THE

MESSENGER OF MATHEMATICS.

EDITED BY

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VOL. XLIII.

[MAY 1913—APRIL 1914].

Cambridge: BOWES & BOWES.

London: MACMILLAN & CO. LTD.

Glasgow: JAMES MACLEHOSE & SONS.

1914.

CONTENTS OF VOL. XLIII.

ERRATA.

p. 113, l. 2, *read* $\psi = \int_{-1}^{1} F(\mu) \sqrt{\left(\frac{r_1^2}{1+\mu} + \frac{r_2^2}{1-\mu} - \tfrac{1}{2}c^2\right)}\, d\mu.$

p. 133, l. 10, *read* $Y_3 = Y_1' f'(Y')$.

" l. 11, " $Y_4 = Y_2' f'(Y') + Y_1'^2 f''(Y')$, &c.

" l. 22, " $X_2 = X_1'' F'(X'')$.

" l. 23, " $X_3 = X_2'' F'(X'') + X_1''^2 F''(X'')$, &c.

MESSENGER OF MATHEMATICS.

PRODUCT-DETERMINANTS OF THE SAME FORM AS ONE OF THEIR FACTORS.

By *Thomas Muir, LL.D.*

1. FROM Lagrange's interesting observation that the quadrinomial

$$x^2 + py^2 + qz^2 + pqw^2$$

is homogenetic,—that is to say, that the product of two such expressions is a similar expression—Samuel Roberts was led to the equally interesting result

$$\begin{vmatrix} x & py & qz & pqw \\ -y & x & -qw & qz \\ -z & pw & x & -py \\ -w & -z & y & x \end{vmatrix} = (x^2 + py^2 + qz^2 + pqw^2)^2,$$

and to the corresponding identity for the case of a determinant of the eighth order. Roberts might, however, have taken further advantage of his opportunity; and on this and other accounts it seems desirable to have a re-examination of the subject. In doing so, it is best to treat the case of the fourth order in what may seem unnecessary detail, the reason being that the space requisite for the proper treatment of the quite similar case of the eighth order would be excessive.

2. Instead of Roberts' determinant, let us consider the more general form

$$\begin{vmatrix} x & bcy & caz & abw \\ -y & x & -aw & az \\ -z & bw & x & -by \\ -w & -cz & cy & x \end{vmatrix}, \text{ or } R_4 \text{ say.}$$

Performing on this the operations

$$\text{row}_2 \times bc, \quad \text{row}_3 \times ca, \quad \text{row}_4 \times ab,$$

we obtain

$$a^2b^2c^2 \times R_4 = \begin{vmatrix} x & bcy & caz & abw \\ -bcy & bcx & -abcw & abcz \\ -caz & abcw & cax & -abcy \\ -abw & -abcz & abcy & abx \end{vmatrix} \qquad \text{(I.)}$$

—a determinant skew with repect to the principal diagonal. From this, by noting that the final expansion can contain no terms involving an odd number of diagonal elements, we learn that R_4 is not altered by changing x into $-x$.

Again, by performing the operations

$$\text{row}_2 \times (-c), \quad \text{row}_3 \times (c), \quad \text{row}_4 \times (-1),$$

there results

$$c^2 . R_4 = \begin{vmatrix} x & bcy & acz & abw \\ cy & -cx & acw & -acz \\ -cz & bcw & cx & -bcy \\ w & cz & -cy & -x \end{vmatrix} \qquad \text{(II.)}$$

—a determinant skew with respect to the secondary diagonal, and thus independent of the sign of w.

Lastly, by interchanging the 2nd row with the 1st, and the 4th with the 3rd, we obtain

$$R_4 = \begin{vmatrix} -y & x & -aw & az \\ x & bcy & caz & abw \\ -w & -cz & cy & x \\ -z & bw & x & -by \end{vmatrix},$$

where the y's are confined to the one diagonal and the z's to the other, and where, by suitable multiplications, we can show that a^2R_4 is skew with respect to the primary diagonal, and b^2R_4 with respect to the secondary diagonal, and thus deduce that R_4 is unaffected by changing the sign of y or the sign of z. (III.)

3. Considerable variety is possible in the expressing of R_4 by means of a skew determinant. Thus, instead of (I.), we might substitute

$$R_4 = \begin{vmatrix} x & cy & az & bw \\ -cy & \frac{c}{b}x & -aw & cz \\ -az & aw & \frac{a}{c}x & -by \\ -bw & -cz & by & \frac{b}{a}x \end{vmatrix} \qquad (\mathrm{I}'.)$$

and instead of (II.)

$$R_4 = \begin{vmatrix} x & by & acz & abw \\ cy & -x & acw & -acz \\ -z & \frac{b}{c}w & x & -by \\ w & z & -cy & -x \end{vmatrix} \qquad (\mathrm{II}'.)$$

4. Now let us form the determinant which is the same function of ξ, η, ζ, ω as R_4 is of x, y, z, w, and distinguish the two as

$$R_4(x, y, z, w), \quad R_4(\xi, \eta, \zeta, \omega).$$

We shall then have, from § 2,

$$R(\xi, \eta, \zeta, \omega), = R_4(\xi, -\eta, -\zeta, -\omega).$$

and hence, by the interchange of rows and columns,

$$R(\xi, \eta, \zeta, \omega) = R_4'(\xi, -\eta, -\zeta, -\omega), \text{ say.}$$

It thus follows that

$$R_4(x, y, z, w) \times R_4(\xi, \eta, \zeta, \omega)$$

$$= R_4(x, y, z, w) \times R'(\xi, -\eta, -\zeta, -\omega)$$

$$= \begin{vmatrix} x & bcy & caz & abw \\ -y & x & -aw & az \\ -z & bw & x & -by \\ -w & -cz & cy & x \end{vmatrix} \cdot \begin{vmatrix} \xi & \eta & \zeta & \omega \\ -bc\eta & \xi & -b\omega & c\zeta \\ -ca\zeta & a\omega & \xi & -c\eta \\ -ab\omega & -a\zeta & b\eta & \xi \end{vmatrix};$$

and, the multiplication being performed in row-by-row fashion, we find the product to be

$$\begin{vmatrix} X & bcY & caZ & abW \\ -Y & X & -aW & aZ \\ -Z & bW & X & -bY \\ -W & -cZ & cY & X \end{vmatrix},$$

where

$$\begin{aligned} X &\equiv x\xi + bcy\eta + caz\zeta + abw\omega, \\ Y &\equiv -x\eta + y\xi - az\omega + aw\zeta, \\ Z &\equiv -x\zeta + by\omega + z\xi - bw\eta, \\ W &\equiv -x\omega - cy\zeta + cz\eta + w\xi; \end{aligned} \qquad \text{(IV.)}$$

in other words, the form of the product is exactly the same as that of the first factor.

5. It is seen that X, Y, Z, W may be easily remembered in the form

$$\begin{aligned} &(\text{1st row of } R_4 \between \xi, \eta, \zeta, \omega), \\ &-(\text{2nd row of } R_4 \between \quad ,, \quad), \\ &-(\text{3rd row of } R_4 \between \quad ,, \quad), \\ &-(\text{4th row of } R_4 \between \quad ,, \quad). \end{aligned} \qquad \text{(V.)}$$

Also that Y, Z, W, being equal to

$$\begin{aligned} &-(x\eta - \xi y) \; - a\,(z\omega - \zeta w), \\ &-(x\zeta - \xi z) \; - b\,(w\eta - \omega y), \\ &-(x\omega - \xi w) - c\,(y\zeta - \eta z), \end{aligned}$$

respectively, must vanish when

$$\xi, \eta, \zeta, \omega = x, y, z, w.$$

Making this substitution, we have, from § 4,

$$\{R_4(x, y, z, w)\}^2 = (x^2 + bcy^2 + caz^2 + abw^2)^4,$$

and therefore

$$\begin{vmatrix} x & bcy & caz & abw \\ -y & x & -aw & az \\ -z & bw & x & -by \\ -w & -cz & cy & x \end{vmatrix} = (x^2 + bcy^2 + caz^2 + abw^2)^2. \qquad \text{(VI.)}$$

6. Using (VI.) along with (IV.), we obtain

$$(x^2+bcy^2+caz^2+abw^2)\,(\xi^2+bc\eta^2+ca\zeta^2+ab\omega^2)$$
$$=X^2+bcY^2+caZ^2+abW^2, \qquad \text{(VII.)}$$

which degenerates into Lagrange's identity, mentioned in § 1, on putting a or b or c equal to 1, and is seen to be included in the same by putting $w=cw'$ and $\omega=c\omega'$.

7. From the result of the substitution made in § 5 it follows that *the elements of the adjugate of* $R_4(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w})$ *are proportional to the elements of* $R_4'(\mathrm{x},-\mathrm{y},-\mathrm{z},-\mathrm{w})$ (VIII.)
This, however, is best viewed as a special case of the theorem that *If**

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \times_{rr} \begin{vmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ r_1 & r_2 & r_3 & r_4 \\ s_1 & s_2 & s_3 & s_4 \end{vmatrix} \equiv \begin{vmatrix} H & \cdot & \cdot & \cdot \\ \cdot & H & \cdot & \cdot \\ \cdot & \cdot & H & \cdot \\ \cdot & \cdot & \cdot & H \end{vmatrix},$$

then the ratio of any element of the adjugate of $|\mathrm{a}_1\mathrm{b}_2\mathrm{c}_3\mathrm{d}_4|$ *to the corresponding element of* $|\mathrm{p}_1\mathrm{q}_2\mathrm{r}_3\mathrm{s}_4|$ *is constant, namely, is equal to* $|\mathrm{a}_1\mathrm{b}_2\mathrm{c}_3\mathrm{d}_4| \div \mathrm{H}$. (IX.)
For proof, let us take B_3 as an example of an element of the adjugate in question. Then

$$HB_3 = \begin{vmatrix} a_1 & a_2 & \cdot & a_4 \\ b_1 & b_2 & H & b_4 \\ c_1 & c_2 & \cdot & c_4 \\ d_1 & d_2 & \cdot & d_4 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & q_1a_1+q_2a_2+q_3a_3+q_4a_4 & a_4 \\ b_1 & b_2 & q_1b_1+q_2b_2+q_3b_3+q_4b_4 & b_4 \\ c_1 & c_2 & q_1c_1+q_2c_2+q_3c_3+c_4c_4 & c_4 \\ d_1 & d_2 & q_1d_1+q_2d_2+q_3d_3+q_4\partial_4 & d_4 \end{vmatrix}$$

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* This hypothesis involves also the identity

$$|p_1q_2r_3s_4| \cdot |a_1b_2c_3d_4| = \begin{vmatrix} H & \cdot & \cdot & \cdot \\ \cdot & H & \cdot & \cdot \\ \cdot & \cdot & H & \cdot \\ \cdot & \cdot & \cdot & H \end{vmatrix},$$

so that the result might have been stated in a dual form.

$$= \begin{vmatrix} a_1 & a_2 & q_3a_3 & a_4 \\ b_1 & b_2 & q_3b_3 & b_4 \\ c_1 & c_2 & q_3c_3 & c_4 \\ d_1 & d_2 & q_3d_3 & d_4 \end{vmatrix} = q_3 |a_1b_2c_3d_4|,$$

and therefore $\qquad B_3 \div q_3 = |a_1b_2c_3d_4| \div H,$

as was to be proved.

If, in addition, it be given that

$$|p_1q_2r_3s_4| = |a_1b_2c_3d_4|$$

(which, of course, need not imply that any element of the one is equal to the corresponding element of the other), we obtain, from the first datum,

$$|a_1b_2c_3d_4| = H^{\frac{4}{2}},$$

and thence $\qquad B_3 = b_3 \,.\, H^{\frac{4}{2}-1}. \qquad$ (X.)

8. Conversely to (IX.), *if* $|\mathrm{p}_1\mathrm{q}_2\mathrm{r}_3\mathrm{s}_4|$ *have its elements proportional to the elements of the adjugate of* $|\mathrm{a}_1\mathrm{b}_2\mathrm{c}_3\mathrm{d}_4|$, *the common ratio being* ρ, *then*

$$|\mathrm{p}_1\mathrm{q}_2\mathrm{r}_3\mathrm{s}_4| = |\mathrm{A}_1\mathrm{B}_2\mathrm{C}_3\mathrm{D}_4| \cdot \rho^4,$$

$$= |\mathrm{a}_1\mathrm{b}_2\mathrm{c}_3\mathrm{d}_4|^{4-1} \cdot \rho^4;$$

and, if addition, it be given that $|\mathrm{p}_1\mathrm{q}_2\mathrm{r}_3\mathrm{s}_4| = |\mathrm{a}_1\mathrm{b}_2\mathrm{c}_3\mathrm{d}_4|$ *we shall have*

$$\rho = |\mathrm{a}_1\mathrm{b}_2\mathrm{c}_3\mathrm{d}_4|^{\frac{1}{4}(2-4)}. \qquad \text{(XI.)}$$

9. The determinant of the eighth order, R_8 say, is

$$\begin{vmatrix} x_1 & bcx_2 & acx_3 & abx_4 & dx_5 & bcdx_6 & acdx_7 & abdx_8 \\ -x_2 & x_1 & -ax_4 & ax_3 & -dx_6 & dx_5 & -adx_8 & adx_7 \\ -x_3 & bx_4 & x_1 & -bx_2 & -dx_7 & bdx_8 & dx_5 & -bdx_6 \\ -x_4 & -cx_3 & cx_2 & x_1 & dx_8 & cdx_7 & -cdx_6 & -dx_5 \\ -x_5 & bcx_6 & acx_7 & -abx_8 & x_1 & -bcx_2 & -acx_3 & abx_4 \\ -x_6 & -x_5 & -ax_8 & -ax_7 & x_2 & x_1 & ax_4 & ax_3 \\ -x_7 & bx_8 & -x_5 & bx_6 & x_3 & -bx_4 & x_1 & -bx_2 \\ -x_8 & -cx_7 & cx_6 & x_5 & -x_4 & -cx_3 & cx_2 & x_1 \end{vmatrix}$$

where the first four-line minor is $R_4(x_1, x_2, x_3, x_4)$, the last four-line minor is $R_4(x_1, -x_3, -x_3, -x_4)$ with its last row and last column changed in sign; the minor occupying the bottom left-hand quarter is $R_4(-x_5, x_6, x_7, x_8)$ with its last column changed in sign; and the minor occupying the remaining quarter is $R_4(x_5, x_6, x_7, x_8)$ with a d annexed as a multiplier to each element and the signs of the last row changed.

The properties of R_8 are exactly similar to those of R_4. (a) It can be expressed as a skew determinant with any one of the eight x's confined to the diagonal. (b) It is not altered in substance by changing the sign of one or more of the x's. (c) If $R_8(x_1, x_2, \ldots, x_8)$ be multiplied by the conjugate of $R_8(\xi_1, -\xi_2, -\xi_3, \ldots, -\xi_8)$, the product-determinant is of the same form as R_8, the new variables $X_1, X_2, \ldots, X_8$ being

$$\begin{array}{l} (\text{1st row of } R_8 \between \xi_1, \xi_2, \ldots, \xi_8), \\ -(\text{2nd row of } R_8 \between \quad ,, \quad), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -(\text{8th row of } R_8 \between \quad ,, \quad). \end{array}$$

(d) The value of R_8 is

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2 + dx_5^2 + bcdx_6^2 + acdx_7^2 + abdx_8^2)^4.$$

(e) The octonomial which is the fourth root of R_8 is homogenetic, the fact in detail being

$$\begin{aligned}
&(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2 + dx_5^2 + bcdx_6^2 + acdx_7^2 + abdx_8^2) \\
&\cdot(\xi_1^2 + bc\xi_2^2 + ca\xi_3^2 + ab\xi_4^2 + d\xi_5^2 + bcd\xi_6^2 + acd\xi_7^2 + abd\xi_8^2) \\
= {} &(x_1\xi_1 + bcx_2\xi_2 + cax_3\xi_3 + abx_4\xi_4 + dx_5\xi_5 + bcdx_6\xi_6 + acdx_7\xi_7 + abdx_8\xi_8)^2 \\
&+ bc\,(x_2\xi_1 - x_1\xi_2 + ax_4\xi_3 - ax_3\xi_4 + dx_6\xi_5 - dx_5\xi_6 + adx_8\xi_7 - adx_7\xi_8)^2 \\
&+ ca\,(x_3\xi_1 - bx_4\xi_2 - x_1\xi_3 + bx_2\xi_4 + dx_7\xi_5 - bdx_8\xi_6 - dx_5\xi_7 + bdx_6\xi_6)^2 \\
&+ ab\,(x_4\xi_1 + cx_3\xi_2 - cx_2\xi_3 - x_1\xi_4 - dx_8\xi_5 - cdx_7\xi_6 + cdx_6\xi_7 + dx_5\xi_8)^2 \\
&+ d\,(x_5\xi_1 - bcx_6\xi_2 - acx_7\xi_3 + abx_8\xi_4 - x_1\xi_5 + bcx_2\xi_6 + acx_3\xi_7 - abx_4\xi_8)^2 \\
&+ bcd\,(x_6\xi_1 + x_5\xi_2 + ax_8\xi_3 + ax_7\xi_4 - x_2\xi_5 - x_1\xi_6 - ax_4\xi_7 - ax_3\xi_8)^2 \\
&+ acd\,(x_7\xi_1 - bx_8\xi_2 + x_5\xi_3 - bx_6\xi_4 - x_3\xi_5 + bx_4\xi_6 - x_1\xi_7 + bx_2\xi_8)^2 \\
&+ abd\,(x_8\xi_1 + cx_7\xi_2 - cx_6\xi_3 - x_5\xi_4 + x_4\xi_5 + cx_3\xi_6 - cx_2\xi_7 - x_1\xi_8)^2.
\end{aligned}$$

(f) The elements of the adjugate of $R_8(x_1, \ldots, x_8)$ are proportional to the conjugate elements of

$$R_8(x_1, -x_2, -x_3, \ldots, -x_8).$$

10. When $a=b=c=1$, the four-line R is a skew orthogonant; and so also is the eight-line R when $a=b=c=d=1$. Further, the sums of the rows of the former are the factors of the determinant

$$\begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix},$$

and the sums of the rows of the latter are the factors of the like determinant (so-called Puchta's)

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ x_2 & x_1 & x_4 & x_3 & x_6 & x_5 & x_8 & x_7 \\ x_3 & x_4 & x_1 & x_2 & x_7 & x_8 & x_5 & x_6 \\ x_4 & x_3 & x_2 & x_1 & x_8 & x_7 & x_6 & x_5 \\ x_5 & x_6 & x_7 & x_8 & x_1 & x_2 & x_3 & x_4 \\ x_6 & x_5 & x_8 & x_7 & x_2 & x_1 & x_4 & x_3 \\ x_7 & x_8 & x_5 & x_6 & x_3 & x_4 & x_1 & x_2 \\ x_8 & x_7 & x_6 & x_5 & x_4 & x_3 & x_2 & x_1 \end{vmatrix}.$$

11. On the historical side of the subject it may be well to recall that Lagrange drew attention to the identity

$$(x^2-ay^2)(\xi^2-a\eta^2)=(x\xi+ay\eta)^2-a(x\eta+y\xi)^2$$

in the *Miscellanea Taurinensia*, vol. iv (1666–1769);* that Euler, in 1770, in the St Petersburg *Novi Commentarii*, vol. xv, p. 75, expressed the product of $x^2+y^2+z^2+w^2$ and $\xi^2+\eta^2+\zeta^2+\omega^2$ as the sum of four squares; that Lagrange, in the Berlin *Nouveaux Mémoires*,† of the same year, extended Euler's result by showing that

$$\begin{aligned}(x^2-ay^2-bz^2+abw^2)(\xi^2-a\eta^2-b\zeta^2+a\omega b^2)\\ =(x\xi+ay\eta\pm bz\zeta\pm abw\omega)^2\\ -a(x\eta+y\xi\pm bz\omega\pm w\zeta)^2\\ -b(x\zeta-ay\omega\pm z\xi\mp aw\eta)^2\\ +ab(\eta\zeta-x\omega\pm w\xi\mp z\eta)^2;\end{aligned}$$

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* *Œuvres*, vol. i., pp. 671–731.

† *Œuvres*, vol. iii., pp. 189–201.

that, in 1860, Souillart, probably from seeing a suggestive re-statement of Euler's old paper in the *Nouvelles Annales*, vol. xv., pp. 403–407, published, in the same serial (vol. xix., pp. 320–322), the related determinant;* and that, as already stated, Roberts, in 1879 (*Mess. of Math.*, vol. viii., pp. 138–140), did the same for the more general result of Lagrange.

The existence of a theorem like Euler's for the sum of eight squares was first established by J. T. Graves, the date apparently being 1843; and the non-existence of a corresponding theorem for the sum of sixteen squares was more or less satisfactorily proved by J. R. Young, in 1847; but on this special branch of the subject a paper by S. Roberts, prefaced by a historical sketch, in the *Quart. Jour. of Math.*, vol. xvi., pp. 159–170, will be found fully informative.

Capetown, S.A.
9th March, 1913.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XXXVI.

On the asymptotic values of certain integrals.

1. In this note I propose to apply the ideas and methods of Paul du Bois-Reymond's *Infinitärcalcül*, which I have discussed at length elsewhere,† to the determination of the asymptotic values of certain integrals of the types

$$\int^{x} \phi(t)\, e^{i\psi(t)}\, dt, \qquad \int_{x}^{\infty} \phi(t)\, e^{i\psi(t)}\, dt,$$

where ϕ and ψ are logarithmico-exponential functions (l-functions). I shall confine myself to the case in which the integral up to infinity is divergent or oscillatory, so that we

* See also Brioschi's attempt, in 1855, in *Crelle's Journ.*, vol. lii., pp. 133–141; and my commentary on it in the *Mess. of Math.*, vol. xxxvii., pp. 107–111.

† "Orders of infinity," *Camb. Math. Tracts*, No. 12; "Properties of logarithmico-exponential functions," *Proc. Lond. Math. Soc.*, vol. x., p. 54; "Oscillating Dirichlet's Integrals," *Quarterly Journal*, vol. xliv., p. 1.

must take x as the upper limit. The reader will find no difficulty in obtaining the corresponding results in the other case.

2. When $\psi \equiv 0$, this problem is solved completely in my paper in the *Proc. Lond. Math. Soc.* already referred to.* If $\psi \prec 1$ or $\psi \sim A$, $e^{i\psi}$ tends to a limit, and the results are the same as when $\psi \equiv 0$. If $\phi \prec \psi'$, the integral up to infinity is convergent; and if $\phi \sim A\psi'$ it is easy to see that

$$\int^x \phi e^{i\psi}\, dt \sim Ae^{i\psi}.$$

We may therefore suppose $\psi \succ 1$, $\phi \succ \psi'$.

The integral $$\int^\infty \phi\, dt$$

is certainly divergent. We write

$$\int^x \phi\, dt = \Phi.$$

Then we can determine (in virtue of the results of my former paper) an L-function ϕ_1 such that $\Phi \sim \phi_1$, and $l\Phi \sim l\phi_1$. We must now distinguish three cases, according as

$$(a)\ \psi \prec l\Phi, \qquad (b)\ \psi \sim Al\Phi, \qquad (c)\ \psi \succ l\Phi.$$

3. In case (a), we have

$$\int^x \phi e^{i\psi}\, dt = C + \Phi e^{i\psi} - i\int^x \Phi\psi' e^{i\psi}\, dt.$$

Now $$\int^x \Phi\psi' e^{i\psi}\, dt = O\int^x \Phi\psi'\, dt.$$

But, since $\psi \prec l\Phi \sim l\phi_1$, we have $\psi' \prec \phi_1'/\phi_1$, and so

$$\int^x \Phi\psi'\, dt \sim \int^x \phi_1\psi'\, dt = o\int^x \phi_1'\, dt = o(\phi_1) = o(\Phi).$$

Thus

(1) $$\int^x \phi e^{i\psi}\, dt \sim \Phi e^{i\psi}.$$

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* *l.c.*, pp. 73–75.

4. In Case (b), we have

$$\psi \sim Al\Phi, \quad \psi = Al\Phi + \psi_1 \qquad (\psi_1 \prec \psi),$$

$$\int^x \phi e^{i\psi} dt = \int^x \Phi^{Ai}\Phi' e^{i\psi_1} dt$$

$$= \text{const.} + \frac{\Phi^{1+Ai}}{1+Ai} . e^{i\psi_1} - \frac{i}{1+Ai} . \int^x \Phi^{1+Ai}\psi_1' e^{i\psi_1} dt.$$

The last integral is

$$\int^x \Phi\psi_1' e^{i\psi} dt = O\int^x \Phi\psi_1' dt = O\int^x \phi_1\psi_1' dt$$

$$= o\int^x \phi_1' dt = o(\phi_1) = o(\Phi).$$

Hence

$$(2) \qquad \int^x \phi e^{i\psi} dt \sim \frac{\Phi^{1+Ai}}{1+Ai} . e^{i\psi_1} = \frac{\Phi e^{i\psi}}{1+Ai}.$$

5. Case (c) requires a slightly more complicated treatment. In this case, we write

$$\int^x \phi e^{i\psi} dt = C + \frac{\phi}{i\psi'} e^{i\psi} - \frac{1}{i}\int^x e^{i\psi} \frac{d}{dt}\left(\frac{\phi}{\psi'}\right) dt.$$

We shall prove that

$$\int^x e^{i\psi} \frac{d}{dt}\left(\frac{\phi}{\psi'}\right) dt = o\left(\frac{\phi}{\psi'}\right),$$

and it will then follow that

$$(3) \qquad \int^x \phi e^{i\psi} dt \sim \frac{\phi}{i\psi'} e^{i\psi}.$$

In the first place, it is clear that

$$\int^x e^{i\psi} \frac{d}{dt}\left(\frac{\phi}{\psi'}\right) dt = O\int^x \frac{d}{dt}\left(\frac{\phi}{\psi'}\right) dt = O\left(\frac{\phi}{\psi'}\right),$$

and so $$\int^x \phi e^{i\psi} dt = O\left(\frac{\phi}{\psi'}\right);$$

and the same result, of course, is true of the real and imaginary parts of the integral.* Again,

$$\psi \succ l\Phi \sim l\phi_1, \quad \psi' \succ \phi_1'/\phi_1,$$

and so, as $\phi_1' \sim \phi$, we have $\phi/\psi' \prec \phi_1$, and therefore

$$\frac{d}{dt}\left(\frac{\phi}{\psi'}\right) \prec \phi.$$

Hence we may write

$$\frac{d}{dt}\left(\frac{\phi}{\psi'}\right) = \phi\eta,$$

where $\eta \prec 1$. As η is an L-function, it is ultimately monotonic, say, for $t > \xi$. Then

$$\int_a^x \frac{d}{dt}\left(\frac{\phi}{\psi'}\right) \cos\psi \, dt = \left(\int_a^\xi + \int_\xi^x\right) \phi\eta \cos\psi \, dt$$

$$= O(1) + \eta(\xi) \int_\xi^{\xi'} \phi \cos\psi \, dt.$$

And $$\int_\xi^{\xi'} \phi \cos\psi \, dt = O\frac{\phi(\xi')}{\psi'(\xi')} + O\frac{\phi(\xi)}{\psi'(\xi)} = O\frac{\phi(x)}{\psi'(x)}.$$

Hence $$\int_a^x \frac{d}{dt}\left(\frac{\phi}{\psi'}\right) \cos\psi \, dt = o\left(\frac{\phi}{\psi'}\right).$$

The corresponding integral containing a sine may be discussed in a precisely similar way; and so the proof of (3) is completed.

6. We have thus found the complete solution of the problem for integrals;† it may be stated as follows.

Determine, by the rules given in "Properties of logarithmico-exponential functions," an L-function ϕ_1 *such that*

$$\phi_1 \sim \Phi = \int^x \phi \, dt.$$

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* For all this argument compare "Oscillating Dirichlet's Integrals," pp. 14–19.

† Assuming that the integral up to infinity is not convergent. When it is, analogous results hold for the integral

$$\int_x^\infty \phi e^{i\psi} \, dt.$$

As I stated in § 1, I leave the formulation of these results to the reader.

Then the integral $\int^{x} \phi e^{i\psi}\, dt$ *is asymptotically equivalent to one or other of the functions*

$$\phi_1 e^{i\psi}, \quad \frac{\phi_1 e^{i\psi}}{1+Ai}, \quad \frac{\phi}{i\psi'} e^{i\psi}$$

according as

$$\psi \prec l\phi_1, \quad \psi \sim Al\phi_1, \quad \psi \succ l\phi_1.$$

7. We are naturally led to consider the corresponding problem for the series of the type

$$\sum^{n} \phi(\nu)\, e^{i\psi(\nu)}.$$

It would be futile to expect a *complete* solution here. It is clear, for example, that the behaviour of such a series as

$$\sum^{n} e^{\alpha i \nu^2} \qquad (\alpha > 0)$$

is not determined by any such simple rules as the foregoing; it depends, in fact, in an exceedingly intricate way, on the arithmetic nature of α*.

The problem may, however, be solved in a number of interesting cases in which it is possible to establish asymptotic relations between the series and the corresponding integral. A number of results in this direction have been proved by Dr. Bromwich and myself,† and for the moment I confine myself to referring to them. I propose, in another note, to reconsider the question with the aid of the methods of the *Infinitärcalcül.*

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* See Hardy and Littlewood. "Some problems of Diophantine Approximation," *Transactions of the Fifth International Congress of Mathematicians,* Cambridge, 1912.

† Bromwich, *Proc. Lond. Math. Soc.*, vol. vi., p. 327; Hardy, *ibid.*, vol. ix., p. 126.

ON FINITE ABELIAN GROUPS OF SUBSTITUTIONS, ESPECIALLY OF ORTHOGONAL SUBSTITUTIONS.

By *H. Bryon Heywood.*

The following work was done at the suggestion of Prof. Harold Hilton, who is responsible for § 2. Its main object is the classification of finite Abelian groups of orthogonal substitutions, which will be found in § 4. This is preceded in § 1 by a summary of some general results on finite Abelian groups which are necessary for the later articles, in § 2 by the simultaneous reduction of such groups of substitutions to a special canonic form, and in § 3 by some results upon these groups depending upon § 1 and § 2.

§ 1. *Remarks on finite Abelian groups.*

Consider any finite Abelian group, G, and let a base* of G be $A, B, C, \ldots K$, where these letters represent permutable operations of order $a, b, c, \ldots, k$ respectively. Then there exists between $A, B, C, \ldots, K$ no relation of the form

$$A^{a'}B^{b'}C^{c'}\ldots K^{k'} = E \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where the integers $a', b', c', \ldots, k'$ are less than the corresponding orders, E is the identical operation, and all operations of the group can be represented by the formula

$$\Theta = A^{\alpha}B^{\beta}C^{\gamma}\ldots K^{\kappa} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\begin{cases} \alpha = 0, 1, 2, \ldots, a-1 \\ \beta = 0, 1, 2, \ldots, b-1 \\ \gamma = 0, 1, 2, \ldots, c-1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \kappa = 0, 1, 2, \ldots, k-1 \end{cases}$$

once and once only, the order of the group being

$$n = abc\ldots k.$$

For Θ we shall use the notation

$$\Theta = \left(\frac{\alpha}{a}, \frac{\beta}{b}, \frac{\gamma}{c}, \ldots, \frac{\kappa}{k}\right) \quad \ldots\ldots\ldots\ldots (3),$$

and we note that two operations follow the law of combination

$$\Theta\Theta' = \left(\frac{\alpha+\alpha'}{a}, \frac{\beta+\beta'}{b}, \frac{\gamma+\gamma'}{c}, \ldots, \frac{\kappa+\kappa'}{k}\right) \ldots. (4).$$

* We here adopt the definition given in Weber's *Algebra*, vol. ii., p. 48.

If any of these fractions are improper, the integral parts may be rejected. The base will be

$$\left.\begin{array}{l} A=(1/a, \quad 0, \quad 0, \ldots, \quad 0) \\ B=(\quad 0, 1/b, \quad 0, \ldots, \quad 0) \\ C=(\quad 0, \quad 0, 1/c, \ldots, \quad 0) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ K=(\quad 0, \quad 0, \quad 0, \ldots, 1/k) \end{array}\right\} \ldots\ldots\ldots\ldots (5).$$

It is easy to deduce other bases from the one we have chosen. Let

$$\begin{array}{l} a=p_1q_1r_1\ldots, \\ b=p_2q_2r_2\ldots, \\ c=p_3q_3r_3\ldots, \\ \ldots\ldots\ldots\ldots\ldots, \end{array}$$

where $p_1, p_2, p_3, \ldots$; $q_1, q_2, q_3, \ldots$; $r_1, r_2, r_3, \ldots$ are the prime invariants; $p_1, p_2, p_3, \ldots$ being powers of the prime p; $q_1, q_2, q_3, \ldots$ powers of the prime q, and so on.

All these prime invariants occur once and once only as factors in the denominators of the fractions belonging to the operations of any given base, no two prime invariants involving the same prime, p say, occurring in the same operation. We thus obtain a new base A', B', C', ..., L' by making a new distribution of the prime invariants among the operations of a base.

We should obtain the base with the maximum number of operations by putting one prime invariant into each operation. This is the base indicated in Weber's *Algebra*. *If p is the prime which occurs most often among the prime invariants (say ϖ times), then the minimum number of operations that a base may have is ϖ*; ways of constructing such a base will occur at once to the reader: a base of this kind occurs in the proof of the well-known fundamental theorem concerning Abelian groups.* This base is obtained by forming an operation by associating the greatest prime invariants corresponding to each of the several primes p, q, r, ..., then a second operation by associating the greatest invariants remaining in the same way, and so on.

If a finite Abelian group G can be generated by any number n of operations, then any sub-group S of G can be generated by n operations or less.†

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* H. Hilton, *An Introduction to the Theory of Groups of Finite Order*, p. 126.
† Frobenius and Stickelberger, *Crelle*, vol. lxxxvi. (1879), p. 232.

§ 2. *Reduction of a finite Abelian group of substitutions to a canonical form.**

If we have an Abelian group of orthogonal substitutions with linear (simple) invariant-factors,† we can transform it into a group of multiplications, each of the type

$$\left.\begin{array}{llll} x_1'=\alpha x_1, & x_2'=\beta x_2, & x_3'=\gamma x_3, & \dots \\ y_1'=\alpha^{-1}y_1, & y_2'=\beta^{-1}y_2, & y_3'=\gamma^{-1}y_3, & \dots \\ X_1'=\pm X_1, & X_2'=\pm X_2, & X_3'=\pm X_3, & \dots \end{array}\right\},$$

having the invariant

$$x_1y_1+x_2y_2+x_3y_3+\dots+X_1^2+X_2^2+X_3^2+\dots.\ddagger$$

If we have any Abelian group of substitutions with linear invariant-factors, we can transform it into a group of multiplications. For we can transform one of them, S, into a canonical form (*i.e.*, into a multiplication, since the invariant-factors of S are linear) and each of the rest into a direct product of substitutions with only one distinct characteristic root.§ But a substitution with linear invariant-factors and only one distinct characteristic-root is a similarity.

Let now the Abelian group of substitutions with linear invariant-factors be also orthogonal.

Suppose, for the sake of illustration, that when the Abelian group is transformed into a group of multiplications, one of these multiplications, S, is

$$(\alpha x_1,\ \alpha x_2,\ \alpha x_3,\ \alpha x_4,\ \alpha^{-1}x_5,\ \alpha^{-1}x_6,\ \alpha^{-1}x_7,\ \alpha^{-1}x_8),$$

while another, T, is

$$(ax_1,\ ax_2,\ ax_3,\ bx_4,\ a^{-1}x_5,\ b^{-1}x_6,\ a^{-1}x_7,\ a^{-1}x_8).$$

Transform the group by

$$(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_8,\ x_7,\ x_6).$$

This does not alter S and transforms T into

$$(ax_1,\ ax_2,\ ax_3,\ bx_4,\ a^{-1}x_5,\ a^{-1}x_6,\ a^{-1}x_7,\ b^{-1}x_8),$$

while the other substitutions remain multiplications.

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* By Professor Hilton

† For instance, a real orthogonal Abelian group.

‡ It is immediately evident that any multiplication with this invariant is of the type stated.

§ *Quarterly Journal*, vol. xl., p. 171; *Messenger*, vol. xli., p. 116.

The quadratic invariant of non-zero determinant common to every substitution of the group has now become the sum of quadratic functions on $x_1, x_2, x_3, x_5, x_6, x_7$, and on x_4, x_8; since it is an invariant of both S and T.

Suppose now one of the other substitutions, U, is

$$(Ax_1\ \ Ax_2,\ Bx_3,\ Cx_4,\ B^{-1}x_5,\ A^{-1}x_6,\ A^{-1}x_7,\ C^{-1}x_8).$$

Transform the group by

$$(x_1,\ x_2,\ x_3,\ x_4,\ x_7,\ x_6,\ x_5,\ x_8).$$

This leaves S and T unaltered and transforms U into

$$(Ax_1,\ Ax_2,\ Bx_3,\ Cx_4,\ A^{-1}x_5,\ A^{-1}x_6,\ B^{-1}x_7,\ C^{-1}x_8).$$

The quadratic invariant must now become the sum of quadratic functions on x_1, x_2, x_5, x_6, on x_3, x_7, and on x_4, x_8; since it is an invariant of S, T, and U.

If now, for example, every substitution of the group is of the same form as U, we transform the variables x_1, x_2, x_5, x_6 so that the quadratic function on x_1, x_2, x_5, x_6 becomes $x_1x_5+x_2x_6$. No substitution of the group is altered thereby, for the transform of a similarity is a similarity.

Continuing in this way, the theorem at the beginning of this section is proved.

§ 3. *Theorem on finite Abelian groups of substitutions.*

From the last article it appears that any finite Abelian group of substitutions of degree ϖ can be transformed into a group of substitutions of the form

$$x_1' = e^{2\pi i(h/m)}x_1,\ \ x_2' = e^{2\pi i(k/n)}x_2,\ \ \ldots,\ \ x_\varpi' = e^{2\pi i(l/p)}x_\varpi,$$

where h/m, k/n, ..., l/p are proper fractions.

If we represented this substitution by the notation

$$\Theta = \left(\frac{h}{m},\ \frac{k}{n},\ \ldots,\ \frac{l}{p}\right),$$

it is clear that the product of two such substitutions will be represented by

$$\Theta\Theta' = \left(\frac{h}{m},\ \frac{k}{n},\ \ldots,\ \frac{l}{p}\right)\left(\frac{h'}{m},\ \frac{k'}{n},\ \ldots,\ \frac{l'}{p}\right)$$
$$= \left(\frac{h+h'}{m},\ \frac{k+k'}{n},\ \ldots,\ \frac{l+l'}{p}\right).*$$

* The denominators m, n, ... may be taken the same for the two substitutions, the fractions having been reduced to a common denominator.

In other words, we have again met with the notation of the first article, and the group under consideration is a sub-group of what we shall in future call the *main group* generated by the base

$$A=\left(\frac{1}{m},\ 0,\ 0,\ \ldots,\ 0\right),$$

$$B=\left(0,\ \frac{1}{n},\ 0,\ \ldots,\ 0\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

or it is the main group itself. The order of the main group is $mn\ldots p$.

The propositions of the first article give us at once the following results.

A finite Abelian group of substitutions of degree ϖ can always be generated by ϖ substitutions.

Frequently the group may be generated by a smaller number of substitutions. In fact, ϖ generating substitutions would not be necessary except in the case where the numbers $m, n, \ldots, p$ had a common factor, and if, of the ϖ numbers $m, n, \ldots, p$, only ϖ_1 had a common factor, then only ϖ_1 generating substitutions, or less, would be needed.

§ 4. *The classification of finite Abelian groups of orthogonal substitutions.*

By the theorem of § 2 any finite Abelian group of orthogonal substitutions can be transformed so that each of its substitutions is of the form

$$\left.\begin{aligned} x_1' &= e^{2\pi i(h/m)}x_1 \\ y_1' &= e^{-2\pi i(h/m)}y_1 \end{aligned}\right\},\quad \left.\begin{aligned} x_2' &= e^{2\pi i(k/n)}x_2 \\ y_2' &= e^{-2\pi i(k/n)}y_2 \end{aligned}\right\},\quad \ldots,$$

where h, m, k, n, ... are integers, to which may be added a certain number of equations such as

$$X_1' = \pm X_1,\ \ X_2' = \pm X_2,\ \ldots.$$

As this group is a particular case of § 3, we might use the same notation as before; however, as the first and second equations (bracketed) are paired in all the substitutions of the group, these can be made to correspond to a single fraction; the same remark applies to the third and fourth, and so on up to the equations in X_1, X_2, ... : the last cannot be paired and must each be associated with a separate fraction. A substitution will thus be denoted by the notation

$$\left(\frac{h}{m}, \frac{k}{n}, \ldots, \frac{l}{p}; \tfrac{1}{2}, 0, \ldots\right),$$

the fractions after the semicolon all being either zero or $\frac{1}{2}$ (since $-1 = e^{2\pi i \frac{1}{2}}$), and the law of combination being the same as before.

Geometrically interpreted the substitution is a transformation of "axes" in space of ϖ dimensions, ϖ being the degree of the substitution; a pair of equations such as

$$x_1' = e^{2\pi i (h/m)} x_1, \quad y_1' = e^{-2\pi i (k/m)} y_1$$

corresponds to a rotation through an angle $2\pi(h/m)$ about an "axis," while the equation $X_1' = -X_1$ corresponds to a reflexion in a "plane."*

It will be remarked that the determinant of the substitution is $+1$ when there is an even number of equations of the form $X_1' = -X_1$, and it is -1 when there is an odd number.

We shall now discuss the finite Abelian groups of substitutions of the several degrees.

Degree 1.—Only two groups occur. One contains the single substitution $x' = x$; the other contains the pair $x' = -x$ and $x' = x$.

Degree 2.—There are two types of group. The first is a cyclic group of rotations (0), $\left(\frac{1}{m}\right)$, $\left(\frac{2}{m}\right)$, $\ldots$, $\left(\frac{m-1}{m}\right)$, whose order is m. The second is the special group containing reflexions $(;\, 0, 0)$, $(;\, \frac{1}{2}, 0)$, $(;\, 0, \frac{1}{2})$, $(;\, \frac{1}{2}, \frac{1}{2})$, and its four sub-groups: two substitutions are necessary to generate it.

Degree 3.—The first type of substitution is of the form $\left(\frac{h}{m}; 0\right)$ or $\left(\frac{h}{m}; \frac{1}{2}\right)$. A main group of substitutions of this type can be displayed in the form

$$(0;\, \tfrac{1}{2}),\ \left(\frac{1}{m}; \tfrac{1}{2}\right),\ \left(\frac{2}{m}; \tfrac{1}{2}\right),\ \ldots,\ \left(\frac{m-1}{m}; \tfrac{1}{2}\right),$$

$$(0;\, 0),\ \left(\frac{1}{m}; 0\right),\ \left(\frac{2}{m}; 0\right),\ \ldots,\ \left(\frac{m-1}{m}; 0\right),$$

---

* Rotation about $x_1 = 0$, $y_1 = 0$; reflexion about $X_1 = 0$. These loci of course are only axis and plane respectively in the three-dimensional case. In four-dimensional space the first would be a plane and the second a space of three dimensions. We retain the words "axis" and "plane" (in inverted commas) for convenience.

and consists of $2m$ substitutions. The first row have a modulus -1, and the second row a modulus $+1$. The group may be generated by a pair; for example, by $\left(\frac{1}{m};\ 0\right)$ and $(0;\ \frac{1}{2})$, when m is even. When m is odd the group is cyclic and is generated, for example, by $\left(\frac{1}{m};\ \frac{1}{2}\right)$.

The second type of group of degree 3 consists of reflexions. It is limited to the main group

$$(;\ 0, 0, 0),\ (;\ \tfrac{1}{2}, 0, 0),\ (;\ 0, \tfrac{1}{2}, 0),\ (;\ 0, 0, \tfrac{1}{2}),$$

$$(;\ 0, \tfrac{1}{2}, \tfrac{1}{2}),\ (;\ \tfrac{1}{2}, 0, \tfrac{1}{2}),\ (;\ \tfrac{1}{2}, \tfrac{1}{2}, 0),\ (;\ \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}),$$

and its sub-groups. Three substitutions, for example $(;\ \frac{1}{2}, 0, 0)$, $(;\ 0, \frac{1}{2}, 0)$, $(;\ 0, 0, \frac{1}{2})$, will be needed to generate the main group.

Degree 4.—The first type consists of substitutions of the form $\left(\frac{h}{m}, \frac{k}{n}\right)$; that is to say, a pair of rotations. The main group or any of its sub-groups can be generated by a pair of substitutions when m and n contain a common factor, and by a single substitution when m and n are prime to each other.

The second type contains the form $\left(\frac{h}{m};\ \frac{1}{2}, \frac{1}{2}\right)$; that is, a set of rotations about any one "axis" with a pair of reflexions about the two perpendicular "planes," and the forms in which one or both of the two last fractions is replaced by a zero. A main group would be

$$\left(\frac{h}{m};\ \tfrac{1}{2}, \tfrac{1}{2}\right),\ \left(\frac{h}{m};\ \tfrac{1}{2}, 0\right),\ \left(\frac{h}{m};\ 0, \tfrac{1}{2}\right),\ \left(\frac{h}{m};\ 0, 0\right)$$

$$(h = 0, 1, 2, \ldots, m-1).$$

The degree is $4m$, and the main group can be generated by three substitutions when m is even and by two when m is odd.

The third consists wholly of reflexions about the four coordinate "planes." There is a single main group consisting of 16 substitutions $(;\ 0, 0, 0, 0)$, $(;\ \frac{1}{2}, 0, 0, 0)$, etc., which can be generated by four substitutions.

Degree 2ϖ.—We pass now to the general case. The first type for a group of even degree 2ϖ would be a group of substitutions consisting of rotations about ϖ of the $\varpi(2\varpi-1)$

"axes" of coordinates. The ϖ axes are the intersections of the 2ϖ coordinate "planes" paired off in any way, but all the substitutions of the given group would consist of rotations about *this set* of "axes" and no other. The type of substitution would be $\left(\frac{h}{m}, \frac{k}{n}, \ldots, \frac{l}{p}\right)$, and if not more than ϖ_1 at a time of the integers $m, n, \ldots, p$ had a common factor, then ϖ_1 substitutions (or less perhaps for sub-groups of the main group) would be necessary to generate a group. The order of the main group would be $mn\ldots p$.

We must next consider groups of substitutions which consist of rotations about certain of the "axes" of coordinates (the same for all substitutions of the group) and reflexions about the "planes" which are perpendicular to all these special axes. If there are rotations about ϖ' "axes," there would be reflexions about $(2\varpi-2\varpi')$ "planes," and such a substitution would be, for example, $\left(\frac{h}{m}, \frac{k}{n}, \ldots, \frac{l}{p}; \frac{1}{2}, \frac{1}{2}, \ldots\right)$. If $m, n, \ldots, p$ were all even, say, a maximum number of $2\varpi-\varpi'$ substitutions might be necessary to generate the group; if some were odd, a smaller number would always be sufficient.

In particular, the substitutions may consist wholly of reflexions about the 2ϖ "axes" of coordinates. There would be a main group of $2^{2\varpi}$ substitutions, to generate which 2ϖ substitutions would be needed.

Degree $2\varpi+1$.—A separate discussion of this case is hardly necessary. The first type would consist of rotations about ϖ "axes," together perhaps with a reflexion about the single "plane" perpendicular to them all. A maximum number of $\varpi+1$ generators would be necessary when the denominators of the elements were all even. There would be, as before, intermediate types consisting partly of rotations and partly of reflexions, and a final type with a single main group of $2^{2\varpi+1}$ substitutions consisting entirely of reflexions.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XXXVII.

