MENSURATION

AND

PRACTICAL GEOMETRY;

CONTAINING

TABLES OF WEIGHTS AND MEASURES, VULGAR AND DECIMAL FRACTIONS,

MENSURATION OF AREAS, LINES, SURFACES, AND SOLIDS,

LENGTHS OF CIRCULAR ARCS, AREAS OF SEGMENTS AND ZONES OF A CIRCLE, BOARD AND TIMBER MEASURE, CENTRES OF GRAVITY, &c., &c.

TO WHICH IS APPENDED A

TREATISE ON THE CARPENTER'S SLIDE-RULE AND

GAUGING.

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CIVIL AND MARINE ENGINEER.

SECOND EDITION.

NEW YORK:

HARPER & BROTHERS, PUBLISHERS,

FRANKLIN SQUARE.

1863.

TO

B. H. BARTOL, ESQ.,

A TRIBUTE TO AN EARLY AND ESTEEMED FRIEND OF

THE AUTHOR.

New York, *January*, 1863.

PREFACE.

THE following work is designed for the use of Students, Mechanics, and Engineers, and the purpose of the author has been to present a full set of rules whereby may be readily determined the Lines, Areas, Surfaces, Solidities or Volumes, and Centres of Gravity of various Regular and Irregular Figures.

In the progress of the work, it was essayed to present a familiar rule for the surface, volume, and centre of gravity of every figure, but, in consequence of the impracticability of reducing the rules in some cases to a formula at all consistent with the design of the work, such cases as presented this difficulty were abandoned: as it occurs, however, the number of them has been very essentially reduced by the aid and advice of Professors A. E. Church, U. S. Military Academy, West Point, and G. B. Docharty, N. Y. Free Academy.

In consequence of the great number of works on Mensuration that have been submitted to the public, the inference is a fair one that the author, in this case, has not presented any novelties whereby he may anticipate any particular notice or attention; he trusts, however, that a reference to the result of his labors will show that, in the essential points of the extent of the figures submitted, as well in their number as variety of section, and in the introduction of rules for determining their centres of gravity, he has submitted some features of so new and useful a purpose as to entitle him to the attention of those upon whom he confidently relies for patronage.

New York, April 1st, 1858.

CONTENTS.

EXPLANATIONS OF CHARACTERS

Used in the following Calculations, etc., etc.

$=$	*Equal to*, as 12 inches$=$1 foot, or $8\times8=16\times4$.
$+$	*Plus*, or *more*, signifies addition; as $4+6+5=15$.
$-$	*Minus*, or *less*, signifies subtraction; as $15-5=10$.
$\times$	*Multiplied by*, or *into*, signifies multiplication; as $8\times9=72$. $a\times d$, $a.d$, or ad, also signify that a is to be multiplied by d.
$\div$	*Divided by*, signifies division; as $72\div9=8$.
$:$ $::$ $:$	*Is to*, *So is*, *To*, signifies Proportion, as $2:4::8:16$; that is, as 2 *is to* 4, *so is* 8 *to* 16.
$\surd$	*Radical sign*, which, prefixed to any number, signifies that the *square root* of that number is required; as $\surd 9$, or $\sqrt{a\times b}$. The degree of the root is indicated by the number placed over the sign, which is termed the *index* of the *root*, or *radial;* as $\sqrt[3]{}$, $\sqrt[4]{}$, etc.
^{2}, ^{3}, ^{4},	added or *set superior* to a number, signifies that that number is to be *squared*, *cubed*, etc.; thus 4^2 means that 4 is to be multiplied by 4; 4^3, that it is to be *cubed*, as 4^3 is$=4\times4\times4=64$. The *power*, or number of times a number is to be multiplied by itself, is shown by the number added, as ^{2}, ^{3}, ^{4}, ^{5}, etc.
$\overline{\quad}$	The *vinculum*, or *bar*, signifies that the numbers are to be taken together; as $\overline{8-2}+6=12$, or $3\times\overline{5+3}=24$.
$.$	*Decimal point*, signifies, when prefixed to a number, that that number has a unit (1) for its denominator; as .1 is $\frac{1}{10}$, .155 is $\frac{155}{1000}$, etc.
$\backsim$	*Difference*, signifies, when placed between two quantities, that their difference is to be taken.

$^{\circ}\ '\ ''\ '''$	Signify *Degrees, minutes, seconds,* and *thirds.*
$\angle$	Signifies *angle.*
$\perp$	Signifies *perpendicular.*
$\triangle$	Signifies *triangle,*
$\square$	Signifies *square,* as $\square$ inches.
$>\ \urcorner$	Signify *inequality,* or *greater than,* and are put between two quantities; as $a \urcorner b$ reads a greater than b.
$<\ \llcorner$	Signify the reverse; as $a \llcorner b$ reads a less than b.
$\therefore$	Signifies *therefore* or *hence.*
$\because$	Signifies *because.*
( ) [ ]	*Parentheses* and *brackets* signify that all the figures within them are to be operated upon as if they were only one; thus $(3+2)\times 5=25$; $[3+2]\times 5=25$.
p or π	Is used to express the ratio of the circumference of a circle to its diameter $=3.1415926$, etc.
A A′ A″ A‴	Signify A, A *prime,* A *second,* A *third,* etc.
$\pm\ \mp$	Signify that the formula is to be adapted to two distinct cases.
a^{-1}, a^{-2}, a^{-3}, etc.	Denote inverse powers of a, and are equal to $\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}$, etc.
$sin.^{-1}a$, $cos.^{-1}a$, etc.	Signify the *arc* or *angle,* the sine or cosine, etc., of which is a, the *arc* or *angle* being expressed in terms of the radius, as the unit, unless otherwise stated. If A° denotes the *arc* or *angle* in degrees, in terms of the radius it is $A=\frac{\pi A^{\circ}}{180^{\circ}}$. If $sin.\ A=a$, $\therefore sin.^{-1}a=A$, etc.
$^{\frac{1}{2}}, ^{\frac{1}{3}}$, etc.	Set superior to a number, signify the square or cube root, etc., of the number; as $2^{\frac{1}{2}}$ signifies the square root of 2.
$^{\frac{4}{2}}, ^{\frac{4}{3}}, ^{\frac{6}{3}}$, etc.	Set superior to a number, signify the square or cube root, etc., of the 4th power, etc., etc.
$^{1\cdot 7}, ^{3\cdot 6}$, etc.	Set superior to a number, signify the tenth root of the 17th power, etc., etc.

HASWELL'S MENSURATION.

MEASURES AND WEIGHTS

Used in this Work.

MEASURES OF LENGTH.

LINEAL.

12 inches = 1 foot.	40 rods = 1 furlong.
3 feet = 1 yard.	8 furlongs = 1 mile.
$5\frac{1}{2}$ yards = 1 rod or pole.	1 degree = 69.77 statute miles.

1 geographical mile = 2046.58 yards or 6139.74 feet.

CIRCLES.

60 thirds = 1 second.	60 minutes = 1 degree.
60 seconds = 1 minute.	360 degrees = 1 circle.

1 day is .002739 of a year.
1 minute is .000694 of a year.

MISCELLANEOUS.

6 points = 1 line.	1 hand = 4 inches.
12 lines = 1 inch.	1 span = 9 inches.
1 palm = 3 inches.	6 feet = 1 fathom.

1 yard = .000568 of a mile.
1 foot = .000199 "
1 inch = .0000158 "

Gunter's Chain is 4 poles or 22 yards in length, and has 100 equal links of .666 of a foot, or 7.92 inches.

80 chains = 1 mile.

Foreign.

GREAT BRITAIN.	Imperial yard	=39.1393 *imperial* inches.
	Mile	=1760 U. S. yards.
FRANCE.	Metre	=39.37079 inches, or 3.2809 feet.
	Foot (old system)	=12.7925 inches.
	Common league	=4264.16 U. S. yards.
AUSTRIA.	Foot	=12.445 inches.*
	Mile	=8296.66 U. S. yards.
CHINA.	Foot, builder's	=12.71 inches.
	" mathematic	=13.12 "
	" surveyor's	=12.58 "
	" tradesman's	=13.32 "
	Li	=629 U. S. yards.
COPENHAGEN.	Foot	=12.35 inches.
GENOA.	Foot	=9.72 "
HAMBURGH.	Foot	=11.29 "
	Mile	=8244 U. S. yards.
LISBON.	Foot	=12.96 inches.
MEXICO.	Foot	=11.1284 inches.
	Common league	=4636.83 U. S. yards.
PRUSSIA.	Foot	=12.361 inches.
	Mile	=8468 U. S. yards.
ROME.	Foot	=11.60 inches.
	Mile	=2025 U. S. yards.
RUSSIA.	Foot	=21.1874 inches.*
	Versta	=1167 U. S. yards.
SPAIN.	Foot	=11.1284 inches.*
	Judicial league	=4636.83 U. S. yards.
SWEDEN.	Foot	=11.6865 inches.*
	Mile	=11700 U. S. yards.
TURKEY.	Pick	=17.905 inches.
	Berri	=1826 U. S. yards.

MEASURES OF SURFACE.

SQUARE.

144 inches =1 foot.
9 feet =1 yard.
$272\frac{1}{4}$ feet } =1 rod or pole.
$30\frac{1}{4}$ yards }

40 rods =1 rood.
4 roods =1 acre.
640 acres =1 mile.

* U. S. Ordnance Manual, 1850.

10 square chains		= 1 acre.
4840 " yards		
160 " poles		
100000 " links		
69.5701 yards square		
220 × 198 feet		
208.710321 feet square		
235.5041 feet diameter, or		
43560 square feet		

MISCELLANEOUS.

24 sheets = 1 quire. 20 quires = 1 ream.

Drawing Paper.

Cap	13 × 16 in.	Columbier	33¾ × 23 in.
Demy	19½ × 15½ "	Atlas	33 × 26 "
Medium	22 × 18 "	Doub. Elephant	40 × 26 "
Royal	24 × 19 "	Theorem	34 × 28 "
Super-royal	27 × 19 "	Antiquarian	52 × 31 "
Imperial	29 × 21¼ "	Emperor	40 × 60 "
Elephant	27¾ × 22¼ "	Uncle Sam	48 × 120 "

Foreign.

FRANCE. *Old System,*		1 square inch =	1.13587 U. S. inch.
New System, 1 are =		100 square metres =	119.603 square yards.
AMSTERDAM.	Morgen	= 9722	" "
BERLIN.	" great	= 6786	" "
HAMBURGH.	"	= 11545	" "
PORTUGAL.	Geira	= 6970	" "
PRUSSIA.	Morgen	= 3053	" "
ROME.	Pezza	= 3158	" "
RUSSIA.	Desiatina	= 13066.6	" "
SPAIN.	Fanegada	= 5500	" "
SWITZERLAND.	Taux	= 7855	" "

Square Foot in U. S. Square Inches.

AMSTERDAM	124.255	BREMEN	129.504
ANTWERP	126.337	COLOGNE	117.288
BERLIN	148.603	DANTZIC	127.690
BOLOGNA	224.700	DENMARK	152.670

Dresden	124.099	Rhineland	152.670
France	163.558	Riga	116.425
Geneva	369.024	Rome	137.358
Hamburgh	127.441	Spain	123.832
Leipsic	123.432	Sweden	136.515
Lisbon	167.547	Venice	187.182
Milan	243.984	Vienna	155.002

MEASURES OF CAPACITY.

LIQUID AND DRY.

7.21875 cub. ins.	=1 gill.	4 quarts	=1 gallon.
4 gills	=1 pint.	2 gallons	=1 peck.
2 pints	=1 quart.	4 pecks	=1 bushel.

MISCELLANEOUS.

1 chaldron=36 bushels, or 57.244 cubic feet, when heaped in the form of a cone.

NOTE.—The standard U. S. bushel is the *Winchester* (British), and it measures 2150.42 cubic inches, and contains 543391.89 troy grains, or 77.627413 pounds avoirdupois of distilled water at its maximum density.

Its dimensions are 18½ inches in diameter inside, 19½ inches outside, and 8 inches deep. When *heaped*, the cone must not be less than 6 inches high, and it contains 2986.4765 cubic inches.

The standard U. S. gallon=231 cubic inches, and contains 58372.1754 troy grains (8.3389 pounds avoirdupois) of distilled water at its maximum density (39°.83).

Foreign.

GREAT BRITAIN. The imperial gallon measures 277.274 cubic inches.
Imperial bushel 2218.192 cubic inches, and when heaped in the form of a true cone (6 inches high) it contains 2815.4872 cubic inches.
1 chaldron=58.68 cubic feet, and weighs 3136 lbs.

FRANCE. *Old System.* 1 Pinte=0.931 litre, or 56.817 cubic inches.
New System. 1 Litre=61.027 U. S. inches.

Amsterdam.	Anker	2331	cub. in.	Mudde	6786	cub. in.
Antwerp.	Stoop	168	"	Viertel	4705	"
Bremen.	Stubgens	194.5	"	Scheffel	4339	"
Constantinople.	Almud	319	"	Kislos	2023	"

COPENHAGEN.	Anker	2335	cub. in.	Toende	8489	cub. in.
GENOA.	Pinte	90.5	"	Mina	7366	"
LISBON.	Almudi	1010	"	Alqueire	827	"
ROME.	Boccali	80	"	Quarti	4226	"
RUSSIA.	Vedro	750.58	"	Chetwert	12800	"
SPAIN.	Quartillos	30.5	"	Catrize	41269	"
	Arroba	4.2455	gall.	Fanega	1.593	bush.
SWEDEN.	Kann	160	cub. in.	Tunnar	8940	cub. in.
TRIPOLI.	Mattari	1376	"	Caffiri	19780	"
VIENNA.	Eimer	3443	"	Metzen	3753	"

MEASURES OF SOLIDITY.

CUBIC.

1728 inches = 1 foot. | 27 feet = 1 yard.

1 foot = 7.4806 gallons.
128 feet = 1 cord.
24.75 " = 1 perch.

Foreign.

FRANCE. Stere (1 cubic metre) = 61027.1 U. S. inches, or 35.3166 cubic feet.

NOTE.—For the solid measures of other foreign countries, take the cube of the measures given in the preceding tables.

MEASURES OF WEIGHT.

AVOIRDUPOIS.

16 drams = 1 ounce.
16 ounces = 1 pound.
112 pounds = 1 cwt.
20 cwt. = 1 ton.

TROY WEIGHT.

24 grains = 1 pennyweight.
20 pennyweights = 1 ounce.
12 ounces = 1 pound.

The pound, ounce, and grain are the same in apothecaries' and troy weights.

7000 troy grains	= 1	lb. avoirdupois.	
175 " pounds	= 144	lbs.	"
175 " ounces	= 192	oz.	"
437.5 troy grains	= 1	oz.	"
1 " pound	= .8228	lb.	"

The standard U. S. pound contains 7000 troy grains, or 27.7015 cubic inches of distilled water at its maximum density.

Foreign.

Great Britain.	Pound avoirdupois = 27.7274 cubic inches of distilled water at the temperature of 62°. Hence, 22.815689 cubic inches weigh 1 troy pound.			
France.	1 Gramme	= 15.43316 troy grains.		
Alexandria.	1 Rottoli	= .9346	pounds	avoirdupois.
Amsterdam.	1 Pound	= 1.0893	"	"
Austria.	1 "	= 1.2351	"	"
Bengal.	1 Seer	= 1.8667	"	"
Bremen.	1 Pound	= 1.0997	"	"
Cairo.	1 Rottoli	= .9523	"	"
China.	1 Catty	= 1.3253	"	"
Constantinople.	1 Oke	= 2.8129	"	"
Copenhagen.	1 Pound	= 1.1014	"	"
Corsica.	1 "	= .7591	"	"
Genoa.	1 " (heavy)	= 1.0768	"	"
Japan.	1 Catty	= 1.3000	"	"
Prussia.	1 Pound	= 1.0333	"	"
Rome.	1 "	= .7479	"	"
Russia.	1 "	= .9020	"	"
Spain.	1 "	= 1.0152	"	"
Sweden.	1 "	= .9376	"	"
Tripoli.	1 Rottoli	= 1.1200	"	"
Venice.	1 Pound (heavy)	= 1.0555	"	"

MISCELLANEOUS.

1 cubic foot of anthracite coal from 50 to 55 lbs.
1 cubic foot of bituminous coal from 45 to 55 lbs.
1 cubic foot of Cumberland coal = 53 lbs.

1 cubic foot of charcoal	= 18.5 lbs.	(hard wood.)
1 cubic foot of charcoal	= 18. "	(pine wood.)
1 cord Virginia pine	=2700 "	
1 cord Southern pine	=3300 "	
1 stone	= 14 "	

Coals are usually purchased at the conventional rate of 28 bushels (5 pecks) to a ton=43.56 cubic feet.

MEASURES OF VALUE.

1 eagle	=258	troy grains.
1 dollar	=412.5	" "
1 cent	=168	" "

The standard of gold and silver is 900 parts of pure metal and 100 of alloy in 1000 parts of coin.

ANCIENT MEASURES.

MEASURES OF LENGTH.

Scripture.

	Feet.	Inches.		Feet.	Inches.
A digit	=0	0.912	A cubit	=1	9.888
A palm	=0	3.648	A fathom	=7	3.552
A span	=0	10.944			

Grecian.

	Feet.	Inches.		Feet.	Inches.
A digit	=0	$0.7554\frac{11}{16}$	A stadium	= 604	4.5
A pous (foot)	=1	0.0875	A mile	=4835	
A cubit	=1	$1.5984\frac{3}{8}$			

A Greek or Olympic foot=12.108 inches.
A Pythic or natural foot = 9.768 "

Jewish.

	Feet.		Feet.
A cubit	= 1.824	A mile	= 7296
A Sabbath day's journey	=3648.	A day's journey	=175104 (or 33 miles 864 feet).

Roman.

	Inches.		Feet.	Inches.
A digit	= .72575	A cubit	= 1	5.406
An uncia (inch)	= .967	A passus	= 4	10.02
A pes (foot)	=11.604	A mile	=4835	

Miscellaneous.

	Feet.		Feet.
Arabian foot	=1.095	Hebrew foot	=1.212
Babylonian foot	=1.140	" cubit	=1.817
Egyptian "	=1.421	" sacred cubit	=2.002

VULGAR FRACTIONS.

A FRACTION, or broken number, is one or more parts of a UNIT.

ILLUSTRATION.—12 inches are 1 foot.

Here 1 *foot* is the unit, and 12 *inches* its parts; 3 *inches*, therefore, are the *one fourth* of a foot, for three is the quarter or fourth of 12.

A *Vulgar Fraction* is a fraction expressed by *two numbers* placed one above the other, with a line between them, as 50 cents is the $\frac{1}{2}$ of a dollar.

The upper number is called the *Numerator*, because it shows the number of parts used.

The lower number is called the *Denominator*, because it denominates, or gives name to the fraction.

The *Terms* of a fraction express both numerator and denominator; as 6 and 9 are the terms of $\frac{6}{9}$.

A *Proper* fraction has the numerator equal to, or less than the denominator; as $\frac{1}{4}$, $\frac{7}{8}$, &c.

An *Improper* fraction is the reverse of a proper one; as $\frac{2}{1}$, $\frac{3}{2}$, &c.

A *Mixed* fraction is a compound of a whole number and a fraction; as $5\frac{7}{8}$, &c.

A *Compound* fraction is the fraction of a fraction; as $\frac{1}{2}$ of $\frac{3}{4}$, $\frac{5}{8}$ of $\frac{7}{8}$, &c.

A *Complex* fraction is one that has a fraction for its numerator or denominator, or both; as $\frac{\frac{1}{2}}{6}$, or $\frac{5}{\frac{3}{4}}$, or $\frac{\frac{1}{2}}{\frac{3}{4}}$, or $\frac{3\frac{1}{2}}{6}$, &c.

A Fraction denotes division, and its value is equal to the quotient obtained by dividing the numerator by the denominator; thus $\frac{12}{4}$ *is equal to* 3, $\frac{21}{5}$ *is equal to* $4\frac{1}{5}$, *and* $\frac{\frac{1}{2}}{6}$ *is equal to* $\frac{1}{12}$.

REDUCTION OF VULGAR FRACTIONS.

To find the greatest Number that will divide two or more Numbers without a Remainder.

RULE.—Divide the greater number by the less; then divide the divisor by the remainder; and so on, dividing always the last divisor by the last remainder, until nothing remains.

When there are more than two numbers, find the greatest common measure of two of them, and then that for this common measure and the remaining number.

EXAMPLE.—What is the greatest common measure of 1908 and 936?

```
936)1908(2
    1872
      36)936(26
         72
         216
         216
```

Hence 36 *is the greatest common measure.*

Ex. 2. What is the greatest common measure of 246 and 372? *Ans.* 6.

Ex. 3. What is the greatest common measure of 1728, 864, and 3456?

```
864)3456(4
    3456
```

864 *is the greatest common measure of* 3456 *and* 864.

```
864)864(1
    864
```

Hence 864 *is the greatest common measure of the three numbers.*

Ex. 4. What is the greatest common measure of 216 and 288? *Ans.* 72.

To find the least common Multiple of two or more Numbers.

RULE.—Divide by any number that will divide two or more of the given numbers without a remainder, and set the quotients with the undivided numbers in a line beneath.

Divide the second line as before, and so on, until there are no two numbers that can be divided; then the continued product of the divisors and quotients will give the multiple required.

EXAMPLE.—What is the least common multiple of 40, 50, and 25?

$$\begin{array}{r|rrr} 5) & 40 & 50 & 25 \\ \hline 5) & 8 & 10 & 5 \\ \hline 2) & 8 & 2 & 1 \\ \hline & 4 & 1 & 1 \end{array}$$

Then $5 \times 5 \times 2 \times 4 = 200$, *Ans.*

To reduce Fractions to their lowest Terms.

RULE.—Divide the terms by any number that will divide them without a remainder, or by their greatest common measure at once.

EXAMPLE.—Reduce $\frac{720}{960}$ of a foot to its lowest terms.

$\frac{720}{960} \div 10 = \frac{72}{96} \div 8 = \frac{9}{12} \div 3 = \frac{3}{4}$, or 9 *inches.*

Ex. 2. Reduce $\frac{136}{204}$ to its lowest terms. *Ans.* $\frac{2}{3}$.

To reduce a Mixed Fraction to its equivalent Improper Fraction.

NOTE.—*Mixed and improper fractions are the same; thus* $5\frac{1}{2} = \frac{11}{2}$.

RULE.—Multiply the whole number by the denominator of the fraction, and to the product add the numerator, then set that sum above the denominator.

EXAMPLE.—Reduce $23\frac{2}{6}$ to a fraction.

$$\frac{23 \times 6 + 2}{6} = \frac{140}{6} = \textit{the answer.}$$

Ex. 2. Reduce $20\frac{3}{6}$ inches to a fraction. *Ans.* $\frac{123}{6}$.

Ex. 3. Reduce $5\frac{7}{8}$ to a fraction. *Ans.* $\frac{47}{8}$.

Ex. 4. Reduce $183\frac{5}{21}$ to a fraction. *Ans.* $\frac{3848}{21}$.

Ex. 5. Reduce $125\frac{1}{2}$ to a fraction. *Ans.* $\frac{251}{2}$.

To reduce an Improper Fraction to its equivalent Whole or Mixed Number.

RULE.—Divide the numerator by the denominator, and the quotient will be the whole or mixed number required.

EXAMPLE.—Reduce $\frac{12}{3}$ to its equivalent number.

$\frac{12}{3}$ or $12 \div 3 = 4$, *the answer.*

Ex. 2. Reduce $\frac{112}{14}$ to its equivalent number. *Ans.* 8.

To reduce a Whole Number to an equivalent Fraction having a given Denominator.

RULE.—Multiply the whole number by the given denominator, and set the product over the said denominator.

EXAMPLE.—Reduce 8 to a fraction whose denominator shall be 9.

$8 \times 9 = 72$; *then* $\frac{72}{9}$ = *the answer.*

Ex. 2. Reduce 12 to a fraction whose denominator shall be 13. *Ans.* $\frac{156}{13}$.

To reduce a Compound Fraction to an equivalent Simple one.

RULE.—Multiply all the numerators together for a numerator, and all the denominators together for a denominator.

NOTE.—*When there are terms that are common, they may be omitted.*

EXAMPLE.—Reduce $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{2}{3}$ to a simple fraction.

$\frac{1}{2} \times \frac{3}{4} \times \frac{2}{3} = \frac{6}{24} = \frac{1}{4}$, *Ans.*

Or, $\frac{1}{\cancel{2}} \times \frac{\cancel{3}}{4} \times \frac{\cancel{2}}{\cancel{3}} = \frac{1}{4}$, *by canceling the 2's and 3's.*

Ex. 2. Reduce $\frac{1}{2}$ of $\frac{3}{4}$ to a simple fraction.

$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$, *Ans.*

Ex. 3. Reduce $\frac{3}{7}$ of $\frac{4}{5}$ to a simple fraction. *Ans.* $\frac{12}{35}$.

Ex. 4. Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{7}$ of $\frac{7}{9}$ of $\frac{8}{11}$ to a simple fraction. *Ans.* $\frac{20}{99}$.

Ex. 5. Reduce 2, and $\frac{2}{5}$ of $\frac{5}{6}$ to a fraction. *Ans.* $\frac{20}{30}$.

Ex. 6. Reduce $2\frac{1}{2}$ and $\frac{1}{2}$ of $\frac{2}{3}$ to a fraction. *Ans.* $\frac{5}{6}$.

To reduce Fractions of different Denominations to equivalent ones having a common Denominator.

RULE.—Multiply each numerator by all the denominators except its own for the new numerators, and multiply all the denominators together for a common denominator.

NOTE.—*In this, as in all other operations, whole numbers, mixed, or compound fractions, must first be reduced to the form of simple fractions.*

EXAMPLE.—Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ to a common denominator.

$$\left.\begin{array}{l} 1\times3\times4=12 \\ 2\times2\times4=16 \\ 3\times2\times3=18 \\ \hline 2\times3\times4=24 \end{array}\right\} \tfrac{12}{24},\ \tfrac{16}{24},\ \tfrac{18}{24},\ \textit{Ans.}$$

The operation may be performed mentally; thus,

Reduce $\frac{1}{8}$, $\frac{3}{2}$, $\frac{6}{8}$, and $\frac{5}{2}$ to a common denominator.

$$\tfrac{1}{8}=\tfrac{1}{8}. \qquad \tfrac{3}{2}=\tfrac{12}{8}. \qquad \tfrac{6}{8}=\tfrac{6}{8},\ \textit{and}\ \tfrac{5}{2}=\tfrac{20}{8}.$$

Ex. 2. Reduce 2, and $\frac{2}{5}$ of $\frac{5}{6}$ to a common denominator.

Ans. $\frac{60}{30}$, $\frac{12}{30}$, $\frac{25}{30}$.

Ex. 3. Reduce $\frac{5}{6}$, $2\frac{3}{5}$, and 4 to a common denominator.

Ans. $\frac{25}{30}$, $\frac{78}{30}$, $\frac{120}{30}$.

Ex. 4. Reduce $\frac{1}{2}$, $\frac{3}{4}$, $1\frac{1}{2}$, and $5\frac{1}{3}$ to a common denominator.

Ans. $\frac{24}{48}$, $\frac{36}{48}$, $\frac{72}{48}$, $\frac{256}{48}$.

NOTE 1. When the denominators of two given fractions have a common measure, divide them by it; then multiply the terms of each given fraction by the quotient arising from the other denominator.

Ex. 5. Reduce $\frac{3}{25}$ and $\frac{3}{35}$ to a common denominator.

$$\frac{\frac{3}{25}}{5}\ \textit{and}\ \frac{\frac{3}{35}}{7}=\tfrac{15}{175}\ \textit{and}\ \tfrac{21}{175},\ \textit{Ans.}$$

NOTE 2. When the less denominator of two fractions exactly divides the greater, multiply the terms of that which has the less denominator by the quotient.

Ex. 6. Reduce $\frac{3}{7}$ and $\frac{5}{14}$ to a common denominator.

$$\tfrac{3}{7}\ \textit{and}\ \frac{\frac{5}{14}}{2}=\tfrac{6}{14}\ \textit{and}\ \tfrac{5}{14},\ \textit{Ans.}$$

To reduce Complex Fractions to Simple ones.

RULE.—Reduce the two parts both to simple fractions, then multiply the numerator of each by the denominator of the other.

EXAMPLE.—Simplify the complex fraction $\frac{2\frac{2}{3}}{4\frac{4}{5}}$.

$$2\tfrac{2}{3}=\tfrac{8}{3} \qquad 8\times 5=40$$
$$4\tfrac{4}{5}=\tfrac{24}{5} \qquad 3\times 24=\overline{72}=\tfrac{5}{9},\ \textit{Ans.}$$

Ex. 2. Simplify the complex fraction $\frac{\frac{5}{2}}{3}$.

$$\frac{\frac{5}{2}}{3}=\frac{5}{2} \quad \frac{3}{\cdot}=\frac{5}{6},\ \textit{Ans.}$$

Ex. 3. Simplify the complex fraction $\frac{3\frac{2}{5}}{4\frac{1}{2}}$. *Ans.* $\frac{34}{45}$.

To find the Value of a Fraction in parts of a whole Number.

RULE.—Multiply the whole number by the numerator, and divide by the denominator; then, if any thing remains, multiply it by the parts in the next inferior denomination, and divide by the denominator as before, and so on as far as necessary; so shall the quotients placed in order be the value of the fractions required.

EXAMPLE.—What is the value of $\frac{1}{2}$ of $\frac{2}{3}$ of \$9?

$\frac{1}{2}$ of $\frac{2}{3}=\frac{2}{6}$, *and* $\frac{2}{6}$ of $\frac{9}{\cdot}=\frac{18}{6}=\$3$, *Ans.*

Ex. 2. Reduce $\frac{3}{4}$ of a pound to avoirdupois ounces.

```
    3
    1
4)  3(0 lbs.
    16 ounces in a lb.
4)48
  12 ounces, Ans.
```

Ex. 3. Reduce $\frac{3}{10}$ of a day to hours.

$\frac{3}{10}\times\frac{24}{\cdot}=\frac{72}{10}=7\frac{2}{10}$ *hours, Ans.*

Ex. 4. Reduce $\frac{4}{5}$ of a pound troy to ounces and pennyweights.

```
          4
         12
        5)48
  ounces 9   3
            20
          5)60
pennyweights 12=9 oz., 12 dwts., Ans.
```

Ex. 5. What is the value of $\frac{7}{8}$ of an acre?

Ans. 3 *roods,* 20 *poles.*

Ex. 6. What is the value of $\frac{2}{5}$ of \$4 83?

Ans. \$1 $93\frac{1}{5}$.

To reduce a Fraction from one Denomination to another.

RULE.—Multiply the number of parts in the next less denomination by the numerator *if the reduction is to be to a less name,* but multiply by the denominator *if to a greater.*

EXAMPLE.—Reduce $\frac{1}{4}$ of a dollar to the fraction of a cent.

$\frac{1}{4}\times\frac{100}{1}=\frac{100}{4}=\frac{25}{1}$, *Ans.*

Ex. 2. Reduce $\frac{1}{6}$ of an avoirdupois pound to the fraction of an ounce.

$\frac{1}{6}\times\frac{16}{1}=\frac{16}{6}=\frac{8}{3}$, *Ans.*

Ex. 3. Reduce $\frac{2}{7}$ of a cwt. to the fraction of a lb.

$\frac{2}{7}\times\frac{112}{\cdot}=\frac{224}{7}=\frac{32}{1}$, *Ans.*

Ex. 4. Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of a mile to the fraction of a foot.

$\frac{2}{3}$ of $\frac{3}{4}=\frac{6}{12}\times\frac{5280}{1}=\frac{31680}{12}=\frac{2640}{1}$, *Ans.*

Ex. 5. Reduce $\frac{1}{4}$ of a square foot to the fraction of an inch.

Ans. $\frac{36}{1}$.

ADDITION OF VULGAR FRACTIONS.

RULE.—If the fractions have a common denominator, add all the numerators together, and place their sum over the denominator.

NOTE.—*If the fractions have not a common denominator, they must be reduced to one, and compound and complex must be reduced to simple fractions.*

EXAMPLE.—Add $\frac{1}{4}$ and $\frac{3}{4}$ together.

$\frac{1}{4}+\frac{3}{4}=\frac{4}{4}=1$, *Ans.*

Ex. 2. Add $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{6}{10}$ to $2\frac{1}{8}$ of $\frac{3}{4}$.

$\frac{1}{2}\times\frac{3}{4}\times\frac{6}{10}=\frac{18}{80}$, *and* $2\frac{1}{8}$ of $\frac{3}{4}=\frac{17}{8}\times\frac{3}{4}=\frac{51}{32}$.

Then, $\frac{18}{80}+\frac{51}{32}=\frac{576}{2560}+\frac{4080}{2560}=1\frac{2096}{2560}=1\frac{131}{160}$, *Ans.*

Ex. 3. Add $\frac{3}{5}$ and $\frac{5}{6}$ together. *Ans.* $1\frac{13}{30}$.

Ex. 4. Add $\frac{5}{8}$, $7\frac{1}{2}$, and $\frac{1}{3}$ of $\frac{3}{4}$ together. *Ans.* $8\frac{3}{8}$.

Ex. 5. Add $\frac{1}{7}$ of an eagle, $\frac{2}{9}$ of a dollar, and $\frac{5}{12}$ of a cent together. *Ans.* $165\frac{375}{756}$.

SUBTRACTION OF VULGAR FRACTIONS.

Rule.—Prepare the fractions, when necessary, the same as for other operations, then subtract the one numerator from the other, and set the remainder over the common denominator.

Example.—What is the difference between $\frac{5}{6}$ and $\frac{1}{6}$?

$$\frac{5}{6}-\frac{1}{6}=\frac{4}{6}, \text{ Ans.}$$

Ex. 2. Subtract $\frac{6}{8}$ from $\frac{3}{9}$.

$$\left.\begin{array}{l} 6\times 9=54 \\ 3\times 8=24 \\ 8\times 9=\overline{72} \end{array}\right\} =\frac{54}{72}-\frac{24}{72}=\frac{30}{72}, \text{ Ans.}$$

Ex. 3. Subtract $\frac{5}{12}$ from $\frac{7}{13}$. *Ans.* $\frac{19}{156}$.

Ex. 4. Subtract $\frac{3}{5}$ of $\frac{1}{20}$ from $\frac{2}{7}$ of $5\frac{1}{6}$ of 1. *Ans.* $\frac{3037}{2100}$.

MULTIPLICATION OF VULGAR FRACTIONS.

Rule.—Prepare the fractions, when necessary, as previously required; multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.

Example.—What is the product of $\frac{3}{4}$ and $\frac{3}{9}$?

$$\frac{3}{4}\times\frac{3}{9}=\frac{9}{36}=\frac{1}{4}, \text{ Ans.}$$

Ex. 2. What is the product of 6 and $\frac{2}{3}$ of 5?

$$6\times\frac{2}{3} \text{ of } 5=6\times\frac{10}{3}=\frac{60}{3}=20, \text{ Ans.}$$

Ex. 3. What is the product of $\frac{2}{3}$, $3\frac{1}{4}$, 5, and $\frac{3}{4}$ of $\frac{3}{5}$?

$$\frac{\cancel{2}}{\cancel{3}}\times\frac{13}{\underset{2}{\cancel{4}}}\times\frac{\cancel{5}}{1}\times(\tfrac{3}{4} \text{ of } \tfrac{3}{5})\frac{\overset{3}{\cancel{9}}}{\underset{4}{\cancel{20}}}=\frac{13}{2}\times\frac{3}{4}=\frac{39}{8}=4\frac{7}{8}, \text{ Ans.}$$

Ex. 4. What is the product of 5, $\frac{2}{3}$, $\frac{2}{7}$ of $\frac{3}{5}$ and $4\frac{1}{6}$?

Ans. $2\frac{8}{21}$.

DIVISION OF VULGAR FRACTIONS.

Rule.—Prepare the fractions, when necessary, as previously required; then divide the numerator by the numerator, and the denominator by the denominator, if they will exactly divide; but if not, invert the terms of the divisor, and multiply the dividend by it, as in multiplication.

Example.—Divide $\frac{25}{9}$ by $\frac{5}{3}$.

$$\tfrac{25}{9}\div\tfrac{5}{3}=\tfrac{5}{3}=1\tfrac{2}{3}, \textit{ Ans.}$$

Ex. 2. Divide $\frac{5}{9}$ by $\frac{2}{15}$.

$$\tfrac{5}{9}\div\tfrac{2}{15}=\tfrac{5}{9}\times\tfrac{15}{2}=\tfrac{75}{18}=4\tfrac{3}{18}, \textit{ Ans.}$$

Ex. 3. Divide $\frac{7}{16}$ by $\frac{3}{4}$. *Ans.* $\frac{7}{12}$.

Ex. 4. Divide $\frac{3}{5}$ by 2. *Ans.* $\frac{3}{10}$.

Ex. 5. Divide $\frac{2}{3}$ of $\frac{1}{3}$ by $\frac{5}{7}$ of $7\frac{3}{5}$. *Ans.* $\frac{7}{171}$.

RULE OF THREE IN VULGAR FRACTIONS.

Rule.—Prepare the fractions, when necessary, as previously required; invert the first term, and multiply it and the second and third terms continually together; the product will be the result required.

Example.—If $\frac{3}{8}$ of a barrel cost $\frac{2}{5}$ of a dollar, what will $\frac{5}{16}$ of a barrel cost?

$$\tfrac{3}{8}:\tfrac{2}{5}::\tfrac{5}{16}:— \qquad \frac{\cancel{8}}{3}\times\frac{2}{\cancel{5}}\times\frac{\cancel{5}}{\underset{2}{\cancel{16}}}=\tfrac{2}{6}=\$0\ 33+, \textit{ Ans.}$$

Ex. 2. What will $3\frac{3}{8}$ ounces of silver cost at $\frac{19}{60}$ of a pound sterling per ounce?

$$3\tfrac{3}{8}=\tfrac{27}{8}.\quad \tfrac{19}{60} \text{ of } \tfrac{20}{\cdot}=\tfrac{380}{60}=\tfrac{38}{6}=\tfrac{19}{3}.$$

$$\frac{1}{\cdot}\times\frac{19}{\cancel{3}}\times\frac{\overset{9}{\cancel{27}}}{8}=\frac{171}{8} \textit{ shillings.}$$

Then $\frac{171}{8}\times\frac{12}{1}=\frac{2052}{8}=256\frac{4}{8}$ *pence, or* £1 1*s.* $4\frac{1}{2}$*d., Ans.*

Ex. 3. What part of a ship is worth \$60,120, when $\frac{1}{8}$ of her cost \$17,535? $\frac{501}{1169}\div 167=\frac{3}{7}$, *Ans.*

DECIMALS.

A DECIMAL is a fraction which has for its denominator a UNIT with as many ciphers annexed as the numerator has places; it is usually expressed by setting down the numerator only, with a point on the left of it. Thus, $\frac{4}{10}$ is .4, $\frac{85}{100}$ is .85, $\frac{0075}{10000}$ is .0075, and $\frac{125}{100000}$ is .00125. When there is a deficiency of figures in the numerator, ciphers are prefixed to make up as many places as there are ciphers in the denominator.

MIXED NUMBERS consist of a whole number and a fraction, as, 3.25, which is the same as $3.\frac{25}{100}$, or $\frac{325}{100}$.

Ciphers on the right hand make no alteration in the value of a decimal, for .4, .40, .400 are all of the same value, each being equal to $\frac{4}{10}$.

ADDITION OF DECIMALS.

RULE.—Set the numbers under each other, according to the value of their places, as in whole numbers, in which position the decimal points will stand directly under each other. Then, beginning at the right hand, add up all the columns as in whole numbers, and place the point directly below all the other points.

EXAMPLE.—Add together 25.125, 56.19, 1.875, and 293.7325.

$$\begin{array}{r@{.}l}
25&125\\
56&19\\
1&875\\
293&7325\\
\hline
376&9225 \textit{ the Sum.}
\end{array}$$

Ex. 2. Add together 27.62, .358, 17.3, .007, and 173.1.
Sum 218.385.

Ex. 3. Add together .001, .09, and .909. *Sum* 1.000.

Ex. 4. Add together 87.5, 56.25, 37.5, and 43.75.
Sum 225.

SUBTRACTION OF DECIMALS.

RULE.—Set the numbers under each other, as in addition; then subtract as in whole numbers, and point off the decimals as in the last rule.

EXAMPLE.—Subtract 15.150 from 89.1759.

$$\begin{array}{r} 89.1759 \\ 15.150 \\ \hline 74.0259 \end{array}\text{, the Rem.}$$

Ex. 2. Subtract 96.50 from 100. *Rem.* 3.50.

Ex. 3. Subtract 3.1416 from 4.5236. *Rem.* 1.3820.

Ex. 4. Subtract 14.56789 from 2486.173.

Rem. 2471.60511.

MULTIPLICATION OF DECIMALS.

RULE.—Set the factors, and multiply them together the same as if they were whole numbers; then point off in the product just as many places of decimals as there are decimals in both the factors. But if there are not so many figures in the product as there are decimal places required, supply the deficiency by prefixing ciphers.

EXAMPLE.—Multiply 1.56 by .75.

$$\begin{array}{r} 1.56 \\ .75 \\ \hline 780 \\ 1092 \\ \hline 1.1700 \end{array}\text{, the Product.}$$

Ex. 2. Multiply 79.25 by .460. *Product* 36.455.

Ex. 3. Multiply 79.347 by 23.15.

Product 1836.88305.

Ex. 4. Multiply .385746 by .00464.

Product .00178986144.

BY CONTRACTION.

To contract the Operation so as to retain only as many Decimal places in the Product as may be thought necessary.

RULE.—Set the unit's place of the multiplier under the figure of the multiplicand whose place is the same as is to be retained for the last in the product, and dispose of the rest of the figures in the contrary order to what they are usually placed in. Then, in multiplying, reject all the figures that are more to the right hand than each multiplying figure, and set down the products, so that their right-hand figures may fall in a column directly below each other; and observe to increase the first figure in every line with what would have arisen from the figures omitted; thus, add 1 for every result from 5 to 14, 2 from 15 to 24, 3 from 25 to 34, 4 from 35 to 44, &c., &c., and the sum of all the lines will be the product as required.

EXAMPLE.—Multiply 13.57493 by 46.20517, and retain only four places of decimals in the product.

```
13.574 93
71 502.64
---------
54 299 72
 8 144 96+2 for 18
   271 50+2  "  18
     6 79+4  "  35
       14+1  "   5
        9+2  "  21
---------
627.23 20
```

6 *is the unit of the multiplier, and* 9 *is the figure of the multiplicand whose place is the same as is to be retained for the last in the product.*

Ex. 2. Multiply 27.14986 by 92.41035, and retain only five places of decimals. *Product* 2508.92806.

Ex. 3. Multiply 480.14936 by 2.72416, and retain only four places of decimals. *Product* 1308.0035.

Ex. 4. Multiply 325.701428 by .7218393, and retain only three places of decimals. *Product* 235.103.

Ex. 5. Multiply 81.4632 by 7.24651, retaining only three places of decmials. *Product* 590.324.

DIVISION OF DECIMALS.

RULE.—Divide as in whole numbers, and point off in the quotient as many places for decimals as the decimal places in the dividend exceed those in the divisor; but if there are not so many places, supply the deficiency by prefixing ciphers.

EXAMPLE.—Divide 53.00 by 6.75.

```
6.75)53.00(7.85+
     47 25
     -----
      5 750
      5 400
      -----
        3500
        3375
        ----
         125
```

Here 2 ciphers were annexed to carry out the division.

Ex. 2. Divide 45.5 by 2100. *Quotient* .0216+.

Ex. 3. Divide 12 by .7854. *Quotient* 15.278.

Ex. 4. Divide .061 by 79000.

Quotient .00000077215+.

Ex. 5. Divide 2.7182818 by 3.1415927.

Quotient .865256—.

Ex. 6. Divide .00128 by 8.192. *Quotient* .000156.

BY CONTRACTION.

RULE.—Take only as many figures of the divisor as will be equal to the number of figures, both integers and decimals, to be in the quotient, and find how many times they may be contained in the first figures of the dividend, as usual.

Let each remainder be a new dividend; and for every such dividend leave out one figure more on the right-hand side of the divisor, carrying for the figures cut off as in Contraction of Multiplication.

NOTE.—*When there are not so many figures in the divisor as are required to be in the quotient, continue the first operation until the number of figures in the divisor be equal to those remaining to be found in the quotient, after which begin the contraction.*

EXAMPLE.—Divide 2508.92806 by 92.41035, retaining only four places of decimals in the quotient.

```
92.4103|5)2508.928|06(27.1498
          1848 207+1
           660 721
           646 872+2
            13 849
             9 241
             4 608
             3 696
               912
               832+4
                80
                74+2
                 6
```

Ex. 2. Divide 4109.2351 by 230.409, retaining only four decimals in the quotient. *Quotient* 17.8345.

Ex. 3. Divide 37.10438 by 5713.96, retaining only five decimals in the quotient. *Quotient* .00649.

Ex. 4. Divide 913.08 by 2137.2, retaining only three decimals in the quotient. *Quotient* .427.

REDUCTION OF DECIMALS.

To reduce a Vulgar Fraction to its equivalent Decimal.

RULE.—Divide the numerator by the denominator, as in division of decimals, annexing ciphers to the numerator as far as necessary, and the quotient will be the decimal required.

EXAMPLE.—Reduce $\frac{4}{5}$ to a decimal.

```
5)4.0
   .8, Quotient.
```

Ex. 2. Reduce $\frac{35}{700}$ to a decimal.

$$\begin{array}{l} 700)35.00(.05,\ \textit{Quotient.} \\ \quad\ \ \underline{35\ 00} \end{array}$$

Ex. 3. Reduce $\frac{5}{8}$ to a decimal. *Quotient* .625.

Ex. 4. Reduce $\frac{15}{16}$ to a decimal. *Quotient* .9375.

Ex. 5. Reduce $\frac{6}{192}$ to a decimal. *Quotient* .03125.

To find the Value of a Decimal in Terms of an inferior Denomination.

RULE.—Multiply the decimal by the number of parts in the next lower denomination, and cut off as many places for a remainder to the right hand as there are places in the given decimal.

Multiply that remainder by the parts in the next lower denomination, again cutting off for a remainder, and so on through all the denominations of the decimal.

Then the several denominations pointed off on the left hand will give the result required.

EXAMPLE.—What is the value of .875 dollar?

$$\begin{array}{r} .875 \\ \underline{100} \\ 87.500\ \textit{cents,} \\ \underline{10} \\ 5.000\ \textit{mills.} \end{array}$$

Ans. 87 *cents*, 5 *mills.*

Ex. 2. What is the content of .140 cubic foot in inches?

$$\begin{array}{r} .140 \\ \underline{1728}\ \textit{cubic inches in a cubic foot.} \\ 241.920 \end{array}$$

Ans. 241.920 *cubic inches.*

Ex. 3. What is the value of .00129 of a foot?

Ans. .01548 *inches.*

Ex. 4. What is the value of 1.075 ton in pounds?

Ans. 2408.

Ex. 5. Reduce .0125 lb. troy to pennyweights.

Ans. 3 *dwts.*

Ex. 6. Reduce .95 mile to its equivalent decimals in its lower denominations.

$$\begin{array}{r l}
.95 & \\
8 & \textit{furlongs.} \\
\hline
7.60 & \\
40 & \textit{rods.} \\
\hline
24.00 &
\end{array}$$

Ans. 7 *furlongs and* 24 *rods.*

Ex. 7. Reduce .05 mile to its equivalent decimals.

Ans. 0 *furlongs and* 16 *rods.*

Ex. 8. Reduce $\frac{4}{5}$ of a mile to its equivalent decimals.

Ans. 6 *furlongs and* 16 *rods.*

Ex. 9. Reduce $\frac{1}{9}$ of a cubic yard to its equivalent decimals.

Ans. 2.9999 *feet*+.

Ex. 10. Reduce $\frac{1}{3}$ of a degree to its equivalent decimals.

Ans. 19 *minutes and* 59.999 *seconds*+.

To reduce a Decimal to its equivalent in a higher Denomination.

RULE.—Divide by the number of parts in the next higher denomination, continuing the operation as far as required.

EXAMPLE.—Reduce 1 inch to the decimal of a foot.

$$\begin{array}{r|l}
12 & 1.00000 \\
\hline
 & .08333,\ \&c.,\ \textit{Ans.}
\end{array}$$

Ex. 2. Reduce 14 minutes to the decimal of a day.

$$\begin{array}{r|l}
60 & 14.00000 \\
\hline
24 & .23333 \\
\hline
 & .00972,\ \&c.,\ \textit{Ans.}
\end{array}$$

Ex. 3. Reduce $14''\ 12'''$ to the decimal of a minute.

$$\begin{array}{r|l}
 & 14''\ 12''' \\
 & 60 \\
\hline
60 & 852.''' \\
\hline
60 & 14.2'' \\
\hline
 & .23666',\ \&c.,\ \textit{Ans.}
\end{array}$$

NOTE.—*When there are several numbers, to be reduced all to the decimal of the highest.*

Reduce them all to the lowest denomination, and proceed as for one denomination.

Ex. 4. Reduce 5 feet 10 inches and 3 barleycorns to the decimal of a yard.

$$\begin{array}{r|lll}
 & \text{Feet.} & \text{Inches.} & \text{Bc.} \\
 & 5 & 10 & 3 \\
 & 12 & & \\
\hline
 & 70 & & \\
 & 3 & & \\
\hline
3 & 213. & & \\
\hline
12 & 71. & & \\
\hline
3 & 5.9166 & & \\
\hline
 & 1.9722, \text{\&c., } Ans. & &
\end{array}$$

Ex. 5. Reduce 1 dwt. to the decimal of a pound troy.
Ans. .004166+ *lb.*

Ex. 6. Reduce 1 yard to the decimal of a mile.
Ans. .000568+ *mile.*

Ex. 7. Reduce 8 feet 6 inches to the decimal of a mile.
Ans. .0016098 *mile.*

Ex. 8. Reduce $4\frac{1}{2}$ miles to the decimal of 80 miles.
Ans. .05625.

Ex. 9. Reduce 14′, 18″, and 36‴ to the decimal of a degree. *Ans.* .2385 *degree.*

Ex. 10. Reduce 17 yards, 1 foot, and 5.98848 inches to the decimal of a mile. *Ans.* .009943 *mile.*

RULE OF THREE IN DECIMALS.

Rule.—Prepare the terms by reducing vulgar fractions to decimals, compound numbers to decimals of the highest denominations, and the first and third terms to the same denomination; then proceed as in whole numbers.

Example.—If $\frac{1}{2}$ a ton of iron cost $\frac{3}{4}$ of a dollar, what will .625 of a ton cost?

$$\left.\begin{array}{l} \frac{1}{2}=.5 \\ \frac{3}{4}=.75 \end{array}\right\} \qquad \begin{array}{r} .5 : .75 :: .625 \\ .625 \\ \hline .5).46875 \\ \hline .9375, \; Ans. \end{array}$$

Ex. 2. If $\frac{3}{8}$ of a yard cost $\frac{2}{5}$ of a dollar, what will $\frac{5}{16}$ of a yard cost? *Ans.* .3333+ *dollar.*

Ex. 3. If $\frac{7}{16}$ of a mile cost \$15.75, what will $\frac{9}{10}$ of a furlong cost? *Ans.* \$4 05.

DUODECIMALS.

In Duodecimals, or Cross Multiplication, the dimensions are taken in feet, inches, and twelfths of an inch.

Rule.—Set down the dimensions to be multiplied together, one under the other, so that feet may stand under feet, inches under inches, &c.

Multiply each term of the multiplicand, beginning at the lowest, by the feet in the multiplier, and set the result of each directly under its corresponding term, carrying 1 for every 12 from 1 term to the other.

In like manner, multiply all the multiplicand by the inches of the multiplier, and then by the twelfth parts, setting the result of each term one place farther to the right hand for every multiplier. The sum of the products is the result required.

Example.—Multiply 1 foot 3 inches by 1 foot 1 inch.

Feet.	Inches.	
1	3	
1	1	
1	3	
	1	3
1	4	3

Proof.—1 foot 3 inches is 15 inches, 1 foot 1 inch is 13 inches; and $15 \times 13 = 195$ square inches. Now the above product reads 1 foot, 4 inches, and 3 twelfths of an inch, and

1 foot	=144	square inches.
4 inches	= 48	"
3 twelfths	= 3	"
	195	"

which is the product required.

Ex. 2. How many square feet, inches, &c., are there in a platform 35 feet $4\frac{1}{2}$ inches long, and 12 feet $3\frac{1}{3}$ inches wide?

Feet.	Inches.	Twelfths.		
35	4	6		
12	3	4		
424	6			
8	10	1	6	
	11	9	6	0
434	3	11	0	0

Or 434 *feet,* 3 *inches, and* 11 *twelfths.*

Ex. 3. Multiply 20 feet $6\frac{1}{2}$ inches by 40 feet 6 inches.

Ans. 831 *feet,* 11 *inches,* 3 *twelfths, which is equal to* 831 *square feet and* 135 *square inches.*

By decimals, 40 ft. 6 in. = 40.5
20 ft. $6\frac{1}{2}$ in. = 20.541666, &c.
831.937499 *square feet.*
144
134.999856 *square inches.*

Table showing the value of Duodecimals in Square Feet and Decimals of an Inch.

	Sq. Feet.		Sq. Inches.
1 Foot	1	or	144.
1 Inch	$\frac{1}{12}$	"	12.
1 Twelfth	$\frac{1}{144}$	"	1.
$\frac{1}{12}$ of 1 Twelfth	$\frac{1}{1728}$	"	.083333, &c.
$\frac{1}{12}$ of $\frac{1}{12}$ of 1 Twelfth	$\frac{1}{20736}$	"	.006944, &c.

Application of this Table.

What number of square inches are there in a floor $100\frac{1}{2}$ feet broad, and 25 feet, 6 inches, and 6 twelfths long?

$100\frac{1}{2}$ *feet*		=	100.5 *feet.*
25 "	(25×144)	=	3600. *inches.*
6 *inches*	(6×12)	=	72. "
6 *twelfths*		=	6. "
25 *feet* 6 *inches* 6 *twelfths*		=	3678. *inches.*

$$
\begin{array}{r l}
12)\overline{3678.} & \textit{inches.} \\
12)\overline{\ 306.5} & \\
\overline{25.54166} & \textit{feet.}
\end{array}
$$

As the 3678 *are square inches, it is necessary to divide by* 144 *to produce square feet, and the operation is more readily performed by dividing twice by* 12 ($12\times12=144$) *than by* 144 *in one division.*

$$
\begin{array}{l r}
\textit{Then} & 25.54166 \\
 & 100.5 \\
\textit{feet} & \overline{2566.936830} \\
 & 12 \\
\textit{inches} & \overline{11.241960} \\
 & 12 \\
\textit{twelfths} & \overline{2.903520}
\end{array}
$$

$$
\begin{array}{l l r l}
\text{Or, } \ldots\ldots\ldots\ldots & 2566\ \textit{feet.} & & \\
11\times12 = & & 132. & \textit{inches.} \\
2 = & & 2. & \text{“} \\
.9\ (.9\div12) = & & .75 & \text{“} \\
\hline
 & 2566\ \textit{feet,} & 134.75 & \textit{inches.}
\end{array}
$$

INVOLUTION.

INVOLUTION is the multiplying any number into itself a certain number of times. The products obtained are called POWERS. The number is called the ROOT, or first power.

When a number is multiplied by itself once, the product is the square of that number; twice, the cube; three times, the biquadrate, &c. Thus, of the number 5,

5 is the Root, or 1st power.
$5\times5=25$ " Square, or 2d power, and is expressed 5^2.
$5\times5\times5=125$ " Cube, or 3d power, and is expressed 5^3.
$5\times5\times5\times5=625$ " Biquadrate, or 4th power, and is expressed 5^4.

The little figure denotes the power, and is called the INDEX or EXPONENT.

EXAMPLE.—What is the cube of 9?

$$9\times9\times9=729,\ \textit{Ans.}$$

Ex. 2. What is the cube of $\frac{3}{4}$? *Ans.* $\frac{27}{64}$.

Ex. 3. What is the 4th power of 1.5? *Ans.* 5.0625.

Ex. 4. What is the square of 4.16? *Ans.* 17.3056.

Ex. 5. What is the square of $\frac{2}{3}$? *Ans.* $\frac{4}{9}$.

Ex. 6. What is the third power of $\frac{5}{9}$? *Ans.* $\frac{125}{729}$.

Ex. 7. What is the fourth power of .025?

Ans. .000000390625.

Ex. 8. What is the fifth power of 5? *Ans.* 3125.

Ex. 9. What is the fifth power of .05?

Ans. .0000003125.

EVOLUTION.

EVOLUTION is finding the ROOT of any number.

The sign $\sqrt{}$ placed before any number indicates the square root of that number is required or shown.

The same character expresses any other root by placing the index above it.

$$\text{Thus, } \sqrt{25}=5, \qquad 4+2=\sqrt{36}.$$
$$\sqrt[3]{27}=3, \text{ and } \sqrt[3]{64}= \quad 4.$$

Roots which only approximate are called Surd Roots, but those which are exact are called Rational Roots.

TO EXTRACT THE SQUARE ROOT.

RULE.—Point off the given number from the place of units into periods of two figures each by setting a point over the place of units, another over that of hundreds, and so on over every second figure, both to the left hand in integers and to the right hand in decimals.

Find the greatest square in the left-hand period, and place its root in the quotient; subtract the square number from the left-hand period, and to the remainder bring down the next period for a new dividend.

Double the root already found for a divisor; find how many times this *incomplete* divisor is contained in the dividend exclusive of the right-hand figure; with the consideration that the result, or root, is to be the units' figure of the *complete* divisor, place the result in the quotient, and at the right hand of the divisor.

Multiply this divisor by the last quotient figure, and subtract the product from the dividend; bring down the next period, and proceed as before.

EXAMPLE.—What is the square root of 2?

1	2.000000(1.414, &c., *Ans.*
1	1
24	100
4	96
281	400
1	281
2824	11900
4	11296
2828	604

Ex. 2. What is the square root of 144?

1	144(12, *Ans.*
1	1
22	044
	44

Ex. 3. What is the square root of 17.3056?

4	17.3056(4.16, *Ans.*
4	16
81	130
1	81
826	4956
	4956

Ex. 4. What is the square root of .000729? *Ans.* .027.

SQUARE ROOTS OF VULGAR FRACTIONS.

RULE.—Reduce the fractions to their lowest terms, and that fraction to a decimal, then proceed as in whole numbers and decimals.

NOTE.—*When the terms of the fractions are squares, take the root of each and set one above the other; as, the square root of* $\frac{25}{36}$ *is* $\frac{5}{6}$.

EXAMPLE.—What is the square root of $\frac{5}{12}$?

$$\tfrac{5}{12}=.41666666, \text{ \&c.}$$

$$\begin{array}{r|l}
6 & .41666666(.6454, \textit{Ans.} \\
6 & 36 \\
\hline
124 & 566 \\
4 & 496 \\
\hline
1285 & 7066 \\
5 & 6425 \\
\hline
12904 & 64166 \\
4 & 51616 \\
\hline
12908 & 12550
\end{array}$$

Ex. 2. What is the square root of $\frac{9}{12}$?

Ans. 0.8660254.

Ex. 3. What is the square root of $17\frac{3}{8}$?

$$17\tfrac{3}{8}=\overline{17\times 8}+3=\tfrac{139}{8}, \text{ and } \tfrac{139}{8}=17.375,$$

which is the integer and decimal from which the root is to be extracted. *Ans.* 4.1683, &c.

TO EXTRACT THE CUBE ROOT

By the ordinary Method.

Point off the given number from the place of units into periods of three figures each, by setting a point over the place of units, and another also over every third figure from thence to the left hand in integers, and to the right hand in decimals.

Find the greatest cube in the left-hand period, and set its root in the quotient; subtract the cube number from the left-hand period, and to the remainder bring down the second period for a new dividend.

To three times the square of the root just found add three times the root itself, setting this one figure place more to the right than the former; add these products together, and call their sum the divisor. Divide the new dividend, less the last figure to the right hand, by the divisor for the next figure of the root, which annex to the former, calling this last figure e, and the part of the root before found a.

Add the following three products together, viz., thrice a square multiplied by e, thrice a multiplied by e square, and e cube, setting each of them one figure place more to the right than the former, and call their sum the subtrahend; which must not exceed the dividend, but if it does, then make the last figure e less, and repeat the operation for finding the subtrahend till it be less than the dividend.

From the dividend take the subtrahend, and to the remainder join the next period of the given number for a new dividend, to which form a new divisor from the whole root already formed, and from thence another figure of the root as already directed, and so on until the extraction is complete.

Example.—Extract the cube root of 48228.544.

$$
\begin{array}{rr|l}
3\times 3^2 = & 27 & 4\dot{8}22\dot{8}.54\dot{4}(36.4,\ \textit{root.} \\
3\times 3 = & 09 & 27 \\
\cline{2-3}
\textit{divisor} & 279 & 21228\ \textit{dividend.} \\
\cline{2-2}
3\times 3^2\times 6 = & 162\ \ & \\
3\times 3\ \times 6^2 = & 324\ & \left.\right\} 19656\ \textit{subtrahend.} \\
6^3 = & 216 & \\
\hline
3\times 36^2 = & 3888 & \quad 1572544\ \textit{dividend.} \\
3\times 36 = & 108 & \\
\cline{2-2}
\textit{divisor} & 38988 & \\
\cline{2-2}
3\times 36^2\times 4 = & 15552\ \ & \\
3\times 36\ \times 4^2 = & 1728\ & \left.\right\} \quad 1572544\ \textit{subtrahend.} \\
4^3 = & 64 & \\
\hline
 & & \qquad 000\ \textit{remainder.}
\end{array}
$$

Ex. 2. Extract the cube root of 46656. *Ans.* 36.

Ex. 3. Extract the cube root of 46656000. *Ans.* 360.

Ex. 4. Extract the cube root of 2. *Ans.* 1.259921.

Ex. 5. Extract the cube root of $\frac{2}{128}$. *Ans.* .25.

TO EXTRACT THE CUBE ROOT.

Rule.—From a table of Roots take the nearest cube to the given number, and call it the assumed cube.

Then, as the given number added to twice the assumed cube is to the assumed cube added to twice the given number, so is the root of the assumed cube to the required cube root, *nearly.*

By using, in like manner, the root thus found as an assumed cube, and proceeding as above, another root will be found still nearer, and in the same manner as far as may be necessary.

EXAMPLE.—What is the cube root of 10517.9?

Nearest cube 10648, root 22.

10648.	10517.9
2	2
21296	21035.8
10517.9	10648.
31813.9 :	31683.8 :: 22 : 21.9+, *Ans.*

To find the fourth Root of a Number.

RULE.—Extract the square root twice.

EXAMPLE.—What is the 4th root of 625? *Ans.* 5.

To find the sixth Root of a Number.

RULE.—Take the cube root of its square root.

EXAMPLE.—What is the $\sqrt[6]{}$ of 441?

$\sqrt{441}=21$, and $\sqrt[3]{21}=2.758923$, *Ans.*

To find the eighth Root of a Number.

RULE.—Extract the square root thrice.

EXAMPLE.—What is the 8th root of 390625? *Ans.* 5.

To extract any Root whatever.

Let P represent the number,
n " the index of the power,
A " the assumed power, r its root.
R " the required root of P.

Then, as the sum of $n+1\times A$ and $n-1\times P$ is to the sum of $n+1\times P$ and $n-1\times A$, so is the assumed root r to the required root R.

EXAMPLE.—What is the cube root of 1500?

The nearest cube is 1331, root 11.

$$P=1500,\ n=3,\ A=1331,\ r=11;$$

then,

$$\begin{array}{l} \overline{n+1}\times A=5324,\ \overline{n+1}\times P=6000 \\ \overline{n-1}\times P=\underline{3000},\ \overline{n-1}\times A=\underline{2662} \\ \qquad\quad 8324 \quad : \quad 8662 :: 11 : 11.446+,\ \textit{Ans.} \end{array}$$

PROPERTIES OF NUMBERS.

1. A *Prime Number* is that which can only be measured (divided without a remainder) by 1 or unity.

2. A *Composite Number* is that which can be measured by some number greater than unity.

3. A *Perfect Number* is that which is equal to the sum of all its divisors or aliquot parts; as $6=\frac{6}{6}, \frac{6}{3}, \frac{6}{2}$.

4. If the sum of the digits constituting any number be divisible by 3 or 9, the whole is divisible by them.

5. A square number can not terminate with an *odd* number of ciphers.

6. No square number can terminate with two equal digits, except two *ciphers* or two *fours*.

7. No number the last digit of which is 2, 3, 7, or 8, is a square number.

GEOMETRY.

Definitions.

For the Definitions of the SURFACES *of* FIGURES, SOLIDS, LINES, &c., &c., *see Mensuration of Areas, Lines, and Surfaces, Mensuration of Solids and Conic Sections.*

A *Point* has position, but not magnitude.

A *Line* is length without breadth, and is either *Right*, *Curved*, or *Mixed.*

A *Right Line* is the shortest distance between two points.

A *Mixed Line* is composed of a right and a curved line.

A *Superficies* has length and breadth only, and is plane or curved.

A *Solid* has length, breadth, and thickness.

An *Angle* is the opening of two lines having different directions, and is either *Right*, *Acute*, or *Obtuse.*

A *Right Angle* is made by a line perpendicular to another falling upon it.

An *Acute Angle* is less than a right angle.

An *Obtuse Angle* is greater than a right angle.

An *Arc* is any part of the circumference of a circle.

A *Chord* is a right line joining the extremities of an arc.

The *Radius* of a circle is a line drawn from the centre to the circumference.

A *Semicircle* is half a circle.

A *Quadrant* is a quarter of a circle.

A *Secant* is a line that cuts a circle, lying partly within and partly without it.

A *Cosecant* is the secant of the complement of an arc.

A *Sine* of an arc is a line running from one extremity of an arc perpendicular to a diameter passing through the other extremity, and the sine of an angle is the sine of the arc that measures that angle.

The *Versed Sine* of an arc or angle is the part of the diameter intercepted between the sine and the arc.

The *Cosine* of an arc or angle is the part of the diameter intercepted between the sine and the centre.

A *Tangent* is a right line that touches a circle without cutting it.

A *Cotangent* is the tangent of the complement of the arc.

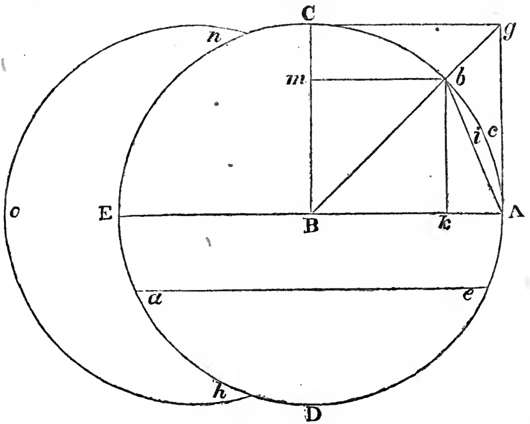
The *Circumference* of every circle is supposed to be divided into 360 equal parts, called *Degrees;* each degree into 60 *Minutes*, and each minute into 60 *Seconds*, and so on.

The *Complement* of an angle is what remains after subtracting the angle from 90 degrees.

The *Supplement* of an angle is what remains after subtracting the angle from 180 degrees.

NOTE.—*A Triangle is also called a Trigon, and a Square a Tetragon.*

To exemplify these definitions, let A *c b*, in the following diagram, be an assumed arc of a circle described with the radius A B.



