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MENSURATION FOR BEGINNERS.

MENSURATION FOR BEGINNERS

WITH NUMEROUS EXAMPLES.

BY

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PREFACE.

THE applications of Mensuration are numerous and important, extending from practical questions of every day life to the highest investigations of science. Mensuration has therefore justly obtained a prominent position in teaching and in examinations. The value of Mensuration is increased by its close and necessary connection with Arithmetic and Geometry. The operations of Arithmetic are here illustrated by a new class of examples; and some of these operations, such as the extraction of the square root and of the cube root, thus gain their true significance and interest. Many of the principal facts of Geometry are introduced and applied, so as to furnish a good introduction to the study of Euclid's Elements, or some substitute for it, to those who have not the opportunity for that study.

The subjects included in the present work are those which have usually found a place in Elementary Treatises on Mensuration. The mode of treatment has been determined by the fact that the work is intended for the use of beginners. Accordingly it is divided into short independent chapters, which are followed by appropriate examples. A knowledge of the elements of Arithmetic is all that is assumed; and in connection with most of the Rules of Mensuration it has been found practicable to give such explanations and illustrations as will supply the place of formal mathematical demonstrations, which would have been unsuitable to the character of the work.

The Examples amount on the whole to nearly twelve hundred in number; some of them are taken from printed examination papers, but most of them are original, and have been expressly constructed with reference to the most important points, and to the usual difficulties of beginners. The miscellaneous collection of Examples at the end of the book is arranged in sets of ten in each set.

Although great care has been taken to ensure accuracy, it can hardly be hoped that a book involving so large an amount of numerical calculation will be free from error. Any corrections or remarks relating to the book will be most thankfully received.

I. TODHUNTER.

CAMBRIDGE,
April, 1869.

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INTRODUCTION.

MENSURATION gives rules for estimating *lengths, areas*, and *volumes*.

We shall assume that the beginner in Mensuration is familiar with the elements of Arithmetic, including the process for the extraction of the square root of a number.

We shall also assume that he is familiar with the use of certain convenient symbols, namely that + denotes addition, − denotes subtraction, × denotes multiplication, ÷ denotes division, and √ denotes the square root.

Some knowledge of Geometry is also necessary; and accordingly the first three Chapters of the book treat of that subject. The beginner should at once read carefully the first Chapter, in which various terms are defined, which we shall have to employ hereafter; he will however probably find that he is already familiar with the meaning of many of these terms from the common use of them. He can then proceed to the fourth and the following Chapters, referring to the second and the third as occasion may require.

FIRST SECTION. GEOMETRY.

I. DEFINITIONS.

1. The words *point* and *line* are too well known to require any definition; but a caution must be given with respect to the strict sense in which these words are used in Geometry.

A point is represented in a printed book by a spot of ink, which may be very small, but still *has some size;* we must not however suppose that a point in Geometry has any size.

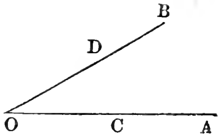
Lines may be straight or curved. A line is represented in a printed book by a band of ink, which may be very narrow, but still *has some breadth;* we must not however suppose that a line in Geometry has any breadth.

2. The word *surface* is also in common use. Surfaces may be flat or curved. A flat surface is usually called a *plane* surface in Geometry. We must not suppose that a surface in Geometry has any thickness.

3. Thus we may say that a point has neither length, breadth, nor thickness; a line has only length; a surface has length and breadth. A solid body has length, breadth, and thickness. We shall not consider solid bodies until we arrive at the Fourth Section of the book; in the first Three Sections we shall consider only lines and figures on a plane surface.

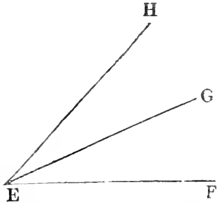
4. An angle is the inclination of two straight lines to one another which meet together, but are not in the same straight line.

Thus the two straight lines *AO*, *BO*, which meet at *O*, form an angle there. The angle is not altered by altering the lengths of the straight lines which form it: thus *CO* and *DO* form the same angle as *AO* and *BO*.



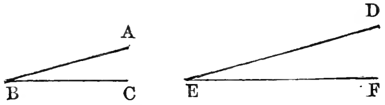
5. If there is only one angle formed at a point, the angle may be denoted by a single letter placed at the angular point: thus, for example, the angle in the preceding Article may be called the angle *O*.

If several angles are formed at the same point each angle is denoted by three letters, the middle letter being at the angular point, and the other two letters being on the straight lines which form the angle, namely one letter on each straight line. Thus *FEG* denotes the angle formed by *FE* and *GE*; *GEH* denotes the angle formed by *GE* and *HE*; *FEH* denotes the angle formed by *FE* and *HE*.



6. If one angle can be so placed on another that the straight lines which form the one angle fall exactly on the straight lines which form the other angle, the two angles are said to be *equal.*

Thus, if we place *BC* on *EF* so that *B* is on *E*, and then find that *BA* falls on *ED*, the angle *ABC* is said to

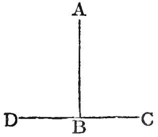


be equal to the angle *DEF*. It is necessary to have this distinct notion of what is meant by the *equality of angles;* and the beginner may actually try whether two assigned angles are equal by cutting them out in paper and placing one on the other. But in the theory of Geometry methods are devised by which angles can be shewn to be equal without actually placing one on the other.

7. Suppose in the diagram to Art. 5 that the angle FEG is equal to the angle GEH; then the whole angle FEH is *twice* the angle FEG. Similarly it is easy to understand what is meant by the statement that a certain angle is *three times* another angle, or *four times* another angle; and so on.

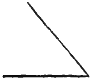
8. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it.

Thus, in the figure, if the angle ABC is equal to the angle ABD each of them is a right angle, and AB is perpendicular to DC.



An obtuse angle is an angle which is greater than a right angle.

An acute angle is an angle which is less than a right angle.



9. Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet.

10. A rectilineal figure is a figure which is bounded by straight lines; the boundaries of the figure are called sides.

A triangle is a rectilineal figure with three sides.

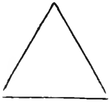
A quadrilateral is a rectilineal figure with four sides.

Any rectilineal figure with more than four sides is called a polygon; if it has five sides it is called a pentagon, if it has six sides it is called a hexagon, and so on.

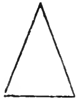
A regular polygon is one which has all its sides equal, and all its angles equal.

11. The following names are used for various kinds of triangles:

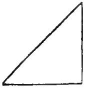
An equilateral triangle is one which has all its sides equal.



An isosceles triangle is one which has two of its sides equal.



A right-angled triangle is one which has a right angle.



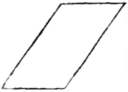
In a right-angled triangle it is found convenient to use the word *sides* only for the two straight lines which *include* the right angle: the straight line which is opposite the right angle is called the *hypotenuse*.

An obtuse-angled triangle is one which has an obtuse angle.

An acute-angled triangle is one which has three acute angles.

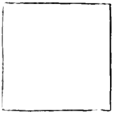
12. The following names are used for various kinds of quadrilaterals :

A parallelogram has its opposite sides parallel and equal.

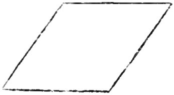


A rectangle is a parallelogram with all its angles right angles.

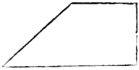
A square is a rectangle with all its sides equal.



A rhombus is a parallelogram which has all its sides equal, but its angles are not right angles.



A trapezoid has two sides parallel.

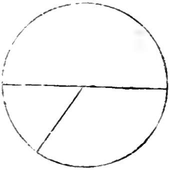


13. Any side of a triangle may be called the base; then the height of the triangle is the perpendicular drawn from the opposite angular point on the base.

Any side of a parallelogram may be called the base; then the height of the parallelogram is the perpendicular drawn from any point in the opposite side on the base.

14. A diagonal of a quadrilateral is a straight line joining two opposite angles. A straight line joining two angles of a polygon which are not adjacent is also called a diagonal of the polygon.

15. A circle is a plane figure bounded by one line which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another: this point is called the centre of the circle.



A radius of a circle is a straight line drawn from the centre to the circumference.

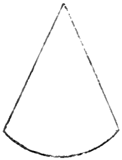
A diameter of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

An arc of a circle is any part of the circumference.

A chord of a circle is the straight line which joins the ends of an arc.

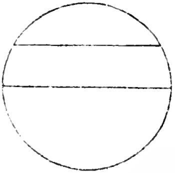
A segment of a circle is the figure bounded by a chord and the arc it cuts off.

A sector of a circle is the figure bounded by two radii and the arc between them.



The angle formed by the two radii is called the angle of the sector.

A zone of a circle is the portion of the circle contained between two parallel chords.



II. THEOREMS.

16. We shall now state some important geometrical facts which it will be advantageous to remember. These facts, with many others, are *demonstrated* in Euclid's Elements of Geometry, that is they are shewn by strict reasoning to be necessarily true: and all who have the opportunity should study Euclid's demonstrations. But the present work may be used by those who have not yet applied themselves to demonstrative Geometry; and so they may for some time be satisfied with understanding the meaning of the statements which we shall make and committing them to memory.

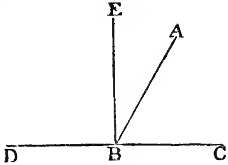
When the demonstration is of a very simple character we shall however give the substance of it, and an attempt to master this, even if at first not entirely successful, will prove very beneficial. Some of the statements for which we give no demonstration will appear almost self-evident; others may be verified by repeated practical measurement; so that at last a confidence in the truth of all may be gained, approaching to absolute conviction.

17. We have selected only a few of the most important geometrical facts out of the large collection which has been formed by the investigations of mathematicians; but these specimens will be sufficient to suggest some idea of the extent and variety of the results which follow by strict connexion from a few elementary principles, and may tempt the student to increase his knowledge of the subject hereafter.

It will be found that Arts. 18...21 relate to angles, Arts. 22...27 relate to triangles, Arts. 28...30 relate to the equivalence of areas, Arts. 31...33 relate to properties of the circle, and Arts. 34...38 to similar triangles.

18. *Let the straight line* AB *make with the straight line* CD *on one side of it the angles* ABC *and* ABD: *these angles will be together equal to two right angles.*

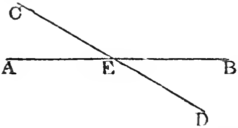
For let BE be at right angles to DC. Then the angle ABD is the sum of the angles ABE and EBD; so that the sum of the two angles *ABC* and *ABD* is equal to the sum of the three angles *ABC*, *ABE*, and *EBD*.



But the angle *EBD* is a right angle; and the sum of the angles *ABC* and *ABE* is *EBC*, which is also a right angle.

Therefore the angles *ABC* and *ABD* are together equal to two right angles.

19. *Let two straight lines* AB *and* CD *cut one another at* E; *the angle* AEC *will be equal to the angle* BED, *and the angle* AED *will be equal to the angle* BEC.

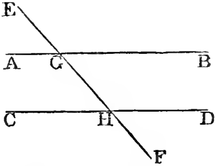


For the angles *AEC* and *CEB* are together equal to two right angles, by Art. 18; and so also are the angles *CEB* and *BED*. Thus the angles *AEC* and *CEB* are together equal to the angles *CEB* and *BED*. Therefore the angle *AEC* must be equal to the angle *BED*.

In a similar manner we can shew that the angle *AED* is equal to the angle *BEC*.

The angles *AEC* and *BED* are called *vertically opposite angles;* and so also are the angles *AED* and *BEC*.

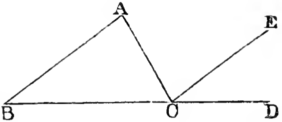
20. *Let the straight line* EF *cut the parallel straight lines* AB, CD: *the angle* EGB *will be equal to the angle* GHD, *and the two angles* BGH *and* GHD *will be together equal to two right angles.*



21. Since the angle *EGB* is equal to the angle *AGH* by Art. 19, it follows from the first part of the preceding Article that the angle *AGH* is equal to the angle *GHD*: these angles are called *alternate angles.*

So also the angle *BGH* is equal to the alternate angle *GHC*.

22. *Let* BC *a side of the triangle* ABC *be produced to* D ; *the exterior angle* ACD *will be equal to the two interior and opposite angles.*



For suppose *CE* to be parallel to *BA*. Then the angle *ECD* is equal to the angle *ABC* by Art. 20; and the angle *ACE* is equal to the angle *BAC* by Art. 21. Thus the whole angle *ACD* is equal to the sum of the two angles *ABC* and *BAC*.

23. *The three angles of any triangle are together equal to two right angles.*

For by Art. 22 the sum of the angles *ABC* and *BAC* is equal to the angle *ACD*. Thus the sum of the three angles *ABC*, *BAC*, and *ACB* is equal to the sum of the two angles *ACD* and *ACB*; that is to two right angles, by Art. 18.

24. *If two sides of a triangle are equal the angles opposite to them will also be equal.*

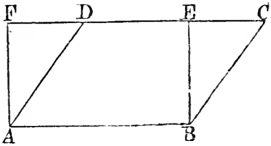
25. *If two angles of a triangle are equal the sides opposite to them will also be equal.*

26. *If two sides of one triangle are equal to two sides of another triangle, each to each, and the angle contained by the two sides of the one equal to the angle contained by the two sides of the other, the triangles will be equal in all respects.*

27. *If two angles of one triangle are equal to two angles of another triangle, each to each, and the side adjacent to the two angles of the one equal to the side adjacent to the two angles of the other, the triangles will be equal in all respects.*

28. *A parallelogram is equivalent to a rectangle on the same base and between the same parallels.*

Let *ABCD* be a parallelogram, and *ABEF* a rectangle, on the same base *AB*, and between the same parallels AB and FC; the parallelogram is equivalent to the rectangle, that is the two figures are of the same size.

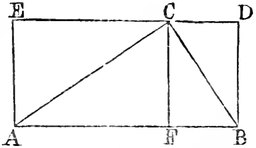


It is in fact easy to admit that the triangle BEC is equal to the triangle AFD; and hence it follows that $ABCD$ is equivalent to $ABEF$.

Instead of saying that the parallelogram and the rectangle are between the same parallels, we may say that they have the same height: see Art. 13.

29. *A triangle is equivalent to half a rectangle having the same base and height.*

Let ABC be a triangle, and $ABDE$ a rectangle, on the same base AB, and having the same height: the triangle is equivalent to half the rectangle.

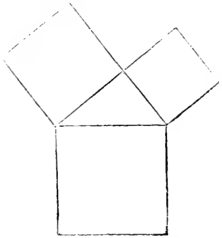


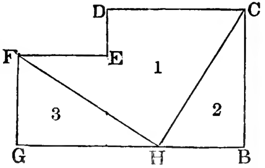
Let CF be the perpendicular from C on AB. It is easy to admit that the triangle BFC is equal to the triangle CDB, and that the triangle AFC is equal to the triangle CEA; and hence it follows that ABC is equivalent to half $ABDE$.

Hence two triangles which have the same base and equal heights are equivalent.

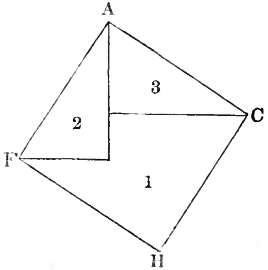
30. *In any right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides.*

The figure represents a right-angled triangle having squares described on its hypotenuse and its sides: the largest square is equal in size to the sum of the other two. This statement is one of the most important in Geometry; and we will shew how its truth may be illustrated.



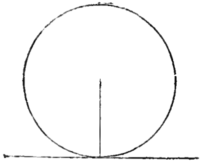


Let *BCDEFG* be a figure composed of two squares placed side by side: take *GH* equal to *BC*, and draw the straight lines *CH* and *FH*. Cut the whole figure out in paper or cardboard, and then divide it into the three pieces marked 1, 2, and 3. Fit the pieces together in the manner indicated by the figure *HCAF*. It will be found that a single square is thus obtained, each side being equal to *FH*.

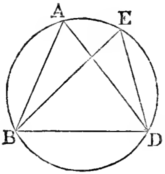


Hence we see that the square described on *FH* is equal to the sum of the squares described on *FG* and *GH*.

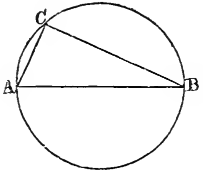
31. *If a straight line touch a circle the radius drawn to the point of contact will be perpendicular to the straight line.*



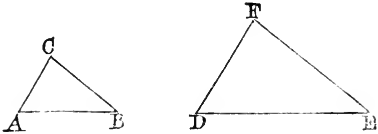
32. *Let* BAD *and* BED *be angles in the same segment* BAED *of a circle: these angles will be equal.*



33. *Let* AB *be a diameter of a circle, and* C *any point on the circumference; draw the straight lines* AC *and* BC: *then the angle* ACB *will be a right angle.*



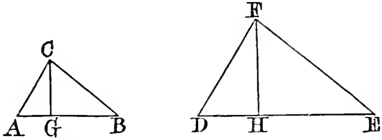
34. *Let* ABC *and* DEF *be two triangles such that the angle* A *is equal to the angle* D, *the angle* B *to the*



angle E, *and the angle* C *to the angle* F: *then the sides opposite the equal angles will be proportionals.*

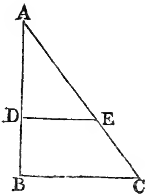
That is, if *EF* be double of *BC*, then *FD* is double of *CA*, and *DE* is double of *AB*; if *EF* be three times *BC*, then *FD* is three times *CA*, and *DE* is three times *AB*; and so on. The two triangles are said to be *similar*.

35. *Let* ABC *and* DEF *be similar triangles, the angles* C *and* F *being corresponding angles; let* CG *and* FH



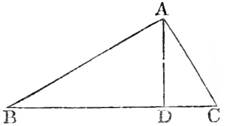
be perpendiculars from C *and* F *on the opposite sides: then* CG *will be to* AB *as* FH *is to* DE.

36. *Let* ABC *be a triangle, and* DE *a straight line parallel to the side* BC, *and meeting the other sides* AB *and* AC: *then the triangles* ABC *and* ADE *will be similar.*

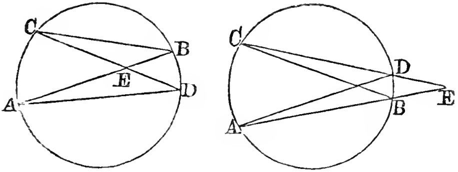


See Art. 20.

37. *Let* ABC *be a right-angled triangle, and let* AD *be the perpendicular from the right angle on the hypotenuse: then the triangles* DBA *and* DAC *will be similar to the triangle* ABC.



38. *Let* AB *and* CD *be two chords of a circle; let them meet, produced if necessary, at* E; *join* BC *and*



AD: *then the triangles* AED *and* BEC *will be similar, the angles* EAD *and* ECB *being equal, and the angles* EDA *and* EBC *being equal.*

See Art. 32

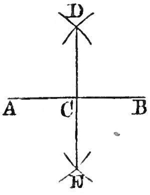
III. PROBLEMS.

39. We shall now give the solutions of a few problems which occur in practice when it is necessary to draw figures accurately. We suppose that a ruler and compasses are employed; these instruments will be sufficient for our purpose. Other instruments are often useful, such as a square and parallel rulers; but they are not absolutely necessary.

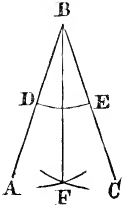
The solutions which we shall give of the problems depend mainly on the principles stated in Chapter II; there will not be much difficulty in verifying practically the correctness of the results, and those who make themselves acquainted with the elements of demonstrative Geometry will perceive the rigorous exactness of the processes.

40. *To divide a given straight line into two equal parts.*

Let AB be the given straight line. From the centres A and B with a radius greater than half AB describe arcs cutting each other at D and E. Join DE, cutting AB at C. Then AC will be equal to CB. The straight line DE will be at right angles to AB, so that we see how to draw a straight line which shall be at right angles to a given straight line and shall also divide it into two equal parts.



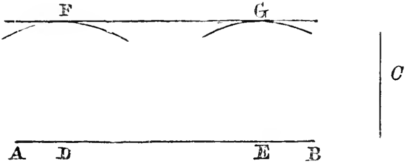
41. *To divide a given angle into two equal parts.*



Let ABC be the given angle. From the centre B with any radius describe an arc, cutting BA at D, and BC at E. From D and E as centres with any sufficient radius describe arcs cutting each other at F. Join BF. Then the angle ABF will be equal to the angle CBF.

42. *To draw a straight line parallel to a given straight line, and at a given distance from it.*

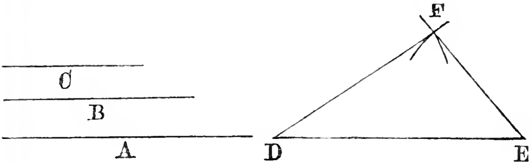
Let AB be the given straight line, and let C be equal to the given distance. From any two points D and E in



AB as centres, with a radius equal to C, describe arcs. Draw a straight line FG touching these arcs. Then FG will be parallel to AB, and at a distance from it equal to C.

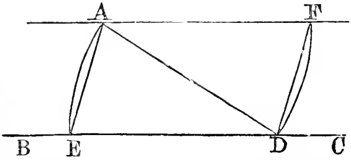
43. *To make a triangle having its sides equal to three given straight lines.*

Let A, B, and C be the given straight lines.



Draw a straight line DE equal to one of the given straight lines, A. From the centre D, with a radius equal to B, describe an arc; and from the centre E, with a radius equal to C, describe another arc. Let these arcs cut each other at F. Join DF and EF. Then DEF will be the triangle required.

44. *Through a given point to draw a straight line parallel to a given straight line.*

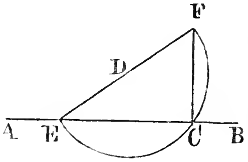


Let A be the given point, and BC the given straight line. Take any point D in BC. From the centre D, with the radius DA, describe an arc, cutting BC at E, and draw the chord AE. From the centre A, with the radius AD, describe an arc, and draw the chord DF equal to the chord AE. Join AF. Then AF will be parallel to BC.

45. *To draw a straight line at right angles to a given straight line from a given point in it.*

Let AB be the given straight line, and C the given point in it.

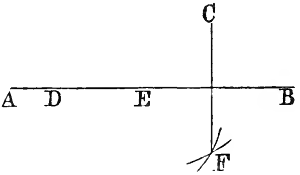
From any point D without the straight line as centre, with radius DC, describe a circle, cutting the given straight line at E. Join ED, and produce it to meet the circumference again at F. Join CF. Then CF will be at right angles to AB.



46. *To draw a perpendicular to a given straight line from a given point without it.*

Let AB be the given straight line, and C the given point without it.

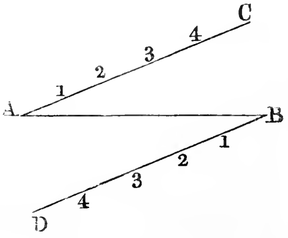
Take any two points D and E in AB. From the centre D, with the radius DC, describe an arc; and from the centre E, with the radius EC, describe another arc. Let these arcs cut each other again at F. Join CF. Then CF will be perpendicular to AB.



47. *To divide a given straight line into any number of equal parts.*

Let AB be the given straight line. Suppose it is to be divided into *five* equal parts.

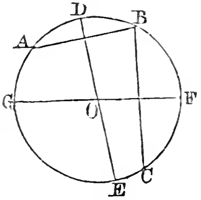
From A draw any straight line AC, and from B draw the straight line BD parallel to AC. Set off along AC *four* lengths, all equal, and mark the points of division 1, 2, 3, 4. Set off



along BD four lengths all equal to the former lengths, and mark the points of division 1, 2, 3, 4. Draw straight lines joining 1 to 4, 2 to 3, 3 to 2, and 4 to 1. These straight lines will divide AB into *five* equal parts.

We should proceed in a similar way whatever be the number of equal parts into which AB is to be divided.

48. *To find the centre of a given circle.*



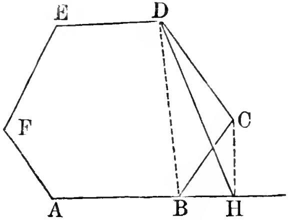
Draw any chord AB, and divide it into two equal parts by the straight line DE at right angles to it. Then the centre of the circle is in DE.

Draw any other chord BC, and divide it into two equal parts by the straight line FG at right angles to it. Then the centre of the circle is in FG.

Thus the centre of the circle is at O, the point of intersection of DE and FG.

This process also shews how to describe a circle which shall pass through three given points A, B, C.

49. *Any polygon being given it is required to construct an equivalent polygon with one side fewer.*



Let $ABCDEF$ be a polygon. Join BD. Through C draw a straight line parallel to BD meeting AB produced at H. Join DH.

Then the triangle BCD is equivalent to the triangle BHD, by Art. 29; and therefore the polygon $ABCDEF$ is equivalent to the polygon $AHDEF$, that is to a polygon with one side fewer.

By repeated operations of this kind we can construct a triangle equivalent to any given polygon.

50. *To construct and use a decimal diagonal scale.*

Take a straight line AB of any convenient length; produce it until the whole straight line is ten times AB, and take BC, CD, ... all equal to AB. Draw any straight line parallel to this, and in it set off ten distances ab, bc, cd, ... all equal to AB. Draw Aa, Bb, Cc, Dd, ... Divide Aa into ten equal parts, and through the points of division draw straight lines parallel to AB. Divide BA into ten equal parts, and mark the points of division by the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. Divide ba into ten equal parts.

Join the points of division of BA and ba by diagonal straight lines; that is join B with the point adjacent to b, join 1 with the next point in ba; join 2 with the next point; and so on.

Thus the scale is constructed.

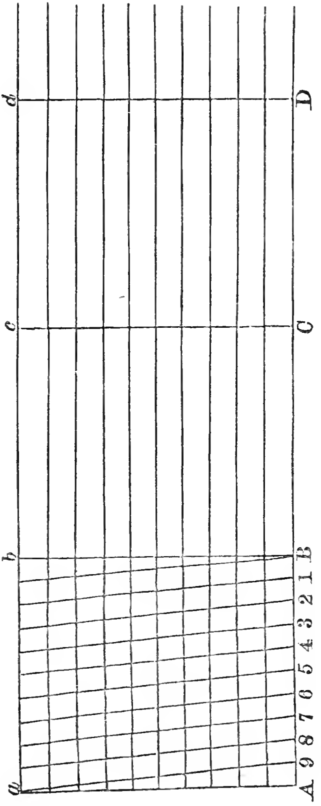
It will be seen that our figure does not represent all the scale, the part beyond Dd being omitted for want of space. It is usual, but not absolutely necessary, to make Aa perpendicular to AB.

The scale is used for finding straight lines of given lengths, and for finding the lengths of given straight lines.

Suppose, for example, that AB represents one inch, and let it be required to find the straight line which represents 2·57 inches. Place one foot of a pair of compasses at D, and open the compasses until the other foot comes to 5; thus we have the length 2·5, which is nearly what is required. Now move one foot of the compasses along Dd, and the other foot along the diagonal straight line which begins at 5, the compasses being at the same time spread to the necessary extent: when both feet are on the seventh straight line parallel to AD the distance between the feet will represent 2·57 inches.

If instead of representing one inch, AB represents 10 inches, then the distance just found will represent 25·7 inches; and if AB represents 100 inches, then the distance just found will represent 257 inches.

Next suppose we have to find the length of a given straight line. Open the compasses so that the feet may fall on the ends of the given straight line. Move one foot of the compasses along one of the direct straight lines Bb, Cc, Dd, ..., and the other foot along one of the diagonal straight lines, until by trial it is found that the two feet fall on two points of division on one of the straight lines parallel to AD: then the length will be known. For example; suppose one foot of the compasses on Cc, and the other on the diagonal straight line which begins at 9, and both feet on the fifth straight line parallel to AB; then the required length is 1·95 times the length of AB.



51. In a similar manner we can construct a *duodecimal* diagonal scale; we shall only have to change the number *ten*, which occurs in the process already given, to the number *twelve*. Then by the same method as we found a straight line of length 2·57 we should, with the aid of the duodecimal diagonal scale, find a straight line of length $2+\frac{5}{12}+\frac{7}{144}$; so that if AB represents one foot we should find a straight line representing the length 2 feet $5\frac{7}{12}$ inches. Similarly the length of a distance determined as in the second example of Art. 50 will now be to the length of AB as $1+\frac{9}{12}+\frac{5}{144}$ is to 1; so that if AB represents one foot, the distance represents 1 foot $9\frac{5}{12}$ inches.

SECOND SECTION. LENGTHS.

IV. TABLES OF LINEAL MEASURE.

52. The student is probably already acquainted with the Table of Measures of Length; but for convenience we will give it here:

12 inches make 1 foot.
3 feet make 1 yard.
6 feet make 1 fathom.
$16\frac{1}{2}$ feet or $5\frac{1}{2}$ yards make 1 rod or pole.
40 poles make 1 furlong.
8 furlongs make 1 mile.

Hence we obtain the following results:

Inches.	Feet.	Yards.	Poles.	Furlongs.	Mile.
12	1				
36	3	1			
198	$16\frac{1}{2}$	$5\frac{1}{2}$	1		
7920	660	220	40	1	
63360	5280	1760	320	8	1

53. In measuring land a chain is used, called *Gunter's Chain*, which is 22 yards long, and consists of 100 equal links; each link is therefore ·22 of a yard long, that is 7·92 inches. Thus 25 links make a pole, 10 chains or 1000 links make a furlong, and 80 chains or 8000 links make a mile.

V. RIGHT-ANGLED TRIANGLE.

54. When we know the lengths of two of the three straight lines which form a right-angled triangle, we can calculate the length of the third straight line. We shall now give the rules for this purpose, which depend on the theorem of Art. 30, as will be more clearly seen hereafter. See Art. 138.

55. *The sides of a right-angled triangle being given, to find the hypotenuse.*

RULE. *Add the squares of the sides and extract the square root of the sum.*

56. Examples:

(1) One side is 8 feet, and the other side is 6 feet.

The square of 8 is 64, and the square of 6 is 36; the sum of 64 and 36 is 100; the square root of 100 is 10. Thus the hypotenuse is 10 feet.

(2) One side is 2 feet, and the other side is 10 inches.

2 feet are 24 inches; the square of 24 is 576, and the square of 10 is 100; the sum of 576 and 100 is 676: the square root of 676 is 26. Thus the hypotenuse is 26 inches.

57. In the Example just solved one side was given expressed in *feet*, and the other side expressed in *inches;* before we applied the rule for finding the hypotenuse we turned the feet into inches, so that both the sides might be expressed in the same denomination. In like manner *before using any rule in mensuration, it is necessary to express all the given lengths in the same denomination.* We may work with all the lengths expressed in inches, or with all expressed in feet, or with all expressed in yards, or with all expressed in any other denomination; but we must not work with some of the lengths expressed in one denomination, and some expressed in another.

58. In the two Examples solved in Art. 56, the square root could be found exactly, and so the length of the hypotenuse was determined accurately. But it may happen that the square root cannot be found exactly; in such a case we can continue the process for extracting the square root to as many decimal places as we think necessary.

59. Examples :

(1) One side is 3 feet 4 inches, and the other side is 2 feet 8 inches.

3 feet 4 inches = 40 inches, 2 feet 8 inches = 32 inches.

32	40	2624·0000 (51·22
32	40	25
64	1600	101) 124
96	1024	101
1024	2624	1022) 2300
		2044
		10242) 25600
		20484
		5116

Thus if we proceed to two decimal places we find that the hypotenuse is approximately 51·22 inches.

(2) One side is 2·4 feet, and the other side is 1·2 yards.

1·2 yards = 3·6 feet.

2·4	3·6	18·7200 (4·32
2·4	3·6	16
96	216	83) 272
48	108	249
5·76	12·96	862) 2300
	5·76	1724
	18·72	576

Thus if we proceed to two decimal places we find that the hypotenuse is approximately 4·32 feet; or, taking the nearest figure, we may say that it is 4·33 feet.

60. *The hypotenuse and one side of a right-angled triangle being given, to find the other side.*

RULE. *From the square of the hypotenuse subtract the square of the given side, and extract the square root of the remainder.*

Or, *Multiply the sum of the hypotenuse and the side by their difference, and extract the square root of the product.*

61. Examples:

(1) The hypotenuse is 10 feet, and one side is 8 feet.

The square of 10 is 100, and the square of 8 is 64 ; take 64 from 100, and the remainder is 36; the square root of 36 is 6. Thus the other side is 6 feet.

Or thus: the sum of the hypotenuse and the given side is 18 ; their difference is 2 ; the product of 18 and 2 is 36 : the square root of 36 is 6.

(2) The hypotenuse is 26 inches, and one side is 10 inches.

The square of 26 is 676, and the square of 10 is 100; take 100 from 676, and the remainder is 576 ; the square root of 576 is 24. Thus the other side is 24 inches.

Or thus: the sum of the hypotenuse and the given side is 36 ; their difference is 16; the product of 36 and 16 is 576 : the square root of 576 is 24.

62. We have given two forms of the Rule in Art. 60; the first form is more obviously connected with the Rule in Art. 55 ; the second form is generally more convenient in practice, as requiring less work.

63. In the two Examples solved in Art. 61 the square root could be found exactly, and so the length of the side was determined accurately. But it may happen that the square root cannot be found exactly; in such a case we can continue the process for extracting the square root to as many decimal places as we think necessary.

64. Examples:

(1) The hypotenuse is 1 foot 9 inches, and one side is 14 inches.

1 foot 9 inches = 21 inches.

$21 + 14 = 35,$

$21 - 14 = 7,$

$35 \times 7 = 245.$

$$\dot{2}4\dot{5}\cdot 0\dot{0}0\dot{0}\,(15\cdot 65$$

```
        1
  25)145
      125
  306) 2000
       1836
 3125) 16400
       15625
         775
```

Thus if we proceed to two decimal places, we find that the required side is approximately 15·65 inches.

(2) The hypotenuse is 2·7 yards, and one side is 3·4 feet.

2·7 yards = 8·1 feet.

$8\cdot 1 + 3\cdot 4 = 11\cdot 5,$

$8\cdot 1 - 3\cdot 4 = 4\cdot 7.$

```
 11·5
  4·7
 805
460
54·05
```

$$5\dot{4}\cdot 05\dot{0}0\,(7\cdot 35$$

```
       49
 143)  505
       429
1465)  7600
       7325
        275
```

Thus if we proceed to two decimal places, we find that the required side is approximately 7·35 feet.

65. We will now solve some exercises which depend on the Rules already given.

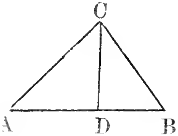
(1) One side of a right-angled triangle is 408 feet; the sum of the hypotenuse and the other side is 578 feet: required the hypotenuse and the other side.

By Art. 60 the square of 408 is equal to the product of the sum of the hypotenuse and the other side into their difference; therefore if the square of 408 be divided by 578, the quotient will be the difference of the hypotenuse and the other side. In this way we find that the difference of the hypotenuse and the other side is 288.

Thus the sum of the hypotenuse and the other side is 578, and their difference is 288. Add and divide by 2; thus we obtain 433, which is the hypotenuse. Subtract 433 from 578 and we obtain 145, which is the other side.

(2) Each side of an equilateral triangle is 1 foot: required the height of the triangle.

Let ABC be the triangle; CD the height. CD will divide AB into two equal parts; thus $AD = \frac{1}{2}$. We shall find CD by the second form of the rule in Art. 60,



$$1 + \frac{1}{2} = \frac{3}{2}, \qquad 1 - \frac{1}{2} = \frac{1}{2}, \qquad \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}.$$

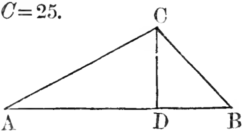
The square root of $\frac{3}{4} = \frac{1}{2}$ of the square root of 3. The square root of 3 cannot be found exactly; if we proceed to three decimal places we obtain 1·732, and half of this is ·866. Thus the height is ·866 feet approximately.

(3) The base of a triangle is 56 feet, the height is 15 feet, and one side is 25 feet: required the other side.

Let $AB = 56$, $CD = 15$, $BC = 25$.

We first find BD by Art. 60. $25 + 15 = 40$, $25 - 15 = 10$, $40 \times 10 = 400$: the square root of 400 is 20. Thus $DB = 20$.

Therefore $AD = 56 - 20 = 36$.



Then we find AC by Art. 55. The square of 36 is 1296, and the square of 15 is 225; the sum of 1296 and 225 is 1521: the square root of 1521 is 39. Thus $AC = 39$ feet.

EXAMPLES. V.

Determine the hypotenuse from the given sides in the following right-angled triangles:

1. 532 feet, 165 feet. 2. 7584 feet, 3937 feet.

3. 278 feet 8 inches, 262 feet 6 inches.

4. Half a mile, 345 yards 1 foot.

Determine in feet, as far as two decimal places, the hypotenuse from the given sides in the following right-angled triangles:

5. 437 feet, 342 feet. 6. 4395 feet, 3874 feet.

7. 314 feet 3 inches, 228 feet 9 inches.

8. A quarter of a mile, 427 yards 2 feet.

Determine the other side from the given hypotenuse and side in the following right-angled triangles:

9. 725 feet, 644 feet.

10. 16417 feet, 14208 feet.

11. 269 feet 5 inches, 250 feet 8 inches.

12. 340 yards 1 foot, 1 furlong.

Determine in feet, as far as two decimal places, the other side from the given hypotenuse and side, in the following right-angled triangles:

13. 647 feet, 431 feet. 14. 4987 feet, 3765 feet.

15. 424 feet 3 inches, 276 feet 6 inches.

16. 5 furlongs, 916 yards 2 feet.

17. The sides of a triangle are 22620 feet and 12815 feet, and the height is 11484 feet: find the base.

18. One side of a right-angled triangle is 3925 feet; the difference between the hypotenuse and the other side is 625 feet: find the hypotenuse and the other side.

19. A ladder 25 feet long stands upright against a wall: find how far the bottom of the ladder must be pulled out from the wall so as to lower the top one foot.

20. A ladder 40 feet long is placed so as to reach a window 24 feet high on one side of a street, and on turning the ladder over to the other side of the street it reaches a window 32 feet high: find the breadth of the street.

21. The bottom of a ladder is placed at a point 14 feet from a house, and the top of the ladder rests against the house at 48 feet from the ground; and on turning the ladder over to the other side of the street its top rests at 40 feet from the ground: find the breadth of the street.

22. Find to ten decimal places the diagonal of a square of which the side is one inch.

23. A side of a square is 110 feet: find the diagonal.

24. The radius of a circle is 82·66 feet; the perpendicular drawn from the centre on the chord is 71·1 feet: find the chord.

25. A footpath goes along two adjacent sides of a rectangle; one side is 196 yards, and the other is 147 yards: find the saving in distance made by proceeding along the diagonal instead of along the two sides.

26. The span of a roof is 28 feet; each of its slopes measures 17 feet from the ridge to the eaves: find the height of the ridge above the eaves.

27. The side of a square is 8 feet: find the radius of the circle described round the square.

28. The radius of a circle is 6 feet: find the side of a square inscribed in the circle.

29. The radius of a circle is 7 feet: find the perpendicular from the centre on a chord 8 feet long.

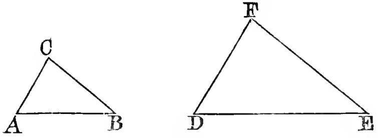
30. The radius of a circle is 17 inches; the perpendicular from the centre on a chord is 13 inches: find the chord.

31. The radius of a circle is divided into six equal parts, and at the five points of division straight lines are drawn at right angles to the radius to meet the circumference: find the lengths of these straight lines, in inches to three decimal places, that of the radius being one foot.

32. The radius of a circle is 7 feet; from a point at the distance of 12 feet from the centre a straight line is drawn to touch the circle: find the length of this straight line.

VI. SIMILAR FIGURES.

66. Let ABC and DEF be two similar triangles. Then AB is to BC as DE is to EF; see Art. 34.



Thus if two sides of one triangle be given, and one of the corresponding sides of a similar triangle, the other corresponding side of the second triangle can be found. The process will be that which is known in Arithmetic as Proportion or the Rule of Three.

67. Examples:

(1) Suppose $AB=5$, $BC=6$, $DE=7$,

$$5 : 6 :: 7 : EF.$$

$$\text{Thus } EF = \frac{6 \times 7}{5} = \frac{42}{5} = 8\tfrac{2}{5}.$$

(2) Suppose $AB=5$, $AC=4$, $DE=7$,

$$5 : 4 :: 7 : DF.$$

$$\text{Thus } DF = \frac{4 \times 7}{5} = \frac{28}{5} = 5\tfrac{3}{5}.$$

68. Similar triangles frequently present themselves in the theory and practice of mathematics.

For example, we found in Art. 65 that if the side of an equilateral triangle be 1 foot the height is ·866... feet. Now this proportion will always hold between the side and the height of an equilateral triangle; so that if the side of an equilateral triangle be 7 feet the height will be 7 × ·866 .. feet.

Again, we have said that the triangles *AED* and *BEC* in the diagram of Art. 38 are similar; so that *EA* is to *ED* as *EC* is to *EB*. Hence it follows by the usual theory of Proportion that the product of *EA* into *EB* is equal to the product of *EC* into *ED*; this is a very remarkable and very important property of the circle.

69. By the aid of similar triangles we can determine the height of an object when we have measured the length of its shadow.

For example, suppose that a stick is fixed upright in the ground, and that the height of the portion above the ground is 3 feet and the length of the shadow 4 feet. Also suppose we find at the same time that the length of the shadow of a certain tree is 52 feet. Then we determine the height of the tree by the proportion

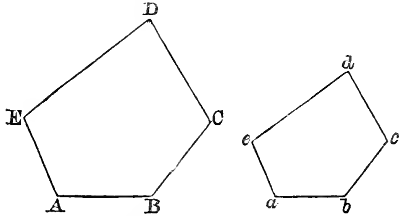
$$4 : 3 :: 52 : \text{the height}.$$

Thus the height is $\frac{52 \times 3}{4}$ feet, that is 39 feet.

70. From similar triangles we pass naturally to the consideration of similar rectilineal figures.

Similar rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

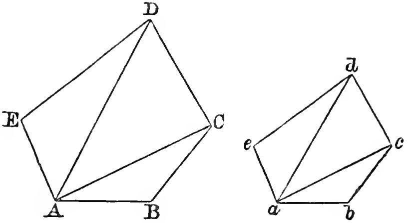
71. Take for example two five-sided figures *ABCDE* and *abcde*; these are similar if the angles at *A*, *B*, *C*, *D*, *E*



are equal to the angles at *a*, *b*, *c*, *d*, *e* respectively, and the sides about the equal angles proportional, that is, AB to BC as ab to bc, and BC to CD as bc to cd, and so on.

72. Thus to ensure the similarity of rectilineal figures we must have two properties, namely, *equality* of angles and *proportionality* of sides. Theory demonstrates that if two triangles have one of these properties they will necessarily have the other; and it is easy to test this practically. For example, let two triangles be drawn on paper, such that the sides of one are twice or three times as long as the sides of the other; cut the triangles out, and apply one triangle on the other; it will be found that the corresponding angles are equal. But in the case of rectilineal figures having more than three sides, either of the properties may exist singly without the other. For example, take a square and any rectangle which is not a square; here the angles of one figure are respectively equal to the angles of the other, but the sides are not proportional. Again, take a square and a rhombus; here the sides of one figure are all in the same proportion to the sides of the other figure, but the angles of one figure are not equal to the angles of the other figure.

73. Similar rectilineal figures can always be divided into the same number of similar triangles. Thus, for example, by drawing the straight lines CA, DA, ca, da,



we can divide the five-sided figures of Art. 71 into three pairs of similar triangles.

74. The statement made in Art. 66 with respect to similar triangles holds for any two similar rectilineal figures; that is, if two straight lines situated in one figure be given, and a straight line corresponding to one of them in a similar figure, the straight line corresponding to the other can be found by Proportion.

75. Similar figures may occur which are bounded by curved lines as well as those which are bounded by straight lines. Thus, two maps of different sizes may represent the same country; the two maps will then be similar. For example, one map may be on the scale of an inch to a mile, and the other map on the scale of half an inch to a mile: then any line drawn on the first map will be twice as long as the corresponding line drawn on the second map.

76. A good notion of similar figures may be conveyed, by saying that they are exactly alike in form although they may differ in size.

All circles are similar figures.

77. We will now solve some exercises which depend on the similarity of figures.

(1) In the diagram of Art. 73 suppose $AE = 2$ inches, $AC = 4\frac{1}{2}$ inches, $ae = 1\frac{1}{4}$ inches: find ac.

$$2 : 4\tfrac{1}{2} :: 1\tfrac{1}{4} : ac,$$

$$\frac{4\frac{1}{2} \times 1\frac{1}{4}}{2} = \frac{45}{16} = 2\tfrac{13}{16}.$$

Thus $ac = 2\frac{13}{16}$ inches.

(2) In the preceding exercise find the proportion of AD to ad.

Since $AE = 2$ and $ae = 1\frac{1}{4}$, any straight line as AD bears to the corresponding straight line ad the proportion of 2 to $1\frac{1}{4}$, that is of 2 to $\frac{5}{4}$, that is of $\frac{8}{4}$ to $\frac{5}{4}$, that is of 8 to 5.

(3) In the diagram of Art. 37 if $BC=15$, and $BA=12$, find BD.

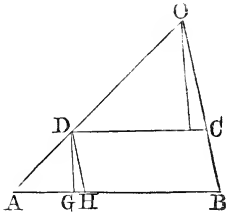
The triangles ABC and DBA are similar: thus

$$BC : BA :: BA : BD;$$

that is, $$15 : 12 :: 12 : BD.$$

Thus $$BD = \frac{12 \times 12}{15} = \frac{144}{15} = \frac{48}{5} = 9\tfrac{3}{5}.$$

(4) $ABCD$ is a trapezoid. The distance of the parallel sides AB and CD is 3 feet; $AB=10$ feet; $DC=6$ feet. Let AD and BC produced meet at O. It is required to find the perpendicular distance of O from DC.



Draw DG perpendicular to AB, and DH parallel to BC. Then $BH=DC$: thus $AH=10-6=4$. Also $DG=3$. Now the triangles ADH and DOC are similar. Therefore by Art. 35,

$$AH : GD :: DC : \text{required distance.}$$

Thus the required distance $= \frac{3 \times 6}{4} = \frac{9}{2} = 4\tfrac{1}{2}$.

EXAMPLES. VI.

1. In the diagram of Art. 36 if $AD=5$ inches, $DE=4$, and $AB=7$, find BC.

2. The side of an equilateral triangle is 2 feet 6 inches: find the height.

3. The shadow of a man 6 feet high standing upright was measured and found to be 8 feet 6 inches: the shadow of a flag-staff, measured at the same time, was found to be 56 feet 8 inches: determine the height of the flag-staff.

4. A stick 3 feet long is placed upright on the ground, and its shadow is found to be 4 feet 6 inches long: find the length of the shadow of a pole which is 45 feet high.

5. A country is 500 miles long: find the length of a map which represents the country on the scale of one-eighth of an inch to a mile.

6. The distance between two towns is 31 miles, and the distance between their places on a map is $7\frac{3}{4}$ inches: find the scale on which the map is drawn.

7. The distance between two towns is 54 miles, and the distance between their places on a map is $6\frac{3}{4}$ inches: find the distance between two other towns if the distance between their places on the map is $8\frac{1}{2}$ inches.

8. In the diagram of Art. 36 if $BC=20$ inches, $DE=16$, and $BD=6$, find BA.

9. In the diagram of Art. 36 if $AD=8$ inches, $DE=7$, and $BD=3$, find BC.

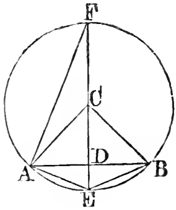
10. In the diagram of Art. 36 if $DE=7$ inches, $BC=10$, and $BD=2$, find DA.

11. The parallel sides of a trapezoid are respectively 16 and 20 feet, and the perpendicular distance between them is 5 feet; the other two sides are produced to meet: find the perpendicular distance of the point of intersection from the longer of the two parallel sides.

12. The parallel sides of a trapezoid are respectively 8 feet and 14 feet; two straight lines are drawn across the figure parallel to these so that the four are equidistant: find the lengths of the straight lines.

VII. CHORDS OF A CIRCLE.

78. Let AB be any chord of a circle, C the centre of the circle; suppose CD drawn perpendicular to AB, and produced to meet the circumference at E.



Then D is the middle point of the chord ADB, and E is the middle point of the arc AEB. AB is the chord of the arc, and AE, or EB, is the chord of half the arc. DE is the height of the arc.

79. Produce EC to meet the circumference at F. The angle EAF is a right angle, by Art. 33. Hence, by Art. 37, the triangles EAF and EDA are similar, so that ED is to EA as EA is to EF. Therefore,

$$ED \times EF = EA \times EA.$$

Also, by Art. 68,

$$ED \times DF = AD \times DB.$$

The present Chapter consists of applications of these two important results. We shall put the applications in the form of Rules for the sake of convenient reference, but any person who masters these two results will find it unnecessary to commit the Rules to memory.

80. *Having given the height of an arc and the chord of half the arc, to find the diameter of the circle.*

RULE. *Divide the square of the chord of half the arc by the height of the arc, and the quotient will be the diameter of the circle.*

81. Examples.

(1) The height of an arc is 4 inches, and the chord of half the arc is 12 inches.

$\frac{12 \times 12}{4} = 36$: thus the diameter is 36 inches.

(2) The height of an arc is 1 foot 4 inches, and the chord of half the arc is 4 feet.

$\frac{4 \times 4}{1\frac{1}{3}} = 4 \times 4 \times \frac{3}{4} = 12$: thus the diameter is 12 feet.

82. *Having given the chord of half an arc and the diameter of the circle, to find the height of the arc.*

Rule. *Divide the square of the chord of half the arc by the diameter of the circle, and the quotient will be the height of the arc*

83. Examples.

(1) The chord of half an arc is 12 inches, and the diameter of the circle is 36 inches.

$\frac{12 \times 12}{36} = 4$: thus the height of the arc is 4 inches.

(2) The chord of half an arc is 4 feet, and the diameter of the circle is 12 feet.

$\frac{4 \times 4}{12} = \frac{4}{3} = 1\frac{1}{3}$: thus the height of the arc is $1\frac{1}{3}$ feet.

84. *Having given the height of an arc and the diameter of the circle, to find the chord of half the arc.*

Rule. *Multiply the diameter of the circle by the height of the arc; the square root of the product will be the chord of half the arc.*

85. Examples.

(1) The height of an arc is 4 inches, and the diameter of the circle is 36 inches.

$36 \times 4 = 144$; the square root of 144 is 12: thus the chord of half the arc is 12 inches.

(2) The height of an arc is $1\frac{1}{3}$ feet, and the diameter of the circle is 12 feet.

$\frac{4}{3} \times 12 = 16$; the square root of 16 is 4: thus the chord of half the arc is 4 feet.

86. *Having given the chord of an arc and the height of the arc, to find the diameter of the circle.*

RULE *Divide the square of half the chord by the height, and the quotient will be the remaining part of the diameter; so that the sum of the quotient and the given height will be the diameter.*

87. Examples.

(1) The chord of an arc is 8 feet, and the height of the arc is 2 feet.

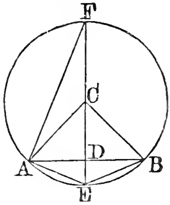
$\frac{4 \times 4}{2} = 8$; thus the remaining part of the diameter is 8 feet: therefore the diameter is 10 feet.

(2) The chord of an arc is 21 feet, and the height of the arc is 4 feet.

$\frac{10{\cdot}5 \times 10{\cdot}5}{4} = \frac{110{\cdot}25}{4} = 27{\cdot}5625$; thus the remaining part of the diameter is 27·5625 feet: therefore the diameter is 31·5625 feet.

88. By the aid of the Rules given in the present Chapter, and in Chapter V., we can solve various other exercises which may be proposed with respect to the diagram of Art. 78, as we will now shew.

89. *Having given the height of an arc and the diameter of the circle, to find the chord of the arc.*



Here we know ED and EF; therefore we know DF: and then we can find AD by Art. 79.

90. Examples.

(1) The height of an arc is 9 feet, and the diameter of the circle is 25 feet.

Here $ED = 9$, $DF = 16$; therefore the square of $AD = 9 \times 16 = 144$: thus $AD = 12$ feet. Therefore $AB = 24$ feet.

(2) The height of an arc is 2 feet, and the diameter of the circle is 10 feet.

Here $ED = 2$, $DF = 8$; therefore the square of $AD = 2 \times 8 = 16$: thus $AD = 4$ feet. Therefore $AB = 8$ feet.

91. *Having given the chord of an arc and the diameter of the circle, to find the height of the arc.*

Here we know AC, which is half the given diameter, and AD which is half the given chord. We first obtain CD by Art. 60; and subtracting this from CE we have DE.

92. Examples.

(1) The chord of an arc is 24 feet, and the diameter of the circle is 25 feet.

Here $AC = 12\frac{1}{2}$ feet, and $AD = 12$ feet.

$12\frac{1}{2} + 12 = 24\frac{1}{2}$; $12\frac{1}{2} - 12 = \frac{1}{2}$; $24\frac{1}{2} \times \frac{1}{2} = \frac{49}{4}$; the square root of $\frac{49}{4}$ is $\frac{7}{2}$, that is $3\frac{1}{2}$; and $12\frac{1}{2} - 3\frac{1}{2} = 9$: therefore $DE = 9$ feet.

(2) The chord of an arc is 8 feet, and the diameter of the circle is 10 feet.

Here $AC=5$ feet, and $AD=4$ feet.

$5+4=9$, $5-4=1$; the square root of 9 is 3; and $5-3=2$: therefore $DE=2$ feet.

93. *Given the chord of an arc and the diameter of the circle, to find the chord of half the arc.*

Here we know AC and AD. We first obtain DE, as in Art. 91; and then we find AE either by Art. 55 or by Art. 84.

94. Examples.

(1) The chord of an arc is 14 inches, and the diameter of the circle is 50 inches.

Here $AC=25$, $AD=7$; thus by Art. 60 we shall obtain $CD=24$. Therefore $DE=1$. Then by Art. 55, or by Art. 84, we find that AE is the square root of 50. Proceeding to four decimal places we obtain $AE=7·0710$. Thus the chord of half the arc is about 7·071 inches. If we proceed to seven decimal places we obtain 7·0710678.

(2) The chord of an arc is 58 inches, and the diameter of the circle is 200 inches.

Here $AC=100$, $AD=29$: thus by Art. 60 we shall find that CD is the square root of 9159: proceeding to four decimal places we obtain $CD=95·7027$ nearly. Therefore we take $DE=4·2973$. We may now calculate AE by the rule of Art. 55 or by that of Art. 84; if AC, AD, and DE were all known exactly, the two rules would give precisely the same result; but in the present case DE is not known exactly, and we shall find that the two rules give results which differ slightly. The rule of Art. 84 is the simplest, and by this we find that AE is the square root of 859·46; hence we obtain $AE=29·32$ very nearly. Thus the chord of half the arc is about 29·32 inches.

95. *Given the chord of half an arc and the diameter of the circle, to find the chord of the arc.*

Here we know AE and EF. We first obtain ED by Art. 82, and then AD by Art. 60.

96. Examples.

(1) The chord of half an arc is 12 inches, and the diameter of the circle is 36 inches.

As in Art. 83 we find that $ED = 4$. Then, by Art. 60, we find that AD is the square root of 128: hence we obtain $AD = 11{\cdot}314$ nearly, and therefore $AB = 22{\cdot}628$ nearly. Thus the chord of the arc is nearly 22·628 inches.

(2) The chord of half an arc is 4 feet, and the diameter of the circle is 12 feet.

As in Art. 83 we find that $ED = 1\frac{1}{3}$. Then by Art. 60 we see that AD is the square root of $16 - \frac{16}{9}$, that is of $\frac{128}{9}$; therefore AD is $\frac{1}{3}$ of the square root of 128: hence proceeding to three decimal places we obtain $AD = 3{\cdot}771$, and therefore $AB = 7{\cdot}542$. Thus the chord of the arc is about 7·542 feet.

