Delta\boxed{1,1}=-0\ .0420284; & \Delta\boxed{5,5}=-0\ .1824735; \\ \Delta\boxed{2,2}=-0\ .0706826; & \Delta\boxed{6,6}=-0\ .0093732; \\ \Delta\boxed{3,3}=-0\ .1465990; & \Delta\boxed{7,7}=-0\ .0017907.\end{array}\right\}\quad(288)$$

Whence we get,

$$\left.\begin{array}{ll}\boxed{0,0}=g-\ 5''.5862842; & \boxed{4,4}=g-\ 7''.5123754; \\ \boxed{1,1}=g-11\ .3567966; & \boxed{5,5}=g-18\ .7786864; \\ \boxed{2,2}=g-13\ .1428556; & \boxed{6,6}=g-\ 2\ .7756254; \\ \boxed{3,3}=g-17\ .6994635; & \boxed{7,7}=g-\ 0\ .6497476.\end{array}\right\}\quad(289)$$

From these quantities we get the following equations,

$$\left.\begin{aligned}
\boxed{0,0}\boxed{1,1}&=g^2-16.9430808.g+63.4422934092;\\
\boxed{0,0}\boxed{2,2}&=g^2-18.7291398.g+73.4197265812;\\
\boxed{0,0}\boxed{3,3}&=g^2-23.2857477.g+98.8742332985;\\
\boxed{1,1}\boxed{2,2}&=g^2-24.4996522.g+149.2607377924;\\
\boxed{1,1}\boxed{3,3}&=g^2-29.0562601.g+201.0092068986;\\
\boxed{2,2}\boxed{3,3}&=g^2-30.8423191.g+232.6214929780;
\end{aligned}\right\}\quad(290)$$

$$\left.\begin{aligned}
\boxed{4,4}\boxed{5,5}&=g^2-26.2910618.g+141.0725417557;\\
\boxed{4,4}\boxed{6,6}&=g^2-10.2880008.g+20.8515399746;\\
\boxed{4,4}\boxed{7,7}&=g^2-8.1621230.g+4.8811478864;\\
\boxed{5,5}\boxed{6,6}&=g^2-21.5543118.g+52.1225989505;\\
\boxed{5,5}\boxed{7,7}&=g^2-19.4284340.g+12.2014064196;\\
\boxed{6,6}\boxed{7,7}&=g^2-3.4253730.g+1.80345594215.
\end{aligned}\right\}\quad(291)$$

$$\left.\begin{aligned}
\boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=g^4-47.7853999.g^3+818.62769096.g^2&\\
-5898.0322091.g+14758.0410108;&
\end{aligned}\right\}\quad(292)$$

$$\left.\begin{aligned}
\boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=g^4-29.7164348.g^3+232.93269093.g^2&\\
-530.6408472.g+254.418113702.&
\end{aligned}\right\}\quad(293)$$

We shall therefore obtain the following

Fundamental Equations for $\mu^{IV}=+\frac{1}{100}$; *or for* $m^{IV}=\frac{1}{1037.504}$.

$$\left.\begin{aligned}
A&=g^2-40.3084942.g\quad+194.2847527;\\
A'&=g^2-23.30847617.g\quad+99.1027623;\\
A''&=g^2-18.08032078.g\quad+69.61059287;\\
A_1&=g^2-14.626095329.g\quad+46.1708070;\\
A_2&=g^2-10.02759291.g\quad+6.38342153;\\
A_3&=g^2-26.365472005.g\quad+83.71712664;
\end{aligned}\right\}\quad(294)$$

$$\left.\begin{aligned}
D&=g^2-47.971972067.g\quad+703.129607;\\
D'&=g^2-55.11012591.g\quad+662.354530;\\
D''&=g^2-31.78521949.g\quad+249.0357671;\\
D_1&=g^2-48.29148549.g\quad+202.685210;\\
D_2&=g^2-51.237157667.g\quad+34.5983143;\\
D_3&=g^2-3.4298710566.g+1.7430000948.
\end{aligned}\right\}\quad(295)$$

$$\left.\begin{aligned}
B=\{g-34.7052507\}b;\quad B'&=\{g-17.713826807\}b;\\
B''&=\{g-12.49329597\}b;
\end{aligned}\right\}\quad(296)$$

$$\left.\begin{aligned}
C&=\{23.7784900-g\}[9.1763990]b';\\
C'&=\{17.72858328-g\}[8.8694654]b',\\
C''&=-[0.2445917]b';\\
C'''&=-[0.2654598]b'.
\end{aligned}\right\}\quad(297)$$

$$\left.\begin{array}{l} E\ =+[9.6231944]b''; \\ E'=\{24.1934821\ -g\}[8.5847028]b''; \\ E''=\{17.70073515-g\}[9.6375865]b''; \\ E'''=+[0.9518913]b''. \end{array}\right\} \quad (298)$$

$$\left.\begin{array}{l} F\ =-[7.6499091]b'''; \\ F'\ =+[8.9577383]b'''; \\ F''=\{14.08448433-g\}[0.8407439]b'''; \\ F'''=\{37.3815426\ -g\}[9.4809266]b'''. \end{array}\right\} \quad (299)$$

$$\left.\begin{array}{l} B_1=\{g-4.7961712\}b_1;\quad B_2=\{g-\ 0.6813625937\}b_1; \\ \qquad\qquad\qquad\qquad\qquad B_3=\{g-18.848037412\ \}b_1; \end{array}\right\} \quad (300)$$

$$\left.\begin{array}{l} C_1=\ \{4.374464644\ -g\}[9.2840950]b_2; \\ C_2=\ \{0.6897012996-g\}[9.1824311]b_2; \\ C_3=-[0.6832317]b_2; \\ C_4=-[0.6834824]b_2; \end{array}\right\} \quad (301)$$

$$\left.\begin{array}{l} E_1=+[7.8267723]b_3; \\ E_2=\{43.9170208\ \ -g\}[8.2017618]b_3; \\ E_3=\{\ 0.649794819-g\}[0.7610455]b_3; \\ E_4=+[1.0655652]b_3; \end{array}\right\} \quad (302)$$

$$\left.\begin{array}{l} F_1=-[7.0339474]b_4; \\ F_2=+[0.2724895]b_4; \\ F_3=\{\ 2.780036238-g\}[1.7737962]b_4; \\ F_4=\{50.5474564\ \ -g\}[9.1128486]b_4; \end{array}\right\} \quad (303)$$

$$\left.\begin{array}{l} g^4-47.7853999.g^3+793.57041154.g^2 \\ \quad -5313.7879554.g+12250.8363744 \end{array}\right\}=(\chi,\chi_1,\chi_2,\chi_3); \quad (304)$$

$$\left.\begin{array}{l} g^4-29.7164348.g^3+174.4588860.g^2 \\ \quad -327.5400877.g+142.7433842 \end{array}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7). \quad (305)$$

The values of b_1, b_2, b_3, and b_4 are given by equations (119); and the values of b, b', b'', and b''' are given by equations (118), by simply adding log. $(1+\mu'')=$ $[0.0043214]$, to the coefficients of N''.

Putting equations (286) and (287), equal to nothing, they will give,

$$\left.\begin{array}{ll} g=\ 5''.48175018; & g_4=\ 0''.618590378; \\ g_1=\ 7\ .29977391; & g_5=\ 2\ .73702595; \\ g_2=17\ .09470611; & g_6=\ 3\ .72450143; \\ g_3=17\ .90916970; & g_7=22\ .63631704. \end{array}\right\} \quad (306)$$

The equations just computed will now give the following values:

For the root g, we get,

$$g=5''.48184344;$$

$N' =+0.04775507N$	log.	98.6790194;
$N'' =+0.03031151N$	"	98.4816076;
$N''' =+0.004615663N$	"	97.6642342;
$N^{IV} =-0.00004974188N$	"	95.6967222n;
$N^{V} =-0.00004429498N$	"	95.6463545n;
$N^{VI} =+0.00001941068N$	"	95.2880406;
$N^{VII} =+0.000000351368N$	"	93.5457621.

For the root g_1, we get,

$$g_1=7''.3005026;$$

$N_1' =-0.7657460N_1$	log.	99.8840847n;
$N_1'' =-0.5821200N_1$	"	99.7650126n;
$N_1''' =-0.09563476N_1$	"	98.9806158n;
$N_1^{IV} =+0.0003996932N_1$	"	96.6017268;
$N_1^{V} =+0.0004077858N_1$	"	96.6104318;
$N_1^{VI} =-0.00010221725N_1$	"	96.0095242n;
$N_1^{VII} =-0.000004220228N_1$	"	94.6253359n.

For the root g_2, we get,

$$g_2=17''.0948281;$$

$N_2' =-\ 7.697880N_2$	log.	0.8863712n;
$N_2'' =+\ 7.832065N_2$	"	0.8938764;
$N_2''' =+13.95241N_2$	"	1.1446493;
$N_2^{IV} =-\ 0.0007072570N_2$	"	6.8495773n;
$N_2^{V} =-\ 0.004093204N_2$	"	7.6120634n;
$N_2^{VI} =+\ 0.0002502615N_2$	"	6.3983940;
$N_2^{VII} =+\ 0.00001841771N_2$	"	5.2652356.

For the root g_3, we get,

$$g_3=17''.90922396;$$

$N_3 =-\ 8.366200N_3$	log.	0.9225281n,
$N_3'' =+10.511163N_3$	"	1.0216508;
$N_3''' =-56.62883N_3$	"	1.7530376n;
$N_3^{IV} =+\ 0.0008158954N_3$	"	6.9116345;
$N_3^{V} =+\ 0.008084869N_3$	"	7.9076730;
$N_3^{VI} =-\ 0.0004567941N_3$	"	6.6597205n;
$N_3^{VII} =-\ 0.00003392040N_3$	"	5.5304609n.

For the root g_4, we get,

$$g_4 = 0''.61859008;$$

$N_4 = +\ 0.1213314 N_4^{IV}$	log.	9.0839733;
$N_4' = +\ 0.1844663 N_4^{IV}$	"	9.2659172;
$N_4'' = +\ 0.2137640 N_4^{IV}$	"	9.3299345;
$N_4''' = +\ 0.3456726 N_4^{IV}$	"	9.5386650;
$N_4^{V} = +\ 1.125699 N_4^{IV}$	"	0.0514222;
$N_4^{VI} = +\ 24.58977 N_4^{IV}$	"	1.3907546;
$N_4^{VII} = +159.0425 N_4^{IV}$	"	2.2015132.

For the root g_5, we get,

$$g_5 = 2''.7369612;$$

$N_5 = +\ 0.2919622 N_5^{IV}$	log.	9.4653267;
$N_5' = +\ 0.2861383 N_5^{IV}$	"	9.4565760;
$N_5'' = +\ 0.2997059 N_5^{IV}$	"	9.4766953;
$N_5''' = +\ 0.3993596 N_5^{IV}$	"	9.6013642;
$N_5^{V} = +\ 0.9082830 N_5^{IV}$	"	9.9582212;
$N_5^{VI} = +15.32706 N_5^{IV}$	"	1.1854589;
$N_5^{VII} = -\ 1.494938 N_5^{IV}$	"	0.1746230n.

For the root g_6, we get,

$$g_6 = 3''.7233071;$$

$N_6 = +0.5627043 N_6^{IV}$	log.	9.7502802;
$N_6' = +0.3824702 N_6^{IV}$	"	9.5825976;
$N_6'' = +0.3767387 N_6^{IV}$	"	9.5760403;
$N_6''' = +0.4346798 N_6^{IV}$	"	9.6381695;
$N_6^{V} = +0.7887484 N_6^{IV}$	"	9.8969386;
$N_6^{VI} = -1.044537 N_6^{VI}$	"	0.0189238n;
$N_6^{VII} = +0.03309702 N_6^{IV}$	"	8.5197888.

For the root g_7, we get,

$$g_7 = 22''.6365781;$$

$N_7 = -0.006141989 N_7^{IV}$	log.	7.7883092n;
$N_7' = +0.01915236 N_7^{IV}$	"	8.2822223;
$N_7'' = -0.1513593 N_7^{IV}$	"	9.1800092n;
$N_7''' = -0.9658338 N_7^{IV}$	"	9.9849024n;
$N_7^{V} = -3.128146 N_7^{IV}$	"	0.4952871n;
$N_7^{VI} = +0.1158790 N_7^{IV}$	"	9.0640050;
$N_7^{VII} = +0.008773014 N_7^{IV}$	"	7.9431488.

35. These numbers are now to be substituted in equations (134) and (135). We must also add the logarithm of $(1+\mu^{IV})$ to those of $\frac{m^{IV}}{n^{IV}a^{IV}}$, $\frac{m^{IV}}{n^{IV}a^{IV}}h^{IV}$, and $\frac{m^{IV}}{n^{IV}a^{IV}}l^{IV}$, in order to obtain the numbers which are to be used in this computation.

For the root $g=5''.481843$, we get,

$$x=+\frac{1850169}{10^{20}}N;\quad y=+\frac{70884.5}{10^{20}}N;\quad z=\frac{10443875}{10^{20}}N^2.$$

Whence $\beta=87^\circ\ 48'\ 21''.4$; log. $N=9.2486682$.

$N\ =+0.1772834$; $N^{IV}=-0.000008818$;
$N'\ =+0.0084662$; $N^{V}\ =-0.000007853$;
$N''=+0.0053737$; $N^{VI}=+0.000003441$;
$N'''=+0.00081828$; $N^{VII}=+0.00000006229$.

For the root $g_1=7''.300503$, we get,

$$x_1=+\frac{1630157}{10^{20}}N_1;\quad y_1=+\frac{4425088}{10^{20}}N_1;\quad z_1=\frac{179817860}{10^{20}}N_1^2.$$

Whence $\beta_1=20^\circ\ 13'\ 23''.8$; log. $N_1=8.4187229$.

$N_1\ =+0.0262254$; $N_1^{IV}=+0.000010482$;
$N_1'\ =-0.0200820$; $N_1^{V}\ =+0.000010694$;
$N_1''=-0.0152664$; $N_1^{VI}=-0.000002681$;
$N_1'''=-0.00250806$; $N_1^{VII}=-0.00000011068$.

For the root $g_2=17''.094828$, we get,

$$x_2=-\frac{15347790}{10^{20}}N_2;\quad y_2=+\frac{36663280}{10^{20}}N_2;\quad z_2=\frac{2.973721}{10^{10}}N_2^2.$$

Whence $\beta_2=337^\circ\ 17'\ 6''.3$; log. $N_2=7.1259941$.

$N_2\ =+0.0013366$; $N_2^{IV}=-0.0000009453$;
$N_2'\ =-0.0102888$; $N_2^{V}\ =-0.0000054709$;
$N_2''=+0.0104682$; $N_2^{VI}=+0.00000033449$;
$N_2'''=+0.0186485$; $N_2^{VII}=+0.00000002462$.

For the root $g_3=17''.909224$, we get,

$$x_3=+\frac{147160530}{10^{20}}N_3;\quad y_3=-\frac{160644600}{10^{20}}N_3;\quad z_3=\frac{14.88588}{10^{10}}N_3^2.$$

Whence $\beta_3=137^\circ\ 30'\ 30''.1$; log. $N_3=7.1654027$.

$N_3\ =+0.0014635$; $N_3^{IV}=+0.0000011941$;
$N_3'\ =-0.0122442$; $N_3^{V}\ =+0.000011832$;
$N_3''=+0.0153834$; $N_3^{VI}=-0.00000066853$;
$N_3'''=-0.0828782$; $N_3^{VII}=-0.00000004964$.

For the root $g_4=0''.6185901$, we get,

$$x_4=+\frac{3.376737}{10^{10}}N_4^{IV};\quad y_4=+\frac{1.377873}{10^{0}}N_4^{IV};\quad z_4=\frac{57802.14}{10^{10}}N_4^{IV2}.$$

Whence $\beta_4=67^\circ\ 48'\ 7''.8$; log. $N_4^{IV}=95.7999963$.

$N_4\ =+0.000007655$; $N_4^{IV}=+0.0000630952$;
$N_4'\ =+0.000011639$; $N_4^{V}\ =+0.000071026$;
$N_4''=+0.0000134875$; $N_4^{VI}=+0.00155150$;
$N_4'''=+0.000021810$; $N_4^{VII}=+0.01003482$.

For the root $g_5=2''.736961$, we get,

$$x_5=+\frac{0.6488003}{10^{10}}N_5^{IV};\ y_5=-\frac{0.1681608}{10^{10}}N_5^{IV};\ z_5=\frac{346.3853}{10^{10}}N_5^{IV2}.$$

Whence $\beta_5=104°\ 31\ 50''\ 1$; log. $N_5^{IV}=97.2866700$.

$N_5\ =+0.00056493$; $N_5^{IV}=+0.00193495$;
$N_5'\ =+0.00055366$; $N_5^{V}\ =+0.00175748$;
$N_5''=+0.00057992$; $N_5^{VI}=+0.0296571$;
$N_5'''=+0.00077274$; $N_5^{VII}=-0.00289263$.

For the root $g_6=3''.723307$, we get,

$$x_6=-\frac{0.4594222}{10^{10}}N_6^{IV};\ y_6=+\frac{0.8649404}{10^{10}}N_6^{IV};\ z_6=+\frac{22.67899}{10^{10}}N_6^{IV2}.$$

Whence $\beta_6=27°\ 58'\ 31''.7$; log. $N_6^{IV}=8.6353288$.

$N_6\ =+0.0243002$; $N_6^{IV}=+0.0431846$;
$N_6'\ =+0.0165168$; $N_6^{V}\ =+0.0340618$;
$N_6''=+0.0162693$; $N_6^{VI}=-0.0451079$;
$N_6'''=+0.0187715$; $N_6^{VII}=+0.0014293$.

For the root $g_7=22''.636578$, we get,

$$x_7=-\frac{1.021673}{10^{10}}N_7^{IV};\ y_7=+\frac{0.7951379}{10^{10}}N_7^{IV};\ z_7=+\frac{83.56643}{10^{10}}N_7^{IV2}.$$

Whence $\beta_7=307°\ 53'\ 32''.2$; log. $N_7^{IV}=8.1901156$.

$N_7\ =-0.0000952$; $N_7^{IV}=+0.0154923$;
$N_7'\ =+0.0002967$; $N_7^{V}\ =-0.0484622$;
$N_7''=-0.0023449$; $N_7^{VI}=+0.0017952$;
$N_7'''=-0.0149630$; $N_7^{VII}=+0.0001359$.

36. We shall now suppose that $\mu^{V}=+\frac{1}{40}$; and the mass of *Saturn* will become, $m^{V}=\frac{1+\frac{1}{40}}{3501.6}=\frac{1}{3416.195}$. Using this value of *Saturn's* mass, we shall find,

$$\left.\begin{array}{ll}\Delta\boxed{0,0}=-0''.0019316, & \Delta\boxed{4,4}=-0''.1849094,\\ \Delta\boxed{1,1}=-0\ .0049722, & \Delta\boxed{5,5}=-0\ .\qquad\quad ,\\ \Delta\boxed{2,2}=-0\ .0081629, & \Delta\boxed{6,6}=-0\ .0348880,\\ \Delta\boxed{3,3}=-0\ .0157850, & \Delta\boxed{7,7}=-0\ .0051937.\end{array}\right\}\quad(307)$$

Whence we get,

$$\left.\begin{array}{ll}\boxed{0,0}=g-\ 5''.5721874; & \boxed{4,4}=g-\ 7''.6972848;\\ \boxed{1,1}=g-11\ .3197404; & \boxed{5,5}=g-18\ .5962129;\\ \boxed{2,2}=g-13\ .0803359; & \boxed{6,6}=g-\ 2\ .8011402;\\ \boxed{3,3}=g-17\ .5686495; & \boxed{7,7}=g-\ 0\ .6531506.\end{array}\right\}\quad(308)$$

From these quantities we get the following equations,

$$\left.\begin{aligned}
\boxed{0,0}\boxed{1,1}&=g^2-16.8919278.g+\ 63.0757148282;\\
\boxed{0,0}\boxed{2,2}&=g^2-18.6525233.g+\ 72.8860828897;\\
\boxed{0,0}\boxed{3,3}&=g^2-23.1408369.g+\ 97.8958073789;\\
\boxed{1,1}\boxed{2,2}&=g^2-24.4000763.g+148.0660067328;\\
\boxed{1,1}\boxed{3,3}&=g^2-28.8883899.g+198.8725515186;\\
\boxed{2,2}\boxed{3,3}&=g^2-30.6489854.g+229.8038367694.
\end{aligned}\right\}\quad(309)$$

$$\left.\begin{aligned}
\boxed{4,4}\boxed{5,5}&=g^2-26.2934977.g+143.1403468927;\\
\boxed{4,4}\boxed{6,6}&=g^2-10.4984250.g+\ 21.5611738841;\\
\boxed{4,4}\boxed{7,7}&=g^2-\ 8.3504354.g+\ \ 5.0274861855;\\
\boxed{5,5}\boxed{6,6}&=g^2-21.3973531.g+\ 52.0905995219;\\
\boxed{5,5}\boxed{7,7}&=g^2-19.2493635.g+\ 12.1461276134;\\
\boxed{6,6}\boxed{7,7}&=g^2-\ 3.4542908.g+\ \ 1.8295664023.
\end{aligned}\right\}\quad(310)$$

$$\left.\begin{aligned}
\boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=g^4-47.5409132.g^3+810.60000012.g^2\\
-5815.0364817.g+14495.04127448
\end{aligned}\right\}\quad(311)$$

$$\left.\begin{aligned}
\boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=g^4-29.7477885.g^3+235.7953005.g^2\\
-542.55408336.g+261.88476948
\end{aligned}\right\}\quad(312)$$

We shall therefore obtain the following

Fundamental Equations for $\mu^{\mathrm{v}}=+\frac{1}{40}$; *or, for* $m^{\mathrm{v}}=\frac{1}{3416.195}$.

$$\left.\begin{aligned}
A&=g^2-40.2318777.g\ \ +193.4460881;\\
A'&=g^2-23.16356537.g\ +\ 98.1230397;\\
A''&=g^2-18.02916778.g\ +\ 69.22796584;\\
A_1&=g^2-14.813573502.g+\ 47.2122074;\\
A_2&=g^2-10.19774833.g\ +\ \ 6.52926208;\\
A_3&=g^2-26.369591592.g+\ 84.94828460.
\end{aligned}\right\}\quad(313)$$

$$\left.\begin{aligned}
D&=g^2-47.872396167.g\ +700.738213;\\
D'&=g^2-54.94225571.g\ \ +656.812394;\\
D''&=g^2-31.59188579.g\ \ +246.0948532;\\
D_1&=g^2-48.76298515.g\ \ +206.032362;\\
D_2&=g^2-51.852306416.g\ +\ 35.1877225;\\
D_3&=g^2-\ 3.4587888566.g+\ \ 1.7691277905.
\end{aligned}\right\}\quad(314)$$

$$\left.\begin{aligned}
B=\{g-34.6427310\}b;\quad B'&=\{g-17.58301281\}b;\\
B''&=\{g-12.4562398\}b;
\end{aligned}\right\}\quad(315)$$

$$\left.\begin{aligned}
C&=\{23.7159703-g\}[9.1763990]b;\\
C'&=\{17.5977693-g\}[8.8694654]b';\\
C''&=-[0.2445917]b';\\
C'''&=-[0.2654598]b';
\end{aligned}\right\}\quad(316)$$

$$\left.\begin{aligned}E &= +[9.6231944]b''; \\ E' &= \{24.1564259-g\}[8.5847028]b''; \\ E'' &= \{17.56992116-g\}[9.6375865]b''; \\ E''' &= +[0.9518913]b'';\end{aligned}\right\} \tag{317}$$

$$\left.\begin{aligned}F &= -[7.6499091]b'''; \\ F' &= +[8.9577383]b'''; \\ F'' &= \{14.02196463-g\}[0.8407439]b'''; \\ F''' &= \{37.3444864-g\}[9.4809266]b'''.\end{aligned}\right\} \tag{318}$$

$$\left.\begin{aligned}B_1=\{g-4.8216860\}b_1;\quad B_2&=\{g-0.6847655937\}b_1; \\ B_3&=\{g-18.667297687\}b_1;\end{aligned}\right\} \tag{319}$$

$$\left.\begin{aligned}C_1 &= \{4.399979444-g\}[9.2840950]b_2; \\ C_2 &= \{0.6931042996-g\}[9.1824311]b_2; \\ C_3 &= -[0.6939556]b_2; \\ C_4 &= -[0.6942063]b_2;\end{aligned}\right\} \tag{320}$$

$$\left.\begin{aligned}E_1 &= +[7.8267723]b_3; \\ E_2 &= \{44.3630057-g\}[8.2017618]b_3; \\ E_3 &= \{\ 0.653197819-g\}[0.7610455]b_3; \\ E_4 &= +[1.0655652]b_3;\end{aligned}\right\} \tag{321}$$

$$\left.\begin{aligned}F_1 &= -[7.0339474]b_4; \\ F_2 &= +[0.2724895]b_4; \\ F_3 &= \{\ 2.805551038-g\}[1.7737962]b_4; \\ F_4 &= \{51.159202116-g\}[9.1128486]b_4;\end{aligned}\right\} \tag{322}$$

$$\left.\begin{aligned}&g^4-47.5409132.g^3+785.54272070.g^2 \\ &-5234.4038677.g+12012.62971359\end{aligned}\right\}(=\chi,\ \chi_1,\ \chi_2,\ \chi_3); \tag{323}$$

$$\left.\begin{aligned}&g^4-29.7477885.g^3+176.4527823.g^2 \\ &-334.71804394.g+146.97863935\end{aligned}\right\}(=\chi_4,\ \chi_5,\ \chi_6,\ \chi_7). \tag{324}$$

The values of b_1, b_2, b_3, and b_4 are given by equations (119); and the values of b, b', b'', and b''' are given by equations (118), by simply adding log. $(1+\mu^{\mathrm{v}})=[0.0107239]$, to the coefficients of N^{v}.

Putting equations (323) and (324) equal to nothing, they will give,

$$\left.\begin{aligned}g &= 5''.46587505; & g_4 &= 0''.622281636; \\ g_1 &= 7\ .25377728; & g_5 &= 2\ .76237256; \\ g_2 &= 17\ .02345660; & g_6 &= 3\ .78742019; \\ g_3 &= 17\ .79780427; & g_7 &= 22\ .57571411.\end{aligned}\right\} \tag{325}$$

The equations just computed will now give the following values:—

For the root g, we get,

$g = 5''.4659743,$	
$N' = +0.04853754N$	log. 8.6860778;
$N'' = +0.03095683N$	" 8.4907565;
$N''' = +0.004748442N$	" 7.6765512;
$N^{IV} = -0.00005254717N$	" 5.7205493n;
$N^{V} = -0.00004692140N$	" 5.6713710n;
$N^{VI} = +0.00002117625N$	" 5.3258490;
$N^{VII} = +0.000000363425N$	" 3.5604147.

For the root g_1, we get,

$g_1 = 7''.2545148,$	
$N_1' = -0.7512055N_1$	log. 9.8757588n;
$N_1'' = -0.5726608N_1$	" 9.7578974n;
$N_1''' = -0.09479172N_1$	" 8.9767704n;
$N_1^{IV} = +0.0004012718N_1$	" 6.6034387;
$N_1^{V} = +0.0004102662N_1$	" 6.6130658;
$N_1^{VI} = -0.0001060996N_1$	" 6.0257136n;
$N_1^{VII} = -0.000004281016N_1$	" 4.6315468n.

For the root g_2, we get,

$g_2 = 17''.0235807,$	
$N_2' = -\ 7.650735N_2$	log. 0.8837032n;
$N_2'' = +\ 7.722107N_2$	" 0.8877358;
$N_2''' = +15.22875N_2$	" 1.1826643;
$N_2^{IV} = -\ 0.0007070363N_2$	" 96.8494417n;
$N_2^{V} = -\ 0.004279910N_2$	" 97.6314346n;
$N_2^{VI} = +\ 0.0002688755N_2$	" 96.4295512;
$N_2^{VII} = +\ 0.00001969787N_2$	" 95.2944192.

For the root g_3, we get,

$g_3 = 17''.7978505,$	
$N_3' = -\ 8.286660N_3$	log. 0.9183798n;
$N_3'' = +10.238030N_3$	" 1.0102164;
$N_3''' = -50.30837N_3$	" 1.7016403n;
$N_3^{IV} = +\ 0.006648115N_3$	" 7.8226985;
$N_3^{V} = +\ 0.006915978N_3$	" 7.8398536;
$N_3^{VI} = -\ 0.0004030984N_3$	" 6.6054110n;
$N_3^{VII} = -\ 0.00002979690N_3$	" 5.4741710n.

For the root g_4, we get,

$$g_4 = 0''.62228135;$$

$N_4 = +\ 0.1215129 N_4^{IV}$	log.	9.0846223;
$N_4' = +\ 0.1845219 N_4^{IV}$	"	9.2660478;
$N_4'' = +\ 0.2136944 N_4^{IV}$	"	9.3297932;
$N_4''' = +\ 0.3455668 N_4^{IV}$	"	9.5385321;
$N_4^{V} = +\ 1.133130 N_4^{IV}$	"	0.0542796;
$N_4^{VI} = +\ 24.593006 N_4^{IV}$	"	1.3908116;
$N_4^{VII} = +160.63011 N_4^{IV}$	"	2.2058270.

For the root g_5, we get,

$$g_5 = 2''.7623108;$$

$N_5 = +\ 0.2979185 N_5^{IV}$	log.	9.4740975;
$N_5' = +\ 0.2890480 N_5^{IV}$	"	9.4609699;
$N_5'' = +\ 0.3020508 N_5^{IV}$	"	9.4800800;
$N_5''' = +\ 0.4005457 N_5^{IV}$	"	9.6026522;
$N_5^{V} = +\ 0.9164754 N_5^{IV}$	"	9.9621209;
$N_5^{VI} = +15.69060 N_5^{IV}$	"	1.1956394;
$N_5^{VII} = -\ 1.514217 N_5^{IV}$	"	0.1801882n.

For the root g_6, we get,

$$g_6 = 3''.7862093;$$

$N_6 = +0.6004756 N_6^{IV}$	log.	9.7784954;
$N_6' = +0.3937991 N_6^{IV}$	"	9.5952747;
$N_6'' = +0.3856959 N_6^{IV}$	"	9.5862450;
$N_6''' = +0.4382820 N_6^{IV}$	"	9.6417536;
$N_6^{V} = +0.794009 N_6^{IV}$	"	9.8998255;
$N_6^{VI} = -1.022221 N_6^{IV}$	"	0.0095448n;
$N_6^{VII} = +0.03054184 N_6^{IV}$	"	8.4848952.

For the root g_7, we get,

$$g_7 = 22''.5759799;$$

$N_7 = -0.006116321 N_7^{IV}$	log.	7.7864903n;
$N_7' = +0.01912057 N_7^{IV}$	"	8.2815008;
$N_7'' = -0.1503075 N_7^{IV}$	"	9.1769806n;
$N_7''' = -0.9432520 N_7^{IV}$	"	9.9746277n;
$N_7^{V} = -3.002313 N_7^{IV}$	"	0.4774559n;
$N_7^{VI} = +0.1143927 N_7^{IV}$	"	9.0583982;
$N_7^{VII} = +0.00864081 N_7^{IV}$	"	7.9365546.

37. These numbers are now to be substituted in equations (134) and (135). We must also add the logarithm of $1+\mu^{v}$, to those of $\frac{m^{v}}{n^{v}a^{v}}$, $\frac{m^{v}}{n^{v}a^{v}}h^{v}$, and $\frac{m^{v}}{n^{v}a^{v}}l^{v}$, in order to obtain the numbers to be used in this computation.

For the root $g=5''.465974$, we get,

$$x=+\frac{1834742}{10^{20}}N;\quad y=+\frac{50945.0}{10^{20}}N;\quad z=\frac{10464937}{10^{20}}N^2.$$

Whence $\beta=88°\ 24''\ 34''.2$, log. $N=9.2440059$.

$N\ \ =+0.1753904$; $N^{IV}=-0.000009216$;
$N'\ \ =+0.0085130$; $N^{V}\ =-0.000008230$;
$N''\ =+0.0054295$; $N^{VI}=+0.000003714$;
$N'''=+0.00083283$; $N^{VII}=+0.00000006374$.

For the root $g_1=7''.254515$, we get,

$$x_1=+\frac{1718569}{10^{20}}N_1;\quad y_1=+\frac{4394108}{10^{20}}N_1;\quad z_1=\frac{173815680}{10^{20}}N_1^2.$$

Whence $\beta_1=21°\ 21'\ 39''.0$; log. $N_1=8.4336898$.

$N_1\ \ =+0.0271450$; $N_1^{IV}=+0.000010892$;
$N_1\ \ =-0.0203915$; $N_1^{V}\ =+0.000011137$;
$N_1''=-0.0155449$; $N_1^{VI}=-0.0000028801$;
$N_1'''=-0.00257312$; $N_1^{VII}=-0.00000011621$.

For the root $g_2=17''.023581$, we get,

$$x_2=-\frac{18688360}{10^{20}}N_2;\quad y_2=+\frac{40529780}{10^{20}}N_2;\quad z_2=\frac{3.058100}{10^{10}}N_2^2.$$

Whence $\beta_2=335°\ 14'\ 43''.7$; log. $N_2=7.1641841$.

$N_2\ \ =+0.0014594$; $N_2^{IV}=-0.000001032$;
$N_2'\ =-0.0111657$; $N_2^{V}\ =-0.000006246$;
$N_2''=+0.0112699$; $N_2^{VI}=+0.0000003924$;
$N_2'''=+0.0222253$; $N_2^{VII}=+0.00000002875$.

For the root $g_3=17''.797850$, we get,

$$x_3=+\frac{132834470}{10^{20}}N_3;\quad y_3=-\frac{143133400}{10^{20}}N_3;\quad z_3=\frac{12.343316}{10^{10}}N_3^2.$$

Whence $\beta_3=137°\ 8'\ 14''.1$; log. $N_3=7.1992139$.

$N_3\ \ =+0.00158203$; $N_3^{IV}=+0.0000010517$;
$N_3'\ =-0.0131097$; $N_3^{V}\ =+0.000010491$;
$N_3''=+0.0161968$; $N_3^{VI}=-0.0000006377$;
$N_3'''=-0.0795892$; $N_3^{VII}=-0.00000004714$.

For the root $g_4=0''.6222813$, we get,

$$x_4=+\frac{3.414548}{10^{10}}N_4^{IV};\quad y_4=+\frac{1.395571}{10^{10}}N_4^{IV};\quad z_4=\frac{58943.27}{10^{10}}N_4^{IV2}.$$

Whence $\beta_4=67°\ 46'\ 10''.2$; log. $N_4^{IV}=95.7964432$.

$N_4\ \ =+0.000007604$; $N_4^{IV}=+0.000062581$;
$N_4'\ =+0.000011548$; $N_4^{V}\ =+0.000070912$;
$N_4''=+0.000013373$; $N_4^{VI}=+0.00153906$;
$N_4'''=+0.000021626$; $N_4^{VII}=+0.0100524$.

For the root $g_5=2''.762311$, we get,

$$x_5=+\frac{0.6624109}{10^{10}}N_5^{IV};\ y_5=-\frac{0.1988506}{10^{10}}N_5^{IV};\ z_5=\frac{361.8939}{10^{10}}N_5^{IV2}.$$

Whence $\beta_5=106°\ 42'\ 33''.6$; log. $N_5^{IV}=7.2812824$.

$N_5\ =+0.00056935$; $N_5^{IV}=+0.00191110$;
$N_5'\ =+0.00055240$; $N_5^{V}\ =+0.00175147$;
$N_5''=+0.00057725$; $N_5^{VI}=+0.0299862$;
$N_5'''=+0.00076548$; $N_5^{VII}=-0.00289381$.

For the root $g_6=3''.786209$, we get,

$$x_6=+\frac{0.4675181}{10^{10}}N_6^{IV};\ y_6=+\frac{0.8555864}{10^{10}}N_6^{IV};\ z_6=\frac{22.61242}{10^{10}}N_6^{IV2}.$$

Whence $\beta_6=28°\ 39'\ 12''.7$; log. $N_6^{IV}=8.6346522$.

$N_6\ =+0.0258909$; $N_6^{IV}=+0.0431174$;
$N_6'\ =+0.0169796$; $N_6^{V}\ =+0.0342356$;
$N_6''=+0.0166303$; $N_6^{VI}=-0.0440755$;
$N_6'''=+0.0188976$; $N_6^{VII}=+0.0013169$.

For the root $g_7=22''.575980$, we get,

$$x_7=-\frac{1.004018}{10^{10}}N_7^{IV};\ y_7=+\frac{0.7872757}{10^{10}}N_7^{IV};\ z_7=\frac{79.68195}{10^{10}}N_7^{IV2}.$$

Whence $\beta_7=308°\ 6'\ 3''.2$; log. $N_7^{IV}=8.2044480$.

$N_7\ =-0.0000979$; $N_7^{IV}=+0.0160121$;
$N_7'\ =+0.0003062$; $N_7^{V}\ =-0.0480733$;
$N_7''=-0.00240674$; $N_7^{VI}=+0.0018317$;
$N_7'''=-0.0151034$; $N_7^{VII}=+0.0001384$.

38. We shall now suppose that $\mu^{VI}=+\frac{1}{20}$; and the mass of *Uranus* will become $m^{VI}=\frac{1+\mu^{VI}}{24905}=\frac{1}{23719}$. Using this value of the mass of *Uranus*, we shall find,

$$\left.\begin{array}{ll}\Delta\boxed{0,0}=-0''.0000666; & \Delta\boxed{4,4}=-0''.0037881;\\ \Delta\boxed{1,1}=-0\ .0001705; & \Delta\boxed{5,5}=-0\ .0139141;\\ \Delta\boxed{2,2}=-0\ .0002778; & \Delta\boxed{6,6}=\ \ 0\ .\qquad\ ;\\ \Delta\boxed{3,3}=-0\ .0005261; & \Delta\boxed{7,7}=-0\ .0130554.\end{array}\right\}\ (326)$$

Whence we get,

$$\left.\begin{array}{ll}\boxed{0,0}=g-\ \ 5''.5703224; & \boxed{4,4}=g-\ \ 7''.5161635;\\ \boxed{1,1}=g-11\ .3149387; & \boxed{5,5}=g-18\ .6101270;\\ \boxed{2,2}=g-13\ .0724508; & \boxed{6,6}=g-\ \ 2\ .7662522;\\ \boxed{3,3}=g-17\ .5533906; & \boxed{7,7}=g-\ \ 0\ .6610123.\end{array}\right\}\ (327)$$

From these quantities we get the following equations:—

$$\left.\begin{aligned}
\boxed{0,0}\boxed{1,1}&=g^2-16.8852611.g+63.0278564952;\\
\boxed{0,0}\boxed{2,2}&=g^2-18.6427732.g+73.8177655141;\\
\boxed{0,0}\boxed{3,3}&=g^2-23.1237130.g+97.7780448551;\\
\boxed{1,1}\boxed{2,2}&=g^2-24.3873895.g+147.9139794608;\\
\boxed{1,1}\boxed{3,3}&=g^2-28.8683793.g+198.6155386162;\\
\boxed{2,2}\boxed{3,3}&=g^2-30.6258414.g+229.4658349917;
\end{aligned}\right\}\quad(328)$$

$$\left.\begin{aligned}
\boxed{4,4}\boxed{5,5}&=g^2-26.1262905.g+139.8767572878;\\
\boxed{4,4}\boxed{6,6}&=g^2-10.2824157.g+20.7916038174;\\
\boxed{4,4}\boxed{7,7}&=g^2-8.1771758.g+4.9682765223;\\
\boxed{5,5}\boxed{6,6}&=g^2-21.3763792.g+51.4803047560;\\
\boxed{5,5}\boxed{7,7}&=g^2-19.2711393.g+12.3015228516;\\
\boxed{6,6}\boxed{7,7}&=g^2-3.4272645.g+1.82852672910.
\end{aligned}\right\}\quad(329)$$

$$\left.\begin{aligned}
\boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=g^4&-47.5111025.g^3+809.61901994.g^2\\
&-5804.87167416.g+14462.73971842;
\end{aligned}\right\}\quad(330)$$

$$\left.\begin{aligned}
\boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=g^4&-29.5535550.g^3+231.24699197.g^2\\
&-527.16726514.g+255.76838948.
\end{aligned}\right\}\quad(331)$$

We shall therefore obtain the following

Fundamental Equations for $\mu^{VI}=+\frac{1}{20}$; *or, for* $m^{VI}=\frac{1}{23719}$.

$$\left.\begin{aligned}
A&=g^2-40.22212762.g+193.3374230;\\
A'&=g^2-23.14644147.g+98.0051228;\\
A''&=g^2-18.02250108.g+69.17798438;\\
A_1&=g^2-14.698591492.g+46.9394976;\\
A_2&=g^2-10.02448873.g+6.47860075;\\
A_3&=g^2-26.200650614.g+83.09379428;
\end{aligned}\right\}\quad(332)$$

$$\left.\begin{aligned}
D&=g^2-47.859709367.g+700.433897;\\
D'&=g^2-54.92219511.g+656.158133;\\
D''&=g^2-31.56874179.g+245.7424731;\\
D_1&=g^2-48.19349485.g+205.563726;\\
D_2&=g^2-51.079862967.g+35.0495614;\\
D_3&=g^2-3.4319830985.g+1.7650873387.
\end{aligned}\right\}\quad(333)$$

$$\left.\begin{aligned}
B=\{g-34.6348459\}b;\quad B'&=\{g-17.567753907\}b;\\
B''&=\{g-12.45143805\}b;
\end{aligned}\right\}\quad(334)$$

$$\left.\begin{aligned}
C&=\{23.7080852-g\}[9.1763990]b';\\
C'&=\{17.58251038-g\}[8.8694654]b';\\
C''&=-[0.2445917]b';\\
C'''&=-[0.2654598]b';
\end{aligned}\right\}\quad(335)$$

$$\left.\begin{aligned} E &= +[9.6231944]b''; \\ E' &= \{24.1516242 -g\}[8.5847028]b''; \\ E'' &= \{17.55466226-g\}[9.6375865]b''; \\ E''' &= +[0.9518913]b''; \end{aligned}\right\} \quad (336)$$

$$\left.\begin{aligned} F &= -[7.6499091]b'''; \\ F' &= +[8.9577383]b'''; \\ F'' &= \{14.01407953-g\}[0.8407439]b'''; \\ F''' &= \{37.3396847 -g\}[9.4809266]b'''; \end{aligned}\right\} \quad (337)$$

$$\left.\begin{aligned} B_1=\{g-4.88782529\}b_1;\ B_2=\{g-\ 0.692627293\}b_1; \\ B_3=\{g-18.679478012\}b_1; \end{aligned}\right\} \quad (338)$$

$$\left.\begin{aligned} C_1 &= \{4.445033406 -g\}[9.2840950]b_2; \\ C_2 &= \{0.7009659996-g\}[9.1824311]b_2; \\ C_3 &= -[0.6832317]b_2; \\ C_4 &= -[0.6834824]b_2; \end{aligned}\right\} \quad (339)$$

$$\left.\begin{aligned} E_1 &= +[7.8479616]b_3; \\ E_2 &= \{43.7484614 -g\}[8.2017618]b_3; \\ E_3 &= \{\ 0.661059519-g\}[0.7610455]b_3; \\ E_4 &= +[1.0867545]b_3; \end{aligned}\right\} \quad (340)$$

$$\left.\begin{aligned} F_1 &= -[7.0339474]b_4; \\ F_2 &= +[0.2724895]b_4; \\ F_3 &= \{\ 2.77088358-g\}[1.7737962]b_4; \\ F_4 &= \{50.3788970 -g\}[9.1128486]b_4; \end{aligned}\right\} \quad (341)$$

$$\left.\begin{aligned} g^4-47.5111025.g^3 +784.56174052.g^2 \\ -5224.66611233.g+11983.3131284 \end{aligned}\right\} =(\chi, \chi_1, \chi_2, \chi_3); \quad (342)$$

$$\left.\begin{aligned} g^4-29.5535550.g^3 +173.33949329.g^2 \\ -325.77538456.g+143.38934238 \end{aligned}\right\} =(\chi_4, \chi_5, \chi_6, \chi_7). \quad (343)$$

The values of b_1, b_2, b_3, and b_4 are given by equations (119); and the values of b, b', b'', and b''' are given by equations (118), by simply adding log. $(1+\mu^{VI})=[0.0211893]$, to the coefficients of N^{VI}.

Putting equations (342) and (343), equal to nothing, they will give,

$$\left.\begin{aligned} g &= 5''.46378145; & g_4 &= 0''.628004507; \\ g_1 &= 7\ .24790553; & g_5 &= 2\ .72658456; \\ g_2 &= 17\ .01456031; & g_6 &= 3\ .72632806; \\ g_3 &= 17\ .78485521; & g_7 &= 22\ .47263787. \end{aligned}\right\} \quad (344)$$

The equations just computed will now give the following values:—

For the root g, we get,

$g=5''.4638782$;

$N' \;=+0.04863950N$	log. 8.6869890;
$N'' \;=+0.03104071N$	" 8.4919316;
$N''' =+0.004765953N$	" 7.6781498;
$N^{IV}=-0.0000511782N$	" 5.7090852n;
$N^{V} \;=-0.0000456230N$	" 5.6591837n;
$N^{VI}=+0.00001999627N$	" 5.3009490;
$N^{VII}=+0.000000318386N$	" 3.5029538.

For the root g_1, we get,

$g_1=7''.2486360$;

$N_1' \;=-0.749378N_1$	log. 9.8747004n;
$N_1'' \;=-0.5714613N_1$	" 9.7569868n;
$N_1''' =-0.0946740N_1$	" 8.9762302n;
$N_1^{IV}=+0.0003968524N_1$	" 6.5986290;
$N_1^{V} \;=+0.0004049153N_1$	" 6.6073642;
$N_1^{VI}=-0.00010217875N_1$	" 6.0093606n;
$N_1^{VII}=-0.000004032583N_1$	" 4.6055833n.

For the root g_2, we get,

$g_2=17''.0146874$;

$N_2' \;=-\;7.644965N_2$	log. 0.8833755n;
$N_2'' \;=+\;7.708896N_2$	" 0.8869922;
$N_2''' =+15.37836N_2$	" 1.1869102;
$N_2^{IV}=-\;0.0007260315N_2$	" 6.8609555n;
$N_2^{V} \;=-\;0.004354339N_2$	" 7.6389222n;
$N_2^{VI}=+\;0.0002668697N_2$	" 6.4262993;
$N_2^{VII}=+\;0.00001945905N_2$	" 5.2891217.

For the root g_3, we get,

$g_3=17''.7849019$;

$N_3' \;=-\;8.277612N_3$	log. 0.9179050n;
$N_3'' \;=+10.208104N_3$	" 1.0089451;
$N_3''' =-49.64280N_3$	" 1.6958563n;
$N_3^{IV}=+\;0.0006759756N_3$	" 6.8299311;
$N_3^{V} \;=+\;0.006899148N_3$	" 7.8387954;
$N_3^{VI}=-\;0.0003922849N_3$	" 6.5936016n;
$N_3^{VII}=-\;0.00002886839N_3$	" 5.4604225n.

For the root g_4, we get,

$$g_4=0''.62800419;$$

$N_4 \;=+\;\;0.1217541 N_4^{IV}$	log. 9.0854837;
$N_4' \;=+\;\;0.1847141 N_4^{IV}$	" 9.2665000;
$N_4'' \;=+\;\;0.2138538 N_4^{IV}$	" 9.3301171;
$N_4''' \;=+\;\;0.3456976 N_4^{IV}$	" 9.5386963;
$N_4^{V} \;=+\;\;1.127368 N_4^{IV}$	" 0.0520657;
$N_4^{VI} =+\;23.96983 N_4^{IV}$	" 1.3796650;
$N_4^{VII}=+153.5574 N_4^{IV}$	" 2.1862708.

For the root g_5, we get,

$$g_5=2''.72651823;$$

$N_5 \;=+\;0.2923210 N_5^{IV}$	log. 9.4658600;
$N_5' \;=+\;0.286536 N_5^{IV}$	" 9.4571790;
$N_5'' \;=+\;0.2999964 N_5^{IV}$	" 9.4771160;
$N_5''' \;=+\;0.3994943 N_5^{IV}$	" 9.6015106;
$N_5^{V} \;=+\;0.9107594 N_5^{IV}$	" 9.9594036;
$N_5^{VI} =+14.68646 N_5^{IV}$	" 1.1669170;
$N_5^{VII}=-\;1.519462 N_5^{IV}$	" 0.1816899n.

For the root g_6, we get,

$$g_6=3''.7251312;$$

$N_6 \;=+0.5714777 N_6^{IV}$	log. 9.7569992;
$N_6' \;=+0.3858590 N_6^{IV}$	" 9.5864288;
$N_6'' \;=+0.3794987 N_6^{IV}$	" 9.5792102;
$N_6''' \;=+0.4358324 N_6^{IV}$	" 9.6393195;
$N_6^{V} \;=+0.7893470 N_6^{IV}$	" 9.8972680;
$N_6^{VI} =-1.030466 N_6^{IV}$	" 0.0130335n;
$N_6^{VII}=+0.0357074 N_6^{IV}$	" 8.5527582.

For the root g_7, we get,

$$g_7=22''.4729002;$$

$N_7 \;=-0.00622292 N_7^{IV}$	log. 7.7939944n;
$N_7' \;=+0.02016987 N_7^{IV}$	" 8.3047032;
$N_7'' \;=-0.1518602 N_7^{IV}$	" 9.1814440n;
$N_7''' \;=-0.9593122 N_7^{IV}$	" 9.9819600n;
$N_7^{V} \;=-3.093542 N_7^{IV}$	" 0.4904560n;
$N_7^{VI} =+0.1154757 N_7^{IV}$	" 9.0624906;
$N_7^{VII}=+0.008683434 N_7^{IV}$	" 7.9386915.

39. These numbers are now to be substituted in equations (134) and (135). We must also add the logarithm of $1+\mu^{VI}$, to those of $\frac{m^{VI}}{n^{VI}a^{VI}}$, $\frac{m^{VI}}{n^{VI}a^{VI}}h^{VI}$, and $\frac{m^{VI}}{n^{VI}a^{VI}}l^{VI}$, in order to obtain the numbers which are to be used in this computation.

For the root $g=5''.463878$, we get,

$$x=+\frac{1846778}{10^{20}}N;\ y=+\frac{61805.2}{10^{20}}N;\ z=\frac{10467663}{10^{20}}N^2.$$

Whence $\beta=88^\circ\ 4'\ 59''.6$; log. $N=9.2468080$.

$N\ =+0.1765257$; $N^{IV}=-0.0000090343$;
$N'\ =+0.0085861$; $N^{V}\ =-0.0000080536$;
$N''\ =+0.0054795$; $N^{VI}=+0.0000035299$;
$N'''=+0.00084131$; $N^{VII}=+0.0000000562$.

For the root $g_1=7''.248636$, we get,

$$x_1=+\frac{1657215}{10^{20}}N_1;\ y_1=\frac{4358217}{10^{20}}N_1;\ z_1=\frac{173063980}{10^{20}}N_1^2.$$

Whence $\beta_1=20^\circ\ 49'\ 9''.6$; log. $N_1=8.4304273$.

$N_1\ =+0.0269418$; $N_1^{IV}=+0.000010692$;
$N_1'\ =-0.0201896$; $N_1^{V}\ =+0.000010909$;
$N_1''\ =-0.0153962$; $N_1^{VI}=-0.000002753$;
$N_1'''=-0.00255069$; $N_1^{VII}=-0.0000001086$.

For the root $g_2=17''.014687$, we get,

$$x_2=-\frac{18857840}{10^{20}}N_2,\ y_2=+\frac{40818170}{10^{20}}N_2,\ z_2=\frac{3.068621}{10^{10}}N_2^2.$$

Whence $\beta_2=335^\circ\ 12'\ 11''.5$; log. $N_2=7.1659197$.

$N_2\ =+0.0014652$; $N_2^{IV}=-0.0000010637$;
$N_2'=-0.0112020$; $N_2^{V}\ =-0.0000063803$;
$N_2''=+0.0112957$; $N_2^{VI}=+0.00000039104$;
$N_2'''=+0.0225336$; $N_2^{VII}=+0.00000002851$.

For the root $g_3=17''.784902$, we get,

$$x_3=+\frac{131044100}{10^{20}}N_3;\ y_3=-\frac{141058200}{10^{20}}N_3;\ z_3=\frac{12.091655}{10^{10}}N_3^2.$$

Whence $\beta_3=137^\circ\ 6'\ 27''.7$; log. $N_3=7.2020253$.

$N_3\ =+0.0015923$; $N_3^{IV}=+0.0000010764$;
$N_3'=-0.0131805$; $N_3^{V}\ =+0.0000109855$;
$N_3''=+0.0162544$; $N_3^{VI}=-0.0000006246$;
$N_3'''=-0.0790463$; $N_3^{VII}=-0.0000000460$.

For the root $g_4=0''.6280042$, we get,

$$x_4=+\frac{3.295099}{10^{10}}N_4^{IV};\ y_4=\frac{1.261747}{10^{10}}N_4^{IV};\ z_4=\frac{53937.18}{10^{10}}N_4^{IV2}.$$

Whence $\beta_4=69^\circ\ 2'\ 50''.4$; log. $N_4^{IV}=95.8156910$.

$N_4\ =+0.000007965$; $N_4^{IV}=+0.000065417$;
$N_4'=+0.000011808$; $N_4^{V}\ =+0.000073749$;
$N_4''=+0.000013989$; $N_4^{VI}=+0.00156804$;
$N_4'''=+0.000022614$; $N_4^{VII}=+0.01004537$.