A *c b*, an Arc of the circle A C E D.
A *b*, the Chord of that arc.
e D *a*, a Segment of the circle.
A B, the Radius.
A B *b*, a Sector.
A D E B, a Semicircle.
C B E, a Quadrant.
A *e a* E, a Zone.
n o h, a Lune.
B *g*, the Secant of the arc A *c b*.
b k, the Sine of do.
A *k*, the Versed Sine of do.

B *k*, the Cosine of the arc A *c b*.
A *g*, the Tangent of do.
C B *b*, the Complement, and *b* B E, the Supplement of the arc A *c b*.
C *g*, the *Cotangent* of the arc, written cot.
B *g*, the *Cosecant* of the arc, written cosec.
m C, the *Coversed sine* of the arc, or, by convention, of the angle A B *b*; written coversin.

The *Vertex* of a figure is its top or upper point. In conic sections it is the point through which the generating line of the conical surface always passes.

The *Altitude*, or height of a figure, is a perpendicular let fall from its vertex to the opposite side, called the base.

The *Measure* of an angle is an arc of a circle contained between the two lines that form the angle, and is estimated by the number of degrees in the arc.

A *Segment* is a part cut off by a plane, parallel to the base.

A *Frustrum* is the part remaining after the segment is cut off.

The *Perimeter* of a figure is the sum of all its sides.

A *Problem* is something proposed to be done.

A *Postulate* is something required.

A *Theorem* is something proposed to be demonstrated.

A *Lemma* is something premised, to render what follows more easy.

A *Corollary* is a truth consequent upon a preceding demonstration.

A *Scholium* is a remark upon something going before it.

MENSURATION OF AREAS, LINES, AND SURFACES.

OF FOUR-SIDED FIGURES.

Parallelograms.

Definition. *Quadrilaterals having their opposite sides parallel.*

To ascertain the Area of a Square, a Rectangle, a Rhombus, or a Rhomboid.

RULE.—Multiply the length by the breadth or height, and the product will be the area.

Or, *length* $\times$ *breadth* $=$ *area.*

NOTE.—The surface of any Quadrilateral is equal to half the product of the diagonals $\times$ the sine of their angle.

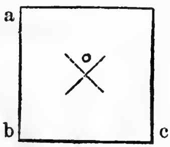
Or, $a\,b \times b\,c = area$.

Or, $l \times b = area$, *l representing the length and b the breadth.*

Square.

Definition. *A plane superficies having equal sides and angles.*

Fig. 1.



EXAMPLE.—The sides *a b*, *b c*, *fig.* 1, are 5 feet 6 inches (5.5); what is the area.

$5.5 \times 5.5 = 30.25$ *square feet.*

Centre of Gravity. Is in its geometrical centre,* *o*.

* The geometrical centre of any Parallelogram, regular Polygon, Circle, Ellipse, &c., &c., is the point of intersection of two or more of their respective diagonals, radii, or diameters.

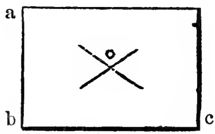
The side of a Square is equal to the square root of its area.

Example.—What is the side of a square when the area of it is 1024 feet? *Ans.* 32 *feet.*

Rectangle.

Definition. *A plane Superficies with parallel sides and equal angles.*

Fig. 2.



Example.—The side *a b*, *fig.* 2, is 5 feet, and *b c* 7 feet 3 inches (7.25); what is the area?

$5 \times 7.25 = 36.25$ *square feet.*

The length of a Rectangle is equal to the area divided by its breadth.

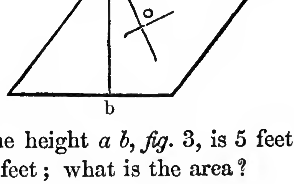
Example.—What is the length of a rectangle, the area being 2048 feet and the breadth 32? *Ans.* 64 *feet.*

Centre of Gravity. Is in its geometrical centre, *o*.

Rhombus (Lozenge).

Definition. *A plane superficies with equal sides, but its angles not right angles.*

Fig. 3.



Example.—The height *a b*, *fig.* 3, is 5 feet 9 inches (5.75), the length *a c*, 7 feet; what is the area?

$5.75 \times 7 = 40.25$ *square feet.*

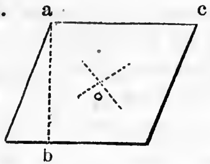
Note.—The opposite angles of a Rhombus are equal.

Centre of Gravity. Is in its geometrical centre, *o*.

Rhomboid.

Definition. *A plane superficies with parallel sides, but its angles not right angles.*

Fig. 4.



Example.—The breadth *a b*, *fig.* 4, is 5 feet, and the length *a c*, 6 feet; what is the area?

$5 \times 6 = 30$ *square feet.*

The opposite angles of a Rhomboid are equal.

Centre of Gravity. Is in its geometrical centre, *o*.

Gnomon.

Definition. *The space included between the lines forming two parallelograms, of which the smaller is inscribed within the larger, so that the angles of each are common to both.*

To ascertain the area of a Gnomon.

Rule.—Find the areas of the two parallelograms, and subtract the less from the greater; the difference will give the area required.

Or, $a - a' = area$.

Example.—The dimensions of a gnomon are 10 by 10 and 6 by 6 inches; what is its area?

$10 \times 10 = 100. \quad 6 \times 6 = 36.$

Then, $100 - 36 = 64 =$ *difference of areas of the parallelograms = the area required.*

Ex. 2. The dimensions of two concentric parallelograms are 15×8 and 12×6 feet; what is the area of the gnomon?

Ans. 48 *feet.*

Centre of Gravity. Is in its geometrical centre.

Triangles.

Definition. *Plane superficies having three sides and angles, and are designated*

Right angled, when one of the angles is a right angle.

Acute angled, when all its angles are less than a right angle.

Obtuse angled, when one angle is greater than a right angle.

Equilateral, when the sides are equal.

Isosceles, when two of the sides are equal.

Scalene, when all the sides are unequal.

NOTES.—Equiangular triangles are similar, or have their like sides proportional.

The *Hypothenuse* is that side of a right-angled triangle which is opposite to the right angle.

The perpendicular height of a triangle is equal to twice its area divided by its base.

To ascertain the Area of a Triangle, Figs. 5, 6, 7.

RULE.—Multiply the base $a\ b$, by the height $c\ d$, and half the product is the area.

Or, $\dfrac{a\ b \times c\ d}{2} = area.$

Or, $\dfrac{b \times h}{2} = area$, *$b$ representing the base, and h the height.*

Fig. 5.

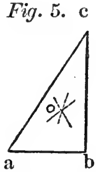


Fig. 6.

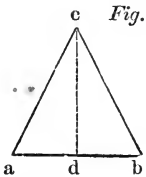
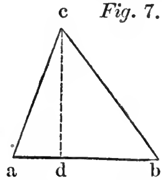


Fig. 7.



EXAMPLE.—The base $a\ b$, *fig.* 5, is 4 feet, and the height $c\ b$, 6 feet; what is the area?

$4 \times 6 = 24$, *and* $24 \div 2 = 12$ *square feet.*

Ex. 2. The base $a\ b$, *fig.* 6, is 5 feet 6 inches, and the height $c\ d$, 5 feet; what is the area?

5 feet 6 inches = 5.5.

$5.5 \times 5 = 27.5$, *and* $27.5 \div 2 = 13.75$ *square feet.*

Ex. 3. The base *a b*, *fig.* 7, is 6 feet 3 inches, and the height *c d* is 5 feet 8 inches; what is the area?

6 feet 3 inches $=6.25$, and 5 feet 8 inches $=5.666$.

Then, $6.25\times5.666=35.413$, *and* $\frac{35.413}{2}=17.706$ *square feet.*

Ex. 4. The base *a b*, *fig.* 7, is 18 feet 4 inches, and the height *c d*, 10 feet 11 inches; what is the area?

18 ft. 4 in. $=18.333$, and 10 ft. 11 in. $=10.916$.

$18.333\times10.916=200.123$, and $\frac{200.123}{2}=100.0625$ *feet.*

Ex. 5. What is the area of a triangle when the base of it is 20 feet and the height 10.25? *Ans.* 102.5 *feet.*

Ex. 6. When the height of a triangle is 16.75 feet and the base 6.24, what is its area? *Ans.* 52.26 *feet.*

Ex. 7. What is the area of a right-angled triangle when its two shortest sides are 26 and 28 feet? *Ans.* 364 *feet.*

Ex. 8. The base of an equilateral triangle is 10 feet, what is its area?

$10\div2=5$, *or the length of the base of two equal right-angled triangles, of which the length of the remaining sides of the triangle is the hypothenuse.*

Therefore, $10^2-5^2=75$, *and* $\sqrt{75}=8.66$, *the height of each of the right-angled triangles.*

Hence, $8.66\times5=43.3$ *square feet.*

Ex. 9. The equal sides of an isosceles triangle are 50 feet, and its base 28; how many square yards does it contain?

Ans. $74\frac{2}{3}$ *yards.*

To ascertain the area of a Triangle by the length of its Sides (Figs. 6 and 7).

Rule.—From half the sum of the three sides subtract each side separately; then multiply the half sum and the three remainders continually together, and the square root of the product is the area.

Or, $\sqrt{s\,(s-a)\times(s-b)\times(s-c)}=$ *area*, *a*, *b*, *c being the sides, and* $s=\frac{a+b+c}{2}$.

When all the Sides are equal.

RULE.—Square the length of a side, and multiply one fourth of the product by 1.732, and this product will give the area required.

Or, $\frac{S^2}{4} \times 1.732 = area.$

EXAMPLE.—The sides of a triangle are 30, 40, and 50; what is the area in square feet?

$$\frac{30+40+50}{2} = \frac{120}{2} = 60, \text{ or } \textit{half sum of the sides.}$$

$$\left.\begin{array}{l} 60-30=30 \\ 60-40=20 \\ 60-50=10 \end{array}\right\} \textit{remainders.}$$

Whence $30 \times 20 \times 10 \times 60 = 360000$, and $\sqrt{360000} = 600$ *square feet.*

Ex. 2. How many acres are there in a triangle, the sides of which are 49, 50.25, and 26 chains? *Ans.* 62.1875.

Ex. 3. What is the area of a triangle, the three sides being 26, 28, and 30 feet? *Ans.* 336 *feet.*

Ex. 4. What is the area of a triangle when its sides are each 50 feet?

$\frac{50^2}{4} = 625 =$ *one fourth of the square of the length of a side,*

and $625 \times 1.732 = 1082.5$ *feet.*

Ex. 5. What is the area of a right-angled triangle when its sides are 50, 50, and 70.7107 feet? *Ans.* 1250 *feet.*

Ex. 6. What is the area of a triangle in square yards when its sides are 500, 400, and 300 feet?

Ans. 6666.66 *yards.*

To ascertain the length of one side of a right-angled Triangle, the length of the other two sides being given (Fig 5).

When the two legs are given, To ascertain the hypothenuse.

RULE.—Add together the squares of the two legs *a b*, *b c*, and extract the square root of the sum.

Or, $\sqrt{a\,b^2+b\,c^2}=$ *hypothenuse.*

Or, $\sqrt{b^2+h^2}$, *b representing the base, and h the height.*

EXAMPLE.—The base $a\,b$ is 30 inches, and the height $b\,c$ 40; what is the length of the hypothenuse?

$$30^2+40^2=2500, \textit{ and } \sqrt{2500}=50 \textit{ inches.}$$

Ex. 2. The base of a right-angled triangle is 38.5 feet, and the perpendicular 18 feet; what is the hypothenuse?

Ans. 42.5 *feet.*

Ex. 3. The height of a church steeple is 103 feet, and the length of its shadow is 320 feet; what is the distance from the point of the shadow to the top of the steeple?

Ans. 336.17 *feet.*

Ex. 4. The base of a triangle is 14 feet, and the height of it 48; what is the length of its hypothenuse?

Ans. 50 *feet.*

Ex. 5. The sides of a triangle are 6 yards, 1 foot, and 11.4 inches; what is the length of its hypothenuse?

Ans. 9 *yards,* 1 *foot, and* .2135 *in.*

To ascertain the other Leg (Fig. 5), the Hypothenuse and one of the Legs being given.

RULE.—Subtract the square of the given leg from the square of the hypothenuse, and the square root of the remainder is the length of the leg required.

$$\text{Or, } \sqrt{hyp.^2 - \begin{cases} b^2 = h. \\ h^2 = b. \end{cases}}$$

$$\text{Or, } \sqrt{a\,c^2 - \begin{cases} a\,b^2 = b\,c. \\ b\,c^2 = a\,b. \end{cases}}$$

EXAMPLE.—The base of a triangle is 30 feet, and the hypothenuse 50; what is the height of it?

$$50^2-30^2=2500-900, \textit{ and } 2500-900=1600.$$

$$\textit{Then } \sqrt{1600}=40 \textit{ feet.}$$

Ex. 2. The hypothenuse of a triangle is 50 feet, and the perpendicular 40; what is the base?

$$50^2-40^2=2500-1600, \textit{ and } 2500-1600=900.$$

$$\textit{Then } \sqrt{900}=30 \textit{ feet.}$$

Ex. 3. It is required to find the length of a ladder, the lower end placed 15 feet from the face of a wall, and the upper end resting on it at a height of 26 feet.

Ans. 30.017 *feet.*

Ex. 4. A pole 50 feet in length, being placed in a street, reached the sill of a window 30 feet from the ground on one side, and being turned over without removing the foot from its location, it reached a window on the opposite side $45\frac{1}{2}$ feet high; what was the breadth of the street? *Ans.* 60.734 *feet.*

Ex. 5. A ladder is to be placed so as to reach the top of a wall 33.75 feet high, and the foot of it can not be set nearer to the base of the wall than 18 feet; what must be the length of the ladder? *Ans.* 38.25 *feet.*

Ex. 6. The base of an isosceles triangle is 25 feet, and the sides of it are 32.5 feet; what is its perpendicular height?

Ans. 30 *feet.*

Ex. 7. If a ladder 100 feet in length was set upright against a vertical wall, and then set out at its foot 10 feet from the wall, how far would the top of the ladder fall?

Ans. .50125 *inches.*

Ex. 8. The width of the wall plates of a house is 48 feet, and the height of the ridge is 10 feet; what must be the length of the rafters? *Ans.* 26 *feet.*

When any two of the dimensions of a triangle and one of the corresponding dimensions of a similar figure are given, and it is required to find the other corresponding dimensions of the last figure.

Let A B C, *a b c, be two similar triangles, Figs.* 8 *and* 9.

Fig. 8.

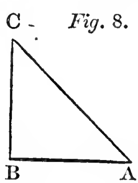
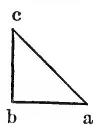


Fig. 9.



Then A B : B C :: *a b* : *b c, or a b* : *b c* :: A B : B C.

NOTE.—The same proportion holds with respect to the similar lineal parts of any other similar figures, whether plane or solid.

Example.—The shadow of a cone 4 feet in length, set vertical, was 5 feet; at the same time, the shadow of a tree was found to be 83 feet; what was the height of the tree, both shadows being on level ground?

$$a\,b : b\,c :: \mathrm{A\,B} : \mathrm{B\,C}$$

$$5 : 4 :: 83 : 66\tfrac{2}{5}$$

$$\begin{array}{r} 4 \\ 5\overline{)332} \\ \hline 66\tfrac{2}{5}\ \textit{feet.} \end{array}$$

Ex. 2. The side of a square is 5 feet, its diagonal 7.071 feet; what will be the side of a square, the diagonal of which is 4 feet? *Ans.* 2.828 *feet.*

Ex. 3. The length of the shadow of a spire is 151.5 feet, while the shadow of a post 8 feet high is 6 feet; what is the height of the spire? *Ans.* 202 *feet.*

To ascertain the length of a Side when the Hypothenuse of a right-angled Triangle of equal sides alone is given.

Rule.—Divide the hypothenuse by 1.414213, and the quotient will give the length of a side.

Or, $\dfrac{hyp.}{1.414213} = \textit{the length of a side.}$

Example.—The hypothenuse of a right-angled triangle is 300 feet; what is the length of its sides?

$$300 \div 1.414213 = 212.1320\ \textit{feet.}$$

Ex. 2. The diagonal of a square is 28.28426 feet; what is the length of a side of it? *Ans.* 20 *feet.*

To ascertain the Perpendicular or Height of a Triangle when the Base and Area alone are given.

Rule.—Divide twice the area of the triangle by its base, and the result is the length of the perpendicular.

Or, $\dfrac{2\,a}{b} = h.$

Example.—The area of a triangle is 10 feet, and the length of its base 5; what is its perpendicular?

$$10 \times 2 = 20, \text{ and } 20 \div 5 = 4\ \textit{feet.}$$

Ex. 2. The base of an isosceles triangle is 25 feet, and its area 375 feet; what is the height of its perpendicular?

Ans. 30 *feet.*

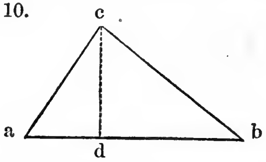
To ascertain the Perpendicular or Height of a Triangle when the two Sides and the Base are given (Fig. 10).

RULE.—As the base is to the sum of the sides, so is the difference of the sides to the difference of the divisions of the base. Half this difference being added to or subtracted from half the base will give the two divisions thereof. Hence, as the sides and their opposite division of the base constitute a right-angled triangle, the perpendicular thereof is readily found by preceding rules.

Or, $\dfrac{b\,c+c\,a\times b\,c\backsim c\,a}{b\,a}=b\,d\backsim d\,a.$

Or, $\dfrac{a\,c^2+a\,b^2-b\,c^2}{2\,a\,b}=a\,d$, *whence* $\sqrt{a\,c^2-a\,d^2}=d\,c.$

Fig. 10.



EXAMPLE.—The three sides of a triangle, *a*, *b*, *c*, *Fig.* 10, are 9.928, 8, and 5 feet; what is the length of the perpendicular on the longest side?

As 9.928 : 8+5 :: 8∾5 : 3.928, *the difference of the divisions of the base.*

Then $3.928\div2=1.964$, *which, added to* $\dfrac{9.928}{2}=4.964+1.964=6.928$, *the length of the longest division of the base.*

Hence we have a right-angled triangle with its base 6.928, *and its hypothenuse* 8; *consequently, its remaining side or perpendicular is* $\sqrt{(8^2-6.928^2)}=4$ *feet.*

Ex. 2. The three sides of a triangle are 42, 40, and 26 feet; what is the height thereof?

Ans. 24 *feet.*

Centres of Gravity. On a line drawn from any angle to the middle of the opposite side, at two thirds of the distance from the angle, as at *o*, *fig.* 5.

TRAPEZIUMS AND TRAPEZOIDS.

Trapezium.

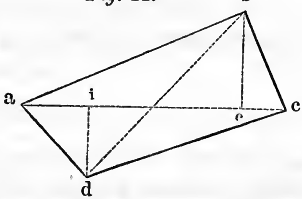
Definition. *Quadrilaterals having unequal sides.*

To ascertain the Area of a Trapezium (*Fig.* 11).

RULE.—Multiply the diagonal $a\,c$ by the sum of the two perpendiculars falling upon it from the opposite angles, and half the product is the area.

Or, $\frac{a\,c \times \overline{b+d}}{2} = area.$

Fig. 11.



EXAMPLE.—The diagonal $a\,c$, *fig.* 11, is 125 feet, and the perpendiculars b and d, 50 and 37 feet; what is the area?

$125 \times \overline{50+37} = 10875$, *and* $10875 \div 2 = 5437.5$ *square feet.*

Ex. 2. What is the area of a plot of ground, the diagonal being 12.5 poles, and the perpendiculars 50 and 37 poles?

Ans. 543.75 *poles.*

Ex. 3. What is the area of a trapezium, the diagonal being 42 feet, one perpendicular 18 feet, and the other 16?

Ans. 714 *feet.*

When the two opposite angles are supplements to each other, that is, when a trapezium can be inscribed in a circle, the sum of its opposite angles being equal to two right angles, or 180°.

RULE.—From half the sum of the four sides subtract each side severally; then multiply the four remainders continually together, and the square root of the product will be the area.

Example.—In a trapezium the sides are 15, 13, 14, and 12, and the diagonal 16 inches; required its area, its opposite angles being supplements to each other.

$$15+13+14+12=54, \text{ and } \frac{54}{2}=27.$$

$$\begin{array}{l} 27 \quad 27 \quad 27 \quad 27 \\ 15 \quad 13 \quad 14 \quad 12 \\ \overline{12} \times \overline{14} \times \overline{13} \times \overline{15} = 32760, \textit{ and } \sqrt{32760} = 180.997, \textit{ Ans.} \end{array}$$

Centre of Gravity. Draw the two diagonals, and find the centres of gravity of each of the four triangles thus formed; join each opposite pair of their centres, and the intersection of the two lines is the centre of gravity.

Trapezoid.

Definition. *A Quadrilateral with only one pair of opposite sides parallel.*

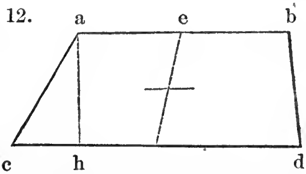
To ascertain the Area of a Trapezoid (Fig. 12).

Rule.—Multiply the sum of the parallel sides $a\,b$, $d\,c$, by $a\,h$, the perpendicular distance between them, and half the product is the area.

Or, $\dfrac{a\,b+d\,c\times a\,h}{2}=$ *area.*

Or, $\dfrac{s+s'\times h}{2}=$ *area,* $s+s'$ *representing the sides.*

Fig. 12.



Example.—The parallel sides are 100 and 132 feet, and the distance between them 62.5 feet; what is the area?

$100+132\times62.5=14500$, *and* $14500\div2=7250$ *square feet.*

Ex. 2. Required the area in feet, the distance between the parallel sides being 12.5 feet, and the sides being 20 and 26.4 yards. *Ans.* 870 *feet.*

Ex. 3. How many square yards are there in a trapezoid the breadth of which is 65 feet, and the two sides 28 and 38.5 feet? *Ans.* 240.1388.

Ex. 4. What is the area of a trapezoid the height of which is 54.25 feet, and the sides 28 feet $1\frac{1}{2}$ inches and 30 feet $4\frac{1}{2}$ inches? *Ans.* 1586.8125.

Centre of Gravity. On a line, e, joining the middle points of the parallel sides $a\,b$, $d\,c$, the distance from

$$d\,c=\frac{e}{3}\times\left(\frac{c\,d+2\,a\,b}{c\,d+a\,b}\right).$$

POLYGONS.

Definition. *Plane figures having more than four sides, and are either regular or irregular, according as their sides or angles are equal or unequal, and they are named from the number of their sides and angles. Thus,*

A Trigon (triangle) has	3	*sides.*	*An Octagon has*	8	*sides.*
Tetragon (square)	4	"	*A Nonagon*	9	"
Pentagon	5	"	*Decagon*	10	"
Hexagon	6	"	*An Undecagon*	11	"
Heptagon	7	"	*A Dodecagon*	12	"

&c., &c., &c.

All of which being composed of isosceles triangles, they may be similarly measured.

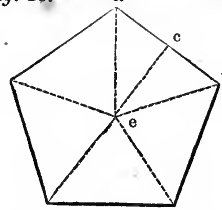
Regular Polygons.

To ascertain the Area of a regular Polygon (Fig. 13).

Rule.—Multiply the length of a side, $a\,b$, by the perpendicular distance to the centre $e\,c$, and the product multiplied by the number of sides and divided by 2 will be the area.

Or, $\frac{a\,b\times c\,e\times n}{2}$=*area, where n represents the number of sides.*

Fig. 13.



EXAMPLE.—What is the area of a pentagon, the side *a b* being 5 feet, and the distance *c e* 4.25 feet?

$5 \times 4.25 \times 5$ $(n) = 106.25$ = *product of length of a side, the distance to the centre, and the number of sides.*

Then $106.25 \div 2 = 53.125$ *feet, the result required.*

Ex. 2. What is the area of a hexagon when its side is 14.6 and its perpendicular 12.64 feet?

Ans. 553.632 *square feet.*

Ex. 3. What is the area of an octagon when its sides are 4.9705 and its perpendicular 6 feet? *Ans.* 119.292 *feet.*

To ascertain the Area of a Regular Polygon when the length of a Side only is given.

RULE.—Multiply the square of the side by the multiplier opposite to the name of the polygon in the following table, and the product will be the area.

No. of Sides.	Name of Polygon.	Area.	A. Radius of Circumscribed Circle.	B. Length of the Side.	C. Radius of Circumscribing Circle.	D. Radius of Inscribed Circle.
3	Trigon	0.4330	2.	1.7320	.5773	.2887
4	Tetragon	1.	1.414	1.4142	.7071	.5
5	Pentagon	1.7205	1.238	1.1756	.8506	.6882
6	Hexagon	2.5981	1.156	1.	1.	.8660
7	Heptagon	3.6339	1.110	.8677	1.1524	1.0383
8	Octagon	4.8284	1.083	.7653	1.3066	1.2071
9	Nonagon	6.1818	1.064	.6840	1.4619	1.3737
10	Decagon	7.6942	1.051	.6180	1.6180	1.5388
11	Undecagon	9.3656	1.042	.5634	1.7747	1.7028
12	Dodecagon	11.1962	1.037	.5176	1.9319	1.8660

EXAMPLE.—What is the area of a square when the length of its sides is 7.0710678 inches?

$7.0710678^2 = 50$, *and* $50 \times 1. = 50$ *inches, Ans.*

Ex. 2. What is the area of a hexagon when the length of its sides is 5 inches? *Ans.* 64.9525 *inches.*

Ex. 3. Required the area of an octagon, the length of its sides being 3.8265 inches. *Ans.* 70.698 *inches.*

To ascertain the Radius of a Circle that contains a given Polygon when the length of a Perpendicular from the Centre alone is given.

RULE.—Multiply the distance from the centre to a side of the polygon by the unit in column A, and the product will be the radius of a circle that will circumscribe the figure.

EXAMPLE.—What is the radius of a circle that contains a hexagon, the distance to the centre being 4.33 inches?

$$4.33 \times 1.156 = 5 \textit{ inches.}$$

Ex. 2. What is the radius of a circle that contains an octagon, the distance from a side to the centre being 4.6168 feet? *Ans.* 5 *feet.*

Ex. 3. What is the radius of a circle that contains a square, the distance from a side to the centre being 3.5355 feet? *Ans.* 5 *feet.*

To ascertain the length of a Side of a Polygon that is contained in a given Circle when the radius of the Circle is given.

RULE.—Multiply the radius of the circle by the unit in column B, and the product will be the length of the side of the figure which the circle will contain.

EXAMPLE.—What is the length of the side of a pentagon contained in a circle 8.5 feet in diameter?

$$\frac{8.5}{2} = 4.25 \textit{ radius, and } 4.25 \times 1.1756 = 5 \textit{ feet.}$$

Ex. 2. What is the length of the side of a hexagon when the diameter of the circumscribing circle is 20 feet? *Ans.* 10 *feet.*

Ex. 3. What is the length of the side of an octagon when the diameter of the circle is 10 feet? *Ans.* 3.8265 *feet.*

To ascertain the Radius of a Circumscribing Circle when the length of a Side is given.

RULE.—Multiply the length of a side of the polygon by the unit in column C, and the product will give the radius of the circumscribing circle.

EXAMPLE.—What is the radius of a circle that will contain a hexagon, a side being 5 inches?

$5 \times 1 = 5$ *inches, Ans.*

Ex. 2. What is the radius of a circle that will contain a pentagon, a side of it being 5.878 inches. *Ans.* 5 *inches.*

Ex. 3. What is the radius of a circle that will contain an octagon, when each of its sides is 7.653 feet?

Ans. 10 *feet.*

Ex. 4. Each side of a pentagon is required to be 9 feet; what are the radii of the circumscribing and inscribed circles?

$9 \times .8506 = 7.6554 = 7$ *feet* 7.8648 *inches, radius of circumscribing circle.*

$9 \times .6882 = 6.1938 = 6$ *feet* 2.3256 *inches, radius of inscribed circle.*

To ascertain the Radius of a Circle that can be inscribed in a given Polygon when the length of a Side is given.

RULE.—Multiply the length of a side of the polygon by the unit in column D, and the product is the radius of an inscribed circle.

EXAMPLE.—What is the radius of the circle that is bounded by a hexagon, its sides being 5 inches?

$5 \times .866 = 4.33$ *inches, Ans.*

Ex. 2. The sides of an octagon are 8 inches; what is the radius of the inscribed circle? *Ans.* 9.657 *inches.*

Ex. 3. The sides of a pentagon are 5.878 inches; what is the radius of its inscribed circle? *Ans.* 4.045 *inches.*

To ascertain the Length of a Side and Radius of a regular Polygon when the Area alone is given.

RULE.—Multiply the square root of the area of the polygon by the multiplier in column E of the following table for the length of the side; by the multiplier in column G of the same table for the radius of the circumscribing circle, and by the multiplier in column H, also in the same table, for the radius of the inscribed circle or perpendicular.

No. of Sides.	Name of Polygon.	E. Length of the Side.	G. Radius of Circumscribing Circle.	H. Radius of Inscribed Circle.	Angle.	Angle of Polygon
3	Trigon	1.5197	.8774	.4387	120°	60°
4	Tetragon	1.	.7071	.5	90	90
5	Pentagon	.7624	.6485	.5247	72	108
6	Hexagon	.6204	.6204	.5373	60	120
7	Heptagon	.5246	.6045	.5446	51 25′	$128\frac{4}{7}$
8	Octagon	.4551	.5946	.5493	45	135
9	Nonagon	.4022	.5880	.5525	40	140
10	Decagon	.3605	.5833	.5548	36	144
11	Undecagon	.3268	.5799	.5564	32 43′	$147\frac{3}{11}$
12	Dodecagon	.2989	.5774	.5577	30	150

EXAMPLE.—The area of a square (tetragon) is 16 inches; what is the length of its side?

$$\sqrt{16}=4, \textit{ and } 4\times 1=4 \textit{ inches.}$$

Ex. 2. The area of an octagon is 70.698 yards; what is the length of the diameter of its circumscribing circle?

Ans. 10 *yards.*

Ex. 3. The area of a square is 50 inches; what is the length of the radius of its inscribed circle?

Ans. 3.5355 *inches.*

Ex. 4. The area of a hexagon is 64.9525 inches; what is the length of its sides? *Ans.* 5 *inches.*

Ex. 5. The area of a decagon is 144 inches; what are the lengths of its sides, and of the radii of its circumscribing and inscribed circles?

$$\left.\begin{array}{l}\sqrt{144}=12, \textit{ and } 12\times.3605=4.326 \textit{ inches,}\\ \sqrt{144}=12, \textit{ and } 12\times.5833=6.9996 \quad ``\\ \sqrt{144}=12, \textit{ and } 12\times.5548=6.6576 \quad ``\end{array}\right\}\textit{Ans.}$$

Additional uses of the foregoing Table.

The sixth and seventh columns of the table will greatly facilitate the construction of these figures with the aid of a sector. Thus, if it is required to describe an octagon, opposite to it, in column sixth, is 45; then, with the chord of 60 on the sector as radius, describe a circle, taking the length 45 on the same line of the sector; mark this distance off on the circumference, which, being repeated around the circle, will give the points of the sides.

The seventh column gives the angle which any two adjoining sides of the respective figures make with each other.

REGULAR BODIES.

*To ascertain the Surface or Linear Edge of any Regular Solid Body.**

RULE.—Multiply the square of the linear edge, or the radius of the circumscribed or inscribed circle, by the units in the following table, under the head of the dimension used, and the product will be the surface or edge required.

Number of sides.	Names of figures.	Surface.	Radius of circum. circle.	Radius of inscribed circle.
4	Tetrahedron	1.73205	1.63294	4.89898
6	Hexahedron	6.	1.15470	2.
8	Octahedron	3.46410	1.41421	2.44949
12	Dodecahedron	20.64578	.71364	.89806
20	Icosahedron	8.66025	1.05146	1.32317

EXAMPLE.—What is the surface of a hexahedron or cube having sides of 5 inches?

$5^2 \times 6 = 25 \times 6 = 150$ *inches, Ans.*

Ex. 2. What is the linear edge of a hexahedron circumscribed by a circle of 10 feet radius? *Ans.* 11.547 *feet.*

Centre of Gravity. In all regular polygons and bodies it is at their geometrical centre.

Irregular Polygons.

Definition. *Figures with unequal sides.*

To ascertain the Area of an Irregular Polygon (*Fig.* 14).

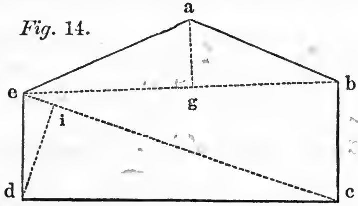
RULE.—Draw diagonals to divide the figures into triangles

* See Appendix (p. 258–61) for additional rules both for Polygons and Regular Bodies.

and quadrilaterals: find the areas of these separately, and the sum of the whole is the area.*

NOTE.—*To ascertain the area of mixed or compound figures, or such as are composed of rectilineal and curvilineal figures together, compute the areas of the several figures of which the whole is composed, then add them together, and the sum will be the area of the compound figure: and in this manner may any irregular surface or field of land be measured, by dividing it into trapeziums and triangles, and computing the area of each separately.*

Fig. 14.



EXAMPLE.—What is the content of *Fig.* 14?

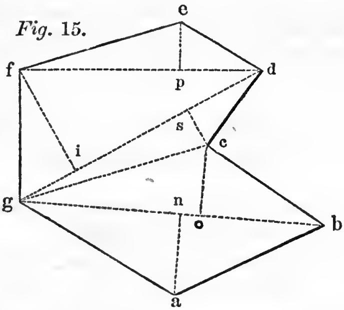
$e\,b = 125$ inches $b\,c = 35.7$ inches $a\,e = 25$ inches
$a\,g = 20$ " $e\,c = 130$ "   and $d\,i = 20$ "

$$\left.\begin{array}{l} e\,b \times a\,g = 125 \times 20 \quad = 2500 \quad \div 2 = 1250 \\ e\,b \times b\,c = 125 \times 35.7 = 4462.5 \div 2 = 2231.25 \\ e\,c \times d\,i = 130 \times 20 \quad = 2600 \quad \div 2 = 1300 \end{array}\right\} \begin{array}{c} \textit{Ans.} \\ 4781.25 \\ \textit{ins.} \end{array}$$

Ex. 2. Required the area of the irregular figure $a\ b\ c\ d\ e\ f\ g\ a$, *Fig.* 15.

$g\,b = 30.5$
$g\,d = 29$
$f\,d = 24.8$
$a\,n = 11.2$

Fig. 15.



$f\,i = 11$
$e\,p = 4$
$c\,o = 6$
$c\,s = 6.6$

* Polygons containing one or more re-entering angles are called Re-entering, as *Fig.* 15. The term *re-entering* is opposed to *salient.* It is a property of a salient polygon that no right line can be drawn external to it that will cut its perimeter in more than two points, while in a re-entering polygon such line may cut it in more than two points.

$\frac{an+co}{2} \times gb = \frac{11.2+6}{2} \times 30.5 = 8.6 \times 30.5 = 262.3 =$ *area of the trapezium* $abcg$.

$\frac{fi+cs}{2} \times gd = \frac{11+6.6}{2} \times 29 = 8.8 \times 29 = 255.2 =$ *area of the trapezium* $gcdf$.

$\frac{fd \times ep}{2} = \frac{24.8 \times 4}{2} = 99.2 = 49.6 =$ *area of the triangle* fde.

Then $262.3 + 255.2 + 49.6 = 567.1 =$ *area of the figure required.*

Ex. 3. Required the area of an irregular polygon.

$eb = 100$ feet	$ec = 110$ feet
$ag = 18$ "	$bc = 12$ "
$ae = 45$ "	$di = 15$ "

Ans. 2385 *feet.*

Ex. 4. In a pentangular field, beginning at the south side and running toward the east, the first side is 2735 links, the second 3115, the third 2370, the fourth 2925, and the fifth 2220; also the diagonal from the first to the third is 3800 links, and that from the third to the fifth 4010; what is the area of the field? *Ans.* 117 *ac.* 2 *ro.* 39 *po.*

NOTE.—As this figure consists of three triangles, all of the sides of which are given, by calculating their areas according to the rule, p. 51, the sum will be the area of the whole figure accurately, without drawing perpendiculars from the angles to the diagonals.

The same thing may also be done in most other cases of this kind.

When any part of the Figure is bounded by a Curve, the Area may be found as follows:

RULE.—Erect any number of perpendiculars upon the base, at equal distances, and find their lengths.

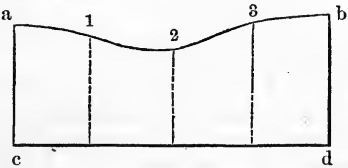
Add the lengths of the perpendiculars thus found together, and their sum divided by their number will give the *mean breadth.* Multiply the mean breadth by the length of the base, and it will give the area of that part of the figure required.

To ascertain the Area of a long Irregular Figure (*Fig.* 16).

RULE.—Take the breadth at several places and at equal distances apart; add them together, and divide their sum by the number of breadths for the mean breadth; multiply this by the length of the figure, and the product is the area.

Or, $\frac{b+b'+b''}{2} = \mathrm{B} \times l = area.$

Fig. 16.



EXAMPLE.—What is the area of *fig.* 16 in square feet?

$a\ c = 50$ *inches,*	$3 = 54$ *inches,*
$1 = 52$ "	$b\ d = 60$ "
$2 = 48$ "	$c\ d = 150$ "

$50+52+48+54+60=264$, *and* $264 \div 5 = 52.8$.

Then $52.8 \times 150 = 7920$, *which* $\div\ 144 = 55$ *square feet, Ans.*

To ascertain the Centre of Gravity of any Plane Figure.

RULE.—Divide it into triangles, and find the centre of gravity of each; connect two centres together, and find their common centre; then connect this common centre and the centre of a third, and find the common centre, and so on, always connecting the last found common centre to another centre till the whole are included, and the last common centre will be that which is required.

Illustration. Where is the centre of gravity of *Fig.* 17?

Fig. 17.

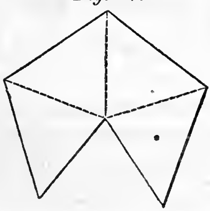


Fig. 18.

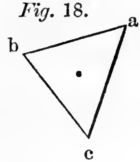


Fig. 19.

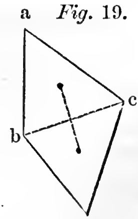


Fig. 20.

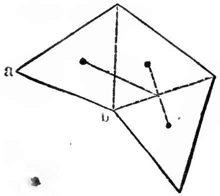
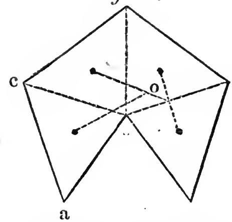


Fig. 21.



The point • represents the centre of gravity of each triangle, *a b c*, *Fig.* 18, of which the figure is composed.

Connect two centres of gravity, as in *Fig.* 19; join their common centre to the centre of gravity of the next triangle, as in *Fig.* 20; join their common centre *o* to the centre of the remaining triangle, *Fig.* 21, and the common centre of this last connection is the required centre of gravity of the figure.

CIRCLE.

Definition. *A plane figure bounded by a true curve, called its Circumference or Periphery.*

The Diameter of a Circle is a right line drawn through its centre, bounded by its periphery.

The Radius of a Circle is a right line drawn from the centre of it to its circumference.

The Circumference of a Circle is assumed to be divided into 360 equal parts, called *degrees;* each degree is divided into 60 parts, called *minutes;* each minute into 60 parts, called *seconds;* and each second into 60 parts, called *thirds,* and so on.

To ascertain the Circumference of a Circle (*Fig.* 22).

Rule.—Multiply the diameter *a b* by 3.1416,* and the product is the circumference.

* The exact proportion of the diameter of a circle to its circumference has never yet been ascertained. Nor can a square or any other right-lined figure be found that shall be equal to a given circle. This is the celebrated problem called the squaring of the circle, which has exercised the abilities of mathematicians for ages, and been the occa-

Or, *as* 7 *is to* 22, *so is the diameter to the circumference.*
Or, *as* 113 *is to* 355, *so is the diameter to the circumference.*

sion of so many disputes. Several persons of eminence, have, at different times, pretended that they had discovered the exact quadrature; but their errors have soon been detected, and it is now conceded as a thing impracticable of attainment.

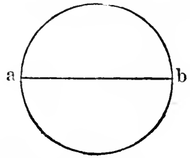
Though the relation between the diameter and circumference can not be accurately expressed, it may yet be approximated to great exactness. In this manner was the problem solved by Archimedes, about two thousand years ago, who discovered the proportion to be nearly as 7 to 22. This he effected by showing that the perimeter of a circumscribed regular polygon of 192 sides is to the diameter in a less ratio than that of $3\frac{10}{70}$ to 1, and that the perimeter of an inscribed polygon of 96 sides is to the diameter in a greater ratio than that of $3\frac{10}{71}$ to 1, and from thence inferred the ratio above mentioned, as may be seen in his book *De Dimensione Circuli.*

The proportion of Vieta and Metius is that of 113 to 355, which is something more than the former. This is a commodious proportion; for, being reduced into decimals, it is correct as far as the sixth figure inclusively. It was derived from the pretended quadrature of a M. Van Eick, which first gave rise to the discovery.

But the first who ascertained this ratio to any great degree of exactness was Van Ceulen, a Dutchman, in his book *De Circulo et Adscriptis.* He found that if the diameter of a circle was 1, the circumference would be 3.141592653589793238462643383279502884 nearly; which is exactly true to 36 places of decimals, and was effected by the continual bisection of an arc of a circle, a method so extremely troublesome and laborious that it must have cost him incredible pains. It is said to have been thought so curious a performance, that the numbers were cut on his tomb-stone in St. Peter's Church-yard at Leyden. This last number has since been confirmed and extended to double the number of places by the late ingenious Mr. Abraham Sharp, of Little Horton, near Bedford, in Yorkshire.

But since the invention of Fluxions, and the Summation of Infinite Series, there have been several methods discovered for doing the same thing with much more ease and expedition. The late Mr. John Machin, Professor of Astronomy in Gresham College, has by these means given a quadrature of the circle which is true to 100 places of decimals; and M. de Lagny, M. Euler, &c., have carried it still farther. All of which proportions are so extremely near the truth, that, except the ratio could be completely obtained, we need not wish for a greater degree of accuracy.—BONNYCASTLE.

Fig. 22.



EXAMPLE.—The diameter of a circle, *fig.* 22, is 1.25 inches; what is its circumference?

$1.25 \times 3.1416 = 3.927$ *inches.*

Ex. 2. The diameter of a circle is 17 feet; what is its circumference? *Ans.* 53.4072 *feet.*

To ascertain the Diameter of a Circle (*Fig.* 22).

Divide the circumference by 3.1416, and the quotient is the diameter.

Or, *as* 22 *is to* 7, *so is the circumference to the diameter.*

To ascertain the Area of a Circle (*Fig.* 22).

RULE.—Multiply the square of the diameter by .7854, and the product is the area.

Or, *multiply the square of the circumference by* .07958.

Or, *multiply half the circumference by half the diameter.*

Or, *multiply the square of the radius by* 3.1416.

Or, $p\,r^2 = area$, *where* p *represents the ratio of the circumference to the diameter, and* r *the radius.*

EXAMPLE.—The diameter of a circle is 8 inches; what is the area of it?

8^2 or $8 \times 8 = 64$, *and* $64 \times .7854 = 50.2656$ *inches.*

Ex. 2. What is the area of a circle, the radius being $39\frac{1}{4}$ yards? *Ans.* 4839.8311 *yards.*

Ex. 3. What is the area of a circle in feet, the diameter being $15\frac{3}{4}$ poles? *Ans.* 3214.6587 *feet.*

Ex. 4. What is the circumference of a circle, the area of which is an acre? *Ans.* 246 *yds.*, 1 *ft.*, $10\frac{1}{3}$ *in.*

Centre of Gravity. Is in its geometrical centre. *Semicircle.*

$\frac{4\,r}{3.\,p} =$ *distance from centre.*

USEFUL FACTORS.

In which p represents the Circumference of a Circle the Diameter of which is 1.

Then

$p = 3.14159265358979323846 26+$	$\frac{1}{4p} = .079577$
$2p = 6.283185307179+$	$\sqrt{p} = 1.772453$
$4p = 12.566370614359+$	$\frac{1}{2}\sqrt{p} = .886226$
$\frac{1}{2}p = 1.570796326794+$	$2\sqrt{p} = 3.544907$
$\frac{1}{4}p = 0.785398163397+$	$\sqrt{\frac{2}{p}} = .797884$
$\frac{4}{3}p = 4.188790$	$\sqrt{\frac{1}{p}} = .564189$
$\frac{1}{6}p = .523598$	$\frac{360}{p} = 114.591559$
$\frac{1}{8}p = .392699$	$\frac{2}{3}p = 2.094395$
$\frac{1}{12}p = .261799$	$\frac{6}{p} = 1.909859$
$\frac{1}{360}p = .008726$	$36p = 113.097335$
$\frac{1}{p} = .318309$	
$\frac{2}{p} = .636619$	
$\frac{4}{p} = 1.273239$	

In which the Diameter of a Circle is 10.

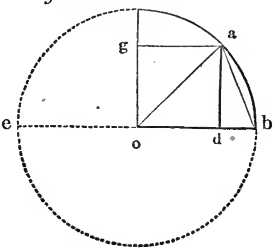
1. Chord of the arc of the semicircle	=10.
2. Chord of half the arc of the semicircle	= 7.071067
3. Versed sine of the arc of the semicircle	= 5.
4. Versed sine of half the arc of the semicircle	= 1.464466
5. Chord of half the arc of the half of the arc of the semicircle	= 3.82683
6. Half the chord of the chord of half the arc	= 3.535533
7. Length of arc of semicircle	=15.707963
8. Length of half the arc of the semicircle	= 7.853981
Square of the chord of half the arc of the semicircle (2.)	=50.
Square root of versed sine of half the arc (4.)	= 1.210151
Square of versed sine of half the arc (4.)	= 2.144664
Square of the chord of half the arc of half the arc of the semicircle (5.)	=14.64467
Square of half the chord of the chord of half the arc (6.)	=12.5

In all the following calculations, p is taken at 3.1416, $\frac{1}{4}p$ at .7854, $\frac{1}{6}p$ at .5236, and whenever the decimal figure next to the one last taken exceeds 5, one is added. Thus, 3.14159 for four places of decimals is set down 3.1416.

Circular Arc.

Definition. *A part of the Circumference of a Circle.*

Fig. 23.



The Sine of an Arc is a line running from one extremity of an arc perpendicular to a diameter joining the other extremity, as *a d*, *fig.* 23.

The Sine of an Angle is the sine of the arc that measures that angle.

The Versed Sine of an Arc or Angle is the part of the diameter intercepted between the sine and the arc, as *d b*. It is frequently written *versed sine of half the arc.*

The Complement of an Arc or Angle is what remains after subtracting the angle from 90°, as *a o c*, *Fig.* 23.

The Supplement of an Arc or Angle is what remains after subtracting the angle from 180°, as *a o e*, *Fig.* 23.

The Coversed Sine is the sine of the complement of the arc, as *c g*, *Fig.* 23.

Note.—For other illustrations of these definitions, see figure, page 46.

To ascertain the Length of an Arc of a Circle (*Fig.* 24) *when the Number of Degrees and the Radius are given.*

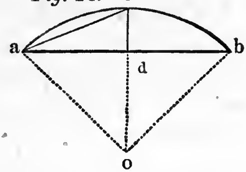
Rule 1. As 180 is to the number of degrees in the arc, so is 3.1416 the radius to its length.

Rule 2. Multiply the radius, *o a*, of the circle by .01745329, and the product by the degrees in the arc.

If the length is required for minutes, multiply the radius by .000290889; if for seconds, multiply by .000004848.

(See Table of length of Arcs, page 130.)

Fig. 24.



EXAMPLE.—The number of degrees in an arc, *o a b*, are 90, and the radius, *o b*, 5 inches; what is the length of the arc?

$180 : 90 :: 5 \times 3.1416 : 7.854$ *inches, Ans.*

Ex. 2. The radius of an arc is 10, and the measure of its angle 44° 30′ 30″; what is the length of the arc?

$10 \times .01745329 = .1745329$, *which* $\times 44 = 7.6794476$, *the length for* 44°.

$10 \times .000290889 = .00290889$, *which* $\times 30 = .0872667$, *the length for* 30′.

$10 \times .000004848 = .00004848$, *which* $\times 30 = .0014544$, *the length for* 30″.

Then, $\left.\begin{array}{r} 7.6794476 \\ .0872667 \\ .0014544 \end{array}\right\} = 7.7681687$, *the length required.*

Or, *reduce the minutes and seconds to the decimal of a degree, and multiply by it.*

See Rule, page 34. 30′ 30″ = .5083, and .1745329 *from above* $\times 44.5083 = 7.768163$.

Ex. 3. The degrees in the arc of a circle are 90, and the diameter of the circle is 20 feet; what is the length of the arc? *Ans.* 15.708.

Ex. 4. The degrees in the arc of a circle are 32° 38′ 42″, and the radius of it is 25 inches; what is the length of the arc?

32° 38′ 42″ = 32.645 degrees.

Ans. 14.2441 *inches.*

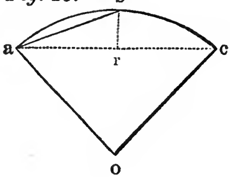
Ex. 5. The degrees in the arc of a circle are 147° 21′ 18″, and the radius of its circle is 25 feet; what is the length of the arc? *Ans.* 64.2959 *feet.*

When the Chord of half the Arc and the Chord of the Arc are given.

RULE.—From 8 times the chord, *a b*, of half the arc, *Fig.* 25, subtract the chord, *a c*, of the arc, and one third of the remainder will be the length nearly.

Or, $\frac{8\,c' - c}{3}$, *c′ representing the chord of half the arc, and c the chord of the arc.*

Fig. 25.



EXAMPLE.—The chord of half the arc, *a b*, is 30 inches, and the chord of the arc 48; what is the length of the arc?

$30 \times 8 = 240 = 8$ *times the chord of half the arc.*

$240 - 48 = 192$, *and* $192 \div 3 = 64$ *inches, Ans.*

Ex. 2. The chord of half the arc is 60 feet, and the chord of the arc 96; what is the length of the arc?

Ans. 128 *feet.*

Ex. 3. The chord of half the arc is 17.67765 feet, and the chord of the arc 25; what is the length of the arc?

Ans. 38.80706* *feet.*

Ex. 4. The chord of half the arc is 3.8268 inches, and the chord of the arc 7.071; what is the length of the arc?

Ans. 7.8478.†

When the Chord of the Arc and the Versed Sine of the Arc are given.

RULE.—Multiply the square root of the sum of the square of the chord, *a c*, *Fig.* 25, and four times the square of the

* The exact length is 39.27 feet.

† The exact length is 7.854 inches.

versed sine, $b\ r$ (equal to twice the chord of half the arc), by ten times the square of the versed sine; divide this product by the sum of fifteen times the square of the chord and thirty-three times the square of the versed sine; then add this quotient to twice the chord, $a\ b$, of half the arc,* and the sum will be the length of the arc very nearly.

Or, $\dfrac{\sqrt{c+4\,v^2}\times 10\,v^2}{15\,c^2+33v^2}+\sqrt{c^2+4\,v^2}$, *c representing the chord, and v the versed sine.*

To ascertain the Versed Sine when the Chord of Half the Arc and Chord of the Arc are given.

Rule.—From the square of the chord, $a\ b$, of half the arc, subtract the square of half the chord of the arc $a\ c$ ($=a\ r^2$), and the square root of the remainder is the versed sine ($b\ r$).

Or $\sqrt{c'^2-(c\div 2)^2}=$*versed sine.*

To ascertain the Length of the Chord of Half the Arc.

Rule 1. Divide the square root of the sum of the square of the chord of the arc and four times the square of the versed sine by two, and the result will be the length.

Or, $\sqrt{c^2+4\,v^2}\div 2=$*chord of half the arc.*

Rule 2. From the sum of the squares of half the chord of the arc and the versed sine, take the square root, and the result will be the length.

Or, $\sqrt{\left(\dfrac{c}{2}\right)^2+v^2}=$*chord of half the arc.*

Rule 3. Multiply the diameter by the versed sine, and the square root of their product will be the length.

Or, $\sqrt{d\times v}=$*chord of half the arc.*

To ascertain the Versed Sine when the Chord of Half the Arc and the Diameter are given.

Rule.—Divide the square of the chord of half the arc by the diameter, and the quotient will be the versed sine.

Or, $(c'^2\div d)=$*versed sine.*

* The square root of the sum of the square of the chord and four times the square of the versed sine is equal to twice *the chord of half the arc.*

When the Chord of the Arc and the Diameter are given.

RULE.—From the square of the diameter subtract the square of the chord and extract the square root of the remainder; subtract this root from the diameter, and half the remainder is the versed sine.

Or, $\frac{\sqrt{d^2-c^2}-d}{2}$ = *versed sine.*

When the Versed Sine is greater than a Semidiameter.

Proceed as before, but add the square root of the remainder (of the squares of the diameter and chord) to the diameter, and half the sum is the versed sine.

Or, $\frac{\sqrt{d^2-c^2}+d}{2}$ = *versed sine.*

To ascertain the Diameter.

RULE 1. Divide the square of the chord of half the arc by the versed sine, and the result will be the diameter.

Or, $\frac{c'^2}{v}$ = *diameter.*

RULE 2. Add the square of half the chord of the arc to the square of the versed sine; divide this sum by the versed sine, and the result will give the diameter.

Or, $\frac{\left(\frac{c}{2}\right)^2+v^2}{v}$ = *diameter.*

To ascertain the Chord of the Arc when the Chord of Half the Arc and the Versed Sine are given.

RULE.—From the square of the chord of half the arc subtract the square of the versed sine, and twice the square root of the remainder will give the chord.

Or, $\sqrt{(c'^2-v^2)}\times 2$ = *chord of the arc.*

EXAMPLE.—The chord of half the arc is 60 inches, and the versed sine 36; what is the length of the chord of the arc?

$60^2-36^2=2304$, *and* $\sqrt{2304}\times 2=96$, *Ans.*

When the Diameter and Versed Sine are given.

RULE.—Multiply the versed sine by 2, and subtract the product from the diameter; then subtract the square of the remainder from the square of the diameter, and the square root of the remainder will give the chord.

Or, $\sqrt{(v \times 2 - d)^2 - d^2} = c$.

If the diameter and chord of half the arc only are given, find the versed sine, as per rule, p. 75, then proceed as above.

EXAMPLE.—The diameter of a circle is 100 feet, and the versed sine of half the arc is 36; what is the length of the chord of the arc?

$36 \times 2 - 100 = 28$, *then* $28^2 - 100^2 = 9216$, *and* $\sqrt{9216} = 96$, *Ans.*

EXAMPLE.—The chord of an arc is 80 inches, and its versed sine 30; what is the length of the arc?

$80^2 = 6400 =$ *square of the chord.*

$30^2 = \ \ 900 =$ *square of the versed sine.*

$\sqrt{(6400 + 900 \times 4)} = 100 =$ *square root of the square of the chord and four times the square of the versed sine, which is, twice the chord of half the arc.*

Then, $100 \times \overline{30^2 \times 10} = 900000 =$ *product of ten times the square of the versed sine and the root above obtained.*

$80^2 \times 15 = 96000 = 15$ *times the square of the chord.*

$30^2 \times 33 = 29700 = 33$ *times the square of the versed sine.*

125700

And $\dfrac{100 \times 30^2 \times 10}{125700} = \dfrac{900000}{125700} = 7.1599$, *which, added to* 100, *or twice the chord of half the arc* $= 107.1599$ *feet.*

Ex. 2. The chord of an arc is 7.07107 inches, and the versed sine 1.46447; what is the length of the arc?

$7.07107^2 \quad = 50 \quad =$ *the square of the chord.*

$1.46447^2 \times 4 = \ \ 8.5787 = 4$ *times the square of the versed sine.*

$\sqrt{58.5787} = 7.6536 =$ *twice the chord of half the arc.*

$1.46447^2 \times 10 = 21.4467 = 10$ *times the square of the versed sine.*

$7.07107^2 \times 15 = 750. \quad = 15$ *times the square of the chord.*

$1.46447^2 \times 33 = \ \ 70.7742 = 33$ *times the square of the versed sine.*

820.7742

Then, $\dfrac{7.6536 \times 21.4467}{820.7742} + 7.6536 = 7.8536$ *inches.*

Ex. 3. The chord of an arc is 96 feet, and the versed sine 36; what is the length of the arc? *Ans.* 128.5918 *feet.*

Ex. 4. The chord of an arc is 40 inches, and the versed sine 15; what is the length of the arc? *Ans.* 53.58 *inches.*

Ex. 5. The chord of an arc is 48 inches, and the versed sine 18; what is the length of the arc? *Ans.* 64.2959 *inches.*

Ex. 6. The chord of an arc is 60 inches, and its versed sine 10; what is the length of the arc?

Ans. 64.3493 *inches.*

Ex. 7. The versed sine of an arc is 2.5658, and the chord 31.6228 inches; what is the length of the arc?

Ans. 32.1747 *inches.*

Ex. 8. The chord of an arc is 7.071, the chord of half the arc is 3.8268, and the diameter of the circle 10 inches; what are the lengths of the versed sine and arc?

By Note, page 75.

$3.8268^2 = 14.6444 =$ *square of half the chord of the arc.*

$7.071 \div 2 = 3.5356$, *and* $3.5356^2 = 12.5 =$ *square of half the chord of the arc.*

Then, $\sqrt{14.6444 - 12.5} = 1.4644 =$ *versed sine.*

Or, by preceding rule, page 75,

$$\frac{3.8268^2}{10} = \frac{14.6444}{10} = 1.4644 = \textit{versed sine.}$$

Length of arc by rule, p. 74, *Example* 2, *p.* 77, 7.8536 *inches.*

Ex. 9. The chord of an arc is 96, and the versed sine 36 inches; what is the chord of half the arc? *Ans.* 60 *inches.*

Ex. 10. The chord of an arc is 70.7107, and the versed sine 14.6447 inches; what are the lengths of

The arc?	*Ans.* 78.54 *inches.*
The chord of half the arc?	*Ans.* 38.268 "
And the diameter of the circle?	*Ans.* 100. "

When the Diameter and Versed Sine are given.

Rule—.Multiply twice the chord of half the arc, *a b*, *Fig.* 25, by 10 times the versed sine, *b r;* divide the product by 27 times the versed sine subtracted from 60 times the diameter, and add the quotient to twice the chord of half the arc; the sum will be the length of the arc very nearly.

Or, $\frac{c' \times 2 \times 10\, v}{60d - 27v} + 2c'$, *d representing the diameter.*

Example.—The diameter of a circle is 100 feet, and the versed sine of the arc 25; what is the length of the arc?

$\sqrt{25 \times 100} = 50 =$ *chord of half the arc.*

$50 \times 2 \times \overline{25 \times 10} = 25000 =$ *twice the chord of half the arc by* 10 *times the versed sine.*

$\overline{100 \times 60} - \overline{25 \times 27} = 5325 = 27$ *times the versed sine subtracted from* 60 *times the diameter.*

Then $\frac{25000}{5325} = 4.6948$, *and* $4.6948 + \overline{50 \times 2} = 104.6948$ *feet.*

Ex. 2. The diameter of a circle is 10 inches, and the versed sine 1.46447; what is the length of the arc?

$\sqrt{1.46447 \times 10} = 3.8268 =$ *chord of half the arc.*

$3.8268 \times 2 \times \overline{10 \times 1.46447} = 112.0847 =$ *twice the chord of half the arc by* 10 *times the versed sine.*

$\overline{10 \times 60} - \overline{1.46447 \times 27} = 560.459 = 27$ *times the versed sine subtracted from* 60 *times the diameter.*

Then $112.0847 \div 560.459 = .19998$, *and* .19998 *added to* $\overline{3.8268 \times 2}$ (*twice the chord of half the arc*) $= 7.8536$ *inches.*

Ex. 3. The diameter of a circle is 10 inches, and the versed sine 1.46447; what is the length of the arc?

Ans. 7.854 *inches.*

Ex. 4. The diameter of a circle is 41.66 feet, and the versed sine 15; what is the length of the arc?

Ans. 53.5799 *feet.*

Ex. 5. The diameter of a circle is 200 feet, and the versed sine of the arc 72; what is the length of the arc?

Ans. 257.1837 *feet.*

Ex. 6. The diameter of a circle is 80 feet, and the chord of half the arc is 16.018; what is the versed sine and what is the length of the arc?

Ans. Versed sine, 3.2072; *Length of arc,* 32.2539 *feet.*

Centre of Gravity. Multiply the radius of the circle by the chord of the arc, and divide the product by the length of the arc; the quotient is the distance of it from the centre of the circle.

Or, $\frac{r \times c}{l}$ = *distance from the centre of the circle.*

EXAMPLES UNDER THE SEVERAL RULES.

1. The degrees in the arc of a sector are 30° 38′ 42″, and the radius of the circle 50; what is the length of the arc?

Ans. 26.7429.

2. The chord of an arc is 70.71067, and the chord of half the arc 38.268; what is the length of the arc?

Ans. 78.536.

3. The chord of an arc is 48, and the versed sine 18; what is the length of the arc? *Ans.* 64.2959.

4. The chord of an arc is 50, the radius 40, and the versed sine 8.775; what is the length of the arc? *Ans.* 54.0096.

5. The diameter of a circle is 10, and the versed sine 5; what is the chord of the arc, the chord of half the arc; and the length of the arc?

Ans. { *Chord of the arc,* 10.
Chord of half the arc, 7.07107.
Length of the arc, 15.708.

6. The diameter of a circle is 100, and the chord of half the arc 60; what is the versed sine? *Ans.* 36.

7. The diameter of a circle is 100, and the chord of the arc 60; what is the versed sine, the chord of half the arc, and the length of the arc?

Ans. { *Versed sine,* 10.
Chord of half the arc, 31.6228.
Length of the arc, 64.3493.

8. The chord of an arc is 96, and the versed sine 36; what are the lengths of the chord of half the arc, the diameter, and the arc?

Ans. { *Chord of half the arc,* 60.
Diameter of the circle, 100.
Length of the arc, 128.5918.

Proportions of the Circle, its Equal, Inscribed, and Circumscribed Squares.

CIRCLE.

1.	Diameter	$\times .8862$	$=$Side of an equal square.
2.	Circumference	$\times .2821$	
3.	Diameter	$\times .7071$	$=$Side of the inscribed square.
4.	Circumference	$\times .2251$	
5.	Area	$\times .9003$	

SQUARE.

6.	A Side	$\times 1.4142$	$=$Diameter of its circumscribing circle.
7.	"	$\times 4.443$	$=$Circumference of its circumscribing circle.
8.	"	$\times 1.128$	$=$Diameter of an equal circle.
9.	"	$\times 3.545$	$=$Circumference of an equal circle.
10.	Square inches	$\times 1.273$	$=$Round inches.

NOTE.—The square described within a circle is one half the area of one described without it.

Sector of a Circle.

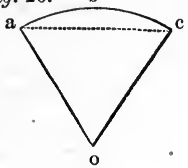
Definition. *A part of a Circle bounded by an arc and two radii.*

To ascertain the Area of a Sector of a Circle when the Degrees in the Arc are given (Fig. 26).

RULE.—As 360 is to the number of degrees in the sector, so is the area of the circle of which the sector is a part to the area of the sector.

Or, $\frac{d \times a}{360}$ $=$*area, d representing the degrees in the arc and a the area of the circle.*

Fig. 26.



EXAMPLE.—The radius of a circle, *o a*, is 5 inches, and the number of degrees of the sector is 22° 30′; what is the area?

Area of a circle of 5 *inches radius is* 78.54 *inches.*

Then, as 360° : 22° 30′ :: 78.54 : 4.90875, *Ans.*

Ex. 2. The degrees in the arc of a sector are 147° 21′ 18″, and the area of the circle is 1963.5 feet; what is the area of the sector?

To reduce 147° 21′ 18″ to a decimal.

$$\begin{array}{r} 21 \quad 18 \\ 60 \\ \hline 60)1278 \\ \hline 60)213 \\ \hline .355 \end{array}$$

Then .355+147=147.355.

Ans. 803.6987 *feet.*

Ex. 3. The degrees in the arc of a sector are 32° 38′ 42″, and the area of the circle is 1963.5 inches; what is the area of the sector? *Ans.* 178.0512 *inches.*

Ex. 4. The degrees in the arc of a sector are 90°, and the area of the circle is 981.75 inches; what is the area of the sector? *Ans.* 245.4375 *inches.*

Ex. 5. The degrees in the arc of a sector are 90°, the versed sine 1.46446 feet, and the chord of half the arc is 3.8268; what is its area?

$3.8268^2 = 14.6446$, *the square of the chord of half the arc.*

$14.6446 \div 1.46446 = 10 =$ *the square of the chord of half the arc* ÷ *the versed sine, which is the diameter.*

$10^2 \times .7854 = 78.54$, *the area of the whole circle.*

Then, as 360° : 90° :: 78.54 : 19.635 *feet.*

NOTE.—Divide the area by .7854, and the square root of the quotient is the diameter of the circle.

Illustration. The area of a circle is 176.715; *what is its diameter?*
176.715÷.7854=225, *and* $\sqrt{225} = 15$, *Ans.*

When the Length of the Arc, &c., are given (Fig. 26).

RULE.—Multiply the length of the arc, *a c b*, by half the length of the radius, *a o*, and the product is the area.

Or, $a \times \frac{r}{2}$ = *area, a representing the arc, and r the radius.*

EXAMPLE.—The length of the arc of a sector is 7.854 inches, and a radius of it is 5; what is its area?

$$7.854 \times \frac{5}{2} = 7.854 \times 2.5 = 19.635 \textit{ inches.}$$

Ex. 2. The length of the arc of a sector is 10.472 inches, and a radius 5; what is its area? *Ans.* 26.18 *inches.*

Ex. 3. The length of the arc of a sector is 14.19 inches, the diameter of the circle being 100; what is its area?

Ans. 354.75 *inches.*

Ex. 4. The radius of a circle is 25 feet, and the versed sine, *b r*, of the arc of a sector is 18; what is the area of the sector?

By Rule, page 78, the length of the arc is 64.2959.

Ans. 803.69875 *feet.*

NOTE.—If the diameter or a radius is not given, see Rules, page 76.

Ex. 5. What is the area of a sector when the versed sine of its arc is 15, and the chord 40 inches?

$\overline{40 \div 2}^2 = 400$ = *square of half the chord of the arc.*

$15^2 = 225$ = *square of the versed sine.*

Then, $\frac{400+225}{15} = 41.666$, *the diameter, and* $\frac{41.666}{2} = 20.833$, *the radius.*

Length of arc by Rule, page 78, 53.58, *and* $53.58 \times \frac{20.833}{2} =$ 558.116 *inches.*

Ex. 6. The radius of a circle is 50 inches, and the versed sine of the arc of a sector of it is 25; what is its area?

Ans. 2617.37 *inches.*

Ex. 7. The diameter of a circle is 100 feet, the versed sine of the arc of a sector is 36, and the chord of half the arc is 60; what is the area of the sector? *Ans.* 3214.795 *feet.*

Ex. 8. The length of the arc of a sector is 104.6948 inches, the chord of the arc is 86.6024, and the versed sine of it is 25; what is the area of the sector?

Ans. 2617.37 *inches.*

Centre of Gravity. Multiply twice the chord of the arc by the radius of the sector, and divide their product by three times the length of the arc; the quotient is the distance from the centre of the circle.

Or, $\frac{2\,c\,r}{3\,l}$ = *distance from centre of circle; r representing radius, and l the length of the arc.*

Example.—Where is the centre of gravity of the sector given in example 5?

$40 \times 2 = 80$ = *twice the chord of the arc.*

$80 \times 20.833 = 1666.64$ = *product of twice the chord and the radius.*

$53.58 \times 3 = 160.74$ = *three times the length of the arc.*

Then, $1666.64 \div 160.74 = 10.369$, *the distance from the centre of the circle.*

Segment of a Circle.