On the region of convergence of Borel's integral.

1. BOREL'S integral, associated with a power series

$$(1) \qquad \Sigma a_n x^n,$$

is

$$(2) \qquad f(x) = \int_0^\infty e^{-t} u(tx)\, dt,$$

where

$$(3) \qquad u(x) = \Sigma \frac{a_n x^n}{n!}.$$

If the series (1) has a positive radius of convergence, the region of convergence of the integral (2) is Borel's "polygon of summability"; the integral is convergent everywhere inside, and nowhere outside, the polygon, and represents the analytic function $f(x)$ defined in the ordinary way by the series (1).

Let us suppose now that the radius of convergence of (1) is zero. If (2) converges for $x = x_0$, it converges uniformly along the straight line $(0, x_0)$.* And if it represents an analytic function $f(x)$ in a region D, that region must extend up to the origin, and the origin must be a singular point of $f(x)$.

My object in this note is to show by examples how Borel's integral may converge in two different regions of the plane, having only the origin as a common boundary point, and represent, in these two regions, two different analytic functions.

2. I consider first the series

$$1 + 0 - \frac{2!}{1!} + 0 + \frac{4!}{2!} + 0 - \ldots,$$

in which $\qquad a_{2\nu} = (-1)^\nu \dfrac{2\nu!}{\nu!}, \quad a_{2\nu+1} = 0.$

Here Borel's integral is

$$f(x) = \int_0^\infty e^{-t-x^2t^2}\, dt,$$

---

* See Note XXXI., vol. xl., p. 161.

which is plainly convergent if

$$-\tfrac{1}{4}\pi \leqq \operatorname{am} x \leqq \tfrac{1}{4}\pi$$

or

$$\tfrac{3}{4}\pi \leqq \operatorname{am} x \leqq \tfrac{5}{4}\pi,$$

i.e., in two quadrants abutting at the origin.

Suppose x real and positive. Then

$$f(x) = \frac{1}{x}\int_0^\infty e^{-(u/x)-u^2}\,du,$$

or, if $x = 1/y$,

$$\begin{aligned} f(x) &= y\int_0^\infty e^{-yu-u^2}\,du = ye^{\frac{1}{4}y^2}\int_{\frac{1}{2}y}^\infty e^{-v^2}\,dv \\ &= ye^{\frac{1}{4}y^2}\left\{\tfrac{1}{2}\sqrt{\pi} - \int_0^{\frac{1}{2}y} e^{-v^2}\,dv\right\} \\ &= F(y) \end{aligned}$$

say. The function $F(y)$ is an integral function of y. Thus, in the quadrant which includes the positive real axis,

$$f(x) = F(1/x).$$

In the other quadrant it is plain that

$$f(x) = F(-1/x),$$

which differs from $F(1/x)$ by

$$\frac{\sqrt{\pi}}{x}\,e^{-\frac{1}{4}x^2}.$$

Thus $f(x)$ is equal to different analytic functions in the two regions.

3. As a second example I shall consider the series in which

$$a_n = \sum_0^\infty \frac{(-1)^\nu \nu^n}{\nu!}.$$

Here

$$\begin{aligned} u(x) = \sum_0^\infty \frac{x^n}{n!}\sum_0^\infty \frac{(-1)^\nu \nu^n}{\nu!} &= \sum_0^\infty \frac{(-1)^\nu}{\nu!}\,e^{\nu x} \\ &= e^{-e^x}. \end{aligned}$$

Thus Borel's integral is

$$f(x) = \int_0^\infty e^{-t-e^{tx}}\,dt. \tag{4}$$

If $x = \xi + i\eta$,

$$\left|e^{-e^{tx}}\right| = e^{-e^{\xi t}\cos\eta t}.$$

It is easy to see that, if $\xi > 0$, the integral (4) is convergent if and only if $\eta = 0$. On the other hand, if $\xi \leqq 0$, it is convergent for all values of η. Thus the integral is convergent

(i) along the positive real axis and (ii) in the half-plane to the left of the imaginary axis.

First suppose $x=\xi>0$. Then, putting $e^t=u$, we obtain

$$f(x)=\int_0^\infty e^{-t-e^{\xi t}}dt = \int_1^\infty e^{-u^\xi}\frac{du}{u^2}$$
$$=\frac{1}{\xi}\int_1^\infty e^{-w}w^{-(1/\xi)-1}\,dw$$
$$=-y\int_1^\infty e^{-w}w^{y-1}\,dw,$$

where $y=-1/\xi=-1/x$. If, in the ordinary notation of the theory of the Gamma function, we write

$$\Gamma(s)=\int_0^\infty e^{-w}w^{s-1}\,dw=\int_0^1 e^{-w}w^{s-1}\,dw+\int_1^\infty e^{-w}w^{s-1}\,dw$$
$$=P(s)+Q(s),$$

when the real part of s is positive, then $Q(s)$ is an integral function of s; and, for x real and positive, we have

(5) $$f(x)=\frac{1}{x}\,Q\left(-\frac{1}{x}\right).$$

Secondly, suppose $\xi<0$, and $x=\xi=-\lambda$. Then

$$f(x)=\int_0^\infty e^{-t-e^{-\lambda t}}dt=\int_1^\infty e^{-u^{-\lambda}}\frac{du}{u^2}$$
$$=\frac{1}{\lambda}\int_0^1 e^{-w}w^{(1/\lambda)-1}\,dw$$
$$=y\int_0^1 e^{-w}w^{y-1}\,dw,$$

where $y=1/\lambda=-1/\xi=-1/x$. Thus, for real negative values of x,

(6) $$f(x)=-\frac{1}{x}\,P\left(-\frac{1}{x}\right).$$

The function $P(s)$ is regular for all values of s save negative integral values (including zero), where it has simple poles. Thus

$$-\frac{1}{x}\,P\left(-\frac{1}{x}\right)$$

is regular in the half-plane which we are considering, and it is clear that equation (6) is valid throughout this half-plane. The equations (5) and (6) show that Borel's integral converges, for different values of x, to two different analytic functions.

A CANONICAL FORM OF THE BINARY SEXTIC.

By *E. K. Wakeford*, Trinity College, Cambridge.

The natural canonical form for the binary sextic would be $x^6+y^6+z^6+30\kappa x^2y^2z^2$, where x, y, z are linear forms satisfying an identity $lx+my+nz\equiv 0$, but apparently the general binary sextic has not been previously reduced to this form (see Elliott, *Algebra of Quantics*, §224). The object of the following work, in the arrangement of which I have been kindly assisted by Mr. P. W. Wood, M.A., of Emmanuel College, is to demonstrate the possibility of such a reduction of the general sextic, and to point out in how many ways the reduction is possible.

Let the sextic be $S\equiv(b_0b_1b_2b_3b_4b_5b_6\between X, Y)^6$, and suppose that the required reduction is possible, x, y, z being the linear factors of the cubic $C\equiv(a_0a_1a_2a_3\between X, Y)^3$. We have then the identity

$$(b_0b_1b_2b_3b_4b_5b_6\between X, Y)^6\equiv x^6+y^6+z^6+30\kappa x^2y^2z^2.$$

If we operate on the right-hand side of this identity with the operator $O\equiv(a_0a_1a_2a_3\between\partial/\partial Y, -\partial/\partial X)^3$, we shall annihilate the terms x^6, y^6, and z^6, and be left with the result of operating on $30\kappa x^2y^2z^2$ alone. This is (Elliott, §49, Ex. 3) a numerical multiple of the cubicovariant of C, which we shall denote by $T=(A_0A_1A_2A_3\between X, Y)^3$. Now the coefficients $A_0A_1A_2A_3$ may be proved by actual substitution to satisfy the equations

$$\left.\begin{aligned} a_2A_0-2a_1A_1+\ a_0A_2\qquad\ &=0\\ a_3A_0-\ a_2A_1-\ a_1A_2+a_0A_3&=0\\ a_3A_1-2a_2A_2+a_1A_3&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\text{(i)}.$$

Hence, if we operate with O upon the left-hand side of our original identity, the coefficients $A_0'A_1'A_2'A_3'$ of the resulting cubic must also satisfy these equations. These coefficients are

$$A_0'\equiv 120\,(a_0b_3-3a_1b_2+3a_2b_1-a_3b_0),$$

$$A_1'\equiv 120\,(a_0b_4-3a_1b_3+3a_2b_2-a_3b_1),$$

$$A_2'\equiv 120\,(a_0b_5-3a_1b_4+3a_2b_3-a_3b_2),$$

$$A_3'\equiv 120\,(a_0b_6-3a_1b_5+3a_2b_4-a_3b_3),$$

If we substitute these expressions for $A_0A_1A_2A_3$ respectively in the equations (i) above, we obtain three homogeneous quadratic equations in $a_0a_1a_2a_3$. There must be at least one system of ratios, say $a_0 : a_1 : a_2 : a_3 = c_0 : c_1 : c_2 : c_3$, which satisfies these equations.

We shall prove that the required reduction of the sextic is possible if we take x, y, z to be the factors of $(c_0c_1c_2c_3 \between X, Y)^3$. We have so far proved that if the reduction is possible at all, the forms x, y, z must be found in this way.

Let $(C_0'C_1'C_2'C_3' \between X, Y)^3$ be the result of operating with $(c_0c_1c_2c_3 \between \partial/\partial Y, -\partial/\partial X)^3$ on S. Then

$$C_0' \equiv 120\,(c_0b_3 - 3c_1b_2 + 3c_2b_1 - c_3b_0),$$

$$C_1' \equiv 120\,(c_0b_4 - 3c_1b_3 + 3c_2b_2 - c_3b_1),$$

etc. (see $A_0'A_1'A_2'A_3'$ above).

Hence
$$\left.\begin{array}{rrrrl} c_2C_0' - & 2c_1C_1' + & c_0C_2' & & = 0 \\ c_3C_0' - & c_2C_1' - & c_1C_2' + & c_0C_3' & = 0 \\ & c_3C_1' - & 2c_2C_2' + & c_1C_3' & = 0 \end{array}\right\} \text{..........(ii)},$$

since $c_0c_1c_2c_3$ are known to satisfy the quadratic equations formed by substituting for $C_0'C_1'C_2'C_3'$ in equations (ii) their values in terms of $c_0c_1c_2c_3$.

Now the coefficients $C_0C_1C_2C_3$ of the cubicovariant of $(c_0c_1c_2c_3 \between X, Y)^3$ also satisfy these equations, and indeed the cubicovariant may be written in the form

$$\begin{vmatrix} X^3 & 3X^2Y & 3XY^2 & Y^3 \\ c_2 & -2c_1 & c_0 & 0 \\ c_3 & -c_2 & -c_1 & c_0 \\ 0 & c_3 & -2c_2 & c_1 \end{vmatrix}.$$

Hence $C_0' : C_1' : C_2' : C_3' = C_0 : C_1 : C_2 : C_3$, for otherwise the cubicovariant of $(c_0c_1c_2c_3 \between X, Y)^3$ would vanish identicallly, equations (ii) not being independent.

In that case $(c_0c_1c_2c_3 \between X, Y)^3$ would be a perfect cube, $\equiv x^3$ suppose, and $(C_0' : C_1' : C_2' : C_3' \between X, Y)^3$ a cubic containing the factor x twice, $\equiv x^2y$ suppose.

Then we should have

$$(\partial S/\partial y^3) = x^2y,$$

whence $\quad S = x^2(ax^4 + bx^3y + cx^2y^2 + \frac{1}{24}y^4)$ suppose,

showing that S would contain a square factor. This is not so in the general case, so that we may disregard it.

Hence, in general, the result of operating with

$$(c_0 c_1 c_2 c_3 \between \partial/\partial Y, -\partial/\partial X)^3$$

on S is to give a numerical multiple of the cubicovariant of $(c_0 c_1 c_2 c_3 \between X, Y)^3$; and so, if we choose κ aright, we shall obtain

$$(c_0 c_1 c_2 c_3 \between \partial/\partial Y, -\partial/\partial X)^3 (S - 30\kappa x^2 y^2 z^2) = 0.$$

Now if $x \equiv \alpha_1 X + \beta_1 Y$, $y \equiv \alpha_2 X + \beta_2 Y$, $z \equiv \alpha_3 X + \beta_3 Y$ are the factors of $(c_0 c_1 c_2 c_3 \between X, Y)^3$, the operator

$$(c_0 c_1 c_2 c_3 \between \partial/\partial Y, -\partial/\partial X)^3$$

is the same as the three combined operators

$$\left(\alpha_1 \frac{\partial}{\partial Y} - \beta_1 \frac{\partial}{\partial X}\right) \left(\alpha_2 \frac{\partial}{\partial Y} - \beta_2 \frac{\partial}{\partial X}\right) \left(\alpha_3 \frac{\partial}{\partial Y} - \beta_3 \frac{\partial}{\partial X}\right).$$

Accordingly, by a known theorem, the general solution of $(c_0 c_1 c_2 c_3 \between \partial/\partial Y, -\partial/\partial X)^3 u = 0$ is $P + Q + R$, where P, Q, and R are the general solutions of

$$\left(\alpha_1 \frac{\partial}{\partial Y} - \beta_1 \frac{\partial}{\partial X}\right) u = 0,$$

$$\left(\alpha_2 \frac{\partial}{\partial Y} - \beta_2 \frac{\partial}{\partial X}\right) u = 0, \text{ and } \left(\alpha_3 \frac{\partial}{\partial Y} - \beta_3 \frac{\partial}{\partial X}\right) u = 0$$

respectively.

(We may take x, y, z to be all different, for otherwise the resulting canonical form would contain too few constants, implicit and explicit, to be general.)

We find therefore that

$$S - 30\kappa x^2 y^2 z^2 = ax^6 + by^6 + cz^6.$$

Hence, by taking suitable numerical multiples of x, y, z, and κ, we reduce S to the form

$$x^6 + y^6 + z^6 + 30\kappa x^2 y^2 z^2,$$

there existing a relation of the form

$$lx + my + nz \equiv 0.$$

We may also write S as

$$ax^6 + by^6 + cz^6 + 30\kappa x^2y^2z^2,$$

where $$x + y + z \equiv 0.$$

If we regard $(a_0a_1a_2a_3)$ as the coordinates of a point in space, the three quadratic equations mentioned above represent three quadrics. We should therefore expect eight distinct solutions of the problem. By considering any common point of the quadrics in particular, it can easily be shown that the tangent planes to the three quadrics at this point do not in general intersect in a line so that the eight points of intersection of the quadrics are all distinct. Hence the reduction is in general possible in eight distinct ways.

The existence of the canonical form (long suspected) is thus established: the form does not appear to lend itself easily to the formation of invariants and covariants.

Note on a previous paper.

On p. 143 of vol. xlii. I published a theorem, which I believed to be new, relating to three triangles circumscribing a conic. My attention has since been drawn to the fact that this theorem with its converse is to be found as Ex. 862 (p. 360) in C. Taylor's *Ancient and modern Geometry of Conics*.

The more general result that the three sets of the six sides of the complete quadrangles formed by the common points of any three conics taken in pairs touch a class cubic leads easily both to the theorem I published and to its converse. This class cubic is the Cayleyan contravariant of the cubic of which the conics are polar conics.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XXXVIII.

On the definition of an analytic function by means of a definite integral.

1. The two theorems proved in this note are in no way of a novel character, and the first of them is actually stated without proof in Osgood's *Lehrbuch der Funktionentheorie.** The second I have never seen quite in the form in which I give it here. The theorems have so many important applications that it seems worth while to state them explicitly and with proofs.

2. I must first define the meaning of the expressions 'regular curve' and 'region,' which occur in the enunciations of the theorems. I do not propose to use these terms in the most general senses possible: I wish indeed to use them in the simplest and narrowest senses possible so long as the theorems retain sufficient generality to admit the ordinary applications.

An *elementary arc* is a set of points in the plane (ξ, η) defined by two equations

$$\xi = \phi(t), \quad \eta = \psi(t) \quad (t_0 \leqq t \leqq t_1),$$

where ϕ and ψ are functions with continuous derivatives which can only vanish for $t = t_0$ or $t = t_1$, and then not simultaneously.

The points

$$\{\phi(t_0), \psi(t_0)\}, \quad \{\phi(t_1), \psi(t_1)\},$$

are the *first* and *last* points of the arc.

A *regular curve* is the set formed by a finite succession of elementary arcs in which the first point of each arc is the last of its predecessor. If the last point of the last arc coincides with the first of the first, the curve is *closed*. If no point of the curve belongs to more than one of its arcs, the curve is *simple*.

A simple closed regular curve divides the points of the plane which do not lie upon it into two classes, *interior* and

* Vol. i., p. 260.

exterior points. The points interior to such a curve will be said to constitute a *region*. The points of a region, together with those of its boundary, constitute a *domain*.

3. THEOREM 1. *Suppose that $f(x, y)$ is a function of the two complex variables x and y, continuous when x varies along a regular curve C and y over a region S. Suppose also that $f(x, y)$ is, for each particular value of x, analytic throughout S. Then*

$$F(y) = \int_C f(x, y)\, dx$$

is analytic throughout S, and

$$F'(y) = \int_C \frac{\partial f}{\partial y}\, dx.$$

We prove first that $F(y)$ is continuous in S. Let Σ be a domain which lies entirely inside S. Then $f(x, y)$ is continuous, and so uniformly continuous, when x varies on C and y in Σ. Hence it follows in the ordinary manner that, if y and $y+h$ lie in Σ,

$$F(y+h) - F(y) = \int_C \{f(x, y+h) - f(x, y)\}\, dx$$

tends to zero with $|h|$. Thus $F(y)$ is continuous in Σ, and so in S.

Now let Γ be a simple closed regular curve lying inside S and including the point y inside it. Then

$$f(x, y) = \frac{1}{2\pi i} \int_\Gamma \frac{f(x, u)}{u - y}\, du,$$

and so

$$F(y) = \frac{1}{2\pi i} \int_C dx \int_\Gamma \frac{f(x, u)}{u - y}\, du.$$

In this equation we may invert the order of integration. In order to prove this we observe first that C and Γ are formed by the union of a finite number of elementary arcs C_i and Γ_j, and that it is obviously sufficient to show that the inversion is permissible when x varies on C_i and u on Γ_j. The arcs C_i, Γ_j are defined by equations of the form

$$x = \phi(t) + i\psi(t) \qquad (t_0 \leqq t \leqq t_1),$$

$$u = \Phi(w) + i\Psi(w) \qquad (w_0 \leqq w \leqq w_1),$$

where $\phi, \psi, \ldots$ are functions which satisfy the conditions of § 2. We substitute for x and u in terms of t and w, and separate the real and imaginary parts of the resulting repeated integrals in t and w. Each of these is the repeated integral of a continuous function of t and w, and the inversion of the order of integration is therefore legitimate.

Inverting the order of integration we obtain

$$F(y) = \frac{1}{2\pi i}\int_\Gamma \frac{du}{u-y}\int_C f(x, u)\, dx = \frac{1}{2\pi i}\int_\Gamma \frac{F(u)}{u-y}\, du$$

Hence

$$\frac{F(y+h)-F(y)}{h} = \frac{1}{2\pi i}\int_\Gamma \frac{F(u)\, du}{(u-y)(u-y-h)},$$

and, $F(u)$ being continuous on Γ, it is easy to show in the ordinary way* that the integral tends, as $h \to 0$, to the limit

$$\frac{1}{2\pi i}\int_\Gamma \frac{F(u)}{(u-y)^2}\, du.$$

Hence $F(y)$ is analytic inside Γ and so inside S. Finally

$$\begin{aligned} F'(y) &= \frac{1}{2\pi i}\int_\Gamma \frac{F(u)}{(u-y)^2}\, du \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{du}{(u-y)^2}\int_C f(x, u)\, dx \\ &= \frac{1}{2\pi i}\int dx \int_\Gamma \frac{f(x, u)}{(u-y)^2}\, du \\ &= \int_C \frac{\partial f}{\partial y}\, dx, \end{aligned}$$

the inversion of the order of integration being justified in precisely the same way as before. Thus the proof of the theorem is completed.

4. It usually happens in applications that the integral

$$\int_C f(x, y)\, dx,$$

by which $F(y)$ is defined, is an *infinite* integral; the contour C stretches to infinity, or f has infinities on C.

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* Cf. Osgood, *l.c.*, p. 241.

Let $C(R)$ denote the aggregate of points of C for which $|x| < R$; and let us suppose that, for any given R, $C(R)$ is a regular curve. Further, let us suppose that C contains a finite number of points ξ_i which we will call *exceptional* points; and let us denote by $C(R, \delta)$ the aggregate of points of $C(R)$ for which $|x - \xi_i| \geqq \delta$. Then $C(R, \delta)$ will consist of a finite number of regular curves $C_j(R, \delta)$. We suppose that each of these curves satisfies, in conjunction with S and f, all the conditions of Theorem 1.

Further, let us suppose that, as $\delta \to 0$, $R \to \infty$, the sum of integrals

$$\sum_{(j)} \int_{C_j(R,\delta)} f(x, y)\, dx$$

tends to a limit, uniformly for all values of y in any domain Σ such as was considered in § 3. This limit we denote by

$$\int_C f(x, y)\, dx;$$

and we say that this integral is *uniformly convergent* in Σ.

We can now state the following theorem:

Theorem 2. *Let C be a contour such that the contour $C(R, \delta)$, formed by the points of C for which*

$$|x| \leqq R, \quad |x - x_i| \geqq \delta,$$

is composed of a finite number of regular curves, each of which, together with the region S and the function $f(x, y)$, satisfies the conditions of Theorem 1. *Further, let the integral*

$$\int_C f(x, y)\, dx$$

be uniformly convergent in any domain Σ interior to S. Then the conclusions of Theorem 1 *still remain true.*

5. In sketching the proof of this theorem I shall confine myself, for the sake of simplicity of statement, to the most important case, viz., that in which C is the positive real axis, and there is one exceptional point, namely, the origin. In this case $C(R, \delta)$ is the segment (δ, R), and our definition of uniform convergence reduces to the ordinary definition for infinite integrals of a function of a real variable. The proof follows exactly the same lines as that of Theorem 1, the condition of uniform convergence being required (i) in proving

that $F(y)$ is continuous and (ii) in justifying the inversions of the order of integration. A few words should perhaps be added on the latter point. We reduce the problem, as in § 3, to an inversion problem concerning real integrals. In this case t is x, and x ranges from 0 to ∞.

The theorem to which we finally appeal is that of de la Vallée-Poussin which asserts that

$$\int_0^\infty dx \int_{w_0}^{w_1} \chi(x, w)\, dw = \int_{w_0}^{w_1} dw \int_0^\infty \chi(x, w)\, dx$$

whenever (i) the inversion is legitimate when the limits $(0, \infty)$ are replaced by any positive numbers, and (ii) the integral with respect to x is uniformly convergent.

6. In order to give a simple application of Theorem 2, let us consider the equation

$$\int_0^\infty e^{-yx^2}\, dx = \tfrac{1}{2}\sqrt{(\pi/y)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which holds when y is real and positive. Let S be any region for all points of which $R(y) > 0$. Then the real parts of the points of Σ have a positive lower limit λ, and the integral in (1) may be seen to be uniformly convergent in Σ by comparison with

$$\int_0^\infty e^{-\lambda x^2}\, dx.$$

Hence the conditions of Theorem 2 are satisfied, and (1) holds for all values of y whose real part is positive. Putting $y = \alpha + i\beta$, and equating real parts, we obtain

$$\int_0^\infty e^{-\alpha x^2} \cos \beta x^2\, dx = \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)} \sqrt{\left\{\frac{\sqrt{(\alpha^2+\beta^2)}+\alpha}{(\alpha^2+\beta^2)^3}\right\}}.$$

Making $\alpha \rightarrow 0$, and using Abel's continuity theorem for infinite integrals,* we obtain

$$\int_0^\infty \cos \beta x^2\, dx = \tfrac{1}{2}\sqrt{(\pi/2\beta)}.$$

The corresponding integral involving a sine may of course be evaluated similarly.

* Bromwich, *Infinite Series*, p. 434.

FACTORISATION OF $N=(y^4 \mp 2)$ & $(2y^4 \mp 1)$.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's College, London.

[The author's acknowledgments are due to Mr H. J. Woodall, A.R.C.Sc., for reading the Proof sheets, and for suggestions; also to M. L. Valroff for numerous additions to the Tables.]

1. *Introduction.* In this Paper it is proposed to develop the factorisation of the numbers (N) of the four types

$$N_i=y^4-2,\quad N_{ii}=y^4+2,\quad N_{iii}=2y^4-1,\quad N_{iv}=2y^4+1\ldots(1).$$

These numbers are closely connected, so that it is convenient to consider them together: they are also closely related to the numbers $(2^n \mp 1)$, and their factorisation depends in fact largely on a prior knowledge of that of the latter kind of numbers, as will appear later.

2. *Notation.* All symbols denote integers. p denotes an (odd) prime.

ω, Ω denote *odd* numbers; ε, E denote *even* numbers; i, I denote integers.

y_i, y_{ii}, y_{iii}, y_{iv} denote the roots (y) of the numbers N_i, N_{ii}, N_{iii}, N_{iv}, respectively; but the subscripts will often *be omitted* when not required to distinguish the four kinds.

3. *Linear and* 2^{ic} *forms of N.* These are shewn below:—

		N_i	N_{ii}	N_{iii}	N_{iv}
Linear	$y=\omega$	$16\varpi-1$	$16\varpi+3$	$32\varpi+1$	$32\varpi+3$.........(2*a*),
	$y=\varepsilon$	$2(8\varpi-1)$	$2(8\varpi+1)$	$32\varpi-1$	$32\varpi+1$ ....(2*b*),
Quadratic		e^2-2f^2	c^2+2d^2	$2f'^2-e'^2$	c^2+2d^2...........(2*c*).

3*a*. *Divisors* 2, 3. N_{iii}, N_{iv} are always *odd*........(2*d*).

$y=\omega$ gives N_i, N_{ii} *odd*; $y=\varepsilon$ gives N_i, N_{ii} *even* and $=2\Omega$.......(2*e*),

$y=3i$ gives N_{ii}, $N_{iv} \neq 3I$; $y \neq 3i$ gives N_{ii}, $N_{iv}=3I$...................(2*f*).

4. *Use of Factor-Tables.* Complete factorisation of these numbers may be obtained by use of the large* Factor-Tables alone up to the following limits—

N_i up to $y=\omega \not> 55$; $y=\varepsilon \not> 66$;

N_{ii} up to $y=\omega=3i \not> 55$; $y=\varepsilon=3i \not> 66$; $y=\omega \neq 3i \not> 73$; $y=\varepsilon \neq 3i \not> 88$;

N_{iii} up to $y \not> 47$; N_{iv} up to $y=3i \not> 47$; $y \neq 3i \not> 62$.

Beyond these limits it is (in general) necessary to search for special factors. The research of these occupies Art. 6—16*d* of this Paper.

* These extend to 10017000.

5. *Case of* $y=2^x$. In this case the numbers (N) take the forms:—

$$N_i=2(2^{4x-1}-1),\quad N_{ii}=2(2^{4x-1}+1),\quad N_{iii}=(2^{4x+1}-1),\quad N_{iv}=(2^{4x+1}+1),$$

the complete factorisation of which is known up to the following high limits—

Number N	$=$	N_i	N_{ii}	N_{iii}	N_{iv}
All values up to	$x=$	18	16	17	16
Highest value	$x=$	22	34	29	26

It is seen that the power of 2 (viz. $4x\mp1$) entering into the binomial N is always *odd*, and that the numbers N_i, N_{iii}, include Mersenne's Numbers (given by $4x\mp1=$ prime).

[It is not to be expected that the study of the present numbers will lead to the discovery of divisors of the as yet unfactorised Mersenne's Numbers. In fact the known factorisation of N when $y=2^x$ afford the principal help to factorisation of all other cases (see Art. 14)].

6. *Linear and* 2^{ic} *forms of Factors* (p). Since

$$N_i=e^2-2f^2,\quad N_{ii}=c^2+2d^2,\quad N_{iii}=2f'^2-e'^2,\quad N_{iv}=2d^2+c^2 \ldots \text{see } (2c),$$

it follows that—(excepting the small factor 2 of N_i, N_{ii})—

All factors of N_i, N_{iii} are of form $p=e^2-2f^2=2f'^2-e'^2$.........(3a).

All factors of N_{ii}, N_{iv} are of form $p=c^2+2d^2$........................(3b).

Hence, also—(excepting the factor 2)—

All factors of N_i, N_{iii} are of form $p=8\varpi+1,\ 7$...................(4a).

All factors of N_{ii}, N_{iv} are of form $p=8\varpi+1,\ 3$....... ....(4b).

7. *Non-divisors* (p). These are of two linear forms $p=8\varpi+1,\ 5$.

1°. Since $p=8\varpi+5\neq e^2-2f^2$ & $\neq 2f'^2-e'^2$, it follows that

All primes of form $p=8\varpi+5$ are non-divisors...............(5).

2°. There is also a limited class of primes $p=8\varpi+1$, which are non-divisors.

For	$N_i\equiv0$	$N_{ii}\equiv0$	$N_{iii}\equiv0$	$N_{iv}\equiv0$	(mod p)
involve	$y^4\equiv+2$	$y^4\equiv-2$	$2y^4\equiv+1$	$2y^4\equiv-1$	
whence	$(2/p)_4=+1$	$(\bar{2}/p)_4=+1$	$(2/p)_4=+1$	$(\bar{2}/p)_4=+1$...........(6).	

But, when—as here—$p=8\varpi+1$, then $(2/p)_4=(\bar{2}/p)_4$ always.

Hence $(\pm2/p)_4=+1$ is a condition* when $p=8\varpi+1$...(6a), so that—

All primes $p=8\varpi+1$ having $(\pm2/p)_4=-1$ are *non-divisors*......(6b).

* This condition excludes about one-half of the total number of primes $p=8\varpi+1$ from the list of possible divisors: this partially accounts for the comparative infrequency of divisors of that form in Mersenne's Numbers.

Another way of putting this is

All primes $p=8\varpi+1$, $=a^2+b^2$, with $b=4\omega$, are *non-divisors*......(6c).

8. *Complete set of Divisors.* It will be seen from what precedes (Art. 6, 7)--

All primes $p=8\varpi+7$ are factors of some N_i and N_{iii}..................(7a).

All primes $p=8\varpi+3$ are factors of some N_{ii} and N_{iv}..................(7b).

All primes $p=8\varpi+1$, having $(2/p)_4=+1$, are factors of some of each kind of N_i, N_{ii}, N_{iii}, N_{iv} ....(7c),

All other (odd) primes are non-divisors..................................(7d).

9. *Congruence-solutions.* The most powerful aid to the factorisation of these numbers (N) is a Table of solutions—(y) of the four congruences—

$N_i=y^4-2\equiv 0$, $N_{ii}=y^4+2\equiv 0$, $N_{iii}=2y^4-1\equiv 0$, $N_{iv}=2y^4+1\equiv 0$
(mod p and p^κ)...(8).

Such a Table is given—(see Tab. I., II.)—at end of this Paper, complete for all primes and prime-powers p and $p^\kappa \not> 1000$.

The mode of solution of these Congruences, and their mutual connexion, are explained in Art. 10—16*d*. Three Methods are available.

Method I. From known factorisations (Art. 10).

Method II. By use of primitive roots (Art. 11—11*b*).

Method III. By use of residues of powers of 2 (Art. 14—16*d*).

Art. 12, 13 contain general properties of the roots applicable to all the Methods.

10. Method I. *From known factorisations.* Every actual factorisation—complete or partial—of any of the numbers (N) shows one root (y) for each of the prime factors, or prime-power factors, found in N. Thus the factorisations explained in Art. 4, 5 furnish (at sight) one, or more, roots (y) of each of the primes and prime-power factors (p and p^κ) found. A Congruence-Table showing the roots (y) so found for those moduli (p and p^κ) can thus be started. It will, of course, be very incomplete; it does, however, yield a certain number of roots more simply than either of the other more powerful Methods described below.

11. Method II. *Use of primitive roots.* (g). Let g be a primitive root (>2) of the prime p, and—for shortness—write

$p-1=\xi$, so that $g^\xi\equiv+1$, $g^{\frac{1}{2}\xi}\equiv-1$ (mod p)................(9).

Now, find α, β, such that

$$g^{\alpha}\equiv+2,\quad g^{\beta}\equiv-2 \pmod p), \ldots, \text{[always possible]} \ldots\ldots\ldots (10),$$

$$\text{And let } y_i\equiv g^{x_i},\quad y_{ii}\equiv g^{x_{ii}},\quad y_{iii}\equiv g^{x_{iii}},\quad y_{iv}\equiv g^{x_{iv}} \pmod p \ldots\ldots (11),$$

where the subscripts under x, y indicate that the x, y belong to the 1st, 2nd, 3rd, or 4th of the Congruences (8) respectively: but these will be written more simply as

$$y\equiv g^{x} \pmod p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11a),$$

when the subscripts are not really required for the sake of distinction

Then the four Congruences (8) may be expressed in terms of x, α, β; and solutions (x) are thence obtained as follows—

Congruences	$N_i\equiv0$	$N_{ii}\equiv0$	$N_{iii}\equiv0$	$N_{iv}\equiv0$
	$g^{4x}\equiv g^{\alpha}$	$g^{4x}\equiv g^{\beta}$	$g^{\alpha+4x}\equiv g^{\xi}$	$g^{\beta+4z}\equiv g^{\xi}$
$x=$	$\frac14(m\xi+\alpha)$	$\frac14(m\xi+\beta)$	$\frac14(m\xi-\alpha)$	$\frac14(m\xi-\beta)$......(12).

Here m is to be an integer determined so that x may be an integer $<\xi$.

Now, it will be found that

$$p=8\varpi+7,\ 3 \text{ have one of } \alpha, \beta \textit{ odd}, \text{ and one } \textit{even} \ldots\ldots\ldots\ldots (13a),$$

$$p=8\varpi+1, \text{ with } (2/p)_4=+1, \text{ has } \alpha=4i,\ \beta=4i' \ldots\ldots\ldots\ldots (13b).$$

From this, it results that, in each of the four Congruences

m has 2 values, giving 2 values of x and y, when $p=8\varpi+7$, 3.......(14a),

m has 4 values, giving 4 values of x and y,

when $p=8\varpi+1$, with $(2/p)_4=+1$...(14b),

whence it follows that, in each Congruence,

Every prime $p=8\varpi+7$, 3 has *two* roots $y\ (<p)$........................(14c).

Every prime $p=8\varpi+1$, with $(2/p)_4=+1$, has *four* roots $y\ (<p)$...(14d).

The set of exponents, say x, x', x'', x''', of any one Congruence (8) are quite simply connected as follows

$p=8\varpi+7$, 3 has only x and $x''=x+\frac12\xi$..(15a).

$p=8\varpi+1$, with $(2/p)_4=+1$ has x, $x'=x+\frac14\xi$, $x''=x+\frac12\xi$, $x'''=x+\frac34\xi$...(15b),

so that, when one (x) has been found, the rest follow at once.

Also, in each Congruence

$$y\equiv g^{x},\ y''\equiv g^{x''};\ \text{ and (when } p=8\varpi+1)\ y'\equiv g^{x'},\ y'''\equiv g^{x'''} \ldots\ldots (16).$$

The above process suffices for the computation of the whole of the roots (y) of each of the four Congruences (8) for every prime for which a primitive root (g) is known.

It involves of course considerable labor when Tables of Residues of g^x are not available, viz.

Finding α, β; x, x', x'', x'''; and Residues of $g^x, g^{x'}, g^{x''}, g^{x'''}$.

[The finding of α, β from (10) is often a difficult matter, involving much tentative work: that of determining m in (12) is comparatively easy: the rest of the process, viz. finding the Residues of g^x in (16) is a direct process, but is laborious when x is large].

11*a*. *Prime-power Moduli* (p^κ). With slight modification the above process—described for prime moduli (p)—suffices also when the modulus is a prime-power (p^κ). The chief changes are—

Use p^κ instead of p; write $\xi = p^{\kappa-1}(p-1)$ instead of $\xi = p-1$.

11*b*. *Use of the Canon Arithmeticus.* This Table gives the complete set of Residues (R) of g^x, and also the exponents (x) yielding the Residues (R) for all moduli p and $p^\kappa \not> 1000$. Hereby the required values of α, β, and the Residues of g^x, $g^{x'}$, &c., can be picked out *at sight*: so that the complete set of solutions (y) of the four Congruences (8) can be thereby found for all (p and p^κ) moduli up to the limit of 1000.

12. *Connexion of roots of a Congruence.* Let y, y', y'', y''' denote the roots of the same Congruence; [only y, y'' are real when $p = 8\varpi + 7, 3$].

Now, since $\pm y$ satisfy the same Congruence, the roots of any one Congruence evidently occur in pairs, connected by the relations

$$y + y'' = p = y' + y''' \quad \text{.................................} (17).$$

And since y, $y\iota$—(where $\iota^2 = -1$)—satisfy the same Congruence, the roots may also be arranged in pairs, connected by the relations

$$y' \equiv \eta y, \; y \equiv \eta' y'; \; y''' \equiv \eta y'', \; y'' \equiv \eta' y''' \pmod p \text{............} (18),$$

where η, η' are roots of $\eta^2 + 1 \equiv 0 \pmod p$.................(19),

[and η, η' are real when $p = 8\varpi + 1$].

12*a*. Hence, for each Congruence, it suffices to compute *one* root—(say y)—by the Rule $y \equiv g^x \pmod p$ of Art. 11, and the remaining roots are then given more simply as follows:—

When $p = 8\varpi + 7, 3$; the other root $y'' = p - y$..........................(20*a*),

When $p = 8\varpi + 1$; one new root is $y' \equiv \eta y$, and the other roots are $y'' = p - y$, $y''' = p - y'$............ ....(20*b*).

[Note that, when the roots (η) of $\eta^2 + 1 \equiv 0 \pmod p$ are known, it is usually much easier to compute the Residue of $y' = \eta y$, than that of $y' = g^{x'} \pmod p$.]

13. *Connexion of different Congruences.* The four Congruences (8) may be arranged in pairs in two ways, modulo p.

(1) *Reciprocal Congruences,* $(N_i\equiv0,\ N_{iii}\equiv0)$, $(N_{ii}\equiv0,\ N_{iv}\equiv0)$.

(2) *Conjugate Congruences,* $(N_i\equiv0,\ N_{ii}\equiv0)$, $(N_{iii}\equiv0,\ N_{iv}\equiv0)$.

The roots of the four Congruences are denoted by y_i, y_{ii}, y_{iii}, y_{iv}, as in Art. 2.

13*a*. *Reciprocal Congruences.* One of each of the roots y_i, y_{iii} may be paired together, and one of each of the roots y_{ii}, y_{iv} may be paired together, each pair in such a way that

$$y_i y_{iii}\equiv\pm1, \text{ and } y_{ii}y_{iv}\equiv\pm1 \pmod p \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

so that these form reciprocal pairs modulo p; and the Congruences to which they belong may for this reason be styled *Reciprocal.*

[Note that y_i, y_{iii} exist for $p=8\varpi+7$; y_{ii}, y_{iv} exist for $p=8\varpi+3$...(21*a*),

and that y_i, y_{iii}; y_{ii}, y_{iv} exist for $p=8\varpi+1$, with $(2/p)_4=+1$...(21*b*).

13*b*. *Conjugate Congruences* $[p=8\varpi+1]$.

$$\text{Since } y_{ii}^4\equiv-y_i^4, \text{ and } y_{iv}^4\equiv-y_{iii}^4 \pmod p\ldots\ldots\ldots\ldots\ldots(22),$$

it is clear that the roots may be paired in such a way that

$$y_{ii}\equiv\zeta y_i,\ y_i\equiv\zeta' y_{ii};\ \ y_{iv}\equiv\zeta y_{iii},\ y_{iii}\equiv\zeta' y_{iv} \pmod p\ldots\ldots.(22a),$$

where ζ, ζ' are roots of $\zeta^4+1\equiv0 \pmod p$..................(23).

[The roots ζ, ζ' are real when $p=8\varpi+1$].

13*c*. *Use of above (for computing).* By these properties the labor of finding the complete set of roots of the four Congruences (8) may be much reduced. It will suffice to find *one root of only one* of the four Congruences (8) by the Rule of $y\equiv g^x \pmod p$ of Art. 11. *One root of each of the other* Congruences (8) may then be found by the use of Rules (21), (22*a*). The other roots of each of the Congruences may then be found from the single known root of each by the Rules of Art. 12*a*.

[The solution of (21), (22*a*) is usually less laborious than the calculations of the Residue $y\equiv g^x$. To use Rule (22*a*) the solutions of $\zeta^4+1\equiv0 \pmod p$ must of course be known.]

14. Method III. *Use of Residues of* 2^x.

It will here be shown that *one* root (y) of each of two of the four Congruences (8) may be found from the Residues of the powers of 2 (*i.e.* from 2^x), whenever the Haupt-Exponent* (say ξ) of 2 is either an odd number, or twice an odd number.

[This condition ($\xi=\omega$ or 2ω) always occurs when $p=8\varpi+7$ or 3, and also occurs usually—(but not always)—when $p=8\varpi+1$ with $(2/p)_4=+1$.]

Four Cases must be distinguished.

Case (1). $\xi=4x-1$; (2). $\xi=4x+1$; (3). $\frac{1}{2}\xi=4x-1$; (4). $\frac{1}{2}\xi=4x+1$.

Case (1). $\xi=4x-1$.

Take $y_i\equiv 2^{x_i}$ (mod p), where $x_i=x=\frac{1}{4}(\xi+1)$ ....(24*a*),

and $y_{iii}\equiv 2^{x_{iii}}$ (mod p), where $x_{iii}=\xi-x_i=\frac{1}{4}(3\xi-1)$............(24*b*).

Then $N_i=y_i^4-2\equiv 2^{4x_i}-2\equiv 2(2^\xi-1)\equiv 0$ (mod p)..............(25*a*),

$N_{iii}=2y_{iii}^4-1=2^{4x_{iii}+1}-1=2^{3\xi}-1\equiv 0$ (mod p)..........(25*b*).

Case (2). $\xi=4x+1$.

Take $y_{iii}=2^{x_{iii}}$ (mod p), where $x_{iii}=x=\frac{1}{4}(\xi-1)$.................(26*a*),

and $y_i\equiv 2^{x_i}$ (mod p), where $x_i=\xi-x=\frac{1}{4}(3\xi+1)$ ....(26*b*).

Then $N_{iii}=2y_{iii}^4-1\equiv 2^{4x_{iii}+1}-1=2^\xi-1\equiv 0$ (mod p)...........(27*a*),

$N_i=y_i^4-2\equiv 2^{4x_i}-2=2(2^{3\xi}-1)\equiv 0$ (mod p)............(27*b*).

It is seen that in both Cases 1°, 2°,

$x_i+x_{iii}=\xi$, and $y_iy_{iii}\equiv +1$ (mod p)......................(28).

Case (3). $\frac{1}{2}\xi=4x-1$.

Take $y_{ii}\equiv 2^{x_{ii}}$ (mod p), where $x_{ii}=x=\frac{1}{4}(\frac{1}{2}\xi+1)$.................(29*a*),

and $y_{iv}\equiv 2^{x_{iv}}$ (mod p), where $x_{iv}=\frac{1}{2}\xi-x=\frac{1}{4}(\frac{3}{2}\xi-1)$...........(29*b*).

Then $N_{ii}=y_{ii}^4+2\equiv 2^{4x_{ii}}+2=2(2^{\frac{1}{2}\xi}+1)\equiv 0$ (mod p)............(30*a*),

and $N_{iv}=2y_{iv}^4+1=2^{4x_{iv}+1}+1=2^{3\cdot\frac{1}{2}\xi}+1\equiv 0$ (mod p)..........(30*b*).

Case (4). $\frac{1}{2}\xi=4x+1$.

Take $y_{iv}=2^{x_{iv}}$ (mod p), where $x_{iv}=x=\frac{1}{4}(\frac{1}{2}\xi-1)$...............(31*a*),

and $y_{ii}=2^{x_{ii}}$ (mod p), where $x_{ii}=\frac{1}{2}\xi-x=\frac{1}{4}(\frac{3}{2}\xi+1)$...........(31*b*).

Then $N_{iv}=2y_{iv}^4+1=2^{4x_{iv}+1}+1=2^{\frac{1}{2}\xi}+1\equiv 0$ (mod p)............(32*a*),

and $N_{ii}=y_{ii}^4+2=2^{4x_{ii}}+2=2(2^{3\cdot\frac{1}{2}\xi}+1)\equiv 0$ (mod p) ....(32*b*).

And, it is seen that in both Cases (2), (4),

$x_{ii}+x_{iv}=\frac{1}{2}\xi$, $y_{ii}y_{iv}\equiv -1$ (mod p)....................(33).

Thus it has been shown that, whenever the Haupt-Exponent (ξ) of 2 is $\xi=\omega$ or 2ω, the Residues of 2^x suffice to give one root (y) of each of two out the four Congruences (8), viz. of the Reciprocal Congruences (Art. 13*a*),

i.e. of $N_i\equiv 0$, $N_{iii}\equiv 0$; or of $N_{ii}\equiv 0$, $N_{iv}\equiv 0$.

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* *Haupt-Exponent*, *i.e.* the least exponent (ξ) giving $2^\xi\equiv +1$ (mod p). This is the German term: it is sometimes styled *Gaussien* by the French.

14a. *Completion of Solutions.* One root of each of two Reciprocal Congruences having been found by the above Rules of Art. 14, a single root of each of the Congruences conjugate to that pair may be found by Rule ($22a$) of Art. $13b$ (when $p=8\varpi+1$).

Otherwise, it really suffices to find one root of any one of the four Congruences (8) by these Rules (of Art. 14). After which one root of each of the other Congruences may be found by the Rules of Art. $13a$, b combined.

The remaining roots of each of the Congruences may then be found by the Rules of Art. $13c$).

[Note that when one root of any one Congruence has been found by the Rules of Art. 14, the calculation of the reciprocal root by the solution of (21) of Art. $13a$ is usually less laborious than by the Rules of Art. 14].

$14b$. *Use of the Binary-Canon.* This Table gives the complete set of Residues (R) of 2^x, and also the exponents (x) yielding the Residues (R) for all moduli p and $p^\kappa \not> 1000$. Hereby the required Residues of 2^x of the formulæ of Art. 14 can be picked out at sight: thus giving (at sight) one root of each of two Reciprocal Congruences for every prime $p \not> 1000$ which has $\xi=\omega$ or 2ω.

14c. *Failing Cases.* There is a limited class of divisors $p=8\varpi+1$, with $(2/p)_4=+1$, in which the above process fails, viz. when $\xi=4x$. In these cases—(which are few* in number)—the roots (y) of the four Congruences are not congruent to any power of 2, so that the process fails.

15. *Contrast of Methods* II., III. Up to the limit of p and $p^\kappa \not> 1000$, the Canon Arithmeticus gives *all* the results required by the formulæ of Art. 11 [*i.e.* all the roots of all the Congruences (8)] with so little trouble that Method II. is to be preferred (as the Binary Canon gives only one root of each of two Reciprocal Congruences).

When, however p or $p^\kappa > 1000$, the use of the powers of 2—(by Method III.)—has considerable advantages (when the Haupt-Exponent (ξ) of 2 is $\xi=\omega$ or 2ω), viz.

(1) A primitive root (y) is not needed.

(2) The solution of $g^x \equiv \pm2$ is unnecessary.

(3) The value of x is given *explicitly* by the formulæ of Art. 14.

(4) The final reduction of y as the Residue of 2^x is usually far easier than that of g^x:—[g is often an inconvenient base].