For the root $g_5=2''.726518$, we get,

$$x_5=+\frac{0.6486453}{10^{10}}N_5^{IV};\ y_5=-\frac{0.1822083}{10^{10}}N_5^{IV};\ z_5=\frac{337.9588}{10^{10}}N_5^{IV2}.$$

Whence $\beta_5=105°\ 41'\ 25''.3$; log. $N_5^{IV}=7.2996358$.

$$\begin{aligned} N_5\ &=+0.00058277; & N_5^{IV}&=+0.00199359;\\ N_5'\ &=+0.00057135; & N_5^{V}\ &=+0.00181568;\\ N_5''&=+0.00059807; & N_5^{VI}&=+0.0292788;\\ N_5'''&=+0.00079643; & N_5^{VII}&=-0.0030292. \end{aligned}$$

For the root $g_6=3''.725131$, we get,

$$x_6=+\frac{0.4579378}{10^{10}}N_6^{IV},\ y_6=+\frac{0.8593662}{10^{10}}N_6^{IV},\ z_6=\frac{22.55046}{10^{10}}N_6^{IV2}.$$

Whence $\beta_6=28°\ 3'\ 7''.9$; log. $N_6^{IV}=8.6352986$.

$$\begin{aligned} N_6\ &=+0.0246773; & N_6^{IV}&=+0.0431816;\\ N_6'\ &=+0.0166620; & N_6^{V}\ &=+0.0340853;\\ N_6''&=+0.0163874; & N_6^{VI}&=-0.0444971;\\ N_6'''&=+0.0188199; & N_6^{VII}&=+0.00154190. \end{aligned}$$

For the root $g_7=22''.4729002$, we get,

$$x_7=-\frac{1.010089}{10^{10}}N_7^{IV};\ y_7=+\frac{0.7868583}{10^{10}}N_7^{IV};\ z_7=\frac{81.93419}{10^{10}}N_7^{IV2}.$$

Whence $\beta_7=307°\ 55'\ 6''.8$; log. $N_7^{IV}=8.1938807$.

$$\begin{aligned} N_7\ &=-0.0000972; & N_7^{IV}&=+0.0156272;\\ N_7'\ &=+0.0003152; & N_7^{V}\ &=-0.0483433;\\ N_7''&=-0.0023732; & N_7^{VI}&=+0.0018046;\\ N_7'''&=-0.0149913; & N_7^{VII}&=+0.0001357. \end{aligned}$$

40. We shall now suppose $\mu^{VII}=+\frac{1}{10}$; and the mass of *Neptune* will become, $m^{VII}=\frac{1+\frac{1}{10}}{18780}=\frac{1}{17072.73}$. Using this value of *Neptune's* mass, we shall find,

$$\left.\begin{aligned} \Delta\boxed{0,0}&=-0''.0000460; & \Delta\boxed{4,4}&=-0''.0024017;\\ \Delta\boxed{1,1}&=-0\ .0001177; & \Delta\boxed{5,5}&=-0\ .0068740;\\ \Delta\boxed{2,2}&=-0\ .0001914; & \Delta\boxed{6,6}&=-0\ .04333257;\\ \Delta\boxed{3,3}&=-0\ .0003611; & \Delta\boxed{7,7}&=\ 0. \end{aligned}\right\}\quad(345)$$

Whence we get,

$$\left.\begin{aligned} \boxed{0,0}&=g-\ 5''.5703018; & \boxed{4,4}&=g-\ 7''.5147771;\\ \boxed{1,1}&=g-11\ .3148859; & \boxed{5,5}&=g-18\ .6030869;\\ \boxed{2,2}&=g-13\ .0723644; & \boxed{6,6}&=g-\ 2\ .8095779;\\ \boxed{3,3}&=g-17\ .5532256; & \boxed{7,7}&=g-\ 0\ .6479569. \end{aligned}\right\}\quad(346)$$

From these quantities we get the following equations,

$$\left.\begin{aligned}
\boxed{0,0}\boxed{1,1}&=g^2-16.8851877.g+63.0273292956;\\
\boxed{0,0}\boxed{2,2}&=g^2-18.6426662.g+72.8170149476;\\
\boxed{0,0}\boxed{3,3}&=g^2-23.1235274.g+97.7767641555;\\
\boxed{1,1}\boxed{2,2}&=g^2-24.3872503.g+147.9123116292;\\
\boxed{1,1}\boxed{3,3}&=g^2-28.8681115.g+198.6127448410;\\
\boxed{2,2}\boxed{3,3}&=g^2-30.6255900.g+229.4621614386;
\end{aligned}\right\}\quad(347)$$

$$\left.\begin{aligned}
\boxed{4,4}\boxed{5,5}&=g^2-26.1178640.g+139.7980514254;\\
\boxed{4,4}\boxed{6,6}&=g^2-10.3243550.g+21.1133516636;\\
\boxed{4,4}\boxed{7,7}&=g^2-8.1627340.g+4.8692516739;\\
\boxed{5,5}\boxed{6,6}&=g^2-21.4126648.g+52.2668218260;\\
\boxed{5,5}\boxed{7,7}&=g^2-19.2510438.g+12.0539985182;\\
\boxed{6,6}\boxed{7,7}&=g^2-3.4575348.g+1.82048538639.
\end{aligned}\right\}\quad(348)$$

$$\left.\begin{aligned}
\boxed{0,0}\boxed{1,1}\boxed{2,2}\boxed{3,3}=g^4-47.5107777.g^3&+809.60832632.g^2\\
-5804.7608117.g&+14462.38720986;
\end{aligned}\right\}\quad(349)$$

$$\left.\begin{aligned}
\boxed{4,4}\boxed{5,5}\boxed{6,6}\boxed{7,7}=g^4-29.5753988.g^3&+231.92196050.g^2\\
-530.90381750.g&+254.50030966.
\end{aligned}\right\}\quad(350)$$

We shall therefore obtain the following

Fundamental Equations for $\mu^{VII}=+\frac{1}{10}$; *or for* $m^{VII}=\frac{1}{17072.73}$.

$$\left.\begin{aligned}
A\phantom{{}_1}&=g^2-40.22202062.g+193.3362269;\\
A'&=g^2-23.14625587.g+98.0038404;\\
A''&=g^2-18.02242768.g+69.17743373;\\
A_1&=g^2-14.639503502.g+46.4149812;\\
A_2&=g^2-10.01320843.g+6.38683557;\\
A_3&=g^2-26.192224114.g+83.01495699;
\end{aligned}\right\}\quad(351)$$

$$\left.\begin{aligned}
D\phantom{{}_1}&=g^2-47.859570167.g+700.430559;\\
D'&=g^2-54.9219773.g+656.151042;\\
D''&=g^2-31.56849039.g+245\ 7386441;\\
D_1&=g^2-48.14983849.g+203.402186;\\
D_2&=g^2-51.063762837.g+34.5618232;\\
D_3&=g^2-3.4620415785.g+1.7536927202.
\end{aligned}\right\}\quad(352)$$

$$\left.\begin{aligned}
B=\{g-34.6347594\}b;\quad B'&=\{g-17''.567588907\}b;\\
B''&=\{g-12\ .451385272\}b;
\end{aligned}\right\}\quad(353)$$

$$\left.\begin{aligned}
C&=\{23.7079988-g\}[9.1763990]b';\\
C'&=\{17.58234538-g\}[8.8694654]b';\\
C''&=-[0.2445917]b';\\
C'''&=-[0.2654598]b';
\end{aligned}\right\}\quad(354)$$

$$\left.\begin{aligned} E &= +[9.6231944]b''; \\ E' &= \{24.1515714 - g\}[8.5847028]b''; \\ E'' &= \{17.55449726 - g\}[9.6375865]b''; \\ E''' &= +[0.9518913]b''; \end{aligned}\right\} \quad (355)$$

$$\left.\begin{aligned} F &= -[7.6499091]b'''; \\ F' &= +[8.9577383]b'''; \\ F'' &= \{14.01399313 - g\}[0.8407439]b'''; \\ F''' &= \{37.3396319 - g\}[9.4809266]b'''; \end{aligned}\right\} \quad (356)$$

$$\left.\begin{aligned} B_1 = \{g - 4.8301237\}b_1; \quad B_2 &= \{g - 0.682733393\}b_1; \\ B_3 &= \{g - 18.672437912\}b_1; \end{aligned}\right\} \quad (357)$$

$$\left.\begin{aligned} C_1 &= \{4.408417144 - g\}[9.2840950]b_2; \\ C_2 &= \{0.6919059696 - g\}[9.1824311]b_2; \\ C_3 &= -[0.6832317]b_2; \\ C_4 &= -[0.6834824]b_2; \end{aligned}\right\} \quad (358)$$

$$\left.\begin{aligned} E_1 &= +[7.8267723]b_3; \\ E_2 &= \{43.7414213 - g\}[8.2017618]b_3; \\ E_3 &= \{0.648008841 - g\}[0.7610455]b_3; \\ E_4 &= +[1.0655652]b_3; \end{aligned}\right\} \quad (359)$$

$$\left.\begin{aligned} F_1 &= -[7.0753401]b_4; \\ F_2 &= +[0.3138822]b_4; \\ F_3 &= \{2.813958738 - g\}[1.7737962]b_4; \\ F_4 &= \{50.37185686 - g\}[9.1128486]b_4. \end{aligned}\right\} \quad (360)$$

$$\left.\begin{aligned} g^4 - 47.5107777.g^3 + 784.55104690.g^2 \\ -5224.5598798.g + 11982.99308371 \end{aligned}\right\} = (\chi, \chi_1, \chi_2, \chi_3); \quad (361)$$

$$\left.\begin{aligned} g^4 - 29.5753988.g^3 + 174.01843537.g^2 \\ -327.77137981.g + 142.39163768 \end{aligned}\right\} = (\chi_4, \chi_5, \chi_6, \chi_7). \quad (362)$$

The values of b_1, b_2, b_3, and b_4 are given by equations (119); and the values of b, b', b'', and b''' are given by equations (118), by simply adding log. $(1+\mu^{vii}) =$ [0.0413927], to the coefficients of N^{vii}.

Putting equations (361) and (362), equal to nothing, they will give,

$$\left.\begin{aligned} g &= 5''.463758326; & g_4 &= 0''.6142757037; \\ g_1 &= 7\ .247841113; & g_5 &= 2\ .770720524; \\ g_2 &= 17\ .014463042; & g_6 &= 3\ .723848045; \\ g_3 &= 17\ .784715218; & g_7 &= 22\ .466554527. \end{aligned}\right\} \quad (363)$$

The equations just computed will now give the following values:

For the root g, we get,

$$g=5''.4638549;$$

$N' =+0.04864067N$	log. 98.6869996;
$N'' =+0.03104168N$	" 98.4919453;
$N''' =+0.004766193N$	" 97.6781717;
$N^{IV}=-0.00005107290N$	" 95.7081905n;
$N^{V} =-0.0000455608N$	" 95.6585913n;
$N^{VI}=+0.00002028325N$	" 95.3071375;
$N^{VII}=+0.0000003449032N$	" 93.5376972.

For the root g_1, we get,

$$g_1=7''.2485708;$$

$N_1' =-0.7476334N_1$	log. 9.8746886n;
$N_1'' =-0.5714480N_1$	" 9.7569768n;
$N_1''' =-0.09467274N_1$	" 8.9762250n;
$N_1^{IV}=+0.0003964014N_1$	" 6.5981351;
$N_1^{V} =+0.000404752N_1$	" 6.6071885;
$N_1^{VI}=-0.0001031014N_1$	" 6.0132647n;
$N_1^{VII}=-0.00000414784N_1$	" 4.6178221n.

For the root g_2, we get,

$$g_2=17''.0145899;$$

$N_2' =-\ 7.644900N_2$	log. 0.8833718n;
$N_2'' =+\ 7.708750N_2$	" 0.8869839;
$N_2''' =+15.37999N_2$	" 1.1869559;
$N_2^{IV}=-\ 0.0007239023N_2$	" 6.8596800n;
$N_2^{V} =-\ 0.004358588N_2$	" 7.6393458n;
$N_2^{VI}=+\ 0.0002678406N_2$	" 6.4278763;
$N_2^{VII}=+\ 0.0000195659N_2$	" 5.2924994.

For the root g_3, we get,

$$g_3=17''.7847617;$$

$N_3' =-\ 8.277515N_3$	log. 0.9179000n;
$N_3'' =+10.20779N_3$	" 1.0089319;
$N_3''' =-49.63566N_3$	" 1.6957938n;
$N_3^{IV}=+\ 0.0006721358N_3$	" 96.8274570;
$N_3^{V} =+\ 0.006905492N_3$	" 97.8391946;
$N_3^{VI}=-\ 0.0003936316N_3$	" 96.5950900n;
$N_3^{VII}=-\ 0.00002907741N_3$	" 95.4635558n.

For the root g_4, we get,

$$g_4 = 0''.6142754;$$

$N_4 = +\ 0.1212222 N_4^{IV}$	log.	9.0835822;	
$N_4' = +\ 0.1842843 N_4^{IV}$	"	9.2654883;	
$N_4'' = +\ 0.2134764 N_4^{IV}$	"	9.3293498;	
$N_4''' = +\ 0.3454170 N_4^{IV}$	"	9.5383437;	
$N_4^{V} = +\ 1.127956 N_4^{IV}$	"	0.0522920;	
$N_4^{VI} = +\ 24.27190 N_4^{IV}$	"	1.3851038;	
$N_4^{VII} = +145.26667 N_4^{IV}$	"	2.1621660.	

For the root g_5, we get,

$$g_5 = 2''.7706502;$$

$N_5 = +\ 0.2995950 N_5^{IV}$	log.	9.4765344;
$N_5' = +\ 0.2898491 N_5^{IV}$	"	9.4621720;
$N_5'' = +\ 0.3027071 N_5^{IV}$	"	9.4810226;
$N_5''' = +\ 0.4009023 N_5^{IV}$	"	9.6030386;
$N_5^{V} = +\ 0.9054583 N_5^{IV}$	"	9.9568685;
$N_5^{VI} = +14.64440 N_5^{IV}$	"	1.1656714;
$N_5^{VII} = -\ 1.406558 N_5^{IV}$	"	0.1481577n.

For the root g_6, we get,

$$g_6 = 3''.7226566;$$

$N_6 = +0.5703212 N_6^{IV}$	log.	9.7561195;
$N_6' = +0.3855324 N_6^{IV}$	"	9.5860609;
$N_6'' = +0.3792427 N_6^{IV}$	"	9.5789172;
$N_6''' = +0.4357298 N_6^{IV}$	"	9.6392173;
$N_6^{V} = +0.7895770 N_6^{IV}$	"	9.8973944;
$N_6^{VI} = -1.083630 N_6^{VI}$	"	0.0348810n;
$N_6^{VII} = +0.0356831 N_6^{IV}$	"	8.5524627.

For the root g_7, we get,

$$g_7 = 22''.4668172;$$

$N_7 = -0.006229914 N_7^{IV}$	log.	7.7944821n;
$N_7' = +0.02023760 N_7^{IV}$	"	8.3061590;
$N_7'' = -0.1519652 N_7^{IV}$	"	9.1817441n;
$N_7''' = -0.9604665 N_7^{IV}$	"	9.9824823n;
$N_7^{V} = -3.092542 N_7^{IV}$	"	0.4903156n;
$N_7^{VI} = +0.1157082 N_7^{IV}$	"	9.0633640;
$N_7^{VII} = +0.008726764 N_7^{IV}$	"	7.9408532.

41. These numbers are now to be substituted in equations (134) and (135). We must also add the logarithm of $1+\mu^{VII}$, to those of $\frac{m^{VII}}{n^{VII}a^{VII}}$, $\frac{m^{VII}}{n^{VII}a^{VII}}h^{VII}$, and $\frac{m^{VII}}{n^{VII}a^{VII}}l^{VII}$, in order to obtain the numbers which are to be used in this computation.

For the root $g=5''.463855$, we get,

$$x=+\frac{1847131}{10^{20}}N;\quad y=+\frac{63087.5}{10^{20}}N;\quad z=\frac{10467693}{10^{20}}N^2.$$

Whence $\beta=88°\ 2'\ 37''.9$; log. $N=9.2468998$.

$N=+0.1765630$; $N^{IV}=-0.0000090176$;
$N'=+0.0085881$; $N^{V}=-0.0000080444$;
$N''=+0.0054808$; $N^{VI}=+0.0000035813$;
$N'''=+0.00084153$; $N^{VII}=+0.0000000609$.

For the root $g_1=7''.248571$, we get,

$$x_1=+\frac{1656245}{10^{20}}N_1;\quad y_1=+\frac{4351968}{10^{20}}N_1;\quad z_1=+\frac{173055600}{10^{20}}N_1^2.$$

Whence $\beta_1=20°\ 50'\ 7''.8$; log. $N_1=8.4298727$.

$N_1=+0.0269075$; $N_1^{IV}=+0.000010666$;
$N_1'=-0.0201633$; $N_1^{V}=+0.000010891$;
$N_1''=-0.0153761$; $N_1^{VI}=-0.00002774$;
$N_1'''=-0.0025474$; $N_1^{VII}=-0.0000001116$.

For the root $g_2=17''.014590$, we get,

$$x_2=-\frac{18874270}{10^{20}}N_2;\quad y_2=+\frac{40848060}{10^{20}}N_2;\quad z_2=\frac{3.068734}{10^{10}}N_2^2.$$

Whence $\beta_2=335°\ 12'\ 0''.8'$; log. $N_2=7.1662319$.

$N_2=+0.0014663$; $N_2^{IV}=-0.0000010862$;
$N_2'=-0.0112099$; $N_2^{V}=-0.000006540$;
$N_2''=+0.0113035$; $N_2^{VI}=+0.0000004019$;
$N_2'''=+0.0225521$; $N_2^{VII}=+0.0000000294$.

For the root $g_3=17''.784762$, we get,

$$x_3=+\frac{131051610}{10^{20}}N_3;\quad y_3=-\frac{141079200}{10^{20}}N_3;\quad z_3=\frac{12.088926}{10^{10}}N_3^2.$$

Whence $\beta_3=137°\ 6'\ 45''.3$; log. $N_3=7.2021538$.

$N_3=+0.0015928$; $N_3^{IV}=+0.0000010705$;
$N_3'=-0.0131842$; $N_3^{V}=+0.000010999$;
$N_3''=+0.0162587$; $N_3^{VI}=-0.000000627$;
$N_3'''=-0.0790583$; $N_3^{VII}=-0.0000000463$.

For the root $g_4=0''.6142754$, we get,

$$x_4=+\frac{3.385189}{10^{10}}N_4^{IV};\quad y_4=+\frac{1.399525}{10^{10}}N_4^{IV};\quad z_4=\frac{53091.32}{10^{10}}N_4^{IV2}.$$

Whence $\beta_4=67°\ 32'\ 18''.7$; log. $N_4^{IV}=95.8388230$.

$N_4=+0.000008364$; $N_4^{IV}=+0.000068996$;
$N_4'=+0.000012715$; $N_4^{V}=+0.000077824$;
$N_4''=+0.000014729$; $N_4^{VI}=+0.00167466$;
$N_4'''=+0.000023832$; $N_4^{VII}=+0.01002282$.

For the root $g_5=2''.7706502$, we get,

$$x_5=+\frac{0.6382110}{10^{10}}N_5^{IV};\ y_5=-\frac{0.1345391}{10^{10}}N_5^{IV};\ z_5=-\frac{318.2862}{10^{10}}N_5^{IV2}.$$

Whence $\beta_5=101°\ 54'\ 14''.5$; log. $N_5^{IV}=7.3115881$.

$N_5\ =+0.00061393$;	$N_5^{IV}=+0.00204922$;
$N_5'\ =+0.00059396$;	$N_5^{V}\ =+0.00185548$;
$N_5''=+0.00062031$;	$N_5^{VI}=+0.0300096$;
$N_5'''=+0.00082154$;	$N_5^{VII}=-0.00288234$.

For the root $g_6=3''.722656$, we get,

$$x_6=-\frac{0.4577640}{10^{10}}N_6^{IV};\ y_6=+\frac{0.8595159}{10^{10}}N_6^{IV};\ z_6=+\frac{22.63367}{10^{10}}N_6^{IV2}.$$

Whence $\beta_6=28°\ 2'\ 20''.5$; log. $N_6^{IV}=8.6337215$.

$N_6\ =+0.0245381$;	$N_6^{IV}=+0.0430251$;
$N_6'\ =+0.0165876$;	$N_6^{V}\ =+0.0339716$;
$N_6''=+0.0163169$;	$N_6^{VI}=-0.0466232$;
$N_6'''=+0.0187473$;	$N_6^{VII}=-0.0015353$.

For the root $g_7=22''.466817$, we get,

$$x_7=-\frac{1.009767}{10^{10}}N_7^{IV};\ y_7=+\frac{0.7872124}{10^{10}}N_7^{IV};\ z_7=+\frac{81.89131}{10^{10}}N_7^{IV2}.$$

Whence $\beta_7=307°\ 56'\ 23''.7$; log. $N_7^{IV}=8.1940957$.

$N_7\ =-0.0000974$;	$N_7^{IV}=+0.0156349$;
$N_7'\ =+0.0003164$;	$N_7^{V}\ =-0.0483516$;
$N_7''=-0.0023759$;	$N_7^{VI}=+0.0018091$;
$N_7'''=-0.0150168$;	$N_7^{VII}=+0.0001364$.

42. We have thus obtained the values of all the constants, corresponding to the separate variations of the planetary masses. If we now subtract the values of the constants which correspond to the assumed masses from the values which result from the supposition that each planetary mass receives in succession a finite increment, and divide the difference of the constants by the supposed increment of mass, and connect together the different results, we shall have the following system of equations for the determination of the constants which correspond to any other assumed finite variation of the masses. The unit of the coefficients of μ, μ', μ'', &c., in the values of N, N', N'', &c., N_1, N_1', N_1'', &c., are the seventh decimal place of these coefficients.

$$g=5''.463803-0''.079418\mu+2''.68094\mu'+0''.91484\mu''+0''.034565\mu'''+1''.8040\mu^{IV}$$
$$+0''.08684\mu^{V}+0''.00150\mu^{VI}+0''.00052\mu^{VII};$$

$$\beta=88°\ 0'\ 38''-2939''\mu+50632''\mu'-20688''\mu''-3815''\mu'''-73640''\mu^{IV}+57456''\mu^{V}$$
$$+5236''\mu^{VI}+1202''\mu^{VII};$$

$$N=+0.1766064-50121\mu-420200\mu'+205480\mu''+7970\mu'''+676800\mu^{IV}$$
$$-486480\mu^{V}-16180\mu^{VI}-4360\mu^{VII};$$

$$N'=+0.0085906+59050\mu+146200\mu'-44980\mu''-4746\mu'''-124400\mu^{IV}-31040\mu^{V}$$
$$-900\mu^{VI}-250\mu^{VII};$$

$$N''=+0.0054825+37020\mu+118060\mu'-19180\mu''-4279\mu'''-108800\mu^{IV}-21200\mu^{V}$$
$$-600\mu^{VI}-170\mu^{VII};$$

$$N'''=+0.0008418+5672\mu+20302\mu'-2300\mu''-561.7\mu'''-23540\mu^{IV}-3596\mu^{V}$$
$$-102\mu^{VI}-29\mu^{VII};$$

$$N^{IV}=-0.0000090-72\mu-52\mu'+27\mu''+3.5\mu'''+182\mu^{IV}-86.4\mu^{V}-6.8\mu^{VI}-1.7\mu^{VII};$$

$$N^{V}=-0.0000080-63\mu-62\mu'+19\mu''+2.9\mu'''+181\mu^{IV}-78.4\mu^{V}-4.0\mu^{VI}-1.1\mu^{VII};$$

$$N^{VI}=+0.0000035+30\mu-9\mu'-20\mu''-1.7\mu'''-76\mu^{IV}-78.9\mu^{V}+2.6\mu^{VI}+6.4\mu^{VII};$$

$$N^{VII}=+0.0000000.6+0.4\mu+1.4\mu'+0.2\mu''-0.0\mu'''-0.8\mu^{IV}-0.2\mu^{V}-1.4\mu^{VI}-0.2\mu^{VII}.$$

$$g_1=7''.248427+0''.193515\mu+0''.08894\mu'+1''33906\mu''+0''.155640\mu'''+5''.2076\mu^{IV}$$
$$+0''.2429\mu^{V}+0''.00418\mu^{VI}+0''.00144\mu^{VII};$$

$$\beta_1=20°\ 50'\ 19''+41718''\mu+171946''\mu'-86900''\mu''+7848''\mu'''-221553''\mu^{IV}+75188''\mu^{V}$$
$$-1396''\mu^{VI}-115''\mu^{VII};$$

$$N_1=+0.0268838+5871\mu+793180\mu'-124700\mu''-29868\mu'''-658400\mu^{IV}$$
$$+104480\mu^{V}+11600\mu^{VI}+2370^{VII};$$

$$N_1'=-0.0201444-27607\mu-17180\mu'+72020\mu''+8397\mu'''+62400\mu^{IV}-98840\mu^{V}$$
$$-9120\mu^{VI}-1890^{VII};$$

$$N_1''=-0.0153619-25311\mu-25560\mu'+18800\mu''+8920\mu'''+95500\mu^{IV}-73200\mu^{V}$$
$$-6860\mu^{VI}-1420^{VII};$$

$$N_1'''=-0.0025451-4573\mu-8070\mu'-14568\mu''+1018\mu'''+3730\mu^{IV}-11212\mu^{V}$$
$$-1120\mu^{VI}-231\mu^{VII};$$

$$N_1^{IV}=+0.0000106+9.3\mu+43.6\mu'+10\mu''-4.4\mu'''-162\mu^{IV}+99.4\mu^{V}+9.6\mu^{VI}$$
$$+2.2\mu^{VII};$$

$$N_1^{V}=+0.0000109+11.4\mu+45.2\mu'+22.2\mu''-3.1\mu'''-181\mu^{IV}+104.8\mu^{V}+6.8\mu^{VI}$$
$$+1.6\mu^{VII};$$

$$N_1^{VI}=-0.0000027--14.6\mu-10.6\mu'+3.6\mu''+1.8\mu'''+63\mu^{IV}-54.4\mu^{V}-1.8\mu^{VI}$$
$$-3.0\mu^{VII};$$

$$N_1^{VII}=-0.0000001-0.1\mu-0.5\mu'-0.2\mu''+0.0\mu'''+1.5\mu^{IV}-1.6\mu^{V}+0.7\mu^{VI}+0.1\mu^{VII}.$$

$$g_2=17''.014373+0''.075742\mu+3''.72988\mu'+4''.06318\mu''+0''.006792\mu'''+8''.0455\mu^{IV}$$
$$+0''.36832\mu^{V}+0''.00628\mu^{VI}+0''.00217\mu^{VII};$$

$$\beta_2=335^\circ\ 11'\ 31''+483''\mu-336590''\mu'-239676''\mu''-27647''\mu'''+753490''\mu^{IV}+7692''\mu^{V}$$
$$+802''\mu^{VI}+294''\mu^{VII};$$

$$N_2=+0.0014673+1200\mu+61784\mu'+47930\mu''+11002\mu'''-130700\mu^{IV}-3160\mu^{V}$$
$$-420\mu^{VI}-90\mu^{VII};$$

$$N_2'=-0.0112171-9383\mu-345400\mu'-339820\mu''-83919\mu'''+928300\mu^{IV}+20560\mu^{V}$$
$$+3020\mu^{VI}+720\mu^{VII};$$

$$N_2''=+0.0113105+6512\mu+410140\mu'+196140\mu''+77329\mu'''-842300\mu^{IV}-16240\mu^{V}$$
$$-2960\mu^{VI}-700\mu^{VII};$$

$$N_2'''=+0.0225719+49397\mu+3069020\mu'+1992300\mu''+155278\mu'''-3928400\mu^{IV}$$
$$-138640\mu^{V}-7660\mu^{VI}+1980\mu^{VII};$$

$$N_2^{IV}=-0.0000011-1.2\mu-77\mu'-43.4\mu''-18.1\mu'''+114^{IV}+10.8\mu^{V}-1.0\mu^{VI}-2.7\mu^{VII};$$

$$N_2^{V}=-0.0000064-10.4\mu-658\mu'-447.6\mu''-110.1\mu'''+930\mu^{IV}+62\mu^{V}+4.2\mu^{VI}$$
$$-13.9\mu^{VII};$$

$$N_2^{VI}=+0.0000004-0.6\mu+38\mu'+25.4\mu''+6.7\mu'''-58\mu^{IV}+0.0\mu^{V}-0.2\mu^{VI}+1.0\mu^{VII};$$

$$N_2^{VII}=+0.0000000.3+0.0\mu+2.8\mu'+1.9\mu''+0.5\mu'''-4.2\mu^{IV}+0.0\mu^{V}-0.1\mu^{VI}+0.1^{VII}.$$

$$g_3=17''.784456+0''.034383\mu+2''.31316\mu'+2''.93202\mu''+0.231407\mu'''+12''.4768\mu^{IV}$$
$$+0''.53576\mu^{V}+0''.00892\mu^{VI}+0''.00306\mu^{VII};$$

$$\beta_3=137^\circ\ 6'\ 36''.5-4099''\mu-113430''\mu'-73350''\mu''+352''\mu'''+143360''\mu^{IV}+3904''\mu^{V}$$
$$-176''\mu^{VI}+88''\mu^{VII};$$

$$N_3=+0.0015934+1401\mu+63100\mu'+48280\mu''+8277\mu'''-129900\mu^{IV}-4560\mu^{V}$$
$$-220\mu^{VI}-60\mu^{VII};$$

$$N_3'=-0.0131892-11200\mu-351120\mu'-410400\mu''-73086\mu'''+945000\mu^{IV}$$
$$+31800\mu^{V}+1740\mu^{VI}+500\mu^{VII};$$

$$N_3''=+0.0162641+8420\mu+440180\mu'+185420\mu''+95264\mu'''-880700\mu^{IV}$$
$$-26920\mu^{V}-1940^{VI}-540\mu^{VII};$$

$$N_3'''=-0.0790650+46942\mu+3069920\mu'+1932860\mu''+161058\mu'''-3813200\mu^{IV}$$
$$-209680\mu^{V}+3740\mu^{VI}+670\mu^{VII};$$

$$N_3^{IV}=+0.0000011-1.3\mu-80.4\mu'-63.4\mu''+3.2\mu'''+128.9\mu^{IV}-5.4\mu^{V}+2.2\mu^{VI}$$
$$+0.5\mu^{VII};$$

$$N_3^{V}=+0.0000110-10.3\mu-662.2\mu'-413\mu''+71\mu'''+823^{IV}-207\mu^{V}-4.8\mu^{VI}$$
$$-1.1\mu^{VII};$$

$$N_3^{VI}=-0.0000006+0.6\mu+38.6\mu'+24.9\mu''-3.8\mu'''-42.7\mu^{IV}-4.8\mu^{V}+0.2\mu^{VI}$$
$$-0.1\mu^{VII};$$

$$N_3^{VII}=-0.0000000.5+0.0\mu+2.8\mu'+1.8\mu''-0.3\mu'''-3.3\mu^{IV}-0.3\mu^{V}+0.1\mu^{VI}+0.0\mu^{VII}.$$

$$g_4 = 0''.6166849 + 0\mu + 0\mu' + 0''.00002\mu'' + 0''.000007\mu''' + 0''.1905\mu^{IV} + 0''.22384\mu^{V} + 0''.22638\mu^{VI} - 0''.0241\mu^{VII};$$

$$\beta_4 = 67^\circ\ 56'\ 35'' + 16''\mu + 34''\mu' + 78''\mu'' + 16''\mu''' - 50700''\mu^{IV} + 25000''\mu^{V} + 79500''\mu^{VI} - 14560''\mu^{VII};$$

$$N_4 = +0.0000077 - 1\mu - 6\mu' - 4\mu'' + 0\mu''' - 58\mu^{IV} - 44\mu^{V} + 50\mu^{VI} + 65\mu^{VII};$$

$$N_4' = +0.0000117 - 5\mu - 11\mu' - 4\mu'' + 0\mu''' - 83\mu^{IV} - 70\mu^{V} + 17\mu^{VI} + 100\mu^{VII};$$

$$N_4'' = +0.0000136 - 1\mu - 20\mu' - 2\mu'' + 1\mu''' - 90\mu^{IV} - 83\mu^{V} + 82\mu^{VI} + 115\mu^{VII};$$

$$N_4''' = +0.0000219 - 0\mu - 2\mu' - 8\mu'' + 0\mu''' - 136\mu^{IV} - 128\mu^{V} + 134\mu^{VI} + 189\mu^{VII};$$

$$N_4^{IV} = +0.0000636 + 0\mu + 0\mu' - 1\mu'' - 1\mu''' - 487\mu^{IV} - 400\mu^{V} + 368\mu^{VI} + 541\mu^{VII};$$

$$N_4^{V} = +0.0000717 + 0\mu + 0\mu' - 0\mu'' - 0\mu''' - 682\mu^{IV} - 318\mu^{V} + 408\mu^{VI} + 612\mu^{VII};$$

$$N_4^{VI} = +0.0015578 + 3\mu - 4\mu' + 12\mu'' - 3\mu''' - 6250\mu^{IV} - 7476\mu^{V} + 2058\mu^{VI} + 11691\mu^{VII};$$

$$N_4^{VII} = +0.0100389 + 17\mu - 22\mu' + 84\mu'' - 15\mu''' - 4110\mu^{IV} + 5388\mu^{V} + 1288\mu^{VI} - 1611\mu^{VII}.$$

$$g_5 = 2''.727659 + 0''.000019\mu + 0''.00026\mu' + 0''.00050\mu'' + 0''.00016\mu''' + 0''.9302\mu^{IV} + 1''.38608\mu^{V} - 0''.02282\mu^{VI} + 0''.42991\mu^{VII};$$

$$\beta_5 = 105^\circ\ 3'\ 53'' + 19''\mu + 940''\mu' + 1860''\mu'' + 566''\mu''' - 192300''\mu^{IV} + 236840''\mu^{V} + 45040''\mu^{VI} - 113780''\mu^{VII};$$

$$N_5 = +0.0005685 - 50\mu - 2656\mu' - 1488\mu'' - 57\mu''' - 3612\mu^{IV} + 322\mu^{V} + 2844\mu^{VI} + 4539\mu^{VII};$$

$$N_5' = +0.0005571 - 65\mu - 620\mu' - 822\mu'' - 30\mu''' - 3436\mu^{IV} - 1881\mu^{V} + 2850\mu^{VI} + 3686\mu^{VII};$$

$$N_5'' = +0.0005832 - 41\mu - 896\mu' - 394\mu'' - 19\mu''' - 3314\mu^{IV} - 2392\mu^{V} + 2968\mu^{VI} + 3708\mu^{VII};$$

$$N_5''' = +0.0007765 - 9\mu - 272\mu' - 498\mu'' - 22\mu''' - 3807\mu^{IV} - 4428\mu^{V} + 3976\mu^{VI} + 4499\mu^{VII};$$

$$N_5^{IV} = +0.0019436 - 1\mu - 80\mu' - 160\mu'' - 55\mu''' - 8674\mu^{IV} - 13008\mu^{V} + 9994\mu^{VI} + 10560\mu^{VII};$$

$$N_5^{V} = +0.0017694 - 1\mu - 62\mu' - 124\mu'' - 43\mu''' - 11920\mu^{IV} - 7172\mu^{V} + 9256\mu^{VI} + 8608\mu^{VII};$$

$$N_5^{VI} = +0.0297330 + 1\mu - 240\mu' - 420\mu'' - 182''' - 75800\mu^{IV} + 101280\mu^{V} - 90840\mu^{VI} + 27660\mu^{VII};$$

$$N_5^{VII} = +0.0029105 + 0\mu + 28\mu' + 54\mu'' + 22\mu''' + 17870\mu^{IV} + 6676\mu^{V} - 23736\mu^{VI} + 2816\mu^{VII}.$$

$$g_6 = 3''.716607 + 0''.000071\mu + 0''.00318\mu' + 0''.00632\mu'' + 0''.002015\mu''' + 0''.6700\mu^{IV} \quad 2''.78408\mu^{V} + 0''.17048\mu^{VI} + 0''.06049\mu^{VII};$$

$$\beta_6 = 28^\circ\ 8'\ 46'' + 19''\mu + 24''\mu' + 90''\mu'' - 0''.5\mu''' - 61400''\mu^{IV} + 73080''\mu^{V} - 6760''\mu^{VI} - 3850''\mu^{VII};$$

$$N_6 = +0.0244939 - 642\mu - 233400\mu' - 121880\mu'' - 4595\mu''' - 193700\mu^{IV} + 558800\mu^{V} \quad 36680\mu^{VI} + 4420\mu^{VII};$$

$$N_6' = +0.0166053 - 531\mu - 21540\mu' - 41360\mu'' - 1758\mu''' - 88500\mu^{IV} + 149720\mu^{V} + 11340\mu^{VI} + 1770\mu^{VII};$$

$$N_6'' = +0.0163413 - 454\mu - 24920\mu' - 20000\mu'' - 1358\mu''' - 72000\mu^{IV} + 115600\mu^{V} + 9220\mu^{VI} - 2440\mu^{VII};$$

$$N_6''' = +0.0187954 - 104\mu - 6080\mu' - 10600\mu'' - 110\mu''' - 23900\mu^{IV} + 40840\mu^{V} + 4900\mu^{VI} - 4810\mu^{VII};$$

$$N_6^{IV} = +0.0431601 + 31\mu + 40\mu' + 140\mu'' + 78\mu''' + 24500\mu^{IV} - 17080\mu^{V} + 4300\mu^{VI} - 13500\mu^{VII};$$

$$N_6^{V} = +0.0341011 + 26\mu + 100\mu' + 260\mu'' + 111\mu''' - 39300\mu^{IV} + 53800\mu^{V} - 3160\mu^{VI} - 12950\mu^{VII};$$

$$N_6^{VI} = -0.0448614 - 0\mu + 1380\mu' + 2680\mu'' + 834\mu''' - 236100\mu^{IV} + 314360\mu^{V} + 72860\mu^{VI} - 176180\mu^{VII};$$

$$N_6^{VII} = +0.0014205 - 1\mu - 100\mu' - 200\mu'' - 66\mu''' + 8800\mu^{IV} - 41440\mu^{V} + 24280\mu^{VI} + 11480\mu^{VII}.$$

$$g_7 = 22''.460848 + 0''.000029\mu + 0''.00130\mu' + 0''.00274\mu'' + 0''.001098\mu''' + 17''.5730\mu^{IV} \quad 4''.60528\mu^{V} + 0''.24104\mu^{VI} + 0''.05969\mu^{VII};$$

$$\beta_7 - 307^\circ\ 56'\ 50'' + 0''\mu + 20''\mu' + 42''\mu'' + 2''.7\mu''' - 19800''\mu^{IV} + 22120''\mu^{V} - 2060''\mu^{VI} - 265''\mu^{VII}$$

$$N_7 = -0.0000975 - 13\mu - 1280\mu' - 1360\mu'' + 97\mu''' + 2300\mu^{IV} - 160\mu^{V} + 60\mu^{VI} + 10\mu^{VII};$$

$$N_7' = +0.0003175 + 147\mu + 10160\mu' + 18400\mu'' - 911\mu''' - 20800\mu^{IV} - 4520\mu^{V} - 460\mu^{VI} - 110\mu^{VII};$$

$$N_7'' = -0.0023780 - 167\mu - 20400\mu' - 7560\mu'' + 3508\mu''' + 33100\mu^{IV} - 11480\mu^{V} + 960\mu^{VI} + 210\mu^{VII};$$

$$N_7''' = -0.0150371 - 201\mu - 10040\mu' - 46960\mu'' - 867\mu''' + 74100\mu^{IV} - 26520\mu^{V} + 9160\mu^{VI} + 2030\mu^{VII};$$

$$N_7^{IV} = +0.0156383 + 1\mu + 40\mu' + 60\mu'' - 14\mu''' - 146000\mu^{IV} + 149520\mu^{V} - 2220\mu^{VI} - 340\mu^{VII};$$

$$N_7^{V} = -0.0483504 - 2\mu + 40\mu' + 0\mu'' + 142\mu''' - 111800\mu^{IV} + 110840\mu^{V} + 1420\mu^{VI} - 120\mu^{VII};$$

$$N_7^{VI} = +0.0018058 + 0\mu + 0\mu' + 0\mu'' - 7\mu''' - 10600\mu^{IV} + 10360\mu^{V} - 240\mu^{VI} + 330\mu^{VII};$$

$$N_7^{VII} = +0.0001365 + 0\mu + 0\mu' + 0\mu'' - 1\mu''' - 600\mu^{IV} + 760\mu^{V} - 160\mu^{VI} - 10\mu^{VII}.$$

CHAPTER II.

ON THE SECULAR VARIATIONS OF THE NODES AND INCLINATIONS OF THE ORBITS.

1. THE secular variations of the nodes, and the inclinations of the orbits, are determined by the integration of a system of differential equations which are entirely similar in form to those from which the eccentricities and perihelia were obtained.

If we denote by ϕ, ϕ', ϕ'', &c., the inclinations, and by θ, θ', θ'', &c., the longitudes of the nodes of the planets, *Mercury*, *Venus*, the *Earth*, &c., and put

$$\left.\begin{array}{lll}\tan\phi\sin\theta=p, & \tan\phi'\sin\theta'=p' & \tan\phi''\sin\theta''=p''\text{ &c.,}\\ \tan\phi\cos\theta=q, & \tan\phi'\cos\theta'=q' & \tan\phi''\cos\theta''=q''\text{ &c.;}\end{array}\right\}\quad(364)$$

we shall have the following system of differential equations for the determination of p, p', p'', &c., q, q', q'', &c.

$$\left.\begin{array}{l}
\frac{dq}{dt}=\quad\left\{(0,1)+(0,2)+(0,3)+\text{&c.}\right\}p-(0,1)p'-(0,2)p''-(0,3)p'''-\text{&c.,}\\
\frac{dp}{dt}=-\left\{(0,1)+(0,2)+(0,3)+\text{&c.}\right\}q+(0,1)q'+(0,2)q''+(0,3)q'''+\text{&c.,}\\
\frac{dq'}{dt}=\quad\left\{(1,0)+(1,2)+(1,3)+\text{&c.}\right\}p'-(1,0)p-(1,2)p''-(1,3)p'''-\text{&c.,}\\
\frac{dp'}{dt}=-\left\{(1,0)+(1,2)+(1,3)+\text{&c.}\right\}q'+(1,0)q+(1,2)q''+(1,3)q'''+\text{&c.,}\\
\frac{dq''}{dt}=\quad\left\{(2,0)+(2,1)+(2,3)+\text{&c.}\right\}p''-(2,0)p-(2,1)p'-(2,3)p'''-\text{&c.,}\\
\frac{dp''}{dt}=-\left\{(2,0)+(2,1)+(2,3)+\text{&c.}\right\}q''+(2,0)q+(2,1)q'+(2,3)q'''-\text{&c.}\\
\text{&c.}
\end{array}\right\}\quad(E)$$

To integrate these equations, we shall suppose

$$\left.\begin{array}{lll}p=N\sin(gt+\beta), & p'=N'\sin(gt+\beta), & p''=N''\sin(gt+\beta)\text{, &c.,}\\ q=N\cos(gt+\beta), & q'=N'\cos(gt+\beta), & q''=N''\cos(gt+\beta)\text{, &c.}\end{array}\right\}\quad(365)$$

If we substitute these values of p, q, p', q', &c., in equations (E), they will become

$$\left.\begin{array}{l}
-gN\;=\{(0,1)+(0,2)+0,3+\text{&c.}\}N\;-(0,1)N'-(0,2)N''-(0,3)N'''-\text{&c.;}\\
-gN'=\{(1,0)+(1,2)+1,3+\text{&c.}\}N'-(1,0)N-(1,2)N''-(1,3)N'''-\text{&c.;}\\
-gN''=\{(2,0)+(2,1)+2,3+\text{&c.}\}N''-(2,0)N-(2,1)N'-(2,3)N'''-\text{&c.;}\\
\text{&c.}
\end{array}\right\}\quad(E')$$

These equations are similar in form to equations (B), except that g is negative. They will produce, by eliminating N', N'', &c., an equation in g, of the eighth degree.

One of the roots of this equation will evidently be equal to nothing, since equations (E′) will be satisfied by supposing $g=0$, and $N=N'=N''$, &c.

2. We shall now suppose,

$$\left.\begin{array}{l}(0,0)=g+(0,1)+(0,2)+(0,3)+\&c.,\\(1,1)=g+(1,0)+(1,2)+(1,3)+\&c.,\\(2,2)=g+(2,0)+(2,1)+(2,3)+\&c.,\\\&c.;\end{array}\right\}\qquad(366)$$

$$\left.\begin{array}{l}-b\ =(0,4)N^{IV}+(0,5)N^{V}+(0,6)N^{VI}+(0,7)N^{VII},\\-b'\ =(1,4)N^{IV}+(1,5)N^{V}+(1,6)N^{VI}+(1,7)N^{VII},\\-b''=(2,4)N^{IV}+(2,5)N^{V}+(2,6)N^{VI}+(2,7)N^{VII},\\-b'''=(3,4)N^{VI}+(3,5)N^{V}+(3,6)N^{VI}+(3,7)N^{VII},\end{array}\right\}\qquad(367)$$

$$\left.\begin{array}{l}-b_1=(4,0)N\ +(4,1)N'+(4,2)N''+(4,3)N''',\\-b_2=(5,0)N\ +(5,1)N'+(5,2)N''+(5,3)N''',\\-b_3=(6,0)N\ +(6,1)N'+(6,2)N''+(6,3)N''',\\-b_4=(7,0)N\ +(7,1)N'+(7,2)N''+(7,3)N'''.\end{array}\right\}\qquad(368)$$

If we now substitute these quantities in equations (E′), they will become

$$\left.\begin{array}{l}(0,0)N\ \ -(0,1)N'\ -(0,2)N''\ -(0,3)N'''\ +b\ =0,\\(1,1)N'\ -(1,0)N\ \ -(1,2)N''\ -(1,3)N'''\ +b'=0,\\(2,2)N''\ -(2,0)N\ \ -(2,1)N'\ -(2,3)N'''\ +b''=0,\\(3,3)N'''\ -(3,0)N\ \ -(3,1)N'\ -(3,2)N''\ +b'''=0,\end{array}\right\}\qquad(E'')$$

$$\left.\begin{array}{l}(4,4)N^{IV}-(4,5)N^{V}-(4,6)N^{VI}-(4,7)N^{VII}+b_1=0,\\(5,5)N^{V}-(5,4)N^{IV}-(5,6)N^{VI}-(5,7)N^{VII}+b_2=0,\\(6,6)N^{VI}-(6,4)N^{IV}-(6,5)N^{V}-(6,7)N^{VII}+b_3=0,\\(7,7)N^{VII}-(7,4)N^{IV}-(7,5)N^{V}-(7,6)N^{VII}+b_4=0.\end{array}\right\}\qquad(E''')$$

These equations are similar to equations (B″) and (B‴) of § 9; and we may make use of equations (31–64) for their solution, if we suppose $\boxed{0,0}=(0,0)$, $\boxed{1,1}=(1,1)$, &c.; $\boxed{0,1}=-(0,1)$, $\boxed{0,2}=-(0,2)$, &c.; $\boxed{1,0}=-(1,0)$, $\boxed{1,2}=-(1,2)$, &c., in these equations. We have given the values of (0,1), (0,2), &c., (1,0), (1,2), &c., in § 7. The values of (0,0), (0,1), (2,2), &c., are given by means of the corresponding values $\boxed{0,0}$, $\boxed{1,1}$, $\boxed{2,2}$, &c., in equations (67), by simply changing the sign of the numerical terms of the second member.

3. We shall now reduce equations (31–64) to numbers. The values of the products (0,0)(1,1), (0,0)(2,2), &c. are given by means of equations (68–79) by simply changing the sign of the coefficients of g.

Computation of A.

$$
\begin{aligned}
&(0,0)(2,2)=g^2+18.6424288.g+\ 72.81534747\\
&-(0,0)(1,2)(2,3)\div(1,3)=\ +19.3779798.g+107.9403044\\
&-(2,2)(1,0)(0,3)\div(1,3)=\ +\ 0.0482175.g+\ \ 0.6303080\\
&+(1,0)(0,2)(2,3)\div(1,3)=\ \qquad\qquad\qquad +\ \ 0.4427946\\
&+(1,2)(2,0)(0,3)\div(1,3)=\ \qquad\qquad\qquad +\ \ 0.0738673\\
&-(2,0)(0,2)\qquad\qquad\ \ =\ \qquad\qquad\qquad -\ \ 0.0350059\\
&\text{Sum of terms}\quad A=g^2+38.0686261.g+181.867616
\end{aligned}
$$

Computation of A'.

$$
\begin{aligned}
&(0,0)(3,3)=g^2+23.1231203.g+97.7739453\\
&-(0,0)(1,3)(3,2)\div(1,2)=\ +\ 0.0270293.g+\ 0.1505601\\
&-(3,3)(1,0)(0,2)\div(1,2)=\ +\ 0.0228504.g+\ 0.4010900\\
&+(1,0)(0,3)(3,2)\div(1,2)=\ \qquad\qquad\qquad +\ 0.0013033\\
&+(1,3)(3,0)(0,2)\div(1,2)=\ \qquad\qquad\qquad +\ 0.0001030\\
&-(3,0)(0,3)\qquad\qquad\ \ =\ \qquad\qquad\qquad -\ 0.0002174\\
&\text{Sum of terms}\quad A'=g^2+23.1730000.g+98.3267843
\end{aligned}
$$

Computation of A''.

$$
\begin{aligned}
&(0,0)(1,1)=g^2+16.8850240.g+63.0261532\\
&-(0,0)(2,1)(1,3)\div(2,3)=\ +\ 1.8241484.g+10.1609735\\
&-(1,1)(2,0)(0,3)\div(2,3)=\ +\ 0.0038119.g+\ 0.0431310\\
&+(2,0)(0,1)(1,3)\div(2,3)=\ \qquad\qquad\qquad +\ 0.0416825\\
&+(2,1)(1,0)(0,3)\div(2,3)=\ \qquad\qquad\qquad +\ 0.0879559\\
&-(1,0)(0,1)\qquad\qquad\ \ =\ \qquad\qquad\qquad -\ 0.5272484\\
&\text{Sum of terms}\quad A''=g^2+18.7129843.g+72.8326477
\end{aligned}
$$

Computation of D.

$$
\begin{aligned}
&(1,1)(2,2)=g^2+24.3869412.g+147.9086074\\
&-(1,1)(0,2)(2,3)\div(0,3)=\ +\ 9.1832668.g+103.9065348\\
&-(2,2)(0,1)(1,3)\div(0,3)=\ +10.9347838.g+142.941385\\
&+(0,1)(1,2)(2,3)\div(0,3)=\ \qquad\qquad\qquad +211.894020\\
&+(0,2)(2,1)(1,3)\div(0,3)=\ \qquad\qquad\qquad +\ 16.751639\\
&-(2,1)(1,2)\qquad\qquad\ \ =\ \qquad\qquad\qquad -\ 35.348315\\
&\text{Sum of terms}\quad D=g^2+44.5049918.g+588.053871
\end{aligned}
$$

Computation of D'.

$$
\begin{aligned}
&(1,1)(3,3)=g_2+28.8676327.g+198.606593\\
&-(1,1)(0,3)(3,2)\div(0,2)=\ +\ 0.0570369.g+\ \ 0.645360\\
&-(3,3)(0,1)(1,2)\div(0,2)=\ +23.0739263.g+405.013500\\
&+(0,1)(1,3)(3,2)\div(0,2)=\ \qquad\qquad\qquad +\ \ 0.623672\\
&+(0,3)(3,1)(1,2)\div(0,2)=\ \qquad\qquad\qquad +\ \ 0.104041\\
&-(3,1)(1,3)\qquad\qquad\ \ =\ \qquad\qquad\qquad -\ \ 0.049305\\
&\text{Sum of terms}\quad D'=g^2+51.9985959.g+604.943861
\end{aligned}
$$

Computation of D''.

$$(2,2)(3,3) = g^2 + 30.6250375.g + 229.4540814$$
$$-(2,2)(0,3)(3,1) \div (0,1) = \quad + \ 0.0045090.g + \ 0.0589430$$
$$-(3,3)(0,2)(2,1) \div (0,1) = \quad + \ 1.5319589.g + \ 26.8902683$$
$$+(0,3)(3,2)(2,1) \div (0,1) = \quad + \ 0.0873762$$
$$+(0,2)(2,3)(3,1) \div (0,1) = \quad + \ 0.0414048$$
$$-(3,2)(2,3) \quad = \quad - \ 0.5237732$$
$$\text{Sum of terms} \quad D'' = g^2 + 32.1615054.g + 256.0083004$$

Computation of A_1.

$$(4,4)(6,6) = g^2 + 10.2786276.g + 20.7811250$$
$$-(4,4)(5,6)(6,7) \div (5,7) = \quad + \ 1.7539697.g + 13.1764792$$
$$-(6,6)(5,4)(4,7) \div (5,7) = \quad + \ 6.3754170.g + 17.6360114$$
$$+(5,4)(4,6)(6,7) \div (5,7) = \quad + \ 8.7135093$$
$$+(5,6)(6,4)(4,7) \div (5,7) = \quad + \ 0.0911343$$
$$-(6,4)(4,6) \quad = \quad - \ 0.0710140$$
$$\text{Sum of terms} \quad A_1 = g^2 + 18.4080144.g + 60.3272452$$

Computation of A_2.

$$(4,4)(7,7) = g^2 + 8.16033230.g + 4.8676955$$
$$-(4,4)(5,7)(7,6) \div (5,6) = \quad +0.06449760.g + 0.4845300$$
$$-(7,7)(5,4)(4,6) \div (5,6) = \quad +4.96787889.g + 3.218971$$
$$+(5,4)(4,7)(7,6) \div (5,6) = \quad +0.411199$$
$$+(5,7)(7,4)(4,6) \div (5,6) = \quad +0.003351$$
$$-(7,4)(4,7) \quad = \quad -0.004301$$
$$\text{Sum of terms} \quad A_2 = g^2 + 13.1927088.g + 8.981445$$

Computation of A_3.

$$(4,4)(5,5) = g^2 + 26.1085883.g + 139.7017323$$
$$-(4,4)(6,5)(5,7) \div (6,7) = \quad + \ 0.2214108.g + \ 1.663321$$
$$-(5,5)(6,4)(4,7) \div (6,7) = \quad + \ 0.0519589.g + \ 0.966238$$
$$+(6,4)(4,5)(5,7) \div (6,7) = \quad + \ 1.099942$$
$$+(6,5)(5,4)(4,7) \div (6,7) = \quad + \ 1.411586$$
$$-(5,4)(4,5) \quad = \quad -134.964267$$
$$\text{Sum of terms} \quad A_3 = g^2 + 26.3819580.g + \ 9.878552$$

Computation of D_1.

$$(5,5)(6,6) = g^2 + 21.3624651.g + \ 51.441815$$
$$-(5,5)(4,6)(6,7) \div (4,7) = \quad + \ 1.3667356.g + \ 25.416106$$
$$-(6,6)(4,5)(5,7) \div (4,7) = \quad +21.1694805.g + \ 58.560122$$
$$+(4,5)(5,6)(6,7) \div (4,7) = \quad + \ 37.130625$$
$$+(4,6)(6,5)(5,7) \div (4,7) = \quad + \ 0.302610$$
$$-(6,5)(5,6) \quad = \quad - \ 0.388348$$
$$\text{Sum of terms} \quad D_1 = g^2 + 43.8986812.g + 172.462930$$

Computation of D_2.

$$(5,5)(7,7)=g^2+19.2441698.g+12.0495445$$
$$-(5,5)(4,7)(7,6)\div(4,6)= +\ 0.0827715.g+\ 1.539238$$
$$-(7,7)(4,5)(5,6)\div(4,6)= +27.1673827.g+17.603294$$
$$+(4,5)(5,7)(7,6)\div(4,6)- \qquad +\ 1.752231$$
$$+(4,7)(7,5)(5,6)\div(4,6)= \qquad +\ 0.018327$$
$$-(7,5)(5,7) \quad = \qquad -\ 0.014281$$
$$\text{Sum of terms}\quad D_2=g^2+46.4943240.g+32.948353$$

Computation of D_3.

$$(6,6)(7,7)=g^2+3.4142091.g+1.79241220$$
$$-(6,6)(4,7)(7,5)\div(4,5)= +0.0006746.g+0.00186605$$
$$-(7,7)(4,6)(6,5)\div(4,5)= +0.0142946.g+0.00926231$$
$$+(4,7)(7,6)(6,5)\div(4,5)= \qquad +0.00118319$$
$$+(4,6)(6,7)(7,5)\div(4,5)= \qquad +0.00092197$$
$$-(7,6)(6,7) \qquad -0.11312682$$
$$\text{Sum of terms}\quad D_3=g^2+3.4291782.g+1.69251890$$

Computation of B, B', and B''.

$$(2,2)=g+13.072173$$
$$-(1,2)(2,3)\div(1,3)= +19.37798$$
$$\text{Sum} =B\div b=g+32.45015$$

$$(3,3)=g+17.5528645$$
$$-(1,3)(3,2)\div(1,2)= +\ 0.0270293$$
$$\text{Sum} =B'\div b=g+17.5798938$$

$$(1,1)=g+11.314768$$
$$-(2,1)(1,3)\div(2,3)= +\ 1.824148$$
$$\text{Sum} =B''\div b=g+13.138916.$$

Computation of B_1, B_2, and B_3.

$$(6,6)=g+2.7662522$$
$$-(5,6)(6,7)\div(5,7)= +1.7539698$$
$$\text{Sum} =B_1\div b_1=g+4.5202220;$$

$$(7,7)=g+0.6479569$$
$$-(5,7)(7,6)\div(5,6) \quad +0.0644976$$
$$\text{Sum} =B_2\div b_1=g+0.7124545;$$

$$(5,5)=g+18.5962129$$
$$-(6,5)(5,7)\div(6,7)= +\ 0.2214108$$
$$\text{Sum} =B_3\div b_1=g+18.8176237.$$

Computation of C, C', C'', and C'''.

$$-(2,2)=-g-13.0721730$$
$$(0,2)(2,3)\div(0,3)= -\ 9.183266$$
$$\text{Sum} =-\{g+22.255439\}$$
$$\log.\{(0,3)\div(1,3)\}=[9.4381189]$$
$$\text{Therefore } C=-\{g+22.255439\}\times [9.4381189]b'.$$

$$-(3,3)=-g-17.5528645$$
$$(0,3)(3,2)\div(0,2)= -\ 0.0570356$$
$$\text{Sum} =-\{g+17.6099001\}$$
$$\log.\{(0,2)\div(2,1)\}=[9.1138076]$$
$$\text{Therefore } C'=-\{g+17.6099001\}\times [9.1138076]b'.$$

Computation of C_1, C_2, C_3, and C_4.

$$-(6,6)=-g-\ 2.7662522$$
$$(4,6)(6,7)\div(4,7)= -\ 1.3667356$$
$$\text{Sum} =-\{g+\ 4.1329878\}$$
$$\log.\{(4,7)\div(5,7)\}=[9.5433087]$$
$$\text{Therefore } C_1=-\{g+\ 4.1329878\}\times [9.5433087]b_2.$$

$$-(7,7)=-g-\ 0.6479569$$
$$(4,7)(7,6)\div(4,6)= -\ 0.0827715$$
$$\text{Sum} =-\{g+\ 0.7307284\}$$
$$\log.\{(4,6)\div(5,6)\}=[9.4349711]$$
$$\text{Therefore } C_2=-\{g+\ 0.7307284\}\times [9.4349711]b_2.$$

$(0,3)(2,1)\div(2,3) = -\ 0.5002407$
$-(0,1) = +\ 2.9986729$
Sum $= C''\div b' = +\ 2.4984322$
Therefore $C'' = +[0.3976676]b'$.

$(0,2)(3,1)\div(3,2) = -\ 0.2370651$
$-(0,1) = +\ 2.9986729$
Sum $= C'''\div b' = +\ 2.7616078$
Therefore $C''' = +[0.4411620]b'$.

Computation of E, E', E'', and E'''.

$(0,3)(1,2)\div(1,3) = -\ 1.818323$
$-(0,2) = +\ 0.861707$
Sum $= E\div b'' = -\ 0.956616$
Therefore $E = -[9.9807377]b''$.