Definition. *A part of a circle bounded by an arc and a chord.*

To ascertain the Area of a Segment of a Circle, Fig. 27, when the Chord and Versed Sine of the Arc, and Radius or Diameter of the Circle are given.

When the Segment is less than a Semicircle, as a b c, Fig. 27.

Rule.—Find the area of the sector having the same arc as the segment; then find the area of the triangle formed by the chord of the segment and the radii of the sector, and the difference of these areas will be the area required.

Note.—Subtract the versed sine from the radius; multiply the remainder by one half of the chord of the arc, and the product will be the area of the triangle.

Or, $a - a'$ = *area of segment; a representing area of the sector, and a′ the area of the triangle.*

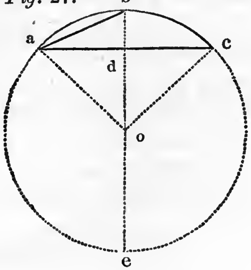
When the Segment is greater than a Semicircle, as a e c, Fig. 27.

Rule.—Find, by the preceding rule, the area of the lesser portion of the circle, *a b c;* subtract it from the area of the whole circle, and the remainder is the area required.

Or, $c - c' =$ *area of segment; c representing area of circle, and c' area of the lesser portion.*

(See Table of Areas, page 134.)

Fig. 27.



EXAMPLE.—The chord, *a c*, *Fig.* 27, is 14.142, the diameter, *b e*, is 20, and the versed sine, *b d*, is 2.929 inches; what is the area of the segment?

By Rule, page 75, $\frac{14.142}{2} = 7.071 =$ *half the chord of the arc.*

$\sqrt{7.071^2 + 2.929^2} = 7.654 =$ *the square root of the sum of the squares of half the chord of the arc and versed sine, which is the chord a b of half the arc a b c.*

By Rule, page 78.

$7.654 \times 2 \times \overline{10 \times 2.929} = 448.371 =$ *twice the chord of half the arc by* 10 *times the versed sine.*

$\overline{20 \times 60} - \overline{2.929 \times 27} = 1120.917 = 60$ *times the diameter subtracted from* 27 *times the versed sine.*

Then, $448.371 \div 1120.917 = .400$, *and* .400 *added to* $\overline{7.654 \times 2}$ (*twice the chord of half the arc*) $= 15.708$ *inches, the length of the arc. By Rule, p.* 82, $15.708 \times \frac{10}{2} = 78.54 =$ *the arc multiplied by half the length of radius, which is the area of the sector.*

$10 - 2.929 = 7.071 =$ *the versed sine subtracted from a radius, which is the height of the triangle a o c, and* $7.071 \times \frac{14.142}{2} = 50 =$ *area of the triangle.*

Consequently, $78.54 - 50 = 28.54$ *inches.*

Ex. 2. The chord of the arc of a segment is 86.6024, the versed sine 25, and the radius 50 feet; what is the area of the segment? *Ans.* 1534.84 *feet.*

Ex. 3. The chord of the arc of a segment is 28 feet, the diameter of the circle 100, and the versed sine of the arc 2; what is the area of the segment? *Ans.* 37.4852 *feet.*

Ex. 4. The diameter of a circle is 50 feet, the chord of the arc of a segment of it is 30, and its versed sine 5; what is the length of the arc, and what the area of the segment?

Ans. Arc, 32.1746 *feet; Segment,* 102.183 *feet.*

Ex. 5. The chord of a segment is 56 inches, its versed sine 4, and the radius of the circle 200 inches; what is its area?

Ans. 149.941 *inches.*

When the Chords of the Arc, and of the half of the Arc, and the Versed Sine are given.

Rule.—To the chord, $a\ c$, of the whole arc, add the chord, $a\ b$, of half the arc and one third of it more; multiply this sum by the versed sine, $b\ d$, and this product, multiplied by .40426, will give the area nearly.

Or, $c+c'+\frac{c'}{3}\times v\times .40426=$ *area nearly.*

Example.—The chord of a segment is 28 feet, the chord of half the arc is 15, and the versed sine 6; what is the area of the segment?

$28+15+\frac{15}{3}=48=$ *the chord of the arc added to the chord of half the arc and* $\frac{1}{3}$ *of it more.*

$48\times 6=288=$ *product of above sum and the versed sine.*

Then, $288\times .40426=116.427$ *feet, the area required.*

Ex. 2. The chord of a segment is 40 feet, its versed sine 10, and the chord of half the arc is 22.36; what is its area?

Ans. 282.226 *feet.*

Ex. 3. What is the area of a segment, its half chord being 14, the chord of half its arc 14.142, and its versed sine 2 yards?

Ans. 37.884 *yards.*

Ex. 4. The chord of a segment is 150 feet, the chord of half the arc is 106.066, and the versed sine 75; what is the area? *Ans.* 8835.739 *feet.**

When the Chord of the Arc (or Segment) and the Versed Sine only are given.

RULE.—Find the chord of half the arc, and proceed as before.

EXAMPLE.—The chord of the arc of a segment is 28 yards, and its versed sine 2; what is the length of the chord of half the arc, and what the area of the segment in feet?

$\frac{28}{2}=14$, *and* $14^2+2^2=200=$ *sum of square of half the chord and the versed sine.*

$\sqrt{200}=14.14213=$ *square root of preceding sum* $=$ *chord of half the arc.*

Area of segment 113.6525 *feet.*

Ex. 2. The chord of a segment is 48 inches, its versed sine 32, and the diameter of the circle 50; what is its area?

The versed sine being greater than half the diameter, the segment is consequently greater than a semicircle.

Area of circle, $50^2\times.7854=1963.5$

Area of lesser portion, versed sine 18. $=$ 640.3478

Ans. 1323.1522 *inches.*

Ex. 3. The chord of an arc is 86.6024, and its versed sine 25 feet; what is the area of the segment in feet and inches? *Ans.* 1549 *feet*, .156 *inches.*

Centre of Gravity. Divide the cube of the chord of the segment by twelve times the area, and the quotient is the distance from the centre of the circle.

Or, $\frac{c^3}{12\,a}=d$, *when c represents the chord of the segment, a the area, and d the distance from the centre of the circle.*

EXAMPLE.—The chord of a segment is 14.14213, its radius 10, and its area 28.53; where is its centre of gravity?

* The exact area is 8835.729.

$14.14213^3 = 2828.426 =$ *cube of the chord.*

$28.53 \times 12 = 342.36 = 12$ *times the area.*

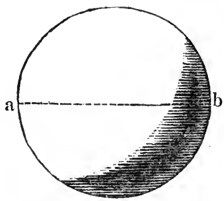
Then, $2828.426 \div 342.36 = 8.261$ *from the centre of the circle, and* $10 - 8.261 = 1.739$ *from the base of the segment.*

Sphere.

Definition. *A figure, the surface of which is at a uniform distance from the centre.*

To ascertain the Convex Surface of a Sphere (Fig. 28).

Fig. 28.



RULE.—Multiply the diameter, *a b*, by the circumference, *a b c d*, and the product will give the surface required.

Or, $d \times c =$ *surface, d representing diameter, and c the circumference.*

Or, $4\, p\, r^2 =$ *surface.** Or, $p\, d^2 =$ *surface.*

EXAMPLE.—What is the surface of a sphere of 10 inches diameter?

$$10 \times 31.416 = 314.16 \textit{ inches.}$$

Ex. 2. The diameter of a sphere is 17 inches; what is the surface of it in feet? *Ans.* 6.305 *square feet.*

Ex. 3. If the circumference of a sphere is 50.2656 inches, what is its surface in feet? *Ans.* 5.585 *feet.*

Centre of Gravity. Is in its geometrical centre.

* *p* or π represents in this, and in all cases where it is used, the ratio of the circumference of a circle to its diameter, or 3.1416.

Segment of a Sphere.

Definition. *A section of a sphere.*

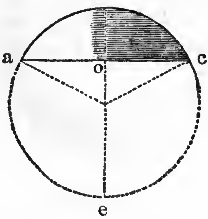
To ascertain the Surface of a Segment of a Sphere, Fig. 29.

Rule.—Multiply the height, *b o*, by the circumference of the sphere, and the product, added to the area of the base, *a o c*, is the surface required.

Or, $h \times c + b =$ *surface, when h represents the height, c the circumference of the sphere, and b area of base.*

Or, $2\,p\,r\,h =$ *convex surface alone.*

Fig. 29.



Example.—The height, *b o*, of a segment, *a b c*, is 36 inches, and the diameter, *b e*, of the sphere, 100; what is the convex surface, and what the whole surface?

$36 \times \overline{100 \times 3.1416} = 11309.76 =$ *height of segment multiplied by the circumference of the sphere.*

Then, to ascertain the area of the base. The diameter and versed sine being given; the diameter of the base of the segment, being equal to the chord of the arc, is, by rule, page 76,

$$36 \times 2 - 100 = 28.$$

$$\sqrt{28^2 - 100^2} = 96.$$

$96^2 \times .7854 = 7238.2464 =$ *convex surface*, and $7238.2464 + 11309.76 = 18548.0064 =$ *convex surface added to area of base = the whole surface.*

Note.—When the convex surface of a figure alone is required, the area or areas of the base or ends must be omitted.

Ex. 2. The height of a segment is 10, and the diameter of the sphere 100 feet; what is the surface? *Ans.* 5969.04 *feet.*

Ex. 3. The diameter of a sphere is 200 inches, and the height of a segment of it is 1 foot 8 inches; what is its surface in feet? *Ans.* 162.4733 *feet.*

When the diameter of the Base of the Segment and the Height of it are alone given.

RULE.—Add the square of half the diameter of the base to the square of the height; divide this sum by the height, and the result will give the diameter of the sphere.

Or, $\overline{d \div 2}^2 + h^2 \div h = diameter.$

EXAMPLE.—The semi-diameter of the base of a segment of a sphere is 48 feet, and the height of it is 36; what is the surface of the segment in square yards?

Ans. 2060.8896 *yards.*

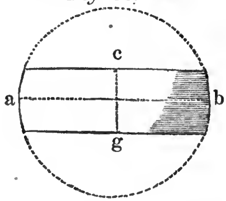
Centre of Gravity of Convex Surface. At the middle of its height.

Spherical Zone (or Frustrum of a Sphere).

Definition. *The part of a sphere included between two parallel chords.*

To ascertain the Surface of a Spherical Zone, Fig. 30.

Fig. 30.



RULE.—Multiply the height, $c\ g$, by the circumference of the sphere, and the product added to the area of the two ends is the surface required.

Or, $h \times c + a + a' = surface.$

Or, $2\,p \times r \times h = convex\ surface.$

EXAMPLE.—The diameter of a sphere, *a b*, from which a segment is cut, is 25 inches, and the height of it, *c g*, is 8; what is its convex surface?

$25 \times 3.1416 \times 8 = 628.32 =$ *height* $\times$ *circumference of sphere* $=$ *convex surface.*

Ex. 2. The height of a zone is 36 inches, and the radius of the sphere is 50 inches; what is its convex surface?

Ans. 11309.76 *inches.*

When the Diameter of the Sphere is not given.

Multiply the mean length of the two chords by half their difference, divide this product by the breadth of the zone, and to the quotient add the breadth. To the square of this sum add the square of the lesser chord, and the square root of their sum will be the diameter of the circle.

Ex. 3. The greater and lesser chords of a segment of a sphere are 96 and 60, and the height of the segment is 26; what is its convex surface and what its surface?

$\left.\begin{array}{ll}\textit{Convex surface,} & 8168.160 \\ \textit{Surface,} & 18233.846\end{array}\right\}$ *Ans.*

Centre of Gravity. At the middle of its height.

Spheroids or Ellipsoids.

Definition. *Figures generated by the revolution of a semi-ellipse about one of its diameters.*

When the revolution is about the transverse diameter they are Prolate, and when it is about the conjugate they are Oblate.

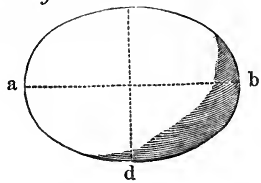
To ascertain the Surface of a Spheroid (Fig. 31).

When the Spheroid is Prolate.

RULE.—Square the diameters, *a b* and *c d*, and multiply the square root of half their sum by 3.1416, and this product by the conjugate diameter.

Or, $\sqrt{\frac{d^2 + d'^2}{2}} \times 3.1416 \times d =$ *surface, d representing conjugate diameter.*

Fig. 31.



EXAMPLE.—A prolate spheroid has diameters of 10 and 14 inches; what is its surface?

$10^2+14^2=296=$ *sum of squares of diameters.*

$296\div2=148$, *and* $\sqrt{148}=12.1655=$ *square root of half the sum of the squares of the diameters.*

$12.1655\times3.1416\times10=382.191=$ *product of root above obtained* $\times$ 3.1416, *and that product by the conjugate diameter.*

Ex. 2. A prolate spheroid has diameters of 16 and 22 inches; what is its surface? *Ans.* 966.879 *inches.*

When the Spheroid is Oblate.

RULE.—Square the diameters, *a b* and *c d*, and multiply the square root of half their sum by 3.1416, and this product by the transverse diameter.

Or, $\sqrt{\frac{d^2+d'^2}{2}}\times3.1416\times d'=$ *surface,* d' *representing transverse diameter.*

EXAMPLE.—An oblate spheroid has diameters of 14 and 10 inches; what is its surface?

$14^2+10^2=296=$ *sum of squares of diameters.*

$296\div2=148$, *and* $\sqrt{148}=12.1655=$ *square root of half the sum of the squares of the diameters.*

$12.1655\times3.1416\times14=535.0679=$ *product of root above obtained* $\times$ 3.1416, *and that product by the transverse diameter.*

Ex. 2. An oblate spheroid has diameters of 22 and 16 inches; what is its surface? *Ans.* 1329.4585 *inches.*

Centre of Gravity. Is in their geometrical centres.

NOTE.—For centre of gravity of semi-spheroids, see Appendix, p. 281.

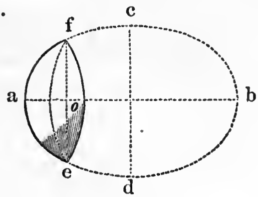
To ascertain the Convex Surface of a Segment of a Spheroid (Fig. 32).

RULE.—Square the diameters, and take the square root of half their sum. Then, as the diameter from which the segment is cut, is to this root, so is the height of the segment to the proportionate height required.

Multiply the product of the other diameter and 3.1416 by the proportionate height of the segment, and this last product will give the surface required.

$$\text{Or, } \frac{\sqrt{\frac{(d^2+d'^2)}{2}}\times h}{d \text{ or } d'} \times d' \text{ or } d \times 3.1416 = \textit{surface.}$$

Fig. 32.



EXAMPLE.—The height, *a o*, of a segment, *e f*, of a prolate spheroid, *Fig.* 32, is 4 inches, the diameters being 10 and 14 inches; what is the convex surface of it?

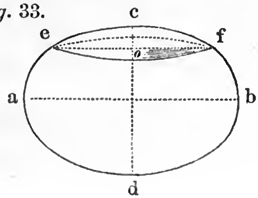
Square root of half the sum of the squares of the diameters, as by previous examples, page 92, 12.1655.

Then, 14 : 12.1655 :: 4 : 3.4758 = *height of segment, proportionate to the mean of the diameters.*

$10 \times 3.1416 \times 3.4758 = 109.1957 =$ *remaining diameter* $\times$ 3.1416, *and again by proportionate height of segment.*

Ex. 2. The height of a segment of a prolate spheroid, *Fig.* 32, is 6 inches, the diameters being 15 and 21 inches; what is the convex surface of it? *Ans.* 252.9663 *inches.*

Fig. 33.



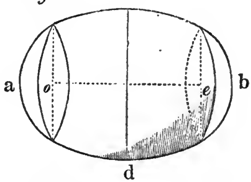
Ex. 3. The height, *c o*, of a segment, *e f*, of an oblate spheroid, *Fig.* 33, is 5 inches, the diameters being 14 and 10; what is the convex surface? *Ans.* 267.5339 *inches.*

To ascertain the Convex Surface of a Frustrum or Zone of a Spheroid (*Fig.* 34).

Rule.—Proceed as by previous rule, page 93, for the surface of a segment, and obtain the proportionate height of the frustrum. Then multiply the product of the diameter parallel to the base of the frustrum and 3.1416 by the proportionate height of the frustrum, and it will give the surface required.

Or, d *or* $d' \times 3.1416 \times h =$ *surface.*

Fig. 34.



Example.—The middle frustrum, *o e*, of a prolate spheroid, *Fig.* 34, is 6 inches, the diameters of the spheroid being 10 and 14 inches; what is its convex surface?

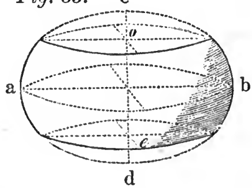
Mean diameter, as per example, page 92, *is* 12.1655.

Diameter parallel to base of frustrum is 10.

As 14 : 12.1655 :: 6 : 5.2138 = *proportionate height of frustrum.*

$10 \times 3.1416 \times 5.2138 = 163.7967 =$ *surface.*

Fig. 35.



Ex. 2. The middle frustrum of an oblate spheroid, *o e*, *Fig.* 35, is 2 inches in height, the diameters of the spheroid, as in

the preceding examples, being 10 and 14; what is its convex surface? *Ans.* 107.0136 *inches.*

Centre of Gravity. Zone. Is in its geometrical centre.

Segment and Frustrum of Spheroid. See Appendix, p. 281.

Circular Zone.

Definition. *A part of a circle included between two parallel chords.*

To ascertain the Area of a Circular Zone (Fig. 36).

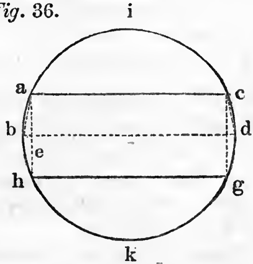
RULE.—To the area of the trapezoid, *a b c d*, or of the parallelogram, *a h c g*, as the case may be, add the area of the segments, *a b*, *c d*, or *a h*, *c g*, and the sum is the area.

Or, subtract the areas of the segments *a i c*, *h k g*, from the area of the circle.

Or, $a + a' = s$, *a representing area of trapezoid, or parallelogram, and a' area of segments.*

(See Table of Areas of Zones, page 130.)

Fig. 36.



When the Diameter of the Circle is not given.

Multiply the mean length of the two chords by half their difference; divide this product by the breadth of the zone, and to the quotient add the breadth.

To the square of this sum add the square of the lesser chord, and the square root of their sum will be the diameter of the circle.

EXAMPLE.—The greater chord, *b d*, is 96 inches, the lesser, *a c*, is 60, and the breadth of the zone, *a e*, is 26; what is its area?

$\frac{96+60}{2}=78=$ *mean length of chords.* $\frac{96-60}{2}=18=$ *half their difference.*

$\frac{78\times18}{26}=54=$ *product of chords and their difference, divided by the breadth of the zone.*

$54+26=80=$ *sum of above quotient and breadth of zone.*

$80^2+60^2=10000=$ *sum of square of above sum and lesser chord.*

Then, $\sqrt{10000}=100=$ *diameter required.*

$\frac{96+60}{2}=78$, *and* $78\times26=2028=$ *area of trapezoid.*

To ascertain the Area of the Segments.

It is necessary, *first*, to ascertain the chord of their arcs; *second*, the versed sine of their arcs.

To ascertain the Chord. The breadth of the zone is the perpendicular, *a e*, of the triangle, of which either chord, *a b*, *c d*, is the hypothenuse. Further, half the difference of the chords *a c* and *b d* of the zone is the length of the base, *b e*, of this triangle.

Hence, having the base and the perpendicular, the hypothenuse or chord of the arc of the segment is readily found.

Thus, $26=$ *breadth of the zone or perpendicular of triangle.*

$\frac{96-60}{2}=18=$ *length of base of triangles.*

Then, $18^2+26^2=1000$, and $\sqrt{1000}=31.6228=$ *chord of arc of segments a b, c d.*

To ascertain the Versed Sine. From the square of the radius subtract the square of half the chord, and the square root of the remainder subtracted from the radius is the versed sine.

Thus, $100\div2=50$, and $50^2=2500=$ *square of radius.*

$31.6228\div2=15.8114$, and $15.8114^2=250=$ *square of half the chord.*

$2500-250=2250$, and $\sqrt{2250}=47.4342=$ *square root of the difference of the squares of the radius and half the chord.*

Then, $50-47.4342=2.5658=$ *versed sine.*

Having obtained the chord of the arcs (31.6228), their versed sines (2.5658), and the diameter of the circle (100), then, by rule, page 75,

$\sqrt{100\times2.5658}=16.0181=$ *chord of half the arc.*

And by rule, page 78, to ascertain the length of an arc,

$16.0181\times2\times\overline{10\times2.5658}=821.9848=$ *twice the chord of half the arc by* 10 *times the versed sine.*

$\overline{100\times60}-\overline{2.5658\times27}=5830.7234=27$ *times the versed sine subtracted from* 60 *times the diameter.*

$821.9848\div5930.7234=.1385=$ *quotient of above product and remainder,* and $.1385+32.0362$ $(16.0181\times2)=32.1747$ $=$ *length of the arc.*

$32.1747\times\overline{50\div2}=804.3675=$ *the product of the length of the arc and half the radius of the circle* $=$ *area of sector.*

And $804.3675-\dfrac{31.6228\times47.4342}{2}=54.3664=$ *area of the triangle subtracted from the area of the sector* $=$ *area of each segment.*

$54.3564\times2 \quad = \quad 108.7328=$ *area of segments.*

Area of trapezoid $=2028.$

$\overline{2136.7328}=$ *area of zone.*

Ex. 2. The greater chord is 24, the lesser 15, and the breadth of the zone 6.5 inches; what is its area?

(Length of arc 8.0437.) *Ans.* 133.5467 *inches.*

Ex. 3. The lesser chord is 96, the greater 100, and the breadth of the zone 14 inches; what is the area of it?

(Length of arc 14.189.) *Ans.* 1381.4851 *inches.*

Ex. 4. The chords of a zone are 96 inches, and its height 28; what is its area?

(Length of arc 28.379.) *Ans.* 2762.95 *inches.*

Centre of Gravity. Find the centres of gravity of the trapezoid and the segments comprising the zone; draw a line (equally dividing the zone) perpendicular to the chords; connect the two centres of the segments by a line cutting the perpendicular to the chords; then will the centre of gravity of the figure be on the perpendicular, toward the lesser chord, at such proportionate distance of the difference between the centres of gravity of the trapezoid and line connecting the centres of the segments as the area of the two segments bears to the area of the trapezoid.

Cylinder.

Definition. *A figure formed by the revolution of a right-angled parallelogram around one of its sides.*

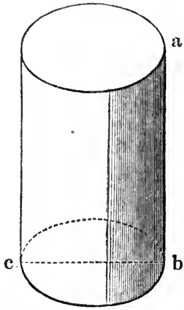
To ascertain the Surface of a Cylinder (Fig. 37).

RULE.—Multiply the length, $a\ b$, by the circumference, and the product added to the area of the two ends will be the surface required.

Or, $l \times c + 2\ a = s$, *where a represents area of end.*

NOTE.—When the internal surface alone is wanted, the areas of the ends are to be omitted.

Fig. 37.



EXAMPLE.—The diameter of a cylinder, $b\ c$, is 30 inches, and its length, $a\ b$, 50 inches; what is its surface?

$30 \times 3.1416 = 94.2480$ *inches = circumference.*

$94.248 \times 50 = 4712.4$ = *area of body.*

And $30^2 \times .7854 = 706.86$ = *area of one end.*

$706.86 \times 2 = 1413.72$ = *area of both ends.*

Then, $4712.4 + 1413.72 = 6126.12$ = *surface required.*

Ex. 2. The diameter of a cylinder is 100 inches, and its length 12 feet; what is its surface? *Ans.* 423.243 *feet.*

Ex. 3. The diameter of a hollow cylinder is 36 inches, and its length 10 feet; what is its internal surface?

Ans. 94.248 *feet.*

Centre of Gravity. Is in its geometrical centre.

Prisms.

Definition. *Figures the sides of which are parallelograms, and the ends equal and parallel.*

NOTE.—When the ends are triangles, they are called *triangular prisms;* when they are square, they are called *square* or *right prisms;* when they are pentagons, *pentagonal prisms*, &c., &c.

To ascertain the Surface of a Prism (Figs. 38 *and* 39).

RULE.—Find the areas of the ends and sides as by the rules for the mensuration of squares, triangles, &c., and add them together; the sum will be the surface of the figure.

Or, $2\,a+a'=s$, *where* a *represents the area of the ends, and* a' *the area of the sides.*

Fig 38.

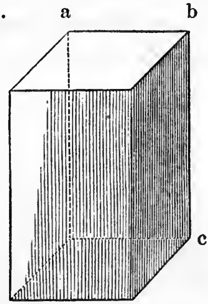
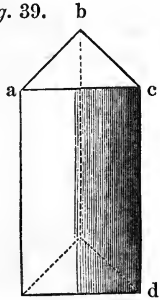


Fig. 39.



EXAMPLE.—The side $a\,b$, *Fig.* 38, of a square prism is 12 inches, and the length, $b\,c$, 30; what is the surface?

$12\times12=144=$ *area of one end.*

$144\times2=288=$ *area of both ends.*

$12\times30=360=$ *area of one side.*

$360\times4=1440=$ *area of four sides.*

Then, $288+1440=1728$ *inches, the surface required.*

Ex. 2. What is the surface of a triangular prism, the sides $a\,b$, $b\,c$, and $c\,a$, *Fig.* 39, being each 12 inches, and the length, $c\,d$, 30 inches?

$12\div2=6$, *and* $\sqrt{6^2-12^2}=10.3923=$ *width of prism.*

Hence, $10.3923\times12\div2=62.3538=$ *area of each end.*

$12 \times 30 = 360 =$ *area of one side.*

$62.3538 \times 2 = 124.7076 =$ *area of ends,*

Then, $360 \times 3 = 1080 =$ *area of sides.*

and $124.7076 + 1080 = 1204.7076$ *inches = surface required.*

Ex. 3. What is the surface of a rhomboidal prism, the depth of it being 5 feet 9 inches, the width 7 feet, and the length 10 feet? *Ans.* 402.5 *feet.*

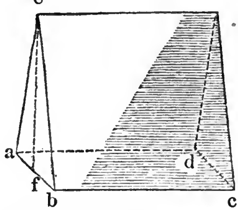
Centre of Gravity. *When the ends are parallelograms,* it is in their geometrical centre.

When the ends are triangles, trapeziums, etc., it is in the middle of their length at the same distance from the base as that of the triangle or trapezoid which is a section of them.

Wedge.

Definition. *A wedge is a prolate triangular prism, and its surface is found by the rule for that of a right prism.*

Fig. 40.



EXAMPLE.—The back of a wedge, *a b c d*, *Fig.* 40, is 20 by 2 inches, and its end, *e f*, 20 by 2 inches; what is its surface?

$20^2 + \overline{2 \div 1}^2 = 401 =$ *sum of the squares of half the base, a f, and the height, e f, of the triangle, e f a.*

$\sqrt{401} = 20.025 =$ *square root of above sum = length of e a.*

Then, $20.025 \times 20 \times 2 = 801 =$ *area of sides.*

And $20 \times 2 = 40 =$ *area of back,* and $\overline{20 \times 2 \div 2} \times 2 = 40 =$ *area of ends.*

Hence, $801 + 40 + 40 = 881 =$ *surface required.*

Centre of Gravity. See rule for prisms.

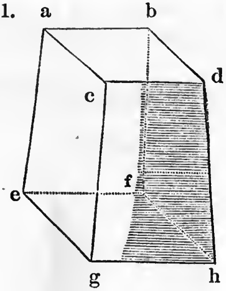
Prismoids.

Definition. *Figures alike to a prism, but having only one pair of their sides parallel.*

To ascertain the Surface of a Prismoid (*Fig.* 41).

Rule.—Find the area of the ends and sides as by the rules for squares, triangles, &c., and add them together.

Fig. 41.



Example.—The ends of a prismoid, *e f g h* and *a b c d*, *Fig.* 41, are 10 and 8 inches square, and its slant height 25; what is its surface?

$$10 \times 10 = 100 = \textit{area of base.}$$
$$8 \times 8 = 64 = \textit{area of top.}$$
$$\frac{10+8}{2} \times 25 = 225, \textit{ and } 225 \times 4 = 1000 = \textit{area of sides.}$$

Then, $100 + 64 + 1000 = 1164 = \textit{surface required.}$

Ex. 2. The ends of a prismoid are 15 and 12 inches square, and its slant height 40; what is its surface?

Ans. 2529 *inches.*

Ex. 3. The ends of a prismoid are 12x16 and 14x18 inches, and its vertical height is 33; what is its surface?

Ans. 2424 *inches.*

Centre of Gravity. Is at the same distance from its base as that of the trapezoid or trapezium which is a section of it.

Ungulas.

Definition. *Cylindrical ungulas are frustrums of cylinders. Conical ungulas* are frustrums of cones.*

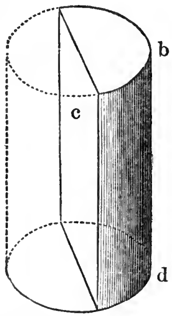
To ascertain the Curved Surface of an Ungula, Figs. 42, 43, 44, 45, *and* 46.

1. *When the Section is parallel to the Axis of the Cylinder, Fig.* 42.

RULE.—Multiply the length, $a\ b\ c$, of the arc line of one end by the height, $b\ d$, and the product will be the *curved surface* required.

Or, $c \times h = s$, *where c represents length of arc line.*

Fig. 42.



EXAMPLE.—The diameter of a cylinder from which an ungula is cut is 10 inches, its length 50, and the versed sine or depth of the ungula is 5 inches; what is the curved surface of it?

$$10 \div 2 = 5 = \textit{radius of cylinder}.$$

Hence the radius and versed sine are equal; the arc line, therefore, of the ungula is one half the circumference of the cylinder, which is $31.416 \div 2 = 15.708$,

and $15.708 \times 50 = 785.400$ *inches.*

Ex. 2. The base line of the section of a cylindrical ungula is 48 inches, the height or versed sine of the arc is 20, and

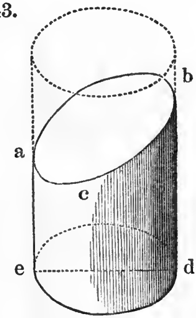
* For mensuration of conical ungulas, see Conic Sections, p. 253.

the length of the ungula is 20.5 feet; what is its curved surface? *Ans.* 109.8388 *feet.*

2. *When the Section passes obliquely through the opposite Sides of the Cylinder, Fig.* 43.

Rule.—Multiply the circumference of the base of the cylinder by half the sum of the greatest and least heights, $d\ b$ and $e\ a$, of the ungula, and the product will give the *curved surface* required.

Fig. 43.



Example.—The diameter of a cylindrical ungula is 10 inches, and the greater and less heights are 25 and 15 inches; what is its surface?

$$10 \textit{ diameter} = 31.416 \textit{ circumference.}$$
$$25+15=40, \textit{ and } 40 \div 2 = 20.$$
Hence, $31.416 \times 20 = 62.8320$ *inches.*

Ex. 2. The circumference of an ungula is 60.75 inches, and the mean height of it 13 feet; what is its surface? *Ans.* 65.8125 *feet.*

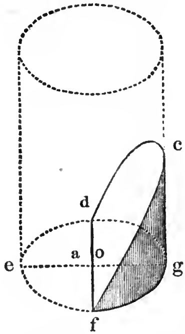
3. *When the Section passes through the Base of the Cylinder and one of its Sides, and the Versed Sine does not exceed the Sine, Fig.* 44.

Rule.—Multiply the sine, $a\ d$, of half the arc, $d\ g$, of the base, $d\ g\ f$, by the diameter, $e\ g$, of the cylinder, and from this product subtract the product* of the arc and cosine, $a\ o$. Mul-

* When the cosine is 0, this product is 0.

tiply the difference thus found by the quotient of the height, $g\,c$, divided by the versed sine, $a\,g$, and the product will be the *curved surface* required.

Fig. 44.



EXAMPLE.—The sine, $a\,d$, of half the arc of the base of an ungula is 5, the diameter of the cylinder is 10, and the height of the ungula 10 inches; what is the curved surface?

$5\times10=50=$*sine of half the arc by the diameter.*

Length of arc, the versed sine and radius being equal, under rule, page $78=15.708$.

Again, as the versed sine and the radius are equal, the cosine is 0.

Hence, when the cosine is 0, *the product is* 0. $50-0=50=$*the difference of the product before obtained and the product of the arc and the cosine.*

$50\times\overline{10\div5}=50\times2=100=$*the difference multiplied by the height divided by the versed sine, which is the surface required.*

Ex. 2. The sine of half the arc of the base of an ungula is 12 inches, the versed sine is 9, the diameter of the cylinder is 25, and the height of the ungula is 18; what is its curved surface?

$12\times25=300=$*product of sine and diameter.*

Arc of base of ungula, by rule, p. 77, *the versed sine being* 9, *is* 32.14795.

Then, $32.14795\times\overline{12.5-9}=112.51782$,

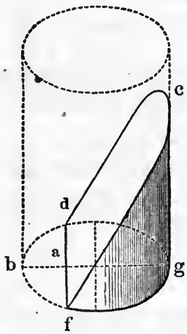
and $300-112.51782=187.48218$, *which, multiplied by* $18\div9$ $=374,96436$ *inches.*

4. *When the Section passes through the Base of the Cylinder, and the Versed Sine exceeds the Sine, a g, Fig.* 45.

Rule.—Multiply the sine of half the arc of the base by the diameter of the cylinder, and to this product add the product of the arc and the excess of the versed sine over the sine of the base.

Multiply the sum thus found by the quotient of the height divided by the versed sine, and the product will be the curved surface required.

Fig. 45.



Example.—The sine, *a d*, of half the arc of an ungula is 12 inches, the versed sine, *a g*, is 16, the height, *c g*, 16, and the diameter of the cylinder 25 inches; what is the *curved surface?*

$12 \times 25 = 300 =$ *sine of half the arc by the diameter of the cylinder.*

Length of arc of base, by rule, p. 74 = *arc of d b f—circumference of base* = 46.392.

Then $46.392 \times \overline{16 - 12.5} = 162.372 =$ *product of arc and the excess of the versed sine over the sine.*

$300 + 162.372 = 462.372 =$ *the sum of the above products.*

$16 \div 16 = 1 =$ *quotient of height divided by the versed sine.*

$462.372 \times 1 = 462.372$ *inches* = *the sum of the products and the height divided by the versed sine* = *the curved surface required.*

Ex. 2. The sine of half the arc of the base of an ungula is 0, the diameter of the cylinder is 10 inches, and the height of the ungula is 20 inches; what is its *curved surface?*

NOTE.—The sine of the arc being 0, the versed sine is equal to the diameter (10), and the sine of the base is $10 \div 2 = 5$.

$0 \times 10 = 0 =$ *product of sine of half the arc and diameter of the cylinder.*

$0 + (31.416$ *(length of arc)* $\times \overline{10 \backsim 5}) = 157.08 =$ *the sum of the product above obtained and the product of the arc and the excess of the versed sine over the sine.*

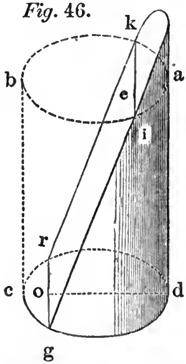
$157.08 \times 20 \div 10 = 314.16 =$ *the above sum* $\times$ *the height* $\div$ *the versed sine* $=$ *the result required.*

5. *When the Section passes obliquely through both Ends of the Cylinder, a b c d, Fig.* 46.

RULE.—Conceive the section to be continued till it meets the side of the cylinder produced; then, as the difference of the versed sines of the arcs of the two ends of the ungula is to the versed sine of the arc of the less end, so is the height of the cylinder to the part of the side produced.

Find the surface of each of the ungulas thus found by the rules 3 and 4, and their difference will be the *curved surface* required.

Fig. 46.



EXAMPLE.—The versed sines *a e*, *d o*, and sines *i k*, *g r*, of the arcs of the two ends of an ungula, *Fig.* 46, are respectively 5 and 2.5, and 5 and 4.25; the height of the ungula within the cylinder, cut from one having 10 inches diameter, is 5 inches; what is the height of the ungula produced beyond the cylinder?

$5 \backsim 2.5 = 2.5$, *and* $2.5 : 2.5 :: 5 : 5 =$ *height of ungula produced beyond the cylinder.*

Ex. 2. The versed sines of the base and arc of an ungula cut from a cylinder of 6 inches diameter are 6 and 2 inches, and its height within the cylinder is 4 inches; what is the distance it extends above or beyond the cylinder?

Ans. 2 *inches.*

Lune.

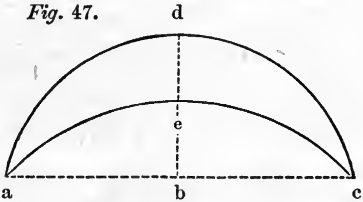
Definition. *The space between the intersecting arcs of two eccentric circles.*

To ascertain the Area of a Lune, Fig. 47.

RULE.—Find the areas of the two segments from which the lune is formed, and their difference will be the area required.

Or, $s - s' = a$, *when* s *and* s' *represents the areas of the segments.*

Fig. 47.



EXAMPLE.—The length of the chord $a\,c$ is 20, the height $e\,d$ is 3, and $e\,b$ 2 inches; what is the area of the lune?

By Rule 2, *p.* 76, *the diameters of the circles of which the lune is formed are thus found:*

For $a\,d\,c$, $\quad \dfrac{10^2+(3+2)^2}{5} = 25.$

For $a\,e\,c$, $\quad \dfrac{10^2+2^2}{2} = 52.$

Then, by rule for the areas of segments of a circle, page 87, *the area of a d c is* 70.5577 *in.*

" *a e c* 27.1638 *in.*

their difference 43.3939 *in., the area of the lune required.*

Ex. 2. The chord of a lune is 40, and the heights of the segments 10 and 4 inches; what is its content?

Ans. 173.5752 *inches.*

Ex. 3. The chord of a lune is 6 feet 8 inches, and the heights of the arcs 1.666 feet and 8 inches; what is its area?

Ans. 694.2996 *inches.*

Ex. 4. The chord of a lune is 86.6024 inches, and the heights of the segments 25 and 15 inches; what is its area?

Ans. 653.3551.

Centre of Gravity. On a line connecting the centres of gravity of the two arcs at a point proportionate to the respective areas of the arcs.

NOTE.—If semicircles be described on the three sides of a right-angled triangle as diameters, two lunes will be formed, and their united areas will be equal to that of the triangle.

Cycloid.

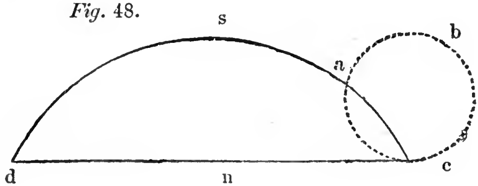
Definition. *A curve generated by the revolution of a circle on a plane.*

To ascertain the Area of a Cycloid, Fig. 48.

RULE.—Multiply the area of the generating circle *a b c* by 3, and the product will give the area required.

Or, $a \times 3 = area$.

Fig. 48.



EXAMPLE.—The generating circle of a cycloid has an area of 115.45 inches; what is the area of the cycloid?

$115.45 \times 3 = 346.35$ *inches.*

Ex. 2. The area of a circle describing a cycloid is 1.625 feet; what is the area of the cycloid in inches?

Ans. 702 *inches.*

Ex. 3. The diameter of a circle describing a cycloid is 66.5 feet; what is the area of the cyloid in inches?

Ans. 1500434.064 *inches.*

To ascertain the Length of a Cycloidal Curve, Fig. 48.

RULE.—Multiply the diameter of the generating circle by 4, and the product will give the length of the curve.

Or, $d \times 4 =$ *length of curve.*

EXAMPLE.—The diameter of the generating circle of a cycloid, *Fig.* 48, is 8 inches; what is the length of the curve $d\ s\ c$?

$8 \times 4 = 32 =$ *product of diameter and* $4 =$ *the length required.*

Ex. 2. The diameter of the generating circle is 20 inches; what is the length of the cycloidal curve?

Ans. 80 *inches.*

Centre of Gravity. At a distance from the centre, n, of the chord, $d\ c$, of the curve $d\ s\ c = \frac{5}{6}$ of the radius of the generating circle.

NOTE.—The curve of a cycloid is the line of swiftest descent; that is, a body will fall through the arc of this curve, from one point to another, in less time than through any other path.

RINGS.

Circular Rings.

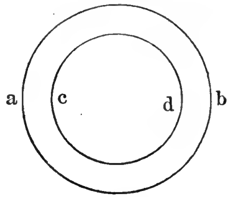
Definition. *The space between two concentric circles.*

To ascertain the Sectional Area of a Circular Ring, Fig. 49.

RULE.—From the area of the greater circle, $a\ b$, subtract that of the less, $c\ d$, and the difference will be the area of the ring.

Or, $a - a' =$ *area.*

Fig. 49.



EXAMPLE.—The diameters of the circles forming a ring are each 10 and 15 inches; what is the area of the ring?

Area of 15 = 176.7146
" 10 = 78.5400
98.1746 *inches.*

Ex. 2. The diameters of a circular ring are 10.75 and 18.25 inches; what is its area? *Ans.* 171.82 *inches.*

Centre of Gravity. Is in its geometrical centre.

Cylindrical Rings.

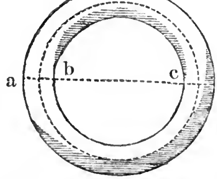
Definition. *A ring formed by the curvature of a cylinder.*

To ascertain the Convex Surface of a Cylindrical Ring, Fig. 50.

RULE.—To the thickness of the ring, *a b*, add the inner diameter, *b c;* multiply this sum by the thickness and the product by 9.8696, and it will give the surface required.

Or, $d+d' \times d \times 9.8696 = surface.$

Fig. 50.



EXAMPLE.—The thickness of a cylindrical ring, *a b*, is 2 inches, and the inner diameter, *b c*, is 18; what is the surface of it?

$2+18=20=$ *thickness of ring added to the inner diameter.*
$20\times2\times9.8696=394.784=$ *the sum above obtained* $\times$ *the thickness of the ring, and that product by* 9.8696, *the result required.*

Ex. 2. The thickness of a ring of metal of 20 inches diameter (internal) is 2 inches; what is the surface of it?

Ans. 434.2624 *inches.*

Link.

Definition. *An elongated ring.*

To ascertain the Convex Surface of a Link, Figures 51.

Rule.—Multiply the circumference of a section of the body, *a b*, of the link by the length of its axis, and the product will give the surface required.

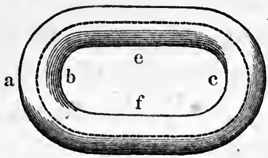
Or, $c\times l=$ *surface.*

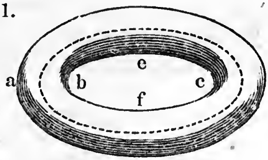
Note.—*To ascertain the Circumference or Length of the Axes.*

When the Ring is elongated. To the less diameter add its thickness, multiply the sum by 3.1416; multiply the difference of the diameters by 2, and the sum of these products will give the result required.

When the Ring is elliptical. Square the diameters of the axes of the ring, and multiply the square root of half their sum by 3.1416; the product will give the length of the body of the ring.

Figs. 51.





Example.—The link of a chain is 1 inch in diameter of body, *a b*, and its inner diameters, *b c* and *e f*, are 12.5 and 2.5 inches; what is its circumference.

$2.5+1\times3.1416=10.9956=$ *length of axis of ends.*
$12.5-\overline{2.5\times2}\times2=15=$ *length of sides of body.*
Then, $10.9956+15=25.9956=$ *length of axis of link, which,* $\times\,3.1416$ (cir. of 1 in.) $=81.6678$ *inches.*

Centres of Gravity. Are in their geometrical centres.

Cones.

Definition. *A figure described by the revolution of a right-angled triangle about one of its legs.*

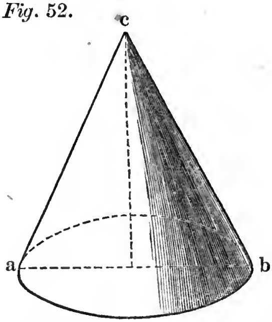
For Sections of a Cone, see Conic Sections, page 228.

To ascertain the Surface of a Cone, Fig. 52.

RULE.—Multiply the perimeter, or circumference of the base, by the slant height, or side of the cone, and half the product added to the area of the base will be the surface.

Or, $\frac{p \times h}{2} = \textit{surface}$.

Fig. 52.



EXAMPLE.—The diameter, *a b*, of the base of a cone is 3 feet, and the slant height, *a c*, 15; what is the surface of the cone?

Perimeter of 3 *feet*$=9.4248$, *and* $\frac{9.4248 \times 15}{2} = 70.686 =$ *surface of side of cone.*

Area of 3 *feet*$=7.068$, *and* $70.686 + 7.068 = 77.754 =$ *surface required.*

Ex. 2. The diameter of the base of a cone is 6.25 inches, and the slant height 18.75; what is the surface of it?

Ans. 214.757 *inches.*

Ex. 3. The diameter of the base of a cone is 20 inches, and the slant height 14.142; what is the surface of the cone?

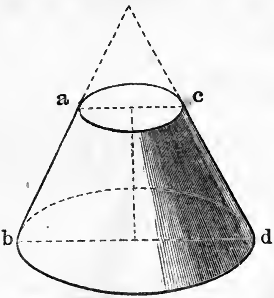
Ans. 758.445 *inches.*

To ascertain the Surface of the Frustrum of a Cone, Fig. 53.

RULE.—Multiply the sum of the perimeters of the two ends by the slant height of the frustrum, and half the product added to the areas of the two ends will be the surface required.

Or, $\frac{p+p' \times h}{2}+a+a'=surface.$

Fig. 53.



EXAMPLE.—The frustrum, *a b c d*, *Fig.* 53, has a slant height of 26 inches, and the circumferences of its ends are 15.7 and 22. inches respectively; what is its surface?

$$\overline{15.7+22.}\times 26 \div 2 = 490.1 = surface\ of\ sides.$$

$$\left(\frac{15.7}{3.1416}\right)^2 \times .7854 + \left(\frac{22.}{3.1416}\right)^2 \times .7854 = 58.12 = areas\ of\ ends.$$

Then, $490.1+58.12=548.22=surface.$

Ex. 2. What is the surface of the frustrum of a cone, the diameters of the ends, *a c* and *b d*, being 4 and 8 feet, and the length of the slant sides 20 feet? *Ans.* 439.824 *feet.*

Ex. 3. What is the surface of the frustrum of a cone, the diameter of the ends being 6.66 and 10 feet, and the length of the slant side 3.73 feet?

Ans. 210.989.

Centres of Gravity. Cone or Frustrum.—At the same distance from the base as in that of the triangle or parallelogram, which is a right section of them.

Pyramids.

Definition. *A figure, the base of which has three or more sides, and the sides of which are plane triangles.*

To ascertain the Surface of a Pyramid, Fig. 54.

RULE.—Multiply the perimeter of the base by the slant height, and half the product added to the area of the base will be the surface.

Or, $\frac{p \times h}{2} + a = surface.$

Fig. 54.

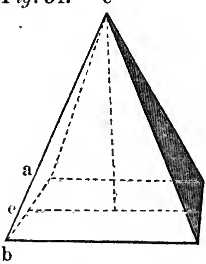
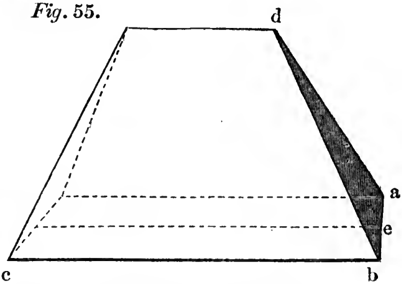


Fig. 55.



EXAMPLE.—The side of a quadrangular pyramid, *a b*, *Fig.* 54, is 12 inches, and its slant height, *c e*, 40; what is its surface?

$$12 \times 4 = 48 = perimeter\ of\ base.$$

$$\frac{48 \times 40}{2} = 960 = area\ of\ sides.$$

Then $\quad 12 \times 12 + 960 = 1104 = surface.$

Ex. 2. The sides of a hexagonal pyramid are 12.5 inches, and its slant height 62.5; what is its surface?

Ans. 2749.703 *inches.*

Ex. 3. The sides, *a b*, *b c*, of an oblong quadrangular pyramid, *Fig.* 55, are 15 and 17.5 inches, and its slant height, *d e*, 36; what is its surface? *Ans.* 1432.5 *inches.*

Ex. 4. The sides of an octagonal pyramid are 4 feet 2 inches, and its slant height 6 feet 9 inches; what is its surface? *Ans.* 196.326 *square feet.*

Ex. 5. What is the surface of a pentagonal pyramid, its slant height being 12 feet, and each side of its base 2 feet?

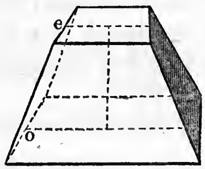
Ans. 66.882.

To ascertain the Surface of the Frustrum of a Pyramid, Fig. 56.

RULE.—Multiply the sum of the perimeters of the two ends by the slant height or side, and half the product added to the area of the ends will be the surface.

Or, $\frac{p+p' \times h}{2}+a+a'=surface.$

Fig. 56.



EXAMPLE.—The sides, *a b, c d, Fig.* 56, of a quadrangular pyramid are 10 and 9 inches, and its slant height, *e o*, 20; what is its surface?

$$10 \times 4 = 40$$
$$9 \times 4 = 36$$
$$\overline{76} = sum\ of\ perimeters.$$

$$76 \times 20 = 1520,\ and\ \frac{1520}{2} = 760 = area\ of\ sides.$$

$$10 \times 10 = 100,\ and\ 9 \times 9 = 81.$$

Then $100+81+760=941$, *the surface.*

Ex. 2. The ends of a frustrum of a quadrangular pyrami are 15 and 9 inches, and its slant height 40; what is its surface?

Ans. 2226 *inches.*

Ex. 3. The ends of a frustrum of a triangular pyramid are 20 and 10 inches, and its slant height 50; what is its surface?

Ans. 2466.5 *inches.*

Ex. 4. The sides of a frustrum of a hexagonal pyramid are 15 and 25 inches, and its slant height 20; what is its surface?

Ans. 32.002 *square feet.*

Centres of Gravity. Pyramid or Frustrum.—At the same distance from the base as in that of the triangle or parallelogram, which is a right section of them.

Helix (Screw).

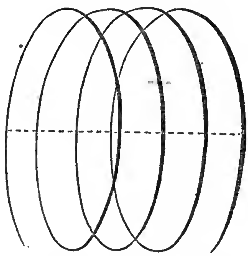
Definition. *A line generated by the progressive rotation of a point around an axis and equidistant from its centre.*

To ascertain the Length of a Helix, Fig. 57.

RULE.—To the square of the circumference described by the generating point, add the square of the distance advanced in one revolution, and the square root of their sum multiplied by the number of revolutions of the generating point will give the length of the line required.

Or, $\sqrt{(c^2+h^2)} \times n$ = *length of line, n representing the number of revolutions.*

Fig. 57.



EXAMPLE.—What is the length of a helical line running 3.5 times around a cylinder of 22 inches in circumference and advancing 16 inches in each revolution?

$22^2+16^2=740$ = *sum of squares of circumference and of the distance advanced.*

$\sqrt{740} \times 3.5 = 95.21$ = *square root of above sum × number of revolutions = length of line required.*

Ex. 2. What is the length of the helical line described by a point in a screw in one revolution at a radius from its axis of 11.3 inches, the progression of the line or pitch of the screw being 17 inches? *Ans.* 73.

Ex. 3. What is the length of the helical line described by a point on the periphery of a screw of 10 feet in diameter, having a pitch of 20 feet? *Ans.* 37.242.

Centre of Gravity. Is in its geometrical centre.

Spirals.

Definition. *Lines generated by the progressive rotation of a point around a fixed axis.*

A Plane Spiral is when the point rotates around a central point.

A Conical Spiral is when the point rotates around an axis or a cone.

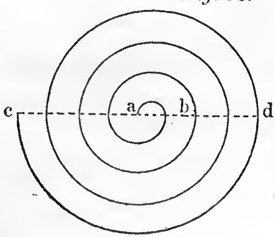
To ascertain the Length of a Plane Spiral Line, Fig. 58.

RULE.—Add together the greater and less diameters,* divide their sum by two, multiply the quotient by 3.1416, again by the number of revolutions, and the product will give the length of the line required.

Or, when the circumferences are given, take their mean length, multiply it by the number of revolutions, and the product will give the length required.

Or, $\frac{d+d'}{2} \times 3.1416 \times n =$ *length of line*, n *representing the number of revolutions.*

Fig. 58.



* When the spiral is other than a line, measure the diameters of it from the middle of the material composing it.

EXAMPLE.—The less and greater diameters of a plane spiral spring, as *a b*, *c d*, *Fig.* 58, are 2 and 20 inches, and the number of revolutions 10; what is the length of it?

$$\frac{2+20}{2}=11=\textit{sum of diameters}\div 2.$$

$$11\times 3.1416=34.5576=\textit{above quotient}\times 3.1416.$$

$34.5576\times 10=345.576=$ *above product* $\times$ *number of revolutions* $=$ *the length of line.*

Ex. 2. The greater and less diameters of a plane spiral are 4 and 30 inches, and the number of revolutions 5; what is the length of it? *Ans.* 267.036 *inches.*

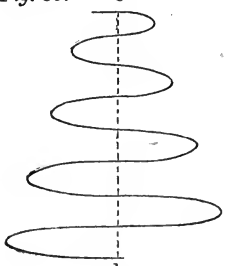
To ascertain the Length of a Conical Spiral, Fig. 59.

RULE.—Add together the greater and less diameters;* divide their sum by two, and multiply its quotient by 3.1416.

To the square of the product of this circumference and the number of revolutions of the spiral, add the square of the height of its axis, and the square root of the sum will be the length required.

Or, $\sqrt{\left(\frac{d+d'}{2}\times 3.1416\times n\right)^2+l^2}=$ *length of spiral.*

Fig. 59.



EXAMPLE.—The greater and less diameters of a conical spiral, *Fig.* 59, are 20 and 2 inches, its height, *c d*, 10, and the number of revolutions 10; what is the length of it?

* *See Note to* Rule, *page* 117.

$20+2\div 2=11\times 3.1416=34.5576=$ *sum of diameters* $\div 2$ *and* $\times 3.1416$.

$34.5576\times 10=345.576$, *and* $345.576^2=119422.77=$ *square of the product of the circumference and number of revolutions.*

$\sqrt{119422.77+10^2}=345.72=$ *the square root of the sum of the above product and the square of the height of the spiral* $=$ *the result required.*

Ex. 2. The greater and less diameters of a conical spiral are 1.5 and 8.75 feet, its height 6 feet, and the number of its revolutions is 5; what is the length of it? *Ans.* 80.725 *inches.*

Ex. 3. The greater and less diameters of a conical spiral are 3 and 9 feet, its height 12.5 feet, and the number of its revolutions 10; what is the length of it? *Ans.* 188.91.

Centres of Gravity. Plane Spiral.—It is in its geometrical centre.

Conical Spiral.—It is at a distance from the base $\frac{1}{4}$ of the line joining the vertex and centre of gravity of the base.

Note.*—This rule is applicable to winding engines where it is required to ascertain the length of a rope, its thickness, the number of revolutions, diameter of drum, etc., etc.

Illustration.—The diameter over the roll of a flat rope upon the drum of a winding engine shaft is 134.5 inches, the diameter of the drum is 94.5 inches, and the number of revolutions 20; what is the length of the rope and what is its thickness?

Ans. Length of the rope, 7194.247 *inches.*

Area of 134.5 *in.*	=	14208.049
" " 94.5	=	7013.802
		7194.247

Then, $7194.247\div 20=359.712=$ *area of rope* $\div$ *revolutions* $=$ *area of each thickness.*

$134.5-94.5\div 2+94.5=114.5$ *and* $114.5\times 3.1416=359.712$ $=$ *circumference of the mean diameter of each thickness.*

Hence, $359.712\div 359.712=1$ *inch* $=$ *the width or thickness of the rope.*

* For Rules to ascertain the elements of Winding Engines, see Haswell's Engineers' and Mechanics' Pocket-book, p. 263–4.

SPINDLES.

Definition. *Figures generated by the revolution of a plane area, when the curve is revolved about a chord perpendicular to its axis, or about its double ordinate, and they are designated by the name of the arc or curve from which they are generated, as Circular, Elliptic, Parabolic, etc., etc.*

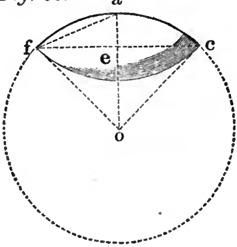
Circular Spindle.

To ascertain the Convex Surface of a Circular Spindle, Fig. 60.

Rule.—Multiply the length, $f\ c$, by the radius, $o\ c$, of the revolving arc; multiply this arc, $f\ a\ c$, by the central distance, $o\ e$, or distance between the centre of the spindle and centre of the revolving arc; subtract this product from the former, double the remainder, multiply it by 3.1416, and the product will be the surface required.

Or, $l \times r - (a \times \sqrt{r^2 - \left(\frac{c}{2}\right)^2}) \times 2\,p$; *a representing the length of the arc, c the chord, and p* 3.1416.

Fig. 60.



Example.—What is the surface of a circular spindle, *Fig.* 60, the length of it, $f\ e$, being 14.142 inches, the radius of its arc, $o\ c$, 10, and the central distance, $o\ e$, 7.071?

$14.142 \times 10 = 141.42 =$ *length* $\times$ *radius.*

Length of arc, by rules, p. 76, 78 = 15.708.

$15.708 \times 7.071 = 111.0713 =$ *length of arc* $\times$ *central distance.*

$141.42 - 111.0713 = 30.3487 =$ *difference of products.*

$30.3487 \times 2 = 60.6974 \times 3.1416 = 190.687 =$ *the remainder doubled* $\times$ 3.1416, *which is the result required.*

Ex. 2. The length of a circular spindle is 28.284 feet, the radius of its arc 20, and the distance between the centre of the spindle and the centre of the revolving arc is 14.142; what is the surface of it? *Ans.* 762.7484 *feet.*

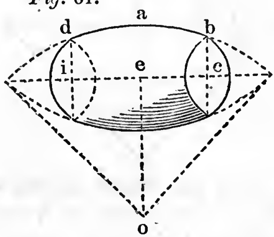
Centre of Gravity. Is in its geometrical centre.

To ascertain the Convex Surface of a Zone of a Circular Spindle, Fig. 61.

Rule.—Multiply the length, *i c*, by the radius, *o a*, of the revolving arc; multiply the arc, *d a b*, by the central distance *o e*; subtract this product from the former, double the remainder, multiply it by 3.1416, and the product will be the surface required.

Or, $l \times r - (a \times \sqrt{r^2 - \left(\frac{c}{2}\right)^2}) \times 2\,p$, *l representing the length of the zone.*

Fig. 61.



Example.—What is the convex surface of the zone of a circular spindle, *Fig.* 61, the length of it being 7.653 inches, the radius of its arc 10, the central distance 7.071, and the length of its side or arc, *d b*, 7.854 inches?

$7.653 \times 10 = 76.53 =$ *length* $\times$ *radius.*

$7.854 \times 7.071 = 55.5356 =$ *length of arc* $\times$ *central distance.*

$76.53 - 55.5356 = 20.9944 =$ *difference of products.*

$20.9944 \times 2 = 41.9888 \times 3.1416 = 131.912 =$ *the remainder doubled* $\times$ 3.1416, *which is the result required.*

Ex. 2. The zone of a circular spindle is 23 inches in length, the radius of its arc 30, its central distance 21.2, and the length of its side 23.56; what is its convex surface?

Ans. 1197.2255 *inches.*

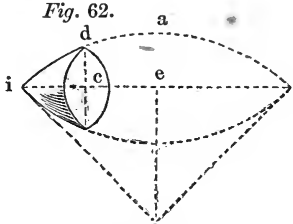
Centre of Gravity. Is in its geometrical centre.

To ascertain the Convex Surface of a Segment of a Circular Spindle, Fig. 62.

RULE.—Multiply the length, *i c*, by the radius of the revolving arc, *o a;* multiply the arc by the central distance, *o e;* subtract this product from the former, double the remainder, multiply it by 3.1416, and the product will be the surface required.

Or, $l \times r - (a \times \sqrt{r^2 - \left(\frac{c}{2}\right)^2}) \times 2\,p$, *l representing the length of the segment.*

Fig. 62.



EXAMPLE.—What is the convex surface of a segment of a circular spindle, *Fig.* 62, the length of it being 3.2495 inches, the radius of its arc 10, the central distance 7.071, and the length of its side, *i d*, 3.927 inches?

$3.2495 \times 10 = 32.495 =$ *length* $\times$ *radius.*

$3.927 \times 7.071 = 27.7678 =$ *length of arc* $\times$ *central distance.*

$32.495 - 27.7678 = 4.7272 =$ *difference of products.*

$4.7272 \times 2 = 9.4544 \times 3.1416 \doteq 29.702$, *which is the result required.*

Ex. 2. The segment of a circular spindle is 14.142 feet in length, the radius of its arc is 20, and the distance between the plane of the segment, *i e*, and the centre of the revolving arc, *o e*, *Fig.* 62, is 14.142, and the length of its side, *i d*, is 15.708; what is its convex surface? *Ans.* 381.3745 *feet.*

For Surface of a Circular Spindle, Zone, or Segment.

General Formula. $S=2(l\,r-a\,c)\,p$, *l representing length of spindle, segment, or zone, a the length of its revolving arc, r the radius of the generating circle, and c the central distance.*

Illustration.—The length of a circular spindle is 14.142 inches, the length of its revolving arc is 15.708, the radius of its generating circle is 10, and the distance of its centre from the centre of the circle from which it is generated is 7.071; what is its surface?

$2\times(14.142\times10-15.708\times7.071)\times3.1416=190.6869=$ *result required.*

Centre of Gravity. See Appendix, page 283.

NOTE.—The surface of the frustrum of a spindle is obtained by the division of the surface of a zone.

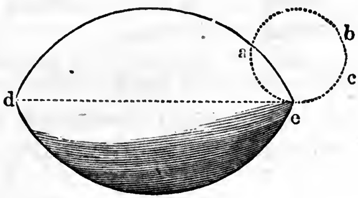
Cycloidal Spindle.

To ascertain the Convex Surface of a Cycloidal Spindle, Fig. 63.

RULE.—Multiply the area of the generating circle by 64, and divide it by 3; the quotient will give the surface required.

Or, $\dfrac{a\times64}{3}=$ *surface.*

Fig. 63.



EXAMPLE.—The area of the generating circle, *a b c*, of a cycloidal spindle, *d e*, is 32 inches; what is the surface of the spindle?

$32 \times 64 = 2048 = \textit{area of circle} \times 64.$

$2048 \div 3 = 682.667 = \textit{above product} \div 3 = \textit{surface required.}$

Ex. 2. The diameter of the generating circle of a cycloidal spindle is 20.375 inches; what is the surface of the spindle?

Ans. 6955.7483 *in.*

Ex. 3. The diameter of the generating circle of a cycloidal spindle is 14.5 inches; what is the surface of the spindle?

Ans. 3522.773 *in.*

Ex. 4. The radius of the generating circle of a cycloidal spindle is 8.5 inches; what is the surface of the spindle in square feet?

Ans. 33.633 *feet.*

Centre of Gravity. Is in its geometrical centre.

Note.—The area of a cycloidal spindle is twice the area of the cycloid, to ascertain which, see rule, page 108.

Elliptic, Parabolic, and Hyperbolic Spindles.

The rules to ascertain the surfaces of either an Elliptic, Parabolic, or Hyperbolic Spindle, or of zones or segments of them, are of a character to preclude their being given in such a form as would be consistent with the design of this work; hence they are omitted here.

See Appendix, p. 278, 279.

Ellipsoid, Paraboloid, or Hyperboloid of Revolution.

Definition. *Figures alike to a cone generated by the revolution of a conic section around its axis.*

Note.—These figures are usually known as Conoids.

When they are generated by the revolution of an ellipse, they are called ellipsoids, and when by a parabola, paraboloids, &c., &c., &c.

The revolution of an arc of a conic section around the axis of the curve will give a segment of a conoid.

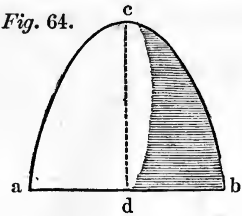
*Ellipsoid.**

To ascertain the Convex Surface of an Ellipsoid, Fig. 64.

RULE.—Add together the square of the base *a b* and four times the square of the height *c d*; multiply the square root of half their sum by 3.1416, and this product by the radius of the base. The product will give the surface required.

Or, $\sqrt{\dfrac{b^2+4h^2}{2}} \times 3.1416 \times r =$ *surface, h representing the height of the ellipsoid.*

Fig. 64.



EXAMPLE.—The base *a b* of the ellipsoid, *Fig.* 64, is 10 inches, and the height *c d* 7; what is its surface?

$10^2+\overline{7^2\times 4}=296=$ *sum of the square of the base and 4 times the square of the height.*

$296\div 2=148$, and $\sqrt{148}=12.1655=$ *square root of half the above sum.*

$12.1655\times 3.1416\times\dfrac{10}{2}=191.0957=$ *product of root above obtained* $\times$ 3.1416, *and that product by the radius of the base = the surface required.*

Ex. 2. The base *a b* of an ellipsoid is 14 inches, and the height *c d* 5; what is the convex surface of it?

Ans. 267.534 *in.*

To ascertain the Convex Surface of a Segment, Frustrum, or Zone of an Ellipsoid.

See rules for the convex surface of a segment, frustrum, or zone of an ellipsoid, p. 93–95.

* An ellipsoid is a semi-spheroid. (See p. 91–94.)

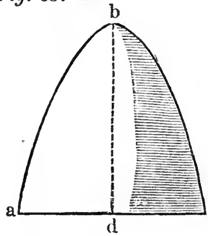
Paraboloid.

To ascertain the Convex Surface of a Paraboloid, Fig. 65.

RULE.—From the cube of the square root of the sum of four times the square of the height, *b d*, and the square of the radius of the base, *d a*, subtract the cube of the radius of the base; multiply the remainder by the quotient of 3.1416 times the radius of the base divided by six times the square of the height, and the product will give the surface required.

Or, $[(\sqrt{4h^2+r^2})^3-r^3]\times\frac{r\times p}{6\times h^2}=$ *convex surface.*

Fig. 65.



EXAMPLE.—The axis *b d* of a paraboloid, *Fig.* 65, is 40 inches, the radius *a d* of its base is 18 inches; what is its convex surface?

$40^2\times4=6400=4$ *times the square of the height.*

$6400+18^2=6724=$ *sum of the above product and the square of the radius of the base.*

$(\sqrt{6724})^3-18^3=545536=$ *the remainder of the cube of the radius of the base subtracted from the cube of the square root of the preceding sum.*

$3.1416\times18\div(6\times40^2)=.0058905=$ *the quotient of* 3.1416 *times the radius of the base* $\div$ 6 *times the square of the height.*

Hence $545536\times.0058905=3213.48=$ *the product of the above remainder and the preceding quotient* $=$ *the result required.*

Ex. 2. The axis *b d* of a paraboloid is 20 inches, and the diameter of its base *a c* is 60 inches; what is its convex surface? *Ans.* 3848.46 *in.*

Ex. 3. The axis of a parabolic conoid is 18 inches, the radius of its base 40 inches; what is its convex surface?

Ans. 5937.16 *in.*

Centre of Gravity.— See Appendix, p. 284.

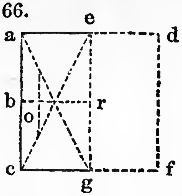
Any Figure of Revolution.