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* Only 5 cases of $p<1000$, viz. $p=113$, 257, 353, 577, 593. All Fermat's Primes $F_n=(2^{2^n}+1)$, >17 fall under this class.

16. *Tables available for Method* III. This Method (by use of Residues of 2^x) is greatly facilitated by the use of suitable Tables:

The following data are required:—

(1) Haupt-Exponents (ξ) of 2, modulo p.

(2) Solutions (η) of $\eta^2+1\equiv 0$ (mod p).

(3) Solutions (ζ) of $\zeta^4+1\equiv 0$ (mod p).

It will be useful to show how far Tables are now available (or likely to be available shortly) for the above.

16*a*. *Haupt-Exponents* (ξ) *of* 2.

The values of these (ξ) have been computed* for all primes p up to the limit $p \not> 100000$, and are being published in a series of Papers in the *Quarterly Journal of Pure and Applied Mathematics* in the form of Tables of the values—(not of ξ, but)—of ν the reciprocal of ξ; *i.e.* of $\nu=(p-1)\div\xi$, whence the value of ξ can be at once deduced as $\xi=(p-1)\div\nu$.

See Journal quoted.	Vol. XXXVII.,	1905, p. 142;	up to $p \not> 10^4$.	
	Vol. XLII.,	1911, pp. 248, 249;	$p>10^4$ up to 3.10^4.	
	Vol. XLIV.,	1912, pp. 45—47;	$p>3.10^4$ up to 6.10^4.	
	Vol. XLIV.,	1913, pp. 240, 241;	$p>6.10^4$ up to 8.10^4.	
	Vol. XLV.,	(to appear shortly)	$p>8.10^4$ up to 10^5.	

16*b*. *Solutions* (η) *of* $\eta^2+1\equiv 0$ (mod p).

These can be obtained from the 2^{ic} partition $p=(a^2+b^2)$ by reduction of the formulæ—

$$\eta\equiv\pm(a\pm mp)/b, \text{ or} \equiv\mp(b\pm mp)/a, \text{ (mod } p)\ldots\ldots\ldots\ldots(34),$$

where the value of m is to be determined so that η may be an integer.

A Table of the values of (a, b) for all primes $p=4\varpi+1 \not> 10^5$ is given in the author's Tables of *Quadratic Partitions*.†

A Table of the actual roots (η) of $\eta^2+1\equiv 0$ (mod p and p^κ) up to p and $p^\kappa \not> 10^5$ has been prepared by the author, and is in course of publication.

16*c*. *Solutions* (ζ) *of* $\zeta^4+1\equiv 0$ (mp).

These can be obtained from the 2^{ic} partitions $p=a^2+b^2=c^2+2d^2$ by the reduction of either of the following formulæ (in which the sets of $\pm$ signs, being independent, give four roots), viz.

$$\zeta\equiv\pm\frac{d\pm mp}{c}(\eta\pm 1), \text{ or} \equiv\mp\frac{c\pm mp}{2d}.(\eta\pm 1), \equiv 0, \text{ (mod } p)\ldots\ldots(35),$$

where η is a root of $\eta^2+1\equiv 0$, (mod p), ..., [see Art. 16*b*],

and the value of m is to be determined so that ζ may be an integer.

A Table of the values of (a, b), (c, d) for all primes $p=8\varpi+1 \not> 10^5$ is given in the author's Table of *Quadratic Partitions*.†

A Table of the actual roots (ζ) of $\zeta^4+1\equiv 0$ (mod p and p^κ) up to p and $p^\kappa \not> 5.10^4$ has been prepared by the author and is in course of publication.

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* By the present author and Mr. H. J. Woodall, A.R.C.Sc., in collaboration.

† Published by Fr. Hodgson, London, 1904.

16d. *Binary Canon Extension.*

This is a Table* showing the Residues (both $\pm R$) of 2^x up to $x=100$ for all p and $p^\kappa \not> 10^4$; and up to $x=36$ for all p and $p^\kappa \not> 12000$. Thus this gives at sight the Residues y of 2^x required by the formulæ of Art. 14 up to $p \not> 10^4$.

17. *Allied-Forms* (N, $\mathbf{N}$). Take a new set of forms

$$\mathbf{N}_i = 8Y_i^4 - 1,\ \ \mathbf{N}_{ii} = 8Y_{ii}^4 + 1;\ \ \mathbf{N}_{iii} = Y_{iii}^4 - 8,\ \ \mathbf{N}_{iv} = Y_{iv}^4 + 8 \ldots\ldots (36),$$

and let their bases Y be connected with the bases (y) of the other set of forms (N) by the relations

$$y_i = 2Y_i,\ \ y_{ii} = 2Y_{ii};\ \ Y_{iii} = 2y_{iii},\ \ Y_{iv} = 2y_{iv} \ldots\ldots\ldots\ldots\ldots\ldots (37).$$

Hereby the two sets (N, $\mathbf{N}$) are connected thus

$$N_i = 2\mathbf{N}_i,\ \ N_{ii} = 2\mathbf{N}_{ii};\ \ \mathbf{N}_{iii} = 2\mathbf{N}_{iii},\ \ \mathbf{N}_{iv} = 2N_{iv} \ldots\ldots\ldots\ldots (38).$$

Hence the solutions (y) of the four Congruences (8) of Art. 9 suffice to give also the solutions (Y) of the four new Congruences

$$\mathbf{N}_i = 0,\ \ \mathbf{N}_{ii} \equiv 0,\ \ \mathbf{N}_{iii} = 0,\ \ \mathbf{N}_{iv} \equiv 0 \ (\text{mod } p \text{ or } p^\kappa) \ldots\ldots\ldots\ldots (39),$$

by the simple relation (37); so that the Tables (I., II.) at end of this Paper of solutions (y) of the Congruences (8) can be easily used as a Table of solutions (Y) of the allied Congruences (37).

Also by the relations (38) the factorisation of either set (N or $\mathbf{N}$) suffices to give that of other set ($\mathbf{N}$ or N).

18. The rest of this Paper deals chiefly (Art. 19—27) with various properties of the four numbers of type N, as follows :

Common factors,	Art. 19	Form $Y^\mu \mp 1$,	Art. 24
Square forms,	,, 20	Trinomial forms,	,, 25
Dimorphism,	,, 21	Isomorph Products,	,, 26
Equality,	,, 21*a*	Problem,	,, 27
Dimorph sums,	,, 22	Octavan forms,	,, 28
Factorisable sums,	,, 23	General forms,	,, 29

and ends with explanation (Art. 30) of the various Factorisation-Tables at end of the Paper.

19. *Common Factors.* Since $\alpha < \xi$, and $\beta < \xi$, and also $\alpha \neq \beta$ (in Art. 11), it follows from (12) that no two of x_i, x_{ii}, x_{iii}, x_{iv}, can be equal for the same p, and therefore also—

No two of y_i, y_{ii}, y_{iii}, y_{iv}, can be equal for the same p.........(40*a*).

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* Prepared by the present author and Mr. H. J. Woodall, A.R.C.Sc., in collaboration. It is at present only in MS.

Hence also—

No two of N_i, N_{ii}, N_{iii}, N_{iv}, formed with the same y, can contain any common factor >3 ....(40*b*).

20. *Square Forms.* Since $y^4 \sim z^2$ cannot $=2$, it is clear that

N_i and N_{ii} cannot $=\square$ ....(41*a*).

Next, let (τ_r', v_r'), and (τ_r, v_r) be the r^{th} of the successive solutions of

$$\tau'^2-2v'^2=-1,\quad \tau^2-2v^2=+1\ ;\ [r=1, 2, 3, \&c.].$$

Then $N_{iii}=2y_{iii}^4-1=\tau'^2$ requires $\tau'^2-2(y_{iii}^2)^2=-1$,

so that (τ', y_{iii}^2) must be a solution of $\tau'^2-2v'^2=-1$, $[v_r'=y_{iii}^2]$.

Here $\tau'=239$, $v'=169=13^2$ gives the only known solution, viz.

$y_{iii}=13$, $N_{iii}=2.13^4-1=239^2$.............. ....(41*b*).

If any other solution exist, it must be in very high numbers, $[r>39$ giving $y>10^{14}]$.

Also $N_{iv}=2y_{iv}^4-1$ requires $\tau^2-2(y_{iv}^2)^2=+1$, so that

(τ, y_{iv}^2) must be a solution of $\tau^2-2y^2=+1$, $[v_r=y_{iv}^2]$.

But it is known that

$$v_{2r-1}=2\tau_r'v_r' \text{ always, and } v_{2r}=2\tau_r v_r \text{ always.}$$

Now τ_r', v_r' are always *odd*, and τ_r, v_r are always *mutually prime*, so that neither v_{2r-1}, v_{2r} can $=\square$; hence no v_r can $=\square$. Thus, finally

N_{iv} cannot $=\square$.................................. (41*c*).

It may be noted also that

$N_i+N_{ii}=y_i^4+y_{ii}^4$, $\frac{1}{2}(N_{iii}+N_{iv})=y_{iii}^4+y_{iv}^4$................(42*a*),

$N_i\sim 2N_{iii}=y_i^4\sim 4y_{iii}^4$, $N_{ii}\sim 2N_{iv}=y_{ii}^4\sim 4y_{iv}^4$ ....(42*b*),

$N_i+2N_{iv}=y_i^4+4y_{iv}^4$, $N_{ii}+2N_{iii}=y_{ii}^4+4y_{iii}^4$ ....(42*c*).

As it is known that none of the six dexters of Results can be square forms, it follows that

None of the six sinisters of (42*a*, *b*, *c*) case $=\square$ ....(42).

21. *Dimorphism impossible.* It is easily seen that

$N_i=N_i'$, $N_{ii}=N_{ii}'$, $N_{iii}=N_{iii}'$, $N_{iv}=N_{iv}'$... ....(43),

are *impossible* (except with $y=y'$ in each case).

21*a*. *Equality of different types.* It is easily seen that

$$\left.\begin{array}{l} N_i = N_{ii} \text{ involves } y_i^4 - y_{ii}^4 = +4 \\ N_{iii} = N_{iv} \text{ involves } y_{iii}^4 - y_{iv}^4 = +1 \\ N_i = N_{iii} \text{ involves } y_i^4 - 2y_{iii}^4 = +1 \\ N_{ii} = N_{iv} \text{ involves } y_{ii}^4 - 2y_{iv}^4 = -1 \\ N_i - N_{iv} \text{ involves } y_i^4 - 2y_{iv}^4 - +3 \\ N_{ii} - N_{iii} \text{ involves } y_{ii} - 2y_{iii}^4 = -3 \end{array}\right\} \text{which are all impossible...(44).}$$

Hence it follows that—

The same number N cannot be expressed in two different types..(44*a*).

22. *Dimorph Sums of N.* Let a_r, b_r, c_r, d_r denote the roots (y) of four (different) numbers A_r, B_r, C_r, D_r of same type (N_r), [$r = i$, ii, iii, iv.]

Then

$$a_r^4 + b_r^4 = c_r^4 + d_r^4 \text{ involve } A_r + B_r = C_r + D_r \;[r = i,\ ii,\ iii,\ iv] \text{....(45}a\text{)},$$

$$a_i^4 + b_{ii}^4 = c_{iii}^4 + d_{iv}^4 \text{ involve } A_i + B_{ii} = \tfrac{1}{2}(C_{iii} + D_{iv}) \text{.................(45}b\text{)},$$

so that every solution of $a^4 + b^4 = c^4 + d^4$ in integers gives a solution of (45*a*, *b*).

[This equation was solved by Euler, see *Comment-Arithm.*, Vol. I., pp. 473–476, &c. A Table of the solutions was given in the author's Paper on *Diophantine Factorisation of Quartans* in the *Messenger of Mathematics*, Vol. XXXVIII., 1908, p. 86. The lowest solution known is $134^4 + 133^4 = 59^4 + 158^4$.]

23. *Factorisable Sums of N.* Since, for the same modulus ($p = 8\varpi + 1$)—see Art. 13, Result (22),

$$y_i^4 + y_{ii}^4 \equiv 0, \text{ and } y_{iii}^4 + y_{iv}^4 \equiv 0 \pmod p, \textit{ always.}$$

These give at once the pairs (N_i, N_{ii}), (N_{iii}, N_{iv}), such that

$$N_i + N_{ii} \equiv 0, \text{ and } N_{iii} + N_{iv} \equiv 0 \pmod p \text{.....................(46).}$$

And, since each such prime p has four roots (y) of each kind (y_i, y_{ii}, y_{iii}, y_{iv}), all satisfying the congruences (8), and all $<p$, it follows that 16 different solutions of each of the congruences (46) exist for each such prime (p) with every root $y < p$.

[*Cor.* The Tables (I., II.) of Solutions of the four congruences (8), given at end of this Paper, supply 32 solutions (a, b) of the congruence

$$a^4 + b^4 \equiv 0 \pmod{p = 8\varpi + 1}, \;[a \,\&\, b < p],$$

for all primes $p = 8\varpi + 1 < 1000$ with $(2/p)_4 = 1$: thus this Table suffices to show factors ($\not> 1000$) of the Quartans $N = a^4 + b^4$; it is, however, not exhaustive up to that limit as it includes only primes (p) such that $(2/p)_4 = +1$.]

24. *Form* N *or* $\frac{1}{2}N=(Y^\mu\mp1)$.

Take $\quad y=2^\lambda.\eta^\mu$, with μ *odd* ....(47*a*).

And, take $\quad \lambda=\frac{1}{4}(k\mu\mp1)$, so that $4\lambda\pm1=k\mu$.....(47*b*),

where k is determined so that λ may be an integer (always possible).

Write $Y=2^k.\eta^4$, and note that $y^4=2^{k\mu\mp1}.\eta^\mu$ ....(47*c*).

I. Let $\lambda=\frac{1}{4}(k\mu+1)$; then—

i. $N_i=y^4-2=2\{(2^k\eta^4)^\mu-1\}$; $\frac{1}{2}N_i=Y^\mu-1$..............(48*a*),

ii. $N_{ii}=y^4+2=2\{(2^k\eta^4)^\mu+1\}$; $\frac{1}{2}N_{ii}=Y^\mu+1$..............(48*b*).

II. Let $\lambda=\frac{1}{4}(k\mu-1)$; then—

iii. $N_{iii}=2(2^{k\mu-1}.\eta^{4\mu})-1=(2^k\eta^4)^\mu-1=Y^\mu-1$..............(48*c*),

iv. $N_{iv}=2(2^{k\mu-1}.\eta^{4\mu})+1=(2^k\eta^4)^\mu+1=Y^\mu+1$..............(48*d*).

Thus it has been shown that

$\frac{1}{2}N$ or $N=(Y^\mu\mp1)$, whenever $y=2^{\frac{1}{4}(k\mu\mp1)}.\eta^\mu$................(49).

And, since μ is *odd*, each of the above forms of N is *composite*, and has $(Y\mp1)$ as an algebraic divisor: and, if μ be a product of odd primes, then $p(Y^\mu\mp1)$ will have several such algebraic divisors.

This form $\frac{1}{2}N$ or $N=(Y^\mu-1)$ is of some importance as regards factorisability, as the procedure for its factorisation is well known. Unfortunately the values of y increase rapidly in magnitude as $\mu=3, 5, 7$, &c., increases.

Ex. Subjoined is a short Table showing the bases $y=2^\lambda\eta^\mu$ of the factorisable N arising from small values of $\mu=3, 5, 7, \ldots$, and the auxiliary bases $Y=2^k\eta^4$ useful in factorising N. It will be noted that

$Y>y$, when $\mu=3$; $Y<y$, when $\mu>3$..................(50).

N_{iii} and N_{iv}	μ, k, λ	3, 3, 2	3, 3, 2	3, 3, 2	3, 3, 2	3, 3, 2	
	η; y	3; 108	5; 500	7; 1372	9; 2916	11; 5324	
	Y	648	5000	$2^3.7^4$	$2^3.9^4$	$2^3.11^4$	
	μ, k, λ	3, 7, 5	3, 7, 5	3, 11, 8	5, 1, 1	5, 1, 1	
	η; y	3; 864	5; 4000	3; 6912	3; 486	5; 6250	
	Y	$2^7.3^4$	$2^7.5^4$	$2^{11}.3^4$	162	2.5^4	
N_i and N_{ii}	μ, k, λ	3, 1, 1	3, 1, 1	3, 1, 1	3, 1, 1	3, 1, 1	
	η; y	3; 54	5; 250	7; 686	9; 1458	11; 2662	
	Y	162	1250	4802	2.9^4	2.11^4	
	μ, k, λ	3, 5, 4	3, 5, 4	3, 5, 4	3, 9, 7	5, 3, 4	7, 1, 2
	η; y	3; 432	5; 2000	7; 5488	3; 3456	3; 3888	3; 8748
	Y	$2^5.3^4$	$2^5.5^4$	$2^5.7^4$	$2^9.3^4$	648	162

☞ See Tab. IV. (at end of this Paper) for the factorisation of several of these high numbers (N).

25. *Trinomial Forms.* By aid of the relations

$$2=3-1;\quad 2^3=3^2-1;\quad 2^3=7+1;$$

it is possible to express some of the numbers N or $\frac{1}{2}N$ in the trinomial forms

$$\tfrac{1}{2}N \text{ or } N=Y^{n'}\mp Y^{n}\mp 1,\quad \text{where } Y=3 \text{ or } 7,\ n=4m \ldots\ldots\ldots\ldots(50).$$

Thus—

$$y=3^m \text{ gives } N_{iii}=2.3^{4m}-1=3^{4m+1}-3^{4m}-1 \ldots\ldots\ldots\ldots(50a),$$

$$N_{iv}=2.3^{4m}+1=3^{4m+1}-3^{4m}+1 \ldots\ldots\ldots\ldots(50b),$$

$$y=2.3^m \text{ gives } \tfrac{1}{2}N_i=8.3^{4m}-1=3^{4m+2}-3^{4m}-1 \ldots\ldots\ldots\ldots(50c),$$

$$\tfrac{1}{2}N_{ii}=8.3^{4m}+1=3^{4m+2}-3^{4m}+1 \ldots\ldots\ldots\ldots(50d),$$

$$y=2.7^m \text{ gives } \tfrac{1}{2}N_i=8.7^{4m}-1=7^{4m+1}+7^{4m}-1 \ldots\ldots\ldots\ldots(50e).$$

$$\tfrac{1}{2}N_{ii}=8.7^{4m}+1=7^{4m+1}+7^{4m}+1 \ldots\ldots\ldots\ldots(50f).$$

25a. *Use of Canons of Residues of* Y^x.

The author has compiled extensive Tables (at present in MS.) showing the Residues (say $\pm R$) of Y^x moduli p and p^κ for the small bases $Y=3, 5, 7, 11$ for the range of powers $x=1$ to 24, and of moduli p and $p^\kappa = \ngtr 10000$.

With these Tables it is easy to pick out at sight the divisors p and $p^\kappa \ngtr 10^4$ of $N=(y^4\mp 2)$, where $y=Y^m$. And, it is easy also in the case of $N=(2y^4\mp 1)$, because

$$N=2y^4\mp 1=Y^{4m}-(-Y^{4m})\mp 1,$$

and the Tables give side by side the values of R, and $(p-R)$ or $(p^\kappa-R)$, when R is the + Residue of Y^{4m} (mod p or p^κ).

The divisors of the trinomial forms of Art. 25 can also be picked out at sight from these Tables.

26. *Isomorph Product.* The question arises whether the product N_r of two numbers L_r, M_r of the same type as N_r, or the product of their halves $\frac{1}{2}L_r$, $\frac{1}{2}M_r$, can be a number *isomorph* (*i.e.* of same type) *with them*, *i.e.*

$$\text{Can } L_r.M_r=N_r \text{ or } \tfrac{1}{2}L_r.\tfrac{1}{2}M_r=\tfrac{1}{2}N_r,\ [r=i, ii, iii, iv]? \ldots\ldots(51).$$

It does not seem easy to settle this question completely. It may, however, be shown to be *impossible* when L_r, M_r are both *prime* (if $M_r > L_r > 1$). This is a special case of the following Theorem, so that its proof is included therein.

26a. *Valroff's* Theorem.*

"If $(2x^2\pm 1)(2y^2\pm 1)=2z^2\pm 1$, [all signs +, or all −], then one of the factors is *always composite* (except when x or $y=1$, or $x=y$)." ....(52).

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* This Theorem (with the − signs only) was proposed as Question 339 in the *Journal Sphinx-Œdipe* for 1912, p. 60: this solution (by the present author) appears on pp. 78, 79 of the same volume.

The two cases with the signs all +, or all −, require separate treatment.

Case I. (with + signs). For shortness, write

$$L=2x^2+1,\ M=2y^2+1,\ N=2z^2+1,\ LM=N\ldots\ldots\ldots\ldots(53).$$

And, if possible, let L, M be *both prime*: in this case the above forms of L, M are *both unique*.

Here $$LM=2U^2+T^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(54),$$

where $$T=2xy\mp 1,\ \ U=x\pm y\ldots\ldots\ldots(54a).$$

And, it is at once seen that

$$T=\pm 1 \text{ is impossible, except when } xy=0, \text{ or } x=y=1\ldots\ldots\ldots(55).$$

This proves the Theorem for the + signs.

Case II. (with − signs). For shortness, write

$$L=2x^2-1,\ M=2y^2-1,\ N=2z^2-1,\ LM=N\ldots\ldots\ldots\ldots\ldots(55),$$

and, if possible, let L, M be *both prime*.

And, let (τ_1', v_1'), (τ_2', v_2'), ..., (τ_ρ', v_ρ') be the successive solutions of the "unit-form"

$$\tau'^2-2v'^2=-1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(56).$$

Now, since M is *prime*, it can be expressed *in only one way* in the infinite series of forms

$$M=t_1^2-2u_1^2=t_2^2-2u_2^2=\ldots=t_\rho^2-2u_\rho^2\ldots\ldots(57),$$

and each pair (t_ρ, u_ρ) can be expressed in terms of the original $(y, 1)$ by means of the members (τ_ρ', v_ρ') of the "unit-form" (56); thus

$$t_\rho=2v_\rho' y+j\tau_\rho',\ \ u_\rho=\tau_\rho' y+jv_\rho',\ [j=\pm 1]\ldots\ldots(58).$$

And, since L also is *prime*, the product $LM=(2x^2-1)$ (t^2-2u^2) consists of *only two infinite series* of product-forms of type

$$LM=2U_\rho^2-T_\rho^2,\ [\rho=1, 2, 3, \ldots]\ldots\ldots\ldots\ldots(59),$$

where T_ρ, U_ρ are given by

$$T_\rho=2u_\rho x+Jt_\rho,\ \ U_\rho=t_\rho x+Ju_\rho,\ [J=\pm 1]\ldots\ldots(60).$$

And the question is finally whether it is possible that

$$T_\rho=2u_\rho x+Jt_\rho=\pm 1, \text{ for some value of } \rho\ldots\ldots(61).$$

This gives $$x=-(Jt_\rho\mp 1)\div 2u_\rho\ldots\ldots\ldots\ldots\ldots\ldots(61a).$$

Now t_ρ is always $<2u_\rho$ when $\rho>1$, so that $x\neq$ integer if $\rho>1$: but t_ρ is always $>2u_\rho$ when $\rho=1$.

Now $\rho=1$ gives $\tau_1'=1$, $v_1'=1$, $t_1=2y+j$, $u_1=y+j$, by (58);

whence $$x=-\frac{J(2y+j)\mp 1}{2(y+j)} \quad\text{......................(61}b\text{)},$$

the only integral values of which are

$x=1$, y arbitrary, which involve $z=y$, $L=1$, $M=N$.

$x=2$, $y=2$, which involve $z=5$, $L=M=7$, $N=49$.

This proves the Theorem for the $-$ sign.

Note that the above proof depends essentially on the forms $(2x^2\pm 1)$, $(2y^2\pm 1)$ of L, M being *unique*: for if either of them were expressible in some other form, say $M=(2w^2\pm v^2)$, different from—(*i.e.* not equivalent to)—the original form $(2y^2\pm 1)$, then the T, U of the product-form of L, M would be different from those used above, and the equation $T=\pm 1$ would become *possible*. An example of each Case will suffice to show this.

I $(2.3^2+1)(2.5^2+1)=19.51=969=2.22^2+1$.

Here $51=3.19=2.5^2+1=7^2+2.1^2$; and the result (2.22^2+1) arises from the conformal* multiplication of $(2.3^2+1)(7^2+2.1^2)$.

II. $(2.2^2-1)(2.14^2-1)=7.391=2737=2.37^2-1$.

Here $391=17.23=2.14^2-1=2.16^2-11^2=21^2-2.5^2$; and the result (2.37^2-1) arises from the conformal* multiplication of (2.2^2-1) by $(21^2-2.5^2)$; this latter form is *not equivalent* to (2.14^2-1).

Note further that Valroff's Theorem is true only when L, M, N are *all three of same type*: thus, if L, M are of same type and both prime, they may yield a product-form N of the reciprocal type. An example of each Case (I., II.) will suffice to show this—

I. $(1^2+2)(3^2+2)=3.11=33=2.4^2+1$; [3, 11 are primes].

II. $(3^2-2)(5^2-2)=7.23=161=2.3^4-1$; [7, 23 are primes].

27. *Problem.* In modification of the Question of Art. 26, the following simpler Problem may be proposed.

Write $L_1=2x_1^2\mp 1$, $L_2=2x_2^2\mp 1$, $L_3=2x_3^2\mp 1$, &c...............(62),

and $N=2y^4\mp 1$..(62a),

where L_1, L_2, L_3 are all quadratic functions, and N a quartic.

The question is—

Can $N=L_1L_2L_3$, ..., [like signs throughout]?. ....(63).

When there are only two factors L_1, L_2, then Valroff's Theorem (Art. 26a) shows that

$N=L_1L_2$ requires one (or both) of L_1, L_2 *composite*.........(64).

No examples have, however, been found.

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* *Conformal multiplication* means multiplication with preservation of (quadratic) form.

When there are more than two factors (L_1, L_2, &c.) the problem is certainly possible, as the following examples show (though no general Rule has been found).

$$2.32^4-1=49.127.337=(2.5^2-1)\,(2.8^2-1)\,(2.13^2-1),$$
$$2.15^4+1=19.73.73\quad=2\,(3^2+1)\,(2.6^2+1)\,(2.6^2+1).$$

28. *Octavan Forms.* Consider the numbers

$$N_i=y^8-2,\ \ N_{ii}=y^8+2,\ \ N_{iii}=2y^8-1,\ \ N_{iv}=2y^8+1.$$

These, being only a special form of the 4-tan numbers (N_i, ..., N_{iv}) of Art. 1, wherein $y=Y^2$, are subject (*mutatis mutandis*) to all the general Rules of those numbers.

The chief modifications are—

In Results 6, 6*a*, *b*, 7*c*, &c., and elsewhere, change $(2/p)_4=1$ into $(2/p)_8=1$.

In (12), change $4x$ into $8x$ and $\frac{1}{4}$ into $\frac{1}{8}$.

In (14*b*, *d*), change 4 values (or roots) into 8 values (or roots).

In (15*b*), the 8 exponents (x, x', &c.) are found by repeated addition of $\frac{1}{8}\xi$ instead of $\frac{1}{4}\xi$.

&c. &c. &c.

But, for practical factorisation of these 8-van forms, it often suffices—(so long as y is small)—to convert them into the 4-tan forms by writing $y=Y^2$, upon which the congruence-solutions (y) of the 4-tan congruences (Art. 9) can be used.

29. *General Forms.* The numbers (N) above discussed have all been 4-tic forms of determinant ± 2. By a quite similar procedure the factorisation of the set of 4-tic forms (N)

$$N_i=y^4-q,\ \ N_{ii}=y^4+q,\ \ N_{iii}=qy^4-1,\ \ N_{iv}=qy^4+1,$$

with determinant $\pm q$, may be effected.

The prime divisors (p) must be of the same linear and 2^{ic} forms as those of ($Y^2\mp q$), ($qY^2\mp 1$), the forms of which have been discussed by Legendre, with the condition—

When $p=4\varpi+1$; then $(q/p)_4$ must $=+1$ for N_i & N_{iii},

$(\bar{q}/p)_4$ must $=+1$ for N_{ii} & N_{iv}.

Ex. When $q=3$; the forms of the prime divisors (p) are

	N_i and N_{iii}		N_{ii} and N_{iv}	
2^{ic}	$3u^2-t^2$	t^2-3u^2		t^2+3u^2
Linear	$12\varpi+11$	$12\varpi+1$ & $(3/p)_4=+1$	$12\varpi+7$	$12\varpi+1$ & $(3/p)_4=+1$

30. *Factorisation-Tables.* Four Tables—(III.—VI.)—of the factorisation of these numbers (N) are presented at end of this Paper. In all these Tables the following signs are used.

(1) A semi-colon (;) on right shows *complete* factorisation (into prime factors).

(2) A full-point (.) on right shows that there are other (undetermined) factors.

(3) A semi-colon (;) in middle separates algebraic factors.
[These occur only in the case of $N=(Y^m-1)$ [m odd] of Art. 24.]

(4) The signs †, ‡, §, ¶ show the limits (as stated below) up to which the search for factors has been pushed, with the aid of various MS. Tables in the author's possession. These often suffice to determine High Prime factors ($>10^7$).

† up to 1000; ‡ up to 10000; § up to 32000; ¶ up to 50000.

[The author's acknowledgments are due to Mr. L. Valroff for 29 of the factorisations of $N_{ii}=(2y^4-1)$, including 19 High Primes ($>10^7$) marked V in the Tables].

30*a*. *Table III.* ($y \not> 100$).

This gives the factorisation of the numbers of the four kinds (N_i to N_{iv}) up to the limit $y \not> 100$. The factorisation is *complete* (into prime factors) up to the following limits

$y=66$ for N_i; $y=62$ for N_{ii}; $y=50$ for N_{iii}; $y=62$ for N_{iv}.

N_i, N_{iii} are so closely related that they are placed together; and N_{ii}, N_{iv} are so closely related that they are placed together. The search for factors (p) has been pushed in all cases up to $p \not> 1000$, and in a few cases wherein $y=2^\lambda.\eta^\mu$ [$\eta=3, 5, 7, 11$], much further, viz.

	N_i	N_{ii}	N_{iii}	N_{iv}
$y=$	80, 88;	98;	48, 50, 56;	48, 96.

Only two cases of the kind $N=(Y^m\mp1)$ of Art. 24 occur in this Table, viz. of $N_i=(54^4-2)$, $N_{ii}=(54^4+2)$.

30*b*. *Table IV.*

This Table gives the factorisation of selected numbers of the four kinds N_i to N_{iv}) in which $y=2^\lambda.\eta^\mu$, [$\eta=3, 5, 7, 11$] from $y>100$ up to 1000.

In this Table several cases occur of the kind $N=(Y^m\mp1)$ of Art. 24 as follows:—

	N_i and N_{ii}	N_{iii} and N_{iv}
$y=$	250, 432, 686;	108, 486, 500, 864.

The search for factors (p) has been pushed in most cases up to $p \not> 10000$, and in some cases further: it is thought worth while recording the results, although in many cases very incomplete.

30*c*. *Table V.*

This Table gives the factorisation of a few selected cases of high numbers (N) with $y>10^3$ but $<10^4$: in most of which N is of the kind $N=(Y^m\mp1)$ of Art. 24, which admits of factorisation to very high limits.

30*d*. *Table VI.*

This Table gives the factorisation of the four numbers $N=(y^8\mp2)$, $(2y^8\mp1)$ up to $y=32$ inclusive.

Congruence-Solutions (y). TAB. I.

$y^4-2\equiv0$ & $2y^4-1\equiv0$ (mod $p=8\varpi+7$). $y^4+2\equiv0$ & $2y^4+1\equiv0$ (mod $p=8\varpi+3$).

p	y^4-2 y	y	$2y^4-1$ y	y
7	2,	5	3,	4
23	8,	15	3,	20
31	15,	16	2,	29
47	17,	30	11,	36
71	15,	56	19,	52
79	3,	76	26,	53
103	48,	55	15,	88
127	4,	123	32,	95
151	16,	135	66,	85
167	34,	133	54,	113
191	33,	158	81,	110
199	47,	152	72,	127
223	98,	125	66,	157
239	92,	147	13,	226
263	68,	195	58,	205
271	63,	208	43,	228
311	114,	197	30,	281
359	78,	281	23,	326
367	153,	214	12,	355
383	157,	226	161,	222
431	107,	324	145,	286
439	36,	403	61,	378
463	156,	307	92,	371
479	115,	364	25,	454
487	86,	401	17,	470
503	58,	445	26,	477
599	222,	377	143,	456
607	297,	310	280,	327
631	57,	574	155,	476
647	6,	641	108,	539
719	18,	701	40,	679
727	235,	492	99,	628
743	66,	677	349,	394
751	142,	609	238,	513
823	221,	602	108,	715
839	395,	444	274,	565
863	227,	636	422,	441
887	33,	854	215,	672
911	120,	791	372,	539
919	408,	511	232,	687
967	368,	599	360,	607
983	131,	852	15,	968
991	301,	690	214,	777
49	23,	26	17,	32
343	121,	222	17,	326
529	146,	383	250,	279
961	356,	605	467,	494

p	y^4+2 y	y	$2y^4+1$ y	y
3	1,	2	1,	2
11	5,	6	2,	9
19	5,	14	4,	15
43	4,	39	11,	32
59	6,	53	10,	49
67	28,	39	12,	55
83	3,	80	28,	55
107	41,	66	47,	60
131	40,	91	36,	95
139	28,	111	5,	134
163	54,	109	3,	160
179	82,	97	24,	155
211	84,	127	103,	108
227	62,	165	11,	216
251	51,	200	64,	187
283	134,	149	19,	264
307	97,	210	19,	288
331	16,	315	62,	269
347	169,	178	154,	193
379	96,	283	75,	304
419	177,	242	116,	303
443	181,	262	93,	350
467	47,	420	159,	308
491	224,	267	217,	274
499	145,	354	117,	382
523	102,	421	241,	282
547	249,	298	134,	413
563	167,	396	118,	445
571	62,	509	175,	396
587	67,	520	184,	403
619	258,	361	12,	607
643	163,	480	71,	572
659	228,	431	211,	448
683	8,	675	256,	427
691	156,	535	31,	660
739	100,	639	303,	436
787	264,	523	158,	629
811	402,	409	232,	579
827	201,	626	144,	683
859	220,	639	82,	777
883	29,	854	274,	609
907	153,	754	83,	824
947	392,	555	244,	703
971	241,	730	278,	693
9	2,	7	4,	5
27	7,	20	4,	23
81	34,	47	31,	50
121	49,	72	42,	79
243	47,	196	31,	212
361	81,	280	156,	205
729	196,	533	212,	517

Congruence-Solutions (*moduli* $p=8\varpi+1\ \ (2/p)_4=+1$).

TAB. II.

$y^4-2\equiv 0$ y y y y	$2y^4-1\equiv 0$ y y y y	p	$y^4+2\equiv 0$ y y y y	$2y^4+1\equiv 0$ y y y y
18, 25, 48, 55	4, 35, 38, 69	73	31, 34, 39, 42	15, 33, 40, 58
5, 8, 81, 84	11, 18, 71, 78	89	7, 29, 60, 82	38, 43, 46, 51
27, 47, 66, 86	12, 46, 67, 101	113	34, 55, 58, 79	10, 37, 76, 103
28, 71, 162, 205	25, 105, 128, 208	233	80, 103, 133, 153	67, 95, 138, 166
35, 46, 211, 222	22, 95, 162, 235	257	73, 117, 140, 184	88, 123, 134, 169
46, 91, 190, 235	55, 105, 176, 226	281	50, 121, 160, 231	72, 118, 163, 209
31, 158, 179, 206	18, 32, 305, 319	337	14, 50, 287, 323	24, 155, 182, 313
48, 102, 251, 305	45, 125, 228, 308	353	80, 170, 183, 273	27, 75, 278, 326
52, 278, 299, 325	110, 245, 332, 467	577	135, 222, 355, 442	13, 265, 312, 564
04, 294, 299, 489	119, 268, 325, 474	593	149, 206, 387, 444	95, 199, 394, 498
45, 216, 385, 556	64, 187, 414, 537	601	123, 251, 350, 478	170, 215, 386, 431
23, 201, 416, 494	132, 306, 311, 485	617	174, 179, 438, 443	39, 162, 455, 578
51, 355, 526, 830	190, 407, 474, 691	881	217, 284, 597, 664	152, 203, 678, 729
9, 110, 827, 928	104, 230, 707, 833	937	126, 334, 603, 811	409, 418, 519, 528

Factorisation of $N=(y^4\mp 2)$ & $(2y^4\mp 1)$. TAB. V.

$N_i=(y^4-2)$		y	$N_{ii}=(y^4+2)$	
257.8713.1401559 ;		1331	3.83.6163.2045129 ;	
2.13121 ; 7.1609.15289 ;		1458	2.11.1193 ; 19.43.210739 ;	
2.7.2857 ; 31.12903871 ;	‡	2000	2.3.59.113 ; 3.19.937.7489 ;	
233.	‡	2401	9.4243.	‡
2.7.47.89 ; 857464807 ;	§	2662	2.43.227 ; 3.73 163.24029 ;	
2087.	‡	3125	81.179.	‡
2.113.367 ; 49.31.73.15511 ;		3456	2.67.619 ; 1719.885313 ;	
2.647 ; 281.628044881 ;	§	3888	2.11.59 ; 1601.109961081 ;	§
2.76831 ;	†	5488	2.9.8537 ; 3.	†
2.7.23 ; 7.71.113.323859367 ;	§	8748	2.163 ; 337.	¶

$N_{iii}=(2y^4-1)$		y	$N_{iv}=(2y^4+1)$	
23.31.12241.719177 ;		1331	3.107.233.2129.39419 ;	
19207 ; 368966473 ;	§	1372	3.19.337 ; 3.73 601.2803 ;	
.	‡	2401	3.107.257 ;	‡
73.719 ; 7.31.12696049 ;	‡	2916	52489 ; 2754937657 ;	¶
7.23	‡	3125	3.	‡
79999 ; 7.	†	4000	27.2963 ; 3.73.	†
117127 ; 49.	†	5324	3.39043 ; 3.19.	†
1249 ; 311.	†	6250	3^2.139 ; 121.691.	†
165887 ; 7.	‡	6912	19.8731 ; 43.	‡

Factorisation of $N=(y^4\mp2)$ & $(2y^4\mp1)$. TAB. IIIA.

$N_i=y^4-2$	$N_{iii}=2y^4-1$	y	$N_{ii}=y^4+2$	$N_{iv}=2y^4+1$
−1;	1;	1	3;	3;
2.7;	31;	2	2.3; 3;	3; 11;
79;	7; 23;	3	83;	163;
2.127;	7.73;	4	2.3; 43;	3; 3; 3.19;
7.89;	1249;	5	3.11.19;	9.139;
2.647;	2591;	6	2.11.59;	2593;
2399;	4801;	7	27.89;	3.1601;
2.23.89;	8191;	8	2.3; 683;	3; 2731;
7.937;	13121;	9	6563;	11.1193;
2.4999;	7.2857;	10	2.3.1667;	3.59.113;
14639;	7.47.89;	11	9.1627;	3.43.227;
2.7.1481;	113.367;	12	2.10369;	67.619;
28559;	239.239;	13	3.9521;	9.11.577;
2.19207;	76831;	14	2.3.19.337;	9.8537;
23.31.71;	103.983;	15	50627;	19.73.73;
2.7; 31; 151;	131071;	16	2.3; 11; 3.331;	3; 43691;
47.1777;	343.487;	17	3.11.2531;	3.55681;
2.73.719;	7.89.337;	18	2.52489;	209953;
7.18616;	71.3671;	19	3.43441;	3.283.307;
2.79999;	23.13913;	20	2.27.2963;	3.11.9697;
194479;	388961;	21	194483;	388963;
2.117127;	257.1823;	22	2.3.39043;	9.52057;
49.5711;	359.1559;	23	3.93281;	27.19.1091;
2.165887;	7.94793;	24	19.8731;	11.179.337;
73.5351;	7.233.479;	25	9.43403;	3.260417;
2.49.4663;	23.79.503;	26	2.3.76163;	3.304651;
113.4703;	1062881;	27	11.48313;	353.3011;
2.233.1319;	1229311;	28	2.3.11.67.139;	3.83.4937;
707279;	31.45631;	29	9.89.883;	3.471521;
2.7.47.1231;	311.5209;	30	2.405001;	1620001;
23.40153;	7.263863;	31	3.73.4217;	243.11.691;
2.524287;	7; 127; 7.337;	32	2.3; 174763;	3; 3.43; 5419;
7.191.887;	31.76511;	33	19.62417;	73.32491;
2.167.4001;	2672671;	34	2.81.73.113;	3.19.46889;
257.5839;	73.41113;	35	3.500209;	3.11.90947;
2.439.1913;	47.71743;	36	2.839809;	131.25643;
7.267737;	1201.3121;	37	3.624721;	3.113.11057;
2.23.45329;	7.73.8161;	38	2.9.11.10531;	3.89.15619;
2313439;	7.660983;	39	11.43.67.73;	617.7499;
2.7.182857;	719.7121;	40	2.3.131.3257;	9.73.7793;
2825759;	1231.4591;	41	3.107.8803;	9.627947;
2.1555847;	6223391;	42	2.73.21313;	121.19.2707;
3418799;	23.271.1097;	43	9.19.19993;	3.89.25609;
2.7.267721;	7496191;	44	2.3.624683;	3.2498731;
601.6823;	7.353.3319;	45	4100627;	8201251;
2.31.257.281;	7.113.11321;	46	2.3.746243;	3.11.89.3049;
7.31.113.199;	9759361;	47	243.43.467;	3.107.30403;
2.73.103.353;	10616831;	48	2.2654209;	1523.6971;
5764799;	23.501287;	49	3.121.15881;	9.59.21713;
2.3124999;	12499999;	50	2.3.11.281.337;	81; 154321;

Factorisation of $N=(y^4\mp2)$ & $(2y^4\mp1)$. TAB. IIIB.

$N_i=y^4-2$	$N_{iii}=2y^4-1$	y	$N_{ii}=y^4+2$	$N_{iv}=2y^4+1$
7.881.1007 ;	13530401 ; V	51	251.26953 ;	89.152027 ;
2.3655807 ;	7.71.29423 ;	52	2.9.19.21379 ;	3.4874411 ;
7890479 ;	7.79.28537 ;	53	3 59.44579 ;	3.11.19.25169 ;
2.7.23 ; 26407 ;	167.101833 ;	54	2.163 ; 26083 ;	43.395491 ;
73.103.1217 ;	281.65129 ;	55	3.113.26993 ;	3.67.83.1097 ;
2.71.69257 ;	1193.16487 ;	56	2.9.546361 ;	3.6556331 ;
631.16729 ;	2473.8537 ;	57		11.1919273 ;
2.7.503.1607 ;	47.263.1831 ;	58	2.3.113.16691 ;	27.73.11483 ;
	7.3462103 ;	59	3.1777.2273 ;	9.2692747 ;
2.6479999 ;	7.31.119447 ;	60	2.11.89.6619 ;	107.242243 ;
49.23.85999 ;	439.63079 ;	61	27.11.46619 ;	3.19.485819 ;
2.7388167 ;	29552671 ; V	62	2.3.19.227.571 ;	3.331.29761 ;
271.58129 ;	31505921 ; V	63		
2.47.178481 ;	31 ; 601.1801 ;	64	2.3 ; 2796203 ;	3 ; 11 ; 251.4051 ;
7.2550089 ;	35701249 ; V	65	9.59.33617 ;	3.
2.113.113.743 ;	49.23.151.223 ;	66	2.107.88667 ;	
	7.113.50951 ;	67	3.587.11443 ;	9.233.19219 ;
2.7.263.5807 ;	42762751 ; V	68	2.3.3563563 ;	9.11.431917 ;
	73.621017 ;	69		59.768377 ;
2.	48019999 ; V	70	2.9.1333889 ;	3.
233.109063 ;	89.571049 ;	71	3.11.19 40529 ;	3.643.26347 ;
2.49.274223 ;	23.199 11743 ;	72	2.121.111049 ;	19.281.10067 ;
	7.8113783 ;	73	3.257.36833 ;	3.
2.	7.8567593 ;	74	2.27.555307 ;	3.
49.645727 ;	63281249 ; V	75		11 43.353.379 ;
2.79.211153 ;	66724351 ; V	76	2.3.5560363 ;	9.113 65609 ;
23.31.47.1049 ;	73.963097 ;	77	3.	27.2603929 ;
2.31.359.1663 ;	89.831799 ;	78	2.	
7.5564297 ;	77900161 ; V	79	9.113.38299 ;	3.121.67.3203 ;
2.20479999 ;	7.1033.11329 ;	80	2.3.83.233.353 ;	3.19.1437193 ;
89.483671 ;	49.191.9199 ;	81	19.19.119243 ;	‡
2.7.79.40879 ;	90424351 ; V	82	2.3.11.43.89.179 ;	3.859.35089 ;
	47.2019503 ;	83	9.11.479377 ;	3.907.34883 ;
2.23.89.12161 ;	94574271 ; V	84	2.211.117979 ;	
	151.691399 ;	85	3.	27.3866713 ;
2.7.71.113.487 ;	4217.25943 ;	86	2.3.83.109841 ;	9.11.1105067 ;
	7.16368503 ; V	87		
2.1399.21433 ;	7.103.166351 ;	88	2.27.1110547 ;	3.73.257.2131 ;
7.8963177 ;	23.5455847 ;	89	3.	3.
2.	71.1848169 ;	90	2.19.43.40153 ;	11.
73.281.3343 ;	31.4424191 ;	91	3.131.174491 ;	3.19.2406139 ;
2.239.149873 ;	463.309457 ;	92	2.9.1987.2003 ;	3.
7.	149610401 ; V	93	11.6800473 ;	443.337721 ;
2.89.438623 ;	7.1783.12511 ;	94	2.3.11.1182953 ;	9.
	7.23.31.127.257 ;	95	3.67 405227 ;	9.131.233.593 ;
2.7.6066761 ;	169869311 ; V	96	2.89 379.1259 ;	169869313 ; §
89.994711 ;	2719.65119 ;	97	9.179 179.307 ;	3.11.43.124777 ;
2.73.223.2833 ;	184473631 ; V	98	2.3.15372803 ; ‡	3 2857 21523 ;
	727.264263 ;	99		19.
2.7.23.310559 ;	89.1447.1553 ;	100	2.3.19.739.1187 ;	3.66666667 ;

Factorisation of $N = (y^4 \mp 2) \& (2y^4 \mp 1)$. TAB. IV.