$-(1,1) = -g - 11.3147682$
$+(0,1)(1,3)\div(0,3) = -10.9347838$
Sum $= -\{g+22.2495520\}$
log. $\{(0,3)\div(2,3)\} = [8.9723624]$
Therefore $E' = -\{g+22.249552\}\times[8.9723624]b''$.

$-(3,3) = -g - 17.5528645$
$+(0,3)(3,1)\div(0,1) = -\ 0.0045090$
Sum $= -\{g+17.5573735\}$
log. $\{(0,1)\div(2,1)\} = [9.7501125]$
Therefore $E'' = -\{g+17.5573735\}\times[9.7501125]b''$.

$(0,1)(3,2)\div(3,1) = -10.89986$
$-(0,2) = +\ 0.86171$
Sum $= E'''\div b'' = -10.03815$
Therefore $E''' = -[1.0016537]b''$.

Computation of F, F', F'', and F'''.

$(0,2)(1,3)\div(1,2) = -\ 0.01326047$
$-(0,3) = +\ 0.02798148$
Sum $= F\div b''' = +\ 0.01472101$
Therefore $F = +[8.1679376]b'''$.

$(0,1)(2,3)\div(2,1) = -\ 0.1677337$
$-(0,3) = +\ 0.0279815$
Sum $= F'\div b''' = -\ 0.1397522$
Therefore $F' = -[9.1453586]b'''$.

$(4,7)(6,5)\div(6,7) = -\ 0.0773584$
$-(4,5) = +\ 7.3963746$
Sum $= C_3\div b_2 = +\ 7.3190162$
Therefore $C_3 = +[0.8644527]b_2$.

$(4,6)(7,5)\div(7,6) = -\ 0.0602795$
$-(4,5) = +\ 7.3963746$
Sum $= C_4\div b_2 = +\ 7.3360951$
Therefore $C_4 = +[0.8654649]b_2$.

Computation of E_1, E_2, E_3, and E_4.

$(4,7)(5,6)\div(5,7) = -\ 0.09722854$
$-(4,6) = +\ 0.07576285$
Sum $E_1\div b_3 = -\ 0.02146569$
Therefore $E_1 = -[8.3317448]b_3$.

$-(5,5) = -g - 18.5962129$
$(4,5)(5,7)\div(4,7) = -21.1694805$
Sum $= -\{g+39.7656934\}$
log. $\{(4,7)\div(6,7)\} = [8.7437718]$
Therefore $E_2 = -\{g+39.7656934\}\times[8.7437718]b_3$.

$-(7,7) = -g - 0.6479569$
$(4,7)(7,5)\div(4,5) = -\ 0.00067458$
Sum $= -\{g+\ 0.64863148\}$
log. $\{(4,5)\div(6,5)\} = [0.7242832]$
Therefore $E_3 = -\{g+0.6486315\}\times[0.7242832]b_3$.

$(4,5)(7,6)\div(7,5) = -\ 9.296196$
$-(4,6) = +\ 0.075763$
Sum $= E_4\div b_3 = -\ 9.220433$
Therefore $E_4 = -[0.9647514]b_3$.

Computation of F_1, F_2, F_3, and F_4.

$(4,6)(5,7)\div(5,6) = -\ 0.01871457$
$-(4,7) = +\ 0.02401693$
Sum $= F_1\div b_4 = +\ 0.00530236$
Therefore $F_1 = +[7.7244692]b_4$.

$(4,5)(6,7)\div(6,5) = -\ 2.296302$
$-(4,7) = +\ 0.024017$
Sum $= F_2\div b_4 = -\ 2.272285$
Therefore $F_2 = -[0.3564628]b_4$.

$$-(2,2)=-g-13.072173$$
$$(0,2)(2,1)\div(0,1)=-\ 1.531958$$
$$\text{Sum}\ =-\{g+14.604131\}$$
$$\log.\ \{(0,1)\div(3,1)\}=[0.7927855]$$
$$\text{Therefore}\ F''=-\{g+14.604131\}\times[0.7927855]b''.$$

$$-(1,1)=-g-11.3147682$$
$$(0,1)(1,2)\div(0,2)=-23.0739263$$
$$\text{Sum}\ =-\{g+34.3886945\}$$
$$\log.\ \{(0,2)\div(3,2)\}=[9.6907241]$$
$$\text{Therefore}\ F'''=-\{g+34.3886945\}\times[9.6907241]b'''.$$

$$-(6,6)=-g+2.7662522$$
$$(4,6)(6,5)\div(4,5)=+\ 0.0142946$$
$$\text{Sum}\ =-\{g+2.7805468\}$$
$$\log.\ \{(4,5)\div(7,5)\}=[1.5514854]$$
$$\text{Therefore}\ F_3=-\{g+2.7805468\}\times[1.5514854]b_4.$$

$$-(5,5)=-g-18.5962129$$
$$(4,5)(5,6)\div(4,6)=-27.1673827$$
$$\text{Sum}\ =-\{g+45.7635956\}$$
$$\log.\ \{(4,6)\div(7,6)\}\ [9.4626364]$$
$$\text{Therefore}\ F_4=-\{45.7635956+g\}\times[9.4626364]b_4.$$

Computation of the Equations of the fourth degree.

$-(2,1)(1,2)(0,0)(3,3)=$	$-35.3483155.g^2$	$-817.363351.g$	-3456.144265
$-(3,2)(2,3)(0,0)(1,1)=$	$-\ 0.5237732.g^2$	$-\ 8.843924.g$	$-\ 33.01142$
$-(1,0)(0,1)(2,2)(3,3)=$	$-\ 0.5272484.g^2$	$-\ 16.147000.g$	$-\ 120.97930$
$-(3,1)(1,3)(0,0)(2,2)=$	$-\ 0.0493054.g^2$	$-\ 0.919173.g$	$-\ 3.59019$
$-(2,0)(0,2)(1,1)(3,3)=$	$-\ 0.0350059.g^2$	$-\ 1.010537.g$	$-\ 6.95240$
$-(3,0)(0,3)(1,1)(2,2)=$	$-\ 0.0002174.g^2$	$-\ 0.005302.g$	$-\ 0.03216$
$+2(3,2)(2,1)(1,3)(0,0)=$		$-\ 1.910880.g$	$-\ 10.64409$
$+2(2,0)(0,1)(1,2)(3,3)=$		$-\ 1.615446.g$	$-\ 28.35570$
$+2(3,1)(1,0)(0,3)(2,2)=$		$-\ 0.004755.g$	$-\ 0.06216$
$+2(3,0)(0,2)(2,3)(1,1)=$		$-\ 0.003993.g$	$-\ 0.04518$
$+\ (1,0)(0,1)(3,2)(2,3)=$			$+\ 0.27616$
$+\ (2,1)(1,2)(3,0)(0,3)=$			$+\ 0.00768$
$+\ (2,0)(0,2)(3,1)(1,3)=$			$+\ 0.00173$
$-2(3,0)(0,1)(1,2)(2,3)=$			$-\ 0.09214$
$-2(3,1)(1,0)(0,2)(2,3)=$			$-\ 0.04366$
$-2(3,0)(0,2)(2,1)(1,3)=$			$-\ 0.00728$
Sum of preceding terms $=$	$-36.4838658.g^2$	$-847.824361.g$	-3659.67438

Add $(0,0)(1,1)(2,2)(3,3)=$

$$g^4+47.5100615.g^3+809.5847278.g^2+5804.515976.g+14461.60808$$

Sum is the value of equation (53).

Whence we get

$$\left.\begin{array}{l} g^4+47.5100615.g^3+773.1008620.g^2 \\ +4956.691615.g+10801.93370 \end{array}\right\}=(\chi,\chi_1,\chi_2,\chi_3). \quad (369)$$

In like manner we get

$-(5,4)(4,5)(6,6)(7,7)=$	$-134.9642669.g^2$	$-460.796228.g$	-241.911599
$-(6,5)(5,6)(4,4)(7,7)=$	$-\ 0.3883479.g^2$	$-\ 3.169047.g$	$-\ 1.890359$
$-(7,6)(6,7)(4,4)(5,5)=$	$-\ 0.1131268.g^2$	$-\ 2.953581.g$	$-\ 15.804014$
$-(6,4)(4,6)(5,5)(7,7)=$	$-\ 0.0710140.g^2$	$-\ 1.366608.g$	$-\ 0.855687$
$-(7,5)(5,7)(4,4)(6,6)=$	$-\ 0.0142805.g^2$	$-\ 0.146774.g$	$-\ 0.296764$
$-(7,4)(4,7)(5,5)(6,6)=$	$-\ 0.0043007.g^2$	$-\ 0.091874.g$	$-\ 0.221237$

$$+2(6,4)(4,5)(5,6)(7,7)= \quad -\ 3.858530.g-\ 2.500161$$
$$+2(7,5)(5,4)(4,7)(6,6)= \quad -\ 0.182088.g-\ 0.503701$$
$$+2(7,6)(6,5)(5,7)(4,4)= \quad -\ 0.050095.g-\ 0.376332$$
$$+2(7,4)(4,6)(6,7)(5,5)= \quad -\ 0.011756.g-\ 0.218615$$
$$+\ (5,4)(4,5)(7,6)(6,7)= \quad +\ 15.268077$$
$$+\ (6,5)(5,6)(7,4)(4,7)= \quad +\ 0.001670$$
$$+\ (6,4)(4,6)(7,5)(5,7)= \quad +\ 0.001014$$
$$-2(7,4)(4,5)(5,6)(6,7)= \quad -\ 0.319377$$
$$-2(7,5)(5,4)(4,6)(6,7)= \quad -\ 0.248866$$
$$-2(7,4)(4,6)(6,5)(5,7)= \quad -\ 0.002603$$

Sum of preceding terms $=-135.5553368.g^2-472.626581.g-249.878554$

Add $(4,4)(5,5)(6,6)(7,7)=$

$$g^4+29.5227974.g^3+230.6343243.g^2+523.7682780.g+250.403089$$

Sum is value of equation (54).

Whence we get

$$\left.\begin{array}{r} g^4-29.5227974.g^3+95.0789875.g^2 \\ +51.141697.g+0.524535 \end{array}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7). \quad (370)$$

Equations (55–64) reduced to numbers are as follows, in which the numbers inclosed in brackets are logarithms.

$$N'=[0.5618811]\frac{AN+f}{D}=[0.8861924]\frac{A'N+f'}{D'}; \quad (371)$$

$$N''=[1.0276376]\frac{A''N+f''}{D}=[0.2498875]\frac{A'N+f'''}{D''}; \quad (372)$$

$$N'''=[9.2072145]\frac{AN+f^{IV}}{D''}=[0.3092759]\frac{A''N+f^{V}}{D'}; \quad (373)$$

$$N=\frac{[9.4381189]Df'-[9.1138076]D'f}{[9.1138076]AD'-[9.4381189]A'D}; \quad (374)$$

$$N=\frac{[8.9723624]Df'''-[9.7501125]D''f''}{[9.7501125]A''D''-[8.9723624]A'D}; \quad (375)$$

$$N=\frac{[0.7927855]D''f^{V}-[9.6907241]D'f^{IV}}{[9.6907241]AD'-[0.7927855]A''D''}; \quad (376)$$

$$N^{V}=[0.4566913]\frac{A_1N^{IV}+f_1}{D_1}=[0.5650289]\frac{A_2N^{IV}+f_2}{D}; \quad (377)$$

$$N^{VI}=[1.2562282]\frac{A_3N^{IV}+f_3}{D_1}=[9.2757168]\frac{A_2N^{IV}+f_4}{D_3}; \quad (378)$$

$$N^{VII}=[8.4485146]\frac{A_1N^{IV}+f_5}{D_3}=[0.5373636]\frac{A_3N^{IV}+f_6}{D_2}; \quad (379)$$

$$N^{IV}=\frac{[9.5433087]D_1f_2-[9.4349711]D_2f_1}{[9.4349711]A_1D_2-[9.5433087]A_2D_1}; \quad (380)$$

$$N^{IV}=\frac{[8.7437718]D_1f_4-[0.7242832]D_3f_3}{[0.7242832]A_3D_3-[8.7437718]A_2D_1}; \quad (381)$$

$$N^{IV}=\frac{[1.5514854]D_3f_6-[9.4626364]D_2f_5}{[9.4626364]A_1D_2-[1.5514854]A_3D_3}. \tag{382}$$

$$\left.\begin{aligned} &\left\{[9.9546226]D'f\ -[0.2789339]Df\ \right\}\frac{1}{N} \\ =&\left\{[9.3015115]Df'''-[0.0792616]D''f''\right\}\frac{1}{N} \\ =&\left\{[8.9337058]D'f^{IV}-[0.0357672]D''f^{V}\right\}\frac{1}{N} \end{aligned}\right\}=(\chi,\chi_1,\chi_2,\chi_3); \tag{383}$$

$$\left.\begin{aligned} &\left\{[0.5477106]D_2f_1-[0.6560482]D_1f_2\right\}\frac{1}{N^{IV}} \\ =&\left\{[8.0240547]D_1f_4-[0.0045661]D_3f_3\right\}\frac{1}{N^{IV}} \\ =&\left\{[7.9147050]D_2f_5-[0.0035540]D_3f_6\right\}\frac{1}{N^{IV}} \end{aligned}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7). \tag{384}$$

If we repeat and number the formulæ which we have computed, we shall have the following

Fundamental Equations for the adopted masses.

$$A\ =g^2+38.0686261.g+181.867616; \tag{385}$$
$$A'=g^2+23.1730000.g+\ 98.3267843; \tag{386}$$
$$A''=g^2+18.7129843.g+\ 72.8326477; \tag{387}$$
$$A_1=g^2+18.4080144.g+\ 60.3272452; \tag{388}$$
$$A_2=g^2+13.1927088.g+\ \ 8.981445; \tag{389}$$
$$A_3=g^2+26.3819580.g+\ \ 9.878552; \tag{390}$$

$$D\ =g^2+44.5049918.g+588.053871; \tag{391}$$
$$D'=g^2+51.9985959.g+604.943861; \tag{392}$$
$$D''=g^2+32.1615054.g+256.0083004; \tag{393}$$
$$D_1=g^2+43.8986812.g+172.462930; \tag{394}$$
$$D_2=g^2+46.4943240.g+\ 32.948353; \tag{395}$$
$$D_3=g^2+\ 3.4291782.g+\ \ 1.69251890; \tag{396}$$

$$\left.\begin{aligned} B=\{g+32.45015\}b;\quad B'&=\{g+17.5798938\}b; \\ B''&=\{g+13.138916\}b; \end{aligned}\right\} \tag{397}$$

$$\left.\begin{aligned} C\ &=-\{g+22.255439\}\ [9.4381189]b'; \\ C'\ &=-\{g+17.6099001\}[9.1138076]b'; \\ C''&=+[0.3976676]b'; \\ C'''&=+[0.4411620]b'; \end{aligned}\right\} \tag{398}$$

$$\left.\begin{aligned} E\ &=-[9.9807377]b''; \\ E'\ &=-\{g+22.249552\}\ [8.9723624]b''; \\ E''&=-\{g+17.5573735\}[9.7501125]b''; \\ E'''&=-[1.0016537]b''; \end{aligned}\right\} \tag{399}$$

$$\left.\begin{aligned}
F &= +[8.1679376]b''';\\
F' &= -[9.1453586]b''';\\
F'' &= -\{g+14.604131\}\,[0.7927855]b''';\\
F''' &= -\{g+34.3886945\}[9.6907241]b''';
\end{aligned}\right\} \quad (400)$$

$$\left.\begin{aligned}
B_1=\{g+4.5202220\}b_1;\ B_2 &= \{g+\ 0.7124545\}b_1;\\
B_3 &= \{g+18.8176237\}b_1;
\end{aligned}\right\} \quad (401)$$

$$\left.\begin{aligned}
C_1 &= -\{g+4.1329878\}[9.5433087]b_2;\\
C_2 &= -\{g+0.7307284\}[9.4349711]b_2;\\
C_3 &= +[0.8644527]b_2;\\
C_4 &= +[0.8654649]b_2;
\end{aligned}\right\} \quad (402)$$

$$\left.\begin{aligned}
E_1 &= -[8.3317448]b_3;\\
E_2 &= -\{g+39.7656934\}[8.7437718]b_3;\\
E_3 &= -\{g+\ 0.6486315\}[0.7242832]b_3;\\
E_4 &= -[0.9647514]b_3;
\end{aligned}\right\} \quad (403)$$

$$\left.\begin{aligned}
F_1 &= +[7.7244692]b_4;\\
F_2 &= -[0.3564628]b_4;\\
F_3 &= -\{g+\ 2.7805468\}[1.5514854]b_4;\\
F_4 &= -\{g+45.7635956\}[9.4626364]b_4.
\end{aligned}\right\} \quad (404)$$

We shall also have

$$\left.\begin{aligned}
-b &= \{\ 1.6028375 \ldots [0.2048895]\}N^{IV}+[8.8879781]N^{V}\\
&\qquad +[7.1246469]\ N^{VI}+[6.6629969]N^{VII};\\
-b' &= \{\ 4.2028443 \ldots [0.6235433]\}N^{IV}+[9.2986071]N^{V}\\
&\qquad +[7.5327484]\ N^{VI}+[7.0706089]N^{VII};\\
-b'' &= \{\ 7.0682646 \ldots [0.8493128]\}N^{IV}+[9.5139049]N^{V}\\
&\qquad +[7.7447989]\ N^{VI}+[7.2820328]N^{VII};\\
-b''' &= \{14.659896\ \ldots [1.1661309]\}N^{IV}+[9.8003037]N^{V}\\
&\qquad +[8.0220791]\ N^{VI}+[7.5575701]N^{VII}.
\end{aligned}\right\} \quad (405)$$

$$\left.\begin{aligned}
-b_1 &= \{0.000094200\ \ \ldots [95.9740497]\}N+[7.6245464]N'\\
&\qquad +[\ 7.9450513]\ N''+[7.4917417]N''';\\
-b_2 &= \{0.0000112026\ \ldots [95.0493193]\}N+[6.6917912]N'\\
&\qquad +[\ 7.0018244]\ N''+[6.5180955]N''';\\
-b_3 &= \{0.00000096881 \ldots [93.9862386]\}N+[5.6261830]N'\\
&\qquad +[\ 5.9329689]\ N''+[5.4401214]N''';\\
-b_4 &= \{0.00000020167 \ldots [93.3046628]\}N+[4.9441177]N'\\
&\qquad +[\ 5.2502770]\ N''+[4.7556866]N'''.
\end{aligned}\right\} \quad (406)$$

4. If we now suppose the second members of equations (369) and (370) to be equal to nothing, we shall obtain the following values of g, g_1, g_2, &c.

$$\left.\begin{aligned}
g &= -\ 5''.1223003, & g_4 &= -\ 0''.01045922,\\
g_1 &= -\ 6\ .5863879, & g_5 &= -\ 0\ .66298299,\\
g_2 &= -17\ .3924594, & g_6 &= -\ 2\ .91695653,\\
g_3 &= -18\ .4089138, & g_7 &= -25\ .93239866.
\end{aligned}\right\} \quad (407)$$

By means of equations (385–407) and (371–384), we now obtain the following values.

For the root g, we get,

$$g = -5''.126112,$$

$N' = +0.1227919N$	log. 9.0891696,
$N'' = +0.08795864N$	" 8.9442784,
$N''' = +0.01757422N$	" 8.2448760,
$N^{IV} = -0.0002076885N$	" 6.3174124n,
$N^{V} = -0.000264229N$	" 6.421980n,
$N^{VI} = +0.000231553N$	" 6.364650,
$N^{VII} = +0.00000640325N$	" 4.806400.

For the root g_1, we get,

$$g_1 = -6''.592128,$$

$N_1' = -0.2764636N_1$	log. 9.4416380n,
$N_1'' = -0.2229392N_1$	" 9.3481864n,
$N_1''' = -0.04673056N_1$	" 8.6696010n,
$N_1^{IV} = +0.0003357396N_1$	" 6.5260026,
$N_1^{V} = +0.0004742548N_1$	" 6.6760116,
$N_1^{VI} = -0.0002454315N_1$	" 6.3899302n,
$N_1^{VII} = -0.00001482107N_1$	" 5.1708794n.

For the root g_2, we get,

$$g_2 = -17''.393390,$$

$N_2' = -\ 5.563101N_2$	log. 0.7453170n,
$N_2'' = +\ 4.563350N_2$	" 0.6592838,
$N_2''' = +33.24578N_2$	" 1.5217364,
$N_2^{IV} = -\ 0.001649287N_2$	" 7.2172964n,
$N_2^{V} = -\ 0.01403826N_2$	" 8.1473134n,
$N_2^{VI} = +\ 0.001367028N_2$	" 7.1357772,
$N_2^{VII} = +\ 0.000157230N_2$	" 6.1965360.

For the root g_3, we get,

$$g_3 = -18''.408914,$$

$N_3' = -\ 6.098710N_3$	log. 0.7852380n,
$N_3'' = +\ 6.655888N_3$	" 0.8232060,
$N_3''' = -10.22309N_3$	" 1.0095825n,
$N_3^{IV} = -\ 0.0001958273N_3$	" 5.2918731n,
$N_3^{V} = -\ 0.0001514003N_3$	" 6.1801268,
$N_3^{VI} = +\ 0.00001260627N_3$	" 5.1005864,
$N_3^{VII} = +\ 0.00000140198N_3$	" 4.1467414.

For the root g_4, we get,

$g_4 = 0$, and

$$N_4 = N_4' = N_4'' = N_4''' = N_4^{IV} = N_4^{V} = N_4^{VI} = N_4^{VII}.$$

For the root g_5, we get,

$$g_5 = -0''.661666,$$

$N_5 \;= +1.232212 N_5^{IV}$	log. 0.0906854,
$N_5' \;= +1.131314 N_5^{IV}$	" 0.0535832,
$N_5'' \;= +1.108223 N_5^{IV}$	" 0.0446270,
$N_5''' = +1.049477 N_5^{IV}$	" 0.0209706,
$N_5^{V} \;= +0.9653287 N_5^{IV}$	" 9.9846752,
$N_5^{VI} = -0.9379252 N_5^{IV}$	" 9.9721682n,
$N_5^{VII} = -9.829270 N_5^{IV}$	" 0.9925212n.

For the root g_6, we get,

$$g_6 = -2''.916082,$$

$N_6 \;= +\ 3.557327 N_6^{IV}$	log. 0.5511238,
$N_6' \;= +\ 2.059163 N_6^{IV}$	" 0.3136906,
$N_6'' \;= +\ 1.845317 N_6^{IV}$	" 0.2660709,
$N_6''' = +\ 1.314187 N_6^{IV}$	" 0.1186570,
$N_6^{V} \;= +\ 0.8164588 N_6^{IV}$	" 9.9119342,
$N_6^{VI} = -20.11300 N_6^{IV}$	" 1.3034768n,
$N_6^{VII} = +\ 2.161663 N_6^{IV}$	" 0.3347880.

For the root g_7, we get,

$$g_7 = -25''.934567,$$

$N_7 \;= -0.04207851 N_7^{IV}$	log. 8.6240604n,
$N_7' \;= -0.04653315 N_7^{IV}$	" 8.6677625n,
$N_7'' \;= -0.4328950 N_7^{IV}$	" 9.6363826n,
$N_7''' = -1.468114 N_7^{IV}$	" 0.1667598n,
$N_7^{V} \;= -2.490709 N_7^{IV}$	" 0.3963230n,
$N_7^{VI} = +0.1093424 N_7^{IV}$	" 9.0387886,
$N_7^{VII} = +0.01225277 N_7^{IV}$	" 8.0882343.

5. Having thus determined all the roots of the equation of the eighth degree, together with the ratios of the constant quantities N', N'', N''', &c., corresponding to each root, the complete integrals of equations (E) will be

$$\left.\begin{aligned}
q &= N\cos(gt+\beta) + N_1\cos(g_1t+\beta_1) + N_2\cos(g_2t+\beta_2) + \&c.,\\
q' &= N'\cos(gt+\beta) + N_1'\cos(g_1t+\beta_1) + N_2'\cos(g_2t+\beta_2) + \&c.,\\
q'' &= N''\cos(gt+\beta) + N_1''\cos(g_1t+\beta_1) + N_2''\cos(g_2t+\beta_2) + \&c.,\\
&\&c.;\\
p &= N\sin(gt+\beta) + N_1\sin(g_1t+\beta_1) + N_2\sin(g_2t+\beta_2) + \&c.,\\
p' &= N'\sin(gt+\beta) + N_1'\sin(g_1t+\beta_1) + N_2'\sin(g_2t+\beta_2) + \&c.,\\
p'' &= N''\sin(gt+\beta) + N_1''\sin(g_1t+\beta_1) + N_2''\sin(g_2t+\beta_2) + \&c.,\\
&\&c.
\end{aligned}\right\} \quad (F)$$

The analysis of § 16 will conduct us to the following equations for the determination of the arbitrary constants corresponding to each root.

$$\left.\begin{aligned}&Np\frac{m}{na}+N'p'\frac{m'}{n'a'}+N''p''\frac{m''}{n''a''}+N'''p'''\frac{m'''}{n'''a'''}+\text{\&c.}\\&=\left\{N^2\frac{m}{na}+N'^2\frac{m'}{n'a'}+N''^2\frac{m''}{n''a''}+N'''^2\frac{m'''}{n'''a'''}+\text{\&c.}\right\}\sin(gt+\beta)\end{aligned}\right\}\quad(408)$$

$$\left.\begin{aligned}&Nq\frac{m}{na}+N'q'\frac{m'}{n'a'}+N''q''\frac{m''}{n''a''}+N'''q'''\frac{m'''}{n'''a'''}+\text{\&c.}\\&=\left\{N^2\frac{m}{na}+N'^2\frac{m'}{n'a'}+N''^2\frac{m''}{n''a''}+N'''^2\frac{m'''}{n'''a'''}+\text{\&c.}\right\}\cos(gt+\beta)\end{aligned}\right\}\quad(409)$$

And since, for the root g_4, we have $N_4=N_4'=N_4''=N_4'''$, &c., the system of equations similar to equations (133) will be divisible by $N_4\ N_4'$ &c., we shall have the following equations of condition:—

$$\left.\begin{aligned}&N\frac{m}{na}+N'\frac{m'}{n'a'}+N''\frac{m''}{n''a''}+N'''\frac{m'''}{n'''a'''}+\text{\&c.}=0;\\&N_1\frac{m}{na}+N_1'\frac{m'}{n'a'}+N_1''\frac{m''}{n''a''}+N_1'''\frac{m'''}{n'''a'''}+\text{\&c.}=0;\\&N_2\frac{m}{na}+N_2'\frac{m'}{n'a'}+N_2''\frac{m''}{n''a''}+N_2'''\frac{m'''}{n'''a'''}+\text{\&c.}=0;\\&\text{\&c.}\end{aligned}\right\}\quad(410)$$

Now putting the first members of equations (408) and (409), equal to x and y respectively, and the coefficient of sin or cos of $(gt+\beta)$, in the same equations equal to z, we shall have

$$\left.\begin{aligned}&x=z\sin(gt+\beta);\quad y=\cos(gt+\beta);\\&\text{whence }\tan(gt+\beta)=x\div y.\end{aligned}\right\}\quad(411)$$

6. In order to find the values of x and y, which are to be used in equations (411), we shall assume the following values of ϕ, ϕ', ϕ'', &c., θ, θ', θ'', &c., corresponding to the beginning of the year 1850, at which epoch $t=0$.

Mercury,	$\phi\ =7°$	$0'$	$8''.2$,	$\theta\ =\ 46°$	$33'$	$3''.2$;
Venus,	$\phi'\ =3$	23	34.4,	$\theta'\ =\ 75$	20	42.9;
The Earth,	$\phi''\ =0$	0	0.0,	$\theta''=\ 0$	0	0 ;
Mars,	$\phi'''\ =1$	51	2.3,	$\theta'''=\ 48$	23	36.8;
Jupiter,	$\phi^{IV}=1$	18	40.3,	$\theta^{IV}=\ 98$	54	20.5;
Saturn,	$\phi^{V}\ =2$	29	22.4,	$\theta^{V}\ =112$	19	20.6;
Uranus,	$\phi^{VI}=0$	46	29.9,	$\theta^{VI}=\ 73$	14	14.4;
Neptune,	$\phi^{VII}=1$	47	0.9,	$\theta^{VII}=130$	7	45.3;

Now, since $q=\tan\phi\cos\theta$, and $p=\tan\phi\sin\theta$, we find,

$$\begin{aligned}&p\ =+0.0891690, && \log.p\ =8.9502138;\\&p'\ =+0.0573577, && \log.p'\ =8.7585915;\\&p''\ =\ 0. && \log.p''\ =-\infty\\&p'''\ =+0.0241597, && \log.p'''\ =8.3830911;\end{aligned}$$

$p^{IV}=+0.0226127,$	$\log. p^{IV}=8.3543534;$
$p^{V}=+0.0402201,$	$\log. p^{V}=8.6044432;$
$p^{VI}=+0.0129519,$	$\log. p^{VI}=8.1123323;$
$p^{VII}=+0.0238090,$	$\log. p^{VII}=8.3767411;$
$q=+0.0844678,$	$\log. q=8.9266911;$
$q'=+0.0149991,$	$\log. q'=8.1760651;$
$q''=\ 0.$	$\log. q''=-\infty\ ;$
$q'''=+0.0214548,$	$\log. q'''=8.3315249;$
$q^{IV}=-0.0035434,$	$\log. q^{IV}=7.5494159n;$
$q^{V}=-0.0165138,$	$\log. q^{V}=8.2178478n;$
$q^{VI}=+0.0039012,$	$\log. q^{VI}=7.5911972;$
$q^{VII}=-0.0200698,$	$\log. q^{VII}=8.3025434n.$

Now, adding the logarithms of $m\div na$, $m'\div n'a'$, &c., which are given in § 5, we shall obtain the following values of the logarithms of the constants for the given epoch, which enter into the values of x and y.

$\log. p\frac{m}{na}=85.9443779;$	$\log. q\frac{m}{na}=85.9208552;$
$\log. p'\frac{m'}{n'a'}=86.9845985;$	$\log. q'\frac{m'}{n'a'}=86.4020721;$
$\log. p''\frac{m''}{n''a''}=-\infty$	$\log. q''\frac{m''}{n''a''}=-\infty$
$\log. p'''\frac{m'''}{n'''a'''}=85.9337058;$	$\log. q'''\frac{m'''}{n'''a'''}=85.8821396;$
$\log. p^{IV}\frac{m^{IV}}{n^{IV}a^{IV}}=89.5793573;$	$\log. q^{IV}\frac{m^{IV}}{n^{IV}a^{IV}}=88.7744198n;$
$\log. p^{V}\frac{m^{V}}{n^{V}a^{V}}=89.4372661;$	$\log. q^{V}\frac{m^{V}}{n^{V}a^{V}}=89.0506707n;$
$\log. p^{VI}\frac{m^{VI}}{n^{VI}a^{VI}}=88.2449047;$	$\log. q^{VI}\frac{m^{VI}}{n^{VI}a^{VI}}=87.7237696;$
$\log. p^{VII}\frac{m^{VII}}{n^{VII}a^{VII}}=88.7292393;$	$\log. q^{VII}\frac{m^{VII}}{n^{VII}a^{VII}}=88.6550416n.$

7. These quantities are now to be substituted in equations (408) and (409), in connection with the values of N', N'', N''', &c., corresponding to the different roots.

For the root $g=-5''.126112$, we find,

$$x=+\frac{0.612517}{10^{14}}N;\ y=+\frac{1.5865662}{10^{14}}N;\ z=\frac{14.046564}{10^{14}}N^2.$$

Whence $\beta=211°\ 6'\ 26''.8$; and $\log. N=9.0830567$.

Therefore, for the root g, we have the following values,

$N=+0.121076,$	$N^{IV}=-0.00002517,$
$N'=+0.0148671,$	$N^{V}=-0.00003200,$
$N''=+0.0106496,$	$N^{VI}=+0.00002804,$
$N'''=+0.0021278,$	$N^{VII}=+0.000000775.$

In like manner we shall find for the root $g_1=6''.592128$,

$$x_1=+\frac{0.692850}{10^{14}}N_1;\quad y_1=-\frac{0.6389472}{10^{14}}N_1;\quad z_1=\frac{33.24236}{10^{14}}N_1^2.$$

Whence $\beta_1=132°\ 40'\ 57''.8$; and log. $N_1=8.4525867$.

Therefore, for the root g_1, we have,

$N_1 =+0.028352$, $\qquad N_1^{IV}=+0.00000952$,
$N_1' =-0.007838$, $\qquad N_1^{V} =+0.00001345$,
$N_1'' =-0.006321$, $\qquad N_1^{VI}=-0.00000696$,
$N_1''' =-0.001325$, $\qquad N_1^{VII}=-0.00000042$.

For the root $g_2=-17''.393390$, we find,

$$x_2=-\frac{6.863228}{10^{13}}N_2,\quad y_2=+\frac{2.889470}{10^{13}}N_2,\quad z_2=\frac{488.6204}{10^{12}}N_2^2.$$

Whence $\beta_2=292°\ 49'\ 53''.2$; and log. $N_2=7.1829906$.

$N_2 =+0.001524$, $\qquad N_2^{IV}=-0.00000251$,
$N_2' =-0.008478$, $\qquad N_2^{V} =-0.00002140$,
$N_2''=+0.0069546$, $\qquad N_2^{VI}=+0.00000208$,
$N_2'''=+0.0506672$, $\qquad N_2^{VII}=+0.00000024$.

For the root $g_3=-18''.408914$, we get,

$$x_3=-\frac{6.724398}{10^{13}}N_3;\quad y_3=-\frac{2.217061}{10^{13}}N_3;\quad z_3=\frac{1925.3617}{10^{13}}N_3^2.$$

Whence $\beta_3=251°\ 45'\ 8''.6$; and log. $N_3=7.5655490$.

$N_3 =+0.003677$, $\qquad N_3^{IV}=-0.000000072$,
$N_3' =-0.022428$, $\qquad N_3^{V} =-0.000000557$,
$N_3''=+0.024477$, $\qquad N_3^{VI}=+0.0000000463$,
$N_3'''=-0.0375951$, $\qquad N_3^{VII}=+0.000000005$.

For the root $g_4=0''$, we get,

$$x_4=+\frac{0.7256453}{10^{10}}N_4;\quad y_4=-\frac{0.2113461}{10^{10}}N_4;\quad z_4=\frac{27.24403}{10^{10}}N_4^2.$$

Whence $\beta_4=106°\ 14'\ 18''.0$; log. $N_4=8.4431335$.

And $N_4=N_4'=N_4''=N_4'''=N_4^{IV}=N_4^{V}=N_4^{VI}=N_4^{VII}=+0.02774173$.

For the root $g_5=-0''.661666$, we get,

$$x_5=+\frac{0.1016992}{10^{10}}N_5^{IV};\quad y_5=-\frac{0.2717211}{10^{10}}N_5^{IV};\quad z_5=\frac{24.19165}{10^{9}}N_5^{IV}.$$

Whence $\beta_5=20°\ 31'\ 24''.6$; and log. $N_5^{IV}=7.0789383$.

$N_5 =+0.001478$, $\qquad N_5^{IV}=+0.0011994$,
$N_5' =+0.001357$, $\qquad N_5^{V} =+0.0011577$,
$N_5''=+0.001329$, $\qquad N_5^{VI}=-0.00112485$,
$N_5'''=+0.001259$, $\qquad N_5^{VII}=-0.0117882$.

For the root $g_6 = -2''.916082$, we get,

$$x_6 = +\frac{0.3678921}{10^{10}} N_6^{IV}; \quad y_6 = -\frac{0.3544802}{10^{10}} N_6^{IV}; \quad z_6 = \frac{580.9484}{10^{10}} N_6^{IV2}.$$

Whence $\beta_6 = 133^\circ\ 56'\ 10''.8$; and log. $N_6^{IV} = 6.9441833$.

$$\begin{array}{ll} N_6\ = +0.003128, & N_6^{IV} = +0.0008794, \\ N_6' = +0.001811, & N_6^{V}\ = +0.0007180, \\ N_6'' = +0.001623, & N_6^{VI} = -0.0176872, \\ N_6''' = +0.001156, & N_6^{VII} = +0.0019010. \end{array}$$

For the root $g_7 = -25''.934567$, we get,

$$x_7 = -\frac{0.2996623}{10^{10}} N_7^{IV}; \quad y_7 = +\frac{0.2203054}{10^{10}} N_7^{IV}; \quad z_7 = \frac{59.03157}{10^{10}} N_7^{IV2}.$$

Whence $\beta_7 = 306^\circ\ 19'\ 21''.2$; log. $N_7^{IV} = 7.7993771$.

$$\begin{array}{ll} N_7\ = -0.0002652, & N_7^{IV} = +0.00630053, \\ N_7' = -0.0002932, & N_7^{V}\ = -0.0156928, \\ N_7'' = -0.0027275, & N_7^{VI} = +0.0006890, \\ N_7''' = -0.0092499, & N_7^{VII} = +0.00007720. \end{array}$$

If these values be substituted in equations (F), we shall have the complete values of q, q', q'', &c., p, p', p'', &c., from which we can obtain the inclination of the orbits of all the planets to the fixed ecliptic of 1850, and the longitudes of the nodes, on the same plane and referred to the equinox of 1850, by the formulæ

$$\tan \phi = \sqrt{p^2 + q^2}; \quad \tan \theta = p \div q. \tag{412}$$

8. If we now substitute in equations (F), the values of q and p, we shall get

$$q = \tan\phi \cos\theta = N \cos(gt+\beta) + N_1 \cos(g_1 t + \beta_1) + N_2 \cos(g_2 t + \beta_2) + \text{\&c.}; \tag{413}$$

$$p = \tan\phi \sin\theta = N \sin(gt+\beta) + N_1 \sin(g_1 t + \beta_1) + N_2 \sin(g_2 t + \beta_2) + \text{\&c.} \tag{414}$$

Multiplying equations (413) by $\sin(gt+\beta)$, and (414) by $-\cos(gt+\beta)$, we shall get, by adding their products, and reducing

$$\tan\phi \sin(\theta - gt - \beta) = N_1 \sin\{(g_1 - g)t + \beta_1 - \beta\} + N_2 \sin\left.\begin{cases}(g_2-g)t \\ +\beta_2 - \beta\} + \text{\&c.}\end{cases}\right\} \quad 415$$

If we multiply (413) by $\cos(gt+\beta)$, and (414) by $\sin(gt+\beta)$, we shall get, by adding their products, and reducing

$$\tan\phi \cos(\theta - gt - \beta) = N + N_1 \cos\{(g_1 - g)t + \beta_1 - \beta\} + N_2 \cos\left.\begin{cases}(g_2-g)t \\ +\beta_2 - \beta\} + \text{\&c.}\end{cases}\right\} \tag{416}$$

Dividing equation (415) by (416) we eliminate $\tan\phi$, and find,

$$\tan(\theta - gt - \beta) = \left.\frac{N_1 \sin\{(g_1-g)t+\beta_1-\beta\} + N_2 \sin\{(g_2-g)t+\beta_2-\beta\} + \text{\&c.}}{N + N_1 \cos\{(g_1-g)t+\beta_1-\beta\} + N_2 \cos\{(g_2-g)t+\beta_2-\beta\} + \text{\&c.}}\right\} \tag{417}$$

When the sum $N_1 + N_2 + N_3 +$ &c. of the coefficients of the cosines of the denominator, taken positively, is less than N, $\tan(\theta - gt - \beta)$ cannot become infinite; the angle $(\theta - gt - \beta)$ cannot become a right angle: consequently, the mean motion of the node will, in this case, be equal to gt. The analysis of § 19 being applied to equation (416), will show that

$$\left.\begin{aligned} &\textit{maximum}\ \tan\phi = N + N_1 + N_2 + N_3 + \text{\&c.};\\ \text{and}\ &\textit{minimum}\ \tan\phi = N - \{N_1 + N_2 + N_3 + \text{\&c.}\} \end{aligned}\right\} \tag{418}$$

We shall now substitute the numbers which we have already computed, in these equations, for the purpose of determining the maximum and minimum values of the inclinations of the different orbits to the fixed ecliptic of 1850, and the mean motions of the nodes of the different planets on that plane.

9. For the planet *Mercury*, we have,

Maximum $\tan\phi = N + N_1 + N_2 + N_3 + \text{\&c.} = 0.187242$. One-half of this is 0.093621, which being less than N, it follows that N exceeds the sum of all the remaining terms; consequently, the mean motion of Mercury's node is equal to g, or $-5''.126112$. The maximum inclination of his orbit to the ecliptic of 1850 is 10° 36′ 20″; and the minimum inclination is 3° 47′ 8″.

The substitution of the numbers for the other planets shows that the minimum inclinations of all the other planetary orbits to the ecliptic of 1850 are equal to nothing; consequently, the mean motions of the nodes on that plane are indeterminate. The maximum inclinations of the different orbits are as follows:—

	Max. inclination.			Max. inclination.	
Venus,	4°	51′	*Jupiter*,	2°	4′
Earth,	4	41	*Saturn*,	2	36
Mars,	7	28	*Uranus*,	2	42.5
			Neptune,	2	22.7

Having thus given the solution of the fundamental equations for the assumed masses, it now remains to determine the coefficients depending on the variation of the masses. This we shall do by using the same finite variations of the masses as were employed in finding the similar coefficients of the variations of the constants on which the eccentricities and perihelia depend.

10. If we now suppose that $\mu = +1.5$, we shall obtain the values of the fundamental quantities which are to be used in the computation by simply making all the terms of equations (153–169) positive. We shall then obtain the following

Fundamental Equations for $\mu = +\frac{3}{2}$: *or for* $m = \frac{1}{1946300.4}$.

$$\left.\begin{aligned} A &= g^2 + 38.201888.g + 183.882325;\\ A' &= g^2 + 23.2189305.g + 98.9957893;\\ A'' &= g^2 + 18.9824430.g + 73.7725468;\\ A_1 &= g^2 + 18.4081571.g + 60.3279033;\\ A_2 &= g^2 + 13.1928504.g + 8.981549;\\ A_3 &= g^2 + 26.3821161.g + 9.881338; \end{aligned}\right\} \tag{419}$$

$$\left.\begin{aligned}
D &= g^2+44.8296684.g+595.295403;\\
D' &= g^2+52.2739916.g+609.992181;\\
D'' &= g^2+32.2340961.g+257.2490518;\\
D_1 &= g^2+43.8986994.g+172.463056;\\
D_2 &= g^2+46.4943411.g+\ 32.948378;\\
D_3 &= g^2+\ 3.4291799.g+\ \ 1.69252064.
\end{aligned}\right\} \tag{420}$$

$$\left.\begin{aligned}
B=\{g+32.51109\}b;\quad B' &= \{g+17.5915487\}b;\\
B'' &= \{g+13.402657\}b;
\end{aligned}\right\} \tag{421}$$

$$\left.\begin{aligned}
C &= -\{g+22.316375\}\ [9.4381189]b';\\
C' &= -\{g+17.6215550\}[9.1138076]b';\\
C'' &= +[0.3976676]b';\\
C''' &= +[0.4411620]b';
\end{aligned}\right\} \tag{422}$$

$$\left.\begin{aligned}
E &= -[9.9807377]b'';\\
E' &= -\{g+22.5132928\}[8.9723624]b'';\\
E'' &= -\{g+17.5690284\}[9.7501125]b'';\\
E''' &= -[1.0016537]b'';
\end{aligned}\right\} \tag{423}$$

$$\left.\begin{aligned}
F &= +[8.1679376]b''';\\
F' &= -[9.1453586]b''';\\
F'' &= -\{g+14.665067\ \}[0.7927855]b''';\\
F''' &= -\{g+34.6524353\}[9.6907241]b''';
\end{aligned}\right\} \tag{424}$$

$$\left.\begin{aligned}
B_1=\{g+4.5202234\}b_1;\quad B_2 &= \{g+\ 0.7124548\}b_1;\\
B_3 &= \{g+18.8176405\}b_1;
\end{aligned}\right\} \tag{425}$$

$$\left.\begin{aligned}
C_1 &= -\{g+4.1329892\ \}[9.5433087]b_2;\\
C_2 &= -\{g+0.73072874\}[9.4349711]b_2;\\
C_3 &= +[0.8644527]b_2;\\
C_4 &= +[0.8654649]b_2;
\end{aligned}\right\} \tag{426}$$

$$\left.\begin{aligned}
E_1 &= -[8.3317448]b_3;\\
E_2 &= -\{g+39.7657102\ \}[8.7437718]b_3;\\
E_3 &= -\{g+\ 0.64863178\}[0.7242832]b_3;\\
E_4 &= -[0.9647509]b_3;
\end{aligned}\right\} \tag{427}$$

$$\left.\begin{aligned}
F_1 &= +[7.7244692]b_4;\\
F_2 &= -[0.3564628]b_4;\\
F_3 &= -\{g+27.805482\}\ [1.5514854]b_4;\\
F_4 &= -\{g+45.7636124\}[9.4626364]b_4.
\end{aligned}\right\} \tag{428}$$

$$\left.\begin{aligned}
&g^4+47.8463930.g^3+784.2708337.g^2\\
&+5058.994655.g+10978.29258
\end{aligned}\right\}=(\chi, \chi_1, \chi_2, \chi_3); \tag{429}$$

$$\left.\begin{aligned}
&g^4+29.5229572.g^3+95.0823272.g^2\\
&+51.151359.g\ +0.529116
\end{aligned}\right\}=(\chi_4, \chi_5, \chi_6, \chi_7). \tag{430}$$

The values of b, b', b'', and b''' are given by equations (405), and the values of b_1, b_2, b_3, and b_4 are given by equations (406), by merely multiplying the coefficients of N by $1+\mu=2.5$.

If we put equations (429) and (430) equal to nothing, they will give

$$
\begin{array}{ll}
g_1=-\ 4''.8105312, & g_4=-\ 0''.0105499, \\
g_2=-\ 7\ .0595858, & g_5=-\ 0\ .6629939, \\
g_3=-17\ .4356542, & g_6=-\ 2\ .9169622, \\
g_4=-18\ .5406218, & g_7=-25\ .9324509.
\end{array}
$$

The solutions of equations (419–430) will now give the following values—remembering that the coefficients of equations (371–384) remain unchanged.

For the root g, we get,

$$
\begin{array}{lll}
g & =-4''.815328, & \\
N' & =+0.2099057N & \log.\ 9.3210242, \\
N'' & =+0.1471310N & \text{“}\ \ 9.1677040, \\
N''' & =+0.0292551N & \text{“}\ \ 8.4662020, \\
N^{IV} & =-0.000390605N & \text{“}\ \ 6.5917376n, \\
N^{V} & =-0.000486221N & \text{“}\ \ 6.6868340n, \\
N^{VI} & =+0.000495923N & \text{“}\ \ 6.6954144, \\
N^{VII} & =+0.000008719136N & \text{“}\ \ 4.9404735.
\end{array}
$$

For the root g_1, we get,

$$
\begin{array}{lll}
g_1 & =-7''.064535, & \\
N_1' & =-0.400690N_1 & \log.\ 9.6028084n, \\
N_1'' & =-0.3382646N_1 & \text{“}\ \ 9.5292564n, \\
N_1''' & =-0.0725079N_1 & \text{“}\ \ 8.8603854n, \\
N_1^{IV} & =+0.000451081N_1 & \text{“}\ \ 6.6542543, \\
N_1^{V} & =+0.000660230N_1 & \text{“}\ \ 6.8196951, \\
N_1^{VI} & =-0.000300091N_1 & \text{“}\ \ 6.4772529n, \\
N_1^{VII} & =-0.0000202801N_1 & \text{“}\ \ 5.3070710n.
\end{array}
$$

For the root g_2, we get,

$$
\begin{array}{lll}
g_2 & =-17''.436558, & \\
N_2' & =-\ 5.522000N_2 & \log.\ 0.7420964n, \\
N_2'' & =+\ 4.240878N_2 & \text{“}\ \ 0.6274557, \\
N_2''' & =+37.23564N_2 & \text{“}\ \ 1.5709589, \\
N_2^{IV} & =-\ 0.001749727N_2 & \text{“}\ \ 7.2429702n, \\
N_2^{V} & =-\ 0.01522711N_2 & \text{“}\ \ 8.1826175n, \\
N_2^{VI} & =+\ 0.001476315N_2 & \text{“}\ \ 7.1691790, \\
N_2^{VII} & =+\ 0.0001698568N_2 & \text{“}\ \ 6.2300830.
\end{array}
$$

For the root g_3, we get,

$$g_3 = -18''.540625,$$

$N_3' = -6.10629 N_3$	log. $0.7857773n$,
$N_3'' = +6.47848 N_3$	" 0.8114730,
$N_3''' = -8.653764 N_3$	" $0.9372052n$,
$N_3^{IV} = -0.0000396789 N_3$	" $5.5985600n$,
$N_3^{V} = -0.000582834 N_3$	" $6.7655450n$,
$N_3^{VI} = +0.0000498987 N_3$	" 5.6980890,
$N_3^{VII} = +0.00000572183 N_3$	" 4.7575350.

For the root $g_4 = 0''$, we get,

$$N_4' = N_4'' = N_4''' = N_4^{IV} = \&c.$$

For the root g_5, we get,

$$g_5 = -0''.661636,$$

$N_5 = +1.235046 N_5^{IV}$	log. 0.0916830,
$N_6' = +1.135283 N_5^{IV}$	" 0.0551040,
$N_5'' = +1.110556 N_5^{IV}$	" 0.0455402,
$N_6''' = +1.049956 N_5^{IV}$	" 0.0211710,
$N_5^{V} = +0.965323 N_5^{IV}$	" 9.9846725,
$N_5^{VI} = -0.938071 N_5^{IV}$	" $9.9722356n$,
$N_5^{VII} = -9.829980 N_5^{IV}$	" $0.9925526n$.

For the root g_6, we get,

$$g_6 = -2''.916041,$$

$N_6 = +\ 3.695208 N_6^{IV}$	log. 0.5676388,
$N_6' = +\ 2.162242 N_6^{IV}$	" 0.3349042,
$N_6'' = +\ 1.91106 N_6^{IV}$	" 0.2812740,
$N_6''' = +\ 1.327425 N_6^{IV}$	" 0.1230098,
$N_6^{V} = +\ 0.816341 N_6^{VI}$	" 9.9118716,
$N_6^{VI} = -20.12041 N_6^{IV}$	" $1.3036368n$,
$N_6^{VII} = +\ 2.162624 N_6^{IV}$	" 0.3349790.

For the root g_7, we get,

$$g_7 = -25''.934626,$$

$N_7 = -0.0421230 N_7^{IV}$	log. $8.6245191n$,
$N_7' = -0.0455785 N_7^{IV}$	" $8.6587603n$,
$N_7'' = -0.4351129 N_7^{IV}$	" $9.6386020n$,
$N_7''' = -1.469776 N_7^{IV}$	" $0.1672216n$,
$N_7^{V} = -2.490698 N_7^{IV}$	" $0.3963210n$,
$N_7^{VI} = +0.1080897 N_7^{IV}$	" 9.0337842,
$N_7^{VII} = +0.01225267 N_7^{IV}$	" 8.0882305.

11. Substituting these values in equations (408) and (409), we shall get the following values:—

For the root $g=-4''.815328$, we get,

$$x=+\frac{1.524075}{10^{14}}N;\quad y=+\frac{3.435486}{10^{14}}N;\quad z=\frac{36.65228}{10^{14}}N^2.$$

Whence $\beta=23^\circ\ 55'\ 15''.6$; and log. $N=9.2108901$.

$N=+0.162514$,	$N^{IV}=-0.0000635$,
$N'=+0.034037$,	$N^{V}=-0.0000790$,
$N''=+0.023913$,	$N^{VI}=+0.0000806$,
$N'''=+0.004754$,	$N^{VII}=+0.0000014$.

For the root $g_1=-7''.064535$, we get,

$$x_1=+\frac{1.725880}{10^{14}}N_1;\quad y_1=-\frac{46}{10^{20}}N_1;\quad z_1=+\frac{75.88155}{10^{14}}N_1^2.$$

Whence $\beta_1=90^\circ\ 0'\ 7''.1$; and log. $N_1=8.3568764$.

$N_1=+0.022744$,	$N_1^{IV}=+0.0000103$,
$N_1'=-0.009114$,	$N_1^{V}=+0.0000150$,
$N_1''=-0.007694$,	$N_1^{VI}=-0.0000068$,
$N_1'''=-0.001649$,	$N_1^{VII}=-0.000000046$.

For the root $g_2=-17''.043656$, we get,

$$x_2=-\frac{6.709998}{10^{13}}N_2;\quad y_2=+\frac{3.568591}{10^{13}}N_2;\quad z_2=\frac{5820.032}{10^{13}}N_2^2.$$

Whence $\beta_2=298^\circ\ 0'\ 19''.7$; and log. $N_2=7.1158842$.

$N_2=+0.001306$,	$N_2^{IV}=-0.00000228$,
$N_2'=-0.007211$,	$N_2^{V}=-0.00001988$,
$N_2''=+0.005538$,	$N_2^{VI}=+0.00000193$,
$N_2'''=+0.048623$,	$N_2^{VII}=+0.00000022$.

For the root $g_3=-18''.540625$, we get,

$$x_3=-\frac{6.589852}{10^{13}}N_3;\quad y_3=-\frac{1.924650}{10^{13}}N_3;\quad z_3=\frac{1774.3737}{10^{13}}N_3^2.$$

Whence $\beta_3=253^\circ\ 43'\ 8''.0$; and log. $N_3=7.5876055$.

$N_3=+0.003869$,	$N_3^{IV}=-0.000000154$,
$N_3'=-0.023626$,	$N_3^{V}=-0.000002255$,
$N_3''=+0.025066$,	$N_3^{VI}=+0.0000001931$,
$N_3'''=-0.033482$,	$N_3^{VII}=+0.0000002213$.

For the root $g_4=0$, we get,

$$x_4=+\frac{0.7257772}{10^{10}}N_4;\quad y_4=-\frac{0.2112211}{10^{10}}N_4;\quad z_4=\frac{27.24551}{10^{10}}N_4^2.$$

Whence $\beta_4=106^\circ\ 13'\ 35''.2$; $N_4=+0.0277436$, log. $N_4=8.4431625$.

For the root $g_5=-0''.661636$, we get,

$$x_5=+\frac{0.1018241}{10^{10}}N_5^{IV};\ y_5=+\frac{0.2719087}{10^{10}}N_5^{IV};\ z_5=\frac{24.19508}{10^9}N_5^{IV2}.$$

Whence $\beta_5=20°\ 32'\ 21''.0$; and log. $N_5^{IV}=7.0792052$.

$N_5\ =+0.001482,$ $N_5^{IV}=+0.00120007,$

$N_5'\ =+0.001362,$ $N_5^{V}\ =+0.00115846,$

$N_5''=+0.001333,$ $N_5^{VI}=-0.00112554,$

$N_5'''=+0.001260,$ $N_5^{VII}=-0.0117964.$

For the root $g_6=-2''.916041$, we get,

$$x_6=+\frac{0.3683810}{10^{10}}N_6^{IV},\ y_6=-\frac{0.3540483}{10^{10}}N_6^{IV},\ z_6=\frac{581.3949}{10^{16}}N_6^{IV}.$$

Whence $\beta_6=133°\ 51'\ 48''.4$; and log. $N_6^{IV}=6.9438946$.

$N_6\ =+0.003247,$ $N_6^{IV}=+0.0008788,$

$N_6'\ =+0.001900,$ $N_6^{V}\ =+0.0007174,$

$N_6''=+0.001679,$ $N_6^{VI}=-0.0176820,$

$N_6'''=+0.001166,$ $N_6^{VII}=+0.0019005.$

For the root $g_7=-25''.934657$, we get,

$$x_7=-\frac{0.2996641}{10^{10}}N_7^{IV};\ y_7=+\frac{0.2202989}{10^{10}}N_7^{IV};\ z_7=\frac{59.03124}{10^{16}}N_7^{IV2}.$$

Whence $\beta_7=306°\ 19'\ 17''.6$; and log. $N_7^{IV}=7.7993764$.