To ascertain the Convex Surface of any Figure of Revolution, Figs. 66, 67, and 68.

Rule.—Multiply the length of the generating line by the circumference described by its centre of gravity, and the product will give the surface required.

Or, $l \times 2r \times p = surface$; *r representing radius of centre of gravity.*

Fig. 66.



Example.—If the generating line $a\,c$ of the cylinder $a\,c\,d\,f$, 10 inches in diameter, *Fig.* 66, is 10, then the centre of gravity of it will be in b, the radius of which is $b\,r = 5$.

Hence $10 \times \overline{5 \times 2} \times 3.1416 = 314.16 =$ *the convex surface of the cylinder.*

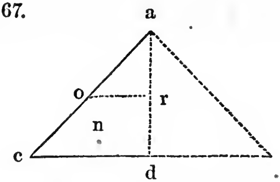
Again, If the generating line is $c\ a\ e\ g$, and it is ($e\ a = 5$, $a\ c = 10$, and $c\ g = 5$) $= 20$, then the centre of gravity o will be in the middle of the line joining the centres of gravity of the triangles $e\ a\ c$ and $a\ c\ g = 3.75$.

Hence $20 \times \overline{3.75 \times 2} \times 3.1416 = 471.24 =$ *the entire surface of the cylinder.*

Proof. { *Convex surface as above* 314.16
Area of each end, $10^2 \times .7854 = 78.54$,
and $78.54 \times 2 =$ 157.08

471.24

Fig. 67.



Ex. 2. If the generating elements of a cone, *Fig.* 67, are $a\ d=10$, $d\ c=10$, and $a\ c$ the generating line$=14.142$, the centre of gravity of which is in o, and $o\ r=5$.

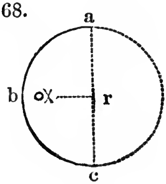
Then, $14.142\times\overline{5\times2}\times3.1416=444.285=$ *the convex surface of the cone,* and $\overline{10\times2}^2\times.7854=314.16=$ *area of base.*

Hence, $444.285+314.16=758.445=$ *whole surface of cone.*

Again. If the generating line is $a\ c\ d=24.142$, then the centre of gravity will be in n, in the middle of the line, joining the angle of the generating line and the base $a\ d$ at $r=5$.

Hence, $24.142\times\overline{5\times2}\times3.1416=758.445=$ *whole surface of cone.*

Fig. 68.



Ex. 3. If the generating elements of a sphere, *Fig.* 68, are $a\ c=10$, $a\ b\ c$ will be 15.708, the centre of gravity of which is in o, and by rule, page 80, $o\ r=3.183$.

Hence $15.708\times\overline{3.183\times2}\times3.1416=314.16=$ *the surface of the sphere.*

To ascertain the Area of an Irregular Figure.

Rule.—Take a uniform piece of board or pasteboard, weigh it, cut out the figure of which the area is required, and weigh it; then, as the weight of the board or pasteboard is to the entire surface, so is the weight of the figure to its surface.

CAPILLARY TUBE.

To ascertain the Diameter of a Capillary Tube.

RULE.—Weigh the tube when empty, and again when filled with mercury; subtract the one weight from the other; reduce the difference to troy grains, and divide it by the length of the tube in inches. Extract the square root of this quotient, and multiply it by .0192245, and the product will be the diameter of the tube in inches.

Or, $\sqrt{\frac{w}{l}} \times .0192245 =$ *diameter; w representing difference in weights in Troy grains, and l the length of the tube.*

EXAMPLE.—The difference in the weights of a capillary tube when empty and when filled with mercury is 90 grains, and the length of the tube is 10 inches; what is the diameter of it?

$90 \div 10 = 9 =$ *weight of mercury* $\div$ *length of tube*; $\sqrt{9} = 3$, and $3 \times .0192245 = .0576735 =$ *the square root of the above quotient* $\times .0192245 =$ *diameter of tube required.*

Proof.—The weight of a cubic inch of mercury is 3442.75 Troy grains, and the diameter of a circular inch of equal area to a square inch is 1.128 (p. 81).

If, then, 3442.75 *grains occupy* 1 *cubic inch,* 90 *grains will require* .0261419 *cubic inch, which,* $\div$ 10 *for the height of the tube, gives* .00261419 *inch for the area of the section of the tube.*

Then $\sqrt{.00261419} = .051129 =$ *side of the square of a column of mercury of this area.*

Hence $.051129 \times 1.128$, *which is the ratio between the side of a square and the diameter of a circle of equal area* $= .0576735$.

F 2

LENGTHS OF CIRCULAR ARCS.

Table of the Lengths of Circular Arcs, the Diameter of a Circle being Unity, and assumed to be divided into 1000 *equal Parts.*

Height.	Length.	Height.	Length.	Height.	Length.	Height.	Length.
.100	1.0265	.134	1.0472	.168	1.0737	.202	1.1055
.101	1.0270	.135	1.0479	.169	1.0745	.203	1.1065
.102	1.0275	.136	1.0486	.170	1.0754	.204	1.1075
.103	1.0281	.137	1.0493	.171	1.0762	.205	1.1085
.104	1.0286	.138	1.0500	.172	1.0771	.206	1.1096
.105	1.0291	.139	1.0508	.173	1.0780	.207	1.1106
.106	1.0297	.140	1.0515	.174	1.0789	.208	1.1117
.107	1.0303	.141	1.0522	.175	1.0798	.209	1.1127
.108	1.0308	.142	1.0529	.176	1.0807	.210	1.1137
.109	1.0314	.143	1.0537	.177	1.0816	.211	1.1148
.110	1.0320	.144	1.0544	.178	1.0825	.212	1.1158
.111	1.0325	.145	1.0552	.179	1.0834	.213	1.1169
.112	1.0331	.146	1.0559	.180	1.0843	.214	1.1180
.113	1.0337	.147	1.0567	.181	1.0852	.215	1.1190
.114	1.0343	.148	1.0574	.182	1.0861	.216	1.1201
.115	1.0349	.149	1.0582	.183	1.0870	.217	1.1212
.116	1.0355	.150	1.0590	.184	1.0880	.218	1.1223
.117	1.0361	.151	1.0597	.185	1.0889	.219	1.1233
.118	1.0367	.152	1.0605	.186	1.0898	.220	1.1245
.119	1.0373	.153	1.0613	.187	1.0908	.221	1.1256
.120	1.0380	.154	1.0621	.188	1.0917	.222	1.1266
.121	1.0386	.155	1.0629	.189	1.0927	.223	1.1277
.122	1.0392	.156	1.0637	.190	1.0936	.224	1.1289
.123	1.0399	.157	1.0645	.191	1.0946	.225	1.1300
.124	1.0405	.158	1.0653	.192	1.0956	.226	1.1311
.125	1.0412	.159	1.0661	.193	1.0965	.227	1.1322
.126	1.0418	.160	1.0669	.194	1.0975	.228	1.1333
.127	1.0425	.161	1.0678	.195	1.0985	.229	1.1344
.128	1.0431	.162	1.0686	.196	1.0995	.230	1.1356
.129	1.0438	.163	1.0694	.197	1.1005	.231	1.1367
.130	1.0445	.164	1.0703	.198	1.1015	.232	1.1379
.131	1.0452	.165	1.0711	.199	1.1025	.233	1.1390
.132	1.0458	.166	1.0719	.200	1.1035	.234	1.1402
.133	1.0465	.167	1.0728	.201	1.1045	.235	1.1414

Table of Circular Arcs—Continued.

Height.	Length.	Height.	Length.	Height.	Length.	Height.	Length.
.236	1.1425	.274	1.1897	.312	1.2422	.350	1.3000
.237	1.1436	.275	1.1908	.313	1.2436	.351	1.3016
.238	1.1448	.276	1.1921	.314	1.2451	.352	1.3032
.239	1.1460	.277	1.1934	.315	1.2465	.353	1.3047
.240	1.1471	.278	1.1948	.316	1.2480	.354	1.3063
.241	1.1483	.279	1.1961	.317	1.2495	.355	1.3079
.242	1.1495	.280	1.1974	.318	1.2510	.356	1.3095
.243	1.1507	.281	1.1989	.319	1.2524	.357	1.3112
.244	1.1519	.282	1.2001	.320	1.2539	.358	1.3128
.245	1.1531	.283	1.2015	.321	1.2554	.359	1.3144
.246	1.1543	.284	1.2028	.322	1.2569	.360	1.3160
.247	1.1555	.285	1.2042	.323	1.2584	.361	1.3176
.248	1.1567	.286	1.2056	.324	1.2599	.362	1.3192
.249	1.1579	.287	1.2070	.325	1.2614	.363	1.3209
.250	1.1591	.288	1.2083	.326	1.2629	.364	1.3225
.251	1.1603	.289	1.2097	.327	1.2644	.365	1.3241
.252	1.1616	.290	1.2120	.328	1.2659	.366	1.3258
.253	1.1628	.291	1.2124	.329	1.2674	.367	1.3274
.254	1.1640	.292	1.2138	.330	1.2689	.368	1.3291
.255	1.1653	.293	1.2152	.331	1.2704	.369	1.3307
.256	1.1665	.294	1.2166	.332	1.2720	.370	1.3323
.257	1.1677	.295	1.2179	.333	1.2735	.371	1.3340
.258	1.1690	.296	1.2193	.334	1.2750	.372	1.3356
.259	1.1702	.297	1.2206	.335	1.2766	.373	1.3373
.260	1.1715	.298	1.2220	.336	1.2781	.374	1.3390
.261	1.1728	.299	1.2235	.337	1.2796	.375	1.3406
.262	1.1740	.300	1.2250	.338	1.2812	.376	1.3423
.263	1.1753	.301	1.2264	.339	1.2827	.377	1.3440
.264	1.1766	.302	1.2278	.340	1.2843	.378	1.3456
.265	1.1778	.303	1.2292	.341	1.2858	.379	1.3473
.266	1.1791	.304	1.2306	.342	1.2874	.380	1.3490
.267	1.1804	.305	1.2321	.343	1.2890	.381	1.3507
.268	1.1816	.306	1.2335	.344	1.2905	.382	1.3524
.269	1.1829	.307	1.2349	.345	1.2921	.383	1.3541
.270	1.1843	.308	1.2364	.346	1.2937	.384	1.3558
.271	1.1856	.309	1.2378	.347	1.2952	.385	1.3574
.272	1.1869	.310	1.2393	.348	1.2968	.386	1.3591
.273	1.1882	.311	1.2407	.349	1.2984	.387	1.3608

Table of Circular Arcs—Continued.

Height.	Length.	Height.	Length.	Height.	Length.	Height.	Length.
.388	1.3625	.417	1.4132	.445	1.4644	.473	1.5176
.389	1.3643	.418	1.4150	.446	1.4663	.474	1.5196
.390	1.3660	.419	1.4168	.447	1.4682	.475	1.5215
.391	1.3677	.420	1.4186	.448	1.4700	.476	1.5235
.392	1.3694	.421	1.4204	.449	1.4719	.477	1.5254
.393	1.3711	.422	1.4222	.450	1.4738	.478	1.5274
.394	1.3728	.423	1.4240	.451	1.4757	.479	1.5293
.395	1.3746	.424	1.4258	.452	1.4775	.480	1.5313
.396	1.3763	.425	1.4276	.453	1.4794	.481	1.5332
.397	1.3780	.426	1.4295	.454	1.4813	.482	1.5352
.398	1.3797	.427	1.4313	.455	1.4832	.483	1.5371
.399	1.3815	.428	1.4331	.456	1.4851	.484	1.5391
.400	1.3832	.429	1.4349	.457	1.4870	.485	1.5411
.401	1.3850	.430	1.4367	.458	1.4889	.486	1.5430
.402	1.3867	.431	1.4386	.459	1.4908	.487	1.5450
.403	1.3885	.432	1.4404	.460	1.4927	.488	1.5470
.404	1.3902	.433	1.4422	.461	1.4946	.489	1.5489
.405	1.3920	.434	1.4441	.462	1.4965	.490	1.5509
.406	1.3937	.435	1.4459	.463	1.4984	.491	1.5529
.407	1.3955	.436	1.4477	.464	1.5003	.492	1.5549
.408	1.3972	.437	1.4496	.465	1.5022	.493	1.5569
.409	1.3990	.438	1.4514	.466	1.5042	.494	1.5585
.410	1.4008	.439	1.4533	.467	1.5061	.495	1.5608
.411	1.4025	.440	1.4551	.468	1.5080	.496	1.5628
.412	1.4043	.441	1.4570	.469	1.5099	.497	1.5648
.413	1.4061	.442	1.4588	.470	1.5119	.498	1.5668
.414	1.4079	.443	1.4607	.471	1.5138	.499	1.5688
.415	1.4097	.444	1.4626	.472	1.5157	.500	1.5708
.416	1.4115						

To find the Length of an Arc of a Circle by the foregoing Table.

Rule.—Divide the height by the base, find the quotient in the column of heights, and take the length of that height from the next right-hand column. Multiply the length thus obtained by the base of the arc, and the product will be the length of the arc required.

EXAMPLE.—What is the length of an arc of a circle, the span or base being 100 feet, and the height 25 feet?

25÷100=.25, *and* .25, *per table,* =1.1591, *which, multiplied by* 100, =115,9100 *feet.*

NOTE.—*When, in the division of a height by the base, the quotient has a remainder after the third place of decimals, and great accuracy is required,*

Take the length for the first three figures, subtract it from the next following length, multiply the remainder by the said fraction, and add the product to the first length; the sum will be the length for the whole quotient.

EXAMPLE.—What is the length of an arc of a circle, the base of which is 35 feet, and the height or versed sine 8 feet?

8÷35=.2285714; *the tabular length for* .228=1.1333, *and for* .229=1.1344, *the difference between which is* .0011. *Then,* .5714×.0011=.00062854.

Hence, .228=1.1333

.0005714= .00062854

1.13392854, *the sum by which the base of the arc is to be multiplied; and* 1.13392854×35=39 *feet* .6874989, *which,* ×12 *for inches*=8.25, *making the length of the arc* 39 *feet* 8.25 *inches.*

AREAS OF SEGMENTS OF A CIRCLE.

Table of the Areas of the Segments of a Circle, the Diameter of which is Unity, and assumed to be divided into 1000 *equal Parts.*

Versed sine.	*Seg. Area.*	Versed sine.	*Seg. Area.*	Versed sine.	*Seg. Area.*	Versed sine.	*Seg. Area.*
.001	.00004	.034	.00827	.067	.02265	.100	.04087
.002	.00012	.035	.00864	.068	.02315	.101	.04148
.003	.00022	.036	.00901	.069	.02336	.102	.04208
.004	.00034	.037	.00938	.070	.02417	.103	.04269
.005	.00047	.038	.00976	.071	.02468	.104	.04310
.006	.00062	.039	.01015	.072	.02519	.105	.04391
.007	.00078	.040	.01054	.073	.02571	.106	.04452
.008	.00095	.041	.01093	.074	.02624	.107	.04514
.009	.00113	.042	.01133	.075	.02676	.108	.04575
.010	.00133	.043	.01173	.076	.02729	.109	.04638
.011	.00153	.044	.01214	.077	.02782	.110	.04700
.012	.00175	.045	.01255	.078	.02835	.111	.04763
.013	.00197	.046	.01297	.079	.02889	.112	.04826
.014	.00220	.047	.01339	.080	.02943	.113	.04889
.015	.00244	.048	.01382	.081	.02997	.114	.04953
.016	.90268	.049	.01425	.082	.03052	.115	.05016
.017	.00294	.050	.01468	.083	.03107	.116	.05080
.018	.00320	.051	.01512	.084	.03162	.117	.05145
.019	.00347	.052	.01556	.085	.03218	.118	.05209
.020	.00375	.053	.01601	.086	.03274	.119	.05274
.021	.00403	.054	.01646	.087	.03330	.120	.05338
.022	.00432	.055	.01691	.088	.03387	.121	.05404
.023	.00462	.056	.01737	.089	.03444	.122	.05469
.024	.00492	.057	.01783	.090	.03501	.123	.05534
.025	.00523	.058	.01830	.091	.03558	.124	.05600
.026	.00555	.059	.01877	.092	.03616	.125	.05666
.027	.00587	.060	.01924	.093	.03674	.126	.05733
.028	.00619	.061	.01972	.094	.03732	.127	.05799
.029	.00653	.062	.02020	.095	.03790	.128	.05866
.030	.00686	.063	.02068	.096	.03849	.129	.05933
.031	.00721	.064	.02117	.097	.03908	.130	.06000
.032	.00756	.065	.02165	.098	.03968	.131	.06067
.033	.00791	.066	.02215	.099	.04027	.132	.06135

Table of Areas of Segments of a Circle—Continued.

Versed sine.	*Seg. Area.*	Versed sine.	*Seg. Area.*	Versed sine.	*Seg. Area.*	Versed sine.	*Seg. Area.*
.133	.06203	.171	.08929	.209	.11908	.247	.15095
.134	.06271	.172	.09004	.210	.11990	.248	.15182
.135	.06339	.173	.09080	.211	.12071	.249	.15268
.136	.06407	.174	.09155	.212	.12153	.250	.15355
.137	.06476	.175	.09231	.213	.12235	.251	.15441
.138	.06545	.176	.09307	.214	.12317	.252	.15528
.139	.06614	.177	.09384	.215	.12399	.253	.15615
.140	.06683	.178	.09460	.216	.12481	.254	.15702
.141	.06753	.179	.09537	.217	.12563	.255	.15789
.142	.06822	.180	.09613	.218	.12646	.256	.15876
.143	.06892	.181	.09690	.219	.12728	.257	.15964
.144	.06962	.182	.09767	.220	.12811	.258	.16051
.145	.07033	.183	.09845	.221	.12894	.259	.16139
.146	.07103	.184	.09922	.222	.12977	.260	.16226
.147	.07174	.185	.10000	.223	.13060	.261	.16314
.148	.07245	.186	.10077	.224	.13144	.262	.16402
.149	.07316	.187	.10155	.225	.13227	.263	.16490
.150	.07387	.188	.10233	.226	.13311	.264	.16578
.151	.07459	.189	.10312	.227	.13394	.265	.16666
.152	.07531	.190	.10390	.228	.13478	.266	.16755
.153	.07603	.191	.10468	.229	.13562	.267	.16844
.154	.07675	.192	.10547	.230	.13646	.268	.16931
.155	.07747	.193	.10626	.231	.13731	.269	.17020
.156	.07820	.194	.10705	.232	.13815	.270	.17109
.157	.07892	.195	.10784	.233	.13900	.271	.17197
.158	.07965	.196	.10864	.234	.13984	.272	.17287
.159	.08038	.197	.10943	.235	.14069	.273	.17376
.160	.08111	.198	.11023	.236	.14154	.274	.17465
.161	.08185	.199	.11102	.237	.14239	.275	.17554
.162	.08258	.200	.11182	.238	.14324	.276	.17643
.163	.08332	.201	.11262	.239	.14409	.277	.17733
.164	.08406	.202	.11343	.240	.14494	.278	.17822
.165	.08480	.203	.11423	.241	.14580	.279	.17912
.166	.08554	.204	.11503	.242	.14665	.280	.18002
.167	.08629	.205	.11584	.243	.14751	.281	.18092
.168	.08704	.206	.11665	.244	.14837	.282	.18182
.169	.08779	.207	.11746	.245	.14923	.283	.18272
.170	.08853	.208	.11827	.246	.15009	.284	.18361

Table of Areas of Segments of a Circle—Continued.

Versed sine.	Seg. Area.	Versed sine.	Seg. Area	Versed sine.	Seg. Area	Versed sine.	Seg. Area.
.285	.18452	.323	.21947	.361	.25551	.399	.29239
.286	.18542	.324	.22040	.362	.25647	.400	.29337
.287	.18633	.325	.22134	.363	.25743	.401	.29435
.288	.18723	.326	.22228	.364	.25839	.402	.29533
.289	.18814	.327	.22321	.365	.25936	.403	.29631
.290	.18905	.328	.22415	.366	.26032	.404	.29729
.291	.18995	.329	.22509	.367	.26128	.405	.29827
.292	.19086	.330	.22603	.368	.26225	.406	.29925
.293	.19177	.331	.22697	.369	.26321	.407	.30024
.294	.19268	.332	.22791	.370	.26418	.408	.30122
.295	.19360	.333	.22886	.371	.26514	.409	.30220
.296	.19451	.334	.22980	.372	.26611	.410	.30319
.297	.19542	.335	.23074	.373	.26708	.411	.30417
.298	.19634	.336	.23169	.374	.26804	.412	.30515
.299	.19725	.337	.23263	.375	.26901	.413	.30614
.300	.19817	.338	.23358	.376	.26998	.414	.30712
.301	.19908	.339	.23453	.377	.27095	.415	.30811
.302	.20000	.340	.23547	.378	.27192	.416	.30909
.303	.20092	.341	.23642	.379	.27289	.417	.31008
.304	.20184	.342	.23737	.380	.27386	.418	.31107
.305	.20276	.343	.23832	.381	.27483	.419	.31205
.306	.20368	.344	.23927	.382	.27580	.420	.31304
.307	.20460	.345	.24022	.383	.27677	.421	.31403
.308	.20553	.346	.24117	.384	.27775	.422	.31502
.309	.20645	.347	.24212	.385	.27872	.423	.31600
.310	.20738	.348	.24307	.386	.27969	.424	.31699
.311	.20830	.349	.24403	.387	.28067	.425	.31798
.312	.20923	.350	.24498	.388	.28164	.426	.31897
.313	.21015	.351	.24593	.389	.28262	.427	.31996
.314	.21108	.352	.24689	.390	.28359	.428	.32095
.315	.21201	.353	.24784	.391	.28457	.429	.32194
.316	.21294	.354	.24880	.392	.28554	.430	.32293
.317	.21387	.355	.24976	.393	.28652	.431	.32391
.318	.21480	.356	.25071	.394	.28750	.432	.32490
.319	.21573	.357	.25167	.395	.28848	.433	.32590
.320	.21667	.358	.25263	.396	.28945	.434	.32689
.321	.21760	.359	.25359	.397	.29043	.435	.32788
.322	.21853	.360	.25455	.398	.29141	.436	.32887

Table of Areas of Segments of a Circle—Continued.

Versed sine.	*Seg. Area.*	Versed sine.	*Seg. Area.*	Versed sine.	*Seg. Area.*	Versed sine.	*Seg. Area.*
.437	.32987	.453	.34577	.469	.36172	.485	.37770
.438	.33086	.454	.34676	.470	.36272	.486	.37870
.439	.33185	.455	.34776	.471	.36371	.487	.37970
.440	.33284	.456	.34875	.472	.36471	.488	.38070
.441	.33384	.457	.34975	.473	.36571	.489	.38170
.442	.33483	.458	.35075	.474	.36671	.490	.38270
.443	.33582	.459	.35174	.475	.36781	.491	.38370
.444	.33682	.460	.35274	.476	.36871	.492	.38470
.445	.33781	.461	.35374	.477	.36971	.493	.38570
.446	.33880	.462	.35474	.478	.37071	.494	.38670
.447	.33980	.463	.35573	.479	.37170	.495	.38770
.448	.34079	.464	.35673	.480	.37276	.496	.38870
.449	.34179	.465	.35773	.481	.37370	.497	.38970
.450	.34278	.466	.35872	.482	.37470	.498	.39070
.451	.34378	.467	.35972	.483	.37570	.499	.39170
.452	.34477	.468	.36072	.484	.37670	.500	.39270

To find the Area of a Segment of a Circle by the above Table.

RULE.—Divide the height or versed sine by the diameter of the circle, and find the quotient in the column of versed sines. Take the area noted in the next column, and multiply it by the square of the diameter, and it will give the area required.

EXAMPLE.—Required the area of a segment, its height being 10, and the diameter of the circle 50 feet.

$10 \div 50 = .2$, *and* .2, *per table*, $= .11182$; *then*, $.11182 \times 50^2 = 279.55$ *feet.*

NOTE.—*When, in the division of a height by the base, the quotient has a remainder after the third place of decimals, and great accuracy is required,*

Take the length for the first three figures, subtract it from the next following length, multiply the remainder by the said fraction, and add the product to the first length; the sum will be the length for the whole quotient.

EXAMPLE.—What is the area of a segment of a circle, the diameter of which is 10 feet, and the height of it 1.575 feet?

$1.575 \div 10 = .1575$; *the tabular area for* $.157 = .07892$, *and for* $.158 = .07964$, *the difference between which is* $.00072$.

Then, $.5 \times .00072 = .000360$.

Hence,

$$\begin{array}{rl} .157 & = .07892 \\ .0005 & = .00036 \\ \hline & .07928 \end{array}$$

$.07928$, *the sum by which the square of the diameter of the circle is to be multiplied; and* $.07928 \times 10^2 = 7.928$ *feet.*

AREAS OF THE ZONES OF A CIRCLE.

Table of the Areas of the Zones of a Circle, the Diameter of which is Unity, and assumed to be divided into 1000 *equal Parts.*

Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.
.001	.00100	.034	.03397	.067	.06680	.100	.09933
.002	.00200	.035	.03497	.068	.06780	.101	.10031
.003	.00300	.036	.03597	.069	.06878	.102	.10129
.004	.00400	.037	.03697	.070	.06977	.103	.10227
.005	.00500	.038	.03796	.071	.07076	.104	.10325
.006	.00600	.039	.03896	.072	.07175	.105	.10422
.007	.00700	.040	.03996	.073	.07274	.106	.10520
.008	.00800	.041	.04095	.074	.07373	.107	.10618
.009	.00900	.042	.04195	.075	.07472	.108	.10715
.010	.01000	.043	.04295	.076	.07570	.109	.10813
.011	.01100	.044	.04394	.077	.07669	.110	.10911
.012	.01200	.045	.04494	.078	.07768	.111	.11008
.013	.01300	.046	.04593	.079	.07867	.112	.11106
.014	.01400	.047	.04693	.080	.07966	.113	.11203
.015	.01500	.048	.04793	.081	.08064	.114	.11300
.016	.01600	.049	.04892	.082	.08163	.115	.11398
.017	.01700	.050	.04992	.083	.08262	.116	.11495
.018	.01800	.051	.05091	.084	.08360	.117	.11592
.019	.01900	.052	.05190	.085	.08459	.118	.11690
.020	.02000	.053	.05290	.086	.08557	.119	.11787
.021	.02100	.054	.05389	.087	.08656	.120	.11884
.022	.02200	.055	.05489	.088	.08754	.121	.11981
.023	.02300	.056	.05588	.089	.08853	.122	.12078
.024	.02400	.057	.05688	.090	.08951	.123	.12175
.025	.02500	.058	.05787	.091	.09050	.124	.12272
.026	.02599	.059	.05886	.092	.09148	.125	.12369
.027	.02699	.060	.05986	.093	.09246	.126	.12469
.028	.02799	.061	.06085	.094	.09344	.127	.12562
.029	.02898	.062	.06184	.095	.09443	.128	.12659
.030	.02998	.063	.06283	.096	.09540	.129	.12755
.031	.03098	.064	.06382	.097	.09639	.130	.12852
.032	.03198	.065	.06482	.098	.09737	.131	.12949
.033	.03298	.066	.06580	.099	.09835	.132	.13045

Table of the Zones of a Circle—Continued.

Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.
.133	.13141	.171	.16761	.209	.20274	.247	.23655
.134	.13238	.172	.16855	.210	.20365	.248	.23742
.135	.13334	.173	.16948	.211	.20456	.249	.23829
.136	.13430	.174	.17042	.212	.20546	.250	.23915
.137	.13527	.175	.17136	.213	.20637	.251	.24002
.138	.13623	.176	.17230	.214	.20727	.252	.24089
.139	.13719	.177	.17323	.215	.20818	.253	.24175
.140	.13815	.178	.17417	.216	.20908	.254	.24261
.141	.13911	.179	.17510	.217	.20998	.255	.24347
.142	.14007	.180	.17603	.218	.21088	.256	.24433
.143	.14103	.181	.17697	.219	.21178	.257	.24519
.144	.14198	.182	.17790	.220	.21268	.258	.24604
.145	.14294	.183	.17883	.221	.21358	.259	.24690
.146	.14390	.184	.17976	.222	.21447	.260	.24775
.147	.14485	.185	.18069	.223	.21537	.261	.24861
.148	.14581	.186	.18162	.224	.21626	.262	.24946
.149	.14677	.187	.18254	.225	.21716	.263	.25021
.150	.14772	.188	.18347	.226	.21805	.264	.25116
.151	.14867	.189	.18440	.227	.21894	.265	.25201
.152	.14962	.190	.18532	.228	.21983	.266	.25285
.153	.15058	.191	.18625	.229	.22072	.267	.25370
.154	.15153	.192	.18717	.230	.22161	.268	.25455
.155	.15248	.193	.18809	.231	.22250	.269	.25539
.156	.15343	.194	.18902	.232	.22335	.270	.25623
.157	.15438	.195	.18994	.233	.22427	.271	.25707
.158	.15533	.196	.19086	.234	.22515	.272	.25791
.159	.15628	.197	.19178	.235	.22604	.273	.25875
.160	.15723	.198	.19270	.236	.22692	.274	.25959
.161	.15817	.199	.19361	.237	.22780	.275	.26043
.162	.15912	.200	.19453	.238	.22868	.276	.26126
.163	.16006	.201	.19545	.239	.22956	.277	.26209
.164	.16101	.202	.19636	.240	.23044	.278	.26293
.165	.16195	.203	.19728	.241	.23131	.279	.26376
.166	.16290	.204	.19819	.242	.23219	.280	.26459
.167	.16384	.205	.19910	.243	.23306	.281	.26541
.168	.16478	.206	.20001	.244	.23394	.282	.26624
.169	.16572	.207	.20092	.245	.23481	.283	.26706
.170	.16667	.208	.20183	.246	.23568	.284	.26789

Table of the Zones of a Circle—Continued.

Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.
.285	.26871	.323	.29886	.361	.32656	.399	.35122
.286	.26953	.324	.29962	.362	.32725	.400	.35182
.287	.27035	.325	.30039	.363	.32794	.401	.35242
.288	.27117	.326	.30114	.364	.32862	.402	.35302
.289	.27199	.327	.30190	.365	.32931	.403	.35361
.290	.27280	.328	.30266	.366	.32999	.404	.35420
.291	.27362	.329	.30341	.367	.33067	.405	.35479
.292	.27443	.330	.30416	.368	.33135	.406	.35538
.293	.27524	.331	.30491	.369	.33203	.407	.35596
.294	.27605	.332	.30566	.370	.33270	.408	.35654
.295	.27686	.333	.30641	.371	.33337	.409	.35711
.296	.27766	.334	.30715	.372	.33404	.410	.35769
.297	.27847	.335	.30790	.373	.33471	.411	.35826
.298	.27927	.336	.30864	.374	.33537	.412	.35883
.299	.28007	.337	.30938	.375	.33604	.413	.35939
.300	.28088	.338	.31012	.376	.33670	.414	.35995
.301	.28167	.339	.31085	.377	.33735	.415	.36051
.302	.28247	.340	.31159	.378	.33801	.416	.36107
.303	.28327	.341	.31232	.379	.33866	.417	.36162
.304	.28406	.342	.31305	.380	.33931	.418	.36217
.305	.28486	.343	.31378	.381	.33996	.419	.36272
.306	.28565	.344	.31450	.382	.34061	.420	.36326
.307	.28644	.345	.31523	.383	.34125	.421	.36380
.308	.28723	.346	.31595	.384	.34190	.422	.36434
.309	.28801	.347	.31667	.385	.34253	.423	.36488
.310	.28880	.348	.31739	.386	.34317	.424	.36541
.311	.28958	.349	.31811	.387	.34380	.425	.36594
.312	.29036	.350	.31882	.388	.34444	.426	.36646
.313	.29115	.351	.31954	.389	.34507	.427	.36698
.314	.29192	.352	.32025	.390	.34569	.428	.36750
.315	.29270	.353	.32096	.391	.34632	.429	.36802
.316	.29348	.354	.32167	.392	.34694	.430	.36853
.317	.29425	.355	.32237	.393	.34756	.431	.36904
.318	.29502	.356	.32307	.394	.34818	.432	.36954
.319	.29580	.357	.32377	.395	.34879	.433	.37005
.320	.29656	.358	.32447	.396	.34940	.434	.37054
.321	.29733	.359	.32517	.397	.35001	.435	.37104
.322	.29810	.360	.32587	.398	.35062	.436	.37153

Table of the Zones of a Circle—Continued.

Height.	Area.	Height.	Area.	Height.	Area.	Height.	Area.
.437	.37202	.453	.37931	.469	.38549	.485	.39026
.438	.37250	.454	.37973	.470	.38583	.486	.39050
.439	.37298	.455	.38014	.471	.38617	.487	.39073
.440	.37346	.456	.38056	.472	.38650	.488	.39095
.441	.37393	.457	.38096	.473	.38683	.489	.39117
.442	.37440	.458	.38137	.474	.38715	.490	.39137
.443	.37487	.459	.38177	.475	.38747	.491	.39156
.444	.37533	.460	.38216	.476	.38778	.492	.39175
.445	.37579	.461	.38255	.477	.38808	.493	.39192
.446	.37624	.462	.38294	.478	.38838	.494	.39208
.447	.37669	.463	.38332	.479	.38867	.495	.39223
.448	.37714	.464	.38369	.480	.38895	.496	.39236
.449	.37758	.465	.38406	.481	.38923	.497	.39248
.450	.37802	.466	.38443	.482	.38950	.498	.39258
.451	.37845	.467	.38479	.483	.38976	.499	.39266
.452	.37888	.468	.38514	.484	.39001	.500	.39270

To find the Area of a Zone by the above Table.

Rule 1.—*When the zone is less than a semicircle,* divide the height by the diameter, and find the quotient in the column of heights. Take out the area opposite to it in the next column on the right hand, and multiply it by the square of the longest chord; the product will be the area of the zone.

Example.—Required the area of a zone, the diameter of which is 50, and its height 15?

$15 \div 50 = .300$; *and* .300, *as per table,* $= .28088$.

Hence, $.28088 \times 50^2 = 702.2$, *the area of the zone.*

Rule 2.—*When the zone is greater than a semicircle,* take the height on each side of the diameter of the circle, and find, by Rule 1, their respective areas; add the areas of these two portions together, and the sum will be the area of the zone.

Example.—Required the area of a zone, the diameter of the circle being 50, and the height of the zone on each side of the diameter of the circle 20 and 15 respectively.

$20 \div 50 = .400$; $.400$, *as per table*, $= .35182$; *and* $.35182 \times 50^2 = 879.55$.

$15 \div 50 = .300$; $.300$, *as per table*, $= .28088$; *and* $.28088 \times 50^2 = 702.2$.

Hence, $879.55 + 702.2 = 1581.65$, *the result required.*

NOTE.—*When, in the division of a height by the chord, the quotient has a remainder after the third place of decimals, and great accuracy is required,*

Take the area for the first three figures, subtract it from the next following area, multiply the remainder by the said fraction, and add the product to the first area; the sum will be the area for the whole quotient.

EXAMPLE.—What is the area of a zone of a circle, the greater chord being 100 feet, and the breadth of it 14 feet 3 inches?

14 *feet* 3 *inches* $= 14.25$, *and* $14.25 \div 100 = .1425$; *the tabular length for* $.142 = .14007$, *and for* $.143 = .14103$, *the difference between which is* $.00096$.

Then, $.5 \times .00096 = .000480$.

Hence,

$$\begin{array}{rl} .142 & = .14007 \\ .0005 & = \underline{.00048} \\ & .14055 \end{array}$$

$.14055$, *the sum by which the square of the greater chord is to be multiplied; and* $.14055 \times 100^2 = 1405.5$ *feet.*

PROMISCUOUS EXAMPLES.

1. If a load of wood is 8 feet long, 3 feet 10 inches wide, and 6 feet 6 inches high, what are its contents?
Ans. 1.72 *cords.*

2. Add $\frac{4}{7}$ of a ton to $\frac{9}{10}$ of a cwt. *Ans.* 12.329 *cwt.*

3. What are the contents of a board 25 feet long and 3 feet wide? *Ans.* 75 *feet.*

4. What is the difference between the contents of two floors; one being 37 feet long and 27 feet wide, and the other 40 feet long and 20 feet wide? *Ans.* 199 *feet.*

5. How many yards of paper that is 30 inches wide will it require to cover the wall of a room that is $15\frac{1}{2}$ feet long, $11\frac{1}{4}$ feet wide, and $7\frac{3}{4}$ feet high? *Ans.* 55.2833 *yards.*

6. If $\frac{1}{5}$ of a post stands in the mud, $\frac{1}{4}$ in the water, and 10 feet above the water, what is the length of the post?
Ans. 18.182 *feet.*

7. What fraction is that to which if $\frac{2}{7}$ of $\frac{5}{9}$ be added the sum will be 1? *Ans.* $\frac{53}{63}$.

8. If the earth make one complete revolution in 23 hours 56 minutes 3 seconds, in what time does it move one degree?
Ans. 3 *min.* 59.3417 *seconds.*

9. From a plank 26 inches broad a square yard and a half is to be sawed off; what distance from the end must the line be struck? *Ans.* 6.23 *feet.*

10. What is the side of a triangle that may be inscribed in a circle, the circumference of which is 1000 feet?
Ans. 275.6556 *feet.*

11. How large a square field can be made in a circle of 100 rods in diameter?
Ans. 22 *rods* 2 *yards* 2 *feet* 4.5 *inches.*

12. A rectangular field is 12 rods 2 yards 2 feet and 3 inches in length, by 9 rods and 1 yard in breadth; what is its area in square yards? *Ans.* 3471.875 *yards.*

13. The sides of a triangular plot of ground are 24, 36, and 48 feet; what is its area in square feet?

Ans. 418.282 *feet.*

14. In turning a chaise within a ring of a certain diameter, the outer wheel made two turns while the inner wheel made but one, the wheels being four feet in diameter, and five feet asunder on the axle-tree; what was the circumference of the track described by the outer wheel? *Ans.* 62.832 *feet.*

15. The ball on the top of a church is 6 feet in diameter; what did the gilding of it cost at 8 cents per square inch?

Ans. 1302.884 *dollars.*

16. A roof of a house is 24 feet 8 inches by 14.5 feet, and is to be covered with lead weighing 8 lbs. per foot; what will be the weight of the lead required?

Ans. 2861.324 *lbs.*

17. The area of an equilateral triangle, whose base falls on the diameter, and its vertex in the middle of the arc of a semicircle, is equal to 100; what is the diameter of the semicircle?

Ans. 26.32148.

18. The distance of the centres of two circles, the diameters of which are each 50, is equal to 30; what is the area of the space inclosed between the two circles by arcs of their circumferences? *Ans.* 559.115.*

19. In the latitude of London, the distance around the earth, measured on that parallel, is about 15,550 miles; now, as the earth revolves in 23 hours and 56 minutes, at what rate per hour does the city of London move from west to east? *Ans.* 649.7214 *miles per hour.*

20. A father left his son an estate, $\frac{1}{4}$ of which he ran through in 8 months; $\frac{3}{7}$ of the remainder lasted him 12 months longer, when he had barely $820 left; what sum did his father leave him? *Ans.* $1913.34.

21. There is a segment of a circle the chord of which is 60 feet, its versed sine 10 feet; what will be the versed sine of that segment of the same circle when the chord is 90 feet?

Ans. 28.2055.

* By Table of Areas of the Segments of a Circle, p. 134.

22. If a line 144 feet long will reach from the top of a fort to a point on the opposite side of a river that is 64 feet wide, what is the height of the fort above that point?

Ans. 128.99 *feet.*

23. A certain room is 20 feet long, 16 feet wide, and 12 feet high; how long must a line be to extend from one of the lower corners to an opposite upper corner? *Ans.* 28.2843 *feet.*

24. Two ships sail from the same port; one sails due north 128 miles, the other due east 72 miles; how far are the ships from each other? *Ans.* 146.86+ *miles.*

25. There are two columns in the ruins of Persepolis left standing upright; one is 70 feet above the plain, and the other 50; in a straight line between these stands an ancient statue 5 feet in height, the head of which is 100 feet from the summit of the higher, and 80 feet from the top of the lower column; required the distance between the tops of the two columns? *Ans.* 143.543 *feet.*

26. The height of a tree growing in the centre of a circular island 100 feet in diameter is 160 feet, and a line extending from the top of it to the further shore is 400 feet; what is the breadth of the stream, assuming the land on each side of the water to be level? *Ans.* 316.6065 *feet.*

27. A ladder 70 feet long is so placed as to reach a window 40 feet from the ground on one side of a street, and without removing it at the foot, will reach a window 30 feet high on the other side; what is the breadth of the street?

Ans. 120.6911 *feet.*

28. If a tree stand on a horizontal plane 80 feet in height, at what height from the ground must it be cut off so that the top of it may fall on a point 40 feet from the bottom of the tree, the end where it was cut off resting on the stump?

Ans. 30 *feet.*

29. Four men, A, B, C, D, bought a grindstone, the diameter of which was 4 feet; they agreed that A should grind off his share first, and that each man should have it alternately until he had worn off his share; how much will each man grind off? *Ans.* A 3.215+, B 3.81+, C 4.97+, D 12 *inches.*

30. The classification of a school is as follows, viz., $\frac{1}{16}$ of the boys are taught geometry, $\frac{3}{8}$ grammar, $\frac{3}{10}$ arithmetic, $\frac{3}{20}$ writing, and 9 reading; what is the number in each branch?

Ans. { 5 *geometry*, 30 *grammar*, 24 *arithmetic*, 12 *writing*, and 9 *reading*.

31. A certain general has an army of 141,376 men. How many must he place in rank and file to form them into a square? *Ans.* 376.

32. If the area of a circle be 1760 yards, how many feet must the side of a square measure to contain that quantity?

Ans. 125.8571 *feet.*

33. If the diameter of a round stick of timber be 24 inches, how large a square stick may be hewn from it?

Ans. 16.97 *inches.*

34. To set out an orchard of 2400 mulberry trees so that the length shall be to the breadth as 3 to 2, and the distance of each tree one from the other 7 yards, how many trees must there be in the length of the orchard, and how many in its breadth, and how many square yards of ground do they stand on?

Ans. { *Trees in length*, 60. *Trees in breadth*, 40. *Square yards*, 117,600.

35. Suppose the expense of paving a semicircular plot of ground, at 30 cents per square foot, amounted to \$25.63, what is the diameter of it? *Ans.* 14.75 *feet.*

36. Two sides of an obtuse-angled triangle are 20 and 40 poles; what must be the length of the third side, that the triangle may contain just an acre?

Ans. 58.876 *poles.*

37. If two sides of an obtuse-angled triangle, the area of which is $=60\times\sqrt{3}$, are 12 and 20, what is the third side?

Ans. 28.

38. If an area of 63 feet is cut off from a triangle, the three sides of which are 13, 20, and 21 feet, by a line parallel to the longest side or base of the triangle, what are the lengths of the sides of the triangle which will include that area?

Operation.—A triangle of the above dimensions has an area of 126. See Rule, p. 51.

Then, as 126 (=*area of triangle*) : $\frac{126}{2}$ (=*area of required triangle*) : : hyp.^{2} (400) : hyp.$'^2$ (200)=*square of hyp. of required triangle*; and $\sqrt{200}$ =14.142=*square root of square of hyp.*=*hyp.*$'$ *of required triangle.*

Hence, as hyp. (20) : base (21) : : hyp.$'$ (14.142) : base of required triangle (14.849).

Consequently, 14.849 is the base, 14.142 is the hyp., and by Rule, p. 53, $\sqrt{14.142^2 - 14.849^2} = 4.527$, *the length of the remaining side.*

39. Seven men bought a grindstone of 60 inches in diameter, each paying $\frac{1}{7}$ part of the cost; what part of the diameter can each grind down for his share?

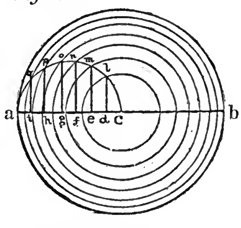
Ans. *The* 1*st*, 4.4508; 2*d*, 4.8400; 3*d*, 5.3535; 4*th*, 6.0765; 5*th*, 7.2079; 6*th*, 9.3935; *and the* 7*th*, 22.6778.

This problem may be thus constructed, *Fig.* 69:

On the radius, *a c*, describe a semicircle; also divide *a c* into as many equal parts, *c d*, *d e*, *e f*, &c., as there are shares, and erect the perpendiculars *d l*, *e m*, *f n*, &c., meeting the semicircle described on *a c* in *l*, *m*, *n*, *o*, *p*, *q*.

Then, with the centre *c* and radii *c l*, *c m*, *c n*, &c., describe circles, and the diameter which each is to grind down will be thus shown.

Fig. 69.



For, the square of the chords or radii *c l*, *c m*, *c n*, &c., are as the cosines *c d*, *c e*, *c f*, &c.

40. A gentleman has a garden 100 feet long and 80 feet broad, and a gravel walk is to be made of an equal width

half around it; what must be the width of the walk so that it will take up just half the ground?*

Operation.—$100 \times 80 = 8000 =$ *area of garden.*

$\frac{100+80}{2} = 90 =$ *half length of half the sides of garden.*

$90^2 - \frac{8000}{2} = 4100 =$ *area of walk if the garden was a square.*

Hence, $\sqrt{4100} = 64.0312$, and $64.0312 - 90 = 25.9688 =$ *width of the walk.*

41. In the midst of a meadow, well covered with grass,
It just took an acre to tether an ass;
How long was a line, that, reaching all round,
Restricted his grazing to an acre of ground?

Ans. 39.2507 yards.

42. A maltster has a kiln that is 16 feet 6 inches square; it is necessary to pull it down, and build a new one that will dry three times as much at a time as the old one did; what must be the length of its side? *Ans.* 28.58 *feet.*

43. In a round garden containing 75 square rods, how large a square garden can be laid out?

Ans. 47.7475 *square rods.*

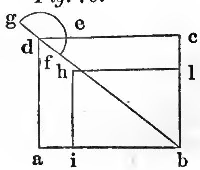
44. If a circular garden contain 75 square rods, what must be the side of a square field that would inclose it?

Ans. 9.772 *rods.*

45. There is a circular field 25 rods in diameter; what is the difference of the areas of the inscribed and circumscribed squares, and how much do they differ from the areas of the field?

Ans. { 312.5 *rods, the difference of the squares;* 134.1261 *rods, the circumscribed square more than the area;* 178.374 *rods, inscribed square less than the area.*

Fig. 70.



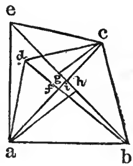
* This problem may be constructed thus: Let *a b c d* represent the garden; make *c e* = *c b*, and with the centre *d* and radius *d e* describe the semicircle *g e f*. Make *b i* = ½ *b g*, *b l* = ½ *b f*, and complete the rectangle *i b l h*, and the result is obtained.

46. Two persons start from the same place; one goes south 4 miles per hour, the other west 5 miles per hour; how far apart are they in 9 hours? *Ans.* 57.6281 *miles.*

47. The four sides of a field, *Fig.* 71, the diagonals of which are equal to each other, are 25, 35, 31, and 19 poles; what is its area?

Construction.—In this question, the sums of the squares of the opposite sides of the trapezium being equal ($d\,c^2$ (19)$+a\,b^2$ (35)$=1586$, and $d\,a^2$ (25)$+c\,b^2$ (31)$=1586$), the figure may be constructed as follows:

Fig. 71.



Draw $a\,b$ and $a\,e$ at right angles, and each equal to the longest of the given sides (35); join $b\,e$, and from the points e and b, with radii equal to the two remaining opposite sides (25 and 31) respectively, describe arcs intersecting in c on the farther side of $b\,e$; join $a\,c$, and draw $b\,f$ at right angles to it. With the centre c, and radius equal to the remaining side (19), describe an arc cutting $b\,f$ produced in d. Join $a\,d$ and $c\,d$; then will $a\,b\,c\,d$ be the figure required.

Hence, in the two triangles $a\,b\,d$ and $e\,a\,c$, we have the two sides $b\,a$, $a\,d$ equal the two $a\,e$, $e\,c$, each to each; and the angles $a\,b\,d$ and $e\,a\,c$ equal (each being the complement of $b\,a\,f$), and $e\,c$ and $a\,d$ similarly situated; wherefore $b\,d=a\,c$.

Calculation.—On $b\,e$ let fall the perpendiculars $c\,g$ and $a\,h$.

Now $b\,e^2=a\,b^2+a\,e^2=35^2\times 2$; $b\,e=\sqrt{35^2\times 2}=35\sqrt{2}=49.4975$.

$a\,h=b\,h=\frac{1}{2}b\,e=24.7487$.

By formula, p. 56, $b\,g=\dfrac{b\,c^2+b\,e^2-c\,e^2}{2\,b\,e}=\dfrac{31^2+\overline{2\times 35^2}-25^2}{2\times 49.4975}=\dfrac{2786}{98.995}$ $=28.1428$.

$g\,h=b\,g-b\,h=28.1428-24.7487=3.3941$.

$c\,g=\sqrt{b\,c^2-b\,g^2}=\sqrt{31^2-28.1428^2}=\sqrt{168.9828}=12.9993$.

By similar triangles (see Note, p. 50), $a\,h+c\,g$ (37.748)$:g\,h$ (3.3941) $::a\,h$ (24.7487)$:h\,i=2.2253$.

$a\,i=\sqrt{a\,h^2+h\,i^2}=\sqrt{24.7487^2+2.2253^2}=\sqrt{612.4981+4.9520}=\sqrt{617.4501}=24.8485$.

Again, by similar triangles, $h\ i$ (2.2253) : $h\ g$ (3.3941) :: $a\ i$ (24.8485) : $a\ c=37.8997=b\ d$; *now, by Rule*, p. 57, *the area of the trapezium** $a\ b\ c\ d$ $=\frac{a\ c\times \overline{b\ f+f\ d}}{2}=\frac{a\ c\times b\ d}{2}=\frac{a\ c^2}{2}=\frac{37.8997^2}{2}=718.1936$ *poles*$=4$ *ac.* 1 *ro.* 38 *po.* 5 *yds.* 7.7076 *feet, Ans.*

48. A messenger traveling 8 miles an hour was sent to Mexico with dispatches for the army; after he had gone 51 miles, another was sent with countermanding orders who could go 19 miles as quick as the former could 16; how long will it take the latter to overtake the former, and how far must he travel?

Operation.—If the first messenger travels 8 miles in an hour, and the second 19 while the first travels 16, the second travels 1.5 miles an hour $\left(\frac{19-16}{2}\right)$ faster than the first.

Then, $51\div1.5=34$ *hours*, and $34\times9.5=323=$ *number of miles traveled by second messenger when he overtook the first.*

Hence, $34+\frac{51}{8}=40.375$, and $40.375\times8=323$ miles $=$ *the distance reached by first messenger.*

49. The hour and minute hands of a watch are exactly together at 12 o'clock; when are they next together?

Operation.—The velocities of the two hands of a watch are to each other as 12 to 1; therefore the difference of velocities is $12-1=11$.

Then, as $11 : \left\{\begin{matrix}12\times1\\12\times2\end{matrix}\right\} :: 1 : \left\{\begin{matrix}1\ h.\ 5\ m.\ 27\frac{3}{11}\ sec.,\ 1st\ time.\\2\ h.\ 10\ m.\ 54\frac{6}{11}\ sec.,\ 2d\ time.\end{matrix}\right.$

50. A person being asked the hour of the day, replied, The time past noon is equal to $\frac{4}{5}$ of the time till midnight; what was the time? *Ans.* 20 *minutes past* 5.

Operation.—If the time required is $\frac{4}{5}$ of the time to midnight, then the whole time from noon to midnight (12 *hours* or 720 *minutes*) is divided into $\frac{9}{5}$ $(\frac{4}{5}+\frac{4}{5}+\frac{1}{5})$.

Hence, if $\frac{9}{5}=720$, $\frac{1}{5}=80$ *minutes*, and $\frac{4}{5}=320$ *minutes*, or 5 *hours* and 20 *minutes, the Ans.*

51. A person being asked the time of day, replied that $\frac{1}{7}$ of the time from noon was equal to $\frac{1}{11}$ of the time to midnight; what was the time? *Ans.* 40 *minutes past* 4.

* $a\ c$ and $b\ d$ being equal.

52. What is the radius of a circular acre?

Operation.—Side of a square (p. 81)$\times$1.128=diameter of an equal circle.

The side of a square acre (p. 13) is 208.710321, which, $\times$1.128= 235.5, and 235.5$\div$2 *to obtain radius*=117.75 *feet.*

53. The time of the day is between 4 and 5, and the hour and minute hands are exactly together; what is the time?

Operation.—The speed of the hands is as 1 to 11.

4 *hours*$\times$60=240, and 240$\div$11=21 *min.* and $49\frac{1}{11}$ *sec.*, which, added to 4, =4 *hours* 21 *min.* and $49\frac{1}{11}$ *sec.*

54. A person being asked what o'clock it was, replied that it was between 5 and 6; but, to be more particular, the minute-hand was as far beyond the 6 as the hour-hand wanted of being to the 6; that is, that the hour and minute hands made equal acute angles with a line passing from the 12 through the 6; required the time.

Operation.—5 *hours*=300 *minutes*, and 6 *hours*=360 *minutes.*

Then, 300+the time by the hour-hand past 5=360−the time by the minute-hand past 6.

As the relative speed of the hour and minute hands is as 1 to 12, 300+1,=360−12.

Consequently, $\frac{360-300}{1+12}=\frac{60}{13}=4$, $36\frac{12}{13}$=the space between the hour and minute hands, which, $\div$2 to obtain the half space (each side of the 6), gives 2 *min.* $18\frac{6}{13}$ *sec.*, which, added to 5 *hours* and 30 *min.*, =5 *hours* 32 *min.* and $18\frac{6}{13}$ *sec.*, *the time required.*

55. Two persons, A and B, start at the same time to meet each other when apart 100 miles; after 7 hours they meet, when it appears that A had ridden $1\frac{1}{2}$ miles per hour faster than B; at what rate per hour did each ride?

Ans. A 7.893, B 6.392 *miles per hour.*

56. Swift can travel 7 miles in $\frac{8}{9}$ of an hour, but Slow can travel only 5 miles in $\frac{7}{11}$ of an hour; both started from one point at the same time to walk a distance of 12 miles; how much sooner will Swift arrive than Slow?

Ans. 12.467 *seconds.*

57. At a certain time between two and three o'clock, the minute-hand of a clock was between three and four; within an hour after, the hour and minute hands had exactly changed places with each other; what was the precise time when the hands were in the first position?

Ans. 2 *h.* 15 *m.* $56\frac{92}{143}$ *sec.*

58. If a traveler were to leave New Haven at 8 o'clock on a morning, and walk toward Albany at the rate of 3 miles an hour, and another traveler were to set out from Albany at 4 o'clock in the evening, and walk toward New Haven at the rate of 4 miles an hour, whereabout on the road would they meet, supposing the distance to be 130 miles?

Ans. 69.4286 *miles from New Haven.*

59. A thief, escaping from an officer, has 40 miles the start, and travels at the rate of 5 miles an hour; the officer in pursuit travels at the rate of 7 miles an hour; how far must he travel before he overtakes the thief?

Ans. 20 *hours, and* 140 *miles.*

60. If 12 oxen graze $3\frac{1}{3}$ acres of grass in 4 weeks, and 21 oxen 10 acres in 9 weeks, how many oxen would it require to graze 24 acres in 18 weeks, the grass to be growing?

Operation.—Each ox grazes a certain quantity in each week, which we suppose to be 100 pounds, and of the whole quantity grazed in each case, a part must have grown during the time of grazing.

Then, by the first condition,
$12\times4\times100=4800$ *lbs.=whole quantity on* $3\frac{1}{3}$ *acres for* 4 *weeks.*
$4800\div3\frac{1}{3}=1440$ *lbs.=whole quantity on* 1 *acre for* 4 *weeks.*
By the second condition,
$21\times9\times100=18900$ *lbs.=whole quantity on* 10 *acres for* 9 *weeks.*
$18900\div10=1890$ *lbs.=whole quantity on* 1 *acre for* 9 *weeks.*
$1890-1440=450$ *lbs.=the quantity grown on an acre for* $9-4=5$ *weeks.*
$450\div\overline{9-4}=90$ *lbs.=the quantity which grows on each acre for* 1 *week.*
$90\times3\frac{1}{3}\times4=1200$ *lbs.=quantity grown on* $3\frac{1}{3}$ *acres for* 4 *weeks.*
$4800-1200=3600$ *lbs.=original quantity of grass on* $3\frac{1}{3}$ *acres.*
$3600\div3\frac{1}{3}=1080$ *lbs.=original quantity on* 1 *acre.*
And by the last condition,
$24\times1080=25920$ *lbs.=original quantity on* 24 *acres.*
$24\times90\times18=38880$ *lbs.=quantity which grows on* 24 *acres in* 18 *weeks.*

25920 + 38880 = 64800 *lbs. = whole quantity on* 24 *acres for* 18 *weeks.*
64800 ÷ 18 = 3600 *lbs. = quantity to be grazed from* 24 *acres each week.*
3600 ÷ 100 = 36 = *number of oxen required to graze the whole.*

61. A tract of land, exactly square, is inclosed by a three-railed fence; the length of each rail is 15 feet, and the number of rails in the fence is equal to the number of acres inclosed; required the area of this tract in acres, and the length of its side in feet.

Operation.—If the tract of land was inclosed by one rail, then, 15 ÷ (4 × 3) = 1.25 feet, the length of its side.

Then, if 43,560 square feet make an acre, as 1.25^2 : 43,560 :: 1 rail : 27878.4, *the number of rails in the fence*, or *the number of acres in the tract;* and (27878.4 × 15) ÷ (4 × 3) = 34,848, *the length of the side in feet.*

62. What is the radius of a circular acre?

Operation.—Side of a square × 1.128 = diameter of an equal circle. By table, p. 13, 208.710321 = the side of a square acre.

Then, 208.710321 × 1.128 = 235.50, which, ÷ 2 (for radius), = 117.75 *feet.*

63. There is an island 20 miles in circumference, and three men start together to travel the same way about it; A goes 2 miles per hour, B 4 miles per hour, and C 6 miles per hour; in what time will they come together again?

Ans. 10 *hours.*

64. A hare starts 12 rods before a hound, but is not perceived by him till she has been off $1\frac{1}{4}$ minutes; she runs at the rate of 36 rods a minute, and the dog, on view of her, makes after her at the rate of 40 rods a minute; how long will the course hold, and what distance will the dog run?

Ans. $14\frac{1}{4}$ *minutes, and he will run* 570 *rods.*

MENSURATION OF SOLIDS.

OF CUBES AND PARALLELOPIPEDONS.

Cube.

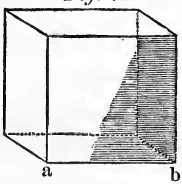
Definition. *A solid contained by six equal square sides.*

To ascertain the Contents of a Cube, Fig. 72

RULE.—Multiply a side of the cube by itself, and that product again by a side, and this last product will give the contents required.

Or, $s^3=S$, *s representing the length of a side, and* S *the contents.*

Fig. 72.



EXAMPLE.—The side *a b* of the cube, *Fig.* 72, is 12 inches; what are the contents of it?

$12 \times 12 \times 12 = 1728$ *inches.*

Ex. 2. The side of a cube is 15 inches; what are its contents in feet and inches?

Ans. 1.953125 *feet*, or 1 *foot* and $11\frac{4375}{10000}$ *inches.*

Ex. 3. The sides of a cube are 12.5 feet; what are its contents in cubic feet and yards?

Ans. $\begin{cases} 1953.125 \textit{ cubic feet.} \\ 72.338 \textit{ cubic yards.} \end{cases}$

Centre of Gravity. Is in its geometrical centre.

Parallelopipedon.

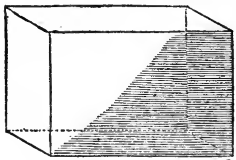
Definition. *A solid contained by six quadrilateral sides, every opposite two of which are equal and parallel.*

To ascertain the Contents of a Parallelopipedon, Fig. 73.

Rule.—Multiply the length by the breadth, and that product again by the depth, and this last product will give the contents required.

Or, $l \times b \times d = S$.

Fig. 73.



Example.—The length *a b*, *Fig.* 73, is 15, the breadth *c d* 12, and the depth *c b* is 11 inches; what are the contents?

$15 \times 12 \times 11 = 1980$ *inches.*

Ex. 2. The length of a parallelopipedon is 15 feet, and each side of it is 21 inches; what are its contents?

Ans. 45.9375 *feet.*

Ex. 3. The dimensions of a parallelopipedon are 20 feet in length, 11.5 in breadth, and 7 in depth; what are its contents in feet? *Ans.* 1610 *feet.*

Centre of Gravity. Is in its geometrical centre.

PRISMS, PRISMOIDS, AND WEDGES.

Prisms.

Definition. *Solids, the ends of which are equal, similar, and parallel planes, and the sides of which are parallelograms.*

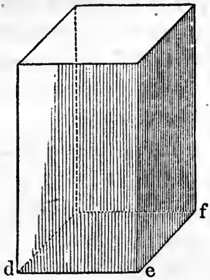
Note.—When the ends of a prism are triangles, it is called a *triangular prism;* when rhomboids, a *rhomboidal prism;* when squares, a *square prism;* when rectangles, a *rectangular prism,* &c.

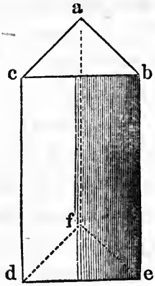
To ascertain the Contents of a Prism, Figs. 74 *and* 75.

RULE.—Multiply the area of the base $d\ e\ f$ by the height, and the product will give the contents required.

Or, $a \times h = S$.

Fig. 74. *Fig.* 75.





EXAMPLE.—A triangular prism, $a\,b\,c\,d\,e\,f$, *Fig.* 75, has sides of 2.5 feet, and a length $c\ d$ of 10 feet; what are its contents?

(By Rule, p. 52) $\frac{2.5^2}{4} = 1.5625$, which, $\times 1.732 = 2.70625 =$ *area of end* $a\ b\ c$, and $2.70625 \times 10 = 27.0625 =$ *feet.*

Ex. 2. A side of the end of a triangular prism is 18 inches, and the length of the prism is 9 feet; what are its contents?

Ans. 8.7682 *feet.*

Ex. 3. What is the solidity of a prism 15 feet in length, the ends of which are hexagonal, with sides 16 inches in length?

(By Rule, p. 60) $16^2 \times 2.5981 = 665.1136 =$ *square of a side multiplied by the tabular number for the area of a hexagon; then,* $665.1136 \times 15 = 9976.704$, *the contents required.*

Ex. 4. The sides of an octagonal prism are 3 feet, and its height 6.75 feet; what are its contents in feet and yards?

Ans. $\begin{cases} 293.3388 \textit{ feet.} \\ 10.8644 \textit{ yards.} \end{cases}$

Centre of Gravity. For rule, see Mensuration of Areas, Lines, and Surfaces, p. 100.

Prismoids.

Definition. *Figures alike to a prism, but having only one pair of their sides parallel.*

NOTE.—Prismoids, alike to prisms, derive their designation from the figure of their ends, as triangular, square, rectangular, pentagonal, &c.

To ascertain the Contents of a Prismoid, Fig. 76.

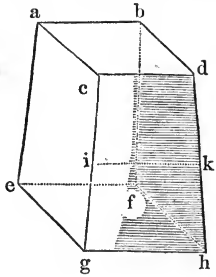
RULE.—To the sum of the areas of the two ends, *a b c d*, *e f g h*, add four times the area of the middle section at *i k* parallel to them; multiply this sum by $\frac{1}{6}$ of the height, and the product will give the contents required.*

Or, $a+a'+4m\times\overline{h\div6}=S$, *a and a′ representing areas of ends, and m area of middle section.*

Or, $(b\times a+4m\times n+d\times c)\times\overline{h\div6}=S$, *a b, c d representing dimensions of ends, and m n of the middle section.*

NOTE.—The length and breadth of the middle section are respectively equal to half the sum of the lengths and breadths of the two ends.

Fig. 76.



EXAMPLE.—What are the contents of a rectangular prismoid, *Fig.* 76, the lengths and breadths of the two ends being 7 and 6, and 3 and 2 inches, and the height 15 feet?

$7\times6+\overline{3\times2}=42+6=48=$ *sum of the areas of the two ends.*

$7+3\div2=10\div2=5=$ *length of the middle section.*

$6+2\div2=8\div2=4=$ *breadth of the middle section.*

$5\times4\times4=80=$ *four times the area of the middle section.*

Then, $48+80\times\frac{15\,ft.}{6}=128\times30=3840$ *cubic inches.*

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* This is a general rule, and applies equally to figures of proportionate or dissimilar ends.

Ex. 2. What is the capacity of a prismoid, the ends of which are respectively 6 by 8 and 9 by 12 inches, and the height of it is 5 feet?* *Ans.* 2.6389 *feet.*

Ex. 3. What are the contents of a prismoid when the ends of it are respectively 40.75 by 27.5 inches and 20.5 by 14.75 inches, and the length of it is 23.625 inches?

Ans. 9.1392 *cubic feet.*

Centre of Gravity. For rule, see Mensuration of Areas, Lines, and Surfaces, p. 101.

Wedge.

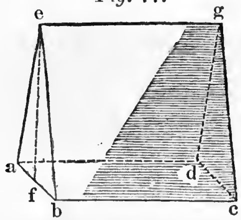
Definition. *A prolate triangular prism.*

To ascertain the Contents of a Wedge, Fig. 77.

Rule.—To the length of the edge, $e\,g$, add twice the length of the back; multiply this sum by the perpendicular height, $e\,f$, and then by the breadth of the back, and $\frac{1}{6}$ of the product will be the solidity required.

Or, $(l+\overline{l' \times 2} \times h \times b) \div 6 = S.$

Fig. 77.



Example.—The back of a wedge, $a\,b$, $d\,c$, is 20 by 2 inches, and its height, $e\,f$, 20 inches; what are its contents?

$20+\overline{20 \times 2}=60=$ *length of the edge added to twice the length of the back.*

$60 \times 20 \times 2=2400=$ *above sum multiplied by the height, and that product by the breadth of the back.*

$2400 \div 6=400=\frac{1}{6}$ *of above product = the contents required.*

---

* An excavation or embankment of a road, when terminated by parallel cross sections, is a rectangular prismoid.

Ex. 2. The back of a wedge is 15 inches by 3 broad, the edge of it 15 inches in length, and the height 30; what are its contents in inches? *Ans.* 675 *inches.*

Ex. 3. The back of a wedge is 64 inches by 9 broad, the length of the edge is 42 inches, and the height is 2 feet 4 inches; how many cubic feet are contained in it?

Ans. 4.1319 *cubic feet.*

Ex. 4. The height of a wedge is 15 inches, the edge 7, and the base 9 by $3\frac{1}{2}$; what are its contents?

Ans. 218.75 *cubic inches.*

Centre of Gravity. For rule, see Mensuration of Areas, Lines, and Surfaces, p. 100.

NOTE.—When a wedge is a true prism, as represented by *Fig.* 77, the contents of it are equal to the area of an end multiplied by its length.

REGULAR BODIES (*Polyhedrons*).

Definition. *A regular body is a solid contained under a certain number of similar and equal plane faces,* all of which are equal regular polygons.*

NOTE 1. The whole number of regular bodies which can possibly be formed is five.

NOTE 2. A sphere may always be inscribed within, and may always be circumscribed about a regular body or polyhedron, which will have a common centre.

1. *The Tetrahedron, or Pyramid, Fig.* 78, *which has four triangular faces.*
2. *The Hexahedron, or Cube, Fig.* 79, *which has six square faces.*
3. *The Octahedron, Fig.* 80, *which has eight triangular faces.*
4. *The Dodecahedron, Fig.* 81, *which has twelve pentagonal faces.*
5. *The Icosahedron, Fig.* 82, *which has twenty triangular faces.*

* The angle of the adjacent faces of a polygon is called the diedral angle.