97. *Given the chord of an arc and the chord of half the arc, to find the diameter of the circle.*

Here we know AD and AE We first obtain ED by Art. 60, and then EF by Art. 80.

98. Examples.

(1) The chord of an arc is 48 inches, and the chord of half the arc is 26 inches.

Here $AD = 24$, $AE = 26$; thus, by Art. 60, we obtain $ED = 10$. Then, by Art. 80, we have $EF = \frac{26 \times 26}{10} = 67{\cdot}6$.

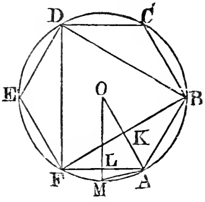
Thus the diameter of the circle is 67·6 inches.

(2) The chord of an arc is 20 inches, and the chord of half the arc is 10·5 inches.

Here $AD = 10$, $AE = 10{\cdot}5$; thus, by Art. 60, we find that ED is the square root of 10·25: proceeding to four decimal places we obtain $ED = 3{\cdot}2015$. Then, by Art. 80, we have $EF = \frac{10{\cdot}5 \times 10{\cdot}5}{3{\cdot}2015}$, which we find $= 34{\cdot}437$ very nearly. Thus the diameter of the circle is very nearly 34·437 inches.

99. As an exercise we will calculate the length of the side of an equilateral triangle inscribed in a circle, and also the length of the side of a regular polygon of twelve sides.

Describe a circle. If we draw in succession chords equal to the radius AB, BC, CD, ... we shall find that exactly six of them can be placed in the entire circumference. In other words, if a regular hexagon be inscribed in a circle, a side of the hexagon is exactly equal to the radius of the circle.



Draw the straight lines FB, BD, DF; thus we form an equilateral triangle.

Suppose the radius of the circle 1 inch: required FB. This is an example of Art. 95. Let O be the centre of the circle; draw OA cutting BF at K.

We find that $AK = \frac{1}{2}$; thus BK is the square root of $\frac{3}{4}$, that is $\frac{1}{2}$ of the square root of 3. Therefore BF is the square root of 3: proceeding to seven places of decimals we obtain $BF = 1{\cdot}7320508$.

Thus the side of the equilateral triangle inscribed in the circle is 1·7320508 inches.

Again, let OL be perpendicular to AF, and produce it to meet the circumference again at M. Join AM. Then AM is one of the sides of a regular polygon of twelve sides inscribed in the circle. We can calculate AM by Art. 93.

$AL = \frac{1}{2}$, $OA = 1$; thus $OL = \frac{1}{2}$ of the square root of 3 $= {\cdot}8660254$: therefore $LM = {\cdot}1339746$. Then AM is the square root of ·2679492, which we shall find to be ·51764 very nearly.

Thus the side of a regular polygon of twelve sides inscribed in the circle is ·51764 inches very nearly.

EXAMPLES. VII.

1. The height of an arc is 15 inches, and the chord of half the arc is 4 feet 6 inches: find the diameter of the circle.

2. The height of an arc is 2·28 feet, and the chord of half the arc is 7·15 feet: find the diameter of the circle.

3. The chord of half an arc is 3 feet 4 inches, and the diameter of the circle is 25 feet: find the height of the arc.

4. The chord of half an arc is 6·43 feet, and the diameter of the circle is 23·65 feet: find the height of the arc.

5. The height of an arc is 1 foot 3 inches, and the diameter of the circle is 11 feet 3 inches: find the chord of half the arc.

6. The height of an arc is 3·24 feet, and the diameter of the circle is 28·76 feet: find the chord of half the arc.

7. The chord of an arc is 20 feet, and the height of the arc is 4 feet: find the diameter of the circle.

8. The chord of an arc is 15·78 feet, and the height of the arc is 2·8 feet: find the diameter of the circle.

9. The chord of an arc is 15 inches, and the diameter of the circle is 20 inches: find the chord of half the arc.

10. The chord of an arc is 80 inches, and the diameter of the circle is 100 inches: find the chord of half the arc.

11. The chord of half an arc is 2 feet 6 inches, and the diameter of the circle is 4 feet 2 inches: find the chord of the arc.

12. The chord of half an arc is 2·4 feet, and the diameter of the circle is 16 feet: find the chord of the arc.

13. The chord of an arc is 12 yards, and the chord of half the arc is 19 feet 6 inches: find the diameter of the circle.

14. The chord of an arc is 49 feet, and the chord of half the arc is 25 feet: find the diameter of the circle.

VIII. CIRCUMFERENCE OF A CIRCLE.

100. We often require to know the proportion which the length of the circumference of a circle bears to the length of the diameter: the proportion cannot indeed be stated exactly, but it can be stated with sufficient accuracy for any practical purpose.

101. *The diameter of a circle being given, to find the circumference.*

RULE. *Multiply the diameter by* $3\frac{1}{7}$, *that is by* $\frac{22}{7}$; *in other words, multiply the diameter by 22, and divide the product by 7.*

102. Examples.

(1) The diameter of a circle is 4 feet 8 inches.

$$4 \text{ feet } 8 \text{ inches} = 56 \text{ inches},$$

$$56 \times \frac{22}{7} = 8 \times 22 = 176.$$

Thus the circumference is about 176 inches, that is, about 14 feet 8 inches.

(2) The diameter of a circle is 4·256 feet.

$$\begin{array}{r} 4{\cdot}256 \\ 22 \\ \hline 8512 \\ 8512 \\ \hline 7\,)\,93{\cdot}632 \\ \hline 13{\cdot}376 \end{array}$$

Thus the circumference is about 13·376 feet.

103. The Rule of Art. 101 makes the circumference a little greater than it ought to be. The circumference of a circle is in fact less than $3\frac{10}{70}$ times the diameter, but greater than $3\frac{10}{71}$ times. The rule of multiplying the diameter by $3\frac{1}{7}$ is generally found sufficiently accurate in practice.

104. We may if we please put the Rule of Art. 101 in the form of a proportion, and say, *as* 7 *is to* 22 *so is the diameter of any circle to the circumference.*

105. The following proportion is still more accurate: *as* 113 *is to* 355 *so is the diameter of any circle to the circumference.* This rule also makes the circumference a little greater than it ought to be; but the error is excessively small, being at the rate of rather less than a foot in nineteen hundred miles.

106. We may also put the proportion in the following form: *the diameter of any circle is to the circumference as* 1 *is to* 3·141592653589793...; the calculation of this proportion has been carried to more than 600 places of decimals. We may use as many as we please of the figures which have been obtained: it is very common to take 3·1416 as a sufficient approximation.

In modern mathematical books the Greek letter π is generally used to denote the number 3·14159265...

107. Accordingly the Rule for finding the circumference of a circle when the diameter is given, may be stated thus: *multiply the diameter by* $3\frac{1}{7}$; *or, if greater accuracy is required, multiply the diameter by* 3·1416.

The latter form of the Rule also makes the circumference a little greater than it ought to be; but the error will not be so much as $\frac{1}{400000}$ part of the circumference: so that the error will be at the rate of less than a foot in seventy-five miles.

108. When we are told to multiply the diameter by 3·1416, we may, if we please, multiply 3·1416 by the diameter. A similar remark applies to all rules relating to the multiplication of numbers.

109. Examples.

(1) The diameter of a circle is 42·7 inches.

$$\begin{array}{r} 31416 \\ \underline{42{\cdot}7} \\ 219912 \\ 62832 \\ \underline{125664} \\ 134{\cdot}14632 \end{array}$$

Thus the circumference is nearly 134·14632 inches.

(2) The diameter of a circle is 8000 miles.

$$\begin{array}{r} 3{\cdot}1416 \\ \underline{8000} \\ 25132{\cdot}8000 \end{array}$$

Thus the circumference is nearly 25132·8 miles.

110. The beginner should exercise himself in actually measuring the diameter and circumference of some circle, as for example, a wheel. Although he may not be able to obtain very accurate results, yet he may convince himself that the circumference is about $3\frac{1}{7}$ times the diameter.

111. *The circumference of a circle being given, to find the diameter.*

RULE. *Divide the circumference by* $3\frac{1}{7}$, *that is by* $\frac{22}{7}$; *in other words, multiply the circumference by* 7, *and divide the product by* 22. *Or, if greater accuracy is required, divide the circumference by* 3·1416.

112. Examples.

(1) The circumference of a circle is 50 feet.

$$\begin{array}{r|l} & 50 \\ & \underline{7} \\ 2 & 350 \\ 11 & 175 \\ \hline & 15{\cdot}9 \end{array}$$

Thus the diameter is about 15·9 feet

(2) The circumference of a circle is 360 feet.

```
3·1416 ) 360·00 ( 114·59
         31416
          45840
          31416
          144240
          125664
           185760
           157080
            286800
            282744
              4056
```

Thus the diameter is about 114·59 feet.

113. We will now solve some exercises which depend on the Rules already given.

(1) Find the diameter of a carriage wheel which is turned round 1000 times in travelling a mile.

Here 1000 times the circumference of the wheel is equal to 1760 yards; thus the circumference is 1·76 yards. Then, by Art. 111, the diameter is $\frac{7}{22} \times 1·76$ yards, that is, $7 \times ·08$ yards, that is, ·56 of a yard.

(2) Suppose that the distance of the earth from the sun is about 95000000 miles, and that the earth describes a circle round the sun in $365\frac{1}{4}$ days: find the number of miles described by the earth in one minute.

The circumference of the circle described by the earth is about $2 \times 95000000 \times 3·1416$ miles, that is, about 596904000 miles. In $365\frac{1}{4}$ days there are 525960 minutes. Divide the number of miles by the number of minutes; thus we obtain very nearly 1135 miles.

EXAMPLES. VIII.

Assuming that the circumference of a circle is $3\frac{1}{7}$ times the diameter, find the circumferences of the circles with the following diameters:

1. 14 feet.
2. 86 yards 1 foot.
3. 213 yards 2 feet 8 inches.
4. 1 furlong 60 yards.

Assuming that the circumference of a circle is 3·1416 times the diameter, find the circumferences of the circles with the following diameters:

5. 27 feet.
6. 61 yards 2 feet.
7. 555 yards 1 foot 6 inches.
8. 1 furlong 80 yards.

Assuming that the circumference of a circle is $3\frac{1}{7}$ times the diameter, find the diameters of the circles with the following circumferences:

9. 66 yards.
10. 10 chains.
11. 3 furlongs 4 chains.
12. 1 mile.

Assuming that the circumference of a circle is 3·1416 times the diameter, find the diameters of the circles with the following circumferences:

13. 1 foot.
14. 25 feet.
15. 108 yards 1 foot.
16. 1 furlong.

17. Suppose that the planet Mercury describes in 88 days a circle round the Sun, of which the radius is 37000000 miles, find the number of miles described by the planet in one second.

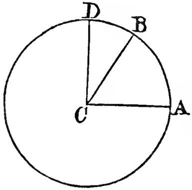
18. The diameter of a carriage wheel is 28 inches: find how many turns the wheel makes in travelling half a mile.

19. A road runs round a circular shrubbery; the outer circumference is 600 feet, and the inner circumference is 480 feet: find the breadth of the road.

20. The difference between the diameter and the circumference of a circle is 10 feet: find the diameter.

IX. ARC OF A CIRCLE.

114. Let C be the centre of a circle, AB any arc of the circle, AD a quarter of the circumference. The length of AB is to the length of AD in the same proportion as the angle ACB is to the angle ACD, that is, as the angle ACB is to a right angle. Therefore the length of AB is to the circumference of the circle in the same proportion as the angle ACB is to four right angles.



115. Angles are usually expressed in degrees, 90 of which make a right angle; and consequently in four right angles there are 360 degrees. A degree is subdivided into 60 minutes, and a minute into 60 seconds.

116. Symbols are used as abbreviations of the words *degrees, minutes,* and *seconds.* Thus $6^{0}\ 23'\ 47''$ is used to denote 6 degrees, 23 minutes, 47 seconds.

117. *The number of degrees in the angle subtended by an arc of a circle at the centre being given, to find the length of the arc.*

Rule. *As* 360 *is to the number of degrees in the angle, so is the circumference of the circle to the length of the arc.*

118. Examples.

(1) The circumference of a circle is 48 inches, and the angle subtended by the arc at the centre is 54 degrees.

360 : 54 :: 48 : the required length,

$$\frac{54 \times 48}{360} = \frac{54 \times 4}{30} = \frac{18 \times 4}{10} = 7{\cdot}2.$$

Thus the length of the arc is 7·2 inches

(2) The circumference of a circle is 25000 miles, and the angle subtended by the arc at the centre is one degree.

360 : 1 :: 25000 : the required length,

```
36)2500(69·4
   216
   ---
    340
    324
    ---
     160
     144
     ---
      16
```

Thus the length of the arc is about 69·4 miles.

119. *The length of an arc of a circle being given, to find the number of degrees in the angle subtended by the arc at the centre of the circle.*

RULE. *As the circumference of the circle is to the length of the arc, so is* 360 *to the number of degrees in the angle.*

120. Examples.

(1) The circumference of a circle is 50 feet, and the arc is 8 feet.

50 : 8 :: 360 : the required number of degrees,

$$\frac{8 \times 360}{50} = \frac{288}{5} = 57\tfrac{3}{5}.$$

Thus the angle is $57\frac{3}{5}$ degrees.

(2) The circumference of a circle is 25000 miles, and the arc is 750 miles.

25000 : 750 :: 360 : the required number of degrees,

$$\frac{750 \times 360}{25000} = \frac{75 \times 36}{250} = \frac{3 \times 36}{10} = 10{\cdot}8.$$

Thus the angle is 10·8 degrees.

121. *The chord of an arc being known, and also the chord of half the arc, to find the length of the arc.*

RULE. *From eight times the chord of half the arc subtract the chord of the whole arc, and divide the remainder by three.*

This Rule is not exact; it gives the length of the arc smaller than it ought to be. If the arc subtend at the centre of the circle an angle of 45 degrees, the error is about $\frac{1}{20000}$ of the length of the arc: the error increases rapidly as the angle increases, and diminishes rapidly as the angle diminishes.

122. Examples.

(1) The chord of an arc is 14 inches, and the radius of the circle is 25 inches.

By Art. 94 the chord of half the arc is about 7·0710678 inches.

$$\begin{array}{r|r} & 7{\cdot}0710678 \\ & 8 \\ \hline & 56{\cdot}5685424 \\ & 14\phantom{{\cdot}5685424} \\ \hline 3 & 42{\cdot}5685424 \\ \hline & 14{\cdot}1895141 \end{array}$$

Thus we obtain for the length of the arc 14·1895141 inches.

(2) The chord of an arc is 58 inches, and the radius of the circle is 100 inches.

By Art. 94 the chord of half the arc is about 29·32 inches.

$$\begin{array}{r|r} & 29{\cdot}32 \\ & 8 \\ \hline & 234{\cdot}56 \\ & 58\phantom{{\cdot}56} \\ \hline 3 & 176{\cdot}56 \\ \hline & 58{\cdot}85 \end{array}$$

Thus we obtain for the length of the arc 58·85 inches.

123. The error which arises from the use of the Rule in Art. 121 is, as we have said, much less for a small arc than for a large arc. It may therefore be expedient in some cases to calculate by the Rule the length of half the arc, and to double this result instead of calculating the length of the whole arc immediately. We should proceed thus: *from eight times the chord of one fourth of the arc subtract the chord of half the arc; multiply the remainder by two, and divide the product by three.*

124. The following Rule for finding the length of an arc is much more accurate than that in Art. 121, and may be used when a very close approximation is required: *to* 256 *times the chord of one fourth of the arc add the chord of the arc; subtract* 40 *times the chord of half the arc, and divide the remainder by* 45.

This Rule gives the length of the arc a little larger than it ought to be. If the arc subtend at the centre of the circle an angle of 45 degrees the error is less than $\frac{1}{80000000}$ of the arc: the error increases rapidly as the angle increases, and diminishes rapidly as the angle diminishes.

125. We will now solve some exercises.

(1) The radius of a circle is 1 foot: find the perimeter, that is the length of the whole boundary, of a sector of 60 degrees.

Since the radius is 1 the circumference of the circle is $2 \times 3{\cdot}1416$, that is, $6{\cdot}2832$; then

$$360 : 60 :: 6{\cdot}2832 : \text{the length of the arc.}$$

Thus the length of the arc is 1·0472. Add the length of the two radii, that is 2; thus the whole perimeter is 3·0472 feet.

(2) The perimeter of a sector of 60 degrees is 20 feet: find the radius.

Use the result of the preceding exercise. Thus we have the proportion

$$3{\cdot}0472 : 20 :: 1 : \text{the required radius.}$$

Hence the required radius $= \frac{20}{3{\cdot}0472}$ feet $= 6{\cdot}5634$ feet very nearly.

EXAMPLES. IX.

1. The radius of a circle is 10 inches, and the angle subtended by an arc at the centre is 72^0: find the length of the arc.

2. The radius of a circle is 19 feet 7 inches, and the angle subtended by an arc at the centre is $10^0\ 24'$: find the length of the arc.

3. The radius of a circle is 2 feet, and the length of an arc is 15 inches : find the angle subtended at the centre by the arc.

4. The radius of a circle is 1 foot, and the length of an arc is equal to the radius : find the angle subtended at the centre by the arc.

5. The chord of an arc is 36 inches, and the chord of half the arc is 19 inches : find the arc.

6. The chord of an arc is 56 inches, and the radius of the circle is 197 inches : find the arc.

7. The chord of an arc is 6 inches, and the radius of the circle is 9 inches : find the arc.

8. The radius of a circle is 5 inches : find the perimeter of a sector, the angle of which is 90^0.

9. The radius of a circle is 16 inches : find the perimeter of a segment, the arc of which subtends an angle of 90^0 at the centre of the circle.

10. The radius of a circle is 1 inch : find the perimeter of a semicircle.

11. The perimeter of a semicircle is 100 feet : find the radius.

12. The radius of a circle is 25 inches, and the angle subtended by an arc at the centre is $32^0\ 31'\ 12''\cdot4$: find the length of the arc.

THIRD SECTION. AREAS.

X. TABLE OF SQUARE MEASURE.

126. It will be convenient to place here the Table of Measures of Area, which is usually called the Table of Square Measure.

144 square inches make 1 square foot.
9 square feet make 1 square yard.
36 square feet make 1 square fathom.
$272\frac{1}{4}$ square feet or $30\frac{1}{4}$ square yards make 1 square rod or pole.
1600 square poles make 1 square furlong.
64 square furlongs make 1 square mile.

Hence we obtain the following results:

Square Inches.	Square Feet.	Square Yards.	Square Rods.	Square Furlongs.	Square Mile.
144	1				
1296	9	1			
39204	$272\frac{1}{4}$	$30\frac{1}{4}$	1		
62726400	435600	48400	1600	1	
4014489600	27878400	3097600	102400	64	1

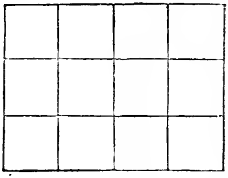
127. The following terms are also used in expressing areas: a *square link*, a *square chain*, a *rood*, and an *acre*.

A square chain contains 22×22, that is, 484 square yards. A rood is 40 poles, that is, 1210 square yards. An acre is 4 roods, that is, 4840 square yards: thus an acre is equal to 10 square chains.

A square chain contains 100×100, that is, 10000 square links; so that an acre is equal to 100000 square links.

XI. RECTANGLE.

128. Suppose we have a rectangle which is 4 inches long and 3 inches broad. Draw straight lines, an inch apart, parallel to the sides. The rectangle is thus divided into 12 equal figures, each of which is a square being an inch long and an inch broad: such a square is called a *square inch.* The rectangle then contains 12 square inches; this fact is also expressed thus: the area of the rectangle is 12 square inches.



The number 12 is the product of the numbers 4 and 3, which denote respectively the length and the breadth of the rectangle.

129. If a rectangle be 8 inches long and 5 inches broad, we can shew in the same manner that its area is 8 times 5 square inches, that is 40 square inches. Similarly, if a rectangle be 9 inches long and 7 inches broad, its area is 9 times 7 square inches, that is, 63 square inches. And so on.

130. In the same manner, if a rectangle be 4 feet long and 3 feet broad, its area is 12 *square feet;* that is, the rectangle might be divided into 12 equal figures, each being a foot long and a foot broad. If a rectangle be 4 yards long and 3 yards broad, its area is 12 *square yards.* And so on.

131. The beginner should observe very carefully the way in which areas are measured; it is a case of the general principle which applies to all measurable things. For example, when we measure lengths we fix on some length for a standard, as an inch or a foot, and we compare other lengths with the standard; thus when we say that a

certain line is 17 inches long, we mean that the line is 17 times as long as our standard, which is one inch. In like manner when we measure areas we fix on some area for a standard, and we compare other areas with the standard. The most convenient standard is found to be the area of a square; it may be a square inch, or a square foot, or any other square.

132. In order then to find the area of a rectangle we must express the length and the breadth in terms of the same denomination; and then the product of the numbers which denote the length and the breadth will denote the area. If the length and the breadth are both expressed in inches, the area will be expressed in *square inches;* if the length and the breadth are both expressed in feet, the area will be expressed in *square feet;* and so on.

133. The student will now be able to understand the way in which we estimate the areas of figures, and to use correctly the rules which will be given; the rules will be stated with brevity, but this will present no difficulty to those who have read the foregoing explanations.

134. *To find the area of a rectangle.*

RULE. *Multiply the length by the breadth, and the product will be the area.*

Sometimes the words *base* and *height* are used respectively for the *length* and *breadth* of a rectangle.

135. Examples.

(1) The length of a rectangle is 3 feet 4 inches, and its breadth is 2 feet 6 inches.

3 feet 4 inches = 40 inches, 2 feet 6 inches = 30 inches; $40 \times 30 = 1200$.

Thus the area is 1200 square inches.

Or thus: 3 feet 4 inches = $3\frac{1}{3}$ feet, 2 feet 6 inches = $2\frac{1}{2}$ ft.: $3\frac{1}{3} \times 2\frac{1}{2} = \frac{10}{3} \times \frac{5}{2} = \frac{25}{3} = 8\frac{1}{3}$.

Thus the area is $8\frac{1}{3}$ square feet.

(2) The length of a rectangle is half a mile, and its breadth is 220 yards.

Half a mile $=880$ yards ; $880 \times 220 = 193600$.

Thus the area is 193600 square yards.

Or thus : 220 yards $= \frac{1}{8}$ of a mile, $\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$.

Thus the area is $\frac{1}{16}$ of a square mile.

136. If we know the area of a rectangle, and also its length, we can find the breadth by dividing the number which expresses the area by the number which expresses the length: and similarly if we know the area and the breadth we can find the length. Of course we must take care to use corresponding denominations for the area and the known length or breadth: see Art. 132.

137. Examples.

(1) The area of a rectangle is 96 square inches, and its length is 1 foot 4 inches.

1 foot 4 inches $=16$ inches ; $\frac{96}{16} = 6$.

Thus the breadth is 6 inches.

(2) The area of a rectangle is 10 square feet, and its breadth is 1 yard.

1 yard $= 3$ feet; $\frac{10}{3} = 3\frac{1}{3}$.

Thus the length is $3\frac{1}{3}$ feet, that is 3 feet 4 inches.

138. A square is a rectangle having its length and breadth equal ; hence to find the area of a square we *multiply the number which denotes the length of a side of the square by itself.* For example, if the length of the side of a square be 7 inches, the area of the square is 7 times 7 square inches, that is, 49 square inches. Thus we see the reason for using the term the *square of a number* to denote the product of the number into itself; and we understand the connexion of the Rules in Chapter V. with the theorem of Art. 30.

139. The statements made in the Table given in Chapter X. will be easily understood and remembered by the aid of the explanations in the present Chapter. Take, for example, the first statement that 144 square inches make 1 square foot: a square foot is a rectangle 12 inches long and 12 inches broad; and therefore by the method of Art. 128, we see that a square foot contains 12×12 square inches, that is 144 square inches.

140. If we know the area of a square we can find the length of a side of the square by *extracting the square root of the number which denotes the area.* For example, suppose the area of a square to be 121 square inches; the square root of 121 is 11: thus the length of a side of the square is 11 inches. Again, suppose the area of a square to be 150 square inches. Here the square root cannot be exactly found, and so the length of the side cannot be determined accurately: if we proceed to three decimal places we obtain 12·247 inches for the required length.

141. The student must distinguish carefully between *square feet* and *feet square.* For example; by three square feet we mean an area which can be divided into three others each of which is a square foot; by three feet square is meant a square the side of which is three feet long, so that the figure contains nine square feet. Similarly by four feet square is meant a square the side of which is four feet long, so that the figure contains sixteen square feet.

142. We will now solve some exercises which depend on the Rules of the present Chapter.

(1) A room is 18 feet 6 inches long and 11 feet 3 inches broad: find the expense of carpeting the room, supposing the carpet to be 30 inches wide, and to cost 6 shillings per yard.

We first find the length of carpet required. The length of the room is $18\frac{1}{2}$ feet, and the breadth is $11\frac{1}{4}$ feet; hence the area of the room is $\frac{37}{2} \times \frac{45}{4}$ square feet, that is $\frac{1665}{8}$ square feet. The breadth of the carpet is $2\frac{1}{2}$ feet.

Hence, by Art. 136, the required length of carpet is obtained by dividing $\frac{1665}{8}$ by $2\frac{1}{2}$; so that it is $\frac{1665}{8} \times \frac{2}{5}$ feet, that is $\frac{333}{4}$ feet. We have now a simple example in Arithmetic: if one yard costs 6 shillings, find the cost of $\frac{333}{4}$ feet. The result is $\frac{1}{3} \times \frac{333}{4} \times 6$ shillings, that is $\frac{333}{2}$ shillings, that is £8. 6*s.* 6*d.*

It is scarcely necessary to remark that in actual experience rather more carpet would be required than our solution indicates, in order to allow for the waste which arises from arranging the pieces with due regard to the correspondence of the pattern.

(2) A room is 18 feet 6 inches long, 11 feet 3 inches broad, and 10 feet high: find the entire area of the four walls.

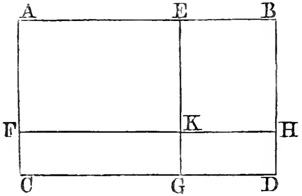
There are *two* walls each containing $\frac{37}{2} \times 10$ square feet, and *two* others each containing $\frac{45}{4} \times 10$ square feet: therefore the entire area is equivalent to that of a rectangle the height of which is 10 feet, and the base $37 + \frac{45}{2}$ feet; thus the entire area is $\frac{119}{2} \times 10$ square feet, that is 595 square feet.

(3) A rectangular grass plot is 160 feet long and 100 feet broad; a gravel walk 4 feet wide surrounds the grass plot: find the area of the walk.

The extreme length of the rectangle including the walk is 168 feet, and the extreme breadth is 108 feet; therefore the area of this rectangle is 168×108 square feet, that is 18144 square feet. The area of the grass plot is 160×100 square feet, that is 16000 square feet. The area of the walk is therefore 18144 − 16000 square feet, that is 2144 square feet.

(4) A rectangle is divided into four rectangles by two straight lines drawn parallel to the sides at given distances from them: find the areas of the four rectangles.

Let $ABDC$ be the rectangle; suppose, for example, that AB is 16 inches, and AC is 9 inches; and that AE is 10 inches and AF is 7 inches. Through E let EKG be drawn parallel to AC; and through F let FKH be drawn parallel to AB.



Then $EB = 6$ inches, and $FC = 2$ inches.
The area of $AEKF = 10 \times 7$, that is 70, square inches.
The area of $EBHK = 6 \times 7$, that is 42, square inches.
The area of $FKGC = 10 \times 2$, that is 20, square inches.
The area of $KHGD = 6 \times 2$, that is 12, square inches.

The sum of these four areas is 144 square inches, and is equal, as it should be, to 16×9 square inches.

This exercise is very simple, but very instructive; it affords a visible representation of an important arithmetical proposition, which in the present case stands thus: the product of the sum of 10 and 6 into the sum of 7 and 2 is equal to the sum of 10×7, 6×7, 10×2, and 6×2.

(5) A rectangle is $\frac{5}{8}$ of an inch long, and $\frac{3}{7}$ of an inch wide: find the area.

By the Rule of Art. 134 the area is $\frac{5}{8} \times \frac{3}{7}$ of a square inch, that is $\frac{15}{56}$ of a square inch. But in the demonstration of the Rule in Art. 128, we supposed that the length and the breadth were expressed in *whole* numbers, so that it is expedient to shew explicitly that the Rule will also hold when fractions occur; this we shall now do.

Reduce the fractions to a common denominator, thus they become $\frac{35}{56}$ and $\frac{24}{56}$ of an inch. Now let us take $\frac{1}{56}$ of an inch as the unit of length; then the length of the rectangle is 35, and the breadth is 24 of these units: and therefore the area of the rectangle is 35×24 square units. And a square inch contains 56×56 of these square units. Thus the area of the rectangle is $\frac{35 \times 24}{56 \times 56}$ of a square inch; that is $\frac{5 \times 3}{8 \times 7}$ of a square inch: that is $\frac{15}{56}$ of a square inch.

EXAMPLES. XI.

Find the area in square yards of squares having the following lengths of sides:

1. 14 yards. 2. 24 yards. 3. $27\frac{1}{2}$ yards.
4. $30\frac{1}{4}$ yards.

Find the area in square yards and square feet of squares having the following lengths of sides:

5. 10 yards 2 feet. 6. 12 yards 1 foot.
7. 18 yards 2 feet. 8. 20 yards 1 foot.

Find the area in square yards, feet, and inches of squares having the following lengths of sides:

9. 3 yards 2 feet 4 inches. 10. 5 yards 2 feet 8 inches.
11. 8 yards 1 foot 9 inches. 12. 14 yards 1 foot 10 inches.

Find the area in acres, roods, and poles of squares having the following lengths of sides:

13. 4 chains 50 links. 14. 7 chains 25 links.
15. 12 chains 45 links. 16. 26 chains 56 links.

Find the areas of squares having the following diagonals:

17. 255 feet. 18. 88 yards 2 feet 3 inches.
19. 12 chains 25 links. 20. 18 chains 36 links.

Find the sides of squares having the following areas:

21. 1764 square yards.
22. 7225 square yards.
23. 74529 square yards.
24. $\frac{1}{16}$ square mile.
25. 160 acres.
26. $2\frac{1}{2}$ acres.
27. 64·064016 square feet.
28. 3 acres 1 rood 13 poles $5\frac{3}{4}$ sq. yards.

Find in feet to three decimal places the sides of squares having the following areas:

29. 120 square feet.
30. 287 square feet.
31. 478 square yards 1 square foot.
32. 526 square yards 2 square feet 90 square inches.
33. 150 acres.
34. $2\frac{3}{4}$ acres.
35. Find the diagonal of a square whose area is 7 square inches.
36. The area of a chess board having 8 squares along each side is 100 square inches: find the length of the side of one of its squares.

Find the area in square feet of rectangles having the following dimensions in feet:

37. 14 by 20.
38. 24 by 18.
39. $15\frac{1}{2}$ by 18.
40. $18\frac{1}{4}$ by $20\frac{1}{2}$.

Find the area in square yards and feet of rectangles having the following dimensions:

41. 5 yards 2 feet by 6 yards.
42. 7 yards 1 foot by 8 yards 2 feet.
43. 10 yards 1 foot by 12 yards 1 foot.
44. 9 yards 2 feet by 18 yards 2 feet.

Find the area in square yards, feet, and inches, of rectangles having the following dimensions:

45. 2 yards 1 foot by 3 yards 1 foot 3 inches.
46. 3 yards 1 foot 4 inches by 4 yards 2 feet.
47. 4 yards 2 feet 8 inches by 4 yards 2 feet 10 inches.
48. 6 yards 1 foot 9 inches by 8 yards 2 feet 11 inches.

Find the area in acres, roods, and poles of rectangles having the following dimensions:

49. 5 chains 14 links by 6 chains 25 links.

50. 7 chains 4 links by 8 chains 12 links.

51. 9 chains 24 links by 10 chains 36 links.

52. 10 chains 80 links by 12 chains 40 links.

Find the breadth in the following rectangles, having given the area and the length:

53. Area 1056 square feet, length 11 yards.

54. Area an acre, length 110 yards.

55. Area a square mile, length 5 miles.

56. Area 1000 acres, length $2\frac{1}{2}$ miles.

57. Area $2\frac{5}{8}$ acres, length $115\frac{1}{2}$ yards.

58. Area $5\frac{1}{4}$ acres, length 32 chains.

59. Area 7 acres 1 rood 15 poles, length 453 yards 2 feet 3 inches.

60. A plank is 18 inches broad: find what length must be cut off that the area may be a square yard.

61. A rectangle is 9 inches by 18 inches: find what decimal it is of a square yard.

62. Express as a fraction of an acre the rectangle which is 121 yards long and 25 yards broad.

63. A street is a quarter of a mile long: find the number of square yards in a pavement $4\frac{1}{2}$ feet wide down one side of the street.

64. A rectangular garden is to be cut from a rectangular field, so as to contain three quarters of an acre; one side of the field is taken for one side of the plot, and measures $2\frac{1}{2}$ chains: find the length of the other side.

65. The diagonal of a rectangle is 458 feet, and one side is 442 feet: find the area.

66. The sides of four squares being respectively 1, 2, 4, and 10 feet; find the side of the square which is equal to the sum of the four.

67. The sides of three squares being 5, 6, and 7 feet: find the side of the square which is equal to the sum of the three.

68. The window of a house is 8 feet 2 inches by 5 feet 3 inches: find the number of panes of glass in it, each measuring 14 inches by 9.

69. A lawn measures 150 feet by 120 feet: find how many pieces of turf are required to cover it, each piece being 3 feet 4 inches by 1 foot 3 inches.

70. Find how many slates measuring 16 inches by 12 inches will be required to cover a roof which measures 24 feet by 18 feet.

71. Find how many bricks measuring 9 inches by $4\frac{1}{2}$ inches will be required to cover a space of 18 feet by 12 feet 9 inches.

72. Find how many planks 12 feet long by 10 inches wide will be required to floor a room which is 24 feet by 20 feet.

73. Find how many planks 12 feet 6 inches long by $9\frac{1}{2}$ inches wide will be required to floor a room which is 50 feet by 16 feet.

74. Find how many persons can stand in a room measuring 15 feet by 9 feet; supposing each person to require a space of 27 inches by 18 inches.

75. A procession is formed of 504 ranks of men, 14 in a rank: if the men were arranged in a solid square, find how many there would be in a side.

76. If one stalk of wheat will grow on nine square inches of ground, find how many stalks will grow on an acre.

77. Find how many trees there are in a wood half a mile long and a quarter of a mile wide, supposing on an average four trees grow on each square chain.

78. A country in the form of a rectangle 600 miles long by 200 miles broad supports a population of 20,000,000: find the average number of acres required to support one person.

79. A room is 25 feet by 18 feet; in the central part is a Turkey carpet which measures 21 feet by 15 feet: find how many yards of oilcloth 27 inches wide will be required to cover the rest of the floor.

80. The side of a square is 85 yards, and a path 10 yards wide goes round the square outside it: find how many stones 1 foot 4 inches long by 10 inches wide will be required to pave the path.

81. A rectangular court measures 63 feet by 36 feet; a path 4 feet 6 inches wide goes round the court outside it: find how many bricks measuring 9 inches by $4\frac{1}{2}$ inches will be required to pave the path.

82. It is found that 1296 bricks, each measuring 9 inches by $4\frac{1}{2}$ inches, have been employed in paving a certain court yard: find how many tiles 6 inches square will be required for a pavement one-ninth of the size.

83. If the adjacent sides of one rectangle be 9 and 16, and of another 36 and 25, compare the sides of the squares respectively equal to these rectangles.

84. Find what length of wall paper 27 inches wide will be required for a room 18 feet long, 12 feet broad, and 10 feet 6 inches high.

85. Find how many square feet of paper will cover the walls of a room which is 24 feet 10 inches long, 16 feet broad, and 18 feet 6 inches high.

86. A rectangle measures 48 feet by 28 feet: find the area of a square which has the same perimeter as the rectangle.

87. A rectangle contains 1323 square feet; and it is three times as long as it is broad: find its sides.

88. Seven sheets of note paper together weigh one ounce; each sheet measures 9 inches by $6\frac{7}{8}$ inches: find the weight of a sheet of the same kind of paper which measures $18\frac{2}{3}$ inches by 11.

89. Shew by examples that if a square and a rectangle have equal perimeters the area of the square is greater than that of the rectangle.

90. Shew by examples that if a square and a rectangle have equal perimeters, the area of the square exceeds the area of the rectangle by the area of a square the side of which is half the difference of the sides of the rectangle.

91. Find the rent at £1. 13*s*. per acre of a rectangular field of which the length is 1 furlong 20 poles, and the breadth 10 poles 1 yard.

92. Find the rent at £4. 10*s*. an acre of a piece of land 4235 yards long and 280 yards wide.

93. A rectangular court measures 18 feet 6 inches by 12 feet 3 inches: find the expense of paving it at 4 pence the square foot.

94. The diagonal of a square court yard is 30 yards: find the cost of gravelling the court at the cost of a shilling for nine square yards.

95. Find the expense of paving an area which measures 32 feet 3 inches by 16 feet 6 inches at 6*s*. 4*d*. per square yard.

96. The length of a street is 1 furlong 92 yards 1 foot 6 inches, and its breadth is 22 yards 8 inches: find the cost of paving it at $8\frac{1}{2}d$. per square yard.

97. In a rectangular court which measures 96 feet by 84 feet there are four rectangular grass plots measuring each $22\frac{1}{2}$ feet by 18 feet: find the cost of paving the remaining part of the court at $8\frac{1}{2}d$. per square yard.

98. Find the expense of paving a road of a uniform breadth of 4 yards round the inside of a rectangular piece of ground the length of which is 85 yards and breadth 56 yards, the cost of paving a square yard being 1*s*. 2*d*.

99. Find the side of a square court yard the expense of paving which is £38. 10*s*. 5*d*. at 3*s*. 9*d*. per square yard.

100. Determine the side of a square garden that cost £57. 15*s*. $2\frac{3}{4}d$. for trenching at $2\frac{3}{4}d$. per square yard.

101. The rent of a square field at £2. 14*s*. 6*d*. per acre amounts to £27. 5*s*.: find the cost of putting a paling round the field at 9*d*. per yard.

Find how many yards of carpet will be required for rooms of the following dimensions:

102. 18 feet by 16 feet; the carpet being 1 yard wide.

103. 24 feet by 16 feet 6 inches; the carpet being 1 yard wide.

104. 21 feet by 15 feet; the carpet being 27 inches wide.

105. 17 feet 3 inches by 9 feet 9 inches; the carpet being 27 inches wide.

106. 28 feet by 23 feet 9 inches; the carpet being 30 inches wide.

107. 27 feet 3 inches by 22 feet 6 inches; the carpet being 30 inches wide.

Find the expense of carpeting rooms, the dimensions and the cost of the carpet being the following:

108. 12 feet 4 inches by 16 feet 3 inches; 1*s*. 6*d*. per square foot.

109. 24 feet 8 inches by 16 feet 3 inches; 13*s*. 6*d*. per square yard.

110. 23 feet 9 inches by 16 feet 3 inches; 2*s*. 9*d*. per square yard.

Find the expense of carpeting rooms, the dimensions of the room and the width and the cost of the carpet being the following:

111. 34 feet by 18 feet 6 inches; carpet 2 feet wide at 4*s*. 6*d*. per yard.

112. 18 feet 9 inches by 17 feet 6 inches; carpet 2 feet wide at 4*s*. 9*d*. per yard.

113. 15 feet 9 inches by 12 feet 5 inches; carpet 1 yard 18 inches wide at 6 shillings per yard.

114. 18 feet 6 inches by 12 feet 6 inches; carpet 27 inches wide at 3 shillings per yard.

115. 15 feet 9 inches by 12 feet 5 inches; carpet 27 inches wide at 4 shillings per yard.

116. 21 feet 8 inches by 16 feet 6 inches; carpet 27 inches wide at 3*s.* $4\frac{1}{2}$*d.* per yard.

117. 17 feet 6 inches by 17 feet 6 inches; carpet 2 feet 4 inches wide at 3*s.* 9*d.* per yard.

118. Supposing the cost of a carpet in a room 25 feet long at 5*s.* per square yard to be £6. 5*s.*: find the breadth of the room.

119. Find the quantity of carpeting required for the central portion of a room, this portion being 13 feet 6 inches wide and 18 feet 9 inches long. Find also the expense, the carpet being 27 inches wide, and 4*s.* 6*d.* per yard. If between the edge of the carpet and the walls there is a distance all round of $2\frac{1}{2}$ feet, find how much of the area of the floor will remain uncovered.

120. Find how many yards of paper will be required for the walls of a room which is 23 feet long, 18 feet wide, and 12 feet high; the paper being a yard wide.

121. Find how many yards of paper will be required for the walls of a room which is 24 feet long, 19 feet 6 inches wide, and 14 feet high; the paper being three quarters of a yard wide.

122. A room is 34 feet long, $18\frac{1}{2}$ feet wide, and 12 feet high: find the expense of papering the walls at 1*s.* 6*d.* per square yard.

123. Find the expense of papering a room 6 yards 1 foot 1 inch long, 6 yards 0 feet 4 inches broad, 12 feet high, with paper a foot wide at 9*d.* per yard.

124. A room is 24 feet long, 15 feet broad, and 11 feet high: find the expense of painting the walls at 3*d.* per square foot; allowing for a fire place which is 4 feet 6 inches by 3 feet, a door which is 7 feet by 4 feet, and two windows each 6 feet 6 inches by 5 feet.

XII. PARALLELOGRAM.

143. We have shewn in Art. 28 that a parallelogram is equivalent to a rectangle having the same base and height: this is the reason of the rule now to be given.

144. *To find the area of a parallelogram.*

RULE. *Multiply the base by the height and the product will be the area.*

145. Examples:

(1) The base of a parallelogram is 5 feet, and its height is 3 feet.

$5 \times 3 = 15$. Thus the area is 15 square feet.

(2) The base of a parallelogram is 3 feet 9 inches, and its height is 2 feet 3 inches.

3 feet 9 inches = 45 inches, 2 feet 3 inches = 27 inches.

$45 \times 27 = 1215$. Thus the area is 1215 square inches.

Or thus: 3 feet 9 inches = $3\frac{3}{4}$ feet, 2 feet 3 inches = $2\frac{1}{4}$ feet. $3\frac{3}{4} \times 2\frac{1}{4} = \frac{15}{4} \times \frac{9}{4} = \frac{135}{16} = 8\frac{7}{16}$. Thus the area is $8\frac{7}{16}$ square feet.

146. If we know the area of a parallelogram, and also one of the two dimensions, the base or the height, we can find the other: see Art. 136.

147. We will now solve some exercises.

(1) The area of a rhombus is 180 square feet, and each side is 15 feet long: find the height.

$$\frac{180}{15} = 12. \quad \text{Thus the height is 12 feet.}$$

(2) Two adjacent sides of a parallelogram are 8 feet and 16 feet respectively; the area is two-thirds that of a square which has the same perimeter: find the height of the parallelogram.

The perimeter of the parallelogram is $16 + 32$ feet, that is 48 feet; hence the side of a square having the same perimeter is 12 feet, and therefore the area of the square is 144 square feet. Thus the area of the parallelogram is $\frac{2}{3}$ of 144 square feet, that is 96 square feet. If we take the side which is 8 feet long for base, the height is $\frac{96}{8}$, that is 12 feet. If we take the side which is 16 feet long for base, the height is $\frac{96}{16}$, that is 6 feet.

(3) Each side of a rhombus is 18 feet, and one of the diagonals also is 18 feet: find the area.

By drawing this diagonal the rhombus is divided into two equilateral triangles; and the height of each triangle, by Art. 68, is $18 \times {}^{\cdot}866...$ feet. Also this height is equal to the height of the rhombus. Thus the area of the rhombus in square feet is $18 \times 18 \times {}^{\cdot}866...$, that is about $280{\cdot}6$ square feet.

EXAMPLES. XII.

Find the areas of the parallelograms having the following bases and heights:

1. Base 14 yards, height 5 yards.

2. Base 15 yards 2 feet, height 11 yards 1 foot.

3. Base 16 yards 2 feet 3 inches, height 14 yards 2 feet 8 inches.

4. Base 14 chains 16 links, height 9 chains 48 links.

Find the heights of the parallelograms having the following areas and bases:

5. Area 1125 square feet, base 15 yards.

6. Area $3\frac{1}{2}$ acres, base 242 yards.

7. Area 93 square feet 140 square inches, base 5 yards 1 foot 7 inches.

8. Area 160 square yards 3 square feet 33 square inches, base 13 yards 1 foot 9 inches.

9. The base of a parallelogram is 4 feet 6 inches, and its height is 2 feet 8 inches; the side adjacent to the base is 3 feet: find the length of the perpendicular on this side from any point in the opposite side.

10. The adjacent sides of a parallelogram are 8 feet and 16 feet, and its area is half that of a square having the same perimeter: find the perpendicular distance between each pair of opposite sides.

11. Each side of a rhombus is 24 feet, and one of the diagonals also is 24 feet: find the area.

12. Each side of a rhombus is 32 feet, and each of the larger angles is equal to twice each of the smaller angles: find the area.

XIII. TRIANGLE.

148. We have shewn in Art. 29 that a triangle is equivalent to half a rectangle having the same base and height: this is the reason of the rule now to be given.

149. *To find the area of a triangle.*

RULE. *Half the product of the base into the height will be the area.*

It is obvious that we may multiply together the base and half the height, or multiply together the height and half the base, or multiply together the base and the height and take half the product.

150. Examples:

(1) The base of a triangle is 3 yards, and its height is 4 feet 6 inches.

$$3 \text{ yards} = 9 \text{ feet}, \ 4 \text{ feet } 6 \text{ inches} = 4\tfrac{1}{2} \text{ feet}.$$

$$9 \times 4\tfrac{1}{2} = 9 \times \frac{9}{2} = \frac{81}{2}; \ \frac{1}{2} \text{ of } \frac{81}{2} = \frac{81}{4} = 20\tfrac{1}{4}.$$

Thus the area is $20\frac{1}{4}$ square feet.

(2) The base of a triangle is 45 feet, and its height is 36 feet.

Half of 36 is 18; $45 \times 18 = 810$. Thus the area is 810 square feet.

151. If we know the area of a triangle, and also one of the two dimensions, the base or the height, we can find the other. For if twice the number expressing the area be divided by the number expressing the height, the quotient is *the base;* and if twice the number expressing the area be divided by the number expressing the base, the quotient is *the height.*

152. *The three sides of a triangle being given, to find the area.*

RULE. *From half the sum of the three sides subtract each side separately; multiply the half sum and the three remainders together: the square root of the product will be the area.*

153. Examples:

(1) The sides of a triangle are 2 feet 2 inches, 2 feet 4 inches, and 2 feet 6 inches respectively.

2 feet 2 inches = 26 inches, 2 feet 4 inches = 28 inches, 2 feet 6 inches = 30 inches.

$$26 + 28 + 30 = 84, \qquad \tfrac{1}{2} \text{ of } 84 = 42;$$

$$42 - 26 = 16, \qquad 42 - 28 = 14, \qquad 42 - 30 = 12.$$

$42 \times 16 \times 14 \times 12 = 112896$. The square root of 112896 is 336. Thus the area is 336 square inches.

(2) The sides of a triangle are 24, 25, and 26 feet respectively.

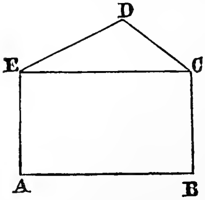
$$24 + 25 + 26 = 75, \qquad \tfrac{1}{2} \text{ of } 75 = 37{\cdot}5.$$

$$37{\cdot}5 - 24 = 13{\cdot}5, \qquad 37{\cdot}5 - 25 = 12{\cdot}5, \qquad 37{\cdot}5 - 26 = 11{\cdot}5.$$

$37{\cdot}5 \times 13{\cdot}5 \times 12{\cdot}5 \times 11{\cdot}5 = 72773{\cdot}4375$. The square root of 72773·4375 cannot be found exactly; if we proceed to three decimal places we obtain 269·766: so that the area is about 269·766 square feet.

154. We will now solve some exercises.

(1) Find the area of the gable end of a house, the breadth being 24 feet, the distance of the eaves from the ground 30 feet, and the perpendicular height of the roof 10 feet.



The figure is composed of a rectangle and a triangle. *AB* or *EC* is the breadth; the eaves are the junctions of the walls and the roof, as at *E* and *C*, so that *AE* or *BC* is the height of the eaves from the ground. The perpendicular height of the roof is the perpendicular from D on EC.

The highest part of the roof is called the ridge, so that D is on the ridge. The triangle CDE is called the gable top.

Here the area of the rectangle $ABCE$ is 24×30 square feet, that is 720 square feet; and the area of the triangle is 24×5 square feet, that is 120 square feet. Thus the whole area is 840 square feet.

(2) The side of an equilateral triangle is one foot; find the area.

$$\text{Half the sum of the sides} = \frac{3}{2}; \quad \frac{3}{2} - 1 = \frac{1}{2}.$$

$\frac{3}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{16}$. The number of square feet in the area is therefore equal to the square root of $\frac{3}{16}$, that is to $\frac{1}{4}$ of the square root of 3.

We may also obtain this result thus. It is shewn in Art. 65 that the height of the triangle is half of the square root of 3, and therefore by Art. 149, the area is one-fourth of the square root of 3. Thus the area is approximately ·433 of a square foot; or to seven decimal places ·4330127.

(3) The sides of a right-angled triangle are 8 feet and 15 feet respectively: find the perpendicular from the right angle on the hypotenuse, and the two parts into which it divides the base.

By Art. 149 the area of the triangle is 60 square feet.

By Art. 55 the length of the hypotenuse is 17 feet.

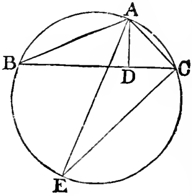
By Art. 151 the perpendicular is $\frac{120}{17}$ feet, that is $7\frac{1}{17}$ feet.

Then, by Art. 60, the shorter of the two parts into which the perpendicular divides the base, in feet, is the square root of $8 \times 8 - \frac{120}{17} \times \frac{120}{17}$, that is the square root of

$64-\frac{14400}{289}$, that is the square root of $\frac{4096}{289}$, that is $\frac{64}{17}$. Therefore the other part, in feet, is $17-\frac{64}{17}$, that is $\frac{225}{17}$.

(4) Having given the sides of a triangle to find the diameter of the circle described round the triangle.

The investigation which we shall now give is valuable not only for the result which will be obtained, but also for the illustration which it affords of the method by which geometrical truths are demonstrated.



Let ABC be the triangle, AE a diameter of the circle described round the triangle, AD the perpendicular from A on the base BC. Join CE.

By Art. 33, the angle ACE is a right angle; so that this angle is equal to the angle ADB.

By Art. 32, the angle AEC is equal to the angle ABD. Therefore, by Art. 23, the angle BAD must be equal to the angle EAC.

Therefore, by Art. 34, the triangles ABD and AEC are similar; so that AB is to AD as AE is to AC; and therefore $AB \times AC = AD \times AE$.

$$\text{Thus } AE = \frac{AB \times AC}{AD} = \frac{AB \times AC \times BC}{AD \times BC}.$$

Hence we have the following result: *the diameter of the circle described round a triangle is equal to the product of the sides of the triangle divided by twice the area of the triangle.*

Suppose, for example, that the sides of the triangle are 26 inches, 28 inches, and 30 inches respectively. By Art. 153 the area is 336 square inches. Thus the diameter of the circle described round the triangle in inches

$$=\frac{26 \times 28 \times 30}{2 \times 336} = \frac{65}{2} = 32\tfrac{1}{2}.$$

EXAMPLES. XIII.

Find the areas of the triangles having the following dimensions:

1. Base 18 feet, height 8 feet.

2. Base 8 yards 1 foot, height 5 yards 2 feet.

3. Base 10 yards 2 feet 6 inches, height 7 yards 1 foot 3 inches.

4. Base 14 chains 15 links, height 12 chains 24 links.

Find the areas of the right-angled triangles having the following dimensions:

5. Hypotenuse 421, side 29.

6. Hypotenuse 730, side 152.

Find approximately the areas of the right-angled triangles having the following dimensions:

7. Hypotenuse 10, side 7.

8. Hypotenuse 13, side 9.

Find the areas of the triangles having the following sides:

9. 5, 5, 6.

10. 65, 65, 112.

11. 85, 85, 154.

12. 373, 373, 504.

13. 68, 75, 77.

14. 20, 493, 507.

15. 105, 116, 143.

16. 111, 175, 176.

17. 43, 875, 888.

18. 319, 444, 455.

19. 533, 875, 888.

20. 3501, 3604, 3605.

Calculate to three decimal places the areas of the triangles having the following sides:

21. 2, 3, 4.	22. 6, 7, 9.
23. 7, 8, 13.	24. 15, 16, 17.
25. 23, 33, 40.	26. 17, 63, 73.

27. The sides of a triangle are 11, 24, and 31: shew that the area is $66\sqrt{3}$.

28. The sides of a triangle are 61, 62, and 63: shew that the area is $744\sqrt{5}$.

29. The sides of a triangle are 68, 75, and 77; a straight line is drawn across the triangle parallel to the longest side, and dividing each of the other sides into two equal parts: find the area of the two parts into which the triangle is divided.

30. The sides of a triangle are 111, 175, and 176; two straight lines are drawn across the triangle parallel to the longest side, and dividing each of the other sides into three equal parts: find the areas of the three parts into which the triangle is divided.

31. The sides of a triangle are 13, 14, and 15 feet: find the perpendicular from the opposite angle on the side of 14 feet.

32. The sides of a triangle are 51, 52, and 53 feet: find the perpendicular from the opposite angle on the side of 52 feet, and find the area of the two triangles into which the original triangle is thus divided.

33. The side of a square is 100 feet; a point is taken inside the square which is distant 60 feet and 80 feet respectively from the two ends of a side: find the areas of the four triangles formed by joining the point to the four corners of the square.

34. ABC is a triangle, and AD is the perpendicular from A on BC. If $AD = 13$ feet, and the lengths of the perpendiculars from D on AB and AC be 5 feet and 10·4 feet respectively, find the sides and the area of the triangle.

35. The base of a triangular field is 1166 links, and the height is 738 links; the field is let for £24 a year: find at what price per acre the field is let.

36. The sides of a triangular field are 350, 440, and 750 yards; the field is let for £26. 5*s*. a year: find at what price per acre the field is let.

37. Find to the nearest square inch the area of a triangle whose sides are 5, 6, and 7 feet.

38. A field is in the form of a right-angled triangle, the two sides which contain the right angle being 100 yards and 200 yards: find its area. If the triangle be divided into two parts by a straight line drawn from the right angle perpendicular to the opposite side, find the area of each part.

39. The sides of a triangle are in the proportion of the numbers 5, 12, and 13; and the perimeter is 50 yards: find the area.

40. The sides of a triangle are in the proportion of the numbers 13, 14, 15; and the perimeter is 70 yards: find the area.

41. Find the cost of painting the gable end of a house at 1*s*. 9*d*. per square yard; the breadth being 27 feet, the distance of the eaves from the ground 33 feet, and the perpendicular height of the roof 12 feet.

Find the diameters of the circles described round the triangles having the following sides:

42. 293, 285, 68.

43. 136, 125, 99.

44. 123, 122, 49.

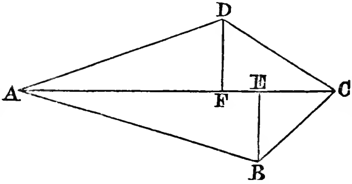
45. 267, 244, 161.

XIV. QUADRILATERALS.

155. A quadrilateral can be divided into two triangles by drawing a diagonal; then the area of each triangle can be found, and the sum of the areas of the triangles will be the area of the quadrilateral.

156. Examples:

(1) The diagonal AC of a quadrilateral $ABCD$ is 12 feet; the perpendicular BE is 3 feet, and the perpendicular DF is 4 feet.