$N_7\ =-0.000265,$ $N_7^{IV}=+0.00630052,$

$N_7'\ =-0.000287,$ $N_7^{V}\ =-0.0156923,$

$N_7''=-0.002741,$ $N_7^{VI}=+0.0006891,$

$N_7'''=-0.007355,$ $N_7^{VII}=+0.00007720.$

12. For an increment of $\frac{1}{20}$, in the mass of *Venus*, we have the preliminary computations by merely making all the coefficients positive in equations (193–208). We shall then obtain the following

Fundamental Equations for $\mu'=+\frac{1}{20}$; *or for* $m'=\frac{1}{371428.6}$.

$$\left.\begin{aligned}A\ &=g^2+38.4851145.g+188.270582;\\ A'\ &=g^2+23.3470945.g+101.1013581;\\ A''&=g^2+18.9541253.g+\ \ 75.3044562;\\ A_1&=g^2+18.4082271.g+\ \ 60.3282264;\\ A_2&=g^2+13.1929198.g+\ \ \ \ 8.981600;\\ A_3&=g^2+26.3821932.g+\ \ \ \ 9.882701.\end{aligned}\right\}\quad(431)$$

$$\left.\begin{aligned}D\ &=g^2+45.3182858.g+610.942268;\\ D'\ &=g^2+53.1764531.g+626.087193;\\ D''&=g^2+32.4522201.g+261.0475918;\\ D_1&=g^2+43.8987079.g+172.463115;\\ D_2&=g^2+46.4943490.g+\ \ 32.948389;\\ D_3&=g^2+\ \ 3.4291807.g+\ \ \ \ 1.69252138.\end{aligned}\right\}\quad(432)$$

$$\left.\begin{array}{l} B=\{g+32.71670\}b; \quad B'=\{g+17.6040547\}b; \\ \qquad\qquad\qquad\qquad B''=\{g+13.230123\}b; \end{array}\right\} \tag{433}$$

$$\left.\begin{array}{l} C\ =-\{g+22.521994\}\ [9.4381189]b'; \\ C'=-\{g+17.6340610\}[9.1138076]b'; \\ C''=+[0.4188569]b'; \\ C'''=+[0.4623513]b'; \end{array}\right\} \tag{434}$$

$$\left.\begin{array}{l} E\ =-[9.9807377]b''; \\ E'=-\{g+22.7962912\}[8.9723624]b''; \\ E''=-\{g+17.5815344\}[9.7501125]b''; \\ E'''=-[1.0016537]b''; \end{array}\right\} \tag{435}$$

$$\left.\begin{array}{l} F\ =+[8.1679376]b'''; \\ F'=-[9.1453586]b'''; \\ F''=-\{g+14.870686\}\ [0.7927855]b'''; \\ F'''=-\{g+35.542391\}\ [9.6907241]b'''; \end{array}\right\} \tag{436}$$

$$\left.\begin{array}{l} B_1=\{g+4.5202241\}b_1; \quad B_2=\{g+\ 0.7124549\}b_1; \\ \qquad\qquad\qquad\qquad B_3=\{g+18.8176481\}b_1; \end{array}\right\} \tag{437}$$

$$\left.\begin{array}{l} C_1=-\{g+4.1329899\}[9.5433087]b_2; \\ C_2=-\{g+0.7307288\}[9.4349711]b_2; \\ C_3=+[0.8644527]b_2; \\ C_4=+[0.8654649]b_2; \end{array}\right\} \tag{438}$$

$$\left.\begin{array}{l} E_1=-[8.3317448]b_3; \\ E_2=-\{g+39.7657180\}[8.7437718]b_3; \\ E_3=-\{g+0.64863188\}[0.7242832]b_3; \\ E_4=-[0.9647509]b_3; \end{array}\right\} \tag{439}$$

$$\left.\begin{array}{l} F_1=+[7.7244692]b_4; \\ F_2=-[0.3564628]b_4; \\ F_3=-\{g+2.7805489\}[1.5514854]b_4; \\ F_4=-\{g+45.763620\}[9.4626364]b_4; \end{array}\right\} \tag{440}$$

$$\left.\begin{array}{l} g^4+47.9507108.g^3+787.5462101.g^2 \\ +5098.436146.g+11226.89493 \end{array}\right\}=(\chi,\chi_1,\chi_2,\chi_3). \tag{441}$$

$$\left.\begin{array}{l} g^4+29.5230351.g^3+95.0839594.g^2 \\ +51.156083.g\ +\ 0.531357 \end{array}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7). \tag{442}$$

The values of b, b', b'', and b''' are given by equations (405); and the values of b_1, b_2, b_3, and b_4 are given by equations (406), by merely multiplying the coefficients of N' by $1+\mu'=1.05$.

If we put equations (441) and (442) equal to nothing, they will give

$$\begin{array}{ll} g\ =-\ 5''.1965445, & g_4=-\ 0''.0105949, \\ g_1=-\ 6\ .6295555, & g_5=-\ 0\ .6629993, \\ g_2=-17\ .4583971, & g_6=-\ 2\ .9169649, \\ g_3=-18\ .6662138, & g_7=-25\ .9324761. \end{array}$$

The solutions of equations (431–442) will now give the following values:—

For the root g, we get,

$$g=-5''.201065,$$

$N' = +0.1374398N$	log. 9.1381124,
$N'' = +0.1000556N$	" 9.0002416,
$N''' = +0.0201689N$	" 8.3046820,
$N^{IV} = -0.000233031N$	" 6.3674143n,
$N^{V} = -0.000298026N$	" 6.4742546n,
$N^{VI} = +0.000252528N$	" 6.4023092,
$N^{VII} = +0.00000754208N$	" 4.8774912.

For the root g_1, we get,

$$g_1=-6''.6347695,$$

$N_1' = -0.2372900N_1$	log. 9.3752793n,
$N_1'' = -0.1935676N_1$	" 9.2868326n,
$N_1''' = -0.0409512N_1$	" 8.6122666n,
$N_1^{IV} = +0.000292034N_1$	" 6.4654333,
$N_1^{V} = +0.000413818N_1$	" 6.6168096,
$N_1^{VI} = -0.000211538N_1$	" 6.3253889n,
$N_1^{VII} = -0.0000129222N_1$	" 5.1113360n.

For the root g_2, we get,

$$g_2=-17''.459300,$$

$N_2' = -\ 5.234937N_2$	log. 0.7189114n,
$N_2'' = +\ 4.210629N_2$	" 0.6243470,
$N_2''' = +40.01022N_2$	" 1.6021710,
$N_2^{IV} = -\ 0.00184238N_2$	" 7.2653793n,
$N_2^{V} = -\ 0.01622134N_2$	" 8.2100866n,
$N_2^{VI} = +\ 0.001569054N_2$	" 7.1956378,
$N_2^{VII} = +\ 0.000180556N_2$	" 6.2566120.

For the root g_3, we get,

$$g_3=-18''.666221,$$

$N_3' = -5.839411N_3$	log. 0.7663690n,
$N_3'' = +6.569047N_3$	" 0.8175024,
$N_3''' = -7.879027N_3$	" 0.8964726n,
$N_3^{IV} = -0.0000520881N_3$	" 5.7167383n,
$N_3^{V} = -0.000963122N_3$	" 6.9836813n,
$N_3^{VI} = +0.0000818396N_3$	" 5.9129634,
$N_3^{VII} = +0.00000942162N_3$	" 4.9741254.

For the root g_4, we get,

$$g_4=0'', \text{ and } N_4=N_4'=N_4''=, \text{\&c.}$$

For the root g_5, we get,

$g_5 = -0''.661664,$

$N_5 = +1.229553 N_5^{IV}$	log. 0.0897471,
$N_5' = +1.131670 N_5^{IV}$	" 0.0537194,
$N_5'' = +1.108860 N_5^{IV}$	" 0.0448769,
$N_5''' = +1.049663 N_5^{IV}$	" 0.0210502,
$N_5^{V} = +0.9653253 N_5^{IV}$	" 9.9846737,
$N_5^{VI} = -0.9380050 N_5^{IV}$	" 9.9722051n,
$N_5^{VII} = -9.829623 N_5^{IV}$	" 0.9925368n.

For the root g_6, we get,

$g_6 = -2''.9160753,$

$N_6 = +\ 3.484775 N_6^{IV}$	log. 0.5421747,
$N_6' = +\ 2.063762 N_6^{IV}$	" 0.3146597,
$N_6'' = +\ 1.853037 N_6^{IV}$	" 0.2678842,
$N_6''' = +\ 1.316456 N_6^{IV}$	" 0.1194064,
$N_6^{V} = +\ 0.8164332 N_6^{IV}$	" 9.9119207,
$N_6^{VI} = -20.11465 N_6^{IV}$	" 1.3035125n,
$N_6^{VII} = +\ 2.159171 N_6^{IV}$	" 0.3342872.

For the root g_7, we get,

$g_7 = -25''.934665,$

$N_7 = -0.0423304 N_7^{IV}$	log. 8.6266524n,
$N_7' = -0.0418973 N_7^{IV}$	" 8.6221861n,
$N_7'' = -0.4430757 N_7^{IV}$	" 9.6464778n,
$N_7''' = -1.470350 N_7^{IV}$	" 0.1674208n,
$N_7^{V} = -2.490680 N_7^{IV}$	" 0.3963178n,
$N_7^{VI} = +0.1093404 N_7^{IV}$	" 9.0387805,
$N_7^{VII} = +0.01225256 N_7^{IV}$	" 8.0882264.

13. If we now substitute these values in equations (408) and (409), we shall obtain the following quantities:—

For the root $g = -5''.201065$, we get,

$$x = +\frac{0.638032}{10^{14}} N; \quad y = \frac{1.6964973}{10^{14}} N; \quad z = \frac{15.329701}{10^{14}} N^2.$$

Whence $\beta = 20^\circ\ 36'\ 29''.8$; and log. $N = 9.0727385$.

$N = +0.118233,$	$N^{IV} = -0.00002755,$
$N' = +0.016251,$	$N^{V} = -0.00003524,$
$N'' = +0.011804,$	$N^{VI} = +0.00002986,$
$N''' = +0.002385,$	$N^{VII} = +0.0000008918.$

For the root $g_1 = -6''.634795$, we get,

$$x_1 = +\frac{0.637032}{10^{14}} N_1; \quad y_1 = -\frac{0.4606865}{10^{14}} N_1; \quad z_1 = \frac{27.744169}{10^{14}} N_1^2.$$

Whence $\beta_1 = 125^\circ\ 52'\ 26''.4$; and log. $N_1 = 8.4523391$.

$N_1 = +0.028336$,	$N_1^{IV} = +0.000008275$,
$N_1' = -0.006724$,	$N_1^{V} = +0.000011726$,
$N_1'' = -0.005485$,	$N_1^{VI} = -0.000005994$,
$N_1''' = -0.001160$,	$N_1^{VII} = -0.0000003662$.

For the root $g_2 = -17''.459300$, we get,

$$x_2 = -\frac{6.884438}{10^{13}} N_2, \quad y_2 = +\frac{3.578728}{10^{13}} N_2, \quad z_2 = \frac{6546.046}{10^{13}} N_2^2.$$

Whence $\beta_2 = 297^\circ\ 28'\ 0''.1$; and log. $N_2 = 7.0738291$.

$N_2 = +0.001185$,	$N_2^{IV} = -0.000002184$,
$N_2' = -0.006205$,	$N_2^{V} = -0.000019227$,
$N_2'' = +0.004991$,	$N_2^{VI} = +0.000001860$,
$N_2''' = +0.047424$,	$N_2^{VII} = +0.000000214$.

For the root $g_3 = -18''.666222$, we get,

$$x_3 = -\frac{6.787567}{10^{13}} N_3; \quad y_3 = -\frac{1.953449}{10^{13}} N_3; \quad z_3 = \frac{1727.1619}{10^{13}} N_3^2.$$

Whence $\beta_3 = 253^\circ\ 56'\ 39''.4$; and log. $N_3 = 7.6116607$.

$N_3 = +0.004089$,	$N_3^{IV} = -0.000000213$,
$N_3' = -0.023880$,	$N_3^{V} = -0.000003939$,
$N_3'' = +0.026863$,	$N_3^{VI} = +0.000000335$,
$N_3''' = -0.032221$,	$N_3^{VII} = +0.0000000385$.

For the root $g_4 = 0''$, we get,

$$x_4 = +\frac{0.7256935}{10^{10}} N_4; \quad y_4 = -\frac{0.2113335}{10^{10}} N_4; \quad z_4 = \frac{27.24487}{10^{10}} N_4^2.$$

Whence $\beta_4 = 106^\circ\ 14'\ 11''.1$; and log. $N_4 = 8.4431447$.

$N_4 = N_4' = N_4'' = N_4''' = N_4^{IV} = N_4^{V} = N_4^{VI} = N_4^{VII} = 0.02774244$.

For the root $g_5 = -0''.661664$, we get,

$$x_5 = +\frac{0.1017327}{10^{10}} N_5^{IV}; \quad y_5 = +\frac{0.2717512}{10^{10}} N_5^{IV}; \quad z_5 = \frac{241.9334}{10^{10}} N_5^{IV2}.$$

Whence $\beta_5 = 20^\circ\ 31'\ 39''.5$; and log. $N_5^{IV} = 7.0789678$.

$N_5 = +0.001475$,	$N_5^{IV} = +0.00119941$,
$N_5' = +0.001357$,	$N_5^{V} = +0.00115783$,
$N_5'' = +0.001330$,	$N_5^{VI} = -0.00112485$,
$N_5''' = +0.001259$,	$N_5^{VII} = -0.0117895$.

For the root $g_6=-2''.916075$, we get,

$$x_6=+\frac{0.3678204}{10^{10}}N_6^{IV};\ y_6=-\frac{0.3543521}{10^{10}}N_6^{IV};\ z_6=\frac{581.0177}{10^{10}}N_6^{IV2}.$$

Whence $\beta_6=133^\circ\ 55'\ 53''.6$; and log. $N_6^{IV}=6.9440119$.

$N_6\ =+0.003063$, $\quad N_6^{IV}=+0.000879047$,
$N_6'\ =+0.001814$, $\quad N_6^{V}\ =+0.000717683$,
$N_6''=+0.001629$, $\quad N_6^{VI}=-0.0176817$,
$N_6'''=+0.001157$, $\quad N_6^{VII}=+0.00189801$.

For the root $g_7=-25''.934665$, we get,

$$x_7=-\frac{0.2996518}{10^{10}}N_7^{IV};\ y_7=+\frac{0.2202706}{10^{10}}N_7^{IV};\ z_7=\frac{59.03076}{10^{10}}N_7^{IV2}.$$

Whence $\beta_7=306^\circ\ 19'\ 9''.1$; and log. $N_7^{IV}=7.7993490$.

$N_7\ =-0.0002667$, $\quad N_7^{IV}=+0.00630012$,
$N_7'\ =-0.0002640$, $\quad N_7^{V}\ =-0.0156916$,
$N_7''=-0.0027914$, $\quad N_7^{VI}=+0.00068886$,
$N_7'''=-0.0092634$, $\quad N_7^{VII}=+0.000077193$

14. For an increment of $\frac{1}{20}$, to the mass of the earth, we have the preliminary computations by merely making all the coefficients positive in equations (232–237). We shall then obtain the following

Fundamental Equations for $\mu''=+\frac{1}{20}$; *or for* $m''=1\div351132.4$.

$$\left.\begin{aligned}
A\ &=g^2+39.080610.g\ +188.728586;\\
A'\ &=g^2+23.3039078.g+\ 99.5792048;\\
A''&=g^2+19.0875991.g+\ 75.2609947;\\
A_1&=g^2+18.4084593.g+\ 60.3292965;\\
A_2&=g^2+13.1931503.g+\ 8.981769;\\
A_3&=g^2+26.3824488.g+\ 9.887223;
\end{aligned}\right\}\quad(443)$$

$$\left.\begin{aligned}
D\ &=g^2+45.2956845.g+610.444623;\\
D'\ &=g^2+52.4179477.g+613.831274;\\
D''&=g^2+32.3259257.g+258.6223612;\\
D_1&=g^2+43.8987357.g+172.463309;\\
D_2&=g^2+46.4943751.g+\ 32.948430;\\
D_3&=g^2+\ 3.4291834.g+\ 1.69252419.
\end{aligned}\right\}\quad(444)$$

$$\left.\begin{aligned}
B=\{g+33.41905\}b;\quad B'&=\{g+17.6677162\}b;\\
B''&=\{g+13.490445\}b;
\end{aligned}\right\}\quad(445)$$

$$\left.\begin{aligned}
C\ &=-\{g+22.714602\}\ [9.4381189]b';\\
C'\ &=-\{g+17.6977225\}[9.1138076]b';\\
C''&=+[0.3976676]b';\\
C'''&=+[0.4411620]b';
\end{aligned}\right\}\quad(446)$$

$$\left.\begin{aligned}E &= -[0.0019270]b''; \\ E' &= -\{g+22.6010814\}[8.9723624]b''; \\ E'' &= -\{g+17.6451959\}[9.7501125]b''; \\ E''' &= -[1.0228430]b'';\end{aligned}\right\} \quad (447)$$

$$\left.\begin{aligned}F &= +[8.1679376]b'''; \\ F' &= -[9.1453586]b'''; \\ F'' &= -\{g+14.680729\}[0.7927855]b'''; \\ F''' &= -\{g+34.7402239\}[9.6907241]b''';\end{aligned}\right\} \quad (448)$$

$$\left.\begin{aligned}B_1 = \{g+4.5202263\}b_1;\ B_2 &= \{g+0.71245534\}b_1; \\ B_3 &= \{g+18.8176739\}b_1;\end{aligned}\right\} \quad (449)$$

$$\left.\begin{aligned}C_1 &= -\{g+4.1329921\}[9.5433087]b_2; \\ C_2 &= -\{g+0.7307293\}[9.4349711]b_2; \\ C_3 &= +[0.8644527]b_2; \\ C_4 &= +[0.8654649]b_2;\end{aligned}\right\} \quad (450)$$

$$\left.\begin{aligned}E_1 &= -[8.3317448]b_3; \\ E_2 &= -\{g+39.7657436\}[8.7437718]b_3; \\ E_3 &= -\{g+0.6486324\}[0.7242832]b_3; \\ E_4 &= -[0.9647509]b_3;\end{aligned}\right\} \quad (451)$$

$$\left.\begin{aligned}F_1 &= +[7.7244692]b_4; \\ F_2 &= -[0.3564628]b_4; \\ F_3 &= -\{g+2.7805511\}[1.5514854]b_4; \\ F_4 &= -\{g+45.7636454\}[9.4626364]b_4.\end{aligned}\right\} \quad (452)$$

$$\left.\begin{aligned}&g^4+47.9724987.g^3+787.7904009.g^2 \\ &+5093.460923g+11190.32489\end{aligned}\right\} = (\chi, \chi_1, \chi_2, \chi_3); \quad (453)$$

$$\left.\begin{aligned}&g^4+29.5232934.g^3+95.0893749.g^2 \\ &+51.171764.g + 0.538790\end{aligned}\right\} = (\chi_4, \chi_5, \chi_6, \chi_7). \quad (454)$$

The values of b, b', b'', and b''' are given by equations (405); and the values of b_1, b_2, b_3, and b_4 are given by equations (406), by merely multiplying the coefficients of N'' by $1+\mu''=1.05$.

If we put the equations (453) and (454) equal to nothing, they will give

$$\begin{aligned}g &= -\ 5''.1662610, & g_4 &= -\ 0''.0107428, \\ g_1 &= -\ 6\ .6253960, & g_5 &= -\ 0\ .6630168, \\ g_2 &= -17\ .5129033, & g_6 &= -\ 2\ .9169739, \\ g_3 &= -18\ .6679386, & g_7 &= -25\ .9325598.\end{aligned}$$

The solutions of equations (443–454) will now give the following values:—

For the root g, we get,

$g=-5''.170230,$	
$N' =+0.1213375N$	log. 9.0839952,
$N'' =+0.0874828N$	" 8.9419226,
$N''' =+0.01800280N$	" 8.2553402,
$N^{IV}=-0.0002097093N$	" 6.3216176n,
$N^{V} =-0.0002676246N$	" 6.4275260n,
$N^{VI} =+0.0002299092N$	" 6.3615564,
$N^{VII}=+0.00000665906N$	" 4.8234126.

For the root g_1, we get,

$g_1=-6''.631323,$	
$N_1' =-0.2725185N_1$	log. 9.4353960n,
$N_1'' =-0.2210735N_1$	" 9.3445366n,
$N_1''' =-0.04781081N_1$	" 8.6795260n,
$N_1^{IV}=+0.000340372N_1$	" 6.5319537,
$N_1^{V} =+0.000482200N_1$	" 6.6832273,
$N_1^{VI}=-0.0002467402N_1$	" 6.3922400n,
$N_1^{VII}=-0.0000150588N_1$	" 5.1777901n.

For the root g_2, we get,

$g_2=-17''.5137898,$	
$N_2' =- 5.562224N_2$	log. 0.7452485n,
$N_2'' =+ 4.103496N_2$	" 0.6131540,
$N_2''' =+38.24098N_2$	" 1.5825290,
$N_2^{IV}=- 0.001733292N_2$	" 7.2388717n,
$N_2^{V} =- 0.01569197N_2$	" 8.1956776n,
$N_2^{VI}=+ 0.001509355N_2$	" 7.1787913,
$N_2^{VII}=+ 0.0001737467N_2$	" 6.2399165.

For the root g_3, we get,

$g_3=-18''.667944,$	
$N_3' =-6.186960N_3$	log. 0.7914773n,
$N_3'' =+6.338887N_3$	" 0.8020130,
$N_3''' =-8.476533N_3$	" 0.9282183n,
$N_3^{IV}=-0.00004491846N_3$	" 5.6524248n,
$N_3^{V} =-0.0007949226N_3$	" 6.9003248n,
$N_3^{VI}=+0.00006740752N_3$	" 5.8287083,
$N_3^{VII}=+0.00000774975N_3$	" 4.8892878.

For the root g_4, we get,

$g_4=0''$, and

$N_4=N_4'=N_4''=N_4'''=$ &c.

For the root g_5, we get,

$$g_5 = -0''.661665,$$

$N_5 = +1.230450 N_5^{IV}$	log. 0.0900640,
$N_5' = +1.130316 N_5^{IV}$	" 0.0531999,
$N_5'' = +1.107793 N_5^{IV}$	" 0.0444583,
$N_5''' = +1.049700 N_5^{IV}$	" 0.0210647,
$N_5^{V} = +0.9653247 N_5^{IV}$	" 9.9846734,
$N_5^{VI} = -0.9380148 N_5^{IV}$	" 9.9722097n,
$N_5^{VII} = -9.829793 N_5^{IV}$	" 0.9925444n.

For the root g_6, we get,

$$g_6 = -2''.916080,$$

$N_6 = +\ 3.509603 N_6^{IV}$	log. 0.5452580,
$N_6' = +\ 2.043376 N_6^{IV}$	" 0.3103480,
$N_6'' = +\ 1.836884 N_6^{IV}$	" 0.2640819,
$N_6''' = +\ 1.315752 N_6^{IV}$	" 0.1191740,
$N_6^{V} = +\ 0.8164404 N_6^{IV}$	" 9.9119245,
$N_6^{VI} = -20.11413 N_6^{IV}$	" 1.3035012n,
$N_6^{VII} = +\ 2.161811 N_6^{IV}$	" 0.3348177.

For the root g_7, we get,

$$g_7 = -25''.934782,$$

$N_7 = -0.04268995 N_7^{IV}$	log. 8.6303257n,
$N_7' = -0.03535223 N_7^{IV}$	" 8.5484168n,
$N_7'' = -0.4372758 N_7^{IV}$	" 9.6407554n,
$N_7''' = -1.478718 N_7^{IV}$	" 0.1698852n,
$N_7^{V} = -2.490648 N_7^{IV}$	" 0.3963122n,
$N_7^{VI} = +0.1093379 N_7^{IV}$	" 9.0387706,
$N_7^{VII} = +0.01225226 N_7^{IV}$	" 8.0882161.

15. If we now substitute these values in equations (408) and (409), we shall obtain the following quantities:—

For the root $g = -5''.170189$, we get,

$$x = +\frac{0.581731}{10^{14}} N; \quad y = +\frac{1.5880256}{10^{14}} N; \quad z = \frac{14.050229}{10^{14}} N^2.$$

Whence $\beta = 20^\circ\ 7'\ 8''.3$, and log. $N = 9.0805176$.

$N = +0.120370$,	$N^{IV} = -0.00002524$,
$N' = +0.014605$,	$N^{V} = -0.00003221$,
$N'' = +0.010530$,	$N^{VI} = +0.00002767$,
$N''' = +0.002167$,	$N^{VII} = +0.0000008015$.

For the root $g_1 = -6''.631323$, we get,

$$x_1 = +\frac{0.768973}{10^{14}} N_1; \quad y_1 = -\frac{0.6414600}{10^{14}} N_1; \quad z_1 = \frac{33.22066}{10^{14}} N_1^2.$$

Whence $\beta_1 = 129° \ 50' \ 4''.4$; and log. $N_1 = 8.4791971$.

$N_1 = +0.030144$,	$N_1^{IV} = +0.00001026$,
$N_1' = -0.008215$,	$N_1^{V} = +0.00001454$,
$N_1'' = -0.006664$,	$N_1^{VI} = -0.000007438$,
$N_1''' = -0.001441$,	$N_1^{VII} = -0.0000004539$.

For the root $g_2 = -17''.5137898$, we get,

$$x_2 = -\frac{6.914692}{10^{13}} N_2; \quad y_2 = +\frac{3.461327}{10^{13}} N_2; \quad z_2 = \frac{6089.390}{10^{13}} N_2^2.$$

Whence $\beta_2 = 296° \ 35' \ 29''.2$; and log. $N_2 = 7.1037540$.

$N_2 = +0.001270$,	$N_2^{IV} = -0.000002201$,
$N_2' = -0.007063$,	$N_2^{V} = -0.000019926$,
$N_2'' = +0.005211$,	$N_2^{VI} = +0.0000019167$,
$N_2''' = +0.048560$,	$N_2^{VII} = +0.0000002206$.

For the root $g_3 = -18''.667944$, we get,

$$x_3 = -\frac{6.844087}{10^{13}} N_3; \quad y_3 = -\frac{2.032363}{10^{13}} N_3; \quad z_3 = \frac{1783.3974}{10^{13}} N_3^2.$$

Whence $\beta_3 = 253° \ 27' \ 40''.1$; and log. $N_3 = 7.6024178$.

$N_3 = +0.004003$,	$N_3^{IV} = -0.00000018$,
$N_3' = -0.024768$,	$N_3^{V} = -0.00000318$,
$N_3'' = +0.025376$,	$N_3^{VI} = +0.00000027$,
$N_3''' = -0.033934$,	$N_3^{VII} = +0.000000031$.

For the root $g_4 = 0''$, we get,

$$x_4 = +\frac{0.7256453}{10^{10}} N_4; \quad y_4 = -\frac{0.2113461}{10^{10}} N_4; \quad z_4 = \frac{27.24508}{10^{10}} N_4^2.$$

Whence $\beta_4 = 106° \ 14' \ 18''.0$; and $N_4 = N_4' = N_4'' =$ &c.,
$= +0.02774066$, log. 8.4431168.

For the root $g_5 = -0''.6616647$, we get,

$$x_5 = +\frac{0.1016673}{10^{10}} N_5^{IV}; \quad y_5 = +\frac{0.2717444}{10^{10}} N_5^{IV}; \quad z_5 = \frac{241.9412}{10^{10}} N_5^{IV2}.$$

Whence $\beta_5 = 20° \ 30' \ 57''.6$; and log. $N_5^{IV} = 7.0789098$.

$N_5 = +0.001476$,	$N_5^{IV} = +0.00119925$,
$N_5' = +0.001356$,	$N_5^{V} = +0.00115768$,
$N_5'' = +0.001329$,	$N_5^{VI} = -0.00112471$,
$N_5''' = +0.001259$,	$N_5^{VII} = -0.01178812$

For the root $g_6 = -2''.916080$, we get,

$$x_6 = +\frac{0.3678561}{10^{10}} N_6^{IV};\quad y_6 = -\frac{0.3544989}{10^{10}} N_6^{IV};\quad z_6 = \frac{581.0130}{10^{10}} N_6^{IV}.$$

Whence $\beta_6 = 133^\circ\ 56'\ 26''.3$; and log. $N_6^{IV} = 6.9441240$.

$$\begin{array}{ll} N_6 = +0.003086, & N_6^{IV} = +0.000879273, \\ N_6' = +0.001797, & N_6^{V} = +0.000717874, \\ N_6'' = +0.001615, & N_6^{VI} = -0.0176858, \\ N_6''' = +0.001157, & N_6^{VII} = +0.00190082. \end{array}$$

For the root $g_7 = -25''.934782$, we get,

$$x_7 = -\frac{0.2996357}{10^{10}} N_7^{IV};\quad y_7 = +\frac{0.2203002}{10^{10}} N_7^{IV};\quad z_7 = \frac{59.02984}{10^{10}} N_7^{IV}.$$

Whence $\beta_7 = 306^\circ\ 19'\ 27''.6$; and log. $N_7^{IV} = 7.7993611$.

$$\begin{array}{ll} N_7 = -0.0002690, & N_7^{IV} = +0.00630030, \\ N_7' = -0.0002227, & N_7^{V} = -0.01569181, \\ N_7'' = -0.0027550, & N_7^{VI} = +0.000688861, \\ N_7''' = -0.0093164, & N_7^{VII} = +0.0000771929. \end{array}$$

16. For an increment of $\frac{1}{10}$, to the mass of the earth, we have the preliminary computations by merely making all the coefficients positive in equations (251–256). We shall then obtain the following

Fundamental Equations for $\mu'' = +\frac{1}{10}$; *or for* $m'' = 1 \div 335172$.

$$\left.\begin{array}{l} A = g^2 + 40.092595.g + 195.673045; \\ A' = g^2 + 23.4348156.g + 100.8391921; \\ A'' = g^2 + 19.4622137.g + \ 77.7179078; \\ A_1 = g^2 + 18.4089042.g + \ 60.3313475; \\ A_2 = g^2 + 13.1935918.g + \ \ 8.982096; \\ A_3 = g^2 + 26.3829396.g + \ \ 9.895893. \end{array}\right\} \quad (455)$$

$$\left.\begin{array}{l} D = g^2 + 46.0863772.g + 633.139825; \\ D' = g^2 + 52.8372995.g + 622.776921; \\ D'' = g^2 + 32.4903462.g + 261.2498776; \\ D_1 = g^2 + 43.8987902.g + 172.463686; \\ D_2 = g^2 + 46.4944263.g + \ 32.948508; \\ D_3 = g^2 + \ 3.4291886.g + \ \ 1.69252948. \end{array}\right\} \quad (456)$$

$$\left.\begin{array}{l} B = \{g + 34.38795\}b; \quad B' = \{g + 17.7555387\}b \\ \qquad\qquad\qquad\qquad\quad B'' = \{g + 13.801975\}b \end{array}\right\} \quad (457)$$

$$\left.\begin{array}{l} C = -\{g + 27.173766\}\ [9.4381149]b'; \\ C' = -\{g + 17.7855450\}[9.1138076]b'; \\ C'' = +[0.3976676]b'; \\ C''' = +[0.4411620]b'. \end{array}\right\} \quad (458)$$

$$\left.\begin{aligned}E &=-[0.0221304]b'';\\ E' &=-\{g+22.6010814\}[8.9723624]b'';\\ E'' &=-\{g+17.7330184\}[9.7501125]b'';\\ E''' &=-[1.0430464]b''.\end{aligned}\right\}\quad(459)$$

$$\left.\begin{aligned}F &=+[8.1679376]b''';\\ F' &=-[9.1453586]b''';\\ F'' &=-\{g+14.757327\}\ [0.7927855]b''';\\ F''' &=-\{g+35.0517532\}[9.6907241]b'''.\end{aligned}\right\}\quad(460)$$

$$\left.\begin{aligned}B_1=\{g+4.5202306\}b_1;\quad B_2&=\{g+\ 0.7124563\}b_1;\\ B_3&=\{g+18.8177241\}b_1;\end{aligned}\right\}\quad(461)$$

$$\left.\begin{aligned}C_1 &=-\{g+4.1329964\}[9.5433087]b_2;\\ C_2 &=-\{g+0.7307302\}[9.4349711]b_2.\\ C_3 &=+[0.8644527]b_2;\\ C_4 &=+[0.8654649]b_2.\end{aligned}\right\}\quad(462)$$

$$\left.\begin{aligned}E_1 &=-[8.3317448]b_3;\\ E_2 &=-\{g+39.7657938\}[8.7437718]b_3;\\ E_3 &=-\{g+\ 0.6486333\}[0.7242832]b_3;\\ E_4 &=-[0.9647514]b_3.\end{aligned}\right\}\quad(463)$$

$$\left.\begin{aligned}F_1 &=+[7.7244692]b_4;\\ F_2 &=-[0.3564628]b_4;\\ F_3 &=-\{g+\ 2.7805554\}[1.5514854]b_4;\\ F_4 &=-\{g+45.7636960\}[9.4626364]b_4.\end{aligned}\right\}\quad(464)$$

$$\left.\begin{aligned}&g^4+48.4349358.g^3+802.5743024.g^2\\ &+5231.898890.g+11585.83042\end{aligned}\right\}=(\chi,\chi_1,\chi_2,\chi_3).\quad(465)$$

$$\left.\begin{aligned}&g^4+29.5237894.g^3+95.0997623.g^2\\ &+51.201831.g\ +\ 0.553045\end{aligned}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7).\quad(466)$$

The values of b, b', b'', and b''' are given by equations (405); and the values of b_1, b_2, b_3, and b_5 are given by equations (406), by merely multiplying the coefficients of N'' by $1+\mu''=1+\frac{1}{10}$.

If we put equations (465) and (466) equal to nothing, they will give,

$$\begin{aligned}g &=-\ 5''.2095599; & g_4 &=-\ 0''.0110263;\\ g_1 &=-\ 6.6631448; & g_5 &=-\ 0.6630507;\\ g_2 &=-17.6257463; & g_6 &=-\ 2.9169913;\\ g_3 &=-18.9364848; & g_7 &=-25.9327210.\end{aligned}$$

Equations (455–466) will now give the following values:—

For the root g, we get,

$g = -5''.2136546$;		
$N' = +0.1200684N$	log.	9.0794288;
$N'' = +0.0871180N$	"	8.9401080;
$N''' = +0.01844217N$	"	8.2658120;
$N^{IV} = -0.0002119225N$	"	96.3261770n;
$N^{V} = -0.000271274N$	"	96.4334082n;
$N^{VI} = +0.000228589N$	"	96.3590546;
$N^{VII} = +0.000006910472N$	"	94.8395077.

For the root g_1, we get,

$g_1 = -6''.6692717$;		
$N_1 = -0.2683056N_1$	log.	9.4286296n;
$N_1'' = -0.2189168N_1$	"	9.3402791n;
$N_1''' = -0.0487843N_1$	"	8.6882800n;
$N_1^{IV} = +0.0003444146N_1$	"	96.5370815;
$N_1^{V} = +0.000489310N_1$	"	96.6895841;
$N_1^{VI} = -0.000247679N_1$	"	96.3938898n;
$N_1^{VII} = -0.00001526832N_1$	"	95.1837912n.

For the root g_2, we get,

$g_2 = -17''.6265859$;		
$N_2' = -\ 5.55453N_2$	log.	0.7446477n;
$N_2'' = +\ 3.67668N_2$	"	0.5654561;
$N_2''' = +43.0731N_2$	"	1.6342060;
$N_2^{IV} = -\ 0.00179671N_2$	"	97.2544790n;
$N_2^{V} = -\ 0.0172969N_2$	"	98.2379690n;
$N_2^{VI} = +\ 0.00164470N_2$	"	97.2160878;
$N_2^{VII} = +\ 0.000189469N_2$	"	96.2775388.

For the root g_3, we get,

$g_3 = -18''.9364959$;		
$N_3' = -6.27941N_3$	log.	0.7979189n;
$N_3'' = +6.06766N_3$	"	0.7830214;
$N_3''' = -7.19812N_3$	"	0.8572192n;
$N_3^{IV} = -0.0000461221N_3$	"	95.6639090n;
$N_3^{V} = -0.00129815N_3$	"	97.1133248n;
$N_3^{VI} = +0.000107625N_3$	"	96.0319138;
$N_3^{VII} = +0.0000124176N_3$	"	95.0940376.

For the root g_4, we get,

$$g_4 = 0'',\ N_4 = N_4' = N_4'' = \&c.$$

For the root g_5, we get,

$$g_5 = -0''.6616623;$$

$N_5 = +1.228747 N_5^{IV}$	log. 0.0894623;
$N_5' = +1.12937 N_5^{IV}$	" 0.0528360;
$N_5'' = +1.107383 N_5^{IV}$	" 0.0442980;
$N_5''' = +1.049923 N_5^{IV}$	" 0.0211576;
$N_5^{V} = +0.965320 N_5^{IV}$	" 9.9846716;
$N_5^{VI} = -0.938100 N_5^{IV}$	" 9.9722493n;
$N_5^{VII} = -9.83017 N_5^{IV}$	" 0.9925608n.

For the root g_6, we get,

$$g_6 = -2''.9160771;$$

$N_6 = +\ 3.46445 N_6^{IV}$	log. 0.5396342;
$N_6' = +\ 2.02862 N_6^{IV}$	" 0.3072002;
$N_6'' = +\ 1.82900 N_6^{IV}$	" 0.2622141;
$N_6''' = +\ 1.317308 N_6^{IV}$	" 0.1196872;
$N_6^{V} = +\ 0.816421 N_6^{IV}$	" 9.9119142;
$N_6^{VI} = -20.11546 N_6^{IV}$	" 1.3035300n;
$N_6^{VII} = +\ 2.161956 N_6^{IV}$	" 0.3348464.

For the root g_7, we get,

$$g_7 = -25''.9350099;$$

$N_7 = -0.0434072 N_7^{IV}$	log. 8.6375619n;
$N_7' = -0.0232236 N_7^{IV}$	" 8.3659300n;
$N_7'' = -0.442046 N_7^{IV}$	" 9.6454670n;
$N_7''' = -1.489419 N_7^{IV}$	" 0.1730168n;
$N_7^{V} = -2.490578 N_7^{IV}$	" 0.3962995n;
$N_7^{VI} = +0.1093334 N_7^{IV}$	" 9.0387530;
$N_7^{VII} = +0.0122517 N_7^{IV}$	" 8.0881970.

17. If we now substitute these values in equations (408) and (409), we shall obtain the following quantities:—

For the root $g = -5''.2136546$, we get,

$$x = +\frac{5.51372}{10^{15}}; \quad y = +\frac{15.903915}{10^{15}}; \quad z = \frac{140.64952}{10^{15}}.$$

Whence $\beta = 19° \ 7' \ 7''.0$; log. $N = 9.0780045$.

$N = +0.1196753$;	$N^{IV} = -0.000025364$;
$N' = +0.0143703$;	$N^{V} = -0.000032468$;
$N'' = +0.0104268$;	$N^{VI} = +0.000027359$;
$N''' = +0.00220757$;	$N^{VII} = +0.00000082705$.

For the root $g_1 = -6''.6692717$, we get,

$$x_1 = +\frac{8.43328}{10^{15}}; \quad y_1 = -\frac{6.419195}{10^{15}}; \quad z_1 = \frac{331.3455}{10^{15}}.$$

Whence $\beta_1 = 127° \ 16' \ 40''.5$; log. $N_1 = 98.5049615$.

$N_1 = +0.0319861$; $N_1^{IV} = +0.000011017$;
$N_1' = -0.0085822$; $N_1^{V} = +0.000015651$;
$N_1'' = -0.0070024$; $N_1^{VI} = -0.000007922$;
$N_1''' = -0.0015604$; $N_1^{VII} = -0.0000004884$.

For the root $g_2 = -17''.626586$, we get,

$$x_2 = -\frac{6.947485}{10^{13}}; \quad y_2 = +\frac{4.015769}{10^{13}}; \quad z_2 = \frac{7425.592}{10^{13}}.$$

Whence $\beta_2 = 300° \ 0' \ 37''.6$; log. $N_2 = 97.0339306$.

$N_2 = +0.0010813$; $N_2^{IV} = -0.000001943$;
$N_2' = -0.0060059$; $N_2^{V} = -0.000018702$;
$N_2'' = +0.00397545$; $N_2^{VI} = +0.000001778$;
$N_2''' = +0.0465733$; $N_2^{VII} = +0.0000002049$.

For the root $g_3 = -18''.9364959$, we get,

$$x_3 = -\frac{6.960810}{10^{13}}; \quad y_3 = -\frac{1.901617}{10^{13}}; \quad z_3 = \frac{1696.1782}{10^{13}}.$$

Whence $\beta_3 = 254° \ 43' \ 12''.9$; log. $N_3 = 97.6288181$.

$N_3 = +0.0042542$; $N_3^{IV} = -0.0000001962$;
$N_3' = -0.0267139$; $N_3^{V} = -0.0000055201$;
$N_3'' = +0.0258130$; $N_3^{VI} = +0.0000004576$;
$N_3''' = -0.0306223$; $N_3^{VII} = +0.0000000528$.

For the root $g_4 = 0''$, we get,

$$x_4 = +\frac{0.7256453}{10^{10}}; \quad y_4 = -\frac{0.2113461}{10^{10}}; \quad z_4 = \frac{27.24612}{10^{10}}.$$

Whence $\beta_4 = 106° \ 14' \ 18''.0$; log. $N_4 = 98.4431002$.

$$N_4 = N_4' = N_4'' = N_4''' = N_4^{IV} = N_4^{V} = N_4^{VI} = N_4^{VII} = 0.0277396.$$

For the root $g_5 = -0''.6616623$, we get,

$$x_5 = +\frac{0.1016437}{10^{10}}; \quad y_5 = +\frac{0.2717608}{10^{10}}; \quad z_5 = \frac{241.9590}{10^{10}}.$$

Whence $\beta_5 = 20° \ 30' \ 37''.9$; log. $N_5^{IV} = 97.0788886$.

$N_5 = +0.0014735$; $N_5^{IV} = +0.0011992$;
$N_5' = +0.0013543$; $N_5^{V} = +0.0011576$;
$N_5'' = +0.0013280$; $N_5^{VI} = -0.0011248$;
$N_5''' = +0.0012591$; $N_5^{VII} = -0.0117880$.

For the root $g_6=-2''.9160771$, we get,

$$x_6=+\frac{0.3678168}{10^{10}},\quad y_6=-\frac{0.3545173}{10^{10}},\quad z_6=\frac{581.0884}{10^{10}}.$$

Whence $\beta_6=133°\ 56'\ 42''.7$; log. $N_6^{IV}=96.9440543$.

$$\begin{array}{ll} N_6\ =+0.0030457; & N_6^{IV}\ =+0.0008812; \\ N_6'\ =+0.0017834; & N_6^{V}\ =+0.00071774; \\ N_6''=+0.0016079; & N_6^{VI}\ =-0.0176842; \\ N_6'''=+0.0011581; & N_6^{VII}=-0.0018963. \end{array}$$

For the root $g_7=-25''.9350099$, we get,

$$x_7=-\frac{0.2996051}{10^{10}};\quad y_7=+\frac{0.2202944}{10^{10}};\quad z_7=\frac{59.02778}{10^{10}}.$$

Whence $\beta_7=306°\ 19'\ 35''.1$; log. $N_7^{IV}=97.7993434$.

$$\begin{array}{ll} N_7\ =-0.0002735; & N_7^{IV}\ =+0.0063000; \\ N_7'\ =-0.0001463; & N_7^{V}\ =-0.0156907; \\ N_7''=-0.0027849; & N_7^{VI}\ =+0.0006888; \\ N_7'''=-0.0093834; & N_7^{VII}=+0.00007719. \end{array}$$

18. If we now suppose the mass of *Mars* to be doubled, we shall have the preliminary computations by making all the coefficients of equations (269–274) positive. We shall then obtain the following

Fundamental Equations for $\mu'''=+1$; or for $m'''=1\div1340318.5$

$$\left.\begin{array}{l} A\ =g^2+38.394808.g\ +184.259392; \\ A'\ =g^2+23.2280108.g+\ 98.9712014; \\ A''=g^2+18.8430013.g+\ 73.7719022; \\ A_1=g^2+18.4111446.g+\ 60.3416523; \\ A_2=g^2+13.1958172.g+\ \ 8.983726; \\ A_3=g^2+26.3853904.g+\ \ 9.939432. \end{array}\right\}\quad(467)$$

$$\left.\begin{array}{l} D\ =g^2+44.9052274.g+596.989960; \\ D'\ =g^2+52.1576683.g+608.070284; \\ D''=g^2+32.4642145.g+260.9092264; \\ D_1=g^2+43.8990384.g+172.465387; \\ D_2=g^2+46.4946594.g+\ 32.948855; \\ D_3=g^2+\ 3.4292114.g+\ \ 1.69255259. \end{array}\right\}\quad(468)$$

$$\left.\begin{array}{ll} B=\{g+32.74835\}b; & B'=\{g+17.6069231\}b; \\ & B''=\{g+13.240952\}b; \end{array}\right\}\quad(469)$$

$$\left.\begin{array}{l} C\ =-\{g+22.553639\}\ [9.4381189]b'; \\ C'\ =-\{g+17.6669357\}[9.1138076]b'; \\ C''=+[0.3976676]b'; \\ C'''=+[0.4411620]b'. \end{array}\right\}\quad(470)$$

$$\left.\begin{aligned}
E &=-[9.9807377]b'';\\
E' &=-\{g+22.3515875\}[8.9723624]b'';\\
E'' &=-\{g+17.5618825\}[9.7501125]b'';\\
E''' &=-[1.0016537]b''.
\end{aligned}\right\}\quad(471)$$

$$\left.\begin{aligned}
F &=+[8.4689676]b''';\\
F' &=-[9.4463886]b''';\\
F'' &=-\{g+14.902331\}[0.7927855]b''';\\
F''' &=-\{g+34.490730\}[9.6907241]b'''.
\end{aligned}\right\}\quad(472)$$

$$\left.\begin{aligned}
B_1=\{g+4.5202496\}b_1;\quad B_2&=\{g+\ 0.7124602\}b_1;\\
B_3&=\{g+18.8179534\}b_1.
\end{aligned}\right\}\quad(473)$$

$$\left.\begin{aligned}
C_1 &=-\{g+4.1330154\ \}[9.5433087]b_2;\\
C_2 &=-\{g+0.73073414\}[9.4349711]b_2;\\
C_3 &=+[0.8644527]b_2;\\
C_4 &=+[0.8654649]b_2.
\end{aligned}\right\}\quad(474)$$

$$\left.\begin{aligned}
E_1 &=-[8.3317448]b_3;\\
E_2 &=-\{g+39.7660231\ \}[8.7437718]b_3;\\
E_3 &=-\{g+\ 0.64863718\}[0.7242832]b_3;\\
E_4 &=-[0.9647514]b_3.
\end{aligned}\right\}\quad(475)$$

$$\left.\begin{aligned}
F_1 &=+[7.7244692]b_4;\\
F_2 &=-[0.3564628]b_4;\\
F_3 &=-\{g+\ 2.7805744\}[1.5514854]b_4;\\
F_4 &=-\{g+45.7639253\}[9.4626364]b_4.
\end{aligned}\right\}\quad(476)$$

$$\left.\begin{aligned}
&g^4+47.9382786.g^3+787.7033176.g^2\\
&\quad+5108.828105.g+11271.52356
\end{aligned}\right\}=(\chi,\chi_1,\chi_2,\chi_3);\quad(477)$$

$$\left.\begin{aligned}
&g^4+29.5262630.g^3+95.1517832.g^2\\
&\quad+51.352675.g\ +0.624611
\end{aligned}\right\}=(\chi_4,\chi_5,\chi_6,\chi_7).\quad(478)$$

The values of b, b', b'', and b''' are given by equations (405), and the values of b_1, b_2, b_3, and b_4 are given by equations (406), by merely multiplying the coefficients of N''' by $1+\mu'''=2$.

If we put equations (477) and (478) equal to nothing, they will give

$$\begin{aligned}
g_1&=-\ \ 5''.1830604, & g_4&=-\ 0''.0124492,\\
g_2&=-\ \ 6\ .7024834, & g_5&=-\ 0\ .6632208,\\
g_3&=-17\ .3261252, & g_6&=-\ 2\ .9170763,\\
g_4&=-18\ .7266096, & g_7&=-25\ .9335168.
\end{aligned}$$

The solutions of equations (467–478) will now give the following values:—

For the root g, we get,

$$
\begin{array}{lll}
g & =-5''.186718, & \\
N' & =+0.1139908N & \log. 9.0568698, \\
N'' & =+0.0802224N & \text{“} \ 8.9042956, \\
N''' & =+0.01622926N & \text{“} \ 8.2102986, \\
N^{IV} & =-0.000196341N & \text{“} \ 6.2930111n, \\
N^{V} & =-0.0002508766N & \text{“} \ 6.3994600n, \\
N^{VI} & =+0.000213949N & \text{“} \ 6.3303095, \\
N^{VII} & =+0.00000630073N & \text{“} \ 4.7993910.
\end{array}
$$

For the root g_1, we get,

$$
\begin{array}{lll}
g_1 & =-6''.708872, & \\
N_1' & =-0.3009834N_1 & \log. 9.4785424n, \\
N_1'' & =-0.2389096N_1 & \text{“} \ 9.3782336n, \\
N_1''' & =-0.05086216N_1 & \text{“} \ 8.7063948n, \\
N_1^{IV} & =+0.0003713773N_1 & \text{“} \ 6.5698154, \\
N_1^{V} & =+0.0005292475N_1 & \text{“} \ 6.7236588, \\
N_1^{VI} & =-0.0002649250N_1 & \text{“} \ 6.4231228n, \\
N_1^{VII} & =-0.00001649956N_1 & \text{“} \ 5.2174723n.
\end{array}
$$

For the root g_2, we get,

$$
\begin{array}{lll}
g_2 & =-17''.327793, & \\
N_2' & =-\ 5.526427N_2 & \log. 0.7424444n, \\
N_2'' & =+\ 4.250740N_2 & \text{“} \ 0.6284646, \\
N_2''' & =+21.16062N_2 & \text{“} \ 1.3255284, \\
N_2^{IV} & =-\ 0.002056417N_2 & \text{“} \ 7.3131111n, \\
N_2^{V} & =-\ 0.01697217N_2 & \text{“} \ 8.2297373n, \\
N_2^{VI} & =+\ 0.001664117N_2 & \text{“} \ 7.2211838, \\
N_2^{VII} & =+\ 0.000191324N_2 & \text{“} \ 6.2817698.
\end{array}
$$

For the root g_3, we get,

$$
\begin{array}{lll}
g_3 & =-18''.726623, & \\
N_3' & =-6.280437N_3 & \log. 0.7979898n, \\
N_3'' & =+7.147022N_3 & \text{“} \ 0.8541251, \\
N_3''' & =-8.117785N_3 & \text{“} \ 0.9094376n, \\
N_3^{IV} & =+0.0000583566N_3 & \text{“} \ 5.7660900, \\
N_3^{V} & =+0.00177227N_3 & \text{“} \ 7.2485300, \\
N_3^{VI} & =-0.0001516876N_3 & \text{“} \ 6.1809500n, \\
N_3^{VII} & =-0.0000176279N_3 & \text{“} \ 5.2462000n.
\end{array}
$$

For the root g_4, we get,

$g_4=0''$; and $N_4=N_4'=N_4''=$&c.

For the root g_5, we get,

$$g_5=-0''.661664,$$

$N_5 =+1.229368N_5^{IV}$	log.	0.0896819,
$N_5' =+1.129025N_5^{IV}$	"	0.0527036,
$N_5'' =+1.105860N_5^{IV}$	"	0.0436997,
$N_5''' =+1.049158N_5^{IV}$	"	0.0208410,
$N_5^{V} =+0.9653170N_5^{IV}$	"	9.9846700,
$N_5^{VI}=-0.9382057N_5^{IV}$	"	9.9722980n,
$N_5^{VII}=-9.830882N_5^{IV}$	"	0.9925925n.

For the root g_6, we get,

$$g_6=-2''.916095,$$

$N_6 =+\ 3.479802N_6^{IV}$	log.	0.5415546,
$N_6' =+\ 2.020979N_6^{IV}$	"	0.3055617,
$N_6'' =+\ 1.810077N_6^{IV}$	"	0.2576970,
$N_6''' =+\ 1.308654N_6^{IV}$	"	0.1168248,
$N_6^{V} =+\ 0.8164215N_6^{IV}$	"	9.9119144,
$N_6^{VI}=-20.11574N_6^{IV}$	"	1.3035360n,
$N_6^{VII}=+\ 2.156995N_6^{IV}$	"	0.3348492.

For the root g_7, we get,

$$g_7=-25''.933517,$$

$N_7 =-0.04094460N_7^{IV}$	log.	8.6121966n,
$N_7' =-0.04815456N_7^{IV}$	"	8.6826374n,
$N_7'' =-0.4073554N_7^{IV}$	"	9.6099734n,
$N_7''' =-1.473038N_7^{IV}$	"	0.1682137n,
$N_7^{V} =-2.490000N_7^{IV}$	"	0.3961992n,
$N_7^{VI}=+0.1092910N_7^{IV}$	"	9.0385844,
$N_7^{VII}=+0.01224673N_7^{IV}$	"	8.0880201.

19. If we now substitute these values in equations (408) and (409), we shall obtain the following quantities:—

For the root $g=-5''.186718$, we get,

$$x=+\frac{6.16813}{10^{15}}N;\ y=+\frac{15.530471}{10^{15}}N;\ z=\frac{134.20637}{10^{15}}N^2.$$

Whence $\beta=21°\ 39'\ 31''.5$; and log. $N=9.0952088$.

$N\ =+0.124511,$	$N^{IV}=-0.000024449,$
$N'\ =+0.014194,$	$N^{V}\ =-0.000031240,$
$N''\ =+0.0099895,$	$N^{VI}=+0.000026642,$
$N'''\ =+0.0020212,$	$N^{VII}=+0.00000078455.$

For the root $g_1=-6''.708872$, we get,

$$x_1=+\frac{6.90464}{10^{15}}N_1;\quad y_1=-\frac{8.259874}{10^{15}}N_1;\quad z_1=\frac{372.8292}{10^{15}}N_1^2.$$

Whence $\beta_1=140°\ 6'\ 26''.4$; and log. $N_1=8.4605329$,

$N_1\ =+0.028876$, $\qquad N_1^{IV}=+0.000010724$,
$N_1'\ =-0.0086913$, $\qquad N_1^{V}\ =+0.000015283$,
$N_1''=-0.0068988$, $\qquad N_1^{VI}=-0.0000076500$,
$N_1'''=-0.0014686$, $\qquad N_1^{VII}=-0.00000047644$.

For the root $g_2=-17''.327793$, we get,

$$x_2=-\frac{6.999264}{10^{13}}N_2;\quad y_2=+\frac{3.944520}{10^{13}}N_2;\quad z_2=\frac{4077.099}{10^{13}}N_2^2.$$

Whence $\beta_2=299°\ 24'\ 14''.3$; and log. $N_2=7.2945933$.

$N_2\ =+0.0019706$, $\qquad N_2^{IV}=-0.0000040523$,
$N_2'\ =-0.0108902$, $\qquad N_2^{V}\ =-0.000033445$,
$N_2''=+0.0083764$, $\qquad N_2^{VI}=+0.0000032793$,
$N_2'''=+0.0416986$, $\qquad N_2^{VII}=+0.00000037702$.

For the root $g_3=-18''.726623$, we get,

$$x_3=-\frac{6.863743}{10^{13}}N_3;\quad y_3=-\frac{2.942093}{10^{13}}N_3;\quad z_3=\frac{2202.062}{10^{13}}N_3^2.$$

Whence $\beta_3=246°\ 47'\ 52''.7$; and log. $N_3=7.5303588$.

$N_3\ =+0.0033912$, $\qquad N_3^{IV}=+0.0000001979$,
$N_3'\ =-0.0212985$, $\qquad N_3^{V}\ =+0.0000060102$,
$N_3''=+0.0242373$, $\qquad N_3^{VI}=-0.0000005144$,
$N_3'''=-0.0275294$, $\qquad N_3^{VII}=-0.00000005978$.

For the root $g_4=0''$, we get,

$$x_4=+\frac{0.7257311}{10^{10}}N_4;\quad y_4=-\frac{0.2112699}{10^{10}}N_4;\quad z_4=\frac{27.24758}{10^{10}}N_4^2.$$

Whence $\beta_4=106°\ 13'\ 51''.5$; and log. $N_4=8.4431120$, $N_4=0.02774035$.

For the root $g_5=-0.6616640$, we get,

$$x_5=+\frac{0.1016924}{10^{10}}N_5^{IV};\quad y_5=+\frac{0.2718731}{10^{10}}N_5^{IV};\quad z_5=+\frac{241.9923}{10^{10}}N_5^{IV2}.$$

Whence $\beta_5=20°\ 30'\ 42''.3$; and log. $N_5^{IV}=7.0790118$.

$N_5\ =+0.0014747$, $\qquad N_5^{IV}=+0.00119953$,
$N_5'\ =+0.0013543$, $\qquad N_5^{V}\ =+0.00115794$,
$N_5''=+0.0013265$, $\qquad N_5^{VI}=-0.00112520$,
$N_5'''=+0.0012585$, $\qquad N_5^{VII}=-0.01179220$.