If the following figures are made of pasteboard, and the lines be so cut that the parts may be turned up and secured together, they will represent the five bodies.

Fig. 78.

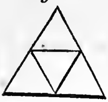


Fig. 79.

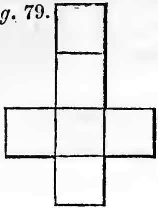


Fig. 80.

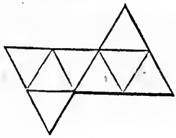


Fig. 81.

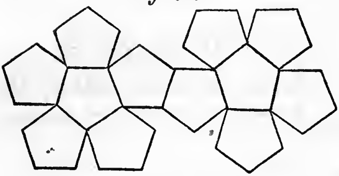
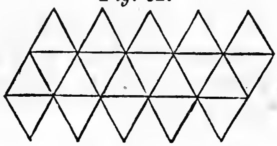


Fig. 82.



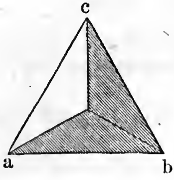
Tetrahedron.

To ascertain the Contents of a Tetrahedron, Fig. 83.

RULE.—Multiply $\frac{1}{12}$ of the cube of the linear side by the square root of 2 (1.414213), and the product will be the contents required.

Or, $\frac{l^3}{12} \times \sqrt{2} = S$, *l representing the length of a side.*

Fig. 83.



EXAMPLE.—The linear side of a tetrahedron, *a b c*, *Fig.* 83, is 4; what are its contents?

$\frac{4^3}{12} \times \sqrt{2} = \frac{64}{12} \times 1.414 = 7.5413 =$ *result required.*

Ex. 2. Required the contents of a tetrahedron, the side of which is 6. *Ans.* 25.452.

Centre of Gravity. Is in the common centre of the centres of gravity of the triangles made by a section through the centre of each side of the figures.

*Hexahedron.**

To ascertain the Contents of a Hexahedron (Cube), see *Fig.* 72, and Rule, p. 155.

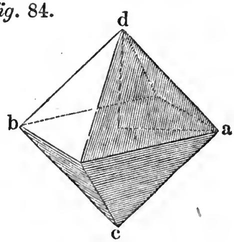
Octahedron.

To ascertain the Contents of an Octahedron, Fig. 84.

RULE.—Multiply $\frac{1}{3}$ of the cube of the linear side by the square root of 2 (1.414213), and the product will be the contents required.

Or, $\frac{l^3}{4} \times \sqrt{2} = S.$

Fig. 84.



EXAMPLE.—What are the contents of the octahedron *a b c d*, *Fig.* 84, the linear side of which is 4?

$\frac{4^3}{3} \times \sqrt{2} = \frac{64}{3} \times 1.414 = 30.1649 =$ *result required.*

Ex. 2. Required the contents of an octahedron, the side of which is 8? *Ans.* 241.3226.

Centre of Gravity. Is in its geometrical centre.

* A hexahedron and a cube are identical figures, being solids having the same number of similar and equal plane faces.

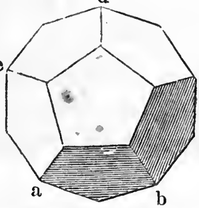
Dodecahedron.

To ascertain the Contents of a Dodecahedron, Fig. 85.

Rule.—To 21 times the square root of 5 add 47, and divide the sum by 40; then, the square root of the quotient being multiplied by 5 times the cube of the linear side, will give the contents required.

Or, $\sqrt{\frac{\sqrt{5}\times21+47}{40}}\times\overline{l^3\times5}=S.$

Fig. 85.



Example.—The linear side of the dodecahedron *a b c d e* is 3; what are its contents?

$\sqrt{\frac{\sqrt{5}\times21+47}{40}}\times27\times5=\sqrt{\frac{2.23606\times21+47}{40}}\times135=206.901=$ *result required.*

Ex. 2. The linear side of a dodecahedron is 1; what is the capacity of it? *Ans.* 7.6631.

Centre of Gravity. Is in its geometrical centre.

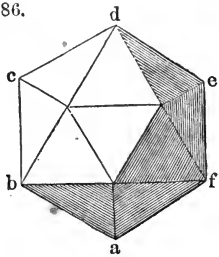
Icosahedron.

To ascertain the Contents of an Icosahedron, Fig. 86.

Rule.—To 3 times the square root of 5 add 7, and divide the sum by 2; then, the square root of this quotient being multiplied by $\frac{5}{6}$ of the cube of the linear side, will give the contents required.

Or, $\sqrt{\frac{\sqrt{5}\times3+7}{2}}\times\frac{l^3\times5}{6}=S.$

Fig. 86.



EXAMPLE.—The linear side of the icosahedron *a b c d e f* is 3; what are its contents?

$$\sqrt{\frac{\sqrt{5}\times 3+7}{2}\times\frac{3^3\times 5}{6}}=\sqrt{\frac{2.23606\times 3+7}{2}\times\frac{27\times 5}{6}}=\sqrt{6.85409\times 22.5}$$
$=58.9056=$ *result required.*

Ex. 2. Required the contents of an icosahedron, the linear side of which is 1? *Ans.* 2.1817.

Centre of Gravity. Is in its geometrical centre.

REGULAR BODIES.

To ascertain the Contents of any regular Solid Body.

When the Linear Edge is given.

RULE.—Multiply the cube of the linear edge by the multiplier in column A in the table on the following page, and the product will be the contents required.

EXAMPLE.—What is the capacity of a hexahedron having sides of 3 inches?

$3^3\times 1=27=$ *tabular volume multiplied by cube of edge = contents required.*

When the radius of the Circumscribing Sphere is given.

RULE.—Cube the radius of the circumscribing sphere, and multiply it by the multiplier opposite to the figure in column B.

EXAMPLE.—The radius of the circumscribing sphere of a hexahedron is 1.732 inches; what is the volume of it?

$1.732^3\times 1.5396=$ *product of cube of radius and tabular multiplier* $=8=$ *result required.*

When the radius of the Inscribed Sphere is given.

RULE.—Cube the radius, and multiply it by the multiplier opposite to the figure in column C.

EXAMPLE.—The radius of the inscribed sphere of a hexahedron is 1 inch; what is its volume?

$1^3 \times 8$=*product of cube of radius and tabular multiplier*=8=*result required.*

No. of sides.	Figures.	A. By linear edge.	B. By radius of circum. sphere.	C. By radius of inscribing sphere.	Angle between two adjacent faces.
4	Tetrahedron	0.11785	.51320	13.85641	70° 31′ 42″
6	Hexahedron	1.	1.53960	8.	90
8	Octahedron	0.47140	1.33333	6.92820	109 28 18
12	Dodecahedron	7.66312	2.78516	5.55029	116 33 54
20	Icosahedron	2.18169	2.53615	5.05406	138 11 23

NOTE.—For further rules to ascertain the elements of polyhedrons, see Appendix, p. 262.

Centre of Gravity. Is in their geometrical centre.

Cylinder.

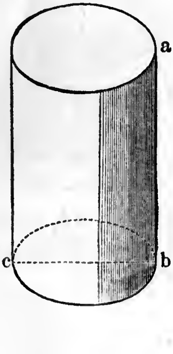
Definition. *A figure formed by the revolution of a right-angled parallelogram around one of its sides.*

To ascertain the Contents of a Cylinder, Fig. 87.

RULE.—Multiply the area of the base by the height, and the product will give the contents required.

Or, $a \times h$=S.

Fig. 87.



Example.—The diameter of a cylinder, $b\ c$, is 3 feet, and its length, $a\ b$, 7 feet; what are its contents?

Area of 3 feet$=7.068$. *Then,* $7.068 \times 7 = 49.476 =$ *result required.*

Ex. 2. What are the contents of a cylinder, the height of which is 5 feet, and the diameter 2 feet?

Ans. 15.708 *feet.*

Ex. 3. The circumference of the base of a cylindrical column is 20.42 feet, and the height of the column is 9.695 feet; what is its volume? *Ans.* 320.515 *feet.*

Centre of Gravity. Is in its geometrical centre.

Cone.

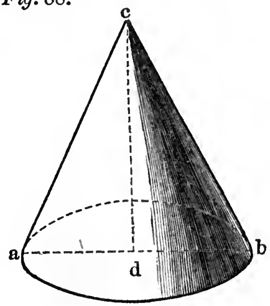
Definition. *A figure described by the revolution of a right-angled triangle about one of its legs.*

To ascertain the Contents of a Cone, Fig. 88.

Rule.—Multiply the area of the base by the perpendicular height, and one third of the product will be the contents required.

Or, $\frac{a \times h}{3} = S$.

Fig. 88.



Example.—The diameter, $a\ b$, of the base of a cone is 15 inches, and the perpendicular height, $c\ d$, 32.5 inches; what are the contents of the cone?

Area of 15 inches=176.7146.

Then, $\frac{176.7146 \times 32.5}{3} = 1914.4082$ *cubic inches.*

Ex. 2. The diameter of the base of a cone is 20, and its height 24 inches; what are its contents in cubic inches?

Ans. 2513.28 *inches.*

Ex. 3. What are the contents of a cone when the diameter of its base is 1.5 feet, and its height 15 feet? *Ans.* 8.8358 *feet.*

Ex. 4. The diameter of the base of a cone is 12.732 feet, and the height of it is 50 feet; what is its volume?

Ans. 2121.9386 *feet.*

Centre of Gravity. It is at a distance from the base $\frac{1}{4}$ of the line joining the vertex and centre of gravity of the base.

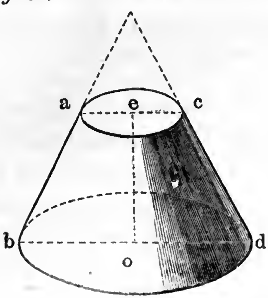
To ascertain the Contents of the Frustrum of a Cone, Fig. 89.

RULE.—Add together the squares of the diameters of the greater and less ends and the product of the two diameters; multiply their sum by .7854, and this product by the height; then divide this last product by three, and the quotient will give the contents required.

Or, *add together the squares of the circumferences of the greater and less ends and the product of the two circumferences; multiply their sum by* .07958, *and this product by the height; then, divide this last product by three, and the quotient will give the contents required.*

Or, $d^2 + d'^2 + \overline{d \times d'} \times \frac{.7854 \times h}{3} = \text{S}.$

Fig. 89.



EXAMPLE.—What are the contents of the frustrum of a cone, the diameters of the greater and less ends, $b\,d$, $a\,c$, being respectively 5 and 3 feet, and the perpendicular height, $e\,o$, 9 feet?

$5^2+3^2+\overline{5\times 3}=49=$ *the sum of the squares of the diameters and the product of the diameters.*

$49\times .7854=38.4846=$ *the above sum by .7854.*

$\frac{38.4846\times 9}{3}=115.4538=$ *the last product* $\times$ *the height and divided by three, which is the result required.*

Ex. 2. What are the contents of the frustrum of a cone, the diameters of the ends being respectively 2 and 4 feet, and the height 9 feet? *Ans.* 65.9736 *feet.*

Ex. 3. The frustrum of a cone is 12 inches in height, and has diameters of 7 and $9\frac{1}{2}$ inches; what are the contents of it? *Ans.* 646.3842 *inches.*

Centre of Gravity. It is at a distance from the base $\frac{1}{4}$ of the smaller end $=\frac{1}{4}$ *height* $\times\frac{(R+r)^2+2R^2}{(R+r)^2-Rr}$; R *and* r *radii of the greater and less ends.*

Pyramid.

Definition. *A figure, the base of which has three or more sides, and the sides of which are plane triangles.*

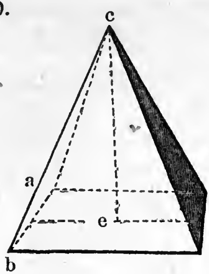
NOTE.—The volume of a pyramid is equal to one third of that of a prism having equal bases and altitude.

To ascertain the Contents of a Pyramid, Fig. 90.

RULE.—Multiply the area of the base by the perpendicular height, and one third of the product will be the contents required.

Or, $\frac{a\times h}{3}=S.$

Fig. 90.



EXAMPLE.—What are the contents of a hexagonal pyramid, *a b c*, *Fig.* 90, a side, *a b*, being 40 feet, and its height, *c e*, 60 feet?

$40^2 \times 2.5981$ (tabular multiplier, p. 60)$=4156.96=$*area of base.*

$\frac{4156.96 \times 60}{3}=83139.2=$*one third of the area of the base* $\times$ *the height* $=$ *the contents required.*

Ex. 2. The height of a quadrangular pyramid is 67 feet, and the width of its base is 16.5 feet; what are its contents in cubic feet? *Ans.* 6080.25.

Ex. 3. What are the contents of a pentagonal pyramid, its height being 12 feet, and each of its sides 2 feet?

Ans. 27.528 *feet.*

Centre of Gravity. It is at a distance from the base $\frac{1}{4}$ of the line joining the vertex and centre of gravity of the base.

To ascertain the Contents of the Frustrum of a Pyramid, Fig. 91.

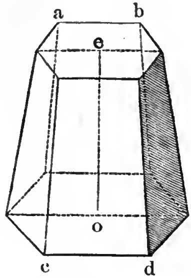
RULE.—Add together the squares of the sides of the greater and less ends and the product of these two sides; multiply the sum by the tabular multiplier for areas in Table, p. 60, and this product by the height; then, divide the last product by three, and the quotient will give the contents required.

Or, $(s^2+s'^2+\overline{s \times s'}) \times$ *tab. mult.* $\times \frac{h}{3}=$S, *s and s′ representing the lengths of the sides.*

NOTE.—When the areas of the ends are known, or can be obtained without reference to a tabular multiplier, use the following.

$\overline{a+a'+\sqrt{a\times a'}}\times\frac{h}{3}=S$, *a and a′ representing areas of the ends.*

Fig. 91.



EXAMPLE.—What are the contents of the frustrum of a hexagonal pyramid, *Fig.* 91, the lengths of the sides of the greater and less ends, *a b*, *c d*, being respectively 3.75 and 2.5 feet, and its perpendicular height, *e o*, 7.5 feet?

$3.75^2+2.5^2=20.3125$ = *sum of the squares of sides of greater and less ends.*

$20.3125+\overline{3.75\times 2.5}=29.6875$ = *above sum added to the product of the two sides.*

$29.6875\times 2.5981\times 7.5=578.48$ = *the last sum × tab. mult., and again by the height, which,* $\div 3=192.82$ *feet.*

Ex. 2. The frustrum of a hexagonal pyramid has sides of 4 and 3 feet, and a height of 9 feet; what are its contents?

Ans. 294.3891 *feet.*

When the Ends of a Pyramid are not those of a Regular Polygon, or when the Areas of the Ends are given.

RULE.—Add together the areas of the two ends and the square root of their product; multiply the sum by the height, and one third of the product will be the contents.

Or, $\overline{a+a'+\sqrt{a\times a'}}\times\frac{h}{3}=S.$

EXAMPLE.—What are the contents of an irregular-sided frustrum of a pyramid, the areas of the two ends being 22 and 88 inches, and the length 20 inches?

$22+88=110=$ *sum of areas of ends.*
$22\times 88=1936$, and $\sqrt{1936}=44=$ *square root of product of areas.*
$\frac{110+44\times 20}{3}=1026.66=$ *one third of sum of above sum and product* $\times$ *the height = feet.*

Ex. 2. The areas of the ends of an irregular-sided frustrum of a pyramid are 81 and 100 inches, and the length 25 inches; what are its contents? *Ans.* 2258.33 *inches.*

Centre of Gravity. It is at a distance from the centre of the smaller end $=\frac{1}{4}$ *height* $\times\frac{(R+r)^2+2R^2}{(R+r)^2-Rr}$; R *and* r *radii of the greater and less ends.*

Spherical Pyramid.

A Spherical Pyramid is that part of a sphere included within three or more adjoining plane surfaces meeting at the centre of the sphere. The spherical polygon defined by these plane surfaces of the pyramid is called the base, and the lateral faces are sectors of circles.

To ascertain the Elements of Spherical Pyramids, see Docharty and Hackley's Geometry.

Cylindrical Ungulas.

Definition. *Cylindrical ungulas are frustrums of cylinders. Conical ungulas are frustrums of cones.**

To ascertain the Contents of a Cylindrical Ungula, Fig. 92.

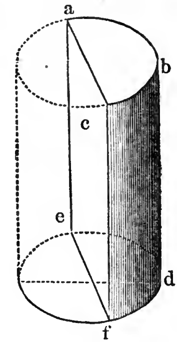
1. *When the section is parallel to the axis of the cylinder.*

RULE—Multiply the area of the base by the height of the cylinder, and the product will be the contents required.

Or, $a\times h=S$.

* For Mensuration of Conical Ungulas, see Conic Sections, p. 253.

Fig. 92.



EXAMPLE.—The area of the base, *d e f*, *Fig.* 92, of a cylindrical ungula is 15.5 ins., and its height 20; what are its contents?

$15.5 \times 20 = 310 =$ *product of area and height* $=$ *result required.*

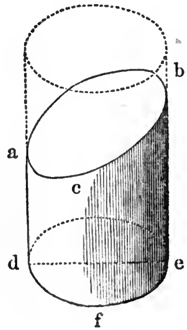
Ex. 2. The area of the base of a cylindrical ungula is 168.25 inches, and the height of it 22; what are its contents in cubic feet? *Ans.* 2.148 *cubic feet.*

2. *When the section passes obliquely through the opposite sides of the cylinder*, *Fig.* 93.

RULE.—Multiply the area of the base of the cylinder by half the sum of the greatest and least lengths of the ungula, and the product will be the contents required.

Or, $a \times \overline{l + l'} \div 2 = S$.

Fig. 93.



Example.—The area of the base *d e f* of a cylindrical ungula is 25 inches, and the greater and less heights of it, *a d*, *b e*, are 15 and 17 inches; what are its contents?

$25 \times \frac{15+17}{2} = 400$ = *product of half the sum of the heights and the area of the base = result required.*

Ex. 2. The area of the base of a cylindrical ungula is 75.8 inches, and the greater and less heights of it are 4.25 and 5.65 feet; what are its contents in cubic feet?

Ans. 2.6056 *cubic feet.*

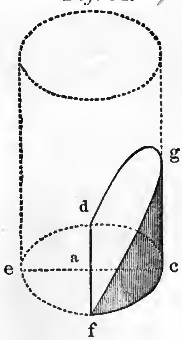
3. *When the section passes through the base of the cylinder and one of its sides, and the versed sine does not exceed the sine, Fig.* 94.

Rule.—From two thirds of the cube of the sine, *a d*, of the arc, *d c f*, of the base, subtract the product of the area of the base and the cosine,* *a e*, of the half arc.

Multiply the difference thus found by the quotient arising from the height divided by the versed sine, and the product will give the contents required.

Or, $\frac{s^3 2}{3} - \overline{a \times c} \times \frac{h}{v\,s} = S$, *v s representing the versed sine.*

Fig. 94.



Example.—The sine *a d* of half the arc of the base of an ungula, *Fig.* 94, is 5, the diameter of the cylinder is 10, and the height of the ungula 10; what are the contents of it?

* When the cosine is 0, the product is 0.

$\frac{2}{3}$ of $5^3=83.333$=*two thirds of the cube of the sine.*
As the versed sine and radius of the base are equal, the cosine is 0.
Hence, area of base $\times$ cosine$=0$.
$83.333-0\times\frac{10}{5}=166.666$=*difference of* $\frac{2}{3}$ *of cube of the sine and the product of area of base and the cosine,* $\times$ *the height* $\div$ *the versed sine=the contents required.*

Ex. 2. The sine of half the arc of the base of an ungula is 12 inches, the diameter of the cylinder is 25, and the height of the ungula 18; what are its contents?

Ans. 1190.34375 *inches.*

For rules to ascertain the area of base, see pp. 85–87.

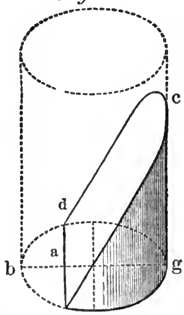
4. *When the section passes through the base of the cylinder, and the versed sine exceeds the sine, Fig.* 95.

RULE.—To two thirds of the cube of the sine of half the arc of the base, add the product of the area of the base and the excess of the versed sine over the sine of the base.

Multiply the sum thus found by the quotient arising from the height divided by the versed sine, and the product will be the contents required.

Or, $\overline{\frac{2\,s^3}{3}+a\times(v\,s\backsim s)}\times\frac{h}{v\,s}=S.$

Fig. 95.



EXAMPLE.—The sine $a\,d$ of half the arc of an ungula, *Fig.* 95, is 12 inches, the versed sine $a\,g$ is 16, the height $c\,g$ 20, and the diameter of the cylinder 25 inches; what are the contents?

$\frac{2}{3}$ of $12^3 = 1152 =$ *two thirds of cube of sine of the arc of the base.*
Area of base (see Rules, p. 84–134) $= 331.78$.

$1152 + (331.78 \times \overline{16 - 12.5}) = 2313.23 =$ *sum of* $\frac{2}{3}$ *of the cube of the sine of the base, and product of area of base, and difference between the versed sine and sine of the base.*

$2313.23 \times \overline{20 \div 16} = 2891.5375 =$ *product of above sum and the height, divided by the versed sine = result required.*

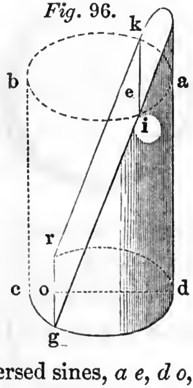
5. *When the section passes obliquely through both ends of the cylinder, Fig.* 96.

RULE.—Conceive the section to be continued till it meets the side of the cylinder produced; then, as the difference of the versed sines of the arcs of the two ends of the ungula is to the versed sine of the arc of the less end, so is the height of the cylinder to the part of the side produced.

Find the contents of each of the ungulas by rules 3 and 4, and their difference will be the contents required.

Or, $\frac{v' \times h}{v - v'} = h'$, *v and v′ representing the versed sines of the arcs of the two ends, h the height of the cylinder, and h′ the height of the part produced.*

Fig. 96.



EXAMPLE.—The versed sines, *a e*, *d o*, and sines, *i k*, *g r*, of the arcs of the two ends of an ungula, *Fig.* 96, are assumed to be respectively 8.5 and 25, and 11.5 and 0 inches, the length of the ungula within the cylinder, cut from one having 25 inches diameter is 20 inches; what is the height of the ungula produced beyond the cylinder, and what the contents of the ungula?

25∾8.5 : 8.5 : : 20 : 10.303 = *height of ungula produced beyond the cylinder.*
Lower ungula, the sine, *g r*, being 0, the versed sine = the diameter.
Base of ungula being a circle of 25 inches diameter, area = 490.874.
The versed sine and diameter of the base being equal (25), the sine is 0.
490.874 × 25 = 6135.925 = *product of area of base and excess of versed sine over the sine of the base.*
30.303 ÷ 25 = 1.2121 = *quotient of height ÷ versed sine.*
Then, 6135.925 × 1.2121 = 7437.3547 = *product of above product and quotient = the result required.*

SPHERE.

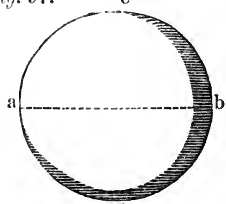
Definition. *A solid, the surface of which is at a uniform distance from the centre.*

To ascertain the Contents of a Sphere, Fig. 97.

RULE.—Multiply the cube of the diameter by .5236, and the product will be the contents required.

Or, $d^3 \times .5236 = S$, *d representing the diameter.*

Fig. 97.



EXAMPLE.—What are the contents of a sphere, *Fig.* 97, its diameter, *a b*, being 10 inches?

$10^3 = 1000$, and $1000 \times .5236 = 523.6$ *cubic inches.*

Ex. 2. The diameter of a sphere is 17 inches; what are its contents? *Ans.* 1.4887 *cubic feet.*

Ex. 3. What are the contents of a globe 10.5 feet in diameter? *Ans.* 606.132 *cubic feet.*

Centre of Gravity. Is in its geometrical centre.

NOTE.—$.5236 = \frac{1}{6}$ of 3.1416.

Segment of a Sphere.

Definition. *A section of a sphere.*

To ascertain the Contents of a Segment of a Sphere, Fig. 98.

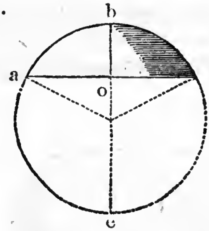
RULE 1. To three times the square of the radius of its base add the square of its height; multiply this sum by the height, and the product, multiplied by .5236, will give the contents required.

Or, $(3r^2+h^2)\times h\times .5236=S$.

2. From three times the diameter of the sphere subtract twice the height of the segment; multiply this remainder by the square of the height, and the product, multiplied by .5236, will give the contents required.

Or, $3\,d-2\,h\times h^2\times .5236=S$.

Fig. 98.



EXAMPLE.—The segment of a sphere, *Fig.* 98, has a radius, *a o*, of 7 inches for its base, and a height, *b o*, of 4 inches; what are its contents?

$7^2\times 3+4^2=163=$*the sum of three times the square of the radius and the square of the height.*

$163\times 4\times .5236=341.3872=$*the above sum multiplied by the height, and by* .5236=*inches.*

Ex. 2. The radius of a spherical segment is 48 inches, and the height 12 inches; what are its contents?

Ans. 44334.2592 *cubic inches.*

Ex. 3. The height of a spherical segment is 2 inches, and the diameter of the sphere 6 inches; what are its contents?

Ans. 29.322 *cubic inches.*

Centre of Gravity. Distance from the centre, $3.1416\, v^2\left(r-\frac{v}{2}\right)^2$ $\div s$, *v being the versed sine, s the contents of the segment, and r the radius of the sphere.*

Distance from the vertex $\left(\frac{8r-3h}{12r-4h}\right)h$, *$h$ representing the height or versed sine of the segment.*

Of a Hemisphere. Distance from the centre $\frac{3}{8}r$.

Spherical Zone (or Frustrum of a Sphere).

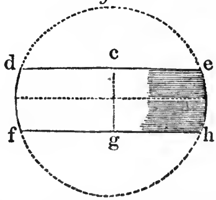
Definition. *The part of a sphere included between two parallel chords.*

To ascertain the Contents of a Spherical Zone, Fig. 99.

Rule.—To the sum of the squares of the radii $d\ c$ and $f\ g$ of the two ends, $d\ f$, $e\ h$, add one third of the square of the height, $c\ g$, of the zone; multiply this sum by the height, and again by 1.5708, and it will give the contents required.

Or, $r^2+r'^2+\frac{h^2}{3}\times h\times 1.5708=\mathrm{S}$.

Fig. 99.



Example.—What are the contents of a spherical zone, *Fig.* 99, the greater and less diameters, $f\ h$ and $d\ e$, being 20 and 15 inches, and the distance between them, or height of the zone, being 10 inches.

$10^2+7.5^2=156.25$=*sum of the squares of the radii of the two ends.*

$156.25+\frac{10^2}{3}=189.58$=*the above sum added to one third of the square of the height.*

$189.58\times 10\times 1.5708=2977.9226$=*the last sum multiplied by the height and again by* 1.5708=*feet.*

Ex. 2. A zone of a sphere has the radii of its ends each 6 inches, and its height is 8 inches; what are its contents?

Ans. 1172.86 *cubic inches.*

Ex. 3. What are the contents of the zone of a sphere, the radii of its ends being 10 and 12 inches, and the height of it 4 inches?

Ans. 1566.6112 *cubic inches.*

Centre of Gravity. Right Zone. Is in its geometrical centre.

Of a Frustrum. $\frac{8r-3h}{13r-4h}=d$, *representing the distance from the vertex of the frustrum.*

Spheroids (Ellipsoids).

Definition. *Solids generated by the revolution of a semi-ellipse about one of its diameters.*

When the revolution is about the transverse diameter, they are Prolate, and when about the conjugate, they are Oblate.

To ascertain the Contents of a Spheroid, Figs. 100 and 101.

RULE.—Multiply the square of the revolving axis, $c\ d$, by the fixed axis, $a\ b$, and this product by .5236, and it will give the contents required.

Or $a^2 \times a' \times .5236 = S$, *a and a′ representing the axes.*

Or, $\frac{4}{3}\ 3.1416 \times r^2 \times r' = S$, *r and r′ representing the semi-axes.*

NOTE.—The contents of a spheroid are equal to two thirds of a cylinder that will circumscribe it.

Fig. 100.

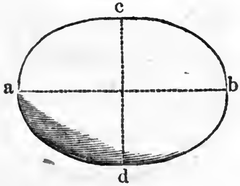
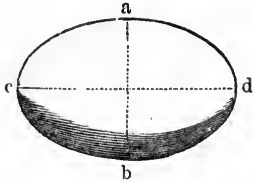


Fig. 101.



EXAMPLE.—In a prolate spheroid, *Fig.* 100, the fixed axis $a\ b$ is 14, and the revolving axis $c\ d$ 10; what are its contents?

$10^2 \times 14 = 1400 =$ *product of square of revolving axis and fixed axis.*

$1400 \times .5236 = 733.04 =$ *above product by .5236 = result required.*

Ex. 2. The axes of a prolate spheroid are 100 and 60 inches; what are its contents? *Ans.* 188496 *inches.*

Ex. 3. The axes of an oblate spheroid, *Fig.* 101, are 10 and 14 inches; what are its contents? *Ans.* 1026.256 *inches.*

Ex. 4. What are the contents of an oblate spheroid, its transverse axis, *c d*, being 24, and its conjugate, *a b*, 18 inches? *Ans.* 5428.685 *inches.*

Ex. 5. What are the contents of an oblate spheroid, the axes of which are 50 and 30 inches? *Ans.* 22.7257 *cubic feet.*

Centre of Gravity. Is in their geometrical centre.

Segments of Spheroids.

To ascertain the Contents of the Segment of a Spheroid.

When the base, e f, is Circular, or parallel to the revolving axis, as c d, Figs. 102 *and* 102*.

RULE.—Multiply the fixed axis, *a b*, by three, the height of the segment, *a o*, by two, and subtract the one product from the other; multiply the remainder by the square of the height of the segment, and the product by .5236.

Then, as the square of the fixed axis is to the square of the revolving axis, so is the last product to the contents of the segment.

Or, $\frac{(3\,a-\overline{2\,h}\times h^2\times .5236)\times a'^2}{a^2}=\mathrm{S}$, *a and a′ representing the fixed and revolving axes.*

Fig. 102.

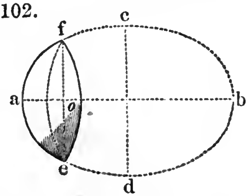
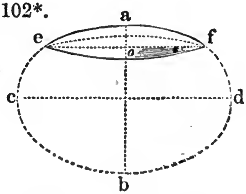


Fig. 102*.



EXAMPLE.—In a prolate spheroid, *Fig.* 102, the fixed or transverse axis, *a b*, is 100, and the revolving or conjugate, *c d*, 60; what are the contents of a segment of it, its height, *a o*, being 10 inches?

$100 \times 3 - \overline{10 \times 2} = 280$ = *twice the height of the segment subtracted from three times the fixed axis.*

$280 \times 10^2 \times .5236 = 14660.8$ = *product of above remainder, the square of the height, and* .5236.

Then, $100^2 : 60^2 :: 14660.8 : 5277.888$ = *the result required.*

Ex. 2. The height of a segment of a prolate spheroid, *Fig.* 102, is 5 inches; what are its contents, the transverse axis being 4 feet 2 inches, and the conjugate 2.5 feet?

Ans. 659.736 *cubic inches.*

Ex. 3. The height of a segment of an oblate spheroid, *Fig.* 102*, is 10 inches, the transverse diameter being 100, and the conjugate 60; what are its contents?

Ans. 23271.111 *cubic inches.*

When the base, e f, is Elliptical, or perpendicular to the revolving axis, as c d, Figs. 103 *and* 103*.

Rule.—Multiply the revolving axis, *c d*, by three, the height of the segment, *c o*, by two, and subtract the one from the other; multiply the remainder by the square of the height of the segment, and the product by .5236.

Then, as the revolving axis is to the fixed axis, so is the last product to the contents.

Or, $\frac{(3\,a' - \overline{2\,h} \times h^2 \times .5236) \times a}{a'} = S$; *a representing the fixed and a′ the revolving axes.*

Fig. 103.

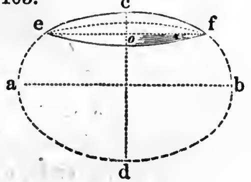
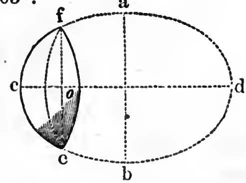


Fig. 103*.



Example.—The diameters, *c d* and *a b*, of an oblate spheroid, *Fig.* 103*, are 100 and 60 inches, and the height of a segment, *c o*, thereof is 12 inches; what are its contents?

$100 \times 3 - \overline{12 \times 2} = 276$ = *twice the height of the segment subtracted from three times the revolving axis.*

$276 \times 12^2 \times .5236 = 20809.9584$ = *product of above remainder, the square of the height, and* .5236.

Then, $100 : 60 :: 20809.9584 : 12485.975$ = *the result required.*

Ex. 2. The segment of a prolate spheroid, *Fig.* 103, is 20 inches in height, the revolving diameter of the spheroid being 10 feet, and the fixed axis 16 feet 8 inches; what are its contents in cubic inches. *Ans.* 111701.333 *inches.*

Ex. 3. The segment of an oblate spheroid, *Fig.* 103*, is 2 feet, and the axes of the spheroid are 200 and 120 inches; what are its contents in cubic feet? *Ans.* 57.8054 *feet.*

Frustra of Spheroids.

To ascertain the Contents of the Middle Frustrum of a Spheroid.

When the ends, e f and g h, are circular, or parallel to the revolving axis, as c d, Figs. 104 *and* 104*.

Rule.—To twice the square of the revolving axis, *c d*, add the square of the diameter of either end, *e f* or *g h*; multiply this sum by the length, *i o*, of the frustrum, and the product again by .2618, and it will give the contents required.

Or, $2\,a'^2 + d^2 \times l \times .2618 = S$.

Fig. 104.

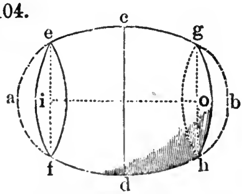
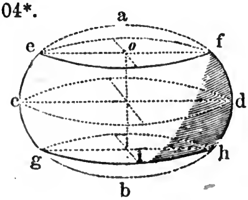


Fig. 104*.



Example.—The middle frustrum, *i o*, of a prolate spheroid, *Fig.* 104, is 36 inches in length, the diameters of it being, in the middle, *c d*, 50 inches, and at its ends, *e f* and *g h*, 40; what are its contents?

$50^2 \times 2 + 40^2 = 6600$ = *sum of twice the square of the middle diameter added to the square of the diameter of the ends.*

$6600 \times 36 \times .2618 = 62203.68$ = *product of the above sum, the length of the frustrum, and* .2618 = *the result required.*

Ex. 2. What are the contents of the middle frustrum of an oblate spheroid, *Fig.* 104*, the transverse diameter being 100 inches, the diameters of the ends of the frustrum each 80 inches, and the length of it 3 feet 4 inches?

Ans. 276460.8 *cubic inches.*

Ex. 3. The middle frustrum of a prolate spheroid, *Fig.* 104, is 80 inches in length, the diameter of it being, in the middle, 60 inches, and at its ends 38.5 inches; what are its contents in cubic feet? *Ans.* 105.232 *cubic feet.*

Ex. 4. The diameter of the middle frustrum of a prolate spheroid, *Fig.* 104, is 100 inches, that of the end of the frustrum is 60 inches, and the length of it is 8 feet; what are its contents? *Ans.* 593134.08 *cubic inches.*

When the ends, e f and g h, are elliptical, or perpendicular to the revolving axis, as c d, Figs. 105 *and* 105*.

Rule.—To twice the product of the transverse and conjugate diameters of the middle section, *a b*, add the product of the transverse and conjugate of either end, *e f* or *g h;* multiply this sum by the length, *o i*, of the frustrum, and the product by .2618, and it will give the contents required.

Or, $\overline{a'\times a}\times 2+\overline{t\times c'}\times l\times .2618=S$.

Fig. 105.

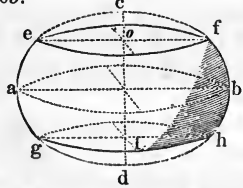
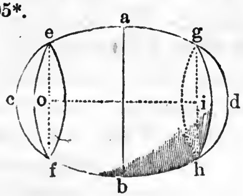


Fig. 105*.



Example.—In the middle frustrum of a prolate spheroid, *Fig.* 105, the diameters of its middle section are 50 and 30 inches, and of its ends 40 and 24 inches; what are its contents, its length being 18 inches?

$50\times 30\times 2=3000=$ *twice the product of the transverse and conjugate diameters.*

$3000+\overline{40\times 24}=3960=$ *sum of the above product and the product of the transverse and conjugate diameters of the ends.*

$3960\times 18\times .2618=18661.104=$ *the preceding product into the length* $.2618=$ *the contents required.*

Ex. 2. The diameters of the middle frustrum of an oblate spheroid, *Fig.* 105,* are 100 and 60 inches in the middle, and 60 and 40 inches at the ends, and the length of it is 6 feet 8 inches; what are its contents? *Ans.* 301593.6 *cubic inches.*

Ex. 3. The middle frustrum of a prolate spheroid is 3 feet 4 inches in height, and has diameters of 8 feet 4 inches and 5 feet in its middle, and of 6 feet 8 inches and 4 feet at its ends; what are its contents in cubic feet?

Ans. 95.993 *cubic feet.*

Centres of Gravity. Frustrum. See Appendix, p. 282.

Middle Frustra, or Zones. Is in their geometrical centre.

Semi-spheroids.—Distance from the centre of the spheroid.

Prolate. $\frac{3}{8}a$. *Oblate.* $\frac{3}{8}b$.

Segments.—Prolate. $\frac{3}{4}\frac{(a+d)^2}{2a+d}$. *Oblate.* $\frac{3}{4}\frac{(b+d')^2}{2b+d'}$, *b representing the semi-conjugate axis, a the semi-transverse, and d, d′ the distances of the base of the segments from the centre of the spheroid.*

Cylindrical Ring.

Definition. *A ring formed by the curvature of a cylinder.*

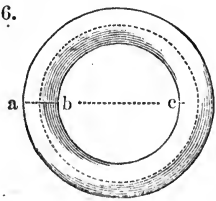
To ascertain the Contents of a Cylindrical Ring, Fig. 106.

Rule.—To the diameter of the body of the ring, *a b*, add the inner diameter of the ring, *b c*; multiply the sum by the square of the diameter of the body, the product by 2.4674, and it will give the contents required.

Or, $d+d' \times d^2 \times 2.4674 = S$, *d and d′ representing the diameter of the body and inner diameter.*

Or, $a \times l = S$, *a representing area of section of body, and l the length of the axis of the body.*

Fig. 106.



EXAMPLE.—What are the contents of an anchor-ring, the diameter of the metal being 3 inches, and the inner diameter of the ring 8 inches?

$3+8\times 3^2=99$=*product of sum of diameters and the square of diameter of body of ring.*

$99\times 2.4674=244.2726$=*above product* $\times 2.4674$=*the contents required.*

Ex. 2. The diameter of the body of a cylindrical ring is 2 inches, and the inner diameter of the ring is 12 inches; what are its contents? *Ans.* 138.1744 *cubic inches.*

Ex. 3. The dimensions of a cylindrical ring are 14 inches in diameter of body and 16 inches in the ring; what are its contents? *Ans.* 14508.312 *cubic inches.*

Centre of Gravity. Is in its geometrical centre.

LINKS.

Definition. *Elongated or elliptical rings.*

Elongated or Elliptical Links.

To ascertain the Contents of an Elongated or Elliptical Link, Figs. 107 and 108.

RULE.—Multiply the area of a section of the body, $a\ b$, of the link by its length, or the circumference of its axis, and the product will give the contents required.

Or, $a\times l=S$, *a representing area of section of body, and l the length of the axis of the body.*

NOTE.—By rule, p. 111, the circumference or length of the axis of an elongated link=the sum, of 3.1416 times the sum of the less diameter added to the thickness of the ring, and the product of twice the remainder of the less diameter subtracted from the greater.

NOTE.—By rule, p. 111, the circumference or length of the axis of an elliptical ring=the square root, of half the sum of the diameters squared $\times 3.1416$.

Fig. 107.

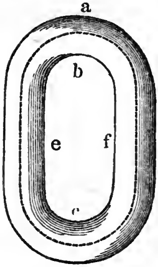
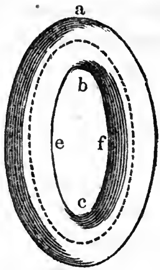


Fig. 108



EXAMPLE.—The elongated link of a chain, *Fig.* 107, is 1 inch in diameter of body, and its inner diameters, *b c* and *e f*, are 10 and 2.5 inches; what are its contents?

Area of 1 inch $=.7854$.

$2.5+1\times3.1416=10.9956=3.1416$ *times the sum of the less diameter and thickness of the ring = length of axis of ends.*

$10-2.5\times2=15=$ *twice the remainder, of the less diameter subtracted from the greater = length of sides of body.*

Then, $10.9956+15=25.9956=$ *length of axis of link.*

Hence, $.7854\times25.9956=20.417=$ *area of link × its length = result required.*

Ex. 2. The elongated link of a chain cable is 1.5 inches in diameter of body, and its inner diameters are 3.5 and 4.5 inches; what are its contents? *Ans.* 38.8584 *inches.*

Ex. 3. The elliptical link of a chain, *Fig.* 108, is 1 inch in diameter, *a b*, of body, and its inner diameters, *b c* and *e f*, are 10 and 2.5 inches; what are its contents?

$\overline{2.5+1}^2+\overline{10+1}^2=133.25=$ *diameters of axes squared.*

$\sqrt{\dfrac{133.25}{2}}\times3.1416=25.643=$ *square root of diameters squared* $\times3.1416$ $=$ *circumference of axis of ring.*

Area of 1 inch $=.7854$.

Then, $25.643\times.7854=20.14=$ *result required.*

Ex. 4. An elliptical link has diameters of 11.5 and 5.5 inches, its diameter of body being 2.5 inches; what are its contents? *Ans.* 185.8272 *inches.*

Centres of Gravity. Is in their geometrical centres.

Spherical Sector.

Definition. *A figure generated by the revolution of a sector of a circle about a straight line drawn through the vertex of the sector as an axis.*

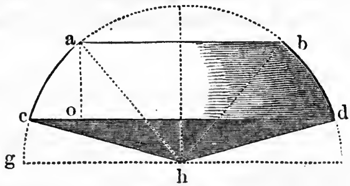
NOTE.—The arc of the sector generates the surface of a zone termed the base of the sector of a sphere, and the radii generates the surfaces of two cones, having a vertex in common with the sector at the centre of the sphere.

To ascertain the Contents of a Spherical Sector, Fig. 109.

RULE.—Multiply the surface of the zone, which is the base of the sector, by one third of the radius of the sphere, and the product will give the contents required.

Or, $a \times \frac{r}{3} = S$, *a representing the area of the base.*

Fig. 109.



EXAMPLE.—What are the contents of a spherical sector, *Fig.* 109, generated by the sector *c a h*, the height of the zone *a b c d* being, *a o*, 12 inches, and the radius, *g h*, of the sphere 15 inches?

$12 \times 94.248 = 1130.976 =$ *height of zone* $\times$ *circumference of sphere* $=$ *surface of zone* (see p. 90).

$1130.976 \times \frac{30}{6} = 5654.88 =$ *product of surface of zone and* $\frac{1}{6}$ *of diameter* ($= \frac{1}{3}$ *of radius*) $=$ *result required.*

Ex. 2. The diameter of a sphere is 10 inches, and the height of the zone or base of a sector is 5 inches; what are its contents? *Ans.* 261.5 *inches.*

Centre of Gravity. Distance from the centre $= \frac{3}{4}(r - \frac{1}{2}h)$.

Distance from the vertex $= \dfrac{2r + 3h}{8}$, *h = the height of the zone.*

NOTE.—The surface of a spherical sector=the sum of the areas of the zone and the two cones.

SPINDLES.

Definition. *Figures generated by the revolution of a plane area bounded by a curve, when the curve is revolved about a chord perpendicular to its axis, or about its double ordinate, and they are designated by the name of the arc from which they are generated, as Circular, Elliptic, Parabolic, etc.*

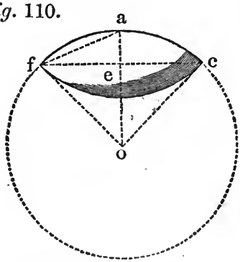
Circular Spindle.

To ascertain the Contents of a Circular Spindle, Fig. 110.

RULE.—Multiply the central distance, *o e*, by half the area of the revolving segment, *a c e f*. Subtract the product from one third of the cube of half the length, *f e*; then multiply the remainder by 12.5664, and the product will give the contents required.

Or, $c\times\frac{a}{2}-\frac{(l\div2)^3}{3}\times12.5664=S$, *l representing the length of the spindle, and a the area of the revolving segment.*

Fig. 110.



EXAMPLE.—What are the contents of a circular spindle, when the central distance, *o e*, *Fig.* 110, is 7.071067 inches, the length, *f c*, 14.14213, and the radius, *o c*, 10 inches?

NOTE.—The area of the revolving segment, *f c*, being=the side of the square that can be inscribed in a circle of 20, is $\overline{20^2\times.7854-14.14213^2}\div4=28.54$.

$7.071067 \times 14.27 = 100.9041 =$ *central distance* $\times$ *half area of revolving segment.*

$100.9041 - \frac{7.07167^3}{3} = 16.947 =$ *remainder of above product and* $\frac{1}{3}$ *of cube of half the length.*

$16.947 \times 12.5664 = 212.9628 =$ *result required.*

Ex. 2. The central distance of a circular spindle is 3 inches, the area of the revolving segment is 11.5 inches, and the length of the spindle 8 inches; what are the contents of it?

Ans. 51.3086 *inches.*

Ex. 3. The length of a circular spindle is 24 inches, and its diameter 18; what are the contents of it?

Ans. 3739.585 *inches.*

The chord of the arc, 24, and the versed sine ($\frac{18}{2}$), 9, being given, the diameter of the circle is found by rules p. 75, 76.

$\sqrt{\left(\frac{24}{2}\right)^2 + 9^2} = 15 =$ *chord of half the arc.*

Hence $\frac{15^2}{9} = 25 =$ *diameter.*

Area of revolving segments, per table, p. 136, 137=159.094.

Centre of Gravity. Is in its geometrical centre.

*Frustrum or Zone of a Circular Spindle.**

To ascertain the Contents of a Frustrum or Zone of a Circular Spindle, Fig. 111.

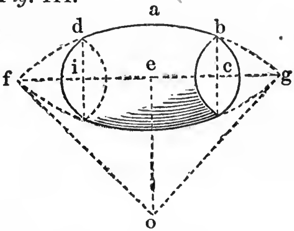
Rule.—From the square of half the length, $f\,g$, of the whole spindle, take $\frac{1}{3}$ of the square of half the length, $i\,c$, of the frustrum, and multiply the remainder by the said half length of the frustrum; multiply the central distance, $o\,c$, by the revolving area which generates the frustrum; subtract this product from the former, and the remainder, multiplied by 6.2832, will give the contents required.

Note.—The revolving area of the frustrum can be obtained by dividing its plane into a segment of a circle and a parallelogram.

* The middle frustrum of a circular spindle is one of the various forms of casks.

Or, $\overline{l\div2}^{\,2}-\frac{\overline{l'\div2}^{\,2}}{3}\times\frac{l}{2}-(c\times a)\times 6.2832=S$, *$l$ and l' representing the lengths of the spindle and of the frustrum, a area of the revolving section of frustrum, and c the central distance.*

Fig. 111.



EXAMPLE.—The length of the middle frustrum of a circular spindle, $i\ c$, is 6 inches, the length of the spindle, $f\ g$, is 8 inches, the central distance, $o\ e$, is 3 inches, and the area of the revolving or generating segment is 10 inches; what are the contents of the frustrum?

$(8\div2)^2-\frac{(6\div2)^2}{3}=13\times3=39=$*product of half the length of the frustrum and the remainder of $\frac{1}{3}$ the square of half the length of the frustrum subtracted from the square of half the length of the spindle.*

$39-\overline{3\times10}=9=$*product of the central distance and the area of the segment subtracted from preceding product.*

$9\times6.2832=56.5488=$*last product* $\times\,6.2832=$*result required.*

Ex. 2. The length of a circular spindle is 1.333 feet, the length of the middle frustrum of it is 1 foot, the central distance is 6 inches, and the area of the revolving segment is 40 inches; what are the contents of the frustrum?

Ans. 452.3904 *cubic inches.*

Ex. 3. The length of the middle frustrum of a circular spindle is 12 inches, the length of the spindle being 24, the central distance is 3.5, and the area of the revolving segment is 96 inches; what are the contents of the frustrum?

Ans. 2865.1392 *cubic inches.*

Centre of Gravity. Is in its geometrical centre.

Segment of a Circular Spindle.

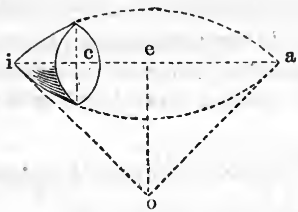
To ascertain the Contents of a Segment of a Circular Spindle, Fig. 112.

RULE.—Subtract the length of the segment, *i c*, from the half length, *i e*, of the spindle; double the remainder, and ascertain the contents of a middle frustrum of this length.

Subtract the result from the contents of the whole spindle, and half the remainder will give the contents of the segment required.*

Or, $C - c \div 2 = S$, *C and c representing the contents of the spindle and middle frustrum.*

Fig. 112.



EXAMPLE.—The length of a circular spindle, *i a*, *Fig.* 112, is 14.14213, the central distance, *o e*, is 7.07107, the radius of the arc, *o a*, is 10, and the length of the segment, *i c*, is 3.53553 inches; what are its contents?

$\frac{14.14213}{2} - 3.53553 \times 2 = 7.07107 =$ *double the remainder, of the length of the segment subtracted from half the length of the spindle = length of the middle frustrum.*

NOTE.—The area of the revolving or generating segment of the whole spindle is 28.54 inches, and that of the middle frustrum is 19.25.

The contents of the whole spindle is	212.9628	*cubic in.*
" " *middle frustrum is*	162.8982	" "
Hence	50.0646 ÷ 2 =	

25.0323 = *the contents required.*

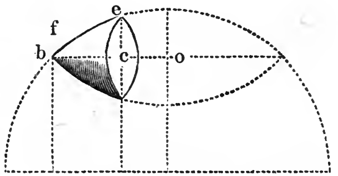
---

* This rule is applicable to the segment of any spindle or any conoid, the volume of the figure and frustrum being first obtained.

*Centre of Gravity.**

$$\frac{\frac{1}{2}(r^2+a^2)(g^2-l^2)-\frac{1}{4}(g^4-l^4)+\frac{2}{3}a[a^3-(r^2-l^2)^{\frac{3}{2}}]}{(r^2+a^2)(g-l)-\frac{1}{3}(g^3-l^3)-2aS}=\textit{distance from centre}$$
of spindle.

When $g=b\ o$.
$l=o\ c$.
$r=$ *radius of circle*, $a\ d$.
$a=a\ o=\sqrt{r^2-g^2}$.
$S=$ *area of the semi-zone*, $e\ f\ g\ h$.



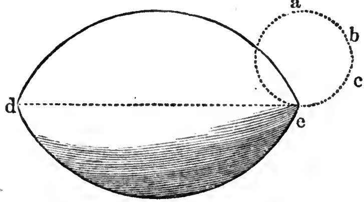
Cycloidal Spindle.†

To ascertain the Contents of a Cycloidal Spindle, Fig. 113.

RULE.—Multiply the product of the square of twice the diameter of the generating circle, *a b c*, and 3.927 by its circumference, and this product divided by 8 will give the contents required.

Or, $\frac{\overline{2d}^2\times 3.927\times \overline{d\times 3.1416}}{8}=S$, *d representing the diameter of the circle, a half width of the spindle.*

Fig. 113.



EXAMPLE.—The diameter of the generating circle, *a b c*, of a cycloid, *Fig.* 113, is 10 inches ; what are the contents of the spindle, *d e* ?

* By Professor A. E. Church, U. S. M. A.
† The contents of a cycloidal spindle are equal to $\frac{5}{8}$ of its circumscribing cylinder.

$\overline{10\times2}^{2}\times3.927=1570.8$ = *product of twice the diameter squared and* 3.927.

$1570.8\times\overline{10\times3.1416}\div8=6168.5316$ *cubic inches = product of the preceding product and the circumference divided by* 8 = *result required.*

Ex. 2. The diameter of the generating circle of a cycloid is 56.5 inches; what are the contents of its spindle in cubic feet?

Ans. 643.9292 *cubic feet.*

Ex. 3. The diameter of the generating circle of a cycloid is 6 feet; what are the contents of its spindle in cubic feet?

Ans. 1332.4028 *cubic feet.*

Elliptic Spindle.

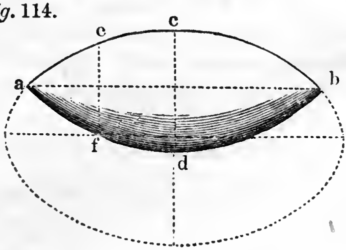
To ascertain the Contents of an Elliptic Spindle, Fig. 114.

RULE.—To the square of its diameter, *c d*, add the square of twice the diameter *e f* at $\frac{1}{4}$ of its length; multiply the sum by the length, *a b*, the product by .1309, and it will give the contents required.

NOTE.—For all such solids, this rule is exact when the body is formed by a conic section, or a part of it, revolving about the axis of the section, and will always be very near when the figure revolves about another line.

Or, $\overline{d^2+2d'^2}\times l\times.1309=S$, *d and d′ representing the diameters as above.*

Fig. 114.



EXAMPLE.—The length of an elliptic spindle, *a b*, *Fig.* 114, is 75 inches, its diameter *c d* 35, and the diameter *e f* at $\frac{1}{4}$ of its length 25; what are its contents?

$35^2 + \overline{25 \times 2}^2 = 3725$ = *sum of squares of diameter of spindle and of twice its diameter at ¼ of its length.*

$3725 \times 75 = 279375$ = *above sum × the length of the spindle.*

Then $279375 \times .1309 = 36570.1875$ = *result required.*

Ex. 2. The length of an elliptic spindle is 100 inches, its diameter 20, and the diameter at ¼ of its length 15; what are its contents? *Ans.* 17017 *cubic inches.*

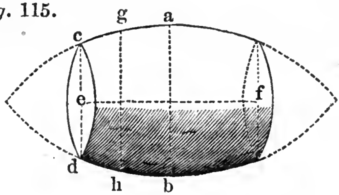
Centre of Gravity. Is in its geometrical centre.

To ascertain the Contents of the Middle Frustrum or a Zone of an Elliptic Spindle, Fig. 115.

Rule.—Add together the squares of the greatest and least diameters, *a b*, *c d*, and the square of double the diameter *g h* in the middle between the two; multiply the sum by the length, *e f*, the product by .1309, and it will give the contents required.*

Or, $d^2 + d'^2 + (2 \times d'')^2 \times l \times .1309 = S$, *d*, *d'*, *and d'' representing the different diameters.*

Fig. 115.



Example.—The greatest and least diameters, *a b* and *c d*, of the frustrum of an elliptic spindle, *Fig.* 115, are 68 and 50 inches, its middle diameter, *g h*, 60, and its length, *e f*, 75; what are its contents?

$68^2 + 50^2 + \overline{60 \times 2}^2 = 21524$ = *sum of squares of greatest and least diameters and of double the middle diameter.*

$21524 \times 75 \times .1309 = 211311.87$ = *product of above sum, the length, and* .1309 = *the result required.*

* For all such solids, this rule is exact when the body is formed by a conic section, or a part of it, revolving about the axis of the section, and will always be very near when the figure revolves about another line.

Ex. 2. The greatest and least diameters of the zone of an elliptic spindle are 20 and 5 inches, its middle diameter 16, and its length 42; what are its contents?

Ans. 7966.312 *cubic inches.*

Ex. 3. The greatest, middle, and least diameters of the zone of an elliptic spindle are 25, 23.5, and 20 inches, and its length 42.5; what are its contents? *Ans.* 17991.55 *cubic inches.*

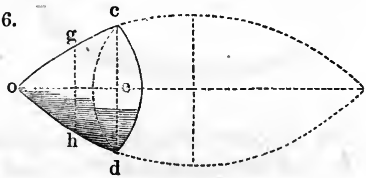
Centre of Gravity. Is in its geometrical centre.

To ascertain the Contents of a Segment of an Elliptic Spindle, Fig. 116.

RULE.—Add together the square of the diameter of the base, *c d*, of the segment, and the square of double the diameter *g h*, in the middle between the base and vertex; multiply the sum by the length, *o e*, of the segment, the product by .1309, and it will give the contents required.*

Or, $d^2+2\,d''^2\times l\times .1309=S$, *d and d″ representing the diameters.*

Fig. 116.



EXAMPLE.—The diameters, *c d* and *g h*, of the segment of an elliptic spindle, *Fig.* 116, are 20 and 12 inches, and the length, *o e*, is 16 inches; what are its contents?

$20^2+\overline{12\times2}^2=976=$ *sum of squares of diameter at base and of double the diameter in the middle.*

$976\times16\times.1309=2044.134=$ *product of above sum, the length of the segment, and* .1309 = *the result required.*

Ex. 2. The diameters of the segment of an elliptic spindle are 25 and 20 inches, and the length of it is 42.5 inches; what are its contents? *Ans.* 12378.231 *cubic inches.*

Centre of Gravity. At two thirds of the length, measuring from the end.

* See note at bottom of page 194.

Parabolic Spindle.

To ascertain the Contents of a Parabolic Spindle, Fig. 117.

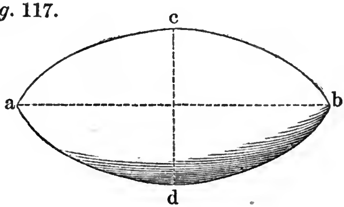
Rule 1.—Multiply the square of the diameter, *a b*, by the length, *c d*, the product by .41888,* and it will give the contents required.

Or, $d^2 \times l \times .41888 = S$.

Rule 2.—To the square of its diameter add the square of twice the diameter at $\frac{1}{4}$ of its length; multiply the sum by the length, the product by .1309, and it will give the contents required.†

Or, $d^2 + 2d'^2 \times l \times .1309 = S$, *d and d′ representing the diameters as above.*

Fig. 117.



Example.—The diameter of a parabolic spindle, *a b*, *Fig.* 117, is 40 inches, and its length, *c d*, 20; what are its contents?

$40^2 \times 20 = 32000 =$ *square of diameter* $\times$ *the length.*

Then $32000 \times .41888 = 13404.16 =$ *above product* $\times .41888 =$ *result required.*

Again, If the middle diameter at one fourth of its length is 29.65, then, by Rule 2,

$40^2 + \overline{29.65 \times 2}^2 \times 20 \times .1309 = 13394.97 =$ *result required.*

Ex. 2. The length of a parabolic spindle is 15.75 feet, and its diameter is 3 feet; what are its contents?

Ans. 59.376 *cubic feet.*

* $\frac{8}{15}$ of .7854.

† See note at bottom of page 194.

Ex. 3. The length of a parabolic spindle is 40 feet, and its diameter is 80 feet; what are its contents in cubic feet?

Ans. 107233.28 *cubic feet.*

Centre of Gravity. Is in its geometrical centre.

To ascertain the Contents of the Middle Frustrum of a Parabolic Spindle, Fig. 118.

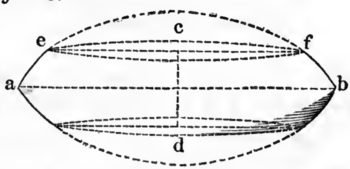
Rule 1.—Add together 8 times the square of the greatest diameter, *a b*, 3 times the square of the least diameter, *e f*, and 4 times the product of these two diameters; multiply the sum by the length, *c d*, the product by .05236, and it will give the contents required.

Or, $d^2 \times 8 + \overline{d'^2 \times 3} + \overline{d \times d' \times 4} \times l \times .05236 = S.$

Rule 2.—Add together the squares of the greatest and least diameters, *a b*, *e f*, and the square of double the diameter in the middle between the two; multiply the sum by the length, *c d*, the product by .1309, and it will give the contents required.

Or, $d^2 + d'^2 + (2d'')^2 \times l \times .1309 = S$, *$d''$ representing the diameter between the two.*

Fig. 118.



Example.—The middle frustrum of a parabolic spindle, *Fig.* 118, has diameters, *a b* and *e f*, of 40 and 30 inches, and its length, *c d*, is 10 inches; what are its contents?

$40^2 \times 8 + 30^2 \times 3 + \overline{40 \times 30} \times 4 = 20300 =$ *the sum of* 8 *times the square of the greatest diameter,* 3 *times the square of the least diameter, and* 4 *times the product of these diameters.*

$20300 \times 10 \times .05236 = 10629.08 =$ *result required.*

Ex. 2. The middle frustrum of a parabolic spindle has diameters of 20 and 15 feet, and its length 5 feet; what are its contents? *Ans.* 1328.635 *cubic feet.*

Ex. 3. The middle frustrum of a parabolic spindle has diameters of 80 and 56.5 feet, and its length is 20 feet; what are its contents? *Ans.* 82578.789 *cubic feet.*

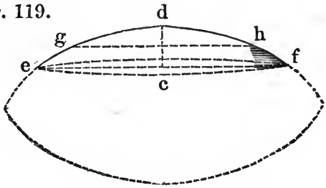
Centre of Gravity. Is in its geometrical centre.

To ascertain the Contents of a Segment of a Parabolic Spindle, Fig. 119.

RULE.—Add together the square of the diameter of the base, *e f*, of the segment, and the square of double the diameter, *g h*, in the middle between the base and vertex; multiply the sum by the length, *c d*, of the segment, the product by .1309, and it will give the contents required.

Or, $d^2+2\,d''^2\times l\times .1309=S$.

Fig. 119.



EXAMPLE.—The segment of a parabolic spindle, *Fig.* 119, has diameters, *e f* and *g h*, of 15 and 8.75 inches, and the length, *c d*, is 2.5 inches; what are its contents?

$15^2+\overline{8.75\times2}^2=531.25$ = *sum of square of base and of double the diameter in the middle of the segment.*

$531.25\times2.5\times.1309=173.852$ = *product of above sum, length of segment, and* .1309 = *the result required.*

Ex. 2. The segment of a parabolic spindle has diameters of 30 and 20 inches, and the length of it is 30 inches; what are its contents in cubic feet? *Ans.* 5.6814 *cubic feet.*

Ex. 3. The segment of a parabolic spindle has diameters of 56.5 and 40 feet, and the length of it is 10 feet; what are its contents in cubic feet?

Ans. 12556.255 *cubic feet.*

Ex. 4. The segment of a parabolic spindle has diameters of 28 and 20 feet, and the length of it is 20 feet; what are its contents? *Ans.* 6241.312 *cubic feet.*

Ex. 5. The segment of a parabolic spindle has diameters of 42 and 30 feet, and the length of it is 60 feet; what are its contents? *Ans.* 42128.856 *cubic feet.*

*Centre of Gravity.**

$$\frac{\frac{b^6}{6}-\frac{d^6}{6}-a\,p\,(b^4-d^4)+2a^2p^2(b^2-d^2)}{\frac{b^5}{5}-\frac{d^5}{5}-\frac{4\,a\,p}{3}(b^3-d^3)+4a^2p^2(b-d)} = \textit{distance from centre}$$

of spindle, a representing semi-diameter of spindle, b the half length, d the distance of the base of the segment from the centre of the spindle, and p one half parameter, which is equal to $\frac{b^2}{2a}$.

Illustration of rules, p. 196, 197, and 198:

Solidity of spindle (Ex. 3, p. 197) by rule 2, 80 feet in diameter by 40 feet in length, the diameter at ¼ of its length being 56.5 feet..=100368.884 *cubic feet.*

Solidity of middle frustrum (Ex. 3, p. 198) by rule 2, 20 feet in length, the diameter at ¼ of its length being 69.25 feet..= 75331.641

Solidity of segment (Ex. 3, p. 199) 10 feet in length =12556.255, which × 2, for two end segments......= 25112.510

100444.151

Solidity of spindle as above..................................=100368.884

Difference, arising from the impracticability of obtaining the middle diameters of the frustrum and segment, from a figure of so minute a scale as that of the example taken for illustration..........= 75.267

* By Professor A. E. Church, U. S. M. A.

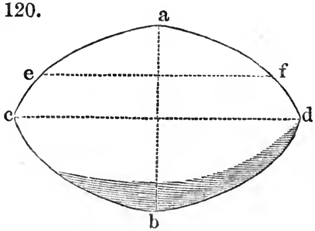
Hyperbolic Spindle.

To ascertain the Contents of a Hyperbolic Spindle, Fig. 120.

RULE.—To the square of its diameter, *c d*, add the square of double the diameter, *e f*, at $\frac{1}{4}$ of its length; multiply the sum by the length, *a b*, the product by .1309, and it will give the contents required.*

Or, $d^2+2d'^2 \times l \times .1309 = S$, *d and d′ representing the diameters as above.*

Fig. 120.



EXAMPLE.—The length, *a b*, *Fig.* 120, of a hyperbolic spindle is 100 inches, and its diameters, *c d* and *e f*, are 150 and 110 inches; what are its contents?

$150^2+\overline{110\times 2}^2\times 100=7090000$=*product of the sum of the squares of the greatest diameter and of twice the diameter at $\frac{1}{4}$ of the length of the spindle and the length.*

Then, $7090000\times .1309=928081$=*the result required.*

Ex. 2. The length of a hyperbolic spindle is 120 inches, and its diameters in the middle and at $\frac{1}{4}$ of its length are 100 and 80 inches; what are its contents?

Ans. 323.614 *cubic feet.*

Centre of Gravity. Is in its geometrical centre.

To ascertain the Contents of the Middle Frustrum of a Hyperbolic Spindle, Fig. 121.

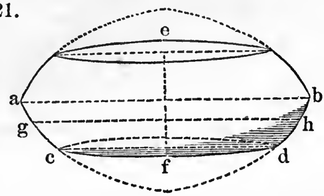
RULE.—Add together the squares of the greatest and least diameters, *a b*, *c d*, and the square of double the diameter, *g h*, in the middle between the two; multiply this sum by the

* See note at bottom of page 194.

length, *e f*, the product by .1309, and it will give the contents required.*

Or, $d^2+d'^2+(2\times d'')^2\times l\times.1309=S$.

Fig. 121.



EXAMPLE.—The diameters *a b* and *c d*, of the middle frustrum of a hyperbolic spindle, *Fig.* 121, are 150 and 110 inches, the diameter *g h*, 140 inches, and the length, *e f*, 50; what are its contents?

$150^2+110^2+\overline{140\times2}^2=113000$=*sum of squares of greatest and least diameters and of double the middle diameter.*

$113000\times50\times.1309=739585$=*product of above sum* × *the length* × .1309 =*result required.*

Ex. 2. The diameters of the middle frustrum of a hyperbolic spindle are 16 and 10 inches, the diameter at $\frac{1}{4}$ of its length is 13.5, and the length of it is 10; what are its contents in cubic inches? *Ans.* 1420.265 *cubic inches.*

Ex. 3. The diameters of the middle frustrum of a hyperbolic spindle are 16 and 12 feet, the diameter at $\frac{1}{4}$ of its length 14.5, and the length of it 20 feet; what are its contents?

Ans. 3248.938 *cubic feet.*

Centre of Gravity. Is in its geometrical centre.