The area of the triangle $ACB = \frac{1}{2} \times 12 \times 3 = 18$;

the area of the triangle $ACD = \frac{1}{2} \times 12 \times 4 = 24$;

$$18 + 24 = 42.$$

Thus the area of the quadrilateral is 42 square feet.

(2) A diagonal of a quadrilateral is 88 yards, and the perpendiculars on it from the opposite angles are 30 yards and 25 yards respectively.

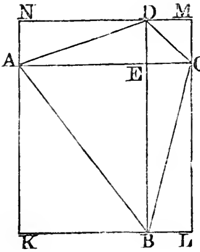
$$\frac{1}{2} \times 88 \times 30 = 1320; \qquad \frac{1}{2} \times 88 \times 25 = 1100.$$

$1320 + 1100 = 2420$. Thus the area of the quadrilateral is 2420 square yards, that is half an acre.

157. It is obvious that in Examples like those of the preceding Article, instead of calculating separately the areas of the two triangles, we may find the area of the quadrilateral by using the following rule: *multiply the sum of the perpendiculars by the diagonal, and take half the product.*

Thus in the first example of Art. 156 the sum of the perpendiculars is 7 feet; and therefore the area in square feet $= \frac{1}{2} \times 12 \times 7 = 42$; in the second example the sum of the perpendiculars is 55 yards, and therefore the area in square yards $= \frac{1}{2} \times 88 \times 55 = 2420$.

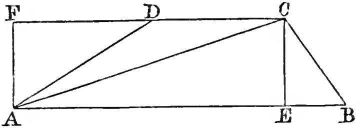
158. In the particular case in which the diagonals of a quadrilateral intersect at right angles the rule just given amounts to this: *take half the product of the two diagonals.*



By the aid of a figure the truth of this rule becomes self-evident. Let $ABCD$ be a quadrilateral such that its diagonals AC and BD intersect at right angles; let E be the point of intersection. Through A and C draw straight lines parallel to BD; through B and D draw straight lines parallel to AC. Thus a rectangle $KLMN$ is formed. Now it is easy to see that, the triangle AEB is equal to the triangle BKA, the triangle BEC is equal to the triangle CLB, the triangle CED is equal to the triangle DMC, and the triangle DEA is equal to the triangle AND. Thus the quadrilateral $ABCD$ is half of the rectangle $KLMN$; and therefore the area of the quadrilateral is equal to half the product of AC and BD.

159. The diagonals of a *rhombus* intersect at right angles; and therefore the rule of the preceding Article may always be applied to a *rhombus.*

160. It is usual to give a special rule for finding the area of a *trapezoid.*



Let $ABCD$ be a quadrilateral having the sides AB and CD parallel. From C draw CE perpendicular to AB; and from A draw AF perpendicular to CD. Then

$$\text{the area of the triangle } ABC = \frac{1}{2} AB \times CE;$$

$$\text{the area of the triangle } ADC = \frac{1}{2} CD \times AF.$$

Now we may admit that $AF = CE$; and therefore the area of the quadrilateral is equal to the product of CE into half the sum of AB and CD. Thus we obtain the Rule which will now be given.

161. *To find the area of a trapezoid.*

RULE *Multiply the sum of the two parallel sides by the perpendicular distance between them, and half the product will be the area.*

162. Examples:

(1) The two parallel sides of a trapezoid are 2 feet 6 inches, and 3 feet 4 inches respectively; and the perpendicular distance between them is 1 foot 8 inches.

2 feet 6 inches $= 2\frac{1}{2}$ feet, 3 feet 4 inches $= 3\frac{1}{3}$ feet.

1 foot 8 inches $= 1\frac{2}{3}$ feet; $2\frac{1}{2} + 3\frac{1}{3} = 5\frac{5}{6}$,

$$\frac{1}{2} \times 5\tfrac{5}{6} \times 1\tfrac{2}{3} = \frac{1}{2} \times \frac{35}{6} \times \frac{5}{3} = \frac{175}{36} = 4\tfrac{31}{36}.$$

Thus the area of the trapezoid is $4\frac{31}{36}$ square feet.

(2) The two parallel sides of a trapezoid are 4·32 feet and 5·48 feet respectively; and the perpendicular distance between them is 2·18 feet.

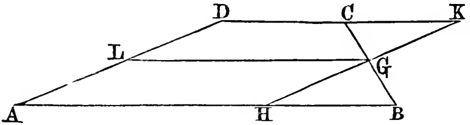
$$4·32+5·48=9·8,\ \frac{1}{2}\text{ of }9·8=4·9,$$

$$2·18\times4·9=10·682.$$

Thus the area of the trapezoid is 10·682 square feet.

163. We have established the rule for finding the area of a trapezoid in a very simple manner in Art. 160; there is also another process which we will give as it is interesting and instructive.

Let *ABCD* be a quadrilateral having the sides *AB* and *CD* parallel. Through *G* the middle point of *BC* draw the straight line *HGK* parallel to *AD*, meeting the parallel sides of the trapezoid at *H* and *K* respectively.

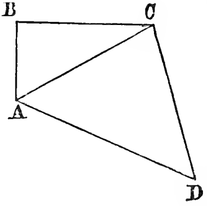


Then the triangles *BGH* and *CGK* are equal; and thus the trapezoid *ABCD* is equivalent to the parallelogram *AHKD*. And since *HB* is equal to *CK*, it follows that *AH* is equal to half the sum of *AB* and *CD*. Thus the trapezoid is equivalent to a parallelogram having its base equal to half the sum of the parallel sides of the trapezoid, and its height equal to the perpendicular distance between those sides. Hence we have the rule given in Art. 161.

Through *G* draw a straight line parallel to *AB* meeting *AD* at *L*. Then *L* is the middle point of *AD*, and *LG*=*AH*; so that half the sum of the parallel sides is equal to the straight line which joins the middle points of the other sides.

164. We will now solve some exercises.

(1) $ABCD$ is a quadrilateral; $AB=3$ feet, $BC=4$ feet, $CD=6$ feet, $DA=7$ feet; and the angle ABC is a right angle: find the area of the quadrilateral.



By Art. 55 we have AC equal to the square root of $9+16$, that is to the square root of 25: so that $AC=5$.

The area of the triangle $ABC = \frac{1}{2} \times 4 \times 3 = 6$.

The area of the triangle ACD can now be found by Art. 152.

$$5+6+7=18,\ \frac{1}{2} \text{ of } 18=9,\ 9-5=4,\ 9-6=3,\ 9-7=2,$$

$$9 \times 4 \times 3 \times 2 = 216.$$

The square root of 216 cannot be found exactly; if we proceed to three decimal places we obtain 14·697. Thus the area of the quadrilateral is about 20·697 square feet.

(2) The diagonals of a rhombus are 80 and 60 feet respectively: find the area; find also the length of a side, and the height of the rhombus.

$\frac{1}{2} \times 80 \times 60 = 2400$. Thus the area is 2400 square feet.

The diagonals of a rhombus intersect at the middle point of each; thus to find the side of the rhombus we must determine the hypotenuse of a right-angled triangle the sides of which are 40 and 30 feet respectively. By Art. 55 the hypotenuse is the square root of 2500; so that the side of the rhombus is 50 feet.

$\frac{2400}{50} = 48$. Thus the height of the rhombus is 48 feet.

EXAMPLES. XIV.

Find the areas of the quadrilaterals having the following dimensions:

1. Diagonal 50·08 feet; perpendiculars 10·12 and 8·4 feet.

2. Diagonal 54 feet; perpendiculars 23 feet 9 inches and 18 feet 3 inches.

3. Diagonal 10 chains 14 links; perpendiculars 6 chains 27 links and 8 chains 6 links.

4. Diagonal 3 chains 27 links; perpendiculars 2 chains 15 links and 1 chain 75 links.

5. Diagonal 18 yards 2 feet, sum of the perpendiculars 16 yards 1 foot.

6. The area of a quadrilateral is 37 acres 1 rood 16 poles; one diagonal is 25 chains: find the sum of the perpendiculars on this diagonal from the two opposite angles.

Find the areas of the trapezoids which have the following dimensions:

7. Parallel sides 3 feet and 5 feet; perpendicular distance 10 feet.

8. Parallel sides 10 feet and 12 feet; perpendicular distance 4 feet.

9. Parallel sides 14 yards and 20 yards; perpendicular distance 12 yards.

10. Sum of the parallel sides 625 links; perpendicular distance 160 links.

11. Sum of the parallel sides 1225 links; perpendicular distance 240 links.

12. Parallel sides 750 links and 1225 links; perpendicular distance 1540 links.

13. The area of a trapezoid is $3\frac{1}{8}$ acres; the sum of the two parallel sides is 242 yards: find the perpendicular distance between them.

14. The area of a trapezoid is 8 acres 2 roods 17 poles; the sum of the parallel sides is 297 yards: find the perpendicular distance between them.

15. In Example 7 a straight line is drawn across the figure parallel to the parallel sides and midway between them: find the area of the two parts into which the trapezoid is divided.

16. In Example 9 two straight lines are drawn across the figure parallel to the parallel sides and dividing each of the other sides into three equal parts: find the areas of the three parts into which the trapezoid is divided.

17. The diagonals of a quadrilateral are 26 feet and 24 feet respectively, and they are at right angles: find the area.

18. The diagonals of a rhombus are 88 yards and 110 yards respectively: find the area.

19. The diagonals of a rhombus are 64 yards and 36 yards respectively: find its area and the cost of turfing it at 4 pence per square yard.

20. The area of a rhombus is 52204 square feet, and one diagonal is 248 feet: find the other.

21. $ABCD$ is a quadrilateral; $AB=28$ feet, $BC=45$ feet, $CD=51$ feet, $DA=52$ feet; the diagonal $AC=53$ feet: find the area.

22. $ABCD$ is a quadrilateral; $AB=48$ chains, $BC=20$ chains, the diagonal $AC=52$ chains, and the perpendicular from D on $AC=30$ chains: find the area.

23. The sides of a quadrilateral taken in order are 27, 36, 30, and 25 feet respectively; and the angle contained by the first two sides is a right angle: find the area.

24. The sides of a quadrilateral taken in order are 5, 5, 4, and 3 feet respectively; and the angle contained by the first two sides is 60^0: find the area.

25. A railway platform has two of its opposite sides parallel and its other two sides equal; the parallel sides are 80 feet and 92 feet respectively; the equal sides are 10 feet each: find the area.

26. $ABCD$ is a quadrilateral; $AB = 845$ feet, $BC = 613$ feet, $CD = 810$ feet; AB is parallel to CD, and the angle at A is a right angle: find the area.

27. $ABCD$ is a quadrilateral; the sides AB and DC are parallel. $AB = 165$ feet, $CD = 123$ feet; the perpendicular distance of AB and DC is 100 feet. E is a point in AB such that AE is equal to half the difference of AB and CD: find the area of the triangle EBC, and of the quadrilateral $AECD$.

28. The diagonals of a rhombus are 88 and 234 feet respectively: find the area; find also the length of a side, and the height of the rhombus.

29. The area of a rhombus is 354144 square feet, and one diagonal is 672 feet: find the other diagonal; find also the length of a side, and the height of the rhombus.

30. Two adjacent sides of a quadrilateral are 228 feet and 704 feet respectively, and the angle contained by them is 90^0; the other two sides of the quadrilateral are equal, and the angle contained by them is 60^0: shew that the area of the quadrilateral in square feet is

$$80256 + 136900\sqrt{3}.$$

XV. RECTILINEAL FIGURE.

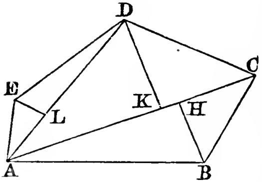
165. *To find the area of any rectilineal figure.*

RULE. *Divide the figure into convenient parts, find the area of every part, and the sum will be the area of the figure.*

In general the parts into which the rectilineal figure can be most conveniently divided will all be triangles: but in some cases we may have a square, a parallelogram, or a trapezoid, as one of the parts.

166. Examples:

(1) *ABCDE* is a five-sided figure: *BH* and *DK* are perpendiculars on *AC*, and *EL* is a perpendicular on *AD*. The following lengths are in feet:

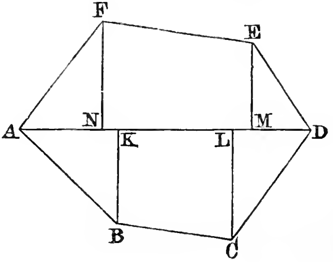


$AC = 10{\cdot}4$, $AD = 8{\cdot}7$,
$BH = 4{\cdot}8$, $DK = 6{\cdot}5$,
$EL = 3{\cdot}2$.

The area of the triangle $ABC = \frac{1}{2} \times 10{\cdot}4 \times 4{\cdot}8 = 24{\cdot}96$;
the area of the triangle $ACD = \frac{1}{2} \times 10{\cdot}4 \times 6{\cdot}5 = 33{\cdot}8$;
the area of the triangle $AED = \frac{1}{2} \times 8{\cdot}7 \times 3{\cdot}2 = 13{\cdot}92$.

Thus the area of the rectilineal figure in square feet $= 24{\cdot}96 + 33{\cdot}8 + 13{\cdot}92 = 72{\cdot}68$.

(2) *ABCDEF* is a six-sided figure: *BK*, *CL*, *EM*,



FN are perpendiculars on AD. The following lengths are in feet:

$$BK=3,\ CL=4,\ EM=4{\cdot}7,\ FN=5{\cdot}1;$$

also $AK=3{\cdot}4$, $KL=3{\cdot}2$, $LD=4{\cdot}1$, $AN=3{\cdot}3$, $NM=5{\cdot}3$.

It follows from these lengths that $AD=10{\cdot}7$, and that $AM=8{\cdot}6$; hence $MD=10{\cdot}7-8{\cdot}6=2{\cdot}1$.

The area of the triangle AKB $=\frac{1}{2}\times 3{\cdot}4\times 3=5{\cdot}1$,
the area of the trapezoid $BKLC$ $=\frac{1}{2}\times 7\times 3{\cdot}2=11{\cdot}2$,
the area of the triangle DLC $=\frac{1}{2}\times 4{\cdot}1\times 4=8{\cdot}2$,
the area of the triangle ANF $=\frac{1}{2}\times 3{\cdot}3\times 5{\cdot}1=8{\cdot}415$,
the area of the trapezoid $FNME$ $=\frac{1}{2}\times 9{\cdot}8\times 5{\cdot}3=25{\cdot}97$,
the area of the triangle EMD $=\frac{1}{2}\times 2{\cdot}1\times 4{\cdot}7=4{\cdot}935$.

Thus the area of the rectilineal figure in square feet $=5{\cdot}1+11{\cdot}2+8{\cdot}2+8{\cdot}415+25{\cdot}97+4{\cdot}935=63{\cdot}82$.

167. We will now solve some exercises.

(1) The side of a regular hexagon is one foot: find the area.

By the aid of the figure in Art. 99, we see that a regular hexagon can be divided into six equilateral triangles; this can be done by drawing straight lines from O to A, B, C, D, E, and F. Now, by Art. 154, the area of each equilateral triangle in square feet is $\frac{1}{4}$ of the square root of 3; therefore the area of the hexagon in square feet is $\frac{6}{4}$ of the square root of 3, that is, $\frac{3}{2}$ of the square root of 3.

(2) A regular polygon of twelve sides is inscribed in a circle of which the radius is one foot: find the area of the polygon.

In Art. 99 it is shewn that AM is the side of a regular polygon of twelve sides inscribed in the circle; so that the area of the polygon is twelve times the area of the triangle OAM. The area of the triangle $OAM=\frac{1}{2}\times OM\times AL$; now $OM=1$ foot, $AL=\frac{1}{2}$ of $AF=\frac{1}{2}$ a foot. Thus the area of the triangle $OAM=\frac{1}{4}$ of a square foot. Therefore the area of the polygon $=\frac{12}{4}$ square feet $=3$ square feet.

EXAMPLES. XV.

1. *ABCDE* is a five-sided figure; the following lengths are in feet: $AC=16$, $AD=12$, the perpendiculars from B and D on AC are 8·4 and 4·6 respectively, and the perpendicular from E on AD is 5 feet: find the area.

2. *ABCDE* is a five-sided figure; BK, CL, EM are perpendiculars on AD; the following lengths are in feet: $AD = 15{\cdot}3$, $BK = 7{\cdot}6$, $CL = 5{\cdot}5$, $EM = 4{\cdot}3$, $AK = 2{\cdot}7$, $DL = 3{\cdot}9$: find the area.

3. *ABCDEF* is a six-sided figure; BK, CL, EM, FN are perpendiculars on AD; the following lengths are in feet: $AD = 18{\cdot}4$, $BK = 5$, $CL = 7$, $EM = 6$, $FN = 4$, $AK = 4{\cdot}7$, $AN = 4{\cdot}1$, $DL = 5{\cdot}3$, $DM = 4{\cdot}9$: find the area.

4. *ABCDEF* is a figure having six equal sides; $AB = 57{\cdot}8$ feet, $BF = 64{\cdot}4$ feet, and the portion *BCEF* forms a rectangle: find the area.

5. *ABCDE* is a five-sided figure, having the angle at E a right angle; the following lengths are in feet: $AB = 14$, $BC = 7$, $CD = 10$, $DE = 12$, $EA = 5$, $AC = 17$. Find the area.

6. Find the area of a regular hexagon each side of which is 20 feet.

7. Find the area of a regular hexagon which is inscribed in a circle, the diameter of which is 100 feet.

8. The length of the side of a field, which is in the form of a regular hexagon, is 10 chains: find the area.

9. The radius of a circle is one foot: find the area of a regular polygon of eight sides inscribed in the circle.

10. Find the area of a regular polygon of 24 sides inscribed in a circle, the radius of which is one foot.

XVI. CIRCLE.

168. *To find the area of a circle.*

RULE. *Multiply the square of the radius by* $\frac{22}{7}$; *or, if greater accuracy is required, multiply the square of the radius by* 3·1416.

169. Examples:

(1) The radius of a circle is 5 feet.

The square of 5 is 25; and $25 \times \frac{22}{7} = \frac{550}{7} = 78\frac{4}{7}$. Thus the area of the circle is about $78\frac{4}{7}$ square feet.

(2) The radius of a circle is 3 miles.

The square of 3 is 9; and $9 \times 3{\cdot}1416 = 28{\cdot}2744$. Thus the area of the circle is about 28·2744 square miles.

170. Both the rules in Art. 168 make the area of the circle a little greater than it ought to be; but the second rule is sufficiently accurate for most practical purposes. If a more accurate result is required we must take as many decimal places of the number 3·1415926... as may be necessary.

171. *The area of a circle being given, to find the radius.*

RULE. *Divide the area by* $\frac{22}{7}$, *and extract the square root of the quotient; or, if greater accuracy is required, divide the area by* 3·1416, *and extract the square root of the quotient.*

172. Examples:

(1) The area of a circle is 100 square feet.

$100 \div \frac{22}{7} = 100 \times \frac{7}{22} = \frac{700}{22} = 31{\cdot}8181...$; the square root of this is 5·64... Thus the radius of the circle is 5·64 feet.

(2) The area of a circle is an acre.

An acre is 4840 square yards; dividing 4840 by 3·1416 we have the quotient 1540·61...; the square root of this is 39·25... Thus the radius of the circle is about 39·25 yards.

173. *To find the area of a circular ring, that is of the space between the circumferences of two concentric circles.*

Rule. *Find the area of each circle, and subtract the area of the inner circle from the area of the outer circle.*

Or, *Multiply the sum of the radii by their difference and the product by* $\frac{22}{7}$, *or if greater accuracy is required by* 3·1416.

Thus the area is half the product of the sum of the circumferences into the difference of the radii; or half the product of the difference of the circumferences into the sum of the radii.

174. Examples:

(1) The radii of the two circles are 10 feet and 12 feet respectively.

The area of the inner circle in square feet

$$= 10 \times 10 \times 3{\cdot}1416 = 314{\cdot}16;$$

the area of the outer circle in square feet

$$= 12 \times 12 \times 3{\cdot}1416 = 452{\cdot}3904,$$

$$452{\cdot}3904 - 314{\cdot}16 = 138{\cdot}2304.$$

Thus the area of the ring is 138·2304 square feet.

Or thus, $12+10=22, \quad 12-10=2,$

$$22 \times 2 \times 3{\cdot}1416 = 138{\cdot}2304.$$

(2) The radii of the two circles are 3 yards and 5 feet respectively.

$$3 \text{ yards} = 9 \text{ feet}; \quad 9+5=14, \quad 9-5=4,$$

$$14 \times 4 \times 3{\cdot}1416 = 175{\cdot}9296.$$

Thus the area of the ring is 175·9296 square feet.

175. If one circle fall entirely within the other, it is obvious that the rule of Art. 173 will give the area of the space between the circumferences of the two circles, even when the circles are not concentric.

176. The rule given in Art. 168 for finding the area of a circle is that to which the beginner's attention should be chiefly, if not entirely directed. Other rules may be given, which are of course equivalent to that: they are of small practical importance, but three such rules will be placed here for use if required.

Multiply the radius by the circumference, and take half the product.

Multiply the square of the diameter by ·7854.

Divide the square of the circumference by 4 × 3·1416; *or multiply the square of the circumference by* ·07958.

177. The first of the three Rules given in the preceding Article is of interest in connexion with the theory of our subject. The Rule amounts to the statement that the area of a circle is equal to the area of a triangle which has the circumference of the circle for its base, and the radius of the circle for its height. A strict demonstration of this statement would be unsuitable for beginners; but it is easy to give a notion of the grounds on which the statement rests.

Suppose we inscribe in a circle a regular polygon with a large number of sides. Then the three following facts are sufficiently obvious: the area of this polygon will not differ much from the area of the circle; the perimeter of the polygon will not differ much from the circumference of the circle; and the perpendicular drawn from the centre of the circle on a side of the polygon will not differ much from the radius of the circle. By joining the angular points of the polygon with the centre of the circle, the polygon is divided into a set of equal triangles; and this set of triangles is equivalent to a single triangle having the perimeter of the polygon for its base, and the perpendicular from the centre of the circle on a side of the polygon for its height. Hence the truth of the statement becomes apparent.

As an illustration of the three facts which we have noticed with respect to a regular polygon of a large number of sides inscribed in a circle, we may refer to the results which have been obtained for a regular polygon of *twelve* sides: see Arts. 99 and 167. Suppose the radius of the circle to be one foot: then the perpendicular from the centre of the circle on a side of the polygon is about ·866 of a foot; the perimeter of the polygon is $12 \times \cdot 51764$ feet, that is, 6·2117 feet very nearly, while the perimeter of the circle is 6·2832 feet; the area of the polygon is 3 square feet, while the area of the circle is 3·1416 square feet.

Again, from Example 10 of Chapter xv. it appears that the area of a regular polygon of *twenty-four* sides inscribed in a circle of one foot radius is 3·1058 square feet.

From these results we may readily allow that by taking a regular polygon of a very large number of sides, the differences between the quantities which are compared will become almost imperceptible.

178. We will now solve some exercises.

(1) The diameter of a circular courtyard is 80 feet; a gravel walk a yard wide runs round it on the inside: find the area of the walk.

The outer boundary of the walk is the circumference of a circle of 40 feet radius; the inner boundary is the circumference of a circle of 37 feet radius. Hence by Art. 173 the area of the gravel walk in square feet

$$=77 \times 3 \times 3 \cdot 1416 = 725 \cdot 7096.$$

(2) The radius of a circle is 15 inches: find the radius of a circle the area of which is three quarters of the area of this circle.

By the rule for finding the area of a circle we see that the areas of two circles are in the same proportion as the squares of their radii. Hence we have the proportion

$1 : \frac{3}{4}$ :: the square of 15 : the square of the required radius.

Therefore the square of the required radius

$$=\frac{3}{4}\times 15\times 15=168{\cdot}75.$$

Hence the required radius is the square root of this number; proceeding to two decimal places we obtain 12·99. Thus the required radius is very nearly 13 inches.

(3) The radius of a circle is 20 inches: it is required to draw three concentric circles in such a manner that the whole area may be divided into four equal parts.

This amounts to three exercises like that just solved. The area of the inner circle is to be a quarter of that of the given circle. Hence, proceeding as before, we shall find that the radius of the inner circle is equal to the square root of 100.

The area between the inner circle and the second circle is also to be a quarter of that of the given circle. Therefore the whole area of the second circle is to be half that of the given circle. Hence, proceeding as before, we shall find that the radius of the second circle is equal to the square root of 200

Similarly, we shall find that the radius of the third circle is equal to the square root of 300.

Thus the radii of the three circles in inches will be found to be respectively 10, 14·14, and 17·32, by proceeding to two decimal places.

EXAMPLES. XVI.

Assuming that the circumference of a circle is $3\frac{1}{7}$ times the diameter, find in square feet the areas of the circles with the following radii:

1. 21 feet.
2. 16 yards 2 feet.
3. One furlong.

Assuming that the circumference of a circle is 3·1416 times the diameter, find in square feet the areas of the circles with the following radii:

4. 25 feet.
5. 992 feet.
6. A quarter of a mile.

Assuming that the circumference of a circle is $3\frac{1}{7}$ times the diameter, find in feet the radii of the circles with the following areas:

7. 100 square feet.
8. One rood.
9. 5 acres 3 roods 8 poles.

Assuming that the circumference of a circle is 3·1416 times the diameter, find in feet the radii of the circles with the following areas:

10. 500 square feet.
11. 6 acres 2 roods 11 poles.
12. A square mile.

[In all future Examples unless anything is stated to the contrary, we shall assume that the circumference of a circle is 3·1416 times the diameter.]

13. The radius of the inner circle of a ring is 14 feet, and the radius of the outer circle is 16 feet: find the area.

14. The radius of the inner circle of a ring is 14 yards 2 feet, and the radius of the outer circle is 18 yards 2 feet: find the area.

15. A circle of radius 10·15 feet falls entirely within another circle of radius 13·35 feet: find the area between the circles.

16. The radius of the inner boundary of a ring is 14 inches; the area of the ring is 100 square inches: find the radius of the outer boundary.

17. The radius of the outer boundary of a ring is 18 feet, the area of the ring is 300 square feet: find the radius of the inner boundary.

18. The area of a quarter of a circle is 7 square yards: find the radius of the circle.

19. The circumference of a circle is 700 feet: find the area.

20. The circumference of a circle is half a mile: find the area.

21. The area of a circle is half an acre: find the circumference.

22. The area of a circle is equal to that of a rectangle which is 400 feet by 256: find the circumference of the circle.

23. The radius of a circle is 8 feet: find the radius of another circle of half the area.

24. The radius of a circle is 18 inches: find the radius of another circle of one-fifth the area.

25. A circle of 10 inches radius is divided into three parts by two concentric circles: find the radii of these circles, so that the three parts may be of equal area.

26. A room 25 feet 3 inches long, and 14 feet 6 inches wide, has a semicircular bow 21 feet in diameter thrown out on one side: find the area of the whole room.

27. If a pressure of 15 lbs. on every square inch be applied to a circular plate 3 feet in diameter, find the total pressure to the nearest hundredweight.

28. Find the expense of paving a circular court 40 feet in diameter at 2*s.* 3*d.* per square foot.

29. The inner diameter of a circular building is 68 feet 10 inches, and the thickness of the wall is 22 inches: find how many square feet of ground the base of the wall occupies.

30. In a circular riding-school of 100 feet in diameter a circular ride, within the outer edge, is to be made of a uniform width of 10 feet: find the cost of doing this at 4*d.* per square foot.

31. A circular grass-plot whose diameter is 40 yards contains a gravel walk, one yard wide, running round it one yard from the edge: find what it will cost to turf the grass-plot at 4*d.* per square yard.

32. A road runs round a circular shrubbery; the outer circumference is 500 feet and the inner circumference is 420 feet: find the area of the road.

33. Find the side of a square which is equivalent in area to a circle of 80 feet radius.

34. Find the radius of a circle which is equivalent in area to a square the side of which is 80 feet.

35. The side of a square is 16 feet; a circle is inscribed in the square so as to touch all its sides: find the area between the circle and the square.

36. The side of a square is 18 feet; a circle is described round the square: find the area between the circle and the square.

37. The sides of a right-angled triangle are 27 feet and 43 feet respectively: find the area of the circle described on the hypotenuse as diameter.

38. The area of a semicircle is 645 square feet: find the length of the perimeter of the semicircle.

39. The radius of a circle is 1 foot; an equilateral triangle is inscribed in the circle: find the area between the circle and the triangle. (See Art. 99.)

40. The sides of a right-angled triangle are 370 feet and 168 feet respectively: find the area of the circle which has the hypotenuse of this triangle for diameter.

41. A rectangle is 8 feet long and 7 feet broad: find the area of the circle which has the same perimeter.

42. The sides of a triangle are 13, 14, and 15 feet: find the area of the circle which has the same perimeter.

If a circle has the same perimeter as a rectangle the circle has the greater area; verify this statement in the following cases:

43. Rectangle 18 feet by 10.

44. Rectangle 27 feet by 13.

If a circle has the same perimeter as a triangle the circle has the greater area; verify this statement in the following cases:

45. Sides of a triangle 9, 10, 17 feet.

46. Sides of a triangle 11, 16, 19 feet.

If a circle has the same area as a rectangle, the circle has the less perimeter; verify this statement in the following cases:

47. Rectangle 15 feet by 12.

48. Rectangle 24 feet by 21.

If a circle has the same area as a triangle the circle has the less perimeter; verify this statement in the following cases:

49. Sides of a triangle 5, 6, 7 feet.

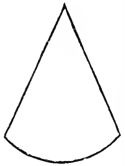
50. Sides of a triangle 12, 15, 17 feet.

51. A circle is 4 feet in circumference: find the area of a square inscribed in it.

52. A circle is 7 feet in circumference: find the area of a square inscribed in it.

XVII. SECTOR OF A CIRCLE AND SEGMENT OF A CIRCLE.

179. *To find the area of a sector of a circle.*



Rule. *As 360 is to the number of degrees in the angle of the sector so is the area of the circle to the area of the sector.*

180. Examples:

(1) The radius of a circle is 25 feet, and the angle of the sector is 80 degrees.

The area of the circle $= 25 \times 25 \times 3{\cdot}1416 = 1963{\cdot}5$.

$$360 : 80 :: 1963{\cdot}5 : \text{required area},$$

$$\frac{80 \times 1963{\cdot}5}{360} = \frac{8 \times 1963{\cdot}5}{36} = \frac{2 \times 1963{\cdot}5}{9} = 436{\cdot}3\ldots$$

Thus the area of the sector is about 436·3 square feet.

(2) The radius of a circle is 12 feet, and the angle of the sector is 75 degrees.

The area of the circle $= 12 \times 12 \times 3{\cdot}1416$.

$$360 : 75 :: 12 \times 12 \times 3{\cdot}1416 : \text{the required area},$$

$$\frac{75 \times 12 \times 12 \times 3{\cdot}1416}{360} = \frac{75 \times 12 \times 3{\cdot}1416}{30} = 30 \times 3{\cdot}1416 = 94{\cdot}248.$$

Thus the area of the sector is 94·248 square feet.

181. The following is another rule for finding the area of a sector of a circle: *multiply the arc by the radius and take half the product.*

The truth of this rule will be obvious from the remarks made in Art. 177.

182. Examples:

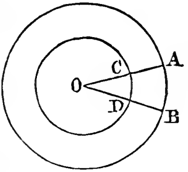
(1) The radius of a circle is 4 feet, and the arc of the sector is equal to the radius.

$\frac{1}{2} \times 4 \times 4 = 8$. Thus the area of the sector is 8 square feet.

(2) The radius of a circle is 2 feet 6 inches, and the arc of the sector is 1 foot 5 inches.

$\frac{1}{2} \times 30 \times 17 = 255$. Thus the area of the sector is 255 square inches.

183. Suppose we require the area of a figure which is the difference of two sectors having a common angle. Let OAB be one sector, and OCD the other sector; so that $ABDC$ is the figure of which the area is required. We may proceed thus:



We may calculate separately the area of each sector and subtract the less from the greater.

Or, we may calculate the area of the entire ring between the two circles, of which AB and CD are arcs; and then use the proportion, *as 360 is to the number of degrees in the angle at O so is the area of the ring to the required area.*

Or we may use this Rule: *multiply the sum of the arcs by the difference of the radii and take half the product.* Or this: *multiply the difference of the arcs by the sum of the radii and take half the product.*

184. Examples:

(1) The radii are 15 feet and 10 feet, and the arcs 6 feet and 4 feet respectively.

By Art. 181, the area of the larger sector in square feet $= \frac{1}{2} \times 15 \times 6 = 45$, and the area of the smaller sector in square feet $=\frac{1}{2}\times 10\times 4=20$: thus the required area in square feet $=45-20=25$.

Or, using the third Rule of the preceding Article, we have the sum of the arcs $=10$ feet, and the difference of the radii $=5$ feet; thus the required area in square feet $=\frac{1}{2}\times 10\times 5=25$.

(2) The radii are 7 feet and 5 feet respectively, and the angle at O is 45 degrees.

Here we use the second Rule of the preceding Article.

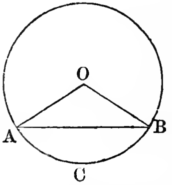
By Art. 173 the area of the entire ring in square feet is $12\times 2\times 3{\cdot}1416$, that is, $75{\cdot}3984$. Then

$$360 : 45 :: 75{\cdot}3984 : \text{the required area.}$$

Thus the required area in square feet

$$=\frac{75{\cdot}3984}{8}=9{\cdot}4248.$$

185. Any chord AB of a circle, which is not a diameter, divides the circle into two segments, one greater than a semicircle, and the other less.



When we know the area of the lesser segment, we can, by subtracting this from the area of the circle, determine the area of the greater segment; so that it is sufficient to give a Rule for finding the area of a segment less than a semicircle.

In the Examples, unless the contrary is expressly stated, we shall always refer to the lesser segment.

Let O be the centre of the circle; then it is obvious that the *segment* ABC is equal to the difference of the *sector* $OACB$ and the *triangle* OAB. Hence we have the Rule which we shall now give.

186. *To find the area of a segment which is less than a semicircle.*

RULE. *Find the area of the sector which has the same arc, and subtract the area of the triangle formed by the radii and the chord.*

187. Examples:

(1) The radius of a circle is 10 inches, and the angle of the sector is 60 degrees: find the area of the segment.

The area of the circle in square inches $= 10 \times 10 \times 3·1416 = 314·16$. By Art. 179, the area of the sector in square inches $= \frac{314·16}{6} = 52·36$. The triangle in this case is equilateral, and by Art. 152 its area in square inches is the square root of $15 \times 5 \times 5 \times 5$, that is, about 43·30.

Thus the area of the segment in square inches

$$= 52·36 - 43·30 = 9·06.$$

(2) The radius of a circle is 4 feet, and the angle of the sector is a right angle: find the area of the segment.

The area of the circle in square feet $= 4 \times 4 \times 3·1416$; therefore the area of the sector in square feet $= 4 \times 3·1416 = 12·5664$. The area of the triangle in square feet $= \frac{1}{2} \times 4 \times 4 = 8$.

Thus the area of the segment in square feet $= 4·5664$.

188. In the examples of the preceding Article we were able to find the areas of the triangles, and thus to deduce the areas of the segments. But in general, if we know only the radius of the circle and the angle, we cannot find the area of the triangle by methods given in the present book, though we could find it with the aid of Trigonometry.

189. *To find the area of a segment of a circle, the chord of the arc and the height being known.*

RULE. *Add together one-fourth of the square of the chord, and two-fifths of the square of the height, and multiply the square root of the sum by four-thirds of the height.*

This Rule is not exact; it gives the area of the segment greater than it ought to be, but the error is very small, provided the angle of the corresponding sector be small; when this angle is 60 degrees, the error is less than $\frac{1}{20000}$ part of the area, and when this angle is 90 degrees, the error is less than $\frac{1}{4000}$ part of the area.

190. Examples:

(1) The chord is 12 inches, and the height is 1 inch.

$$\frac{1}{4} \times 12 \times 12 = 36, \qquad \frac{2}{5} \times 1 \times 1 = \frac{2}{5};$$

$$36 + \frac{2}{5} = 36{\cdot}4: \text{ the square root of this is } 6{\cdot}0332,$$

$$\frac{4}{3} \times 1 \times 6{\cdot}0332 = 8{\cdot}0443.$$

Thus the area of the segment is about 8·0443 square inches.

(2) The chord is 20 inches, and the height is 1·4 inches.

$$\frac{1}{4} \times 20 \times 20 = 100, \qquad \frac{2}{5} \times 1{\cdot}4 \times 1{\cdot}4 = {\cdot}784;$$

the square root of 100·784 = 10·0391,

$$\frac{4}{3} \times 1{\cdot}4 \times 10{\cdot}0391 = 18{\cdot}73965.$$

Thus the area of the segment is about 18·74 square incnes.

191. If the angle of the corresponding sector be so large that the Rule is not sufficiently accurate, we may divide the segment into a triangle and two equal smaller segments; we can then calculate the area of the triangle by an exact Rule, and calculate the area of the smaller segments by the Rule in Art. 189. See the figure in Art. 78, where the segment *ADBE* is made up of the triangle *ABE*, and the segments having the chords *AE* and *EB*.

192. We will now solve some exercises.

(1) The radius of a circle is 25 inches, and the chord of the sector is 14 inches: find the area of the sector.

This exercise cannot be solved exactly, but only approximately. By Art. 122 the length of the arc is about 14·1895141 inches; and thus the area of the sector in square inches $= \frac{1}{2} \times 25 \times 14·1895141 = 177·36892$.

(2) The radius of a circle is 25 inches, and the chord of a segment is 14 inches: find the area of the segment.

We have just found for the area of the sector 177·36892. The area of the triangle can be obtained by Art. 152, the sides being 25, 25, and 14 respectively: the area will be found to be 168 square inches. Thus the area of the segment is about 9·36892 square inches.

Or we may calculate the area of the segment by the Rule in Art. 189.

We must first determine the height. With the figure of Art. 78 we have $AC = 25$, $AB = 14$. Thus $AD = 7$; and then, by Art. 60, we get $CD = 24$. Therefore $DE = 1$.

$$\frac{1}{4} \times 14 \times 14 = 49, \qquad \frac{2}{5} \times 1 \times 1 = \frac{2}{5}.$$

$$49 + \frac{2}{5} = 49·4: \text{ the square root of this is } 7·02851\ldots$$

$$\frac{4}{3} \times 1 \times 7·02851 = 9·37135.$$

Thus, by this Rule, we obtain for the area of the segment 9·37135... square inches; this differs but slightly from the former result.

EXAMPLES. XVII.

Find in square feet the areas of the sectors of circles having the following dimensions:

1. Radius 24 feet, angle 25^0.

2. Radius 12 feet, angle 120^0.

3. Radius 48 feet, angle 28^0.

4. Two concentric circles have radii of 10 feet and 15 feet respectively: find the area of the figure bounded by these circles, and by radii inclined at an angle of 40^0 to each other.

5. Two concentric circles have radii of 10 feet and 18 feet respectively: find the area of the figure bounded by these circles, and by radii inclined at an angle of 50^0 to each other.

6. The area of a sector is 150 square feet; the angle of the sector is 50^0: find the radius.

7. The area of a sector is 230 square feet; the angle of the sector is 40^0: find the whole perimeter of the sector.

8. The area of a sector is 45 square feet; the radius is 8 feet: find the angle.

9. The area of a sector is 94 square feet; the radius is 16 feet: find the arc.

10. The area of a sector is 357 square feet; the arc is 96 feet: find the radius.

11. The area of a sector is 125 square feet; the area of the circle is 400 square feet: find the angle.

12. The area of a sector is 115 square feet; the area of the circle is 700 square feet: find the arc.

13. The chord of a sector is 58 inches; the radius is 100 inches: find the area of the sector.

14. The chord of a sector is 6 inches; the radius is 9 inches: find the area of the sector.

15. The radius is 10 feet; the angle of the sector is 30^0: find the area of the segment.

16. The radius is 10 feet; the angle of the sector is 120^0: find the area of the segment.

17. The radius of a circle is 10 feet; two parallel chords are drawn each equal to the radius: find the area of the zone between the chords.

18. The radius of a circle is 12 feet; two parallel chords are drawn on the same side of the centre, one subtending an angle of 60^0 at the centre, and the other an angle of 90^0: find the area of the zone between the chords.

19. The radius of a circle is 12 feet; two parallel chords are drawn on opposite sides of the centre, one subtending an angle of 60^0 at the centre, and the other an angle of 90^0: find the area of the zone between the chords.

20. The radius of a circle is 15 feet: find the areas of the two parts into which it is divided by a chord equal to the radius.

Find, by Art. 189, in square feet to three decimal places the areas of the segments of circles having the following dimensions:

21. Chord 17·3205 feet; height 5 feet.

22. Chord 14·1421 feet; height 2·9289 feet.

23. Chord 10 feet; height 1·3397 feet.

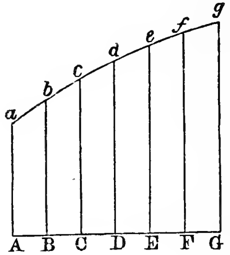
24. Chord 5·1764 feet; height ·3407 feet.

25. Chord 2·6105 feet; height ·0855 feet.

XVIII. SIMPSON'S RULE.

193. We shall now give a very important Rule by which the areas of certain figures may be approximately found.

Let there be an area bounded by the straight line AG, the straight lines Aa and Gg at right angles to AG, and the curve ag.



Divide AG into any *even* number of equal parts AB, BC, CD, ... ; at the points of division draw straight lines Bb, Cc, Dd,... at right angles to AG, to meet the curve. The straight lines Aa, Bb, Cc, ... Gg are called *ordinates*. Then the area may be approximately found by the following RULE: *Add together the first ordinate, the last ordinate, twice the sum of all the other odd ordinates, and four times the sum of all the even ordinates; multiply the result by one-third of the common distance between two adjacent ordinates.*

194. In the figure there are seven ordinates; the even ordinates are Bb, Dd, and Ff; the other odd ordinates besides the first and last are Cc and Ee.

195. The Rule given in Art. 193 is sometimes called the *method of equidistant ordinates;* but it is more usually called *Simpson's Rule*, although it was not invented by Simpson.

196. Examples:

(1) Suppose there are seven ordinates, the common distance being 1 foot; and that these ordinates are respectively 4·12, 4·24, 4·36, 4·47, 4·58, 4·69, and 4·80 feet.

4·12	4·36	4·24	8·92
4·80	4·58	4·47	17·88
8·92	8·94	4·69	53·60
	2	13·40	3 | 80·40
	17·88	4	26·80
		53·60	

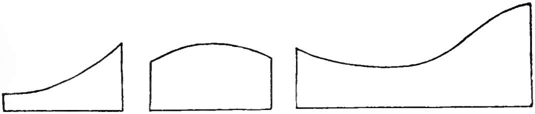
Thus the area is about 26·8 square feet.

(2) Suppose there are five ordinates, the common distance being 2 feet; and that these ordinates are respectively 1·26, 1·44, 1·59, 1·71, and 1·82 feet.

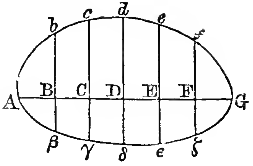
1·26	1·59	1·44	3·08
1·82	2	1·71	3·18
3·08	3·18	3·15	12·60
		4	18·86
		12·60	2
			3 | 37·72
			12·57

Thus the area is about 12·57 square feet.

197. In the figure of Art. 193, the curve is concave towards the straight line AG; moreover the ordinates continually increase from one end of this straight line to the other. But Simpson's Rule is applicable to areas in which the curve has other shapes, as in the following figures:



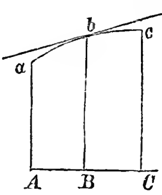
The result will in general be more accurate the more ordinates are used; and the Rule ought not to be trusted if the curve be very irregular.



198. If the area be bounded by the straight line AG and the curve AdG, the same Rule applies; only here the first and last ordinates are nothing, and so do not occur in the calculation.

If the area be bounded by the closed curve $AdG\delta A$, the same Rule may be applied: we have now to take as the ordinates the breadths $b\beta$, $c\gamma$, $d\delta$,...

199. The beginner would not be able to follow a strict investigation of Simpson's Rule; but he may see without difficulty that there are grounds for confidence in the approximate accuracy of the Rule.



We will call the portion of the area between two consecutive ordinates a *piece*; and we will consider the first two pieces of the figure in Art. 193.

Suppose the straight line ab drawn; we thus obtain a trapezoid, and it is obvious from the diagram that the first piece is greater than this trapezoid; so that the area of the first piece is greater than the product of AB into half the sum of Aa and Bb.

Therefore twice the area of the first piece is greater than the product of AB into the sum of Aa and Bb.

Similarly, twice the area of the second piece is greater than the product of BC into the sum of Bb and Cc.

Thus twice the area of the first two pieces is *greater* than the product of AB into the sum of Aa and Cc and twice Bb.

Again: suppose the straight line bt to touch the curve at b, and let Aa and Bb be produced to meet this straight line. Thus another trapezoid is formed; and by Art. 163, the area of this trapezoid is equal to the product of AC

into Bb, that is, to the product of AB into twice Bb. And it is obvious from the diagram that the sum of the first two pieces is less than this trapezoid.

Thus the area of the first two pieces is *less* than the product of AB into twice Bb.

Hence we may expect that when we combine these two results, the errors will to some extent balance each other; and that three times the area of the first two pieces will be very nearly equal to the product of AB into the sum of Aa and Cc and four times Bb.

By proceeding in this way with the other pairs of pieces in the figure of Art 193, we may obtain a sufficient confidence in Simpson's Rule. With respect to the figures in Art. 197, the process would be similar, though not absolutely the same; the main fact is that we combine two results, one of which is too large, and the other too small, and trust that the errors will to some extent balance each other.

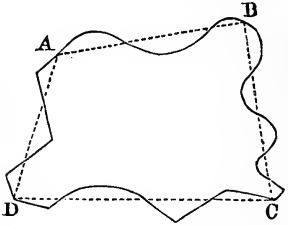
200. If ag were a straight line, instead of a curve, in the figure of Art. 193, Simpson's Rule would give the *exact* value of the area; but in such a case the whole figure would form a trapezoid, and the area is most easily found by taking the product of AG into half the sum of Aa and Gg. Also if ag is a curve *of a certain form* the Rule will give an *exact* result; but we cannot explain in an elementary manner what this form is.

201. It may happen that the boundary of a figure is a curve which is too irregular to allow of the immediate application of Simpson's Rule; in such a case we may adopt the following method: Draw a rectilineal polygon, differing as little as possible from the figure, and determine the area of this polygon exactly; then by Simpson's Rule calculate separately the areas of the portions which lie between the polygon and the figure; add these to the area of the polygon or subtract them from the area of the polygon, according as they fall without or within the polygon, and the final result will be approximately the area of the figure.

202. In Land Surveying it is often necessary to determine the area of a figure which has its boundary composed in an irregular manner of curves and numerous short

straight lines; it is found in practice that the area can be obtained with ease and sufficient accuracy by a method of adjustment of the boundary to a more commodious form.

Thus, suppose a field to be represented on a plan by the figure *ABCD*. Draw a straight line from *A* to *B*; the small portions lost and gained will obviously balance each other, exactly or very nearly, and the area will thus remain almost or quite unchanged. Similarly, straight lines may be drawn from *B* to *C*, from *C* to *D*, and from *D* to *A*, with a like balance of loss and gain. Thus we have a four-sided rectilineal figure equivalent to the original figure; and the area can be easily obtained.



The skill and judgment of the surveyor will be exercised in drawing the straight lines, so that the greatest possible accuracy may be secured. A piece of transparent horn with straight edges is very useful in drawing the straight lines; the horn is placed over the irregular boundary and is shifted about until there appears to be an equal portion on each side of the edge between the edge and the boundary.

203. An experimental method of determining the area enclosed by an irregular boundary may be noticed.

Suppose a field to be represented on a plan by the figure *ABCD* of Art. 202. Cut the figure out of stout paper or cardboard of uniform thickness, and weigh it in a very delicate balance. Also cut out a square inch from the same paper or cardboard, and weigh it. Then by proportion we can find how many square inches the figure *ABCD* contains. And from observing the scale on which the plan is drawn, we shall know the area on the ground which corresponds to a square inch on the plan; and thus finally we can determine the area of the field.

An interesting application has been made of this process to determine the proportion of the water to the land on the surface of the earth. *Camb. Phil. Trans.* Vol. VI.

EXAMPLES. XVIII.

Apply Simpson's Rule to find in square feet the areas of figures having the following dimensions:

1. Ordinates 3, 8, 15, 24, 35, 48, 63 feet; common distance 1 foot.

2. Ordinates 4, 14, 36, 76, 140 feet; common distance 1 foot.

3. Ordinates 0, 20, 32, 36, 32, 20, 0 feet; common distance 2 feet.

4. Ordinates 0, 1·25, 4, 6·75, 8, 6·25, 0 feet; common distance 1 foot.

5. Ordinates 6·082, 6·164, 6·245, 6·325, 6·403, 6·481, 6·557 feet; common distance 1 foot 6 inches.

6. Ordinates 2·714, 2·759, 2·802, 2·844, 2·884 feet; common distance 9 inches.

7. Ordinates 14·2, 14·9, 15·3, 15·1, 14·5, 14·1, 13·7 feet; common distance 3 feet.

8. Ordinates 0, 1·11, 2·48, 4·17, 6·24, 8·75, 11·76, 15·33, 19·52 feet; common distance 1 foot.

9. Ordinates 10·204, 9·804, 9·434, 9·090, 8·771, 8·475, 8·197, 7·937, 7·692 feet; common distance 1 foot.

10. Ordinates 2·4849, 2·5649, 2·6391, 2·7081, 2·7726, 2·8332, 2·8904, 2·9444, 2·9957 feet; common distance 1 foot.

11. Ordinates 0, ·4359, ·6, ·7141, ·8, ·8660, ·9165, ·9539, ·9798, ·9950, 1 foot; common distance ·1 of a foot.

12. Ordinates $\frac{10}{10}$, $\frac{10}{11}$, $\frac{10}{12}$, $\frac{10}{13}$, $\frac{10}{14}$, $\frac{10}{15}$, $\frac{10}{16}$, $\frac{10}{17}$, $\frac{10}{18}$, $\frac{10}{19}$, $\frac{10}{20}$ feet; common distance ·1 of a foot.

XIX. SIMILAR FIGURES.

204. In Chapter VI. we have drawn attention to the nature of Similar Figures; and we have now to point out the relation which holds between the *areas* of Similar Figures. We shall state a most important proposition, and then proceed to apply it to various problems.

205. *The areas of similar figures are as the squares of corresponding lengths.*

For example, suppose we have two similar triangles, and that the side of one triangle is three times the corresponding side of the other; then the area of the larger triangle is nine times the area of the smaller triangle, the number 9 being the square of the number 3. And it is easy to see the reason for this fact: the larger triangle has its base three times the base of the smaller, and, because the triangles are similar, the height of the larger triangle is also three times the height of the smaller; but the area is half the product of the base into the height; and therefore the area of the larger triangle is 9 times the area of the smaller triangle.

In like manner, if two triangles are similar, and the side of one triangle is five times the corresponding side of the other, the area of the larger triangle is twenty-five times the area of the smaller triangle.

206. We have found in Art. 154, that the area of an equilateral triangle, of which the side is 1 foot, is ·4330127 square feet: suppose we require the area of an equilateral triangle, of which the side is 7 feet.

The square of 1 is 1, and the square of 7 is 49; therefore we have the proportion

$$1 : 49 :: \cdot 4330127 : \text{required area.}$$

Thus the required area in square feet

$$= 49 \times \cdot 4330127 = 21 \cdot 2176223.$$

In the same way we may proceed with other examples, and it is obvious that we shall have the following RULE for finding the area of any equilateral triangle: *Multiply the square of the length of a side by* ·4330127.

207. Circles are similar figures; and the areas of two circles are in the same proportion as the squares of their radii, see Art. 178. So also sectors of circles *having the same angle* are similar figures; and the corresponding segments are similar figures; the areas of two similar sectors are in the same proportion as the squares of the radii, and so also are the areas of two similar segments.

208. Suppose we require the radius of a circle, such that the area of the segment corresponding to an angle of 60^0 shall be 20 square inches.

In Art. 187 we have found that if the radius is 10 inches, the area of the segment corresponding to this angle is 9·06 square inches; thus we have the proportion

9·06 : 20 :: 100 : the square of the required radius.

Therefore the square of the required radius

$$=\frac{20 \times 100}{9 \cdot 06}=220 \cdot 75\,;$$

the square root of this number = 14·857.

Thus the required radius is 14·857 inches.

Again; suppose we require the radius of a circle such that the area of the segment corresponding to an angle of 90^0 shall be 10 square feet.

By Art. 187 we have the proportion

4·5664 : 10 :: 16 : the square of the required radius.

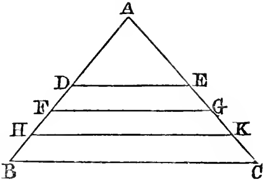
Therefore the square of the required radius

$$=\frac{10 \times 16}{4 \cdot 5664}=35 \cdot 0385\,;$$

the square root of this number = 5·92.

Thus the required radius is 5·92 feet.

209. *ABC* is a triangle; the side *AB* is 10 feet: it is required to divide the triangle into four equal parts by straight lines parallel to *BC*.



This problem resembles one solved in Art. 178. Suppose *DE* the straight line nearest to *A*. Then the area of *ADE* will be one-fourth of the area of *ABC*.

Thus we have the proportion

$1 : \frac{1}{4}$ :: the square of *AB* : the square of *AD*.

Therefore the square of $AD = \frac{1}{4}$ of the square of $AB = \frac{1}{4} \times 100 = 25$; the square root of this number $= 5$. Thus $AD = 5$ feet.

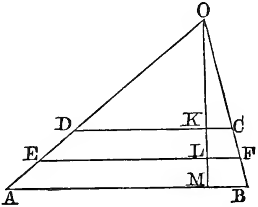
In like manner, if *FG* be the next straight line

$1 : \frac{2}{4}$ :: the square of *AB* : the square of *AF*.

Therefore the square of $AF = \frac{2}{4}$ of the square of $AB = \frac{2}{4} \times 100 = 50$; thus the number of feet in $AF =$ the square root of $50 = 7{\cdot}0710678$.

In like manner, if *HK* be the next straight line, we find that the number of feet in $AH =$ the square root of $75 = 8{\cdot}6602540$.

210. *ABCD* is a trapezoid; the perpendicular distance of the parallel sides *AB* and *DC* is 3 feet; $AB = 10$ feet, $DC = 6$ feet: it is required to divide the trapezoid into two equal parts by a straight line parallel to *AB*. Produce *AD* and *BC* to meet at *O*. Let *EF* denote the required straight line. Draw *OM* perpendicular to *AB*, meeting *EF* at *L*, and *DC* at *K*.



Since EF divides the trapezoid into two equal parts, the triangle OEF will be equal to half the sum of the triangles OAB and ODC. The three triangles ODC, OEF, and OAB are similar; and their areas are therefore as the squares of the corresponding lengths OK, OL, and OM.

Hence the square of OL must be equal to half the sum of the squares of OK and OM.

Now by Art. 77, we have $OK = 4{\cdot}5$; therefore $OM = 7{\cdot}5$. The square of $4{\cdot}5 = 20{\cdot}25$; the square of $7{\cdot}5 = 56{\cdot}25$. Thus the square of OL = half of $76{\cdot}5 = 38{\cdot}25$; and therefore the number of feet in OL = the square root of $38{\cdot}25 = 6{\cdot}1846$.

The number of feet in $KL = 6{\cdot}1846 - 4{\cdot}5 = 1{\cdot}6846$. Thus the position of EF is determined.

211. We will now solve some exercises.

(1) A plan of an estate is drawn on the scale of 1 inch to 20 feet: find what space on the plan will correspond to 8000 square yards of the estate.