For the root $g_6=-2''.9160954$, we get,

$$x_6=+\frac{0.3679185}{10^{10}}N_6^{IV};\quad y_6=-\frac{0.3544211}{10^{10}}N_6^{IV};\quad z_6=\frac{581.1008}{10^{10}}N_6^{IV2}.$$

Whence $\beta_6=133°\ 55'\ 46''.2$; and log. $N_6^{IV}=6.9440507$.

$$\begin{array}{ll} N_6\ =+0.0030592, & N_6^{IV}=+0.00087912, \\ N_6'\ =+0.0017767, & N_6^{V}\ =+0.00071774, \\ N_6''=+0.0015913, & N_6^{VI}=-0.0176843, \\ N_6'''=+0.0011505, & N_6^{VII}=+0.00190064. \end{array}$$

For the root $g_7=-25''.936797$, we get,

$$x_7=-\frac{0.2995996}{10^{10}}N_7^{IV};\quad y_7=+\frac{0.2201126}{10^{10}}N_7^{IV};\quad z_7=\frac{59.01481}{10^{10}}N_7^{IV2}.$$

Whence $\beta_7=306°\ 18'\ 15''.6$; and log. $N_7^{IV}=7.7993080$.

$$\begin{array}{ll} N_7\ =-0.0002579, & N_7^{IV}=+0.00629953, \\ N_7'\ =-0.00030335, & N_7^{V}\ =-0.0156858, \\ N_7''=-0.00256615, & N_7^{VI}=+0.00068848, \\ N_7'''=-0.0092794. & N_7^{VII}=+0.000077149. \end{array}$$

20. For an increment of $+\frac{1}{100}$, to the mass of the *Jupiter*, we shall obtain the preliminary computations by merely making all the coefficients positive in equations (280–293). We shall then obtain the following

Fundamental Equations for $\mu^{IV}=+\frac{1}{100}$; *or for* $m^{IV}=1\div1037.504$.

$$\left.\begin{array}{l} A\ =g^2+38.155337.g\ +182.786001; \\ A'=g^2+23.3356274.g+\ 99.4308553; \\ A''=g^2+18.7710411.g+\ 73.2781865; \\ A_1=g^2+18.4811418.g+\ 60.7217122; \\ A_2=g^2+13.2441783.g+\ 9.040174; \\ A_3=g^2+26.5649511.g+\ 9.944072. \end{array}\right\}\quad(479)$$

$$\left.\begin{array}{l} D\ =g^2+44.6177028.g+590.564857; \\ D'=g^2+52.1872233.g+610.731488; \\ D''=g^2+32.3787870.g+259.4006143; \\ D_1=g^2+44.0905279.g+173.591533; \\ D_2=g^2+46.6785882.g+\ 33.163967; \\ D_3=g^2+\ 3.4403421.g+\ 1.70359455. \end{array}\right\}\quad(480)$$

$$\left.\begin{array}{l} B=\{g+32.52083\}b;\quad B'=\{g+17.7264928\}b; \\ \qquad\qquad\qquad\qquad\quad B''=\{g+13.180944\}b. \end{array}\right\}\quad(481)$$

$$\left.\begin{array}{l} C\ =-\{g+22.326122\}\ [9.4381189]b'; \\ C'\ =-\{g+17.7564991\}[9.1138076]b'; \\ C''=+[0.3976676]b'; \\ C'''=+[0.4411620]b'. \end{array}\right\}\quad(482)$$

$$\left.\begin{aligned} E &= -[9.9807377]b''; \\ E' &= -\{g+22.2915804\}[8.9723624]b''; \\ E'' &= -\{g+17.7039725\}[9.7501125]b''; \\ E''' &= -[1.0016537]b''. \end{aligned}\right\} \quad (483)$$

$$\left.\begin{aligned} F &= +[8.1679376]b'''; \\ F' &= -[9.1453586]b'''; \\ F'' &= -\{g+14.674814\}[0.7927855]b'''; \\ F''' &= -\{g+34.4307229\}[9.6907241]b'''. \end{aligned}\right\} \quad (484)$$

$$\left.\begin{aligned} B_1=\{g+4.5295952\}b_1; \quad B_2 &= \{g+\ 0.7142452\}b_1; \\ B_3 &= \{g+19.0000972\}b_1. \end{aligned}\right\} \quad (485)$$

$$\left.\begin{aligned} C_1 &= -\{g+4.1423610\}[9.5433087]b_2; \\ C_2 &= -\{g+0.73251914\}[9.4349711]b_2; \\ C_3 &= +[0.8644527]b_2; \\ C_4 &= +[0.8654649]b_2. \end{aligned}\right\} \quad (486)$$

$$\left.\begin{aligned} E_1 &= -[8.3317448]b_3; \\ E_2 &= -\{g+39.9481669\}[8.7437718]b_3; \\ E_3 &= -\{g+0.6504222\}[0.7242832]b_3; \\ E_4 &= -[0.9647514]b_3. \end{aligned}\right\} \quad (487)$$

$$\left.\begin{aligned} F_1 &= +[7.7244692]b_4; \\ F_2 &= -[0.3564628]b_4; \\ F_3 &= -\{g+2.7899200\}[1.5514854]b_4; \\ F_4 &= -\{g+45.946069\}[9.4626364]b_4. \end{aligned}\right\} \quad (488)$$

$$\left.\begin{aligned} g^4+47.7853999.g^3+782.14382516.g^2 \\ +5044.303371.g+11057.20317 \end{aligned}\right\} = (\chi, \chi_1, \chi_2, \chi_3). \quad (489)$$

$$\left.\begin{aligned} g^4+29.7164348.g^3+96.0269583.g^2 \\ +51.793880.g\ +\ 0.533011 \end{aligned}\right\} = (\chi_4, \chi_5, \chi_6, \chi_7). \quad (490)$$

The values of b_1, b_2, b_3, and b_4 are given by equations (406); and the values of b, b', b'', and b''' are given by equations (405), by merely multiplying the coefficients of N^{IV} by $1+\mu^{IV}=1.01$.

If we now put equations (489) and (490) equal to nothing, they will give

$$\begin{aligned} g &= -\ 5''.1493604, & g_4 &= -\ 0''.0104945, \\ g_1 &= -\ 6\ .6300170, & g_5 &= -\ 0\ .6646769, \\ g_2 &= -17\ .5198308, & g_6 &= -\ 2\ .9259522, \\ g_3 &= -18\ .4861917, & g_7 &= -26\ .1153112. \end{aligned}$$

The solutions of equations (479–490) will now give the following values:—

For the root g, we get,

$$g = -5''.153064,$$

$N \;\; = +0.1198583 N$	log. 9.0786680,
$N'' = +0.08550057 N$	" 8.9319690,
$N''' = +0.01695429 N$	" 8.2292796,
$N^{IV} = -0.0002019713 N$	" 6.3052894n,
$N^{V} = -0.0002567140 N$	" 6.4094494n,
$N^{VI} = +0.0002241425 N$	" 6.3505243,
$N^{VII} = +0.000006320077 N$	" 4.8007224.

For the root g_1, we get,

$$g_1 = -6''.635874,$$

$N_1' = -0.2841568 N_1$	log. 9.4535581n,
$N_1'' = -0.2283353 N_1$	" 9.3585790n,
$N_1''' = -0.04747286 N_1$	" 8.6764453n,
$N_1^{IV} = +0.000343187 N_1$	" 6.5355307,
$N_1^{V} = +0.0004843167 N_1$	" 6.6851294,
$N_1^{VI} = -0.0002492740 N_1$	" 6.3966770n,
$N_1^{VII} = -0.0000151962 N_1$	" 5.1817351n.

For the root g_2, we get,

$$g_2 = -17''.520779,$$

$N_2' = -\; 5.627642 N_2$	log. 0.7503265n,
$N_2'' = +\; 4.727661 N_2$	" 0.6746463,
$N_2''' = +31.11766 N_2$	" 1.4930069,
$N_2^{IV} = -\; 0.001595785 N_2$	" 7.2029742n,
$N_2^{V} = -\; 0.01334776 N_2$	" 8.1254084n,
$N_2^{VI} = +\; 0.001291801 N_2$	" 7.1111957,
$N_2^{VII} = +\; 0.0001489017 N_2$	" 6.1728995.

For the root g_3, we get,

$$g_3 = -18''.486192,$$

$N_3' = -\; 6.134120 N_3$	log. 0.7877523n,
$N_3'' = +\; 6.742900 N_3$	" 0.8288466,
$N_3''' = -11.29640 N_3$	" 1.0529402n,
$N_3^{IV} = +\; 0.000000955724 N_3$	" 93.9803324,
$N_3^{V} = +\; 0.0001854001 N_3$	" 96.2681100,
$N_3^{VI} = -\; 0.0000169937 N_3$	" 95.2302878n,
$N_3^{VII} = -\; 0.000002026216 N_3$	" 94.3066957n.

For the root g_4, we get,

$$g_4 = 0'', \text{ and } N_4 = N_4' = N_4'' =, \&c.$$

For the root g_5, we get,

$$g_5=-0''.6633671,$$

$N_5\ =+1.231158N_5^{IV}$	log. 0.0903139,
$N_5'\ =+1.130398N_5^{IV}$	" 0.0532315,
$N_5''\ =+1.107341N_5^{IV}$	" 0.0442814,
$N_5'''\ =+1.049016N_5^{IV}$	" 0.0207820,
$N_5^{V}\ =+0.9653980N_5^{IV}$	" 9.9847064,
$N_5^{VI}=-0.9445346N_5^{IV}$	" 9.9752179n,
$N_5^{VII}=-9.899907N_5^{IV}$	" 0.9956311n.

For the root g_6, we get,

$$g_6=-2''.925086,$$

$N_6\ =+\ 3.536719N_6^{IV}$	log. 0.5486005,
$N_6'\ =+\ 2.047060N_6^{IV}$	" 0.3111304,
$N_6''\ =+\ 1.834513N_6^{IV}$	" 0.2635208,
$N_6'''\ =+\ 1.310090N_6^{IV}$	" 0.1173012,
$N_6^{V}\ =+\ 0.8166500N_6^{IV}$	" 9.9120360,
$N_6^{VI}=-20.25015N_6^{IV}$	" 1.3064432n,
$N_6^{VII}=+\ 2.169824N_6^{IV}$	" 0.3364244.

For the root g_7, we get,

$$g_7=-26''.117474,$$

$N_7\ =-0.04211995N_7^{IV}$	log. 8.6244879n,
$N_7'\ =-0.04851416N_7^{IV}$	" 8.6858685n,
$N_7''\ =-0.4329533N_7^{IV}$	" 9.6364410n,
$N_7'''\ =-1.477216N_7^{IV}$	" 0.1694440n,
$N_7^{V}\ =-2.515433N_7^{IV}$	" 0.4006128n,
$N_7^{VI}=+0.1096052N_7^{IV}$	" 9.0398312,
$N_7^{VII}=+0.01229432N_7^{IV}$	" 8.0897044.

21. If we now substitute these values in equations (408) and (409), we shall obtain the following quantities:—

For the root $g=-5''.153064$, we get,

$$x=+\frac{0.616929}{10^{14}}N;\ y=+\frac{1.5676783}{10^{14}}N;\ z=\frac{13.836148}{10^{14}}N^2.$$

Whence $\beta=21°\ 28'\ 52''.1$, and log. $N=9.0855076$.

$N\ =+0.121761,$	$N^{IV}=-0.0000245922,$
$N'\ =+0.014594,$	$N^{V}\ =-0.0000312577,$
$N''\ =+0.0104106,$	$N^{VI}=+0.0000272918,$
$N'''\ =+0.0020644;$	$N^{VII}=+0.000000769538.$

For the root $g_1 = -6''.6358742$, we get,

$$x_1 = +\frac{0.685590}{10^{14}} N_1; \quad y_1 = -\frac{0.6767428}{10^{14}} N_1; \quad z_1 = \frac{34.48222}{10^{14}} N_1^2.$$

Whence $\beta_1 = 134° \ 37' \ 40''.5$; and log. $N_1 = 8.4461821$.

$N_1 \ = +0.027937$, $N_1^{IV} = +0.00000958766$,
$N_1' = -0.0079385$, $N_1^{V} \ = +0.0000135304$,
$N_1'' = -0.0063791$, $N_1^{VI} = -0.0000069640$,
$N_1''' = -0.00132626$, $N_1^{VII} = -0.00000042454$.

For the root $g_2 = -17''.520779$, we get,

$$x_2 = -\frac{69.06713}{10^{14}} N_2; \quad y_2 = +\frac{26.31099}{10^{14}} N_2; \quad z_2 = \frac{4443.491}{10^{13}} N_2^2.$$

Whence $\beta_2 = 290° \ 51' \ 15''.3$; and log. $N_2 = 7.2209728$.

$N_2 \ = +0.0016633$, $N_2^{IV} = -0.0000026543$,
$N_2' = -0.0093605$, $N_2^{V} \ = -0.000022202$,
$N_2'' = +0.0078636$, $N_2^{VI} = +0.0000021487$,
$N_2''' = +0.0517583$, $N_2^{VII} = +0.00000024767$.

For the root $g_3 = -18''.486192$, we get,

$$x_3 = -\frac{67.51436}{10^{14}} N_3; \quad y_3 = -\frac{23.46891}{10^{14}} N_3; \quad z_3 = \frac{20391.171}{10^{14}} N_3^2.$$

Whence $\beta_3 = 250° \ 49' \ 54''.7$; and log. $N_3 = 7.5447249$

$N_3 \ = +0.0035053$, $N_3^{IV} = +0.00000000335$,
$N_3' = -0.0215019$, $N_3^{V} \ = +0.0000006499$,
$N_3'' = +0.0236359$, $N_3^{VI} = -0.00000005957$,
$N_3''' = -0.0395973$, $N_3^{VII} = -0.00000000710$.

For the root $g_4 = 0''$, we get,

$$x_4 = +\frac{7.294416}{10^{11}} N_4; \quad y_4 = -\frac{2.119410}{10^{11}} N_4; \quad z_4 = \frac{274.1191}{10^{11}} N_4^2.$$

Whence $\beta_4 = 106° \ 12' \ 4''.8$; and $N_4 = N_4' = N_4'' =$ &c.,
$= +0.02771087$, log. 8.4426502.

For the root $g_5 = -0''.6633671$, we get,

$$x_5 = +\frac{1.016103}{10^{11}} N_5^{IV}; \quad y_5 = +\frac{2.742753}{10^{11}} N_5^{IV}; \quad z_5 = \frac{245.2401}{10^{10}} N_5^{IV2}.$$

Whence $\beta_5 = 20° \ 19' \ 41''.0$; and log. $N_5^{IV} = 7.0765225$

$N_5 \ = +0.0014684$, $N_5^{IV} = +0.00119268$,
$N_5' = +0.0013482$, $N_5^{V} \ = +0.00115141$,
$N_5'' = +0.0013207$, $N_5^{VI} = -0.00112652$,
$N_5''' = +0.0012511$, $N_5^{VII} = -0.0118074$.

For the root $g_6=-2''.925086$, we get,

$$x_6=+\frac{0.3697416}{10^{10}}N_6^{IV};\ y_6=-\frac{0.3561999}{10^{10}}N_6^{IV};\ z_6=\frac{588.7466}{10^{10}}N_6^{IV}.$$

Whence $\beta_6=133°\ 55'\ 52''.8$; and log. $N_6^{IV}=6.9405338$.

$N_6\ =+0.0030841,$ $N_6^{IV}=+0.00087204,$
$N_6'\ =+0.0017851,$ $N_6^{V}\ =+0.00071215,$
$N_6''=+0.0015998,$ $N_6^{VI}=-0.0176594,$
$N_6'''=+0.0011424,$ $N_6^{VII}=+0.0018922.$

For the root $g_7=-26''.117473$, we get,

$$x_7=-\frac{0.3026288}{10^{10}}N_7^{IV};\ y_7=+\frac{0.2224873}{10^{10}}N_7^{IV};\ z_7=\frac{60.04189}{10^{10}}N_7^{IV}.$$

Whence $\beta_7=306°\ 19'\ 21''.6$; and log. $N_7^{IV}=7.7962857$.

$N_7\ =-0.0002635,$ $N_7^{IV}=+0.00625584,$
$N_7'\ =-0.0003035,$ $N_7^{V}\ =-0.0157362,$
$N_7''=-0.0027085,$ $N_7^{VI}=+0.00068567,$
$N_7'''=-0.0092412,$ $N_7^{VII}=+0.000076911.$

22. For an increment of $+\frac{1}{40}m^{V}$, to the assumed mass of *Saturn*, we shall obtain the preliminary computations by merely making all the coefficients positive in equations (307–312). We shall then obtain the following

Fundamental Equations for $\mu^{V}=+\frac{1}{40}$; *or for* $m^{V}=1\div 3416.195$.

$$\left.\begin{aligned}A\ &=g^2+38.078720.g\ +181.976176;\\ A'&=g^2+23.1907166.g+\ 98.4490593;\\ A''&=g^2+18.7198881.g+\ 72.8857520;\\ A_1&=g^2+18.6278118.g+\ 61.6540448;\\ A_2&=g^2+13.3828119.g+\ 9.178963;\\ A_3&=g^2+26.5724027.g+\ 10.089397.\end{aligned}\right\}\qquad(491)$$

$$\left.\begin{aligned}D\ &=g^2+44.5181269.g+588.346190;\\ D'&=g^2+52.0193531.g+605.574324;\\ D''&=g^2+32.1854533.g+256.3822746;\\ D_1&=g^2+44.4628062.g+176.258864;\\ D_2&=g^2+47.1787023.g+\ 33.673551;\\ D_3&=g^2+\ 3.4692599.g+\ 1.72977087.\end{aligned}\right\}\qquad(492)$$

$$\left.\begin{aligned}B=\{g+32.45831\}b;\quad B'&=\{g+17.5956788\}b;\\ B''&=\{g+13.143888\}b.\end{aligned}\right\}\qquad(493)$$

$$\left.\begin{aligned}C\ &=-\{g+22.263602\}\ [9.4381189]b';\\ C'\ &=-\{g+17.6256851\}[9.1138076]b';\\ C''&=+[0.3976676]b';\\ C'''&=+[0.4411620]b'.\end{aligned}\right\}\qquad(494)$$

$$\left.\begin{aligned} E &= -[9.9807377]b''; \\ E' &= -\{g+22.2545242\}[8.9723624]b''; \\ E'' &= -\{g+17.5731585\}[9.7501125]b''; \\ E''' &= -[1.0016537]b''. \end{aligned}\right\} \tag{495}$$

$$\left.\begin{aligned} F &= +[8.1679376]b'''; \\ F' &= -[9.1453586]b'''; \\ F'' &= -\{g+14.612294\}\;[0.7927855]b'''; \\ F''' &= -\{g+34.3936667\}[9.6907241]b'''. \end{aligned}\right\} \tag{496}$$

$$\left.\begin{aligned} B_1=\{g+4.5551100\}b_1; \quad B_2 &= \{g+\;0.7176484\}b_1; \\ B_3 &= \{g+18.8231590\}b_1. \end{aligned}\right\} \tag{497}$$

$$\left.\begin{aligned} C_1 &= -\{g+4.1678758\}[9.5433087]b_2; \\ C_2 &= -\{g+0.7359221\}[9.4349711]b_2. \\ C_3 &= +[0.8751766]b_2; \\ C_4 &= +[0.8761888]b_2. \end{aligned}\right\} \tag{498}$$

$$\left.\begin{aligned} E_1 &= -[8.3317448]b_3; \\ E_2 &= -\{g+40.2949304\}[8.7437718]b_3; \\ E_3 &= -\{g+\;0.6538252\}[0.7242832]b_3; \\ E_4 &= -[0.9647514]b_3. \end{aligned}\right\} \tag{499}$$

$$\left.\begin{aligned} F_1 &= +[7.7244692]b_4; \\ F_2 &= -[0.3564628]b_4; \\ F_3 &= -\{g+\;2.8154348\}[1.5514854]b_4; \\ F_4 &= -\{g+46.44278\}\;\;[9.4626364]b_4. \end{aligned}\right\} \tag{500}$$

$$\left.\begin{aligned} g^4+47.5409132.g^3+774.11613432.g^2 \\ +4966.568399.g+10830.80683 \end{aligned}\right\} =(\chi, \chi_1, \chi_2, \chi_3). \tag{501}$$

$$\left.\begin{aligned} g^4+29.7477885.g^3+96.8557913.g^2 \\ +52.577272.g\;+\;0.536670 \end{aligned}\right\} =(\chi_4, \chi_5, \chi_6, \chi_7). \tag{502}$$

The values of b_1, b_2, b_3, and b_4 are given by equations (406); and the values of b, b', b'', and b''' are given by equations (405), by merely multiplying the coefficients of N^v by $1+\mu^v=1.025$.

If we now put equations (501) and (502) equal to nothing, they will give,

$$\begin{aligned} g &= -\;5''.1255241; & g_4 &= -\;0''.0104061; \\ g_1 &= -\;6\;.5914740; & g_5 &= -\;0\;.6688791; \\ g_2 &= -17\;.4063959; & g_6 &= -\;2\;.9523070; \\ g_3 &= -18\;.4175192; & g_7 &= -26\;.1161961. \end{aligned}$$

Substituting these roots in succession in equations (491–502), we shall get the following values:—

For the root g, we get,

$$g = -5''.1293390,$$

$N' = +0.1224435N$	log. 9.0879358,
$N'' = +0.08766774N$	" 8.9428398,
$N''' = +0.0175016N$	" 8.2430770,
$N^{IV} = -0.000205739N$	" 6.3133160n,
$N^{V} = -0.000261587N$	" 6.4176164n,
$N^{VI} = +0.000236310N$	" 6.3734826,
$N^{VII} = +0.0000062338N$	" 4.7947524.

For the root g_1, we get,

$$g_1 = -6''.5971682,$$

$N_1' = -0.277326N_1$	log. 9.4429910n,
$N_1'' = -0.223546N_1$	" 9.3493660n,
$N_1''' = -0.0468220N_1$	" 8.6704496n,
$N_1^{IV} = +0.000334045N_1$	" 6.5238055,
$N_1^{V} = +0.000471677N_1$	" 6.6736443,
$N_1^{VI} = -0.000250301N_1$	" 6.3984620n,
$N_1^{VII} = -0.0000148755N_1$	" 5.1724700n.

For the root g_2, we get,

$$g_2 = -17''.4073173,$$

$N_2' = -\ 5.57006N_2$	log. 0.7458600n,
$N_2'' = +\ 4.57184N_2$	" 0.6609136,
$N_2''' = +33.0320N_2$	" 1.5189348,
$N_2^{IV} = -\ 0.00161379N_2$	" 7.2078461n,
$N_2^{V} = -\ 0.0137070N_2$	" 8.1369422n,
$N_2^{VI} = +\ 0.00136814N_2$	" 7.1361306,
$N_2^{VII} = +\ 0.000156952N_2$	" 6.1957676.

For the root g_3, we get,

$$g_3 = -18''.4175195,$$

$N_3' = -\ 6.10257N_3$	log. 0.7855130n,
$N_3'' = +\ 6.66495N_3$	" 0.8237970,
$N_3''' = -10.3261N_3$	" 1.0139370n,
$N_3^{IV} = -\ 0.000017665N_3$	" 95.2471266n,
$N_3^{V} = -\ 0.000117075N_3$	" 96.0684623n,
$N_3^{VI} = +\ 0.00000985785N_3$	" 94.9937824,
$N_3^{VII} = +\ 0.00000108146N_3$	" 94.0340085.

For the root g_4, we get;

$$g_4 = 0'', \quad N_4 = N_4' = N_4'' = N_4''' = \&c.$$

For the root g_5, we get,

$$g_5=-0''.6675803,$$

$N_5 \;=+1.234453N_5^{IV}$	log. 0.0914746;
$N_5' \;=+1.132464N_5^{IV}$	" 0.0540246;
$N_5'' \;=+1.109145N_5^{IV}$	" 0.0449882;
$N_5''' \;=+1.049894N_5^{IV}$	" 0.0211454;
$N_5^{V} \;=+0.965528N_5^{IV}$	" 9.9847648;
$N_5^{VI}=-0.925706N_5^{IV}$	" 9.9664731n;
$N_5^{VII}=-9.910500N_5^{IV}$	" 0.9960956n.

For the root g_6, we get,

$$g_6=-2''.9514286,$$

$N_6 \;=+\;3.63816N_6^{IV}$	log. 0.5608818;
$N_6' \;=+\;2.08409N_6^{IV}$	" 0.3189167;
$N_6'' \;=+\;1.86452N_6^{IV}$	" 0.2705668;
$N_6''' \;=+\;1.31997N_6^{IV}$	" 0.1205640;
$N_6^{V} \;=+\;0.816989N_6^{IV}$	" 9.9122164;
$N_6^{VI}=-20.18151N_6^{IV}$	" 1.3049538n;
$N_6^{VII}=+\;2.139190N_6^{IV}$	" 0.3302494.

For the root g_7, we get,

$$g_7=-26''.1183694,$$

$N_7 \;=-0.0416526N_7^{IV}$	log. 8.6196418n;
$N_7' \;=-0.0489363N_7^{IV}$	" 8.6896315n;
$N_7'' \;=-0.426719N_7^{IV}$	" 9.6301420n;
$N_7''' \;=-1.440405N_7^{IV}$	" 0.1584846n;
$N_7^{V} \;=-2.42982N_7^{IV}$	" 0.3855740n;
$N_7^{VI}=+0.108637N_7^{IV}$	" 9.0359788;
$N_7^{VII}=+0.012173N_7^{IV}$	" 8.0853954.

23. If we now substitute these values in equations (408) and (409), we shall obtain the following quantities:—

For the root $g=-5''.1293390$, we get,

$$x=+\frac{606572}{10^{20}}; \quad y=+\frac{1589168.6}{10^{20}}; \quad z=\frac{140213200}{10^{20}};$$

Whence $\beta=20°\ 53'\ 20''.5$; and log. N=9.0839064.

$$N\;=+0.1213127; \qquad N^{IV}=-0.00002496;$$
$$N'\;=+0.0148551; \qquad N^{V}\;=-0.000031737;$$
$$N''\;=+0.0106362; \qquad N^{VI}=+0.000028671;$$
$$N'''=+0.00212364; \qquad N^{VII}=+0.0000007563.$$

For the root $g_1 = -6''.5971682$, we get,

$$x_1 = +\frac{702346}{10^{20}};\quad y_1 = -\frac{650773.9}{10^{20}};\quad z_1 = \frac{33379780}{10^{20}};$$

Whence $\beta_1 = 132^\circ\ 49'\ 3''.9$; log. $N_1 = 8.4576530$.

$N_1 = +0.0286850$; $N_1^{IV} = +0.000009582$;
$N_1' = -0.00795527$; $N_1^{V} = +0.000013530$;
$N_1'' = -0.00641252$; $N_1^{VI} = -0.000007180$;
$N_1''' = -0.00134306$; $N_1^{VII} = -0.0000004267$.

For the root $g_2 = -17''.4073173$, we get,

$$x_2 = -\frac{68779790}{10^{20}};\quad y_2 = +\frac{28706100}{10^{20}};\quad z_2 = \frac{4.840416}{10^{10}};$$

Whence $\beta_2 = 292^\circ\ 39'\ 13''.8$; log. $N_2 = 7.1874476$.

$N_2 = +0.0015397$; $N_2^{IV} = -0.000002485$;
$N_2' = -0.0085764$; $N_2^{V} = -0.000021105$;
$N_2'' = +0.0070528$; $N_2^{VI} = +0.0000021066$;
$N_2''' = +0.0508607$; $N_2^{VII} = +0.0000002417$.

For the root $g_3 = -18''.4175195$, we get,

$$x_3 = -\frac{67277160}{10^{20}};\quad y_3 = -\frac{22295350}{10^{20}};\quad z_3 = \frac{1.9362943}{10^{10}};$$

Whence $\beta_3 = 251^\circ\ 39'\ 34''.1$; log. $N_3 = 97.5635231$.

$N_3 = +0.0036604$; $N_3^{IV} = -0.00000006466$;
$N_3' = -0.0223376$; $N_3^{V} = -0.00000042853$;
$N_3'' = +0.0243961$; $N_3^{VI} = +0.000000036083$;
$N_3''' = -0.0377972$; $N_3^{VII} = +0.00000000039585$.

For the root $g_4 = 0''$, we

$$x_4 = +\frac{0.7324877}{10^{10}};\quad y_4 = -\frac{0.2141555}{10^{10}};\quad z_4 = \frac{27.41415}{10^{10}};$$

Whence $\beta_4 = 106^\circ\ 17'\ 49''.9$; log. $N_4 = 98.4446362$.

$$N_4 = N_4' = N_4'' = N_4''' = N_4^{IV} = N_4^{V} = N_4^{VI} = N_4^{VII} = +0.0278379$$

For the root $g_5 = -0''.6675803$, we get,

$$x_5 = +\frac{0.1042216}{10^{10}};\quad y_5 = +\frac{0.2727222}{10^{10}};\quad z_5 = \frac{245.6575}{10^{10}};$$

Whence $\beta_5 = 20^\circ\ 55'\ 5''.7$; log. $N_5^{IV} = 97.0749906$.

$N_5 = +0.00146716$; $N_5^{IV} = +0.00118851$;
$N_5' = +0.0013459$; $N_5^{V} = +0.00114755$;
$N_5'' = +0.00131823$; $N_5^{VI} = -0.00110001$;
$N_5''' = +0.00124781$; $N_5^{VII} = -0.0117785$.

For the root $g_6=-2''.9514286$, we get,

$$x_6=+\frac{0.3712512}{10^{10}};\quad y_6=-\frac{0.3561690}{10^{10}};\quad z_6=\frac{584.6008}{10^{10}};$$

Whence $\beta_6=133°\ 48'\ 43''.9$; log. $N_6^{IV}=96.9445043$.

$$\begin{aligned}
N_6 &=+0.0032017; & N_6^{IV} &=+0.000880044;\\
N_6' &=+0.0018341; & N_6^{V} &=+0.000718986;\\
N_6'' &=+0.0016409; & N_6^{VI} &=-0.0177606;\\
N_6''' &=+0.0011616; & N_6^{VII} &=+0.0018826.
\end{aligned}$$

For the root $g_7=-26''.1183694$, we get,

$$x_7=-\frac{0.2996395}{10^{10}};\quad y_7=+\frac{0.2202906}{10^{10}}.\quad z_7=\frac{57.99673}{10^{10}};$$

Whence $\beta_7=306°\ 19'\ 22''.0$; log. $N_7^{IV}=97.8070262$.

$$\begin{aligned}
N_7 &=-0.00002671; & N_7^{IV} &=+0.00641248;\\
N_7' &=-0.00031380; & N_7^{V} &=-0.0155812;\\
N_7'' &=-0.0027363; & N_7^{VI} &=+0.00069663;\\
N_7''' &=-0.0092366; & N_7^{VII} &=+0.000078059.
\end{aligned}$$

24. For an increment of $\frac{1}{20}m^{VI}$, to the assumed mass of *Uranus*, we shall obtain the preliminary computations by merely making all the coefficients positive in equations (326-331). We shall then obtain the following

Fundamental Equations for $\mu^{VI}=+\frac{1}{20}$; *or for* $m^{VI}=\frac{1}{23719}$.

$$\left.\begin{aligned}
A &=g^2+38.068970.g+181.871338;\\
A' &=g^2+23.1735927.g+98.3308977;\\
A'' &=g^2+18.7132214.g+72.8344734;\\
A_1 &=g^2+18.4995010.g+61.4402060;\\
A_2 &=g^2+13.2095523.g+9.147128;\\
A_3 &=g^2+26.3996602.g+10.055139.
\end{aligned}\right\}\qquad(503)$$

$$\left.\begin{aligned}
D &=g^2+44.5054401.g+588.063848;\\
D' &=g^2+51.9992925.g+604.964956;\\
D'' &=g^2+32.1623093.g+256.0208612;\\
D_1 &=g^2+43.9809321.g+175.644437;\\
D_2 &=g^2+46.5212935.g+33.556164;\\
D_3 &=g^2+3.4429483.g+1.72374142.
\end{aligned}\right\}\qquad(504)$$

$$\left.\begin{aligned}
B=\{g+32.45043\}b;\quad B' &=\{g+17.5804200\}b;\\
B'' &=\{g+13.139086\}b.
\end{aligned}\right\}\qquad(505)$$

$$\left.\begin{aligned}
C &=-\{g+22.255718\}[9.4381189]b';\\
C' &=-\{g+17.6104269\}[9.1138076]b';\\
C'' &=+[0.3976676]b';\\
C''' &=+[0.4411620]b'.
\end{aligned}\right\}\qquad(506)$$

$$\left.\begin{aligned} E &= -[9.9807377]b''; \\ E' &= -\{g+22.2497227\}[8.9723624]b''; \\ E'' &= -\{g+17.5578996\}[9.7501125]b''; \\ E''' &= -[1.0016537]b''. \end{aligned}\right\} \quad (507)$$

$$\left.\begin{aligned} F &= +[8.1679376]b'''; \\ F' &= -[9.1453586]b'''; \\ F'' &= -\{g+14.604409\}[0.7927855]b'''; \\ F''' &= -\{g+34.3888650\}[9.6907241]b'''. \end{aligned}\right\} \quad (508)$$

$$\left.\begin{aligned} B_1=\{g+4.6079205\}b_1;\ B_2&=\{g+0.7255099\}b_2; \\ B_3&=\{g+18.8315378\}b_2. \end{aligned}\right\} \quad (509)$$

$$\left.\begin{aligned} C_1 &= -\{g+4.2013246\}[9.5433087]b_2; \\ C_2 &= -\{g+0.7437838\}[9.4349711]b_2; \\ C_3 &= +[0.8644527]b_2; \\ C_4 &= +[0.8654649]b_2. \end{aligned}\right\} \quad (510)$$

$$\left.\begin{aligned} E_1 &= -[8.3529341]b_3; \\ E_2 &= -\{g+39.7796075\}[8.7437718]b_3; \\ E_3 &= -\{g+0.6616869\}[0.7242832]b_3; \\ E_4 &= -[0.9859407]b_3. \end{aligned}\right\} \quad (511)$$

$$\left.\begin{aligned} F_1 &= +[7.7244692]b_4; \\ F_2 &= -[0.3564628]b_4; \\ F_3 &= -\{g+2.7812615\}[1.5514854]b_4; \\ F_4 &= -\{g+45.77751\}[9.4626364]b_4. \end{aligned}\right\} \quad (512)$$

$$\left.\begin{aligned} &g^4+47.5111025.g^3+773.1351541.g^2 \\ &+4957.025771g+10802.91193 \end{aligned}\right\} =(\chi, \chi_1, \chi_2, \chi_3); \quad (513)$$

$$\left.\begin{aligned} &g^4+29.5535550.g^3+95.6630308.g^2 \\ &+52.197094.g + 0.534229 \end{aligned}\right\} =(\chi_4, \chi_5, \chi_6, \chi_7). \quad (514)$$

The values of b_1, b_2, b_3, and b_4 are given by equations (406); and the values of b, b', b'', and b''' are given by equations (405), by merely multiplying the coefficients of N^{VI} by $1+\mu^{VI}=1.05$.

If we now put the equations (513) and (514) equal to nothing, they will give

$$\begin{aligned} g &= -\ 5''.1224110; & g_4 &= -\ 0''.0104337; \\ g_1 &= -\ 6\ .5865615; & g_5 &= -\ 0\ .6748372; \\ g_2 &= -17\ .3929254; & g_6 &= -\ 2\ .9245347; \\ g_3 &= -18\ .4092044; & g_7 &= -25\ .9437494. \end{aligned}$$

Substituting these roots in succession in equations (503–514), we shall get the following values:—

For the root g, we get,

$$g=-5''.1262710;$$

$N' =+0.1227677N$	log. 9.0890840;
$N'' =+0.0879386N$	" 8.9441794;
$N''' =+0.0175667N$	" 8.2446904;
$N^{IV}=-0.000208367N$	" 6.3188297n;
$N^{V} =-0.00026463N$	" 6.4226422n;
$N^{VI}=+0.000232174N$	" 6.3658128;
$N^{VII}=+0.00000549425N$	" 4.7599088.

For the root g_1, we get,

$$g_1=-6''.5922959;$$

$N_1' =-0.276494N_1$	log. 9.4416856n;
$N_1'' =-0.222958N_1$	" 9.3482238n;
$N_1''' =-0.0467327N_1$	" 8.6696208n;
$N_1^{IV}=+0.000336574N_1$	" 6.5270807;
$N_1^{V} =+0.000474690N_1$	" 6.6764098;
$N_1^{VI}=-0.000245839N_1$	" 6.3906513n;
$N_1^{VII}=-0.0000143346N_1$	" 5.1563868n.

For the root g_2, we get,

$$g_2=-17''.3938608;$$

$N_2' =-\ 5.56333N_2$	log. 0.7453350n;
$N_2'' =+\ 4.56392N_2$	" 0.6593380;
$N_2''' =+33.2389N_2$	" 1.5216464;
$N_2^{IV}=-\ 0.00165952N_2$	" 7.2199837n;
$N_2^{V} =-\ 0.0140236N_2$	" 8.1468598n;
$N_2^{VI}=+\ 0.00136629N_2$	" 7.1355420;
$N_2^{VII}=+\ 0.000156224N_2$	" 6.1937476.

For the root g_3, we get,

$$g_3=-18''.4092048;$$

$N_3' =-\ 6.098831N_3$	log. 0.7852466n;
$N_3'' =+\ 6.65618N_3$	" 0.8232250;
$N_3''' =-10.22645N_3$	" 1.0097248n;
$N_3^{IV}=-\ 0.0000196421N_3$	" 95.2931878n;
$N_3^{V} =-\ 0.000150200N_3$	" 96.1766688n;
$N_3^{VI}=+\ 0.0000125074N_3$	" 95.0971660;
$N_3^{VII}=+\ 0.00000139286N_3$	" 94.1439073.

For the root g_4, we get,

$$g_4=0'',\ N_4=N_4'=N_4''=N_4'''=\ \&c.$$

For the root g_5, we get,

$$g_5=-0''.6735122;$$

$N_5 \;=+1.23718N_5^{IV}$	log. 0.0924317;
$N_5' \;=+1.133973N_5^{IV}$	" 0.0546030;
$N_5'' \;=+1.110407N_5^{IV}$	" 0.0454822;
$N_5''' \;=+1.05043N_5^{IV}$	" 0.0213676;
$N_5^{V} \;=+0.964612N_5^{IV}$	" 9.9843526;
$N_5^{VI}=-0.937483N_5^{IV}$	" 9.9719633n;
$N_5^{VII}=-9.79928N_5^{IV}$	" 0.9911943n.

For the root g_6, we get,

$$g_6=-2''.9236175;$$

$N_6 \;=+\;3.57518N_6^{IV}$	log. 0.5532976;
$N_6' \;=+\;2.06480N_6^{IV}$	" 0.3148787;
$N_6'' \;=+\;1.84969N_6^{IV}$	" 0.2670992;
$N_6''' \;=+\;1.285565N_6^{IV}$	" 0.1190940;
$N_6^{V} \;=+\;0.816008N_6^{IV}$	" 9.9116944;
$N_6^{VI}=-19.1632N_6^{IV}$	" 1.2824678n;
$N_6^{VII}=+\;2.16794N_6^{IV}$	" 0.3360471.

For the root g_7, we get,

$$g_7=-25''.9459126;$$

$N_7 \;=-0.0420490N_7^{IV}$	log. 8.6237552n;
$N_7' \;=-0.0466848N_7^{IV}$	" 8.6691754n;
$N_7'' \;=-0.432476N_7^{IV}$	" 9.6359617n;
$N_7''' \;=-1.466224N_7^{IV}$	" 0.1662004n;
$N_7^{V} \;=-2.491785N_7^{IV}$	" 0.3965106n;
$N_7^{VI}=+0.1093547N_7^{IV}$	" 9.0388375;
$N_7^{VII}=+0.0122059N_7^{IV}$	" 8.0865696.

25. If we now substitute these values in equations (408) and (409), we shall obtain the following quantities:—

For the root $g=-5''.1262710$, we get,

$$x=+\frac{610397}{10^{20}};\quad y=+\frac{1587285.2}{10^{20}};\quad z=\frac{14044917}{10^{20}}.$$

Whence $\beta=21^\circ\ 1'\ 11''.9$; log. $N=9.0833149$.

$N\;=+0.1211476$;	$N^{IV}=-0.000025246$;
$N'=+0.0148742$;	$N^{V}\;=-0.000032062$;
$N''=+0.0106545$;	$N^{VI}=+0.00002813$;
$N'''=+0.00212865$;	$N^{VII}=+0.000000697$.

For the root $g_1=-6''.5922959$, we get,

$$x_1=+\frac{696641}{10^{20}};\quad y_1=-\frac{640902.9}{10^{20}};\quad z_1=\frac{33247150}{10^{20}}.$$

Whence $\beta_1=132^\circ\ 41'\ 2''.3$; log. $N_1=8.4538431$.

$N_1=+0.0284342$; $N_1^{IV}=+0.000009570$;
$N_1'=-0.0078621$; $N_1^{V}=+0.000013498$;
$N_1''=-0.0063398$; $N_1^{VI}=-0.0000069904$;
$N_1'''=-0.0013288$; $N_1^{VII}=-0.0000004076$.

For the root $g_2=-17''.3938608$, we get,

$$x_2=-\frac{68615050}{10^{20}};\quad y_2=+\frac{28882030}{10^{20}};\quad z_2=\frac{0.4884728}{10^{10}}.$$

Whence $\beta_2=292^\circ\ 49'\ 39''.4$; log. $N_2=97.1830006$.

$N_2=+0.0015241$; $N_2^{IV}=-0.0000025292$;
$N_2'=-0.0084788$; $N_2^{V}=-0.000021373$;
$N_2''=+0.0069557$; $N_2^{VI}=+0.0000020823$;
$N_2'''=+0.0050658$; $N_2^{VII}=+0.00000023809$.

For the root $g_3=-18''.4092048$, we get,

$$x_3=-\frac{0.006724489}{10^{10}};\quad y_3=-\frac{0.002217478}{10^{10}};\quad z_3=\frac{1.9257107}{10^{10}}.$$

Whence $\beta_3=251^\circ\ 44'\ 58''.0$; log. $N_3=97.5654837$.

$N_3=+0.0036769$; $N_3^{IV}=-0.00000007222$;
$N_3'=-0.0224249$; $N_3^{V}=-0.0000005523$;
$N_3''=+0.0244742$; $N_3^{VI}=+0.000000046$;
$N_3'''=-0.0376018$; $N_3^{VII}=+0.00000000512$.

For the root $g_4=0''$, we get,

$$x_4=+\frac{0.7265241}{10^{10}};\quad y_4=-\frac{0.2110814}{10^{10}};\quad z_4=\frac{27.31188}{10^{10}}.$$

Whence $\beta_4=106^\circ\ 12'\ 1''.7$; log. $N_4=98.4424954$.

$N_4=N_4'=N_4''=N_4'''=N_4^{IV}=N_4^{V}=N_4^{VI}=N_4^{VII}=0.0277010$.

For the root $g_5=-0''.6735122$, we get,

$$x_5=+\frac{0.1020975}{10^{10}};\quad y_5=+\frac{0.2702021}{10^{10}};\quad z_5=\frac{240.6404}{10^{10}}.$$

Whence $\beta_5=20^\circ\ 42'\ 11''.2$; log. $N_5^{IV}=97.0793126$.

$N_5=+0.0014850$; $N_5^{IV}=+0.00120036$;
$N_5'=+0.0013612$; $N_5^{V}=+0.0011579$;
$N_5''=+0.0013329$; $N_5^{VI}=-0.0011251$;
$N_5'''=+0.0012609$; $N_5^{VII}=-0.0117624$.

For the root $g_6=-2''.9236175$, we get,

$$x_6=+\frac{0.3679656}{10^{10}};\quad y_6=-\frac{0.3547543}{10^{10}};\quad z_6=\frac{555.2999}{10^{10}}.$$

Whence $\beta_6=133^\circ\ 57'\ 9''.9$; log. $N_6^{IV}=96.9640000$.

$$\begin{array}{ll} N_6\ =+0.0032908; & N_6^{IV}=+0.00092045; \\ N_6'\ =+0.0019005; & N_6^{V}\ =+0.00075109; \\ N_6''=+0.0017026; & N_6^{VI}=-0.01763875; \\ N_6'''=+0.0012109; & N_6^{VII}=+0.00199548. \end{array}$$

For the root $g_7=-25''.9459126$, we get,

$$x_7=-\frac{0.3001025}{10^{10}};\quad y_7=+\frac{0.2204575}{10^{10}};\quad z_7=\frac{59.06886}{10^{10}}.$$

Whence $\beta_7=306^\circ\ 18'\ 4''.6$; log. $N_7^{IV}=97.7996217$.

$$\begin{array}{ll} N_7\ =-0.0002651; & N_7^{IV}=+0.00630408; \\ N_7'\ =-0.0002943; & N_7^{V}\ =-0.0157084; \\ N_7''=-0.0027264; & N_7^{VI}=+0.00068938; \\ N_7'''=-0.0092432; & N_7^{VII}=+0.00007695. \end{array}$$

26. For an increment of $+\frac{1}{10}m^{VII}$, to the assumed mass of *Neptune*, we shall obtain the preliminary computations by merely making all the coefficients positive in equations (345–350). We shall then obtain the following

Fundamental Equations for $\mu^{VII}=+\frac{1}{10}$; or for $m^{VII}=1\div 17072.73$.

$$\left.\begin{array}{l} A\ =g^2+38.068863.g\ +181.870184; \\ A'=g^2+23.1734071.g+\ 98.3296126; \\ A''=g^2+18.7131480.g+\ 72.8339084; \\ A_1=g^2+18.4537418.g+\ 60.9399039; \\ A_2=g^2+13.2015603.g+\ \ 9.072649; \\ A_3=g^2+26.3912337.g+\ \ 9.975760. \end{array}\right\}\quad(515)$$

$$\left.\begin{array}{l} D\ =g^2+44.5053009.g+588.060747; \\ D'=g^2+51.9990747.g+604.958351; \\ D''=g^2+32.1620579.g+256.0169344; \\ D_1=g^2+43.9488809.g+174.214515; \\ D_2=g^2+46.5094752.g+\ 33.282984; \\ D_3=g^2+\ 3.4725714.g+\ \ 1.70970867. \end{array}\right\}\quad(516)$$

$$\left.\begin{array}{ll} B=\{g+32.45034\}b; & B'=\{g+17.5802549\}b; \\ & B''=\{g+13.139034\}b. \end{array}\right\}\quad(517)$$

$$\left.\begin{array}{l} C\ =-\{g+22.255630\}\ [9.4381189]b'; \\ C'\ =-\{g+17.6102612\}[9.1138076]b'; \\ C''=+[0.3976676]b'; \\ C'''=+[0.4411620]b' \end{array}\right\}\quad(518)$$

$$\left.\begin{aligned} E &= -[9.9807377]b''; \\ E' &= -\{g+22.2496697\}[8.9723624]b''; \\ E'' &= -\{g+17.5577346\}[9.7501125]b''; \\ E''' &= -[1.0016537]b''. \end{aligned}\right\} \quad (519)$$

$$\left.\begin{aligned} F &= +[8.1679376]b'''; \\ F' &= -[9.1453586]b'''; \\ F'' &= -\{g+14.604322\}\ [0.7927855]b'''; \\ F''' &= -\{g+34.3888122\}[9.6907241]b'''. \end{aligned}\right\} \quad (520)$$

$$\left.\begin{aligned} B_1=\{g+4.5635477\}b_1;\quad B_2 &= \{g+\ 0.7189043\}b_1; \\ B_3 &= \{g+18.8244977\}b_1. \end{aligned}\right\} \quad (521)$$

$$\left.\begin{aligned} C_1 &= -\{g+4.1763135\ \}[9.5433087]b_2; \\ C_2 &= -\{g+0.73900559\}[9.4349711]b_2; \\ C_3 &= +[0.8644527]b_2; \\ C_4 &= +[0.8654649]b_2. \end{aligned}\right\} \quad (522)$$

$$\left.\begin{aligned} E_1 &= -[8.3317448]b_3; \\ E_2 &= -\{g+39.7725674\ \}[8.7437718]b_3; \\ E_3 &= -\{g+\ 0.64869894\}[0.7242832]b_3; \\ E_4 &= -[0.9647514]b_3. \end{aligned}\right\} \quad (523)$$

$$\left.\begin{aligned} F_1 &= +[7.7658619]b_4; \\ F_2 &= -[0.3978555]b_4; \\ F_3 &= -\{g+\ 2.8238725\}[1.5514854]b_4; \\ F_4 &= -\{g+45.770470\}\ [9.4626364]b_4. \end{aligned}\right\} \quad (524)$$

$$\left.\begin{aligned} g^4+47.5107777.g^3+773.1244605.g^2 \\ +4956.921649.g+10802.60734 \end{aligned}\right\} = (\chi, \chi_1, \chi_2, \chi_3); \quad (525)$$

$$\left.\begin{aligned} g^4+29.5753988.g^3+96.3534529.g^2 \\ +52.082669.g+0.529879 \end{aligned}\right\} = (\chi_4, \chi_5, \chi_6, \chi_7). \quad (526)$$

The values of b_1, b_2, b_3, and b_4 are given by equations (406), and the values of b, b', b'', and b''' are given by equations (405), by merely multiplying the coefficients of N^{VII} by $1+\mu^{VII}=1.10$.

If we now put equations (525) and (526) equal to nothing, they will give

$$\begin{aligned} g_1 &= -\ 5''.1223768; & g_4 &= -\ 0''.01037221; \\ g_2 &= -\ 6\ .5865075; & g_5 &= -\ 0\ .66492499; \\ g_3 &= -17\ .3927801; & g_6 &= -\ 2\ .96207581; \\ g_4 &= -18\ .4091132; & g_7 &= -25\ .93802580. \end{aligned}$$

Substituting these roots in succession in equations (515–526), we get the following values:—

For the root g, we get,

$g=-5''.1262327,$		
$N'=+0.122771N$	log.	9.0890969,
$N''=+0.0879417N$	"	8.9441950,
$N'''=+0.0175679N$	"	8.2447196,
$N^{IV}=-0.000207983N$	"	6.3180273n,
$N^{V}=-0.000264402N$	"	6.4222642n,
$N^{VI}=+0.0002360322N$	"	6.3729713,
$N^{VII}=+0.0000061622$	"	4.7897356.

For the root g_1, we get,

$g_1=-6''.5922343,$		
$N_1'=-0.276482N_1$	log.	9.4416674n,
$N_1''=-0.222950N_1$	"	9.3482076n,
$N_1'''=-0.0466245N_1$	"	8.6696144n,
$N_1^{IV}=+0.00033610N_1$	"	6.5264690,
$N_1^{V}=+0.000474472N_1$	"	6.6762012,
$N_1^{VI}=-0.000248245N_1$	"	6.3948800n,
$N_1^{VII}=-0.0000147153N_1$	"	5.1677685n.

For the root g_2, we get,

$g_2=-17''.3937133,$		
$N_2'=-\ 5.56327N_2$	log.	0.7453298n,
$N_2''=+\ 4.56374N_2$	"	0.6593212,
$N_2'''=+33.24130N_2$	"	1.5216780,
$N_2^{IV}=-\ 0.0016539N_2$	"	7.2185079n,
$N_2^{V}=-\ 0.0140311N_2$	"	8.1470920n,
$N_2^{VI}=+\ 0.00137024N_2$	"	7.1367973,
$N_2^{VII}=+\ 0.00015714N_2$	"	6.1962851.

For the root g_3, we get,

$g_3=-18''.4091136,$		
$N_3'=-\ 6.09878N_3$	log.	0.7852438n,
$N_3''=+\ 6.65608N_3$	"	0.8232187,
$N_3'''=-10.2254N_3$	"	1.0096800n,
$N_3^{IV}=-\ 0.0000195968N_3$	"	95.2921852n,
$N_3^{V}=-\ 0.000150608N_3$	"	96.1778487n,
$N_3^{VI}=+\ 0.0000125707N_3$	"	95.0993576,
$N_3^{VII}=+\ 0.00000139394N_3$	"	94.1442453.

For the root g_4, we get,

$$g_4=0'';\ N_4=N_4'=N_4''=N_4'''=\&c.$$

For the root g_5, we get,

$$g_5 = -0''.6634935,$$

$N_5 = +1.23293 N_5^{IV}$	log. 0.0909496,
$N_5' = +1.13171 N_5^{IV}$	" 0.0537360,
$N_5'' = +1.10855 N_5^{IV}$	" 0.0447544,
$N_5''' = +1.04961 N_5^{IV}$	" 0.0210270,
$N_5^{V} = +0.965270 N_5^{IV}$	" 9.9846490,
$N_5^{VI} = -0.921756 N_5^{IV}$	" 9.9646159n,
$N_5^{VII} = -8.94419 N_5^{IV}$	" 0.9515409n.

For the root g_6, we get,

$$g_6 = -2''.9612185,$$

$N_6 = +\ 3.66750 N_6^{IV}$	log. 0.5643695,
$N_6' = +\ 2.09390 N_6^{IV}$	" 0.3209559,
$N_6'' = +\ 1.87228 N_6^{IV}$	" 0.2723698,
$N_6''' = +\ 1.32235 N_6^{IV}$	" 0.1213462,
$N_6^{V} = +\ 0.813392 N_6^{IV}$	" 9.9102996,
$N_6^{VI} = -20.4503 N_6^{IV}$	" 1.3106998n,
$N_6^{VII} = +\ 2.15783 N_6^{IV}$	" 0.3340172.

For the root g_7, we get,

$$g_7 = -25''.9401904,$$

$N_7 = -0.042064 N_7^{IV}$	log. 8.6239109n,
$N_7' = -0.0466092 N_7^{IV}$	" 8.6684721n,
$N_7'' = -0.432691 N_7^{IV}$	" 9.6361780n,
$N_7''' = -1.46720 N_7^{IV}$	" 0.1664891n,
$N_7^{V} = -2.49115 N_7^{IV}$	" 0.3963997n,
$N_7^{VI} = +0.109524 N_7^{IV}$	" 9.0395107,
$N_7^{VII} = +0.0122518 N_7^{I}$	" 8.0882005.

27. If we now substitute these values in equations (408) and (409) we shall obtain the following quantities:—

For the root $g = -5''.1262327$, we get,

$$x = +\frac{611711}{10^{20}}; \quad y = +\frac{1586934.1}{10^{20}}; \quad z = \frac{14045156}{10^{20}}.$$

Whence $\beta = 21°\ 4'\ 40''.0$; log. $N = 9.0831058$.

$N = +0.1210893$;	$N^{IV} = -0.00002519$;
$N' = +0.0148664$;	$N^{V} = -0.00003202$;
$N'' = +0.0106498$;	$N^{VI} = +0.00002858$;
$N''' = +0.0021278$;	$N^{VII} = +0.0000007462$.

For the root $g_1 = -6''.5922343$, we get,

$$x_1 = +\frac{693376}{10^{20}}; \quad y_1 = -\frac{638974.9}{10^{20}}; \quad z_1 = \frac{33245220}{10^{20}}.$$

Whence $\beta_1 = 132^\circ\ 39'\ 44''.3$; log. $N_1 = 98.4527380$.

$N_1 = +0.0283621$; $N_1^{IV} = +0.000009533$;
$N_1' = -0.0078418$; $N_1^{V} = +0.000013457$;
$N_1'' = -0.0063234$; $N_1^{VI} = -0.0000070408$;
$N_1''' = -0.0013254$; $N_1^{VII} = -0.00000041736$.

For the root $g_2 = -17''.3937133$, we get,

$$x_2 = -\frac{68626660}{10^{20}}; \quad y_2 = +\frac{28878680}{10^{20}}; \quad z_2 = \frac{0.4885253}{10^{10}}.$$

Whence $\beta_2 = 292^\circ\ 49'\ 18''.4$; log. $N_2 = 97.1830089$.

$N_2 = +0.0015241$; $N_2^{IV} = -0.000002521$;
$N_2' = -0.0084789$; $N_2^{V} = -0.000021385$;
$N_2'' = +0.0069555$; $N_2^{VI} = +0.0000020884$;
$N_2''' = +0.0506625$; $N_2^{VII} = +0.0000002395$.

For the root $g_3 = -18''.4091136$, we get,

$$x_3 = -\frac{67244580}{10^{20}}; \quad y_3 = -\frac{22173520}{10^{20}}; \quad z_3 = \frac{1.9255992}{10^{10}}.$$

Whence $\beta_3 = 251^\circ\ 45'\ 1''.1$; log. $N_3 = 97.5655046$.

$N_3 = +0.0036771$; $N_3^{IV} = -0.00000007206$;
$N_3' = -0.0224258$; $N_3^{V} = -0.0000005538$;
$N_3'' = +0.0244750$; $N_3^{VI} = +0.00000004622$;
$N_3''' = -0.0375997$; $N_3^{VII} = +0.000000005126$.

For the root $g_4 = 0''$, we get,

$$x_4 = +\frac{0.7310062}{10^{10}}; \quad y_4 = -\frac{0.2158651}{10^{10}}; \quad z_4 = \frac{27.46919}{10^{10}}$$

Whence $\beta_4 = 106^\circ\ 27'\ 6''.5$; log. $N_4 = 98.4432303$;

$$N_4 = N_4' = N_4'' = N_4''' = N_4^{IV} = N_4^{V} = N_4^{VI} = N_4^{VII} = +0.02774790.$$

For the root $g_5 = -0.6634935$, we get,

$$x_5 = +\frac{0.1014674}{10^{10}}; \quad y_5 = +\frac{0.2722315}{10^{10}}; \quad z_5 = \frac{222.4744}{10^{10}}.$$

Whence $\beta_5 = 20^\circ\ 26'\ 30''.0$; log. $N_5^{IV} = 97.1159056$.