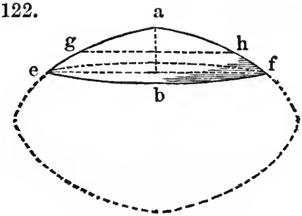
To ascertain the Contents of a Segment of a Hyperbolic Spindle, Fig. 122.

RULE.—Add together the square of the diameter of the base, *e f*, of the segment, and the square of double the diameter, *g h*, in the middle between the base and vertex; multiply the sum by the length, *a b*, of the segment, the product by .1309, and it will give the contents required.

* See note at bottom of page 194.

Or, $\overline{d^2+d''^2} \times l \times .1309 = S$, *d and d″ representing the diameters.*

Fig. 122.



EXAMPLE.—The segment of a hyperbolic spindle, *Fig.* 122, has diameters, *e f* and *g h*, of 110 and 65 inches, and its length, *a b*, 25; what are its contents?

$110^2 + \overline{65 \times 2}^2 = 29000$ = *sum of squares of diameter of base and of double the middle diameter.*

$29000 \times 25 \times .1309 = 94902.5$ = *product of above sum* × *the length* × .1309 = *result required.*

Ex. 2. The segment of a hyperbolic spindle has diameters of 8 inches at its base and 6 at half its length, its length being 10; what are its contents?

Ans. 272.272 *cubic inches.*

Ex. 3. The segment of a hyperbolic spindle has diameters of 50 and 31 inches at its base and half its length, its length being 25; what are its contents?

Ans. 20760.74 *cubic inches.*

Centre of Gravity. At $\frac{3}{4}$ of the height, measured from the vertex.

ELLIPSOID, PARABOLOID, AND HYPERBOLOID OF REVOLUTION* (CONOIDS).

Definition. *Figures like to a cone, described by the revolution of a conic section around and at a right angle to the plane of their fixed axes.*

* These figures have been known as conoids. For the definition of a conoid, see Conic Sections, page

Ellipsoid of Revolution (Spheroid).

An ellipsoid of revolution is a semi-spheroid. (See p. 180–184.)

*Paraboloid of Revolution.**

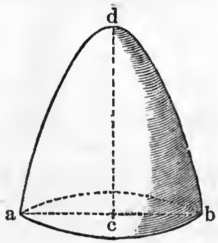
To ascertain the Contents of a Paraboloid of Revolution, Fig. 123.

RULE.—Multiply the area of the base, *a b*, by half the altitude, *d c*, and the product will give the contents required.

NOTE.—This rule will hold for any segment of the paraboloid, whether the base be perpendicular or oblique to the axis of the solid.

Or, $a \times \frac{h}{2} = \mathrm{S}$.

Fig. 123.



EXAMPLE.—The diameter, *a b*, of the base of a paraboloid of revolution, *Fig.* 123, is 20 inches, and its height, *d c*, 20 inches; what are its contents?

Area of 20 inches diameter of base$=314.16$.

$314.16 \times \frac{20}{2} = 3141.6 =$ *area of base* $\times$ *half the height=result required.*

Ex. 2. The diameter of the base of a paraboloid of revolution is 11.5 inches, and its height 7; what are its contents?

Ans. 363.5411 *cubic inches.*

* The contents of a paraboloid of revolution are$=\frac{1}{2}$ of its circumscribing cylinder.

Ex. 3. The diameter of the base of a paraboloid of revolution is 11 feet 3 inches, and its height 8 feet; what are its contents in cubic feet? *Ans.* 397.608 *cubic feet.*

Centre of Gravity. At two thirds of the height, measured from the vertex.

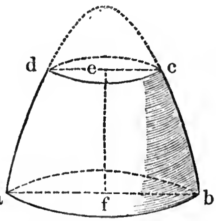
Frustrum of a Paraboloid of Revolution.

To ascertain the Contents of a Frustrum of a Paraboloid of Revolution, Fig. 124.

RULE.—Multiply the sum of the squares of the diameters, *a b* and *d c*, by the height, *e f*, of the frustrum, this product by .3927, and it will give the contents nearly.

Or, $d^2 + d'^2 \times h \times .3927 = S$.

Fig. 124.



EXAMPLE.—The diameters, *a b* and *d c*, of the base and vertex of the frustrum of a paraboloid of revolution, *Fig.* 124, are 20 and 11.5 inches, and its height, *e f*, 12.6; what are its contents?

$20^2 + 11.5^2 = 532.25$ = *sum of squares of the diameters.*

$532.25 \times 12.6 \times .3927 = 2633.5837$ = *product of above sum, the height, and* .3927 = *the result required.*

Ex. 2. The diameters of the frustrum of a paraboloid of revolution are 30 and 58 inches, and the height 18; what are its contents in cubic feet? *Ans.* 17.4424 *cubic feet.*

Ex. 3. The diameters of the frustrum of a paraboloid of revolution are 48 and 60 inches, and the height 18; what are its contents in cubic feet? *Ans.* 24.151 *cubic feet.*

Centre of Gravity. From the vertex $\frac{1}{3}h\,\frac{2R^2 + r^2}{R^2 + r^2}$, R *and* r *representing radii of base and diameter, and* h *the height.*

Segment of a Paraboloid of Revolution.

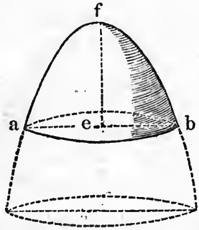
To ascertain the Contents of the Segment of a Paraboloid of Revolution, Fig. 125.

RULE.—Multiply the area of the base, *a b*, by half the altitude, *e f*, and the product will give the contents required.

NOTE.—This rule will hold for any segment of the paraboloid, whether the base be perpendicular or oblique to the axis of the solid.

Or, $a \times \frac{h}{2} = S$.

Fig. 125.



EXAMPLE.—The diameter, *a b*, of the base of a segment of a paraboloid of revolution, *Fig.* 125, is 11.5 inches, and its height, *e f*, is 7.4; what are its contents?

Area of 11.5 inches diameter of base $= 103.869$.

$103.869 \times \frac{7.4}{2} = 384.315 =$ *area of base* $\times$ *half the height* $=$ *the result required.*

Ex. 2. The diameter of the base of a segment of a paraboloid of revolution is 30 inches, and its height 25; what are its contents in cubic inches?

Ans. 8835.73 *cubic inches.*

Ex. 3. The diameter of the base of a segment of a paraboloid of revolution is 48 inches, and the height 15; what are its contents in cubic feet? *Ans.* 7.854 *cubic feet.*

Centre of Gravity. At two thirds of the height, measured from the vertex.

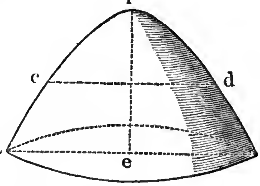
Hyperboloid of Revolution.

To ascertain the Contents of a Hyperboloid of Revolution, Fig. 126.

Rule.—To the square of the radius of the base, *a b*, add the square of the middle diameter, *c d*; multiply this sum by the height, *e f*, the product by .5236, and it will give the contents required.

Or, $r^2+d\times h\times .5236=S$, *d representing the middle diameter.*

Fig. 126.



Example.—The base, *a b*, of a hyperboloid of revolution, *Fig.* 126, is 80 inches, the middle diameter, *c d*, 66, and the height, *e f*, 60; what are its contents?

$\overline{80\div 2}^{\,2}+66^2=5956=$ *sum of square of radius of base and middle diameter.*

$5956\times 50=297800\times .5236=155928.08=$ *product of above sum, the height, and* $.5236=$ *result required.*

Ex. 2. The base of a hyperboloid of revolution is 20 inches, its middle diameter is 16, and its height 20; what are its contents? *Ans.* 3728.032 *cubic inches.*

Ex. 3. The base of a hyperboloid of revolution is 104 inches, the diameter at half its height is 68, and the height of it is 50 inches; what are its contents in cubic feet? *Ans.* 111.0226 *cubic feet.*

Centre of Gravity. Distance from the vertex, $\frac{4b+3h}{6b+4h}\times h$, *b representing the base, and h the height of the figure.*

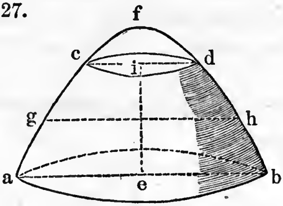
Frustrum of a Hyperboloid of Revolution.

To ascertain the Contents of the Frustrum of a Hyperboloid of Revolution, Fig. 127.

RULE.—Add together the squares of the greatest and least semi-diameters, $e\,b$ and $i\,d$, and the square of the whole diameter, $g\,h$, in the middle of the two; multiply this sum by the height, $i\,e$, the product by .5236, and it will give the contents required.

Or, $d^2+d'^2+d''^2 \times h \times .5236 = S$, *d, d', and d'' representing the several diameters.*

Fig. 127.



EXAMPLE.—The frustrum of a hyperboloid of revolution, *Fig.* 127, is in height, $e\,i$, 50 inches, the diameters of the greater and less ends, $a\,b$ and $c\,d$, are 110 and 42, and that of the middle diameter, $g\,h$, is 80; what are the contents?

$110 \div 2 = 55$ and $42 \div 2 = 21$. Hence, $55^2+21^2+80^2=9866=$ *sum of the squares of the semi-diameters of the ends and that of the whole diameter in the middle.*

$9866 \times 50 = 493300 \times .5236 = 258291.88$ *cubic inches.*

Ex. 2. The height of a frustrum of a hyperboloid of revolution is 1 foot, the greatest and least diameters 10 and 6 inches, and the middle diameter 8.5 inches; what are its contents in cubic feet? *Ans.* .38633 *cubic foot.*

Ex. 3. The height of the frustrum of a hyperboloid of revolution is 10 inches, the radii of the ends 21 and 1 inches, and the middle diameter 25 inches; what are its contents in cubic feet? *Ans.* 3.2331 *cubic feet.*

Centre of Gravity.

$\frac{3}{4}\,\frac{(d+d')(2a^2-d'^2+d^2)}{3a^2-d'^2+d'd+d^2}$ = *distance from centre of the base of the figure, a representing the semi-transverse axis, or distance from centre of the curve to vertex of figure, f; d and d′ the distances from the centre of the curve to the centre of the less and greater diameter of the frustrum.*

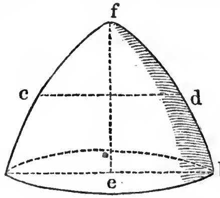
Segment of a Hyperboloid of Revolution.

To ascertain the Contents of the Segment of a Hyperboloid of Revolution, Fig. 128.

RULE.—To the square of the radius of the base, *a e*, add the square of the middle diameter, *c d;* multiply this sum by the height, *e f*, the product by .5236, and it will give the contents required.

Or, $r^2+d^2\times h\times .5236=S$, *r representing radius of base.*

Fig. 128.



EXAMPLE.—The radius, *a e*, of the base of a segment of a hyperboloid of revolution, *Fig.* 128, is 21 inches, its middle diameter, *c d*, is 30, and its height, *e f*, 15; what are its contents?

$21^2+30^2\times 15=20115$ = *the product of the sum of the squares of the radius of the base and the middle diameter multiplied by the height.*

$20115\times .5236=10532.214$ = *result required.*

Ex. 2. The radius of the base of a segment of a hyperboloid of revolution is 55 inches, its middle diameter 70, and its height 65; what are its contents?

Ans. 269719.45 *cubic inches.*

Centre of Gravity. (See rule for hyperboloid, page 206.)

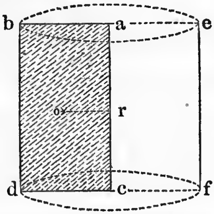
ANY FIGURE OF REVOLUTION.

To ascertain the Contents of any Figure of Revolution, Fig. 129.

RULE.—Multiply the area of the generating surface by the circumference described by its centre of gravity.

Or, $a \times 2r \times p =$ *contents, r representing radius of centre of gravity.*

Fig. 129.



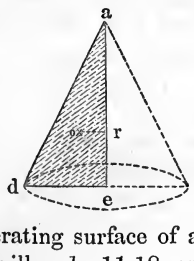
EXAMPLE.—If the generating surface, *a b c d*, of the cylinder, *b e d f*, *Fig.* 129, is 5 inches in width and 10 in height, then will *a b*$=5$ and *b d*$=10$, and the centre of gravity will be in *o*, the radius of which is *r o*$=5 \div 2 = 2.5$.

Hence, $10 \times 5 = 50 =$ *area of generating surface.*

$50 \times \overline{2.5 \times 2} \times 3.1416 = 785.4 =$ *area* $\times$ *circumference of its centre of gravity = the contents of the cylinder.*

Proof. Volume of a cylinder 10 inches in diameter and 10 inches in height. $10^2 \times .7854 = 78.54$, and $78.54 \times 10 = 785.4$.

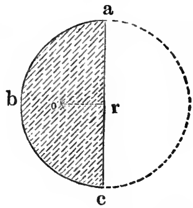
Fig. 130.



Ex. 2. If the generating surface of a cone, *Fig.* 130, is *a e* $=10$, *d e*$=5$, then will *a d*$=11.18$, and the area of the triangle$=10 \times 5 \div 2 = 25$, the centre of gravity of which is in *o*, and *o r*, by rule, page 57, $=1.666$.

Hence, $25 \times \overline{1.666 \times 2} \times 3.1416 = 261.8 =$ *area of generating surface* $\times$ *circumference of its centre of gravity = the contents of the cone.*

Fig. 131.



Ex. 3. If the generating surface of a sphere, *Fig.* 131, is $a\,b\,c$, and $a\,c=10$, $a\,b\,c$ will be $\left(\frac{10^2\times.7854}{2}\right)=39.27$, the centre of gravity of which is in o, and by rule, page 87, $o\,r=2.122$.

Hence, $39.27\times\overline{2.122\times 2}\times 3.1416=523.6=$ *area of generating surface* $\times$ *circumference of its centre of gravity* $=$ *the contents of the sphere.*

To ascertain the Contents of an Irregular Body.

RULE.—Weigh it both in and out of fresh water, and note the difference in pounds; then, as 62.5* is to this difference, so is 1728† to the number of cubic inches in the body.

Or, divide the difference in pounds by 62.5, and the quotient will give the volume in cubic feet.

NOTE.—If salt water is to be used, the ascertained weight of a cubic foot of it, or 64, is to be used for 62.5.

EXAMPLE.—An irregular shaped body weighs 15 pounds in water, and 30 out; what is its volume in cubic inches?

$30-15=15=$ *difference of weights in and out of water.*
$62.5:15::1728:414.72=$ *volume of the body in cubic inches.*
Or, $15\div 62.5=.24$, and $.24\times 1728=414.72=$ *volume of the body.*

Ex. 2. An irregular body weighs 187.5 pounds in water, and 250 out; what is its volume? *Ans.* 1728 *cubic inches.*

Ex. 3. The difference in weights of a bronze gun in and out of water is 625 pounds; what is its volume in cubic feet?

Ans. 10 *cubic feet.*

* The weight of a cubic foot of fresh water.
† The number of inches in a cubic foot.

PROMISCUOUS EXAMPLES.

1. If a stone measures 4 feet 6 inches long, 2 feet 9 inches broad, and 5 feet 4 inches deep, how many cubic feet does it contain? *Ans.* 66 *feet.*

2. The dimensions of a bushel measure are $18\frac{1}{2}$ inches in diameter, and 8 inches deep; what should be the dimensions of a measure of like form that would contain 8 bushels?

Ans. 37 *inches in diameter, and* 16 *inches deep.*

3. If a box, of plank 3.5 inches thick, is 4 feet 9 inches in length, 3 feet 7 inches in breadth, and 2 feet 11 inches in height, how many square feet did it require to make the box, how many cubic feet does it contain, and how many does it measure?

Ans. 70.208 *square feet,* 29.167, *and* 49.644 *cubic feet.*

4. A well 40 feet in depth is to be lined, of which the diameter is 6.5 feet, the thickness of the wall is to be 1.5 feet, leaving the inner diameter of the well 3.5 feet; how many cubic feet of stone will be required? *Ans.* 942.478 *cubic feet.*

5. How many bricks, exclusive of mortar, 8 inches long, 4 inches wide, and 2 inches thick, will it take to build a wall 40 feet long, 20 feet high, and 2 feet thick?

Ans. 43,200 *bricks.*

6. How many bricks, exclusive of mortar, will it take to build the walls of a house which is 80 feet long, 40 feet wide, and 25 feet high, the walls to be 12 inches thick, the bricks being 8 inches long, 4 broad, and 2 thick? *Ans.* 159,300 *bricks.*

7. How many bricks will it require to construct the walls of a house 64 feet long, 32 feet wide, and 28 feet high; the walls are to be 1 foot 4 inches thick, and there are to be three doors 7 feet 4 inches high, and 3 feet 8 inches wide; also 14 windows 3 feet wide and 6 feet high, and 16 windows 2 feet 8 inches wide and 5 feet 8 inches high: each brick is to be 8 inches long, 4 inches wide, and 2 inches thick?

Ans. 171,990 *bricks.*

8. If a garden 100 feet long and 80 feet wide is to be inclosed with a ditch 4 feet wide, how deep must it be dug that the soil taken from it may raise the surface one foot?

Ans. 5.319 *feet.*

9. If a man dig a small square cellar, which will measure 6 feet each way, in one day, how long would it take him to dig a similar one that measured 10 feet each way?

Ans. 4.629 *days.*

10. If a lead pipe $\frac{3}{4}$ of an inch in diameter will fill a cistern in 3 hours, what should be its diameter to fill it in 2 hours? *Ans.* .918 *inch.*

11. What are the contents of a stick of round timber 20 feet long, the diameter at the larger end being 12 inches, and at the smaller end 6 inches?

Ans. 10.9126 *cubic feet.*

12. Required the volume of Bunker Hill Monument, the height of which is 220 feet, by 30 feet square at its base, and 15 feet at its vertex. *Ans.* 115500 *cubic feet.*

13. What are the contents of a spherical segment 3 feet in height, cut from a sphere 10 feet in diameter?

Ans. 113.0976 *cubic feet.*

14. The largest of the Egyptian pyramids is square at its base, and measures 693 feet on a side; its height is 500 feet. Supposing it to come to a point at its vertex, what would be its contents, and how many miles in length of wall, 5 feet in height and 2 feet thick, would it make?

Ans. $\begin{cases} 80041500 \textit{ cubic feet.} \\ 1515.9375 \textit{ miles in length.} \end{cases}$

15. What are the contents of a sphere, the diameter of which is 20 inches? *Ans.* 4188.8 *cubic inches.*

16. What is the weight of an iron spherical shell 5 inches in diameter, the thickness of the metal being 1 inch, estimating a cubic inch of iron to weigh .25 of a pound?

Ans. 12.8282 *pounds.*

17. How many cubic feet of water are there in a pond that measures 200 acres, and is 20 feet deep?

Ans. 174240000 *cubic feet.*

18. If rain was to fall to the depth of 3 inches on a surface of 20000 square acres, what would be the number of hogsheads of water fallen, assuming each hogshead to contain 100 gallons, and each gallon 231 cubic inches?

Ans. { 16292571 *hhds.* 42 *galls.* 3 *qts.* 0 *pts.* 3.43 *gills,* *or* 217800000 *cubic feet.* }

19. The ditch of a fortification is 1000 feet long, 9 feet deep, 20 feet broad at bottom, and 22 at top; how much water will fill the ditch, allowing 231 cubic inches to make a gallon? *Ans.* 1413818.1819 *gallons.*

20. What must be the height of a bin that will contain 600 bushels, its length being 8 feet and breadth 4?*

Ans. 23.333 *feet.*

NOTE.—As a bushel contains very nearly one fourth more than a cubic foot, the dimensions of a bin, etc., for any required number of bushels, may be readily found by adding one fourth to the number of bushels; the result will give the number of cubic feet the bin will contain, or that may be required. Therefore, *when two dimensions of a bin, etc., are given, divide the number of cubic feet by their product, and the quotient will be the other dimension.*

In the above example, then,

$600 \div 4 = 150$, and $600 + 150 = 750 =$ *the number of cubic feet the bin is to contain.*

Then $750 \div \overline{8 \times 4} = 23.4375$ *feet = the height of the bin required.*

21. The length of a bin is 4 feet, its breadth 5 feet 6 inches; what must its height be, by the above rule, that it may contain 272 bushels? *Ans.* 15 *feet* 5.454 *inches.*

22. There are 1000 men besieged in a town with provisions for 5 weeks, allowing each man 16 ounces a day; if they are re-enforced by 500 more, and no relief can be received till the end of 8 weeks, how many ounces must be given daily to each man? *Ans.* 6.66 *ounces.*

23. An officer drew up his company in a square, the number in each rank being equal; on being re-enforced with three times his first number of men, he placed them all in the same form, and then the number in each rank was just double what it was at first; he was again re-enforced with three times his

* The standard United States bushel is 2150.42 cubic inches.

whole number of men, and, after placing them all in the same form as at first, his number in each rank was 40 men; how many men had he at first? *Ans.* 100 *men.*

24. The volume of a sphere is 381.7044 cubic inches; required its radius. *Ans.* 4.5 *inches.*

25. If an iron wire $\frac{1}{10}$ of an inch in diameter will sustain a weight of 450 pounds, what weight might be sustained by a wire an inch in diameter? *Ans.* 45000 *pounds.*

26. The edge of a cube is 36 inches; what is the volume of a sphere that may be inscribed within it?

Ans. 24429.0816 *cubic inches.*

27. In the walls of Balbeck, in Turkey, there are three stones laid end to end, now in sight, one of which is 63 feet long, 12 feet thick, and 12 feet broad; what is the weight, supposing its specific gravity to be 3 times that of water?

Ans. 759.375 *tons.*

28. If two men carry a burden of 200 pounds suspended near the middle of a pole, the ends of which rest on their shoulders, how much of the load is borne by each man, it hanging 6 inches from the middle, and the whole length of the pole being 4 feet?

Ans. 125 *pounds, and* 75 *pounds.*

29. A joist is $8\frac{1}{2}$ inches deep and $3\frac{1}{2}$ broad; what will be the depth of a beam of twice the contents of the joist that is $4\frac{3}{4}$ inches broad? *Ans.* 12.526 *inches.*

30. Bunker Hill Monument is 30 feet square at its base, 15 feet square at its top, and its height is 220 feet; from the bottom to the top, through its centre, is a cylindrical opening 15 feet in diameter at the bottom and 11 feet at the top; how many cubic feet are there in the monument?

Ans. 86068.444 *cubic feet.*

31. If a bell 4 inches in height, 3 inches in diameter (external), and $\frac{1}{4}$ of an inch in thickness, weigh 2 pounds, what should be the dimensions of a bell, of like proportions, that would weigh 2000 pounds?

Ans. $\begin{cases}\text{3 feet 4 inches high, 2 feet 6 inches diameter,}\\ \text{and } 2\frac{1}{2} \text{ inches thick.}\end{cases}$

32. If a round column 7 inches in diameter has a capacity of 4 cubic feet, of what diameter is a column of equal length that contains 10 times as much?

NOTE.—The contents of cylinders, prisms, parallelopipedons, etc., of equal altitudes, are to each other as the squares of their diameters or like sides. The same rule is applicable to frustrums of a cone or pyramid when the altitude is the same and the ends proportional.

Hence, *as* $4:40$, or *as* $1:10::7^2:490$ = *the square of the required diameter*, and $\sqrt{490}=22.1359$, *the diameter required.*

33. A frustrum of a cone is 12 inches in height, and the diameters of the greater and smaller ends 5 and 3 inches respectively. Required the diameter of a frustrum of the same altitude that will contain 3848.46 cubic inches, and have its diameters in the same proportion as the smaller one.

Ans. The greater diameter 25, *and less diameter* 15 *inches.*

34. There is a fish the head of which weighs 15 pounds, his tail weighs as much as his head and half as much as his body, and his body weighs as much as his head and tail. Required the weight of the fish. *Ans.* 72 *pounds.*

35. A certain grocer has only 5 weights; with these he can weigh any quantity by pounds from 1 to 121 pounds. Required the weights. *Ans.* 1, 3, 9, 27, *and* 81 *pounds.*

36. A gentleman has a bowling-green 300 feet long and 200 feet broad, which he desires to raise one foot higher by means of earth to be taken from a ditch that is to go around it; to what depth must the ditch be dug, supposing its breadth to be 8 feet? *Ans.* 7 *feet* 3.21 *inches.*

37. Required a cylindrical vessel 3 feet in depth that shall hold twice as much as a vessel 28 inches deep and 46 inches in diameter; what must be its diameter?

Ans. 57.37 *inches.*

38. A cubic foot of brass is to be drawn into a wire of $\frac{1}{40}$ of an inch in diameter; what will be the length of the wire, assuming there is to be no loss of metal in the operation?

Ans. 97784.5684 *yards.*

39. One end of a pile of wood is perpendicular to the horizon, the other is an inclined plane; the length of the pile at the bottom is 64 feet, at the top 50 feet, in height 12 feet, and

the length of the wood 5 feet; required the number of cords it contains? *Ans.* 26 *cords* 92 *feet.*

40. If a vessel of war, with her ordnance, rigging, and appointments, is depressed so as to displace 50000 cubic feet of water, what is the weight of the vessel?*

Ans. 1428.571 *tons.*

41. The monument erected in Babylon by Queen Semiramis at her husband Ninus's tomb is said to have been one block of solid marble† in the form of a square pyramid, the sides of the base being 20 feet, and the height of the monument 150 feet; if this monument had been sunk in the Euphrates, what weight would it have required to raise its apex to the surface of the water?‡ *Ans.* 2450000 *pounds.*

42. If the pyramid described in the last example were divided into three equal parts by planes parallel to its base, what would be the length of each part, measured from the top? *Ans.* 104.0042, 27.0329, *and* 18.9629 *feet.*

43. There is a mill-hopper in the form of a square pyramid, the contents of which are 13.5 feet, the side of its greater end and depth are in the proportion of 1 to 1.5; but one foot has to be cut off its length to make a passage for the grain from the hopper to the mill-stone. Required its contents in corn measure? *Ans.* 10.446 *bushels.*

44. A crucible is in the form of a conic frustrum; the bottom of it is 2 inches in diameter, the top 3, and the depth 6.7365; this crucible is filled with melted metal, of which it is required to make a sphere; what is the diameter of the mold? *Ans.* 4 *inches.*

45. Suppose a cistern has two pipes, and that one can fill it in $8\frac{1}{2}$ hours, the other in $4\frac{3}{4}$; in what time can both fill it together? *Ans.* 3 *hours* 2 *min.* 49.5 *sec.*

46. A certain cistern has three pipes; the first will empty it in 20 minutes, the second in 40 minutes, and the third in 75 minutes; in what time would they all empty it?

Ans. 11 *min.* 19 *sec.* 15 *thirds.*

* A cubic foot of sea water weighs 64 pounds.

† The weight of a cubic foot of marble is assumed to be 185 pounds.

‡ The weight of a cubic foot of fresh water is 62.5 pounds.

47. A reservoir of water has two supply cocks; the first will fill it in 40 minutes, and the second in 50; it has also a discharging cock, by which it may be emptied, when full, in 25 minutes. Now if all the cocks are opened at once, and the water runs uniformly, how long before the cistern will be filled? *Ans.* 3 *hours* 20 *minutes.*

Operation. If it will fill once in 40 minutes by one cock, and also in 50 minutes by the other cock, it will fill 2.25 times in 50 minutes by both cocks.

Then, 2.25 *times* : 50 *minutes* :: 1 *time* : 22.222 *minutes* = *the time in which it will fill by both cocks.*

Now, as it will empty in 25 minutes, the time gained in filling over emptying will be as 25 to 22.222, or 9 to 8.

Consequently, it will fill 9 times while it empties 8 times; and as the time of filling is 22.222 minutes, $22.222 \times 9 = 199.998$ *minutes = the time required.*

48. A cistern containing 60 gallons of water has three unequal cocks for discharging it; the largest will empty it in one hour, the second in two hours, and the third in three; in what time will the cistern be emptied if they all run together? *Ans.* 32 *min.* 43 *sec.* 26 *thirds.*

Operation.

60 galls. by smallest cock in 3 hours $= \frac{60}{3} = 20$ *galls. in* 1 *hour.*
60 " " second cock in 2 hours $= \frac{60}{2} = 30$ " 1 "
60 " " largest cock in 1 hour $= \frac{60}{1} = 60$ " 1 "

Then, $20 + 30 + 60 = 110$ *gallons in* 1 *hour.*

Hence, 110 *gallons* : 1 *hour* :: 60 *gallons* : .5454 *hours* = 32 *minutes,* 43 *seconds, and* 26 *thirds = the result required.*

49. A reservoir has three pipes; the first can fill it in 12 days, the second in 11 days, and the third can empty it in 14 days; in what time will it be filled if they are all running together? *Ans.* 9 *days* 17 *hours* 24 *min.*

50. If a pipe $1\frac{1}{2}$ inches in diameter will fill a cistern in 50 minutes, how long would it require a pipe that is 2 inches in diameter to fill the same cistern? *Ans.* 28 *min.* 7.5 *sec.*

51. If a pipe 6 inches in diameter will draw off a certain quantity of water in 4 hours, in what time would it take 3 pipes of four inches in diameter to draw off twice the quantity? *Ans.* 6 *hours.*

52. A water tub contains 147 gallons; the supply pipe gives 14 gallons in 9 minutes; the tap discharges 40 gallons in 31 minutes; now, supposing the tap, the tub being empty, to be carelessly left open, and the water to be turned on at 2 o'clock in the morning; a servant at 5, finding the water running, shuts the tap. Required the time in which the tub will be filled after this discovery.

Ans. 6 *hours* 3 *min.* 48.7 *sec.*

53. If the diameter of the earth is 7930 miles, and that of the moon 2160, required the ratio of their surfaces and their solidities, assuming them to be spheres.

NOTE.—The surfaces of all similar solids are to each other as the squares of their like dimensions, such as diameters, circumferences, linear sides, etc., etc.; and their solidities are as the cubes of those dimensions.

Operation. Hence, the surface of the moon : the surface of the earth $::2160^2:7930^2$, and $\frac{2160^2}{7930^2}$*$=\frac{4665600}{62884900}=\frac{1}{13.47}$, or as $1:13.47$.

Also, the solidity of the moon : solidity of the earth $::2160^3:7930^3$, and $\frac{2160^3}{7930^3}=\frac{1}{49.483}$, or as $1:49.483$.

54. A sugar-loaf is to be divided equally among three persons by sections parallel to the base; it is required to find the height of each person's share, assuming the loaf to be a cone the height of which is 20 inches.

Operation. By similar cones, †$3:1::20^3:\frac{20^3}{3}=2666.667=$ *the cube of the height of the upper section; hence,* $\sqrt[3]{2666.667}=13.867$, *the upper part.*

Also, $3:2::20^3:\frac{2\times20^3}{3}=5333.333$, and $\sqrt[3]{5333.333}-13.867=3.604$, *the middle part;* consequently, the lower part will be $20-\overline{13.867+3.604}=2.529$ *inches.*

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* The ratio of one quantity to another may be obtained by dividing the antecedent by the consequent.

† This proportion, as well as all others of the kind, may be expressed thus: $\sqrt[3]{3}:\sqrt[3]{1}::20:\frac{20}{\sqrt[3]{3}}=$the height of the upper section; and in some instances this is the most convenient method.

55. A ship has a leak by which she would fill and sink in 15 hours, but by means of her pumps she can be pumped out, if full, in 16 hours; now, if the pumps are worked from the time the leak begins, how long before the ship will sink?

Ans. 240 *hours.*

Operation. She will fill $\frac{1}{15}$ in an hour; then, if $\frac{1}{16}$ is pumped out, the water gains $\frac{1}{15}-\frac{1}{16}=\frac{1}{240}$ of the ship per hour.

56. Three men bought a piece of tapering timber, which was the frustrum of a square pyramid; one side of the base was 3 feet, one side of the top 1 foot, and the length 18 feet; what is the length of each man's piece, assuming they are to divide equally?

Operation. By similar triangles, $3-1:18::1:9=$*the length of the piece cut off from the end, by which the piece is deficient of being a pyramid.*

Then, by rules, pages 168, 169, the contents of the piece of timber (a frustrum of a pyramid) and the piece cut off (a pyramid) are 78 and 3 cubic feet. Also, $78\div3=26=$*the contents of each person's share*, which, added to the contents of the piece cut off$=26+3=29=$*the contents of the first division (including the piece cut off)*, and $26\times2+3=55=$*the contents of the second division (including the piece cut off).*

Now, by similar pyramids, $3:29::9^3$ (*length of piece cut off*)$:7047$, and $\sqrt{7047}=19.172$; hence, $19.172-9=10.172=$*the length of the first division of the frustrum.*

Again, $29:55::19.172^3$ (*length of first division*)$:13365$, and $\sqrt[3]{13365}=23.731$; hence, $23.731-19.172=4.559=$*the length of the second division of the frustrum.*

Hence,

Length of timber....................................		18.000
" *first division*..............................	10.172	
" *second division*..............................	4.559	14.731
" *third division*....................................		3.269

57. Assuming the earth to be a sphere, and a quarter of its radius 1000 miles, what is the area of its surface, its volume, and weight, the mean density of it being 353.75 pounds per cubic foot?

Ans. { *Surface*, 201062400 *square miles.*
Volume, 268083200000 *cubic miles.*

58. The sides of the base of an irregular tetrahedron are 21, 20, and 13 feet, and its height 9; what is its volume?

By rules, pages 56 and 60, the area of the base of the figure is 12 feet.

Ans. 756 *cubic feet.*

59. A regular tetrahedron contains 1.8414 cubic feet; required its side and surface.

Ans. Side, 30 *inches; surface,* 1558.845 *square inches.*

60. Required the volume of the frustrum of a triangular pyramid, the base of which has a side of 9 inches, its vertex 4, and its lateral edge 5.

Operation. It is required to ascertain the perpendicular height or length of the frustrum.

1. $9^2-(9\div 2)^2=60.75$, and $\sqrt{60.75}=7.7942$=*perpendicular of triangle of base.*

2. By rule, p. 62, to ascertain centre of base (an equilateral triangle), $9\times .5773=5.1957$=*radius of circumscribing circle of base, or centre of base.*

3. The base and vertex of the frustrum have sides of 9 and 4, and the slant height is 5; hence, $9\backsim 4:5::4:4$=*slant height of end of pyramid if continued;* and $5+4=9$=*height of whole pyramid.*

4. If 5.196=radius of base of pyramid, and 9=its slant height, then, $\sqrt{(5.196^2-9^2)}=7.348$=*height of whole pyramid;* and $9:5::7.348:4.082$ =*height of frustrum of pyramid.*

Then, by rule, p. 169, $9^2+4^2+\overline{9\times 4}\times .433\times 4.082\div 3=78.3595$=*result required.*

61. The sides of the base of an irregular tetrahedron are 12, 15, and 17 inches, and its height 9; required its volume.

Ans. 263.248 *cubic inches.*

62. How large a cube may be inscribed in a sphere 40 inches in diameter? *Ans.* 23.094 *inches.*

63. How many cubic inches are contained in a cube that may be inscribed in a sphere 20 inches in diameter?

Ans. 1480.2936 *cubic inches.*

By rule, page 61.

64. If a stone is put into a vessel of 14 cubic feet in capacity, and it then requires but 2.5 quarts of water to fill it, what is the volume of the stone?

Ans. 13 *feet* 1560 *inches.*

65. A cone, the diameter of which is 12 inches and altitude 10, being put into a vessel filled with rain water, with its base upward, was depressed to a point where the area of its section, parallel to the base, was eighty inches; required the weight of the cone.

As $12 : 10 :: \sqrt{(80 \div .7854)} : 8.41$ *inches = depth of the cone in water;* then, $80 \times 8.41 \times \frac{1}{3} = 224.266$, *cubic inches, the volume of the part immersed. Consequently,* as $1728 : 224.266 :: 1000^* : 129.784$ *ounces.*

66. Into a vessel filled with rain water, suppose there be put a cone of dry wood, having a volume of one cubic foot, with the lesser end downward, and its axis perpendicular to the surface of the water, and if a plane passing through the centre of gravity of the piece, parallel to its base, coinciding with the water's surface, is found to rest in equilibrium, required the quantity of water that will run over from the vessel, and the specific gravity of the cone.

Operation.—The centre of gravity of a cone is distant from the vertex $\frac{3}{4}$ of its axis; *hence,* $4^3 : 3^3 :: 1728 : 729$ = *cubic inches immersed in the water.*

Consequently, the quantity of water run over will be 729 cubic inches, and the specific gravity of the water will be to that of the cone as 729 to 1728, or as 27 to 64, *Ans.*

67. A right cone cost \$1363, at \$120 per cubic foot, the diameter of its base being to its altitude as 5 to 8. It is required to have its *convex* surface divided in the same ratio by a plane parallel to the base, the upper part to be the greater; what is the slant height of each part?

$1363 \div 120 = 11.358$ = *the volume of the cone in feet;* and $5^2 \times .7854 \times \frac{8}{3} = 52.36$ = *volume of a cone similar to it, the altitude of which is* 8.

Again, the surface of similar solids being as the squares of their like dimensions, $\sqrt{(5+8)} : \sqrt{8} :: \sqrt{(2.5^2+8^2)}$ *the side of the said similar cone* $: \sqrt{\frac{562}{13}}$ = *the slant height of the upper part of this cone when its surface is divided in the ratio proposed.*

Consequently, $\sqrt{70\frac{1}{4}} - \sqrt{\frac{562}{13}} = \sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}}$ = *the slant height of the under part of it.*

Then, as similar solids are to each other as the cubes of their like dimensions, $\sqrt[3]{52.36} : \sqrt{11.358} :: \sqrt{\frac{562}{13}} : 3.9506$ = *the length of the slant height of the upper part.*

---

* Weight of a cubic foot of fresh water in ounces.

Again, $\sqrt[3]{52.36} : \sqrt[3]{11.358} :: \sqrt{\frac{562}{13}} - \sqrt{\frac{562}{8}} : 1.0855 =$ *the length of the slant height of the lower part.*

68. An elliptic inclosure has diameters of 840 and 612 links within its wall, which is 14 inches thick; required the area it incloses and covers.

Ans. { *It incloses* 4 *acres and* 6 *poles, and covers* 1760.5 *square feet.*

69. A block of marble, in the form of a square pyramid, weighs 18 tons, the perpendicular height being twice the diagonal of the base; required its dimensions and volume in cubic feet.

A cubic inch of the marble is assumed to weigh 1.6 ounces avoirdupois.

Ans. { *Side,* 4.35325 *feet.* 233.334 *cubic feet.*

70. A pail containing 2.1215 cubic feet is 12 inches in depth; what are its top and bottom diameters, they being in the proportion of 5 to 3?

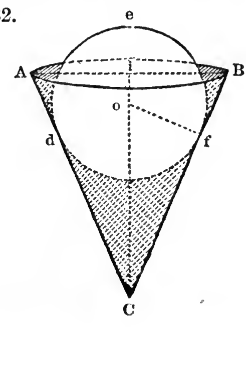
Ans. 14.64 *and* 24.4 *inches.*

71. If a sphere, the diameter of which is 4 inches, is depressed in a conical glass full of water, the diameter of which is 5 inches and altitude 6, it is required to know the quantity of water which will run over? *Ans.* 26.272 *inches.*

By Construction, Fig. 132.

Draw a section of the glass, as A B C, and of the sphere, *d e f.*

Fig. 132.



Then will A B $=5$; C $i=6$; B $i=5\div 2=2.5$; A C and B C $=\sqrt{(\text{C } i^2+\text{B } i^2)}=6.5$; and $o\,f=4\div 2$ (*half diameter of sphere*) $=2$.

The triangles B i C and $o\,f$ C are similar, for they have the common angle C, and the right angles i and f; *hence*, their remaining angles are equal.

Therefore, B i : B C :: $o\,f$: o C; that is, as $2.5:6.5::2:5.2=$ *the depth of the glass from where the centre of the sphere rests.*

Consequently, $5.2-6=.8$, and $2+.8=2.8$ *inches of the sphere immersed*, and $4-2.8=1.2$ *inches of it above the glass.*

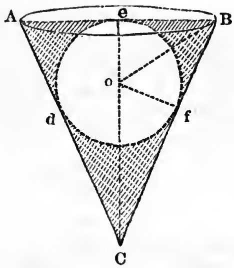
Hence, the volume of the segment immersed (by rule 2, p. 177) deducted from that of the sphere $=26.2722$ *cubic inches* $=$ *the result required.*

72. If a sphere is depressed in a conical glass full of water, the diameter of which is 5 inches and altitude 6, so that its upper edge is in a line with the rim of the glass, while its side rests upon the side of the glass; required the quantity of water that will overflow from the glass.

By Construction, Fig. 133.

Draw a section of the glass, as A B C, and of the sphere, as $d\,e\,f$.

Fig. 133.



Then will C $e=6$, B C $=\frac{5}{2}=2.5$, and, by preceding example, B C $=6.5$.

Hence, $o\,e$ and $o\,f$ are equal, and the line B o, bisecting B $e\,o\,f$, is a hypothenuse common to both triangles, B $e\,o$, B $f\,o$.

Consequently, B e and B f are equal to 2.5, and B C$-$B $f=$C $f=6.5-2.5=4$.

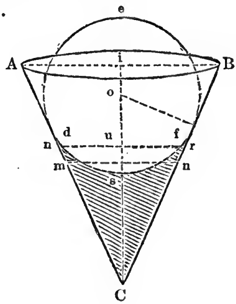
$\therefore 6$ (C e) : 2.5 (e B) :: 4 (C f) : 1.667 ($f\,o$), and $1.667\times 2=3.334$ *inches, the diameter of the sphere*, the volume of which (by rule, p. 176) $=19.387$ *inches* $=$ *the quantity of water that will overflow from the glass.*

73. If a sphere, the diameter of which is 4 inches, is depressed in a conical glass $\frac{1}{5}$ full of water, the diameter of which is 5 inches and altitude 6, it is required how much of the vertical axis of the sphere is immersed in water.

By Construction, Fig. 134.

Draw a section of the glass, as A B C, and of the sphere, *d e f*.

Fig. 134.



Let *m n* be the original level of the water, and *n r* the level when the sphere is immersed.

Then will the cone *n* C *r*=*cone m* C *n*+*the volume of the segment of the sphere d s f*=$\frac{1}{5}$ *of cone* A B C+*the volume of the segment d s f.*

A C=$\sqrt{\text{C}\,i^2(6)+\text{A}\,i^2(5\div2)}$=6.5=*length of slant side.*

As A *i* (2.5) : A C (6.5) : : *o f*, *radius of sphere* (2) : C *o* (5.2)=*distance from the centre of sphere at rest and the bottom of the glass.*

C *s*=C *o*−*o s*=5.2−2=3.2.

Contents of cone (by rule, p. 166)=39.27, $\frac{1}{5}$ of which=7.854.

Put x=*s u*=*the immersed part of the axis of the sphere*, and C *u*=C *s* +*s u*=3.2+x.

Then, as similar solids are to each other as the cubes of their like dimensions,

$6^3:(3.2+x)^3::39.27:$ cone *n* C *r*; $\therefore$ cone *n* C $r=\dfrac{(3.2+x)^3}{216}\times 39.27$.

Segment *d s f* (by rule 2, p. 177)=$(4\times3-2x)\times x^2\times .5236$.

Since Cone *n* C *r*=cone *m* C *n*+segment *d s f*, $\therefore \dfrac{(3.2+x)^3}{216}\times39.27=$ $7.854+(4\times3-2x)\times x^2\times .5236$.

And $25\times(3.2+x)^3=5\times216+4\times6^2\times(6-x)\times x^2$.

Cube $3.2+x$ and $25\ (3.2^3+3\times3.2^2x+3\times3.2x^2+x^3)=5\times6^3+4\times6^2$ $(6-x)\times x^2$.

*Actually multiplying the terms in the first member by 25,**

$\frac{16^3}{5}+3\cdot16^2x+3\cdot5\cdot16x^2+25x^3=5\cdot6^3+4\cdot6^2(6-x)x^2$.†

But $4\cdot6^2(6-x)x^2=4\cdot6^3x^2-4\cdot6^2x^3$.

$\therefore\frac{16^3}{5}+3\cdot16^2x+3\cdot5\cdot16x^2+25x^3=5\cdot6^3+4\cdot6^3x^2-4\cdot6^2x^3$.

Multiplying by 5 *throughout,*

$16^3+3\cdot5\cdot16^2x+3\cdot5^2\cdot16x^2+5^3x^3=5^2\,6^3+4\cdot5\cdot6^3x^2-4\cdot5\cdot6^2x^3$.

By transposition,

$(5^3-4\cdot5\cdot6^2)x^3+(3\cdot5^2\cdot16-4\cdot5\cdot6^3)x^2+3\cdot5\cdot16^2x=5^2\cdot6^3-16^3$.

But, $5^3+4\cdot5\cdot6^2=\frac{845}{5}=169=\left(\frac{169}{13}\right)^2\times5=13^2\times5$.

$3\cdot5^2\cdot16-4\cdot5\cdot6^3=-3120=-\frac{3120}{3},\ \frac{1040}{5},\ \frac{208}{13}=16=3\cdot5\cdot13\cdot16$
$=3120$.

$5^2\cdot6^3-16^3=1304=\frac{1304}{8}=163=8\cdot163$.

$3\cdot5\cdot16^2=\frac{3840}{3},\ \frac{1280}{5},\ \frac{256}{16},\ 16=3\cdot5\cdot16^2=3840$.

$\therefore5\cdot13^2x^3-3\cdot5\cdot13\cdot16x^2+3\cdot5\cdot16^2x=8\cdot163$.

$\therefore5(13^2x^3-3\cdot13\cdot16x^2+3\cdot16^2x)=8\cdot163$.

Or, $13^2x^3-3\cdot13\cdot16x^2+3\cdot16^2x=\frac{8\cdot163}{5}$.

Multiplying both members by 13,

$13^3x^3-3\cdot13^2\cdot16x^2+3\cdot13\cdot16^2x=\frac{8\cdot13\cdot163}{5}=\frac{16952}{5}=3390$.

Subtract 16^3 *from both numbers.*

$13^3x^3-3\cdot13^2\cdot16x^2+3\cdot13\cdot16^2x-16^3=\frac{8\cdot13\cdot163}{5}-16^3=$
$\frac{8\cdot13\cdot163-5\cdot16^3}{5}=\frac{-3528}{5}=-705\cdot6$.

The first member is now a perfect cube, *the root of which is*
$13x-16=\sqrt[3]{-705.6}=8.90265$.
$13x=16-8.90265=7.09735$.

$x=\frac{7.09735}{13}=.54595$ *inch.*

PROOF. C $s=3.2$, and $s\,u=.54595$.
Hence, C $u=3.2+.54595=3.74595$.

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* By Professor G. B. Docharty, New York.

† See Explanation of Characters, page 10, for use of a period between two factors.

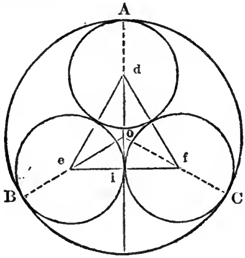
Then, $6^3 : 39.27 :: 3.74595^3 : 9.5563 =$ *volume of water in cone*, $n\,C\,r$, from which is to be deducted the volume of segment $d\,s\,f$, and the remainder should be equal to $\frac{1}{5}$ of $39.27 = 7.854$.

Thus, volume of segment (by rule 2, p. 177)$=1.7023$, and $9.5563 - 1.7023 = 7.854 =$ *result required.*

74. A lady having three daughters had a farm of 450.758 acres, in a circular form, with her dwelling-house in the centre. Being desirous of having her daughters near her, she gave to them three equal parcels of land as large as could be made in three equal circles within the periphery of her farm, one to each, with a dwelling-house in the centre of each; that is, there were to be three equal circles as large as could be drawn within the periphery of the farm; required the diameters of the farm and of the three parcels.

$\sqrt{450.758 \times 43560 \div .7854} = 5000$ *feet = diameter of farm.*

By Construction, Fig. 135.



Draw the given circle, with o as its centre, and divide its periphery into three equal parts, as at A B C; connect A o, B o, and C o, and assume $d\,e\,f$ as the centres of the required circles.

As the three circles required touch one another and the given circle, the points, as A, B, and C, the centres, $d\,e\,f$, of the required circles, and o, are necessarily in right lines. Connect d, e, and f.

Then, as $d\,e\,f$ and $e\,o\,f$ are isosceles triangles, the angle d and the base $e\,f$ are bisected at right angles in i by the line $d\,i$, and $e\,o$, in like manner, bisects the angle e.

The triangles, $e\,d\,i$, $e\,o\,i$, are equiangular:

Hence, $e\,d : e\,i = (\frac{1}{2}e\,d) :: e\,o : o\,i = (\frac{1}{2}e\,o)$; $\therefore e\,o = 2\,o\,i$.

Put B $e=x=(e\ i)$, R=B $o=2500=$*radius of given circle*$=\left(\frac{5000}{2}\right)$;

then, $e\ o=R-x$. $\therefore \frac{1}{2} e\ o=o\ i=\frac{R-x}{2}$.

$e\ o^2=e\ i^2+o\ i^2$, *or* $(R-x)^2=x^2+\frac{(R-x)^2}{4}$.

Or, $R^2-2Rx+x^2=x^2+\frac{R^2-2Rx+x^2}{4}$;

Or, $4R^2-8Rx=R^2-2Rx+x^2$.

Transposing the formulæ,

$x^2+6Rx=3R^2$: $x^2+6Rx+9R^2=3R^2+9R^2=12R^2$.

$x+3R=\pm\sqrt{12R^2}$: $x=\pm\sqrt{12R^2}-3R$.

$x=\pm R\sqrt{12}-3R$: $x=\pm R(\sqrt{12}-3)=$*radius of required circles.*

$\therefore$ Radius being$=\sqrt{12}-3=3.4641-3=.4641$, *which is a constant multiplier for all like problems.*

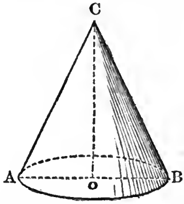
Consequently, $2500\times.4641=1160.25=$A d, B e, and C f=*product of radius of circle and multiplier=radii of each of the circles required.*

75. The weight of a quantity of silt in 30 cubic inches of salt water is 4.21 grains, assuming the weights of silt and salt water to be respectively 125 and 64 pounds per cubic foot; what is the volume of the silt compared to that of the water? *Ans.* 1. *to* 3608.307.

CONIC SECTIONS.

Definition. *Plane figures generated by the cutting of a cone.*

A *Cone* is a figure described by the revolution of a right-angled triangle about one of its legs.



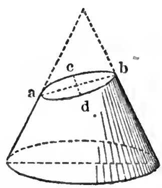
The *axis* (of a cone) is the line about which the triangle revolves, as C *o*.

The *base* is the circle which is described by the revolving base of the triangle, as B *o*.

Notes.—If a cone is cut by a plane through the vertex and base, the section will be a triangle, as A C B.

If a cone is cut by a plane parallel to its base, the section will be a circle.

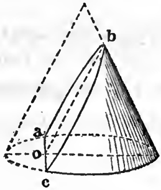
An *Ellipse* is a figure generated by an oblique plane cutting a cone, as *a b c d*.



The *transverse axis* or diameter (of an ellipse) is the longest right line that can be drawn in it, as *a b*.

The *conjugate axis* or diameter is a line drawn through the centre of the ellipse perpendicular to the transverse axis, as *c d*.

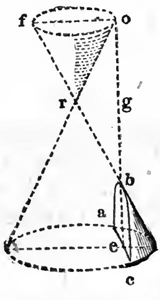
A *Parabola* is a figure generated by a plane cutting a cone parallel to its side, as *a b c.*



The *axis* (of a parabola) is a right line drawn from the vertex to the middle of the base, as *b o.*

NOTE.—A parabola has no conjugate diameter.

An *Hyperbola* is a figure generated by a plane cutting a cone at any angle with the base greater than that of the side of the cone, as *a b c.*



The *transverse axis* or diameter, *o b* (of an hyperbola), is that part of the axis *e b*, which, if continued, as at *o*, would join an opposite cone, *o f r.*

The *conjugate axis* or diameter is a right line drawn through the centre, *g*, of the transverse axis, and perpendicular to it.

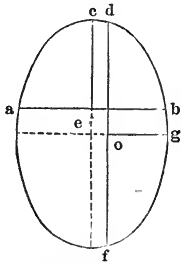
The straight line through the *foci* is the indefinite transverse axis; that part of it between the vertices of the curves, as *o b*, is the definite transverse axis. Its middle point, *g*, is the centre of the curve.

The *eccentricity* of an hyperbola is the ratio obtained by dividing the distance from the centre to either *focus* by the semitransverse axis.

The *asymptotes* of an hyperbola are two right lines to which the curve continually approaches, touches at an infinite distance, but does not pass; they are prolongations of the diagonals of the rectangle constructed on the extremes of the axes.

Two hyperbolas are *conjugate* when the transverse axis of the one is the conjugate of the other, and contrariwise.

An *Ordinate* is a right line from any point of a curve to either of the diameters, as *a e* and *d o*; *a b* and *d f* are double ordinates.



An *abscissa* is that part of the diameter which is contained between the vertex and an ordinate, as *c e*, *g o*.

The *parameter* of any diameter is equal to four times the distance from the *focus* to the vertex of the curve; the parameter of the axis is the least possible, and is termed the parameter of the curve.

The *parameter* of the curve of a conic section is equal to the chord of the curve drawn through the *focus* perpendicular to the axis.

The *parameter* of the transverse axis is the least, and is termed the parameter of the curve.

The *parameter* of a conic section and the *foci* are sufficient elements for the construction of the curve.

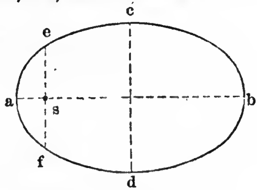
In the *Parabola* the parameter of any diameter is a third proportional to the abscissa and ordinate of any point of the curve, the abscissa and ordinate being referred to that diameter and the tangent at its vertex.

In the *Ellipse* and *Hyperbola*, the parameter of any diameter is a third proportional to the diameter and its conjugate.

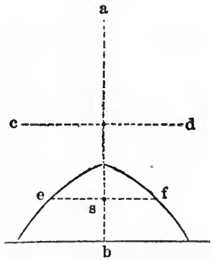
Note.—*To determine the Parameter of an Ellipse or Hyperbola.*

Rule. Divide the product of the conjugate diameter, multiplied by itself, by the transverse, and the quotient is equal to the parameter.

In the annexed figures of an *Ellipse* and *Hyperbola*, the transverse and conjugate diameters, *a b*, *c d*, are each 30 and 20.

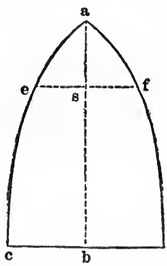


Then, 30 : 20 : : 20 : 13.333 = *parameter*.



Hence, the parameter of the curve = *e f*, a double ordinate passing through the *focus s*.

In a *Parabola*. The abscissa *a b*, and ordinate *c b*, are also equal to 30 and 20.



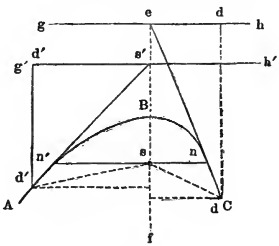
Hence, the parameter of the curve = *e f*.

A *Focus* is a point on the principal axis where the double ordinate to the axis, through the point, is equal to the parameter, as *e f* in the preceding figures.

It may be determined arithmetically thus: Divide the square of the ordinate by four times the abscissa, and the quotient will give the focal distances *a s* and *s* in the preceding figures.

The *Directrix* of a conic section is a straight line, such that the ratio obtained by dividing the distance from any point of the curve to it by the distance from the same point to the *focus* shall be constant.

It is always perpendicular to the principal axis; and if the curve is given, it is constructed as follows:



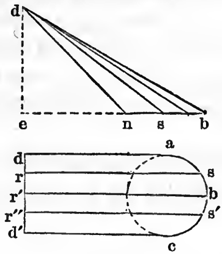
Let A B C represent the curve or curves, *e f* their axis, and *s* the focus.

Through *s* draw *s n* or *n′* perpendicular to the axis till it meets the curve in *n* or *n′*; at *n*, *n′* draw the tangents *n e* or *n′ e′*, cutting the axis at *e* and *e′*; through *e*, *e′* draw *g h* and *g′ h′* perpendicular to the axis, and they will be the *directrices* of two conic sections.

If *d s*, drawn from any point, as *d*, $>$ (is less than) *d d*, the curve is an ellipse; if equal to each other, it is the curve of a parabola; and $<$ (if greater), as *d′ s*, *d′ d′*, it is the curve of an hyperbola.

Ellipsoid, *Paraboloid*, and *Hyperboloid of Revolution*. Figures generated by the revolution of an ellipse, parabola, etc., around their axes. (See p. 124 and 202.)

A *Conoid* is a warped surface generated by a right line being moved in such a manner that it will touch a straight line and curve, and continue parallel to a given plane. The straight line and curve are called *directrices,* the plane a plane *directrix,* and the moving line the *generatrix.*



Thus, let $a\ b\ c$ be a circle in a horizontal plane, and $d\ d'$ the projection of a right line perpendicular to a vertical plane, $d\ e$; if right lines, $d\ a$, $r\ s$, $r'\ b$, $r''\ s$, and $d'\ c$, be moved so as to touch the circle and right line $d\ d'$, and be constantly parallel to the plane $r\ b$, it will generate the conoid $d\ b\ s\ n$.

NOTE.—All the figures which can possibly be formed by the cutting of a cone are mentioned in these definitions, and are the five following, viz., a *triangle*, a *circle*, an *ellipse*, a *parabola*, and an *hyperbola;* but the last three only are termed the *conic sections.*

ELLIPSE.

To describe an Ellipse.

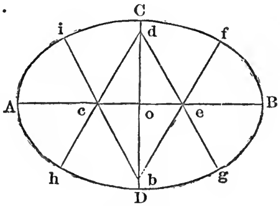
The Transverse and Conjugate Diameters being given, Fig. 1.

RULE 1.—Draw the transverse and conjugate diameters, A B, C D, bisecting each other perpendicularly in *o*.

Make A *e* equal to D C; divide *e* B into three equal parts; set off two of those parts from *o* to *e* and from *o* to *c*; then with the distance *c e* make the two equilateral triangles *c b e* and *c d e*.

These angles are the centres, and the sides being continued are the lines of direction for the several arcs of the ellipse A C B D.

Fig. 1.



NOTE.—Mechanics are oftentimes required to work an architrave, etc., about windows, of this form; they may, by the help of the four centres *c*, *d*, *e*, *b*, and the lines of direction *h d*, *b f*, *d g*, *b i*, describe another ellipse around the former, and at any distance required.

RULE 2.—Draw the line C D equal in length to the transverse diameter; also, E F equal in length to the conjugate diameter, and at right angles with C D.

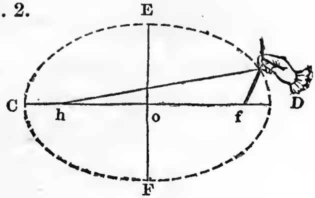
Take the distance C *o* or *o* D, and with it, from the points E and F, intersect the diameter C D at *h* and *f*, which points are the *foci*.

Secure a string at *h* and *f* of such a length that it may just reach to E or F.

Introduce a pencil, and bearing upon the string, carry it around the centre *o*, and it will describe the ellipse required.*

* It is a property of the ellipse that the sum of two lines drawn from the *foci* to meet in any point in the curve is equal to the transverse di-

Fig. 2.

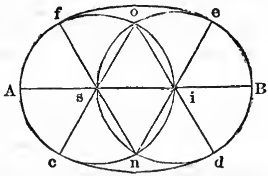


The Transverse Diameter alone being given, Fig. 3.

RULE.—Let A B be the given length.

Divide it into three equal parts, as A *s i b*. Then, with the radius A *s*, describe A *f o i n c*, and from *i* the circle B *d n s o e;* then with *n f* and *o c* describe *f e* and *c d*, and the required ellipse is made.

Fig. 3.



When any three of the four following Terms of an Ellipse are given, viz., the Transverse and Conjugate Diameters, an Ordinate, and its Abscissa, to find the remaining Term.

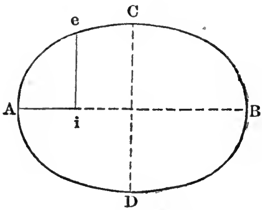
To ascertain the Ordinate, the Transverse and Conjugate Diameters and the Abscissa being given, Fig. 4.

RULE.—As the transverse diameter is to the conjugate, so is the square root of the product of the two abscissas to the ordinate which divides them.

Or, $\frac{c}{t} \times \sqrt{a \times (t - a')} = o$, *t representing the transverse diameter, c the conjugate, a′ the less abscissa, and o the ordinate.*

ameter, and from this the correctness of the above construction is evident.

Fig. 4.



EXAMPLE.—The transverse diameter, A B, of an ellipse, *Fig.* 4, is 25 inches, the conjugate, C D, is 16, and the abscissa A *i*, 7; what is the length of the ordinate *i e?*

$25-7=18=$*second abscissa.*
$\sqrt{7\times18}=11.225=$*square root of the abscissæ.*
Hence, 25 : 16 : : 11.225 : 7.184 *inches, the length of the ordinate required.*

Ex. 2. The transverse diameter or axis of an ellipse is 100 inches, the conjugate 60, one abscissa 20, and the other 80; what is the length of the ordinate? *Ans.* 24 *inches.*

To ascertain the Abscissæ, the Transverse and Conjugate Diameters and the Ordinate being given, Fig. 4.

RULE.—As the conjugate diameter is to the transverse, so is the square root of the difference of the squares of the ordinate and semi-conjugate to the distance between the ordinate and centre, and this distance being added to, or subtracted from the semi-transverse, will give the abscissæ required.

Or, $\frac{t}{c}\times\sqrt{-o^2\left(\frac{c}{2}\right)^2}=\begin{cases}\frac{t}{2}+x=a, \\ \frac{t}{2}-x=a',\end{cases}$ *x representing the distance obtained, and a a' the greater and less abscissæ.*

EXAMPLE.—The transverse diameter, A B, of an ellipse, *Fig.* 4, is 25 inches, the conjugate, C D, 16, and the ordinate *i e* 7.184; what is the abscissa *i* B?

$\sqrt{7.184^2-8^2}=3.519943=$*square root of difference of squares of semi-conjugate and ordinate.*
Hence, as 16 : 25 : : 3.52 : 5.5 $=$*distance between ordinate and centre.*

$$\left.\begin{array}{l} \textit{Then, } \frac{25}{2}=12.5, \text{ and } 12.5+5.5=18=\text{B } i, \\ \frac{25}{2}=12.5, \text{ and } 12.5-5.5=\ 7=\text{A } i, \end{array}\right\} \textit{abscissæ required.}$$

Ex. 2. The transverse diameter, A B, of an ellipse is 50 inches, the conjugate, C D, 32, and the ordinate $i\ e$ 14.368; what are the lengths of the abscissæ?

Ans. 11 *and* 39 *inches.*

To ascertain the Transverse Diameter, the Conjugate, Ordinate, and Abscissa being given, Fig. 4.

Rule.—To or from the semi-conjugate, according as the greater or less abscissa is used, add or subtract the square root of the difference of the squares of the ordinate and semi-conjugate.

Then, as this sum or difference is to the abscissa, so is the conjugate to the transverse.

$$\text{Or, } \frac{a \times c}{\left.\begin{array}{l} c \div 2 + \\ c \div 2 - \end{array}\right\} \sqrt{o^2 - \left(\frac{c}{2}\right)^2}} = t.$$

Example.—The conjugate diameter, C D, of an ellipse, *Fig.* 4, is 16 inches, the ordinate $i\ e$ is 7.184, and the abscissæ B i, i A are 18 and 7; what is the length of the transverse diameter?

$\sqrt{7.184^2 - \left(\frac{16}{2}\right)^2} = 3.52 =$ *square root of difference of squares of ordinate and semi-conjugate.*

$$\left.\begin{array}{l} \frac{16}{2}+3.52 : 18 :: 16 : 25, \\ \frac{16}{2}-3.52 :\ 7 :: 16 : 25, \end{array}\right\} = \textit{transverse diameter required.}$$

Ex. 2. The conjugate diameter of an ellipse is 60 inches, the ordinate 24, and the abscissa 20; what is the length of the transverse diameter?

Ans. 100 *inches.*

To ascertain the Conjugate Diameter, the Transverse, Ordinate, and Abscissa being given, Fig. 4.

RULE.—As the square root of the product of the abscissæ is to the ordinate, so is the transverse diameter to the conjugate.

Or, $\overline{o \times t} \div \sqrt{a \times a'} = c$.

EXAMPLE.—The transverse diameter, A B, of an ellipse, *Fig.* 4, is 25 inches, the ordinate *i e* 7.184, and the abscissæ B *i* and *i* A 18 and 7; what is the length of the conjugate diameter?

$\sqrt{18 \times 7} = 11.225 =$ *square root of product of abscissæ.*
$11.225 : 7.184 :: 25 : 16 =$ *conjugate diameter required.*

Ex. 2. The transverse diameter of an ellipse is 100 inches, the ordinate 24, and the abscissæ 20 and 80; what is the length of the conjugate diameter? *Ans.* 60 *inches.*

To ascertain the Circumference of an Ellipse, Fig. 4.

RULE.—Multiply the square root of half the sum of the squares of the two diameters by 3.1416, and this product will give the circumference nearly.

Or, $\sqrt{\dfrac{d^2 + d'^2}{2}} \times 3.1416 =$ *circumference.*

EXAMPLE.—The transverse and conjugate diameters, A B and C D, of an ellipse, *Fig.* 4, are 24 and 20 inches; what is its circumference?

$\dfrac{24^2 + 20^2}{2} = 488$, and $\sqrt{488} = 22.09 =$ *square root of half the sum of the squares of the diameters.*

Hence, $22.09 \times 3.1416 = 69.398 =$ *the above root* $\times 3.1416 =$ *the result required.*

Ex. 2. The diameters of an ellipse are 30 and 20 inches; what is its circumference? *Ans.* 80.0951 *inches.*

To ascertain the Area of an Ellipse, Fig. 4.

RULE.—Multiply the diameters together, the product by .7854, and the result will give the area required.

Or, multiply one diameter by .7854, and the product by the other.

Or, $d \times d' \times .7854 = area$.

EXAMPLE.—The transverse diameter of an ellipse, A B, *Fig.* 4, is 12 inches, and its conjugate, C D, 9; what is its area?

$12 \times 9 \times .7854 = 84.8232 =$ *product of diameters and* $.7854 =$ *result required.*

Ex. 2. The diameters of an ellipse are 70 and 50 feet; what is its area? *Ans.* 2748.9 feet.

Centre of Gravity. Is in its geometrical centre.

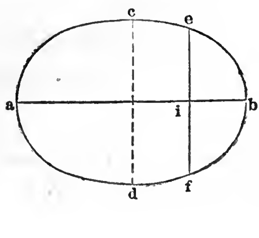
Segment of an Ellipse.

To ascertain the Area of a Segment of an Ellipse when its base is parallel to either axis, as e i f, Fig. 5.

RULE.—Divide the height of the segment *b i* by the diameter or axis, *a b*, of which it is a part, and find in the table of areas of segments of a circle, p. 134–138, a segment having the same versed sine as this quotient; then multiply the area of the segment thus found and the two axes of the ellipse together, and the product will give the area required.

Or, $h \div d \times tab.\ area \times d \,.\, d' = area$.

Fig. 5.



EXAMPLE.—The height, *b i*, *Fig.* 5, is 5 inches, and the axes of the ellipse are 30 and 20; what is the area of the segment?

$\frac{5}{30}$=.1666=*tabular versed sine, the area of which* (p. 135) *is* .08554.

Hence, .08554×30×20=51.324=*the area required.*

Ex. 2. The height of a segment at right angles or perpendicular to the transverse diameter of an ellipse is 6.25 inches, and the diameters are 16 and 25? what is the area of the segment? *Ans.* 61.42 *inches.*

Ex. 3. The height of a segment of an ellipse at right angles to the conjugate diameter is 25 inches, and the diameters are 50 and 70; what is the area of the segment?