The scale is that of 1 inch to 240 inches. The required space will be obtained by dividing 8000 square yards by the square of 240.

$$\frac{8000}{240 \times 240} = \frac{80}{24 \times 24} = \frac{10}{24 \times 3} = \frac{5}{36}.$$

Thus the required space is $\frac{5}{36}$ of a square yard, that is, $\frac{5}{36}$ of 9 square feet, that is, $1\frac{1}{4}$ square feet.

(2) If a square inch on a plan corresponds to 4 square yards of the original, find the scale.

4 square yards $= 4 \times 9 \times 144$ square inches. The square root of $4 \times 9 \times 144 = 72$. Thus the scale is that of 1 inch to 72 inches.

(3) The sides of a rectangle are in the proportion of 4 to 5, and the area is 180 square feet: find the sides.

If the sides of a rectangle are 4 and 5 feet respectively, the area is 20 square feet. Thus we have the proportion

20 : 180 :: the square of 4 : the square of the required corresponding side.

Therefore the square of this side $= \frac{16 \times 180}{20} = 16 \times 9 = 144$; thus this side $= 12$ feet; and therefore the other required side $= 15$ feet.

(4) An equilateral triangle and a circle have the same perimeter: compare their areas.

Suppose each side of the triangle to be 1 foot; then the area is ·43301 square feet. The perimeter of the triangle is 3 feet. If the perimeter of a circle be 3 feet, the area of the circle will be found by Chap. XVI. to be ·7162 square feet. Divide ·7162 by ·43301; the quotient is 1·65... Thus the area of the circle is 1·65...times the area of the equilateral triangle.

We shall obtain the same final result whatever be the length of the side of the equilateral triangle. If, for example, we suppose each side to be 7 feet, we shall obtain for the areas of the triangle and of the circle respectively 49 times the former values, but the proportion of the areas will remain unchanged.

EXAMPLES. XIX.

1. A field containing 3600 square yards is laid down on a plan to a scale of 1 inch to 10 feet: find the number of square inches of the plan it will occupy.

2. A field containing 6 acres is laid down on a plan to a scale of 1 inch to 20 feet: find how much paper it will cover.

3. Determine the scale used in the construction of a plan upon which every square inch of surface represents a square yard.

4. Determine the scale used in the construction of a plan upon which a square foot of surface represents an area of ten acres.

5. A field is ten thousand times as large as the plan which has been made of it: find what length on the plan will represent a length of 20 yards in the field.

6. An estate, which has been surveyed, is one hundred million times as large as the plan which has been made of it: express the scale of the plan in terms of inches to a mile.

7. The sides of a rectangle are in the proportion of 2 to 3; and the area is 210 square feet: find the sides.

8. The sides of a triangle are in the proportion of the numbers 13, 14, and 15; and the area is 24276 square feet: find the sides in feet.

9. The sides of a triangle are in the proportion of the numbers 7, 15, and 20; and the area is 2226 square feet: find the sides in feet.

10. An equilateral triangle and a square have the same perimeter: compare the areas.

11. A square and a regular hexagon have the same perimeter: compare their areas.

12. A circle and a square have the same perimeter: compare their areas.

13. A circle and a regular hexagon have the same perimeter: compare their areas.

14. Find the side of an equilateral triangle, so that the area may be 100 square feet.

15. Find the side of a regular hexagon, which shall be equal in area to an equilateral triangle, each side of which is 150 feet.

16. Find the radius of a circle, such that the area of a segment corresponding to an angle of 90^0 may be 50 square feet.

17. One side of a triangle is 15 feet; it is required to divide the triangle into five equal parts by straight lines parallel to one of the other sides: find the distances from the vertex of the points of division of the given side.

18. An equilateral triangle and a square have the same area: compare their perimeters.

19. The parallel sides of a trapezoid are respectively 16 and 20 feet, and the perpendicular distance between them is 5 feet; it is required to divide the trapezoid into two equal trapezoids: find the distance of the dividing straight line from the shorter of the parallel sides.

20. The side of a square is 12 feet; the square is divided into three equal parts by two straight lines parallel to a diagonal: find the perpendicular distance between the parallel straight lines.

21. A circle and a regular polygon of twelve sides have the same perimeter: shew, by Arts. 99 and 167, that the area of the circle is $\frac{3 \cdot 2154}{3 \cdot 1416}$ times the area of the polygon.

FOURTH SECTION. VOLUMES.

XX. DEFINITIONS.

212. We shall commence this part of the subject with definitions of some terms which we shall have to employ.

Although it is convenient to collect the definitions in one Chapter, it is not necessary for the beginner to study them closely all at once; it will be sufficient to read them with attention, and then to recur to them hereafter as occasion may require.

213. Parallel planes are such as do not meet one another, although produced.

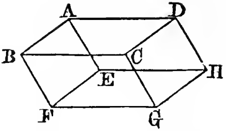
Thus the floor and the ceiling of a room are parallel planes.

214. A straight line is said to be at right angles to a plane, or perpendicular to a plane, when it makes right angles with every straight line which it meets in that plane.

This is the strict geometrical definition; but without regarding it, the student will gain an adequate idea of what is meant by a perpendicular to a plane from considering the illustration afforded by a straight rod when fixed in the ground so as to be upright.

So also the student will readily understand when one plane is perpendicular or at right angles to another, without a strict geometrical definition. Thus the walls of a room are perpendicular to the floor and to the ceiling; and a door while moving on its hinges remains perpendicular to the floor and to the ceiling.

215. A parallelepiped is a solid bounded by six parallelograms, of which every opposite two are equal and in parallel planes.

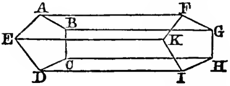


The diagram represents a parallelepiped. $ABCD$ and $EFGH$ are equal parallelograms in parallel planes; $ABFE$ and $DCGH$ are equal parallelograms in parallel planes; and $ADHE$ and $BCGF$ are equal parallelograms in parallel planes.

A parallelepiped is called *rectangular* when the six bounding parallelograms are *rectangles*, and *oblique* when they are not. A common brick furnishes an example of a rectangular parallelepiped. A rectangular parallelepiped which has its six bounding rectangles all equal is called a cube. This will be found equivalent to saying that a cube is a solid bounded by six equal squares, of which every opposite two are in parallel planes.

216. Plane figures which form the boundaries of solids are called *faces* of the solid; the straight lines which form the boundaries of the plane figures are called *edges* of the solid. Thus a parallelepiped has six faces and twelve edges.

217. A prism is a solid bounded by plane rectilineal figures, of which two are equal and in parallel planes, and the rest are parallelograms. The two bounding figures which are equal and in parallel planes are called the *ends* of the prism.



The diagram represents a prism having for its ends the equal pentagons $ABCDE$ and $FGHIK$; these figures are in parallel planes. The other bounding figures of the solid are parallelograms, as $ABGF$, $BCHG$, and so on. Such a prism is called a pentagonal prism; if the ends are hexagons, the prism is called a hexagonal prism; and so on.

A prism is called *right* when the parallelograms between the ends are rectangles, and *oblique* when they are not.

In a right prism the ends are at right angles to the other faces.

Thus we see that a parallelepiped is included among prisms, and that a rectangular parallelepiped and a cube are included among right prisms.

In a rectangular parallelepiped every face is at right angles to the four faces which it meets.

218. A pyramid is a solid bounded by three or more triangles which meet at a point, and by another rectilineal figure.

The point is called the *vertex* of the pyramid, and the rectilineal figure opposite to the vertex is called the *base* of the pyramid.

When three triangles meet at the vertex the base of the pyramid is a triangle. When four triangles meet at the vertex the base of the pyramid is a quadrilateral; the famous pyramids of Egypt, of which the student has probably seen pictures, are of this kind, the bases being squares. When five triangles meet at the vertex the base is a pentagon. And so on.

219. A frustum of a solid is a slice of it, contained between the base and any plane parallel to the base; the base and the opposite face are called the *ends* of the frustum. Thus, if a pyramid be cut into two pieces by any plane parallel to the base, one of the two pieces is a frustum of a pyramid, and the other is a pyramid.

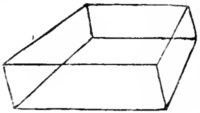
220. A wedge is a solid bounded by five planes; the base is a rectangle, the two ends are triangles, and the other two faces are trapezoids.

The line of intersection of the two trapezoids is called the *edge;* that side of the base which is parallel to the edge is called the *length* of the base.

If the length of the base is equal to the edge, the trapezoids are parallelograms, and the wedge is an oblique triangular prism. If the parallelograms are rectangles, the wedge is a right triangular prism.

221. A prismoid is a solid having for its ends any two parallel plane rectilineal figures of the same number of sides, and having all its faces trapezoids.

If the ends are similar figures, similarly situated, the prismoid is a frustum of a pyramid.



If the ends are rectangles, the prismoid is a frustum of a wedge: the term prismoid is by some writers restricted to this solid.

222. A sphere is a solid having every point of its surface equally distant from a certain point, called the centre of the sphere.

A radius of a sphere is a straight line drawn from the centre to the surface.

A diameter of a sphere is a straight line drawn through the centre, and terminated both ways by the surface.

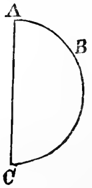
The intersection of a sphere with any plane is a circle; if the plane passes through the centre of the sphere the intersection is called a great circle of the sphere.

A sphere is sometimes called a *globe*, and sometimes a *round body:* marbles and billiard balls are familiar examples of spheres.

If a sphere be cut into two parts by a plane, each part is called a *segment* of a sphere; the *base* of the segment is the circle which is formed by the intersection of the sphere and the plane: if a diameter of the sphere be drawn perpendicular to the base of the segment, the portion intercepted by the segment is called the *height* of the segment.

A *zone* of a sphere is the part of the sphere contained between two parallel planes; the *height* of the zone is the perpendicular distance between the two parallel planes.

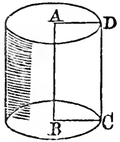
223. Another method of defining a sphere may be given. Let ABC be a semicircle, AC being the diameter. Suppose the semicircle to be cut out in paper or cardboard, and let AC be kept fixed, and the semicircle be turned round AC. The figure ABC, as it turns round, sweeps out a solid, and this solid is a sphere.



It is not *necessary* to introduce this method, because we can give a good definition, and form a clear conception of a sphere without it; but there are other solids for which such a method must be used, and so we have applied it to the sphere in order to render it more readily intelligible.

224. A cylinder is a solid produced by turning a rectangle round one of its sides, which remains fixed.

Thus let *ABCD* be a rectangle; let *AB* be kept fixed, and let the rectangle be turned round *AB*.



Then the figure *ABCD*, as it turns round, sweeps out a solid, which is called a cylinder.

AB is called the *axis* of the cylinder. The circles described by *AD* and *BC* are called the *ends* of the cylinder; either end may be called the *base* of the cylinder.

An uncut lead pencil is an example of a cylinder.

If a cylinder be cut by any plane parallel to the ends, the intersection of the plane and the cylinder is a circle; and each of the two parts into which the cylinder is divided is a cylinder.

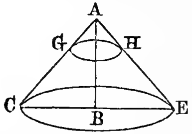
225. The solid which we have called a cylinder is more strictly called a *right circular cylinder;* the word *right* refers to the fact that the axis *AB* is perpendicular to the base, and the word *circular* refers to the fact that the base is a circle. Other cylinders besides right circular cylinders occur in mathematical investigations.

A cylinder bears some resemblance to a prism which has for its base a rectilineal figure with a very large number of sides, each side being very small. A right circular cylinder resembles a *right* prism, the ends of which are regular polygons. And an *oblique* prism will give an idea of cylinders which are not right circular cylinders.

In future, when we use the single word cylinder, we shall mean that what we say is applicable to any cylinder; but it will be sufficient for the beginner to think of a right circular cylinder.

226. A cone is a solid produced by turning a right-angled triangle round one of the sides which contain the right angle, this side remaining fixed.

Thus let ABC be a triangle having a right angle at B; let AB be kept fixed, and let the triangle be turned round AB. Then the figure ABC, as it turns round, sweeps out a solid, which is called a cone.



The point A is called the *vertex* of the cone; AB is called the *axis* of the cone. The circle described by BC as it turns round is called the *base* of the cone. A straight line drawn from the vertex to the circumference of the base, as AC, is called the *slant side* of the cone, and sometimes the *slant height* of the cone.

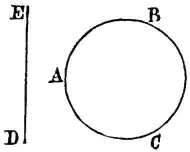
If a cone be cut by any plane parallel to the base, the intersection of the plane and the cone is a circle; for example, GH in the diagram represents such a circle. When a cone is cut into two parts by a plane parallel to the base, the part between the vertex and the plane is a cone; the other part is called a frustum of a cone, see Art. 219. The *slant side* or *slant height* of the frustum of a cone is that portion of the slant side of the cone which is cut off by the frustum; for example, in the diagram GC is the slant side of the frustum cut off by the plane GH.

227. The solid which we have called a cone is more strictly called a *right circular* cone; the word *right* refers to the fact that the axis AB is perpendicular to the base, and the word *circular* refers to the fact that the base is a circle. Other cones besides right circular cones occur in mathematical investigations.

A cone bears some resemblance to a pyramid which has for its base a rectilineal figure, with a very large number of sides, each side being very small. A right circular cone resembles a pyramid in which the base is a regular polygon, and the triangular faces are all equal. And a pyramid, the base of which is not restricted to be a regular polygon, and the triangular faces of which are not all equal, will give an idea of cones which are not right circular cones.

In future, when we use the single word cone, we shall mean that what we say is applicable to any cone; but it will be sufficient for the beginner to think of a right circular cone.

228. A solid ring is a solid produced by turning a circle round any straight line in the plane of the circle which does not cut the circle.



Thus, let ABC be a circle, and DE any straight line in the plane of the circle, which does not cut the circle, let DE be kept fixed, and let the circle be turned round DE. Then the figure ABC, as it turns round, sweeps out a solid, which is called a solid ring, or briefly a ring

229. Any face of a parallelepiped may be called the base; then the *height* of the parallelepiped is the perpendicular drawn to the base from any point of the opposite face.

The height of a pyramid or a cone is the perpendicular drawn to the base from the vertex.

The height of a prism, or a cylinder, or a prismoid, or a frustum of a solid, is the perpendicular drawn to one end from any point of the other end; either end may be called a base.

The height of a wedge is the perpendicular drawn from any point of the edge to the base.

XXI. SOLID MEASURE.

230. A Table of Solid Measure might be given as extensive as the Table of Square Measure of Art. 126; but it will be sufficient to observe that

1728 cubic inches make 1 cubic foot,
27 cubic feet make 1 cubic yard.

231. The connection which subsists between the systems of measures and of weights must be noticed.

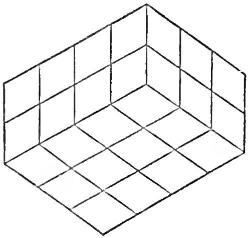
The grain is thus determined: a cubic inch of pure water weighs 252·458 grains.

A pound Avoirdupois contains 7000 grains.

A cubic foot of pure water weighs $1728 \times 252{\cdot}458$ grains, that is, $\frac{16 \times 1728 \times 252{\cdot}458}{7000}$ Avoirdupois ounces: it will be found that this number to three decimal places is 997·137. Thus it is usually sufficient in practice to take 1000 Avoirdupois ounces as the weight of a cubic foot of pure water.

A gallon is a measure which will hold 10 Avoirdupois pounds of pure water, that is 70000 grains. Hence the number of cubic inches in a gallon is $\frac{70000}{252{\cdot}458}$: it will be found that this number to three decimal places is 277·274. Thus it is usually sufficient in practice to take $277\frac{1}{4}$ as the number of cubic inches in a gallon.

XXII. RECTANGULAR PARALLELEPIPED.



232. Suppose we have a rectangular parallelepiped, which is 4 inches long, 3 inches broad, and 2 inches high. Let the rectangular parallelepiped be cut by planes, an inch apart, parallel to the faces; it is thus divided into 24 equal solids, each of which is a cube, being an inch long, an inch broad, and an inch high: such a cube is called a *cubic inch*. The rectangular parallelepiped then contains 24 cubic inches; this fact is also expressed thus: the volume of the rectangular parallelepiped is 24 cubic inches.

The word *content*, or the word *solidity*, may be used instead of the word *volume*.

The number 24 is the product of the numbers 4, 3, and 2, which denote respectively the length, the breadth, and the height of the rectangular parallelepiped.

233. If a rectangular parallelepiped be 8 inches long, 7 inches broad, and 5 inches high, we can shew in the same manner that its volume is 8 times 7 times 5 cubic inches, that is, 280 cubic inches. Similarly, if a rectangular parallelepiped be 15 inches long, 12 inches broad, and 10 inches high, its volume is 15 times 12 times 10 cubic inches, that is, 1800 cubic inches. And so on.

234. In the same manner, if a rectangular parallelepiped be 4 feet long, 3 feet broad, and 2 feet high, its volume is 24 cubic feet; that is, the rectangular parallelepiped might be divided into 24 equal solids, each being a foot long, a foot broad, and a foot high. If a rectangular

parallelepiped be 4 yards long, 3 yards broad, and 2 yards high, its volume is 24 cubic yards. And so on.

235. The beginner will observe, that the way in which *volumes* are measured is another case of the general principle explained in Art. 131. We fix on some volume as a standard, and we compare other volumes with this standard. The most convenient standard is found to be the volume of a cube; it may be a cubic inch, or a cubic foot, or any other cube.

236. In order then to find the volume of a rectangular parallelepiped we must express the length, the breadth, and the height in terms of the same denomination; and the product of the numbers which denote the length, the breadth, and the height, will denote the volume. If the length, the breadth, and the height are all expressed in inches, the volume will be expressed in cubic inches; if the length, the breadth, and the height, are all expressed in feet, the volume will be expressed in cubic feet; and so on.

237. In the example given in Art. 232, we find that the volume is equal to $4 \times 3 \times 2$ cubic inches. Now suppose we take for the base of the rectangular parallelepiped the rectangle which is 4 inches by 3; then the height is 2 inches, and the area of the base is 12 square inches. Thus the number denoting the volume is equal to the product of the numbers denoting the area of the base and the height. If we take for the base the rectangle which is 4 inches by 2; then the height is 3 inches: and, as before, the number denoting the volume is equal to the product of the numbers denoting the area of the base and the height. Or we may take for the base the rectangle which is 3 inches by 2; then the height is 4 inches: and, as before, the number denoting the volume is equal to the product of the numbers denoting the area of the base and the height.

238. The student will now be able to understand the way in which we estimate the volumes of solids, and to use correctly the rules which will be given: the rules will be stated with brevity, but this will present no difficulty to those who have read the foregoing explanations.

239. *To find the volume of a rectangular parallelepiped.*

Rule. *Multiply together the length, the breadth, and the height, and the product will be the volume.*

Or, *Multiply the area of the base by the height, and the product will be the volume.*

240. Examples.

(1) The length of a rectangular parallelepiped is 2 feet 6 inches, the breadth is 1 foot 8 inches, and the height is 9 inches.

2 feet 6 inches = 30 inches, 1 foot 8 inches = 20 inches, $30 \times 20 \times 9 = 5400$.

Thus the volume is 5400 cubic inches.

(2) The area of the base of a rectangular parallelepiped is 15 square feet, and the height is 3 feet 9 inches.

3 feet 9 inches = 3·75 feet.

$15 \times 3{\cdot}75 = 56{\cdot}25$.

Thus the volume is 56·25 cubic feet.

241. If we know the volume of a rectangular parallelepiped, and also the area of its base, we can find the height by dividing the number which expresses the volume by the number which expresses the area of the base; and similarly if we know the volume and the height we can find the area of the base. Of course we must be careful to use corresponding denominations for the volumes and the known area or height: see Art. 132.

242. Examples.

(1) The volume of a rectangular parallelepiped is 576 cubic inches, and the area of the base is half a square foot: find the height.

Half a square foot = 72 square inches; $\frac{576}{72} = 8$.

Thus the height is 8 inches.

(2) The volume of a rectangular parallelepiped is 8 cubic feet, and the height is 1 foot 4 inches: find the area of the base.

1 foot 4 inches $= 1\frac{1}{3}$ feet; $\frac{8}{1\frac{1}{3}} = \frac{8}{1} \times \frac{3}{4} = 6$.

Thus the area of the base is 6 square feet.

243. A cube is a rectangular parallelepiped having its length, breadth, and height equal; hence to find the volume of a cube we multiply the number which denotes the length by itself, and multiply the product by the number again. Thus we see the reason for using the term *cube of a number* to denote the result obtained by multiplying a number by itself, and the product by the number again.

244. The statements made in Art. 230 as to the connection between cubic inches, cubic feet, and a cubic yard, will be easily understood by the aid of the explanations of the present Chapter. Take, for example, the first statement that 1728 cubic inches make 1 cubic foot: a cubic foot is a cube 12 inches long, 12 inches broad, and 12 inches high; and therefore by the method of Art. 232 we see that a cubic foot contains $12 \times 12 \times 12$ cubic inches, that is, 1728 cubic inches.

245. We will now solve some exercises.

(1) Find how many bricks will be required to build a wall 25 yards long, 15 feet high, and 1 foot $10\frac{1}{2}$ inches thick; a brick being 9 inches long, $4\frac{1}{2}$ wide, and 3 deep.

The number of cubic inches in the wall is

$$25 \times 3 \times 12 \times 15 \times 12 \times \frac{45}{2};$$

the number of cubic inches in a brick is $9 \times \frac{9}{2} \times 3$; divide the former number by the latter, and the quotient is 30000, which is therefore the number of bricks required.

(2) A reservoir is 15 feet 4 inches long by 8 feet 3 inches wide: find how many cubic feet of water must be drawn off to make the surface sink 1 foot.

The volume of the water which must be drawn off is that of a rectangular parallelepiped $15\frac{1}{3}$ feet long, $8\frac{1}{4}$ feet wide, and 1 foot deep; therefore the number of cubic feet in the volume is $\frac{46}{3} \times \frac{33}{4} \times 1$, that is $\frac{23 \times 11}{2}$, that is $126\frac{1}{2}$.

(3) Find the length of a cubical vessel which will hold 100 gallons.

The vessel is to hold 27727·4 cubic inches; so that the number of inches in the length will be found by *extracting the cube root* of this number: it will be found that the cube root is 30·267.

(4) A vessel is in the shape of a rectangular parallelepiped; it is without a lid; externally the length is 4 feet, the breadth 3 feet, and the depth 2 feet; the thickness of the material is half an inch: find the number of cubic inches of the material.

Externally the dimensions in inches are 48, 36, and 24; thus the volume is 41472 cubic inches.

Internally the dimensions in inches are 47, 35, and $23\frac{1}{2}$; thus the volume is $38657\frac{1}{2}$ cubic inches.

The difference is $2814\frac{1}{2}$ cubic inches. This is the required result.

EXAMPLES. XXII.

Find the number of cubic feet and inches in cubes having the following lengths:

1. 2 feet 8 inches.
2. 1 fathom.
3. 1 yard 1 foot 9 inches.
4. 1 pole.

Find the number of cubic feet and inches in rectangular parallelepipeds which have the following dimensions:

5. 4 feet 8 inches, 3 feet 6 inches, 2 feet 4 inches.

6. 7 feet 9 inches, 4 feet 6 inches, 2 feet 3 inches.

7. 6 yards 2 feet 7 inches, 3 feet 4 inches, 2 feet 11 inches.

8. 10 yards 9 inches, 5 yards 1 foot 7 inches, 2 yards 8 inches.

Find the number of cubic feet and inches in rectangular parallelepipeds which have the following dimensions:

9. Area of base 16 square feet, height 4 feet 3 inches.

10. Area of base 1000 square inches, height 1 yard.

11. Area of base 12 square feet 80 square inches, height 2 feet 7 inches.

12. Area of base 1 square yard 5 square feet 120 square inches, height 1 yard 1 foot 6 inches.

Find the heights of the rectangular parallelepipeds which have the following volumes and bases:

13. Volume 6 cubic feet, base 8 square feet.

14. Volume 3 cubic feet, base 3 feet 4 inches by 2 feet 6 inches.

15. Volume 124 cubic feet 1668 cubic inches, base 24 square feet 143 square inches.

16. Volume 198 cubic feet 856 cubic inches, base 34 square feet 4 square inches.

Find the areas of the bases of the rectangular parallelepipeds which have the following volumes and heights:

17. Volume 15 cubic feet, height 9 inches.

18. Volume 9 cubic feet 48 cubic inches, height 2 feet 1 inch.

19. Volume 99 cubic feet 428 cubic inches, height 5 feet 10 inches.

20. Volume 296 cubic feet 144 cubic inches, height 8 feet 6 inches.

Find to the nearest gallon the volumes of rectangular parallelepipeds having the following dimensions:

21. 6 feet, 6 feet, 6 feet.

22. 6 feet 8 inches, 6 feet 8 inches, 6 feet 5 inches.

23. 7 feet 4 inches, 7 feet 6 inches, 6 feet 9 inches.

24. 8 feet 6 inches, 8 feet 4 inches, 8 feet 2 inches.

Find to the nearest hundredweight the weights of the water which can be contained in rectangular parallelepipeds having the following dimensions:

25. 5 feet, 5 feet, 5 feet.

26. 5 feet 6 inches, 5 feet 6 inches, 5 feet 3 inches.

27. 6 feet 9 inches, 6 feet 5 inches, 5 feet 10 inches.

28. 9 feet 4 inches, 8 feet 7 inches, 8 feet 2 inches.

29. Shew that a cube 6 inches long is equivalent to the sum of three cubes which are respectively 3 inches, 4 inches, and 5 inches long.

30. Find the number of cubic chains in a rectangular parallelepiped whose edges are 94 chains 50 links, 1 chain 5 links, and $31\frac{1}{2}$ links.

31. The beams of wood used in building a house are 3 inches thick, and 10 inches wide; 200 of them are used which together amount to 1000 cubic feet: find the length of each beam.

32. Find how many bricks will be required to build a wall 90 feet long, 18 inches thick, and 8 feet high; a brick being 9 inches long, $4\frac{1}{2}$ wide, and 3 deep.

33. A certain book is 8 inches long, $5\frac{1}{4}$ inches broad, and $2\frac{1}{4}$ inches thick: find how many such books can be packed in a box which is 3 feet 6 inches long, 3 feet broad, and 2 feet deep.

34. If a cubic foot of gold may be made to cover uniformly 432000000 square inches, find the thickness of the coating of gold.

35. A metre is 39·37 inches: find the number of cubic feet in a cube whose side is a metre.

36. A block of stone is 4 feet long, $2\frac{1}{2}$ feet broad, and $1\frac{1}{4}$ feet thick; it weighs 27 cwt.: find the weight of 100 cubic inches of the stone.

37. If a cubic foot of marble weigh 2·716 times as much as a cubic foot of water, find the weight of a block of marble 9 feet 6 inches long, 2 feet 3 inches broad, and 2 feet thick.

38. Shew that a cubical vessel the length of which is 14·04 inches will hold less than 10 gallons; and that a cubical vessel the length of which is 14·05 inches will hold more than 10 gallons.

39. A reservoir is 24 feet 8 inches long by 12 feet 9 inches wide: find how many cubic feet of water must be drawn off to make the surface sink 1 foot.

40. A cistern is 13 feet 6 inches long by 9 feet 9 inches wide: find through how many inches the surface will sink if 260 gallons of water are drawn off.

41. If gold be beaten out so thin that an ounce will form a leaf of 20 square yards, find how many of these leaves will make an inch thick, the weight of a cubic foot of gold being 10 cwt. 95 lbs.

42. Shew that a cubic fathom of water weighs about 6 tons.

43. If a rectangular parallelepiped has its length, its breadth, and its depth respectively half as large again as another rectangular parallelepiped, shew that the first is more than three times as large as the second.

44. If a rectangular parallelepiped has its length, its breadth, and its depth respectively a quarter as large again as another rectangular parallelepiped, shew that the first is nearly twice as large as the second.

45. If a rectangular parallelepiped has its length, its breadth, and its depth respectively a sixth, a seventh, and an eighth as large again as another; shew that the first is half as large again as the second.

46. A vessel is in the shape of a cube; it is without a lid: if the external length is 3 feet, and the thickness of the material one inch, find the number of cubic inches of the material.

47. A vessel is in the shape of a rectangular parallelepiped; it is without a lid; if externally the length be 6 feet, the breadth 5 feet, and the depth 3 feet, and the thickness of the material half an inch, find the number of cubic inches of the material.

48. The external length, breadth, and height of a closed rectangular wooden box are 18 inches 10 inches and 6 inches respectively, and the thickness of the wood is half an inch; when the box is empty it weighs 15 lbs., and when filled with sand 100 lbs.: find the weight of a cubic inch of wood, and of a cubic inch of sand.

49. A box without a lid is made of wood an inch thick; the external length, breadth, and height of the box are 2 feet 10 inches, 2 feet 5 inches, and 1 foot 7 inches respectively: find what volume the box will hold.

50. A rectangular parallelepiped having a square base is 3 feet 4 inches high; the volume is 40 cubic feet 1440 cubic inches: find the side of the square base.

51. A rectangular parallelepiped has two faces each containing 2 square feet 117 square inches, two faces each containing 2 square feet 12 square inches, and two faces each containing 1 square foot 96 square inches: shew that the volume of the rectangular parallelepiped is 3 cubic feet 216 cubic inches.

The following Examples involve the extraction of the cube root:

52. Find the length of a cube which will be equivalent to 2 cubic feet.

53. Find the length of a cube which will be equivalent to 3000 cubic feet.

54. Find the length of a cubical vessel which will hold 4000 gallons of water.

55. Find the length of a cubical vessel which will hold a ton weight of water.

56. If 100 cubic inches of a certain kind of stone weigh 14 lbs., find the length of a cube of this stone which weighs half a ton.

XXIII. PARALLELEPIPED, PRISM, CYLINDER.

246. *To find the volume of a parallelepiped, a prism, or a cylinder.*

Rule. *Multiply the area of the base by the height, and the product will be the volume.*

247. Examples:

(1) The area of the base of a parallelepiped is 5 square feet, and the height is 9 inches.

$$9 \text{ inches} = \frac{3}{4} \text{ of a foot}, \qquad 5 \times \frac{3}{4} = \frac{15}{4} = 3\tfrac{3}{4}.$$

Thus the volume is $3\frac{3}{4}$ cubic feet.

(2) The base of a prism is a triangle, the sides of which are 1 foot 1 inch, 1 foot 8 inches, and 1 foot 9 inches respectively; and the height of the prism is 1 foot 10 inches.

We must first find the area of the base by Art. 152.

1 foot 1 inch = 13 inches, 1 foot 8 inches = 20 inches, 1 foot 9 inches = 21 inches.

$$13 + 20 + 21 = 54, \qquad \frac{54}{2} = 27,$$

$$27 - 13 = 14, \qquad 27 - 20 = 7, \qquad 27 - 21 = 6.$$

$27 \times 14 \times 7 \times 6 = 15876$. The square root of 15876 is 126. Thus the area of the base is 126 square inches.

1 foot 10 inches = 22 inches.

$126 \times 22 = 2772$. Thus the volume of the prism is 2772 cubic inches.

(3) The radius of the base of a cylinder is 5 inches, and the height is 16 inches.

The area of the base in square inches $= 5 \times 5 \times 3{\cdot}1416 = 78{\cdot}54$.

$$78{\cdot}54 \times 16 = 1256{\cdot}64.$$

Thus the volume is about 1256·64 cubic inches.

248. If we know the volume of a parallelepiped, a prism, or a cylinder, and also the area of the base, we can find the height by dividing the number which expresses the volume by the number which expresses the area of the base; and similarly if we know the volume and the height, we can find the area of the base.

249. Examples:

(1) The volume of a prism is a cubic foot, and the area of the base is 108 square inches: find the height.

$\frac{1728}{108} = 16$. Thus the height is 16 inches.

(2) The volume of a cylinder is 2000 cubic inches, and the height is 4 feet 2 inches: find the area of the base.

$\frac{2000}{50} = 40$. Thus the area of the base is 40 square inches.

250. It will be seen that the Rule in Art. 246 is the same as the second form of the Rule in Art. 239; and it will be instructive to attempt to explain the reason of this coincidence. We suppose that the beginner has convinced himself by the process of Art. 232, that the Rule holds for any rectangular parallelepiped; and we will try to shew that it will also hold for a *right* prism or a *right* circular cylinder.

251. Refer to the diagram of Art. 29. Let there be two right prisms with the same height, one having the triangle ABC for base, and the other having the rectangle $ABDE$ for base. By the method of Art. 29 we can shew that the prism on the triangular base is half the prism on

the rectangular base. Hence it will follow that the Rule of Art. 246 holds for any right prism on a triangular base. Therefore the Rule will also hold for a right prism which has any rectilineal figure for its base; because such a base could be divided into triangles; and the prism could be divided into corresponding prisms with triangular bases, for each of which the Rule holds.

We are thus led to the notion that the volume of a right prism of given height depends only on the *area* of the base, and not on the *shape* of that base; and this may suggest that the Rule will also hold for a right circular cylinder. And we may infer that the Rule will also hold for other solids which are not called cylinders in ordinary language; for example, the shaft of a fluted column.

252. The Rule will also hold for *oblique* prisms and cylinders. The ground on which this rests is the following proposition: *an oblique parallelepiped is equivalent to a rectangular parallelepiped which has the same base and an equal height.* This proposition resembles that in Art. 28; and the mode of demonstration is similar; but instead of *one* adjustment of adding an area and subtracting an equal area, we shall here in general require *two* separate adjustments, of adding a volume and subtracting an equal volume.

253. We will now solve some exercises.

(1) A cubic inch of metal is to be drawn into a wire $\frac{1}{10}$ of an inch thick: find the length of the wire.

The wire will be a cylinder, having the radius of its base $\frac{1}{20}$ of an inch. Thus the area of the base in square inches $= \frac{3{\cdot}1416}{400} = {\cdot}007854$. As the volume is 1 cubic inch, we divide 1 by ·007854; and thus we obtain for the length of the wire 127·3 inches.

(2) Find the volume of a cylindrical shell, the height being 5 feet, the radius of the inner surface 3 inches, and the radius of the outer surface 4 inches.

By a cylindrical shell is meant the solid which remains when from a solid cylinder another cylinder with the same axis or with a parallel axis is removed; such bodies are usually called pipes or tubes.

By Art. 173, the area of the base in square inches $=7 \times 1 \times 3{\cdot}1416 = 21{\cdot}9912$; the height is 60 inches: therefore the volume in cubic inches $= 60 \times 21{\cdot}9912 = 1319{\cdot}472$.

(3) The height of a cylinder is to be equal to the radius of the base, and the volume is to be 500 cubic inches: find the height.

Since the height is equal to the radius of the base, the product of 3·1416 into the *cube* of the number of inches in the radius must be equal to 500: hence the cube of the number of inches in the radius $= \frac{500}{3{\cdot}1416} = 159{\cdot}15$. By extracting the cube root we obtain 5·419. Thus the radius is nearly 5·42 inches.

EXAMPLES. XXIII.

Find in cubic feet and inches the volumes of the prisms having the following dimensions:

1. Base 6 square feet 35 square inches; height 2 feet 6 inches.

2. Base 15 square feet 135 square inches; height 3 feet 11 inches.

3. Base 23 square feet 115 square inches; height 4 feet 7 inches.

4. Base 35 square feet 123 square inches; height 5 feet 5 inches.

Find in cubic feet and inches the volumes of the triangular prisms having the following dimensions:

5. Sides of the base 7, 15, and 20 inches: height 45 inches

6. Sides of the base 16, 25, and 39 inches; height 52 inches.

7. Sides of the base 13, 40, and 51 inches; height 58 inches.

8. Sides of the base 25, 33, and 52 inches; height 62 inches.

Find in cubic feet and decimals the volumes of the cylinders having the following dimensions:

9. Radius of base 2 feet; height 3 feet 6 inches.

10. Radius of base 2 feet 6 inches; height 4 feet 3 inches.

11. Radius of base 3 feet 6 inches; height 5 feet 9 inches.

12. Radius of base 5 feet 4 inches; height 6 feet $4\frac{1}{2}$ inches.

Find the heights of the prisms which have the following volumes and bases:

13. Volume 18 cubic feet 708 cubic inches; base 6 square feet 100 square inches.

14. Volume 28 cubic feet 500 cubic inches; base 7 square feet 103 square inches.

15. Volume 36 cubic feet 349 cubic inches; base 9 square feet 35 square inches.

16. Volume 65 cubic feet 782 cubic inches; base 14 square feet 118 square inches.

Find the radii of the bases of the cylinders which have the following volumes and heights:

17. Volume 10000 cubic inches; height 4 feet 2 inches.

18. Volume 20 cubic feet; height 4 feet $7\frac{1}{2}$ inches.

19. Volume 50 cubic feet; height 5 feet $4\frac{1}{2}$ inches.

20. Volume 100 cubic feet; height 5 feet 10 inches.

Find to the nearest gallon the quantity of water which will be held by cylindrical vessels having the following dimensions:

21. Radius of base 10 inches; height 20 inches.

22. Radius of base 2 feet 6 inches; height 4 feet.

23. Radius of base 5 feet; height 8 feet.

24. Radius of base 7 feet 6 inches; height 10 feet.

25. The height of a prism is 24 feet; the base is a trapezoid, the parallel sides being 18 feet and 12 feet respectively, and the distance between them 5 feet: find the volume.

26. The wall of China is 1500 miles long, 20 feet high, 15 feet wide at the top, and 25 feet wide at the bottom: find how many cubic yards of material it contains.

27. Find the number of cubic feet of earth which must be dug out to form a ditch 1000 feet long, 8 feet deep, 16 feet broad at the bottom, and 20 feet broad at the top.

28. Find to the nearest gallon the quantity of water which will be required to fill a ditch having the following dimensions: length 40 feet, depth 6 feet, breadth at the top 10 feet, breadth at the bottom 8 feet.

29. A ditch is 8 feet deep, 24 feet broad at the top, and 16 feet broad at the bottom: find the length of the ditch if 250000 cubic feet of earth are dug out to make it.

30. A ditch is 4 feet deep, 5 feet broad at the top, and 4 feet broad at the bottom: find the length of the ditch to the nearest foot, if it will hold 10000 gallons of water.

31. Find how many cubic feet of earth must be dug out to make a well 3 feet in diameter and 30 feet deep.

32. Find how many cubic yards of earth must be dug out to make a well 4 feet in diameter and 119 feet deep.

33. Find the number of cubic yards of earth dug out of a tunnel 100 yards long, whose section is a semicircle, with a radius of 10 feet.

34. Find how many pieces of money $\frac{3}{4}$ of an inch in diameter, and $\frac{1}{8}$ of an inch thick, must be melted down in order to form a cube whose edge is 3 inches long.

35. The diameter of a well is 4 feet, and its depth 30 feet: find the cost of excavation at 7*s.* 6*d.* per cubic yard.

36. The diameter of a well is 3 feet 6 inches, and its depth 40 feet: find the cost of excavation at 7*s.* 6*d.* per cubic yard.

37. The diameter of a well is 3 feet 9 inches, and its depth 45 feet: find the cost of excavation at 7*s.* 3*d.* per cubic yard.

38. If 30 cubic inches of gunpowder weigh a pound, find what length of a gun, 6 inches bore, will be filled with 10 lbs. of powder.

39. A cubic foot of brass is to be drawn into a wire $\frac{1}{20}$ of an inch in diameter: find the length of the wire.

40. A cubic foot of brass is to be drawn into a wire ·025 of an inch thick: find the length of the wire.

41. Find the volume of a cylindrical shell, the radius of the inner surface being 5 inches, the radius of the outer surface 6 inches, and the height 7 feet.

42. Find the volume of a cylindrical shell, the radius of the outer surface being 10 inches, the thickness 2 inches, and the height 9 feet.

43. Find the volume of a cylindrical shell, the radius of the inner surface being 12 inches, the thickness 3 inches, and the height 10 feet.

44. An iron pipe is 3 inches in bore, half an inch thick, and 20 feet long: find its weight, supposing that a cubic inch of iron weighs 4·526 ounces.

45. The length of a leaden pipe is 13 feet, its bore is $1\frac{3}{4}$ inches, and its thickness $1\frac{1}{8}$ inches: find its weight, supposing a cubic inch of lead to weigh 6·604 ounces.

46. Find the cost of a leaden pipe of 2 inches bore, which is half an inch thick and 8 yards long, at $2\frac{1}{2}$*d.* per lb., supposing a cubic foot of lead to weigh 11412 ounces.

47. A square iron rod, an inch thick, weighs $10\frac{1}{2}$ lbs.: find the weight of a round iron rod of the same length and thickness.

48. Every edge of a certain triangular prism measures 10 inches: find the volume.

49. The base of a certain prism is a regular hexagon every edge of the prism measures 1 foot: find the volume of the prism.

50. The radius of the inner surface of a leaden pipe is $1\frac{1}{2}$ inches, and the radius of the outer surface is $1\frac{9}{16}$ inches: if the pipe be melted, and formed into a solid cylinder of the same length as before, find the radius.

51. The trunk of a tree is a right circular cylinder, 3 feet in diameter and 20 feet high: find the volume of the timber which remains when the trunk is trimmed just enough to reduce it to a rectangular parallelepiped on a square base.

The following examples involve the extraction of the cube root:

52. The sides of the base of a triangular prism are 52, 51, and 25 inches respectively; and the height is 60 inches: find the length of a cube of equivalent volume.

53. The height of a cylinder is 4 feet 9 inches, the radius of the base is 4 feet 3 inches: find the length of a cube of equivalent volume.

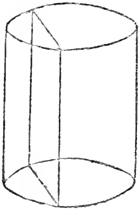
54. Suppose a sovereign to be $\frac{7}{8}$ of an inch in diameter, and $\frac{1}{16}$ of an inch in thickness; if 100000 of them be melted down and formed into a cube, find the length of the cube.

55. The height of a cylinder is to be 10 times the radius of the base, and the volume is to be 25 cubic feet: find the radius.

56. The height of a cylindrical vessel is to be half the radius of the base, and the cylinder is to hold a gallon: find the radius.

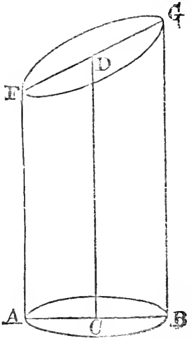
XXIV. SEGMENTS OF A RIGHT CIRCULAR CYLINDER. RING.

254. There are certain segments of a right circular cylinder, the volumes of which can be found by very simple Rules, as we will now shew.



255. Suppose a right circular cylinder cut into two parts by a plane parallel to the axis; then each part has a segment of a circle for base. The volume of each part may be found by the Rule of Art. 246.

256. Suppose a solid has been obtained by cutting a right circular cylinder by a plane, inclined to the axis, which does not meet the base of the cylinder. Let the straight line *CD*, drawn from the centre of the base at right angles to the base to meet the other plane, be called the *height* of the solid. Then the Rule for finding the volume of this solid is the same as that given in Art. 246.



Thus we may say that the height of the solid is the portion of the axis of the cylinder which is contained between the two ends.

257. The preceding Rule may be easily justified. For if we suppose a plane drawn through *D* parallel to the base of the cylinder, it will cut off a wedge-shaped slice, which may be so adjusted to the remaining solid as to form a complete right circular cylinder with the height *CD*.

258. In the diagram of Art. 256, it is easy to see that CD is half the sum of AF and BG; that is, the height is equal to half the sum of the greatest and the least straight lines which can be drawn on the solid parallel to the axis of the cylinder. See Art. 163.

259. Suppose a solid has been obtained by cutting a right circular cylinder by *two* planes, inclined to the axis, which do not meet each other. The volume of the solid will be found by *multiplying the base of the cylinder by the height of the solid;* where by the height of the solid we must understand the portion of the axis of the cylinder which is contained between the two ends. The Rule follows from the fact that the solid may be supposed to be the difference of two solids of the kind considered in Art. 256.

260. Suppose we have a solid like that represented in the diagram of Art. 256, and that it is bent round until A and F meet: we obtain a solid resembling a *solid ring;* and thus a ring may be described roughly as a cylinder bent round until the ends meet. This is not exact, but it will serve to illustrate the Rule which we shall now give.

261. *To find the volume of a solid ring.*

Multiply the area of a circular section of the ring by the length of the ring.

The circular section is sometimes called the *cross* section. The *length* of the ring is the length of the circumference of the circle which passes through the centres of all the cross sections; or it may be described as half the sum of the inner and outer boundaries of the ring. See Art. 258.

262. Examples:

(1) The radius of the circular section of a ring is one inch, and the length of the ring is ten inches.

The area of the circular section of the ring is 3·1416 square inches; therefore the volume of the ring in cubic inches is $10 \times 3{\cdot}1416$, that is, 31·416.

(2) The inner diameter of a ring is 7 inches, and the outer diameter is 8 inches.

The difference of these diameters is twice the diameter of the circular section; therefore the radius of the circular section is $\frac{1}{4}$ of an inch; and the area of the circular section is $\cdot 19635$ square inches.

The inner boundary of the ring is $7 \times 3{\cdot}1416$ inches, and the outer boundary is $8 \times 3{\cdot}1416$ inches; half the sum of these numbers is $23{\cdot}562$ inches, which is therefore the length of the ring.

Therefore the volume of the ring in cubic inches $= \cdot 19635 \times 23{\cdot}562 = 4{\cdot}6264$ very nearly.

EXAMPLES. XXIV.

1. Shew that the length of a ring is equal to the difference of the outer boundary and the circumference of the cross section.

2. Shew that the length of a ring is equal to the sum of the inner boundary and the circumference of the cross section.

Find in cubic inches the volumes of the rings having the following dimensions:

3. Length $20\frac{1}{2}$ inches, radius of cross section $\frac{7}{8}$ of an inch.

4. Length 16 inches, diameter of cross section $1{\cdot}1$ inches.

5. Outer diameter $4{\cdot}8$ inches, inner $4{\cdot}2$ inches.

6. Inner diameter $12{\cdot}3$ inches, diameter of cross section $3{\cdot}2$ inches.

7. Outer diameter 19 inches, diameter of cross section $3\frac{1}{4}$ inches.

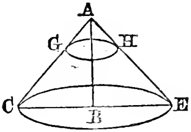
8. Outer boundary 15 inches, circumference of cross section $1{\cdot}6$ inches.

9. The volume of a ring is 800 cubic inches, the radius of the cross section is 2 inches: find the length of the ring.

10. The volume of a ring is 100 cubic inches, and the length is 20 inches: find the inner diameter.

XXV. PYRAMID. CONE.

263. *To find the volume of a pyramid or a cone.*



RULE. *Multiply the area of the base by the height and one-third of the product will be the volume.*

264. Examples:

(1) The base of a pyramid is a square, each side of which is 3 feet 6 inches, and the height of the pyramid is 3 feet 9 inches.

$$3 \text{ feet } 6 \text{ inches} = 3\tfrac{1}{2} \text{ feet},$$

$$3 \text{ feet } 9 \text{ inches} = 3\tfrac{3}{4} \text{ feet}.$$

$$3\tfrac{1}{2} \times 3\tfrac{1}{2} = \frac{7}{2} \times \frac{7}{2} = \frac{49}{4},$$

$$\frac{1}{3} \times \frac{49}{4} \times \frac{15}{4} = \frac{49 \times 5}{4 \times 4} = \frac{245}{16} = 15\tfrac{5}{16}.$$

Thus the volume is $15\frac{5}{16}$ cubic feet.

(2) The radius of the base of a cone is 10 inches, and the height of the cone is 18 inches.

$$10 \times 10 \times 3{\cdot}1416 = 314{\cdot}16,$$

$$\frac{1}{3} \times 18 \times 314{\cdot}16 = 6 \times 314{\cdot}16 = 1884{\cdot}96.$$

Thus the volume is about 1884·96 cubic inches.

265. If we know the volume of a pyramid or a cone, and also the area of the base, we can find the height by dividing three times the number which expresses the volume by the number which expresses the area; and similarly, if we know the volume and the height, we can find the area of the base.

266. Examples :

(1) The volume of a pyramid is a cubic yard, and the area of the base is 18 square feet: find the height.

A cubic yard = 27 cubic feet; $\frac{3 \times 27}{18} = \frac{9}{2} = 4\frac{1}{2}$.

Thus the height is $4\frac{1}{2}$ feet.

(2) The volume of a cone is half a cubic foot, and its height is 27 inches: find the area of the base.

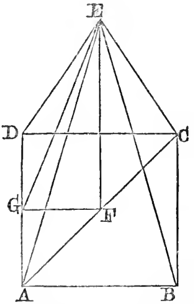
Half a cubic foot = 864 cubic inches; $\frac{3 \times 864}{27} = 96$.

Thus the area of the base is 96 square inches.

267. We will now solve some exercises.

(1) The base of a pyramid is a square, each side of which measures 10 feet; the length of each of the four edges which meet at the vertex is 18 feet: find the volume.

We must determine the height of the pyramid.



Let *ABCD* be the base, and *E* the vertex of the pyramid. Let *EF* be the height of the pyramid, that is, the perpendicular from *E* on the base; then *F* will be the middle point of the diagonal *AC*.

Now, by Art. 55, we shall find that the number of feet in *AC* is $10\sqrt{2}$; and thus the number of feet in *AF* is $5\sqrt{2}$.

In the right-angled triangle *AEF*, the hypotenuse *AE* is 18 feet; and the number of feet in *AF* is $5\sqrt{2}$; therefore, by Art. 60, the number of feet in *EF* is the square root of $324 - 50$, that is, the square root of 274, that is, 16·5529454.

Hence the volume of the pyramid in cubic feet

$$=\frac{1}{3}\times 100\times 16{\cdot}5529454=551{\cdot}76484.$$

(2) The base of a pyramid is a square, each side of which is 10 feet; the length of the straight line drawn from the vertex to the middle point of any side of the base is 13 feet: find the volume.

We must determine the height of the pyramid. Using the same figure as in the preceding Exercise, let G be the middle point of AD. Then EG is 13 feet; and GF is 5 feet: therefore, by Art. 60, the number of feet in EF is the square root of $169-25$, that is, the square root of 144. that is 12.

Hence the volume of the pyramid in cubic feet

$$=\frac{1}{3}\times 100\times 12=400.$$

We may observe that EG is sometimes called the *slant height* of the pyramid.

(3) A corner of a cube is cut off by a plane which meets the edges at distances 3, 4, and 5 inches respectively from their common point: find the volume of the piece cut off.

The piece cut off is a triangular pyramid: we may take for the base the right-angled triangle the sides of which are 3 and 4 inches respectively, and then the height of the pyramid is 5 inches. Hence the volume of the pyramid in cubic inches $=\frac{1}{3}\times\frac{3\times 4}{2}\times 5=10.$

EXAMPLES. XXV.

Find in cubic feet and inches the volumes of the pyramids having the following dimensions:

1. Base 7 square feet 102 square inches; height 2 feet 5 inches.

2. Base 14 square feet 96 square inches; height 3 feet 7 inches.

3. Base 20 square feet 120 square inches; height 5 feet 8 inches.

4. Base 23 square feet 21 square inches; height 4 feet 11 inches.

Find in cubic feet and decimals the volumes of the triangular pyramids having the following dimensions:

5. Sides of the base 4, 5, and 7 feet; height 6 feet.

6. Sides of the base 7, 9, and 11 feet; height 8 feet.

7. Sides of the base 15, 19, and 20 feet; height 22 feet.

8. Sides of the base 23, 27, and 30 feet; height 24 feet.

Find in cubic feet and decimals the volumes of the cones having the following dimensions:

9. Radius of base 2 feet; height 4 feet.

10. Radius of base 3 feet 6 inches; height 5 feet.

11. Radius of base 4·2 feet; height 5·3 feet.

12. Radius of base 10 feet; height 10 feet.

Find the heights of the pyramids which have the following volumes and bases:

13. Volume 17 cubic feet 363 cubic inches; base 2 square feet 143 square inches.

14. Volume 33 cubic feet 309 cubic inches; base 4 square feet 83 square inches.

15. Volume 91 cubic feet 792 cubic inches; base 9 square feet 21 square inches.

16. Volume 114 cubic feet 1152 cubic inches; base 10 square feet 96 square inches.

Find the radii of the bases of the cones which have the following volumes and heights:

17. Volume 4000 cubic inches; height 5 feet.

18. Volume 40 cubic feet; height 5·3 feet.

19. Volume 60·7 cubic feet; height 5·45 feet.

20. Volume 120 cubic feet; height 6·24 feet.

21. The faces of a pyramid on a square base are equilateral triangles, a side of the base being 120 feet: find the volume.

22. Find the volume of a pyramid which stands on a square base, each side of which has 200 feet, each of the edges which meet at the vertex being 150 feet.

23. A pyramid has a square base, the area of which is 20·25 square feet; each of the edges of the pyramid which meet at the vertex is $30\frac{3}{4}$ feet: find the volume.

24. The base of a pyramid is a rectangle 80 feet by 60 feet; each of the edges which meet at the vertex is 130 feet: find the volume.

25. The base of a pyramid is a square, each side of which is 24 feet; the length of the straight line drawn from the vertex to the middle point of any side of the base is 21·8 feet: find the volume.

26. The base of a pyramid is a square, each side of which is 12 feet; the length of the straight line drawn from the vertex to the middle point of any side of the base is 25 feet: find the volume.

27. The base of a pyramid is a rectangle, which is 21 feet by 25 feet; the length of the straight line drawn from the vertex to the middle point of either of the longer sides of the base is 23·3 feet: find the volume.

28. The base of a pyramid is a rectangle, which is 18 feet by 26 feet; the length of the straight line drawn from the vertex to the middle point of either of the shorter sides of the base is 24 feet: find the volume.

29. The slant side of a right circular cone is 25 feet, and the radius of the base is 7 feet: find the volume.

30. The section of a right circular cone by a plane through its vertex perpendicular to the base is an equilateral triangle, each side of which is 12 feet: find the volume of the cone.

31. The slant side of a right circular cone is 41 feet, and the height is 40 feet: find the volume.

32. The slant side of a right circular cone is 55 feet, and the height is 42 feet: find the volume.

33. Find how many gallons are contained in a vessel which is in the form of a right circular cone, the radius of the base being 8 feet, and the slant side 17 feet.

34. A conical wine glass is 2 inches wide at the top and 3 inches deep: find how many cubic inches of wine it will hold.

35. Find the volume of a circular cone, the height of which is 15 feet, and the circumference of the base 16 feet.

36. A cone, 3 feet high and 2 feet in diameter at the bottom, is placed on the ground, and sand is poured over it until a conical heap is formed 5 feet high and 30 feet in circumference at the bottom: find how many cubic feet of sand there are.

37. The volume of a cone is $22\frac{1}{2}$ cubic feet; the circumference of the base is 9 feet: find the height.

38. Find the number of cubic feet in a regular hexagonal room, each side of which is 20 feet in length, and the walls 30 feet high, and which is finished above with a roof in the form of a hexagonal pyramid 15 feet high.

39. Find the volume of the pyramid formed by cutting off a corner of the cube, whose side is 20 feet, by a plane which bisects its three conterminous edges.

40. The edge of a cube is 14 inches; one of the corners of the cube is cut off, so that the part cut off forms a pyramid, with each of its edges terminating in the angle of the cube, 6 inches in length: find the volume of the solid that remains.

41. The great pyramid of Egypt was 481 feet in height, when complete; and its base was a square 764 feet in length: find the volume to the nearest number of cubic yards.

42. The spire of a church is a right pyramid on a regular hexagonal base; each side of the base is 10 feet and the height is 50 feet; there is a hollow part which is also a right pyramid on a regular hexagonal base, the height of the hollow part is 45 feet, and each side of the base is 9 feet: find the number of cubic feet of stone in the spire.

XXVI. FRUSTUM OF A PYRAMID OR CONE.

268. *To find the volume of a frustum of a Pyramid or Cone.*

Rule. *To the areas of the two ends of the frustum add the square root of their product; multiply the sum by the height of the frustum, and one-third of the product will be the volume.*

269. Examples:

(1) The area of one end of a frustum of a pyramid is 18 square inches, and the area of the other end is 98 square inches; and the height of the frustum is 15 inches.