$N_i = (y^4 - 2)$	$N_{iii} = (2y^4 - 1)$	y	$N_{ii} = y^4 + 2$	$N_{iv} = (2y^4 + 1)$
2.31.2194337;	647; 7.73.823;	108	4513 15073;	11.59; 211.1987;
2.4591.17137;	23.13682777; ‡	112	2.3.59.73.6089;	81 11.353203;
223.1094801;	113.353.12241;	125	3.43.1892563;	3. ‡
2.113.2899823;	89.14727191; ‡	160	2.9.11.281.11779;	3.163.2680409;
2 233.1477999.	257.5359903;	162	2.67.5139907;	617.2232569;
2.23.20858969; ‡	281.6829271;	176	2.3.19.43.195739;	9. ‡
2 23.29542489; ‡	7. ‡	192	2.11.113.546643;	‡
2. ‡	7.3847.118831;	200	2.9.251.354139;	3.121.8815427;
2.601.1810967;	‡	216	2. ‡	89.113.227.1907;
2.191.1289.5113;	47.3719 28807;	224	2 3.491.854593;	3.19.89.992561;
2. ‡	7. ‡	242	2.3.19.59.419.1217;	3. ‡
7. ‡	‡	243	‡	19.1811.202667;
2.1249; 7.127.1759;	23.23.31.476401;	250	2.9.139; 3.73.7129	3.8273.314779;
2.5741.599591;	73. ‡	288	2. ‡	11.307.4074449;
2.49. ‡	‡	320	2.3. ‡	27. ‡
‡	31 3271.273001;	343	3. ‡	3.11.
2.7. ‡	257. ‡	352	2.3. ‡	3.5107.2004073;
2. ‡	3607.12056153; ‡	384	2. ‡	19.2267.1009601;
2 79. ‡	‡	392	2.3.227.947.18307;	9. ‡
2.2591; 7 960151;	31.5209.431369;	432	2.2593; 19.283.1249;	‡
2. ‡	79. ‡	448	2.9. ‡	3 659.40750729; ‡
2.73 2207.173137;	7.23; 693025471; ¶	486	2.113. †	163; 11.5531.11251;
2.47.1753.379289;	4999; 7.79.103.439;	500	2.3.11. ‡	3.1667; 3.8331667;
2.359. ‡	7. ‡	640	2.3. ‡	3.11. ‡
2 4801; 23064007; ‡	†	686	2.3.1601; 3.7684801;	3. †
2 89. ‡	7.239. ‡	704	2.9. ‡	3.4001.40928971; ‡
2.7. ‡	73.27823.342569;	768	2. ‡	121. ‡
2.7. ‡	71.71. ‡	800	2.3. ‡	3.1259. ‡
2.8713.31978439; ‡	7.1481; 107505793; ‡	864	2.11.43. ‡	10369; 1867.57571;
2. ‡	‡	896	2.3.121.2131.416593;	9. ‡
2.7.167. ‡	89.983.20071873; ‡	968	2.3. ‡	27.83.139.5637347;
2.2969. ‡	3833. ‡	1000	2.3. ‡	3.43.2347.6605827;

☞ For Table V. see p. 53.

Factorisation of $N=(y^8 \mp 2)$ & $(2y^4 \mp 1)$. TAB. VI.

y^8-2	$2y^8-1$	y	y^8+2	$2y^8+1$
;	1;	1	3;	3;
27;	7.73;	2	2.3; 43;	3; 3; 3.19;
37;	13121;	3	6563;	11.1193;
; 31; 151;	131071;	4	2.3; 11; 3.331;	3.43691;
351;	7.233.479;	5	9.43403;	3.260417;
39.1913;	47.71743;	6	2.839809;	131.25643;
4799;	23.501287;	7	3.121.15881;	9.59.21713;
7.178481;	31; 601.1801;	8	2.3; 2796203;	3; 11; 251.4051;
483671;	49.191.9199;	9	19.19.119243;	‡
23.310559;	89.1447.1553;	10	2.3.19.739.1187;	3.66666667; ‡
1.73.1223;	428717761; V	11	3.281.254281;	9. ‡
‡	7.122851913; V	12	2.6521.32969;	139.827.7481;
	1631461441; V	13	27.83.347.1049;	3.257.1307.1619;
‡	89.33173799; ‡	14	2 729.1012201;	3.11.89441761; ‡
†	†	15	11. †	†
147483647;	7; 23.89; 599479;	16	2.3; 715827883;	3; 3; 683; 67.20857;
†	223. †	17	3.59.227.173617;	3.19. †
431.1826311;	103. †	18	2.11. †	†
†	7.73.233.285287;	19	3 619.9145699;	3.11.337.3054323;
‡	73. ‡	20	2.3. ‡	9. ‡
†	233.863.376199;	21	67.257.281.7817;	19.43. †
‡	7079.15503849; ‡	22	2.9.131.23272211; ‡	3.43. ‡
39. †	7.31.503.14349II;	23	9. †	3. †
‡	431. ‡	24	2.107. ‡	‡
9.1433.63647;	†	25	3.523.97251683; ‡	27.11. †
†	7 257.8081.28729;	26	2.3.11. †	3. †
679.1364071;	71; ‡	27	‡	881. ‡
‡	‡	28	2.3 19. ‡	3. ‡
†	73.13759.996103;	29	3.11.19. †	81. †
†	7.23.26633.306023;	30	2.	11. †
	89.1289.	31	9.337.	3.281.
191; 79.121369;	13367.164511353;	32	2.3; 3; 2731; 22366891;	3; 83.8831418697;

NOTE ON THE SOLUTION OF THE DIFFERENTIAL EQUATION $r=f(t)$.

By *J. R. Wilton, M.A., B.Sc.*

I HAVE been unable to find any reference to solutions of the partial differential equation

$$\frac{\partial^2 z}{\partial x^2}=f\left(\frac{\partial^2 z}{\partial y^2}\right),$$

i.e., $$r=f(t)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

except for a few particular cases. The general solution, though complicated in appearance, is very simple to obtain, and it is difficult to believe that it can have escaped observation; but there is no reference to the solution in any book or memoir to which I have had access. I have not, however, seen Legendre's original memoir on the equation $f(r, s, t)=0$ in *L'Histoire de l'Académie des Sciences* for 1787.

Following Legendre's method, we differentiate equation (1) with regard to y and take q as the new dependent variable, obtaining

$$\frac{\partial^2 q}{\partial x^2}=f'\left(\frac{\partial q}{\partial y}\right)\frac{\partial^2 q}{\partial y^2}.$$

By Legendre's transformation (the principle of duality) this becomes

$$\frac{\partial^2 u}{\partial X^2}=f'(X)\frac{\partial^2 u}{\partial Y^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where $$u=sx+ty-q,\quad X=t,\quad Y=s\quad\ldots\ldots\ldots\ldots(3).$$

To solve equation (2), assume a solution of the form

$$u=\phi(X)\psi(Y).$$

Substituting this value of u in (2), we find

$$\phi''(X)\psi(Y)=f'(X)\phi(X)\psi''(Y),$$

whence, if we assume that $\psi(Y)=\begin{matrix}\cos\\ \sin\end{matrix}\,\mu Y$,

$$\phi''(X)+\mu^2 f'(X)\phi(X)=0\quad\ldots\ldots\ldots\ldots\ldots(4),$$

a differential equation for ϕ which has two linearly independent solutions, ϕ_1 and ϕ_2 say. Hence a particular solution of equation (2) may be expressed in the form

$$u = A\phi_1(X)\cos\mu Y + B\phi_2(X)\sin\mu Y,$$

and by the addition of particular integrals, we obtain the general solution

$$u = \int_a^b F_1(\mu)\,\phi_1(X)\cos\mu Y\,d\mu$$

$$+ \int_a^b F_2(\mu)\,\phi_2(X)\sin\mu Y\,d\mu \quad\text{.............}(5),$$

where F_1 and F_2 are arbitrary functions of μ; ϕ_1 and ϕ_2, of course, depend on μ; and a and b are any constants.

In most cases either of the two integrals in equation (5) will serve as the general solution of (2), but the two integrals are given in order to include those special cases in which equation (2) is soluble by Laplace's method, as extended by Legendre. In order to obtain the solution of (1) from this result, we use the relations

$$x = \frac{\partial u}{\partial Y}, \quad y = \frac{\partial u}{\partial X},$$

$$xY + yX - q$$

$$= u = \int_a^b F_1\phi_1\cos\mu Y\,d\mu + \int_a^b F_2\phi_2\sin\mu Y\,d\mu \ldots(6),$$

$$x = -\int_a^b \mu F_1\phi_1\sin\mu Y\,d\mu + \int_a^b \mu F_2\phi_2\cos\mu Y\,d\mu \ldots(7),$$

$$y = \int_a^b F_1\phi_1'\cos\mu Y\,d\mu + \int_a^b F_2\phi_2'\sin\mu Y\,d\mu \ldots(8).$$

Making use of equations (7) and (8), we see that we may re-write equation (6) in the form

$$q = \int_a^b F_1[(X\phi_1' - \phi_1)\cos\mu Y - Y\phi_1\mu\sin\mu Y]\,d\mu$$

$$+ \int_a^b F_2[(X\phi_2' - \phi_2)\sin\mu Y + Y\phi_2\mu\cos\mu Y]\,d\mu \ldots(9),$$

We have now to integrate this equation with regard to y, and thus to obtain z. In carrying out the analysis of this step, the work will be much simplified by making use of the relations

$$\left.\begin{aligned}
\frac{\partial x}{\partial X} &= -\int_a^b \mu\, F_1\phi_1' \sin\mu Y\, d\mu + \int_a^b \mu\, F_2\phi_2' \cos\mu Y\, d\mu \\
\frac{\partial x}{\partial Y} &= -\int_a^b \mu^2 F_1\phi_1 \cos\mu Y\, d\mu - \int_a^b \mu^2 F_2\phi_2 \sin\mu Y\, d\mu \\
\frac{\partial y}{\partial X} &= f'(X)\frac{\partial x}{\partial Y} \\
\frac{\partial y}{\partial Y} &= \frac{\partial x}{\partial X}
\end{aligned}\right\} \ldots(10),$$

where, in obtaining the third of equations (10), use has been made of (4). Using these relations, we find

$$\frac{\partial(x, y)}{\partial(X, Y)}\,\frac{\partial X}{\partial y} = -\frac{\partial x}{\partial Y},$$

$$\frac{\partial(x, y)}{\partial(X, Y)}\,\frac{\partial Y}{\partial y} = \frac{\partial x}{\partial X}.$$

And therefore

$$q\,\frac{\partial(x, y)}{\partial(X, Y)} = \frac{\partial z}{\partial y}\,\frac{\partial(x, y)}{\partial(X, Y)} = \frac{\partial x}{\partial X}\frac{\partial z}{\partial Y} - \frac{\partial x}{\partial Y}\frac{\partial z}{\partial X}.$$

In order to perform the integration, we assume

$$\frac{\partial z}{\partial X} = p\,\frac{\partial x}{\partial X} + q\,\frac{\partial y}{\partial X} \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

$$\frac{\partial z}{\partial Y} = p\,\frac{\partial x}{\partial Y} + q\,\frac{\partial y}{\partial Y} \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where p is a function of X and Y which is plainly equal to $\partial z/\partial x$, but is as yet undetermined.

From equations (11) and (12) we see, by differentiating the first with regard to Y, the second with regard to X, and equating the results, that

$$\begin{aligned}
\frac{\partial x}{\partial X}\frac{\partial p}{\partial Y} - \frac{\partial x}{\partial Y}\frac{\partial p}{\partial X} &= \frac{\partial q}{\partial X}\frac{\partial y}{\partial Y} - \frac{\partial q}{\partial Y}\frac{\partial y}{\partial X} \\
&= \frac{\partial x}{\partial X}\left(X\frac{\partial y}{\partial X} + Y\frac{\partial y}{\partial Y}\right) - \frac{\partial y}{\partial X}\left(X\frac{\partial x}{\partial X} + Y\frac{\partial x}{\partial Y}\right) \\
&= Y\,\frac{\partial(x, y)}{\partial(X, Y)},
\end{aligned}$$

where use has been made of equations (4) and (10) in obtaining $\partial q/\partial X$. To solve this equation for p assume

$$\frac{\partial p}{\partial Y}=Y\frac{\partial x}{\partial X}+v\frac{\partial x}{\partial Y}=Y\frac{\partial y}{\partial Y}+v\frac{\partial y}{\partial Y}\quad\ldots\ldots\ldots\ldots(13),$$

$$\frac{\partial p}{\partial X}=Yf'(X)\frac{\partial x}{\partial Y}+v\frac{\partial x}{\partial X}=Y\frac{\partial y}{\partial X}+v\frac{\partial x}{\partial X}\ldots(14),$$

where v is a function of X and Y to be determined.

Differentiating equation (13) with regard to X, equation (14) with regard to Y, and equating the right-hand sides, we find

$$\frac{\partial x}{\partial X}\frac{\partial v}{\partial Y}-\frac{\partial x}{\partial Y}\frac{\partial v}{\partial X}=-\frac{\partial y}{\partial X},$$

which is plainly satisfied by

$$v=g(x)+f(X).$$

where g is an arbitrary function of x.

From the form of the equations which have to be satisfied by p and z, it is clear that the arbitrary function of x in the expression for v leads merely to an arbitrary function of x in the expression for z. We therefore put $g=0$, and therefore $v=f(X)$, so that

$$p=Yy+\int f(X)\frac{\partial x}{\partial X}dX$$

$$=Yy+xf(X)-\int xf'(X)\,dX$$

$$=Yy+xf(X)-\int_a^b\frac{1}{\mu}F_1\phi_1'\sin\mu Y\,d\mu+\int_a^b\frac{1}{\mu}F_2\phi_2'\cos\mu Y\,d\mu,$$

using equation (4).

We have now to substitute this value of p in equations (11) and (12), and hence to obtain z. Performing the integration, we find

$$z=h(x)+px+qy-\tfrac{1}{2}Xy^2-Yxy-\tfrac{1}{2}f(X)\,x^2+w,$$

where h is an arbitrary function of x, and w is a function of X and Y, such that

$$\frac{\partial w}{\partial X}=\tfrac{1}{2}\{x^2f'(X)+y^2\},$$

$$\frac{\partial w}{\partial Y}=xy.$$

It is easily seen from equations (10) that these two equations are consistent; wherefore

$$w=\tfrac{1}{2}\int\{[x^2f'(X)+y^2]\,dX+2xy\,dY\}.$$

It is possible to obtain the explicit form of w, but the result is extremely complicated, and as there is no point of interest in the work the integration has not been carried out.

The expression for z contains an arbitrary function of x. We have now to determine this function. Differentiating with regard to x, we find

$$\frac{\partial z}{\partial x}=h'(x)+\left(p\frac{\partial x}{\partial X}+q\frac{\partial y}{\partial X}\right)\frac{\partial X}{\partial x}+\left(p\frac{\partial x}{\partial Y}+q\frac{\partial y}{\partial Y}\right)\frac{\partial Y}{\partial x}.$$

But
$$\frac{\partial(x,y)}{\partial(X,Y)}\frac{\partial X}{\partial x}=\frac{\partial x}{\partial X},$$
$$\frac{\partial(x,y)}{\partial(X,Y)}\frac{\partial Y}{\partial x}=-\frac{\partial y}{\partial X};$$

therefore
$$\frac{\partial z}{\partial x}=p+h'(x),$$

while
$$\frac{\partial z}{\partial y}=q.$$

Thus
$$\frac{\partial^2 z}{\partial x^2}=h''(x)+\frac{\partial p}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial p}{\partial Y}\frac{\partial Y}{\partial x}$$
$$=h''(x)+f(X),$$

and
$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial q}{\partial y}=X.$$

Therefore $h''(x)=0$; and we may take $h(x)=0$.

Hence, finally, the solution of (1) is given by eliminating X and Y from (7) and (8), and

$$z=\tfrac{1}{2}Xy^2+Yxy+\tfrac{1}{2}f(X)x^2+\tfrac{1}{2}\int\{[x^2f'(X)+y^2]\,dX+2xy\,dY\}$$
$$-y\left[\int_a^b F_1\phi_1\cos\mu Y\,d\mu+\int_a^b F_2\phi_2\sin\mu Y\,d\mu\right]$$
$$-x\left[\int_a^b\frac{1}{\mu}F_1\phi_1'\sin\mu Y\,d\mu-\int_a^b\frac{1}{\mu}F_2\phi_2'\cos\mu Y\,d\mu\right]$$

where F_1 and F_2 are arbitrary functions of μ, and ϕ_1 and ϕ_2 are two linearly independent solutions of (4).

The method applied to the solution of equation (2) may be used to solve a somewhat more general equation, namely,

$$r - 2f(x)\,s + g(x)\,t = 0 \quad\dots\dots\dots\dots\dots(15),$$

where f and g are arbitrary functions of x. For assume, as a trial solution,

$$z = \phi(x)\,\psi(y),$$

and therefore $\quad r = \phi''\psi,\quad s = \phi'\psi',\quad t = \phi\psi''$,

so that $\quad \phi''\psi - 2f(x)\,\phi'\psi' - g(x)\,\psi'' = 0$,

or, putting $\psi = e^{\mu Y}$,

$$\phi''(x) - 2\mu f(x)\,\phi'(x) + \mu^2 g(x)\,\phi(x) = 0 \dots\dots(16),$$

an equation which will have two linearly independent solutions, ϕ_1 and ϕ_2 say. The general solution of equation (15) will thus be

$$z = \int_a^b e^{\mu Y}\left[F_1(\mu)\,\phi_1(x) + F_2(\mu)\,\phi_2(x)\right] d\mu.$$

The solution of equation (15) does not, however, enable us to obtain the solution of a more general equation of the type

$$r = f(s, t).$$

ON THE SERIES FOR SINE AND COSINE.

By *F. Jackson*, University College, London.

§ 1. Prof. M. J. M. Hill has given (*Mess. of Math.*, vol. xxxv., pp. 58–69) an inductive proof of the series for $\sin x$ and $\cos x$. The demonstration is based on Le Cointe's identity

$$3^n \sin\frac{x}{3^n} - \sin x = 4\left(\sin^3\frac{x}{3} + 3\sin^3\frac{x}{3^2} + \dots + 3^{n-1}\sin^3\frac{x}{3^n}\right),$$

and consists in proving that, for any positive integral value of r,

$$3^n\left\{\sin\frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3!}\left(\frac{x}{3^n}\right)^3 - \dots + \frac{(-1)^r}{(2r-1)!}\left(\frac{x}{3^n}\right)^{2r-1}\right\}$$

$$\gtrless \left(\sin x - x + \frac{x^3}{3!} - \dots - (-1)^r\frac{x^{2r-1}}{(2r-1)!}\right),$$

the sign of inequality being $>$ or $<$ according as r is odd or even.

If the second member of this inequality is denoted by $\phi(x)$, the first member is

$$3^n \phi\left(\frac{x}{3^n}\right).$$

I think that it has not been noticed that a similar proof can be given which depends upon proving that

$$2^n \phi\left(\frac{x}{2^n}\right) \gtrless \phi(x),$$

the signs of inequality $>$ or $<$ occurring alternately as above.

§ 2. We start with the identity

$$\begin{aligned} 2^n \sin\frac{x}{2^n} - \sin x = {} & 2\sin\frac{x}{2}\left(1-\cos\frac{x}{2}\right) \\ & + 2^2 \sin\frac{x}{2^2}\left(1-\cos\frac{x}{2^2}\right) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + 2^n \sin\frac{x}{2^n}\left(1-\cos\frac{x}{2^n}\right) \ldots\ldots\text{(I.)}, \end{aligned}$$

the truth of which is evident, for multiplying out the right-hand side, we get

$$2\sin\frac{x}{2} - \sin x + 2^2\sin\frac{x}{2^2} - 2\sin\frac{x}{2} + \ldots + 2^n\sin\frac{x}{2^n} - 2^{n-1}\sin\frac{x}{2^{n-1}},$$

whence we get $\qquad 2^n \sin\frac{x}{2^n} - \sin x.$

Now, if x is any acute angle,

$$\sin x < x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(1)},$$

and $\qquad 1 - \cos 2x = 2\sin^2 x,$

therefore $\qquad 1 - \cos 2x < 2x^2,$

therefore $\qquad 1 - \cos x < \frac{x^2}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(2)}.$

From (1) and (2)

$$\sin x\,(1-\cos x) < \frac{x^3}{2} \ldots\ldots\ldots\ldots\ldots\ldots\text{(3)}.$$

Now use (3) in the identity (I.). Therefore

$$2^n \sin \frac{x}{2^n} - \sin x < 2 \cdot \tfrac{1}{2} \left(\frac{x}{2}\right)^3 + 2^2 \cdot \tfrac{1}{2} \left(\frac{x}{2^2}\right)^3 + \ldots + 2^n \tfrac{1}{2} \left(\frac{x}{2^n}\right)^3.$$

The difference between the two members of this inequality depends upon n, but does not tend to zero as n tends to infinity; for it consists of the sum of the expressions

$$\left(\frac{x}{2}\right)^3 - 2 \sin \frac{x}{2} \left(1 - \cos \frac{x}{2}\right)$$

$$2 \left(\frac{x}{2^2}\right)^3 - 2^2 \sin \frac{x}{2^2} \left(1 - \cos \frac{x}{2^2}\right)$$

etc.

Therefore

$$2^n \sin \frac{x}{2^n} - \sin x < \left(\frac{x}{2}\right)^3 \left[1 + 2 \left(\frac{1}{2}\right)^3 + 2^2 \left(\frac{1}{2^2}\right)^3 + \ldots + 2^{n-1} \left(\frac{1}{2^{n-1}}\right)^3\right]$$

$$< \left(\frac{x}{2}\right)^3 \frac{1 - (1/2^{2n})}{1 - (1/2^2)}$$

$$< \frac{x^3}{3!} \left(1 - \frac{1}{2^{2n}}\right);$$

therefore

$$2^n \left[\left(\sin \frac{x}{2^n} - \frac{x}{2^n} + \frac{1}{3!} \left(\frac{x}{2^n}\right)^3\right] < \sin x - x + \frac{x^3}{3!}.$$

If n tends to infinity, the left-hand side tends to zero, while the difference remains positive; therefore

$$0 < \sin x - x + \frac{x^3}{3!}$$

i.e., $$\sin x > x - \frac{x^3}{3!} \quad \text{.......................} (4).$$

Use (4) in $$1 - \cos 2x = 2 \sin^2 x;$$

therefore $$1 - \cos 2x > 2 \left(x - \frac{x^3}{3!}\right)^2$$

$$> 2x^2 - 4 \frac{x^4}{3!} + 2 \left(\frac{x^3}{3!}\right)^2$$

$$> 2x^2 - \frac{4x^4}{3!};$$

therefore $$1 - \cos x > \frac{x^2}{2!} - \frac{x^4}{4!} \text{ .... (5).}$$

Therefore $$\sin x(1-\cos x) > \left(x - \frac{x^3}{3!}\right)\left(\frac{x^2}{2!} - \frac{x^4}{4!}\right)$$
$$> \frac{x^3}{2!} - x^5\left(\frac{1}{4!} + \frac{1}{3!\,2!}\right)$$
$$> \frac{x^3}{3!}.3 - \frac{x^5}{5!}.15$$
$$> \frac{x^3}{3!}(2^2-1) - \frac{x^5}{5!}(2^4-1) \text{ ...... (6).}$$

Use this in the identity (I.). Therefore

$$2^n \sin\frac{x}{2^n} - \sin x > \frac{x^3}{3!}(2^2-1)\left[2.\left(\frac{1}{2}\right)^3 + 2^2\left(\frac{1}{2^2}\right)^3 + \ldots + 2^n\left(\frac{1}{2^n}\right)^3\right]$$
$$- \frac{x^5}{5!}(2^4-1)\left[2.\left(\frac{1}{2}\right)^5 + 2^2\left(\frac{1}{2^2}\right)^5 + \ldots + 2^n\left(\frac{1}{2^n}\right)^5\right].$$

Therefore

$$2^n \sin\frac{x}{2^n} - \sin x > \frac{x^3}{3!}(2^2-1)\frac{1}{2^2}\frac{\{1-(1/2^{2n})\}}{1-(1/2^2)}$$
$$- \frac{x^5}{5!}(2^4-1)\frac{1}{2^4}\frac{\{1-(1/2^{4n})\}}{1-(1/2^4)}$$
$$> \frac{x^3}{3!}\left(1 - \frac{1}{2^{2n}}\right) - \frac{x^5}{5!}\left(1 - \frac{1}{2^{4n}}\right).$$

Therefore

$$2^n\left\{\sin\frac{x}{2^n} - \frac{x}{2^n} + \frac{1}{3!}\left(\frac{x}{2^n}\right)^3 - \frac{1}{5!}\left(\frac{x}{2^n}\right)^5\right\} > \sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!}.$$

Now make n tend to infinity, and reasoning as before it follows that

$$\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!} \text{ .... (7).}$$

Substitute this into

$$1 - \cos 2x = 2\sin^2 x,$$

therefore

$$1 - \cos 2x < 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)^2 < 2x^2 - 2x^4\left(\frac{1}{1!\,3!} + \frac{1}{3!\,1!}\right)$$
$$+ 2x^6\left(\frac{1}{1!\,5!} + \frac{1}{3!\,3!} + \frac{1}{5!\,1!}\right)$$

provided
$$\frac{2x^8}{3!\,5!} > \frac{x^{10}}{5!\,5!},$$

i.e., $x^2 < 40$, which is so. Therefore

$$1 - \cos 2x < \frac{2^2 x^2}{2!} - \frac{x^4}{4!}\, 2 \,.\, \tfrac{1}{2} \,.\, 2^4 + \frac{x^6}{6!}\, 2 \,.\, \tfrac{1}{2} \,.\, 2^6$$

$$< \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!},$$

therefore
$$1 - \cos x < \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} \quad \text{..............} (8).$$

Before proceeding to prove the induction I shall quote a theorem given by Prof. Hill (*loc. cit.*, art. 1), and make two deductions therefrom which will be used in the subsequent work.

§ 3. *Hill's theorem.* If

$$u_1,\ u_3,\ \ldots,\ u_{4r+1},\ u_{4r+3}$$

and
$$v_1,\ v_3,\ \ldots,\ v_{4r+1},\ v_{4r+3}$$

are two series of positive quantities, each in descending order of magnitude, and if U and V are two quantities such that

$$u_1 - u_3 + u_5 - \ldots + u_{4r+1} \geq U \geq u_1 - u_3 + u_5 - \ldots + u_{4r+1} - u_{4r+3},$$

and

$$v_1 - v_3 + v_5 - \ldots + v_{4r+1} \geq V \geq v_1 - v_3 + v_5 - \ldots + v_{4r+1} - v_{4r+3},$$

then UV is less than

$$\begin{aligned} &u_1 v_1 \\ &-(u_1 v_3 + u_3 v_1) \\ &+(u_1 v_5 + u_3 v_3 + u_5 v_1) \\ &- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+(u_1 v_{4r+1} + u_3 v_{4r-1} + \ldots + u_{4r+1} v_1), \end{aligned}$$

but is greater than

$$\begin{aligned} &u_1 v_1 \\ &-(u_1 v_3 + u_3 v_1) \\ &+(u_1 v_5 + u_3 v_3 + u_5 v_1) \\ &- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+(u_1 v_{4r+1} + u_3 v_{4r-1} + \ldots + u_{4r+1} v_1) \\ &-(u_1 v_{4r+3} + u_3 v_{4r+1} + \ldots + u_{4r+3} v_1). \end{aligned}$$

§ 4. Let $$t_r = \frac{x^r}{r!}.$$

Then $t_1, t_3, t_5, \ldots$ are decreasing if
$$x^2 < 6,$$
and $t_2, t_4, t_6, \ldots$ are decreasing if
$$x^2 < 12.$$
These are both satisfied if $x < \frac{1}{2}\pi < \sqrt{3}$.

Therefore, using the theorem of § 3,

$$\begin{aligned}
&(t_1 - t_3 + t_5 - \ldots + t_{4r+1})^2 \\
&< t_1 t_1 \\
&- (t_1 t_3 + t_3 t_1) \\
&+ (t_1 t_5 + t_3 t_3 + t_5 t_1) \\
&- \ldots\ldots\ldots\ldots\ldots\ldots \\
&+ (t_1 t_{4r+1} + t_3 t_{4r-1} + \ldots + t_{4r+1} t_1).
\end{aligned}$$

Now $$t_1 t_{2m+1} + t_3 t_{2m-1} + \ldots + t_{2m+1} t_1$$
$$= \frac{x^{2m+2}}{(2m+2)!} \left(\frac{(2m+2)!}{1!\,(2m+1)!} + \frac{(2m+2)!}{3!\,(2m-1)!} + \ldots + \frac{(2m+2)!}{(2m+1)!\,1!} \right)$$
$$= \frac{x^{2m+2}}{(2m+2)!} \tfrac{1}{2} (1+1)^{2m+2}$$
$$= \frac{1}{2} \frac{(2x)^{2m+2}}{(2m+2)!},$$
therefore $$(t_1 - t_3 + t_5 - \ldots + t_{4r+1})^2$$
$$< \frac{1}{2} \left\{ \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \ldots + \frac{(2x)^{4r+2}}{(4r+2)!} \right\} \quad \ldots\ldots(9).$$

Also $$(t_1 - t_3 + t_5 - \ldots + t_{4r+1})(t_2 - t_4 + \ldots + t_{4r+2})$$
$$\begin{aligned}
&< t_1 t_2 \\
&- (t_1 t_4 + t_3 t_2) \\
&+ (t_1 t_6 + t_3 t_4 + t_5 t_2) \\
&- \ldots\ldots\ldots\ldots\ldots\ldots \\
&+ (t_1 t_{4r+2} + t_3 t_{4r} + \ldots + t_{4r+1} t_2).
\end{aligned}$$

Now $$t_1 t_{2m} + t_3 t_{2m-2} + \ldots + t_{2m-1} t_2$$

$$= \frac{x^{2m+1}}{(2m+1)!} \left\{ \frac{(2m+1)!}{1!\,(2m)!} + \frac{(2m+1)!}{3!\,(2m-2)!} + \ldots + \frac{(2m+1)!}{(2m-1)!\,2!} \right\}$$

$$= \frac{x^{2m+1}}{(2m+1)!} \left\{ \tfrac{1}{2}(1+1)^{2m+1} - 1 \right\}$$

$$= \frac{x^{2m+1}}{(2m+1)!} (2^{2m} - 1),$$

therefore $$(t_1 - t_3 + t_5 - \ldots + t_{4r+1})(t_2 - t_4 + \ldots + t_{4r+2})$$

$$< \frac{x^3}{3!}(2^2 - 1) - \frac{x^5}{5!}(2^4 - 1) + \frac{x^7}{7!}(2^6 - 1) - \ldots + \frac{x^{4r+3}}{(4r+3)!}(2^{4r+2} - 1) \quad \ldots\ldots(10).$$

Similarly, we can show that

$$(t_1 - t_3 + t_5 - \ldots - t_{4r+3})^2$$

$$> \tfrac{1}{2} \left\{ \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \ldots - \frac{(2x)^{4r+4}}{(4r+4)!} \right\} \ldots(11),$$

and $$(t_1 - t_3 + t_5 - \ldots - t_{4r+3})(t_2 - t_4 + t_6 - \ldots - t_{4r+4})$$

$$> \frac{x^3}{3!}(2^2 - 1) - \frac{x^5}{5!}(2^4 - 1) + \frac{x^7}{7!}(2^6 - 1) - \ldots - \frac{x^{4r+5}}{(4r+5)!}(2^{4r+4} - 1) \quad \ldots\ldots(12).$$

§ 5. Let us now assume that

$$\sin x < t_1 - t_3 + t_5 - \ldots + t_{4r+1} \ldots\ldots\ldots\ldots\ldots(13).$$

Use this in $$1 - \cos 2x = 2 \sin^2 x.$$

Therefore, from (9),

$$1 - \cos 2x < \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \ldots + \frac{(2x)^{4r+2}}{(4r+2)!},$$

therefore

$$1 - \cos x < \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots + \frac{x^{4r+2}}{(4r+2)!},$$

i.e., $$< t_2 - t_4 + t_6 - \ldots + t_{4r+2} \ldots\ldots\ldots\ldots\ldots\ldots(14).$$

Therefore, from (13), (14), and (10),

$$\sin x\,(1-\cos x)$$

$$<\frac{x^3}{3!}(2^2-1)-\frac{x^5}{5!}(2^4-1)+\frac{x^7}{7!}(2^6-1)-\ldots+\frac{x^{4r+3}}{(4r+3)!}(2^{4r+2}-1)$$

$$\ldots\ldots(15).$$

Now use (15) in the right-hand side of the identity (I.). Therefore

$$2^n\sin\frac{x}{2^n}-\sin x$$

$$<\frac{x^3}{3!}(2^2-1)\left\{2\left(\frac{1}{2}\right)^3+2^2\left(\frac{1}{2^2}\right)^3+\ldots+2^n\left(\frac{1}{2^n}\right)^3\right\}$$

$$-\frac{x^5}{5!}(2^4-1)\left\{2\left(\frac{1}{2}\right)^5+2^2\left(\frac{1}{2^2}\right)^5+\ldots+2^n\left(\frac{1}{2^n}\right)^5\right\}$$

$$+\frac{x^7}{7!}(2^6-1)\left\{2\left(\frac{1}{2}\right)^7+2^2\left(\frac{1}{2^2}\right)^7+\ldots+2^n\left(\frac{1}{2^n}\right)^7\right\}$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{x^{4r+3}}{(4r+3)!}(2^{4r+2}-1)\left\{2\left(\frac{1}{2}\right)^{4r+3}+2^2\left(\frac{1}{2^2}\right)^{4r+3}+\ldots+2^n\left(\frac{1}{2^n}\right)^{4r+3}\right\}.$$

Therefore

$$2^n\sin\frac{x}{2^n}-\sin x$$

$$<\frac{x^3}{3!}\left(1-\frac{1}{2^{2n}}\right)-\frac{x^5}{5!}\left(1-\frac{1}{2^{4n}}\right)+\frac{x^4}{7!}\left(1-\frac{1}{2^{6n}}\right)-\ldots$$

$$\ldots+\frac{x^{4r+3}}{(4r+3)!}\left(1-\frac{1}{2^{(4r+2)n}}\right),$$

therefore

$$2^n\left\{\sin\frac{x}{2^n}-\frac{x}{2^n}+\frac{1}{3!}\left(\frac{x}{2^n}\right)^3-\ldots+\frac{1}{(4r+3)!}\left(\frac{x}{2^n}\right)^{4r+3}\right\}$$

$$<\sin x-x+\frac{x^3}{3!}-\ldots+\frac{x^{4r+3}}{(4r+3)!}.$$

Now make n tend to infinity. The left-hand side tends to zero; and reasoning as in § 2, we see that

$$0 < \sin x - x + \frac{x^3}{3!} - \ldots + \frac{x^{4r+3}}{(4r+3)!},$$

therefore $$\sin x > t_1 - t_3 + t_5 - \ldots - t_{4r+3} \ldots\ldots\ldots\ldots (16).$$

In a similar way from

$$1 - \cos 2x = 2\sin^2 x,$$

and, using (16) and (11), we get

$$1 - \cos x > t_2 - t_4 + t_6 - \ldots - t_{4r+4} \ldots\ldots\ldots\ldots (17).$$

Then from (16), (17), and (12),

$$\sin x\,(1 - \cos x) > \frac{x^3}{3!}(2^2 - 1) - \frac{x^5}{5!}(2^4 - 1) + \frac{x^7}{7!}(2^6 - 1) - \ldots$$

$$\ldots - \frac{x^{4r+5}}{(4r+5)!}(2^{4r+4} - 1) \ldots\ldots\ldots\ldots (18).$$

Substituting this into the identity (I.), we get

$$2^n \sin\frac{x}{2^n} - \sin x$$

$$> \frac{x^3}{3!}\left(1 - \frac{1}{2^{2n}}\right) - \frac{x^5}{5!}\left(1 - \frac{1}{2^{4n}}\right) + \frac{x^7}{7!}\left(1 - \frac{1}{2^{6n}}\right) - \ldots$$

$$\ldots - \frac{x^{4r+5}}{(4r+5)!}\left(1 - \frac{1}{2^{(4r+4)n}}\right),$$

therefore

$$2^n \left\{\sin\frac{x}{2^n} - \frac{x}{2^n} + \frac{1}{3!}\left(\frac{x}{2^n}\right)^3 - \frac{1}{5!}\left(\frac{x}{2^n}\right)^5 + \frac{1}{7!}\left(\frac{x}{2^n}\right)^7 - \ldots\right.$$

$$\left.\ldots - \frac{1}{(4r+5)!}\left(\frac{x}{2^n}\right)^{4r+5}\right\}$$

$$> \sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots - \frac{x^{4r+5}}{(4r+5)!}.$$

And, reasoning as before, it follows that

$$0 > \sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots - \frac{x^{4r+5}}{(4r+5)!},$$

i.e., $$\sin x < t_1 - t_3 + t_5 - \ldots + t_{4r+5} \ldots\ldots\ldots\ldots (19).$$

Now (19) can be got from (13) by changing r into $r+1$, and we have proved (in § 2) that (13) is true for $r=0$, 1, therefore it must be true for $r=2$, and so on for all positive integral values.

It follows therefore that the inequalities (14), (15), (16) (17), (18) are also true for all positive integral values of r.

Therefore, since the series

$$t_1 - t_3 + t_5 - \ldots,$$

and

$$t_2 - t_4 + t_6 - \ldots,$$

are both absolutely convergent, it follows that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots,$$

and

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots + (-1)^{r-1} \frac{x^{2r}}{(2r)!} + \ldots,$$

i.e., $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots.$$

This completes the demonstration for x an acute angle. By using the series in the expressions for $\sin(x+y)$, $\cos(x+y)$, we can prove that the series are true for all values of x.

QUELQUES REMARQUES SUR LES CONGRUENCES $r^{p-1} \equiv 1 \pmod{p^2}$ et $(p-1)! \equiv -1 \pmod{p^2}$.

par *N. G. W. H. Beeger.*

La première de ces deux congruences s'est introduit dans les recherches récentes sur le dernier théorème de Fermat. Si l'équation

$$x^p + y^p + z^p = 0$$

a lieu pour des valeurs de x, y, et z qui ne sont pas divisibles par p, chaque facteur r de x, y, z doit satisfaire à la congruence :

$$(1) \qquad r^{p-1} \equiv 1 \pmod{p^2}.$$

C'est le théorème de M. Fürtwangler.*

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* *Wiener Berichte,* Abt. IIa, cxxi, p. 589.

Abel a posé la question de satisfaire à la congruence (1) par des nombres $r < p$.* La seule réponse fut une petite table de Jacobi, calculée par Busch.†

M. Cunningham‡ a trouvé quelques cas de la congruence $r^{p-1} \equiv 1 \pmod{p^a}$ avec $r < p^{a-1}$. On les trouve dans la table suivante

r	p	a	r	p	a	r	p	a	
1068	5	6	2819	19	4	53	97	2	
18	7	3	2820	19	4	43	103	2	(Th. Gosset)
19	7	3	333	19	3	58	131	2	
1353	7	5	19	43	2	69	631	2	
1354	7	5	53	59	2	252	997	2	
82681	7	7	11	71	2	175	487	2	(Th. Gosset)
82682	7	7	26	71	2	307	487	2	(,,)
239	13	4	31	79	2	10	487	2	(Desmarest)
158	17	3				100	487	2	(,,)
390112	17	6							

M. Hertzer§ a recherché s'il y a des nombres $r < p$ pour lesquelles on a $r^{p-1} \equiv 1 \pmod{p^3}$ pour $p < 307$. Il a trouvé seulement $r = 68$, $p = 113$, et $r = 3, 9$, $p = 11$. Du reste il a pris $p = 331, 353, 487, 673$ et aussi tous les nombres premiers entre 307 et 753, mais seulement pour a et $p - a$ si a $a < \sqrt{p}$. Ainsi il a encore trouvé :

$$r = 18, 324, 71, \qquad p = 331,$$

$$r = 14, 196, \qquad p = 353,$$

$$r = 100, 175, 307, \qquad p = 487,$$

$$r = 22, 484, \qquad p = 673.$$

Desmarest‖ semble avoir vérifié que pour $r = 10$ les seuls nombres $p < 1000$ sont $p = 3$ et 487.

Proth¶ donne, sans aucune démonstration, comme théorème que la congruence (1) est impossible pour $r = 2$.

M. Cunningham** a vérifié qu'il n'y a pas de nombres $p < 1000$ qui satisfont à $2^{p-1} \equiv 1 \pmod{p^2}$.

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* *Crelle's Journal*, 1828, p. 212.

† *Ibid.*, p. 301.

‡ *Messenger*, vol. xxix., (1900), "Period length of Circulates," table, art. 10, p. 158. Pour ce qui concerne les cas de deux r qui diffèrent l'unité voyez I, 3 du présent mémoire.

§ *Arch. d. Math. u Phys.*, B. 13, 1908, p. 107.

‖ *Théorie des nombres*, Paris, 1852, p. 295.

¶ *C.R.*, vol. lxxxiii., p. 1288.

** *Rep. British Ass.*, 1910, p. 530.

M. Meissner* a poussé plus loin ces calculs et il a trouvé le cas $2^{1092} \equiv 1 \pmod{1093^2}$, et a vérifié qu'il n'y a pas d'autres nombres $p < 2000$. L'assertion de Proth se trouve donc contrarié. La découverte de M. W. Meissner nous montre qu'il est impossible de démontrer le dernier théorème de Fermat par moyen de l'impossibilité de $2^{p-1} \equiv 1 \pmod{p^2}$ laquelle congruence aura lieu, suivant M. Wieferich, si $x^p + y^p + z^p = 0$.

M. Mirimanoff† a étudié le reste de

$$\frac{3^{p-1} - 1}{p} \pmod{p}$$

et tout récent M. Bachmann‡ a donné une nouvelle expression du reste de

$$\frac{2^{p-1} - 1}{p} \pmod{p}.$$

I. 1. La solution de la congruence $x^{p-1} \equiv 1 \pmod{p^2}$ a été donne par M. Worms de Romilly§: Soit ω une raçine primitive de p^2. Formons les restes:

$$x_i \equiv \omega^{ip} \pmod{p^2} \qquad i = 1, 2, \ldots, p-1. \tag{2}$$

Ces $p-1$ restes sont tous incongruents $\pmod{p^2}$ et

$$x_i^{p-1} \equiv \omega^{ip(p-1)} \equiv 1 \pmod{p^2}.$$

Il 'sensuit que tous ces restes sont les raçines de la congruence.

J'ai calculé les raçines de la congruence

$$x^{p-1} \equiv 1 \pmod{p^2}$$

pour tous les nombres premiers $p < 200$. On trouve cette table à la fin du présent mémoire. J'ai pris, pour chaque nombre premier p, une raçine primitive de p, ω.‖ Je recherche d'abord si ω^{p-1} n'est pas congruent de l'unité $\pmod{p^2}$. Alors on sait que ω est raçine primitive de p^2. Je calcule maintenant

$$\pm x_1 \equiv \omega^p \pmod{p^2}$$

où $\pm x_1$ a la plus petite valeur absolu. x_1 est donc raçine. Je prends

$$x_1^2 \equiv \omega^{2p} \equiv \pm x_2 \pmod{p^2}$$

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* *Sitz. Ber. d. K. Preuss Acad.*, vol. xxxv., 1913, p. 663.

† *Jour. f. Math.*, B. 115, p. 295.

‡ *Ibid.*, B. 142, p. 41.

§ *Interm. d. Math.*, 1901, p. 214.

‖ Il y a des tables de raçines primitives, v. p. e., *Acta Math.*, B 17.

où $\pm x_2$ a encore la plus petite valeur absolu. Alors il vient

$$x_1 x_2 \equiv \pm x_3, \quad x_1 x_3 \equiv \pm x_4, \quad \text{etc.},$$

jus qu'à
$$x_1 x_{\frac{1}{2}(p-3)} \equiv \pm x_{\frac{1}{2}(p-1)}.$$

On a une contrôle du calcul parce que

$$\pm x_{\frac{1}{2}(p-1)} \equiv \omega^{\frac{1}{2}(p-1)p} \equiv -1$$

de sorte qu'on doit trouver $x_{\frac{1}{2}(p-1)} = 1$. Ainsi on trouve $\frac{1}{2}(p-1)$ raçines de la congruence. Les autres $\frac{1}{2}(p-1)$ se trouvent en diminuant p^2 des raçines trouvées, car si x_i est raçine, il en est de même de $p^2 - x_i$. Dans la table je n'ai écrit que les $\frac{1}{2}(p-1)$ raçines qui ont les plus petites valeurs.

Exemple—$p = 23$, $\omega = 5$:

$$5^{23} \equiv 28, \qquad -28.130 \equiv 63,$$
$$28^2 \equiv 255, \qquad 28.63 \equiv 177,$$
$$28.255 \equiv 263, \qquad 28.177 \equiv 195,$$
$$28.263 \equiv -42, \qquad 28.195 \equiv 170,$$
$$-28.42 \equiv -118, \qquad 28.170 \equiv -1.$$
$$-28.118 \equiv -130,$$

Les 11 raçines les plus petites seront donc: 28, 255, 42, 118, 130, 63, 177, 195, 170, 1.

Il va sans dire que si x est raçine, il en sera de même de $x + p^2 k$. Donc, on peut déduire autant de cas que l'on veut de la congruence $x^{p-1} \equiv 1 \pmod{p^2}$ par l'addition de Kp^2 au nombres donnés dans la table. On peut trouve p.e.

$$60000^{148} \equiv 1 \pmod{149^2},$$

parce que
$$60000 = -6603 + 3.149^2.$$

2. On peut aisément trouver quelques propriétés des raçines: Formons le produit

$$(x - x_1)(x - x_2)\ldots(x - x_{p-1}) = x^{p-1} - S_1 x^{p-2} + \ldots + S_{p-1}$$

où x_i sont les raçines de $x^{p-1} \equiv 1 \pmod{p^2}$.

On aura donc

$$(x - x_1)(x - x_2)\ldots(x - x_{p-1}) \equiv x^{p-1} - 1 \pmod{p^2}.$$

Il sensuit:

$$S_1 \equiv 0, \; S_2 \equiv 0, \; \ldots, \; S_{p-1} \equiv -1 \pmod{p^2}.$$

Soient $x_1, \ldots, x_{\frac{1}{2}(p-1)}$ les plus petites racines. Les $\frac{1}{2}(p-1)$ autres racines seront donc

$$p^2 - x_1, \ldots, p^2 - x_{\frac{1}{2}(p-1)}.$$

Nous avons démontré que $S_{p-1} \equiv -1 \pmod{p^2}$. Donc

$$x_1 x_2 \ldots x_{\frac{1}{2}(p-1)} (p^2 - x_1)(p^2 - x_2) \ldots (p^2 - x_{\frac{1}{2}(p-1)}) \equiv -1 \pmod{p^2},$$

ou $$x_1^2 x_2^2 \ldots x^2_{\frac{1}{2}(p-1)} \equiv (-1)^{\frac{1}{2}(p+1)} \pmod{p^2}.$$

Si $p = 4n-1$ on aura

$$x_1 x_2 \ldots x_{\frac{1}{2}(p-1)} \equiv \pm 1 \pmod{p^2}.$$

La détermination du signe du deuxième membre semble difficile.