Ans. 1374.415 *inches.*

The area of an elliptic segment may also be found by the following rule:

Ascertain the segment of the circle described upon the same axis to which the base of the segment is perpendicular.

Then, as this axis is to the other axis, so is the circular segment to the elliptical segment.

ILLUSTRATION.—In the above example, the axis to which the base of the segment is perpendicular is the conjugate, 50, and the height of the segment 25. *Also, the area of the segment is one half of that of a circle of* 50 *diameter*=$\frac{1963{:}4954}{2}$=981.7472.

Hence, 50:70::981.75:1374.45=*area of elliptic segment.*

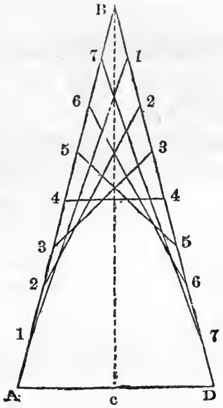
PARABOLA.

1. *To describe a Parabola, the Base and Height being given, Fig.* 6.

OPERATION.—Draw an isosceles triangle, as A B D, the base of which shall be equal to, and its height, B *c*, twice that of the proposed parabola.

Divide each side, A B, D B, into any number of equal parts; then draw lines, 1 1, 2 2, 3 3, etc.; and their intersection will define the curve of a parabola.

Fig. 6.



2. *To describe a Parabola, any Ordinate to the Axis and its Abscissa being given, Fig.* 7.

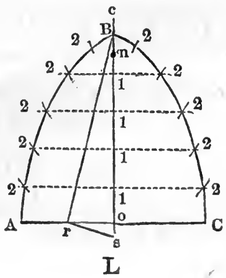
OPERATION.—Bisect the ordinate, as A *o* in *r;* join B *r*, and draw *r s* perpendicular to it, meeting the axis continued to *s*.

Draw B *c* and B *n* each equal to *o s*, and *n* will be the focus of the curve.

Take any number of points, 1 1, etc., on the axis, through which draw the double ordinates 2 1 2, etc., of an indefinite length. Then, with the radii *c n*, *c* 1, etc., and focal centre *n*, describe arcs cutting the corresponding ordinates in the points 2 2, etc., and the curve A B C, drawn through all the points of intersection, will define the parabola required.

NOTE.—The line 2 *n* 2, passing through the focus *n*, is the parameter.

Fig. 7.



To ascertain either Ordinate or Abscissa of a Parabola, the other Ordinate and the Abscissæ, or the other Abscissa and the Ordinates being given, Fig. 8.

RULE.—As either abscissa is to the square of its ordinate, so is the other abscissa to the square of its ordinate.

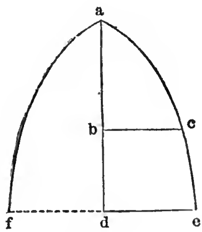
Or, 1. $\dfrac{o^2 \times a'}{a} = o'^2$. 2. $\dfrac{o'^2 \times a}{a'} = o^2$.

3. $\dfrac{o^2 \times a'}{o'^2} = a$. 4. $\dfrac{o'^2 \times a}{o^2} = a'$.

Or, *as the square root of any abscissa is to its ordinate, so is the square root of any other abscissa to its ordinate.*

Hence, $\dfrac{o \times \sqrt{a'}}{\sqrt{a}} = o'$.

Fig. 8.



EXAMPLE.—The abscissa $a\ b$, of the parabola, *Fig.* 8, is 9, its ordinate, $b\ c$, 6; what is the ordinate $d\ e$, the abscissa of which, $a\ d$, is 16?

Hence, $9 : 6^2 :: 16 : 64$, and $\sqrt{64} = 8 =$ *length of ordinate required*
Or, $\sqrt{9} : 6 :: \sqrt{16} : 8 =$ *ordinate as before.*

Ex. 2. The less abscissa of a parabola is 16, its ordinate 10; what is the ordinate, the abscissa of which is 36?

Ans. 15.

Ex. 3. The abscissæ of a parabola are 9 and 16, and their corresponding ordinates 6 and 8; any three of these being taken, it is required to find the fourth.

1. $\frac{6^2 \times 16}{9} = \frac{576}{9} = 64$, and $\sqrt{64} = 8 =$ *ordinate.*

2. $\frac{8^2 \times 9}{16} = \frac{576}{16} = 36$, and $\sqrt{36} = 6 =$ *ordinate.*

3. $\frac{6^2 \times 16}{8^2} = \frac{576}{64} = 9 =$ *less abscissa.*

4. $\frac{8^2 \times 9}{6^2} = \frac{576}{36} = 16 =$ *abscissa.*

Parabolic Curve.

To ascertain the Length of the Curve of a Parabola cut off by a Double Ordinate.

RULE.—To the square of the ordinate add $\frac{4}{3}$ of the square of the abscissa, and the square root of this sum, multiplied by two, will give the length of the curve nearly.

Or, $\sqrt{\left(o^2 + \frac{4\,a^2}{3}\right)} \times 2 =$ *length of curve.*

EXAMPLE.—The ordinate, *d e*, *Fig.* 8, is 8, and its abscissa, *a d*, 16; what is the length of the curve *f a e*?

$8^2 + \frac{4 \times 16^2}{3} = 405.333 =$ *sum of square of the ordinate and* $\frac{4}{3}$ *of the square of the abscissa*, and $\sqrt{405.333} = 20.133$, which $\times 2 = 40.267 =$ *length required.*

NOTE.—This rule can be used only when the abscissa does not exceed half the ordinate. The length of the curve in other cases is to be found by means of hyperbolic logarithms, as shown by writers on fluxions.

Ex. 2. The ordinate is 16 inches, and its abscissa 32; what is the length of the curve? *Ans.* 80.533 *inches.*

Ex. 3. The abscissa is 20, and its ordinate 12 inches; what is the length of the curve? *Ans.* 52.05 *inches.*

Ex. 4. The abscissa is 5 feet, and its ordinate 3; what is the length of the curve? *Ans.* 13.014 *feet.*

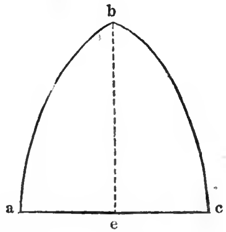
Parabola.

To ascertain the Area of a Parabola, Fig. 9.

RULE.—Multiply the base by the height and $\frac{2}{3}$ of the product will give the area required.*

Or, $\frac{2}{3}\, b \times h = area.$

Fig. 9.



EXAMPLE.—What is the area of the parabola *a b c*, *Fig.* 9, the height, *b e*, being 16 inches, and the base, or double ordinate, *a c*, 16?

$16 \times 16 = 256 =$ *product of base and height*, and $\frac{2}{3}$ of $256 = 170.667 =$ *area required.*

Ex. 2. The height of an abscissa of a parabola is 32 inches, and the base or double ordinate is 16; what is its area?

Ans. 341.333 *square inches.*

Ex. 3. The height of a parabola is 50 inches, and its base 24; what is its area? *Ans.* 800 *square inches.*

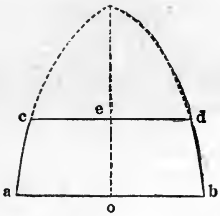
To ascertain the Area of a Frustrum of a Parabola, Fig. 10.

RULE.—Multiply the difference of the cubes of the two ends of the frustrum, *a b*, *c d*, by twice its altitude, *e o*, and divide the product by three times the difference of the squares of the ends.

Or, $\frac{d^3 \backsim d'^3 \times 2h}{d^2 \backsim d'^2 \times 3} = area$, *d and d′ reoresenting the lengths of the base and lesser end.*

* *Corollary.*—A parabola is $\frac{2}{3}$ of its circumscribing parallelogram.

Fig. 10.



EXAMPLE.—The ends of a frustrum of a parabola, *a b* and *c d*, *Fig.* 10, are 10 and 6 inches, and the height, *e o*, is 10 inches; what is its area?

$10^3 \backsim 6^3 \times \overline{10 \times 2} = 15680$ = *difference of cubes of the ends* × *twice the height.*

$15680 \div \overline{10^2 \backsim 6^2 \times 3} = 81.667$ = *preceding product* ÷ 3 *times the difference of the squares of the ends* = *area required.*

Ex. 2. The base and lesser end of a frustrum of a parabola are 30 and 24 inches, and the height 22 inches; what is its area? *Ans.* 596.444 *inches.*

Ex. 3. The base and upper end of a frustrum of a parabola are 30 and 20 inches, and the height 20 inches; what is the area? *Ans.* 506.667 *inches.*

Ex. 4. The ends of the frustrum of a parabola are 5 and 4 feet, and its height 2.5 feet; what is its area? *Ans.* 11.296 *feet.*

Ex. 5. The ends of a frustrum of a parabola are 8.5 and 7 feet, and its height 6 feet; what is its area? *Ans.* 46.645 *feet.*

NOTE.—Any parabolic frustrum is equal to a parabola of the same altitude, the base of which is equal to the base of the frustrum, increased by a third proportional to the sum of the two ends and the lesser end.

Illustration.—In Example 1, the base and end are 10 and 6.

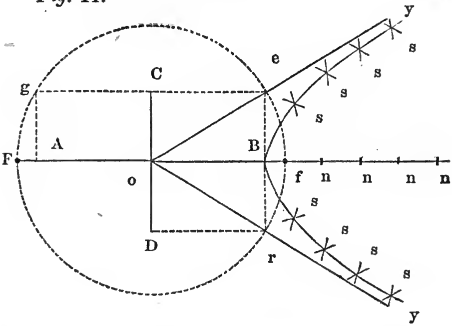
Then, $10 + 6 : 6 :: 6 : 2.25$ = *third proportional to the sum of the two ends and the lesser end.*

Hence, $10 + 2.25 = 12.25$ = *sum of length of base of parabola and third proportional,* the area of which, the height being 10, is = 81.667.

HYPERBOLA.

To describe an Hyperbola, the Transverse and Conjugate Diameters being given, Fig. 11.

Fig. 11.



Let A B represent the transverse diameter, and C D the conjugate.

Draw C *e* parallel to A B, and *e r* parallel to C D; draw *o e*, and with the radius *o e*, with *o* as a centre, describe the circle F *e f r*, cutting the transverse axis produced in F and *f*; then will F and *f* be the foci of the figure.

In *o* B produced take any number of points, *n*, *n*, etc., and from F and *f* as centres, with A *n* and B *n* as radii, describe arcs cutting each other in *s*, *s*, etc. Through *s*, *s*, etc., draw the curve *s s s* B *s s s*, and it will be the hyperbola required.

NOTE.—If straight lines, as *o e y* and *o r y*, are drawn from the centre *o* through the extremities *e r*, they will be the asymptotes of the hyperbola, the property of which is to approach continually to the curve, and yet never to touch it.

When the Foci and the Conjugate Axis are given.

Let F and *f* be the foci, and C D the conjugate axis, as in the preceding figure.

Through C draw *g* C *e* parallel to F and *f*; then, with *o* as a centre and *o* F as a radius, describe an arc cutting *g* C *e* at *g* and *e*; from these points let fall perpendiculars upon the line connecting F and *f*, and the part intercepted between them, as A B, will be the transverse axis.

To ascertain the Ordinate of an Hyperbola, the Transverse and Conjugate Diameters and the Abscissa being given, Fig. 12.

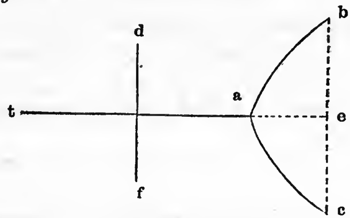
RULE.—As the transverse diameter is to the conjugate, so is the square root of the product of the abscissæ to the ordinate required.

Or, $\frac{c \times \sqrt{a \times a'}}{t} = ordinate.$

NOTE.—1. In hyperbolas, the less abscissa, added to the axis (the transverse diameter), gives the greater.

2. The difference of two lines drawn from the *foci* of any hyperbola to any point in the curve is equal to its transverse diameter.

Fig. 12.



EXAMPLE.—The hyperbola, *a b c*, *Fig.* 12, has a transverse diameter, *a t*, of 120 inches, a conjugate, *d f*, of 72, and the abscissa, *a e*, is 40; what is the length of the ordinate *e c*?

$40+120=160=$ *sum of less abscissa and axis=greater abscissa.*
$120:72::\sqrt{(40\times160)}:48=$ *ordinate required.*

Ex. 2. The transverse diameter of an hyperbola is 25, the conjugate 15, and the less abscissa 6 inches; what is the length of the ordinate? *Ans.* 8.1829 *inches.*

Ex. 3. The transverse diameter of an hyperbola is 5 feet, the conjugate 3, and the less abscissa 1.8 feet; what is the length of the ordinate? *Ans.* 2 *feet.*

Ex. 4. The transverse diameter of an hyperbola is 5 feet, the conjugate 3, and the greater abscissa 8.667 feet; what is the length of the ordinate? *Ans.* 3.67 *feet.*

To ascertain the Abscissæ, the Transverse and Conjugate Diameters and the Ordinate being given, Fig. 12.

RULE.—As the conjugate diameter is to the transverse, so is the square root of the sum of the squares of the ordinate and semi-conjugate to the distance between the ordinate and the centre, or half the sum of the abscissæ.

Then, the sum of this distance and the semi-transverse will give the greater abscissa, and their difference the less abscissa.

Or, $\frac{t\sqrt{o^2+(c\div 2)^2}}{c}=\frac{a+a'}{2}=$ *half the sum of the abscissæ.*

$\frac{a+a'}{2}+\frac{t}{2}=a$, and $\frac{a+a'}{2}-\frac{t}{2}=a'$.

EXAMPLE.—The transverse diameter, $a\ t$, of an hyperbola, *Fig.* 12, is 120 inches, the conjugate, $d\ f$, 72, and the ordinate, $e\ c$, 48; what are the lengths of the abscissæ, $t\ e$ and $a\ e$?

$72:120::\sqrt{48^2+(72\div 2)^2}=60:100=$ *half the sum of the abscissæ.*

$100+(120\div 2)=160=$ *above sum added to the semi-transverse = the greater abscissa,* and

$100-(120\div 2)=40=$ *above sum subtracted from the semi-transverse= the less abscissa.*

Ex. 2. The transverse and conjugate diameters of an hyperbola are 25 and 15 inches, and the ordinate 8.1829; what are the lengths of the abscissæ? *Ans.* 6 *and* 31 *inches.*

To ascertain the Conjugate Diameter, the Transverse Diameter, the Abscissæ, and Ordinate being given, Fig. 12.

RULE.—As the square root of the product of the abscissæ is to the ordinate, so is the transverse diameter to the conjugate.

Or, $\frac{o\times t}{\sqrt{(a\times a')}}=$ *conjugate diameter.*

EXAMPLE.—The transverse diameter, $a\ b$, of an hyperbola, *Fig.* 12, is 120, the ordinate, $e\ c$, 48, and the abscissæ, $t\ e$ and $a\ e$, 160 and 40; what is the length of the conjugate, $d\ f$?

$\sqrt{40\times 160}=80:48::120:72=$ *conjugate required.*

Ex. 2. The transverse diameter of an hyperbola is 25 inches, the ordinate 8.1829, and the abscissæ 6 and 31; what is the length of the conjugate? *Ans.* 15 *inches.*

To ascertain the Transverse Diameter, the Conjugate, the Ordinate, and an Abscissa being given, Fig. 12.

RULE.—Add the square of the ordinate to the square of the semi-conjugate, and extract the square root of their sum.

Take the sum or difference of the semi-conjugate and this root, according as the greater or less abscissa is used.*

Then, as the square of the ordinate is to the product of the abscissa and conjugate, so is the sum or difference above found to the transverse diameter required.

Or, $\left(\overline{a \text{ or } a' \times c} \times (\sqrt{o^2+(c\div 2)^2} \pm c \div 2)\right) \div o^2 =$ *transverse diameter.*

EXAMPLE.—The conjugate diameter, *d f*, of an hyperbola, *Fig.* 12, is 72 inches, the ordinate, *e c*, 48, and the less abscissa, *a e*, 40; what is the length of the transverse diameter, *a t*?

$\sqrt{48^2+(72\div 2)^2}=60=$ *square root of the squares of the ordinate and semi-conjugate.*

$60+\overline{72\div 2}=96=$ *sum of above root and the semi-conjugate* (the less abscissa being used).

$40\times 72=2880=$ *product of abscissa and conjugate.*

$48^2 : 2880 :: 96 : 120 =$ *transverse diameter.*

Ex. 2. The conjugate diameter of an hyperbola is 15 inches, the ordinate 8.1829, and the greater abscissa 31; what is the length of the transverse diameter?

Ans. 25 *inches.*

Ex. 3. The conjugate diameter of an hyperbola is 6 feet 8 inches, the ordinate 4 feet, and the less abscissa 3 feet 4 inches; what is the length of the transverse diameter?

Ans. 142.336 *feet.*

* When the greater abscissa is used, the difference is taken, and contrariwise.

To ascertain the Length of any Arc of an Hyperbola, commencing at the vertex, Fig. 13.

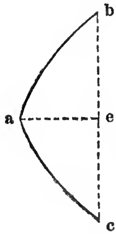
RULE.—To 19 times the transverse diameter add 21 times the parameter of the axis, and multiply the sum by the quotient of the less abscissa divided by the transverse diameter.

To 9 times the transverse diameter add 21 times the parameter, and multiply the sum by the quotient of the less abscissa divided by the transverse diameter.

To each of the products thus found add 15 times the parameter, and divide the former by the latter; then this quotient, being multiplied by the ordinate, will give the length of the arc nearly.

Or, $$\frac{t\times 19+\overline{21\times p}\times\frac{a}{t}+\overline{15\times p}}{t\times\ 9+\overline{21\times p}\times\frac{a}{t}+\overline{15\times p}}\times o=\textit{arc nearly}.$$

Fig. 13.



EXAMPLE.—In the hyperbola *a b c*, *Fig.* 13, the transverse diameter is 120, the conjugate 80, the ordinate *e c* 48, and the less abscissa, *a e*, 40; required the length of the arc *b a c*?

$120:80::80:53.3333=$*parameter.*

$\overline{120\times 19}+\overline{53.3333\times 21}\times\frac{40}{120}=1133.3333=$*product of the sum of* 19 *times the transverse and* 21 *times the parameter, by the quotient of the less abscissa and transverse.*

NOTE.—As the transverse diameter is to the conjugate, so is the conjugate to the parameter. (See rule, p. 231.)

$\overline{120 \times 9} + \overline{53.3333 \times 21} \times \frac{40}{120} = 733.3333 =$ *product of the sum of* 9 *times the transverse and* 21 *times the parameter, by the quotient of the less abscissa and transverse.*

$1133.3333 + \overline{53.3333 \times 15} \div (\overline{733.3333 + 53.3333 \times 15}) = 1.2622 =$ *quotient of former product and* 15 *times the parameter,* $\div$ *latter product and* 15 *times the parameter.*

$1.2622 \times 48 = 60.5856 =$ *above quotient* $\times$ *the ordinate = the length required.*

Ex. 2. The transverse diameter of an hyperbola is 60, the conjugate 36, the ordinate 24, and the less abscissa 20 inches; required the length of the curve.

Ans. 31.324 *feet.*

Ex. 3. The transverse diameter of an hyperbola is 25, the conjugate 15, and the less abscissa 6; required the length of the curve. *Ans.* 10.2777.

Ex. 4. The transverse diameter of an hyperbola is 50, the conjugate 15, and the greater abscissa 31; what is the length of the curve? *Ans.* 5.9395.

To ascertain the Area of an Hyperbola, the Transverse, Conjugate, and less Abscissa being given, Fig. 13.

RULE.—To the product of the transverse diameter and less abscissa add $\frac{5}{7}$ of the square of this abscissa, and multiply the square root of the sum by 21.

Add 4 times the square root of the product of the transverse diameter and less abscissa to the product last found, and divide the sum by 75.

Divide 4 times the product of the conjugate diameter and less abscissa by the transverse diameter, and this last quotient, multiplied by the former, will give the area required nearly.

Or, $\frac{\sqrt{t \times a' + \frac{5}{7} a'^2} \times 21 + (\sqrt{t \times a'} \times 4)}{75} \times \frac{c \times a' \times 4}{t} = area.$

EXAMPLE.—The transverse diameter of an hyperbola, *Fig.* 13, is 60 inches, the conjugate 36, and the less abscissa or height, *a e*, 20; what is the area of the figure?

$60 \times 20 + \frac{5}{7}$ of $20^2 = 1485.7143$ = *sum of the product of the transverse and abscissa and* $\frac{5}{7}$ *of the square of the abscissa.*

$\sqrt{1485.7143} \times 21 = 809.424 = 21$ *times the square root of the above sum.*

$\sqrt{60 \times 20} \times 4 + 809.424 = 947.988$ = *sum of above result and the square root of* 4 *times the product of the transverse and abscissa.*

$947.988 \div 75 = 12.6398$ = *quotient of above result* $\div 75$.

$\frac{36 \times 20 \times 4}{60} \times 12.6398 = 606.7104$ = *product of* 4 *times the product of the conjugate and abscissa* $\div$ *the transverse and the above quotient* = *area required.*

Ex. 2. The axes of an hyperbola are 100 and 60, and the less abscissa 50 inches; what is its area?

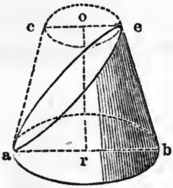
Ans. 22.3636 *square feet.*

Ex. 3. Required the area of an hyperbola, the greater abscissa being 6.25 feet, the ordinate 2.165, and the transverse diameter 4.167. *Ans.* 9.6008 *square feet.*

CONICAL UNGULAS.

To ascertain the Convex Surface of a Conic Ungula, Fig. 14.

Fig. 14.



1. *When the Section passes through the opposite Extremities of the Ends of the Frustrum, as a e.*

RULE.—Let *d represent diameter of greater end, d′ diameter of less end, and h the perpendicular height of the frustrum, as o r.*

Then, $\frac{.7854}{d-d'}\sqrt{4h^2+(d-d')^2}\times d^2-\frac{\sqrt{(d+d'^2\times d\,d')}}{2}$ = *convex surface of elliptic ungula, a e b.*

EXAMPLE.—The diameters *a b*, *c e*, of the frustrum of the cone, *Fig.* 14, are 10 and 5 inches, and the height, *o r*, 10; what is the convex surface of the elliptic ungula *a e b?*

$\frac{.7854}{d-d'}=\frac{.7854}{10-5}=.15708$, and $.15708\times\sqrt{4\,h^2+(d-d')^2}=.15708\times\sqrt{(4\times100+\overline{10-5}^2}=3.2383$, and $3.2383\times d^2-\frac{\sqrt{(\overline{d+d'}^2+d\,d')}}{2}=3.2383\times100-\frac{\sqrt{(\overline{10+5}^2\times10\times5)}}{2}=3.2383\times46.967=152.0932$ = *convex surface required.*

To ascertain the Contents of a Conic Ungula, Fig. 14.

RULE.—Let $n=.7854$.

Then, $\frac{d^2-d'\sqrt{d\,d'}}{d-d'}\times\frac{n\,d\,h}{3}$ = *contents of the greater elliptic ungula a e b.*

$\frac{d\sqrt{d\,d'}-d'^2}{d-d'}\times\frac{n\,d'\,h}{3}$ = *contents of the less elliptic ungula a c e.*

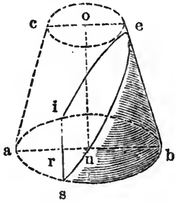
$\frac{(d^{\frac{3}{2}}-d'^{\frac{3}{2}})^2}{d-d'}\times\frac{n\,h}{3}$ = *difference of these ungulas.*

EXAMPLE.—The diameters, height, and section of the frustrum being as in the preceding example, what are its contents?

$$\frac{10^2-5\sqrt{10\times5}}{10-5}=\frac{100-5\times7.071}{5}=\frac{100-35.355}{5}=12.929, \text{ which} \times$$
$\frac{.7854\times10\times10}{3}=\frac{78.44}{3}=26.18$, and $12.929\times26.18=338.4812=$ *contents required.*

To ascertain the Convex Surface of a Conic Ungula, Fig. 15.

Fig. 15.



2. *When the Section cuts off part of the Base, and makes the Angle e r b less than the angle c a b.*

NOTE.—*When a Segment greater than a Semicircle is to be found,*
Subtract the quotient of its versed sine, divided by the diameter of the circle, from unity. Then take the tabular segment (p. 134–137) which corresponds to the remainder from .78539, the whole tabular area, and the remainder will be the tabular segment corresponding to the segment required.

RULE.— $\frac{1}{d-d'}\sqrt{4h^2+(d-d')^2}\times(tab.\ seg.\ i\ b\ s-\frac{d'^2}{d^2}\times\frac{\frac{1}{2}\times(d+d')-a\,r}{d'-a\,r}\times\sqrt{\frac{b\,r}{d'-a\,r}}\times$ *area of segment of circle a b, the height of which is* $d\times\frac{d'-a\,r}{d'}\Big)=$ *convex surface of elliptic ungula i e b s.*

EXAMPLE.—The diameters $a\,b$, $c\,e$, of the frustrum of the cone, *Fig.* 15, are 10 and 5 inches, and the height $o\,n$ 10; what is the convex surface of the elliptic ungula, the base $b\,r$ being 6 inches?

$\frac{1}{d-d'}=\frac{1}{10-5}=.2$, and $.2\times\sqrt{4h^2+(d-d')^2}=.2\times\sqrt{(4\times100+\overline{10-5}^2)}$ $=4.1231.$

Tab. seg. i b s $=\frac{6}{10}=.6$, and $.6-1=.4$, and tab. seg. of $.4\times10^2=29.337$, which $-78.539=49.202$.

$$\frac{d'^2}{d^2}=\frac{25}{100}\cdot\frac{\frac{1}{2}\times(d+d')-a\,r}{d'-a\,r}=\frac{\frac{1}{2}\times(10+5)-4}{5-4}=\frac{3.5}{1}=3.5.$$

$$\sqrt{\frac{b\,r}{d'-a\,r}}=\sqrt{\frac{6}{5-4}}=\frac{6}{1}=2.4495.$$

Area of seg. cir. a b . height $=d\times\frac{d'-a\,r}{d'}=10\times\frac{5-4}{5}=2$, and $2\div10$ $=.2$, *tab. seg.* $=.11823$, which $\times\,10^2=11.1823$.

Then, $4.1231\times(49.202-\overline{\tfrac{25}{100}\times3.5\times2.4495\times11.1823})=104.0464=$ *convex surface required.*

To ascertain the Contents of a Conic Ungula, Fig. 15.

Rule.—Let S = *tabular segment of* $b\,r$, *the versed sine being* $\div\,a\,b$; $s=$ *tabular segment, the versed sine being* $\overline{b\,r-(d-d')}\div d'$.

Then, $\left(S\times d^3-s\times d'^3\times\frac{b\,r}{b\,r-\overline{d-d'}}\sqrt{\frac{b\,r}{b\,r-\overline{d-d'}}}\right)\times\frac{\frac{1}{3}h}{d-d'}=$ *contents of the ungula i e b s.*

Example.—The dimensions of the frustrum and ungula being the same as in the preceding example, what are its contents?

$S=\frac{b\,r}{d}=\frac{6}{10}$, and $\frac{6}{10}=.6\ \therefore 1-.6=.4$, *the tabular segment of which is* (page 135) .29337, and $.78539-.29337=.49202$.

$s=b\,r-(d-d'\div d')=\frac{6-5}{5}=.2$, *the tabular segment of which is* .11182.

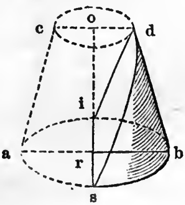
$S\times d^3=.49202\times10^2=492.02$.

$s\times d'^3\times\frac{b\,r}{b\,r-\overline{d-d'}}\sqrt{\frac{b\,r}{b\,r-\overline{d-d'}}}=.11182\times5^3\times\overline{6\times2.4495}=205.427$, which $-\,492.02=286.593$.

$\frac{\frac{1}{3}h}{d-d'}=\frac{3.333}{5}=.6667$, and $286.593\times.6667=191.0715=$ *result required.*

To ascertain the Convex Surface of a Conic Ungula, Fig. 16.

Fig. 16.



3. *When the Section is parallel to one of the Sides of the Frustrum, as* $d\,r$.

$$\frac{1}{d-d'}\sqrt{4\,h^2+(d-d')^2}\times\textit{area of seg. } i\,b\,s-\left(\frac{2}{3}(d-d')\times\sqrt{d'\times(d-d')}\right)$$

$=$ *convex surface of parabolic ungula i d b s.*

EXAMPLE.—The diameters *a b*, *c d*, of the frustrum of the cone, *Fig.* 16, are 10 and 5 inches, and the height *o r* 10; what is the convex surface of the ungula?

$\frac{1}{d-d'}=\frac{1}{10-5}=.2$, and $.2\times\sqrt{4h^2+(d-d')^2}=\sqrt{4\times100+(10-5)^2}=$ $.2\times20.6155=4.1231.$

Area of seg. i b s. $-\left(\frac{2}{3}(d-d')\times\sqrt{d'\times(d-d')}\right)=39.27-\left(\frac{2}{3}(10-5)\times\sqrt{5\times(10-5)}\right)=39.27-16.667=22.603.$

Then, $4.1231\times22.603=93.1944=$ *convex surface required.*

To ascertain the Contents of a Conic Ungula, Fig. 16.

RULE.—Let A=*area of base i b s.*

Then, $\left(\frac{A\times d}{d-d'}-\frac{4}{3}d'\sqrt{(d-d')\times d'}\right)\times\frac{1}{3}h=$ *contents required.*

EXAMPLE.—The diameters of a frustrum of a cone, *Fig.* 16, *a b* and *c d*, are 10 and 5 inches, and the height *o r* 10; what are the contents of the ungula *i d b s*?

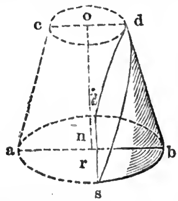
$A=10^2\times.7854\div2=39.27.$

$\frac{A\times d}{d-d'}=\frac{39.27\times10}{10-5}=\frac{392.70}{5}=78.54.$

$\frac{4}{3}d'\sqrt{(d-d')\times d'}=\frac{4}{3}5\sqrt{(10-5)\times5}=6.667\times5=33.335,$ and $33.335-78.54=45.205$, which $\times\frac{1}{3}10=45.205\times3.334=150.7135=$ *contents required.*

To ascertain the Convex Surface of a Conic Ungula, Fig. 17.

Fig. 17.



4. *When the Section cuts off part of the Base, and makes the Angle d r b greater than the Angle c a b.*

$$\frac{1}{d-d'}\times\sqrt{4h^2+(d-d')^2}\times\left(cir.\ seg.\ i\ b\ s-\frac{d'^2}{d^2}\times\frac{b\,r-\frac{1}{2}(d-d')}{b\,r-\overline{d-d'}}\sqrt{\frac{b\,r}{b\,r-\overline{d'-d}}}\right)=\textit{convex surface of hyperbolic ungula } i\,d\,b\,s.$$

EXAMPLE.—The dimensions of the frustrum being the same as in the preceding example, and the height $o\,n$ and base $b\,r$ of the ungula being 10 and 2.5 inches, what are its contents?

$$\frac{d-d'}{1}\times\sqrt{4\,h^2+(d-d')^2}=.2\times 20.6155=4.1231.$$

And (*cir. seg.* $i\,b\,s=\frac{2.5}{10}=.25$, *tab. seg.*$=.153546\times 10^2=15.3546$.

$$\frac{d'^2}{d}=\frac{25}{100}\cdot\frac{b\,r-\frac{1}{2}(d-d')}{b\,r-(d-d')}\sqrt{\frac{b\,r}{b\,r-(d-d')}}=\frac{2.5-\frac{1}{2}\times(10-5)}{2.5-(10-5)}=\frac{6.25}{2.5}$$

$$\sqrt{\frac{2.5}{2.5-(5-10)}}=1.$$

Then, $4.1231\times(15.346-\frac{25}{100}\times\frac{6.25}{2.5}\times 1)=15.346-.625=14.721$, which $\times 4.1231=60.6962=$ *convex surface required.*

To ascertain the Contents of a Conic Ungula, Fig. 17.

Let *the area of the hyperbolic section* $i\,d\,s=A$, *and the area of the circular seg.* $i\,b\,s=a$.

Then, $\frac{\frac{1}{3}h}{d-h}\times(a\times d-A\times\frac{d'\times b\,r}{d\,r}=$ *contents of the hyperbolic ungula* $i\,d\,b\,s$.

NOTE.—The transverse diameter of the hyp. seg. $=\frac{d'\times d\,r}{d-d'-b\,r}$, the conjugate $=d'\sqrt{\frac{b\,r}{d-d'-b\,r}}$, and the abscissa $=d\,r$, from which its area may be found by rule, page 251.

EXAMPLE.—The diameters $a\,b$, $c\,d$, of the frustrum of the cone, *Fig.* 17, are 10 and 5 inches, the heights $o\,n$ and $d\,r$, of an hyperbolic ungula are 10, and the base $b\,r$ 2.5 inches; what are its contents?

A (by rule, page 251)$=53.675$, $a=15.355$.

Then, $\frac{\frac{1}{3}h}{d-h}=\frac{3.334}{10-10}=3.334$, and $\left(a\times d-A\times\frac{d'-b\,r}{d\,r}\right)=(15.355\times 10$ $=153.55-53.675=99.875$, which $\times\frac{5-2.5}{10}=24.9688$, which $\times 3.334=$ $83.246=$ *contents required.*

REGULAR POLYGONS.

(Appendix to Rules, see Table, page 261.)

TO ASCERTAIN THE ELEMENTS OF ANY REGULAR POLYGON.

To ascertain the Radius of the Inscribed or Circumscribing Circles of a Polygon.

1. *When the Length of a Side is given.*

RULE.—Multiply the length of the side by the units in columns A and B of the following table under the head of the element required.

EXAMPLE.—The length of the side of a square (tetragon) is 2 inches; what are the radii of its inscribed and circumscribing circles?

$2 \times .5 = 1.$ = *product of length of side and tabular multiplier = radius of its inscribed circle.*

$2 \times .70711 = 1.41422$ = *product of length of side and tabular multiplier = radius of its circumscribing circle.*

Ex. 2. The length of the side of a hexagon is 2 inches; what is the radius of its inscribed circle?

Ans. 1.73204 *inches.*

2. *When the Area is given.*

RULE.—Extract the square root of the area, and multiply it by the units in columns C and D of the following table under the head of the element required.

EXAMPLE.—The area of a square is 4 inches; what are the radii of its inscribed and circumscribing circles?

$\sqrt{4} \times .5 = 1.$ = *radius of its inscribed circle.*

$\sqrt{4} \times .70711 = 1.41422$ = *radius of its circumscribing circle.*

Ex. 2. The area of a hexagon is 64.952 inches; what is the radius of its circumscribing circle? *Ans.* 5 *inches.*

3. *When the Radius of the Circumscribing Circle is given.*

Rule.—Multiply the radius given by the unit in the column E opposite to the figure for which the radius is required.

Example.—The radius of the circumscribing circle of a square is 2 inches; what is the radius of its inscribed circle?

$2 \times .70711 = 1.41422 =$ *radius of its inscribed circle.*

Ex. 2. The radius of the circumscribing circle of a hexagon is 5 inches; what is the radius of its inscribed circle?

Ans. 4.3301 *inches.*

4. *When the Radius of the Inscribed Circle is given.*

Rule.—Multiply the radius given by the unit in the column F opposite to the figure for which the radius is required.

Example.—The radius of the inscribed circle of a square is 1.41421 inches; what is the radius of its circumscribing circle?

$1.41421 \times 1.41421 = 2 =$ *radius of its circumscribing circle.*

Ex. 2. The radius of the inscribed circle of a hexagon is .866025 inches; what is the radius of its circumscribing circle?

Ans. 1 *inch.*

To ascertain the Area.

1. *When the Radii of the Inscribed or Circumscribing Circles are given.*

Rule.—Square the radius given, and multiply it by the unit in the columns G and H of the following table under the head of the element given.

Example.—The radius of the inscribed circle of a square is 2 inches; what is its area?

$2^2 \times 4 = 16 =$ *area.*

Ex. 2. The radius of the circumscribing circle of a hexagon is 5 inches; what is its area?

Ans. 64.952 *inches.*

3. *When the Length of the Side is given.*

RULE.—Square the length of the side, and multiply it by the unit in column I.

EXAMPLE.—The length of the side of a square is 4 inches; what is its area?

$4^2 \times 1 = 16 =$ *area.*

Ex. 2. The length of the side of a hexagon is 5 inches; what is its area? *Ans.* 64.952 *inches.*

To ascertain the Length of a Side.

1. *When the Radii of the Inscribed or Circumscribing Circles are given.*

RULE.—Multiply the radius given by the unit in the columns K and L of the following table under the head of the radius given.

EXAMPLE.—The radius of the inscribed circle of a square is 2 inches; what is the length of a side?

$2 \times 2 = 4 =$ *length of side.*

Ex. 2. The radius of the circumscribing circle of a hexagon is 5 inches; what is the length of a side?

Ans. 5 *inches.*

2. *When the Area is given.*

RULE.—Extract the square root of the area, and multiply it by the unit in column M of the following table.

EXAMPLE.—The area of a square is 16 inches; what is the length of a side?

$\sqrt{16} \times 1 = 4 =$ *length of side.*

Ex. 2. The area of a hexagon is 64.952 inches; what is the length of a side? *Ans.* 5 *inches.*

NOTE.—For other rules to ascertain the surface or linear edge of a polygon, see page 64.

TABLE OF UNITS FOR ELEMENTS OF ANY REGULAR POLYGON.

Figure.	A. Radii of inscribed circle. By length of side.	B. Radii of circumscribing circle. By length of side.	C. Radii of inscribed circle. By area.	D. Radii of circumscribing circle. By area.	E. Radii of inscribed circle. By circum. circle.	F. Radii of circumscribing circle. By inscrib. circle.	G. Area. By radii of inscribed circle.	H. Area. By radii of circum. circle.	I. Area. By length of side.	K. Length of side. By radii of inscribed circle.	L. Length of side. By radii of circum. circle.	M. Length of side. By area.
Trigon......	.28867	.57735	.43869	.87738	.5	2.	5.19615	1.29904	.43301	3.46410	1.73205	1.51967
Tetragon...	.5	.70711	.5	.70711	.70711	1.41421	4.	2.	1.	2.	1.41421	1.
Pentagon..	.68819	.85065	.52467	.64853	.80902	1.23607	3.63272	2.37764	1.72048	1.45308	1.17557	.76238
Hexagon...	.86602	1.	.53728	.62040	.86602	1.15470	3.46410	2.59808	2.59808	1.15470	1.	.62040
Heptagon..	1.03826	1.15238	.54465	.60452	.90097	1.10992	3.37102	2.73641	3.63391	.96315	.86777	.52458
Octagon....	1.20711	1.30656	.54934	.59460	.92388	1.08239	3.31371	2.82842	4.82843	.82843	.76537	.45509
Nonagon...	1.37374	1.46190	.55252	.58798	.93969	1.06418	3.27573	2.89254	6.18182	.72794	.68404	42020
Decagon ...	1.53884	1.61803	.55477	.58332	.95106	1.05146	3.24920	2.93893	7.69421	.64984	.61803	.36051
Undecagon	1.70284	1.77473	.55642	.57991	.95949	1.04222	3.22989	2.97353	9.36564	.58725	.56346	.32676
Dodecagon	1.86603	1.93185	.55768	.57735	.96593	1.03528	3.21539	3.	11.19615	.53590	.51764	.29886

MENSURATION OF REGULAR BODIES, OR POLYHEDRONS.

To ascertain the Elements of any Regular Body, Figs. 72, 83, 84, 85, *and* 86, p. 72, 161–164.

To ascertain the Radius of a Sphere that will Circumscribe a given Regular Body, and the Radius of one also that may be Inscribed within it.

1. *When the Linear Edge is given.*

Rule.—Multiply the linear edge by the multiplier opposite to the body in the columns A and B in the following table, under the head of the element required.

Example.—The linear edge of a hexahedron or cube is 2 inches; what are the radii of the circumscribing and inscribed spheres?

$2 \times .86602 = 1.73204 =$ *product of edge and tabular multiplier = radius of circumscribing sphere.*

$2 \times .5 = 1 =$ *product of edge and tabular multiplier = radius of inscribed sphere.*

Ex. 2. The linear edge of a hexahedron is 10 inches; what are the radii of its inscribed and circumscribing spheres?

Ans. 5 and 8.6602 *inches.*

2. *When the Surface is given.*

Rule.—Multiply the square root of the surface by the multiplier opposite to the body in the columns C and D in the following table, under the head of the element required.

Example.—The surface of a hexahedron is 25 inches; what are the radii of the circumscribing and inscribed spheres?

$\sqrt{25} \times .35355 = 1.76775 =$ *radius of circumscribing sphere.*

$\sqrt{25} \times .20412 = 1.0206 =$ *radius of inscribed sphere.*

Ex. 2. The surface of a hexahedron is 24 inches; what is the radius of its inscribed sphere? *Ans.* 1 *inch.*

3. *When the Volume is given.*

RULE.—Multiply the cube root of the volume by the multiplier opposite to the body in the columns E and F in the following table under the head of the element required.

EXAMPLE.—The volume of a hexahedron is 8 inches; what are the radii of its circumscribing and inscribed spheres?

$\sqrt[3]{8} \times .86602 = 1.73204 =$ *radius of circumscribing sphere.*
$\sqrt[3]{8} \times .5 = 1 =$ *radius of inscribed sphere.*

4. *When one of the Radii of the Circumscribing or Inscribed Sphere alone is required, the other being given.*

RULE.—Multiply the given radius by the multiplier opposite to the body in the columns G and H in the following table, under the head of the other radius.

EXAMPLE.—The radius of the circumscribing sphere of a hexahedron is 1 inch; what is the radius of its inscribed sphere?

$1 \times .57735 = .57735 =$ *radius required.*

Ex. 2. The radius of the inscribed sphere of a hexahedron is 2 inches; what is the radius of its circumscribing sphere? *Ans.* 3.4641 *inches.*

To ascertain the Linear Edge of a Polyhedron.

1. *When the Radius of the Circumscribing or Inscribed Sphere is given.*

RULE.—Multiply the radius given by the multiplier opposite to the body in the columns I and K in the following table.

EXAMPLE.—The radius of the circumscribing sphere of a hexahedron is 1 inch; what is its linear edge?

$1 \times 1.1547 = 1.1547 =$ *edge required.*

Ex. 2. The radius of the inscribed sphere of a hexahedron is .57735 inch; what is its linear edge?

Ans. 1.1547 *inch.*

2. *When the Surface is given.*

Rule.—Multiply the square root of the surface by the multiplier opposite to the body in the column L in the following table.

Example.—The surface of a hexahedron is 6 inches; what is its linear edge?

$\sqrt{6} \times .40825 = 1 =$ *linear edge.*

Ex. 2. The surface of a hexahedron is 24 inches; what is its linear edge? *Ans.* 2 *inches.*

3. *When the Volume is given.*

Rule.—Multiply the cube root of the volume by the multiplier opposite to the body in the column M in the following table.

Example.—The volume of a hexahedron is .3535 inch; what is its linear edge?

$\sqrt[3]{.3535} \times 1 = .7071 =$ *linear edge.*

Ex. 2. The volume of a hexahedron is 8 inches; what is its linear edge? *Ans.* 2 *inches.*

To ascertain the Surface of a Polyhedron.

1. *When the Radius of the Circumscribing Sphere is given.*

Rule.—Multiply the square of the radius by the multiplier opposite to the body in the column N in the following table.

Example.—The radius of the circumscribing sphere of a hexahedron is .866025 inch; what is its surface?

$.86602^2 \times 8 = 6 =$ *surface required.*

2. *When the Radius of the Inscribed Sphere is given.*

RULE.—Multiply the square of the radius by the multiplier opposite to the body in the column O in the following table.

EXAMPLE.—The radius of the inscribed sphere of a hexahedron is .5 inch; what is its surface?

$.5^2 \times 24 = 6 =$ *surface required.*

3. *When the Linear Edge is given.*

RULE.—Multiply the square of the edge by the multiplier opposite to the body in the column P in the following table.

EXAMPLE.—The linear edge of a hexahedron is 1 inch; what is its surface?

$1^2 \times 6 = 6 =$ *surface required.*

4. *When the Volume is given.*

RULE.—Extract the cube root of the volume, and multiply the square of the root by the multiplier opposite to the body in column Q in the following table.

EXAMPLE.—The volume of a hexahedron is 8 inches; what is its surface?

$\sqrt[3]{8} = 2$, and $2^2 \times 6 = 24 =$ *surface required.*

Ex. 2. The volume of a hexahedron is 353.5533 inches; what is its surface? *Ans.* 300 *inches.*

To ascertain the Volume of a Polyhedron.

1. *When the Linear Edge is given.*

RULE.—Cube the linear edge, and multiply it by the multiplier opposite to the body in column R in the following table.

EXAMPLE.—The linear edge of a hexahedron is 2 inches; what is its volume?

$2^3 \times 1 = 8 =$ *volume required.*

2. *When the Radius of the Circumscribing Sphere is given.*

RULE.—Multiply the cube of the radius given by the multiplier opposite to the body in the column S in the following table.

EXAMPLE.—The radius of the circumscribing sphere of a hexahedron is .5 inch; what is the volume of the figure?

$5^3 \times 1.5396 = .1925$ *cubic inch.*

Ex. 2. The radius of the circumscribing sphere of a hexahedron is 1.73205 inch; what is the volume of the figure?

Ans. 8 *cubic inches.*

3. *When the Radius of the Inscribed Sphere is given.*

RULE.—Multiply the cube of the radius given by the multiplier opposite to the body in the column T in the following table.

EXAMPLE.—The radius of the inscribed sphere of a hexahedron is .5 inch; what is its volume?

$.5^3 \times 8 = 1$ *cubic inch.*

Ex. 2. The radius of the inscribed sphere of a hexahedron is 3.535 inches; what is its volume?

Ans. 353.3932 *cubic inches.*

4. *When the Surface is given.*

RULE.—Cube the surface given, extract the square root, and multiply the root by the multiplier opposite to the body in the column U in the following table.

EXAMPLE.—The surface of a hexahedron is 6 inches; what is its volume?

$\sqrt{6^3} \times .06804 = 1$ *cubic inch.*

Ex. 2. The surface of a hexahedron is 24 inches; what is its volume? *Ans.* 8 *inches.*

Ex. 3. The surface of an octahedron is 125 inches; what is its volume? *Ans.* 102.1743 *cubic inches.*

TABLE OF UNITS FOR ELEMENTS OF ANY REGULAR BODY.

	Figure.	A. Radius of circumscribing sphere. By linear edge.	B. Radius of inscribed sphere. By linear edge.	C. Radius of circumscribing sphere. By surface.	D. Radius of inscribed sphere. By surface.	E. Radius of circumscribing sphere. By volume.	F. Radius of inscribed sphere. By volume.	G. Radius of inscribed sphere. By circumscribing sphere.	H. Radius of circumscribing sphere. By inscribed sphere.	I. Linear edge. By radius of circum. sphere.	K. Linear edge. By radius of inscribed sphere.
4	Tetrahedron.....	.61237	.20412	.46530	.15510	1.24896	.41634	.33333	3.	1.63299	4.89898
6	Hexahedron.....	.86602	.5	.35355	.20412	.86602	.5	.57735	1.73205	1.15470	2.
8	Octahedron......	.70711	.40825	.37992	.21935	.90806	.52456	.57735	1.73205	1 41421	2.44949
12	Dodecahedron..	1.40126	1.11352	.30839	.24507	.71075	.56480	.79465	1.25841	.71364	.89806
20	Icosahedron.....	.95106	.75576	.32318	.25681	.73329	.58271	.79465	1.25841	1.05146	1.32317

	Figure.	L. Linear edge. By surface.	M. Linear edge. By volume.	N. Surface. By radius of circum. sphere.	O. Surface. By radius of inscribed sphere.	P. Surface. By linear edge.	Q. Surface. By volume.	R. Volume. By linear edge.	S. Volume. By radius of circum. sphere.	T. Volume. By radius of inscribed sphere.	U. Volume. By surface.
4	Tetrahedron.....	.75984	2.03955	4.61880	41.56922	1.73205	7.20562	.11785	.51320	13.85641	.05170
6	Hexahedron.....	.40825	1.	8.	24.	6.	6.	1.	1.53960	8.	.06804
8	Octahedron......	.53729	1.28490	6.92820	20.78461	3.46410	5.71910	.47140	1.33333	6.92820	.07311
12	Dodecahedron..	.22008	.50722	10.51462	16.65087	20.64573	5.31161	7.66312	2.78517	5.55029	.08169
20	Icosahedron.....	.33981	.77102	9.57454	15.16217	8.66025	5.14835	2.18170	2.53615	5.05406	.08560

OROPHOIDS* (*Domes, Arched and Vaulted Roofs, etc.*).

DEFINITION. *Figures generated by a curved line running from the periphery or the angles alone of their base to a common centre at the top. When the curve runs from the periphery alone, and their point of connection is in the centre, they are Regular; when the point of connection is eccentric, they are Oblique; when curves run both from the angles and some intermediate point, or there is any combination of elements in their generation, they are Compound or Irregular.*

OROPHOIDS or ARCHED ROOFS are either *Domes, Saloons, Vaults*, or *Groins*.

A Dome is formed by arches springing from a circular or polygonal base, and meeting in the centre at the vertex.

A Saloon (frustrum of an orophoid) is formed by arches springing from a circular or polygonal base, and connecting with a flat roof or ceiling in the middle.

A Vault is formed by arches springing from two opposite bases alone, and meeting in a line at the vertex.

A Groin† is formed by the intersection of one vault with another at any angle.

Orophoids are of the following forms:

1. *Circular Dome, Saloon*, or *Vault*, when generated by an arc of a circle.
2. *Elliptical Dome, Saloon*, or *Vault*, when generated by an arc of an ellipse.

* From ὀροφος, a roof. The absence of any generic term to denote this class of figures has induced the adoption of the above term.

† The curved surface between two adjacent groins is termed the sectroid.

3. *Gothic Dome, Saloon,* or *Vault,* when generated by two circular or elliptical arcs meeting in the centre of the arch.

4. *Curvilinear Dome, Saloon,* or *Vault,* when generated by an irregular curve.

*To ascertain the External or Internal Surface of a Spherical Dome.**

RULE.—Multiply the area of the base by 2, and the product will give the surface required.

Or, $a \times 2 = surface$.

EXAMPLE.—The external diameter of a hemispherical dome is 20 feet; what is the surface of it?

$20^2 \times .7854 = 314.16 = $ *area of base.*
$314.16 \times 2 = 628.32 = $ *twice the area of the base = surface required.*

Ex. 2. The sides of a quadrangular spherical dome are 20 feet; what is the surface of it? *Ans.* 800 *square feet.*

Ex. 3. The internal sides of a hexagonal spherical dome are 10 feet; what is the surface of it?

Ans. 519.62 *square feet.*

To ascertain the Surface of a Side of a Polygonal Spherical Dome.

RULE.—Multiply the base by its length, divide by 1.5708, and the quotient will give the surface required.

Or, $\frac{b \times l}{1.5708} = surface$.

EXAMPLE.—The side of a quadrangular spherical dome is 20 feet; what is the surface of it?

* An elliptical dome or vault is either a semi-spheroid (ellipsoid) or a segment of a spheroid; for rules to ascertain the surface of these, see p. 91–93, and 125.

A parabolic dome or vault is a paraboloid, or a segment of one; for rules to ascertain the surface of which, see page 126.

A Gothic dome or vault is a semi-circular spindle, or a segment of one; for rules to ascertain the surface of which, see p. 120–122.

$20 \div 2 \times 3.1416 = 31.416 \div 2 = 15.708 =$ *radius of dome* $\times\ 3.1416 \div 2 =$ *perimeter or length of side.*

$20 \times 15.708 = 314.16 =$ *product of length of base and length of side.*

$314.16 \div 1.5708 = 200 =$ *above product* $\div\ 1.5708 =$ *surface of side required.*

Ex. 2. The side of a hexagonal spherical dome is 10 feet; what is the surface of it? *Ans.* 86.6025 *feet.*

*To ascertain the Volume of a Spherical Dome.**

RULE.—Multiply the area of its base by $\frac{2}{3}$ of the height, and the product will give the volume required.

Or, $a \times \frac{2}{3} h =$ *volume.*

EXAMPLE.—The diameter of the base of a spherical dome is 20 feet; what is its volume?

$20^2 \times .7854 = 314.16 =$ *area of base.*

$314.16 \times \frac{2}{3}$ of $\frac{20}{2} = 2094.4 =$ *area* $\times \frac{2}{3}$ *of height = volume required.*

Ex. 2. The side of a hexagonal spherical dome is 10 feet; what is its volume? *Ans.* 1732.0506 *cubic feet.*

To ascertain the External or Internal Surface of a Saloon.

RULE.—To the area of the ceiling add the surface of its sides, and the sum will give the surface required.

Or, $a + s =$ *surface.*

EXAMPLE.—A saloon roof has a quadrangular arch of 2 feet radius springing over a rectangular base of 20 by 16 feet; what is its surface?

$20 - \overline{2+2} = 16 =$ *length of ceiling.*

$16 - \overline{2+2} = 12 =$ *width of ceiling.*

$16 \times 12 = 192 =$ *area of ceiling.*

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* An elliptic dome is either a semi-spheroid (ellipsoid) or a segment of a spheroid. For rules to ascertain the volume of them, see p. 179–181.

Parabolic and hyperbolic domes are paraboloids and hyperboloids of revolution. For rules to ascertain the volumes of them, see p. 203–206.

A Gothic dome is a semi-circular spindle or a segment of one. For rules to ascertain the volume of them, see p. 188–191.

$2+2\times 3.1416=12.5664=$ *circumference of arch having a radius of* 2 *feet.*

$12.5664\div 4=3.1416=$ *length of arch at sides.*

$3.1416\times\overline{16+16+12+12}=175.9296=$ *length of arch* $\times$ *length of sides and ends.*

$\frac{2+2\times 4}{4}=4=$ *sum of sides of mitred ends of sides of saloon* $\div 4=$ *side of a quadrangular dome.*

$4\times 4\times 2=32=$ *area of base* $\times\ 2=$ *surface of quadrangular spherical dome.*

$192+175.9296+32=399.9296=$ *surface of saloon required.*

Ex. 2. A saloon roof has a hexagonal arch of 4.33 feet radius springing over an oblong hexagonal base having sides of 20 feet and ends of 10 feet; what is its surface?

Ans. 556.5832 *square feet.*

To ascertain the Volume of a Saloon.

RULE.—Multiply the square of twice the height of the arc by .7854, divide the product by 4, and multiply the quotient by the length of the sides of the ceiling.

Subtract the side of the ceiling from the side of the saloon, ascertain the area of a like figure having a side equal to this remainder (see rule, p. 60), and multiply this area by two thirds of the height of ceiling.

Multiply the area of the ceiling by the height of it; and this product, added to the preceding, will give the volume required.

Or, $\frac{2\,h^2\times .7854}{4}\times l+\frac{2}{3}a'+\overline{a\times h}=$ *volume.*

EXAMPLE.—The sides of a quadrangular circular saloon roof are 20 feet, and the height of the ceiling 2 feet; what is the volume of it?

NOTE.—*When the Saloon is a Circle.*

To ascertain the Volume of the Ring projecting beyond the Diameter of the Ceiling.

Ascertain the centre of gravity of a section of the ring, and multiply the area of this section by the circumference described by its centre of gravity. (See Example 2, page 272.)

$\overline{2\times2}^2\times.7854\div4=3.1416$=*one fourth of product of the square of twice the height of the arc and* .7854.

$3.1416\times(20-\overline{2+2})\times4=201.0624$=*product of preceding quotient and the length of the sides of the ceiling.*

$\overline{2+2}-20=16$, and $16-20=4$=*diameter of ceiling subtracted from diameter of saloon.*

$4^2\times1=16$=*product of square of side and tabular multiplier=area of figure.*

$16\times\frac{2}{3}$ of $2=21.333$=*product of area of base and* $\frac{2}{3}$ *of height of ceiling.*

$16^2\times2=512$=*product of area of ceiling and its height.*

$512+21.333+201.0624=734.3954$=*sum of above products=volume required.*

Ex. 2. A circular room 40 feet diameter, and 25 feet high to the ceiling, is covered with a saloon having a circular arch of 5 feet radius; required the contents of the room in cubic feet. *Ans.* 30779.453 *feet.*

Operation.—Area of $40\times\overline{25-5}=1256.64\times20=25132.8$ = *volume of body of room.*

Area of flat portion of ceiling, $40-\overline{5+5}=706.86\times25\backsim20=3534.3$ =*volume of body of roof.*

Area of quadrant of a circle having a radius of $5=\frac{78.54}{4}=19.635$.

Volume of quadrantal ring of 5 feet height and base=*area thereof* ×*the circumference described by its centre of gravity.* (See p. 209.)

Hence (by rule, p. 84), *the centre of gravity of the ring having the section of a sector of a circle of* 5 *feet radius is* 3.001 *feet from the angle of it.*

Then (by rule, p. 55), *the hypothenuse of the right angle* (3) *being alone given, the length of the side (that is, the distance of the centre of gravity of the sector from the vertical side of it)*=2.122. *Therefore,* $40-\overline{5+5}+\overline{2.122}\times2=34.244$=*diameter of circle described by centre of gravity of quadrantal ring.*

Consequently, circumference of $34.244\times19.635=2112.353$=*volume of quadrantal ring.*

Volume of body of room,		25132.8
" "	roof,	3534.3
"	ring,	2112.353
		30779.453 *cubic feet=volume required.*

Ex. 3. A quadrangular building having sides of 40 and 30 feet is covered with a saloon 25 feet in height from the floor, having an arch of 5 feet radius; what is the volume of the saloon? *Ans.* 29296.83 *cubic feet.*

To ascertain the Surface of a Vault.

RULE.—Multiply the length of the arch by the length of the vault, and the product will give the surface required.

Or, $p \times l = surface$, *p representing the perimeter of the arch.*

EXAMPLE.—What is the concave or internal surface of a circular vault, the width of it being 40 feet and the length 80?

$40 \times 3.1416 \div 2 \times 80 = 5026.56 =$ *product of length of arch and length of vault = result required.*

Ex. 2. The width of an elliptic arched vault is 18 feet, its height 12, and its length 50; what is its internal surface?

Ans. 1666.085 *square feet.*

To ascertain the Volume of a Vault.

RULE.—Multiply the area of a section of the vault by its length, and the product will give the volume required.

Or, $a \times l = volume$.

EXAMPLE.—The width of a semi-circular arched vault is 10 feet, and its length 60; what is its volume?

$10^2 \times .7854 \div 2 = 39.27 =$ *area of semicircle of* 10 *feet span.*
$39.27 \times 60 = 2356.2 =$ *product of area and length = volume required.*

Ex. 2. The width of a semi-elliptic arched vault is 25 feet, its height 17.5, and its length 40 feet; what are its contents?

Ans. 13744.5 *cubic feet.*

To ascertain the Internal Surface of a Circular Groin.

RULE.—Multiply the area of the base by 1.1416, and the product will give the surface required.

Or, $a \times 1.1416 = surface$.

NOTE.—The exact surface or volume of a groin is obtained by subtracting from the sum of the surfaces or volumes of the two vaults composing the groin, the surface or volume of the quadrantal arch formed by them.

Thus, in Example 1, the surface of a circular groin of 12 feet base is as follows:

Surface of vault, $12 \times 3.1416 \div 2 = 18.8496$, which $\times 12$ and $2 = 452.3904 =$ *surface of the two vaults.*

$12^2 \times 2 = 288 =$ *surface of the quadrantal arch.*

Hence, $452.3904 - 288 = 164.3904 =$ *surface.*

EXAMPLE.—What is the surface of a circular groin, a side of its square being 12 feet.

$12^2 \times 1.1416 = 164.3904$ = *product of area of base and* 1.1416 = *surface required.*

Ex. 2. What is the surface of a circular groin, a side of its square being 8 feet? *Ans.* 73.0624 *feet.*

NOTE.—This rule may be observed in elliptical groins, as the error or difference is too small to be regarded in ordinary practice.

To ascertain the Volume of a Circular or an Elliptical Groin.

RULE.—Multiply the area of the base by the height, and the product again by .904, and it will give the volume required.

Or, $a \times h \times .904 = volume$.

EXAMPLE.—What is the volume of the vacuity or space formed by a circular groin, one side of its square being 10 feet?

$10^2 \times 5 \times .904 = 452$ = *product of area of base, the height and* .904 = *volume required.*

Ex. 2. What is the volume of the vacuity formed by an elliptic groin, one side of its square being 24 feet, and its height 9 feet? *Ans.* 4686.336 *cubic feet.*

To ascertain the Internal Surface of a Triangular Groin.

RULE.—Ascertain the length of a side of the arch, multiply it by twice the width of the vault, and the product will give the surface required.*

Or, $l \times b \times 2 = surface$.

EXAMPLE.—The width of a triangular groin is 12 feet, and its height 12; what is its internal surface?

$\sqrt{(12^2 + \overline{12 \div 2}^{\,2})} = 13.4164$ = *length of one side of arch.*

$13.4164 \times 12 \times 2 = 321.9936$ = *product of length of side and twice the base* = *result required.*

* See note, page 273.

To ascertain the Volume of the Materials that form the Groin.

RULE.—Multiply the area of the base by the height, including the work to the top of the groin, and from this product subtract the volume of the vacuity; the result will give the volume required.

A General Rule for the Measurement of the Contents of Arches is thus:

From the volume of the whole, considered as a solid, from the springing of the arch to the outside of it, deduct the vacuity contained between the said springing and the under side of it, and the remainder will give the contents of the solid part.

In measuring works where there are many groins in a range, the cylindrical pieces between the groins, and on their sides, must be computed separately.

When the upper sides of orophoids, whether vaults or groins, are built up solid, above the haunches, to the height of the crown, it is evident that the product of the area of the base and the height will be the whole contents. And for the volume of the vacuity to be deducted, take the area of its base, computing its mean height according to its figure.

BOARD AND TIMBER MEASURE.

To ascertain the Surface of a Board or Plank.

RULE.—Multiply the length by the breadth, and the product will give the surface required.

Or, $l \times b = surface$.

NOTE.—When the piece is tapering, add the breadths of the two ends together, and take half the sum for the mean breadth.

EXAMPLE.—The length of a plank is 16 feet, and its breadth 15 inches; what is its surface?

$16 \times 1.25\,(15) = 20 =$ *product of length and breadth = surface required.*

Ex. 2. The length of a plank is 25 feet, and its breadth 14 inches; what is its surface? *Ans.* 29.167 *square feet.*

Ex. 3. The length of a plank is 18 feet, and its widths at the ends are 17 and 19 inches; what is its surface?

Ans. 27 *square feet.*

To ascertain the Contents of Squared Timber.

RULE.—Multiply the breadth by the thickness, and this product by the length, and it will give the contents required.

Or, $b \times t \times l = contents$.

EXAMPLE.—The length of a piece of square timber is 20 feet, its sides at its less end are 15 inches, and at its greater end 19; what are its contents?

$19 + 15 \div 2 = 17$, and $17^2 \times 20 \div 144 = 40.1388 =$ *product of square of mean side and the length* $\div 144$ *to produce feet = contents required.*

Ex. 2. The ends of a piece of timber are 18 and 22 inches square, and the length of it is 22.5 feet; what are its contents? *Ans.* 62.5 *cubic feet.*

NOTE.—1. If the piece tapers regularly from one end to the other, the breadth and thickness, taken in the middle, will be the mean breadth and thickness.

2. If the piece does not taper regularly, but is thicker in some places than in others, take several different dimensions, and their sum, divided by the number of them, will give the mean dimensions.

To ascertain the Contents of Round or Unsquared Timber.

RULE.—Multiply the square of one fifth of the girth by twice the length, and the product will give the contents nearly.*

Or, $\overline{c \div 5}^2 \times 2\,l = contents.$

EXAMPLE.—The diameter of a round piece of timber is $23\frac{7}{8}$ inches, and its length 18 feet; what are its contents?

75 (circumference of $23\frac{7}{8}$)$\div 5 = 15$, and $15^2 = 225$, and $225 \times \overline{18 \times 2} = 8100$, which $\div 144 = 56.25$ *cubic feet.*†

Ex. 2. The circumference of a round piece of timber is 168 inches, and its length 15; what are its contents?

Ans. 235.2 *cubic feet.*

* The ordinary rule is to square one fourth of the girth, and multiply it by the length.

In order to show the fallacy of taking one fourth of the girth for the side of a mean square, take the following example:

A diameter of 13.5 inches will give an area of 143.13. *Hence*, a piece of round timber of this diameter will have nearly a square foot of area of section.

The circumference for a diameter of $13.5 = 42.41$, and $42.41 \div 4 = 10.6$, and $10.6^2 = 112.36 =$ *area of section of timber.*

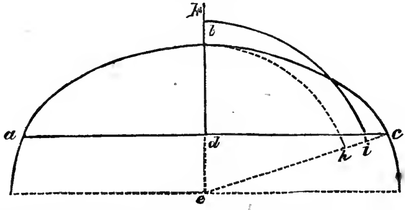
By the above rule, $42.41 \div 5 = 8.48$, and $8.48^2 = 71.91$, which $\times 2 = 143.82 =$ *area of section of timber.*

† When feet are multiplied by inches, $\div 144$ to obtain cubic feet.

MENSURATION OF SURFACES.

To ascertain the Length of an Elliptic Curve which is less than half of the entire Figure, Fig. 1.

Fig. 1.



Geometrically.—Let the curve of which the length is required be *a b c*.

Extend the versed sine *b d* to meet the centre of the curve in *e*.

Draw the line *c e*, and from *e*, with the distance *e b*, describe *b h;* bisect *c h* in *i*, and from *e*, with the radius *e i*, describe *k i*, and it is equal to half the arc *a b c*.

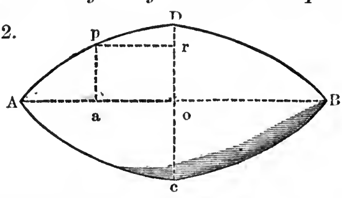
To ascertain the Length when the Curve is greater than half the entire Figure.

Rule.—Find by the above problem the curve of the less portion of the figure, and subtract it from the circumference of the ellipse, and the remainder will be the length of the curve required.

Parabolic Spindle.

To ascertain the Surface of a Parabolic Spindle, Fig.* 2.

Fig. 2.



* By Professor G. B. Docharty, New York Free Academy.

Let $o\,D=p$, $A\,o=q$,

$a\,o=\;p\,r\;=x$, and

$a\,p=\;o\,r\;=y$.