The square root of 18×98 is 42; $18 + 98 + 42 = 158$.

$\frac{1}{3} \times 15 \times 158 = 790$. Thus the volume is 790 cubic inches.

(2) The radius of one end of a frustum of a cone is 5 feet, and the radius of the other end is 3 feet; and the height of the frustum is 8 feet.

The area of one end in square feet $= 25 \times 3{\cdot}1416$; the area of the other end in square feet $= 9 \times 3{\cdot}1416$; the square root of the product of these numbers is 3·1416 multiplied by the square root of 9×25, that is, $3{\cdot}1416 \times 15$.

Add these results and we obtain 3·1416 multiplied by the sum of 25, 9, and 15; that is, $3{\cdot}1416 \times 49$.

Then $\frac{1}{3} \times 8 \times 49 \times 3{\cdot}1416 = 410{\cdot}5024$.

Thus the volume is 410·5024 cubic inches.

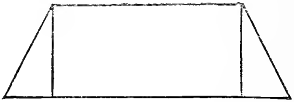
270. It will be seen that in the second example of the preceding Article we have adopted a peculiar arrangement, with the view of saving labour in multiplication. The same arrangement may always be used with advantage in finding the volume of the frustum of a cone, when the radii of the ends are known. In fact, in such a case we may, instead of the rule in Art. 268, adopt the following, which is sub-

stantially the same, but more convenient in form: *add the squares of the radii of the ends to the product of the radii: multiply the sum by the height, and this product by* 3·1416 *: one-third of the result will be the volume.*

271. We will now solve some exercises.

(1) The radii of the ends of a frustum of a right circular cone are 7 inches and 10 inches respectively; and the slant height of the frustum is 5 inches.

We must determine the height of the frustum.



Let the diagram represent a section of the frustum made by a plane, containing the axis of the cone. We see that the slant height is the hypotenuse of a right-angled triangle, of which one side is the height of the frustum, and the other side is the difference of the radii of the ends.

In the present case the slant height is 5 inches, and the difference of the radii of the ends is 3 inches; therefore, by Art. 60, the height of the frustum is 4 inches.

Now $7 \times 7 = 49$, $10 \times 10 = 100$, $7 \times 10 = 70$;

$$49 + 100 + 70 = 219\,; \quad \frac{1}{3} \times 4 \times 219 \times 3{\cdot}1416 = 917{\cdot}3472.$$

Thus the volume is 917·3472 cubic inches.

(2) The ends of a frustum of a pyramid are equilateral triangles, the sides being 3 feet and 4 feet respectively; and the height is 9 feet.

By Art. 206 the area of one end in square feet $= 9 \times {\cdot}433\ldots$; and the area of the other end in square feet $= 16 \times {\cdot}433\ldots$; the square root of the product of these numbers $= 12 \times {\cdot}433\ldots$ Add these three results, and we obtain $37 \times {\cdot}433\ldots$

Then $\frac{1}{3} \times 9 \times 37 \times {\cdot}433\ldots = 48{\cdot}063$.

Thus the volume is rather more than 48 cubic feet.

EXAMPLES. XXVI.

Find in cubic feet the volumes of frustums of pyramids which have the following dimensions:

1. Areas of ends 4·5 square feet and 12·5 square feet; height 1·5 feet.

2. Areas of ends 4 square feet and 5 square feet; height 2 feet 6 inches.

3. Areas of ends 900 square inches and 6·5 square feet; height 2 yards.

4. Areas of ends 7·5 square feet and 8·25 square feet; height 6·125 feet.

Find in cubic feet the volumes of frustums of cones which have the following dimensions:

5. Radii of the ends 3 feet and 4 feet; height $5\frac{1}{4}$ feet.

6. Radii of the ends 4·5 feet and 5·4 feet; height 6·5 feet.

7. Radii of the ends 4·8 feet and 6·4 feet; height 7·2 feet.

8. Radii of the ends 6·375 feet and 5·1 feet; height 10 feet.

9. The slant side of the frustum of a right circular cone is 5 feet, and the radii of the ends are 7 feet and 10 feet: find the volume.

10. Find the cost of the frustum of a right circular cone of marble at 24 shillings per cubic foot, the diameter of the greater end being 4 feet, of the smaller end $1\frac{1}{2}$ feet, and the length of the slant side 8 feet.

11. The ends of the frustum of a pyramid are equilateral triangles, the lengths of the sides being 6 feet and 7 feet respectively; the height of the frustum is 4 feet: find the volume.

12. The ends of the frustum of a pyramid are squares, the lengths of the sides being 20 feet and 30 feet respectively: the length of the straight line which joins the middle point of any side of one end with the middle point of the corresponding side of the other end is 13 feet: find the volume.

13. The shaft of Pompey's pillar, which is situated near Alexandria in Egypt, is a single stone of granite; the height is 90 feet, the diameter at one end is 9 feet, and at the other end 7 feet 6 inches: find the volume.

14. The mast of a ship is 50 feet high; the circumference at one end is 60 inches, and at the other 36: find the number of cubic feet of wood.

15. The radii of the ends of a frustum of a right circular cone are 7 feet and 8 feet respectively; and the height is 3 feet: find the volume of the cone from which the frustum was obtained.

16. The radii of the ends of the frustum of a right circular cone are 7 feet and 8 feet respectively; and the height is 3 feet: find the volumes of the two pieces obtained by cutting the frustum by a plane parallel to the ends and midway between them.

17. The radii of the ends of a frustum of a right circular cone are 7 feet and 8 feet respectively; and the height is 3 feet; the frustum is cut into three, each one foot in height, by planes parallel to the ends: find the number of cubic inches in each of the pieces.

18. The ends of the frustum of a pyramid are regular hexagons, the lengths of the sides being 8 feet and 10 feet respectively; the height of the frustum is 12 feet: find the volumes of the two pieces obtained by cutting the frustum by a plane parallel to the ends and midway between them.

XXVII. WEDGE.

272. *To find the volume of a wedge.*

RULE. *Add the length of the edge to twice the length of the base; multiply the sum by the width of the base, and the product by the height of the wedge; one-sixth of the result will be the volume of the wedge.*

273. Examples:

(1) The edge of a wedge is 12 inches; the length of the base is 16 inches, and the breadth is 7 inches; the height of the wedge is 24 inches.

$$12+16+16=44; \ \frac{1}{6}\times 44\times 7\times 24=1232.$$

Thus the volume of the wedge is 1232 cubic inches.

(2) The edge of a wedge is $5\frac{1}{2}$ inches; the length of the base is 3 inches, and the breadth is 2 inches; the height of the wedge is 4 inches.

$$5\tfrac{1}{2}+3+3=11\tfrac{1}{2}=\frac{23}{2}; \ \frac{1}{6}\times\frac{23}{2}\times 2\times 4=\frac{46}{3}=15\tfrac{1}{3}.$$

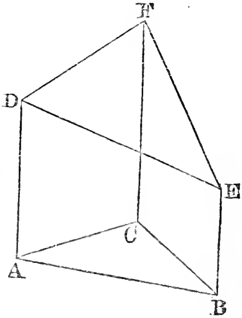
Thus the volume of the wedge is $15\frac{1}{3}$ cubic inches.

274. If the edge of a wedge be equal to the length of the base, the wedge is a triangular prism; so that we are thus furnished with another Rule for finding the volume of such a prism. This Rule is in general not identical with that given in Art. 246, because the dimensions which are supposed to be known are not the same in the two cases. If the prism be a right prism, it is easy to see that the two Rules are really identical.

275. If the edge of a wedge be shorter than the length of the base, the wedge can be divided into an oblique triangular prism and a pyramid on a rectangular base, by drawing a plane through one end of the edge parallel to the triangular face at the other end. And, in like manner, if the edge be longer than the length of the base, the wedge is equal to the excess of a certain triangular prism over a certain pyramid on a rectangular base. It is by this consideration that the Rule of Art. 272 can be shewn to be true.

276. It has been proposed to extend the meaning of the word *wedge* to the case in which the base instead of being a rectangle is a parallelogram or a trapezoid. The Rule for finding the volume will still hold, provided we understand the *length of the base* to be half the sum of the parallel sides, and the *breadth* of the base to be the perpendicular distance between the parallel sides.

277. Suppose a solid has been obtained by cutting a right triangular prism by a plane, inclined to the length of the prism, which does not meet the base of the prism.



The volume of this solid may be found by the following Rule:

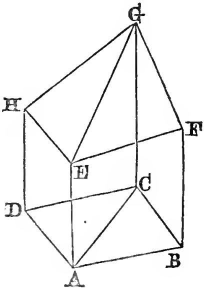
Multiply the area of the base of the prism by one-third of the sum of the parallel edges of the solid.

278. The solid considered in the preceding Article is a wedge in the enlarged meaning of the word noticed in Art. 276: the Rule for finding the volume may be demonstrated by a method similar to that explained in Art. 275. For if through E, the upper end of the shortest of the three edges, we draw a plane parallel to the base ABC, we divide the solid into a right prism and a pyramid; the

volumes of each of these can be obtained by known Rules, and it will be found that the sum of the two volumes will agree with that assigned by the Rule of Art. 277.

279. Suppose a solid has been obtained by cutting a right prism having a parallelogram for base by a plane, inclined to the length of the prism, which does not meet the base of the prism. The volume of this solid may be found by the following Rule:

Multiply the area of the base of the prism by one-fourth of the sum of the four parallel edges of the solid.



280. It is easy to see that $AE+CG=BF+DH$; for each of these is equal to twice the distance between the point of intersection of AC and BD and the point of intersection of EG and FH; and thus the Rule of the preceding Article might be given in this form:

Multiply the area of the base of the prism by half the sum of two opposite edges out of the four parallel edges.

281. The Rule of Art. 280 follows from that of Art. 277. For if we cut the solid into two pieces by a plane which passes through AE and CG, the Rule of Art. 277 will determine the volume of each piece; and it will be found that the sum of the two volumes will agree with that assigned by the Rule of Art. 280.

The Rules of Arts. 277 and 279 may be extended in the same way as the Rule of Art. 256 is extended in Art. 259.

282. We will now solve some exercises.

(1) The edge of a wedge is 18 inches; the length of the base is 20 inches; the area of a section of the wedge made by a plane perpendicular to the edge is 150 square inches: find the volume.

The section made by a plane perpendicular to the edge is a triangle; therefore the product of the base of this triangle into its height is 2×150, that is, 300: this product is the same as that of the breadth of the wedge into the height of the wedge.

$$18 + 20 + 20 = 58; \quad \frac{1}{6} \times 58 \times 300 = 2900.$$

Thus the volume is 2900 cubic inches.

The result can be obtained more readily by the Rule of Art. 277.

$$18 + 20 + 20 = 58; \quad \frac{1}{3} \times 58 \times 150 = 2900.$$

(2) The edge of a wedge is 16 inches; the length of the base is 24 inches, and the breadth is 6 inches; and the height of the wedge is 10 inches. The wedge is divided into a pyramid and a prism by a plane through one end of the edge parallel to the triangular face at the other end. Find the volume of each part.

The length of the base of the pyramid is $24 - 16$ inches, that is, 8 inches; hence, by Art. 263, the volume of the pyramid in cubic inches $= \frac{1}{3} \times 8 \times 6 \times 10 = 160$.

The prism has three parallel edges, each 16 inches long; and, by Art. 274, its volume in cubic inches

$$= \frac{1}{2} \times 16 \times 6 \times 10 = 480.$$

EXAMPLES. XXVII.

1. The edge of a wedge is 2 feet 3 inches; the length of the base 2 feet 3 inches, and the breadth is 8 inches; the height of the wedge is 15 inches: find the volume.

2. The edge of a wedge is 9 feet; the length of the base is 6 feet, and the breadth is 3 feet; the height of the wedge is 2 feet: find the volume.

3. The base of a wedge is a square, a side of which is 15 inches; the edge is 24 inches, and the height of the wedge is 24 inches: find the volume.

4. The base of a prism is an equilateral triangle, each side of which is 4 inches: find the volume of the solid obtained by cutting off a piece of this prism, so that the sum of the three parallel edges is 15 inches.

5. The base of a prism is a rectangle which measures 7 inches by 8: find the volume of the solid obtained by cutting off a piece of this prism, so that the sum of the four parallel edges is 42 inches.

6. The edge of a wedge is 21 inches; the length of the base is 27 inches; the area of a section of the wedge made by a plane perpendicular to the edge is 160 square inches: find the volume.

7. The edge of a wedge is 25 inches; the length of the base is 22 inches; a section of the wedge made by a plane perpendicular to the edge is an equilateral triangle, each side of which is 10 inches: find the volume.

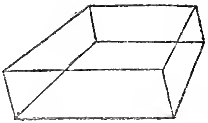
8. The edge of a wedge is 15 inches; the length of the base is 24 inches, and the breadth is 7 inches; the height of the wedge is 22 inches; the wedge is divided into a pyramid and a prism by a plane through one end of the edge parallel to the triangular face at the other end: find the volume of each part.

9. The edge of a wedge is 2 feet 3 inches; the length of the base is 2 feet 9 inches, and the breadth is 8 inches; the height of the wedge is 14 inches; the wedge is divided into two pieces by a plane which passes through a point in the edge, distant 18 inches from one end, and which is parallel to the triangular face at that end: find the volume of each piece.

10. The edge of a wedge is 36 inches; the length of the base is 27 inches, and the breadth 5 inches; the height of the wedge is 12 inches. The wedge is divided by a plane, so that the sum of the three parallel edges in one part is 42 inches: find the volume of each part.

XXVIII. PRISMOID.

283. *To find the volume of a Prismoid.*



RULE. *Add together the areas of the two ends and four times the area of a section parallel to the two ends and midway between them; multiply the sum by the height, and one-sixth of the product will be the volume.*

284. Examples.

(1) The area of one end is 4 square feet, of the other end 9 square feet, and of the middle section 6 square feet; the height is 2 feet.

$$4+24+9=37, \quad \frac{1}{6}\times 37\times 2=\frac{37}{3}=12\tfrac{1}{3}.$$

Thus the volume is $12\frac{1}{3}$ cubic feet.

(2) The area of one end is 224 square inches, of the other end 216 square inches, and of the middle section 221 square inches: the height is 18 inches.

$$224+884+216=1324, \quad \frac{1}{6}\times 1324\times 18=3972.$$

Thus the volume is 3972 cubic inches.

285. The demonstration of the Rule in Art. 283 depends on the fact that a prismoid can be divided into pyramids and wedges, some having their bases in one end of the prismoid, and some in the other, and all having the same height as the prismoid.

286. Each side of the middle section is equal to half the sum of the corresponding sides of the ends. Thus if the ends are rectangles of known dimensions, the area of the middle section can be easily found: for then four times the area of the middle section is equal to the area

of a rectangle, having for each of its dimensions the sum of the corresponding dimensions of the ends.

Each angle of the middle section is equal to the corresponding angle at the ends.

287. If the ends of a prismoid are similar figures similarly situated, the prismoid is a frustum of a pyramid, and therefore the volume might be found by the Rule of Art. 268. By comparing the two Rules, we infer that in this case, four times the area of the middle section is equal to the sum of the areas of the ends added to twice the square root of the product of these areas.

288. It has been proposed to extend the meaning of the term prismoid so as to apply to cases in which the ends are not rectilineal figures. Accordingly, the following definition may be given: *A prismoid has for its ends any two parallel plane figures, and has its other boundary straight.* By having the other boundary straight is meant that a straight line may be placed on the boundary at any point, so as to coincide with the surface from end to end. This definition will include a frustum of a cone, or either of the pieces obtained by cutting a frustum of a cone by a plane which meets both ends.

The Rule in Art. 283 holds for this extended meaning of the term prismoid.

289. The Rule for finding the volume of a prismoid holds for many other solids; but it would not be possible to define these solids in an elementary manner: the advanced student may consult the author's *Integral Calculus*, Art. 192.

290. We will now solve some exercises.

(1) The ends of a prismoid are trapezoids with four parallel edges; the parallel sides of one end are 100 feet and 32 feet respectively, and the distance between them is 28 feet; the corresponding dimensions of the other end are 80 feet, 30 feet, and 26 feet; the distance between the ends is 112 feet: find the volume.

The middle section is a trapezoid; one of the two parallel sides is half of $100+80$ feet, that is, 90 feet; and the other is half of $32+30$ feet, that is, 31 feet: the distance between these two parallel sides is half the sum of the corresponding distances for the two ends, that is, half of $28+26$ feet, that is, 27 feet.

The area of one end in square feet $=\frac{132}{2}\times 28=1848$;

the area of the other end in square feet $=\frac{110}{2}\times 26=1430$;

the area of the middle section in square feet

$$=\frac{121}{2}\times 27=1633\tfrac{1}{2};$$

four times this area $=6534$;

$$1848+1430+6534=9812;\quad \frac{1}{6}\times 9812\times 112=183157\tfrac{1}{3}.$$

Thus the volume is $183157\frac{1}{3}$ cubic feet.

(2) The edge of a wedge is 21 inches; the length of the base is 15 inches, and the breadth 9 inches; the height of the wedge is 6 inches; the wedge is divided into three parts of equal heights by planes parallel to the base: find the volume of each part.

The parts are two prismoids and a wedge; the height of each part is 2 inches.

The first prismoid has one end a rectangle which measures 15 inches by 9; it will be found that the other end is a rectangle which measures in the corresponding manner 17 inches by 6. The volume, by Art. 283, is 239 cubic inches.

The second prismoid has one end a rectangle which measures 17 inches by 6; it will be found that the other end is a rectangle which measures in the corresponding manner 19 inches by 3. The volume, by Art. 283, is 161 cubic inches.

The edge of the wedge is 21 inches; the length of the base is 19 inches, and the breadth 3 inches. The volume, by Art. 272, is 59 cubic inches.

The sum of the three volumes in cubic inches is 239 + 161 + 59, that is, 459: it will be found that this is equal to the volume of the original wedge, as of course it should be.

EXAMPLES. XXVIII.

1. Find the number of cubic feet which must be removed to form a prismoidal cavity; the depth is 12 feet, and the top and the bottom are rectangles, the corresponding dimensions of which are 400 feet by 180, and 350 feet by 150.

2. Find the number of cubic feet which must be removed to form a prismoidal cavity; the depth is 12 feet, and the top and the bottom are rectangles, the corresponding dimensions of which are 400 feet by 180 feet, and 150 feet by 350 feet.

3. Find the volume of a coal waggon the depth of which is 47 inches; the top and the bottom are rectangles, the corresponding dimensions of which are 81 inches by 54 inches, and 42 inches by 30 inches.

4. Find the number of gallons of water required to fill a canal the depth of which is $4\frac{1}{2}$ feet, and the top and the bottom of which are rectangles, the corresponding dimensions of which are 250 feet by 16 feet, and 240 feet by 14 feet.

5. Find the number of cubic feet which must be removed to form a railway cutting in the form of a prismoidal cavity; the ends are trapezoids with four parallel edges; the parallel sides of one end are 144 feet and 36 feet, and the distance between them is 36 feet: the corresponding dimensions of the other end are 108 feet, 36 feet, and 24 feet: the distance between the ends is $137\frac{1}{2}$ feet.

6. The ends of a prismoid are rectangles, the corresponding dimensions of which are 12 feet by 10 feet and

8 feet by 6 feet; the height of the prismoid is 4 feet: the prismoid is divided by a plane parallel to the ends and midway between them: find the volume of each part.

7. The ends of a prismoid are rectangles, the corresponding dimensions of which are 16 feet by 11 feet, and 10 feet by 8 feet; the height of the prismoid is 9 feet: the prismoid is divided into three parts, each 3 feet high, by planes parallel to the ends: find the volume of each of the parts.

8. The ends of a prismoid are rectangles, the corresponding dimensions of which are 20 feet by 16 feet, and 14 feet by 12 feet; the height of the prismoid is 5 feet: the prismoid is cut into two wedges by a plane which passes through one of the longer sides of one end, and the opposite longer side of the other end: find the volume of each part.

9. The edge of a wedge is 24 inches, the length of the base 8 inches, and the breadth is 7 inches; the height of the wedge is 16 inches; the wedge is divided into two parts by a plane parallel to the base midway between the edge and the base: find the volume of each part.

10. The edge of a wedge is 27 inches; the length of the base is 18 inches, and the breadth is 15 inches; the height of the wedge is 12 inches; the wedge is divided into three parts of equal height by two planes parallel to the base: find the volume of each part.

11. The ends of a prismoid are rectangles, the corresponding dimensions of which are 18 feet by 10 feet, and 12 feet by 16 feet; the height of the prismoid is 9 feet; a section is made by a plane parallel to the ends at the distance of 3 feet from the larger end: shew that the section is a square.

12. Find the volumes of the two parts in the preceding Example.

XXIX. SPHERE.

291. *To find the volume of a Sphere.*

RULE. *Multiply the cube of the diameter by one-sixth of* 3·1416, *that is, by* ·5236.

292. Examples.

(1) The diameter of a sphere is 10 inches.

The cube of 10 is 1000; $\cdot 5236 \times 1000 = 523\cdot 6$.

Thus the volume of the sphere is about 523·6 cubic inches.

(2) The diameter of a sphere is $3\frac{1}{2}$ feet.

The cube of 3·5 is 42·875; $42\cdot 875 \times \cdot 5236 = 22\cdot 44935$.

Thus the volume of the sphere is very nearly 22·45 cubic feet.

293. The volume of a spherical *shell* will of course be obtained by subtracting the volume of a sphere having its diameter equal to the inner diameter of the shell from the volume of a sphere having its diameter equal to the outer diameter of the shell. Thus we obtain the Rule which will now be given.

294. *To find the volume of a spherical shell.*

RULE. *Subtract the cube of the inner diameter from the cube of the outer diameter, and multiply the result by* ·5236.

295. Examples.

(1) The outer diameter of a spherical shell is 9 inches, and the thickness of the shell is 1 inch.

Here the inner diameter will be 7 inches.

The cube of 9 is 729; the cube of 7 is 343; $729 - 343 = 386$; $\cdot 5236 \times 386 = 202\cdot 1096$.

Thus the volume of the shell is very nearly 202·11 cubic inches.

(2) The inner diameter of a spherical shell is 10 inches, and the thickness of the shell is $1\frac{1}{2}$ inches.

Here the outer diameter will be 13 inches.

The cube of 13 is 2197; the cube of 10 is 1000; $2197 - 1000 = 1197$; $\cdot 5236 \times 1197 = 626\cdot 7492$.

Thus the volume of the shell is very nearly 626·75 cubic inches.

296. If one sphere fall entirely within the other, it is obvious that the Rule of Art. 294 will give the volume of the space between the surfaces of the two spheres, even when the spheres are not concentric.

297. We will now solve some exercises.

(1) The circumference of a great circle of a sphere is 28 inches: find the volume of the sphere.

We first determine the diameter of the sphere; by Art. 111 this will be about 8·9 inches: then by Art. 291 we shall obtain for the volume of the sphere about 369·12 cubic inches.

(2) Find the weight of a leaden ball 5 inches in diameter, supposing a cubic inch of lead to weigh 6·6 ounces.

The cube of 5 is 125; $125 \times \cdot 5236 = 65\cdot 45$. Thus the volume of the ball is 65·45 cubic inches; and therefore its weight in ounces $= 65\cdot 45 \times 6\cdot 6 = 431\cdot 97$.

(3) If a cubic inch of gold weighs 11·194 ounces, find the diameter of a ball of gold which weighs 1000 ounces.

The number of cubic inches in the ball will be $\frac{1000}{11\cdot 194}$, that is, about 89·334; this number then is equal to the product of the cube of the diameter into ·5236. Thus to obtain the cube of the diameter we must divide 89·334 by ·5236; the quotient will be found to be 170·615. The cube root of this number will be the diameter; we shall find that this cube root is 5·546. Thus the diameter of the ball is about 5·55 inches

EXAMPLES. XXIX.

Find the volumes of spheres having the following diameters:

1. 11 inches. 2. 8 feet. 3. 24 feet. 4. 32·5 feet.

Find to the nearest hundredth of a cubic foot the volumes of spheres having the following circumferences of great circles:

5. 6 feet. 6. 8 feet. 7. 10 feet. 8. 12 feet.

Find in cubic inches the volumes of spherical shells having the following dimensions:

9. External diameter 5 inches, internal 4.

10. External diameter 8 inches, internal 6.

11. External diameter 10 inches, internal 7.

12. External diameter 16 inches, internal 12.

13. Find how many gallons a hemispherical bowl, 2 feet 4 inches in diameter, will hold.

14. Find how long it will take to fill a hemispherical tank of 10 feet diameter, from a cistern which supplies by a pipe 6 gallons of water per minute.

15. A solid is in the form of a right circular cylinder with hemispherical ends; the extreme length is 29 feet and the diameter is 3 feet: find the volume.

16. A solid is in the form of a right circular cylinder with hemispherical ends; the extreme length is 22 feet and the diameter is 2 feet 6 inches: find what will be the weight of water equal in bulk to this solid.

17. A sphere, $4\frac{1}{2}$ inches in diameter, is cut out of a cube of wood, the edge of which is $4\frac{1}{2}$ inches: find the quantity of wood which is cut away.

18. Find the weight of a spherical shot of iron, 6 inches in diameter, supposing a cubic inch of iron to weigh 4·2 ounces.

19. If a sphere of lead, 4 inches in diameter, weighs 221·3 ounces, find the weight of a sphere of lead 5 inches in diameter.

20. Find the weight of gunpowder required to fill a hollow sphere of 7 inches diameter, supposing that 30 cubic inches of gunpowder weigh one lb.

21. Find the weight of gunpowder required to fill a hollow sphere of 9 inches diameter.

22. Find the weight of a spherical shell one inch thick, the external diameter of which is 10 inches, composed of a substance a cubic foot of which weighs 216 lbs.

23. Find the weight of a spherical shell $1\frac{1}{2}$ inches thick, the external diameter of which is 11 inches, composed of iron weighing 4 cwt. to the cubic foot.

24. The external diameter of a shell is 8·4 inches and the internal diameter is 7·2 inches: find the weight of the shell if it is composed of a substance of which a cubic foot weighs 7860 ounces.

25. Find the weight of a shell $1\frac{5}{8}$ inches thick, the external diameter of which is 13 inches, composed of metal a cubic inch of which weighs 4·4 ounces.

26. Find the weight of a shell $3\frac{1}{2}$ inches thick, the external diameter of which is 1 foot $5\frac{1}{2}$ inches, composed of metal a cubic foot of which weighs 480 lbs.

27. If an iron ball, 4 inches in diameter, weigh 9 lbs., find the weight of an iron shell 2 inches thick, whose external diameter is 20 inches.

28. If a shell, the external and internal diameters of which are 5 inches and 3 inches, weighs $8\frac{1}{2}$ lbs., find the weight of a shell composed of the same substance, the external and internal diameters of which are $7\frac{1}{2}$ inches and $4\frac{1}{2}$ inches.

29. Shew that the weight of a cone, 7 inches high on a circular base, of which the radius is 2 inches, is equal to that of a spherical shell of the same material, of which the external diameter is 4 inches and the thickness is 1 inch.

30. Find the weight of a pyramid of iron, such that its height is 8 inches and its base is an equilateral triangle, each side being 2 inches, supposing a ball of iron 4 inches in diameter to weigh 9 lbs.

31. The radius of the base of a cone is 4 inches: find the height, so that the volume may be equal to that of a sphere with diameter 4 inches.

32. The height of a cone is 12 inches: find the radius of its base, so that the volume may be equivalent to that of a sphere with diameter 6 inches.

33. The circumference of the base of a cone is 32 feet: find the height so that the volume may be equivalent to that of a sphere with diameter 10 feet.

34. A solid is composed of a hemisphere and a cone on opposite sides of the same circular base; the diameter of this base is 5 feet, and the height of the cone is 5 feet: find the volume of the solid.

35. Find how many times larger the Earth is than the Moon, taking the diameter of the Earth as 7900 miles, and that of the Moon as 2160 miles.

The following examples involve the extraction of the cube root:

36. Find the length of a cube which shall be equivalent in volume to a sphere 20 inches in diameter.

37. Find the diameter of a sphere which shall be equivalent in volume to a cube 20 inches in length.

38. Find the diameter of a sphere which shall be equivalent in volume to a cylinder, the radius of the base of which is 8 inches and the height 12 inches.

39. If 30 cubic inches of gunpowder weigh one lb., find the internal diameter of a hollow sphere which will hold 15 lbs.

40. If a leaden ball of one inch in diameter weigh $\frac{3}{14}$ lb., find the diameter of a leaden ball which weighs 588 lbs.

41. If a cubic inch of metal weigh 6·57 ounces, find the diameter of a sphere of the metal which weighs 220·16 ounces.

42. A Stilton cheese is in the form of a cylinder, and a Dutch cheese in the form of a sphere. Determine the diameter of a Dutch cheese which weighs 9 lbs., when a Stilton cheese, 14 inches high and 8 inches in diameter, weighs 12 lbs.

XXX. ZONE AND SEGMENT OF A SPHERE.

298. *To find the volume of a zone of a sphere.*

RULE. *To three times the sum of the squares of the radii of the two ends, add the square of the height; multiply the sum by the height, and the product by* ·5236 : *the result will be the volume.*

299. Examples.

(1) The radii of the ends are 8 inches and 11 inches, and the height is 2 inches.

$$64 + 121 = 185; \quad 3 \times 185 = 555; \quad 555 + 4 = 559;$$
$$559 \times 2 \times \cdot 5236 = 585 \cdot 3848.$$

Thus the volume is about 585 cubic inches.

(2) The radius of each end is 20 inches, and the height is 9 inches.

$$400 + 400 = 800; \quad 3 \times 800 = 2400; \quad 2400 + 81 = 2481;$$
$$2481 \times 9 \times \cdot 5236 = 11691 \cdot 4644.$$

Thus the volume is nearly 11691·5 cubic inches.

300. The Rule given in Art. 298 will serve to find the volume of a *segment* of a sphere, if we remember that the radius of one end of a segment is nothing; but it may be convenient to state the Rule for this case explicitly.

301. *To find the volume of a segment of a sphere.*

RULE. *To three times the square of the radius of the base add the square of the height; multiply the sum by the height, and the product by* ·5236 : *the result will be the volume.*

302. Examples.

(1) The radius of the base is 5 inches, and the height 3 inches.

$3 \times 25 = 75;\ 75 + 9 = 84;\ 84 \times 3 \times {}^{\cdot}5236 = 131{}^{\cdot}9472.$

Thus the volume is nearly 132 cubic inches.

(2) The diameter of the base of a segment is $3\frac{1}{2}$ feet, and the height is 9 inches.

$$9 \text{ inches} = \frac{3}{4} \text{ of a foot; the square of } \frac{3}{4} = \frac{9}{16};$$

$$\text{the radius of the base} = \frac{7}{4} \text{ feet; the square of } \frac{7}{4} = \frac{49}{16};$$

$$3 \times \frac{49}{16} = \frac{147}{16}; \qquad \frac{9}{16} + \frac{147}{16} = \frac{156}{16} = \frac{39}{4};$$

$\frac{3}{4} \times \frac{39}{4} \times {}^{\cdot}5236 = 3{}^{\cdot}828825.$ Thus the volume is nearly 3·83 cubic feet.

303. We will now solve some exercises.

(1) The height of a segment of a sphere is 3 inches, and the diameter of the sphere is 14 inches: find the volume of the segment.

We must first determine the square of the radius of the base of the segment. Using the diagram of Art. 78 we have $ED = 3$ inches, and $EF = 14$ inches. By Art. 89 we shall find that the square of $AD = 33$.

$33 \times 3 = 99;\ 99 + 9 = 108;\ 108 \times 3 \times {}^{\cdot}5236 = 169{}^{\cdot}6464.$

Thus the volume is about 170 cubic inches.

(2) The radius of the base of a segment is 24 inches, and the radius of the sphere is 25 inches: find the volume.

We must first determine the height of the segment. Using the diagram of Art. 78, we have $AD = 24$ inches, and $AC = 25$ inches. By Art. 60 we shall find that $CD = 7$ inches. Therefore $DE = 18$ inches. The square of $24 = 576$; $576 \times 3 = 1728$; the square of $18 = 324$; $1728 + 324 = 2052$.

$$2052 \times 18 \times {}^{\cdot}5236 = 19339{}^{\cdot}6896.$$

Thus the volume is nearly 19340 cubic inches.

EXAMPLES. XXX.

1. The radii of the ends of a zone of a sphere are 7 inches and 8 inches; and the height is 3 inches: find the volume.

2. The radii of the ends of a zone of a sphere are 8 inches and 12 inches; and the height is 6 inches: find the volume.

3. The height of a segment of a sphere is 6 feet and the diameter of the base is 8 feet: find the volume.

4. The height of a segment of a sphere is 2 feet 8 inches and the diameter of the base is 8 feet: find the volume.

5. The height of a segment of a sphere is 4 feet, and the diameter of the sphere is 12 feet: find the volume.

6. The height of a segment of a sphere is 5 feet and the diameter of the sphere is 15 feet: find the volume.

7. The radius of the base of a segment of a sphere is 12 feet and the radius of the sphere is 13 feet: find the volume of the segment.

8. The radius of the base of a segment of a sphere is 8 feet and the radius of the sphere is 17 feet: find the volume of the segment.

9. The diameter of a sphere is 20 feet: find the volumes of the two segments into which the sphere is divided by a plane, the perpendicular distance of which from the centre is 5 feet.

10. The diameter of a sphere is 18 feet: the sphere is divided into two segments, one of which is twice as high as the other: find the volume of each.

11. The radius of the base of a segment of a sphere is 1 inch, and the radius of the sphere is $2\frac{1}{4}$ inches: find the volume of the segment.

12. Find the weight of an iron dumb-bell, consisting of two spheres of $4\frac{1}{2}$ inches diameter, joined by a cylindrical bar, 6 inches long and 2 inches in diameter; an iron ball 4 inches in diameter weighing 9 lbs.

13. The diameter of a sphere is 9 feet; the sphere is divided into three parts of equal height by two parallel planes: find the volume of each part.

14. A sphere, 16 inches in diameter, is divided into four parts of equal height by three parallel planes: find the volume of each part.

15. Find the volume of a zone of a sphere, supposing the ends to be on the same side of the centre of the sphere, and distant respectively 10 inches and 15 inches from the centre; and the radius of the sphere to be 20 inches.

16. Find the volume of a zone of a sphere, supposing the ends to be on opposite sides of the centre of the sphere, and distant respectively 10 inches and 15 inches from the centre; and the radius of the sphere to be 20 inches.

17. A bowl is in the shape of a segment of a sphere; the depth of the bowl is 9 inches, and the diameter of the top of the bowl is 3 feet: find to the nearest pint the quantity of water the bowl will hold.

18. Verify by calculating various cases the following statement: if the height of a segment of a sphere is three-fourths of the radius of the sphere the volume of the segment is three-fourths of the volume of a sphere which has its radius equal to the height of the segment.

XXXI. IRREGULAR SOLIDS.

304. We will now explain methods by which we may, in some cases, determine the volumes of solids that are not included in any of the Rules which have been given.

305. Suppose the solid is one which will sink in water, but will not be injured by water.

Put the solid inside a vessel of convenient shape, such as a rectangular parallelepiped or a cylinder. Pour water into the vessel until the solid is quite covered; and note the level at which the water stands. Remove the solid and note the level at which the water then stands. The volume of the solid is of course equal to the volume of the water which would be contained in the vessel between the two levels; and this can be easily calculated.

Or we might state the process thus: fill the vessel full of water; put the solid into it gently and measure the volume of the water which runs over.

306. If the solid is composed entirely of the same substance we may estimate its volume by means of its weight thus: weigh the solid, also weigh a cubic inch of the same substance as the solid; divide the weight of the solid by the weight of the cubic inch, and the quotient will be the number of cubic inches in the solid. If instead of ascertaining the weight of a cubic inch of the substance we ascertain the weight of any known volume of the substance, we can determine by a proportion the volume of the proposed solid.

Some examples of the principle that volumes of solids of the same substance are in the same proportion as their weights, have been given at the end of Chap. XXIX.

307. A Rule resembling that given in Chapter XVIII. may be used for finding approximately the volumes of certain solids;

Divide the length of the solid into any even number of equal parts; and ascertain the areas of sections of the solid through the points of division perpendicular to the length of the solid. Add together the first area, the last area, twice the sum of all the other odd areas, and four times the sum of all the even areas; multiply the sum by one-third of the common distance between two adjacent sections.

308. The preceding Rule will in general be more accurate the greater the number of sections that are made; and the rule ought not to be trusted if the solid be very irregular in form. The areas required in the Rule may themselves be conveniently determined approximately, in some cases, by Chapter XVIII. The Rule is employed by Civil Engineers for calculating quantities of earthwork, and by Naval Architects for calculating the volumes of water displaced by ships. To ensure accuracy, Naval Architects often perform the calculation in two ways; namely, from a series of horizontal sections, and from a series of transverse sections.

EXAMPLES. XXXI.

1. The radius of the base of a cylindrical vessel is 10 inches; a block of stone is placed in the vessel and is covered with water; on removing the block the level of the water sinks 6 inches: find the volume of the block.

2. If a cubic foot of marble weighs 2716 ounces, find the volume of a block of marble which weighs 4 tons 8 cwt.

3. A cask full of water weighs 3 cwt.; the cask when empty weighs 40 lbs.: find to the nearest gallon the capacity of the cask.

4. Five equidistant sections of a solid are taken, the common distance being 3 feet; the areas of these sections in square feet are 3·72, 5·28, 6·96, 8·77, and 10·72: find the volume of the solid between the extreme sections.

5. Five equidistant sections of a solid are taken, the common distance being 6 inches; these sections are all circles, and their circumferences are respectively 57 inches, 63 inches, 69 inches, 76 inches, 83 inches: find the volume of the solid between the extreme sections.

XXXII. SIMILAR SOLIDS.

309. Similar solids are such as are alike in form though they may differ in size.

In common language the fact that one solid is similar to another is often expressed by saying that one is a *model* of the other.

310. All cubes are similar solids. All spheres are similar solids.

311. It is easy in various cases to give tests for determining whether two solids, which are called by the same name, are similar. For example :

If the three edges of one rectangular parallelepiped which meet at a point are respectively double, or treble, or any number of times, the three edges of another which meet at a point, the two rectangular parallelepipeds are similar.

If the height and the diameter of the base of one right circular cone are respectively double, or treble, or any number of times, the height and the diameter of the base of another, the two right circular cones are similar. The same test will serve for two right circular cylinders.

We may express these statements thus : Two rectangular parallelepipeds are similar when their edges are proportionals. Two right circular cones, or two right circular cylinders, are similar when their heights and the diameters of their bases are proportionals.

312. The following most important proposition holds with respect to similar solids :

The volumes of similar solids are as the cubes of corresponding lengths :

For example, suppose the diameter of one sphere to be 5 inches, and the diameter of another sphere to be 4 inches; the volume of the first sphere is to the volume of the second as the cube of 5 is to the cube of 4, that is, as 125 is to 64: so that the larger sphere is almost double the other.

Persons who have not had their attention drawn to such a fact, often find a difficulty in realizing the rapid rate at which the volumes of similar solids increase, as some assigned dimension of them is increased.

313. We will now solve some exercises.

(1) The edge of a cube is 1 foot: find the number of feet in the edge of another cube of double the volume.

The cube of the required number is to the cube of 1 as 2 is to 1; so that the required number is the cube root of 2: this will be found to be 1·2599210. Thus we see that a cube with its edge 1·26 feet is rather more than double a cube with its edge 1 foot.

(2) The height of a pyramid is 12 feet: it is required to cut off a frustum which shall be a fourth of the pyramid.

Since the frustum is to be a fourth of the pyramid, the remaining pyramid will be three-fourths of the original pyramid; and these two pyramids are similar. Therefore the cube of the height of the remaining pyramid must be three-fourths of the cube of the height of the original pyramid, that is, $\frac{3}{4}$ of 1728, that is, 1296; thus the height in feet of the remaining pyramid is the cube root of 1296: it will be found that this is 10·9027.

Hence the height of the frustum in feet is 12 − 10·9027, that is, 1·0973.

(3) The diameters of the ends of a frustum of a cone are 6 feet and 10 feet, and the height of the frustum is 3 feet: it is required to divide the frustum into two equal parts by a plane parallel to the base.

In the diagram of Art. 210, let AB and CD be the diameters of the ends. As in that Article we find $OK = 4{\cdot}5$, and $OM = 7{\cdot}5$. Let OL denote the perpendicular distance of the required plane from O. Thus we shall find that the cube of OL must be equal to half the sum of the cubes of OK and OM. The cube of $4{\cdot}5 = 91{\cdot}125$; the cube of $7{\cdot}5 = 421{\cdot}875$. Thus the cube of $OL = \frac{1}{2}$ of $513 = 256{\cdot}5$; and therefore the number of feet in OL = the cube root of $256{\cdot}5$; it will be found that this is $6{\cdot}3537$.

Hence the distance of the required plane from the smaller end of the frustum in feet $= 6{\cdot}3537 - 4{\cdot}5 = 1{\cdot}8537$.

(4) A frustum of a circular cone is trimmed just enough to reduce it to a frustum of a pyramid with square ends: find how much of the volume is removed.

This Exercise may be conveniently placed here although not strictly connected with the subject of similar solids.

Suppose that the radius of one end is 2 feet, and the radius of the other end 3 feet; and that the height is 12 feet. By Art. 268 the volume of the frustum of the cone in cubic feet $= \frac{1}{3} \times 19 \times 3{\cdot}1416 \times 12 = 238{\cdot}7616$. When the frustum of the cone is trimmed the ends become squares, the diagonals of which are 4 feet and 6 feet respectively: by Art. 268 the volume of the frustum of a pyramid in cubic feet $= \frac{1}{3} \times 38 \times 12 = 152$. The volume removed is therefore $86{\cdot}7616$ cubic feet; and the fractional part of the original volume is $\frac{86{\cdot}7616}{238{\cdot}7616}$, that is $\cdot 36 \ldots$: thus rather more than $\frac{9}{25}$ of the original volume is removed.

Now on examining this process it will be immediately seen that the result will be the same whatever be the *height* of the frustum; and by trial it will be found that the result will also be the same whatever be the *radii of the ends* of the frustum of a cone.

Thus the result is true for any frustum of a right circular cone.

EXAMPLES. XXXII.

1. If a cannon ball, $3\frac{1}{2}$ inches in diameter, weigh 6 lbs., find the weight of a ball of the same metal $5\frac{1}{4}$ inches in diameter.

2. If the model of a steam engine weigh 80 lbs., find the weight of the engine itself, supposing it made of the same substance as the model and of nine times its lineal dimensions.

3. The heights of two similar right circular cylinders are 7 inches and 10 inches respectively: shew that a similar cylinder 11·03 inches high is less than the sum of the two, and a similar cylinder 11·04 inches is greater than the sum of the two.

4. The height of a pyramid is 16 inches, and its volume is 400 cubic inches: the pyramid is divided into two parts by a plane parallel to the base and distant 4 inches from it: find the volumes of the parts.

The next four examples involve the extraction of the cube root:

5. If a cannon ball, $3\frac{1}{2}$ inches in diameter, weigh 6 lbs., find the diameter of a ball of the same metal which weighs 20 lbs.

6. The height of a right circular cylinder is 4 feet: find the height of a similar cylinder of nine times the volume.

7. The diameters of the ends of the frustum of a cone are respectively 20 feet and 16 feet, and the height of the frustum is 5 feet; the frustum is divided into two equal parts by a plane parallel to the ends: find the distance of the plane from the smaller end.

8. If the frustum in the preceding Example is divided into three equal parts by planes parallel to the ends, find the distances of the planes from the smaller end.

9. A pyramid on a regular hexagonal base is trimmed just enough to reduce it to a cone: shew that rather less than $\frac{1}{10}$ of the original volume is removed.

10. A frustum of a pyramid on a square base is trimmed just enough to reduce it to a frustum of a cone: shew that rather more than $\frac{1}{5}$ of the original volumeis removed.

11. Every edge of a pyramid on a square base is 1 foot: shew that the volume of the pyramid is $\frac{\sqrt{2}}{6}$ of a cubic foot; and that the volume of any pyramid on a square base which has all its edges equal may be obtained by mutiplying the cube of an edge by $\frac{\sqrt{2}}{6}$.

12. Every edge of a pyramid on a triangular base is 1 foot: shew that the volume of the pyramidis $\frac{\sqrt{2}}{12}$ of a cubic foot; and that the volume of any pyramid on a triangular base which has all its edges equal maybe obtained by multiplying the cube of an edge by $\frac{\sqrt{2}}{12}$.

FIFTH SECTION. AREAS OF THE SURFACES OF SOLIDS.

XXXIII. PLANE SURFACES.

314. We now proceed to the measurement of the surfaces of solids; this subject is properly connected with the second Section of our work, but we have thought it more convenient for beginners to treat of the *volumes* of solids before treating of the *areas of the surfaces* of solids.

315. The area of any *plane* surface of a solid must be found by the Rules given in the third Section of the work. We will mention the various cases that can arise.

The faces of a rectangular parallelepiped are all rectangles; the faces of any other parallelepiped are parallelograms, two or four of which may be rectangles. The ends of a prism are triangles or other rectilineal figures; the other faces are rectangles or parallelograms according as the prism is right or oblique. The base of a pyramid is a triangle or other rectilineal figure, and the other faces are triangles. The ends of a prismoid, or of a frustum of a pyramid, are triangles or other rectilineal figures, and the other faces are trapezoids. Two of the faces of a wedge are triangles; each of the other three faces is a trapezoid, or a parallelogram, or a rectangle. In all these cases the surfaces are plane rectilineal figures, and their areas can be found by Rules already given.

A Rule has also been given for finding the area of a circle; and the following cases will occur for the application of the Rule: the ends of a circular cylinder, the base of a circular cone, the ends of a frustum of a circular cone, the base of a segment of a sphere, and the ends of a zone of a sphere.

In fact any Rule given in the third Section of the book might find an application in the present Section. Thus, for example, the Rule for finding the area of a segment of a circle will enable us to find the areas of the ends of the segments of cylinders which are considered in Art. 255.

316. Examples:

(1) Find the area of the whole surface of a cube which is 8 inches long.

Each face of the cube is a square containing 64 square inches; and there are six faces: thus the area of the whole surface in square inches is 6×64, that is 384.

(2) A pyramid stands on a square base which is 10 inches long, and each of the four faces which meet at the vertex is an equilateral triangle: find the area of the whole surface of the pyramid.

The area of the base is 100 square inches. The area of each of the triangular faces is about 43·3 square inches, by Art. 206; therefore the area of the four triangular faces is about 173·2 square inches. Thus the area of the whole surface of the pyramid is about 273·2 square inches.

(3) A vessel is to be made in the form of a rectangular parallelepiped without a lid; externally the length is 4 feet, the breadth 3 feet, and the height is 2 feet: find the area of the whole external surface.

The surface consists of two rectangles each measuring 4 feet by 2 feet, two rectangles each measuring 3 feet by 2 feet, and one rectangle measuring 4 feet by 3 feet: the total area is 40 square feet.

Suppose that the vessel is to be formed of metal half an inch thick; and that we have to find how much metal is required. The answer for practical purposes is this: we shall require 40 square feet of the assigned thickness.

And in this manner any such question is usually solved when the thickness of the metal is small compared with the dimensions of the vessel.

The *exact* method of solution has been given in Art. 245, from which it appears that the exact result is $2814\frac{1}{2}$ cubic inches. Now a sheet of metal of 40 square feet in area and half an inch thick will be found to contain 2880 cubic inches. Thus we see that the approximate result is slightly in excess of the exact result. The thinner the material of which the vessel is composed, the smaller will be the difference between the approximate result and the true result.

(4) A vessel is to be made in the form of a rectangular parallelepiped on a square base without a lid, to hold a cubic foot; the height is to be half the length: find the area of the whole internal surface.

A vessel of the same base but of twice the height would be in the shape of a cube, and would hold 2 cubic feet, that is 3456 cubic inches. Hence the length of a side of the base in inches will be the cube root of 3456: thus we shall obtain for a side of the base 15·119... inches. Since the height is half the length, the area of the base is double that of any of the other four faces; and therefore the area of the whole internal surface is three times that of the base; so that in square inches it is three times the square of 15·119...: it will be found that this is 685·75...

Hence, as in the preceding Example, if the vessel is to be made of metal of an assigned small thickness we shall require about 686 square inches of metal of that thickness.

317. The principle illustrated in the last two Examples may be thus stated: in order to construct a vessel of material having an assigned small thickness we require a sheet of the material equivalent to the external surface of the vessel. Admitting this principle we can give an interesting practical form to some results of calculation. Thus, for instance, by comparing the results of Examples 37...41 at the end of the present Chapter,

and of similar Examples, we can see that the following theorem is true: a vessel of *given capacity* is to be made in the form of a rectangular parallelepiped on a square base; if there is to be no lid the internal surface will be least when the height is half the length. Thus, with the view of saving material, the most advantageous shape is that in which the height is half the length.

In like manner from the results of Examples 42...46, and of similar Examples, we learn that if there is to be a lid the cube is the most advantageous shape.

And suppose that we have to make a vessel in the form of a rectangular parallelepiped on a square base, out of a *given quantity of material:* then if there is to be no lid the capacity will be greatest when the height is half the length; and if there is to be a lid the capacity will be greatest when the vessel is a cube.

EXAMPLES. XXXIII.

Find the area of the whole surface of the cubes which have the following lengths:

1. 2 feet 6 inches.
2. 3 feet 8 inches.
3. 5 feet 10 inches.
4. 6 feet 7 inches.

Find the area of the whole surfaces of rectangular parallelepipeds which have the following dimensions:

5. 2 feet 6 inches, 3 feet, 5 feet.
6. 2 feet 4 inches, 3 feet 6 inches, 4 feet.
7. 2 feet 8 inches, 3 feet 2 inches, 4 feet 10 inches.
8. 2 feet 11 inches, 3 feet 7 inches, 5 feet 2 inches.

Find the area of the whole surface of right triangular prisms having the following dimensions:

9. Sides of the base, 3, 4, and 5 feet; height 8 feet.
10. Sides of the base, 8, 15, and 17 feet; height 10 feet.

11. Sides of the base 1 foot 4 inches, 2 feet 1 inch, 3 feet 3 inches; height 7 feet 6 inches.

12. Sides of the base 2 feet 1 inch, 2 feet 9 inches, 4 feet 4 inches; height 8 feet.

13. Find the area of the whole surface of a pyramid on a square base; each side of the base is 2 feet 7 inches, and the length of the straight line drawn from the vertex to the middle point of any side of the base is 3 feet 5 inches.

14. Find the area of the whole surface of a pyramid on a square base; each side of the base is 3 feet 4 inches, and the length of the straight line drawn from the vertex to the middle point of any side of the base is 5 feet 8 inches.

15. Find the area of the whole surface of a pyramid on a square base; each side of the base is 3 feet 4 inches, and each of the other edges is 8 feet 5 inches.

16. Find the area of the whole surface of a pyramid on a square base; each side of the base is 28 feet, and each of the other edges is 16 feet 1 inch.

17. Find the area of the whole surface of a pyramid on a square base, having its other faces equal; each side of the base is 17 feet 6 inches, and the height of the pyramid is 17 feet 4 inches.

18. Find the area of the whole surface of a pyramid on a square base, having its other faces equal; each side of the base is 29 feet 2 inches, and the height of the pyramid is 24 feet.

19. Find the area of the whole surface of a frustum of a pyramid; the ends are squares, the sides of which are 2 feet and 3 feet respectively, and the distance between the parallel sides of each trapezoidal face is 6 inches.

20. Find the area of the whole surface of a frustum of a pyramid; the ends are squares, the sides of which are 2 feet 3 inches and 4 feet 9 inches respectively, and the distance between the parallel sides of each trapezoidal face is 18 inches.

21. Find the area of the whole surface of a frustum of a pyramid; the ends are squares, the sides of which are 3 feet 4 inches and 3 feet 10 inches respectively; and each of the remaining edges is 5 inches.

22. Find the area of the whole surface of a frustum of a pyramid; the ends are squares, the sides of which are 3 feet 2 inches and 4 feet respectively; and each of the remaining edges is 13 inches.

23. Find the area of the whole surface of a prismoid; the ends are rectangles; one measures 7 feet by 6 feet, and the corresponding dimensions of the other are 4 feet 6 inches and 4 feet 10 inches; each of the remaining edges is 25 inches.

24. The four faces of a triangular pyramid are equilateral triangles, the edge of each being 10 feet: find the area of the whole surface.

25. Find the area of the whole surface of a pyramid on a rectangular base which measures 4 feet 6 inches by 6 feet 8 inches, each of the remaining edges being 6 feet.

26. The area of the whole surface of a cube is 7 square feet 6 square inches: find the volume.

27. The dimensions of a rectangular parallelepiped are 3, 7, and 9 feet: find the edge of the cube of equivalent surface.

28. The edge of a wedge is 12 inches, the length of the base is 10 inches, and its breadth 2 inches; each of the other sides of the trapezoidal faces is 25 inches: find the area of the whole surface.

29. The edge of a wedge is 14 inches, the length of the base is 6 inches, and its breadth 2 inches; each of the other sides of the trapezoidal faces is 9 inches: find the area of the whole surface.

30. Find the area of the whole surface of a right prism, the ends of which are regular hexagons, each edge of the solid being 2 feet.

31. The base of a pyramid is a regular octagon, each side being 4 feet; each of the other edges of the pyramid is 12 feet 1 inch: find the area of the surface excluding the base.

32. The dimensions of a rectangular parallelepiped are 18 inches, 16 inches and 6 inches: find the area of the whole surface of a cube of equivalent volume.

33. The length, breadth, and height of a rectangular parallelepiped are respectively 8, 18 and 21 inches: find its surface. Also find the surface of a rectangular parallelepiped of the same height and volume on a square base.

If two rectangular parallelepipeds have the same height and volume, and one of them have a square base, the whole surface of this will be less than the whole surface of the other: verify this statement by comparing the surfaces of the following rectangular parallelepipeds with the surface of others having respectively the same volume, and height, and square bases:

34. Base 3 feet by 4 feet; height 5 feet.

35. Base 3 feet by 7 feet; height 9 feet.

36. Base 8 feet by 15 feet; height 19 feet.

The following examples involve the extraction of the cube root:

A vessel *without* a lid in the form of a rectangular parallelepiped on a square base is to be made to hold 1000 cubic inches: find in square inches the area of the whole external surface in the following cases:

37. The height equal to the length.

38. The height equal to twice the length.

39. The height equal to half the length.

40. The height equal to three times the length.

41. The height equal to one-third of the length.

A vessel *with* a lid in the form of a rectangular parallelepiped on a square base is to be made to hold 1000 cubic inches: find in square inches the area of the whole internal surface in the following cases:

42. The height equal to the length.

43. The height equal to twice the length.

44. The height equal to half the length.

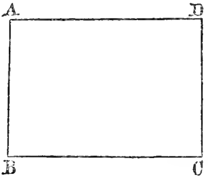
45. The height equal to three times the length.

46. The height equal to one-third of the length.

XXXIV. RIGHT CIRCULAR CYLINDER.

318. The surface of a right circular cylinder consists of two circular ends, and another portion which we shall call the *curved surface.*

319. Let *ABCD* be a rectangle. Cut it out of paper or cardboard. Then let it be bent until the edge *AB* just comes into contact with the edge *DC*. It is easy to see that by proper adjustment, we can thus obtain a thin cylindrical shell; *AB* becomes the height of the shell, and *BC* the circumference of the base. Hence it will follow that the curved surface of a right circular cylinder is equal to a rectangle one dimension of which is the height of the cylinder, and the other dimension the circumference of the base of the cylinder: thus we obtain the Rule which we shall now give.



320. *To find the area of the curved surface of a right circular cylinder.*

Rule. *Multiply the circumference of the base by the height of the cylinder.*

321. Examples.

(1) The radius of the base of a right circular cylinder is 3 feet, and the height is $2\frac{1}{2}$ feet: find the area of the curved surface.