3. Si $p = 6n+1$ il y a toujours deux racines qui diffèrent l'unité.

La preuve peut se donner ainsi:

$$x^{p-1} - 1 = x^{6n} - 1$$
$$= (x+1)(x-1)(x^2+x+1)(x^2-x+1)(x^{6(n-1)}+\ldots),$$
$$(x+1)^{p-1} - 1 = (x+1)^{6n} - 1$$
$$= (x+2)x(x^2+3x+3)(x^2+x+1)\{(x+1)^{6(n-1)}+\ldots\}.$$

Si donc on prend

(3) $$x^2 + x + 1 \equiv 0 \pmod{p^2},$$

on aura à la fois

$$x^{p-1} \equiv 1 \pmod{p^2} \text{ et } (x+1)^{p-1} \equiv 1 \pmod{p^2}.$$

On preut énoncer ce théorème sous la forme suivante: Pour toute raçine primitive ω de p^2 on a:

$$\omega^{\frac{1}{6}(p-1)p} - \omega^{\frac{1}{3}(p-1)p} \equiv 1 \pmod{p^2}.$$

Soit $$x = \omega^{\alpha p} \text{ et } x+1 \equiv \omega^{\beta p},$$

on aura

(4) $$\omega^{\beta p} - \omega^{\alpha p} \equiv 1,$$

et suivant (3): $$\omega^{2\alpha p} + \omega^{\alpha p} + 1 \equiv 0,$$

et aussi: $$(\omega^{\alpha p} - 1)(\omega^{2\alpha p} + \omega^{\alpha p} + 1) \equiv 0,$$

ou $$\omega^{3\alpha p} - 1 \equiv 0.$$

Donc $$\alpha = \tfrac{1}{3}(p-1),$$
et il suit de (4) :

$$(\omega^{\beta p} - 1)^3 \equiv \omega^{(p-1)p} \equiv 1. \tag{5}$$

Du reste on a $$x = \omega^{\beta p} - 1.$$
Substituons cette expression dans (3) :

$$(\omega^{\beta p} - 1)^2 + (\omega^{\beta p} - 1) + 1 \equiv 0 \pmod{p^2},$$
$$\omega^{2\beta p} - \omega^{\beta p} + 1 \equiv 0,$$
$$\omega^{\beta p}(\omega^{\beta p} - 1) \equiv -1,$$
$$\omega^{3\beta p}(\omega^{\beta p} - 1)^3 \equiv -1,$$

et suivant (5) : $$\omega^{3\beta p} \equiv -1,$$
d'où $$\beta = \tfrac{1}{6}(p-1).$$

TABLE DES RAÇINES DE LA CONGRUENCE $x^{p-1} \equiv 1 \pmod{p^2}$.

p	raçines x
3	1
5	1 7
7	1 18 19
11	1 3 9 27 40
13	1 19 22 23 70 80
17	1 38 40 65 75 110 131 134
19	1 28 54 62 68 69 99 116 127
23	1 28 42 63 118 130 170 177 195 255 263
29	1 14 41 60 63 137 190 196 221 236 267 270 374 416
31	1 115 117 145 229 235 333 338 374 388 414 430 439 440 448
37	1 18 76 117 300 324 348 354 356 424 437 473 476 494 581 582 632 678
41	1 51 148 207 313 378 471 487 505 509 540 644 719 744 761 768 776 787 824 834
43	1 19 75 78 210 261 276 288 292 303 361 367 403 423 424 537 588 641 660 764 891
47	1 53 67 71 116 172 202 230 280 295 339 438 479 600 623 629 655 867 874 1042 1064 1085 1088
53	1 338 406 413 451 460 500 521 655 737 752 780 851 856 862 869 895 925 985 1009 1032 1120 1153 1223 1341 1366

p	racines x
59	1 53 137 298 299 300 506 559 612 672 805 806 809 893 946 970 1030 1105 1106 1107 1311 1364 1404 1405 1505 1558 1611 1642 1702
61	1 264 432 498 572 574 601 618 665 673 682 763 838 880 936 1003 1025 1033 1079 1116 1237 1303 1339 1440 1518 1553 1618 1660 1661 1693
67	1 143 248 279 310 448 504 560 567 630 699 700 722 875 897 1078 1121 1199 1218 1301 1342 1457 1471 1528 1719 1831 1857 1868 1910 1994 1996 2154 2216
71	1 11 26 121 223 261 286 438 482 577 606 633 676 681 757 859 969 978 1140 1306 1331 1335 1625 1745 1755 1778 1833 1895 1922 2170 2289 2395 2450 2453 2458
73	1 306 368 527 619 621 672 699 711 734 770 776 786 923 1032 1134 1144 1355 1381 1392 1417 1440 1482 1595 1650 1667 1818 1953 2092 2162 2198 2199 2286 2349 2430 2480
79	1 31 146 147 319 377 439 470 491 604 750 795 810 961 1127 1414 1523 1684 1714 1715 1905 2088 2092 2123 2276 2318 2442 2465 2509 2593 2715 2739 2838 2886 2887 3004 3032 3034 3035
83	1 99 161 260 269 293 380 401 526 562 615 670 821 822 925 1020 1050 1081 1115 1116 1180 1290 1389 1451 1453 1635 1816 1975 2018 2161 2354 2355 2560 2569 2633 2912 3038 3175 3181 3205 3418
89	1 184 254 426 605 707 790 826 842 927 1070 1145 1148 1219 1485 1510 1556 1573 1659 1765 2054 2172 2247 2272 2466 2508 2605 2635 2690 2782 2977 3018 3171 3187 3332 3352 3492 3598 3645 3663 3694 3858 3861 3926
97	1 53 107 138 226 279 382 402 412 525 750 866 978 1147 1428 1497 1626 1651 1667 1873 2022 2040 2095 2114 2252 2488 2569 2764 2809 2822 2961 3018 3020 3079 3234 3238 3667 3670 3738 4031 4045 4052 4069 4230 4337 4431 4620 4621
101	1 181 248 268 298 341 392 417 421 455 472 515 534 539 567 617 629 649 747 1059 1406 1524 1638 1744 1816 1854 2140 2158 2198 2264 2398 2497 2594 2933 2960 2977 3005 3046 3113 3252 3824 4070 4084 4451 4605 4732 4794 4845 4893 4943

p	racines x
103	1 43 147 164 348 350 386 391 470 687 908 1008 1067 1547 1595 1849 1889 2135 2286 2400 2670 2703 2817 2867 2890 2911 3038 3326 3392 3417 3445 3446 3557 3589 3645 3676 3697 4027 4031 4288 4355 4405 4425 4432 4441 4620 4808 4931 5101 5173 5244
107	1 164 317 469 510 695 730 955 1058 1100 1200 1239 1384 1477 1570 1777 1799 1837 1948 2004 2167 2195 2308 2364 2430 2552 2574 2583 2638 2784 2808 2878 2886 3079 3227 3365 3473 3495 3497 3594 3666 3895 3998 5061 5085 5203 5230 5257 5383 5517 5573 5602 5676
109	1 96 291 380 402 410 476 499 500 624 681 837 977 1256 1347 1379 1514 1693 1766 1828 2040 2470 2602 2613 2637 2665 2697 2727 2772 2815 2928 2935 2949 3023 3094 3202 3250 3369 3384 3651 3806 3936 4049 4056 4077 4174 4730 4898 5032 5064 5542 5744 5910 5947
113	1 68 129 174 356 373 620 690 742 937 954 1027 1057 1220 1330 1359 1418 1497 1578 1668 1690 1710 1901 1990 2618 2959 3029 3092 3366 3455 3517 3647 3853 3872 3997 4106 4121 4152 4156 4624 4689 4738 4793 4853 5098 5139 5152 5385 5573 5728 5952 5991 6070 6145 6335 6346
127	1 38 62 497 736 911 1155 1159 1224 1444 1549 1730 1813 1821 1846 1875 1891 2163 2172 2356 2360 2755 2757 3102 3347 3592 3607 3820 3844 4290 4339 4345 4378 4495 4497 4574 4612 4621 4682 4757 4803 4972 4973 5074 5095 5175 5632 5643 5654 6319 6485 6533 6609 6690 6734 6922 7094 7246 7342 7464 7916 7992 8034
131	1 58 111 250 424 659 835 899 1033 1100 1113 1465 1634 1657 1973 2386 2661 2818 3034 3053 3107 3177 3364 3416 3634 3674 3875 3900 4029 4050 4090 4135 4138 4337 4362 4419 4456 4505 4840 4844 5249 5256 5354 5464 5693 5869 6144 6341 6376 6438 6446 6572 6860 6880 7160 7397 7431 7734 7804 8014 8166 8194 8431 8490 8565

p	racines x
137	1 19 161 361 1062 1409 1464 1496 1621 1704 1785 1814 1863 1886 2061 2141 2574 2591 2693 2972 3059 3072 3099 3141 3342 3372 3400 3623 3630 3815 4233 4245 4349 4505 5140 5162 5246 5351 5579 5736 5829 5850 5927 6021 6106 6239 6563 6598 6607 6613 6686 6739 6859 7078 7152 7191 7401 7415 7555 7761 7824 8002 8250 8293 9047 9114 9236 9267
139	1 328 369 398 534 553 914 1105 1172 1263 1563 1793 2000 2004 2250 2299 2343 2414 2485 2752 2883 3096 3129 3327 3483 3598 3659 3802 3836 3839 4002 4336 4503 4593 4659 4703 4878 5051 5106 5278 5345 5431 5487 6001 6158 6159 6734 6915 7207 7406 7481 7495 7546 7563 7576 7706 8342 8474 8523 8525 8553 8588 8798 8809 8878 8881 9003 9280 9344
149	1 313 957 960 1312 1425 1510 1644 1744 1747 1969 2005 2046 2281 2752 2963 3155 3207 3431 3521 3528 3933 3949 4072 4085 4310 4334 4463 4716 4746 4955 5023 5231 5331 5361 5571 5608 5786 5937 6137 6409 6603 6804 6535 6687 7219 7927 7977 8012 8139 8214 8255 8462 8479 8499 9053 9079 9153 9165 9283 9464 9499 9873 9974 10156 10193 10289 10334 10472 10606 10671 10842 10928 11038
151	1 78 206 559 582 598 751 785 850 1042 1149 1582 2000 2103 2257 2430 2866 3072 3166 3269 3291 3387 3539 3607 3671 3793 3863 4171 4269 4427 4462 4901 5339 5386 5596 5887 6021 6024 6084 6124 6362 6678 6733 6772 6922 7132 7173 7212 7308 7489 7608 7733 7734 8684 8809 8884 8922 8949 9033 9072 9109 9183 9391 9426 9443 9690 9825 9928 10075 10348 10426 10531 10616 10915 11195
157	1 226 233 325 358 497 500 519 604 857 892 1081 1101 1636 1778 1859 2010 2184 2318 2336 2489 2860 2941 3360 3510 3698 3868 4350 4400 4413 4492 4583 4759 4919 4921 4991 5021 5383 5400 5486 5609 5888 5951 6212 6239 6457 6745 6895 6896 6961 7029 7162 7386 7444 7932 8222 8440 8757 9022 9467 9497 10043 10058 10245 10307 10535 10578 10644 10769 10923 11018 11378 11390 11453 11639 11910 12050 12273

p	racines x
163	1 65 84 218 266 590 630 710 711 728 1022 1182 1396 1635 1678 1995 2474 2547 2703 2794 2877 2986 3143 3170 3578 3743 4174 4225 4373 4416 4425 4495 4634 4725 4736 4850 5219 5325 5460 5614 5620 5818 6141 6162 6204 6502 6551 6586 6608 6666 6923 6967 6988 7056 7665 7952 8014 8107 8166 8186 8257 8293 8865 8935 8951 9279 9503 9806 10244 10288 10470 10988 11033 11703 11781 12069 12188 12399 12585 12797 13277
167	1 253 350 865 867 1040 1149 1314 1406 1443 1683 1792 2068 2226 2522 2678 2982 3048 3267 3283 3307 3329 3381 3761 3780 3924 4029 4211 4228 4267 4778 4804 4878 4883 5286 5398 5567 5601 5681 6071 6188 6687 6810 6839 6980 7018 7039 7152 7464 7821 8060 8114 8120 8198 8231 8283 8933 9166 9232 9342 9418 9426 9542 9607 9748 9902 10119 10308 10926 10944 11232 11704 11807 11862 12012 12119 12189 12217 12556 12935 13642 13676 13888
173	1 259 340 369 419 753 793 1040 1451 1606 1642 1727 2106 2256 2367 2554 3048 3052 3300 3303 3518 3608 4013 4116 4156 4966 5317 5342 5423 5482 5548 5744 5784 5966 6272 6377 6673 6715 6732 6785 6844 7185 7223 7318 7382 7502 7521 7575 7706 8148 8159 8282 8496 8754 9337 9389 9835 9850 10143 10371 11114 11195 11278 11400 11778 12040 12158 12314 12416 12464 12590 13077 13175 13241 13268 13292 13341 13389 13484 13712 14276 14282 14473 14476 14627 14770
179	1 532 632 649 939 1148 1336 1672 1957 2167 2344 2404 2591 2599 2712 3004 3421 3591 3608 3766 3839 3922 4223 4240 4266 4668 4905 5345 5399 5850 6365 6617 7183 7522 7644 7687 7749 8017 8109 8219 8276 8434 8477 8574 8695 8868 9018 9078 9400 9479 9533 9813 10166 10453 10608 10821 11286 11407 11422 11482 11546 11547 11764 11836 12048 12391 12485 12775 12810 13108 13215 13401 13735 14138 14139 14363 14486 14799 14932 15071 15325 15329 15331 15358 15811 15812 15814 15822 16614

p	racines x
181	1 78 298 313 314 416 420 836 1241 1368 1485 1618 2240 2291 2326 2430 2956 2987 3398 3458 3598 3659 3718 3832 4047 4151 4154 4699 4711 4840 4845 4874 5009 5167 5263 5579 5884 5910 6084 6151 6921 7026 7076 7087 7296 7636 7914 7920 8269 8347 8421 8909 9251 9269 9331 9447 9479 9517 9622 9894 10238 10876 10915 10968 11177 11637 11944 11982 12151 12300 12595 12741 12754 12960 13139 14141 14205 14324 14532 14893 15019 15138 15214 15222 15379 15473 15612 15662 15898 16124
191	1 176 395 714 746 938 979 1249 1293 1372 1802 2017 2044 2528 2721 2822 3442 4170 4208 4300 4643 4669 4798 4840 5066 5232 5275 5379 5385 5505 5700 5796 5867 6277 6609 7156 7378 7993 8364 8453 8682 8807 8883 8961 9301 9378 9793 9818 9935 10101 10322 10686 10826 10988 11124 11177 12150 12537 12777 12824 13070 13074 13361 13466 13573 13895 13979 14062 14385 14571 14586 14628 14788 15623 15966 15991 16072 16077 16107 16221 16242 16260 16276 16375 16612 16752 16846 17317 17319 17379 17423 17583 17652 17830 18213
193	1 276 436 558 813 873 874 894 895 954 1678 1732 1767 1947 2018 2361 2501 2561 2831 2897 3021 3374 3455 3851 4325 5012 5099 5272 6026 6205 6230 6627 6666 7312 7346 7835 8138 8574 8589 9058 9513 10085 10098 10172 10215 10846 10848 11011 11148 11500 11584 11776 11817 11850 12183 12458 12459 13029 13354 13364 13372 13441 13576 13723 13797 13861 13999 14128 14318 14615 14819 14894 15173 15184 15240 15382 15860 15886 16050 16140 16420 17007 17149 17294 17343 17345 17454 17455 17512 17730 17901 18022 18158 18353 18403 18453
197	1 143 284 318 349 556 997 1328 1336 1342 1393 1672 1753 1784 1803 1890 1912 1929 2051 2139 2236 2455 2650 2768 2997 3038 4141 4155 4184 4443 4510 4595 4623 4628 5154 5374 5445 5697 5980 6242 6566 6574 6665 6962 7098 7522 7535 7698 8303 8666 8781 8994 9140 9147 9276 9297 9965 10028 10938 11006 11098 11485 11630 11774 12030 12377 12419 12486 12665 12694 12928 13468 13532 13834 13795 14080 14128 14162 14405 14526 14806 14823 14895 15025 15229 15303 15750 16075 16177 16551 17130 17179 17237 17311 17825 18360 18468 18885

p	racines x
199	1 174 339 579 667 709 1091 1239 1312 1479 1706 1724 1927 1942 2045 2101 2251 2309 2418 2745 3421 3504 3532 3882 3975 4336 4563 4631 4667 5208 5315 5756 6240 6604 6905 7229 8171 9157 9165 9269 9278 9318 9325 9386 9523 9579 9817 9828 10052 10527 10670 10835 11152 11290 11493 11519 11919 12115 12132 13432 13439 13440 13712 13774 13987 14131 14224 14501 14654 14879 14881 15051 15229 15332 15345 15558 15590 15681 16507 16533 16763 16832 16919 17581 18057 18077 18433 18490 18501 18655 18985 19048 19385 19563 19637 19714 19732 19740 19792

II. *La congruence* $(p-1)!+1 \equiv 0 \pmod{p^2}$.

On en connaît seulement les cas $p=5$, 13. La verification directe pour un certain nombre p est impossible. J'ai fait usage de la formule connue des nombres de Bernouilli :

$$(1) \qquad ph_{p-1} - p + 1 \equiv (p-1)! + 1 \pmod{p^2},$$

dans laquelle h_{p-1} est un des nombres de Bernouilli qui sont définis par l'équation symbolique :

$$(h+1)^n = h^n, \qquad h_1 = -\tfrac{1}{2}.$$

On peut prouver la formule (1) de la manière suivante : Je pars de la relation

$$m! = m^m - \binom{m}{1}(m-1)^m + \binom{m}{2}(m-2)^m - \ldots,$$

d'où

$$(p-1)! = (p-1)^{p-1} - \binom{p-1}{1}(p-2)^{p-1} + \binom{p-1}{2}(p-3)^{p-1} - \ldots,$$

$$\begin{aligned}(p-1)!+1 = {} & \{(p-1)^{p-1}-1\} \\ & - \binom{p-1}{1}\{(p-2)^{p-1}-1\} + \ldots - \binom{p-1}{1}(1^{p-1}-1) \\ & + 1 - \binom{p-1}{1} + \binom{p-1}{2} - \ldots - \binom{p-1}{p+2} + 1,\end{aligned}$$

$$\begin{aligned}(p-1)!+1 \equiv {} & \{(p-1)^{p-1}-1\} \\ & + \{(p-2)^{p-1}-1\} + \ldots + (2^{p-1}-1) + (1^{p-1}-1) \pmod{p^2},\end{aligned}$$

$$(p-1)!+1 \equiv 1^{p-1} + 2^{p-1} + \ldots + (p-1)^{p-1} - (p-1) \pmod{p^2}.$$

Soit maintenant $S_{p-1}(p) = 1^{p-1} + 2^{p-1} + \ldots + (p-1)^{p-1}$ la fonction de Bernouilli, on aura

$$S_{p-1}(p) = 1/p\,(h+p)^p \equiv ph_{p-1},$$

donc $$(p-1)! + 1 \equiv ph_{p-1} - p + 1 \pmod{p^2}.$$

Adams a calculé les nombres de Bernouilli jusqu'à h_{124}.* A l'aide de cette table *j'ai vérifié qu'il n'y a pas de nombres* $p < 114$ *qui satisfont à la congruence*

$$(p-1)! + 1 \equiv 0 \pmod{p^2}.$$

Prenons p.e. $p = 61$ on aura

$$61 h_{60} = \frac{121\,52331\,40483\,75557\,20403\,04994\,07982\,02460\,41491}{930930}.$$

Il faut donc chercher le reste r de

$$60 . 930930 + 121\,52331\,40483\,75557\,20403\,04994\,07982\,02460\,41491 \pmod{61^2}.$$

On trouve $r = 2745$. On peut contrôler ce resultat, parce qu'il faut que

$$2745 \equiv 0 \pmod{61},$$

car $(p-1)! + 1 \equiv 0 \pmod{p}$. Le nombre 61 ne satisfait donc pas à la congruence $(p-1)! + 1 \equiv 0 \pmod{p^2}$.

Enfin je veux tirer l'attention sur le quotient de Wilson. On a

$$\frac{2!+1}{3} = 1, \qquad \frac{4!+1}{5^2} = 1, \qquad \frac{6!+1}{7} = 103,$$

$$\frac{10!+1}{11} = 329891, \qquad \frac{12!+1}{13^2} = 283329.$$

Tous ces nombres sont premiers.

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* *Crelle's Journal*, B. 85, p. 269.

ON A NOTE ON THE ELEMENTARY THEORY OF GROUPS OF FINITE ORDER (vol. xlii., pp. 132–134).

By *H. W. Chapman, B.Sc.*

In the above-mentioned paper I proved that if H be any sub-group of a group G it is possible to find a single set of operations $S_1, S_2, \ldots, S_m$ belonging to the group G, such that the group can be written in either of the two forms $S_1H, S_2H, \ldots, S_mH$; $HS_1, HS_2, \ldots, HS_m$. It has since been pointed out to me that the same result, with others, has been given in the *Quarterly Journal of Mathematics*, vol. xli., pp. 382–384, by Professor G. A. Miller.

I may perhaps be allowed to remark that my proof follows directly from first principles, whereas Professor Miller's involves the theory of the representation of a group as transitive.

I also wish to point out that, owing to an oversight in reading the proof, the p^{th} rows in my schemes (A) and (B) are wrongly placed and should be interchanged; also (A) and (B) should be interchanged in the last paragraph.

A NOTE ON LEGENDRE'S FUNCTIONS.

By *A. E. Jolliffe, M.A.*

In vol. vii., series 2, of the *London Mathematical Society's Proceedings*, Professor E. W. Hobson, in a paper entitled "Series of Legendre's Functions," proves, as a lemma, that $|(n\sin\theta)^{\frac{1}{2}}P_n(\cos\theta)|$ is less than some fixed number, independent of n and θ, for all values of θ in the range $(0, \pi)$. As the results based on this lemma are of considerable importance and the proof of it there given is somewhat intricate, a very simple proof may be of some interest.

It can be shown in a variety of ways that, $\pi > \theta > 0$,

$$Q_n(\cos\theta) + \tfrac{1}{2}\pi i\,P_n(\cos\theta) = \int_0^{\pi} \frac{z^{n+1}\sin^{2n+1}\phi\,d\phi}{(1 - z^2\sin^2\phi)^{\frac{1}{2}}} \quad (z \equiv e^{i\theta}),$$

therefore

$$|Q_r(\cos\theta)+\tfrac{1}{2}\pi i P_n(\cos\theta)| \mathrel{\overline{<}} \int_0^\pi \frac{\sin^{2n+1}\phi\, d\phi}{|1-z^2\sin^2\phi|^{\frac{1}{2}}}.$$

Now $|1-z^2\sin^2\phi| = (1-2\cos 2\phi\sin^2\phi+\sin^4\phi)^{\frac{1}{2}}$

$$=\{(1+\sin^2\phi)^2\sin^2\theta+(1-\sin^2\phi)^2\cos^2\theta\}^{\frac{1}{2}} > (1+\sin^2\phi)\sin\theta,$$

therefore

$$|Q_n(\cos\theta)+\tfrac{1}{2}\pi i P_n(\cos\theta)| < \frac{1}{(\sin\theta)^{\frac{1}{2}}}\int_0^\pi \frac{\sin^{2n+1}\phi}{(1+\sin^2\phi)^{\frac{1}{2}}}\,d\phi,$$

i.e., $$< \frac{1}{(\sin\theta)^{\frac{1}{2}}}\int_0^{\frac{1}{2}\pi}\sin^n\psi\, d\psi$$

(by the substitution $\sin^2\phi=\sin\psi$),

i.e., $$< \frac{1}{(\sin\theta)^{\frac{1}{2}}}\left(\frac{\pi}{2n}\right)^{\frac{1}{2}}.$$

Therefore $$|(n\sin\theta)^{\frac{1}{2}}P_n(\cos\theta)| < (2/\pi)^{\frac{1}{2}},$$

and incidentally $$|(n\sin\theta)^{\frac{1}{2}}Q_n(\cos\theta)| < (\pi/2)^{\frac{1}{2}}.$$

An alternative form of the analysis has been suggested to me by Dr. T. J. I'a Bromwich.

We have

$$Q_n(\cos\theta)+\tfrac{1}{2}\pi i P_n(\cos\theta) = \int_0^z \frac{t^n\, dt}{\{(t-z)(t-1/z)\}^{\frac{1}{2}}},$$

where the integration may be taken along the straight line joining the origin to the point z. If t be any point on this path of integration and r the distance of t from the origin, then it is evident at once from a figure that $|t-z|=1-r$ and $|t-z^{-1}|>(1+r)\sin\theta$, therefore

$$|Q_n(\cos\theta)+\tfrac{1}{2}\pi i P_n(\cos\theta)| < \frac{1}{(\sin\theta)^{\frac{1}{2}}}\int_0^1 \frac{r^n\, dr}{(1-r^2)^{\frac{1}{2}}},$$

leading to the same result as before.

NOTES ON EXACT DIFFERENTIAL EXPRESSIONS AND THEIR INTEGRATION WITHOUT QUADRATURES.

By *E. B. Elliott.*

If $D \equiv \dfrac{d}{dx} \equiv \dfrac{\partial}{\partial x} + y_1 \dfrac{\partial}{\partial y} + y_2 \dfrac{\partial}{\partial y_1} + y_3 \dfrac{\partial}{\partial y_2} + \ldots \text{to } \infty,$

where y_r denote $d^r y / dx^r$, the well-known necessary and sufficient condition (Euler's) for a function

$$F_n \equiv F(x;\ y, y_1, \ldots, y_n)$$

to be an exact derivative $D\phi$ is

$$(0, n)\, F_n \equiv \left(\frac{\partial}{\partial y} - D\frac{\partial}{\partial y_1} + D^2 \frac{\partial}{\partial y_2} - \ldots + (-1)^n D^n \frac{\partial}{\partial y_n}\right) F_n = 0.$$

The subject occupied a number of writers in the middle of the last century. See, in particular, Bertrand and Sarrus in vol. xvii. of the *Journal de l'Ecole polytechnique*. Mr. J. E. Campbell has recently given the best form of proof free from all reference to the Calculus of Variations. First he shows the condition to be necessary by obtaining

$$(0, n)\, D\phi = (-1)^n D^{n+1} \frac{\partial}{\partial y_n} \phi$$

from the alternant identities

$$\frac{\partial}{\partial y} D - D \frac{\partial}{\partial y} = 0, \qquad \frac{\partial}{\partial y_r} D - D\frac{\partial}{\partial y_r} = \frac{\partial}{\partial y_{r-1}} \quad (r = 1, 2, \ldots, n),$$

and observing that if an F_n is a $D\phi$, the ϕ must be free from y_n. Then he adopts the method of Sarrus for exhibiting the sufficiency, noticing that if $(0, n)\, F_n = 0$, and in fact if it does not involve y_{2n}, we must have

$$F_n = P_{n-1} y_n + Q_{n-1}$$

(with P_{n-1}, Q_{n-1} not extending beyond y_{n-1})

$$= y_n \frac{\partial}{\partial y_{n-1}} R_{n-1} + Q_{n-1}$$

$$= DR_{n-1} + Q_{n-1} - \left(\frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + \ldots + y_{n-1} \frac{\partial}{\partial y_{n-2}}\right) R_{n-1}$$

$$= DR_{n-1} + F_{n-1},$$

where, as F_n and DR_{n-1} are annihilated by $(0, n)$, F_{n-1} must be, and so by $(0, n-1)$; whence

$$F_{n-1} = DR_{n-2} + F_{n-2}, \text{ \&c., \&c.,}$$

and eventually

$$F_n = D\{R_{n-1} + R_{n-2} + \ldots + R_0 + \int \phi(x)\,dx\}.$$

This integration of an F_n satisfying $(0, n) F_n = 0$ requires quadratures, at most $n+1$ in number.

I have nowhere seen a record of the following observations.

I. *When the condition is satisfied by an F_n of algebraical form in $y, y_1, \ldots, y_n$, F_n can be integrated by differential operations.*

Whatever function be operated on, we have

$$\left(y\frac{\partial}{\partial y} + y_1\frac{\partial}{\partial y_1} + \ldots + y_n\frac{\partial}{\partial y_n}\right) - y(0, n)$$
$$= D\{y(1, n) + y_1(2, n) + \ldots + y_{n-1}(n, n)\},$$

where $$(r, n) \equiv \frac{\partial}{\partial y_r} - D\frac{\partial}{\partial y_{r+1}} + \ldots + (-1)^{n-r} D^{n-r}\frac{\partial}{\partial y_n}.$$

This is readily proved by taking together the first terms, the second terms, &c., of the two operators on the left and using, for $r = 1, 2, \ldots, n$,

$$y_r z + (-1)^{r-1} y z_r = D\{y_{r-1}z - y_{r-2}z_1 + \ldots + (-1)^{r-1} y z_{r-1}\}.$$

Hence a function which is homogeneous of degree i $(\neq 0)$ in $y, y_1, \ldots, y_n$, and which is annihilated by $(0, n)$, is integrated by direct operation on it with

$$\frac{1}{i}\{y(1, n) + y_1(2, n) + \ldots + y_{n-1}(n, n)\}.$$

Now an F_n algebraical in $y, y_1, \ldots, y_n$ can be arranged in a sum of parts homogeneous in them—not necessarily a finite sum if it be not rational and integral. If it have the annihilator $(0, n)$, so must its parts separately. Unless then there is a part of zero dimensions, F_n can be integrated by direct operation.

In the case of F_n rational and integral, a part of zero dimensions involves x only. It must be integrated by a quadrature. In the general case of F_n algebraical, such a part may also involve $y, y_1, \ldots, y_n$. It may be hopeless to look for a direct operator free from transcendents which will integrate it. For instance, regard $y_1/y = D\log y$. It is, however, of theoretical, though not practical, interest to note that simple transcendental transformations will prepare it for treatment like other parts. Put $y = e^z$ in the partial F_n of zero dimensions. It becomes an $f(x; z_1, z_2, \ldots, z_n)$ free from z. This is annihilated by $(0, n)_z$ because it is a derivative, or because $(0, n)_z = y\,(0, n)_y$. Arrange it as a sum of parts homogeneous in $z_1, z_2, \ldots, z_n$. If no part is of zero dimensions, direct operation integrates it as before. To deal with a part of zero dimensions, after removal of terms in x only, put $z_1 = e^{\zeta}$, thus obtaining an $f(x; \zeta_1, \zeta_2, \ldots, \zeta_{n-1})$ with one argument less than before. This is annihilated by $D\,(1, n)_z$, which s $\frac{\partial}{\partial z} - (0, n)_z$, and therefore by $(1, n)_z$, *i.e.*, $(0, n-1)_{z_1}$ (which cannot produce from it a constant other than zero because of its dimensions in $z_1, z_2, \ldots$), so that it is by $(0, n-1)_{\zeta}$. Repeat the reasoning if there is a part of zero dimensions in $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$; and so on. Eventually the whole of F_n, except terms in x only remaining for quadrature, is integrated by direct operation.

An alternative method, using weight (sum of suffixes) instead of degree, is available when F_n does not contain x explicitly. By a method used already, we see that

$$y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + \ldots + y_{n+1}\frac{\partial}{\partial y_n} - y_1(0, n)$$

$$= D\,\{y_1(1, n) + y_2(2, n) + \ldots + y_n(n, n)\}.$$

Here the left-hand member is $D - \frac{\partial}{\partial x} - y_1(0, n)$, so that if $(0, n)\,F_n = 0$, with F_n free from x,

$$F_n = \{y_1(1, n) + y_2(2, n) + \ldots + y_n(n, n)\},$$

since D annihilates no function of $y, y_1, \ldots, y_n$. Now it is easy to verify that

$$y_1\frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_2} + \ldots + ny_n\frac{\partial}{\partial y_n} - \{y_1(1, n) + y_2(2, n) + \ldots + y_n(n, n)\}$$

$$= D\,\{y_1(2, n) + 2y_2(3, n) + \ldots + (n-1)\,y_{n-1}(n, n)\}.$$

Consequently, if the F_n is of weight w throughout,

$$(w-1)F_n = D\{y_1(2, n) + 2y_2(3, n) + \ldots + (n-1)y_{n-1}(n, n)\}.$$

An F_n free from x can be arranged as a sum of isobaric parts, and the parts of different weights can thus be integrated separately by direct operation not involving transcendentals, except a part of unit weight, such as y_1/y. Only for such a part is exponential transformation, as above, necessary.

II. *Exact derivatives when there are several dependent variables.*

The early investigators gave a set of conditions

$$(0, m)_y F = 0, \quad (0, n)_z F = 0, \quad (0, p)_u F = 0, \ \ldots$$

as necessary and sufficient to secure that

$$F(x;\ y, y_1, \ldots, y_m;\ z, z_1, \ldots, z_n;\ u, u_1, \ldots, u_p;\ \ldots)$$

be an exact derivative $D\phi$.

I have not seen it stated that if F is of algebraical form in all its arguments but x, so that it can be arranged as a sum of parts homogeneous in a chosen system of arguments $y, y_1, \ldots, y_m$, and if none of these parts is of zero dimensions in them, the single condition $(0, m)_y F = 0$ suffices. This, and the fact that the integration can be performed by direct operations, can be established as before. We can also, as before, deal with a part of degree zero, and actually involving any of $y, y_1, \ldots, y_m$, by exponential transformations. Eventually the whole of F, but for a residue not involving y and its derivatives at all, is thus integrated. Such a residue R, if there be any involving z and its derivatives, has to obey $(0, n)_z R = 0$; and further direct operations in a second system are necessary. It may be that we thus have to run through all the systems.

III. *Abbreviated conditions.*

Closely connected with II. is the fact that a function $F_n(x; y, y_1, \ldots, y_n)$, which is a $D\phi$ with ϕ free from y_m $(m < n)$, obeys $(0, m) F_n = 0$. As a partial converse, an F_n annihilated by $(0, m)$, which is homogeneous of non-zero degree i in the abbreviated system $y, y_1, \ldots, y_m$, is the derivative of a ϕ free from y_m directly derived from it by operation with the abbreviated

$$\frac{1}{i}\{y(1, m) + y_1(2, m) + \ldots + y_{m-1}(m, m)\}.$$

For instance, finding that $(0, 5)$ annihilates $(5y_{10}+xy_{11})y+xy_6y_5$, we know that this is a derivative, and can directly integrate it, without examining the effect on it of $(0, 11)$.

This still holds if other dependent variables $z, u, \ldots$ and their derivatives are present in F.

IV. *Integrability more than once.*

Bertrand gave r necessary and sufficient conditions for an F_n to be an r^{th} derivative. These are the annihilation of F_n by the operators which occur as co-factors with $1, -\epsilon, \epsilon^2, \ldots, (-1)^{r-1}\epsilon^{r-1}$ in the expansion of

$$\frac{\partial}{\partial y} - (D+\epsilon)\frac{\partial}{\partial y_1} + (D+\epsilon)^2\frac{\partial}{\partial y_2} - \ldots + (-1)^n(D+\epsilon)^n\frac{\partial}{\partial y_n},$$

in powers of the arbitrary constant ϵ. A convenient method of proof uses the fact, easily obtained by use of alternant identities like $\frac{\partial}{\partial y_r}(D+\epsilon) - (D+\epsilon)\frac{\partial}{\partial y_r} = \frac{\partial}{\partial y_{r-1}}$, that operation with the expansion on $(D+\epsilon)\phi$ produces $(-1)^n(D+\epsilon)^{n+1}\frac{\partial}{\partial y_n}\phi$. The fact yields $n+1$ facts upon taking separately the co-factors with different powers of ϵ.

In particular, for F_n to be a second derivative, it is sufficient, and also necessary, that

$$(0, n)\, F_n = 0,$$

and

$$(1, n)'\, F_n \equiv \left\{\frac{\partial}{\partial y_1} - 2D\frac{\partial}{\partial y_2} + 3D^2\frac{\partial}{\partial y_3} - \ldots + (-1)^{n-1} n D^{n-1}\frac{\partial}{\partial y_n}\right\} F_n = 0.$$

It is perhaps worth remarking that, when F_n is free from x, the one condition $(1, n)'\, F_n = 0$ is sufficient, having $(0, n)\, F_n = 0$ as a consequence, provided that F_n, when arranged as a sum of isobaric parts, has no part of weight zero. If $F^{(w)}$ is a part of weight w, a method used more than once above gives us that

$$\{w - y_1(1, n)'\}\, F^{(w)} = D\,\{y_1(2, n)' + y_2(3, n)' + \ldots + y_{n-1}(n, n)'\}\, F^{(w)},$$

where

$$(r, n)' \equiv r\frac{\partial}{\partial y_r} - (r+1)\, D\frac{\partial}{\partial y_{r+1}} + (r+2)\, D^2\frac{\partial}{\partial y_{r+2}} - \ldots + (-1)^{n-r} n D^{n-r}\frac{\partial}{\partial y_n}.$$

Thus, if $(1, n)' F^{(w)} = 0$, $F^{(w)}$ is a $D\phi$ with ϕ obtained directly. Now the second of the $n+1$ facts above is

$$(0, n)\, \phi - (1, n)' D\phi = (-1)^n (n+1) D^n \frac{\partial}{\partial y_n} \phi\,;$$

and the expression on the right here is zero for our present ϕ, which does not extend beyond y_{n-1}. We have then $(0, n)\phi = 0$, so that ϕ is a $D\psi$, and consequently $F^{(w)}$ a $D^2\psi$.

ON THE SOLUTION OF AN EQUATION OF THE FORM $F(r, s, t) = 0$.

By *J. R. Wilton, M.A., B.Sc.*, Assistant Lecturer in Mathematics at the University of Sheffield.

An exhaustive list of cases in which the partial differential equation

$$F(r, s, t) = 0$$

is soluble by Darboux's method (and therefore by that of Legendre), on confining oneself to characteristics of order not higher than the second, has been given by Boer.* For convenience of reference I here reproduce his list.

(1) $ar + bs + ct + (rt - s^2) = e$, where a, b, c, e are constants.

(2) The eliminant of m from

$$r + ms = mF - m^2 F',$$

$$t + s/m = F' + F/m.$$

(3) The eliminant of m and n from

$$r = m^2 F'' - 2mF' + 2F + n^2 G'' - 2nG' + 2G,$$

$$s = -mF'' + F' - nG'' + G',$$

$$t = F'' + G'',$$

where, as throughout this list, F is an arbitrary function of m, G of n.

(4) $r = f(s)$.

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* F. De Boer, *Archives Neérlandaises*, t. xxvii.

(5) The eliminant of m and n from

$$(rt-s^2)/(r+2as+a^2t) = m^2F''-2mF'+2F+n^2G''-2nG'+2G,$$

$$(s+at)/(r+2as+a^2t) = -mF''+F'-nG''+G',$$

$$1/(r+2as+a^2t) = -F''-G'',$$

where a is an arbitrary constant.

(6) The eliminant of m and n from

$$r+2as+a^2t = (a-b)\,(n+2G'/G''+G'^3/G''^2M),$$

$$r+2bs+b^2t = (a-b)\,(m+2F'/F''-F'^3/F''^2M),$$

$$r+(a+b)\,s+abt = (a-b)\,F'^{\frac{3}{2}}G'^{\frac{3}{2}}/F''G''M,$$

where $\quad M = (GG''-2G'^2)/4G''-(FF''-2F'^2)/4F'',$

and a and b are constants.

(7) A set of results of which the final form cannot be found. (Equations 104, p. 400, of Boer's paper.)

(8) To these may be added the equations

$$r = f(t), \quad r+at = f(s),$$

which, though they cannot in general be solved by Darboux's method, will always yield to that of Legendre.

In the present paper it is shown that the equation resulting from the elimination of m between

$$r + ms = f(m) \qquad\qquad (1),$$

$$t + s/m = g(m) \qquad\qquad (2),$$

where f and g are arbitrary functions of their argument m, though not obviously included in the list, is soluble by the same method. It must therefore be included somewhere, probably under case (7).

Differentiating equation (1) with regard to y we obtain

$$\frac{\partial s}{\partial x} + m\frac{\partial s}{\partial y} = (f'-s)\frac{\partial m}{\partial y},$$

which, on making use of (2), reduces to

$$\left(f'-\frac{\partial q}{\partial x}\right)\left(\frac{\partial^2 q}{\partial y^2}+\frac{1}{m}\,\frac{\partial^2 q}{\partial x\,\partial y}\right) = \left(g'+\frac{1}{m^2}\,\frac{\partial q}{\partial x}\right)\left(\frac{\partial^2 q}{\partial x^2}+m\frac{\partial^2 q}{\partial x\,\partial y}\right) \qquad (3),$$

and this by Legendre's transformation (the principle of duality) becomes

$$(f'-x)\left(\frac{\partial^2 v}{\partial x^2}-\frac{1}{m}\frac{\partial^2 v}{\partial x\partial y}\right)+\left(g'+\frac{x}{m^2}\right)\left(m\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y^2}\right)=0 \quad \text{......(4)},$$

where $v=sx+ty-q$, and m is a function of x and y, given by

$$y+x/m=g(m)\text{.....................(5)}.$$

In equation (4) change the independent variables to x and m.

We must replace $\dfrac{\partial v}{\partial x}$ by $\dfrac{\partial v}{\partial x}+\dfrac{m}{x+m^2g'}\dfrac{\partial v}{\partial m}$, $\dfrac{\partial v}{\partial y}$ by $\dfrac{m^2}{x+m^2g'}\dfrac{\partial v}{\partial m}$, and we find

$$\frac{\partial^2 v}{\partial x^2}+\frac{m(f'+m^2g')}{x+m^2g'}\left(\frac{\partial^2 v}{\partial x\,\partial m}-\frac{1}{x+m^2g'}\frac{\partial v}{\partial m}\right)=0,$$

which, in order to avoid a complicated notation, we shall, without danger of confusion, re-write as

$$r+\frac{m(f'+m^2g')}{x+m^2g'}\left(s-\frac{q}{x+m^2g'}\right)=0\text{.........(6)}.$$

The characteristic equations of the first system are

$$dm=0,\qquad dp+\frac{m(f'+m^2g')\,dq}{x+m^2g'}=\frac{m(f'+m^2g')\,q\,dx}{(x+m^2g')^2};$$

while those of the second system are

$$dm=\frac{m(f'+m^2g')\,dx}{x+m^2g'},\qquad dp=\frac{m(f'+m^2g')\,q\,dx}{(x+m^2g')^2}.$$

The second system cannot have two integrable combinations unless $f'+m^2g'=0$, which is Boer's case (2). The first leads plainly to the first order integral

$$p+\frac{m(f'+m^2g')}{x+m^2g'}q=F(m)\text{..............(7)}.$$

Let $$\log h(m)=-\int\frac{dm}{m(f'+m^2g')},$$

and let $$F(m)=H-hH'/h',$$

where H is an arbitrary function of m, so that equation (7) becomes

$$p-qh/[h'(x+m^2g')]=H-hH'/h'.$$

It is easy to verify that the solution of this equation is

$$v=xH+\int m^2g'H'dm+G\left(xh+\int m^2g'h'dm\right),$$

where G is an arbitrary function of its argument

$$xh+\int m^2g'h'dm.$$

Also $$\frac{\partial v}{\partial x}=H+hG',$$

$$\frac{\partial v}{\partial m}=(x+m^2g')(H'+h'G').$$

The solution of equation (3) is therefore given by eliminating m and s from

$$sx+ty-q=sH+\int m^2g'H'dm+G\left(sh+\int m^2g'h'dm\right),$$

$$x=H+mH'+(h+mh')\,G' \qquad\qquad (8),$$

$$y=m^2(H'+h'G') \qquad\qquad (9),$$

where $$t+s/m=g(m).$$

The first of these equations may, by using (8), (9), and (2), be written

$$q=shG'+m^2g(H'+h'G')-G-\int m^2g'H'dm \qquad (10).$$

To find z we have to integrate the equation

$$\frac{\partial z}{\partial y}=q.$$

Let

$$J=\frac{\partial(x,\ y)}{\partial(s,\ m)}=mh^2G''[m(H''+h''G')+2(H'+h'G')],$$

so that $$J\frac{\partial s}{\partial y}=-\frac{\partial x}{\partial m},\qquad J\frac{\partial m}{\partial y}=\frac{\partial x}{\partial s}.$$

Therefore $$Jq=J\left(\frac{\partial z}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial z}{\partial m}\frac{\partial m}{\partial y}\right)$$

$$=\frac{\partial x}{\partial s}\frac{\partial z}{\partial m}-\frac{\partial x}{\partial m}\frac{\partial z}{\partial s}.$$

Hence z is found by integrating the simultaneous equations

$$\frac{ds}{-\frac{\partial x}{\partial m}} = \frac{dm}{\frac{\partial x}{\partial s}} = \frac{dz}{Jq}.$$

One solution is

$$H + mH' + (h + mh')\, G' = x = \text{constant},$$

i.e.,

$$G' = \frac{x - H - mH'}{h + mh'} \text{..................} (11).$$

Also

$$\frac{dz}{dm} = Jq \Big/ \frac{\partial x}{\partial s} = -\, mh\, \frac{dG'}{dm}\, q,$$

where G' is determined as a function of m by equation (11).

Thus

$$z = -\int mhq\, \frac{dG'}{dm}\, dm \text{................} (12),$$

in which the appropriate functions of m, drawn from equations (10) and (11), must be substituted for s, G', and G before integration, while x is to be treated as a constant during the integration, and afterwards to be put equal to

$$H + mH' + G'(h + mh').$$

There is a slight simplification in equation (12) if we take m and $sh + \int m^2 g' h' dm$ as new independent variables, but I have been unable to obtain the explicit form of z when G is arbitrary.

The elimination of m and s from equations (8), (9), and (12) leads to the solution of that equation of the form $F(r, s, t) = 0$ which results from the elimination of m from

$$r + ms + \int (m^2 g' + h / mh')\, dm = 0$$

and

$$t + s/m = g,$$

where h and g are arbitrary functions of m.

ON A METHOD OF REARRANGING THE POSITIVE INTEGERS IN A SERIES OF ORDINAL NUMBER GREATER THAN THAT OF ANY GIVEN FUNDAMENTAL SEQUENCE OF Ω.

By *N. Wiener.*

1. LET I represent the series of positive integers greater than 1 in their order of magnitude.

2. Let p_n stand for the nth prime in order of magnitude. Let A represent a series of positive integers, not necessarily in order of magnitude. Let a and b, respectively, be the ath and bth integers in order of magnitude. Let $a \underset{\rightarrow}{A} b$ mean, "a precedes b in the order determined by A." Construct, now, the series P of the primes of the form p_n, where $p_a \underset{\rightarrow}{P} p_b$, when, and only when, $a \underset{\rightarrow}{A} b$. Let us call this series $P(A)$. It will be seen immediately that $P(A)$ is by definition ordinally similar to A, and hence must have the same ordinal number.

For example, if A be the series

$$1, 3, 5, 7, 9, \ldots, 2, 4, 6, 8, 10, \ldots,$$

$P(A)$ will be the series

$$1, 3, 7, 13, 19, \ldots, 2, 5, 11, 17, 23, \ldots.$$

If A be the series

$$1, 3, 5, 7, 9, \ldots, 2, 6, 10, 14, 18, \ldots, 4, 12, 20, 28, 36, \ldots, 8, 24, \ldots,$$

$P(A)$ will be the series

$$1, 3, 7, 13, 19, \ldots, 2, 11, 23, 41, 59, \ldots, 5, 31, 67, 103, \ldots, 17, 83, \ldots.$$

3. Given a well-ordered series P of primes, it will have a first term, a second term, ..., an nth term. Let the nth term be represented by the symbol P_n. Take, now, those products of k terms of P satisfying the following conditions:

(a) Every such product contains at least one P_n, where n is finite.

(b) If P_a is a factor of such a product, and if when P_b is another factor of that product $a < b$, the product contains a distinct factors, none of which are equal to 1.

(*c*) No factor of the product occurs more than once in the product.

Since all the factors by which the products in question are determined are primes, and since no factor occurs twice in any product, it follows that each group of n factors satisfying (a), (b), and (c) determines one product, and one only, and *vice versa*.

Let us represent a product of the form in question by $P_n \cdot p' \cdot p'' \cdot p''' \dots p^{(n-1)}$, where p', p'', ..., $p^{(n-1)}$ are distinct members of P, and, if $p^{(k)} = P_l$, $n < l$. It will be seen on inspection that any product which will satisfy (a), (b), and (c) may be expressed in this form, and *vice versa*. It is also clear that the order of the terms in the product is a matter of indifference. We may then, without any loss of generality, assume that $p' < p'' < p''' < \dots < p^{(n-1)}$.