$N_5 = +0.0016101$; $N_5^{IV} = +0.00130589$;
$N_5' = +0.0014779$; $N_5^{V} = +0.00126053$;
$N_5'' = +0.0014476$; $N_5^{VI} = -0.00120371$;
$N_5''' = +0.0013738$; $N_5^{VII} = -0.0116801$.

For the root $g_6=-2''.9612185$, we get,

$$x_6=+\frac{0.3722380}{10^{10}};\quad y_6=-\frac{0.3654807}{10^{10}};\quad z_6=\frac{600.4976}{10^{10}}$$

Whence $\beta_6=134°\ 28'\ 30''.7$; log. $N_6^{IV}=96.9388828$.

$N_6=+0.0031860$; $N_6^{IV}=+0.000868726$;
$N_6'=+0.0018190$; $N_6^{V}=+0.00070661$;
$N_6''=+0.0016265$; $N_6^{VI}=-0.0177657$;
$N_6'''=+0.0011488$; $N_6^{VII}=+0.00187456$.

For the root $g_7=-25''.9401904$, we get,

$$x_7=-\frac{0.2997138}{10^{10}};\quad y_7=+\frac{0.2203005}{10^{10}};\quad z_7=\frac{59.04657}{10^{10}}$$

Whence $\beta_7=306°\ 19'\ 2''.1$; log. $N_7^{IV}=97.7993116$.

$N_7=-0.00026499$; $N_7^{IV}=+0.00629958$;
$N_7'=-0.00029362$; $N_7^{V}=-0.0156932$;
$N_7''=-0.00272577$; $N_7^{VI}=+0.00068996$;
$N_7'''=-0.00924274$; $N_7^{VII}=+0.000077181$.

28. Having thus obtained the values of all the constants, corresponding to the separate variation of the planetary masses, it now remains to find the coefficients of these variations. This we shall do by taking the difference between the constants corresponding to the assumed masses and those depending on the assumed variation of mass for each planet, and dividing by the assumed variation. By this means we shall obtain the following table—observing that the coefficients of μ, μ', μ'', &c., in the values of N, N', N'', &c., are given in units of the seventh decimal place of these coefficients.

$$g=-5''.126112+0''.207190\mu-1''.49906\mu'-0''.88154\mu''-0''.060606\mu'''-2''.6952\mu^{IV}$$
$$-0''.129080\mu^{V}-0''.00318\mu^{VI}-0''.00121\mu^{VII};$$

$$\beta=21°\ 6'\ 26''.8+6752''.6\mu-35940''\mu'-71170''\mu''+1984''.7\mu'''+134530''\mu^{IV}$$
$$+31452''\mu^{V}-12596''\mu^{VI}-1070''\mu^{VII};$$

$$N=+0.121076+276274\mu-568000\mu'-140640\mu''+34384\mu'''+688000\mu^{IV}$$
$$+95880\mu^{V}+29840^{VI}+1630\mu^{VII};$$

$$N'=+0.014867+127798\mu+276820\mu'-52360\mu''-6929\mu'''-273100\mu^{IV}-4800\mu^{V}$$
$$+2840\mu^{VI}-70\mu^{VII};$$

$$N''=+0.010650+88422\mu+230840\mu'-23860\mu''-6601\mu'''-239000\mu^{IV}-5360\mu^{V}$$
$$+1960\mu^{VI}+20\mu^{VII};$$

$$N'''=+0.002128+17510\mu+51480\mu'+7640\mu''-1066\mu'''-63400\mu^{IV}-1680\mu^{V}$$
$$+320\mu^{VI}+0\mu^{VII};$$

$$N^{IV}=-0.0000252-255.4\mu-480\mu'-14\mu''+7.2\mu'''+580\mu^{IV}+84\mu^{V}-32\mu^{VI}-2\mu^{VII};$$

$$N^{V}=-0.0000320-313.4\mu-648\mu'-42\mu''+7.6\mu'''+740\mu^{IV}+104.8\mu^{V}-24\mu^{VI}-2\mu^{VII};$$

$$N^{VI}=+0.0000280+350.4\mu+364\mu'-74\mu''-14.0\mu'''-750\mu^{IV}+252\mu^{V}+36\mu^{VI}$$
$$+54\mu^{VII};$$

$$N^{VII}=+0.000000775+4.28\mu+24\mu'+5.2\mu''+0.1\mu'''-5\mu^{IV}-7.6\mu^{V}-31.2\mu^{VI}$$
$$-2.9\mu^{VII}.$$

$$g_1=-6''.592128-0''.314940\mu-0''.85282\mu'-0''.78390\mu''-0''.116744\mu'''-4''.3746\mu^{IV}$$
$$-0''.20160\mu^{V}-0''.00336\mu^{VI}-0''.00106\mu^{VII};$$

$$\beta_1=132^\circ\ 40'\ 57''.8-102434''\mu-490268\mu'-205068''\mu''+26729''\mu'''+700270''\mu^{IV}$$
$$+19444''\mu^{V}+90''\mu^{VI}-735''\mu^{VII};$$

$$N_1=+0.0283520-37374\mu-3360\mu'+358300\mu''+5235\mu'''-41500\mu^{IV}+133120\mu^{V}$$
$$+16400\mu^{VI}+990\mu^{VII}:$$

$$N_1'=-0.0078380-8507\mu+222860\mu'-75320\mu''-8530\mu'''-100200\mu^{IV}-46800\mu^{V}$$
$$-4760\mu^{VI}-350\mu^{VII};$$

$$N_1''=-0.0063210-9157\mu+167160\mu'-68660\mu''-5780\mu'''-58300\mu^{IV}-36680\mu^{V}$$
$$-3800\mu^{VI}-260\mu^{VII};$$

$$N_1'''=-0.0013250-2162\mu+32900\mu'-2360\mu''-1437\mu'''-1400\mu^{IV}-7280\mu^{V}$$
$$-780\mu^{VI}-50\mu^{VII};$$

$$N_1^{IV}=+0.00000952+5\mu-250\mu'+148\mu''+12\mu'''+67\mu^{IV}+24\mu^{V}+10\mu^{VI}+1\mu^{VII};$$

$$N_1^{V}=+0,00001345+10\mu-346\mu'+218\mu''+18\mu'''+80\mu^{IV}+32\mu^{V}+10\mu^{VI}+0\mu^{VII};$$

$$N_1^{VI}=-0.00000696+1\mu+194\mu'-96\mu''-7\mu'''+0\mu^{IV}-88\mu^{V}-6\mu^{VI}-8\mu^{VII};$$

$$N_1^{VII}=-0.000000420-0\mu+10.8\mu'-6.8\mu''-0.6\mu'''+0\mu^{IV}-2.8\mu^{V}+2.4\mu^{VI}+0.3\mu^{VII}.$$

$$g_2=17''.393390-0''.028779\mu-1''.31820\mu'-2''.4080\mu''+0''.065597\mu'''-12''.7389\mu^{IV}$$
$$-0''.55708\mu^{V}-0''.00942\mu^{VI}-0''.00323\mu^{VII};$$

$$\beta_2=292^\circ\ 49'\ 53''.2+12418''\mu+333738''\mu'+270720''\mu''+23660''\mu'''-711800''\mu^{IV}$$
$$-25592''\mu^{V}-276''\mu^{VI}-348\mu^{VII};$$

$$N_2=+0.0015240-1455\mu-67740\mu'-50820\mu''+4466\mu'''+139300\mu^{IV}+6280\mu^{V}$$
$$+0\mu^{VI}+0\mu^{VII};$$

$$N_2'=-0.0084783+8450\mu+454660\mu'+283020\mu''-24120\mu'''-882200\mu^{IV}$$
$$-39240\mu^{V}-10\mu^{VI}-60\mu^{VII};$$

$$N_2''=+0.0069546-9445\mu-392740\mu'-348760\mu''+14218\mu'''+909000\mu^{IV}$$
$$+39280\mu^{V}+220\mu^{VI}+90^{VII};$$

$$N_2'''=+0.0506672-13628\mu-648600\mu'-421340\mu''+79686\mu'''+1091100\mu^{IV}$$
$$+77400\mu^{V}-180\mu^{VI}-47\mu^{VII};$$

$$N_2^{IV}=-0.00000251+1.6\mu+66\mu'+62\mu''-15.4\mu'''-140\mu^{IV}+12\mu^{V}-4\mu^{VI}-1\mu^{VII};$$

$$N_2^{V}=-0.00002140+10.1\mu+434\mu'+294\mu''-120.4\mu'''-800\mu^{IV}-120\mu^{V}+6\mu^{VI}$$
$$+2\mu^{VII};$$

$$N_2^{VI}=+0.00000208-1\mu-44\mu'-32\mu''+12\mu'''+70\mu^{IV}+12\mu^{V}+0\mu^{VI}+1\mu^{VII};$$

$$N_2^{VII}=+0.00000024-0.12\mu-5.2\mu'-4\mu''+1.4\mu'''+8\mu^{IV}+0.8\mu^{V}-0.4\mu^{VI}-0.1\mu^{VII}.$$

$$g_3=-18''.408914-0''.087808\mu-5''.14616\mu'-5''.18060\mu''-0''.317709\mu'''-7''.7278\mu^{IV}$$
$$-0''.34424\mu^{V}-0''.00582\mu^{VI}-0''.00020\mu^{VII};$$

$$\beta_3=251^\circ\ 45'\ 8''.6+4720''\mu+157816''\mu'+123030\mu''-17836''\mu'''-331390''\mu^{IV}$$
$$-12580''\mu^{V}-212''\mu^{VI}-75''\mu^{VII};$$

$$N_3=+0.0036775+1277\mu+82380\mu'+65160\mu''-2863\mu'''-172200\mu^{IV}-6880\mu^{V}$$
$$-120\mu^{VI}-40\mu^{VII};$$

$$N_3'=-0.0224278-7986\mu-290400\mu'-468080\mu''+11283\mu'''+925900\mu^{IV}$$
$$+36080\mu^{V}+580\mu^{VI}+200\mu^{VII};$$

$$N_3'' = +0.0244768+3926\mu+47734\mu'+179920\mu''-2395\mu'''-840900\mu^{IV}-32280\mu^{V}-520\mu^{VI}-180\mu^{VII};$$

$$N_3''' = -0.0375951+27422\mu+1075000\mu'+732200\mu''+100657\mu'''-2002200\mu^{IV}-80840\mu^{V}-1340\mu^{VI}-460\mu^{VII};$$

$$N_3^{IV} = -0.000000072-0.54\mu-28.2\mu'-21.6\mu''+2.70\mu'''+75\mu^{IV}+2.8\mu^{V}+0\mu^{VI}+0\mu^{VII};$$

$$N_3^{V} = -0.000000557-11.32\mu-676.4\mu'-525\mu''+65.7\mu'''+1207\mu^{IV}+51.2\mu^{V}+1\mu^{VI}+0.3\mu^{VII};$$

$$N_3^{VI} = +0.000000046+1.0\mu+57.8\mu'+44.8\mu''-5.6\mu'''-106\mu^{IV}-4\mu^{V}+0\mu^{VI}+0.0\mu^{VII};$$

$$N_3^{VII} = +0.0000000005+0.12\mu+6.6\mu'+5.2\mu''-0.65\mu'''-12\mu^{IV}-0.4\mu^{V}+0\mu^{VI}+0.0\mu^{VII}.$$

$$g_4 = 0''$$

$$\beta_4 = 106^\circ\ 14'\ 18''.0-28''.6\mu-138\mu'+0''\mu''-26''.5\mu'''-13320''\mu^{IV}+8476''\mu^{V}-2726''\mu^{VI}+7685''\mu^{VII};$$

$$N_4 = +0.0277417+13\mu+140\mu'-200\mu''-13\mu'''-30800\mu^{IV}+38480\mu^{V}-8140\mu^{VI}+620\mu^{VII}.$$

$$g_5 = -0''.661666+0''.000020\mu+0''.00004\mu'+0''.00002\mu''+0.000002\mu'''-0''.1701\mu^{IV}-0''.23656\mu^{V}-0''.23692\mu^{VI}-0''.01828\mu^{VII};$$

$$\beta_5 = 20^\circ\ 31'\ 24''.6+24''.3\mu+298''\mu'-540''\mu''-42''.3\mu'''-70360''\mu^{IV}+56804''\mu^{V}+12932''\mu^{VI}-2946''\mu^{VII};$$

$$N_5 = +0.0014778+29\mu-620\mu'-440\mu''-31\mu'''-9400\mu^{IV}-4240\mu^{V}+1440\mu^{VI}+13230\mu^{VII};$$

$$N_5' = +0.0013568+37\mu+100\mu'-260\mu''-25\mu'''-8600\mu^{IV}-4360\mu^{V}+880\mu^{VI}+12110\mu^{VII};$$

$$N_5'' = +0.0013291+24\mu+180\mu'-120\mu''-26\mu'''-8400\mu^{IV}-4360\mu^{V}+760\mu^{VI}+11850\mu^{VII};$$

$$N_5''' = +0.0012586+9\mu+80\mu'+40\mu''-1\mu'''-7500\mu^{IV}-4320\mu^{V}+460\mu^{VI}+11520\mu^{VII};$$

$$N_5^{IV} = +0.00119933+5\mu+16\mu'-16\mu''+2\mu'''-6650\mu^{IV}-4328\mu^{V}+206\mu^{VI}+10656\mu^{VII};$$

$$N_5^{V} = +0.00115773+5\mu+20\mu'-10\mu''+2\mu'''-6320\mu^{IV}-4072\mu^{V}+34\mu^{VII}+10280\mu^{VII};$$

$$N_5^{VI} = -0.00112485-4.6\mu+0\mu'+28\mu''-3.5\mu'''-1670\mu^{IV}+9936\mu^{V}-50\mu^{VI}-7886\mu^{VII};$$

$$N_5^{VII} = -0.0117882-55\mu-260\mu'+16\mu''-40\mu'''-19200\mu^{IV}+3880\mu^{V}+5160\mu^{VI}+10810\mu^{VII}$$

$$g_6 = -2''.916082+0''.000028\mu+0''.00014\mu'+0''.00004\mu''-0''.000013\mu'''-0''.9004\mu^{IV}-1''.41388\mu^{V}-0''.15070\mu^{VI}-0''.45136\mu^{VII};$$

$$\beta = 133^\circ\ 56'\ 10''.8-175''\mu-344''\mu'+310''\mu''-24''.6\mu'''-1800''\mu^{IV}-17876''\mu^{V}+1182''\mu^{VI}+19400''\mu^{VII};$$

$$N_6 = +0.0031283+794\mu-13000\mu'-8480\mu''-691\mu'''-44200\mu^{IV}+29360\mu^{V}+32500\mu^{VI}+5780\mu^{VII};$$

$$N_6'=+0.0018108+596\mu+660\mu'-2820\mu''-341\mu'''-25700\mu^{IV}+9320\mu^{V}+17960\mu^{VI}+820\mu^{VII};$$

$$N_6''=+0.0016228+378\mu+1220\mu'-1540\mu''-315\mu'''-23000\mu^{IV}+7240\mu^{V}+15960\mu^{VI}+370\mu^{VII};$$

$$N_6'''=+0.0011557+73\mu+300\mu'+240\mu''-52\mu'''-15300\mu^{IV}+1560\mu^{V}+11040\mu^{VI}-690\mu^{VII};$$

$$N_6^{IV}=+0.0008794-4\mu-80\mu'-20\mu''-3\mu'''-7400\mu^{IV}+240\mu^{V}+8200\mu^{VI}-1170\mu^{VII};$$

$$N_6^{V}=+0.0007180-4\mu-60\mu'-20\mu''-3\mu'''-5900\mu^{IV}+400\mu^{V}+6620\mu^{VI}-1140\mu^{VII};$$

$$N_6^{VI}=-0.0176872+35\mu+1100\mu'+280\mu''+29\mu'''+27800\mu^{IV}-29360\mu^{V}+9680\mu^{VI}-7850\mu^{VII};$$

$$N_6^{VII}=+0.0019010-4\mu-600\mu'-40\mu''-9\mu'''-8800\mu^{IV}-7360\mu^{V}+18900\mu^{VI}-2640\mu^{VII}.$$

$$g_7=-25''.934567-0''.000039\mu-0''.00196\mu'-0''.00430\mu''-0''.00223\mu'''-18''.2906\mu^{IV}-7''.35208\mu^{V}-0''.22692\mu^{VI}-0''.05623\mu^{VII};$$

$$\beta_7=306^\circ\ 19'\ 21''.2-2''.4\mu-242''\mu'+128''\mu''-65''.6\mu'''+40''\mu^{IV}+32''\mu^{V}-1532''\mu^{VI}-191''\mu^{VII};$$

$$N_7=-0.0002652-1\mu-300\mu'-740\mu''+73\mu'''+1700\mu^{IV}-760\mu^{V}+20\mu^{VI}+20\mu^{VII};$$

$$N_7'=-0.0002932+40\mu+5840\mu'+14100\mu''-102\mu'''-10300\mu^{IV}-8240\mu^{V}-220\mu^{VI}-40\mu^{VII};$$

$$N_7''=-0.0027275-93\mu-12780\mu'-5500\mu''+1614\mu'''+19000\mu^{IV}-3520\mu^{V}+220\mu^{VI}+170\mu^{VII};$$

$$N_7'''=-0.0092499+12631\mu-2700\mu'-13300\mu''-295\mu'''+8700\mu^{IV}+5280\mu^{V}+1340\mu^{VI}+720\mu^{VII};$$

$$N_7^{IV}=+0.00630053+0\mu-80\mu'-40\mu''-10\mu'''-44700\mu^{IV}+44800\mu^{V}+720\mu^{VI}-90\mu^{VII};$$

$$N_7^{V}=-0.0156928+3\mu+240\mu'+200\mu''+70\mu'''-43400\mu^{IV}+44640\mu^{V}-3120\mu^{VI}-40\mu^{VII};$$

$$N_7^{VI}=+0.0006890-1\mu-20\mu'-20\mu''-5\mu'''-3300\mu^{IV}+3040\mu^{V}+80\mu^{VI}+100\mu^{VII};$$

$$N_7^{VII}=+0.00007720+0\mu+0\mu'+0\mu''-0.5\mu'''-290\mu^{IV}+344\mu^{V}-50\mu^{VI}-2\mu^{VII}.$$

We have thus obtained the system of constants and the coefficients of their variations, corresponding to the ecliptic of 1850, as the plane of reference; and we shall now inquire what modifications are necessary in order to refer the same quantities to the invariable plane of the planetary system.

CHAPTER III.

ON THE POSITIONS AND SECULAR VARIATIONS OF THE ORBITS WHEN REFERRED TO THE INVARIABLE PLANE OF THE PLANETARY SYSTEM.

1. WE shall now refer the positions of the orbits to the invariable plane of the planetary system, in order to discover whether there are any laws which control their mutual positions, of a similar nature to those which we have shown to exist relatively to the eccentricities and perihelia. For this purpose it is necessary to first determine the position of the invariable plane with reference to the fixed ecliptic of 1850; and we can then readily refer all the orbits to that plane. The position of this plane is found by the principle, that the sum of the products, formed by multiplying each planetary mass by the projection of the area described by its radius vector, in a given time, is a maximum. If we put γ for the inclination of the invariable plane to the fixed ecliptic of 1850, and Π for the longitude of its ascending node on the same plane, we shall have (*Mécanique Céleste* [1162]),

$$c \tan\gamma \sin\Pi = c''; \quad c \tan\gamma \cos\Pi = c'. \qquad (527)$$

But we have

$$\left.\begin{aligned} c &= m\sqrt{\mu a(1-e^2)}\cos\phi + m'\sqrt{\mu' a'(1-e'^2)}\cos\phi' \\ &\quad + m''\sqrt{\mu'' a''(1-e''^2)}\cos\phi'' + \&c., \\ c' &= m\sqrt{\mu a(1-e^2)}\sin\phi\cos\theta + m'\sqrt{\mu' a'(1-e'^2)}\sin\phi'\cos\theta' \\ &\quad + m''\sqrt{\mu'' a''(1-e''^2)}\sin\phi''\cos\theta'' + \&c., \\ c'' &= m\sqrt{\mu a(1-e^2)}\sin\phi\sin\theta + m'\sqrt{\mu' a'(1-e'^2)}\sin\phi'\sin\theta' \\ &\quad + m''\sqrt{\mu'' a''(1-e''^2)}\sin\phi''\sin\theta'' + \&c. \end{aligned}\right\} \quad (528)$$

If we denote the sun's mass by unity, we shall have

$$\mu = 1 + m, \quad \mu' = 1 + m', \quad \mu'' = 1 + m'', \ \&c.;$$

but we shall also have

$$\sqrt{\mu a} = na^2, \quad \sqrt{\mu' a'} = n'a'^2, \quad \sqrt{\mu'' a''} = n''a''^2, \ \&c.$$

Substituting these values in equations (528), they will become

$$\left.\begin{aligned} c &= mna^2\sqrt{1-e^2}\cos\phi + m'n'a'^2\sqrt{1-e'^2}\cos\phi' \\ &\quad + m''n''a''^2\sqrt{1-e''^2}\cos\phi'' + \&c., \\ c' &= mna^2\sqrt{1-e^2}\sin\phi\cos\theta + m'n'a'^2\sqrt{1-e'^2}\sin\phi'\cos\theta' \\ &\quad + m''n''a''^2\sqrt{1-e''^2}\sin\phi''\cos\theta'' + \&c., \\ c'' &= mna^2\sqrt{1-e^2}\sin\phi\sin\theta + m'n'a'^2\sqrt{1-e'^2}\sin\phi'\sin\theta' \\ &\quad + m''n''a''^2\sqrt{1-e''^2}\sin\phi''\sin\theta'' + \&c. \end{aligned}\right\} \quad (529)$$

Substituting in these equations the values of m, n, a, e, ϕ and θ, given in §§ 5 and 17 of Chapter I, and § 6 of Chapter II, we shall get

$$c=+0.0035274157,\quad c'=-0.00002735230,\quad c''=+0.00009393304. \qquad (530)$$

In finding these quantities n'' has been supposed to equal unity, and the values of n, n', n''', &c. have been multiplied by $\frac{1}{n''}$ in order to preserve the same ratio.

Substituting the values of c, c', and c'', in equations (527), we shall obtain

$$\Pi=106°\ 14'\ 6''.00, \text{ and } \gamma=1°\ 35'\ 19''.376. \qquad (531)$$

If we now denote by ϕ_0, ϕ_0', ϕ_0'', &c., θ_0, θ_0', θ_0'', &c., the respective inclinations and longitudes of ascending nodes of the different planets, on the invariable plane; the values of θ_0, θ_0', θ_0'', &c. being reckoned from the descending node of the fixed ecliptic of 1850, on the invariable plane; we shall have the following equations to determine θ_0, θ_0', &c., ϕ_0, ϕ_0', &c.

$$\left.\begin{aligned} &\sin\phi_0 \sin\theta_0=\sin\phi \sin(\theta-\Pi),\\ &\sin\phi_0 \cos\theta_0=\cos\gamma \sin\phi \cos(\theta-\Pi)-\sin\gamma \cos\phi. \end{aligned}\right\} \qquad (532)$$

These equations will give the following elements:—

Mercury,	ϕ_0 $=6°$	$20'$	$58''.08$,	θ_0 $=287°$	$54'$	$5''.12$,
Venus,	ϕ_0' $=2$	11	13.57,	θ_0' $=307$	14	8.10,
The Earth,	ϕ_0'' $=1$	35	19.376,	θ_0'' $=180$	0	0.00,
Mars,	ϕ_0''' $=1$	40	43.70,	θ_0''' $=248$	56	21.45,
Jupiter,	ϕ_0^{IV} $=0$	19	59.674,	θ_0^{IV} $=210$	7	35.44,
Saturn,	ϕ_0^{V} $=0$	55	30.924,	θ_0^{V} $=16$	34	26.66,
Uranus,	ϕ_0^{VI} $=1$	1	45.27,	θ_0^{VI} $=204$	12	33.78,
Neptune,	ϕ_0^{VII} $=0$	43	24.845,	θ_0^{VII} $=286$	39	55.10.

2. Now putting

$$\left.\begin{aligned} &\tan\phi_0 \sin\theta_0=p_0, &&\tan\phi'_0 \sin\theta_0'=p_0' \text{ \&c.},\\ &\tan\phi_0 \cos\theta_0=q_0, &&\tan\phi'_0 \cos\theta_0'=q_0' \text{ \&c.} \end{aligned}\right\} \qquad (533)$$

we shall get the following values

$$\begin{aligned}
p_0 &=-0.1058879, & q_0 &=+0.0342038,\\
p_0' &=-0.0304057, & q_0' &=+0.0231090,\\
p_0'' &=\ 0. & q_0'' &=-0.0277354,\\
p_0''' &=-0.0273512, & q_0''' &=-0.0105324,\\
p_0^{IV} &=-0.00273067, & q_0^{IV} &=-0.00503058,\\
p_0^{V} &=+0.00460691, & q_0^{V} &=+0.0154792,\\
p_0^{VI} &=-0.00736719, & q_0^{VI} &=-0.0163856,\\
p_0^{VII} &=+0.0126079, & q_0^{VII} &=+0.000734628.
\end{aligned}$$

If we substitute these values in equations (408) and (409), we shall obtain the values of β, β_1, β_2, &c., N, N_1, N_2, &c., corresponding to the invariable plane. But instead of performing this operation separately for each root, we shall proceed in the following manner.

3. If we neglect the squares of e, e', e'', &c., ϕ, ϕ', ϕ'', &c., γ, we may put

$$\cos\gamma=1;\ \sin\gamma=\tan\gamma=\gamma;\ \sin\phi=\tan\phi;\ \sin\phi'=\tan\phi', \text{ \&c.};$$

then we shall have

$$\left.\begin{aligned} p&=\sin\phi\sin\theta, & p'&=\sin\phi'\sin\theta', & p''&=\sin\phi''\sin\theta'' \text{ \&c.},\\ q&=\sin\phi\cos\theta, & q'&=\sin\phi'\cos\theta', & q''&=\sin\phi''\cos\theta'' \text{ \&c.}\end{aligned}\right\} \quad (534)$$

Substituting these values in equations (408) and (409), also putting $t=0$, and remembering that for the root $g_4=0$, we have $N_4=N_4'=N_4''$, &c., we shall have

$$\left.\begin{aligned} \frac{m}{na}\sin\phi\sin\theta+\frac{m'}{n'a'}\sin\phi'\sin\theta'+\text{\&c.}&=\left\{\frac{m}{na}+\frac{m'}{n'a'}+\text{\&c.}\right\}N_4\sin\beta_4;\\ \frac{m}{na}\sin\phi\cos\theta+\frac{m'}{n'a'}\sin\phi'\cos\theta'+\text{\&c.}&=\left\{\frac{m}{na}+\frac{m'}{n'a'}+\text{\&c.}\right\}N_4\cos\beta_4.\end{aligned}\right\} \quad (535)$$

But if we neglect m^2, m'^2, &c., we shall have

$$mna^2=\frac{m}{na},\quad m'n'a'^2=\frac{m'}{n'a'},\quad m''n''a''^2=\frac{m''}{n''a''}, \text{ \&c.},$$

and equations (527) will give, by substituting the values of c,c',c'',

$$\left.\begin{aligned} \frac{m}{na}\sin\phi\sin\theta+\frac{m'}{n'a'}\sin\phi'\sin\theta'+\text{\&c.}&=\left\{\frac{m}{na}+\frac{m'}{n'a'}+\text{\&c.}\right\}\gamma\sin\Pi;\\ \frac{m}{na}\sin\phi\cos\theta+\frac{m'}{n'a'}\sin\phi'\cos\theta'+\text{\&c.}&=\left\{\frac{m}{na}+\frac{m}{n'a'}+\text{\&c.}\right\}\gamma\cos\Pi.\end{aligned}\right\} \quad (536)$$

Comparing equations (535) and (536) we find $\Pi=\beta_4$, and $\gamma=N_4$. Now substituting $\Pi=\beta_4$, $\gamma=N_4$, in equations (532), they will give

$$\left.\begin{aligned} &\sin\phi_0\sin\theta_0=\sin\phi\sin(\theta-\beta_4)=\sin\phi\sin\theta\cos\beta_4-\sin\phi\cos\theta\sin\beta_4\\ &\qquad\qquad\qquad\qquad\qquad\qquad=p\cos\beta_4-q\sin\beta_4=p_0;\\ &\sin\phi_0\cos\theta_0=\sin\phi\cos(\theta-\beta_4)-\gamma=\\ &\sin\phi\cos\theta\cos\beta_4-\sin\phi\sin\theta\sin\beta_4-\gamma=q\cos\beta_4-p\sin\beta_4-N_4=q_0.\end{aligned}\right\} \quad (537)$$

And since the relative values of N, N', N'', &c., N_1, N_1', N_1'', &c., are known we may determine their actual values corresponding to the invariable plane, by the analysis of Chapter II, § 5. We shall therefore suppose

$$\left.\begin{aligned} &q_0=\alpha N\cos(gt+\beta^{(0)})+\alpha_1 N_1\cos(g_1t+\beta_1^{(0)})+\alpha_2 N_2\cos(g_2t+\beta_2^{(0)})+\text{\&c.},\\ &q_0'=\alpha N'\cos(gt+\beta^{(0)})+\alpha_1 N_1'\cos(g_1t+\beta_1^{(0)})+\alpha_2 N_2'\cos(g_2t+\beta_2^{(0)})+\text{\&c.},\\ &\text{\&c.};\\ &p_0=\alpha N\sin(gt+\beta^{(0)})+\alpha_1 N_1\sin(g_1t+\beta_1^{(0)})+\alpha_2 N_2\sin(g_2t+\beta_2^{(0)})+\text{\&c.},\\ &p_0'=\alpha N'\sin(gt+\beta^{(0)})+\alpha_1 N_1'\sin(g_1t+\beta_1^{(0)})+\alpha_2 N_2'\sin(g_2t+\beta_2^{(0)})+\text{\&c.},\\ &\text{\&c.},\end{aligned}\right\} \quad (538)$$

α, α_1, α_2, &c., being the constant factors which are necessary in order to reduce the numbers already calculated to the corresponding ones for the invariable plane; and $\beta^{(0)}$, $\beta_1^{(0)}$, $\beta_2^{(0)}$, &c., being the constants necessary to satisfy the equations for the given epoch.

Equations (408) and (409) will also become

$$\left.\begin{aligned} &\alpha\left\{Np_0\frac{m}{na}+N'p_0'\frac{m'}{n'a'}+N''p_0''\frac{m''}{n''a''}+\&c.\right\} \\ &=\alpha^2\left\{N^2\frac{m}{na}+N'^2\frac{m'}{n'a'}+\&c.\right\}\sin(gt+\beta^{(0)}); \end{aligned}\right\} \quad (539)$$

$$\left.\begin{aligned} &\alpha\left\{Nq_0\frac{m}{na}+N'q_0'\frac{m'}{n'a'}+N''q_0'\frac{m''}{n''a''}+\&c.\right\} \\ &=\alpha^2\left\{N^2\frac{m}{na}+N'^2\frac{m'}{n'a'}+\&c.\right\}\cos(gt+\beta^{(0)}); \end{aligned}\right\} \quad (540)$$

If we substitute in these equations the values of p_0, p_0', &c., q_0, q_0', &c., given by equations (537), they will become

$$\left.\begin{aligned} &\alpha\left\{Np\frac{m}{na}+N'p'\frac{m'}{n'a'}+N''p''\frac{m''}{n'a'}+\&c.\right\}\cos\beta_4 \\ &-\alpha\left\{Nq\frac{m}{na}+N'q'\frac{m'}{n'a'}+N''q''\frac{m''}{n''a''}+\&c.\right\}\sin\beta_4 \\ &=\alpha^2\left\{N^2\frac{m}{na}+N'^2\frac{m'}{n'a'}+N''^2\frac{m''}{n''a''}+\&c.\right\}\sin(gt+\beta^{(0)}); \end{aligned}\right\} \quad (541)$$

$$\left.\begin{aligned} &\alpha\left\{Nq\frac{m}{na}+N'q'\frac{m'}{n'a'}+N''q''\frac{m''}{n''a''}+\&c.\right\}\cos\beta_4 \\ &-\alpha\left\{Np\frac{m}{na}+N'p'\frac{m'}{n'a'}+N''p''\frac{m''}{n''a''}+\&c.\right\}\sin\beta \\ &-\alpha\left\{N\frac{m}{na}+N'\frac{m}{n'a'}+N''\frac{m''}{n''a''}+\&c.\right\}N_4 \\ &=\alpha^2\left\{N^2\frac{m}{na}+N'^2\frac{m'}{n'a'}+N''^2\frac{m''}{n''a''}+\&c.\right\}\cos(gt+\beta^{(0)}). \end{aligned}\right\} \quad (542)$$

Now according to equations (410), the coefficient of N_4 in this equation is equal to nothing; and if we substitute the values of the coefficients of $\alpha\cos\beta_4$, and $\alpha\sin\beta_4$, which are given by equations (408) and (409), both members of equations (541) and (542) will be divisible by the coefficients of $\alpha\sin(gt+\beta^{(0)})$, and $\alpha\cos(gt+\beta^{(0)})$, and we shall find

$$\left.\begin{aligned} \sin(gt+\beta-\beta_4)&=\alpha\sin(gt+\beta^{(0)}); \\ \cos(gt+\beta-\beta_4)&=\alpha\cos(gt+\beta^{(0)}). \end{aligned}\right\} \quad (543)$$

Whence we get

$$\tan(gt+\beta-\beta_4)=\tan(gt+\beta^{(0)})$$

Therefore

$$\beta^{(0)}=\beta-\beta_4, \text{ and } \alpha=1. \quad (544)$$

It therefore follows that in order to apply our numbers to the invariable plane, we have only to diminish the constants β, β_1, β_2, &c., by the longitude of the ascending node of that plane, on the fixed ecliptic of 1850, and neglect the constant term.

Therefore we shall have

$$\left.\begin{aligned}p_0=N\sin(gt+\beta-\beta_4)+N_1\sin(g_1t+\beta_1-\beta_4)+N_2\sin(g_2t+\beta_2-\beta_4)\\+N_3\sin(g_3t+\beta_3-\beta_4)+N_5\sin(g_5t+\beta_5-\beta_4)\\+N_6\sin(g_6t+\beta_6-\beta_4)+N_7\sin(g_7t+\beta_7-\beta_4);\\q_0=N\cos(gt+\beta-\beta_4)+N_1\cos(g_1t+\beta_1-\beta_4)+N_2\cos(g_2t+\beta_2-\beta_4)\\+N_3\cos(g_3t+\beta_3-\beta_4)+N_5\cos(g_5t+\beta_5-\beta_4)\\+N_6\cos(g_6t+\beta_6-\beta_4)+N_7\cos(g_7t+\beta_7-\beta_4);\end{aligned}\right\}\quad(545)$$

$$\left.\begin{aligned}p_0'=N'\sin(gt+\beta-\beta_4)+N_1'\sin(g_1t+\beta_1-\beta_4)+N_2'\sin(g_2t+\beta_2-\beta_4)\\+N_3'\sin(g_3t+\beta_3-\beta_4)+\&c.;\\q_0'=N'\cos(gt+\beta-\beta_4)+N_1'\cos(g_1t+\beta_1-\beta_4)+N_2'\cos(g_2t+\beta_2-\beta_4)\\+N_3'\cos(g_3t+\beta_3-\beta_4)+\&c.\end{aligned}\right\}\quad(546)$$

Substituting for p_0 and q_0 their values given by the first members of equations (537), we shall easily find

$$\left.\begin{aligned}\sin\phi_0\sin(\theta_0-gt-\beta)=N_1\sin\{(g_1-g)t+\beta_1-\beta\}+N_2\sin\{(g_2-g)t+\beta_2-\beta\}\\+N_3\sin\{(g_3-g)t+\beta_3-\beta\}+N_5\sin\{(g_5-g)t+\beta_5-\beta\}\\+N_6\sin\{(g_6-g)t+\beta_6-\beta\}+N_7\sin\{(g_7-g)t+\beta_7-\beta\}\end{aligned}\right\}\quad(547)$$

$$\left.\begin{aligned}\sin\phi_0\cos(\theta_0-gt-\beta)\qquad\qquad\qquad\qquad\\=N+N_1\cos\{(g_1-g)t+\beta_1-\beta\}+N_2\cos\{(g_2-g)t+\beta_2-\beta\}\\+N_3\cos\{(g_3-g)t+\beta_3-\beta\}+N_5\cos\{(g_5-g)t+\beta_5-\beta\}\\+N_6\cos\{(g_6-g)t+\beta_6-\beta\}+N_7\cos\{(g_7-g)t+\beta_7-\beta\}\end{aligned}\right\}\quad(548)$$

From these equations it is easy to show that the mean motion of θ_0 is equal to gt when N exceeds the sum of the coefficients of the cosines, all taken positively. We shall also have

$$\left.\begin{aligned}\textit{maximum}\ \phi_0=N+N_1+N_2+N_3+N_5+N_6+N_7;\\\text{and }\textit{minimum}\ \phi_0=N-(N_1+N_2+N_3+N_5+N_6+N_7).\end{aligned}\right\}\quad(549)$$

4. If we now substitute in these equations the values given in Chapter II, § 7, we shall obtain the following *maxima*, *minima*, and *mean motions*.

	Inclination to invariable plane.		Mean motion of nodes in a Julian year.
	Maximum.	Minimum.	
Mercury,	9° 10′ 41″	4° 44′ 27″	—5″.126076
Venus,	3 16 18	0 0 0	indeterminate
The Earth,	3 6 0	0 0 0	"
Mars,	5 56 2	0 0 0	"
Jupiter,	0 28 56	0 14 23	—25″.934567
Saturn,	1 0 39	0 47 16	—25 .934567
Uranus,	1 7 10	0 54 25	— 2 .916082
Neptune,	0 47 21	0 33 43	— 0 .661666

5. It thus appears that the mean motion of the nodes of *Jupiter* and *Saturn*, on the invariable plane, are exactly the same. This indicates a relation of a permanent character between the positions of the nodes of these two planets, the nature of which we shall now examine.

If we divide equation (547) by (548) we shall obtain an equation similar to (147) and (148). And if in this equation we substitute the numbers corresponding to *Jupiter* and *Saturn*, we find that the mean places of the nodes, on the invariable plane, will be the same if N_7^{IV} and N_7^{V} have the same sign, and that they will differ by 180° if N_7^{IV} and N_7^{V} have different signs. The computed numbers show that the signs of these two quantities are different: it therefore follows that the mean longitudes of the nodes of *Jupiter* and *Saturn*, on the invariable plane of the planetary system, always differ by 180°. We shall find, by the analysis of Chapter I, § 21, that the actual place of *Jupiter's* node may differ from its mean place to the extent of 19° 38′, while that of *Saturn* can deviate from its mean place only to the extent of 7° 7′. It therefore follows that the longitudes of the nodes of these two planets can differ from 180° to the extent of 26° 45′. Their nearest possible approach is therefore 153° 15, while their present distance apart is 166° 27′.

We shall also find that the actual place of *Mercury's* node can differ from its mean place to the extent of 18° 31′; while the nodes of *Uranus* and *Neptune* can respectively deviate from their mean places to the extent of 6° 0′ and 9° 40′.

CHAPTER IV.

ON THE PRECESSION OF THE EQUINOXES AND THE OBLIQUITY OF THE ECLIPTIC.

1. THE analytical formulæ for the precession of the equinoxes and the obliquity of the ecliptic to the equator, referred to a fixed and also to a movable plane, are given by the formulæ [3100, 3101, 3107, and 3110], *Mécanique Céleste.* In order to reduce them to numbers we shall observe that the letter c in the notation of the formulæ corresponds to N'', N_1'', N_2'', &c. in this work. If we denote the mean value of the precession in a Julian year by l, and the mean obliquity of the ecliptic by h, and also put

$$c=\frac{l}{l+g}N'', \quad c_1=\frac{l}{l+g_1}N_1'', \quad c_2=\frac{l}{l+g_2}N_2'', \text{ \&c.},$$

$f=l+g$, $f_1=l+g_1$, $f_2=l+g_2$, $f_3=l+g_3$, $f_4=l+g_4=l$, $f_5=l+g_5$, &c., we shall have the following formulæ for determining the precession and obliquity:

$$\left.\begin{aligned}\psi=lt=\zeta&+c\left\{\cot h-\frac{g}{f}\tan h\right\}\sin(ft+\beta)\\&+c_1\left\{\cot h-\frac{g_1}{f_1}\tan h\right\}\sin(f_1t+\beta_1)\\&+c_2\left\{\cot h-\frac{g_2}{f_2}\tan h\right\}\sin(f_2t+\beta_2)\\&+c_3\left\{\cot h-\frac{g_3}{f_3}\tan h\right\}\sin(f_3t+\beta_3)\\&+c_4\left\{\cot h-\frac{g_4}{f_4}\tan h\right\}\sin(f_4t+\beta_4)\\&+c_5\left\{\cot h-\frac{g_5}{f_5}\tan h\right\}\sin(f_5t+\beta_5)\\&+c_6\left\{\cot h-\frac{g_6}{f_6}\tan h\right\}\sin(f_6t+\beta_6)\\&+c_7\left\{\cot h-\frac{g_7}{f_7}\tan h\right\}\sin(f_7t+\beta_7)\end{aligned}\right\}\begin{array}{c}\text{Precession of the}\\\text{Equinoxes on the}\\(550)\\\text{Fixed Ecliptic.}\end{array}$$

$$\left.\begin{aligned}\varepsilon_1=h-c\cos(ft+\beta)-c_1\cos(f_1t+\beta_1)-c_2\cos(f_2t+\beta_2)\\-c_3\cos(f_3t+\beta_3)-c_4\cos(f_4t+\beta_4)-c_5\cos(f_5t+\beta_5)\\-c_6\cos(f_6t+\beta_6)-c_7\cos(f_7t+\beta_7)\end{aligned}\right\}\begin{array}{c}\text{Obliquity of the}\\(551)\\\text{Equator to the}\\\text{Fixed Ecliptic.}\end{array}$$

$$\left.\begin{aligned}
\psi' = lt + \zeta & - \frac{g}{f} N'' \left\{ \cot h + \frac{l}{f} \tan h \right\} \sin(ft + \beta) \\
& - \frac{g_1}{f_1} N_1'' \left\{ \cot h + \frac{l}{f_1} \tan h \right\} \sin(f_1 t + \beta_1) \\
& - \frac{g_2}{f_2} N_2'' \left\{ \cot h + \frac{l}{f_2} \tan h \right\} \sin(f_2 t + \beta_2) \\
& - \frac{g_3}{f_3} N_3'' \left\{ \cot h + \frac{l}{f_3} \tan h \right\} \sin(f_3 t + \beta_3) \\
& - \frac{g_5}{f_5} N_5'' \left\{ \cot h + \frac{l}{f_5} \tan h \right\} \sin(f_5 t + \beta_5) \\
& - \frac{g_6}{f_6} N_6'' \left\{ \cot h + \frac{l}{f_6} \tan h \right\} \sin(f_6 t + \beta_6) \\
& - \frac{g_7}{f_7} N_7'' \left\{ \cot h + \frac{l}{f_7} \tan h \right\} \sin(f_7 t + \beta_7)
\end{aligned}\right\}
\begin{matrix}\text{Precession of the} \\ \text{Equinoxes on the} \\ (552) \\ \text{Apparent Ecliptic.}\end{matrix}$$

$$\left.\begin{aligned}
\varepsilon = h & + \frac{g}{f} N'' \cos(ft + \beta) + \frac{g_1}{f_1} N_1'' \cos(f_1 t + \beta_1) \\
& + \frac{g_2}{f_2} N_2'' \cos(f_2 t + \beta_2) + \frac{g_3}{f_3} N_3'' \cos(f_3 t + \beta_3) \\
& + \frac{g_5}{f_5} N_5'' \cos(f_5 t + \beta_5) + \frac{g_6}{f_6} N_6'' \cos(f_6 t + \beta_6) \\
& + \frac{g_7}{f_7} N_7'' \cos(f_7 t + \beta_7)
\end{aligned}\right\}
\begin{matrix}\text{Apparent obliquity} \\ (553) \\ \text{of the Ecliptic.}\end{matrix}$$

In equations (550) and (552), ζ is to be determined so that ψ and ψ' shall be equal to nothing when $t=0$. We may determine l and h as follows:—

If we take the differential of equation (552), we shall obtain

$$\left.\begin{aligned}
\frac{d\psi'}{dt} = l & - g N'' \left\{ \cot h + \frac{l}{f} \tan h \right\} \cos(ft + \beta) \\
& - g_1 N_1'' \left\{ \cot h + \frac{l}{f_1} \tan h \right\} \cos(f_1 t + \beta_1) \\
& - g_2 N_2'' \left\{ \cot h + \frac{l}{f_2} \tan h \right\} \cos(f_2 t + \beta_2) \\
& - g_3 N_3'' \left\{ \cot h + \frac{l}{f_3} \tan h \right\} \cos(f_3 t + \beta_3) \\
& - g_5 N_5'' \left\{ \cot h + \frac{l}{f_5} \tan h \right\} \cos(f_5 t + \beta_5) \\
& - g_6 N_6'' \left\{ \cot h + \frac{l}{f_6} \tan h \right\} \cos(f_6 t + \beta_6) \\
& - g_7 N_7'' \left\{ \cot h + \frac{l}{f_7} \tan h \right\} \cos(f_7 t + \beta_7)
\end{aligned}\right\} \qquad (554)$$

Now at the epoch of 1850, at which time we have supposed $t=0$, we have $\varepsilon = 23° \; 27' \; 31.''0$; and $\frac{d\psi'}{dt} = 50''.23572$ according to the investigations of BESSEL.

The first members of equations (553) and (554) are therefore known; and if we substitute in them the values of g, g_1, g_2, &c., N'', N_1'', N_2'', &c., β, β_1, β_2, &c., corresponding to the assumed masses, together with f, f_1, f_2, &c., they will become — the numbers in brackets being logarithms,

$$\left.\begin{aligned}23^\circ\ 27'\ 31''.0=h&-\frac{[8.7069903]}{l+g}-\frac{[8.4509982]}{l+g_1}-\frac{[8.6715146]}{l+g_2}\\&+\frac{[9.1494996]}{l+g_3}-\frac{[6.9157253]}{l+g_5}+\frac{[7.5163249]}{l+g_6}\\&+\frac{[8.6222025]}{l+g_7}.\end{aligned}\right\}\quad(555)$$

$$\left.\begin{aligned}50''.23572+0''.05933222\cot h=l+&\left\{\frac{[8.7069903]}{l+g}+\frac{[8.4509982]}{l+g_1}\right.\\&+\frac{[8.6715146]}{l+g_2}-\frac{[9.1494996]}{l+g_3}+\frac{[6.9157253]}{l+g_5}\\&\left.-\frac{[7.5163249]}{l+g_6}-\frac{[8.6222025]}{l+g_7}\right\}l\tan h.\end{aligned}\right\}\quad(556)$$

If we divide equation (556) by $l\tan h$, and add the quotient to equation (555), we shall get

$$\frac{50''.23572+0''.05933222\cot h}{l\tan h}+23^\circ\ 27'\ 31''.0=h+\cot h.\qquad(557)$$

Whence we get

$$l=\frac{50''.23572+0''.05933222\cot h}{1+(h-84451''.0)\tan h}.\qquad(558)$$

The direct determination of h and l from equations (555) and (556) is troublesome, and it is better to solve them by approximation. A few trials will show that $23^\circ\ 17'\ 16''=83836''$ is a near approximation to the value of h. If we substitute $h=23^\circ\ 17'\ 16''$ in equation (558), we shall get $l=50''.4382997$; and if we substitute this value of l in equation (555) we shall find

$$h=23^\circ\ 27'\ 31''.0-10'\ 14''.4265=23^\circ\ 17'\ 16''.5735.$$

Now, substituting this value of h in equation (558), we shall get

$$l=50''.4382387.$$

Having found h and l, we must substitute them in equations (550–553), and we shall obtain the expressions for the numerical values of the precession and obliquity during all past and future ages.

Adding g, g_1, g_2, &c. to l, we shall get f, f_1, f_2, &c., as follows,

$f=45''.312168$,	$f_4=l=50''.438239$,
$f_1=43\ .846111$,	$f_5=49\ .776573$,
$f_2=33\ .044849$,	$f_6=47\ .522157$,
$f_3=32\ .029325$,	$f_7=24\ .503672$.

We also have

$$\begin{array}{llll} \beta = 21^\circ & 6' & 26''.8 & \qquad \beta_4=106^\circ \;\; 14' \;\; 18''.0 \\ \beta_1=132 & 40 & 56.2 & \qquad \beta_5= 20 \;\; 31 \;\; 24.6 \\ \beta_2=292 & 49 & 53.2 & \qquad \beta_6=133 \;\; 56 \;\; 10.8 \\ \beta_3=251 & 45 & 8.6 & \qquad \beta_7=306 \;\; 19 \;\; 21.2 \end{array}$$

Equations (550–553) reduced to numbers become

$$\left.\begin{aligned} \psi_1=50''.438239t+8915''.6+\;& 5800''.35\sin(ft+\beta) \\ -\;& 3581''.52\sin(f_1t+\beta_1) \\ +\;& 5583''.09\sin(f_2t+\beta_2) \\ +\;& 20438''.28\sin(f_3t+\beta_3) \\ +\;& 13294''.37\sin(f_4t+\beta_4) \\ +\;& 646''.98\sin(f_5t+\beta_5) \\ +\;& 834''.76\sin(f_6t+\beta_6) \\ -\;& 3217''.97\sin(f_7t+\beta_7) \end{aligned}\right\} \begin{array}{l}\text{Precession on}\\ \text{fixed Ecliptic}\\ \quad(559)\\ \text{of 1850.}\end{array}$$

$$\left.\begin{aligned} \varepsilon_1=23^\circ\,17'\,16''.57-2445''.33\cos(ft+\beta)&+1499''.78\cos(f_1t+\beta_1) \\ -2189''.55\cos(f_2t+\beta_2)&-7950''.46\cos(f_3t+\beta_3) \\ -5722''.14\cos(f_4t+\beta_4)&-\;277''.79\cos(f_5t+\beta_5) \\ -355''.257\cos(f_6t+\beta_6)&+1158''.01\cos(f_7t+\beta_7) \end{aligned}\right\} \begin{array}{l}\text{Obliquity of}\\ \quad(560)\\ \text{fixed Ecliptic}\\ \text{of 1850.}\end{array}$$

$$\left.\begin{aligned} \psi'=50''.438239t+8915''.6&+696''.462\sin(ft+\beta) \\ &-552''.463\sin(f_1t+\beta_1) \\ &+2250''.29\sin(f_2t+\beta_2) \\ &+8708''.52\sin(f_3t+\beta_3) \\ &+10''.0558\sin(f_5t+\beta_5) \\ &+57''.102\sin(f_6t+\beta_6) \\ &-1910''.92\sin(f_7t+\beta_7) \end{aligned}\right\} \begin{array}{l}\text{Precession on}\\ \quad(561)\\ \text{apparent Ecliptic.}\end{array}$$

$$\left.\begin{aligned} \varepsilon=23\;17'\;16''.57-248''.520\cos(ft+\beta)&+\;196''.017\cos(f_1t+\beta_1) \\ -755''.057\cos(f_2t+\beta_2)&-2901''.753\cos(f_3t+\beta_3) \\ -\;3''.644\cos(f_5t+\beta_5)&-\;20''.539\cos(f_6t+\beta_6) \\ &+\;595''.433\cos(f_7t+\beta_7) \end{aligned}\right\} \begin{array}{l}\text{Obliquity of}\\ \quad(562)\\ \text{apparent}\\ \text{Ecliptic.}\end{array}$$

If we take the differentials of equations (561) and (562), we shall get

$$\left.\begin{aligned} \frac{d\psi'}{dt}=50''.438239+0''.152998\cos(ft+\beta)&-0''.117438\cos(f_1t+\beta_1) \\ +0''.360530\cos(f_2t+\beta_2)&+1''.352281\cos(f_3t+\beta_3) \\ +0''.0024267\cos(f_5t+\beta_5)&+0''.013156\cos(f_6t+\beta_6) \\ &-0''.227011\cos(f_7t+\beta_7) \end{aligned}\right\} (563)$$

$$\left.\begin{aligned} \frac{d\varepsilon}{dt}=+0''.054595\;\sin(gt+\beta)&-0''.041668\;\sin(g_1t+\beta_1) \\ +0''.120965\;\sin(g_2t+\beta_2)&+0''.450592\;\sin(g_3t+\beta_3) \\ +0''.0008794\sin(g_5t+\beta_5)&+0''.004732\;\sin(g_6t+\beta_6) \\ &-0''.0707357\sin(g_7t+\beta_7) \end{aligned}\right\} \qquad (564)$$

If we put $t=0$, in equations (563) and (564), we shall get the values of precession and variation of obliquity at the beginning of 1850. These values are

$$\frac{d\psi'}{dt}=50''.23572, \qquad \frac{d\varepsilon}{dt}=0''.489682.$$

If we take the sum of the coefficients of the *sines* in the expression of ψ' equation (561), without regard to their signs, we shall get the maximum quantity by which the true place of the equinox can differ from its mean place. This sum is $14185''.81=3°\ 56'\ 25''.81$; it therefore follows that, if we suppose the equinox to have a uniform yearly motion equal to $50''.438239$, its place, when computed for any epoch, will not differ from the true place by an amount exceeding $3°\ 56'\ 26''$. This remark is of especial importance in regard to the computation of the elements of terrestrial physics during past geological periods.

If we divide the number of seconds in the circumference of the circle by the mean motion of the equinoxes, we shall get $1296000''\div50''.438239=25694.8=$ the number of years required for the equinoxes to perform a complete revolution in the heavens.

If we take the sum of the coefficients of the *cosines* in the expression of ε equation (562), without regard to their signs, we shall obtain the maximum quantity by which the obliquity of the ecliptic can differ from its mean value. This sum is $4720''.96=1°\ 18'\ 40''.96$. From this it follows that the obliquity of the ecliptic is always confined within the limits $23°\ 17'\ 16''.57\pm1°\ 18'\ 40''.96$; or, between $24°\ 35'\ 57''.53$ and $21°\ 58'\ 35''.61$. The amount of its oscillations cannot therefore exceed $2°\ 37'\ 22''$.

If we take the sum of the coefficients of $\frac{d\psi'}{dt}$, equation (563), we shall get the maximum quantity by which the annual precession can differ from its mean value. This sum is equal to $2''.225841$; whence it follows that the annual precession is always confined within the limits

$$50''.438239\pm2''.225841.$$

The maximum value of precession in a Julian year is therefore equal to $52''.664080$, and the minimum value of precession during the same time is equal to $48''.212398$. If we divide the difference of these two numbers by the time required for the earth to describe one second of an arc, we shall get the maximum variation of the tropical year, equal to

$$4''.451682\div(3548''.1876)\div86400^s)=108^s.40 \text{ seconds of time.}$$

If we subtract the present value of the precession from its maximum value, we shall get $2''.42836$ for the difference between them. Dividing this number by the time required for the earth to describe one second of arc, we shall get the amount of time by which the present tropical year exceeds the tropical year when it is reduced to its minimum length. The number thus found is $59^s.13$. In like manner we shall find that the tropical year may exceed its present length by

$49^s.27$. The present length of the tropical year is $365^d\ 5^h\ 48^m\ 47^s.26$. Whence we get

$$\text{maximum length of tropical year} = 365^d\ 5^h\ 49^m\ 36^s.53;$$
$$\text{and minimum length of tropical year} = 365\ \ 5\ \ 47\ \ 48.13.$$

Lastly, if we take the sum of the coefficients of the *sines* in the formula for $\frac{d\varepsilon}{dt}$, we shall find that the annual variation of the obliquity is always confined within the limits of

$$0'' \text{ and } \pm 0''.744167.$$

CHAPTER V.

TABULAR VALUES OF THE ELEMENTS OF THE PLANETARY ORBITS.

1. In order to complete the subject of the secular inequalities, and render the preceding investigation of greater practical value, we have reduced all the astronomical elements to tables. The tabulated values for the planets embrace a period of 7200 years; commencing 6400 years before 1850, and continuing till A. D. 2650. These are given at intervals of a century during the entire period. The elements of the earth's orbit, together with the precession of the equinoxes in both longitude and right ascension, and the obliquity of the ecliptic to the equator, are given for an interval of 16,000 years; commencing 8000 years before, and ending 8000 years after the year 1850. They are also given at intervals of a century during the first half of that period, and at intervals of four centuries and eight centuries for the remainder of the period. The tabulated values are computed from the constants corresponding to a mass of the *Earth* which is equal to the assumed mass increased by its *tenth* part, or for $m''=\frac{1}{335172}$ of the sun's mass. This value of the *Earth's* mass corresponds to a solar parallax of about 8″.775, which is but little less than the recent determinations of that element; and it is remarkable that this value of the *Earth's* mass is very nearly equal to that which permits the planetary orbits to attain a greater eccentricity than any other mass moving at the same distance. Slight changes in this value of the mass would therefore produce only very inconsiderable changes in the variation of the elements of the planetary orbits.