The circumference of the base in feet $= 2 \times 3 \times 3{\cdot}1416$ $= 18{\cdot}8496$; $\frac{5}{2} \times 18{\cdot}8496 = 47{\cdot}124$.

Thus the area of the curved surface is about $47{\cdot}124$ square feet.

(2) The diameter of the base of a right circular cylinder is 16 inches, and the height is 25 inches: find the area of the whole surface.

The circumference of the base in inches $= 16 \times 3{\cdot}1416 = 50{\cdot}2656$; $25 \times 50{\cdot}2656 = 1256{\cdot}64$.

Thus the area of the curved surface is about 1256·64 square inches.

The area of the two circular ends in square inches $= 2 \times 64 \times 3{\cdot}1416 = 402{\cdot}1248$,

$$1256{\cdot}64 + 402{\cdot}1248 = 1658{\cdot}7648.$$

Therefore the area of the whole surface is about 1658·7648 square inches.

322. The following inferences may be easily drawn from the Rule of Art. 320:

If the height of the right circular cylinder be equal to the radius of the base, the area of the curved surface is equal to that of the two ends of the cylinder; if the height be twice the radius, the area of the curved surface is twice that of the ends; if the height be three times the radius, the area of the curved surface is three times that of the ends: and so on.

If the height be half the radius the area of the curved surface is half that of the ends; if the height be a third of the radius the area of the curved surface is a third of that of the ends: and so on.

We may sum up these inferences thus: *the height of a right circular cylinder bears the same proportion to the radius of the base as the area of the curved surface bears to the area of the two ends.*

323. By the process of Art. 319 we can easily form other right cylindrical shells the bases of which are not circles, but various oval curves: and thus we see that the area of the curved surface of *any right cylinder* may be found by multiplying the perimeter of the base by the height of the cylinder.

324. We will now solve some exercises.

(1) The area of the whole surface of a right circular cylinder is 20 square feet, and the height of the cylinder is equal to half the radius of the base: find the radius of the base.

By Art. 322 the area of the curved surface is equal to that of one of the ends; and thus three times the area of an end is 20 square feet: therefore the area of one end $= 6\frac{2}{3}$ square feet $= 960$ square inches. We must then find the radius of the base by Art. 171. Divide 960 by 3·1416; the quotient is 305·58; the square root of this is 17·48... Thus the radius of the base is 17·5 inches nearly.

(2) The area of the curved surface of a right circular cylinder is 30 square feet, and the volume is 120 cubic feet: find the radius of the base, and the height of the cylinder.

The product of the height into the area of the base is 120; the product of the height into the circumference of the base is 30: hence, by division, the area of the base divided by the circumference of the base $= 120 \div 30 = 4$. But, by Art. 176, the area of the base divided by the circumference of the base is equal to half the radius: thus half the radius is 4 feet, and therefore the radius is 8 feet.

Hence the circumference of the base in feet $= 16 \times 3{\cdot}1416 = 50{\cdot}2656$; and therefore the height of the cylinder in feet $= 30 \div 50{\cdot}2656 = {\cdot}5967...$

(3) A vessel is to be made in the form of a right circular cylinder, without a lid, to hold a cubic foot; the height is to be equal to the radius of the base: find the area of the whole internal surface.

The volume is to be 1728 cubic inches. Proceeding as in Art. 253, we see that the height in inches will be the cube root of $\frac{1728}{3{\cdot}1416}$: thus we shall obtain for the height 8·1934 inches. The area of the whole internal surface is three times the area of one end; so that in square inches it is three times the product of 3·1416 into the square of 8·1934; it will be found that this is 632·70...

From this result and the last of those obtained in Art. 316 we see that less material is required for a vessel of the shape here considered than for a vessel of the shape there considered.

325. By comparing the results of Examples 21...30 at the end of the present Chapter, and of similar examples, we learn that the most advantageous shape for a right circular cylindrical vessel, if there is to be no lid, is that in which the height is equal to the radius of the base; and if there is to be a lid, that in which the height is twice the radius of the base. For by adopting these shapes we use the least quantity of material to secure a given capacity, and we secure the greatest capacity with a given quantity of material.

EXAMPLES. XXXIV.

Find the area of the curved surface of right circular cylinders having the following dimensions:

1. Height 2 feet 2 inches, circumference of base 4 feet.

2. Height 2 feet 5 inches, circumference of base 4 feet 9 inches.

3. Height 1 foot 10 inches, radius of base 1 foot 5 inches.

4. Height 2 feet 6 inches, radius of base 2 feet 4 inches.

5. Height 40 feet, radius of base 8 feet.

Find the area of the whole surface of right circular cylinders having the following dimensions:

6. Height 3 feet, radius of base 2 feet.

7. Height 5 feet, radius of base 3 feet 6 inches.

8. Height 1 foot 2 inches, radius of base 8 inches.

9. Height 5 feet 6 inches, circumference of base 20 feet.

10. Height 6 feet 3 inches, circumference of base 24 feet.

11. The area of the curved surface of a right circular cylinder is 6 square feet, and the circumference of the base is 3 feet 9 inches: find the height.

12. The area of the curved surface of a right circular cylinder is $5\frac{5}{8}$ square feet, and the radius of the base is $2\frac{1}{4}$ feet: find the height.

13. The area of the whole surface of a right circular cylinder is 14 square feet, and the height of the cylinder is equal to the radius of the base: find the radius of the base.

14. The area of the whole surface of a right circular cylinder is 24 square feet, and the height of the cylinder is twice the radius of the base: find the radius of the base.

15. The area of the whole surface of a right circular cylinder is 30 square feet, and the height of the cylinder is half the radius of the base: find the radius of the base.

16. The area of the curved surface of a right circular cylinder is $2\frac{1}{2}$ square feet, and the volume of the cylinder is $3\frac{1}{4}$ cubic feet: find the radius of the base.

17. The area of the curved surface of a right circular cylinder is 4 square feet, and the volume of the cylinder is 5 cubic feet: find the area of one end.

18. The area of the curved surface of a right circular cylinder is 3 square feet, and the volume is $2\frac{1}{4}$ cubic feet: find the height.

19. The area of the base of a right circular cylinder is 314·16 square inches; the volume is 3141·6 cubic inches: find the area of the curved surface.

20. The area of the base of a right circular cylinder is 1000 square inches; the volume is 5 cubic feet: find the area of the curved surface.

The following examples involve the extraction of the cube root:

A vessel *without* a lid in the form of a right circular cylinder is to hold 31416 cubic inches; find in square inches the area of the whole internal surface in the following cases:

21. The height equal to the radius of the base.

22. The height double the radius of the base.

23. The height half the radius of the base.

24. The height three times the radius of the base.

25. The height one-third of the radius of the base.

A vessel *with* a lid in the form of a right circular cylinder is to hold 31416 cubic inches: find in square inches the area of the whole internal surface in the following cases:

26. The height equal to the radius of the base.

27. The height double the radius of the base.

28. The height half the radius of the base.

29. The height three times the radius of the base.

30. The height one-third of the radius of the base.

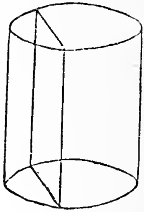
31. The edge of a cube is 10 inches; a right circular cylinder of the same volume as the cube has its height equal to the radius of its base: find the area of the whole surface of the cube, and also of the cylinder.

32. A vessel in the form of a right circular cylinder without a lid is to contain 1000 gallons: find the area of the whole internal surface supposing the vessel to be of the most advantageous shape.

XXXV. SEGMENTS OF A RIGHT CIRCULAR CYLINDER. RING.

326. Simple rules can be given for finding the areas of the curved surfaces of certain segments of a right circular cylinder; as we will now shew.

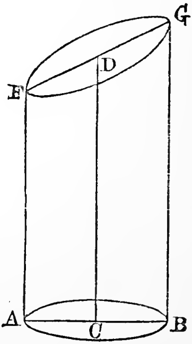
327. Suppose a right circular cylinder cut into two parts by a plane parallel to the axis; the surface of each part consists of two segments of a circle, a rectangle, and another portion which we shall call the *curved surface.*



The area of each of the segments of a circle can be found by Arts. 185 and 186. The area of the rectangle can be found by Art. 134. The area of the curved surface can be found by the following Rule: *multiply the length of the arc of the base by the height of the cylinder.*

Or we may find the area of the rectangle and the curved surface together by the Rule of Art. 323, *multiply the perimeter of the base by the height of the cylinder.*

328. Suppose a solid has been obtained by cutting a right circular cylinder by a plane, inclined to the axis, which does not meet the base of the cylinder. The surface of this solid consists of the base which is a circle, the other end which is also a plane curve, and another portion which we shall call the *curved surface.*



The area of the base can be found by Art. 168. No Rule has been given in this work for finding the area of the other end *exactly;* but the area might be found approximately by Art. 193:

we may remark that this plane curve is called an *ellipse*, and is of great importance in mathematical investigations.

The area of the *curved surface* can be found by the following Rule: *multiply the circumference of the base by the height of the solid.*

329. The *height of the solid* in the preceding Rule is to be understood in the same sense as in Art. 256. The truth of the Rule may be shewn in the manner of Art. 257.

330. Suppose a solid has been obtained by cutting a right circular cylinder by *two* planes inclined to the axis, which do not meet each other. The area of the *curved surface* will be found by *multiplying the circumference of the base of the cylinder by the height of the solid.* The *height of the solid* is to be understood as in Art. 259. The Rule follows from the fact that the solid may be supposed to be the difference of two solids of the kind considered in Art. 256 or Art. 328.

331. *To find the area of the surface of a solid ring.*

RULE. *Multiply the circumference of a circular section of the ring by the length of the ring.*

The *length* of the ring is to be understood as in Art. 261. The Rule may be illustrated in the manner of Art. 260.

332. Examples:

(1) The radius of the circular section of a ring is one inch, and the length of the ring is ten inches.

The circumference of the circular section of the ring is $2 \times 3{\cdot}1416$ inches; therefore the area of the surface of the ring in square inches is $10 \times 2 \times 3{\cdot}1416$, that is $62{\cdot}832$. Thus the area of the surface is 63 square inches nearly.

(2) The inner diameter of a ring is 7 inches, and the outer diameter is 8 inches.

As in Art. 262 we find that the radius of the circular section is $\frac{1}{4}$ of an inch, and the length of the ring is $23{\cdot}562$ inches; therefore the area of the surface of the ring in square inches $= \frac{1}{2} \times 3{\cdot}1416 \times 23{\cdot}562 = 37{\cdot}01\ldots$

EXAMPLES. XXXV.

1. Find the area of the curved surface of the smaller of the two pieces in the diagram of Art. 327, supposing the height of the solid to be 4 feet, the radius of the circle 15 inches, and the chord of the circle equal to the radius.

2. Find the area of the curved surface of the smaller of the two pieces in the diagram of Art. 327, supposing the height of the solid to be 4 feet 2 inches, the radius of the circle to be 8 inches, and the chord to subtend a right angle at the centre of the circle.

3. The radius of the base of a cylinder is 16 inches; a piece is cut off by two planes inclined to the axis of the cylinder, which do not meet each other; the length of the portion of the axis between the two planes is 35 inches: find the area of the curved surface.

Find in square inches the areas of the surfaces of rings having the following dimensions:

4. Length 20 inches; circumference of cross section 4 inches.

5. Length 25 inches; radius of cross section $\frac{3}{4}$ of an inch.

6. Outer diameter 4·7 inches; inner diameter 4·1 inches.

7. Inner diameter 11 inches; diameter of cross section 2 inches.

8. Outer diameter 26 inches; diameter of cross section 4 inches.

9. Outer diameter 25 inches; circumference of cross section 10 inches.

10. Inner diameter 20 inches; circumference of cross section 12 inches.

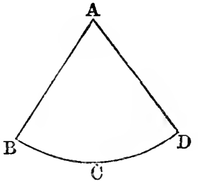
11. The area of the surface of a ring is 100 square inches; the radius of the cross section is 1 inch: find the length of the ring.

12. The area of the surface of a ring is 120 square inches; the length is 20 inches: find the inner diameter.

XXXVI. RIGHT CIRCULAR CONE.

333. The surface of a right circular cone consists of a circular base and another portion which we shall call the *curved* surface.

334. Let $ABCD$ be a sector of a circle. Cut it out of paper or cardboard. Then let it be bent until the edge AB just comes into contact with the edge AD. It is easy to see that by proper adjustment we can thus obtain a thin shell, the outside of which will correspond to the curved surface of a right circular cone: A becomes the vertex of the cone, AB becomes the slant height of the cone, and BCD becomes the circumference of the base of the cone. Hence it will follow that the curved surface of a right circular cone is equal to a sector of a circle, the radius of the sector being the slant height of the cone, and the arc of the sector being the circumference of the base of the cone: thus we obtain the Rule which we shall now give.



335. *To find the area of the curved surface of a right circular cone.*

Rule. *Multiply the circumference of the base by the slant height of the cone, and half the product will be the area of the curved surface.*

336. Examples:

(1) The radius of the base of a right circular cone is 8 inches, and the slant height is 14 inches: find the area of the curved surface.

The circumference of the base in inches $= 2 \times 8 \times 3{\cdot}1416$ $= 50{\cdot}2656$; $\frac{1}{2} \times 14 \times 50{\cdot}2656 = 351{\cdot}8592$.

Thus the area of the curved surface is 352 inches nearly.

(2) The radius of the base of a right circular cone is 4 feet, and the height of the cone is 3 feet: find the area of the whole surface.

We must first find the slant height of the cone; by Art. 55, the slant height in feet is the square root of $16+9$, that is, the square root of 25, that is 5.

$$\frac{1}{2} \times 5 \times 2 \times 4 \times 3{\cdot}1416 = 20 \times 3{\cdot}1416 = 62{\cdot}832.$$

Thus the area of the *curved surface* is 62·832 square feet.

The area of the base in square feet $= 4 \times 4 \times 3{\cdot}1416 = 50{\cdot}2656$. Therefore the area of the whole surface in square feet $= 62{\cdot}832 + 50{\cdot}2656 = 113{\cdot}0976$.

337. The following inferences may be easily drawn from the Rule in Art. 335:

If the slant height of the right circular cone be *twice* the radius of the base the area of the curved surface is *twice* that of the base of the cone; if the slant height be *three times* the radius of the base, the area of the curved surface is *three times* that of the base of the cone; and so on.

We may sum up these inferences thus: *the slant height of a right circular cone bears the same proportion to the radius of the base as the area of the curved surface bears to the area of the base.*

338. We will now solve some exercises.

(1) The area of the whole surface of a right circular cone is 24 square feet, and the slant height is twice the radius of the base: find the radius of the base.

By Art. 337 the area of the curved surface is equal to twice the area of the base; and thus three times the area of the base is 24 square feet: therefore the area of the base is 8 square feet. We must then find the radius of the base by Art. 171. Divide 8 by 3·1416; the quotient is 2·5464...; the square root of this is 1·5958 very nearly. Thus the radius of the base is 1·5958 feet very nearly.

(2) The volume of a right circular cone is 20 cubic feet; the height is twice the radius of the base: find the area of the whole surface.

By Art. 263 we see that the product of the cube of the radius of the base into $\frac{2}{3}$ of 3·1416 is equal to 20; so that the cube of the radius of the base

$$=\frac{60}{2\times 3{\cdot}1416}=\frac{30}{3{\cdot}1416}=9{\cdot}549\ldots;$$

therefore the radius of the base in feet is the cube root of 9·549...: this we shall find to be 2·1215... Therefore the area of the base in square feet is the product of 3·1416 into the square of 2·121: it will be found that this is 14·140...

Now if the radius of the base were 1 foot, and the height of the cone were 2 feet, the slant height would be $\sqrt{5}$ feet, by Art. 55; that is, the slant height would be $\sqrt{5}$ times the radius of the base. And thus in the present case since the height is twice the radius of the base, the slant height is $\sqrt{5}$ times the radius of the base: that is the slant height is 2·236... times the radius of the base.

Hence, by Art. 337, the area of the curved surface in square feet is the product of 2·236... into 14·140... that is 31·617...

Therefore the whole area of the surface in square feet $=14{\cdot}140+31{\cdot}617=45{\cdot}757$.

(3) The volume of a right circular cone is 20 cubic feet; the slant height is three times the radius of the base: find the area of the whole surface.

If the radius of the base were 1 foot, and the slant height were 3 feet, the height would be $\sqrt{8}$ feet by Art. 60; that is, the height would be $\sqrt{8}$ times the radius of the base. And thus in the present case since the slant height is 3 times the radius of the base, the height is $\sqrt{8}$ times the radius of the base.

Then, as in the preceding Exercise, we see that the cube of the radius of the base $= \frac{3 \times 20}{\sqrt{8} \times 3 \cdot 1416}$; it will be found that this is 6·752...; therefore the radius of the base is the cube root of 6·752...: it will be found that this is 1·890...

By Art. 337, the area of the whole surface is 4 times the area of the base; so that in square feet it is 4 times the product of 3·1416 into the square of 1·890...; it will be found that this is 44·888...

The whole surface is less in this case than in the case of the preceding Exercise. In fact, it will be found by comparing the results of Examples 31...40 at the end of the present Chapter, and of similar Examples, that if the whole surface of a right circular cone be given, the volume is greatest when the slant height is three times the radius; and if the volume of a right circular cone be given the whole surface is least when the slant height is three times the radius.

EXAMPLES. XXXVI.

Find in square inches the area of the curved surface of right circular cones having the following dimensions:

1. Slant height 2 feet 3 inches, circumference of base 4 feet 5 inches.

2. Slant height 3 feet 2 inches, circumference of base 5 feet 7 inches.

3. Slant height 2 feet, radius of base 1 foot 9 inches.

4. Slant height 2 feet 8 inches, radius of base 2 feet 10 inches.

5. Slant height 3 feet, radius of base 1 foot 6 inches.

6. Height 2 feet, radius of base 7 inches.

7. Height 3 feet 4 inches, radius of base 9 inches.

8. Height 2 feet 6 inches, radius of base 1 foot 4 inches.

9. Height 5 feet, radius of base 11 inches.

10. Height 4 feet 8 inches, radius of base 2 feet 9 inches.

11. Height 5 feet, perimeter of base 6·2832 feet.

12. Height 12 feet, perimeter of base 10 feet.

Find in square feet the area of the whole surface of right circular cones having the following dimensions:

13. Slant height 4 feet, radius of base 2 feet.

14. Slant height 5·3 feet, radius of base 3·2 feet.

15. Slant height 6 feet, circumference of base 8 feet.

16. Slant height 6·4 feet, circumference of base 9·7 feet.

17. Height 1 foot, radius of base 5 inches.

18. Height 1 foot 9 inches, radius of base 1 foot 8 inches.

19. Height 18 inches, circumference of base 27 inches.

20. Height 4 feet, circumference of base 7 feet.

21. The area of the curved surface of a right circular cone is 750 square inches, and the circumference of the base is 50 inches: find the slant height.

22. The area of the curved surface of a right circular cone is 800 square inches, and the circumference of the base is 64 inches: find the height of the cone.

23. The area of the curved surface of a right circular cone is 12 square feet, and the radius of the base is 1·5 feet: find the slant height.

24. The area of the curved surface of a right circular cone is 25 square feet, and the radius of the base is 2·25 feet: find the height of the cone.

25. The area of the curved surface of a right circular cone is 650 square inches, and the slant height is 25 inches: find the circumference of the base.

26. The area of the curved surface of a right circular cone is 18 square feet, and the slant height is $3\frac{1}{8}$ feet: find the radius of the base.

27. The area of the whole surface of a right circular cone is 15 square feet, and the slant height is three times the radius of the base: find the radius of the base.

28. The area of the whole surface of a right circular cone is 19 square feet, and the slant height is four times the radius of the base: find the radius of the base.

29. Find what length of canvass three-quarters of a yard wide is required to make a conical tent 12 feet in diameter and 8 feet high.

30. Find what length of canvass two-thirds of a yard wide is required to make a conical tent 16 yards in diameter and 10 feet high.

The area of the whole surface of a right circular cone is 100 square feet; find in cubic feet the volume in the following cases:

31. The slant height twice the radius of the base.

32. The slant height three times the radius of the base.

33. The slant height four times the radius of the base.

34. The slant height five times the radius of the base.

35. The slant height six times the radius of the base.

The following examples involve the extraction of the cube root:

The volume of a right circular cone is 314160 cubic inches; find in square inches the whole surface in the following cases:

36. Height equal to the radius of the base.

37. Height equal to twice the radius of the base.

38. Height equal to three times the radius of the base.

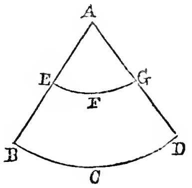
39. Height equal to half the radius of the base.

40. Height equal to a third of the radius of the base.

XXXVII. FRUSTUM OF A RIGHT CIRCULAR CONE.

339. The surface of a frustum of a right circular cone consists of two circular ends and another portion which we shall call the *curved surface.*

340. Let $ABCD$ be a sector of a circle. With A as centre and any radius less than AB describe the arc EFG. Let the piece $BCDGFE$ be cut out of paper or cardboard. Then let it be bent round until the edge EB just comes into contact with the edge GD. It is easy to see that by proper adjustment we can thus obtain a thin shell the outside of which will correspond to the curved surface of a frustum of a right circular cone; EFG becomes the circumference of one end of the frustum, and BCD becomes the circumference of the other end; EB becomes the slant height of the frustum. Hence it will follow that the curved surface of a frustum of a right circular cone is equal to the difference of two sectors of circles which have a common angle; the arcs of the sectors being the circumferences of the ends of the frustum, and the difference of their radii being the slant height of the frustum: thus from the last Rule of Art. 183 we obtain the Rule which will now be given.



341. *To find the area of the curved surface of a frustum of a right circular cone.*

Rule. *Multiply the sum of the circumferences of the two ends of the frustum by the slant height of the frustum, and half the product will be the area of the curved surface.*

342. Examples.

(1) The radius of one end of a frustum of a right circular cone is 10 inches, and the radius of the other end is 15 inches; the slant height is 16 inches: find the area of the curved surface.

The sum of the circumferences in inches is the product of 3·1416 into the sum of 20 and 30, that is, into 50; thus the sum of the circumferences is $50 \times 3·1416$ inches.

$$\frac{1}{2} \times 16 \times 50 \times 3·1416 = 8 \times 50 \times 3·1416 = 400 \times 3·1416 = 1256·64.$$

Thus the area of the curved surface is 1256·64 square inches.

(2) The radius of one end of a frustum of a right circular cone is 5 feet, and the radius of the other end is 8 feet; the slant height is 8 feet: find the area of the whole surface.

The area of the curved surface in square feet

$$= 13 \times 8 \times 3·1416 = 104 \times 3·1416.$$

The area of one end in square feet $= 25 \times 3·1416$, and the area of the other end in square feet $= 64 \times 3·1416$. Hence the area of the whole surface is the product of 3·1416 into the sum of 104, 25, and 64, that is, into 193: thus the area of the whole surface in square feet

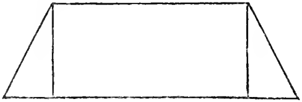
$$= 193 \times 3·1416 = 606·3288.$$

343. It may be inferred from the Rule of Art. 341 that *the slant height of a frustum of a right circular cone bears the same proportion to the difference of the radii of the ends as the area of the curved surface bears to the difference of the areas of the ends.*

344. We will now solve some exercises.

(1) The radii of the ends of a frustum of a right circular cone are 7 inches and 10 inches, and the height of the frustum is 4 inches: find the area of the curved surface.

Let the diagram represent a section of the frustum made by a plane containing the axis of the cone. We see that the slant height is the hypotenuse of a right-angled triangle, of which one side is the height of the frustum, and the other side is the difference of the radii of the ends.



In the present case the height of the frustum is 4 inches, and the difference of the radii is 3 inches; therefore, by Art. 55, the slant height is 5 inches.

Therefore the area of the curved surface in square inches $= 5 \times 17 \times 3{\cdot}1416 = 267{\cdot}036$.

(2) The diameters of the ends of a frustum of a right circular cone are 16 feet and 24 feet respectively; the height of the frustum is equal to the product of these diameters divided by their sum: find the area of the curved surface, and the area of the two ends.

The height of the frustum in feet $= \frac{16 \times 24}{40} = \frac{4 \times 24}{10} = 9{\cdot}6$.

We must determine the slant height. The square of $9{\cdot}6 = 92{\cdot}16$; the difference of the radii of the ends is 4 feet; the square of $4 = 16$; $92{\cdot}16 + 16 = 108{\cdot}16$; the square root of $108{\cdot}16 = 10{\cdot}4$. Thus the slant height is $10{\cdot}4$ feet.

The area of the curved surface in square feet

$$= 10{\cdot}4 \times 20 \times 3{\cdot}1416 = 208 \times 3{\cdot}1416 = 653{\cdot}4528.$$

The area of the two ends is the product of $3{\cdot}1416$ into the sum of the squares of 8 and 12, that is, the product of $3{\cdot}1416$ into the sum of 64 and 144, that is, the product of $3{\cdot}1416$ into 208.

Hence the area of the two ends is equal to the area of the curved surface. It will be found on trial that this is always the case if the height of the frustum of a right circular cone is equal to the product of the diameters of the ends divided by their sum.

(3) The radii of the ends of a frustum of a right circular cone are 8 inches and 10 inches respectively; the slant height is 6 inches: if the frustum be divided into two of equal curved surfaces, find the slant height of each.

We must determine the slant height of the whole cone.

By a method like that of the fourth Exercise of Art. 77, we find that the slant height of the whole cone is 30 inches; and therefore the slant height of the smaller cone is 24 inches.

Then we proceed after the manner of Art. 210 to determine the slant distance from the vertex of the plane which divides the curved surface of the frustum into two equal parts. The square of 24 is 576; the square of 30 is 900; half the sum of these squares is 738: the square root of 738 will be found to be rather more than 27·166. Subtract 24 from this, and the remainder is 3·166. Thus the slant height of one part is rather more than 3·166 inches; and therefore the slant height of the other part is rather less than 2·834 inches.

EXAMPLES. XXXVII.

Find the area of the curved surface of frustums of right circular cones, having the following dimensions:

1. Circumferences of ends 15 inches and 17 inches, slant height 11 inches.

2. Circumferences of ends 19 inches and 23 inches, slant height 13 inches.

3. Radii of ends 7 inches and 9 inches, slant height 5 inches.

4. Radii of ends 2·6 feet and 3·4 feet, slant height 5 feet.

5. Radii of ends 11 and 16 inches, height 12 inches.

6. Radii of ends 4 feet and 3 feet, height 2 feet 11 inches.

7. Radii of ends 4 feet and 5 feet, height 3 feet.

8. Radii of ends $5\frac{1}{2}$ feet and $6\frac{1}{4}$ feet, height $2\frac{1}{2}$ feet.

Find the area of the whole surface of frustums of right cones having the following dimensions:

9. Circumferences of ends 14 and 16 inches, slant height 10 inches.

10. Circumferences of ends 17 and 21 inches, slant height 9 inches.

11. Radii of ends 2 feet and 3 feet, slant height $2\frac{1}{2}$ feet.

12. Radii of ends 3·4 feet and 4·2 feet, slant height 2 feet.

13. Radii of ends 12 and 18 inches, height 8 inches.

14. Radii of ends 12 and 20 inches, height 15 inches.

Find in square feet the area of the curved surface and the area of the two ends, supposing the height of the frustum equal to the product of the diameters of the ends divided by their sum, for frustums with the following dimensions:

15. Diameters of ends 6 feet and 4 feet.

16. Diameters of ends 13 feet and 7 feet.

17. Diameters of ends 20 feet and 12 feet.

18. Diameters of ends 25 feet and 40 feet.

19. The radii of the ends of a frustum are 5 feet and 8 feet, and the slant height is 4 feet: if the frustum be divided into two of equal curved surface, find the slant height of each part.

20. A tent is made in the form of a frustum of a right circular cone surmounted by a cone: find the number of square yards of canvass required for the tent, supposing the diameters of the ends of the frustum to be 28 feet and 16 feet respectively, the height of the frustum 8 feet, and the height of the conical part 6 feet.

XXXVIII. SPHERE.

345. *To find the area of the surface of a sphere.*

Rule. *Multiply the square of the diameter by* 3·1416.

346. Examples.

(1) The diameter of a sphere is 9 inches.

$$9 \times 9 \times 3{\cdot}1416 = 254{\cdot}4696.$$

Thus the area of the surface is 254·47 square inches nearly.

(2) The diameter of a sphere is $3\frac{1}{2}$ feet.

$$3{\cdot}5 \times 3{\cdot}5 \times 3{\cdot}1416 = 38{\cdot}4846.$$

Thus the area of the surface is about 38·4846 square feet.

347. Other modes of expressing the Rule in Art. 345 may be given: *multiply the diameter of the sphere by its circumference*; or, *divide the square of the circumference by* 3·1416. By the circumference of the sphere is meant the circumference of the circle which will produce the sphere in the manner explained by Art. 223, that is, the circumference of a great circle of the sphere.

348. It follows from Arts. 320 and 345, that *the area of the surface of a sphere is equal to the area of the curved surface of a right circular cylinder which has its height and the diameter of its ends equal to the diameter of the sphere.*

349. From Arts. 291 and 345 we may deduce the following important result: *the volume of a sphere is equal to one-third of the product of the area of the surface into the radius.*

350. The result just given may be easily remembered by its resemblance to the Rule for finding the volume of a pyramid or a cone. Let O denote the centre of a sphere; and let P, Q, R denote three points on the surface of the sphere very near to each other. Suppose we cut from the sphere the piece bounded by the planes POQ, QOR, ROP,

and by the portion of the surface of the sphere which they intercept. This piece will resemble a triangular pyramid; so that we may readily admit that the volume of the piece is equal to one-third of the product of the radius of the sphere into the intercepted portion of the surface of the sphere. The whole sphere may be supposed to be cut up into a very large number of very small pieces like that just considered; and thus we are easily led to the result given in Art. 349. The student will observe the resemblance of these remarks to those in Art. 177.

351. The sphere has the following remarkable property: of all solids of a given volume the sphere is that which has the least surface, and of all solids of a given surface the sphere is that which has the greatest volume. The student may verify this statement by such examples as 16...20 at the end of the present Chapter.

352. We will now solve some exercises.

(1) The area of the surface of a sphere is 200 square inches: find the diameter, and the volume of the sphere.

The product of the square of the diameter into 3·1416 is equal to 200; therefore the square of the diameter $=\frac{200}{3 \cdot 1416}=63 \cdot 6618$: the square root of this number will be found to be 7·9789. Thus the diameter is 7·98 inches very nearly.

Then, by Art. 349, the volume of the sphere in cubic feet

$$=\frac{1}{3}\times 200\times\frac{1}{2}\times 7 \cdot 9789=265 \cdot 96\ldots$$

(2) The volume of a sphere is 1000 cubic inches: find the area of its surface.

By Art. 291 the cube of the diameter of the sphere $=\frac{1000}{\cdot 5236}=1909 \cdot 855$; therefore the diameter of the sphere in inches is the cube root of this number: it will be found that this is 12·407. Then by Art. 345 we shall obtain for the area of the surface 483·6 square inches very nearly.

EXAMPLES. XXXVIII.

Find the areas of the surfaces of spheres having the following dimensions:

1. Radius 5 inches.
2. Radius 15 inches.
3. Radius 2·2 feet.
4. Circumference 20 inches.
5. Circumference 4 feet.
6. Circumference 6·4 feet.

Find the diameters of the spheres having the following superficial areas:

7. 400 square inches. 8. 64 square feet. 9. 75 square feet.

Find the volumes of the spheres having the following superficial areas:

10. 20 square feet. 11. 50 square feet. 12. 100 square feet.

13. Find the volume of a sphere when its surface is equal to that of a circle 4 feet in diameter.

14. Find the volume of a sphere when its surface is equal to that of a circle 9 feet in diameter.

15. A cylinder 5 feet long and 3 feet in diameter is closed by a hemisphere at each end: find the area of the whole surface.

16. The radius of the base of a right circular cylinder is 10 inches, and the height is 10 inches; the surface of a sphere is equal to the whole surface of this cylinder: find the volume of each.

17. The surface of a sphere is equal to that of a cube the length of which is one foot: find the volume of each.

18. The surface of a sphere is equal to that of a right circular cylinder the radius of the base of which is one foot, and the height two feet: find the volume of each.

The following examples involve the extraction of the cube root:

19. The volume of a sphere is equal to that of a cube the length of which is one foot: find the surface of each.

20. The volume of a sphere is equal to that of a right circular cylinder the radius of the base of which is one foot, and the height 2 feet: find the surface of each.

XXXIX. ZONE OF A SPHERE. SEGMENT OF A SPHERE.

353. The surface of a zone of a sphere consists of two circular ends and another portion which we shall call the *curved surface.*

The surface of a segment of a sphere consists of a circular base and another portion which we shall call the *curved surface.*

354. *To find the area of the curved surface of a zone of a sphere or of a segment of a sphere.*

Rule. *Multiply the circumference of the sphere by the height of the zone or segment.*

355. Examples.

(1) The height of a segment of a sphere is 6 inches, and the diameter of the sphere is 18 inches: find the area of the curved surface.

$$6 \times 18 \times 3 \cdot 1416 = 339 \cdot 2928.$$

Thus the area of the curved surface is 339·3 square inches nearly.

(2) The ends of a zone of a sphere are distant 2 feet and 4 feet respectively from the centre of a sphere, and are on the same side of the centre; the diameter of the sphere is 14 feet: find the area of the whole surface of the zone.

The area of the curved surface in square feet

$$= 2 \times 14 \times 3 \cdot 1416 = 28 \times 3 \cdot 1416.$$

By Art. 89 the square of the radius of one end

$$= 9 \times 5 = 45;$$

and the square of the radius of the other end

$$=11\times 3=33:$$

thus the area of the two ends in square feet $=78\times 3{\cdot}1416$. Therefore the area of the whole surface in square feet

$$=106\times 3{\cdot}1416=333{\cdot}0096.$$

356. It appears from Arts. 320 and 354 that *the curved surface of a zone of a sphere or of a segment of a sphere is equal to the area of the curved surface of a right circular cylinder which has its height equal to that of the zone or segment, and the diameter of its ends equal to the diameter of the sphere.*

This remarkable result holds also for the surface of a sphere, if by the height of the sphere we understand the diameter of the sphere: see Art. 348.

357. We will now solve some exercises.

(1) The height of a segment of a sphere is 7 inches, and the circumference of the sphere is 64 inches: find the area of the whole surface of the segment.

The area of the curved surface in square inches

$$=7\times 64=448.$$

The diameter of the sphere is $\frac{64}{3{\cdot}1416}$ inches; therefore, by Art. 79, the square of the radius of the base of the segment is obtained by subtracting 7 from $\frac{64}{3{\cdot}1416}$, and multiplying the remainder by 7; so that it is $\frac{7\times 64}{3{\cdot}1416}-49$. The area of the base of the segment in square inches is the product of this result into $3{\cdot}1416$; therefore it is $7\times 64-49\times 3{\cdot}1416$, that is $448-49\times 3{\cdot}1416$. Thus the area of the whole surface in square inches is

$$896-49\times 3{\cdot}1416,$$

that is $$896-153{\cdot}9384,$$

that is $$742{\cdot}0616.$$

The method which we have adopted in solving this exercise may appear rather artificial and difficult to a beginner; but it deserves attention. It will be seen that in effect we establish the following Rule: *the whole surface of a segment of a sphere is equal to twice the excess of the curved surface above a circle which has the height of the segment for its radius.*

(2) A zone of a sphere is the difference of two segments of the heights 13 inches and 9 inches respectively, and the circumference of the sphere is 82 inches: find the area of the whole surface of the zone.

The area of the curved surface in square inches

$$=4\times 82=328.$$

By Art. 79 the square of the radius of one end of the zone is obtained by subtracting 9 from $\frac{82}{3\cdot1416}$, and multiplying the remainder by 9; so that it is $\frac{9\times 82}{3\cdot1416}-81$. The area of this end of the zone in square inches is the product of this result into 3·1416; therefore it is

$$9\times 82-81\times 3\cdot1416.$$

Similarly the area of the other end of the zone in square inches will be found to be $13\times 82-169\times 3\cdot1416$. Hence the area of the whole surface of the zone is equal to the sum of 4×82, 9×82, and 13×82, diminished by the sum of $81\times 3\cdot1416$ and $169\times 3\cdot1416$, that is, to

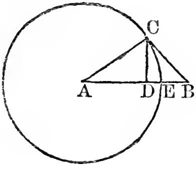
$$2\times 13\times 82-250\times 3\cdot1416.$$

Thus the area of the whole surface is 1346·6 square inches.

It will be seen that in effect we establish the following Rule for finding the area of the whole surface of a zone of a sphere, regarded as the difference of two segments of the sphere: *from twice the curved surface of the large segment subtract the areas of two circles having for their radii respectively the heights of the segments.*

(3) The radius of a sphere is 12 feet; from a point which is at the distance of 15 feet from the centre of the sphere straight lines are drawn to touch the sphere, thus determining a segment of the sphere: find the area of the curved surface of this segment.

Let A be the centre of the sphere, B the point from which straight lines are drawn, BC one of the straight lines, meeting the sphere at C. Draw CD perpendicular to AB. Let E denote the point where AB cuts the sphere. Then the surface of which we have to find the area is the curved surface of the segment of which DE is the height. We shall now find the length of DE.



Taking all the lengths in feet we have

$$AB=15, \qquad AC=AE=12;$$

hence, by Art. 60, we obtain $BC=9$: and, by Art. 151, we shall find that $CD=\frac{36}{5}=7{\cdot}2$. By Art. 60 we shall see that AD is the square root of $144-51{\cdot}84$, that is the square root of $92{\cdot}16$; it will be found that this is $9{\cdot}6$. We may obtain this result more easily by similar triangles; by Art. 37 we have

$$BA : AC :: AC : AD,$$

that is $$15 : 12 :: 12 : AD;$$

therefore $$AD=\frac{12 \times 12}{15}=\frac{48}{5}=9{\cdot}6.$$

Then $$DE=12-9{\cdot}6=2{\cdot}4.$$

Therefore the area of the curved surface of the segment in square feet $=2{\cdot}4 \times 2 \times 12 \times 3{\cdot}1416=180{\cdot}95616$.

An eye placed at B would see exactly that portion of the surface of the sphere of which we have just found the area.

If we wish to know merely what fractional part of the whole surface of the sphere is visible to an eye placed at B, we have only to form the fraction which has the length of DE for the numerator, and the length of the diameter of the sphere for denominator: in the present case the fraction is $\frac{2{\cdot}4}{24}$, that is $\frac{1}{10}$.

EXAMPLES. XXXIX.

Find the areas of curved surfaces of segments of a sphere having the following dimensions:

1. Height of segment 10 inches, circumference of sphere 85 inches.

2. Height of segment $2\frac{1}{4}$ feet, circumference of sphere 20 feet.

3. Height of segment 9 inches, radius of sphere 16 inches.

4. Height of segment 2·4 feet, radius of sphere 3·25 feet.

Find the areas of the whole surfaces of segments of a sphere having the following dimensions:

5. Height of segment 2 feet, radius of sphere 7 feet.

6. Height of segment 8 inches, radius of sphere 25 inches.

7. Height of segment 3 feet, circumference of sphere 27 feet.

8. Height of segment 11 inches, circumference of sphere 90 inches.

Find the areas of the whole surfaces of zones which are the differences of segments having the following dimensions:

9. Radius of sphere 11 feet, heights 3 feet and 10 feet.

10. Radius of sphere 15 inches, heights 6 inches and 9 inches.

11. Circumference of sphere 40 feet, heights 2 feet and 5 feet.

12. Circumference of sphere 75 inches, heights 3 inches and 7 inches.

Find the areas of the whole surfaces of zones of a sphere having the following dimensions:

13. Radius of sphere 9 feet; distances of ends of zone from the centre 2 feet and 3 feet, on opposite sides of the centre.

14. Radius of sphere 16 inches; distances of ends of zone from the centre 5 inches and 9 inches, on the same side of the centre.

15. Circumference of sphere 32 feet; distances of ends of the zone from the centre 3 feet and 4 feet, on opposite sides of the centre.

16. Circumference of sphere 90 inches; distances of ends of the zone from the centre 6 inches and 10 inches, on the same side of the centre.

17. A sphere is 80 feet in diameter: find what fraction of the whole surface will be visible to an eye placed at a distance of 41 feet from the centre.

18. A sphere is 90 feet in diameter: find what fraction of the whole surface will be visible to an eye placed at a distance of 8 feet from the surface.

19. Find at what distance from the surface of a sphere an eye must be placed to see one-sixteenth of the surface.

20. Find at what distance from the surface of a sphere an eye must be placed to see one-eighth of the surface.

SIXTH SECTION. PRACTICAL APPLICATIONS.

XL. INTRODUCTION.

358. We have already given various applications of the Rules of Mensuration to matters of practical interest; for instance the Examples of the calculations of the expense of carpeting the floors of rooms and papering the walls which occur under Chapter XI. Such applications require only a knowledge of the elements of Arithmetic in addition to the principles of Mensuration already explained.

A few more examples will be given at the end of the present Chapter.

359. There are however other applications which require the student to know the meaning of certain technical terms, or to employ certain approximate Rules admitted by custom. We shall consider these cases in the three Chapters which immediately follow the present Chapter

EXAMPLES. XL.

1. A room is 24 feet 3 inches long, 11 feet 9 inches broad, and 11 feet 6 inches high : find the cost of painting the walls at 1*s.* 6*d.* per square foot.

2. Find the cost of painting the four walls of a room which is 32 feet 4 inches long, 15 feet 8 inches broad, and 11 feet 6 inches high at 3 shillings per square foot.

3. Find the cost of painting the four walls of a room whose length is 24 feet 3 inches, breadth 15 feet 8 inches, and height 11 feet 6 inches at 4 shillings per square foot.

4. A room is 24 feet 10 inches long, 16 feet broad, and 12 feet 4 inches high: find the cost of painting the four walls at 9 pence per square foot.

5. The length of a room is 7 yards 1 foot 3 inches, the breadth is 5 yards 2 feet 9 inches, and the height 4 yards 6 inches: find the cost of papering the walls, supposing the paper to be a yard broad and to cost 9*d.* per yard.

6. A cubical box is covered with sheet lead which weighs 4 lbs. per square foot; and 294 lbs. of lead are used: find the size of the box.

7. Find the cost of lining the sides and the bottom of a rectangular cistern 12 feet 9 inches long, 8 feet 3 inches broad, 6 feet 6 inches deep with sheet lead which costs £1. 8*s.* per cwt., and weighs 8 lbs. to the square foot.

8. A cistern open at the top is to be lined with sheet lead which weighs 6 lbs. to the square foot; the cistern is 4 feet 6 inches long, 2 feet 8 inches wide, and holds 42 cubic feet: find the weight of lead required.

9. A box with a lid is made of planking $1\frac{1}{2}$ inches thick; if the external dimensions be 3 feet 6 inches, 2 feet 6 inches, and 1 foot 9 inches, find exactly how many square feet of planking are used in the construction.

10. A flat roof is 17 feet 4 inches long and 13 feet 4 inches wide: find the cost of covering it with sheet lead one-sixteenth of an inch thick, supposing that a cubic inch of lead weighs $6\frac{1}{2}$ ounces avoirdupois, and that 1 lb. of it costs $3\frac{1}{2}d.$

11. Find the cost of painting the wall of a cylindrical room 16 feet high, and 18 feet in diameter, at $7\frac{1}{2}d.$ per square yard.

12. Find the cost of painting a conical spire 64 feet in circumference at the base, and 108 feet in slant height, at $7\frac{1}{2}d.$ per square yard.

13. Find how much it will cost to gild the inner surface of a hemispherical bowl 2 feet 4 inches in diameter at $1\frac{1}{2}d.$ per square inch.

14. A circular room has perpendicular walls 15 feet high, the diameter of the room being 28 feet; the roof is a hemispherical dome: find the cost of plastering the whole surface at 9 pence per square foot.

15. Find the cost of a string moulding round the springing of the dome in the preceding Example at 15*d.* per foot.

16. A rectangular court-yard is 100 feet long and 90 feet broad; a footway goes through the length 6 feet broad; the footway is laid with stone at 4*s.* 6*d.* per square yard, and the remainder is covered with turf at 9*d.* per square yard: find the whole cost.

17. Required the cost of glazing the windows of a house at a shilling per square foot; there being three stories and three windows in each story; the height of the windows of the lower story is 8 feet, of the middle story 7 feet, and of the upper story 5 feet; and the common breadth of all the windows is 4 feet.

18. Find how many square feet of flooring there are in a house of three stories, measuring within the walls 58 feet by 34 feet, deducting the vacancy for the stairs 15 feet 3 inches by 8 feet.

19. A room is 22 feet long, 20 feet wide, and 14 feet 6 inches high: find the expense of covering the walls with paper 30 inches wide at $11\frac{1}{4}d.$ a yard; allowing for two doors each 8 feet by 5 feet 3 inches, a fire-place 6 feet 6 inches by 6 feet, and a window 12 feet by 5 feet 7 inches.

20. A square court-yard is 36 feet long; in the middle of it is a circular basin 13 feet in diameter; and a flower-bed 4 feet wide is left round three sides: find the expense of paving the remainder of the court-yard at 8 shillings per square yard.

XLI. ARTIFICERS' WORK.

360. Artificers compute their work in various ways, but in general a foot or a yard is the standard of length, and therefore a square foot or a square yard is the standard of area, and a cubic foot or a cubic yard is the standard of volume.

361. The work of flooring, roofing, plastering or tiling is often estimated by the number of *squares*, each square consisting of 100 square feet.

362. The measure of the roof of a house can be deduced from the measure of the base when the *pitch* of the roof is known. There are three pitches which have received names.

(1) The *common or true pitch*. In this the length of the rafters is three-fourths of the breadth of the building; and hence the practical rule is to take the flat and half the flat of the house as the measure of the roof.

(2) The *Gothic pitch*. In this the length of the rafters is equal to the breadth of the building, and consequently the roof is equal to twice the flat.

(3) The *pediment pitch*. In this the perpendicular height is $\frac{2}{9}$ of the breadth of the building. In this case the length of the rafters will be nearly $\frac{11}{20}$ of the breadth of the building, so that the roof will be nearly $\frac{11}{10}$ of the flat.

363. Artificers of every kind have various special modes of estimating the charge for workmanship, which have the sanction of custom; but as these modes involve no principle of mensuration it is not expedient to detail them here. We will give as a specimen the mode in which the charge is made for *doors*.

It is usual to add the thickness of the door both to the length and to the breadth, and to take the product of the length and the breadth thus increased for the area. If the door be panelled on only one side this area is increased by its half. If the door be panelled on both sides the area is doubled.

Thus, for example, suppose that a door is 7 feet 5 inches high, 4 feet 3 inches wide, and 1 inch thick. Then the height is taken as 7 feet 6 inches, and the breadth as 4 feet 4 inches; and so the area in square feet is taken as $7\frac{1}{2} \times 4\frac{1}{3}$, that is as $\frac{15}{2} \times \frac{13}{3}$, that is as $\frac{5 \times 13}{2}$, that is as $32\frac{1}{2}$. If the door be panelled on only one side the charge is for $48\frac{3}{4}$ square feet. If the door be panelled on both sides the charge is for 65 square feet.

364. Engineers always estimate brickwork by the cubic yard; but such brickwork as occurs in connexion with ordinary house-building is estimated in a peculiar way which we will now explain.

365. A brick wall which is a brick and a half thick is said to be of the *standard thickness*. Brickwork of the standard thickness is estimated by the number of square yards in the area formed by its height and its length, or by the number of square rods, each square rod consisting of $30\frac{1}{4}$ square yards, that is of $272\frac{1}{4}$ square feet. Thus a standard rod of brickwork is a mass of brickwork which has a surface of a square rod and is a brick and a half thick.

366. *To find the number of standard rods of brickwork in a wall.*

Rule. *Find the area of the surface in square feet, and divide it by* $272\frac{1}{4}$; *the quotient will be the number of rods if the wall be of the standard thickness; if not, multiply the quotient by the number of half bricks in the thickness, and divide the product by* 3.

In practice 272 is generally used instead of $272\frac{1}{4}$.

367. Examples.

(1) Find the number of standard rods of brickwork in a wall 105 feet long, $8\frac{1}{2}$ feet high, and $2\frac{1}{2}$ bricks thick.

$$\frac{105 \times 8\frac{1}{2}}{272} \times \frac{5}{3} = \frac{35 \times 5 \times 8\frac{1}{2}}{272} = \frac{35 \times 5 \times 17}{544} = 5\cdot46....$$

Thus the number of standard rods is nearly $5\frac{1}{2}$.

(2) Find the number of yards of brickwork of the standard thickness contained in a triangular gable-top which is 15 feet high, and the base of which is 20 feet, supposing the thickness 2 bricks.

$$\frac{20 \times 15}{2 \times 9} = \frac{50}{3}; \quad \frac{50}{3} \times \frac{4}{3} = \frac{200}{9} = 22\tfrac{2}{9}.$$

Thus the number of standard yards is $22\frac{2}{9}$.

368. A common brick is $8\frac{1}{2}$ inches long, 4 inches broad, and $2\frac{1}{2}$ inches thick; but, on account of the mortar, when laid in brickwork every dimension is to be taken half an inch greater: thus the dimensions are to be taken as 9 inches, $4\frac{1}{2}$ inches, and 3 inches. The standard rod requires 4500 bricks of the usual size, allowing for waste.

EXAMPLES. XLI.

1. Find the number of standard rods of brickwork in a wall 62 feet 6 inches long, 14 feet 8 inches high, and $2\frac{1}{2}$ bricks thick.

2. Find the number of standard rods of brickwork in a gable-end wall 2 bricks thick, 22 feet long, 27 feet high to the eaves, and 36 feet from the ground to the ridge of the roof.

3. Find the cost of a wall with a triangular gable-top of 10 feet high, the height of the wall being 36 feet, the breadth 24 feet, and the thickness $2\frac{1}{2}$ bricks, at 34*s.* per standard rod.

4. The end wall of a house is 30 feet long; it is 40 feet high to the eaves, and there is a triangular gable of 10 feet high; up to the height of 20 feet the wall is $2\frac{1}{2}$ bricks thick, between the height of 20 feet and 40 feet it is 2 bricks thick, and the gable-top is $1\frac{1}{2}$ bricks thick: find the number of standard yards.

5. Assuming that bricks cost £2 per thousand, that 4500 are required for a rod, that cartage and mortar cost together 22 shillings per rod, and labour £2 per rod: find the cost of building a wall 136 feet long, 18 feet high, and 2 bricks thick.

6. Find the cost of roofing a house of the common pitch at 15 shillings per square; the length being 40 feet, and the breadth 35 feet.

7. Find the cost of roofing a building of the Gothic pitch at 25 shillings per square; the length being 120 feet and the breadth 40 feet.

8. A building 30 feet long by 20 feet broad is to be covered with lead, so that the roof shall be eleven-tenths of the flat: find the cost supposing the lead to weigh 8 lbs. to the square foot and to cost 21 shillings per cwt.

9. A partition measures 45 feet 5 inches, by 8 feet 2 inches: find the cost at £6. 15*s*. per square.

10. The floor of a room measures 44 feet by 24 feet: find the cost of flooring at £6. 5*s*. per square, allowing for two hearths each 7 feet by 4 feet.

11. A room is 34 feet long, 18 feet 6 inches wide, and 12 feet high: find the cost of wainscoting the room at £10 per square.

12. The length of the roof of a house is 50 feet, and the length of a string stretched over the ridge from eaves to eaves is 60 feet: find the cost of the roof at £2. 7*s*. 6*d*. per square.

13. A garden wall is 180 feet long and 7 feet high; the wall is 1 brick thick, but there are 18 piers each $1\frac{1}{2}$ feet wide, and at these the whole thickness is $1\frac{1}{2}$ bricks: find the number of standard yards.

14. A room is 36 feet long, 18 feet broad, and 12 feet high: find the whole cost of plastering the walls at one shilling per square yard, and the ceiling at eighteen-pence.

15. Find the whole cost of flooring two rooms at £5 per square; one room measuring 28 feet by 16 feet, and the other 24 feet by 15 feet 6 inches.

16. A room is 25 feet long, 20 feet broad, and 12 feet high; the walls are to have three coats of paint, each costing 10 shillings per square: find the whole cost.

XLII. TIMBER MEASURE.

369. If a piece of timber be in the shape of any of the bodies considered in the Fourth Section, the volume can be determined by the appropriate Rule there given. If no exact Rule is applicable we may in some cases use with advantage the method of equidistant sections which is given in Art. 307. In two cases which often occur in practice rules are adopted, which although not exact, are recommended by their simplicity: these rules we shall now give.

370. *To find the volume of squared or four-sided timber.*

RULE. *Multiply the mean breadth by the mean thickness, and the product by the length; and take the result for the volume.*

In order to obtain the mean breadth, the actual breadth should be measured at various equidistant points, and the sum of the results divided by the number of the measurements: and in the same way the mean thickness should be obtained.

371. Examples.

(1) The length of a piece of timber is 24 feet, the mean breadth is 1 foot 9 inches, and the mean thickness is 1 foot 6 inches.

$$24 \times 1\tfrac{3}{4} \times 1\tfrac{1}{2} = 24 \times \frac{7}{4} \times \frac{3}{2} = 63.$$

Thus we obtain 63 cubic feet for the volume.

(2) The length of a piece of timber is $16\frac{1}{2}$ feet, the thickness at one end is 1 foot, and at the other end 1 foot 8 inches: the breadth is 2 feet.

Here we take for the mean thickness in feet half the sum of 1 and $1\frac{2}{3}$, that is $1\frac{1}{3}$.

$$16\tfrac{1}{2} \times 2 \times 1\tfrac{1}{3} = \frac{33}{2} \times \frac{2}{1} \times \frac{4}{3} = 44.$$

Thus the volume is 44 cubic feet.

372. If the piece of timber tapers regularly from one end to the other it is usual to take for the mean breadth the breadth at the middle, or, which is the same thing, half the sum of the breadths at the ends; and similarly the mean thickness is estimated. But in this case the piece of timber is really a prismoid, and so the volume might be determined exactly by the Rule of Art. 283. The approximate Rule has the advantage of being simpler than the exact Rule.

If, as in Example (2) of Art. 371, it tapers regularly as to its thickness, and is constant as to its breadth, the Rule gives the exact result. The piece of timber is in this case a prism, the ends of the prism being trapezoids, and the height of the prism being the breadth of the piece of timber. Of course a similar remark applies to the case in which the thickness is constant and the breadth tapers regularly.

373. *To find the volume of round or unsquared timber.*

Rule. *Multiply the square of the mean quarter girt by the length, and take the product for the volume.*

374. Examples.

(1) The length of a piece of unsquared timber is 32 feet, and the girt is 6 feet.

The quarter girt is $\frac{3}{2}$ feet; the square of $\frac{3}{2}$ is $\frac{9}{4}$; $\frac{9}{4} \times 32 = 72$. Thus we obtain 72 cubic feet for the volume.

(2) The length of a piece of unsquared timber is 24 feet; the girt at one end is 5 feet, and at the other end 6 feet.

Here we take for the mean girt $\frac{5+6}{2}$; and so for the mean quarter girt $\frac{11}{8}$; the square of $\frac{11}{8}$ is $\frac{121}{64}$;

$$\frac{121}{64} \times 24 = \frac{121 \times 3}{8} = \frac{363}{8} = 45\tfrac{3}{8}.$$

Thus we obtain $45\frac{3}{8}$ cubic feet for the volume.

375. If the piece of timber be exactly a cylinder in shape, we can determine its volume exactly by the Rule of Art. 246. We shall find that in the case of a right circular cylinder the Rule of Art. 373 gives a result which is rather more than three-fourths of the true result; perhaps the Rule was constructed with the design of making some allowance for the loss of timber which occurs when the piece is reduced by squaring in the ordinary way; see the last Exercise of Art. 313.

If the piece of timber be not a *circular* cylinder the result given by the Rule will generally approach nearer to the true result.