I shall now arrange these products in a series $p(P)$ in accordance with the following rules:

(1)
$$P_n \cdot p' \cdot p'' \dots p^{(n-1)} \underset{\rightarrow}{p(P)} P_m \cdot q' \cdot q'' \dots q^{(m-1)}, \text{ if } n < m;$$

(2)
$$P_n \cdot p' \cdot p'' \dots p^{(n-1)} \underset{\rightarrow}{p(P)} P_n \cdot q' \cdot q'' \dots q^{(n-1)}, \text{ if } p' \underset{\rightarrow}{P} q';$$

(3)
$$P_n \cdot p' \cdot p'' \dots p^{(k-1)} \cdot p^{(k)} \cdot p^{(k+1)} \dots p^{(n-1)} \underset{\rightarrow}{p(P)} P_n \cdot p' \cdot p'' \dots$$
$$\dots p^{(k-1)} \cdot q^{(k)} \cdot q^{(k+1)} \dots q^{(n-1)}, \text{ if } p^{(k)} \underset{\rightarrow}{P} q^{(k)}.$$

I now wish to prove that if the ordinal number of P is α, that of $p(P)$ is α^ω, provided α is a number with no immediate predecessor.

By rule 3, if $p \underset{\rightarrow}{P} q$,

$$P_n \cdot p' \cdot p'' \dots p^{(n-2)} p \underset{\rightarrow}{p(P)} P_n \cdot p' \cdot p'' \dots p^{(n-2)} q.$$

For this to be true, however, it is necessary that (1) neither p nor q should be a P_m, where $m < n$, and (2) that

$$p > p^{(n-2)} > \dots > p'' > p', \quad q > p^{(n-2)} > \dots > p'' > p'$$

by the conventions we decided on in representing a product in the form $P_n \cdot p' \cdot p'' \dots p^{(n-1)}$. That is, p and q are excluded from (1) the $(n-1)$ terms preceding P_n in p, and (2) the finite number of members of P not greater in numerical value than the largest $p^{(k)}$. Except for this finite group of values, p and q may assume any other value in P, and the order of the products $P_n.p'.p''...p^{(n-2)}.p$ and $P_n.p'.p''...p^{(n-2)}.q$, which will actually exist, will be the same as the order in P of p and q. That is, the series of the $P_n.p'.p''...p^{(n-2)}.p$'s will be similar to that of the p's, with a finite number of terms of the latter removed. Since, however, the ordinal number of the p's has no immediate predecessor, it can be shown readily that the removal of a finite number of terms from P will not alter its number, and therefore that the series of the $P_n.p'.p''...p^{(n-2)}.p$'s, arranged as they occur in $p(P)$, where $P_n, p', p'', ..., p^{(n-2)}$ are assigned, and p is allowed to take all possible values, has the ordinal number α.

In a precisely parallel manner, it may be shown that the series of the $P_n.p'.p''...p^{(n-3)}.q.r$'s, arranged as they occur in $p(P)$, where $P_n, p', p'', ..., p^{(n-3)}$ are assigned, q is allowed to take all possible values, and r is given some particular appropriate value for each value of q, has the ordinal number q.

Therefore, the series of the $P_n.p'.p''...p^{(n-3)}.q.r$'s, arranged as they occur in $p(P)$, where $P_n, p', p'', ..., p^{(n-3)}$ are assigned and *both q and r are allowed to take all possible values*, forms a series of the number α of series of the number α, or a series of the number α^2 by the definition of α^2.

Similarly, the series of the $P_n.p'.p''...p^{(n-4)}.q.r.s$'s, arranged as they occur in $p(P)$, where $P_n, p', p'', ..., p^{(n-4)}$ are assigned, and q, r, and s are allowed to take all possible values, forms a series of the number α of the series of the number α^2, or a series of the number α^3.

In a similar manner it can be shown that it follows in general from rules (2) and (3) that if the series of the $P_n.p'.p''...p^{(k)}.p^{(k+1)}...p^{(n-1)}$'s, where $P_n, p', p'', ..., p^{(k)}$ are assigned, and $p^{(k+1)}, ..., p^{(n-1)}$ are allowed to take all possible values, when arranged as they occur in $p(P)$, has the umber α^{n-k-1}, the series of the $P_n.p'.p''...p^{(k-1)}.p^{(k)}...p^{(n-1)}$'s where $P_n, p', p'', ..., p^{(k-1)}$ are assigned, and $p^{(k)}, ..., p^{(n-1)}$ are allowed to take all possible values when arranged as they occur in $p(P)$, has the number α^{n-k}.

Therefore, by mathematical induction, the number of the series of terms $P_n.p'.p''...p^{(n-1)}$, arranged as they occur in $p(P)$, where P_n is given and $p', p'', ..., p^{(n-1)}$ are allowed to assume all possible values, is α^{n-1}.

Now, by (1), $p(P)$ consists in the various series of terms $P_n.p'.p''...p^{(n-1)}$, where $p', p'', ..., p^{(n-1)}$ are allowed to assume all possible values, arranged in the order of magnitude of n.

Therefore the ordinal number of $p(P)$ is

$$\alpha^1+\alpha^2+\alpha^3+\alpha^4+\ldots+\alpha^n+\ldots=\alpha^\omega.$$

As an example of $p(P)$, let P be the series

1, 3, 7, 13, 19, 29, ..., 2, 5, 11, 17, 23, 31, ...,

where every prime whose position in the series of primes is odd belongs in the first part of the series, and every prime whose position in the series of primes is even belongs in the second part. Then $p(P)$ will be the series

3.7, 3.13, 3.19, 3.29, ..., 3.2, 3.5, 3.11, 3.17, 3.23, 3.31, ...
7.13.19, 7.13.29, ..., 7.13.17, 7.13.23, 7.13.31, ...
7.19.29, 7.19.37, ..., 7.19.23, 7.19.31, 7.19.41, ...
7.29.37, ..., 7.29.31, ...
....
....
7.2.13, 7.2.19, 7.2.29, ..., 7.2.5, 7.2.11, 7.2.17, ...
7.5.13, 7.5.19, 7.5.29, ..., 7.5.11, 7.5.17, 7.5.23, ...
....
....
13.19.29.37, 13.19.29.43, ..., 13.19.29.41, 13.19.29.47, ...
13 19.37.43, 13.19.37.53, ..., 13.19.37.47, 13.19.37.59, ...
....
....
13.19.23.29, 13.19.23.37, ..., 13.19.23.31, 13.19.23.41, ...
13.19.31.37, 13.19.31.43, ..., 13.19.31.41, 13.19.31.47, ...
....
....
13.29.37.43, ..., 13.29.37.41, ...
13.29.43.53, ..., 13.29.43.47, ...
....
....
13.29.31.41, ..., 13.29.31.37, ...

and so on indefinitely.

4. Given a set of series $A_1, A_2, A_3, \ldots, A_n, \ldots$, whose numbers are positive integers, construct the series of numbers, S, such that, if $a \underrightarrow{A_n} b$, $2^n(2a-1) \underrightarrow{S} 2^n(2b-1)$, and, if a belongs to A_m and b to A_n (if $m<n$), $2^n(2a-1) \underrightarrow{S} 2^m(2b-1)$.

It is clear that no term in S is repeated, for, if $a \neq b$, $2^n(2a-1) \neq 2^n(2b-1)$, and, if $m > n$, there are no pairs of terms a and b such that $2^n(2a-1) = 2^m(2b-1)$, for, if this could happen, an odd number $2a-1$ would equal an even number $2^{m-n}(2b-1)$. Let us call the series S, obtained from $A_1, A_2, \ldots, A_n, \ldots, S_n(A_n)$. Representing the number of each A_k by α_k, it is easy to see that the ordinal number of $S_n(A_n)$ is $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n + \ldots$. As $\alpha \geqq \alpha_1$, $\alpha_1 + \alpha_2 \geqq \alpha_2$, $\alpha_1 + \alpha_2 + \alpha_3 \geqq \alpha_3, \ldots, \alpha_1 + \alpha_2 + \ldots + \alpha_n \geqq \alpha_n, \ldots$, it is obvious that that the ordinal number of $S_n(A_n) \geqq$ the upper limit of the ordinal number of A_n. As an example of $S_n(A_n)$, let A_n be the series

$$2^{n!}.1,\ 2^{n!}.3,\ 2^{n!}.5,\ 2^{n!}.7,\ \ldots,\ 2^{n!-1}.1,\ 2^{n!-1}.3,\ 2^{n!-1}.5,\ \ldots$$

$$\ldots,\ 2^{n!-2}.1,\ 2^{n!-2}.3,\ 2^{n!-2}.5,\ \ldots,\ \ldots,$$

$$\ldots,\ \ldots,\ 2^{(n-1)!+1}.1,\ 2^{(n-1)!+1}.3,\ \ldots,$$

whose ordinal number is obviously

$$\omega\,[n! - (n-1)!] = \omega\,[(n-1)\,(n-1)!],$$

whose upper limit is $\omega.\omega$. Then $S_n(A_n)$ will be the series

$$2\,(2.2\ - 1),\ 2\,(2.2\,.3 - 1),\ 2\,(2.2.5 - 1),\ \ldots,$$

$$2^2(2.2^2 - 1),\ 2^2(2.2^2.3 - 1),\ \ldots,$$

$$2^2(2.2\ - 1),\ 2^2(2.2\,.3 - 1),\ \ldots,$$

$$2^3(2.2^6 - 1),\ 2^3(2.2^6.3 - 1),\ \ldots,$$

$$2^3(2.2^5 - 1),\ 2^3(2.2^5.3 - 1),\ \ldots,$$

$$2^3(2.2^4 - 1),\ 2^3(2.2^4.3 - 1),\ \ldots,$$

$$2^3(2.2^3 - 1),\ 2^3(2.2^3.3 - 1),\ \ldots,\ 2^4(2.2^{24} - 1),\ \ldots.$$

Its number will be ω^2, which is $\geqq \omega^2$.

5. The number of I, by the definition of ω, is ω.

Let us write $\Phi(A)$ for $p\{P(A)\}$. Then, by (2), (3), the number of $\Phi(I)$ is

$$\omega^\omega.$$

Then, by (2), (3), the number of $\Phi^2(I)$ is

$$(\omega^\omega)^\omega = \omega^{\omega^2}.$$

Then, by (2), (3), the number of $\Phi^3(I)$ is

$$(\omega^{\omega^2})^\omega = \omega^{\omega^3}.$$

Then, by (2), (3), the number of $\Phi^n(I)$ is

$$\omega^{\omega^n}.$$

Then, by (4), the number of $S_n\{\Phi^n(I)\}$ is

$$\omega^\omega + \omega^{\omega^2} + \omega^{\omega^3} + \ldots + \omega^{\omega^n} + \ldots = \omega^{\omega^\omega}.$$

Then, by (2), (3), the number of $\Phi[S_n\{\Phi^n(I)\}]$ is*

$$\left(\omega^{\omega^\omega}\right)^\omega = \omega^{\omega^{\omega+1}}.$$

Then, by (2), (3), the number of $\Phi^2[S_n\{\Phi^n(I)\}]$ is

$$\left(\omega^{\omega^{\omega+1}}\right)^\omega = \omega^{\omega^{\omega+2}}.$$

Then, by (2), (3), the number of $\Phi^m[S_n\{\Phi^n(I)\}]$ is

$$\omega^{\omega^{\omega+m}}.$$

Then, by (4), the number of $S_m(\Phi^m[S_n\{\Phi^n(I)\}])$ is

$$\omega^{\omega^{\omega+1}} + \omega^{\omega^{\omega+2}} + \omega^{\omega^{\omega+3}} + \ldots + \omega^{\omega^{\omega+n}} + \ldots = \omega^{\omega^{\omega.2}}.$$

Let us write $\Psi(A)$ for $S_m\{\Phi^m(A)\}$.

We have shown that the number of $\Psi(I)$ is ω^{ω^ω}, and that that of $\Psi^2(I)$ is $\omega^{\omega^{\omega.2}}$. Similarly, it can be shown that the number of $\Psi^n(I)$ is $\omega^{\omega^{\omega.n}}$. Therefore, by (4), the number of $S_m\{\Psi^m(I)\}$ is equal to $\omega^{\omega^\omega} + \omega^{\omega^{\omega.2}} + \omega^{\omega^{\omega.3}} + \ldots + \omega^{\omega^{\omega.n}} + \ldots$, and is at least $\omega^{\omega^{\omega^2}}$.

$\therefore$ the number of		$\Phi[S_m\{\Psi^m(I)\}]$	is at least		$\left(\omega^{\omega^{\omega^2}}\right)^\omega = \omega^{\omega^{\omega^2+1}}$.
$\therefore$,,	,,	$\Phi^n[S_m\{\Psi^m(I)\}]$	,,	,,	$\omega^{\omega^{\omega^2+n}}$.
$\therefore$,,	,,	$\Psi[S_m\{\Psi^m(I)\}]$	,,	,,	$\omega^{\omega^{\omega^2+\omega}}$.
Similarly, ,,	,,	$\Psi^2[S_m\{\Psi^m(I)\}]$	,,	,,	$\omega^{\omega^{\omega^2+\omega.2}}$.
,,	,,	$\Psi^n[S_m\{\Psi^m(I)\}]$	,,	,,	$\omega^{\omega^{\omega^2+\omega.n}}$.
,,	,,	$S_n(\Psi^n[S_m\{\Psi^m(I)\}])$	,,	,,	$\omega^{\omega^{\omega^2.2}}$.

* We can always apply Φ to any S_n [for S_n, and in consequence $P(S)$, has no last term] and be sure of raising the number of the S to the ωth power.

Let us write $F(A)$ for $S_n\{\Psi^n(A)\}$. We have shown that the number of $F(I)$ is at least $\omega^{\omega^{\omega^2}}$, and that that of $F^2(I)$ is at least $\omega^{\omega^{\omega^2 . 2}}$. Similarly, we may prove that the number of $F^n(I)$ is at least $\omega^{\omega^{\omega^2 . n}}$.

$\therefore$ the number of $S_n\{F^n(I)\}$ is at least $\omega^{\omega^{\omega^2 . \omega}} = \omega^{\omega^{\omega^3}}$.

"	"	$\Phi[S_n\{F^n(I)\}]$	"	"	$\omega^{\omega^{\omega^3+1}}$.
"	"	$\Phi^m[S_n\{F^n(I)\}]$	"	"	$\omega^{\omega^{\omega^3+m}}$.
"	"	$\Psi[S_n\{F^n(I)\}]$	"	"	$\omega^{\omega^{\omega^3+\omega}}$.
"	"	$\Psi^m[S_n\{F^n(I)\}]$	"	"	$\omega^{\omega^{\omega^3+\omega . m}}$.
"	"	$F[S_n\{F^n(I)\}]$	"	"	$\omega^{\omega^{\omega^3+\omega^2}}$.
"	"	$F^m[S_n\{F^n(I)\}]$	"	"	$\omega^{\omega^{\omega^3+\omega^2 . m}}$.
"	"	$S_m(F^m[S_n\{F^n(I)\}])$	"	"	$\omega^{\omega^{\omega^3 . 2}}$.

Let us write $G(A)$ for $S_n\{F^n(A)\}$. We have shown that the number of $G(I)$ is at least $\omega^{\omega^{\omega^3}}$ and that that of $G^2(I)$ is at least $\omega^{\omega^{\omega^3 . 2}}$. It can be shown in the same manner that the number of $G^n(I)$ is at least $\omega^{\omega^{\omega^3 . n}}$. Let us write $H(A)$ for $S_n\{G^n(I)\}$. Then it is obvious that the ordinal number of $H(I)$ is at least $\omega^{\omega^{\omega^4}}$.

In a precisely analogous manner we can construct a series of number at least $\omega^{\omega^{\omega^n}}$, whatever n may be. Let us call this series, in general, $K_n(I)$, where $K_1(I) = \Psi(I)$, $K_2(I) = F(I)$, $K_2(I) = G(I)$. $K_n(I)$ is always constructed according to a perfectly definite method, leaving no possible doubt what step to take after any given step, for after any series you have obtained you form the Φ of that series, and after any series of series, you form its S. Therefore no implicit postulation of Zermelo's axiom is to be found in any of my constructions, so that I can be sure that they always exist. Therefore I can form $S_n\{K_n(I)\}$, and its number will be $\omega^{\omega^{\omega^\omega}}$ at least.

In a precisely parallel manner we can construct a rearrangement not less than $\omega^{\omega^{\omega^{\omega^\omega}}}$, etc. Given rearrangements of I, which will be at least as large as ω, ω^ω, ω^{ω^ω}, $\omega^{\omega^{\omega^\omega}}$, etc., we can, by means of S, construct a rearrangement of I at least as large as an

$$\omega \text{ times } \left\{ \omega^{\omega^{\omega^{\omega^{\omega^{\cdots}}}}} \right. ,$$

or an ϵ.

In general, given a rearrangement L of I, such that the ordinal number of $L(I)$ is at least as large as $\omega^{\omega^{\alpha}}$, $\Phi\{L(I)\}$ will have an ordinal number at least as large as $\omega^{\omega^{\alpha+1}}$. Also, given a sequence of rearrangements $L_1, L_2, L_3, \ldots, L_n, \ldots,$ of I of ordinal numbers at least as large as $\omega^{\omega^{\alpha_1}}, \omega^{\omega^{\alpha_2}}, \ldots,$ $\omega^{\omega^{\alpha_n}}, \ldots,$ respectively, if α_ω be the limit of the sequence $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \ldots,$ then the ordinal number of $S_n(\omega^{\omega^{\alpha_n}}) \geqq \alpha_\omega$. Therefore we can construct a number at least as large as $\omega^{\omega^{\alpha}}$, where α denotes any ordinal number which can be formed from 1 by the repetition of the operations (1) of adding 1 to a previously given ordinal number, and (2) of taking the number of any given infinite sequence of numbers previously obtained. But the class of such numbers is the class of numbers of fundamental sequences of Ω. Therefore, if α is the number of a fundamental sequence of Ω, we can get by our method a not smaller rearrangement of the number-series than $\omega^{\omega^{\alpha}}$. If, then, $\omega^{\omega^{\alpha}} \geqq \alpha$, the proposition I set out to prove is obviously proved. This is clearly true if $\omega^{\alpha} \geqq \alpha$. This can be proved in the following manner:

(1) $\omega^1 \geqq 1$.

(2) Let $\omega^{\alpha} \geqq \alpha$. Then

$$\omega^{\alpha+1} = \omega^{\alpha}.\omega = \omega^{\alpha} + \omega^{\alpha}.\omega \geqq \alpha + \omega^{\alpha}.\omega \geqq \alpha + 1.$$

(3) Let $\omega^{\alpha_n} \geqq \alpha_n$, when α_n takes any one of the infinite series of values $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m, \ldots,$ whose upper limit is α_ω. Since $\alpha_\omega > \alpha_n$, it can be shown that

$$\omega^{\alpha_\omega} = \omega^{\alpha_n+\beta} = \omega^{\alpha_n}\omega^{\beta} = \omega^{\alpha_n} + \omega^{\alpha_n}\omega^{\beta} \geqq \omega^{\alpha_n}.$$

Therefore, $\omega^{\alpha_\omega} \geqq \alpha_n$, whatever n is. Therefore, $\omega^{\alpha_\omega} \geqq \alpha_\omega$.

Therefore, when α is the number of a fundamental sequence of Ω, we have a method of rearranging the positive integers in a series of number $\geqq \alpha + 1$, and hence $> \alpha$.

It will be noted that the method I have developed enables me directly to reorder, not the whole, but a part of the series of the integers in a series of number greater than the number ω^{ω}, but this is of no importance, for let the part of I so arranged be A. Let a_1 be the numerically smallest member of A, a_2 the next, and so on. Then replace each a_n by n. This will give a rearrangement of I similar to the already obtained rearrangement of A.

The interest of the construction of rearrangements of I lies in the fact that all the proofs hitherto given of the existence of numbers greater than those of any given fundamental sequence of Ω have involved the multiplicative axiom.* By actually rearranging I in a series of such a number, we avoid this.

It should be noted that the particular nature of the process Φ we have chosen of increasing the number of a series by rearranging it is a matter of more or less indifference; any other process which, when applied to a series, always gives a larger or equal series would have done quite as well, logically. For example, if $\Phi'(B) = B$, the number of $S_n\{\Phi'^n(B)\} = \omega$, multiplied by the number of B; and, as it can be shown that $\omega . \alpha > \alpha$, it is clear that, by the same sort of proof which we used to show that Φ and S together enable us to construct a rearrangement of I larger in number than any given fundamental sequence of Ω, S alone will enable us to do it. However, at least at first, the use of Φ enables us to increase the ordinal number of the rearrangement of I more rapidly than that of S alone would.

AN ARRANGEMENT OF THE POSITIVE INTEGERS IN THE TYPE ϵ_1.

By *E. K. Wakeford*, Trinity College, Cambridge.

1. "THE numbers" means the positive integers.

Given some arrangement A of the numbers, and the set P of prime numbers, $A(P)$ denotes the result of arranging the prime numbers in the order A by putting instead of a number n of A the nth prime in order of magnitude. Unity will not be considered a prime.

When we speak of the order-type of some number in an arrangement of numbers we mean the order-type of the set of numbers preceding it in the arrangement. For instance, in the sequence 12345... the order-type of n is $(n-1)$, and in the repeated sequence 1357..., 2468... the order type of 2 is ω.

Multiplying together two relatively prime sets of numbers means forming the set consisting of members which are the products of one out of each set. The order is given by taking the first of the second set and multiplying it in turn by each of the first set, then taking the second of the second

* See Whitehead and Russell, *Principia Mathematica*, vol iii., p. 170, lines 6 from bottom to bottom.

set and treating it likewise, and so on. If N_1, N_2 are the order-types of the first two sets, the order-type of the resulting set will be N_1N_2 (where the order of letters may be important).

2. Multiplying the set

$$1\ 2\ 2^2\ 2^3\ \ldots,$$

of type ω, by the set $1\ 3\ 3^2\ 3^3\ \ldots,$

also of type ω, according to the rule above as shown:

$$\begin{array}{llllll} 1 & 2 & 2^2 & 2^3 & 2^4 & \ldots, \\ 3.1 & 3.2 & 3.2^2 & 3.2^3 & 3.2^4 & \ldots, \\ 3^2.1 & 3^2.2 & 3^2.2^2 & 3^2.2^3 & \ldots & \ldots, \end{array}$$

and we have a set of type ω^2.

Now multiplying this by the set

$$1\ 5\ 5^2\ 5^3\ \ldots,$$

which is of type ω, we obtain first the same set of type ω^2 as before, next the numbers of that set multiplied by 5, giving us $\omega^2.2$ numbers, next the number smultiplied by 5^2, giving us $\omega^2.3$ numbers in all, and so on till we get ω^3 numbers.

By taking now the set

$$1\ 7\ 7^2\ 7^3\ \ldots,$$

we obtain in all ω^4 numbers, and so on, till, when all the primes have been taken in order of magnitude, an arrangement of the primes which we shall call $A_1(P)$, we have obtained all the numbers arranged in a set of type ω^ω. Call this arrangement of the numbers A_2. The order type of a particular number Πp^c is $\Sigma\omega^r.c$, where p has order-type r in the arrangement $A_1(P)$. For instance, the order-type of $2^4.3.5^5.11$ is

$$\omega^4 + \omega^2.5 + \omega + 4.$$

3. Now form $A_2(P)$, thus arranging the primes in a type ω^ω. Then take them in succession in this order, instead of the order $A_1(P)$ in which we took them before.

The first ω primes, viz., the 1st, 2nd, 4th, 8th, ... primes, give us ω^ω numbers as before. We then take the 3rd prime, *i.e.*, 5, and multiply each of the numbers of this set by the numbers of the set

$$1\ 5\ 5^2\ 5^3 \ldots.$$

As in the previous work 1 gives us the same ω^ω numbers, 5 gives us another ω^ω numbers, making $\omega^\omega.2$ in all, 5^2 yet another, making $\omega^\omega.3$, and so on till we arrive at $\omega^{\omega+1}$ after using all the powers of five. Then taking the 6th prime, *i.e.*, 13, and multiply the existing set by the set

$$1\ 13\ 13^2\ 13^3\ldots,$$

we thus arrive at a set of type $\omega^{\omega+2}$, and after taking the 12th, 24th, 48th, ... primes we reach a set of type $\omega^{\omega\,2}$.

We notice that we have so far only used $\omega.2$ primes, viz., the first $\omega.2$ primes in $A_2(P)$, and have thus obtained $\omega^{\omega.2}$ numbers. It is almost self-evident that by taking ω^2 primes from $A_2(P)$ we shall obtain ω^{ω^2} numbers, for ω^{ω^2} is the limit of $\omega^\omega, \omega^{\omega.2}, \omega^{\omega.3}, \ldots$, just as ω^2 is the limit of $\omega, \omega.2, \omega.3, \omega.4$, etc. But we have not yet taken nearly all the numbers of $A_2(P)$. After what has been said it should be clear that as we take each prime in turn we add 1 to the index of ω, so that ω^3 primes yield ω^{ω^3} numbers, etc. Finally, by taking all the primes in the order $A_2(P)$, we obtain ω^{ω^ω} numbers, all the numbers being at length obtained. We call this arrangement A_3.

The order-type of a particular number Πp^c is as before $\Sigma\omega^r.c$, but here p has order-type r in the arrangement $A_2(P)$. For instance, the order-type of $2^3.11.13^2$ is $\omega^{\omega^2}+\omega^{\omega+1}.2+3$, for 2, 11, 13 have order types 0, ω^2, and $\omega+1$ respectively in $A_2(P)$.

4. It is not difficult to see that by forming $A_3(P)$ we could obtain the numbers in a type $\omega^{\omega^{\omega^\omega}}$ and so on. In each case the formula for the order-type of any particular number Πp^c in the arrangement A^r is $\Sigma\omega^r.c$, where p has order-type r in the arrangement $A_{n-1}(P)$.

To obtain a set of type $\omega^{\omega^{\omega^{\cdots}}}\equiv\epsilon_1$ we may simply take any ω^2 arrangement of the numbers, operating on the nth ω of them with the operator A_n, as suggested:

$$\begin{aligned}&A_1(1,\ 3,\ 5,\ 7,\ \ldots\ldots\ldots\ldots\ldots)\,\omega,\\ +&A_2(2.1,\ 2.3,\ 2.5,\ 2.7,\ \ldots)\,\omega^\omega,\\ +&A_3(2^2.1,\ 2^2.3,\ 2^2.5,\ 2^2.7,\ \ldots)\,\omega^{\omega^\omega},\\ +&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \text{etc.}\end{aligned}$$

The rule for finding the type of any particular number would not be so elegant. Possibly it could be improved by some device, but it is quite simple even as it stands.

5. This last process, which finds a number for the limit of any sequence of given numbers, corresponds, I believe, to Mr. Wiener's S operator. After reading some of his work I saw that if I rearranged the indices of the primes instead of the primes themselves, I could raise a number to the power of ω, instead of raising ω to the power of a number. This latter process is much quicker at first, but stops at ϵ_1, since $\omega^{\epsilon_1} = \epsilon_1$, while the former (combined with the S process) goes on for ever. If we take the primes in order A, and their indices in order B, the resulting set has order-type B^A, so that by this means any number can be raised to the power of itself. This process combined with the S method goes on for ever. In fact, any method (even adding unity), which must increase any Cantor number, combined with the S method, is bound to go on for ever, and so to prove the existence of any number of the second class that has been defined.

ON THE SERIES FOR SINE AND COSINE.

By *Prof. E. J. Nanson.*

The power series for $\sin x$, $\cos x$ follow at once from the two theorems

$$\sin x \text{ lies between } S_n(x),\ S_{n+1}(x) \ldots\ldots\ldots\ldots (1),$$

$$\cos x \text{ lies between } C_n(x),\ C_{n+1}(x) \ldots\ldots\ldots\ldots (2),$$

where
$$S_n(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

$$C_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^n x^{2n}}{(2n)!}.$$

Since $\cos x$ lies between 1 and $1 - \frac{1}{2}x^2$ for all values of x, it is sufficient, in order to prove (1), (2), to show first that (1) follows from (2) for all values of x, and, second, that if (1) is granted, then it follows that $\cos x$ lies between $C_{n+1}(x)$, $C_{n+2}(x)$.

Now these two results may be proved by similar elementary methods. For we have

$$\frac{\sin 2mx}{\sin x} = \cos x + \cos 3x + \ldots + \cos (2m-1)\, x.$$

Hence, assuming (2), it follows that

$$\frac{\sin 2mx}{\sin x} \text{ lies between } P_n,\ P_{n+1},$$

where
$$P_r = s_0 - \frac{x^2}{2!}s_2 + \ldots + (-1)^n \frac{x^{2n}}{(2n)!}s_{2n},$$

$$s_r = 1^r + 3^r + \ldots + (2m-1)^r$$

$$s_r = \frac{2^r m^{r+1}}{r+1}(1+\epsilon_r), \text{ and } \operatorname*{Lt}_{m\to\infty} \epsilon_r = 0.$$

Hence, if $y = 2mx$, $L = x/\sin x$, it follows that

$$L \sin y \text{ lies between } Q_n,\ Q_{n+1},$$

where

$$Q_n = \frac{y}{1!} - \frac{y^3}{3!}(1+\epsilon_2) + \ldots + (-1)^n \frac{y^{2n+1}}{(2n+1)!}(1+\epsilon_{2n}).$$

Hence, making $m \to \infty$, it follows that

$$\sin y \text{ lies between } S_n(y),\ S_{n+1}(y),$$

so that (1) follows from (2).

Again we have

$$\frac{1-\cos 2mx}{2\sin x} = \sin x + \sin 3x + \ldots + \sin(2m-1)x.$$

Hence, assuming (1), it follows that

$$\frac{1-\cos 2mx}{2\sin x} \text{ lies between } P_n,\ P_{n+1},$$

when
$$P_n = \frac{x}{1!}s_1 - \frac{x^3}{3!}s_3 + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}s_{2n+1},$$

and hence, as before, it follows that

$$L(1-\cos y) \text{ lies between } Q_n,\ Q_{n+1},$$

where

$$Q_n = \frac{y^2}{2!}(1+\epsilon_1) - \frac{y^4}{4!}(1+\epsilon_3) + \ldots + (-1)^n \frac{y^{2n+2}}{(2n+2)!}(1+\epsilon_{2n+1}).$$

Hence, making $m \to \infty$, it follows that

$$1 - \cos y \text{ lies between } R_n,\ R_{n+1},$$

where $$R_n = \frac{y^2}{2!} - \frac{y^4}{4!} + \ldots + (-1)^n \frac{y^{2n+2}}{(2n+2)!},$$

and hence that

$$\cos y \text{ lies between } C_{n+1}(y),\ C_{n+2}(y).$$

Thus the two theorems (1), (2) have been proved for all values of x. Reference may be made to the *Messenger*, vol. xxxv., pp. 58–69, 142–144; vol. xliii., pp. 63–71; and to the *Mathematical Gazette*, vol. iii., pp. 284–288.

SOME SIMPLE TRANSFORMATIONS OF STOKES' CURRENT FUNCTION EQUATION.

By *J. R. Wilton, M.A., B.Sc.*, Assistant Lecturer in Mathematics in the University of Sheffield.

1. One or two of the following elementary transformations of Stokes' current function equation, namely,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = 0 \quad\ldots\ldots\ldots\ldots\ldots (1),$$

may possibly be unfamiliar, though most of them are probably well known to all who have studied the subject of fluid motion symmetrical about an axis.

Using the notation

$$R = \varpi^2,\quad r = \sqrt{(x^2+\varpi^2)},\quad \xi = r + x,\quad \eta = r - x,$$

$$r_1 = [(x+\tfrac{1}{2}c)^2 + \varpi^2]^{\frac{1}{2}},\quad r_2 = [(x-\tfrac{1}{2}c)^2 + \varpi^2]^{\frac{1}{2}},$$

we shall find

$$\begin{aligned}\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} &= \frac{\partial^2\psi}{\partial x^2} + R\frac{\partial^2\psi}{\partial R^2}\\
&= \frac{\partial^2\psi}{\partial r^2} + \frac{4R}{r}\frac{\partial^2\psi}{\partial r\,\partial R} + 4R\frac{\partial^2\psi}{\partial R^2}\\
&= \frac{\partial^2\psi}{\partial r^2} + \frac{2x}{r}\frac{\partial^2\psi}{\partial x\,\partial r} + \frac{\partial^2\psi}{\partial x^2}\\
&= \frac{4}{\xi+\eta}\left(\xi\frac{\partial^2\psi}{\partial\xi^2} + \eta\frac{\partial^2\psi}{\partial\eta^2}\right) \quad\ldots\ldots\ldots\ldots\ldots (2)\\
&= \frac{\partial^2\psi}{\partial r_1^2} + \frac{r_1^2+r_2^2-c^2}{r_1 r_2}\frac{\partial^2\psi}{\partial r_1\,\partial r_2} + \frac{\partial^2\psi}{\partial r_2^2} \ldots (3).\end{aligned}$$

Also, if ϕ is the velocity potential corresponding to the stream function ψ, we have the relations

$$\frac{\partial\phi}{\partial x} = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}, \quad \frac{\partial\phi}{\partial\varpi} = \frac{1}{\varpi}\frac{\partial\psi}{\partial x},$$

whence, if we take ϕ and ψ as independent variables, we find

$$\frac{\partial x}{\partial\phi} = -\frac{1}{2}\frac{\partial R}{\partial\psi}, \quad \frac{\partial x}{\partial\psi} = \frac{1}{2R}\frac{\partial R}{\partial\phi},$$

and therefore

$$\frac{\partial^2 R}{\partial\psi^2} + \frac{\partial^2}{\partial\phi^2}(\log R) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).*$$

A solution of equation (4) is easily seen to be

$$\varpi^2 = R = (a^2 - \psi^2)\sec^2\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The corresponding value of x is, it is easy to verify,

$$x = \psi\tan\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

The stream lines are therefore the hyperboloids

$$\frac{\varpi^2}{a^2 - \psi^2} - \frac{x^2}{\psi^2} = 1,$$

of which $\psi = 0$ consists of that part of the plane $x = 0$ which lies outside the circle $\varpi = a$, so that equations (5) and (6) give, in a form somewhat simpler than the familiar form as given, for instance, in Lamb's *Hydrodynamics*, § 102, 3°, or § 108, 1°, the solution of the problem of the flow of water through a circular hole in an infinite plane.

The velocity is q, given by the equation

$$\frac{1}{q^2} = \frac{1}{4}\left\{\left(\frac{\partial R}{\partial\psi}\right)^2 + \frac{1}{R}\left(\frac{\partial R}{\partial\phi}\right)^2\right\} = \frac{\psi^2 R^2}{(a^2 - \psi^2)^2} + \frac{Rx^2}{\psi^2},$$

and over the circle $\varpi = a$, $x = 0$ this gives

$$\frac{1}{q^2} = a^2 - \varpi^2,$$

which is a well-know result.

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* A solution of equation (4) involving an arbitrary function is given by the equation $R = f(\phi + \iota\psi R^{-\frac{1}{2}})$, but this has no physical meaning as it makes the velocity everywhere infinite.

Another obvious solution of equation (4) is

$$\varpi^2 = R = \psi e^{\phi},$$

with the corresponding value of x,

$$x = \tfrac{1}{2}(\psi - e^{\phi}).$$

The stream lines are

$$2x = \psi - \frac{\varpi^2}{\psi},$$

and the ϕ curves are

$$2x = -e^{\phi} + \frac{\varpi^2}{e^{\phi}},$$

both of which equations represent a system of confocal paraboloids of revolution.

The velocity at any point is given by the equation

$$\frac{4}{q^2} = \varpi^2 + e^{2\phi}.$$

2. The equation resulting from the transformation (2) is remarkably simple, and it has an obvious general solution in the form of an infinite series of terms, each of which consists of the product of a Bessel function of ξ multiplied by a Bessel function of η, one of the two arguments being a pure imaginary. This form of the equation is adapted for giving the solution of problems in which certain boundary conditions have to be satisfied over some one or more of a set of confocal paraboloids.

The transformation (3) is somewhat remarkable in that it does not become indeterminate on putting $c = 0$. In fact, when $c = 0$, the solution is

$$\psi = \sqrt{(r_1^2 - r_2^2)}\,\{F(r_1^2 - r_2^2) + G(r_1/r_2)\},$$

showing that, by a simple transformation, it will be possible to obtain a solution when c is not zero in an ascending series of powers of c. I am not aware, however, that any physical importance can be attached to this fact.

The equation

$$\frac{\partial^2\psi}{\partial r_1^2} + \frac{\partial^2\psi}{\partial r_2^2} + \frac{r_1^2 + r_2^2 - c^2}{r_1 r_2}\,\frac{\partial^2\psi}{\partial r_1\,\partial r_2} = 0 \;\ldots\ldots\ldots (7)$$

has a solution of the form

$$\psi = \int_{-1}^{1} F(\mu) \sqrt{\left(\frac{r_1^2}{1+\mu} + \frac{r_2^2}{1-\mu} - c^2\right)}\, d\mu,$$

which may conceivably lead to some new results, though the task of looking for such would probably be a somewhat thankless one.

An evident set of particular solutions of equation (7) is

$$r_1, \quad r_2, \quad \frac{r_1^2 - c^2}{r_2}, \quad \frac{r_2^2 - c^2}{r_1} \quad \text{..............(8)},$$

of which the two latter are most easily derived by noticing that the solution for a source at the origin of r_1 is

$$\cos\theta_1 = (x + \tfrac{1}{2}c)/r_1,$$

and therefore a solution of equation (7) is

$$r_1 - 2c\cos\theta_1 = (r_2^2 - c^2)/r_1.$$

Each of the four solutions (8) corresponds to a motion due to a semi-infinite line source and a semi-infinite line sink in the same straight line, the strength in every case being 2π per unit length. The third of the four solutions has in addition a point source at the origin of r_2, and the fourth a point sink at the origin of r_1, the strength in each case being $8\pi c$.

The solution

$$\psi = \frac{m}{4\pi(\kappa - 1)} \left\{\kappa(r_1 - r_2) + \frac{r_1^2 - c^2}{r_2} - \frac{r_2^2 - c^2}{r_1}\right\}$$

represents a finite line source of strength m per unit length. together with two equal point sources at its extremities, of which the strength is at our disposal by proper choice of κ.

A few other rather curious results may just be noted.

The equation

$$\psi = \frac{m}{8\pi}(r_1^2 + r_2^2 - c^2)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

represents the motion inside a sphere due to a diametral line source of strength m per unit length with equal point sinks at the extremities. And

$$\psi = \frac{m}{8\pi}(r_1^2 + r_2^2 - c^2)\left(\frac{n}{r_1} - \frac{1}{r_2}\right),$$

represents the motion in the presence of the two intersecting spheres

$$r_1^2 + r_2^2 = c^2, \quad r_1 = nr_2,$$

due to a certain arrangement of line and point sources and sinks. And

$$\psi = \frac{m}{8\pi(n+1)} \left\{ (n+2) r_1 - (2n+1) r_2 - \frac{r_1^2 - c^2}{r_2} + n \frac{r_2^2 - c^2}{r_1} \right\}$$

represents the motion, either inside or outside the sphere $r_1 = nr_2$, due to a finite radial line source of strength m per unit length, together with a point sink at the extremity remote from the surface. The length of the source in comparison with the radius of the sphere is arbitrary, but one end of it is necessarily in contact with the surface.

THE RELATION BETWEEN THE PENCIL OF TANGENTS TO A RATIONAL PLANE CURVE FROM A POINT AND THEIR PARAMETERS.*

By *J. E. Rowe.*

Introduction.

THE relation between the *pencil* of tangents to a rational plane curve from a point and their *parameters* along the curve is a question which arises very early in the study of rational plane curves. After further reading on the subject the student finds himself in a position to predict that no simple relation of this kind is *likely* to exist. In as much as this is quite an unsatisfactory position to hold—one which is not justified by facts, but one which the student is forced to hold because there has never been sufficient research on this subject—I shall outline a method of attacking the problem which is straightforward and which makes it possible, in particular cases at least, to discover interesting relations which *do* exist between the pencil of tangents and their parameters. The rational plane cubic is taken up as the simplest illustrative example; incidentally, it is necessary to give a new geometric interpretation to several combinants of two binary cubics.

* Read before the *American Mathematical Society*, April 26, 1913.

Covariants of rational curves defined by the pencil of tangents from a point.

§ 1. Let the R^n (or the rational plane curve of order n) be written parametrically

(1) $$R^n \equiv x_\iota = a_\iota t^n + nb_\iota t^{n-1} \ldots \quad (\iota = 0, 1, 2).$$

If (1) is cut by the two lines

(2) $$(\zeta x) = \zeta_0 x_0 + \zeta_1 x_1 + \zeta_2 x_2 = 0,$$

(3) $$(\eta x) = \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 = 0,$$

the result is the two binary n-ics

$$u_n \equiv (a\zeta)\, t^n + n\, (b\zeta)\, t^{n-1} \ldots = 0,$$

and $$v_n \equiv (a\eta)\, t^n + n\, (b\eta)\, t^{n-1} \ldots = 0,$$

whose roots are the parameters of the points in which the lines (2) and (3) cut the R^n.

It has been explained in a previous paper* how the combinants of u_n and v_n are transformed into covariant curves of the R^n by substituting (in the combinant of u_n and v_n equated to zero) x_0, x_1, x_2 for the coordinates of the point in which the lines (2) and (3) intersect, which point will be referred to in the sequel as the point x.

Consider the expression

(6) $$u_n K + v_n = 0,$$

a binary n-ic in t; its discriminant equated to zero may be put in the form

(7) $$D_0 K^{2n-2} + D_1 K^{2n-3} \ldots D_{2n-2} = 0.$$

Invariants of (7) are combinants of u_n and v_n. A combinant of u_n and v_n may be defined† as a function of their coefficients (and possibly variable, although not generally containing the variable in what follows) which is unaltered (except by a constant multiplier), not only when the variable is linearly transformed, but also when, for u_n and v_n, linear combinations of u_n and v_n are substituted. Hence, an invariant of (7) is a combinant of u_n and v_n, *i.e.*, it is unaltered if we substitute $lu_n + mv_n$, $l'u_n + m'v_n$ for u_n, v_n. For, by this substitution, we get the same invariant of $(lK + l')\, u_n + (mK + m')\, v_n$, which is equivalent to a linear transformation of K, by which the invariants of (7) are unaltered.

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* J. E. Rowe, "Important covariant curves and a complete system of invariants of the rational quartic curve," *Transactions of the American Mathematical Society*, vol. xii. (July, 1911), pp. 295–6.

† Salmon, *Modern Higher Algebra*, Fourth Edition, p. 161.

The roots of (7) are those values of K which, substituted in

(8) $$(\zeta x)\,K + (\eta x) = 0,$$

yield equations of tangents to the R^n through the point x. Any invariant relation imposed upon the roots of (7) imposes that same invariant relation upon the pencil of tangents from the point x to the R^n. Hence, by the use of the usual translation scheme a combinant of u_n and v_n, derived as an invariant of (7), becomes (equated to zero) the equation of a covariant locus of the R^n defined by a projective relation connecting the pencil of tangents from any point of it to the R^n.

Covariants of the R^n defined by the parameters of tangents from a point.

§ 2. The Jacobian of u_n and v_n is a combinant which, equated to zero, may be written in the form

(9) $$E_0 t^{2n-2} + E_1 t^{2n-3} \ldots E_{2n-2} = 0.$$

The roots of (9) are those values of t which occur as squared factors in members of ths system* of binary n-ics (6); in other words the roots of (9) are the $2n-2$ parameters of the tangents of the R^n from a point of x. Invariants of (9) are, of course, combinants of u_n and v_n. Hence, a combinant of u_n and v_n derived as an invariant of (9) is, equated to zero, the equation of a covariant of the R^n defined by some projective relation connecting the parameters of the $2n-2$ tangents that can be drawn from a point of it to the R^n.

Covariants of R^n from which the pencil of tangents and their parameters are projectively equivalent.

§ 3. To derive these curves it is only necessary to make a comparative study of the invariants of (7) and (9). Suppose that I_2 and I_4 are invariants of (7), and I_2' and I_4' the same invariants of (9), of degree in the coefficients of these equations indicated by their subscripts. Then, if

(10) $$\frac{I_4}{I_2^2} = \frac{I_4'}{I_2'^2},$$

we have, by cross-multiplication and transposition, the curve

(11) $$I_2'^2 I_4 - I_2^2 I_4' = 0.$$

But this curve would arise in the same way from

(12) $$\frac{I_4 + KI_2^2}{I_4 + mI_2^2} = \frac{I_4' + KI_2'^2}{I_4' + mI_2'^2}.$$

* Salmon (*loc. cit.*, p. 4), p. 162.

Hence (11) is the equation of a covariant locus of the R^n such that the pencil of tangents and the parameters of these tangents from a point of it, *i.e.*, a point of (11), are projectively equivalent for a *certain set of invariant relations.* If a definite projective property, say B', is imposed upon the *parameters* of the tangents of the R^n from any point of (11) the *pencil* of tangents will possess this property B' also, and *vice versa*, if only this invariant relation can be imposed by equating to zero an invariant of the form

$$(13) \qquad aI_2^2 + bI_4 = 0.$$

Similarly, I_6 and I_6', together with I_2 and I_2', give rise to the locus

$$(14) \qquad I_2'^3 I_6 - I_2^3 I_6' = 0,$$

from which the pencil of tangents and their parameters are projectively equivalent for the set of invariants

$$(15) \qquad a' I_2^3 + b' I_6 = 0.$$

If a curve is a factor of (11), (14), and

$$(16) \qquad I_2 I_4' - I_2' I_4 = 0,$$

then, for such a curve, the pencil of tangents and their parameters from any point of it to the R^n are projectively equivalent for the vanishing of any invariant of the form

$$(17) \qquad a_1 I^3 + b_1 I_2 I_4 + c_1 I_6.$$

If I_2, I_4, and I_6 constitute the complete system of invariants of the binary form (7), a curve whose equation is a factor of (11), (14), and (16) is such that from any point of it the pencil of tangents to the R^n and their parameters along the R^n are *projectively equivalent in the fullest sense.* The general method of procedure is so obvious that a formal statement is unnecessary. There may be a set of parameters along the R^n, the roots of a binary $(2n-2)$-ic, whose absolute invariants are the same as those of (7); this could be verified by carrying out the process just outlined. Its geometric meaning is that in such a case the pencil of tangents from a point and this set of parameters are projectively equivalent.

The rational plane cubic.

§4. Let the R^3 be written parametrically

$$(18) \qquad v_\iota = a_\iota t^3 + 3b_\iota t^2 + 3c_\iota t + d_\iota \quad (\iota = 0,\ 1,\ 2).$$

Cutting (18) by (2) and (3) gives rise to

$$(19) \quad u_3 \equiv (a\zeta)\, t^3 + 3\,(b\zeta)\, t^2 + 3\,(c\zeta)\, t + (d\zeta) = 0,$$

$$(20) \quad v_3 \equiv (a\eta)\, t^3 + 3\,(b\eta)\, t^2 + 3\,(c\eta)\, t + (d\eta) = 0.$$

The expression corresponding to (7) may be written

$$(21) \qquad D_0K^4 + D_1K^3 + D_2K^2 + D_3K + D_4 = 0,$$

The Jacobian of (19) and (20) becomes

$$(22) \quad |abx|\, t^4 + 2\,|acx|\, t^3 + [\,|adx| + 3\,|bcx|\,]\, t^2 \ldots = 0,$$

by making use of the translation scheme already referred to, which changes combinants of (19) and (20) into covariant loci of the R^3. Several of these combinants have been geometrically interpreted* before. $P=0$, the apolarity condition of (19) and (20), becomes the line on the flexes on the R^3. $Q=0$, the condition that there be a member of the system of binary cubics u_3K+v_3, which contains a cubed factor, is the equation of the three flex tangents. The eliminant of (19) and (20) becomes the point equation of the R^3, and may be expressed in the form $P^3-27Q=0$. We shall give a table of combinants and their geometric interpretation; so far as the geometric interpretation is concerned all of those given in the table are new except the first two, and the reason for their place in the table will appear as we proceed.