2. The eccentricities and places of the perihelia have been computed from the values of h, h', h'', &c., l, l', l'', &c., given by equations (C) (page 32), by substituting the values of N, N', N'', &c., N_1, N_1', N_1'', &c., β, β_1, β_2, &c., g, g_1, g_2, &c., given in Art. 31 (pages 64 and 65), by means of the equations

$$\tan \varpi = h \div l, \quad \tan \varpi' = h' \div l', \text{ \&c.}; \quad e = h \div \sin \varpi = l \div \cos \varpi, \quad e' \div h' \div \varpi' = l' \div \cos \varpi', \text{ \&c.}$$

In like manner the nodes and inclinations of the orbits on the fixed ecliptic of 1850, have been computed from the values of q, p, q', p', &c., given by equations (F) page 111, by substituting the values of N, N', N'', &c., N_1, N_1', N_1'', &c., β, β_1, β_2, &c., g, g_1, g_2, &c., given in Art. 17, pages 134, &c., by means of the equations

$$\left.\begin{aligned} \tan \theta = p \div q, \quad \tan \theta' = p' \div q', \text{ \&c.}; \quad \tan \phi &= p \div \sin \theta = q \div \cos \theta, \\ \tan \phi' &= p' \div \sin \theta' = q' \div \cos \theta', \text{ \&c.} \end{aligned}\right\}$$

3. The inclinations of the orbits of the planets and the longitudes of their ascending nodes on the ecliptic, or variable orbit of the earth, are denoted by ϕ_1, ϕ_1', ϕ_1'', &c., θ_1, θ_1', θ_1'', &c., and have been computed by the equations

$$\tan\theta_1=(p-p'')\div(q-q''); \quad \tan\theta_1'=(p'-p'')\div(q'-q''); \quad \&c.;$$

$$\left.\begin{array}{l}\tan\phi_1=(p-p'')\div\sin\theta_1=(q-q'')\div\cos\theta_1;\\ \tan\phi_1'=(p'-p'')\div\sin\theta_1'=(q'-q'')\div\cos\theta_1',\ \&c.\end{array}\right\}$$

The longitudes of the perihelia and nodes are in every case counted from the mean equinox of 1850.0.

4. The precession of the equinoxes on the fixed ecliptic of 1850, and the inclination of the equator to the same plane, are denoted by ψ and ε_1; and they are determined by equations (550) and (551). Reducing them to numbers, and transforming so as to dispense with the arbitrary constant quantities, they will become

$$\left.\begin{array}{ll}\psi=lt+[4.0317043]\sin\tfrac{1}{2}ft\cos\tfrac{1}{2}ft & +[3.6828039]\sin\tfrac{1}{2}f_1t\cos\tfrac{1}{2}f_1t\\ \quad-[3.5715890]\sin^2\tfrac{1}{2}ft & +[3.8013126]\sin^2\tfrac{1}{2}f_1t\\ & +[3.5080433]\sin\tfrac{1}{2}f_2t\cos\tfrac{1}{2}f_2t\\ & +[3.7464212]\sin^2\tfrac{1}{2}f_2t\\ \quad-[4.0645663]\sin\tfrac{1}{2}f_3t\cos\tfrac{1}{2}f_3t & -[3.8713230]\sin\tfrac{1}{2}f_4t\cos\tfrac{1}{2}f_4t\\ \quad+[4.6281028]\sin^2\tfrac{1}{2}f_3t & -[4.4070533]\sin^2\tfrac{1}{2}f_4t\\ & +[3.0831687]\sin\tfrac{1}{2}f_5t\cos\tfrac{1}{2}f_5t\\ & -[2.6561496]\sin^2\tfrac{1}{2}f_5t\\ \quad-[3.0600136]\sin\tfrac{1}{2}f_6t\cos\tfrac{1}{2}f_6t & -[3.5903084]\sin\tfrac{1}{2}f_7t\cos\tfrac{1}{2}f_7t\\ \quad-[3.0760078]\sin^2\tfrac{1}{2}f_6t & -[3.7238538]\sin^2\tfrac{1}{2}f_7t\end{array}\right\}\quad(565)$$

$$\left.\begin{array}{rl}\varepsilon_1=23^\circ\ 27'\ 31''.00 & +[3.1962358]\sin\tfrac{1}{2}ft\cos\tfrac{1}{2}ft\\ & +[3.6563511]\sin^2\tfrac{1}{2}ft\\ & -[3.4230485]\sin\tfrac{1}{2}f_1t\cos\tfrac{1}{2}f_1t\\ & +[3.3045398]\sin^2\tfrac{1}{2}f_1t\\ & -[3.3390526]\sin\tfrac{1}{2}f_2t\cos\tfrac{1}{2}f_2t\\ & +[3.1006747]\sin^2\tfrac{1}{2}f_2t\\ & -[4.2160834]\sin\tfrac{1}{2}f_3t\cos\tfrac{1}{2}f_3t\\ & -[3.6525469]\sin^2\tfrac{1}{2}f_3t\\ & +[4.0408748]\sin\tfrac{1}{2}f_4t\cos\tfrac{1}{2}f_4t\\ & -[3.5051445]\sin^2\tfrac{1}{2}f_4t\\ & +[2.2889032]\sin\tfrac{1}{2}f_5t\cos\tfrac{1}{2}f_5t\\ & +[2.7159223]\sin^2\tfrac{1}{2}f_5t\\ & +[2.7049217]\sin\tfrac{1}{2}f_6t\cos\tfrac{1}{2}f_6t\\ & -[2.6889275]\sin^2\tfrac{1}{2}f_6t\\ & +[3.2799396]\sin\tfrac{1}{2}f_7t\cos\tfrac{1}{2}f_7t\\ & -[3.1463942]\sin^2\tfrac{1}{2}f_7t\end{array}\right\}\quad(566)$$

The coefficients in these equations are logarithms of seconds of arc; and f, f_1, f_2, &c. have the following values:—

$f=45''.225870$	$l=f_4=50''.439525$
$f_1=43\ .770258$	$f_5=49\ .777863$
$f_2=32\ .812939$	$f_6=47\ .523448$
$f_3=31\ .503029$	$f_7=24\ .504515$

In all cases t denotes Julian years of $365\frac{1}{4}$ days.

5. The general precession of the equinoxes in longitude is very nearly the same as the precession on the apparent ecliptic, which is denoted by ψ', and is given by equation (552). But as the apparent ecliptic is continually shifting its position in space, the motion of precession on such an assumed plane becomes the same as it would be along a warped surface, and very imperfectly represents the general precession at times only a few hundred years from the epoch, although its maximum deviation from the truth can never exceed one-fourth of a degree. But on account of the importance of the subject we shall determine in a rigorous manner the general precession of the equinoxes in both longitude and right ascension. For this purpose we shall consider the spherical triangle formed by the fixed ecliptic of 1850, and the apparent ecliptic and equator of any time t. In this triangle there are known the two angles and the included side; namely, the angle of inclination of the apparent ecliptic to the fixed ecliptic of 1850, which is denoted by ϕ'', and the inclination of the equator to the same plane, which is denoted by ε_1, and the included side which is equal to $\psi+\theta''$. The three remaining parts of the triangle are the distances from the extremities of the known side of the triangle to the point of intersection of the apparent ecliptic and equator, and the angle included by these sides. We shall denote these quantities by $\psi'+\theta''$, ϑ, and ε. ψ' denotes the general precession in longitude, ϑ denotes the planetary precession, which is the distance between the fixed and apparent ecliptics measured on the apparent equator, and ε denotes the apparent obliquity of the ecliptic. The fundamental equations of spherical trigonometry will therefore furnish the following formulæ for the determination of ψ', ϑ, and ε:—

$$\left.\begin{aligned}\sin\varepsilon\sin\vartheta&=-\sin\phi''\sin(\psi+\theta'')\\ \sin\varepsilon\cos\vartheta&=\sin\phi''\cos\varepsilon_1\cos(\psi+\theta'')+\cos\phi''\sin\varepsilon_1\end{aligned}\right\}\quad(567)$$

$$\left.\begin{aligned}\sin\varepsilon\sin(\psi'+\theta'')&=\sin\varepsilon_1\sin(\psi+\theta'')\\ \sin\varepsilon\cos(\psi'+\theta'')&=\sin\varepsilon_1\cos\phi''\cos(\psi+\theta'')+\cos\varepsilon_1\sin\phi''\end{aligned}\right\}\quad(568)$$

The negative sign is given to the first equation because a forward motion of the equinox is a diminution of precession.

The equations (567) give the values of ϑ and ε; and either one of the equations (568) will give the value ψ' when ε has been determined. Since ϑ is always a very small arc, it is determinable with all desirable precision by means of its tangent; but this is not the case with ε. This quantity cannot be determined with extreme precision by means of its *sine*, without using logarithms to more than *seven places* of decimals. But as ε never differs greatly from ε_1, we may compute the difference between ε and ε_1 by eliminating ϑ from equations (567), and we shall find,

$$\sin(\varepsilon-\varepsilon_1)=\frac{\sin^2\phi''\cos^2\varepsilon_1-\sin^2\phi''\sin^2\varepsilon_1\cos^2(\psi+\theta'')+2\cos\phi''\sin\phi''\cos\varepsilon_1\sin\varepsilon_1\cos(\psi+\theta'')}{\sin(\varepsilon+\varepsilon_1)}. \quad (569)$$

Although ε appears in the denominator of this equation it is readily determinable from its *sine* with sufficient precision to be used in finding $\sin(\varepsilon-\varepsilon_1)$ with accuracy.

Since ψ' never differs greatly from ψ, we may readily transform equations (568) so as to give the difference of these quantities, and we shall find

$$\left.\begin{aligned}\sin\varepsilon\sin(\psi-\psi')=\cos\varepsilon_1\sin\phi''\sin(\psi+\theta'')&\\ -2\sin^2\tfrac{1}{2}\phi''\sin\varepsilon_1\sin(\psi+\theta'')\cos(\psi+\theta'')&\end{aligned}\right\} \quad (570)$$

This equation determines $\psi-\psi'$ with all desirable precision, ε having been previously determined.

6. We shall now consider the spherical triangle formed by the fixed ecliptic of 1850, the fixed equator of 1850, and the apparent equator of any time t. Since the inclination of the equator to the fixed ecliptic of 1850 is given by equation (566), we may suppose this quantity to be known for the given time. Then calling ε_1 the inclination of the fixed equator and ecliptic ε_1', the inclination of the apparent equator to the fixed ecliptic, and ψ the total luni-solar precession during the time t; the two angles and the included side of the proposed triangle will be known, and the three remaining parts may readily be determined in the following manner: Let the distances from the intersection of the fixed and apparent equators to the fixed ecliptic be denoted by $90°-z$ and $90°+z'$, and the angle of intersection of the two equators be denoted by Θ, we shall have the following equations for determining these last named quantities:—

$$\left.\begin{aligned}\sin\Theta\cos z&=\sin\psi\sin\varepsilon_1'\\ \sin\Theta\sin z&=\sin\varepsilon_1\cos\varepsilon_1'-\cos\psi\sin\varepsilon_1'\cos\varepsilon_1\\ \cos\Theta&=\cos\varepsilon_1\cos\varepsilon_1'+\sin\varepsilon_1\sin\varepsilon_1'\cos\psi\\ \sin\Theta\cos z'&=\sin\psi\sin\varepsilon_1\\ \sin\Theta\sin z'&=\cos\psi\sin\varepsilon_1\cos\varepsilon_1'-\cos\varepsilon_1\sin\varepsilon_1'\end{aligned}\right\} \quad (571)$$

These equations determine z, z', and Θ rigorously; but since ε_1 is nearly equal to ε_1', they are under a very inconvenient form for accurate computation when ψ is a small angle. They may, however, be readily put under the following form for very accurate and convenient computation:—

$$\left.\begin{aligned}\sin\Theta\cos z&=\sin\psi\sin\varepsilon_1'\\ \sin\Theta\sin z&=2\sin^2\tfrac{1}{2}\psi\sin\varepsilon_1'\cos\varepsilon_1+\sin(\varepsilon_1-\varepsilon_1')\\ \sin^2\tfrac{1}{2}\Theta&=\sin^2\tfrac{1}{2}(\varepsilon_1-\varepsilon_1')+\sin^2\tfrac{1}{2}\psi\sin\varepsilon_1\sin\varepsilon_1'\\ \sin\Theta\cos z'&=\sin\psi\sin\varepsilon_1\\ \sin\Theta\sin z'&=2\sin^2\tfrac{1}{2}\psi\sin\varepsilon_1\cos\varepsilon_1'-\sin(\varepsilon_1-\varepsilon_1')\end{aligned}\right\} \quad (572)$$

The sum of the quantities z and z' might very properly be called the luni-solar precession in right ascension; and if to this we add the planetary precession we shall have the general precession in right ascension at any time t, equal to $z+z'+\vartheta$; all of which quantities being taken from the table with the argument t.

7. Tables I—VIII have been computed as explained in §§ 2 and 3. They show the elements of the planetary orbits at the times given in the first column of each table, and seem to require no explanation as to their uses. ψ, and ε_1, in Tables IX and X, have been computed from the formulæ (565) and (566); ϑ, ψ' and ε have been computed from equations (567), (570), and (569), by using the values of θ'', ϕ'', ψ, and ε_1 given in Tables VIII and IX; and lastly, z, z', and Θ have been computed by means of equations (572), by using the values of ψ, ε, ε_1' given in Table IX.

8. Having explained the method of constructing the tables, we will now explain the manner of using Tables IX and X in connection with θ'' and ϕ'', which are given in Table VIII. The quantities θ'', ϕ'', and ψ' are useful in reducing the longitudes of the celestial bodies from the mean equinox of 1850 to the mean equinox of any other date, and *vice versa.* This transformation is effected by means of the following equations, in which λ and β denote the mean longitude and latitude of a celestial body at the epoch of 1850, and λ' and β' denote the same co-ordinates referred to the mean equinox of any time t, before or after that epoch.

$$\left.\begin{aligned}\cos\beta'\cos(\lambda'-\theta''-\psi')&=\cos\beta\cos(\lambda-\theta'')\\ \cos\beta'\sin(\lambda'-\theta''-\psi')&=\cos\beta\cos\phi''\sin(\lambda-\theta'')+\sin\beta\sin\phi''\\ \sin\beta'&=\sin\beta\cos\phi''-\cos\beta\sin\phi''\sin(\lambda-\theta'')\end{aligned}\right\}\quad(573)$$

For reducing *to* the equinox of 1850 these equations take the following form:—

$$\left.\begin{aligned}\cos\beta\cos(\lambda-\theta'')&=\cos\beta'\cos(\lambda'-\theta''-\psi')\\ \cos\beta\sin(\lambda-\theta'')&=\cos\beta'\cos\phi''\sin(\lambda'-\theta''-\psi')-\sin\phi''\sin\beta'\\ \sin\beta&=\sin\beta'\cos\phi''+\cos\beta'\sin\phi''\sin(\lambda'-\theta''-\psi')\end{aligned}\right\}\quad(574)$$

It is sometimes desirable to find the difference in the longitude or latitude of a star arising from the precession of the equinoxes. This difference may be found by the following formulæ, the employment of which is perhaps more laborious than that of the preceding from which they were derived; but they may in ordinary cases be managed by the use of *five-figure* logarithms, whereas equations (573) and (574) require *seven-figure* logarithms to arrive at accurate results.

$$\lambda'=\lambda+\psi'+\Bigg[\text{arc tan}=\left\{\frac{\tan\beta\sin\phi''\cos(\lambda-\theta'')-2\sin^2\tfrac{1}{2}\phi''\cos(\lambda-\theta'')\sin(\lambda-\theta'')}{1+\tan\beta\sin\phi''\sin(\lambda-\theta'')-2\sin^2\tfrac{1}{2}\phi''\sin^2(\lambda-\theta'')}\right\}\Bigg];\quad(575)$$

$$\beta'=\beta-2\,\text{arc sin}\left\{\frac{\cos\beta\sin\phi''\sin(\lambda-\theta'')+2\sin\beta\sin^2\tfrac{1}{2}\phi''}{2\cos\tfrac{1}{2}(\beta'+\beta)}\right\}$$

$$\lambda=\lambda'-\psi'-\Bigg[\text{arc tan}=\left\{\frac{\tan\beta'\sin\phi''\cos(\lambda'-\theta''-\psi')+2\sin^2\tfrac{1}{2}\phi''\cos(\lambda'-\theta''-\psi')\sin(\lambda'-\theta''-\psi')}{1-\tan\beta'\sin\phi''\sin(\lambda'-\theta''-\psi')-2\sin^2\tfrac{1}{2}\phi''\sin^2(\lambda'-\theta''-\psi')}\right\}\Bigg]\quad(576)$$

$$\beta=\beta'+2\,\text{arc sin}\left\{\frac{\cos\beta'\sin\phi''\sin(\lambda'-\theta''-\psi')-2\sin\beta'\sin^2\tfrac{1}{2}\phi''}{2\cos\tfrac{1}{2}(\beta'+\beta)}\right\}\quad(577)$$

If, in these equations, we neglect quantities of the order $\sin^2 \phi''$, they assume a very convenient form for computation, and at the same time possess sufficient accuracy for computations extending over a period of several hundred years, except for stars situated very near the pole, in which cases some of the preceding equations must be employed if we wish to obtain accurate results.

$$\left.\begin{aligned}\lambda' &= \lambda + \psi' + \phi'' \tan\beta \cos(\lambda - \theta'') \\ \beta' &= \beta - \phi'' \sin(\lambda - \theta'');\end{aligned}\right\} \quad (578)$$

$$\left.\begin{aligned}\lambda &= \lambda' - \psi' - \phi'' \tan\beta' \cos(\lambda' - \theta'' - \psi') \\ \beta &= \beta' + \phi'' \sin(\lambda' - \theta'' - \psi').\end{aligned}\right\} \quad (579)$$

In these equations ϕ'' is to be expressed in seconds of arc.

9. For the reduction of right ascensions and declinations all the necessary data depending on the motion of the equinox are contained in Table X. The quantities in the table are adapted to computation by the following formulæ, in which α and δ denote the mean right ascension and declination of a star at the epoch of 1850; and α' and δ' denote the same co-ordinates referred to the mean equinox of any time t before or after the epoch.

$$\left.\begin{aligned}\cos\delta' \sin(\alpha' - z' - \vartheta') &= \cos\delta \sin(\alpha + z - \vartheta) \\ \cos\delta' \cos(\alpha' - z' - \vartheta') &= \cos\delta \cos\Theta \cos(\alpha + z - \vartheta) - \sin\delta \sin\Theta \\ \sin\delta' &= \cos\delta \sin\Theta \cos(\alpha + z - \vartheta) + \sin\delta \cos\Theta\end{aligned}\right\} \quad (580)$$

For reducing *to* the mean equinox of 1850 these equations take the following form:—

$$\left.\begin{aligned}\cos\delta \sin(\alpha + z - \vartheta) &= \cos\delta' \sin(\alpha' - z' - \vartheta') \\ \cos\delta \cos(\alpha + z - \vartheta) &= \cos\delta' \cos\Theta \cos(\alpha' - z' - \vartheta') + \sin\Theta \sin\delta' \\ -\sin\delta &= \cos\delta' \sin\Theta \cos(\alpha' - z' - \vartheta') - \cos\Theta \sin\delta'\end{aligned}\right\} \quad (581)$$

The first two of equations (580) will very readily give

$$\alpha' = \alpha + (z + z' + \vartheta' - \vartheta) + \left[\text{arc tan} = \left\{ \frac{\tan\delta \sin\Theta \sin(\alpha + z - \vartheta) + 2\sin^2\tfrac{1}{2}\Theta \sin(\alpha + z - \vartheta)\cos(\alpha + z - \vartheta)}{1 - \tan\delta \sin\Theta \cos(\alpha + z - \vartheta) - 2\sin^2\tfrac{1}{2}\Theta \cos^2(\alpha + z - \vartheta)} \right\}\right] \quad (582)$$

Here the term $z + z' + \vartheta' - \vartheta$ appears as the general precession in right ascension common to all the stars, and the last term of the equation gives the correction depending on the place of each particular star.

In like manner the first two of equations (581) will give

$$\alpha = \alpha' - (z + z' + \vartheta' - \vartheta) - \left[\text{arc tan} = \left\{ \frac{\tan\delta' \sin\Theta \sin(\alpha' - z' - \vartheta') - 2\sin^2\tfrac{1}{2}\Theta \sin(\alpha' - z' - \vartheta')\cos(\alpha' - z' - \vartheta')}{1 + \tan\delta' \sin\Theta \cos(\alpha' - z' - \vartheta') - 2\sin^2\tfrac{1}{2}\Theta \cos^2(\alpha' - z' - \vartheta')} \right\}\right] \quad (583)$$

For stars situated near the pole, equations (580) and (581) are preferable to (582) and (583), because when δ or δ' is equal to 90° the terms depending on tan δ or tan δ' become infinite, and the equations (582) and (583) assume an indeterminate form. But this is not the case with equations (580) and (581); for if $\delta = 90°$ equations (580) will give $\sin\delta' = \cos\Theta$, and then we shall find $\cos(\alpha - z' - \vartheta') = -\sin\Theta \div \cos\delta' = -1$, whence $\alpha' - z' - \vartheta' = 180°$, from which α' is easily determined.

Rigorous expressions for the difference of the declinations depending on the precession are not readily deducible from equations (580) and (581) of sufficient simplicity to possess any advantage in computation over the original formulæ, since Θ is not necessarily a small angle.

10. Having given the necessary formulæ for reducing the position of a star which is given in longitude and latitude, or right ascension and declination, with reference to the equinox of 1850, to that of any other date, and the reverse, we shall now give some examples by way of illustration.

Example I. The mean place of polaris, at the beginning of the year 1850, was, $\alpha=16°\ 15'\ 22''.815$, $\delta=88°\ 30'\ 34''.889$; it is required to find its right ascension and declination at the beginning of 1950, 2050, and 2150, and also the maximum declination of the star.

In 1850 the planetary precession was equal to nothing, therefore ϑ disappears from the second member of equations (580). For $t=+100$, Table X, gives $z=0°\ 38'\ 23''.482$, $z'=0°\ 38'\ 36''.527$, $\Theta=0°\ 33'\ 24''.811$, and $\vartheta=-0°\ 0'\ 12''.433$. Whence $\alpha+z=16°\ 53'\ 46''.297$; and the computation is as follows, using the first and second of equations (580):—

$\alpha+z$	sin 9.4633534
δ	cos 8.4151046
$\alpha+z$	cos 9.9808361
Θ	cos 9.9999795
Θ	sin 7.9876415
δ	—sin 9.9998531n
$\cos\delta\cos(\alpha+z)\cos\Theta=+0.02488400$	log. 8.3959202
$-\sin\delta\sin\Theta\quad=-0.00971616$	" 7.9874946n
$\cos\delta'\cos(\alpha'-z'-\vartheta')=+0.01516784$	" 8.1809237
$\cos\delta'\sin(\alpha'-z'-\vartheta')=\cos\delta\sin(\alpha+z)$	" 7.8784580
$\alpha'-z'-\vartheta'=26°\ 29'\ 21''.70$	tan 9.6975343
$z'+\vartheta'=\quad 38\ \ 24.10$	
$\alpha'=27°\ \ 7'\ 45''.80$	
$\alpha'-z'-\vartheta'$	cos 9,9518314
$\cos\delta'\cos(\alpha'-z'-\vartheta')$	log. 8.1809237
$\delta'=89°\ \ 1'\ 44''.277$	cos 8.2290923

Computing in the same way for $t=+200$ and $t=+300$, we shall find the values of α' and δ' as in the following table:—

	α'	δ'
1950	27° 7′ 45″.80	89° 1′ 44″.277
2050	56 51 57.27	89 27 20.654
2150	120 32 36.31	89 28 13.594

The declination evidently attains a maximum value some time between 2050 and 2150. If we compute its place for $t=250$, we shall find $z=1°\ 35'\ 59''.93$, $z'=1°\ 36'\ 26''.63$, $\Theta=1°\ 23'\ 29''.80$, and $\vartheta'=-21''.80$.

Then we find $\alpha'=88° 13' 39''.20$ and $\delta'=89° 32' 32''.202$.

Now having the declination for $t=200$, $t=250$, and $t=300$, we very readily find that the maximum will take place when $t=+252.335$, or a little before the middle of the year 2102. Computing the place of the star for this last value of t, we find

$$\alpha'=89° 53' 2''.78 \text{ and } \delta'=89° 32' 32''.973.$$

The nearest approach of the pole to the star is therefore equal to $27' 27''.027$.

Example II. Let it be required to find the declination of Polaris when $t=-8000$. For the value of t we get $z=-53° 0' 2''.8$, $z'=-54° 34' 37''.2$, $\Theta=-39° 24' 51''.5$, and $\vartheta'=+2° 43' 48''.3$. $\alpha+z=-36° 44' 40''.0$.

Therefore the computation will stand as follows:—

$\alpha+z$	sin 9.7768805n
δ	cos 8.4151046
$\alpha+z$	cos 9.9038016
Θ	cos 9.8879407
Θ	sin 9.8027213n
δ	—sin 9.9998531n
$\cos\delta\cos(\alpha+z)\cos\Theta=+0.0161008$	log. 8.2068469
$-\sin\delta\sin\Theta=+0.6347087$	" 9.8025744
$\cos\delta'\cos(\alpha'-z'-\vartheta')=+0.6508095$	" 9.8134539
$\cos\delta'\sin(\alpha'-z'-\vartheta')$	" 8.1919851n
$\alpha'-z'-\vartheta'=358° 37' 49''.7$	tan 8.3785312n
$z'+\vartheta'=-51\ 50\ 48.9$	
$\alpha'=306° 47' 0''.8$	
$\alpha'-z'-\vartheta'$	cos 9.9998759
$\cos\delta'\cos(\alpha'-z'-\vartheta')$	log. 9.8134539
$\delta'=49° 23' 0$	cos 9.8135780

From this calculation it follows that the present polar star was, 8000 years ago, distant 40° 37′ from the pole.

Example III. In 1850 the place of the star α Cephei was

$$\alpha=318° 45',\ \delta=+61° 56';$$

required its mean place when $t=+5600$ years.

In this example we find

$$z=+36° 55'\ z'+\vartheta'=+37° 35',\ \Theta=28° 44'.$$

Then we get $\alpha'-z'-\vartheta'=249° 44'$, whence $\alpha'=287° 19'$ and $\delta'=+87° 50'$.

It therefore follows that the star α Cephei will be only about 2° distant from the pole in 5600 years; it will therefore constitute the pole star of that period.

The equations for reducing from longitude and latitude to right ascension and declination, and the reverse, are the following:—

$$\left.\begin{aligned}\cos\delta\cos\alpha &= \cos\beta\cos\lambda\\ \cos\delta\sin\alpha &= \cos\varepsilon\cos\beta\sin\lambda - \sin\varepsilon\sin\beta\\ \sin\delta &= \sin\varepsilon\cos\beta\sin\lambda + \cos\varepsilon\sin\beta\end{aligned}\right\} \quad (584)$$

$$\left.\begin{aligned}\cos\beta\cos\lambda &= \cos\delta\cos\alpha\\ \cos\beta\sin\lambda &= \cos\varepsilon\cos\delta\sin\alpha + \sin\varepsilon\sin\delta\\ \sin\beta &= -\sin\varepsilon\cos\delta\sin\alpha + \cos\varepsilon\sin\delta\end{aligned}\right\} \quad (585)$$

Example IV. The right ascension and declination of α Tauri (*Aldebaran*) in 1850 was $\alpha = 66° \, 49' \, 46''.35$, $\delta = +16° \, 12' \, 11''.0$; required its longitude and latitude for $t = -4900$, or, for the beginning of the year B. C. 3050.

Since $\varepsilon = 23° \, 27' \, 31''.0$, in 1850, equations (585) will give the longitude and latitude for the same epoch, as follows:—

$$\lambda = 67° \, 41'' \, 34'.1, \qquad \beta = -5° \, 28' \, 40''.1.$$

And for $t = -4900$. Tables VIII and IX will give

$$\theta'' = 5° \, 22' \, 51''.7, \ \phi'' = 0° \, 41' \, 22''.45, \text{ and } \psi' = -67° \, 40' \, 32''.2.$$

If these quantities be substituted in equations (573), we shall find

$$\lambda' - \theta'' - \psi' = 62° \, 16' \, 45''.5, \ \sin\beta' = -0.106061;$$

whence we get

$$\lambda' = 359° \, 59' \, 5''.0, \text{ and } \beta' = -6° \, 5' \, 17''.9.$$

Therefore the star *Aldebaran*, at the beginning of the year B. C. 3050, was only 55''.0 westward of the vernal equinox, measured on the ecliptic of that date, and coincided with the equinox in the year 3049 B. C.

Example V. The right ascension and declination of *Aldebaran* being, as in the preceding example, at the beginning of 1850, required its right ascension and declination at the beginning of the year B. C. 3050. For $t = -4900$, Table X gives $z = -31° \, 43' \, 26''.4$, $z' = -32° \, 44' \, 31''.8$, $\Theta = -26° \, 14' \, 1''.4$, and $\vartheta' = +1° \, 31' \, 0''0$. If these quantities be substituted in equations (580), we shall find

$$\alpha' - z' - \vartheta' = 33° \, 42' \, 2''.8, \text{ and } \sin.\delta' = -0.0969604.$$

Whence we get

$$\alpha' = 2° \, 28' \, 31''.0, \text{ and } \delta' = -5° \, 33'' \, 51'.0.$$

We might have found these last quantities by means of equations (584) by substituting for λ, β, the values of λ' and β', found in example IV, and using the value of ε corresponding to that epoch, which value is $\varepsilon = 24° \, 3' \, 8''.2$.

From this computation we see that, although the star *Aldebaran* was 55'' *westward* of the equinox when measured on the ecliptic, it was nearly $2\frac{1}{2}$ degrees *eastward* of the equinox when measured on the equator, and instead of being in a northern constellation then, as now, it was in reality in a southern constellation.

11. We will now determine the position of the pole of the equator. The longitude of the pole on the fixed ecliptic of 1850 at any time t will evidently be equal

to $90°-\psi$; and the latitude of the pole will also be equal to $90°-\varepsilon_1'$, or to the complement of the obliquity of the ecliptic of 1850, at the given time. If we then put $\lambda=90°-\psi$, $\beta=90°-\varepsilon_1'$, in equations (584), the resulting α and δ will evidently be the right ascension and declination of the pole of the equator for the time t, referred to the equinox and equator of 1850. Calling the right ascension of the pole A, and the declination D, we shall evidently have

$$\left.\begin{aligned}\cos D\cos A&=\sin\varepsilon_1'\sin\psi\\ \cos D\sin A&=-\sin\varepsilon_1\cos\varepsilon_1'+\cos\varepsilon_1\sin\varepsilon_1'\cos\psi\\ \sin D&=\cos\varepsilon_1\cos\varepsilon_1'+\sin\varepsilon_1\sin\varepsilon_1'\cos\psi\end{aligned}\right\}\quad(586)$$

ε_1 denoting the obliquity in 1850, and ε_1' denoting the obliquity of the fixed ecliptic of 1850 *at the time* for which the computation is made. If we compare these equations with equations (571), we find $\sin A=-\sin z$, and $\sin D=\cos\Theta$.

Therefore $A=-z$, and $D=90°-\Theta$.

Now since z and Θ are already tabulated we can enter Table X with the argument t, and take out the right ascension and declination of the pole by mere inspection.

Example VI. What will be the right ascension and declination of pole 5600 years hence, when referred to the equinox and equator of 1850? Entering Table X with the argument $t=+5600$, we find $z=36°\ 55'\ 6''.4$, and $\Theta=28°\ 44'\ 0''.89$. Therefore $A=323°\ 4'\ 53''.6$, and $D=61°\ 15'\ 59''.11$. The mean place of α Cephei in 1850 was $\alpha=318°\ 45'$, and $\delta=61°\ 56'$. This star will therefore be the pole-star of that period.

TABLE I.—*Elements of the Orbit of Mercury.*

t	e	ϖ	θ	ϕ	θ_1	ϕ_1
—6400	0.205301	65° 13′ 40″	54° 0′ 37″.5	7° 13′ 4″.81	59° 31′ 23″.8	6° 36′ 57″.1
6300	0.205308	65 22 55	53 53 35.6	7 12 52.31	59 19 30.0	6 37 13.0
6200	0.205314	65 32 11	53 46 33.8	7 12 39.81	59 7 35.5	6 37 48.9
6100	0.205321	65 41 26	53 39 32.1	7 12 27.32	58 55 40.3	6 38 14.6
6000	0.205328	65 50 42	53 32 30.4	7 12 14.84	58 43 44.5	6 38 40.1
5900	0.205335	65 59 57	53 25 28.9	7 12 2.37	58 31 48.0	6 39 5.6
5800	0.205341	66 9 13	53 18 27.4	7 11 49.90	58 19 51.1	6 39 30.9
5700	0.205348	66 18 28	53 11 26.3	7 11 37.44	58 7 53.5	6 39 56.1
5600	0.205354	66 27 44	53 4 25.2	7 11 24.99	57 55 55.2	6 40 21.1
5500	0.205361	66 37 0	52 57 24.7	7 11 12.54	57 43 56.4	6 40 46.1
5400	0.205367	66 46 16	52 50 24.1	7 11 0.10	57 31 57.0	6 41 10.9
5300	0.205373	66 55 32	52 43 23.5	7 10 47.67	57 19 57.0	6 41 35.5
5200	0.205379	67 4 48	52 36 22.9	7 10 35.24	57 7 56.3	6 42 0.0
5100	0.205385	67 14 4	52 29 22.3	7 10 22.83	56 55 55.2	6 42 24.4
5000	0.205391	67 23 20	52 22 21.7	7 10 10.42	56 43 53.5	6 42 48.7
4900	0.205397	67 32 36	52 15 21.1	7 9 58.02	56 31 51.2	6 43 12.8
4800	0.205402	67 41 52	52 8 20.5	7 9 45.64	56 19 48.4	6 43 36.8
4700	0.205408	67 51 8	52 1 20.1	7 9 33.27	56 7 45.1	6 44 0.7
4600	0.205414	68 0 24	51 54 19.8	7 9 20.92	55 55 41.2	6 44 24.4
4500	0.205420	68 9 39	51 47 19.4	7 9 8.58	55 43 36.8	6 44 48.0
4400	0.205425	68 18 55	51 40 19.1	7 8 56.25	55 31 31.9	6 45 11.4
4300	0.205431	68 28 10	51 33 18.9	7 8 43.93	55 19 26.5	6 45 34.8
4200	0.205436	68 37 26	51 26 18.7	7 8 31.62	55 7 20.5	6 45 58.0
4100	0.205442	68 46 41	51 19 18.8	7 8 19.32	54 55 14.0	6 46 21.0
4000	0.205447	68 55 57	51 12 18.9	7 8 7.03	54 43 7.1	6 46 43.9
3900	0.205453	69 5 14	51 5 19.2	7 7 54.75	54 31 0.7	6 47 6.7
3800	0.205458	69 14 30	50 58 19.6	7 7 42.49	54 18 51.8	6 47 29.4
3700	0.205463	69 23 46	50 51 20.1	7 7 30.24	54 6 43.4	6 47 51.9
3600	0.205468	69 33 3	50 44 20.6	7 7 18.01	53 54 34.6	6 48 14.3
3500	0.205473	69 42 19	50 37 21.3	7 7 5.79	53 42 25.3	6 48 36.6
3400	0.205478	69 51 35	50 30 22.0	7 6 53.57	53 30 15.6	6 48 58.7
3300	0.205483	70 0 51	50 23 22.7	7 6 41.37	53 18 5.4	6 49 20.6
3200	0.205487	70 10 7	50 16 23.5	7 6 29.19	53 5 54.8	6 49 42.5
3100	0.205492	70 19 23	50 9 24.3	7 6 17.02	52 53 43.8	6 50 4.2
3000	0.205497	70 28 40	50 2 25.2	7 6 4.87	52 41 32.4	6 50 25.8
2900	0.205501	70 37 56	49 55 26.1	7 5 52.74	52 29 20.5	6 50 47.2
2800	0.205506	70 47 12	49 48 27.1	7 5 40.61	52 17 8.3	6 51 8.5
2700	0.205510	70 56 28	49 41 28.1	7 5 28.49	52 4 55.7	6 51 29.6
2600	0.205515	71 5 44	49 34 29.2	7 5 16.40	51 52 42.7	6 51 50.6
2500	0.205520	71 15 0	49 27 30.3	7 5 4.32	51 40 29.3	6 52 11.5
2400	0.205524	71 24 16	49 20 31.4	7 4 52.25	51 28 15.5	6 52 32.2
2300	0.205529	71 33 33	49 13 32.5	7 4 40.20	51 16 1.4	6 52 52.8
2200	0.205533	71 42 50	49 6 33.7	7 4 28.17	51 3 46.8	6 53 13.3
2100	0.205538	71 52 7	48 59 34.9	7 4 16.17	50 51 31.9	6 53 33.6
2000	0.205542	72 1 24	48 52 36.1	7 4 4.17	50 39 16.5	6 53 53.8
1900	0.205547	72 10 40	48 45 37.4	7 3 52.19	50 27 0.9	6 54 13.8
1800	0.205551	72 19 57	48 38 38.7	7 3 40.23	50 14 44.9	6 54 33.8
1700	0.205555	72 29 14	48 31 40.0	7 3 28.28	50 2 28.5	6 54 53.5
1600	0.205559	72 38 31	48 24 41.4	7 3 16.35	49 50 11.8	6 55 13.2
1500	0.205563	72 47 47	48 17 42.7	7 3 4.44	49 37 54.8	6 55 32.6
1400	0.205567	72 57 4	48 10 44.1	7 2 52.56	49 25 37.5	6 55 52.0
1300	0.205571	73 6 21	48 3 45.5	7 2 40.69	49 13 19.8	6 56 11.2
1200	0.205575	73 15 38	47 56 46.9	7 2 28.84	49 1 1.9	6 56 30.3
1100	0.205579	73 24 55	47 49 48.3	7 2 17.00	48 48 43.7	6 56 49.2
1000	0.205583	73 34 12	47 42 49.7	7 2 5.19	48 36 25.0	6 57 8.0
900	0.205586	73 43 29	47 35 51.1	7 1 53.40	48 24 6.0	6 57 26.7
800	0.205590	73 52 46	47 28 52.5	7 1 41.63	48 11 46.7	6 57 45.2
700	0.205594	74 2 2	47 21 53.9	7 1 29.88	47 59 27.2	6 58 3.6
600	0.205597	74 11 19	47 14 55.2	7 1 18.15	47 47 7.4	6 58 21.8
500	0.205601	74 20 36	47 7 56.6	7 1 6.43	47 34 47.7	6 58 39.9
400	0.205604	74 29 53	47 0 57.9	7 0 54.70	47 22 27.0	6 58 57.9
300	0.205608	74 39 10	46 53 59.3	7 0 43.07	47 10 6.5	6 59 15.7
200	0.205611	74 48 26	46 47 0.6	7 0 31.41	46 57 45.6	6 59 33.3
—100	0.205615	74 57 43	46 40 1.9	7 0 19.77	46 45 24.6	6 59 50.9
0	0.205618	75 7 0	46 33 3.2	7 0 8.16	46 33 3.2	7 0 8.2
+100	0.205621	75 16 16	46 26 4.5	6 59 56.57	46 20 42.0	7 0 25.5
200	0.205624	75 25 33	46 19 5.7	6 59 45.00	46 8 20.4	7 0 42.6
300	0.205627	75 34 50	46 12 7.0	6 59 33.45	45 55 58.6	7 0 59.6
400	0.205630	75 44 7	46 5 8.2	6 59 21.92	45 43 36.6	7 1 16.4
500	0.205633	75 53 24	45 58 9.4	6 59 10.42	45 31 14.4	7 1 33.2
600	0.205636	76 2 41	45 51 10.6	6 58 58.93	45 18 52.0	7 1 49.7
700	0.205639	76 11 58	45 44 11.7	6 58 47.47	45 29.4	7 2 6.1
+800	0.205642	76 21 15	45 37 12.8	6 58 36.02	44 54 6.7	7 2 22.4

TABLE II.—*Elements of the Orbit of Venus.*

t	e'	ϖ'	θ'	ϕ'	θ_1'	ϕ_1'
—6400	0.0104237	124° 30′ 10″	92° 31′ 48″	3° 18′ 33″.9	108° 11′ 5″	3° 19′ 42″.1
6300	0.0103630	124 42 13	92 16 15	3 18 45.6	107 39 55	3 19 45.8
6200	0.0103023	124 50 12	92 0 41	3 18 58.5	107 8 45	3 19 49.4
6100	0.0102417	124 58 6	91 45 6	3 19 10.5	106 37 36	3 19 53.1
6000	0.0101812	125 5 54	91 29 30	3 19 22.2	106 6 27	3 19 56.8
5900	0.0101208	125 13 37	91 13 52	3 19 33.7	105 35 20	3 20 0.5
5800	0.0100606	125 21 15	90 58 14	3 19 45.0	105 4 13	3 20 4.1
5700	0.0100005	125 28 47	90 42 34	3 19 56.0	104 33 7	3 20 7.8
5600	0.0099405	125 36 14	90 26 53	3 20 6.7	104 2 1	3 20 11.5
5500	0.0098806	125 43 35	90 11 10	3 20 17.2	103 30 57	3 20 15.2
5400	0.0098209	125 50 51	89 55 27	3 20 27.4	102 59 53	3 20 18.9
5300	0.0097613	125 58 2	89 39 43	3 20 37.5	102 28 50	3 20 22.6
5200	0.0097019	126 5 7	89 23 58	3 20 47.2	101 57 48	3 20 26.2
5100	0.0096426	126 12 6	89 8 11	3 20 56.8	101 26 46	3 20 29.9
5000	0.0095834	126 19 0	88 52 24	3 21 6.0	100 55 45	3 20 33.6
4900	0.0095244	126 25 48	88 36 35	3 21 15.0	100 .24 45	3 20 37.3
4800	0.0094654	126 32 30	88 20 46	3 21 23.8	99 53 46	3 20 41.0
4700	0.0094066	126 39 6	88 4 55	3 21 32.3	99 22 48	3 20 44.7
4600	0.0093480	126 45 36	87 49 3	3 21 40.6	98 51 50	3 20 48.4
4500	0.0092895	126 52 0	87 33 11	3 21 48.6	98 20 53	3 20 52.1
4400	0.0092311	126 58 17	87 17 17	3 21 56.4	97 49 57	3 20 55.8
4300	0.0091729	127 4 28	87 1 22	3 22 4.0	97 19 1	3 20 59.5
4200	0.0091149	127 10 33	86 45 26	3 22 11.2	96 48 7	3 21 3.2
4100	0.0090571	127 16 32	86 29 29	3 22 18.3	96 17 13	3 21 6.8
4000	0.0089994	127 22 24	86 13 31	3 22 25.1	95 46 19	3 21 10.5
3900	0.0089419	127 28 9	85 57 32	3 22 31.6	95 15 27	3 21 14.2
3800	0.0088845	127 33 48	85 41 32	3 22 37.9	94 44 35	3 21 17.9
3700	0.0088273	127 39 20	85 25 31	3 22 44.0	94 13 44	3 21 21.6
3600	0.0087702	127 44 46	85 9 29	3 22 49.8	93 42 54	3 21 25.3
3500	0.0087133	127 50 5	84 53 26	3 22 55.3	93 12 5	3 21 29.0
3400	0.0086566	127 55 17	84 37 22	3 23 0.6	92 41 16	3 21 32.6
3300	0.0086000	128 0 21	84 21 16	3 23 5.7	92 10 28	3 21 36.3
3200	0.0085436	128 5 20	84 5 10	3 23 10.5	91 39 41	3 21 40.0
3100	0.0084873	128 10 11	83 49 3	3 23 15.1	91 8 54	3 21 43.6
3000	0.0084313	128 14 54	83 32 54	3 23 19.4	90 38 9	3 21 47.3
2900	0.0083755	128 19 29	83 16 45	3 23 23.4	90 7 24	3 21 50.9
2800	0.0083198	128 23 57	83 0 34	3 23 27.3	89 36 39	3 21 54.6
2700	0.0082643	128 28 17	82 44 23	3 23 30.8	89 5 56	3 21 58.2
2600	0.0082090	128 32 30	82 28 10	3 23 34.2	88 35 13	3 22 1.9
2500	0.0081539	128 36 35	82 11 57	3 23 37.3	88 4 31	3 22 5.5
2400	0.0080989	128 40 32	81 55 42	3 23 40.1	87 33 49	3 22 9.1
2300	0.0080442	128 44 20	81 39 27	3 23 42.7	87 3 8	3 22 12.8
2200	0.0079896	128 48 0	81 23 10	3 23 45.0	86 32 28	3 22 16.4
2100	0.0079352	128 51 31	81 6 52	3 23 47.1	86 1 49	3 22 20.0
2000	0.0078810	128 54 54	80 50 33	3 23 49.0	85 31 10	3 22 23.6
1900	0.0078271	128 58 8	80 34 14	3 23 50.6	85 0 32	3 22 27.2
1800	0.0077733	129 1 13	80 17 53	3 23 51.9	84 29 55	3 22 30.8
1700	0.0077197	129 4 9	80 1 31	3 23 53.0	83 59 18	3 22 34.4
1600	0.0076663	129 6 56	79 45 8	3 23 53.9	83 28 42	3 22 38.0
1500	0.0076132	129 9 33	79 28 44	3 23 54.5	82 58 7	3 22 41.6
1400	0.0075602	129 12 1	79 12 19	3 23 54.9	82 27 33	3 22 45.2
1300	0.0075074	129 14 19	78 55 53	3 23 55.0	81 56 59	3 22 48.7
1200	0.0074549	129 16 28	78 39 26	3 23 54.9	81 26 26	3 22 52.3
1100	0.0074027	129 18 26	78 22 58	3 23 54.6	80 55 54	3 22 55.8
1000	0.0073506	129 20 14	78 6 30	3 23 53.9	80 25 22	3 22 59.4
900	0.0072987	129 21 52	77 49 59	3 23 53.1	79 54 51	3 23 2.9
800	0.0072470	129 23 21	77 33 29	3 23 52.0	79 24 21	3 23 6.4
700	0.0071956	129 24 39	77 16 57	3 23 50.6	78 53 52	3 23 10.0
600	0.0071444	129 25 46	77 0 24	3 23 49.1	78 23 23	3 23 13.0
500	0.0070934	129 26 44	76 43 50	3 23 47.2	77 52 54	3 23 17.0
400	0.0070427	129 27 31	76 27 14	3 23 45.2	77 22 27	3 23 20.5
300	0.0069922	129 28 7	76 10 38	3 23 42.8	76 52 0	3 23 23.9
200	0.0069419	129 28 33	75 54 0	3 23 40.3	76 21 34	3 23 27.4
—100	0.0068919	129 28 48	75 37 23	3 23 37.5	75 51 8	3 23 30.8
0	0.0068420	129 28 52	75 20 43	3 23 34.4	75 20 42.9	3 23 34.4
+100	0.0067924	129 28 45	75 4 2	3 23 31.1	74 50 18	3 23 37.8
200	0.0067431	129 28 28	74 47 20	3 23 27.6	74 19 55	3 23 41.2
300	0.0066940	129 28 0	74 30 38	3 23 23.8	73 49 32	3 23 44.6
400	0.0066452	129 27 22	74 13 54	3 23 19.7	73 19 9	3 23 48.0
500	0.0065966	129 26 33	73 57 8	3 23 15.4	72 48 48	3 23 51.4
600	0.0065483	129 25 33	73 40 22	3 23 11.0	72 18 26	3 23 54.8
700	0.0065002	129 24 22	73 23 34	3 23 6.2	71 48 6	3 23 58.1
+800	0.0064523	129 23 0	73 6 45	3 23 1.2	71 17 46	3 24 1.5

TABLE III.—*Elements of the Orbit of Mars.*

t	e'''	ϖ'''	θ'''	ϕ'''	θ_1'''	ϕ_1'''
−6400	0.0869446	304° 22′ 34″	63° 53′ 22″	2° 17′ 9″.7	86° 36′ 50″	1° 54′ 44″.0
6300	0.0870448	304 50 33	63 40 30	2 16 50.0	86 2 43	1 54 39.3
6200	0.0871449	305 18 30	63 27 35	2 16 30.2	85 28 32	1 54 34.6
6100	0.0872449	305 46 26	63 14 37	2 16 10.2	84 54 19	1 54 29.9
6000	0.0873449	306 14 19	63 1 36	2 15 50.1	84 20 2	1 54 25.3
5900	0.0874448	306 42 11	62 48 33	2 15 29.8	83 45 41	1 54 20.6
5800	0.0875447	307 10 1	62 35 27	2 15 9.4	83 11 17	1 54 16.1
5700	0.0876445	307 37 49	62 22 18	2 14 48.8	82 36 50	1 54 11.5
5600	0.0877443	308 5 35	62 9 6	2 14 28.1	82 2 19	1 54 7.0
5500	0.0878440	308 33 20	61 55 52	2 14 7.3	81 27 45	1 54 2.4
5400	0.0879437	309 1 2	61 42 35	2 13 46.3	80 53 7	1 53 58.0
5300	0.0880433	309 28 43	61 29 14	2 13 25.1	80 18 26	1 53 53.5
5200	0.0881429	309 56 23	61 15 51	2 13 3.8	79 43 42	1 53 49.1
5100	0.0882424	310 24 0	61 2 24	2 12 42.4	79 8 55	1 53 44.7
5000	0.0883418	310 51 36	60 48 55	2 12 20.8	78 34 4	1 53 40.3
4900	0.0884412	311 19 10	60 35 24	2 11 59.0	77 59 9	1 53 36.0
4800	0.0885405	311 46 43	60 21 49	2 11 37.2	77 24 12	1 53 31.7
4700	0.0886397	312 14 13	60 8 11	2 11 15.1	76 49 11	1 53 27.4
4600	0.0887388	312 41 42	59 54 30	2 10 52.9	76 14 6	1 53 23.2
4500	0.0888378	313 9 10	59 40 47	2 10 30.6	75 38 59	1 53 19.0
4400	0.0889367	313 36 35	59 27 0	2 10 8.1	75 3 48	1 53 14.9
4300	0.0890354	314 3 59	59 13 10	2 9 45.5	74 28 34	1 53 10.7
4200	0.0891341	314 31 21	58 59 17	2 9 22.7	73 53 17	1 53 6.6
4100	0.0892327	314 58 42	58 45 21	2 8 59.8	73 17 56	1 53 2.6
4000	0.0893311	315 20 0	58 31 22	2 8 36.7	72 42 32	1 52 58.5
3900	0.0894294	315 53 18	58 17 20	2 8 13.4	72 7 5	1 52 54.6
3800	0.0895276	316 20 34	58 3 15	2 7 50.0	71 31 35	1 52 50.6
3700	0.0896257	316 47 48	57 49 6	2 7 26.5	70 56 1	1 52 46.7
3600	0.0897236	317 15 0	57 34 54	2 7 2.7	70 20 25	1 52 43.9
3500	0.0898214	317 42 11	57 20 39	2 6 38.9	69 44 44	1 52 39.1
3400	0.0899191	318 9 20	57 6 21	2 6 14.9	69 9 1	1 52 35.4
3300	0.0900166	318 36 28	56 51 59	2 5 50.7	68 33 14	1 52 31.7
3200	0.0901140	319 3 34	56 37 34	2 5 26.3	67 57 24	1 52 28.1
3100	0.0902113	319 30 39	56 23 6	2 5 1.8	67 21 31	1 52 24.5
3000	0.0903084	319 57 42	56 8 34	2 4 37.2	66 45 35	1 52 21.0
2900	0.0904053	320 24 43	55 53 59	2 4 12.4	66 9 36	1 52 17.5
2800	0.0905021	320 51 43	55 39 21	2 3 47.4	65 33 33	1 52 14.0
2700	0.0905987	321 18 41	55 24 39	2 3 22.3	64 57 27	1 52 10.7
2600	0.0906952	321 45 38	55 9 53	2 2 57.0	64 21 18	1 52 7.3
2500	0.0907915	322 12 33	54 55 4	2 2 31.6	63 45 6	1 52 4.0
2400	0.0908876	322 39 26	54 40 12	2 2 6.0	63 8 51	1 52 0.8
2300	0.0909836	323 6 19	54 25 15	2 1 40.2	62 32 32	1 51 57.6
2200	0.0910793	323 33 10	54 10 15	2 1 14.3	61 56 11	1 51 54.5
2100	0.0911748	323 59 59	53 55 12	2 0 48.2	61 19 46	1 51 51.4
2000	0.0912702	324 26 47	53 40 4	2 0 21.9	60 43 18	1 51 48.4
1900	0.0913654	324 53 33	53 24 53	1 59 55.5	60 6 47	1 51 45.5
1800	0.0914603	325 20 18	53 9 38	1 59 28.9	59 30 13	1 51 42.6
1700	0.0915550	325 47 1	52 54 19	1 59 2.2	58 53 35	1 51 39.8
1600	0.0916496	326 13 44	52 38 56	1 58 35.2	58 16 55	1 51 37.1
1500	0.0917440	326 40 24	52 23 29	1 58 8.2	57 40 11	1 51 34.4
1400	0.0918381	327 7 4	52 7 58	1 57 40.9	57 3 25	1 51 31.8
1300	0.0919320	327 33 41	51 52 24	1 57 13.5	56 26 36	1 51 29.2
1200	0.0920257	328 0 18	51 36 46	1 56 46.0	55 49 44	1 51 26.7
1100	0.0921192	328 26 52	51 21 4	1 56 18.3	55 12 50	1 51 24.3
1000	0.0922125	328 53 26	51 5 18	1 55 50.4	54 35 52	1 51 22.0
900	0.0923056	329 19 58	50 49 27	1 55 22.3	53 58 52	1 51 19.7
800	0.0923984	329 46 29	50 33 33	1 54 54.1	53 21 48	1 51 17.5
700	0.0924910	330 12 59	50 17 34	1 54 25.7	52 44 42	1 51 15.3
600	0.0925834	330 39 27	50 1 31	1 53 57.2	52 7 33	1 51 13.2
500	0.0926655	331 5 54	49 45 23	1 53 28.5	51 30 21	1 51 11.2
400	0.0927674	331 32 19	49 29 11	1 52 59.6	50 53 6	1 51 9.3
300	0.0928590	331 58 43	49 12 55	1 52 30.5	50 15 49	1 51 7.5
200	0.0929504	332 25 6	48 56 34	1 52 1.3	49 38 28	1 51 5.7
−100	0.0930415	332 51 28	48 40 8	1 51 31.9	49 1 4	1 51 4.0
0	0.0931324	333 17 47.8	48 23 36.8	1 51 2.30	48 23 36.8	1 51 2.30
+100	0.0932230	333 44 7	48 7 1	1 50 32.6	47 46 6	1 51 0.7
200	0.0933134	334 10 24	47 50 20	1 50 2.6	47 8 33	1 50 59.2
300	0.0934035	334 36 41	47 33 34	1 49 32.5	46 30 57	1 50 57.8
400	0.0934934	335 2 56	47 16 43	1 49 2.2	45 53 18	1 50 56.4
500	0.0935830	335 29 10	46 59 46	1 48 31.7	45 15 36	1 50 55.1
600	0.0936723	335 55 23	46 42 44	1 48 1.1	44 37 50	1 50 53.9
700	0.0937613	336 21 34	46 25 36	1 47 30.3	44 0 2	1 50 52.8
+800	0.0938502	336 47 44	46 8 22	1 46 59.2	43 22 11	1 50 51.7