376. Dr Hutton proposed to use the following Rule instead of the Rule of Art. 373: *Multiply the square of one-fifth of the mean girt by twice the length.* Dr Hutton's Rule makes the volume $\frac{32}{25}$ times as great as the ordinary Rule, and gives a result which is very nearly exact when the piece of timber is exactly a circular cylinder.

377. If the piece of unsquared timber tapers regularly from one end to the other it is usual to take as the mean girt the girt at the middle, or, which is the same thing, half the sum of the girts at the ends. If the ends are exact circles and the piece of timber tapers regularly, it is really a frustum of a cone, and so the volume might be determined exactly by the Rule of Art. 268. The approximate Rule has the advantage of being simpler than the exact Rule.

EXAMPLES. XLII.

Find the number of cubic feet in pieces of timber of the following dimensions:

1. Length $22\frac{1}{2}$ feet; breadth at one end 2 feet 9 inches, at the other 2 feet 3 inches; thickness at the first end 1 foot 10 inches, at the other 1 foot 6 inches.

2. Length 27 feet; mean breadth $3\frac{1}{4}$ feet, mean thickness $1\frac{1}{3}$ feet.

3. Length 32 feet; mean breadth $2\frac{7}{8}$ feet, mean thickness $1\frac{3}{4}$ feet.

4. Length 56 feet; mean girt 5 feet.

5. Length 32 feet; girt at one end 25 inches, at the other 35 inches.

6. Length 24 feet; mean girt 40 inches.

7. A piece of timber is 36 feet long and tapers regularly; its breadth and thickness at one end are 30 inches and 20 inches respectively, and at the other end 24 inches and 18 inches respectively: find the number of cubic feet in the piece by the Rule of Art. 370.

8. Find the number of cubic feet in the piece of timber of the preceding Example by the exact rule of Art. 283.

9. A piece of timber is 40 feet long and tapers regularly; one end is a circle 7 feet in circumference, and the other end is a circle 4 feet in circumference: find the number of cubic feet in the piece by the Rule of Art. 373.

10. Find the number of cubic feet in the piece of timber of the preceding Example by the Rule of Art. 376.

11. Find the number of cubic feet in the piece of timber of Example 9 by the exact rule of Art. 268.

12. If the piece of timber of Example 9 is squared, the ends being made squares as large as possible, find the number of cubic feet in the piece produced. [See Examples 51 and 52 of Chapter XVI.]

XLIII. GAUGING.

378. By gauging is meant estimating the volumes of casks, that is, the volumes of liquids which the casks would hold.

Casks differ in shape, and various rules have been given for estimating their volumes according to the shape which they take exactly or approximately. For example, suppose a cask to be formed of two equal frustums of a cone united at their bases; we can determine the volume exactly by Art. 268: and if a cask be very nearly of this shape, we may estimate the volume approximately by proceeding as if the cask were exactly of this shape.

379. But it is found that a Rule may be given which will serve tolerably well for all the shapes of casks which occur in practice. To apply this Rule we must know three *internal* measurements of the cask, namely the length, the diameter at one end, which is called the *head diameter*, and the diameter at the middle, which is called the *bung diameter*.

380. The dimensions of a cask will always be taken to be *expressed in inches*.

381. *To find the volume of a cask.*

RULE. *Add into one sum* 39 *times the square of the bung diameter*, 25 *times the square of the head diameter, and* 26 *times the product of those diameters; and multiply the sum by the length of the cask: multiply the product by* ·000031473, *and the result may be taken for the volume of the cask in gallons.*

382. Examples.

(1) The length of a cask is 40 inches, the bung diameter is 32, and the head diameter is 24.

$$39 \times 32 \times 32 = 39936,$$

$$25 \times 24 \times 24 = 14400,$$

$$26 \times 32 \times 24 = 19968,$$

$$39936 + 14400 + 19968 = 74304.$$

$$74304 \times 40 \times {\cdot}000031473 = 93{\cdot}54279\ldots$$

Thus the volume of the cask is about $93\frac{1}{2}$ gallons.

(2) The length of a cask is 20 inches, the bung diameter is 16, and the head diameter is 12.

$$39 \times 16 \times 16 = 9984,$$

$$25 \times 12 \times 12 = 3600,$$

$$26 \times 16 \times 12 = 4992,$$

$$9984 + 3600 + 4992 = 18576.$$

$$18576 \times 20 \times {\cdot}000031473 = 11{\cdot}6928\ldots$$

Thus the volume of the cask is about 11·7 gallons.

383. It is sometimes necessary to know the quantity of liquid in a vessel which is only partly filled. The word *ullage* means strictly the portion of a partly filled cask which is not occupied by the liquid: but the word is now applied to the occupied portion as well as to the unoccupied portion, the former being called the *wet ullage* and the latter the *dry ullage*.

384. Two cases may occur; namely, that of a *standing* cask, and that of a *lying* cask. We will consider the former case. The depth of the liquid is called the *wet inches;* the difference between the wet inches and the length of the cask is called the *dry inches*.

385. *To estimate the wet ullage of a standing cask which is less than half full.*

Rule. *Multiply the square of the dry inches by the difference between the bung diameter and the head diameter; and divide the product by the square of the length: subtract this from the bung diameter, and the result may be taken as the mean diameter of the occupied portion of the cask.*

Then proceed as in finding the volume of a cylinder: *Multiply the square of the mean diameter by the wet inches, and the product by* ·0028326; *the result may be taken for the number of gallons in the wet ullage.*

386. *To estimate the wet ullage of a standing cask which is more than half full.*

Apply the method of Art. 385, using *wet* inches instead of *dry* inches, and *dry* inches instead of *wet* inches; we thus obtain the *dry* ullage: subtract the dry ullage from the volume of the cask, and the remainder is the wet ullage.

387. Examples.

(1) The length of a cask is 40 inches, the bung diameter is 32, and the head diameter is 24; the number of wet inches is 10: find the wet ullage.

The number of dry inches is 30; the difference between the bung diameter and the head diameter is 8.

$$\frac{30 \times 30 \times 8}{40 \times 40} = \frac{9}{2}; \qquad 32 - \frac{9}{2} = \frac{55}{2}.$$

Thus we take $\frac{55}{2}$ for the mean diameter.

$$\frac{55}{2} \times \frac{55}{2} \times 10 \times \cdot 0028326 = 21 \cdot 4215\ldots$$

Thus the wet ullage is about 21·4 gallons.

(2) The length of a cask is 20 inches, the bung diameter is 16, and the head diameter is 12; the number of wet inches is 15: find the wet ullage.

Here we first find the *dry* ullage.

$$\frac{15 \times 15 \times 4}{20 \times 20} = \frac{9}{4}; \qquad 16 - \frac{9}{4} = \frac{55}{4}.$$

$$\frac{55}{4} \times \frac{55}{4} \times 5 \times \cdot 0028326 = 2\cdot 6777 \text{ very nearly.}$$

Subtract this from 11·6928, the volume of the whole cask, by Art. 382; the remainder is 9·0151. Thus the wet ullage is about 9 gallons.

388. No satisfactory rule can be given for estimating the ullage of any lying cask. A rule proposed by Hutton amounts to assuming the cask to be a cylinder; the rule is substantially this: *find the area of the segment of a circle which is obtained by the section of the fluid with a plane perpendicular to the length of the cask; multiply this area by the length of the cask, and divide the product by* 277·274; *the result may be taken for the number of gallons in the wet ullage.*

389. The business of gauging is practically performed by excisemen with the aid of instruments called the gauging or diagonal rod, and the gauging or sliding rule. These instruments however are recommended not so much on account of the accuracy of the results to which they lead, as of the expedition with which these results are obtained. The construction and mode of using these instruments can be far more readily understood by actual experience than by any description.

EXAMPLES. XLIII.

Find in gallons the volumes of casks having the following dimensions:

1. Length 50·2, bung diameter 31·5, head diameter 22·7.
2. Length 47·5, bung diameter 28·5, head diameter 21·4.
3. Length 42·5, bung diameter 32·5, head diameter 26·5.
4. Length 30·5, bung diameter 26·5, head diameter 23.
5. Length 46·8, bung diameter 30·5, head diameter 26.
6. Length 34·5, bung diameter 32·3, head diameter 27·6.
7. Length 46·9, bung diameter 31·2, head diameter 26·1.

[The dimensions in the above seven examples are stated in Lubbock's tract on Cask-gaging, 1834, to be the average dimensions respectively of a Port-pipe, a Madeira-pipe, a Sherry-butt, a Sherry-hogshead, a Bourdeaux brandy-puncheon, a Rum-puncheon, a Brandy-piece.]

Find in gallons the wet ullage in the following cases of standing casks:

8. Length 60, bung diameter 36, head diameter 30, wet inches 12.

9. Length 50, bung diameter 32, head diameter 27, wet inches 10.

10. Length 30, bung diameter 27, head diameter 23, wet inches 9.

11. If the length of the cask in Example 1 is increased by ·1 of an inch, shew that the volume will be increased by about ·22 of a gallon.

12. If the head diameter in Example 2 is increased by ·1 of an inch, shew that the volume will be increased by about ·27 of a gallon.

13. If the bung diameter in Example 3 is increased by ·1 of an inch, shew that the volume will be increased by about ·43 of a gallon.

14. If all the dimensions in Example 1 are increased by ·1 of an inch, shew that the volume will be increased by about a gallon.

SEVENTH SECTION. LAND SURVEYING.

XLIV. USE OF THE CHAIN.

390. A VERY important application of some of the rules of mensuration is furnished in Land Surveying; and to this we now proceed.

391. Land is measured with a chain, called Gunter's chain, which is 4 poles, that is, 22 yards long; the chain consists of 100 equal links, so that each link is $\frac{22}{100}$ of a yard long, that is 7·92 inches.

392. A *picket* is a rod stuck into the ground to mark a certain position.

393. A *field-book* is a book in which the results obtained by measurement are recorded.

394. We will now explain how *a straight line is measured with the chain.*

We will suppose that the straight line which is to be measured is the distance between two points each marked by a picket.

Ten small arrows are provided which may be stuck in the ground. Two persons engage in the work, one of whom is called the leader; both place themselves at one of the pickets. The leader takes in his hand the ten arrows and one end of the chain, and walks towards the second picket; the follower keeps the other end of the chain at the first picket. When the leader has stretched the chain to its full length he puts an arrow in the ground to mark the spot to which the chain reached; he then walks towards the second picket carrying with him his end of the chain as before. The follower now comes up to the arrow, and holds his end of the chain at it until the leader has again stretched the chain and stuck the second arrow in the ground. Then the follower takes up the first arrow and walks towards the second. The process is continued until the proposed length is measured.

Whenever the follower has the ten arrows in his hands he records in the field-book that a length of ten chains has been passed over; then he gives the ten arrows again to the leader, and the work is continued. Thus on arriving at the second picket the field-book shews the number of *tens* of chains passed over, the arrows in the follower's hands correspond to the number of additional chains, and the number of links between the last arrow and the second picket can be counted. Thus the required length is found.

395. In measuring with the chain, great care must be taken to preserve the proper direction; and there is in general a double test of the accuracy with which this is effected. When the leader fixes an arrow he should take care that the straight line between this arrow and the first picket will pass through the follower's arrow; and at the same time the follower should take care that the straight line between his arrow and the second picket will pass through the leader's arrow.

396. If a field be in the shape of any rectilineal figure, and we measure the lengths of the appropriate straight lines, we can find the area of the field by the corresponding rule in the Third Section.

397. Examples.

(1) A rectangular field is 8 chains 95 links long, and 3 chains 26 links broad.

8 chains 95 links = 8·95 chains; 3 chains 26 links = 3·26 chains. We use the rule of Art. 134.

2·9177	8·95
4	3·26
3·6708	5370
40	1790
26·8320	2685
	29·1770

The area of the field is 29·177 square chains, that is 2·9177 acres; we may reduce the decimal of an acre to roods and poles: thus we obtain 2 acres 3 roods 27 poles very nearly. See Art. 126.

(2) The sides of a triangular field are 5·2 chains, 5·6 chains, and 6 chains respectively.

We use the rule of Art. 152.

$$5·2 + 5·6 + 6 = 16·8, \qquad \tfrac{1}{2} \text{ of } 16·8 = 8·4;$$

$$8·4 - 5·2 = 3·2, \qquad 8·4 - 5·6 = 2·8, \qquad 8·4 - 6 = 2·4.$$

$8·4 \times 3·2 \times 2·8 \times 2·4 = 180·6336$. The square root of 180·6336 is 13·44.

Thus the area is 13·44 square chains, that is, 1·344 acres, that is, 1 acre 1 rood 15·04 poles.

(3) The radius of a circular grass plot is 2 chains 50 links.

We use the rule of Art. 168.

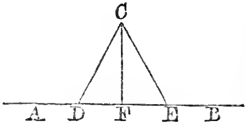
$$2·5 \times 2·5 \times 3·1416 = 19·635.$$

Thus the area is 19·635 square chains, that is, 1·9635 acres, that is, 1 acre 3 roods 34·16 poles.

XLV. PERPENDICULARS.

398. Some of the rules for finding the areas of rectilineal figures require us to know the length of the *perpendicular* from some given point to some given straight line. When the *situation* of such a perpendicular is known, the length of it can be measured in the manner explained in Art. 394; we shall now shew how the situation is determined.

399. *To determine the situation of the perpendicular to a given straight line from a given point without it.*



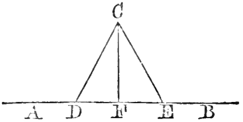
Let AB be the given straight line, and C the given point without it.

Fold a string into two equal parts. Let one person hold the middle at C; let two other persons hold the ends, and stretch the two parts, so that the ends may lie on the straight line AB, say at D and E. Take F the middle point of DE. Then CF is the perpendicular required.

400. The straight line AB in the preceding Article is supposed to be clearly marked on the ground in some manner. This may be effected by stretching a string or chain tightly between A and B, or by placing pickets at short distances in the direction passing through A and B. If, however, the straight line AB has not been thus marked out on the ground, a person standing beyond A must take care that the end of the string is properly placed at D, and then standing beyond B he must take care that the end of the string is properly placed at E.

401. *To determine the situation of the straight line at right angles to a given straight line from a given point in it.*

Let AB be the given straight line, and F the given point in it.



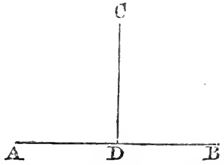
Take D and E points in AB, so that FD and FE may be equal. Fold a string, longer than DE, into two equal parts. Let the ends of the string be fixed at D and E, and let a person take the middle and stretch both the parts. Suppose C the point to which the middle of the string is thus brought. Then FC is at right angles to AB, and is therefore the straight line required.

402. We see then that the situation of any required perpendicular can be determined with the aid of a string; but an instrument called the *Cross* is often employed by land surveyors for this purpose.

403. *The Cross.* This is usually a round piece of wood, about six inches in diameter, having two fine grooves in it, which are at right angles to each other. A staff with a pointed end is stuck upright in the ground, and the cross is fixed on the top of the staff so as to form a small round table.

404. *To determine with the aid of the cross the situation of the perpendicular to a given straight line from a given point without it.*

Let AB be the given straight line, and C the given point without it.



Place pickets at A, B, and C. Select a point in AB which appears by inspection to be at or near the intersection of the required perpendicular with AB; let D denote this point. Fix the staff at D, and place the cross with one of its grooves parallel to AB; so that in looking along the groove in one direction, the picket at A is seen, and in looking along the same groove in the other direction the picket at B is seen. Look along the *other* groove; if the picket at C is seen in the line of this groove, then the point D is the intersection of the perpendicular from C with AB: but if the picket at C is not seen in the line of the groove, the staff must be taken up and moved along AB, to the right or to the left of the assumed position, according as the picket at C appeared to the right or to the left of the groove. By a little trial the proper position will be found for the staff, such that the pickets at A and B can be seen in looking along one groove, and the picket at C in looking along the other groove; and this position of the staff determines the situation of the perpendicular on AB from C.

405. *To determine with the aid of the cross the situation of the straight line at right angles to a given straight line from a given point in it.*

Let AB be the given straight line, and D the given point in it.

Fix the staff at D, and place the cross with one of the grooves parallel to AB. Then the other groove determines the direction of the straight line at right angles to AB.

406. Thus in the preceding Chapter and the present we have explained how the operations are to be performed which will furnish the lengths required for calculating the areas of fields. We will now give some examples of the calculations.

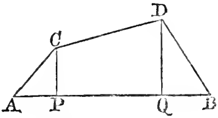
407. Examples.

(1) The base of a triangle is 13·2 chains, and the height is 8·3 chains.

$$\frac{1}{2} \times 13 \cdot 2 \times 8 \cdot 3 = 54 \cdot 78.$$

The area of the field is 54·78 square chains, that is, 5·478 acres, that is, 5 acres 1 rood 36·48 poles.

(2) $ABDC$ is a four-sided field; CP and DQ are perpendiculars on AB.



The following measurements are taken in links:

$$AP=112, \quad AQ=448, \quad AB=626,$$
$$CP=223, \quad DQ=295.$$

Hence we have $PQ=336$, $QB=178$.

The following will be the areas of the parts of the field in square links:

$$\text{the triangle } APC=\frac{1}{2}\times 112\times 223=12488,$$

$$\text{the trapezoid } PQDC=\frac{1}{2}\times 336\times 518=87024,$$

$$\text{the triangle } DQB=\frac{1}{2}\times 178\times 295=26255.$$

The sum of these three numbers is 125767; so that the area of the field is 1·25767 acres, that is, about 1 acre 1 rood 1 pole.

XLVI. THE FIELD-BOOK.

408. Many fields may be conveniently surveyed by measuring a straight line from one corner to another, and also the perpendiculars on it from the other corners. The first straight line is called a *base-line*, or a *chain-line;* and the perpendiculars are called *offsets*. It is often advantageous to take for the base-line the longest straight line which can be drawn in the field; and thus sometimes one of the sides of the field may be the base-line, as in the second example of Art. 407.

We will now explain the method in which the results of the measurements are usually recorded in the field-book.

409. *The field-book.* Each page of the field-book is divided into three columns; the surveyor begins at the bottom of the page and writes upwards.

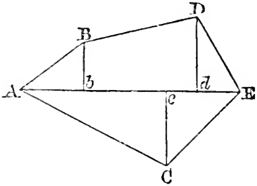
In the middle column are entered lengths obtained in measuring along the base-line, in the right-hand column are entered the lengths of offsets to the right of the base-line, and in the left-hand column the lengths of offsets to the left of the base-line. The offsets are entered against the corresponding distances of the points on the base-line at which they are measured.

The field-book is used to record not only measured lengths, but various particulars, which may be useful in drawing a plan of the estate surveyed. Thus a note is made in the margin of the point at which the chain-line crosses any fence, or foot-path, or ditch, or stream. The positions of adjacent buildings, or of remarkable trees may also be indicated; and if the chain-line passes near a boundary which is rather irregular, the form of the boundary may be traced.

410. Examples.

(1)

	to E	
	1125	
to D 260	750	
	625	250 to C
to B 230	300	
	From A	



The surveyor begins at A and measures towards E; Ab is 300 links, and at b is an offset bB to the left of 230 links; Ac is 625 links, and at c is an offset cC to the right of 250 links: Ad is 750 links, and at d is an offset dD to the left of 260 links; AE is 1125 links.

We can now calculate the areas of the parts; and we shall have the following results in square links:

$$\text{the triangle } AbB = \frac{1}{2} \times 300 \times 230 = 34500\,;$$

$$\text{the trapezoid } bdDB = \frac{1}{2} \times 450 \times 490 = 110250\,;$$

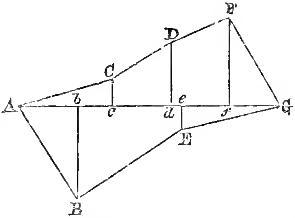
$$\text{the triangle } dED = \frac{1}{2} \times 375 \times 260 = 48750\,;$$

$$\text{the triangle } AEC = \frac{1}{2} \times 1125 \times 250 = 140625.$$

Thus the whole area is 334125 square links, that is, 3·34125 acres, that is, 3 acres 1 rood 14·6 poles.

(2)

	to G	
	1020	
to F 470	990	
	610	50 to E
to D 320	585	
to C 70	440	
	315	350 to B
	From A	



The measurements are taken in links; and we shall have the following results for the areas of the parts in square links:

$$\text{the triangle } AbB = \frac{1}{2} \times 315 \times 350 = 55125,$$

$$\text{the trapezoid } beEB = \frac{1}{2} \times 295 \times 400 = 59000,$$

$$\text{the triangle } eGE = \frac{1}{2} \times 410 \times 50 = 10250,$$

$$\text{the triangle } GfF = \frac{1}{2} \times 30 \times 470 = 7050,$$

$$\text{the trapezium } fdDF = \frac{1}{2} \times 405 \times 790 = 159975,$$

$$\text{the trapezium } dcCD = \frac{1}{2} \times 145 \times 390 = 28275,$$

$$\text{the triangle } cAC = \frac{1}{2} \times 440 \times 70 = 15400.$$

Thus the whole area is 335075 square links, that is, 3·35075 acres, that is, 3 acres 1 rood 16·12 poles.

411. The ends of a chain-line are called stations; they are frequently denoted in the field-book thus: ①, ②, ③,...

The situations of the chain-lines with respect to the points of the compass are often recorded. The record *from ① range E*, indicates that the chain-line commencing at the first station proceeds towards the East. So the record ③ N. 50° W., indicates that the chain-line commencing at the second station proceeds in the direction which makes an angle of 50 degrees with the North direction reckoned towards the West.

Sometimes the situation of the successive chain-lines is sufficiently shewn by the words *right* and *left*. Thus the record *from* ② *on L*, indicates that on arriving at the second station, the surveyor turns to the left hand of the direction in which he has been walking.

If 0 occurs in either offset column, it indicates that the chain-line meets at the corresponding point the boundary of the land which is being surveyed.

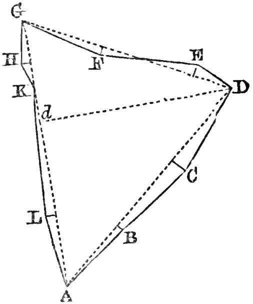
412. In order to provide themselves with a test of the accuracy of their work, surveyors always measure more lengths than would be theoretically sufficient. Thus, for example, suppose that a field bounded by four straight sides is to be surveyed; it would be theoretically sufficient to measure the four sides and one diagonal: for the area of each of the two triangles into which this diagonal divides the figure can then be calculated. But the surveyor will also measure the second diagonal. He will draw a plan of the figure from the measured lengths of the four sides and of the first diagonal, and draw the second diagonal on the plan: he will then examine if the length of the second diagonal as found from the plan by the known scale on which the plan was drawn corresponds with the measured length. If these two lengths do correspond, the surveyor gains confidence in the accuracy of the work: but if the two lengths do not correspond there is an error in the operations with the chain or in the drawing of the plan, and this error must be discovered and corrected.

If the field to be surveyed be in the form of a triangle, the sides will be measured, from which the area can be found, and a plan can be drawn. To test the accuracy of the work, either the perpendicular from an angle to the opposite side, or the straight line drawn from a definite point of one side to a definite point of another will be measured; and this measured length will be compared with the length obtained from the plan.

A length which is measured for the purpose of testing the accuracy of the work is called a *proof*-line, or a *check*-line, or a *test*-line.

413. In surveying a field or a number of fields, we shall have a series of operations and records like those exemplified in Art. 410; namely, one for each chain-line. As an example, we will take the case of a field which approximates to the form of a triangle, so that three chain-lines will occur in the survey.

	⊙ *A*	
	1650	0
	1300	30 *L*
D 1232	*d* 726	
	500	0 *K*
	260	20 *H*
	0	0
	⊙ *G*	
turn	to the	left
	⊙ *G*	
0	1430	
F 10	820	
	600	0
	270	40 *E*
	0	0
	⊙ *D*	
turn	to the	left
	⊙ *D*	
	1540	0
	960	30 *C*
	300	10 *B*
	0	0
from	⊙ *A*	go North East.



The offsets are much exaggerated in the figure for the sake of distinctness.

The sides of the triangle *ADG* are respectively 1540 links, 1430 links, and 1650 links; hence it will be found by Art. 152 that the area of this triangle is 1016400 square links. We proceed to calculate the areas of the small pieces lying between the sides of this triangle and the boundary of the field.

Along *AD* there are offsets to *B* and to *C*; thus we have to estimate a triangle, a trapezoid, and another triangle; the areas are the following in square links:

$$\text{the first triangle} = \frac{1}{2} \times 300 \times 10 = 1500,$$

$$\text{the trapezoid} = \frac{1}{2} \times 660 \times 40 = 13200,$$

$$\text{the second triangle} = \frac{1}{2} \times 580 \times 30 = 8700;$$

the total is 23400.

Along *DG* there is an offset to *E*, and an *inset* to *F*; thus there are two corresponding triangles, the latter of which is to be *subtracted:*

$$\text{the first triangle} = \frac{1}{2} \times 600 \times 40 = 12000,$$

$$\text{the second triangle} = \frac{1}{2} \times 830 \times 10 = 4150;$$

the balance is 7850, to be *added.*

Along *GA* there are offsets to *H* and to *L*, and the boundary meets the chain-line at *K*; thus there are two corresponding triangles:

$$\text{the first triangle} = \frac{1}{2} \times 500 \times 20 = 5000,$$

$$\text{the second triangle} = \frac{1}{2} \times 1150 \times 30 = 17250;$$

the total is 22250.

$$1016400 + 23400 + 7850 + 22250 = 1069900.$$

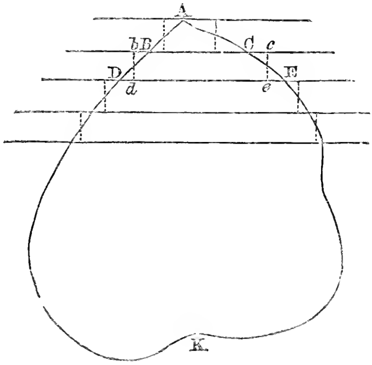
Thus the area of the whole field is 10·699 acres.

The perpendicular is measured as a proof-line, and found to be 1232 links, while *Gd* is 726 links.

414. Instead of the field-book, another method of recording the results of measurement is sometimes adopted. A plan is drawn resembling the field to be surveyed, and the lengths are noted in the plan as they are found against the corresponding parts of the figure.

415. We have hitherto supposed that the boundary of a field which is to be surveyed may be regarded practically as composed of a moderate number of straight lines. But if the boundary is so irregular that this supposition is not admissible, we must employ the principle of adjustment which has been explained in Art. 202: a plan of the field must be drawn and the boundary changed into a rectilineal boundary enclosing an equal area. We will explain a convenient method of applying the principle.

416. Let *ABDKEC* be the plan of the field. Draw on the plan equidistant parallel straight lines; thus dividing the figure into strips of equal width. Consider one of these strips, as *BDEC*. Draw the straight line *bd* at right angles to the parallels, so that the area of



the strip may be the same whether *BD* or *bd* be regarded as its end; if *BD* can be regarded as a straight line, *bd* will pass through its middle point; if *BD* be not a straight line the position of *bd* must be determined as well as possible by the eye. Similarly, draw *ce* at the other end of the strip, so as to leave the area unchanged. Then the area *BDEC* is equivalent to the rectangle *bdec*.

Proceeding in this way we obtain a series of rectangles which are together equivalent to the original figure. The area corresponding to all these rectangles can be easily ascertained, and therefore the area of the original figure. Suppose, for example, that the parallel straight lines are drawn an inch apart; and that the sum of the lengths of all the rectangles is 29 inches: the area of the original figure is 29 square inches. Now suppose that the plan has been drawn on the scale of three chains to an inch; then a square inch of the plan corresponds to nine square chains of the field; therefore the area of the field is 9×29 square chains, that is, 261 square chains.

In practice the process of forming the sum of the lengths of the rectangles is performed by an instrument called the *computation scale.*

EXAMPLES. XLVI.

Draw plans and find the areas of fields from the following notes, in which the lengths are expressed in links:

1.

	to *E*	
	550	
to *D* 100	400	
	350	110 to *C*
to *B* 155	180	
	from *A*	

2.

	to *E*	
	500	
to *C* 50	220	140 to *D*
to *B* 160	100	
	from *A*	

3.

	to *E*	
	300	
	243	180 to *D*
to *C* 136	162	
	66	122 to *B*
	from *A*	

4.

	to *E*	
	450	80 to *D*
	290	90 to *C*
to *B* 200	150	
	from *A*	

5.

	to F	
	800	
to E 120	650	
to D 70	400	
	350	110 to C
to B 150	180	
	from A	

6.

	to G	
	600	
to F 140	560	
to E 150	480	
	470	to D 170
to C 50	380	
	100	to B 150
	from A	

7.

	78	
8	53	4
14	36	8
4	21	5
	⊙	

8.

	102	
9	75	8
4	40	12
17	12	7
	⊙	

9.

	120
19	100
26	80
27	60
25	40
18	20
	⊙

10.

	130	
	110	22
26	90	
	50	40
28	30	
	⊙	

11.

60	380	20
4	260	100
4	180	76
60	100	10
20	80	60
0	0	50
	⊙	

12.

	1394	
270	1112	
220	940	
184	614	
	368	235
	160	62
	38	42
	0	
	⊙	

13. Make a rough sketch of the field ABC, and calculate its area from the accompanying field-book; the chain-lines are all within the field.

10	250 A	
50	200	
0	0	
	⊙ C	
0	390 C	
40	200	
30	100	
10	0	
	⊙ B	
0	560 B	
30	100	
0	0	
	N. 52° W.	
	⊙ A	

14. Make a rough sketch of the field $ABCD$, and calculate its area from the accompanying field-book. The side AD was not measured, but it was a straight line without offsets.

	750 D	0
	400	60
	0	0
	N. 66° E.	
	⊙ C	
	1000 C	0
	800	10
0	660	
20	600	
80	420	
	240	0
	200	10
	0	40
	S. 84° E.	
	⊙ B	
	1000 B	0
	600	80
	0	0
	S. 36° W.	
	⊙ A	

15. Lay down the field *ABCDEFG*, and find its area from the following dimensions:

	to ⊙ *D*	
	1560	
	864	20 *E*
G 690	618	
	from ⊙ *F*	
	to ⊙ *G*	
	1305	
D 690	363	
	from ⊙ *C*	
	to ⊙ *C*	
	1650	
B 362	1230	
	405	390 *G*
Begin	at ⊙ *A*	range East

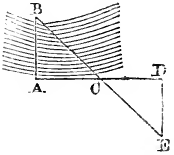
16. Make a rough sketch of a field from the following field-book: find the area of the whole field, assuming that the triangle *BCD* contains 416732 square links, and that the piece at *C* between the boundary and the offsets to *CD* and *CB* respectively contains 300 square links.

	⊙ *B*	
	944	0)
	830	60)
(0	510	
(50	480	
	120	0
	0	20
	⊙ *C*	
	1024 *C*	10
	640	30
	0	0
	⊙ *D*	
	1292 *D*	10
	1040	20
	680	80
(0	312 *B*	
(10	0	40
Begin at	⊙ *A*	

XLVII. PROBLEMS.

417. In our account of Land Surveying we have confined ourselves to illustrating the use of the chain and the cross. In surveys of great extent or of extreme accuracy, instruments are employed for measuring angles, and the calculations are effected by the aid of the science of trigonometry. With these resources also problems relating to the distance of an inaccessible object are usually solved. Nevertheless some of these problems may be treated sufficiently for practical purposes in a simpler manner: we will give examples.

418. *To find the breadth of a river.*



Let A be an object close to the river; B an object on the other side, directly opposite to A, and also close to the river.

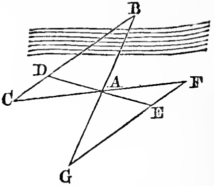
Draw a straight line AC at right angles to AB, of any convenient length, and fix a picket at C. Produce the straight line AC to a point D, such that CD is equal to AC. From D draw a straight line at right angles to AD, and in it find the point E so that BCE may be in a straight line.

Then the triangles CAB and CDE are equal in all respects, and DE is equal to AB. We can measure DE, and thus we find the length of AB, that is, the breadth of the river.

419. The preceding Article requires us to be able to draw a straight line at right angles to another straight line; we have shewn in Chapter XLV. how this may be done. We will however now solve the problem by another method which does not require a right angle.

420. *To find the distance between two points, one of which is inaccessible.*

Let A and B be the two points, B being inaccessible on account of a river, or some other obstacle.



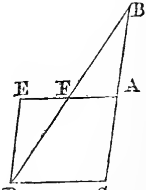
From A measure any straight line AC. Fix a picket at any point D in the direction BC. Produce CA to F, so that AF may be equal to AC; and produce DA to E so that AE may be equal to AD. Fix pickets at F and E. Then find the point G at which the directions of BA and FE intersect; that is, find the point from which A and B appear in one straight line, and E and F appear in another straight line.

Then the triangles DAB and EAG are equal in all respects, and GA is equal to AB. We can measure GA, and thus we find the length of AB.

421. In Articles 418 and 420 we assume that we have the command of sufficient ground to enable us to trace a straight line of the *same length* as that which we wish to measure. These methods would be practically inapplicable if the inaccessible objects were at a considerable distance. We shall therefore give a solution which could be used in such a case.

422. *To find the distance between two points, one of which is inaccessible and remote.*

Let A and B be the two points, B being inaccessible and remote.



Measure any length AC in the direction of BA; and from C in any convenient direction measure CD equal to CA. Take two strings, each equal to CA; fix an end of one at A, and an end of the other at D. Then stretch the strings out, so that they form straight lines, and so that their other ends shall meet at a point; let E be this point. Then $ACDE$ is a rhombus. Place a picket at F, the point at which the directions AE and BD intersect.

The triangles FAB and FED are similar. Therefore EF is to ED as AF is to AB. Thus if we measure EF and FA, we can find AB from this proportion.

XLVIII. DUODECIMALS.

423. Examples relating to square measure and to solid measure are sometimes worked by a method which is called *Cross Multiplication or Duodecimals.* This method is found convenient in practice, and the theory of it is instructive; so that the explanation which we shall now give deserves attention. We shall first consider the case of *square* measure.

424. The student of course knows perfectly well what is meant by a *square foot* and what is meant by a *square inch;* he must now become familiar with another area, namely, a rectangle which is twelve inches long and one inch broad: we will call this a *superficial prime.*

Thus we have the following addition to the Table of square measure:

12 square inches make 1 superficial prime,

12 superficial primes make 1 square foot.

425. Any number of square inches greater than 12 can be separated into superficial primes and square inches. Thus, for example,

17 square inches = 1 superficial prime 5 square inches,

32 square inches = 2 superficial primes 8 square inches,

54 square inches = 4 superficial primes 6 square inches.

Any number of superficial primes greater than 12 can be separated into square feet and superficial primes. Thus, for example,

19 superficial primes = 1 square foot 7 superficial primes,

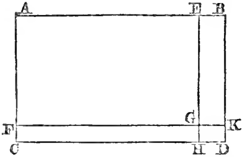
45 superficial primes = 3 square feet 9 superficial primes,

54 superficial primes = 4 square feet 6 superficial primes.

426. A rectangle which measures 1 foot by 1 inch is a superficial prime; hence a rectangle which measures 2 feet by 1 inch contains 2 superficial primes, a rectangle which measures 3 feet by 1 inch contains 3 superficial primes, and so on.

Again, a rectangle which measures 7 feet by 1 inch contains 7 superficial primes; hence a rectangle which measures 7 feet by 2 inches contains 14 superficial primes, a rectangle which measures 7 feet by 3 inches contains 21 superficial primes, and so on. Hence we arrive at a general result which is expressed briefly thus: *the product of feet into inches gives primes.*

427. Required the area of a rectangle which is 8 feet 9 inches long, and 5 feet 6 inches broad.



Let $ABDC$ represent the rectangle; AB being the length, and AC the breadth. Suppose AE to be 8 feet, and AF to be 5 feet; so that EB is 9 inches, and FC is 6 inches. Through E draw EH parallel to AC; and through F draw FK parallel to AB: let G be the point of intersection of these straight lines.

Thus the whole rectangle $ABDC$ is divided into four parts, namely:

the rectangle $EBKG$, which measures 5 feet by 9 inches, and contains therefore 45 superficial primes, that is 3 square feet 9 superficial primes;

the rectangle $AEGF$, which measures 8 feet by 5 feet, and contains therefore 40 square feet;

the rectangle $GKDH$, which measures 9 inches by 6 inches, and contains therefore 54 square inches, that is 4 superficial primes 6 square inches;

the rectangle $FGHC$, which measures 8 feet by 6 inches, and contains therefore 48 superficial primes, that is 4 square feet.

The sum of the first two rectangles is 43 square feet 9 superficial primes; the sum of the second two rectangles is 4 square feet 4 superficial primes 6 square inches: the sum of the four rectangles is therefore 48 square feet 1 superficial prime 6 square inches.

428. The work of the preceding example is recorded thus:

The length is written in one line, and the breadth in another; feet under feet, and inches under inches.

8	9	
5	6	
43	9	
4	4	6
48	1	6

We first multiply by the 5 which stands for feet: 5 times 9 are 45; 45 superficial primes are 3 square feet 9 superficial primes; set down 9 and carry 3; 5 times 8 are 40; 40 and 3 are 43; set down 43 to the left of the place in which the superficial primes were recorded.

Then we multiply by the 6 which stands for inches. 6 times 9 are 54; 54 square inches are 4 superficial primes 6 square inches; set down 6 to the right of the column in which superficial primes are recorded, and carry 4; 6 times 8 are 48; 48 and 4 are 52; 52 superficial primes are 4 square feet 4 superficial primes; set down these in their proper columns.

Then add together the two lines which have been obtained. 6 square inches are brought down; 4 and 9 are 13; 13 superficial primes are 1 square foot 1 superficial prime; set down 1 and carry 1; 1 and 4 and 43 are 48.

Thus the result is 48 square feet 1 superficial prime 6 square inches.

429. We may express the result in other forms.

Thus 1 superficial prime 6 square inches are 18 square inches; therefore the result is 48 square feet 18 square inches.

Again, 1 superficial prime is $\frac{1}{12}$ of a square foot, 6 square inches are $\frac{6}{144}$ of a square foot; therefore the result in square feet is $48+\frac{1}{12}+\frac{6}{144}$, that is $48+\frac{1}{12}+\frac{1}{24}$, that is $48\frac{3}{24}$, that is $48\frac{1}{8}$.

These forms of the result agree with those which we should obtain without the aid of cross multiplication.

Thus: 8 feet 9 inches = 105 inches, 5 feet 6 inches = 66 inches. $105 \times 66 = 6930$;

6930 square inches = 48 square feet 18 square inches.

Or thus: 8 feet 9 inches $= 8\frac{3}{4}$ feet $= \frac{35}{4}$ feet,

5 feet 6 inches $= 5\frac{1}{2}$ feet $= \frac{11}{2}$ feet,

$$\frac{35}{4} \times \frac{11}{2} = \frac{385}{8} = 48\tfrac{1}{8}.$$

430. We will now briefly consider the extension of the method to examples relating to *solid* measure; we must introduce two new terms, *solid primes* and *solid seconds*, the meanings of which are thus assigned:

12 cube inches make 1 solid second,

12 solid seconds make 1 solid prime,

12 solid primes make 1 solid foot.

431. By proceeding as in Art. 426 we shall easily arrive at results which, in conjunction with two results already known, may be expressed briefly thus:

the product of feet into superficial feet is solid feet,

the product of feet into superficial primes is solid primes,

the product of feet into square inches is solid seconds;

the product of inches into superficial feet is solid primes,

the product of inches into superficial primes is solid seconds,

the product of inches into square inches is solid inches.

432. Required the volume of a rectangular parallelepiped which is 8 feet 9 inches long, 5 feet 6 inches broad, and 4 feet 3 inches high.

We have found in Art. 428 that the area of the base is 48 square feet 1 superficial prime 6 square inches: we will now give the remainder of the process:

48	1	6	
4	3	0	
192	6	0	
12	0	4	6
204	6	4	6

We first multiply by the 4 which stands for feet. 4 times 6 are 24; 24 solid seconds are 2 solid primes; set down 0 and carry 2; 4 times 1 are 4; 4 and 2 are 6; set down 6; 4 times 48 are 192; set down 192.

Then we multiply by the 3 which stands for inches. 3 times 6 are 18; 18 solid inches are 1 solid second 6 solid inches; set down 6 to the right of the column in which solid seconds are recorded, and carry 1; 3 times 1 are 3; 3 and 1 are 4; set down 4; 3 times 48 are 144; 144 solid primes are 12 solid feet; set down 0 in the column of solid primes and 12 in the column of solid feet.

Then add together the two lines which have been obtained. Thus the result is 204 solid feet 6 solid primes 4 solid seconds 6 solid inches.

433. As in Art. 429 we may express the result in other forms, and shew that it agrees with what we should obtain without the aid of cross multiplication. If we express the result in terms of solid feet it becomes

$$204 + \frac{6}{12} + \frac{4}{144} + \frac{6}{1728},$$

that is, $204 + \frac{1}{2} + \frac{1}{36} + \frac{1}{288}$, that is $204\frac{153}{288}$.

434. The inches on a carpenter's rule are divided into *twelfths*; and consequently examples relating to areas and volumes may occur in practice in which the dimensions involve twelfths of an inch. These examples are similar in principle to those we have already considered. We have to introduce other new terms; so that the whole series of terms for areas will be assigned by the following table:

a superficial prime is one-twelfth of a square foot,

a superficial second is the same as a square inch, namely, one-twelfth of a superficial prime,

a superficial third is one-twelfth of a superficial second,

a superficial fourth is one-twelfth of a superficial third.

435. Required the area of a rectangle which measures 8 feet 9 inches 10 twelfths by 5 feet 6 inches 7 twelfths.

8	9	10		
5	6	7		
44	1	2		
4	4	11	0	
	5	1	8	10
48	11	2	8	10

The result is 48 square feet 11 superficial primes 2 superficial seconds 8 superficial thirds 10 superficial fourths.

EXAMPLES. XLVIII.

Find by duodecimals the areas of rectangles having the following dimensions:

1. 4 feet, 2 feet 3 inches.

2. 5 feet, 3 feet 4 inches.

3. 3 feet 8 inches, 2 feet 6 inches.

4. 4 feet 5 inches, 3 feet 9 inches.

5. 5 feet 7 inches, 4 feet 10 inches.

6. 5 feet 11 inches, 4 feet 7 inches.

7. 4 feet 3 inches 4 twelfths, 3 feet 3 inches.

8. 4 feet 8 inches 5 twelfths, 3 feet 4 inches.

9. 5 feet 4 inches 8 twelfths, 2 feet 7 inches 3 twelfths.

10. 6 feet 8 inches 7 twelfths, 3 feet 4 inches 5 twelfths.

Find by duodecimals the volumes of rectangular parallelepipeds having the following dimensions:

11. 3 feet, 3 feet, 1 foot 6 inches.

12. 5 feet, 3 feet, 2 feet 3 inches.

13. 4 feet, 3 feet 4 inches, 3 feet 3 inches.

14. 5 feet, 4 feet 8 inches, 3 feet 2 inches.

15. 6 feet 3 inches, 5 feet 3 inches, 3 feet 9 inches.

16. 7 feet 5 inches, 6 feet 7 inches, 3 feet 10 inches.

XLIX. METRICAL SYSTEM.

436. The French system of measures called the *metrical system* is frequently used in English scientific works; so that we shall here explain the system.

437. The standard of length is the *metre* which is equal to 39·37079 English inches The metre was intended to be one-ten-millionth part of the distance from the pole to the equator measured on the Earth's surface; recent investigations shew that the metre is about $\frac{1}{208}$ of an inch shorter than it should have been to correspond to this intention.

The standard of area is the *are*, which is 100 square metres.

The standard of volume is the *stere*, which is a cubic metre.

All the multiples and the sub-divisions of any measure are decimal and are formed in the same manner; the multiples by syllables derived from the Greek, and the sub-divisions by syllables derived from the Latin. Thus

Myriameter = 10000 metres,
Kilometre = 1000 metres,
Hectometre = 100 metres,
Decametre = 10 metres,
Decimetre = $\frac{1}{10}$ metre,
Centimetre = $\frac{1}{100}$ metre,
Millimetre = $\frac{1}{1000}$ metre.

Similarly a hectare = 100 ares, a centiare = $\frac{1}{100}$ are, a dekastere = 10 steres, a decistere = $\frac{1}{10}$ stere.

For liquid measures the standard is a *litre*, which is a cubic decimetre.

For weight the standard is a *gramme*, which is the weight of a cubic centimetre of water: it is equal to 15·432 English grains.

EXAMPLES. XLIX.

1. The diameter of a circle is 15 metres: find the circumference.

2. Find the area of a rectangle which is 407·75 metres long, and 304 metres wide.

3. The parallel sides of a trapezoid are 157·6 metres and 94 metres; and the perpendicular distance between them is 72 metres: find the area.

4. Find in cubic metres the volume of a wall which is 48 metres long, 3·4 metres high, and ·45 of a metre thick.

5. Find to the nearest cubic decimetre the volume of a right cone, the height being 2·4 metres, and the radius of the base ·4 of a metre.

6. A vessel when empty weighs 1·67 kilogrammes, and when full of water weighs 6·9 kilogrammes: find the capacity of the vessel in cubic decimetres.

7. A cistern is 8·5 metres long, 6 metres wide, and 9·2 metres deep: find how many hectolitres of water it will hold.

8. The weight of water which a certain cylinder will hold is 36 kilogrammes; the radius of the cylinder is 15 centimetres: find the height of the cylinder to the nearest centimetre.

9. A telegraph wire is 35 kilometres long, and $2\frac{1}{2}$ millimetres in diameter: find the volume in cubic decimetres.

10. The diameter of an iron ball is ·2 of a metre: find its weight in kilogrammes, supposing that any volume of iron is 7·5 times as heavy as an equal volume of water.

11. Find to the nearest square metre the area of the whole surface of a right cone, the radius of the base being 3 metres, and the slant height 8 metres.

12. Find to the nearest square metre the area of the surface of a sphere of which the diameter is 9 metres.

13. Shew that an acre contains about 40·47 ares.

14. Shew that a cubic yard contains about 765·4 cubic decimetres.

15. Shew that a gallon contains about 4·54 litres.

The following examples involve the extraction of the cube root:

16. A hollow sphere holds a litre: find the radius of the sphere.

17. A vessel in the form of a right circular cylinder with its height equal to the diameter of its base holds a litre: find the height.

18. The volume of a right circular cone with its height equal to the radius of its base is a cubic decimetre: find the height.

MISCELLANEOUS EXAMPLES.

1. The top of a may-pole, being broken off by a blast of wind, struck the ground at a distance of 15 feet from the foot of the pole: find the height of the whole may-pole, supposing the length of the broken piece to be 39 feet.

2. Find how many square feet there are in 1200 square inches.

3. The area of two squares is 100 acres, and the side of one is three times as long as the side of the other: find the area of each.

4. A square contains 2533 feet 64 inches: find its side.

5. The perimeter of a rectangle is 144 yards, and the length is three times the breadth: find the area.

6. Find the expense of carpeting a room 21 feet long and 20 feet broad, with carpet 27 inches wide at 4*s.* 6*d.* per yard.

7. A room is 16 feet $2\frac{1}{2}$ inches long, 15 feet $3\frac{1}{2}$ inches broad, and 12 feet high: find the expense of covering the walls with paper, 9 inches wide, at $2\frac{1}{2}d.$ per yard.

8. A building has 63 windows: 40 of them contain 12 panes each 20 inches by 16; the others contain 9 panes each 16 inches square: find the cost of glazing the whole at 2*s.* 3*d.* per square foot.

9. The sides of a triangle are 890, 990, and 1000 links: find the area.

10. The area of a trapezoid is 475 square feet; the perpendicular distance between the two parallel sides is 19 feet: find the two parallel sides, their difference being 4 feet.

11. Two roads cross at right angles; two men start at the same time from the point where the roads meet, one man walking along one road at the rate of 4 miles an hour,

and the other man walking along the other road at the rate of 3 miles an hour: find how far apart the men are ten minutes after starting.

12. Find how many square yards there are in $\frac{1}{16}$ of a square mile.

13. The perimeter of one square is 748 inches, and that of another is 336 inches: find the perimeter of a square which is equal in area to the two.

14. A square contains 3690 feet 81 inches: find its side.

15. Find how long it will take to walk round a field containing 13 acres 1089 yards in the form of a square at the rate of $2\frac{1}{2}$ miles an hour.

16. Find the expense of carpeting a room 16 feet 8 inches long, and 12 feet broad, with carpet a yard wide at 6 shillings a yard.

17. A room is 24 feet long, 20 feet broad, and 14 feet 3 inches high: find the expense of covering the walls with paper 30 inches wide at $11\frac{1}{4}d.$ per yard; allowing 8 feet by 5 feet 3 inches for each of 4 doors, 10 feet by 6 feet 8 inches for each of 3 windows, and 6 feet 6 inches by 4 feet for a fire-place.

18. The length of a room is double the breadth; the cost of colouring the ceiling at $4\frac{1}{2}d.$ per square yard is £2. 12*s.* 1*d.*, and the cost of painting the four walls at 2*s.* 4*d.* per square yard is £35: find the height of the room.

19. The sides of a triangle are 848, 900, and 988 links: find the area.

20. A rectangle is 41 yards 1 foot 3 inches by 10 yards 10 inches: find how many circles of one inch radius are equivalent in area to this rectangle; assuming that the area of a circle of one inch radius is $\frac{355}{113}$ square inches.

21. Construct a diagonal scale; explain its use; shew where the points of the compasses should be placed to measure off a length 1·37.

22. Determine the excess of 15 feet square over 15 square feet.

23. There are two rectangular fields equal in area; the sides of one field are 945 yards and 1344 yards in length, and the longer side of the other field is 1134 yards: find the length of the shorter side, and express the area of each field in acres, roods, poles, and square yards.

24. Find the side of a square which contains 7367 square feet 52 inches.

25. The area of a rectangular field whose length is three times its breadth is 6 acres 960 yards: find its perimeter. Find also the distance from corner to corner.

26. Find the expense of carpeting a room 18 feet 9 inches long, and 17 feet 6 inches broad, with carpet 27 inches wide at 5*s.* 3*d.* per yard.

27. A room is 15 feet long, 10 feet broad, and 9 feet 9 inches high: find the expense of painting the walls and the ceiling at 1*s.* 9*d.* per square yard.

28. A room three times as long as it is broad is carpeted at 4*s.* 6*d.* per square yard; and the walls are coloured at 9*d.* per square yard; the respective costs being £8. 5*s.* $4\frac{1}{2}$*d.* and 4 guineas: find the dimensions of the room.

29. The perimeter of an isosceles triangle is 306 feet, and each of the equal sides is five-eighths of the third side: find the area.

30. $ABCDE$ is a five-sided figure, and the angles at B, C, and D are right angles: if $AB = 20$ feet, $BC = 18$ feet, $CD = 32$ feet, and $DE = 13$ feet, find the area of the figure, and the length of AE.

31. The span of a bridge, the form of which is an arc of a circle, being 96 feet, and the height being 12 feet, find the radius.

32. A square yard is divided into 576 equal squares: find the length of a side of each.

33. Find the number of acres in the area of the base of the great pyramid of Egypt. See page 154.

34. Find the side of a square whose area is equal to a rectangle 81 feet long and $60\frac{3}{4}$ inches wide.

35. A rectangular field is 300 yards long and 200 broad: find the distance from corner to corner. If a belt of trees 30 yards wide be planted round the field, find the area of the interior space.

36. In a rectangular court which measures 96 feet by 84 feet, there are four rectangular grass-plots measuring each $22\frac{1}{2}$ feet by 18: find the cost of paving the remaining part of the court at $8\frac{1}{2}d.$ per square yard.

37. A room is 30 feet long, 15 feet broad, and 15 feet high: find the expense of covering the walls with paper $4\frac{1}{2}$ feet wide at $4\frac{1}{2}d.$ per yard.

Also find the cost for a room twice as long, twice as broad, and twice as high, the paper being half as wide and costing half as much per yard as before.

38. A room whose height is 11 feet, and length twice its breadth, takes 165 yards of paper 2 feet wide for its four walls: find how much carpet it will require.

39. The sides of a triangle are 25, 39, and 56 feet respectively: find the areas of the two triangles into which it is divided by the perpendicular from the angle opposite the largest side on that side.

40. Make a rough sketch and find the area of a field $ABCD$ from the following measures taken in links:

BM the perpendicular from B on $AC = 740$,
DN the perpendicular from D on $AC = 816$,
$AC = 1220$, $AM = 532$, $AN = 486$.

41. The chord of an arc is 24 feet, and the height of the arc is 5 feet: find the length of the arc.

42. A room is 15 feet 9 inches long: find its breadth, that it may contain 21 square yards.

43. The area of a rectangle is 372 feet $32\frac{5}{6}$ inches square measure; and one side is 20 feet $5\frac{1}{2}$ inches: find the other side.

44. Find the length of the side of a square which is equal in area to a rectangle 972 yards long and 363 yards broad.

45. If a rectangular piece of building land 375 feet 6 inches long, and 75 feet 6 inches broad, cost £118. 2*s.* 6¼*d.*, find the price of a piece of similar land 278 feet 9 inches long and 157 feet broad.

46. Find the cost of paving a street half a mile long and 47 feet broad, at the cost of 7½*d.* per square yard.

47. A room is 20 feet 6 inches long, 15 feet 6 inches broad, and 16 feet high: find the expense of covering the walls with paper, 30 inches wide, at 7½*d.* per yard; allowing for two doors each 8 feet by 3 feet 9 inches, one window 5 feet by 7 feet, two other windows each 5 feet by 4 feet, and a fire-place 4 feet 8 inches by 3 feet.

48. The carpeting of a room twice as long as it was broad at 5 shillings per square yard cost £6. 2*s.* 6*d.*; and the painting of the walls at 9*d.* per square yard cost £2. 12*s.* 6*d.*: find the dimensions of the room.

49. $ABCD$ is a four-sided figure; BC is parallel to AD; $AB=BC=CD=325$ feet; and $AD=733$ feet: find the area.

50. The diameter of a circle is 12 feet: find the area of a square inscribed in it.

51. The sides of one end of a frustum of a triangular pyramid are 12, 15, and 20 inches respectively; the longest side of the other end is 25 inches: find the other sides of this end.

52. The expense of paving a rectangular court which measures 48 feet by 24 feet is £42: find the expense of paving another rectangular court which measures 60 feet by 32 feet.

53. The area of a rectangle is 3075 feet 70⅚ inches square measure, and one side is 81 feet 9¼ inches: find the other side.

54. Assuming that three hectares contain 35881 square yards, and that one hectare contains 10000 square metres, find the length of a metre in terms of a yard.

55. Find how many planks, each 13½ feet long and 10½ inches wide, will be required for the construction of a platform 54 yards long and 21 yards broad. Find the cost of the wood at 5½*d.* per square foot.

56. Find the difference in expense of carpeting a room 17 feet 9 inches long and 12 feet 6 inches broad, with Brussels carpet $\frac{3}{4}$ of a yard wide at 4*s.* 6*d.* per yard, and with Kidderminster carpet $\frac{7}{8}$ yard wide at 3*s.* 6*d.* per yard.

57. A room is 15 feet 9 inches long, 9 feet 3 inches broad, and 10 feet high: find how many postage stamps will be required to cover the walls, allowing for two windows, each 5 feet by 4 feet, and three doors, each 6 feet 8 inches by 3 feet. A postage stamp is $\frac{15}{16}$ inch long, and $\frac{3}{4}$ inch wide.

58. A person has a triangular-shaped garden, the base of which contains 200 yards, and divides it into two equal parts by a hedge parallel to the base: find the length of the hedge.

59. Verify the following statement by examples: the area of the space between two concentric circles is equal to the area of a circle which has for its diameter a chord of the outer circle which touches the inner circle.

60. Find the expense of paving a circular court 30 feet in diameter, at 2*s.* 3*d.* per square foot, leaving in the centre a space in the form of a regular hexagon, each side of which measures 2 feet.