Table of combinants and their geometric interpretation.

Combinant.	In terms of P and Q.	Locus from which tangents to R^3:
S of (22)	$3P^2$.	Have self-apolar parameters
T of (22)	$54Q-P^3$.	Have harmonic parameters
S of (21)	$3P(P^3-24Q)$	Form self-apolar pencil
T of (21)	$-(P^6-36P^3Q+216Q^2)$	Form harmonic pencil
See later discussion	P^3-32Q $P^6-40P^3Q+432Q^2$	Cubic and sextic from which the pencil of tangents and their parameters are projectively equivalent.

By S is meant the invariant of the binary quartic of degree two in its coefficients; by T the catalecticant of the binary quartic, which is also the condition that the roots of the quartic be harmonically separated; these constitute the complete system of the binary quartic. Observe that the S and T of (21)

* W. Gross, *Mathematische Annalen*, vol. 32, (1888), pp. 144–5; Grace and Young, *Algebra of Invariants*, pp. 317–8.

cannot be expressed in terms of the S and T of (22); this leads to the theorem: *an invariant relation cannot be imposed upon the pencil of tangents from a point to the R^3 by imposing a single projective relation upon their parameters along the R^3*

The binary quartic has only one absolute invariant $\dfrac{S^3}{T^2}$; hence, by making the absolute invariants of (20) and (21) equal, we are making the roots of these equations projectively equivalent. After cross-multiplication and transposition this gives rise to a multiple of

$$(24)\quad P^2 Q (P^{12} - 99 P^9 Q + 3656 P^6 Q^2 - 432139 P^3 Q^3 + 72^3 Q^4) = 0,$$

which readily factors into

$$(25)\quad P^3 Q (P^3 - 27 Q)(P^3 - 32 Q)(P^6 - 40 P^2 Q + 432 Q^2) = 0.$$

Hence, not only do the pencil of tangents and their parameters become projectively equivalent from any point of $Q=0$ and $P^3 - 27Q = 0$ (where there is coincidence), but also from a point of either of the curves

$$(26)\qquad P^3 - 32 Q = 0,$$

$$(27)\qquad P^6 - 40 P^3 Q + 432 Q^2 = 0,$$

which are entirely new loci.

It should be noticed that P is a factor of the S of (22) and (21) and that the pencil of tangents to the R^3 from any point of it as well as the parameters of these tangents are self-apolar. Also that the pencil of tangents from a point of P to R^3 is harmonic if only their parameters are harmonic.

The equations of the osculant conic of the R^3 at a point whose parameter is t' may be written

$$(28)\quad x_\iota = (a_\iota t' + b_\iota) t^2 + 2 (b_\iota t' + c_\iota) t + (c_\iota t' + d_\iota) \quad (\iota = 0, 1, 2).$$

If the point equation of this conic is found and t' made equal to t (to show that it has become variable), remembering the translation scheme already used, the result may be written

$$
\begin{aligned}
(28)\qquad & [4\,|abx|\,|bcx| - |acx|^2]\, t^4 \\
& + [4\,|abx|\,|bdx| - 2\,|acx|\,|adx| + 2\,|acx|\,|bcx|]\, t^3 \\
& + [4\,|abx|\,|cdx| + 2\,|acx|\,|bdx| \\
& + 3\,|bcx|^2 - |adx|^2 - 2\,|adx|\,|bcx|]\, t^2 \\
& + [\ldots\ldots\ldots\ldots\ldots\ldots]\, t + [\ldots\ldots\ldots\ldots\ldots\ldots] = 0.
\end{aligned}
$$

If a particular value of t, say t', is substituted in (29), we have the point equation of the osculant conic of the R^3 at the point whose parameter is t'; if the coordinates of a given point are substituted in (29), the result is a binary quartic whose roots are the parameters of the osculant conics which pass through the given point. The S and T of (29) are respectively

$$\frac{P(P^3 - 24Q)}{12}, \tag{30}$$

$$\frac{P^6 - 36P^3Q + 216Q^2}{216}. \tag{31}$$

It is easy to verify that the absolute invariant of (29) is equal to the absolute invariant of (21), which proves the theorem: *The pencil of tangents of the R^3 from a point and the parameters of the osculant conics of the R^3 through the point are projectively equivalent.*

Concluding observations.

§ 5. In the case of the R^3 it has been shown that the curves (26) and (27) are related to the R^3 in such a way that a pencil of tangents drawn from a point of either of these loci to the R^3 has the same projective property as their parameters along the R^3. Similarly, by using the same process it would be necessary to make a comparative study of the invariants of two binary sextics to solve the problem for the R^4, and this would lead to a series of curves from which the pencil of tangents to the R^4 and their parameters are projectively equivalent for a *limiting* set of invariants; the same kind of results may be obtained for the R^5 and higher R^n. It is my belief that the method just outlined may be applied with advantage to the solution of certain problems of construction in connection with invariant pencils of six lines, but I shall be content, for the present, if I have established my point—that interesting relations *do* exist between the pencil of tangents to the R^n and their parameters, and that these relations may be found by the method indicated.

ON THE SOLUTION OF SOME THEOREMS IN ELEMENTARY OPTICS, HYDROSTATICS, &c.

By *J. H. H. Goodwin.*

THE following applications of (1) a theorem of Apollonius and (2) of the well-known theorem that the centre of gravity of a uniform tetrahedron coincides with that of equal weights placed at its corners are, I believe, new. They may possibly be useful as affording simple solutions to three important propositions in Elementary Mathematics.

To find the minimum and greatest deviation when a ray of light passes through a prism in a principal plane.

Let μ be the index of refraction into the prism, ι the angle of the prism; ϕ, ϕ' the angles of incidence and emergence at the first face, and ψ', ψ the angles of incidence and emergence at the second face.

By a well-known theorem the locus of a point whose distances from fixed points O and O' are in the ratio of greater inequality $\mu:1$ is a circle whose centre C is in OO' produced. Let P, Q be points on this circle such that the angles $PO'C$ and $QO'C$ are ϕ and ψ respectively. Then, since

$$\sin PO'C : \sin POO' :: OP : O'P :: \mu : 1,$$

the angle $POO' = \phi'$, and similarly the angle $QOO' = \psi'$. Let the internal bisector of the angle QOP, meet PQ in T. Then the angle $TOO' = \frac{1}{2}(\phi' + \psi') = \frac{1}{2}\iota$ and the line TO is fixed.

Also, since OT bisects the angle QOP, it follows that

$$QT : TP = QO : PO = \mu QO' : \mu PO' = QO' : PO'.$$

Therefore TO' bisects the angle $QO'P$, and the angle

$$TO'C = \tfrac{1}{2}(\phi + \psi),$$

whence the angle

$$O'TO = \tfrac{1}{2}(\phi + \psi) - \tfrac{1}{2}\iota = \tfrac{1}{2}D,$$

where D is the deviation.

Now if ϕ' decreases from the value $\frac{1}{2}\iota$, ψ' increases, since $\phi' + \psi' = \iota$, and both P and Q move along the circle receding from each other, and for both reasons T approaches O along the fixed line TO; and the angle $O'TO$, which measures the semi-deviation, continually increases until OQ become a tangent to the circle, after which the construction becomes imaginary.

The minimum deviation occurs when P and Q coincide and $\phi' = \psi' = \frac{1}{2}\iota$. The greatest deviation occurs when either OP or OQ is a tangent to the circle, *i.e.*, when $OO'P$ or $OO'Q$ respectively is a right angle, *i.e.*, when either ϕ' or ψ' is equal to the critical angle $\sin^{-1}(1/\mu)$.

To find the centre of pressure of a triangle ABC wholly immersed in a homogeneous liquid.

Let A', B', C' be the orthogonal projections of A, B, C on the effective surface, Δ the area of $A'B'C'$; α, β, γ the lengths of AA', BB', CC', and σ the specific gravity of the liquid.

Now it is known that the centre of pressure of ABC is in the vertical through the centre of gravity of the fluid contained in $ABCC'B'A'$. Let the weight of this fluid acting at its centre of gravity be resolved into three components P, Q, R along $A'A$, $B'B$, $C'C$ respectively.

Now $ABCC'B'A'$ may be divided into three tetrahedra $AA'B'C'$, $ABB'C'$, $ACC'B$ of weights, say, w_1, w_2, and w_3; and since the centre of gravity of a tetrahedron is the same as that of weights placed at its four corners, each equal to one-fourth of the weight of the tetrahedron, it follows that the centre of gravity of $ABCC'B'A'$ lies in the resultant of $\frac{1}{4}w_1 + \frac{1}{4}(w_1 + w_2 + w_3)$ along $A'A$ together with two forces along $B'B$ and $C'C$ respectively.

Now $$P + Q + R = w_1 + w_2 + w_3,$$

and $$w_1 = \tfrac{1}{3}\Delta\alpha\sigma,$$

therefore $$P = \tfrac{1}{12}\Delta\alpha\sigma + \tfrac{1}{4}(P + Q + R) \qquad (1).$$

And, *since the values of P, Q, R are unique*, it follows by symmetry that

$$Q = \tfrac{1}{12}\Delta\beta\sigma + \tfrac{1}{4}(P + Q + R), \quad R = \&\text{c}.$$

By addition, we have

$$P + Q + R = \tfrac{1}{3}\Delta\sigma(\alpha + \beta + \gamma).$$

Hence, by (1), $$P = \tfrac{1}{12}\Delta\sigma(2\alpha + \beta + \gamma),$$

with similar expressions for Q and R.

It follows that the centre of pressure of ABC is the centre of gravity of weights proportional to $2\alpha + \beta + \gamma$, $2\beta + \gamma + \alpha$, $2\gamma + \alpha + \beta$ acting at A, B, C respectively.

[Or we may proceed as follows:

$$\begin{aligned} w_3 &\text{ the weight of } ABCC' \\ &= \tfrac{1}{3}(\tfrac{1}{2}CC'.CB\sin BCC') \times (A'C'\sin A'C'B') \times \sigma \\ &= \tfrac{1}{6}CC'.B'C'.A'C'\sin A'C'B' \times \sigma \\ &= \tfrac{1}{3}\gamma\Delta\sigma, \end{aligned}$$

and similarly w_1, $w_2 = \frac{1}{3}\alpha\Delta\sigma$ and $\frac{1}{3}\beta\Delta\sigma$ respectively; whence $P = \frac{1}{12}\Delta\sigma(2\alpha+\beta+\gamma)$ as before.]

To find the moment of inertia of a triangle ABC about any straight line PQ in its plane.

Let the triangle ABC be rotated through a small angle θ about the line PQ into the position $A'B'C'$. Then if dS be a small area of the triangle at R, and if R' be the position of R after the rotation, and if p be the perpendicular from R on PQ, the required moment of inertia is

$$\int p^2 dS = \int \frac{RR'}{\theta} . pdS = \frac{1}{\theta}\int p.(RR'.dS) = \frac{K}{\theta},$$

where K is the moment of the volume of the solid $ABCC'B'A'$ about PQ......(1).

This solid may be divided into the three tetrahedra $AA'B'C'$, $ABB'C$, $AB'C'C$. The bases of these tetrahedra, viz., $A'B'C'$, ABC, $AB'C'$ are, by orthogonal projection, each equal to Δ, the area of ABC, if we neglect θ^2 and their heights to the same order of small quantities are respectively AA', BB', CC', or $\alpha\theta$, $\beta\theta$, $\gamma\theta$, where α, β, γ are the perpendiculars from A, B, C on PQ. Hence their volumes are respectively $\frac{1}{3}\Delta\alpha\theta$, $\frac{1}{3}\Delta\beta\theta$, $\frac{1}{3}\Delta\gamma\theta$; and, since the centre of gravity of a tetrahedron is the same as that of particles, each equal to one-fourth of its weight, placed at the angular points, the moment of the figure $ABCC'B'A'$ about PQ is the same as that of particles each equal to

$$\tfrac{1}{12}\Delta\alpha\theta \text{ at } A, A', B', C',$$
$$\tfrac{1}{12}\Delta\beta\theta \text{ at } A, B, B', C,$$
$$\tfrac{1}{12}\Delta\gamma\theta \text{ at } A, B', C', C,$$

which is

$$(2\alpha+\beta+\gamma)\tfrac{1}{12}\Delta\alpha\theta + (2\beta+\gamma+\alpha)\tfrac{1}{12}\Delta\beta\theta + (2\gamma+\alpha+\beta)\tfrac{1}{12}\Delta\gamma\theta.$$

Hence, by (1), the required moment of inertia is

$$\tfrac{1}{3}\Delta\left[\{\tfrac{1}{2}(\alpha+\beta)\}^2 + \{\tfrac{1}{2}(\beta+\gamma)\}^2 + \{\tfrac{1}{2}(\gamma+\alpha)\}^2\right],$$

which is the moment of inertia about PQ of particles placed at the mid-points of the sides of ABC, each equal to one-third of the mass of the triangle.

A NON-ABELIAN GROUP WHOSE GROUP OF ISOMORPHISMS IS ABELIAN.

By *G. A. Miller*.

In the "Second report on recent progress in the theory of groups of finite order," published in volume ix. (1902) of the *Bulletin of the American Mathematical Society*, it was stated, on page 116, that "no one seems to have investigated the question whether a non-abelian group can have an abelian group of isomorphisms." In the Appendix of Hilton's *Finite Groups* (1908), page 233, the question whether a non-abelian group can have an abelian group of isomorphisms is placed among "a few interesting questions still awaiting solution." In what follows we shall give a very simple example of a non-abelian group which actually has an abelian group of isomorphisms.

Let s_1 be an operator of order 8 and let s_2 be an operator of order 2 which transforms s_1 into its fifth power. The group $\{s_1, s_2\}$, which is generated by s_1 and s_2, is clearly a non-abelian group of order 16. We extend this group by means of an operator s_3 which is of order 2 and is commutative with each of the operators of $\{s_1, s_2\}$. Finally, we extend this group just obtained by adding an operator s_4 which is also of order 2 and which satisfies the following conditions:

$$s_4^{-1}s_1s_4 = s_1, \quad s_4^{-1}s_2s_4 = s_3s_2, \quad s_4^{-1}s_3s_4 = s_3.$$

The group $\{s_1, s_2, s_3, s_4\} \equiv G$ is of order 64 and its central is of order 8, being generated by s_1^2 and s_3.

The co-sets (Nebengruppe) of G with respect to its central will be called *central co-sets* of G, and we shall first prove that each of these central co-sets is invariant under the group of isomorphisms I of G. This is equivalent to proving that every operator of the group of inner isomorphisms of G is invariant under I. This group of inner isomorphisms is of order 8 and it contains seven operators of order 2. Four of these operators correspond to operators of order 8 in G, one corresponds to operators of order 4, while each of the remaining two corresponds to a co-set involving four operators of each of the orders 2 and 4.

One, and only one, central co-set of G is composed of eight operators of order 8, each of which is transformed under the group of inner isomorphisms of G only into itself and into its fifth powers. These are the 8 operators of order 8 contained in $\{s_1, s_3\}$. Hence this central co-set, which we shall denote hereafter by C_1, is invariant under I. It is evident that G is generated by C_1 and its two central co-sets C_2, C_3, each of which involves four operators of each of the orders 2 and 4.

As one of the two central co-sets C_2, C_3 is composed of operators which are commutative with each operator of C_1, while the other co-set does not have this property, it results that each of the two co-sets C_2, C_3 must be invariant under I. As each of the three co-sets C_1, C_2, C_3 is invariant under I and as these co-sets generate G, it has been proved that *every central co-set of G is invariant under the group of isomorphisms of G.*

We shall now prove that each of the three operators of order 2 in the central of G is also invariant under I. In fact, one of these operators is the fourth power of each of the 32 operators of order 8 contained in G, and hence it is invariant under I. A second one of these three operators of order 2 is the commutator of every pair of non-commutative operators of order 2 contained in G. Hence this one is also invariant under I. As two of these three operators of order 2 are invariant under I, the third must also have this property.

Two of the operators of order 4 in the central of G are the squares of the operators of C_1, and hence these two operators are transformed among themselves under I. It is now easy to find the order of I. In fact, the operators of C_1 cannot be transformed in more than eight different ways under I since they must be transformed into themselves multiplied by operators of the central of G. These eight ways correspond to pormutations of the operators of C_1, which constitute the abelian group of order 8 and of type (1, 1, 1).

The four operators of order 2 in each of the central co-sets C_2, C_3 can be transformed into themselves multiplied only by the operators of the central whose orders divide 2. Hence these operators are transformed separately according to the group of order 4 and of type (1, 1). The order of I can therefore not exceed 128. Moreover, G admits 127 isomorphisms of order 2 since the given transformations of the operators of C_1, C_2, C_3 are independent of each other. This completes a proof of the fact that I is abelian. In fact, it is the abelian group of order 128 and of type (1, 1, 1, ...). Hence *G is a non-abelian group of order* 64 *which has an abelian group of isomorphisms of order* 128.

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A GROUP OF ORDER p^m WHOSE GROUP OF ISOMORPHISMS IS OF ORDER p^a.

By *G. A. Miller.*

In the Appendix to Hilton's *Introduction to the theory of groups of finite order* (1908), page 233, the following question appears as number eight of a list of "a few interesting questions still awaiting solution": "Can a group of order p^a have a group of automorphisms whose order is also a power of p?" When $p=2$ the infinite system of abelian groups of order p^m which contain no two equal invariants is composed of groups whose groups of isomorphisms have orders of the form p^a, but when p is an odd prime number there is no abelian group of order p^m whose group of isomorphisms has an order which is of the form p^a. We proceed to construct a non-abelian group of order p^9, p being any odd prime whatever, whose group of isomorphisms has an order of the form p^a.

Let s_1, s_2, and s_3 be three operators of orders p^4, p^3, and p^2 respectively, p being any odd prime number, and suppose that these operators satisfy the following conditions:

$$s_1^{-1}s_3s_1 = s_1^{p^3}s_3, \quad s_2^{-1}s_3s_2 = s_3^{p+1}, \quad s_1^{-1}s_2s_1 = s_2^{p^2+1}.$$

The central of the group G generated by s_1, s_2, s_3, is of index p^3, and it is generated by s_1^p, s_2^p, s_3^p. The central quotient group is abelian because each commutator of G is invariant under G, and this quotient group is of type (1, 1, 1). The sub-group of order p generated by $s_1^{p^3}$ is a characteristic sub-group of G because it is generated by each operator of order p^4 contained in G. The sub-group of order p^2 generated by $s_1^{p^3}$ and $s_2^{p^2}$ is also a characteristic sub-group, because it involves all the operators of order p which are generated separately by the operators of order p^3 contained in G.

The sub-group of order p^3 generated by the three operators $s_1^{p^3}$, $s_2^{p^2}$, s_3^p is composed of all the operators of G whose orders divide p, and hence it is also a characteristic sub-group of G. These three characteristic sub-groups will be denoted by H_1, H_2, and H_3 respectively. By adjoining to H_3 the operator $s_1^{p^2}$ there results a characteristic sub-group H_4 of order p^4, since it involves all the operators of order p^2 contained in G which generate separately the characteristic sub-group H_1.

By extending H_4 by s_2^p and then extending the group thus obtained by s_3 there result two more characteristic sub-groups H_5 and H_6 of orders p^5 and p^6 respectively. The latter of these is composed of all the operators of G whose orders divide p^2, while the former may be distinguished by the fact that it contains no operator of order p^2 which generates s_3^p.

Two additional characteristic sub-groups may be obtained by extending H_6 by means of s_1^p and then extending by means of s_2 the group thus obtained. These two characteristic sub-groups will be denoted by H_7 and H_8 respectively. The former contains the p^{th} powers of the operators of highest order contained in G, while the latter is composed of all the operators of G whose orders divide p^3. The series of characteristic sub-groups H_1, H_2, ..., H_8 satisfies the condition that each includes all those which precede it.

To prove that the order of the group of isomorphisms I of G is of the form p^a it is only necessary to prove that in every possible isomorphism of G the operators of H_α are transformed into themselves multiplied by operators of $H_{\alpha-1}$ ($\alpha = 1, 2, \ldots, 8$ and $H_0 = 1$), and that all the operators of G which are not in H_8 are transformed into themselves multiplied by operators of H_8 by each of the operators of I.* In other words, we have to prove that every isomorphism of G is a p-isomorphism.

We shall now prove that if s_1 is transformed into $t_0 s_1$ under I, then t_0 and s_1 must be commutative. All the operators of order p^4 which transform each operator of H_6 into itself multiplied by a power of s_1 must correspond to each other under I, and these operators constitute a sub-group of order p^8. Some of the operators of this sub-group transform operators of H_8 into themselves multiplied by operators of H_2, while other operators of this sub-group do not have this property. As s_3 is one of the latter operators and as s_2 is not contained in the sub-group of order p^8 composed of those operators of G which transform each operator of H_6 into itself multiplied by an operator of H_1, it has been proved that s_1 corresponds only to operators which are commutative with s_1 in every possible automorphism of G.

Suppose that, in some automorphism of G, s_1 corresponds to s_1^α, where α has one of the values $1, 2, \ldots, p-1$. Let s_3' correspond to s_3 in the same automorphism. Since

$$s_1^{-\alpha} s_3 s_1^\alpha = s_1^{\alpha p^3} s_3 \quad \text{and} \quad s_1^{-\alpha} s_3' s_1^\alpha = s_1^{\alpha p^3} s_3',$$

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* G. A. Miller, *Annals of Mathematics*, vol. 3 (1902), p. 180.

it results that s_3' is equal to the product of s_3 and some operator which is commuative with s_1. That is, s_3 corresponds to itself multiplied by some operator of H_5 whenever s_1 corresponds to s_1^a. In the same automorphism s_2 must correspond to itself multiplied by some operator of H_7, since the operator which corresponds to s_2 must transform the operator which corresponds to s_3 into itself multiplied by s_3^p into some operator of H_2.

It is now easy to prove that $\alpha = 1$. From the fact that s_1 transforms s_2 into itself multiplied by $s_2^{p^2}$, it results that s_1^a transforms the operator which corresponds to s_2 in this automorphism into itself multiplied by $s_2^{p^2}$ into some operator of H_1. This cannot be equal to $s_2^{ap^2}$ into some operator of H_1 unless $\alpha = 1$. Hence s_1 cannot correspond to any power of itself, except the first power, in any automorphism of G. Since these arguments were based upon the way in which s_1 transforms G, it has been proved that, *in all the possible automorphisms of G, s_1 must correspond to itself multiplied by an operator of the central of G.*

It may be observed that the commutator sub-group of H_8, is composed of the powers of s_3^p, and hence s_3 generates a characteristic sub-group of order p. It must therefore correspond to itself multiplied by operators of H_3 in every automorphism of G. As it has been proved that, in every possible automorphism of G, the operators which can correspond to the three generators of G, s_1, s_2, and s_3 respectively, are obtained by multiplying these operators by operators which appear in a smaller sub-group of the series H_1, H_2, ..., H_8, G, it has been proved that the order of the group of isomorphisms of G is of the form p^a for every value of the odd prime number p.

For the sake of simplicity we confined our attention, in what precedes, to the case when p is an odd prime number. In this case the p^{th} power of the product of any two operators of G is the product of their p^{th} powers, while this is not always true when $p = 2$. The series of sub-groups H_1, H_2, ..., H_8 is, however, composed of characteristic sub-groups even when $p = 2$. Hence the order of the group of isomorphisms of G is a power of p even in this special case. We have therefore proved that *G is a group of order p^9 whose group of isomorphisms is of the form p^a when p is any prime number whatever.*

NOTE ON THE EQUATION $s=f(z)$.

By *J. R. Wilton, M.A., B.Sc.*, Assistant Lecturer in Mathematics at the University of Sheffield.

The general form of a pseudo-sphere can be obtained, when the measure of curvature is small, to any required degree of accuracy. If R_1 and R_2 are the principal radii of curvature, we have $R_1R_2=-a^2$, and, as is well known, the problem of determining the form of the pseudo-sphere is, by Bonnet's theorem, reduced to the solution of the differential equation

$$s=\frac{\partial^2 z}{\partial x\,\partial y}=\frac{1}{a^2}\sin z \qquad\text{(1)},$$

where $x=$ constant, $y=$ constant are the asymptotic curves on the surface and z is the angle between them.*

If the measure of curvature is small, a is large, and it is easy in this case to obtain successive approximations to the solution of (1). Thus, if a were infinitely great (corresponding to the particular case of the developable surface), we should have $s=0$ and $z=X+Y$, where X and Y are arbitrary functions of x and y respectively. Substituting this value of z in the right-hand side of (1), we find

$$s=\sin(X+Y)=\sin X\cos Y+\cos X\sin Y,$$

and therefore $$z_1=X+Y+\frac{1}{a^2}(X_1Y_2+X_2Y_1),$$

where $$X_1=\int\sin X\,dx,\quad X_2=\int\cos X\,dx,$$
$$Y_1=\int\sin Y\,dy,\quad Y_2=\int\cos Y\,dy,$$

is a first-order approximation to the solution of (1) when a is large. Substituting $z=z_1+\zeta$ in equation (1), we have

$$\frac{\partial^2\zeta}{\partial x\,\partial y}=\frac{1}{a^4}(X_1Y_2+X_2Y_1)\cos(X+Y),$$

i.e.,

$$z_2=z_1+\frac{1}{a^4}\{\int X_1\cos X\,dx\int Y_2\cos Y\,dy+\int X_2\cos X\,dx\int Y_1\cos Y\,dy$$
$$-\int X_1\sin X\,dx\int Y_2\sin Y\,dy-\int X_2\sin X\,dx\int Y_1\sin Y\,dy\}$$

is a second-order approximation to the value of z. We may

* See, for example, Forsyth's *Differential Geometry*, p. 74. The determination of surfaces of constant *mean* curvature depends on the same equation (see p. 77 footnote).

evidently repeat the process until z is obtained to the required degree of accuracy.

It is clear that the solution will be of the form

$$z = X + Y + \sum_{m=1}^{\infty} \frac{1}{a^{2m}} S_m,$$

$$S_m = \sum_{n=1}^{2m} X_{mn} Y_{mn},$$

where X and Y are arbitrary, but X_{mn} is a function of x whose form depends on that of X, while Y_{mn} is a function of y whose form depends on that of Y. Substituting in (1), we have

$$\sum_{m=1}^{\infty} \frac{1}{a^{2m}} \sum_{n=1}^{2m} X'_{mn} Y'_{mn} = \sin(X+Y) \left\{1 - \frac{\Sigma^2}{2!} + \frac{\Sigma^4}{4!} - \dots\right\}$$

$$+ \cos(X+Y) \left\{\Sigma - \frac{\Sigma^3}{3!} + \dots\right\},$$

where $$\Sigma = \sum_{m=1}^{\infty} \frac{1}{a^{2m}} S_m.$$

It is clear that there are sufficient equations to determine the forms of the functions X_{mn} and Y_{mn}. We have already obtained the first six of each.

It is, however, sufficient to assume for z a form apparently less general. Let

$$z = X + \sum_{n=1}^{\infty} \frac{(y-b)^n}{n!} X_n,$$

where b is a constant and X_n is a function of x, whose form depends on that of X. Substituting this value of z in (1), we find*

$$\sum_{n=1}^{\infty} \frac{(y-b)^{n-1}}{(n-1)!} X_n' = \sin X \left\{1 - \frac{\Sigma^2}{2!} + \frac{\Sigma^4}{4!} - \dots\right\}$$

$$+ \cos X \left\{\Sigma - \frac{\Sigma^3}{3!} + \dots\right\} \dots\dots\dots (2),$$

where $$\Sigma = \sum_{n=1}^{\infty} \frac{(y-b)^n}{n!} X_n.$$

The coefficient of $\dfrac{(y-b)^{n-1}}{(n-1)!}$ on the right-hand side of (2)

* We have put a^2y instead of y in equation (1), so that it becomes $s = \sin z$.

is readily obtained by the multinomial theorem. It is, however, somewhat complicated in form, and it will be sufficient to call it F_n. We then have

$$X_n = \int_c^x F_n\,dx + Y^{(n)}(b),$$

where c is a constant, the same for all values of n, and $Y^{(n)}(b)$, the value of the n^{th} differential coefficient of a function Y when $y=b$, is an arbitrary constant. We thus have, as the general solution of the equation

$$s = \sin z,$$

$$z = X + Y - Y(b) + \sum_{n=1}^{\infty} \int_c^x \frac{(y-b)^n}{n!} F_n\,dx \ldots\ldots (3),$$

where Y is a function of y, expansible by Taylor's theorem in the form

$$Y = Y(b) + \sum_{n=1}^{\infty} \frac{(y-b)^n}{n!} Y^{(n)}(b);$$

and the series on the right-hand side of (3) will be convergent for sufficiently small values of $y-b$.

The relation (3) furnishes us with a formally complete solution of the equation $s=\sin z$, such that

$$z = X \text{ when } y = b,$$

$$z = Y + X(c) - Y(b) \text{ when } x = c.$$

It is, however, on account of its complicated form, of very little practical importance, though of some interest from a purely theoretical standpoint.

It is evident that precisely the same method will furnish us with a solution of the more general equation

$$s = f(z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

provided that $f(z+h)$ may be expanded by Taylor's theorem. As before, we assume

$$z = X + \sum_{n=1}^{\infty} \frac{(y-b)^n}{n!} X_n,$$

and we find

$$\sum_{n=1}^{\infty} \frac{(y-b)^{n-1}}{(n-1)!} X_n' = f(X+\Sigma)$$

$$= f(X) + \Sigma f'(X) + \ldots \frac{\Sigma^n}{n!} f^{(n)}(X) + \ldots.$$

Pick out the coefficient of $(y-b)^{n-1}/(n-1)!$ on the right-hand side; let it be F_n. Exactly as before we have

$$X_n=\int_c^x F_n\,dx+Y^{(n)}(b),$$

and the equation (3), with the new value of F_n, represents the solution of (4).

It is always possible to find the form of F_n, but the result is in general complicated; and it is only in very special cases that the sequence equation for X_n can be solved. For instance, even in the simple case,

$$s=z^2,$$

this sequence equation is

$$X_{n+1}=n!\int_c^x \sum_{m=0}^{n} \frac{X_m}{m!}\frac{X_{n-m}}{(n-m)!}\,dx+Y^{(n+1)}(b),$$

where $X_0=X$.

As a particular example of the method, consider the equation

$$s=z \qquad\qquad (5).$$

On substituting

$$z=X+Y(b)+\sum_{n=1}^{\infty}\frac{(y-b)^n}{n!}X_n$$

in this, we find

$$X_n=\int_c^x\int_c^x\cdots\int_c^x X\,(dx)^n+\frac{(x-c)^n}{n!}Y(b)$$
$$+\frac{(x-c)^{n-1}}{(n-1)!}Y'(b)+\ldots(x-c)\,Y^{(n-1)}(b)+Y^{(n)}(b),$$

and the general solution of (5), such that

$$z=X+Y(b) \text{ when } y=b,$$
$$z=Y+X(c) \text{ when } x=c,$$

is

$$z=X+Y+\sum_{n=1}^{\infty}\frac{(y-b)^n}{n!}\left\{\int_c^x\int_c^x\cdots\int_c^x X\,(dx)^n\right.$$
$$\left.+\sum_{m=1}^{n}\frac{(x-c)^m}{m!}Y^{(n-m)}(b)\right\}.$$

The general solution of equation (5) is, of course, well known, but I am not aware that it has been given in this form.

It is clear that the equation

$$r=f(q) \qquad\qquad (6),$$

which is of some importance in the theory of conduction of heat,* may be treated in the same manner. We take

$$z = Y + \sum_{n=1}^{\infty} \frac{(x-c)^n}{n!} Y_n,$$

and substituting in (6), we have

$$\sum_{n=0}^{\infty} \frac{(x-c)^n}{n!} Y_{n+2} = f\left\{Y' + \sum_{n=1}^{\infty} \frac{(x-c)^n}{n!} Y_n'\right\}$$

$$= f(Y') + \Sigma f'(Y') + \dots \frac{\Sigma^m}{m!} f^{(m)}(Y') + \dots.$$

Whence Y and Y_1 are arbitrary, and by equating the coefficients of powers of $(x-c)$, we find

$$Y_2 = f(Y'),$$
$$Y_3 = Y_1 f'(Y'),$$
$$Y_4 = Y_2 f'(Y') + Y_1^2 f''(Y'), \text{ \&c.}$$

On the other hand, solving for q, we may write (6) in the form

$$q = F(r),$$

and if we assume

$$z = X + \sum_{n=1}^{\infty} \frac{(y-b)^n}{n!} X_n,$$

we have

$$\sum_{n=0}^{\infty} \frac{(y-b)^n}{n!} X_{n+1} = F\left\{X'' + \sum_{n=1}^{\infty} \frac{(y-b)^n}{n!} X_n''\right\}$$

$$= F(X'') + \Sigma F'(X'') + \dots \frac{\Sigma^m}{m!} F^{(m)}(X'') + \dots.$$

Whence X is arbitrary, and

$$X_1 = F(X''),$$
$$X_2 = X_1 F'(X''),$$
$$X_3 = X_2 F'(X'') + X_1^2 F''(X''), \text{ \&c.}$$

Either form of solution is general.

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* Differentiating (6) with regard to y, and taking q as the new dependent variable, we have $\frac{\delta^2 q}{\delta x^2} = f'(q) \frac{\delta q}{\delta y}$, which is the equation of conduction of heat when the conductivity is a function of the temperature q.

SOME THEOREMS CONCERNING DIRICHLET'S SERIES.

By *G. H. Hardy* and *J. E. Littlewood.*

I.

1. The present paper is intended as a supplement to a series of papers published during the last few years in the *Proceedings of the London Mathematical Society.*

These papers have been concerned, in the main, with what we have called "Tauberian" theorems, theorems whose general character is the same as that of Tauber's well-known converse of Abel's theorem on the continuity of a power-series. The most typical Tauberian theorems have, as one of their hypotheses, a hypothesis of the type

$$a_n = O(n^a), \tag{1.1}$$

where a_n is the general term of the series considered. It is a natural conjecture that there must be analogues of these theorems in which this hypothesis is replaced by one as to the convergence of a series of the type

$$\Sigma\, n^\beta\,|a_n|^\gamma\,; \tag{1.2}$$

and the fundamental importance of such hypotheses in the theory of Fourier's series suggests that theorems of this character might prove to be very interesting.

One such theorem has been proved already by Fejér.* Fejér shows that

if (i) *the series* Σa_n *is summable* $(C1)$, (ii) *the series* $\Sigma\, n\,|a_n|^2$ *is convergent, then the series* Σa_n *is convergent.*

This theorem is the analogue, in the direction indicated above, of the simplest case of what we have called the "general Cesàro-Tauber theorem," from which it differs in that the hypothesis that $a_n = O(1/n)$ is replaced by the hypothesis (ii).

2. We do not propose now to work out systematically a whole theory analogous to that contained in our former papers. We shall confine ourselves to proving the analogues of two of our simplest theorems, viz.: (i) *if* $a_n = O(1/n)$ *and* $f(x) = \Sigma a_n x^n$ *tends to a limit as* x *tends to* 1 *through real values less than* 1, *then* Σa_n *is convergent*; (ii) *if* $a_n = O(1/n)$,

* *Comptes Rendus,* 6th January, 1913.

$b_n = O(1/n)$, *and the series* Σa_n, Σb_n *are convergent, then the product series* Σc_n, *formed in accordance with Cauchy's rule for multiplication, is convergent*: or rather of the generalisations of these two theorems which hold for Dirichlet's series and Dirichlet's multiplication.

One preliminary remark is required. In our previous researches there was a sharp distinction between "general" theorems, theorems whose hypotheses involve an O, and "special" theorems, theorems whose hypotheses involve an o. This distinction now disappears: the theorems which we shall prove are of a "special" character, and their proofs involve none of the characteristic difficulties of those of the "general" theorems; nor do they appear to be capable of any generalisation analogous to the passage from the "special" to the "general."

3. In what follows we shall, as usual, denote by (λ_n) an arbitrary increasing sequence of positive numbers, tending to infinity with n, and we shall be concerned with series Σa_n, such that the series

$$\Sigma \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^p |a_n|^{p+1}, \tag{3.1}$$

where p is a positive number, is convergent: this series reduces to $\Sigma n^p |a_n|^{p+1}$ when $\lambda_n = n$. It will be convenient to write $\lambda_0 = 0$.

It should be observed first that *the convergence of the series* (3.1) *for any particular value of p neither implies, nor is implied by, its convergence for any other value of p.* We can see this by considering the special case in which $\lambda_n = n$. Suppose first that

$$a_n = \frac{1}{n(\log n)^\alpha},$$

where $0 < \alpha \leq 1$. Then the series (3.1) is convergent if

$$p > (1/\alpha) - 1,$$

so that its chance of convergence is increased by an increase in p. If on the other hand we suppose that $a_n = \nu^{-\alpha}$ when $n = \nu^\beta$, α and β being positive integers, of which the latter is the greater, and that $a_n = 0$ when n is not a perfect β^{th} power, the series (3.1) assumes the form

$$\Sigma \nu^{p\beta - (p+1)\alpha},$$

and is convergent if

$$p < \frac{\alpha - 1}{\beta - \alpha}.$$

Thus in this case the chance of convergence is diminished by an increase in p.

Secondly, we observe that *if the series* (3.1) *is convergent, the series* $\Sigma a_n e^{-\lambda_n s}$ *is absolutely convergent for all positive values of* s. The proof of this depends on an inequality on which much of our subsequent analysis will depend, viz., the inequality

$$(3.2) \qquad \Sigma\, ab \leq (\Sigma\, a^{p+1})^{1/(p+1)} (\Sigma\, b^{(p+1)/p})^{p/(p+1)},$$

known as the "generalised inequality of Schwarz." In this inequality the a's, the b's, and p are positive.*

We have

$$\sum_1^n |a_\nu|\, e^{-\lambda_\nu s} = \sum_1^n \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}}\right)^{p/(p+1)} |a_\nu| \left(\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}\right)^{p/(p+1)} e^{-\lambda_\nu s}$$

$$\leq \left\{\sum_1^n \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}}\right)^p |a_\nu|^{p+1}\right\}^{1/(p+1)} \left\{\sum_1^n \left(\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}\right) e^{-\{(p+1)/p\}\lambda_\nu s}\right\}^{p/(p+1)}.$$

Also

$$\sum_1^n \left(\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}\right) e^{-\{(p+1)/p\}\lambda_\nu s} \leq \frac{1}{\lambda_1} \sum_1^n (\lambda_\nu - \lambda_{\nu-1})\, e^{-\{(p+1)/p\}\lambda_\nu s}$$

$$< \frac{1}{\lambda_1} \int_0^{\lambda_n} e^{-\{(p+1)/p\} ts}\, dt < \frac{p}{(p+1)\lambda_1 s}.$$

From these inequalities our assertion follows immediately.

4. Theorem A. *Suppose that the series*

$$\Sigma \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^p |a_n|^{p+1}$$

is convergent, and that the series $f(s) = \Sigma a_n e^{-\lambda_n s}$, *then certainly absolutely convergent for* $s > 0$, *tends to a limit* A *as* $s \to 0$. *Then the series* Σa_n *is convergent to the sum* A.

Choose m so that

$$\sum_{m+1}^{\infty} \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}}\right)^p |a_n|^{p+1} < \epsilon^{p+1},$$

and s so that $s = 1/\lambda_n$, where $n > m$. Then, if

$$a_1 + a_2 + \ldots + a_n = A_n,$$

we have

$$A_n - f\left(\frac{1}{\lambda_n}\right) = \sum_1^m a_\nu (1 - e^{-\lambda_\nu s}) + \sum_{m+1}^n a_\nu (1 - e^{-\lambda_\nu s}) - \sum_{n+1}^{\infty} a_\nu e^{-\lambda_\nu s}$$

$$= S_1 + S_2 + S_3,$$

say. Then

* For a proof of the inequality see, *e.g.*, F. Riesz, *Math. Annalen*, vol. 69, p. 455.

$$|S_2| < s \sum_{m+1}^{n} \lambda_\nu |a_\nu|$$

$$\leq s \left\{ \sum_{m+1}^{n} \left(\frac{\lambda_\nu}{\lambda_\nu - \lambda_{\nu-1}} \right)^p |a_\nu|^{p+1} \right\}^{1/(p+1)} \left\{ \sum_{m+1}^{n} \lambda_\nu^{1/p} (\lambda_\nu - \lambda_{\nu-1}) \right\}^{p/(p+1)}$$

$$< \epsilon s \left\{ \sum_{1}^{n} \lambda_\nu^{1/p} (\lambda_\nu - \lambda_{\nu-1}) \right\}^{p/(p+1)} < \epsilon s \lambda_n = \epsilon.$$

Also

$$|S_3| \leq \left\{ \sum_{n+1}^{\infty} \frac{|a_\nu|^{p+1}}{(\lambda_\nu - \lambda_{\nu-1})^p} \right\}^{1/(p+1)} \left\{ \sum_{n+1}^{\infty} (\lambda_\nu - \lambda_{\nu-1}) \, e^{-\{(p+1)/p\}\lambda_\nu s} \right\}^{p/(p+1)}$$

$$< \epsilon \lambda_n^{-p/(p+1)} \left(\int_0^\infty e^{-\{(p+1)/p\} ts} \, dt \right)^{p/(p+1)}$$

$$= \epsilon \left(\frac{p}{p+1} \right)^{p/(p+1)} < \epsilon.$$

Finally it is evident that, if n is large enough in comparison with m, we have $|S_1| < \epsilon$, and so

$$\left| A_n - f\left(\frac{1}{\lambda_n} \right) \right| < 3\epsilon;$$

and the theorem is therefore proved.

In particular the convergence of $\Sigma n^p |a_n|^{p+1}$, and the existence of Abel's limit $\lim \Sigma a_n x^n$ when $x \to 1$, involve the convergence of Σa_n. Finally, since the summability (Cr) of Σa_n involves the existence of Abel's limit, a series Σa_n, such that $\Sigma n^p |a_n|^{p+1}$ is convergent, cannot be summable (Cr) unless convergent. For $p = 1$, $r = 1$, this reduces to Fejér's result.

5. Theorem B. *If Σa_n, Σb_n converge to sums A, B, and*

$$\Sigma \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} \right)^p |a_n|^{p+1}, \quad \Sigma \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} \right)^q |b_n|^{q+1}$$

are convergent, then the Dirichlet's product of the two series, formed according to the rule associated with Dirichlet's series of type (λ_n), converges to the sum AB.

The proof of this theorem is a modification of that of the theorem of which it is the analogue, given in one of our former papers.*

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* *Proc. London Math. Soc.*, vol. 10, p. 399.

We shall use the notation

$$U(x) = \sum_{n \leq x} u_n$$

to denote the sum of those terms of a series $u_1 + u_2 + \ldots$ whose rank is not greater than a positive number x, not necessarily an integer. We shall denote by $\lambda(x)$ a continuous and steadily increasing function of x, which assumes the value λ_n for $x = n$, and by (ν_r) the sequence $(\lambda_m + \lambda_n)$, arranged in ascending order of magnitude.

The product series is Σc_r, where

$$c_r = \sum_{\lambda_m + \lambda_n = \nu_r} a_m b_n.$$

Thus

$$C(r) = \Sigma a_m b_n,$$

where the summation is bounded by the inequalities

$$m \geq 1, \quad n \geq 1, \quad \lambda_m + \lambda_n \leq \nu_r.$$

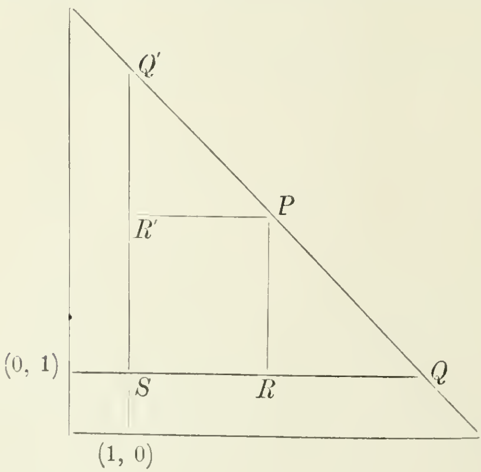
Let us draw the curve whose equation is

$$\lambda(x) + \lambda(y) = \nu_r,$$

and take on it the point P whose coordinates are

$$x_r = \bar{\lambda}(\tfrac{1}{2}\nu_r), \quad y_r = \bar{\lambda}(\tfrac{1}{2}\nu_r),$$

where $\bar{\lambda}$ is the function inverse to λ. Then $C(r)$ is the sum of all products $a_m b_n$ such that (m, n) lies in or on the boundary of the region SQQ', and $A(x_r) B(x_r)$ the sum of all such



that (m, n) lies in or on the boundary of $SRPR'$. Hence

$$C(r) - A(x_r)\,B(x_r) = \sum_{(D)} a_m b_n + \sum_{(D')} a_m b_n,$$

where D and D' denote the regions PQR, $P'Q'R'$, the boundaries of these regions being reckoned as part of them, except in so far as they are formed by the lines PR, PR'. It is plainly sufficient for our purpose to show that (*e.g.*)

$$\sum_{(D)} a_m b_n \to 0.$$

as $r \to \infty$.

Now $$\sum_{(D)} a_m b_n = \sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} a_m B\{\bar{\lambda}(\nu_r - \lambda_m)\},$$

the modulus of which is less than a constant multiple of

$$\sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} |a_m|.$$

We can choose r so that

$$\sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} \left(\frac{\lambda_m}{\lambda_m - \lambda_{m-1}}\right)^p |a_m|^{p+1} < \epsilon^{p+1};$$

and then

$$\sum |a_m| \leq \left\{ \sum \left(\frac{\lambda_m}{\lambda_m - \lambda_{m-1}}\right)^p |a_m|^{p+1} \right\}^{1/(p+1)} \left(\sum \frac{\lambda_m - \lambda_{m-1}}{\lambda_m} \right)^{p/(p+1)}$$

$$< \epsilon \left\{ \sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} \frac{\lambda_m - \lambda_{m-1}}{\lambda_m} \right\}^{p/(p+1)}.$$

But

$$\sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} \frac{\lambda_m - \lambda_{m-1}}{\lambda_m} < 1 + \int_{\frac{1}{2}\nu_r}^{\nu_r} \frac{dt}{t} = 1 + \log 2.$$

Hence $$\sum_{\frac{1}{2}\nu_r < \lambda_m \leq \nu_r} |a_m| \to 0,$$

and so $$\sum_{(D)} a_m b_n \to 0,$$

as $r \to \infty$.

6. A comparison of the argument which precedes with that of our previous paper shows at once that *a series Σa_n for which*

$$\sum \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^p |a_n|^{p+1}$$

is convergent may be multiplied by a series Σb_n *for which*

$$b_n = o\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right).$$

Whether o may be replaced by O in this result we cannot say.

In another of our papers* we showed that our theorem concerning the multiplication, by Cauchy's rule, of series whose general terms are of order $1/n$ is a corollary of another theorem, viz., that *a series* Σa_n, *for which* $a_n = O(1/n)$, *if summable by any of Cesàro's means, is summable* $(C, -1+\delta)$ *for all positive values of* δ. It is naturally suggested that this theorem also has an analogue, and we have in fact proved the following result.

THEOREM C. *If* Σa_n *is summable* (Ck) *for any value of* k, *and*

$$\Sigma n^p |a_n|^{p+1}$$

is convergent, then Σa_n *is summable* $\left(C, -\frac{p}{p+1} + \delta\right)$ *for all positive values of* δ.