TABLE IV.—*Elements of the Orbit of Jupiter.*

t	e^{IV}	ϖ^{IV}	θ^{IV}	ϕ^{IV}	θ_1^{IV}	θ_1^{IV}
—6400	0.0393563	3° 10′ 38″	92° 20′ 14″	1° 32′ 14″.9	124° 29′ 0″	1° 41′ 19″.4
6300	0.0394984	3 15 47	92 21 55	1 31 59.0	124 5 5	1 40 58.0
6200	0.0396406	3 21 2	92 23 45	1 31 43.1	123 41 9	1 40 36.7
6100	0.0397830	3 26 25	92 25 42	1 31 27.2	123 17 13	1 40 15.4
6000	0.0399255	3 31 55	92 27 48	1 31 11.4	122 53 17	1 39 54.1
5900	0.0400681	3 37 32	92 30 2	1 30 55.6	122 29 22	1 39 32.7
5800	0.0402109	3 43 16	92 32 24	1 30 39.9	122 5 26	1 39 11.4
5700	0.0403538	3 49 6	92 34 54	1 30 24.3	121 41 31	1 38 50.0
5600	0.0404967	3 55 3	92 37 34	1 30 8.7	121 17 36	1 38 28.7
5500	0.0406397	4 1 7	92 40 21	1 29 53.1	120 53 40	1 38 7.3
5400	0.0407827	4 7 18	92 43 17	1 29 37.6	120 29 45	1 37 46.0
5300	0.0409258	4 13 35	92 46 22	1 29 22.2	120 5 50	1 37 24.6
5200	0.0410689	4 19 58	92 49 36	1 29 6.9	119 41 55	1 37 3.3
5100	0.0412120	4 26 29	92 52 57	1 28 51.6	119 18 0	1 36 41.9
5000	0.0413550	4 33 5	92 56 28	1 28 36.4	118 54 5	1 36 20.6
4900	0.0414980	4 39 48	93 0 7	1 28 21.3	118 30 10	1 35 59.2
4800	0.0416410	4 46 36	93 3 55	1 28 6.3	118 6 15	1 35 37.8
4700	0.0417839	4 53 31	93 7 52	1 27 51.3	117 42 20	1 35 16.5
4600	0.0419267	5 0 32	93 11 57	1 27 36.5	117 18 24	1 34 55.1
4500	0.0420694	5 7 39	93 16 12	1 27 21.7	116 54 29	1 34 33.8
4400	0.0422121	5 14 52	93 20 34	1 27 7.0	116 30 34	1 34 12.4
4300	0.0423546	5 22 10	93 25 6	1 26 52.4	116 6 39	1 33 51.0
4200	0.0424969	5 29 34	93 29 47	1 26 37.9	115 42 43	1 33 29.7
4100	0.0426391	5 37 4	93 34 36	1 26 23.5	115 18 48	1 33 8.3
4000	0.0427812	5 44 40	93 39 34	1 26 9.3	114 54 52	1 32 47.0
3900	0.0429231	5 52 21	93 44 41	1 25 55.1	114 30 56	1 32 25.6
3800	0.0430648	6 0 8	93 49 57	1 25 41.0	114 7 0	1 32 4.2
3700	0.0432064	6 8 0	93 55 22	1 25 27.1	113 43 5	1 31 42.9
3600	0.0433478	6 15 58	94 0 55	1 25 13.2	113 19 9	1 31 21.6
3500	0.0434890	6 24 1	94 6 38	1 24 59.5	112 55 12	1 31 0.2
3400	0.0436300	6 32 10	94 12 29	1 24 46.0	112 31 16	1 30 38.9
3300	0.0437708	6 40 23	94 18 30	1 24 32.5	112 7 20	1 30 17.6
3200	0.0439113	6 48 42	94 24 39	1 24 19.2	111 43 23	1 29 56.3
3100	0.0440516	6 57 6	94 30 57	1 24 6.0	111 19 26	1 29 35.0
3000	0.0441917	7 5 35	94 37 23	1 23 53.0	110 55 29	1 29 13.7
2900	0.0443315	7 14 9	94 43 59	1 23 40.0	110 31 32	1 28 52.4
2800	0.0444711	7 22 48	94 50 43	1 23 27.3	110 7 34	1 28 31.1
2700	0.0446104	7 31 31	94 57 36	1 23 14.6	109 43 37	1 28 9.9
2600	0.0447494	7 40 20	95 4 37	1 23 2.2	109 19 39	1 27 48.6
2500	0.0448882	7 49 13	95 11 47	1 22 49.9	108 55 41	1 27 27.3
2400	0.0450265	7 58 11	95 19 6	1 22 37.7	108 31 42	1 27 6.1
2300	0.0451645	8 7 14	95 26 34	1 22 25.7	108 7 43	1 26 44.8
2200	0.0453023	8 16 22	95 34 9	1 22 13.8	107 43 44	1 26 23.6
2100	0.0454398	8 25 34	95 41 54	1 22 2.2	107 19 45	1 26 2.4
2000	0.0455769	8 34 50	95 49 46	1 21 50.7	106 55 46	1 25 41.2
1900	0.0457137	8 44 11	95 57 47	1 21 39.3	106 31 46	1 25 20.0
1800	0.0458501	8 53 37	96 5 56	1 21 28.2	106 7 46	1 24 58.8
1700	0.0459862	9 3 6	96 14 13	1 21 17.2	105 43 45	1 24 37.7
1600	0.0461220	9 12 41	96 22 38	1 21 6.4	105 19 44	1 24 16.5
1500	0.0462574	9 22 19	96 31 12	1 20 55.7	104 55 43	1 23 55.4
1400	0.0463924	9 32 2	96 39 53	1 20 45.3	104 31 41	1 23 34.3
1300	0.0465271	9 41 48	96 48 42	1 20 35.1	104 7 38	1 23 13.2
1200	0.0466613	9 51 39	96 57 38	1 20 25.0	103 43 35	1 22 52.1
1100	0.0467952	10 1 34	97 6 43	1 20 15.2	103 19 32	1 22 31.0
1000	0.0469286	10 11 32	97 15 54	1 20 5.5	102 55 28	1 22 10.0
900	0.0470616	10 21 35	97 25 14	1 19 56.1	102 32 23	1 21 48.9
800	0.0471942	10 31 42	97 34 40	1 19 46.8	102 7 18	1 21 27.9
700	0.0473264	10 41 53	97 44 14	1 19 37.8	101 43 13	1 21 6.9
600	0.0474581	10 52 7	97 53 54	1 19 28.9	101 19 7	1 20 45.9
500	0.0475894	11 2 25	98 3 42	1 19 20.3	100 55 1	1 20 24.9
400	0.0477202	11 12 48	98 13 36	1 19 11.8	100 30 54	1 20 3.9
300	0.0478505	11 23 14	98 23 38	1 19 3.6	100 6 46	1 19 43.0
200	0.0479804	11 33 43	98 33 46	1 18 55.6	99 42 38	1 19 22.1
—100	0.0481098	11 43 16	98 44 0	1 18 47.8	99 18 30	1 19 1.2
0	0.0482388	11 54 53.1	98 54 20.5	1 18 40.30	98 54 20.5	1 18 40.3
+100	0.0483672	12 5 34	99 4 47	1 18 33.0	98 30 10	1 18 19.4
200	0.0484952	12 16 17	99 15 20	1 18 25.9	98 6 0	1 17 58.6
300	0.0486227	12 27 4	99 25 58	1 18 19.0	97 41 48	1 17 37.8
400	0.0487496	12 37 55	99 36 42	1 18 12.4	97 17 36	1 17 17.0
500	0.0488760	12 48 49	99 47 31	1 18 6.0	96 53 23	1 16 56.2
600	0.0490020	12 59 46	99 58 25	1 17 59.8	96 29 9	1 16 35.4
700	0.0491274	13 10 47	100 9 25	1 17 53.9	96 4 55	1 16 14.7
+800	0.0492523	13 21 51	100 20 29	1 17 48.2	95 40 39	1 15 54.0

Table V.—*Elements of the Orbit of Saturn.*

t	e^{v}	ϖ^{v}	θ^{v}	ϕ^{v}	θ_1^{v}	ϕ_1^{v}
—6400	0.0708682	62° 8′ 48″	127° 38′ 42″	2° 8′ 28″.6	144° 49′ 21″	2° 41′ 31″.7
6300	0.0706762	62 34 9	127 25 42	2 8 57.1	144 19 58	2 41 23.4
6200	0.0704829	62 59 31	127 12 37	2 9 25.4	143 50 34	2 41 15.0
6100	0.0702882	63 24 55	126 59 29	2 9 53.4	143 21 8	2 41 6.4
6000	0.0700921	63 50 19	126 46 17	2 10 21.1	142 51 41	2 40 57.8
5900	0.0698947	64 15 45	126 33 2	2 10 48.6	142 22 11	2 40 49.1
5800	0.0696958	64 41 12	126 19 43	2 11 15.8	141 52 40	2 40 40.2
5700	0.0694955	65 6 40	126 6 21	2 11 42.7	141 23 7	2 40 31.3
5600	0.0692939	65 32 9	125 52 55	2 12 9.4	140 53 31	2 40 22.3
5500	0.0690909	65 57 39	125 39 27	2 12 35.8	140 23 54	2 40 13.1
5400	0.0688866	66 23 11	125 25 55	2 13 1.9	139 54 14	2 40 3.9
5300	0.0686810	66 48 44	125 12 20	2 13 27.8	139 24 33	2 39 54.5
5200	0.0684740	67 14 19	124 58 41	2 13 53.4	138 54 50	2 39 45.1
5100	0.0682657	67 39 55	124 45 0	2 14 18.7	138 25 5	2 39 35.5
5000	0.0680560	68 5 32	124 31 16	2 14 43.8	137 55 18	2 39 25.8
4900	0.0678450	68 31 11	124 17 28	2 15 8.6	137 25 29	2 39 16.1
4800	0.0676327	68 56 51	124 3 38	2 15 33.0	136 55 37	2 39 6.2
4700	0.0674191	69 22 32	123 49 45	2 15 57.3	136 25 44	2 38 56.2
4600	0.0672041	69 48 15	123 35 49	2 16 21.2	135 55 49	2 38 46.2
4500	0.0669878	70 14 0	123 21 51	2 16 44.9	135 25 51	2 38 36.0
4400	0.0667702	70 39 46	123 7 49	2 17 8.3	134 55 52	2 38 25.7
4300	0.0665513	71 5 34	122 53 46	2 17 31.4	134 25 50	2 38 15.4
4200	0.0663310	71 31 23	122 39 40	2 17 54.2	133 55 47	2 38 4.9
4100	0.0661095	71 57 14	122 25 31	2 18 16.8	133 25 41	2 37 54.3
4000	0.0658868	72 23 6	122 11 20	2 18 39.0	132 55 33	2 37 43.6
3900	0.0656628	72 49 0	121 57 7	2 19 1.0	132 25 24	2 37 32.8
3800	0.0654375	73 14 56	121 42 51	2 19 22.6	131 55 12	2 37 22.0
3700	0.0652110	73 40 53	121 28 33	2 19 44.0	131 24 58	2 37 11.0
3600	0.0649832	74 6 52	121 14 13	2 20 5.1	130 54 42	2 37 0.0
3500	0.0647542	74 32 53	120 59 50	2 20 25.9	130 24 23	2 36 48.8
3400	0.0645239	74 58 55	120 45 26	2 20 46.4	129 54 3	2 36 37.6
3300	0.0642924	75 25 0	120 31 0	2 21 6.6	129 23 40	2 36 26.2
3200	0.0640597	75 51 6	120 16 31	2 21 26.5	128 53 16	2 36 14.8
3100	0.0638257	76 17 14	120 2 1	2 21 46.1	128 22 48	2 36 3.2
3000	0.0635905	76 43 24	119 47 29	2 22 5.4	127 52 19	2 35 51.6
2900	0.0633541	77 9 36	119 32 54	2 22 24.4	127 21 48	2 35 39.9
2800	0.0631165	77 35 50	119 18 18	2 22 43.2	126 51 14	2 35 28.1
2700	0.0628776	78 2 6	119 3 39	2 23 1.6	126 20 37	2 35 16.2
2600	0.0626375	78 28 24	118 48 59	2 23 19.7	125 49 59	2 35 4.2
2500	0.0623962	78 54 44	118 34 17	2 23 37.5	125 19 18	2 34 52.2
2400	0.0621537	79 21 6	118 19 33	2 23 55.1	124 48 35	2 34 40.0
2300	0.0619100	79 47 31	118 4 48	2 24 12.3	124 17 49	2 34 27.7
2200	0.0616652	80 13 57	117 50 2	2 24 29.2	123 47 2	2 34 15.4
2100	0.0614192	80 40 26	117 35 14	2 24 45.8	123 16 12	2 34 3.0
2000	0.0611720	81 6 57	117 20 24	2 25 2.1	122 45 19	2 33 50.4
1900	0.0609237	81 33 30	117 5 33	2 25 18.1	122 14 24	2 33 37.8
1800	0.0606743	82 0 5	116 50 41	2 25 33.8	121 43 27	2 33 25.1
1700	0.0604237	82 26 43	116 35 47	2 25 49.1	121 12 28	2 33 12.3
1600	0.0601720	82 53 24	116 20 52	2 26 4.2	120 41 26	2 32 59.5
1500	0.0599192	83 20 7	116 5 55	2 26 18.9	120 10 22	2 32 46.5
1400	0·0596652	83 46 52	115 50 57	2 26 33.4	119 39 16	2 32 33.5
1300	0.0594101	84 13 40	115 35 57	2 26 47.5	119 8 7	2 32 20.3
1200	0.0591540	84 40 30	115 20 56	2 27 1.3	118 36 56	2 32 7.1
1100	0.0588968	85 7 23	115 5 54	2 27 14.8	118 5 42	2 31 53.8
1000	0.0586384	85 34 18	114 50 51	2 27 28.0	117 34 26	2 31 40.5
900	0.0583789	86 1 16	114 35 46	2 27 40.8	117 3 7	2 31 27.0
800	0.0581183	86 28 17	114 20 40	2 27 53.4	116 31 45	2 31 13.5
700	0.0578567	86 55 21	114 5 34	2 28 5.6	116 0 21	2 30 59.8
600	0.0575940	87 22 28	113 50 27	2 28 17.5	115 28 55	2 30 46.2
500	0.0573302	87 49 38	113 35 18	2 28 29.2	114 57 26	2 30 32.4
400	0.0570653	88 16 50	113 20 8	2 28 40.4	114 25 54	2 30 18.5
300	0.0567994	88 44 6	113 4 57	2 28 51.4	113 54 19	2 30 4.6
200	0.0565325	89 11 25	112 49 46	2 29 2.0	113 22 42	2 29 50.6
—100	0.0562646	89 38 46	112 34 34	2 29 12.4	112 51 3	2 29 36.5
0	0.0559956	90 6 12.0	112 19 20.6	2 29 22.4	112 19 20.6	2 29 22.4
+100	0.0557257	90 33 40	112 4 7	2 29 32.1	111 47 36	2 29 8.2
200	0.0554547	91 1 12	111 48 52	2 29 41.4	111 15 48	2 28 53.9
300	0.0551827	91 28 47	111 33 36	2 29 50.5	110 43 58	2 28 39.5
400	0.0549097	91 56 25	111 18 19	2 29 59.2	110 12 5	2 28 25.0
500	0.0546358	92 24 7	111 3 2	2 30 7.6	109 40 9	2 28 10.5
600	0.0543609	92 51 52	110 47 44	2 30 15.7	109 8 10	2 27 55.9
700	0.0540850	93 19 41	110 32 25	2 30 23.4	108 36 8	2 27 41.2
+800	0.0538082	93 47 34	110 17 6	2 30 30.9	108 4 4	2 27 26.5

TABLE VI.—*Elements of the Orbit of Uranus.*

t	e^{VI}	ϖ^{VI}	θ^{VI}	ϕ^{VI}	θ_1^{VI}	ϕ_1^{VI}
−6400	0.0480857	165° 1′ 47″	70° 3′ 6″	0° 53′ 14″.98	130° 34′ 2″	0° 54′ 30″.40
6300	0.0480544	165 7 18	70 4 45	0 53 8.46	129 46 15	0 54 13.82
6200	0.0480232	165 12 48	70 6 27	0 53 1.94	128 58 12	0 53 57.49
6100	0.0479920	165 18 18	70 8 10	0 52 55.41	128 9 52	0 53 41.42
6000	0.0479608	165 23 47	70 9 56	0 52 48.89	127 21 17	0 53 25.58
5900	0.0479297	165 29 15	70 11 43	0 52 42.36	126 32 26	0 53 10.00
5800	0.0478987	165 34 43	70 13 33	0 52 35.84	125 43 19	0 52 54.66
5700	0.0478677	165 40 10	70 15 25	0 52 29.31	124 53 57	0 52 39.58
5600	0.0478368	165 45 36	70 17 20	0 52 22.79	124 4 19	0 52 24.75
5500	0.0478059	165 51 2	70 19 16	0 52 16.27	123 14 26	0 52 10.18
5400	0.0477751	165 56 27	70 21 15	0 52 9.75	122 24 18	0 51 55.87
5300	0.0477443	166 1 52	70 23 17	0 52 3.23	121 33 54	0 51 41.83
5200	0.0477136	166 7 16	70 25 20	0 51 56.71	120 43 15	0 51 28.05
5100	0.0476829	166 12 39	70 27 26	0 51 50.19	119 52 21	0 51 14.54
5000	0.0476523	166 18 2	70 29 35	0 51 43.68	119 1 12	0 51 1.29
4900	0.0476217	166 23 24	70 31 46	0 51 37.18	118 9 49	0 50 48.30
4800	0.0475912	166 28 46	70 34 0	0 51 30.68	117 18 11	0 50 35.59
4700	0.0475607	166 34 7	70 36 16	0 51 24.19	116 26 19	0 50 23.16
4600	0.0475303	166 39 27	70 38 34	0 51 17.70	115 34 13	0 50 11.00
4500	0.0474999	166 44 46	70 40 56	0 51 11.21	114 41 54	0 49 59.14
4400	0.0474696	166 50 5	70 43 19	0 51 4.73	113 49 21	0 49 47.56
4300	0.0474394	166 55 24	70 45 46	0 50 58.24	112 56 35	0 49 36.28
4200	0.0474092	167 0 42	70 48 15	0 50 51.75	112 3 36	0 49 25.29
4100	0.0473791	167 5 59	70 50 46	0 50 45.27	111 10 24	0 49 14.60
4000	0.0473490	167 11 15	70 53 20	0 50 38.80	110 17 0	0 49 4.21
3900	0.0473190	167 16 31	70 55 57	0 50 32.34	109 23 24	0 48 54.12
3800	0.0472891	167 21 47	70 58 36	0 50 25.88	108 29 37	0 48 44.34
3700	0.0472592	167 27 1	71 1 18	0 50 14.43	107 35 38	0 48 34.87
3600	0.0472294	167 32 15	71 4 3	0 50 13.00	106 41 28	0 48 25.71
3500	0.0471997	167 37 29	71 6 51	0 50 6.59	105 47 7	0 48 16.86
3400	0.0471700	167 42 42	71 9 41	0 50 0.18	104 52 36	0 48 8.33
3300	0.0471405	167 47 54	71 12 34	0 49 53.78	103 57 56	0 48 0.11
3200	0.0471110	167 53 5	71 15 30	0 49 47.39	103 3 5	0 47 52.20
3100	0.0470816	167 58 16	71 18 29	0 49 41.01	102 8 6	0 47 44.61
3000	0.0470523	168 3 27	71 21 30	0 49 34.63	101 12 58	0 47 37.33
2900	0.0470231	168 8 37	71 24 34	0 49 28.26	100 17 42	0 47 30.38
2800	0.0469939	168 13 46	71 27 41	0 49 21.90	99 22 19	0 47 23.74
2700	0.0469648	168 18 55	71 30 51	0 49 15.56	98 26 48	0 47 17.43
2600	0.0469358	168 24 3	71 34 3	0 49 9.23	97 31 10	0 47 11.44
2500	0.0469069	168 29 10	71 37 18	0 49 2.91	96 35 27	0 47 5.77
2400	0.0468780	168 34 17	71 40 36	0 48 56.60	95 39 37	0 47 0.43
2300	0.0468493	168 39 23	71 43 57	0 48 50.31	94 43 42	0 46 55.42
2200	0.0468206	168 44 29	71 47 21	0 48 44.03	93 47 43	0 46 50.72
2100	0.0467921	168 49 34	71 50 47	0 48 37.77	92 51 39	0 46 46.35
2000	0.0467636	168 54 39	71 54 16	0 48 31.52	91 55 32	0 46 42.32
1900	0.0467352	168 59 43	71 57 48	0 48 25.28	90 59 22	0 46 38.62
1800	0.0467069	169 4 47	72 1 24	0 48 19.05	90 3 9	0 46 35.24
1700	0.0466787	169 9 50	72 5 1	0 48 12.83	89 6 53	0 46 32.19
1600	0.0466506	169 14 52	72 8 42	0 48 6.63	88 10 36	0 46 29.47
1500	0.0466227	169 19 54	72 12 26	0 48 0.45	87 14 18	0 46 27.08
1400	0.0465948	169 24 55	72 16 12	0 47 54.28	86 18 0	0 46 25.01
1300	0.0465670	169 29 56	72 20 2	0 47 48.13	85 21 41	0 46 23.27
1200	0.0465393	169 34 56	72 23 54	0 47 41.99	84 25 25	0 46 21.84
1100	0.0465117	169 39 56	72 27 50	0 47 35.88	83 29 10	0 46 20.75
1000	0.0464842	169 44 55	72 31 48	0 47 29.79	82 32 56	0 46 19.99
900	0.0464568	169 49 54	72 35 49	0 47 23.72	81 36 45	0 46 19.55
800	0.0464295	169 54 52	72 39 54	0 47 17.66	80 40 36	0 46 19.44
700	0.0464023	169 59 49	72 44 1	0 47 11.63	79 44 29	0 46 19.65
600	0.0463752	170 4 46	72 48 11	0 47 5.61	78 48 28	0 46 20.17
500	0.0463482	170 9 43	72 52 24	0 46 59.61	77 52 32	0 46 21.01
400	0.0463213	170 14 39	72 56 40	0 46 53.63	76 56 40	0 46 22.17
300	0.0462945	170 19 34	73 0 59	0 46 47.67	76 0 55	0 46 23.64
200	0.0462679	170 24 29	73 5 21	0 46 41.74	75 5 14	0 46 25.43
−100	0.0462413	170 29 24	73 9 46	0 46 35.83	74 9 41	0 46 27.53
0	0.0462149	170 34 17.7	73 14 13.4	0 46 29.93	73 14 13.4	0 46 29.93
+100	0.0461886	170 39 11	73 18 44	0 46 24.06	72 18 54	0 46 32.64
200	0.0461624	170 44 4	73 23 18	0 46 18.21	71 23 42	0 46 35.65
300	0.0461363	170 48 56	73 27 54	0 46 12.38	70 28 39	0 46 38.96
400	0.0461103	170 53 48	73 32 33	0 46 6.57	69 33 46	0 46 42.57
500	0.0460844	170 58 40	73 37 15	0 46 0.79	68 39 1	0 46 46.47
600	0.0460587	171 3 31	73 42 0	0 45 55.03	67 44 26	0 46 50.67
700	0.0460330	171 8 21	73 46 48	0 45 49.30	66 50 0	0 46 55.16
+800	0.0460074	171 13 11	73 51 39	0 45 43.58	65 55 44	0 46 59.93

TABLE VII.—*Elements of the Orbit of Neptune.*

t	e^{VII}	ϖ^{VII}	θ^{VII}	ϕ^{VII}	θ_1^{VII}	ϕ_1^{VII}
—6400	0.0088883	49° 26′ 1″	130° 29′ 8″	1° 46′ 14″.21	149° 31′ 7″	2° 22′ 12″.02
6300	0.0088924	49 26 35	130 28 47	1 46 14.90	149 12 59	2 21 41.70
6200	0.0088965	49 27 10	130 28 27	1 46 15.59	148 54 52	2 21 11.29
6100	0.0089006	49 27 45	130 28 6	1 46 16.28	148 36 45	2 20 40.78
6000	0.0089047	49 28 21	130 27 45	1 46 16.98	148 18 38	2 20 10.17
5900	0.0089088	49 28 57	130 27 25	1 46 17.67	148 0 30	2 19 39.47
5800	0.0089130	49 29 33	130 27 4	1 46 18.36	147 42 23	2 19 8.66
5700	0.0089172	49 30 10	130 26 44	1 46 19.05	147 24 16	2 18 37.76
5600	0.0089214	49 30 48	130 26 24	1 46 19.74	147 6 8	2 18 6.77
5500	0.0089256	49 31 25	130 26 3	1 46 20.43	146 48 1	2 17 35.69
5400	0.0089299	49 32 4	130 25 43	1 46 21.12	146 29 54	2 17 4.50
5300	0.0089342	49 32 42	130 25 22	1 46 21.81	146 11 46	2 16 33.22
5200	0.0089385	49 33 21	130 25 2	1 46 22.51	145 53 39	2 16 1.85
5100	0.0089428	49 34 1	130 24 42	1 46 23.21	145 35 31	2 15 30.39
5000	0.0089472	49 34 41	130 24 21	1 46 23.91	145 17 24	2 14 58.84
4900	0.0089515	49 35 21	130 24 1	1 46 24.61	144 59 16	2 14 27.20
4800	0.0089559	49 36 2	130 23 41	1 46 25.31	144 41 9	2 13 55.47
4700	0.0089603	49 36 44	130 23 20	1 46 26.01	144 23 1	2 13 23.65
4600	0.0089647	49 37 25	130 23 0	1 46 26.72	144 4 53	2 12 51.75
4500	0.0089691	49 38 8	130 22 40	1 46 27.43	143 46 46	2 12 19.76
4400	0.0089736	49 38 50	130 22 20	1 46 28.14	143 28 38	2 11 47.69
4300	0.0089780	49 39 33	130 22 0	1 46 28.85	143 10 30	2 11 15.53
4200	0.0089824	49 40 17	130 21 39	1 46 29.57	142 52 22	2 10 43.30
4100	0.0089868	49 41 1	130 21 19	1 46 30.28	142 34 14	2 10 10.99
4000	0.0089913	49 41 45	130 20 59	1 46 31.00	142 16 6	2 9 38.60
3900	0.0089957	49 42 30	130 20 39	1 46 31.72	141 57 58	2 9 6.12
3800	0.0090002	49 43 15	130 20 19	1 46 32.44	141 39 49	2 8 33.56
3700	0.0090046	49 44 1	130 19 59	1 46 33.16	141 21 40	2 8 0.92
3600	0.0090091	49 44 47	130 19 39	1 46 33.89	141 3 32	2 7 28.20
3500	0.0090135	49 45 33	130 19 19	1 46 34.62	140 45 23	2 6 55.40
3400	0.0090180	49 46 20	130 18 59	1 46 35.35	140 27 14	2 6 22.51
3300	0.0090225	49 47 7	130 18 39	1 46 36.08	140 9 5	2 5 49.54
3200	0.0090270	49 47 54	130 18 19	1 46 36.81	139 50 56	2 5 16.49
3100	0.0090315	49 48 42	130 17 59	1 46 37.54	139 32 46	2 4 43.37
3000	0.0090361	49 49 30	130 17 39	1 46 38.28	139 14 36	2 4 10.17
2900	0.0090406	49 50 18	130 17 19	1 46 39.01	138 56 27	2 3 36.89
2800	0.0090452	49 51 6	130 17 0	1 46 39.75	138 38 17	2 3 3.53
2700	0.0090497	49 51 56	130 16 40	1 46 40.49	138 20 6	2 2 30.10
2600	0.0090542	49 52 45	130 16 20	1 46 41.23	138 1 56	2 1 56.59
2500	0.0090587	49 53 35	130 16 0	1 46 41.97	137 43 45	2 1 23.01
2400	0.0090633	49 54 26	130 15 40	1 46 42.71	137 25 34	2 0 49.36
2300	0.0090678	49 55 16	130 15 21	1 46 43.45	137 7 23	2 0 15.63
2200	0.0090724	49 56 7	130 15 1	1 46 44.20	136 49 12	1 59 41.83
2100	0.0090769	49 56 59	130 14 41	1 46 44.95	136 31 1	1 59 7.96
2000	0.0090815	49 57 51	130 14 21	1 46 45.70	136 12 49	1 58 34.01
1900	0.0090861	49 58 43	130 14 1	1 46 46.45	135 54 37	1 57 59.99
1800	0.0090907	49 59 36	130 13 42	1 46 47.20	135 36 25	1 57 25.90
1700	0.0090953	50 0 30	130 13 22	1 46 47.95	135 18 13	1 56 51.74
1600	0.0090999	50 1 23	130 13 2	1 46 48.71	135 0 0	1 56 17.51
1500	0.0091045	50 2 17	130 12 42	1 46 49.47	134 41 47	1 55 43.20
1400	0.0091091	50 3 12	130 12 22	1 46 50.23	134 23 33	1 55 8.84
1300	0.0091137	50 4 7	130 12 3	1 46 50.99	134 5 20	1 54 34.40
1200	0.0091184	50 5 2	130 11 43	1 46 51.75	133 47 5	1 53 59.91
1100	0.0091230	50 5 58	130 11 23	1 46 52.51	133 28 51	1 53 25.35
1000	0.0091276	50 6 54	130 11 3	1 46 53.28	133 10 36	1 52 50.72
900	0.0091322	50 7 50	130 10 43	1 46 54.04	132 52 20	1 52 16.04
800	0.0091369	50 8 47	130 10 24	1 46 54.81	132 34 5	1 51 41.29
700	0.0091415	50 9 45	130 10 4	1 46 55.58	132 15 48	1 51 6.48
600	0.0091461	50 10 43	130 9 44	1 46 56.36	131 57 32	1 50 31.60
500	0.0091507	50 11 41	130 9 24	1 46 57.13	131 39 15	1 49 56.66
400	0.0091554	50 12 40	130 9 4	1 46 57.91	131 20 58	1 49 21.66
300	0.0091600	50 13 39	130 8 45	1 46 58.69	131 2 41	1 48 46.60
200	0.0091646	50 14 38	130 8 25	1 46 59.47	130 44 23	1 48 11.47
—100	0.0091692	50 15 38	130 8 5	1 47 0.25	130 26 4	1 47 36.29
0	0.0091739	50 16 38.6	130 7 45.3	1 47 1.04	130 7 45.3	1 47 1.04
+100	0.0091785	50 17 39	130 7 25	1 47 1.82	129 49 26	1 46 25.73
200	0.0091832	50 18 40	130 7 6	1 47 2.61	129 31 6	1 45 50.37
300	0.0091879	50 19 42	130 6 46	1 47 3.40	129 12 46	1 45 14.95
400	0.0091926	50 20 44	130 6 26	1 47 4.19	128 54 25	1 44 39.47
500	0.0091972	50 21 46	130 6 6	1 47 4.98	128 36 3	1 44 3.94
600	0.0092019	50 22 49	130 5 46	1 47 5.77	128 17 41	1 43 28.35
700	0.0092066	50 23 52	130 5 27	1 47 6.56	127 59 18	1 42 52.71
+800	0.0092113	50 24 55	130 5 7	1 47 7.36	127 40 55	1 42 17.01

TABLE VIII.—*Elements of the Orbit of the Earth.*

t	e''	ϖ''	θ''	ϕ''
—8000	0.0192055	75° 23′ 45″.8	13° 12′ 44.3	1° 8′ 9.114
7900	0.0191870	75 41 40.1	12 57 32.4	1 7 17.243
7800	0.0191682	75 59 35.6	12 42 20.6	1 6 25.364
7700	0.0191489	76 17 32.4	12 27 9.0	1 5 33.480
7600	0.0191294	76 35 30.6	12 11 57.5	1 4 41.591
7500	0.0191096	76 53 30.1	11 56 46.2	1 3 49.698
7400	0.0190894	77 11 30.9	11 41 35.1	1 2 57.802
7300	0.0190689	77 29 32.9	11 26 24.2	1 2 5.904
7200	0.0190482	77 47 36.2	11 11 13.4	1 1 14.004
7100	0.0190272	78 5 40.7	10 56 2.8	1 0 22.105
7000	0.0190058	78 23 46.4	10 40 52.3	0 59 30.207
6900	0.0189841	78 41 53.3	10 25 42.1	0 58 38.312
6800	0.0189620	79 0 1.4	10 10 32.0	0 57 46.420
6700	0.0189396	79 18 10.7	9 55 22.0	0 56 54.533
6600	0.0189170	79 36 21.1	9 40 12.3	0 56 2.650
6500	0.0188941	79 54 32.7	9 25 2.7	0 55 10.773
6400	0.0188708	80 12 45.5	9 9 53.3	0 54 18.904
6300	0.0188473	80 30 58.6	8 54 44.0	0 53 27.044
6200	0.0188234	80 49 13.1	8 39 34.9	0 52 35.194
6100	0.0187992	81 7 28.8	8 24 25.9	0 51 43.354
6000	0.0187747	81 25 45.7	8 9 17.2	0 50 51.524
5900	0.0187499	81 44 3.9	7 54 8.6	0 49 59.707
5800	0.0187248	82 2 23.3	7 39 0.1	0 49 7.902
5700	0.0186994	82 20 44.0	7 23 1.9	0 48 16.120
5600	0.0186737	82 39 5.9	7 8 43.8	0 47 24.347
5500	0.0186477	82 57 29.1	6 53 35.8	0 46 32.590
5400	0.0186214	83 15 53.7	6 38 28.0	0 45 40.852
5300	0.0185948	83 34 19.6	6 23 20.4	0 44 49.132
5200	0.0185678	83 52 46.7	6 8 13.0	0 43 57.431
5100	0.0185405	84 11 15.0	5 53 5.7	0 43 5.750
5000	0.0185130	84 29 44.6	5 37 58.6	0 42 14.090
4900	0.0184852	84 48 15.4	5 22 51.7	0 41 22.453
4800	0.0184570	85 6 47.5	5 7 44.9	0 40 30.840
4700	0.0184285	85 25 20.8	4 52 38.3	0 39 39.253
4600	0.0183998	85 43 55.2	4 37 31.8	0 38 47.692
4500	0.0183708	86 2 30.8	4 22 25.5	0 37 56.158
4400	0.0183415	86 21 7.6	4 7 19.4	0 37 4.652
4300	0.0183119	86 39 45.6	3 52 13.4	0 36 13.175
4200	0.0182820	86 58 24.8	3 37 7.6	0 35 21.729
4100	0.0182518	87 17 5.1	3 22 1.9	0 34 30.315
4000	0.0182213	87 35 46.7	3 6 56.4	0 33 38.931
3900	0.0181905	87 54 29.6	2 51 51.1	0 32 47.577
3800	0.0181595	88 13 13.7	2 36 46.0	0 31 56.254
3700	0.0181282	88 31 59.0	2 21 41.0	0 31 4.962
3600	0.0180965	88 50 45.6	2 6 36.2	0 30 13.706
3500	0.0180645	89 9 33.4	1 51 31.6	0 29 22.487
3400	0.0180323	89 28 22.4	1 36 27.1	0 28 31.301
3300	0.0179998	89 47 12.6	1 21 22.8	0 27 40.153
3200	0.0179670	90 6 4.0	1 6 18.6	0 26 49.047
3100	0.0179339	90 24 56.8	0 51 14.6	0 25 57.981
—3000	0.0179005	90 43 51.0	0 36 10.8	0 25 6.958

Table VIII.—*Elements of the Orbit of the Earth*—continued.

t	e''	ϖ''	θ''	ϕ''
—3000	0.0179005	90° 43′ 51″.0	0° 36′ 10″.8	0° 25′ 6″.958
2900	0.0178668	91 2 46.5	0 21 7.1	0 24 15.977
2800	0.0178329	91 21 43.4	0 6 3.6	0 23 25.040
2700	0.0177987	91 40 41.6	359 51 0.2	0 22 34.149
2600	0.0177642	91 59 41.1	359 35 57.0	0 21 43.304
2500	0.0177294	92 18 42.0	359 20 53.9	0 20 52.504
2400	0.0176943	92 37 44.2	359 5 51.0	0 20 1.748
2300	0.0176589	92 56 48.0	358 50 48.3	0 19 11.038
2200	0.0176233	93 15 53.3	358 35 45.7	0 18 20.378
2100	0.0175875	93 35 0.0	358 20 43.2	0 17 29.769
2000	0.0175513	93 54 8.3	358 5 41.0	0 16 39.210
1900	0.0175148	94 13 18.0	357 50 38.9	0 15 48.703
1800	0.0174781	94 32 29.3	357 35 36.9	0 14 58.248
1700	0.0174411	94 51 42.0	357 20 35.1	0 14 7.844
1600	0.0174038	95 10 56.3	357 5 33.5	0 13 17.494
1500	0.0173662	95 30 11.6	356 50 32.0	0 12 27.200
1400	0.0173284	95 49 28.1	356 35 30.7	0 11 36.961
1300	0.0172903	96 8 45.6	356 20 29.6	0 10 46.779
1200	0.0172520	96 28 4.1	356 5 28.6	0 9 56.657
1100	0.0172134	96 47 23.9	355 50 27.8	0 9 6.592
1000	0.0171745	97 6 44.9	355 35 27.2	0 8 16.586
900	0.0171353	97 26 7.6	355 20 26.7	0 7 26.642
800	0.0170959	97 45 31.7	355 5 26.4	0 6 36.759
700	0.0170562	98 4 57.5	354 50 25.7	0 5 46.937
600	0.0170163	98 24 24.8	354 35 25.3	0 4 57.178
500	0.0169762	98 43 53.6	354 20 25.4	0 4 7.484
400	0.0169367	99 3 24.0	354 5 25.7	0 3 17.854
300	0.0168949	99 22 55.9	353 50 26.1	0 2 28.290
200	0.0168539	99 42 29.4	353 35 26.6	0 1 38.792
—100	0.0168127	100 2 4.4	353 20 27.4	0 0 49.362
00	0.0167712	100 21 41.0	353 5 28.0	0 0 0.000
+100	0.0167295	100 41 18.9	172 50 28.6	0 0 49.293
200	0.0166875	101 0 58.5	172 35 29.9	0 1 38.516
300	0.0166452	101 20 39.5	172 20 31.2	0 2 27.667
400	0.0166027	101 40 22.0	172 5 32.6	0 3 16.747
500	0.0165599	102 0 6.1	171 50 34.1	0 4 5.754
600	0.0165169	102 19 51.8	171 35 35.8	0 4 54.688
800	0.0164302	102 59 27.8	171 5 39.8	0 6 32.333
1200	0.0162535	104 19 0.7	170 5 48.4	0 9 46.700
1600	0.0160731	105 39 0.3	169 5 59.8	0 12 59.799
2000	0.0158888	106 59 24.9	168 6 13.6	0 16 11.576
2400	0.0157009	108 20 18.9	167 6 29.3	0 19 21.978
3200	0.0153141	111 3 35.3	165 7 6.9	0 25 38.45
4000	0.0149133	113 49 3.6	163 7 51.8	0 31 48.79
4800	0.0144993	116 36 49.1	161 8 46.0	0 37 52.70
5600	0.0140723	119 26 59.1	159 9 46.5	0 43 49.70
6400	0.0136340	122 20 1.0	157 10 53.9	0 49 39.49
7200	0.0131843	125 16 6.1	155 12 7.3	0 55 21.67
+8000	0.0127243	128 15 28.5	153 13 26.4	1 0 55.94

TABLE IX.—*Precession of the Equinoxes and Obliquity of the Ecliptic.*

t	ψ	ϵ_1	ψ'	ϵ
—8000	—112° 24′ 57″.5	24° 24′ 51″.3	—109° 55′ 42.2″	24° 15′ 24″.8
7900	111 2 3.0	24 23 13.0	108 34 12.4	24 15 11.8
7800	109 39 3.5	24 21 35.7	107 12 41.6	24 14 57.9
7700	108 15 59.0	24 19 59.4	105 50 9.8	24 14 43.3
7600	106 52 49.6	24 18 24.3	104 29 37.0	24 14 28.0
7500	105 29 35.4	24 16 50.1	103 8 3.3	24 14 11.8
7400	104 6 16.6	24 15 17.2	101 46 28.5	24 13 55.0
7300	102 42 53.1	24 13 45.5	100 24 52.6	24 13 37.4
7200	101 19 25.3	24 12 15.1	99 3 15.7	24 13 19.0
7100	99 55 53.1	24 10 45.8	97 41 37.6	24 12 59.9
7000	98 32 16.8	24 9 18.0	96 19 58.3	24 12 40.1
6900	97 8 36.2	24 7 51.4	94 58 17.9	24 12 19.6
6800	95 44 51.7	24 6 26.3	93 36 36.5	24 11 58.4
6700	94 21 3.2	24 5 2.5	92 14 54.0	24 11 36.4
6600	92 57 11.0	24 3 40.2	90 53 10.2	24 11 13.8
6500	91 33 15.1	24 2 19.3	89 31 25.3	24 10 50.4
6400	90 9 15.8	24 1 0.0	88 9 39.2	24 10 26.4
6300	88 45 12.9	23 59 42.2	86 47 51.7	24 10 1.7
6200	87 21 6.9	23 58 25.9	85 26 3.2	24 9 36.4
6100	85 56 57.6	23 57 11.2	84 4 13.4	24 9 10.3
6000	84 32 45.3	23 55 58.1	82 42 22.3	24 8 43.6
5900	83 8 30.1	23 54 46.5	81 20 30.1	24 8 16.2
5800	81 44 12.1	23 53 36.5	79 58 36.5	24 7 48.2
5700	80 19 51.3	23 52 28.0	78 36 41.6	24 7 19.5
5600	78 55 28.1	23 51 21.2	77 14 45.3	24 6 50.2
5500	77 31 2.3	23 50 16.1	75 52 47.7	24 6 20.3
5400	76 6 34.3	23 49 12.7	74 30 48.7	24 5 49.8
5300	74 42 4.1	23 48 10.9	73 8 48.3	24 5 18.7
5200	73 17 31.9	23 47 10.8	71 46 46.5	24 4 47.0
5100	71 52 57.8	23 46 12.4	70 24 43.3	24 4 14.6
5000	70 28 21.9	23 45 15.8	69 2 38.5	24 3 41.7
4900	69 3 44.2	23 44 20.7	67 40 32.2	24 3 8.2
4800	67 39 5.1	23 43 27.4	66 18 24.5	24 2 34.2
4700	66 14 24.3	23 42 35.6	64 56 15.0	24 1 59.6
4600	64 49 42.4	23 41 45.6	63 34 4.2	24 1 24.5
4500	63 24 59.3	23 40 57.2	62 11 51.9	24 0 48.8
4400	62 0 15.2	23 40 10.6	60 49 38.0	24 0 12.6
4300	60 35 30.1	23 39 25.5	59 27 22.6	23 59 35.8
4200	59 10 44.2	23 38 42.1	58 5 5.4	23 58 58.6
4100	57 45 57.5	23 38 0.2	56 42 46.6	23 58 20.8
4000	56 21 10.3	23 37 20.0	55 20 26.0	23 57 42.6
3900	54 56 22.5	23 36 41.3	53 58 3.7	23 57 3.8
3800	53 31 34.4	23 36 4.3	52 35 39.6	23 56 24.6
3700	52 6 46.0	23 35 29.8	51 12 13.9	23 55 44.9
3600	50 41 57.6	23 34 55.0	49 50 46.4	23 55 4.8
3500	49 17 9.1	23 34 22.6	48 28 17.2	23 54 24.2
3400	47 52 20.7	23 33 51.8	47 5 46.1	23 53 43.2
3300	46 27 32.4	23 33 22.4	45 43 13.2	23 53 1.8
3200	45 2 44.5	23 32 54.5	44 20 38.4	23 52 20.0
3100	43 37 56.8	23 32 28.0	42 58 1.5	23 51 37.7
—3000	— 42 13 9.7	23 32 2.9	— 41 35 22.8	23 50 55.1

TABLE IX.—*Precession of the Equinoxes and Obliquity of the Ecliptic*—continued.

t	ψ		ψ'	ϵ
−3000	−42° 13′ 9.7	23° 32 2.9	−41° 35 22.8	23° 50 55.1
2900	40 48 23.2	23 31 39.2	40 12 42.3	23 50 12.1
2800	39 23 37.5	23 31 16.9	38 49 59.8	23 49 28.7
2700	37 58 52.4	23 30 55.9	37 27 15.3	23 48 45.0
2600	36 34 8.3	23 30 36.1	36 4 29.0	23 48 0.9
2500	35 9 25.1	23 30 17.6	34 41 40.7	23 47 16.5
2400	33 44 42.95	23 30 0.4	33 18 50.41	23 46 31.7
2300	32 20 1.86	23 29 44.3	31 55 57.95	23 45 46.5
2200	30 55 22.04	23 29 29.4	30 33 3.47	23 45 1.4
2100	29 30 43.50	23 29 15.6	29 10 6.89	23 44 15.5
2000	28 6 6.29	23 29 2.86	27 47 8.21	23 43 29.56
1900	26 41 30.59	23 28 51.20	26 24 7.52	23 42 43.32
1800	25 16 56.32	23 28 40.52	25 1 4.64	23 41 56.83
1700	23 52 23.62	23 28 30.80	23 37 59.64	23 41 10.09
1600	22 27 52.57	23 28 22.01	22 14 52.52	23 40 23.23
1500	21 3 23.21	23 28 14.08	20 51 43.23	23 39 36.05
1400	19 38 55.62	23 28 7.00	19 28 31.79	23 38 48.69
1300	18 14 29.87	23 28 0.72	18 5 18.21	23 38 1.17
1200	16 50 6.00	23 27 55.23	16 42 2.47	23 37 13.46
1100	15 25 44.02	23 27 50.47	15 18 44.50	23 36 25.56
1000	14 1 24.01	23 27 46.26	13 55 24.33	23 35 37.46
900	12 37 6.01	23 27 42.72	12 32 1.95	23 34 49.26
800	11 12 50.06	23 27 39.75	11 8 37.37	23 34 0.98
700	9 48 36.20	23 27 37.31	9 45 10.55	23 33 12.55
600	8 24 24.452	23 27 35.352	8 21 41.494	23 32 23.960
500	7 0 14.858	23 27 33.823	6 58 10.204	23 31 35.302
400	5 36 7.442	23 27 32.680	5 34 36.670	23 30 46.562
300	4 12 2.229	23 27 31.874	4 11 0.886	23 29 57.750
200	2 47 59.240	23 27 31.357	2 47 22.850	23 29 8.877
−100	− 1 23 58.492	23 27 31.082	−1 23 42.555	23 28 19.956
0	0 0 0.000	23 27 31.000	0 0 00.000	23 27 31.000
+100	+ 1 23 56.225	23 27 31.063	+1 23 44.818	23 26 42.020
200	2 47 50.175	23 27 31.223	2 47 31.903	23 25 53.027
300	4 11 41.846	23 27 31.432	4 11 21.255	23 25 4.034
400	5 35 31.238	23 27 31.640	5 35 12.875	23 24 15.054
500	6 59 18.351	23 27 31.800	6 59 6.765	23 23 26.097
600	8 23 3.192	23 27 31.863	8 23 2.924	23 23 37.175
800	11 10 26.10	23 27 31.51	11 11 2.05	23 20 59.49
1200	16 44 45.24	23 27 27.53	16 47 27.49	23 17 44.78
1600	22 18 30.18	23 27 16.70	22 24 29.04	23 14 32.43
2000	27 51 43.22	23 26 56.1	28 2 6.45	23 11 22.40
2400	33 24 27.47	23 26 23.0	33 40 19.39	23 8 15.7
3200	44 28 45.5	23 24 29.1	44 58 29.9	23 2 14.7
4000	55 32 2.9	23 21 17.2	56 18 55.9	22 56 34.9
4800	66 35 7.2	23 16 33.7	67 41 31.2	22 51 21.2
5600	77 38 52.4	23 10 10.4	79 6 6.5	22 46 38.8
6400	88 44 16.7	23 2 4.8	90 32 30.1	22 42 32.0
7200	99 52 19.0	22 52 20.6	102 0 25.8	22 39 4.8
+8000	+111 3 56.2	22 41 7.7	+113 29 33.3	22 36 21.0

TABLE X.—*For Precession in Right Ascension and Declination.*

	z	z'	$z+z'$	Θ	ϑ
−8000	−53° 0′ 2″.8	−54° 34′ 37″.2	−107° 34′ 40″.0	−39° 24′ 51″.5	+2° 43′ 48″.27
7900	52 17 37.9	53 51 58.1	106 9 36.0	39 3 30.2	2 42 13.99
7800	51 35 17.9	53 9 19.7	104 44 37.6	38 41 50.0	2 40 35.30
7700	50 53 3.0	52 26 42.0	103 19 45.0	38 19 51.0	2 38 52.32
7600	50 10 53.0	51 44 5.1	101 54 58.1	37 57 33.4	2 37 5.17
7500	49 28 48.0	51 1 29.0	100 30 17.0	37 34 57.4	2 35 13.97
7400	48 46 48.0	50 18 53.9	99 5 41.9	37 12 3.4	2 33 18.85
7300	48 4 53.0	49 36 20.0	97 41 13.0	36 48 51.6	2 31 19.92
7200	47 23 3.1	48 53 47.4	96 16 50.5	36 25 22.2	2 29 17.35
7100	46 41 18.3	48 11 16.0	94 52 34.3	36 1 35.6	2 27 11.25
7000	45 59 38.6	47 28 46.1	93 28 24.7	35 37 31.8	2 25 1.74
6900	45 18 3.8	46 46 17.6	92 4 21.4	35 13 11.2	2 22 48.98
6800	44 36 34.1	46 3 50.9	90 40 25.0	34 48 34.1	2 20 33.11
6700	43 55 9.3	45 21 25.8	89 16 35.1	34 23 40.6	2 18 14.29
6600	43 13 49.5	44 39 2.5	87 52 52.0	33 58 31.0	2 15 52.63
6500	42 32 34.8	43 56 41.1	86 29 15.9	33 33 5.7	2 13 28.32
6400	41 51 25.0	43 14 21.9	85 5 46.9	33 7 24.9	2 11 1.44
6300	41 10 20.0	42 32 4.7	83 42 24.7	32 41 28.8	2 8 32.18
6200	40 29 19.9	41 49 49.7	82 19 9.6	32 15 17.7	2 6 0.77
6100	39 48 24.7	41 7 37.0	80 56 1.7	31 48 51.7	2 3 27.23
6000	39 7 34.3	40 25 26.7	79 33 1.0	31 22 11.3	2 0 51.77
5900	38 26 48.9	39 43 18.8	78 10 7.7	30 55 16.6	1 58 14.56
5800	37 46 8.2	39 1 13.4	76 47 21.6	30 28 7.9	1 55 35.72
5700	37 5 32.2	38 19 10.4	75 24 42.6	30 0 45.4	1 52 55.44
5600	36 25 0.8	37 37 10.3	74 2 11.1	29 33 9.4	1 50 13.84
5500	35 44 34.0	36 55 12.8	72 39 46.8	29 5 20.0	1 47 31.13
5400	35 4 11.7	36 13 18.3	71 17 30.0	28 37 17.8	1 44 47.44
5300	34 23 53.9	35 31 26.8	69 55 20.7	28 9 2.9	1 42 2.93
5200	33 43 40.5	34 49 38.4	68 33 18.9	27 40 35.6	1 39 17.77
5100	33 3 31.5	34 7 52.9	67 11 24.4	27 11 56.0	1 36 32.12
5000	32 23 26.8	33 26 10.7	65 49 37.5	26 43 4.6	1 33 46.13
4900	31 43 26.4	32 44 31.8	64 27 58.2	26 14 1 4	1 30 59.98
4800	31 3 30.1	32 2 56.0	63 6 26.1	25 44 46.9	1 28 13.80
4700	30 23 37.9	31 21 23.6	61 45 1.5	25 15 21.1	1 25 27.78
4600	29 43 49.7	30 39 54.5	60 23 44.2	24 45 44.4	1 22 42.05
4500	29 4 5.6	29 58 29.1	59 2 34.7	24 15 57.0	1 19 56.77
4400	28 24 25.2	29 17 6.9	57 41 32.1	23 45 59.3	1 17 12.11
4300	27 44 48.8	28 35 48.5	56 20 37.3	23 15 51.4	1 14 28.22
4200	27 5 16.1	27 54 33.6	54 59 49.7	22 45 33.7	1 11 45.26
4100	26 25 47.2	27 13 22.4	53 39 9.6	22 15 6.2	1 9 3.36
4000	25 46 21.8	26 32 14.8	52 18 36.6	21 44 29.4	1 6 22.69
3900	25 7 0.1	25 51 10.7	50 58 10.8	21 13 43.5	1 3 43.39
3800	24 27 41.6	25 10 10.7	49 37 52.3	20 42 48.7	1 1 5.62
3700	23 48 26.2	24 29 14.7	48 17 40.9	20 11 45.3	0 58 29.51
3600	23 9 14.0	23 48 22.5	46 57 36.5	19 40 33.5	0 55 55.21
3500	22 30 5.2	23 7 34.1	45 37 39.3	19 9 13.6	0 53 22.87
3400	21 50 59.3	22 26 49.7	44 17 49.0	18 37 45.9	0 50 52.63
3300	21 11 56.5	21 46 9.1	42 58 5.6	18 6 10.5	0 48 24.62
3200	20 32 56.4	21 5 32.5	41 38 28.9	17 34 27.8	0 45 58.99
3100	19 53 59.1	20 24 59.8	40 18 58.9	17 2 38.0	0 43 35.87
−3000	−19 15 4.4	−19 44 31.1	−38 59 35.5	−16 30 41.3	+0 41 15.38

Table X.—*For Precession in Right Ascension and Declination*—continued.

t	z	z'	$z+z'$	Θ	ϑ
−3000	−19° 15 4.4	−19° 44 31.1	−38° 59′ 35″.5	−16° 30′ 41″.3	+0° 41′ 15.″38
2900	18 36 12.5	19 4 6.5	37 40 19.0	15 58 38.1	0 38 57.66
2800	17 57 22.9	18 23 45.9	36 21 8.8	15 26 28.5	0 36 42.82
2700	17 18 35.8	17 43 29.4	35 2 5.2	14 54 12.8	0 34 30.99
2600	16 39 50.8	17 3 16.8	33 43 7.6	14 21 51.3	0 32 22.29
2500	16 1 8.2	16 23 8.2	32 24 16.4	13 49 24.2	0 30 16.84
2400	15 22 27.5	15 43 3.7	31 5 31.2	13 16 51.7	0 28 14.77
2300	14 43 49.0	15 3 3.0	29 46 52.0	12 44 14.2	0 26 16.14
2200	14 5 12.2	14 23 6.4	28 28 18.6	12 11 31.8	0 24 21.04
2100	13 26 37.1	13 43 13.9	27 9 51.0	11 38 44.8	0 22 29.62
2000	12 48 3.7	13 3 25.3	25 51 29.0	11 5 53.5	0 20 42.00
1900	12 9 31.9	12 23 40.6	24 33 12.5	10 32 58.1	0 18 58.24
1800	11 31 1.5	11 43 59.8	23 15 1.3	9 59 58.8	0 17 18.45
1700	10 52 32.5	11 4 22.9	21 56 55.4	9 26 56.0	0 15 42.70
1600	10 14 4.6	10 24 49.7	20 38 54.3	8 53 49.8	0 14 11.08
1500	9 35 38.2	9 45 20.3	19 20 58.5	8 20 40.5	0 12 43.68
1400	8 57 12.6	9 5 54.6	18 3 7.2	7 47 28.4	0 11 20.55
1300	8 18 47.8	8 26 33.0	16 45 20.5	7 14 13.7	0 10 1.77
1200	7 40 23.7	7 47 14.9	15 27 38.6	6 40 56.6	0 8 47.43
1100	7 2 0.5	7 8 0.4	14 10 0.9	6 7 37.5	0 7 37.59
1000	6 23 37.8	6 28 49.4	12 52 27.2	5 34 16.5	0 6 32.30
900	5 45 15.8	5 49 41.9	11 34 57.7	5 0 53.9	0 5 31.61
800	5 6 54.0	5 10 37.8	10 17 31.8	4 27 30.0	0 4 35.58
700	4 28 32.5	4 31 37.3	9 0 9.8	3 54 4.9	0 3 44.26
600	3 50 11.032	3 52 39.776	7 42 50.808	3 20 39.064	0 2 57.696
500	3 11 49.738	3 13 45.575	6 25 35.313	2 47 12.593	0 2 15.920
400	2 33 28.307	2 34 54.553	5 8 22.860	2 13 45.771	0 1 38.971
300	1 55 6.727	1 56 6.585	3 51 13.312	1 40 18.850	0 1 6.879
200	1 16 44.874	1 17 21.582	2 34 6.456	1 6 52.081	0 0 39.673
−100	−0 38 22.631	−0 38 39.455	−1 17 2.086	−0 33 25.715	+0 0 17.373
0	0 0 0.000	0 0 0.000	0 0 0.000	0 0 0.000	0 0 0.000
+100	+0 38 23.482	+0 38 36.527	+1 17 0.009	+0 33 24.811	−0 0 12.433
200	1 16 47.594	1 17 10.547	2 33 58.141	1 6 48.471	−0 0 19.917
300	1 55 12.512	1 55 42.112	3 50 54.624	1 40 10.724	−0 0 22.444
400	2 33 38.370	2 34 11.277	5 7 49.647	2 13 31.327	−0 0 20.013
500	3 12 5.23	3 12 38.10	6 24 43.33	2 46 50.04	−0 0 12.626
600	3 50 33.13	3 51 2.64	7 41 35.77	3 20 6.62	−0 0 0.292
800	5 7 33.36	5 7 46.46	10 15 19.82	4 26 32.15	+0 0 39.178
1200	7 41 52.11	7 40 52.87	15 22 44.99	6 38 46.37	0 2 56.756
1600	10 16 41.14	10 13 38.94	20 30 20.08	8 49 57.76	0 6 30.865
2000	12 52 7.6	12 46 14.2	25 38 21.8	10 59 50.14	0 11 18.674
2400	15 28 17.5	15 18 48.0	30 47 5.5	13 8 7.12	0 17 16.358
3200	20 43 12.7	20 24 34.2	41 7 46.9	17 18 49.10	0 32 21.726
4000	26 2 13.6	25 32 26.2	51 34 39.7	21 19 51.28	0 50 59.337
4800	31 26 0.6	30 43 56.8	62 9 57.4	25 9 0.00	1 12 10.125
5600	36 55 6.4	36 0 38.9	72 55 45.3	28 44 0.89	1 34 44.92
6400	42 29 53.8	41 24 1.0	83 53 54.8	32 2 40.02	1 57 27.37
7200	48 10 33.3	46 55 21.8	95 5 55.1	35 2 45.43	2 18 55.87
+8000	53 57 1.6	52 35 46.6	+106 32 48.1	+37 42 10.22	+2 37 46.67