Then, from the nature of the curve,

$$p\,r^2 : A\,o^2 :: D\,r : D\,c;$$

that is, $$x^2 : q^2 :: p-y : p.$$

$$p\,x^2=q^2\,p-q^2\,y;$$

$$q^2\,y=p\,(q^2-x^2);$$

$$y=\frac{p}{q^2}(q^2-x^2).$$

And, by differentiating,

$$d\,y=-\frac{2\,p\,x\,d\,x}{q^2};\ \therefore d\,y^2=\frac{4\,p^2\,x^2\,d\,x^2}{q^4}.$$

Let $d\,s$ represent the differential of the surface; *then*,

$$d\,s=2\,\pi\,y\sqrt{d\,x^2+d\,y^2}.$$

Substitute for y its value, $\frac{p}{q^2}(q^2-x^2)$, and for $d\,y^2$ its value, $\frac{4\,p^2\,x^2\,d\,x^2}{q^4}$, we have

$$d\,s=\frac{2\,\pi\,p}{q^2}(q^2-x^2)\sqrt{d\,x^2+\frac{4\,p^2\,x^2\,d\,x^2}{q^4}}.$$

Or, $$d\,s=\frac{2\,\pi\,p}{q^2}(q^2-x^2)\,d\,x\sqrt{\frac{q^4+4\,p^2\,x^2}{q^4}}$$

$$=\frac{2\,\pi\,p}{q^4}(q^2-x^2)\,d\,x\,(q^4+4\,p^2\,x^2)^{\frac{1}{2}}$$

$$=\frac{2\,\pi\,p}{q^2}\,d\,x\,(q^4+4\,p^2x^2)^{\frac{1}{2}}-\frac{2\,\pi\,p}{q^4}\,x^2\,(q^4+4\,p^2\,x^2)^{\frac{1}{2}}\,dx;$$

$$\therefore s=\frac{2\,\pi p}{q^2}\int_o^q dx(q^4+4p^2x^2)^{\frac{1}{2}}-\frac{2\,\pi\,p}{q^2}\int_o^q x^2\,dx(q^4+4\,p^2x^2)^{\frac{1}{2}}$$

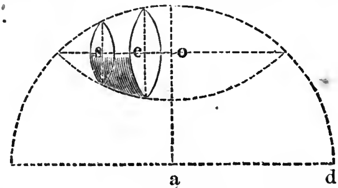
$$=\frac{\pi}{4\,p}\left\{\frac{1}{q}(q^4+4\,p^2\,q^2)^{\frac{1}{2}}+\frac{q}{2\,p}\log.\,(q+\frac{1}{2\,p}\sqrt{q^4+4\,p^2\,q^2})-\frac{1}{2\,q^3}\right.$$
$$(q^4+4\,p^2\,q^2)^{\frac{3}{2}}+\frac{q}{16\,p^2}(q^4+4\,p^2\,q^2)^{\frac{1}{2}}+\frac{q^4}{32\,p^3}\log.\,(q^4+4\,p^3\,q^2)$$
$$\left.-\frac{16\,p^2\,q^2-q^4}{32\,p^3}\log.\,\frac{q^2}{2\,p}.\right\}$$

Which is the formula for the surface of one half of the spindle, or A D c.

The logarithms indicated in the preceding and subsequent formulæ are hyperbolical or Naperian.

If a table of them is not at hand, one of common logarithms may be used instead, by dividing the common logarithm by .4343, and using the quotient when the logarithm is indicated to be used in the formula.

To ascertain the Surface of a Frustrum of a Circular Spindle, Fig. 3.



Let R represent the radius of the circle, an arc of which generates the spindle, as $a\ d$; D the distance of the centre of the spindle from the centre of the circle, as $a\ o$; d and d' the distances of the two bases from the centre of the spindle, as $o\ c$, $o\ s$; $h = d' \mp d$, the altitude, $c\ s$, of the frustrum, or the distance between the bases.

Find z and z' from the formulæ $\sin. z = \frac{d}{R}$, $\sin. z' = \frac{d'}{R}$.

Find Z from the formula $Z = \frac{\pi (z' \mp z)}{180°}$.

Find the surface from the formula $S = 2 \pi R (h - D Z)$.

In the formulæ for h and z, the *upper* sign is used when both bases are on the same side of the centre of the spindle; the *lower* sign when the centre is between them. *Z* is the arc which generates the frustrum expressed in units of the radius.

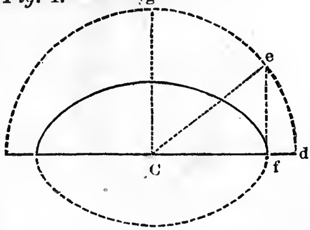
Corollary.—The entire surface of the spindle may be found from the formula $S = 2 \pi R (l - D z)$, l being the length of the spindle, and z being determined from $\sin. \frac{1}{2} z = \frac{l}{2 R}$ multiplied by $\frac{\pi}{180°}$, and D from $D = R \cos. \frac{1}{2} z$.

CENTRES OF GRAVITY OF SURFACES.

Semi-Spheroid or Ellipsoid, Fig. 4.

Fig. 4.

Prolate.



Distance from $C = \frac{r^3 - y^3}{3\,S}$, *in which* C d, *the radius of the auxiliary circle*, $= r = \frac{a^2}{\sqrt{a^2 - b^2}}$, a *and* b *representing the semi-transverse and conjugate axes*, $y = e\,f$, *and* $S =$ *area of segment of plane* $C\,g\,e\,f$.

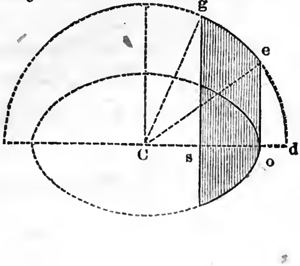
Oblate. $= \frac{(r'^2 + b^2)^{\frac{3}{2}} - r'^3}{\frac{3}{2}(b\sqrt{r'^2 + b^2} - r'^2 \log. r' + r'^2 \log. (\sqrt{r'^2 + b^2} + b))}$,

in which $r' = \frac{b^2}{\sqrt{a^2 - b^2}}$, b *and* a *representing the semi-conjugate and transverse axes.*

Segment of Semi-Spheroid or Ellipsoid, Figs. 5 and 6.

Fig. 5.

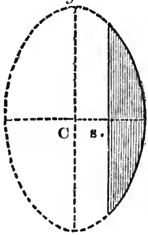
Prolate.



Distance from $C=\frac{(r^2-l^2)^{\frac{3}{2}}-(r^2-a^2)^{\frac{3}{2}}}{3\,S}$, or $=\frac{y^3-y'^3}{3\,S}$, *in which* $y=g\,s$, $y'=e\,o$, *and* $S=$*area of plane* $s\,g\,e\,o$, $l=C\,s$.

Oblate.

Fig. 6.



Distance from C$=$

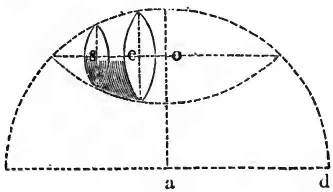
$$\frac{(r'^2+b^2)^{\frac{3}{2}}-(r'^2+l^2)^{\frac{3}{2}}}{\frac{3}{2}(b\sqrt{r'^2+b^2}+r'^2\log.\sqrt{r'^2+b^2}+b-l\sqrt{r'^2+l^2}-r'^2\log.(\sqrt{r'^2+l^2}+l))},$$

in which $r'=\frac{b^2}{\sqrt{a^2-b^2}}$, $l=C\,s$.

Surface of a Frustrum of a Circular Spindle, *Fig.* 7, p. 283.

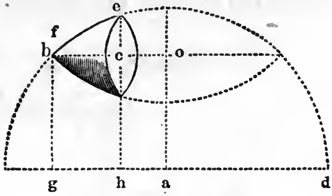
The distance from the centre of the spindle$=\frac{r^2-r'^2}{2(h-D.z)}$, r *and* r' *being the radii of the two bases,* e *and* s*;* h *the distance between the two bases;* D *the distance of the centre of the spindle from the centre of the circle, as* $a\,o$*;* z *the generating arc, expressed in units of the radius.*

Fig. 7.



Surface of a Segment of a Circular Spindle, as b c, Fig. 8.

Fig. 8.



Distance from centre of spindle $=\frac{r^2}{2(h-D.z)}$, *r representing the radius of the base; the other symbols the same as given on page* 282.

NOTE.—This last formula is essentially the same as the following.

Segment of a Circular Spindle.

Distance from centre $=\frac{b^2}{2\left[g-l-a\left(\sin.^{-1}\frac{g}{r}-\sin.^{-1}\frac{l}{r}\right)\right]}$,

sin. $^{-1}\frac{g}{r}$ *and sin.* $^{-1}\frac{l}{r}$ *denoting the arcs, the sines of which are respectively* $\frac{g}{r}$ *and* $\frac{l}{r}$; $g=f\,o$, $l=o\,c$, $r=$ *radius of circle* $=a\,d$, $a=a\,o=\sqrt{r^2-g^2}$, *and* $b=$ *radius of end circle of segment.*

Paraboloid of Revolution.

(See *Fig.* 65, on page 126.)

Distance from vertex $=\frac{1}{10\,p}\,\frac{(p^2+b^2)^{\frac{3}{2}}(3\,b^2-2\,p^2)+2\,p^5}{(p^2+b^2)^{\frac{3}{2}}-p^3}$,

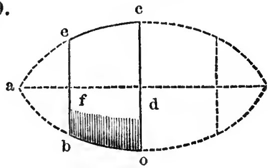
$a=$ *altitude*, $b=$ *radius of base, and* $p=\frac{b^2}{2\,a}$.

MENSURATION OF SOLIDS.

Cycloidal Spindle.

To ascertain the Contents of the Frustrum of a Cycloidal Spindle, b e c o, Fig. 9.

Fig. 9.



$$\frac{p}{6}\left((2\,r-l)^{\frac{1}{2}}(2\,l^{\frac{5}{2}}+5\,l^{\frac{3}{2}}\,r+15\,l^{\frac{1}{2}}\,r^2-15\,r^3\,\text{ver.sin.}^{-1}\frac{l}{r}+15\,p\,r^3)\right)$$

$=$*contents, l representing e f,* $r=\frac{d\,c}{2}$, *and p, as before,* 3.1416,

ver. sin. $^{-1}\frac{l}{r}$, *symbol for the arc, the v. s. of which* $=\frac{l}{r}$.

To ascertain the Contents of the Segment of a Cycloidal Spindle, a b e, Fig. 9.

$$\frac{p}{6}\left(15\,r^3\,\textit{ver. sin.}^{-1}\frac{l}{r}-(2\,r-l)^{\frac{1}{2}}\,(2\,l^{\frac{5}{2}}+5\,l^{\frac{3}{2}}\,r+15\,l^{\frac{1}{2}}\,r^2)\right)=\textit{contents.}$$

Frustra of Spheroids, or Ellipsoids of Revolution, $e\,c\,d\,f$, Figs. 104 *and* 104*, *page* 182.

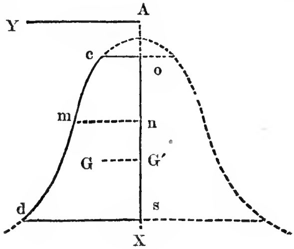
Distance from centre of spheroid.

Prolate $=\frac{3}{4}\,\frac{d\,(2\,a^2-d^2)}{3\,a^2-d^2}$; *Oblate* $=\frac{3}{4}\,\frac{d\,(2\,b^2-d^2)}{3\,b^2-d^2}$; *a representing the semi-transverse axis, b the semi-conjugate, and d the height of the frustrum.*

SOLIDS OF REVOLUTION.

To ascertain the Volume of a Solid of Revolution, Fig. 7.*

Fig. 7.



Let A X, the axis of x, be the axis of revolution, and c m d the generating curve, A n$=x$, m n$=y$, the co-ordinates of any point of the curve, and let the solid be terminated by planes perpendicular to the axis, cutting it at o and s.

Let A o$=a$, and A s$=b$, the abscissæ of these points; o $c=r$, and s $d=r'$, the radii of the two bases. The origin, A, may be taken at any convenient point on A X.

The general formula,† when A X is the axis of revolution, is $V=p\int y^2\,dx$, in which $p=3.1416$; *$\int$ is the symbol of integration, and d that of the differential.*

If, in the expression for V, y^2 or dx be eliminated by means of the equation of the generating curve, and the integration be effected between the limits $x=a$ and $x=b$, or $y=r$ and $y=r'$, the value of V is determined.

Corollary.—If A Y, or the axis of y, is the axis of revolution, *then,*

$$V=p\int x^2\,dy,$$

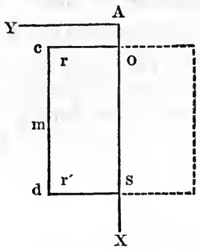
which differs from the preceding simply by the interchange of the letters x and y.

* By Professor J. H. C. Coffin, U. S. N.

† This formula is thus read: *The volume* is equal to p times the integral of y^2 multiplied by the differential of x.

EXAMPLE. *To ascertain the Volume of a Cylinder, Fig.* 8.

Fig. 8.



The generating line, c m d, is a right line parallel to the axis, A X; its equation is $y=r=r'$.

Whence $V = p\,r^2 \int dx$; and integrating between $x=a$ and $x=b$,

$V=p\,r^2(b-a)$; or, letting $h=b-a$ ($=$o s), *the altitude of the cylinder,*

$V=p\,r^2 h$; or, since $p\,r^2=$*the area of the base,*

The volume of a cylinder is equal to the area of its base multiplied by its altitude.

EXAMPLE 2. *To ascertain the Volume of the Frustrum of a Cone with a Circular Base.*

The generating curve is a straight line inclined to the axis. Let $h=b-a$ ($=$o s), the altitude of the frustrum. The equation of the generating line is (the origin being at the vertex),

$y=\dfrac{r'-r}{h}\,x$, whence $dx=\dfrac{h}{r'-r}\,dy$, and

$V=\dfrac{p\,h}{r'-r}\displaystyle\int y^2\,dy$; and integrating between $y=r$ and $y=r'$,

$V=\dfrac{p\,h\,(r'^3-r^3)}{3\,(r'-r)}$; or reducing,

$V=\frac{1}{3}\,h\,p\,(r^2+r'^2+r\,r')$; i. e. (since $p\,r^2$ and $p\,r'^2$ are the two bases, and $p\,r\,r'$ is a mean proportional between the two),

The volume of such a conical frustrum is equal to the sum of the two bases and their mean proportional, multiplied by one third of the altitude.

Or the following rule may be used:

Add together the squares of the radii of the two bases and the product of those radii, and multiply the sum by one third of the altitude and the number 3.1416.

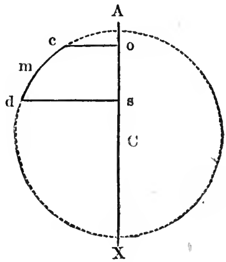
Consequently, If the volume of the cone itself be required,

$r'=o$, and $V=\frac{1}{3}hpr^2$; that is,

The volume of a cone is equal to its base multiplied by one third of its altitude.

EXAMPLE 3. *To ascertain the Volume of a Spherical Segment, Fig.* 9.

Fig. 9.



The generating curve, c m d, is an arc of a circle, and its equation, if the origin be taken at the centre of the sphere, C, is

$y^2=R^2-x^2$ (R being the radius of the sphere); *whence,*

$V=p\int(R^2-x^2)\,dx$; and integrating between $x=a$ and $x=b$,

$V=p\left[R^2(b-a)-\frac{1}{3}(b^3-a^3)\right]$; or, letting $h=b-a$, the altitude of the segment, and substituting for R^2 its value, $\frac{1}{2}(r^2+r'^2)+\frac{1}{2}(a^2+b^2)$,

$V=hp\left[\frac{1}{2}(r^2+r'^2)+\frac{1}{6}h^2\right]$; that is,

The volume of a spherical segment is equal to half the sum of its bases + one sixth the area of the circle, the radius of which is equal to the altitude, multiplied by the altitude.

Or the following rule may be used:

Add together the square of the altitude and three times the squares of the radii of the two bases, and multiply the sum by one sixth of the altitude and by the number 3.1416.

Corollary 1. If one of the bases$=0$, $V=\frac{1}{6}hp(3r^2+h^2)$; that is,

The volume of a spherical segment with a single base is equal to the sum of the area of a circle, the radius of which is the altitude, and three times the base, multiplied by one sixth the altitude.

2. If both bases$=0$, the segment becomes the entire sphere, and

$V=\frac{1}{6}p\,h^3$; or, since $h=2\,R$, *the diameter,*

$V=\frac{2}{3}p\,R^2\times h=\frac{4}{3}p\,R^3$; that is,

The volume of a sphere is equal to two thirds of a cylinder having the same diameter and altitude.

Or, *It is equal to four thirds of the cube of its radius, multiplied by* 3.1416.

2. If both bases$=0$, the segment becomes the entire sphere, and

$V=\frac{1}{6}p\,h^3$; or, since $h=2\,R=D$, *the diameter,*

$V=\frac{2}{3}p\,R^2\times h=\frac{4}{3}p\,R^3=\frac{1}{6}p\,D^3$; that is,

The volume of a sphere is equal to two thirds of a cylinder having the same diameter and altitude.

Or, *It is equal to four thirds of the cube of its radius, multiplied by* 3.1416.

Or, *It is equal to one sixth of the cube of its diameter, multiplied by* 3.1416.

EXAMPLE 4. *To ascertain the Volume of a Segment of a Prolate Spheroid.*

The generating curve, c m d, is an arc of an ellipse; and if the origin be taken at the centre, C, the equation is $y^2=\frac{B^2}{A^2}(A^2-x^2)$, in which

A and B are respectively the semi-transverse and semi-conjugate axes of the ellipse. *Whence,*

$V=p\frac{B^2}{A^2}\int(A^2-x^2)\,dx$; and integrating between $x=a$ and $x=b$,

$V = p\frac{B^2}{A^2}[A^2(b-a)-\frac{1}{3}(b^3-a^3)]$; or, letting $h=b-a$ ($=$os), *the altitude of the segment*, substituting for A^2 its value,

$\frac{1}{2}\frac{A^2}{B^2}(r^2+r'^2)+\frac{1}{2}(a^2+b^2)$, and reducing,

$$V = hp\left[\tfrac{1}{2}(r^2+r'^2)+\frac{B^2}{A^2}\cdot\frac{h^2}{6}\right];$$ that is,

The volume of such a segment is equal to half the sum of the bases + one sixth of the area of the circle the radius of which is the altitude, multiplied by $\left(\frac{B^2}{A^2}\right)$, *the square of the ratio of the axes of the generating ellipse, multiplied by the altitude.*

Or the following rule may be used:

Multiply the squares of the altitude and semi-conjugate axis together; divide by the square of the semi-transverse axis; add together the quotient and three times the squares of the radii of the two bases; and multiply the sum by one sixth of the altitude and by the number 3.1416.

Corollary 1. If one of the bases$=0$, the expression for the volume is $V=hp\left[\frac{r^2}{2}+\frac{B^2}{A^2}\cdot\frac{h^2}{6}\right]$.

2. If both bases$=0$, the segment is the entire spheroid, and the altitude $h=2A$; and

$V=pB^2\times\frac{4}{3}h$; or,

The volume of such a spheroid is equal to $\frac{2}{3}$ *the volume of a cylinder of the same altitude, the base of which is equal to the middle section of the spheroid.*

EXAMPLE 5. *To ascertain the Solidity of a Segment of an Oblate Spheroid.*

The equation is $y^2=\frac{A^2}{B^2}(B^2-x^2)$; *whence*

$V=p\frac{A^2}{B^2}\int(B^2-x^2)\,dx$; and integrating between $x=a$ and $x=b$, and reducing, as in Example 4,

$$V=hp\left[\tfrac{1}{2}(r^2+r'^2)+\frac{A^2}{B^2}\cdot\frac{h^2}{6}\right].$$

EXAMPLE 6. *To ascertain the Volume of a Segment of an Hyperboloid of Revolution.*

The equation is $y^2 = \frac{B^2}{A^2}(x^2 - A^2)$; *whence*

$V = p\frac{B^2}{A^2}\int(x^2 - A^2)\,dx$; and integrating and reducing, as in Examples 4 and 5,

$$V = hp\left[\tfrac{1}{2}(r^2 + r'^2) - \frac{B^2}{A^2}\cdot\frac{h^2}{6}\right].$$

EXAMPLE 7. *To ascertain the Volume of a Segment of a Paraboloid.*

The equation is $y^2 = 2\,P\,x$, in which $2\,P =$ the parameter.

$V = 2\,P\,p\int x\,dx$; or, integrating between $x = a$ and $x = b$,

$V = P\,p\,(b^2 - a^2)$; or, since $r^2 = 2\,P\,a$, and $r'^2 = 2\,P\,b$, and $h = b - a$,

$V = \frac{1}{2}hp\,(r^2 + r'^2)$.

Another method of determining the volume of a solid of revolution is to ascertain the area of the generating surface and the distance of its centre of gravity from the axis of revolution.

Let $A =$ *the area* o s *d* c (*Fig.* 7).

$g =$ *the distance of its centre of gravity*, G, *from the axis* A X; *then*

$V = 2\,p\,g \times A$; that is,

The volume is equal to the area of the generating surface multiplied by the circumference of the circle described by its centre of gravity.

CENTRES OF GRAVITY.

To ascertain the Centre of Gravity of a Solid of Revolution.

The centre of gravity is upon the axis of revolution, and it is necessary to determine only its distance from some particular point, as, for example, the vertex, or the intersection of the axis by a base, or the origin of co-ordinates, A, *Fig.* 7.

As before, Let A X, *the axis of x, be the axis of revolution, etc.* (See page 286.)

Let G′ *be the centre of gravity of the solid, and* $AG' = g$.

The general formula is

$$g = \frac{p \int y^2 x\, dx}{V} = \frac{\int y^2 x\, dx}{\int y^2\, dx}.$$

If y or x and dx be eliminated by means of the equation of the curve which generates the surface, and the integration be effected between the limits $x=a$ and $x=b$, or $y=r$ and $y=r'$, the distance of the centre of gravity, G′, from the origin of co-ordinates is determined.

EXAMPLE. *To asceriain the Centre of Gravity of a Cylinder with a Circular Base.* (See *Fig.* 8, page 287.)

The equation is $y=r=r'$; *whence*

$g = \dfrac{r^2 \int x\, dx}{r^2 \int dx}$, and integrating between the limits $x=a$ and $x=b$,

$$g = \frac{b^2 - a^2}{2(b-a)} = \tfrac{1}{2}(b+a);$$ or, if the origin be taken at o,

$g = \frac{1}{2} b =$ *half the altitude.*

EXAMPLE 2. *To ascertain the Centre of Gravity of the Frustrum of a Cone with a Circular Base.* (See Example 2, p. 287.)

The equation is $y = \dfrac{r'-r}{h} x$; *whence* $x = \dfrac{h}{r'-r} y$, and $dx = \dfrac{h}{r'-r} dy$; and

$$g=\frac{h}{r'-r}\cdot\frac{\int y^3\,d\,y}{\int y^2\,d\,y};\text{ and integrating,}$$

$$g=\frac{3\,h\,(r'^4-r^4)}{4\,(r'-r)\,(r'^3-r^3)};\text{ and reducing,}$$

$$g=\frac{3\,h\,(r'^2+r^2)\,(r'+r)}{4\,(r'^3-r^3)}.$$

Whence, To ascertain the distance of the centre of gravity from the vertex of the cone,

Multiply together the sum of the radii of the two bases, the sum of their squares, and $\frac{3}{4}$ the altitude, and divide the product by the difference of the cubes of those radii.

Corollary 1. If the distance from the greater base (or $b\,G'$) be required,

$$b\,G'=g'=\tfrac{1}{4}\,h.\frac{r'^4-4\,r'\,r^3+3\,r^4}{(r'-r)\,(r'^3-r^3)}=\tfrac{1}{4}\,h.\frac{r'^2+2\,r\,r'+3\,r^2}{r'^2+r\,r'+r^2}$$

$$=\tfrac{1}{4}\,h.\frac{(r'+r)^2+2\,r^2}{(r'+r)^2-r\,r'}.$$

2. For the entire cone, $r=0$, and

$$g=\tfrac{3}{4}\,h,\text{ or } g'=\tfrac{1}{4}\,h;\text{ that is,}$$

The distance of the centre of gravity from the vertex is equal to $\frac{3}{4}$ the altitude; or, from the base, is equal to $\frac{1}{4}$ the altitude.

EXAMPLE 3. *To ascertain the Centre of Gravity of a Spherical Segment.* (See Example 3, page 288.)

Equation of curve, $y^2=R^2-x^2$; *whence* $x\,d\,x=-y\,d\,y$; and

$$g=\frac{p\int-y^3\,d\,y}{V}=\frac{r'^4-r^4}{4\,h[\frac{1}{2}(r'^2+r^2)+\frac{1}{6}h^2]}=\frac{r'^4-r^4}{2\,h[r'^2+r^2+\frac{1}{3}h^2]};$$

Hence, To ascertain the distance of the centre of gravity of the segment from the centre of the sphere,

Take the difference of the 4th powers of the radii of the bases as a dividend; and for the divisor, multiply the sum of the squares of the radii and $\frac{1}{3}$ the square of the altitude by twice the altitude; the quotient is the distance required.

The centre of gravity is between the centre of the sphere and the lesser base.

Corollary. For a segment with a single base, $r=0$; and $g=\frac{r'^4}{2\,h\,r'^2+\frac{1}{3}h^3}=\frac{(\mathrm{R}-\frac{1}{2}h)^2}{\mathrm{R}-\frac{1}{3}h}$, *the distance from the centre of the sphere.*

EXAMPLE 4. *To ascertain the Centre of Gravity of a Frustrum of a Prolate or Oblate Spheroid.*

The distance from the centre of the spheroid is

$$g=\frac{\frac{\mathrm{A}^2}{\mathrm{B}^2}(r'^4-r^4)}{2\,h(r'^2+r^2+\frac{1}{3}\frac{\mathrm{B}^2}{\mathrm{A}^2}h^2)},=\frac{3}{4}\frac{(d\pm d')\,(2\,\mathrm{A}^2-d'^2-d^2)}{3\,\mathrm{A}^2-d'^2\mp d'\,d-d^2}\text{ for a}$$

prolate spheroid.

$$g=\frac{\frac{\mathrm{B}^2}{\mathrm{A}^2}(r'^4-r^4)}{2\,h(r'^2+r^2+\frac{1}{3}\frac{\mathrm{A}^2}{\mathrm{B}^2}h^2)},=\frac{3}{4}\frac{(d\pm d')(2\,\mathrm{B}^2-d'^2-d^2)}{3\,\mathrm{B}^2-d'^2\mp d'\,d-d^2}\text{ for an}$$

oblate spheroid, A *and* B *representing semi-transverse axes, d and d' respectively the distances from the centre of the spheroid to the base and end of the frustrum. If both these are on the same side of the centre, the upper sign is used, but if they are on different sides, the lower sign is used.*

EXAMPLE 5. *To ascertain the Centre of Gravity of a Segment of a Prolate or Oblate Spheroid.*

$$g=\frac{(\mathrm{A}-\frac{1}{2}h)^2}{\mathrm{A}-\frac{1}{3}h},\text{ for a prolate spheroid.}$$

$$g=\frac{(\mathrm{B}-\frac{1}{2}h)^2}{\mathrm{B}-\frac{1}{3}h},\text{ for an oblate spheroid.}$$

Frustrum of Hyperboloid of Revolution.

Distance from centre of hyperboloid $=\frac{3}{4}\frac{(d'+d)(d'^2+d^2-2\,a^2)}{d'^2+d'\,d+d^2-3\,a^2}$,

a=semi-transverse axis, d=distance from centre to base of segment.

Segment of Hyperboloid of Revolution, Fig. 12, p. 247.

Distance from centre of hyperboloid (point of intersection of the diameters $t\,a$ and $d\,f$) $=\frac{3}{4}\,\frac{(d+a)^2}{2\,a+d}$, *a and d as before, and d′ = distance from centre to base of frustrum.*

Frustrum of a Paraboloid of Revolution.

Distance from vertex of paraboloid $=\frac{2}{3}\,\frac{d'^2+d'\,d+d^2}{d'+d}$, *d representing height of paraboloid, and d′ the distance between the frustrum and vertex.*

For the mensuration of SPHERICAL TRIANGLES and PYRAMIDS, SPIRALS, EPI-CYCLES, EPI-CYCLOIDS, CARDIOIDS, HELICOIDS, PELI-COIDS, etc., etc., see *Loomis's Analytical Geometry and Calculus; Davies and Peck's Dictionary of Mathematics; Docharty's Geometry; Hackley's Geometry.*

For a Glossary and Explanation of Geometrical Figures, see *Davies and Peck*, and the *Library of Useful Knowledge*, vols. i. and ii.

CARPENTERS' SLIDE-RULE, OR GUNTER'S LINE.

This instrument is commonly called a Sliding Rule. It is constructed of two pieces of box-wood or ivory, of a foot in length each, connected together by a joint, which enables them to be folded up lengthwise upon an edge.

On one side, the whole rule is divided upon its outer edges into inches and eighths, for the purpose of taking dimensions, and upon its inner edges there are several plane scales, each divided into twelve parts, which are designed for the reduction of dimensions for the purposes of drawing.

On the other side there is a metallic slide, and four lines marked A, B, C, and D, the two middle, B and C, being upon the slide.

Three of these lines, A, B, and C, are doubled in their dimensions; that is, they proceed from 1 to 10 twice; and the fourth line, D, is a single one, running from 1 to 10, and is called the line of *roots*.*

The use of the double lines, A and B, is for operations in arithmetic, and ascertaining the areas of plane figures.

Upon the other part of this face there is a table of *gauge points* for the ascertaining and measurement of the different elements and substances there given, under the respective columns of *Square, Circular, and Globe;* and the *gauge point* to be selected for operation is determined by the denomination in which the dimensions are given; thus, if in three dimensions, and all in feet, that under F.F.F. is to be taken; if one dimension is in feet and the others in inches, that under F.I.I. is to be taken; and if all are in inches alone, that under I.I.I. is to be taken. Again, if of two denominations, or of one only, the *gauge point* is to be taken from under F.I., I.I., or F. and I., as the case may be.

* Upon some rules, the fourth line, D, is divided from 4 to 40, in which case it is termed a *girth-line*, and is then used in the measurement of timber.

The divisions on the first, second, and third lines are marked alike, each beginning at 1, which may be 1, 10, 100, or 1000, etc., or .1, .01, .001, etc.; but, whatever it is assumed to be, the second or middle line of these divisions must be taken at 10 times as many as the first, and the third line must be taken at 10 times as many as the second.

Numeration is the first operation to be acquired upon this instrument, for when that is understood, all other operations will become quite easy. It is first to be observed that the values of the divisions upon the rule are all arbitrary, and the value set upon them must be such as will meet the requirements of the question, which must be determined when a question is proposed.

The principal divisions indicated by figures at 1, 2, 3, 4, and so on to 10, are termed primes, and the next divisions tenths, and these again are, or may be, subdivided into hundredth and thousandth parts.

If the 1 (next to the joint) represents one tenth, then will the middle 1 be a unit, or a whole number, and the other figures toward the right hand are likewise whole numbers, from the middle 1 to 10 at the end; but if the first 1 represents a unit, then the middle 1 will be 10, and the 10 at the right hand 100; if the first 1 represents 10, the middle 1 will be 100, and that at the right hand 1000; always increasing in a tenfold proportion, according to the value set upon the first 1. The figures between 1 and 10 are designated after the same manner; that is, if 1 at the beginning is one tenth, 2 will be two tenths, and the next 2 toward the right hand 2 units; but if 1 at the beginning is 1 unit, then 2 will be 2 units, and the other 2 will be 20; if the first 1 is called 10, then 2 will be 20, and the next 2 is 200, etc. When the 1 at the left hand is taken as .1, or one tenth, the line is read in the following order: 1, 2, 3, 4, 5, 6, 7, 8, 9 tenths, unity or 1; 2, 3, 4, 5, 6, 7, 8, 9, 10; when a higher value is set on them, they will read thus, beginning next the joint, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 300, and so on to 1000.

To multiply two Numbers.

Set 1 on A (1st line) to either of the given numbers on B (2d line), then at the other number on A (1st line) will be found the product on B (2d line).

EXAMPLE.—Multiply 12 by 25.

Under 1 on A put 12 on B, and under 25 on A is 300 on B.

Ex. 2. Multiply 64 by 15. *Ans.* 960.

NOTE.—If the third term runs beyond the end of the line, look for it on the first radius or other part of the line, and increase it ten times.

To divide one Number by another.

Set 1 on A to the divisor on B, then at the dividend on B will be found the quotient on A.

EXAMPLE.—Divide 300 by 25.

Under 1 on A put 25 on B, and at 300 on B is 12 on A.

NOTE.—If the dividend runs beyond the end of the line, diminish it 10 or 100 times, as may be required, to make it fall upon A, and increase the quotient accordingly.

To ascertain a Fourth Proportional.

Set the first term upon A to the second on B, then at the third on A is the fourth on B.

EXAMPLE. — What is the fourth proportional to 8, 20, and 30?

Under 8 on A put 20 on B, then at 30 on A is 75 on B.

To ascertain a Mean Proportional.

Set the first term on C to the same term on D, then at the second on C is the mean on D.

EXAMPLE.—What is the mean proportional between 20 and 80?

Under 20 on C set 20 on D, and at 80 on C is 40 on D.

Rule of Three Direct.

In the Rule of Three Direct, there are three numbers given to find a fourth, that shall have the same proportion to the third as the second has to the first.

The operation is, as the first term upon A is to the second upon B, so is the third term upon A to the fourth upon B.

Or, bring the first term upon B to the second upon A; then against the third upon B is the result upon A.

EXAMPLE.—If a man can walk 20 miles in 5 hours, how long will he require to walk 125 miles at the same rate?

Over 20 upon B put 5 upon A, and against 125 upon B is 31.25, *the result*, upon A.

Rule of Three Inverse.

In this rule, there are three numbers given to find a fourth, that shall have the same proportion to the second as the first has to the third.

NOTE.—If more requires more, or less requires less, the question belongs to the rule of three *Direct;* but if more requires less, or less requires more, it then belongs to the rule of three *Inverse.*

EXAMPLE.—If 6 men can do a certain piece of work in 8 days, how many will it require to perform the same in 3 days?

NOTE.—In inverse proportion, the slide is to be inverted (by withdrawing it and introducing the opposite end of it); then the question will be operated in the same way as in direct proportion.

Invert the slide in the groove, and over 8 upon C set 6 upon A; then at 3 upon C is 16, *the result*, upon A.

Vulgar and Decimal Fractions.

To reduce a Vulgar Fraction to its equivalent Decimal Expression.

The operation is, as the denominator upon A is to 1 upon B, so is the numerator upon A to the decimal required upon B.

EXAMPLE.—Reduce $\frac{1}{4}$ to a decimal.

Set 1 upon B to 4 upon A; then at 1 upon A is .25, *the result*, upon B.

To extract the Square Root.

When the lines C and D are equal at both ends, C is a table of squares, and D a table of roots; consequently, opposite to any number or division upon C is its square root upon D.

Example.—If a tower 30 feet in height is on the side of a river which is 40 feet in width, what must be the length of a ladder that will reach from the opposite side of the river to the top of the tower?

Operation.—Set the slide even at both ends, and over 30 upon D is 90 on C, and at 40 on D is 160 upon C, which, when added to 90, =250; then under 250 upon C is 50 upon D, *the length of the ladder.*

To Square a Number.

Set 1 upon D to 1 upon C; then against the number upon D will be found the square upon C.

To Cube a Number.

Set the number upon C to 1 or 10 upon D, and against the same number upon D will be its cube upon C.

Set 6 upon C to 10 upon D, and at 6 upon D is 216 upon C.

Land Measuring.

The *Gauge points* for measuring land are the number of square chains, square perches, and square yards that are contained in an acre. If the dimensions are given in chains, the *gauge point* is 1, 10, 100, etc., upon A; if in perches, it is 160; but if it is given in yards, the *gauge point* is 4840, which the length upon B must always be set to, and opposite the breadth upon A is the result, in acres and parts, upon B.

Example.—If a field is 20 chains 50 links in length, and 4 chains 40 links broad, how many acres does it contain?

Set 20.5 upon B to 1 upon A, and at or under 4.4 upon A is 9 upon B=*the result in acres.*

Mensuration of Solids.

In measuring and weighing solid bodies, the tables of *Gauge points* upon the rule are always to be made use of, and are thus explained.

1. All *gauge points* are taken on the line A.
2. All lengths must be noted on the line B, and are to be set to the *gauge point* on the line A.
3. All squares and diameters are found on the line D.
4. Opposite the square or diameter on the line D is the content or result on the line C; or, as the length upon B is to the *gauge point* upon A, so is the square or diameter upon D to the content upon C.

There are three *gauge points* for every element that is given in the table for square; F.F.F. signifying that when the length and both the squares are feet, the *gauge point* is to be found under F.F.F. in the same line with that of the element or material to be measured or weighed.

If the length is given in feet, and both the squares are inches, then the *gauge point* is under F.I.I.; but if the dimensions of both length and square are in inches, then the *gauge point* is under I.I.I.

There are two *gauge points* for every thing that is to be measured or weighed of a cylindric form; first, when the length is in feet and the diameter in inches, the *gauge point* is under F.I. If the length is in inches and the diameter in inches, then the *gauge point* is under I.I.

There are two *gauge points* to weigh or measure every thing of a globular figure.

A globe having but one dimension, it must be either in feet or inches; if it is in feet, the *gauge point* is under F.; if it is in inches, the *gauge point* is under I.

NOTE.—In measuring or weighing square timber, stone, metals, or any other bodies that are unequal sided, a mean proportion must be first estimated, in order to obtain a true cube or square.

The general rule for a globe is, as the *gauge point* on A is to the diameter on B, so is the content on C to the diameter on D; or, set the diameter upon B to the *gauge point* upon A, and against the diameter upon D is the result upon C.

EXAMPLE.—If a piece of timber is 16 inches broad, 6 inches thick, and 20 feet long, how many cubic feet does it contain?

Ascertain the mean square by setting 16 upon C to 16 upon D, and opposite to 6 upon C is 9.8 upon D, *the side of a square equal to* 16 *by* 6; having thus ascertained the true square, look for the *gauge point* for cubic feet, and under F.I.I. is 144; set 20, the length, upon B to 144 upon A, and against 9.8 upon D are 13.3 cubic feet upon C.

Round timber is generally measured by the girth, which is ascertained by a line run around the middle of the tree or log, and taking one fifth of the girth for the side of the square. This is not precisely correct, but it is the method commonly practiced, and is performed on this rule, after the girth is taken, as follows.

EXAMPLE.—A round log is 30 feet in length, and 40 inches around its middle; how many cubic feet does it contain?

Here one fifth of the girth is 8 inches.
Multiply the length, 30, by 2, and set the result, 60, upon B to 144 upon A, and against 8 upon D is 26.6, *the result, in feet,* upon C.

The circumference of the above log being 40 inches, the diameter is 12.73 inches.
Set 30, the length, upon B, to 1833, the *gauge point,* upon A, and against 12.73 upon D are 26.5 *feet, the result,* upon C. By this it will be seen that there is but a slight difference between the customary and true methods of measuring.

Cylinder, Globe, and Cone.

EXAMPLE.—If a cylinder is 6 inches long, and 6 inches in diameter, how many cubic inches does it contain?

The *gauge point* for cubic inches is 1273.
Set 6 upon B to 1273 upon A, and against 6 upon D are 169 *cubic inches, the result,* upon C.

Cask Gauging.

The gauging of casks is performed, after a mean diameter is found, exactly in the same manner as in the last examples. Casks are generally reduced to what is termed four *Varieties;* and their mean diameters may be found by multiplying the difference between the head and bung diameters of the *first variety* by .7, the *second* by .63, the *third* by .56, and the *fourth* by .52. The respective products of these numbers, added to the head diameter, will give the mean diameter.

Set the length upon B to the *gauge point* upon A, and over the mean diameter on D is the result upon C.

EXAMPLE.—In a cask of the *first variety,* the head diameter is 24, the bung diameter 28, and the length 30 inches; how many gallons will it contain?

Set 30 upon B to 353, the *gauge point* for the imperial measure, upon A, and against 26.8, the mean diameter, upon D is 61, *the result in gallons,* upon C.

Ex. 2. How many gallons are contained in a cask of the *second variety,* the head diameter 18, the bung diameter 23, and length 28 inches?

Set 28 upon B to the *gauge point* upon A, and against 21.15, the mean diameter, upon D is 35.2, *the result in gallons,* upon C.

Ex. 3. If a cask of the *third variety* is 20 inches at the head, 26 at the bung, and 29 inches long, what will be its contents in gallons?

Set 29 upon B to the *gauge point* upon A, and against 23.36, the mean diameter, upon D is 44.5, *the result in gallons,* upon C.

Ex. 4. A cask of the *fourth variety* is 34 inches long, the head diameter 26, and the bung diameter 32; how many gallons will it hold?

Set 34 upon B to the *gauge point* upon A, and against 29.12, the mean diameter, on D is 81.5, *the result in gallons,* upon C.

Miscellaneous Questions.

Under this head may be introduced a great many original questions, as well as such as could not be introduced in regular order in the foregoing rules.

Of a Circle.

The Diameter being given, to ascertain the Area; or the Area being given, to ascertain the Diameter.

Set .7854, the area of unity, upon C to unity or 10 upon D, then the lines C and D will be a table of areas and diameters; for against any diameter upon D is the area in square inches upon C.

The Circumference being given, to ascertain the Area; or the Area being given, to ascertain the Circumference.

Set .0795 upon C to 1 or 10 upon D, then the lines C and D will be a table of areas and circumferences; for against any circumference upon D is the area in square inches upon C.

The Circumference being given, to ascertain the Diameter; or the Diameter being given, to ascertain the Circumference.

Set 1 upon B to 3.141 upon A, then the lines A and B will be a table of diameters and circumferences; for against any diameter upon B is the circumference upon A.

To ascertain the Side of a Square equal in Area to any given Circle.

Set .886 upon B to 1 upon A, then against any diameter of a circle upon A is the side of a square that will be equal in area upon B.

To ascertain the Side of the greatest Square that can be inscribed in any given Circle.

Set .707 upon B to 1 upon A, and against any diameter of a circle upon A is the side of its greatest inscribed square upon B.

To ascertain the Side of the greatest Equilateral Triangle that can be inscribed in any given Circle.

Set 1 upon B to 115 upon A, and against any diameter of a circle upon A is the length of a side of a triangle upon B.

The Weights of Metals, and various other Results, may be obtained in a similar manner, for the rules of operation of which, reference is given to the Book of Directions usually furnished with the slide-rule.

Illustrations.

What is the weight of a piece of cast iron 3 inches square and 6 feet long?

The cast iron *gauge point* for feet long and inches square is 32. *See the rule.*

Set 6 upon B to 32 upon A, and against 3 upon D is 168, *the result, in pounds,* upon C.

A cylinder is 6 inches in length and 6 inches in diameter; what is its weight in cast iron, wrought iron, and brass?

1st. For cast iron, the *gauge point* is 489.
Set 6 upon B to 489 upon A, and against 6 upon D is 44 upon C.

2d. For wrought iron, the *gauge point* is 453.
Set 6 upon B to 453 upon A, and against 6 upon D is 47.5 upon C.

3d. For brass, the *gauge point* is 424.
Set 6 upon B to 424 upon A, and against 6 upon D is 511, *the result,* upon C.

TABLE OF ADDITIONAL GAUGE POINTS.

	Square.			Cylinder.		Globe.	
	F.F.F.	F.I.I.	I.I.I.	F.I.	I.I.	F.	I.
Oak	174	252	303	320	386	332	578
Mahogany	15	217	2605	276	333	286	49
Box	155	243	269	31	342	296	512
Marble	591	85	102	116	13	113	195
Brick	795	115	138	147	176	152	263
Sulphur	8	115	138	146	126	153	264
Alcohol	193	278	333	354	424	369	637

CASK GAUGING.

The operation of cask gauging is ordinarily performed with the aid of five instruments, viz., a *Gauging Slide-rule*, a *Gauging* or *Diagonal Rod*, *Callipers*, a *Bung Rod*, and a *Wantage Rod*.

THE GAUGING SLIDE-RULE.*

The Gauging Slide-rule is a flat rule, very similar to an ordinary slide-rule, except that it is not jointed, and its being adapted for use for the purpose of measuring and gauging casks, in addition to those of the ordinary computations effected by a slide-rule.

Upon the plain or outer face there are *five* lines; the first three are alike, being equally divided, and all of the same radius,† and each containing twice the length of one.

The *fourth* line is differently divided from the others, and is used in the operation of gauging, in the determination of the contents of casks when *Lying*, by the element of the depth of liquor within them, which is termed the *wet inches*.

The *fifth* line is similarly divided to the *fourth*, and is used in the operation of gauging, in the determination of the contents of casks, when *Standing*, by the element of the depth of liquor within them, which is also termed the *wet inches*.

Note.—The operation of gauging in this manner—that is, by the element of wet inches—is termed *Ullaging*.

Upon the opposite or inner face there are *four* lines; the *first* is divided to represent gallons,‡ the *second* is a line of mean diameters, and the *third* and *fourth* lines are divided into inches and tenths.

* As manufactured by Belcher, Brothers, & Co., New York.

† The first three lines are divided alike to the ordinary carpenters' slide-rule, or Gunter's line, described at page 297, and the operations of multiplication, division, etc., etc., may be performed, by inspection, as there described.

‡ 231 cubic inches, which is the U. S. standard gallon.

The third line is divided each way from the thumb-piece, running from 38 to 63 to the right, and from 0 to 12 to the left.

The use of this line is to measure the diameter of the head of a cask, and is thus operated:

Place the outside edge of the brass shoulder on the right end of the gauge, on the head of a cask, and close to the inside of a stave in line with the centre of the head; move the thumb-piece until its perpendicular or left face is in a line with a point at the other end, which would give the diameter of the head on its inside; then remove the gauge, and on the *fourth* line, under the face of the thumb-piece, read off the diameter of the head in inches and tenths.

If the diameter of the head exceeds 38 inches, then it is to be found on the *third* line, at the left end of the gauge, on the line running from 38 to 63.

On the left of the thumb-piece is a scale of 12 inches and tenths of inches, and above it is a *Scale of* 1*st Varieties;* that is, *varieties* of the first form of casks, the use of which is hereafter explained under the head of *Varieties.*

The *fourth* line is divided into inches and tenths, running from the right to the left, and is used for the purposes of ordinary measurement.

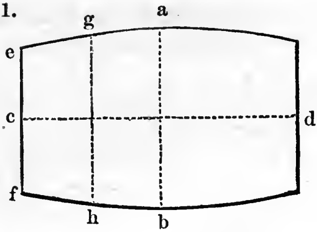
Upon one side of the instrument is a *Scale of* 2*d Varieties,* the use of which, and of all like scales, is to obtain by inspection the *mean diameter* of casks of the different varieties of figure, and which are thus classed.

*Varieties of Casks.**

First Variety.—Casks of the ordinary form, being that of the *Middle Frustrum of a Prolate Spheroid,* as *Fig.* 1.

* The basis of determination of a scale of varieties is that of giving a multiplier whereby the *mean diameter* of a cask may be ascertained, and the operation is effected as shown on page 310.

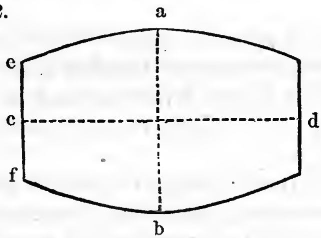
Fig. 1.



Rum puncheons and whiskey barrels are fair exponents of this form, which comprises all casks having a spherical outline of stave.

Second Variety.—Casks of the form of the *Middle Frustrum of a Parabolic Spindle,* as *Fig.* 2.

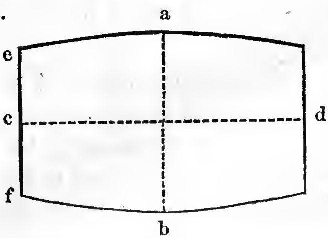
Fig. 2.



Wine casks are exponents of this form, which comprises all casks in which the curve of the staves quickens slightly at the bilge.

Third Variety.—Casks of the form of the *Middle Frustrum of a Paraboloid,* as *Fig.* 3.

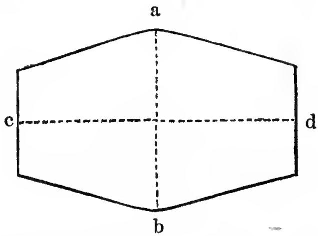
Fig. 3.



Brandy casks and provision barrels are exponents of this form, which comprises all casks in which the curve of the staves quickens at the chime.

Fourth Variety.—Casks of the form of two equal *Frustrums of Cones*, as *Fig.* 4.

Fig. 4.



A gin pipe is an exponent of this form, which comprises all casks in which the curve of the staves quickens sharply at the bilge.

As the rule, however, is provided with but scales of two varieties, it is usual to apply the first scale to all casks in which the middle diameter (intermediate between the bung and head), as *g h*, *Fig.* 1, approaches nearest to that of the bung diameter.

The scale of 2d variety upon the edge of the rule is adapted to all casks in which the middle diameter approaches next or second in order of proportion to that of the bung diameter, as *Fig.* 2.

Scales of 3d and 4th varieties are not given. Such scales, however, are wanted, and are applicable to all casks in which the middle diameter bears a less proportion to the bung diameter than in any of the other varieties.

To ascertain the Mean Diameter of a Cask.

Rule.—Subtract the head diameter from the bung diameter in inches, and multiply the difference by the following units for the four varieties; add the product to the head diameter, and the sum will give the mean diameter of the varieties required.

1st variety	.7	3d variety	.56
2d variety	.63	4th variety	.52

Example.—The bung and head diameters of a cask of the 1st variety are 24 and 20 inches; what is its mean diameter?

$24 - 20 = 4$, and $4 \times .7 = 2.8$, which, added to 20, $= 22.8$ *inches, the mean diameter.*

Ex. 2. The bung and head diameters of a cask of the 2d variety are 23 and 20 inches; what is its mean diameter?

Ans. 21.89 *inches.*

Operation by the Gauging Slide-rule.

Example 1. Subtract 20 from 24, and over 4, *the difference*, on the line of inches of the scale of 1st varieties, read 2.76, which, added to 20, $=22.76$, *the result required.*

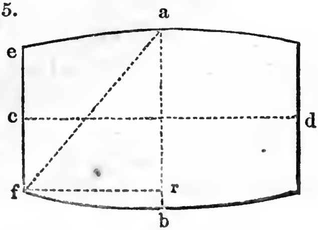
Ex. 2. Subtract 20 from 23, and under 3, *the difference*, on the line of inches of the scale of 2d varieties, read 1.88, which, added to 20, $=$ 21.88, *the result required.*

THE GAUGING OR DIAGONAL ROD.

The Gauging or *Diagonal Rod* is a square rule having four faces, being commonly four feet long. This instrument is used both for gauging and measuring casks, and in gauging, the contents are ascertained from one dimension only, viz., the diagonal of the cask, or the length from the centre of the bung-hole to the junction of the head of the cask with the stave opposite to the bung, being the longest straight line that can be drawn within a cask from the centre of the bung. Accordingly, on two opposite faces of the rule are scales of inches for measuring this diagonal, between which, on a third face, are placed the contents in gallons.

To ascertain the Contents of a Cask by the Gauging or Diagonal Rod, Fig. 5.

Fig. 5.



Operation.—Introduce the pointed end of the rod into the bung-hole of a cask (the plane face of the rod being uppermost) until it reaches the junction of the head and bottom of the cask at its lowest point; then adjust the upper end of the rod, so that the under side (divided into gallons) shall be in the centre of the hole in a line with the under side of the bung stave. Observe this point, by the aid of the divisions of inches and tenths on either of the sides of the rod; withdraw it, and, in a line with the division observed, read off on the under side of the rod the contents of the cask in gallons.

Illustration.—In the preceding figure, the bung diameter, $a\ b$, is 24 inches, the head diameter, $e\ f$, is 20 inches, and the half length, $f\ r$, is 18 inches.

Hence, by Geometry, the height, $a\ r$, of the triangle $a\ f\ r$ is $24-\dfrac{24-20}{2}=22$. *Consequently*, the length$=22$, and the base$=18$, the hypothenuse, $a\ f$, $=28.425$. *Then*, by inspection of the rod, it will be seen that in a line with 28.425 inches is 63, *the number of gallons the cask contains.*

THE CALLIPERS.

The Callipers is a sliding rule adapted to project over the chimes of casks to measure their inner length, and when it is adjusted to the heads of a cask, the inner length of it may be read off, a difference of 2 inches, being an allowance of 1 inch for the thickness of each head, being provided for in the divisions of the rule.

NOTE.—When the thickness of the heads is known to differ from an inch each, the difference above or less than an inch, as the case may be, is to be subtracted from or added to the length indicated by the callipers.

THE BUNG ROD.

The Bung Rod is alike to the diagonal rod in construction, and is a rod for determining the inner diameter of a cask, or the *wet inches* therein; and in order to enable the divisions to be accurately noted, there is a slide running around the rod, with a collar upon its lower end, which is brought to bear upon the under side of the stave at the bung; the rod, with the slide retained in position, is then removed, and the divisions on the rod read off.

NOTE.—It is customary to combine this instrument with the diagonal rod, the inches on the latter answering all the purposes of the measurement required, and the slide is removed or adjusted as may be required.

THE WANTAGE ROD.

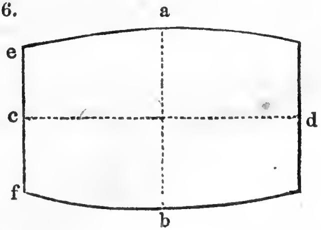
The Wantage Rod is a scale having four equal sides, upon which are divisions adapted to the many varieties of vessels, as barrels, tierces, and hogsheads.

The use of this instrument is to obtain by inspection the number of gallons of liquor deficient in a cask, when the deficiency does not reach to an extent that would class the vessel as an *ullage cask.*

There is a metal collar upon one of its sides, which is introduced into the bung-hole of a cask until its upper side is at the under side of the bung stave; the instrument is then removed, and the wet line indicates the number of gallons (under the designation on the rod, of the cask to which it is applied) the cask is deficient, or *wants* of being full.

To ascertain the Contents of a Cask of the 1*st Variety, Fig.* 6.

Fig. 6.



By Mensuration.

RULE.—To twice the square of the bung diameter, $a\ b$, add the square of the head diameter, $e\ f$; multiply this sum by the length, $c\ d$, of the cask, and the product again by .2618, and it will give the contents in cubic inches, which, being divided by 231, will give the result in gallons.*

Example.—The bung and head diameters of a cask, *a b* and *e f*, are 24 and 20 inches, and the length, *c d*, 36; what are its contents in gallons?

$24^2 \times 2 + 20^2 = 1552$, which $\times 36 = 55872$, and $55872 \times .2618 = 14627.2896$, which $\div 231 = 63.32$, *the gallons required.*

Ex. 2. The bung and head diameters of a cask are 36 and 30 inches, and the length 54; what are its contents in gallons? *Ans.* 213.7 *gallons.*

By the Gauging Slide-rule.

Operation.—Subtract the head diameter from the bung diameter, and look for the difference on the lower line (inches) of the scale of 1st varieties, and above it, on the second line, is the mean difference, which is to be added to the head diameter for the mean diameter.

Thus, $24 - 20 = 4 =$ *difference of diameters.*
Above 4, on 1st line of scale, read 2.75, which, added to $20, = 22.75$, *the mean diameter required.*

Then, set the left end of the slide on the inner face of the rule to the length of the cask (36) on the first line; look for the mean diameter (22.75) on the second line (or top line of slide), and above it, on the first line, read 63.6, *which is the capacity of the cask in gallons.*

Ex. 2. The bung and head diameters of a cask are 36 and 30 inches, and the length 54; what are its contents in gallons?

The difference of diameters is 6, which, by the scale, $= 4.16$.

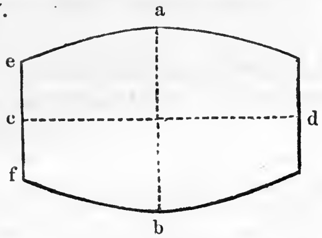
Ans. 214.6 *gallons.*

Ex. 3. The length of a cask is 51 inches, and its bung and head diameters 31 and 26 inches; what are its contents in gallons? *Ans.* 150.6 *gallons.*

* Whenever the contents are required in bushels, divide by 2150.4.

To ascertain the Contents of a Cask of the 2d Variety, Fig. 7.

Fig. 7.



By Mensuration.

RULE.—To the square of a head diameter add double the square of the bung diameter, and from the sum subtract $\frac{4}{10}$ of the square of the difference of the diameters; then multiply the remainder by the length, and the product again by .2618, which, being divided by 231, will give the contents in gallons.

EXAMPLE.—The bung and head diameters of a cask, *a b* and *e f*, are 24 and 18 inches, and the length, *c d*, 36; what are its contents in gallons?

$18^2+\overline{24^2\times 2}=1476$, and $1476-\frac{4}{10}$ of $\overline{24-18}^{\,2}=1461.6$, which $\times 36 = 52617.6$, and $52617.6\times .2618=13775.288$, which $\div 231=59.63$, *the gallons required.*

By the Gauging Slide-rule.

Operation.—Subtract the head diameter from the bung diameter, and look for the difference on the upper line (inches) of the scale of 2d varieties, and below it, on the second line, is the mean difference, which is to be added to the head diameter for the mean diameter.

Thus, $24-18=6=$ *difference of diameters.*

Below 6, on 2d line of scale of 2d varieties, read 3.8, which, added to 18, =21.8, *the mean diameter required.*

Then, set the left end of the slide on the inner face of the rule to the length of the cask (36) on the first line; look for the mean diameter (21.8) on the second line (or top line of slide), and above it, on the first line, read 58.2, *which is the capacity of the cask in gallons.*

Ex. 2. The bung and head diameters of a cask are 36 and 27 inches, and the length 54; what are its contents in gallons?

The difference of diameters is 9, which, by the scale, $=5.77$.

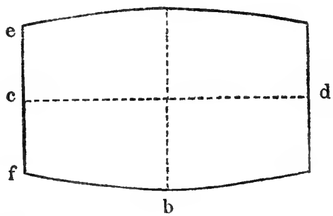
Ans. 197.5 *gallons.*

Ex. 3. The length of a cask is 51 inches, its bung and head diameters 30 and 26 inches; what are its contents in gallons?

Ans. 141.3 *gallons.*

To ascertain the Contents of a Cask of the 3d Variety, Fig. 8.

Fig. 8.



By Mensuration.

Rule.—To the square of the bung diameter add the square of the head diameter; multiply the sum by the length, and the product again by .3927, which, being divided by 231, will give the contents in gallons.

Example.—The bung and head diameters of a cask, *a b* and *e f*, are 24 and 20 inches, and the length, *c d*, 36; what are its contents in gallons?

$24^2+20^2\times 36=35136$, which $\times .3927=13797.907$, and $13797.907\div 231=59.73$, *the result required.*

By the Gauging Slide-rule.

Operation.—Subtract the head diameter from the bung diameter, and multiply the difference by the unit .56, page 310, which is to be added to the head diameter for the mean diameter.

Thus, $20-24=4$, which $\times .56=2.24$, and $20+2.24=22.24$, *the mean diameter required.*

Then, set the left end of the slide on the inner face of the rule to the length of the cask (36) on the first line; look for the mean diameter (22.24) on the second line (or top line of slide), and above it, on the first line, read 60.16, *which is the capacity of the cask in gallons.*

Ex. 2. The bung and head diameters of a cask are 36 and 30 inches, and the length 54; what are its contents in gallons?

The difference of diameters is 6, which, by the rule, $=3.36$.

Ans. 204.8 *gallons.*

Ex. 3. The length of a cask is 51 inches, its bung and head diameters 31 and 26 inches; what are its contents in gallons?

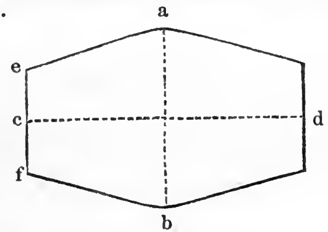
Ans. 144.9 *gallons.*

Ex. 4. The length of a cask is 50 inches, and its bung and head diameters 30 and 25 inches; what are its contents?

Ans. 131.4 *gallons.*

To ascertain the Contents of a Cask of the 4th Variety, Fig. 9.

Fig. 9.



By Mensuration.

RULE.—Add the square of the difference of the diameters to three times the square of their sum; then multiply the sum by the length, and the product again by .06566, and it will give the contents in cubic inches, which, being divided by 231, will give the result in gallons.

EXAMPLE.—The bung and head diameters of a cask, *a b* and *e f*, are 24 and 16 inches, and the length, *c d*, 36; what are its contents in gallons?

$\overline{24-16}^2+\overline{(24+16)^2\times 3}=4864$, which $\times\ 36=175104$, and $175104\times .06566=11497.329$, which$\div 231=49.77$, *the gallons required.*

By the Gauging Slide-rule.

Operation.—Subtract the head diameter from the bung diameter, and multiply the difference by the unit .52, page 310, which is to be added to the head diameter for the mean diameter.

Thus, $24-16=8$, which $\times .52=4.16$, and $16+4.16=20.16$, *the mean diameter required.*

Then, set the left end of the slide on the inner face of the rule to the length of the cask (36) on the first line: look for the mean diameter (20.16) on the second line (or top line of slide), and above it, on the first line, read 49.8, *which are the contents of the cask in gallons.*

Ex. 2. The bung and head diameters of a cask are 36 and 24 inches, and the length 54; what are its contents in gallons?

The difference of diameters is 12, which, by the rule, $=6.24$.

Ans. 168.5 *gallons.*

Ex. 3. The length of a cask is 51 inches, its bung and head diameters 31 and 23 inches; what are its contents in gallons and in bushels? *Ans.* 128.2 *gallons,*

and $\frac{128.2\times 231}{2150.42}=13.77$ *bushels.*

To ascertain the Contents of a Cask when the Dimensions are less than the Divisions on the Scale as numbered, viz., for a Length of less than 25 *inches, and a Mean Diameter of less than* 17.3 *inches.*

Operation.—Determine the mean diameter; then double both the length and the mean diameter; ascertain the contents for those dimensions, and divide the result by 8.

NOTE.—When only one of the dimensions is doubled, divide the result by 4.

EXAMPLE.—The dimensions of a barrel of the 2d variety, having bung and head diameters of 12 and 10 inches, is 16 inches in length; what are its contents?

Mean diameter$=11.26$ inches.

Set the left end of the slide to 32 (16×2) on the 1st line, and over 22.52 (11.26×2), on the second line, is 55.36 on the 1st line, which $\div 8=6.92$, *the contents of the barrel in gallons.*

ULLAGE CASKS.

To ascertain the Contents of Ullage Casks.

When a cask is only partly filled, it is termed an *ullage cask*, and is considered in two positions, viz., as lying on its side, when it is termed a *Segment Lying* (S.L.), or as standing on its end, when it is termed a *Segment Standing* (S.S.).

To Ullage a Lying Cask.

By Mensuration.

Divide the wet inches by the bung diameter; find the quotient in the column of versed sines in the table of circular segments, page 134, and take out its corresponding segment; then multiply this segment by the capacity of the cask in gallons, and the product again by 1.25 for the ullage required.

EXAMPLE.—The capacity of a cask is 90 gallons, the bung diameter being 32 inches; what are its contents, at 8 inches depth?

$8 \div 32 = .25$, the tab. seg. of which is .15355, which $\times 90 = 13.8195$, which $\times 1.25 = 17.2744$, *the contents, in gallons.*

By the Gauging Slide-rule.

Operation.—On the plane face,

Set the bung diameter on 3d line to 100 at the right hand on 4th line, and on this line, under the wet inches on 3d line, take off the number, and set it (by moving the slide) on 3d line to 100 on 4th line, and under the capacity of the cask on the 1st line read the contents in gallons required.

Thus, set 32 on 3d line to 100 on 4th line, and under 8 on 3d line take off 17.75 on 4th line, and set it on 3d line at 100 on 4th line; then under 90 on 1st line read 16.85, *the contents in gallons.*

Ex. 2. The capacity of a cask being 92 gallons, and the bung diameter 32, required the ullage of the segment when the wet inches are 8. *Ans.* 17.85 *gallons.*

Ex. 3. The wet inches in a lying cask are 12 inches, the bung diameter 24, and the capacity of the cask 70 gallons; what are the contents of the cask? *Ans.* 35.2 *gallons.*

To Ullage a Standing Cask.

By Mensuration.

Add together the square of the diameter at the surface of the liquor, the square of the head diameter, and the square of double the diameter taken in the middle between the two; then multiply the sum by the wet inches (length between the surface and nearest end), the product again by .1309, and divide by 231 for the result in gallons.

EXAMPLE.—The diameter at the surface of the liquor is 29 inches, the head diameter of the cask is 24, the diameter taken in the middle of the two is 27, and the depth of the liquor, or wet inches, is 20; what are the contents of the cask?

$29^2+24^2+\overline{27\times 2}^2=4333$, which $\times$ 10=43330, and $43330\times .1309\div 231=24.554$, *the result, in gallons.*

By the Gauging Slide-rule.

Operation.—On the plane face,

Set the length on 3d line to 100 at the right hand on 5th line, and on this line, under the wet inches on 3d line, take off the number, and set it (by moving the slide) on 3d line to 100 on 5th line, and under the capacity of the cask on the 1st line read the contents, in gallons, required.

The capacity of the cask being 94.5 gallons, and the length 35 inches.

Thus, set 35 on 3d line to 100 on 5th line, and under 10 on 3d line take off 26.8 on 5th line, and set it at 100 on 5th line; then under 94.5 on 1st line read 25.3, *the contents, in gallons.*

Ex. 2. The length of a standing cask is 24 inches, the wet inches 12, and the capacity of it 64 gallons; what are its contents in gallons? *Ans.* 32.28 *gallons.*

Ex. 3. The length of a standing cask is 24 inches, the wet inches 16, and the capacity of it 65 gallons; what are its contents in gallons? *Ans.* 44.6 *gallons.*

To ascertain the Contents of a Cask by four Dimensions.

RULE.—Add together the squares of the bung and head diameters, and the square of double the diameter taken in the middle between the bung and head; then multiply the sum by the length of the cask, and the product by .1309; then divide this product by 231 for the result in gallons.

EXAMPLE.—What are the contents of a cask, the length of which is 40 inches, the bung diameter 32, the head diameter 24, and the middle diameter between the bung and the head 28.75 inches?

$32^2+24^2=1600=$ *sum of squares of bung and head diameters.*

$\overline{28.75\times2}^2=3306.25$, and $3306.25+1600=4906.25=$ *sum of squares of bung and head diameters and of double the middle diameter.*

$4906.25\times40=196250=$ *product of above sum and the length of the cask.*

Then, $196250\times.1309=25689.125=$ *number of cubic inches in the cask,* which $\div231=111.2083$ *gallons.*

Ex. 2. The bung and head diameters of a cask are 24 and 16 inches, the middle diameter 20.5, and the length of it 36 inches; what are its contents in gallons?

Ans. 51.26 *gallons.*

To ascertain the Contents of any Cask from three Dimensions only.

RULE.—Add into one sum 39 times the square of the bung diameter, 25 times the square of the head diameter, and 26 times the product of the two diameters; then multiply the sum by the length, and the product by .008726; then divide the quotient by 231 for the result in gallons.

EXAMPLE.—The length of a cask is 35.5 inches, its bung diameter 28.3 inches, and its head diameter 24 inches; what are its contents in gallons?

$28.3^2\times39=31234.71=39$ *times the square of the bung diameter.*

$24^2\times25=14400=25$ *times the square of the head diameter.*

$28.3\times24\times26=17659.2=26$ *times the product of the two diameters.*

$31234.71+14400+17659.2=63293.91$, which $\times$ $35.5=2246933.8=$ *the sum of the above products* $\times$ *the length,* which $\times$ $.008726=19606.744=$ *number of cubic inches,* which $\div231=84.88=$ *gallons required.*

Ex. 2. What are the contents of a cask, the length being 40, and the bung and head diameters 32 and 24 inches?

Ans. 112.273 *gallons.*

Note.—This is the most exact rule of any for three dimensions, and agrees nearly with the result as determined by a diagonal rod.

Illustration.—If a diagonal rod was applied to a cask of the dimensions above given (Ex. 2), the length or distance determined by it would be 34.4.

An inspection of the rod will show 110.1 gallons to be indicated at this point.

Note.—The Divisor for English ale gallons is 282, and for Imperial gallons 277.274.

THE END.