61. Find how many bricks there are in a wall which is 120 bricks long, 15 bricks high, and 2 bricks thick.

62. Find the thickness of a solid whose length is 2 yards, breadth $1\frac{1}{2}$ yards, and solid content 1 cubic yard 6 cubic feet and 1296 cubic inches.

63. A room 18 feet 9 inches long and 13 feet 4 inches broad is flooded with water to a depth of 2 inches: find the weight of the water, supposing a cubic foot of water to weigh $62\frac{1}{2}$ lbs.

64. If gold can be beaten so thin that a grain will form a leaf of 56 square inches, find how many of these leaves will make an inch thick, the weight of a cubic foot of gold being 10 cwt. 95 lbs.

65. A cube contains $2\cdot\dot{3}7\dot{0}$ cubic yards: find how many linear feet there are in an edge.

66. A closed vessel formed of metal one inch thick, and whose external dimensions are 7 feet 3 inches, 6 feet 5 inches, and 4 feet 3 inches, weighs 2 cwt. 2 qrs. 7 lbs.: find the weight of a solid mass of the metal of the same dimensions.

67. A monolith of red granite in the Isle of Mull is said to be about 115 feet long, and to have an average transverse section of 113 square feet. If shaped for an obelisk it would probably lose one-third of its bulk and then weigh about 639 tons. Determine the number of cubic yards in such an obelisk, and the weight in pounds of a cubic foot of granite.

68. The top of a circular table is 7 feet in diameter, and 1 inch thick: find how many cubic feet of wood it contains, and what will be the cost of polishing its upper surface at sixpence per square foot.

69. Find how many gallons of water can be held in a leathern hose 2 inches in bore and 40 feet long.

70. The ends of a frustum of a pyramid are right-angled triangles; the sides containing the right angle of one end are 2 feet and 3 feet; the smallest side of the other end is 8 feet; the height of the frustum is 7 feet: find the volume.

71. Find how many bricks of which the length, breadth, and thickness are 9, $4\frac{1}{2}$, and 3 inches will be required to build a wall of which the length, height, and thickness are 72, 8, and $1\frac{1}{2}$ feet.

72. Find the length of a solid whose thickness is 1 foot, breadth 18 inches, and solid content 3 cubic feet 216 cubic inches.

73. A box is 4 feet long, 2 feet 6 inches wide, and 1 foot 6 inches deep; in it are packed 252 books each 8 inches long, 5 inches wide, and $1\frac{1}{2}$ inches thick: find how many more books each 6 inches long 3 inches wide and $1\frac{1}{4}$ inches thick can be packed in the box.

74. Find the height of a parallelepiped which contains 659 feet 1248 inches, and whose base is 26 feet 6 inches.

75. Find the edge of a cube which contains 10970·645048 cubic inches.

76. The surface of a cube contains 393·66 square feet: find the length of an edge and the cubical content.

77. Find the volume of a cone, the height of which is one yard and the radius of the base one foot.

78. A garden roller of iron is half an inch thick, the length is 30 inches, and the diameter of the inner surface is 20 inches: find the weight, supposing a cubic inch of iron to weigh 4·562 ounces.

79. A square tower 21 feet on each side is to have either a flat roof covered with sheet lead which costs 6 pence per square foot, or a pyramidal roof whose vertical height is 10 feet, covered with slates which cost 18*s.* 9*d.* per hundred, and each of which has an exposed surface of 12 inches by 9 inches. Find the cost in each case.

80. The surface of a certain solid is three times as great as the surface of a similar solid: find the proportion which the volume of the first solid bears to the volume of the second.

81. The three conterminous edges of a rectangular parallelepiped are 36, 75, and 80 inches respectively: find the edge of a cube which will be of the same capacity.

82. A river 30 feet deep and 200 yards wide is flowing at the rate of 4 miles an hour: find how many tons of water run into the sea per minute.

83. Find the number of inches in the side of a cube whose solid content is 5·359375 cubic feet.

84. The content of a cistern is the sum of two cubes whose edges are 10 inches and 2 inches, and the area of its base is the difference of two squares whose sides are $1\frac{1}{9}$ and $1\frac{2}{9}$ feet: find the depth of the cistern.

85. The cost of a cube of metal at £3. 10*s.* 4*d.* per cubic inch is £1206. 4*s.* 4*d.*: find the cost of gilding it over at $\frac{1}{2}$*d.* per square inch.

86. If gold can be beaten out so thin that a grain will form a leaf of 56 square inches, find how many of these leaves will be required to make up the thickness of a sheet of paper; a cubic foot of gold weighing 1215 lbs. Avoirdupois, and 400 sheets of paper making a book one inch thick.

87. A hemispherical basin 15 feet in diameter will hold 120 times as much as a cylindrical tub, the depth of which is 1 foot 6 inches: find the diameter of the tub.

88. A right-angled triangle of which the sides are 3 and 4 inches in length is made to turn round on the longer side: find the volume and the area of the whole surface of the cone thus formed.

89. The height of a frustum of a pyramid is 4 inches; the lower end is a rectangle which is 9 inches by 12; the upper end is a rectangle of which the longer side is 8 inches; find the volume of the frustum.

90. The area of the surface of a sphere is 25 square inches: find the volume of the sphere.

91. Find how many superficial feet of inch plank can be sawn out of a log 20 feet 4 inches long, 1 foot 10 inches wide, and 1 foot 6 inches deep.

92. Find the weight of water in a bath 6 feet long, 3 feet wide, and 1 foot 9 inches deep.

93. Find the length of the side of a cube which contains 344324701729 cubic inches.

94. Find how many cubical packages each having $4\frac{1}{3}$ inches in an edge will fill a box whose length, breadth, and depth are 2 feet 2 inches, 3 feet 3 inches, and 4 feet 4 inches.

95. The flooring of a room 14 feet 3 inches long, and 13 feet 4 inches broad is composed of half-inch planks, each 8 inches wide and 10 feet long: find how many will be required, and the weight of the whole if one cubic inch of wood weighs half an ounce.

96. Supposing a cubical room to contain 46656 cubic feet, find the expense of carpeting it with carpet 27 inches wide at 4*s.* 6*d.* per yard.

97. A pyramidal roof 16 feet high, standing on a square base which is 24 feet on each side, is covered with sheet lead $\frac{1}{16}$ of an inch thick: find the weight of the lead, supposing a cubic inch to weigh 7 ounces.

If the lead is stripped off and cast into bullets, each of which is in the form of a cylinder, $\frac{1}{2}$ inch long and $\frac{6}{11}$ inch in diameter, terminated at one end by a cone of the same diameter and $\frac{3}{8}$ inch high, find how many bullets there will be.

98. A spherical cannon ball, 9 inches in diameter, is melted and cast into a conical mould the base of which is 18 inches in diameter: find the height of the cone.

99. A cylinder, 24 feet long and 4 feet in diameter, is closed by a hemisphere at each end: find the area of the whole surface.

100. A tin funnel consists of two parts; one part is conical, the slant height is 6 inches, the circumference at one end 20 inches, and at the other end $1\frac{1}{4}$ inches; the other part is cylindrical, the circumference being $1\frac{1}{4}$ inches and the length 8 inches: find the number of square inches of tin.

101. Find how many cubes whose edges are $2\frac{3}{4}$ inches, can be cut out of a cube of which the edge is 22 inches.

102. The breadth of a room is two-thirds of its length and three halves of its height; the content of the room is 5832 cubic feet: find the dimensions of the room.

103. Find the length of the side of a cube which contains 733626753859 cubic inches.

104. One solid contains $30\frac{3}{8}$ cubic feet; another solid contains $4\frac{1}{2}$ cubic yards: find what multiple the latter is of the former.

105. Find the number of gallons of water which pass in 10 minutes under a bridge 17 feet 8 inches wide; the stream being 10 feet 11 inches deep, and its velocity 4 miles an hour.

106. A cubic inch of metal expands so that each face is increased by ·0201 of its former area: find the increase of volume.

107. A sphere and a cube have the same surface: shew that the volume of the sphere is 1·3820 times that of the cube.

108. A sphere has the same surface as a right circular cylinder with its height equal to twice the radius of its base: shew that the volume of the sphere is 1·2247 times that of the cylinder.

109. A sphere and a cube have the same volume: shew that the surface of the cube is 1·2407 times that of the sphere.

110. A sphere has the same volume as a right circular cylinder with its height equal to twice the radius of its base: shew that the surface of the cylinder is 1·1447 times that of the sphere.

111. Find to the nearest lb. the weight of the water which a cistern will hold, whose length, breadth, and depth are 5 feet 6 inches, 3 feet 9 inches, and 1 foot 3 inches respectively.

112. A bed of gravel 4 feet 6 inches in depth extends over the whole of a field of 3 acres 3 roods: find the value of the gravel at sixpence per cubic yard.

113. The volume of a cube is 5 cubic feet 621 cubic inches: find the length of a diagonal of the cube.

114. Supposing a brick to be 9 inches long, $4\frac{1}{2}$ inches wide, and 3 inches thick, and to weigh 5 lbs., find the weight of a stack of bricks 10 feet high, 6 feet wide, and 3 feet thick.

115. A pyramid on a square base has four equilateral triangles for its four other faces, each edge being 20 feet: find the volume.

116. The radius of the base of a cylinder is 3 feet, and the height is 20 feet: find the volume.

117. A ditch of a certain length is 4 feet deep, 16 feet broad at the top, and 12 feet broad at the bottom: if the ditch be half filled with water, find the depth of the water.

118. The radius of one end of a frustum of a cone is 18 inches: the radius of the other end is 12 inches; the height is 15 inches: find the volume.

119. If the weight of an iron ball, 4 inches in diameter, is 9 lbs., find the weight of a shell whose external and internal diameters are 7 inches and 3 inches respectively.

120. The area of the whole surface of a right circular cone is 32 square feet, and the slant height is three times the radius of the base: find the volume of the cone.

121. The area of the base of a parallelepiped is a square yard, and the height is 2 feet 6 inches: find the volume.

122. A layer of coal 5 feet thick underlies the whole of an estate of 120 acres: find the value of the coal at 12 shillings per ton, supposing a cubic yard of the coal to weigh a ton.

123. Find the number of cubic inches in a piece of plate glass 5 feet long, 3 feet 4 inches wide, and $\frac{3}{8}$ths of an inch thick.

124. The area of the coal field of South Wales is 1000 square miles, and the average thickness of the coal is 60 feet. If a cubic yard of coal weigh a ton, and the annual consumption of coal in Great Britain be 70000000 tons, find the number of years for which this coal field alone would supply Great Britain with coal at the present rate of consumption.

125. A pyramid is cut into two pieces by a plane parallel to the base, midway between the vertex and the base: shew that one piece is equal to seven times the other.

126. A cylinder is 5 feet in diameter, and its volume is 785·4 cubic feet: find the height of the cylinder in feet.

127. The ends of a prismoid are rectangles, the corresponding dimensions of which are 8 feet by 7 and 10 feet by 6, and the height is 4 feet: find the volume of the prismoid.

128. The height of the frustum of a cone is 7 feet, and the radii of the two ends are 4 feet and 5 feet respectively; the frustum is cut into two pieces by a plane parallel to the ends and distant 3·884 feet from the smaller end: shew that the two pieces are of equal volume.

129. A solid ball 4 inches in radius of a certain material weighs 8 lbs.: find the weight of a spherical shell of that material, the internal diameter of which is 8 inches, and the external diameter 10 inches.

130. The circumference of a great circle of a ball is 15·708 feet: find the whole surface of the ball.

131. A stone wall is built 10 feet 6 inches high and 2 feet 3 inches thick; the excavation from which the stone was taken measures in length, breadth, and depth, 30 feet, 28 feet, and 18 feet respectively; find the length of the wall.

132. A cistern measures in length, depth, and breadth 5 feet, 4 feet, and 3 feet respectively; the cistern after being filled with water can be emptied by a pipe in an hour and a half: find how many gallons are discharged through the pipe in a minute.

133. Find how many cubic feet of deal are contained in 200 planks each 15 feet long, 10 inches wide, and $1\frac{1}{2}$ inches thick.

134. A pond whose area is 4 acres is frozen over with ice to the uniform thickness of 6 inches: if a cubic foot of ice weigh 896 ounces Avoirdupois, find the weight of ice on the pond in tons.

135. Suppose that the coal consumed in a month in England were formed into a square pyramid on a base equal to that of the great pyramid of Egypt: find the height of the pyramid which would be thus formed. See pages 154 and 278.

136. A trench is dug 8 feet deep, 14 feet wide at the top and 10 feet wide at the bottom; and the earth removed is thrown up by the side of the trench so as to form a bank sloping on each side at the same angle to the horizon, the height of the bank being three-fourths of its base: find the height of the bank.

137. Find the volume of a cylinder 40 feet high, the radius being 8 feet.

138. A bucket is in the form of a frustum of a cone; the diameter at the bottom is 1 foot, and at the top 1 foot 3 inches; the depth is 1 foot 6 inches: find to the nearest pound how much more the bucket weighs when full of water than when empty.

139. If 30 cubic inches of gunpowder weigh 1 lb., find the diameter of a hollow sphere which will hold 11 lbs.

140. The height of a zone of a sphere is $2\frac{1}{2}$ feet, and the diameter of the sphere is $6\frac{1}{2}$ feet: find the area of the curved surface.

141. A field contains 1 acre, 2 roods, 16 poles: find how many cubic yards of earth will be required to raise the surface of the field 18 inches.

142. A cubic inch of gold is beaten into gold leaf sufficient to cover 7 square feet: find the thickness of the gold leaf.

143. Find how many cubic feet of air a room will contain which is 24 feet 9 inches long, 18 feet 4 inches broad, and 10 feet 8 inches high.

144. In laying the foundation of a house an excavation is made 40 feet long, 30 feet broad, and 6 feet deep; the earth removed is spread uniformly over a field containing half an acre: find how much the surface of the field will be raised supposing that each cubic foot of earth is increased $\frac{1}{10}$ of a cubic foot by the removal.

145. Find the volume of a pyramid the height of which is 12 inches, and the base an equilateral triangle each side of which is 10 inches.

146. Find what length of wire ·08 of an inch in diameter can be formed out of a cubic inch of metal.

147. A gutter is formed by joining two equal planks so as to have two of their longest edges in contact, the planks are 5 inches wide, and they are fastened together so that the extreme breadth of the gutter is 8 inches: if the gutter is 4 yards long, find how many cubic inches of water it will hold.

148. The radii of the ends of a frustum of a right circular cone are 4 inches and 5 inches respectively; the height of the frustum is 3 inches: find the volume of the whole cone.

149. If a 12 lb. shot have a diameter of 4·4 inches, find the weight of a shot the diameter of which is 3·96 inches.

150. The area of the curved surface of a right circular cylinder is 600 square inches; and the height of the cylinder is 25 inches: find the radius of the base.

151. A rectangular field is 440 yards long, and 154 yards wide: find its area in acres. Also find the areas of the portions into which it is divided by a straight line drawn from the middle point of one side to one of the opposite corners.

152. The walls of a room 21 feet long, 15 feet 9 inches wide, and 11 feet 8 inches high, are painted at the expense of £9. 12*s.* 6*d.*: find the additional expense of painting the ceiling at the same rate.

153. A parallelogram has two sides which are 8 feet 9 inches long, and two which are 7 feet 4 inches long, and a diagonal which is 11 feet 7 inches long: determine whether the parallelogram is a rectangle.

154. From a point in the circumference of a circle two chords are drawn at right angles, and their lengths are 13 and 17 inches respectively: find the area of the circle.

155. The area of the Yorkshire coal field is $937\frac{1}{2}$ square miles, and the average thickness of the coal is 70 feet. If a cubic yard of coal weigh a ton and the annual consumption of coal in England be 70000000 tons, find the number of years for which this coal field alone would supply Great Britain with coal at the present rate of consumption.

156. If the coal consumed in one year in England were piled up into a rectangular stack having for base an area of ten acres, find the height of the stack to the nearest yard.

157. A cubic foot of gold is extended by hammering so as to cover an area of 6 acres: find the thickness of the gold in decimals of an inch, correct to the first two significant figures.

158. Find to three decimal places the number of gallons in a cubic foot.

159. A cubic inch of brass is to be drawn into a wire $\frac{1}{25}$ of an inch in diameter: find the length of the wire to the nearest inch.

160. The sides of a right-angled triangle are 3 inches and 4 inches respectively: find the volume of the double cone formed by the revolution of this triangle round its hypotenuse.

161. The base of an aquarium is a square, the height is half a side of the base, and there is no lid; the glass cost £1. 11*s*. 3*d*. at 15*s*. a square yard: find the number of gallons the aquarium will hold.

162. Supposing a cubic foot of brass to weigh 8500 ounces, find the weight of a yard of brass wire the thickness of which is $\frac{1}{30}$ of an inch.

163. A right-angled triangle of which the sides are 5 and 12 inches in length is made to turn round the hypotenuse: find the volume and the surface of the double cone thus formed.

164. The weights of two globes are as 9 to 25; the weights of a cubic inch of the substances are as 15 to 9: compare the diameters of the globes.

165. A gold wire of ·01 of an inch in thickness is bent into a ring of one inch internal diameter: if the area enclosed by the ring be gilded with a weight of gold equal to the weight of the ring, find the thickness of the gilding.

166. A solid is composed of a cone and a hemisphere on opposite sides of the same circular base, the diameter of which is 2 feet, and the vertical angle of the cone is a right angle: the solid is immersed in a cylinder full of water, whose circular section also has a diameter of 2 feet, so that the vertex of the cone rests on the centre of the cylindrical base, while the highest part of the hemisphere just coincides with the surface of the water: find the quantity of water remaining in the cylinder.

167. A hemispherical ball of 6 feet in diameter is partially buried with its mouth downwards, and in a horizontal position, so that only one-third of the height appears above the ground: find what quantity of earth must be dug out in order to leave the ball entirely uncovered, and just surrounded by a cylindrical wall of earth.

168. The frustum of a right cone is 6 feet high, the radius of the smaller end is 2 feet, and the radius of the larger end is 3 feet: find the position of the plane parallel to the ends which will divide the frustum into two equal parts: find also the volume of each part.

169. Find to the nearest square inch the quantity of leather required to cover a spherical foot ball which measures 23 inches in circumference.

170. Find the volume of a cask in gallons, the length being 47·5 inches, the bung diameter 28·6, and the head diameter 26·5.

171. An area in the form of an equilateral triangle is paved at the rate of 9*d.* per square foot, and it is fenced at the rate of 5 shillings per foot: shew that the cost of fencing is to the cost of paving as $80\sqrt{3}$ is to three times the number of feet in a side.

172. Find the side of an equilateral triangle, supposing it cost as much to pave the area at 9*d.* per square foot as to fence the three sides at 5 shillings per foot.

173. A rectangle is ·202 of an inch longer than a certain square, and ·2 of an inch narrower; but contains the same area: shew by a figure that the number of inches in the side of the square is the product of ·202 into ·2, divided by their difference.

174. In measuring the edges of a cubical box to ascertain its content, an error of ·202 of an inch is made in excess for the length, and of ·2 of an inch in defect for the breadth, the height being properly measured; the calculated volume agrees with the true volume: find the volume in cubic inches.

175. The radius of a circle is $\sqrt{2}$ inches; two parallel straight lines are drawn in it, each an inch from the centre: find the area of the part of the circle between the straight lines.

176. A square hole 2 inches wide is cut through a solid cylinder of which the radius is $\sqrt{2}$ inches, so that the axis of the hole cuts at right angles the axis of the cylinder: find how much of the material is cut out.

177. A vessel is to be made in the form of a rectangular parallelepiped on a square base, and another vessel of the same capacity in the form of a right circular cylinder; the vessels are to have no lids, and each vessel is to be made of the most advantageous shape for the sake

of saving material: shew that the material in the cylindrical vessel is about ·92 of the material in the other vessel.

178. Shew that the same result as in the preceding Example holds if the vessels are to have lids.

179. A pyramid on a square base has every edge 100 feet long: find the edge of a cube of equal volume.

180. Verify by calculating various cases the following statement: a right circular cone is divided into a cone and a frustum of a cone, and the frustum is trimmed just enough to reduce it to a right circular cylinder; if the height of the frustum is one-third of the height of the original cone the volume of the cylinder is greater than in any other case, and is four-ninths of the volume of the original cone.

ANSWERS.

[The answers when not exact are given to the *nearest* figure; so that sometimes they are a little too great and sometimes a little too small.]

V. 1. 557 feet. 2. 8545 feet. 3. 382 ft. 10 in. 4. 945 yards 1 foot. 5. 554·92. 6. 5858·66. 7. 388·69. 8. 1840·78. 9. 333 feet. 10. 8225 feet. 11. 98 ft. 9 in. 12. 259 yds. 2 ft. 13. 482·54. 14. 3270·31. 15. 321·77. 16. 1824·14. 17. 19488 + 5687 ft. 18. 12637, 12012 ft. 19. 7 ft. 20. 32 + 24 ft. 21. 14 + 30 ft. 22. 1·4142135624 in. 23. 155·56 ft. 24. 84·32 ft. 25. 98 yds. 26. 9·64 ft. 27. 5·66 ft. 28. 8·485 ft. 29. 5·74 ft. 30. 21·91 in. 31. 11·832, 11·314, 10·392. 8·944, 6·633. 32. 9·75 ft.

VI. 1. 5·6 inches. 2. 25·98 in. 3. 40 ft. 4. $67\frac{1}{2}$ ft. 5. 5 ft. $2\frac{1}{2}$ in. 6. A quarter of an inch to a mile. 7. 68 miles. 8. 30 in. 9. $9\frac{5}{8}$ in. 10. $4\frac{2}{3}$ in. 11. 25 ft. 12. 10 ft., 12 ft.

VII. 1. $16\frac{1}{5}$ ft. 2. 22·42 ft. 3. $5\frac{1}{3}$ in. 4. 1·74 ft. 5. 3 ft. 9 in. 6. 9·65 ft. 7. 29 ft. 8. 25·03 ft. 9. 8·23 in. 10. 44·72 in. 11. 4 ft. 12. 4·75 ft. 13. 50·7 ft. 14. 125·63 ft.

VIII. 1. 44 ft. 2. 271 yds. 1 ft. 3. 672 yds. 8 in. 4. 4 furlongs. 5. 84·8232 ft. 6. 581·196 ft. 7. 5235·4764 ft. 8. 942·48 yds. 9. 21 yds. 10. 70 yds. 11. 238 yds. 12. 560 yds. 13. ·3183 of a foot. 14. 7·9577 feet. 15. 103·4505 ft 16. 70·028 yds. 17. 30·6. 18. 360. 19. 19·0985 feet. 20. 4·67 feet.

IX. 1. 12·5664 in. 2. 42·656 in. 3. $35^0·81$.
4. $57^0·3$. 5. $38\frac{2}{3}$ in. 6. 56·19 in.
7. 6·117 in. 8. 17·854 in. 9. 47·7602 in.
10. 5·1416 in. 11. 19·45 in. 12. 14·1897 in.

XI. 1. 196. 2. 576. 3. $756\frac{1}{4}$. 4. $915\frac{1}{16}$.
5. 113 yds. 7 ft. 6. 152 yds. 1 ft. 7. 348 yds. 4 ft.
8. 413 yds. 4 ft. 9. 14 yds. 2 ft. 64 in.
10. 34 yds. 6 ft. 16 in. 11. 73 yds. 6 ft. 9 in.
12. 213 yds. 4 ft. 52 in. 13. 2 acres 4 poles.
14. 5 ac. 1 ro. 1 po. 15. 15 ac. 2 ro. ·04 po.
16. 79 ac. 2 ro. 6·9376 po. 17. 32512·5 sq. ft.
18. 3938 sq. yds. 2 ft. 76·5 in. 19. 7 ac. 2 ro. ·5 po.
20. 16 ac. 3 ro. 16·7168 po. 21. 42 yds. 22. 85 yds.
23. 273 yds. 24. 440 yds. 25. 880 yds.
26. 110 yds. 27. 8·004 ft. 28. 127 yds.
29. 10·954. 30. 16·941. 31. 65·597.
32. 68·823. 33. 2556·169. 34. 346·107.
35. 3·742 in. 36. $1\frac{1}{4}$ in. 37. 280.
38. 432. 39. 279. 40. $374\frac{1}{8}$.
41. 34 yds. 42. 63 yds. 5 ft. 43. 127 yds. 4 ft.
44. 180 yds. 4 ft. 45. 7 yds. 8 ft. 108 in.
46. 16 yds. 96 in. 47. 24 yds. 1 ft. 80 in.
48. 59 yds. 87 in. 49. 3 ac. 34 po.
50. 5 ac. 2 ro. 34·6368 po. 51. 9 ac. 2 ro. 11·6224 po.
52. 13 ac. 1 ro. 22·72 po. 53. 32 ft. 54. 44 yds.
55. 352 yds. 56. 1100 yds. 57. 110 yds.
58. $36\frac{3}{32}$ yds. 59. 78 yds. 1 ft. 60. 2 yds.
61. ·125. 62. $\frac{5}{8}$. 63. 660. 64. 3 chains.
65. 53040 sq. ft. 66. 11 feet. 67. 10·488 ft.
68. 49. 69. 4320. 70. 324. 71. 816.
72. 48. 73. $891\frac{6}{19}$. 74. 40. 75. 84.
76. 696960. 77. 3200. 78. 3 84. 79. 20.
80. 30780. 81. 3456. 82. 162. 83. As 2 is to 5.
84. 280 ft. 85. $1510\frac{5}{6}$. 86. 1444 sq. ft.
87. 21, 63 ft. 88. $\frac{64}{135}$ of an ounce. 91. £6. 6*s*.
92. £1102. 10*s*. 93. £3. 15*s*. $6\frac{1}{2}$*d*. 94. £2. 10*s*.
95. £18. 14*s*. $5\frac{1}{2}$*d*. 96. £245. 18*s*. $11\frac{7}{9}$*d*.
97. £25. 7*s*. 2*d*. 98. £62. 1*s*. 4*d*. 99. 43 ft.

100. 71 yds. 101. £33. 102. 32. 103. 44.
104. 46 yds. 2 ft. 105. 24 yds. 33 in. 106. 88 yds. 2 ft.
107. 81 yds. 27 in. 108. £15. 0s. $7\frac{1}{2}d$. 109. £30. 1s. 3d.
110. £5. 17s. $11\frac{5}{48}d$. 111. £23. 11s. 9d.
112. £12. 19s. $9\frac{3}{16}d$. 113. £4. 6s. 11d. 114. £5. 2s. $9\frac{1}{3}d$.
115. £5. 15s. $10\frac{2}{3}d$. 116. £8. 18s. 9d. 117. £8. 4s. $0\frac{3}{4}d$.
118. 9 ft. 119. $28\frac{1}{8}$ sq. yds., £8. 8s. 9d., $20\frac{25}{36}$ sq. yds.
120. 109 yds. 1 ft. 121. 180 yds. 1 ft. 4 in. 122. £10. 10s.
123. £11. 4s. 6d. 124. £9. 7s. $10\frac{1}{2}d$.

XII. 1. 70 sq. yds. 2. 177 sq. yds. 5 ft.
3. 249 sq. yds. 3 ft. 72 in. 4. 13 ac. 1 ro. 27·7888 po.
5. 25 ft. 6. 70 yds. 7. 5 ft. 8 in. 8. 35 ft. 5 in.
9. 4 ft. 10. 9 ft., 4 ft. 6 in. 11. 498·8 sq. ft.
12. 886·8 sq. ft.

XIII. 1. 72 sq. ft. 2. 212·5 sq. ft.
3. 40 sq. yds. 1 sq. ft. 81 sq. in.
4. 8 ac. 2 ro. 25·568 po. 5. 6090. 6. 54264.
7. 24·995. 8. 42·214. 9. 12. 10. 1848.
11. 2772. 12. 69300. 13. 2310. 14. 3570.
15. 6006. 16. 9240. 17. 18060. 18. 66990.
19. 223860. 20. 5515650. 21. 2·905. 22. 20·976.
23. 24·249. 24. 109·982. 25. 379·473.
26. 463·757. 29. 577·5, 1732·5. 30. $1026\frac{2}{3}$, 3080, $5133\frac{1}{3}$.
31. 12 ft. 32. 45 feet; 540, 630 sq. ft.
33. 2400, 2600, 1800, 3200 square feet.
34. $DB=\frac{65}{12}$ ft., $AB=\frac{169}{12}$, $DC=\frac{52}{3}$, $AC=\frac{65}{3}$; area $147\frac{7}{8}$ sq. ft. 35. £5. 11s. $6\frac{3}{4}d$. 36. £2. 15s.
37. 2116. 38. 10000, 2000, 8000 sq. yds.
39. 750 sq. ft. 40. 2100 sq. ft. 41. £10. 4s. 9d.
42. 293. 43. $141\frac{2}{3}$. 44. 125·05. 45. 271·45.

XIV. 1. 463·7408. 2. 1134 sq. ft.
3. 72·6531 sq. ch. 4. 6·3765 sq. ch. 5. 1372 sq. ft.
6. 29·88 ch. 7. 40 sq. ft. 8. 44 sq. ft.
9. 204 sq. yds. 10. 5 sq. ch. 11. 14·7 sq. ch.
12. 152·075 sq. ch. 13. 125 yds. 14. $280\frac{1}{2}$ yds.
15. $17\frac{1}{2}$, $22\frac{1}{2}$ sq. ft. 16. 60, 68, 76 sq. yds.

17. 312 sq. ft. 18. An acre. 19. 1152 sq. yds., £19. 4*s*.
20. 421 ft. 21. 1800 sq. ft. 22. 1260 sq. ch.
23. 839·553 sq. ft. 24. 16·825 sq. ft. 25. 688 sq. ft.
26. 506430 sq. ft. 27. 7200, 7200 sq. ft.
28. 10296 sq. ft.; 125, 82·368 ft.
29. 1054, 625, 566·6304 ft.

XV. 1. 134 sq. ft. 2. 110·865 sq. ft. 3. 150·6 sq. ft.
4. 6813·52 sq. ft. 5. 142·557 sq. ft. 6. 1039·23 sq. ft.
7. 6495·2 sq. ft. 8. 259·81 sq. ch. 9. $2\sqrt{2}$ sq. ft.
10. $6 \times \cdot 51764$ sq. ft.; see Art. 99.

XVI. 1. 1386. 2. $7857\frac{1}{7}$. 3. $1369028\frac{4}{7}$.
4. 1963·5. 5. 3091535·46. 6. 5473923·84.
7. 5·64. 8. 58·86. 9. 283·53. 10. 12·616.
11. 301·79. 12 2978·9. 13. 188·496 sq. ft.
14. 3769·92 sq. ft. 15. 236·24832 sq. ft. 16. 15·094 in.
17. 15·116 feet. 18. 8·956 ft. 19. 38993 sq. ft.
20. 554622 sq. ft. 21. 523·16 ft. 22. 1134·4 ft.
23. 5·657 ft. 24. 8·05 in. 25. 5·77, 8·16 in.
26. 539·3057 sq. ft. 27. 136. 28. £141. 7·44*s*.
29. 407·01. 30. £47. 2·48*s*. 31. £19. 0*s*. $1\frac{1}{2}$*d*.
32. 5857 sq. ft. 33. 141·8 ft. 34. 45·1 ft.
35. 54·9376 sq. ft. 36. 184·9392 sq. ft. 37. 2024·8 sq. ft.
38. 104·2 ft. 39. 1·8426 sq. ft. 40. 129688·3896 sq. ft.
41. 71·62 sq. ft. 42. 140·374 sq. ft. 51. ·81 sq. ft.
52. 2·48 sq. ft.

XVII. 1. 125·664. 2. 150·7968. 3. 562·9747.
4. 43·63 sq. ft. 5. 97·74 sq. ft. 6. 18·54 ft.
7. 69·26 ft. 8. $80^0{\cdot}57$. 9. 11·75 ft. 10. 7·4375 ft.
11. $112^0{\cdot}5$. 12. 15·41 ft. 13. 2942 sq. in.; see Art. 122.
14. 27·53 sq. in. 15. 1·180 sq. ft.; see Art. 167.
16. 61·42 sq. ft.; see Art. 99. 17. 296·04 sq. ft.
18. 28·05 sq. ft. 19. 398·25 sq. ft.
20. 20·382, 686·478 sq. ft. 21. 61·464. 22. 28·546.
23. 9·059. 24. 1·180. 25. ·1489 sq. ft.

XVIII. 1. 162. 2. 192. 3. 288. 4. 27.
5. 56·907. 6. 8·403. 7. 263·9. 8. 59·307.
9. 70·641. 10. 22·0957. 11. ·7817. 12. ·6931.

XIX. 1. 324. 2. 653·4 sq. in. 3. 1 in. to 36.
4. 1 in. to 660. 5. 7·2 in. 6. 6·336 inches to a mile.
7. 11·832, 17·748 ft. 8. 221, 238, 255.
9. 50·96, 109·20, 145·60.
10. The square is 1·299 times the triangle.
11. The hexagon is 1·1547 times the square.
12. The circle is 1·2732 times the square.
13. The circle is 1·1026 times the hexagon.
14. 15·197 ft. 15. 61·237 ft. 16. 13·236 ft.
17. 6·7082, 9·4868, 11·6190, 13·4164 ft. 18. The perimeter of the triangle is 1·14 times that of the square. 19. 2·638 ft. 20. 3·114 ft.

XXII. 1. 18 ft. 1664 inches. 2. 216 ft.
3. 107 ft. 297 in. 4. 4492 ft. 216 in. 5. 38 ft. 192 in.
6. 78 ft. 810 in. 7. 200 ft. 200 in. 8. 3399 ft. 1008 in.
9. 68 ft. 10. 20 ft. 1440 in. 11. 32 ft. 752 in.
12. 66 ft. 1296 in. 13. 9 in. 14. 4·32 in.
15. 5 ft. 16. 5 ft. 10 in. 17. 20 sq. ft.
18. 4 sq. ft. 48 in. 19. 17 sq. ft. 2 in. 20. 34 sq. ft. 120 in.
21. 1346. 22. 1777. 23. 2314. 24. 3605.
25. 70. 26. 89. 27. 141. 28. 365.
30. 31·255875. 31. 24 ft. 32. 15360. 33. 384.
34. ·000004 of an inch. 35. 35·314. 36. 14 lbs.
37. 116109 ounces. 39. $314\frac{1}{2}$. 40. 3·8 in.
41. 291600. 46. 6196. 47. $6810\frac{1}{2}$. 48. $\frac{1}{21}$ lb., $\frac{1}{9}$ lb.
49. 9 cubic ft. 50. 3 ft. 6 in. 52. 1·26 ft.
53. 14·42 ft. 54. 103·51 in. 55. 3·297 ft.
56. 20 in.

XXIII. 1. 15 ft. 1050 in. 2. 62 ft. 729 in.
3. 109 ft. 133 in. 4. 194 ft. 363 in. 5. 1 ft. 162 in.
6. 3 ft. 1056 in. 7. 5 ft. 408 in. 8. 11 ft. 1452 in.
9. 43·9824. 10. 83·44875. 11. 221·28645.
12. 569·6768. 13. 33 in. 14. 44 in. 15. 47 in.
16. 53 in. 17. 7·979 in. 18. 1·1732 ft.
19. 1·7208 ft. 20. 2·336 ft. 21. 23. 22. 489.
23. 3916. 24. 11013. 25. 1800 cub. ft.
26. $117333333\frac{1}{3}$. 27. 144000. 28. 13461.
29. 1562·5 ft. 30. 89. 31. 212·06. 32. 55·4.
33. 1745·3. 34. 489. 35. £5. 4*s*. 9*d*.

36. £5. 7*s*. 37. £6. 13*s*. 6*d*. 38. 10·61 in.
39. 880061 in. 40. 3520244 ft. 41. 1·68 cub. ft.
42. 7·0686 cub. ft. 43. 17·6715 cub. ft. 44. 373 lbs.
45. 654 lbs. 46. £4. 17*s*. 3*d*. 47. 8·2467 lbs.
48. 433 cub. in. 49. 2·598 cub. ft. 50. $\frac{7}{16}$ of an inch.
51. 90 cub. ft. 52. 33·45 in. 53. 6·46 ft.
54. 15·55 in. 55. ·927 ft. 56. 5·61 in.

XXIV. 3. 49·3. 4. 15·2. 5. 1 very nearly.
6. 391·6. 7. 410·5. 8. 2·7. 9. 63·7 in.
10. 3·84 in.

XXV. 1. 6 ft. 362 in. 2. 17 ft. 896 in.
3. 32 ft. 704 in. 4. 45 ft. 1121 in. 5. 19·596.
6. 83·785. 7. 987·798. 8. 2378·571. 9. 16·755.
10. 64·141. 11. 97·905. 12. 1047·2. 13. 17 ft. 3 in.
14. 21 ft. 9 in. 15. 30 ft. 16. 32 ft. 3 in.
17. 7·979 in. 18. 2·6846 ft. 19. 3·2612 ft.
20. 4·2853 ft. 21. 407293·5 cub. ft. 22. 666666$\frac{2}{3}$ cub. ft.
23. 206·448 cub. ft. 24. 192000 cub. ft. 25. 3494·4 cub. ft.
26. 1164·9 cub. ft. 27. 3640 cub. ft. 28. 3147·18 cub. ft.
29. 1231·5 cub. ft. 30. 391·8 cub. ft. 31. 3392·928 cub. ft.
32. 55461·8064 cub. ft. 33. 6265·2. 34. 3·1416.
35. 101·86 cub. ft. 36. 116·224 cub. ft. 37. 10·472 ft.
38. 36373·07. 39. 166$\frac{2}{3}$ cub. ft. 40. 2708 cub. in.
41. 3466145. 42. 1173·46 cub. ft.

XXVI. 1. 12·25. 2. 11·227. 3. 38·2475.
4. 48·216. 5. 203·4186. 6. 501·7292. 7. 714·1736.
8. 1038·4362. 9. 917·3472. 10. £60. 4*s*.
11. 73·323 cub. ft. 12. 7600 cub. ft. 13. 4824·32 cub. ft.
14. 65. 15. 1608·5 cub. ft. 16. 247·8, 283·1 cub. ft.
17. 278873·5, 305413·8, 333160·4.
18. 1127·6, 1408·2 cub. ft.

XXVII. 1. 1620 cub. in. 2. 21 cub. ft.
3. 3240 cub. in. 4. 34·64 cub. in. 5. 588 cub. in.
6. 4000 cub. in. 7. 995·9 cub. in.
8. 462, 1155 cub. in. 9. 1008, 728 cub. in.
10. 420, 480 cub. in.

XXVIII. 1. 744000 cub. ft. 2. 832000 cub. ft.
3. 125067 cub. in. 4. 103110. 5. 336600.
6. $126\frac{2}{3}$, $198\frac{2}{3}$ cub. ft. 7. 473, 371, 281 cub. ft.
8. 480, 720 cub. ft. 9. $261\frac{1}{3}$, $485\frac{1}{3}$ cub. in.
10. 250, 670, 970 cub. in. 12. 1144, 584 cub. ft.

XXIX. 1. 696·9 cub. in. 2. 268·1 cub. ft.
3. 7238·2 cub. ft. 4. 17974·2 cub. ft. 5. 3·65.
6. 8·65. 7. 16·89. 8. 29·18. 9. 31·94.
10. 154·99. 11. 344·01. 12. 1239·88. 13. 20·73.
14. 272 minutes. 15. 197·9208 cub. ft.
16. 6494 lbs. 17. 43·412 cub. in. 18. 29·688 lbs.
19. 432·227 ounces. 20. 5·986 lbs. 21. 12·72 lbs.
22. 31·94 lbs. 23. 111·18 lbs. 24. 522·67 ounces.
25. 2926·2 ounces. 26. 611·12 lbs. 27. 549 lbs.
28. 28·6875 lbs. 30. 1·24 lbs. 31. 2 in. 32. 3 in.
33. 19·28 in. 34. 65·45 cub. ft. 35. Nearly 49 times.
36. 16·12 in. 37. 24·814 in. 38. 16·64 in.
39. 9·5076 in. 40. 14 in. 41. 4 in. very nearly.
42. 10·03 in.

XXX. 1. 546·6384 cub. in. 2. 2073·456 cub. in.
3. 263·8944 cub. ft. 4. 76·95 cub. ft. 5. 234·57 cub. ft.
6. 458·15 cub. ft. 7. 2077·64 cub. ft. 8. 205·25 cub. ft.
9. 654·5, 3534·3 cub. ft. 10. 791·6832, 2261·952 cub. ft.
11. ·375 of a cub. in. 12. 30·5 lbs.
13. Two of 98·96 cub. ft., one of 183·78 cub. ft.
14. Two of 335·1 cub. in., two of 737·2 cub. in.
15. 3796·1 cub. in. 16. 26834·5 cub. in. 17. 143.

XXXI. 1884·96 cub. in. 2. 58·06 cub. ft. 3. 30.
4. 84·56 cub. feet. 5. 9333 cub. in.

XXXII. 1. 20·25 lbs. 2. 58320 lbs.
4. 168·75, 231·25 cub. in. 5. 5·228 in. 6. 8·320 ft.
7. 2·7744 ft. 8. 1·9265, 3·5635 feet.

XXXIII. 1. 37 sq. ft. 72 in. 2. 80 sq. ft. 96 in.
3. 204 sq. ft. 24 in. 4. 260 sq. ft. 6 in. 5. 70 sq. ft.
6. 63 sq. ft. 7. 73 sq. ft. 40 in. 8. 88 sq. ft. 10 in.
9. 108 sq. ft. 10. 520 sq. ft. 11. 51 sq. ft. 96 in.
12. 77 sq. ft. 132 in. 13. 24 sq. ft. 47 in.
14. 48 sq. ft. 128 in. 15. 66 sq. ft. 16 in.
16. 1227 sq. ft. 48 in. 17. 985 sq. ft. 120 in.
18. 2488 sq. ft. 128 in. 19. 18 sq. ft.
20. 48 sq. ft. 90 in. 21. 30 sq. ft. 84 in.
22. 40 sq. ft. 52 in. 23. 104 sq. ft. 84 in.
24. 173·2 sq. ft. 25. 88·2888 sq. ft. 26. 2197 cub. in.
27. 6·083 ft. 28. 619·52 sq. in. 29. 191·13 sq. in.
30. 44·7846 sq. ft. 31. 190 sq. ft. 96 in. 32. 6 sq. ft.
33. 1380, 1296 sq. in. 37. 500. 38. 566·96.
39. 476·22. 40. 624·97. 41. 485·35. 42. 600.
43. 629·96. 44. 634·96. 45. 673·05. 46. 693·36.

XXXIV. 1. 1248 sq. in. 2. 1653 sq. in.
3. 2349·92 sq. in. 4. 5277·89 sq. in. 5. 2010·624 sq. ft.
6. 62·832 sq. ft. 7. 186·9252 sq. ft. 8. 1105·84 sq. in.
9. 173·662 sq. ft. 10. 241·673 sq. ft. 11. 19·2 in.
12. 4·775 in. 13. 12·666 in. 14. 13·541 in.
15. 21·409 in. 16. 2·6 ft. 17. 19·635 sq. ft.
18. 3·82 in. 19. 628·32 sq. in. 20. 968·54 sq. in.
21. 4375. 22. 4593. 23. 4630. 24. 4907.
25. 5055. 26. 5833. 27. 5512. 28. 6944.
29. 5608. 30. 8088. 31. 600, 586 sq. in.
32. 18683 sq. in.

XXXV. 1. 753·98 sq. in. 2. 628·32 sq. in.
3. 3518·59 sq. in. 4. 80. 5. 117·81. 6. 13·03.
7. 256·61. 8. 868·53. 9. 685·4. 10. 897·98.
11. 15·92 in. 12. 4·46 in.

XXXVI. 1. 715·5. 2. 1273. 3. 1583·37.
4. 3418·06. 5. 2035·76. 6. 549·78. 7. 1159·25.
8. 1709·03. 9. 2108·01. 10. 6738·73. 11. 2306·74.
12. 8715·7. 13. 37·699. 14. 85·452. 15. 29·093.
16. 38·527. 17. 1·9635. 18. 21·3803. 19. 2·1378.

20. 18·4322. 21. 30 in. 22. 22·831. 23. 2·5465 ft.
24. 2·729 ft. 25. 52 in. 26. 1·8335 ft.
27. 1·0925 ft. 28. 1·0998 ft. 29. 28 yds. nearly.
30. 327 yds. nearly. 31. 62·688. 32. 66·490.
33. 65·147. 34. 62·688. 35. 60·074. 36. 33989.
37. 28701. 38. 28172. 39. 47335. 40. 60154.

XXXVII. 1. 176 sq. in. 2. 273 sq. in.
3. 251·328 sq. in. 4. 94·248 sq. ft. 5. 1102·70 sq. in.
6. 9764·09 sq. in. 7. 89·41 sq. ft. 8. 96·35 sq. ft.
9. 185·97 sq. in. 10. 229·09 sq. in. 11. 80·11 sq. ft.
12. 139·49 sq. ft. 13. 2412·75 sq. in. 14. 3418·06 sq. in.
15. 40·84. 16. 171·22. 17. 427·26. 18. 1747·515.
19. 2·228, 1·772 feet. 20. 105 nearly.

XXXVIII. 1. 314·16 sq. in. 2. 2827·44 sq. in.
3. 60·821 sq. ft. 4. 127·32 sq. in. 5. 5·093 sq. ft.
6. 13·038 sq. ft. 7. 11·284 in. 8. 4·5135 ft.
9. 4·886 ft. 10. 8·4104 cub. ft. 11. 33·245 cub. ft.
12. 94·0315 cub. ft. 13. 4·1888 cub. ft.
14. 47·713 cub. ft. 15. 75·3984 sq. ft.
16. cylinder 3141·6 cub. in.; sphere 4188·8 cub. in.
17. cube 1 cub. ft.; sphere 1·382 cub. ft.
18. cylinder 6·282 cub. ft.; sphere 7·695 cub. ft.
19. cube 6 sq. ft.; sphere 4·836 sq. ft.
20. cylinder 18·8496 sq. ft.; sphere 16·4666 sq. ft.

XXXIX. 1. 850 sq. in. 2. 45 sq. ft.
3. 904·78 sq. in. 4. 49·009 sq. ft. 5. 163·3632 sq. ft.
6. 2312·22 sq. in. 7. 133·7256 sq. ft. 8. 1599·87 sq. in.
9. 1039·8696 sq. ft. 10. 1328·897 sq. in.
11. 308·8936 sq. ft. 12. 867·79 sq. in.
13. 750·8424 sq. ft. 14. 1677·6144 sq. in.
15. 308·4343 sq. ft. 16. 1221·894 sq. in. 17. $\frac{1}{82}$.
18. $\frac{4}{53}$. 19. $\frac{1}{7}$ of the radius. 20. $\frac{1}{3}$ of the radius.

XL. 1. £62. 2*s*. 2. £165. 12*s*. 3. £183. 12*s*. 4*d*.
4. £37. 15*s*. 5*d*. 5. £4. 3*s*. 4*d*. 6. $3\frac{1}{2}$ ft. long.
7. £37. 16*s* $4\frac{1}{2}$*d*. 8. 373 lbs. 9. $34\frac{3}{4}$.

10. £12. 6*s*. $5\frac{1}{2}$*d*. 11. £3. 2*s*. 10*d*. very nearly. 12. £12.
13. £7. 14*s*. nearly. 14. £95. 13*s*. 3*d*. very nearly.
15. £5. 10*s*. very nearly. 16. £50. 17. £12.
18. 5550. 19. £6. 8*s*. 6*d*. 20. £33. 18*s*. 6*d*.

XLI. 1. 5·62. 2. 3·4. 3. £10. 5*s*. 4. $216\frac{2}{3}$.
5. £145. 4*s*. 6. £15. 15*s*. 7. £120. 8. £49. 10*s*.
9. £25. 0*s*. 8*d*. 10. £62. 10*s*. 11. £126.
12. £71. 5*s*. 13. $100\frac{1}{3}$. 14. £12. 12*s*. 15. £41.
16. £16. 4*s*.

XLII. 1. $93\frac{3}{4}$. 2. 117. 3. 161. 4. $87\frac{1}{2}$.
5. $12\frac{1}{2}$. 6. $16\frac{2}{3}$. 7. $128\frac{1}{4}$. 8. $128\frac{1}{2}$. 9. $75\frac{5}{8}$.
10. $96\frac{4}{5}$. 11. 98·676. 12. 62·8.

XLIII. 1. 110·87. 2. 88·18. 3. 108·54.
4. 54·2. 5. 108·7. 6. 90·03. 7. 112·43.
8. 35·15. 9. 23·49. 10. 15·98.

XLVI. 1. ·7975 of an acre. 2. ·626 of an acre.
3. ·56283 of an acre. 4. ·7165 of an acre.
5. 1·1445 acres. 6. 1·1165 acres. 7. 766 sq. links.
8. 1511 sq. links. 9. 2300 sq. links. 10. 5640 sq. links.
11. ·311 of an acre. 12. 3·61135 acres. 13. ·662 of an acre.
14. 8·606127 acres; it will be found that the angle *ABC* is 60°, and the angle *ACD* is 90°.
15. 16·24425 acres. 16. 5·02922 acres.

XLVIII. 1. 9 sq. ft. 2. 16 sq. ft. 8 pr. 3. 9 sq. ft. 2 pr.
4. 16 sq. ft. 6 pr. 9 in. 5. 26 sq. ft. 11 pr. 10 in.
6. 27 sq. ft. 1 pr. 5 in. 7. 13 sq. ft. 10 pr. 10 in.
8. 15 sq. ft. 8 pr. 0 in. 8 th. 9. 14 sq. ft. 0 pr. 4 in. 10 th.
10. 22 sq. ft. 7 pr. 4 in. 10 th. 11 fo. 11. 13 cub. ft. 6 pr.
12. 33 cub. ft. 9 pr. 13. 43 cub. ft. 4 pr.
14. 73 cub. ft. 10 pr. 8 sec. 15. 123 cub. ft. 0 pr. 6 sec. 9 in.
16. 187 cub. ft. 2 pr. 0 sec. 2 in.

XLIX. 1. 47·124 metres. 2. 12 hectares 39·56 ares.
3. 90·576 ares. 4. 73·44. 5. 402. 6. 5·23. 7. 4692. 8. 51.
9. 172. 10. 31·416. 11. 104. 12. 254. 16. ·062 of a metre.
17. ·108 of a metre. 18. ·098 of a metre.

MISCELLANEOUS EXAMPLES.

1. 75 ft. 2. $8\frac{1}{3}$. 3. 10 ac.; 90 ac. 4. 50 ft. 4 in.
5. 972 sq. yds. 6. £14. 7. £3. 10*s*. 8. £161. 8*s*. 9. 3·96 ac.
10. 23 ft.; 27 ft. 11. $\frac{5}{6}$ of a mile. 12. 193600. 13. 820 in.
14. 60 ft. 9 in. 15. 13·8 minutes. 16. £6. 13*s*. 4*d*.
17. £5. 7*s*. 6*d*. 18. 6 yds. 19. 3·5568 acres. 20. 175602.
22. 210 sq. ft. 23. 1120 yds.; 262 ac. 1 rd. 26 po. $3\frac{1}{2}$ sq. yds.
24. 85 ft. 10 in. 25. 800 yds.; 316·228 yds. 26. £12. 15*s*. $2\frac{1}{2}$*d*.
27. £6. 3*s*. $11\frac{1}{2}$*d*. 28. Length $10\frac{1}{2}$ yds., breadth $3\frac{1}{2}$, height 4.
29. 3468 sq. ft. 30. 546 sq. ft.; 13 ft. 31. 102 ft.
32. $1\frac{1}{2}$ in. 33. Rather more than $13\frac{3}{8}$. 34. 243 in.
35. 360·555 yds.; 33600 sq. yds. 36. £25. 7*s*. 2*d*.
37. £1. 17*s*. 6*d*. Four times as much. 38. 50 sq. yds.
39. 150 sq. ft.; 270 sq. ft. 40. 9·4916 ac. 41. $26\frac{2}{3}$ ft.
42. 12 ft. 43. 18 ft. $2\frac{1}{6}$ in. 44. 594 yds.
45. £182. 6*s*. $11\frac{3}{4}$*d*. 46. £430. 16*s*. 8*d*. 47. £4. 3*s*. 7*d*.
48. 21 ft. long, $10\frac{1}{2}$ ft. broad, 10 ft. high. 49. 133837 sq. ft.
50. 72 sq. ft. 51. 15 in.; $18\frac{3}{4}$ in. 52. £70. 53. 37 ft. $7\frac{1}{3}$ in.
54. 1·0936 yds. 55. 864; £233. 17*s*. 9*d*. 56. £2. 9*s*. $3\frac{2}{3}$*d*.
57. 81920. 58. 141·42 yds. 60. £78. 7*s*. 61. 3600.
62. 15 in. 63. $2604\frac{1}{6}$ lbs. 64. 275625. 65. 4 feet.
66. $3349\frac{1}{2}$ lbs. 67. $320\frac{70}{81}$; 165. 68. 3·20705; 19*s*. 3*d*. 69. 5·44.
70. 147 cub. ft. 71. 12288. 72. 25 in. 73. 480. 74. 25 ft. 4 in.
75. 22·22 in. 76. 8·1 ft., 531·441 cub. ft. 77. 3·1416 cub. ft.
78. 275·4 lbs. 79. £11. 0*s*. 6*d*.; £7. 12*s*. 3*d*. 80. $3\sqrt{3}$ to 1.
81. 5 ft. 82. 176786. 83. 21. 84. 27 in. 85. 12*s*. 3*d*.
86. 689. 87. 2 ft. 6 in. 88. 37·6992 cub. in.; 75·3984 sq. in.
89. 304 cub. in. 90. 11·754 cub. in. 91. 671. 92. 1969 lbs.
93. 7009 in. 94. 648. 95. $28\frac{1}{2}$; $427\frac{1}{2}$ lbs. 96. £43. 4*s*.
97. 3780 lbs.; 59159. 98. $4\frac{1}{2}$ in. 99. 351·8592 sq. ft. 100. $73\frac{3}{4}$.
101. 512. 102. 27, 18, 12 ft. 103. 9019 in. 104. Four times.
105. 4230794. 106. ·030301 of the former volume.
111. 1611. 112. £680. 12*s*. 6*d*. 113. $21\sqrt{3}$ in. 114. 12800 lbs.
115. 1885·6 cub. ft. 116. 565·488 cub. ft. 117. 2·142 ft.
118. 10744·272 cub. in. 119. 44 lbs. 7 oz. 120. 12·036 cub. ft.
121. $22\frac{1}{2}$ cub. ft. 122. £580800. 123. 900. 124. 885. 126. 40.
127. $233\frac{1}{3}$ cub. ft. 129. $7\frac{5}{8}$ lbs. 130. 78·54 sq. ft. 131. 640 ft.
132. 4·15. 133. $312\frac{1}{2}$. 134. 2178. 135. 809 ft. 136. 12 ft.
137. 8042·5 cub. ft. 138. 94. 139. 8·574 in. 140. 51·051 sq. ft.
141. 3872. 142. $\frac{1}{1008}$ of an inch. 143. 4840. 144. $\frac{4}{11}$ of a foot.
145. 173·205 cub. in. 146. 199 in. 147. 1728. 148. 392·7 cub. in.

149. 8·748 lbs. 150. 3·82 in. 151. 14, $10\frac{1}{2}$, $3\frac{1}{2}$.
152. £3. 14*s*. 3*d*. 153. No. 154. 359·7 sq. in. 155. 968.
156. 1446. 157. ·000046. 158. 6·232. 159. 796.
160. 30·16 cub. in. 161. 48·7. 162. ·1545 of an oz.
163. 290 cub. in.; 246 sq. in. 164. As 3 to 5.
165. ·00032 of an in. 166. 3·1416 cub. ft. 167. 8·3776 cub. ft.
168. 3·577 feet from smaller end; 59·69 cub. ft. 169. 168.
170. 103·4. 172. 46·188 ft. 174. 8242·408 cub. in.
175. 5·1416 sq. in. 176. 10·2832 cub. in. 179. 61·77 ft.

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