In order to prove this theorem, we observe† that the necessary and sufficient condition that a series Σa_n, known to be summable $(C, r+1)$, shall be summable (Cr), is that

$$t_n^r = o(n^{r+1}),$$

where

$$t_n^r = \binom{r+n-1}{r} a_1 + \binom{r+n-2}{r} 2a_2 + \ldots + \binom{r}{r} na_n.$$

Plainly

$$t_n^r = O\{n^r |a_1| + (n-1)^r 2|a_2| + \ldots + n|a_n|\}.$$

We divide the expression inside the brackets into the two parts

$$S_1 = \sum_{\nu=1}^{m} (n-\nu+1)^r \nu |a_\nu|, \quad S_2 = \sum_{m+1}^{n} (n-\nu+1)^r \nu |a_\nu|;$$

and we choose m so that

$$\sum_{m+1}^{\infty} \nu^p |a_\nu|^{p+1} < \epsilon^{p+1}.$$

* *Proc. London Math. Soc.*, vol. 11, p. 462.
† *Proc. London Math. Soc.*, vol. 8, p. 304.

Then

$$|S_2| \leq \left(\sum_{m+1}^{n} \nu^p |a_\nu|^{p+1} \right)^{1/(p+1)} \left\{ \sum_{m+1}^{n} (n-\nu+1)^{\{(p+1)r\}/p} \nu^{1/p} \right\}^{p/(p+1)}$$

$$< \epsilon \left\{ \sum_{1}^{n} (n-\nu+1)^{\{(p+1)r\}/p} \nu^{1/p} \right\}^{p/(p+1)} < \epsilon K n^{r+1},$$

where K is a constant. Also

$$|S_1| < n^r \sum_{1}^{m} \nu |a_\nu| < \epsilon n^{r+1},$$

if n is large enough in comparison with m. These inequalities obviously suffice to establish Theorem C.

II.

7. The theorem with which we shall conclude this paper is of a deeper character.

We have shown* that *if* $f(x) = \Sigma a_n x^n$ *is a power series, all of whose coefficients are positive, and which is convergent when* $0 < x < 1$, *and if*

$$f(x) \sim \frac{A}{(1-x)^\alpha} \quad (A > 0,\ \alpha > 0),$$

as $x \rightarrow 1$, *then*

$$A_n = a_1 + a_2 + \ldots + a_n \sim \frac{An^\alpha}{\Gamma(1+\alpha)}.\dagger$$

Further, we showed that *the hypothesis that* $a_n \geq 0$ *may be replaced by the more general hypothesis that* $a_n > -Kn^{\alpha-1}$.

8. We shall now prove

THEOREM D. *If* $f(s) = \Sigma a_n e^{-\lambda_n s}$ *is a Dirichlet's series convergent for* $s > 0$, *of type* (λ_n) *such that*

$$\frac{\lambda_{n+1}}{\lambda_n} \rightarrow 1$$

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* *Proc. London Math. Soc.*, vol. 13. This paper has not yet been published.

† In the paper referred to above we consider relations of the type

$$f(x) \sim \frac{A}{(1-x)^\alpha} \left\{ \log\left(\frac{1}{1-x}\right) \right\}^{\alpha_1} \left\{ \log\log\left(\frac{1}{1-x}\right) \right\}^{\alpha_2} \ldots.$$

The differences introduced into the proof by the adoption of the more general hypothesis are of the nature of trivial complications, and we shall confine ourselves now to the case in which $\alpha_1 = \alpha_2 = \ldots = 0$. The reader will easily satisfy himself of the truth of the more general results which are at once suggested.

as $n \to \infty$, and with positive coefficients; if further

$$f(s) \sim As^{-\alpha} \quad (A>0,\ \alpha \geq 0)$$

as $s \to 0$: then

$$A_n = a_1 + a_2 + \ldots + a_n \sim \frac{A\lambda_n^{\alpha}}{\Gamma(1+\alpha)}$$

as $n \to \infty$.

We shall base our proof on the following lemma.

LEMMA D1. *If the series*

$$F(s) = \Sigma a_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-sx}\, dx$$

is convergent for $s>0$; if further $a_n \geq 0$ and

$$F(s) \sim As^{-\alpha} \quad (A>0,\ \alpha>0)$$

as $s \to 0$, then $$-F'(s) \sim A\alpha s^{-\alpha-1}.$$

Let $$G(s) = \frac{F(s)}{s};$$

then

$$sG'(s) = F'(s) - \frac{F(s)}{s} = -\Sigma a_n \int_{\lambda_n}^{\lambda_{n+1}} xe^{-sx}\, dx - \frac{1}{s}\, \Sigma a_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-sx}\, dx$$

plainly decreases steadily as $s \to 0$. Hence, by a theorem of Landau,*

$$G'(s) \sim \frac{d}{ds}(As^{-\alpha-1}) = -A(\alpha+1)s^{-\alpha-2},$$

$$-F'(s) = -\frac{F(s)}{s} - G'(s) \sim A\alpha s^{-\alpha-1}.$$

There is also another lemma which we shall find useful, although it is of no particular intrinsic interest.

LEMMA D2. *If ζ and ρ are positive, and*

$$\zeta \to 0, \quad \rho \to \infty, \quad \zeta^2\rho \to \infty,$$

then

$$\frac{1}{\Gamma(\rho+1)} \int_0^{\rho(1-\zeta)} e^{-u}u^{\rho}\, du \to 0, \qquad \frac{1}{\Gamma(\rho+1)} \int_{\rho(1+\zeta)}^{\infty} e^{-u}u^{\rho}\, du \to 0.$$

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* *Rendiconti di Palermo*, vol. 26, p. 218; see also *Proc. London Math. Soc.*, vol. 13

Consider the second integral, for example. It is

$$\frac{1}{\Gamma(\rho+1)}\int_{\rho(1+\zeta)}^{\infty} u^{\rho} e^{-u/(1+\zeta)} e^{-\zeta u/(1+\zeta)}\,du$$
$$< \frac{\{\rho(1+\zeta)\}^{\rho} e^{-\rho}}{\Gamma(\rho+1)}\int_{\rho(1+\zeta)}^{\infty} e^{-\zeta u/(1+\zeta)}\,du$$
$$< K\frac{(1+\zeta)^{\rho}}{\zeta\sqrt{\rho}}\,e^{-\zeta\rho}$$
$$= \frac{K}{\zeta\sqrt{\rho}}\,e^{-\rho\{\zeta-\log(1+\zeta)\}}$$
$$< \frac{K}{\zeta\sqrt{\rho}},$$

where K is a constant. That the other integral tends to zero may be proved in a similar manner.

9. Before proceeding to the proof of the main theorem we add the following preliminary remarks.

(i) Our argument will involve three variables, ζ, r, and s. Of these ζ and r are definite functions of one another, and $\zeta \to 0$, $r \to \infty$, $\zeta^2 r \to \infty$. We may, for example, suppose that $\zeta^3 r = 1$. The choice of a value of s will always be subsequent to that of ζ and r.

(ii) We shall make a number of assertions of the type

$$|f(\zeta, r, s)| < \epsilon,$$

or, more generally,

$$\phi(\zeta, r, s, \epsilon) < 0.$$

All such assertions are to be interpreted as follows: "given any positive number ϵ, we can choose r_0 so that, when any definite r greater than r_0 is taken, we can then choose s_0 so that $\phi < 0$ for $0 < s \leq s_0$, or for all such values of s as satisfy some further condition or conditions previously laid down."

It follows, of course, that when ϵ occurs in each of a succession of inequalities it must not be regarded as a definite number having the same value in each inequality.

(iii) We may plainly take $A = 1$.

10. We observe first that

(10.1) $$A_n = O(\lambda_n^{a});$$

since $$A_n < e\sum_{1}^{n} a_\nu e^{-(\lambda_\nu/\lambda_n)} < ef\left(\frac{1}{\lambda_n}\right).$$

Next, we have

$$f(s)=\Sigma a_n e^{-\lambda_n s}=\Sigma A_n(e^{-\lambda_n s}-e^{-\lambda_{n+1}s})=s\Sigma A_n\int_{\lambda_n}^{\lambda_{n+1}}e^{-sx}\,dx,$$

and so
$$\Sigma A_n\int_{\lambda_n}^{\lambda_{n+1}}e^{-sx}\,dx\sim s^{-\alpha-1}.$$

Hence, by Lemma D1,

$$\Sigma A_n\int_{\lambda_n}^{\lambda_{n+1}}x^r e^{-sx}\,dx\sim\frac{\Gamma(\alpha+r+1)}{\Gamma(\alpha+1)}s^{-\alpha-r-1} \tag{10.2}$$

for any value of r.

We shall suppose that r and s are such that

$$\frac{r+\alpha}{s}=\lambda_m,$$

and we shall denote by $\lambda_{m-\nu}$ and $\lambda_{m+\nu}$ the last and first respectively of the λ's such that

$$\lambda_{m-\nu}<(1-\zeta)\lambda_m,\quad \lambda_{m+\nu}>(1+\zeta)\lambda_m. \tag{10.3}$$

It is important to observe that *it is possible to choose r and s so that either $m-\nu$ or $m+\nu$ shall be equal to any assigned large integer p.* For example, $m-\nu=p$ if

$$\lambda_p<\frac{(1-\zeta)(r+\alpha)}{s},\qquad \lambda_{p+1}\geq\frac{(1-\zeta)(r+\alpha)}{s},$$

and we can certainly choose r and s so that these inequalities shall be satisfied. Thus $m-\nu$ and $m+\nu$ may be regarded as variables which assume all integral values, from a certain point onwards, as they tend to ∞.

Now

$$\begin{aligned}\sum_{m+\nu}^{\infty}A_n\int_{\lambda_n}^{\lambda_{n+1}}x^r e^{-sx}\,dx &< K\sum_{m+\nu}^{\infty}\lambda_n^{\alpha}\int_{\lambda_n}^{\lambda_{n+1}}x^r e^{-sx}\,dx\\ &< K\int_{\lambda_{m+\nu}}^{\infty}x^{r+\alpha}e^{-sx}\,dx,\\ &= Ks^{-r-\alpha-1}\int_{s\lambda_{m+\nu}}^{\infty}u^{r+\alpha}e^{-u}\,du,\end{aligned}$$

where K is a constant. The lower limit is greater than $(1+\zeta)(r+\alpha)$. Hence, by Lemma D2, we have

$$\sum_{m+\nu}^{\infty}A_n\int_{\lambda_n}^{\lambda_{n+1}}x^r e^{-sx}\,dx<\epsilon\Gamma(r+\alpha+1)\,s^{-r-\alpha-1}; \tag{10.4}$$

and a similar argument shows that

$$(10.5) \qquad \sum_{1}^{m-\nu-1} A_n \int_{\lambda_n}^{\lambda_{n+1}} x^r e^{-sx} dx < \epsilon \Gamma(r+\alpha+1) s^{-r-\alpha-1},$$

11. From (10.2), (10.4), and (10.5) it follows that

$$(11.11) \qquad \sum_{m-\nu}^{m+\nu-1} A_n \int_{\lambda_n}^{\lambda_{n+1}} x^r e^{-sx} dx > (1-\epsilon) \frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha+1)} s^{-r-\alpha-1},$$

$$(11.12) \qquad \sum_{m-\nu}^{m+\nu-1} A_n \int_{\lambda_n}^{\lambda_{n+1}} x^r e^{-sx} dx < (1+\epsilon) \frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha+1)} s^{-r-\alpha-1}.$$

But, since $a_n \geq 0$, A_n is a steadily increasing function of n. Hence

$$A_{m-\nu} \int_{\lambda_{m-\nu}}^{\lambda_{m+\nu}} x^r e^{-sx} dx < (1+\epsilon) \frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha+1)} s^{-r-\alpha-1},$$

$$A_{m+\nu} \int_{\lambda_{m-\nu}}^{\lambda_{m+\nu}} x^r e^{-sx} dx > (1-\epsilon) \frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha+1)} s^{-r-\alpha-1}.$$

In virtue of Lemma D2, we may replace the limits in these integrals by 0 and ∞. The first inequality then gives

$$A_{m-\nu} < (1+\epsilon) \frac{\Gamma(r+\alpha+1)}{\Gamma(r+1)\Gamma(\alpha+1)} s^{-\alpha},$$

$$A_{m-\nu} < \frac{1+\epsilon}{\Gamma(\alpha+1)} \left(\frac{r}{s}\right)^{\alpha},$$

$$(11.2) \qquad A_{m-\nu} < \frac{1+\epsilon}{\Gamma(\alpha+1)} \lambda_m^{\alpha}.$$

Now $\quad \lambda_{m-\nu} < (1-\zeta)\lambda_m, \quad \lambda_{m-\nu+1} \geq (1-\zeta)\lambda_m,$

and

$$\frac{\lambda_{n+1}}{\lambda_n} \to 1.^*$$

Hence

$$(11.3) \qquad A_{m-\nu} < \frac{1+\epsilon}{\Gamma(\alpha+1)} \lambda_{m-\nu}^{\alpha};$$

and similarly we can show that

$$A_{m+\nu} > \frac{1-\epsilon}{\Gamma(\alpha+1)} \lambda_{m+\nu}^{\alpha}.$$

* It is interesting to observe that this is the only point in the proof at which any use is made of this hypothesis.

It now follows from the remark made early in § 10 that, given any positive ϵ, we can choose p_0 so that

$$\frac{1-\epsilon}{\Gamma(\alpha+1)}\lambda_p^{\alpha} < A_p < \frac{1+\epsilon}{\Gamma(\alpha+1)}\lambda_p^{\alpha}$$

for $p > p_0$, and the proof of the theorem is accordingly completed.

12. It is easy to deduce from Theorem D a more general theorem.

Theorem E. *The conclusion of Theorem D is still valid when $\alpha > 0$ and the condition that a_n is positive is replaced by the more general condition*

$$a_n > -K\lambda_n^{\alpha-1}(\lambda_n - \lambda_{n-1}).$$

Let
$$\phi(s) = \Sigma\lambda_n^{\alpha-1}(\lambda_n - \lambda_{n-1})\, e^{-\lambda_n s}.$$

Then it is easily proved that the series is convergent for $s > 0$ and that

$$\phi(s) \sim \Gamma(\alpha)\, s^{-\alpha}$$

as $s \to 0$.*

The series
$$g(s) = f(s) + K\phi(s) = \Sigma b_n e^{-\lambda_n s},$$

where
$$b_n = a_n + K\lambda_n^{\alpha-1}(\lambda_n - \lambda_{n-1}),$$

satisfies the condition

$$b_n \geq 0, \quad g(s) \sim \{A + K\Gamma(\alpha)\}\, s^{-\alpha}.$$

Hence
$$\sum_1^n b_\nu \sim \left\{\frac{A}{\Gamma(\alpha+1)} + \frac{K}{\alpha}\right\}\lambda_n^{\alpha};$$

and since
$$\sum_1^n \lambda_\nu^{\alpha-1}(\lambda_\nu - \lambda_{\nu-1}) \sim \frac{\lambda_n^{\alpha}}{\alpha},$$

it follows that
$$A_n \sim \frac{A\lambda_n^{\alpha}}{\Gamma(\alpha+1)}.$$

13. Theorem F. *The conclusion of Theorem E is still valid when $\alpha = 0$.*

The proof given in the last section depends essentially on the hypothesis $\alpha > 0$. The result is true when $\alpha = 0$, but the proof is more subtle.†

* Cf. Knopp, "Divergenzcharactere gewisser Dirichlet'scher Reihen," *Acta Mathematica*, vol. 34, pp. 165–204 (especially pp. 191–294).

† Cf. *Proc. London Math. Soc.*, vol. 13.

We have to prove that *if*

$$\text{(i)} \quad a_n > -K\frac{\lambda_n - \lambda_{n-1}}{\lambda_n},$$

$$\text{(ii)} \quad f(x) = \Sigma a_n e^{-\lambda_n s} \to A,$$

as $s \to 0$, *then* Σa_n *is convergent.*

We have $$f(s) = A + o(1),$$
and
$$f''(s) = \Sigma a_n \lambda_n^2 e^{-\lambda_n s} > -K\Sigma\lambda_n(\lambda_n - \lambda_{n-1})\, e^{-\lambda_n s} > -K/s^2.$$
Hence* $$f'(s) = o(1/s),$$
$$\Sigma a_n \lambda_n e^{-\lambda_n s} = o(1/s).$$

To this series we can apply Theorem E; and so we obtain

$$a_1\lambda_1 + a_2\lambda_2 + \ldots + a_n\lambda_n = o(\lambda_n).$$

But this equation, together with condition (ii), secures the convergence of the series Σa_n†; so that the theorem is proved. This theorem is of considerable interest as embodying the widest direct extension at present known of Tauber's original converse of Abel's theorem.‡

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* *Proc. London Math. Soc.*, *l.c. supra.*

† Schnee, *Rendiconti di Palermo*, vol. 27, p. 87.

‡ In our earlier writings on this subject we have made considerable use of the following preliminary lemma: *If* $f(x)$ *has continuous derivatives of the first two orders, and* $f(x) = A + o(1)$, $f''(x) = O(1)$, *as* $x \to \infty$, *then* $f'(x) = o(1)$. Prof. J. Hadamard has very kindly pointed out to us that this result had already been proved independently, in the course of certain dynamical investigations, by himself ("Sur certaines propriétés des trajectories en Dynamique," *Journal de Mathématiques*, ser. 5, vol. 3, 1897, p. 334), and by Herr A. Kneser ("Studien über die Bewegungsvorgänge in der Umgebung instabiler Gleichgewichtslagen, *Journal für Mathematik*, vol. 118, 1897, p. 199). Hadamard and Kneser indeed prove the result, as Prof. Landau asks us to state, in the more general form in which it appears in his paper "Einige Ungleichungen für zweimal differentiierbare Funktionen" (*Proc. London Math. Soc.*, ser. 2, vol. 13, 1913, p. 43), where only the existence and not the continuity of $f''(x)$ is presupposed.

Both in our own writings and in Landau's paper the theorem in question appears only as a preliminary to a series of numbered theorems, the novelty of which is in no way affected by this anticipation.

We take this opportunity of referring also to a recent paper by Mr. A. Rosenblatt ("Über die Multiplikation der unendlichen Reihen," *Bulletin de l'Académie des Sciences de Cracovie*, 1913, p. 603), which contains a number of very interesting generalisations of some of our theorems on the multiplication of series.

ROOTS (y) OF $y^{qp^a} \mp 1 \equiv 0 \pmod{p^\kappa}$.

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[The author's acknowledgments are due to Mr. H. J. Woodall, A.R.C.Sc., for help in reading the proof-sheets.]

1. *Subject.* THE object of this Paper is to develop Rules for computing the *complete set of proper roots* (y) of the Congruences

$$y^{qp^a} \mp 1 \equiv 0 \pmod{p^\kappa} \quad [a \nless 1,\ \kappa > a] \text{.....................(1)},$$

wherein the exponent of y contains p^a. The Rules will be shown to be very simple.

2. *Notation.*

p an *odd* prime.

p^κ the modulus of the Congruences (1).

y denotes a root of $y^{qp^a} - 1 \equiv 0 \pmod{p^\kappa}$.

y' ,, ,, of $y^{qp^a} + 1 \equiv 0 \pmod{p^\kappa}$.

ξ means the Haupt-Exponent (Art. 4) of y modulo p^κ.

μ denotes the *number of proper roots* (y) [see Art. 4] of the Congruences (1).

$\tau(x)$ denotes the Totient of (x), so that

$$\tau(p) = p-1,\ \ \tau(p^a) = (p-1).p^{a-1},\ \ \tau(qp^a) = \tau(q).\tau(p^a) \text{.......(2)}.$$

3. *Fermat's Theorem* [for mod p^κ]. It is well known that—

$$y^{\tau(p^\kappa)} \equiv +1 \pmod{p^\kappa} \text{ always, } [y \text{ prime to } p] \text{...............(3)},$$

and that—

$$y^x \equiv +1 \pmod{p^\kappa} \text{ requires } x = \text{ a factor of } \tau(p^\kappa) \text{.............(4)}.$$

Hence

$$y^{qp^a} \equiv +1 \pmod{p^\kappa} \text{ requires}$$
$$qp^a = \text{ a factor of } (p-1).p^{\kappa-1} \text{............................(5)},$$

which requires—

$$q = \text{a factor of } (p-1),\ [q \text{ may} = 1, \text{ or } (p-1)] \text{..............(5a)},$$
$$a \ngtr \kappa - 1 \text{...(5b)}.$$

4. *Haupt-Exponent* (ξ), *Residue-Index* (ν), *Proper Root* (y).

The Haupt-Exponent* (ξ) of y modulo p^κ is defined to be the *Least value of the Exponent* x satisfying the Congruence

$$y^x - 1 \equiv 0 \pmod{p^\kappa} \qquad \text{(6)},$$

and—(in this case)—y is said to be a *proper root* of that Congruence. Also, since x, ξ are factors of $\tau(p^\kappa)$ by (4) we may write—

$$nx = \nu\xi = \tau(p^\kappa) \qquad \text{(7)},$$

so that—

$$\nu \text{ is the } max. \text{ value of } n, \text{ and } \xi \text{ is the } min. \text{ value of } x \qquad \text{(8)},$$

and here ν is styled the *Residue-Index* of y modulo p^κ.

When $x \neq \xi$ in (6), y is said to be an *improper root* of (6).

5. *Roots y mod. successive prime-powers p^κ.* The finding of roots (y) for successive prime-power moduli p^κ, $p^{\kappa+1}$, $p^{\kappa+2}$, &c., depends on the following general theorem—

"If y be a root of $y^x \equiv +1 \pmod{p^\kappa}$,

then y is also a root of $y^{xp} \equiv +1 \pmod{p^{\kappa+1}}$"...(9).

For, by the hypothesis—

$$y^x = mp^\kappa + 1 \qquad \text{(10)},$$

$$\therefore \quad y^{xp} = (mp^\kappa + 1)^p$$

$$= 1 + p \cdot mp^\kappa + \text{terms containing } p^{2\kappa}$$

$$\equiv +1 \pmod{p^{\kappa+1}} \qquad \text{(11)}.$$

Again, since the general value of y satisfying (11) is

$$Y = mp^\kappa + y \qquad \text{(12)},$$

whereby, $$Y^x \equiv y^x \equiv +1 \pmod{p^\kappa} \qquad \text{(12}a\text{)}.$$

Hence also, Y is the general form of the roots of (12)...(12b).

In what precedes this, y, Y are not necessarily *proper roots* of (9) and (11). But, taking $x = \xi$ the *Haupt-Exponent* of y modulo p^κ,

$$Y^\xi \equiv y^\xi \equiv +1 \pmod{p^\kappa} \qquad \text{(13)},$$

$$\therefore \quad Y^{\xi p} \equiv y^{\xi p} \equiv +1 \pmod{p^{\kappa+1}} \qquad \text{(14)}.$$

and $$Y^{\frac{1}{2}\xi} \equiv y^{\frac{1}{2}\xi} \equiv -1 \pmod{p^\kappa} \ [\text{if } \xi \text{ be } even] \qquad \text{(13}a\text{)}.$$

$$\therefore \quad Y^{\frac{1}{2}\xi p} \equiv y^{\frac{1}{2}\xi p} \equiv -1 \pmod{p^{\kappa+1}} \ [\xi \ even] \qquad \text{(13}b\text{)}.$$

And here y, Y are *proper roots* of (13), and are also—(with rare exceptions)—*proper roots* of (14).

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* This is the German term: it is often styled *Gaussien* by French writers.

6. *Simplest Case* $[q=1,\ \alpha=1]$. The proposed Congruences (1) become simply

$$y^p - 1 \equiv 0, \text{ and } y^p + 1 \equiv 0 \pmod{p^\kappa} \quad\ldots\ldots(15).$$

Here, since $y=+1$, $y'=-1$, are the only proper roots of the above when $\kappa=1$, the general formulæ for the roots (y, y') of the above (15) are—

$$y = mp+1,\ y' = m'p-1 \pmod{p^2} \quad\ldots\ldots(16),$$

$$y = mp^{\kappa-1}+1,\ y' = m'p^{\kappa-1}-1 \pmod{p^\kappa} \quad\ldots\ldots(16a),$$

and the *whole set of proper roots* $(y, y') < p^\kappa$ may be obtained at once from these formulæ by simply taking

$$m, m' = 1, 2, 3, \ldots, (p-1), \text{ in succession} \quad\ldots\ldots(17),$$

excluding $m.\ m'=0$, because $y=+1$, $y'=-1$ are not proper roots of the Congruences (15) when $\kappa>1$. This shows that—

The number of proper roots of (15) is $\mu=p-1$ ......(18).

6*a*. *Properties of Roots* (y, y'). Since p is *odd*, the two Congruences (15) co-exist, and the formulæ (16, 16*a*) show that the roots y, y' may be paired together in two ways, so that—

1°. $m=m'$ gives $y-y'=2$, $y+y'=2mp^{\kappa-1}$, $yy'=m^2p^{2\kappa-2}-1$ ......(19*a*).

2°. $m+m'=p$ gives $y+y'=p^\kappa$, $yy' \equiv -1 \pmod{p^{\kappa-1}}$...(19*b*).

And, as to the sums of the roots—

$$\Sigma(y)-\Sigma(y')=2(p-1)=2\tau(p),\quad \Sigma(y)+\Sigma(y')=(p-1)p^\kappa=\tau(p^{\kappa+1}) \quad ..(20a),$$

$$\Sigma(y)=\tfrac{1}{2}\tau(p^{\kappa+1})+\tau(p),\quad \Sigma(y')=\tfrac{1}{2}\tau(p^{\kappa+1})-\tau(p) \quad\ldots\ldots(20b).$$

7. *Other simple Cases* $[q=1,\ \alpha>1$, but $<\kappa]$. The proposed Congruences (1) become

$$y^{p^\alpha}-1 \equiv 0,\quad y^{p^\alpha}+1 \equiv 0 \pmod{p^\kappa},\ [\alpha<\kappa] \quad\ldots\ldots(21).$$

Here since, as in the previous Case (Art. 6), $y=+1$, $y'=-1$ are the only proper roots when $\kappa=\alpha$, the general formulæ for the roots (y, y') of the above (21) are

$$y=mp+1,\quad y'=m'p-1 \pmod{p^\kappa},\ [\kappa=\alpha+1] \quad\ldots\ldots(22),$$

$$y=mp^{\kappa-\alpha}+1,\ y'=m'p^{\kappa-\alpha}-1 \pmod{p^\kappa} \quad\ldots\ldots(22a),$$

and the *whole set of proper roots* $(y, y') < \kappa$ may be obtained at once from the formulæ by simply taking

$$m,\ m' = 1, 2, 3, \ldots, (p^\alpha-1) \text{ in succession} \quad\ldots\ldots(23),$$

excluding $m=0$, and $m=$ multiple of p, because these values do not yield proper roots of the Congruences (15) when $\kappa=\alpha$. This shows that—

The number of proper roots of (21) is $\mu=p^\alpha-p^{\alpha-1}=\tau(p^\alpha)$ ....(24).

7*a*. *Properties of Roots* (y, y'). Since p is *odd* the two Congruences (15) co-exist, and the formulæ (16, 16*a*) shew that the roots y, y' may be paired together in two ways, so that—

1°. $m = m'$ gives $y - y' = 2$, $y + y' = 2mp^{\kappa-a}$, $yy' = m^2\mu^{2\kappa-2a} - 1$........(25*a*),

2°. $m + m' = p - 1$ gives $y + y' = p^\kappa$, $yy' \equiv -1 \pmod{p^{\kappa-a}}$..(25*b*).

And, as to the sums of the roots—

$$\Sigma(y) - \Sigma(y') = 2\tau(p^a), \quad \Sigma(y) + \Sigma(y') = \tau(p^{\kappa+a}) \text{.............(26a).}$$

$$\Sigma(y) = \tfrac{1}{2}\tau(p^{\kappa+a}) + \tau(p^a), \quad \Sigma(y') = \tfrac{1}{2}\tau(p^{\kappa+a}) - \tau(p^a) \text{............(26a).}$$

8. *More general Case* $[q > 1, \alpha = 1, \kappa > 1]$. The proposed Congruences become

$$y^{qp} - 1 \equiv 0, \quad y^{qp} + 1 \equiv 0 \pmod{p^\kappa}, \quad [\kappa > 1] \text{..................(27),}$$

where q is a factor of $(p-1)$, by (5*a*).

Let η, η' be proper roots of the auxiliary Congruences

$$y^q - 1 \equiv 0, \quad y^q + 1 \equiv 0 \pmod{p^{\kappa-1}} \text{.........................(28),}$$

the modulus $(p^{\kappa-1})$ being therein *one degree lower* than that (p^κ) of the proposed Congruences.

Then—by the general Theorem (9)—the general formulæ for the roots (y, y') of (27) are

$$y = mp + \eta, \quad y' = m'p + \eta' \pmod{p^2} \text{.....................(29),}$$

$$y = mp^{\kappa-1} + \eta, \quad y' = m'p^{\kappa-1} + \eta' \pmod{p^\kappa} \text{... ....(29a),}$$

and the whole set of roots $(y, y') < p^\kappa$ of (27) may be found by taking

$m, m' = 0, 1, 2, 3, \ldots, (p-1)$ [p values] in succession..........(30),

for each sub-root η, η' of the auxiliary Congruences.

But *one* root (y, y') in the set of p roots y, y' arising as above from each sub-root (η, η') is really a proper root of one of the Congruences

$$y^q - 1 \equiv 0, \quad y^q + 1 \equiv 0 \pmod{p^\kappa} \text{.......................(31),}$$

of lower order than the proposed [though with same modulus (p^κ)]; and is therefore to be* rejected (as *not being a proper root* of the proposed Congruences): so that each sub-root (η, η') yields effectively only $(p-1)$ proper roots (y, y'). This shows that—

Number of proper roots (y, y') of (27)

$= (p-1) \times$ number of proper roots (η, η') of (31)......(32),

$= \tau(qp)$ with q *odd* ....(32*a*),

Number of y' is $= \tau(2qp)$ with q *even*....................................(32*b*).

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* It is not possible to recognise these roots *à priori*. A Table of the roots of (31) is in fact required.

8*a*. *Properties of the Roots* (y, y'). Two Cases arise according as q is *odd*, or *even*.

Case i. q *odd* gives the exponent (qp) of (27) *odd*, so that the two Congruences co-exist. The formulæ (29, 29*a*) show that the roots y, y' may be paired together in two ways so that—

1°. $m = m'$ gives $y - y' = \eta - \eta'$, $y + y' = 2mp^{\kappa-1} + (\eta + \eta')$..........(33*a*).

2°. $m + m' = p$ gives $y + y' = p^\kappa$..........................(33*b*).

And, as to the sums of the roots

$\Sigma(y) - \Sigma(y') = \Sigma\{m(\eta - \eta')\} \equiv -2 . \tau(p) \equiv +2 \pmod p$, $\Sigma(y) + \Sigma(y') = \tau(p^{\kappa+1})$..(34*a*)

$\Sigma(y) \equiv +1$, $\Sigma y' \equiv -1 \pmod p$..(34*b*).

Case ii. q *even* gives the exponent (qp) of (27) *even*, so that the only effective Congruence is

$$y^{qp} + 1 \equiv 0 \bmod p^\kappa \qquad (35),$$

and here the roots may be paired by taking the roots equidistant from the ends (of the complete set) so that—

$$y' + \text{the conjugate root } y' = p^\kappa \qquad (36),$$

And $$\Sigma(y') = (p-1)p^\kappa \equiv \tau(p^{\kappa+1}) \qquad (37),$$

9. *Most general Case* $[q > 1,\ \alpha > 1,\ \kappa > \alpha]$. The proposed Congruences are now of the most general kind (1), viz.

$$y^{qp^\alpha} - 1 \equiv 0,\quad y^{qp^\alpha} + 1 \equiv 0 \pmod{p^\kappa},\quad [\kappa > \alpha] \qquad (38),$$

where q is a factor of $(p-1)$, by (5*a*).

Let η, η' be proper roots of the auxiliary Congruences

$$y^q - 1 \equiv 0,\quad y^q + 1 \equiv 0 \pmod{p^{\kappa-1}} \qquad (39),$$

the modulus $(p^{\kappa-1})$ being therein—as in Art. 8—*one degree lower* than that (p^κ) of the proposed Congruences: these roots η, η' will (for shortness' sake) be styled *Sub-roots*.

Then—by the general Theorem (9)—the general formulæ for the roots y, y' of (38) are

$$y = mp + \eta,\quad y' = m'p + \eta' \pmod{p^\kappa},\ [\kappa = \alpha + 1] \qquad (40),$$

$$y = mp^{\kappa-\alpha} + \eta,\quad y' = m'p^{\kappa-\alpha} + \eta' \pmod{p^\kappa} \qquad (40a),$$

and the whole set of roots $(y, y') < p^\kappa$ of (38) may be obtained from these formulæ by taking—

$m, m' = 0, 1, 2, 3, \ldots, (p^\alpha - 1)$, $[p^\alpha$ values$]$, in succession....(41),

for each Sub-root (η, η') of the auxiliaries.

But $p^{\alpha-1}$ roots (y, y') of the set of p^α roots (y, y') arising as above from each Sub-root (η, η') will be found to be really the complete set of roots of all kinds (both proper and improper) of one of the Congruences

$$y^{q\cdot p^{a-1}}-1\equiv 0,\quad y^{q\cdot p^{a-1}}+1\equiv 0 \pmod{p^\kappa}\ldots\ldots\ldots\ldots(42),$$

of lower order than the proposed (though with same modulus (p^κ): and are therefore to be* rejected (as *not being proper roots* of the proposed Congruences): so that each Sub-root (η, η') yields effectively only $p^a-p^{a-1}=\tau(p^a)$ proper roots (y, y'). This shows that—

Number of proper roots (y, y') of (38)

$$=\tau(p^a)\times\text{ number of proper sub-roots }(q, q')\ldots\ldots\ldots\ldots(43),$$

$$=\tau(qp^a)\text{ with } q \textit{ odd}\ldots\ldots\ldots\ldots(43a),$$

Number of proper roots y' is $=\tau(2qp^a)$ with q *even* ....(43b).

9*a*. *Properties of the roots* (y, y'). Two Cases arise according as q is *odd*, or *even*.

Case i. q *odd* gives the exponent (qp^a) of (38) *odd*, so that the two Congruences (38) co-exist. The formulæ (40, 40*a*) show that the roots (y, y') may be paired together in two ways, so that—

1°. $m=m'$ gives $y-y'=\eta-\eta',\ \ y+y'=2mp^{\kappa-a}+(\eta+\eta')$.........(44*a*).

2°. $m+m'=p^a-1$ gives $y+y'=p^\kappa$..........................(44*b*).

And, as to the sums of the roots

$$\Sigma(y)-\Sigma(y')=\Sigma\{m(\eta-\eta')\}\equiv+2 \pmod p,\quad \Sigma(y)+\Sigma(y')=\tau(p^{\kappa+a})\ldots(45a),$$

$$\Sigma(y)\equiv+1,\quad \Sigma(y')\equiv-1 \pmod p\ldots\ldots\ldots\ldots(45b).$$

Case ii. q *even* gives the exponent (qp^a) of (38) *even* so that the only effective Congruence is

$$y^{qp}+1\equiv 0 \pmod{p^\kappa}\ldots\ldots\ldots\ldots(46),$$

and here the roots may be paired by taking roots equi-distant from the ends (of the complete set), so that—

$$y'+\text{ the conjugate } y'=p^\kappa\ldots\ldots\ldots\ldots(47),$$

and $\quad \Sigma(y')=\tau(p^a)\,.\,p^\kappa=\tau(p^{\kappa+a})\ldots\ldots\ldots\ldots(48).$

10. *Divisibility of binomial factors by* p^x—

Let $F(\alpha)$ denote $(y^{qp^\alpha}\mp 1)$ ....(49).

Let $f(\alpha)$ denote $F(\alpha)\div F(\alpha-1)$ ....(50),

so that $F(0)$ means $(y^q\mp 1)$, $F(1)$ means $(y^{qp}\mp 1)$, &c...(50*a*), wherein q may $=1$ or any factor of $(p-1)$.

* It is not possible to recognise these roots *à priori*. A Table of all the roots of (42) is in fact required.

Here the same sign is to be used in the symbols F, f throughout any one research.

Then $f(0), f(1), f(2), \ldots, f(\alpha)$, are the binomial algebraic factors of $F(\alpha)$, so that—

$$F(\alpha) = f(0) \cdot f(1) \cdot f(2) \ldots f(\alpha)\,; \quad [\alpha+1 \text{ factors}] \ldots\ldots\ldots\ldots (51).$$

Now, let y be a *proper root* of $F(\alpha) \equiv 0$ (mod p^κ), $[\kappa > \alpha]$.

Then this involves the following important laws of divisibility of the binomials f by p and its powers.

1°. p is a *divisor of each* of the $(\alpha+1)$ binomials $f(0)$ to $f(\alpha)$ ....(52).

2°. $\kappa = \alpha + 1$ involves that *each* of the $(\alpha+1)$ binomials $f(0)$ to $f(\alpha)$ contains p *once* only...(52*a*).

3°. $\kappa > \alpha + 1$ involves that $f(0)$ contains $p^{\kappa-\alpha}$, and *each* of the α binomials $f(1)$ to $f(\alpha)$ contains p *once* only...(52*b*).

11. *Tabulation of Roots.* The number (μ) of roots (y, y') being, [see Results (18), (24), (32), (43)], $\mu = \tau(q \cdot p^\alpha)$ is so large, even for small values of p and α, as to preclude tabulation except for a few small primes with quite small values of α.

Some space may be saved by the simple relations between certain associated roots y, y'.

1°. $q = 1$ gives $y' = y - 2$ *always*........................(53*a*).

2°. $q = 3$ gives $y' = y - 1$ *always*........................(53*b*).

so that in those two Cases it suffices to tabulate one set (say y), leaving the other set (y') to be inferred from those relations.

12. *Tests.* The following Tests are so simple as to admit of being *very easily applied* to the results.

1°. When $q = 1$ and $\alpha = 1$; then $\Sigma(y) \equiv \tau(p)$, $\Sigma(y') \equiv -\tau(p)$ (mod p^κ)..(54).

2°. When $q = 1$ and $\alpha > 1$; then $\Sigma(y) \equiv \tau(p^\alpha)$, $\Sigma(y') \equiv -\tau(p^\alpha)$ (mod p^κ)..(54*a*).

3°. When q is *odd* and > 1; then $\Sigma(y) \equiv +1$, $\Sigma(y') \equiv -1$ (mod p)......(54*b*).

13. *Auxiliary Congruence Solutions.* The solutions (η, η') of the Auxiliary Congruences (28, 31, 39, 42)

$$y^q - 1 \equiv 0, \quad y^q + 1 \equiv 0 \ (\text{mod } p^{\kappa-1} \text{ and } p^\kappa),$$

$$y^{qp^\alpha} - 1 \equiv 0, \ y^{qp^\alpha} + 1 \equiv 0 \ (\text{mod } p^{\kappa-1} \text{ and } p^\kappa),$$

are required to form the Congruences (1) which are the subject of this Paper.

Tables giving the *complete set of proper roots* modulo p^κ, for all values of q possible to each prime, up to $p = 101$, and $\kappa = 1, 2$ in all cases (and for some of the smaller primes up to $\kappa = 5$) are given in the author's Paper on *Period-Lengths of Circulates* in Vol. XXIX, 1900, pp. 166–179 of this Journal.

Corrigenda in the Tables in Vol.* XXIX.

page 158. In the small Table, cancel two lines;—

line 10, on left [$l = 13$, $r = 44$, $N^{t} = 53^2$, $r_t < 53$].

line 8, on right [$l = 35$, $r = 60$, $N^{t} = 71^2$, $r_t < 71$].

page 174. Table of $\rho^{35} \equiv +1$, mod 71^2. *For* $\rho^{11} \equiv 60$, *Read* 5030.

page 177. Tables of $\rho^{13} \equiv +1$, and $r^{13} \equiv -1$, mod 53^2. Cancel both lines.

For $\rho^{13} \equiv +1$ (mod 53^2),

Read 752, 895, 1689, 460, 413, 1586, 1656, 925, 1777, 2029, 521, 1341.

And, *For* $r^{13} \equiv +1$ (mod 53^2),

Read 2057, 1914, 1120, 2349. 2396, 1223, 1153, 1884, 1032, 780, 2288, 1468.

14. *Present Tables.* In the Tables following, the moduli include the powers of all the small primes $p = 3$ to 19, up to the limit $p^{\kappa} \not> 10^4$.

The Tables give the complete sets of roots (y, y') of the Congruences (1) for these moduli (p^{κ}) for the exponents ξ or $\frac{1}{2}\xi = qp^{\alpha}$ as shown in the scheme below. The last line shows the number (μ) of roots of each Congruence.

Tab.	I.			II.		III.	
Mod p^{κ}	3^2 to 3^8	3^3 to 3^8	3^4 to 3^8	5^2 to 5^5	5^3 to 5^5	7^2 to 7^4	7^3, 7^4
Exponent p^{α}	3	3^2	3^3	5	5^2	7	7^2
μ *of* y, y'	2	6	18	4	20	6	42

Tab.	IV.		V.		
Mod p^{κ}	11^2, 11^3	11^3	13^2, 13^3	17^2, 17^3	19^2, 19^3
Exponent p^{α}	11	11^2	13	17	19
μ *of* y, y'	10	110	12	16	18

Tab.	VI.		VII.		VIII.		
Mod p^{κ}	5^2 to 5^5	5^3 to 5^5	7^2 to 7^4	11^2, 11^3	13^2, 13^3		
Exponent qp^{α}	2.5	2.3^2	3.7	5.11	2.13	3.13	6.13
μ *of* y, y'	.	.	12	40	.	24	.
μ *of* y'	8	40	.	.	26	.	48

* Confirmation of the general correctness of the roots y and y' ($< \frac{1}{2}p^2$) modulo p^2 in these Tables has recently appeared in Dr. N. G. W. H. Beeger's Paper in Vol. XLIII. of this Journal. On pp. 77 to 83, he gives the complete set of the roots ($< \frac{1}{2}p^2$) of the Congruence $x^{p-1} \equiv +1$ (mod p^2), up to $p = 199$. In those Tables the proper and improper roots are *all placed in one list.* The figures agree throughout with the figures in the author's Tables in Vol. XXIX. (after making the corrections stated; these corrections were discovered by the author long ago).

Tables of $y^{p^{a}} \equiv \pm 1 \ (\text{mod } p^{\kappa})$.

TAB. I.

mod p^{κ}	$\xi=3$ y'	y'	$\frac{1}{2}\xi=3$ $y^{\backprime}$	$y^{\backprime}$	$\xi=9$ y'	y'	y'	y'	y'	y'	$\frac{1}{2}\xi=9$ $y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$
3^2	4,	7	2,	5	.	.	.	.	.	.	.	.	.	.	.	[illegible]
3^3	10,	19	8,	17	4,	7,	13,	16,	22,	25	2,	5,	11,	14,	20,	[illegible]
3^4	28,	55	26,	53	10,	19,	37,	46,	64,	73	8,	17,	35,	44,	62,	[illegible]
3^5	82,	163	80,	161	28,	55,	109,	136,	190,	217	26,	53,	107,	134,	188,	21[illegible]
3^6	244,	487	242,	485	82,	163,	325,	406,	568,	649	80,	161,	323,	404,	566,	6[illegible]
3^7	730,	1459	728,	1457	244,	487,	973,	1216,	1702,	1945	242,	485,	971,	1214,	1700,	19[illegible]
3^8	2188,	4375	2186,	4373	730,	1459,	2917,	3646,	5104,	5833	728,	1457,	2915,	3644,	5102,	58[illegible]

mod p^{κ}	$\xi = 27$ y'	y'	y'	y'	y'	y'	y'	y'	y'
3^4	4,	7,	13,	16,	22,	25,	31,	34,	40
	43,	49,	52,	58,	61,	67,	70,	76,	79
3^5	10,	19,	37,	46,	64,	73,	91,	100,	118
	127,	145,	154,	172,	181,	199,	208,	226,	235
3^6	28,	55,	109,	136,	190,	217,	271,	298,	352
	379,	433,	460,	514,	541,	595,	622,	676,	703
3^7	82,	163,	325,	406,	568,	649,	811,	892,	1054
	1135,	1297,	1378,	1540,	1621,	1783,	1864,	2026,	2107
3^8	244,	487,	973,	1216,	1702,	1945,	2431,	2674,	3160
	3403,	3889,	4132,	4618,	4861,	5347,	5590,	6076,	6319

mod p^{κ}	$\frac{1}{2}\xi = 27$ $y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$
3^4	2,	5,	11,	14,	20,	23,	29,	32,	38
	41,	47,	50,	56,	59,	65,	68,	74,	77
3^5	8,	17,	35,	44,	62,	71,	89,	98,	116
	125,	143,	152,	170,	179,	197,	206,	224,	233
3^6	26,	53,	107,	134,	188,	215,	269,	296,	350
	377,	431,	458,	512,	539,	593,	620,	674,	701
3^7	80,	161,	323,	404,	566,	647,	809,	890,	1052
	1133,	1295,	1376,	1538,	1619,	1781,	1862,	2024,	2105
3^8	242,	485,	971,	1214,	1700,	1943,	2429,	2672,	3158
	3401,	3887,	4130,	4616,	4859,	5345,	5588,	6074,	6317

Tables of $y^{p^a} \equiv \pm 1 \pmod{p^\kappa}$.

Tab. II.

mod p	$\xi=5$ y	y	y	y	$\frac{1}{2}\xi=5$ y'	y'	y'	y'
5^2	6,	11,	16,	21	4,	9,	14,	19
5^3	26,	51,	76,	101	24,	49,	74,	99
5^4	126,	251,	376,	501	124,	249,	374,	499
5^5	626,	1251,	1876,	2501	624,	1249,	1874,	2499

mod p^κ	$\xi=25$ y	y	y	y	y	y	y	y	y	y
5^3	6,	11,	16,	21,	31,	36,	41,	46,	56,	61
	66,	71,	81,	86,	91,	96,	106,	111,	116,	121
5^4	26,	51,	76,	101,	151,	176,	201,	226,	276,	301
	326,	351,	401,	426,	451,	476,	526,	551,	576,	601
5^5	126,	251,	376,	501,	751,	876,	1001,	1126,	1376,	1501
	1626,	1751,	2001,	2126,	2251,	2376,	2626,	2751,	2876,	3001

mod p^κ	$\frac{1}{2}\xi=25$ y'	y'	y'	y'	y'	y'	y'	y'	y'	y'
5^3	4,	9,	14,	19,	29,	34,	39,	44,	54,	59
	64,	69,	79,	84,	89,	94,	104,	109,	114,	119
5^4	24,	49,	74,	99,	149,	174,	199,	224,	274,	299
	324,	349,	399,	424,	449,	474,	524,	549,	574,	599
5^5	124,	249,	374,	499,	749,	874,	999,	1124,	1374,	1499
	1624,	1749,	1999,	2124,	2249,	2374,	2624,	2749,	2874,	2999

Tables of $y^{p^a} \equiv \pm 1 \pmod{p^k}$.

TAB. III.

mod p^κ	$\xi=7$						$\frac{1}{2}\xi=7$					
	y	y	y	y	y	y	y'	y'	y'	y'	y'	y'
7^2	8,	15,	22,	29,	36,	43	6,	13,	20,	27,	34,	41
7^3	50,	99,	148,	197,	246,	295	48,	97,	146,	195,	244,	293
7^4	344,	687,	1030,	1373,	1716,	2059	342,	685,	1028,	1371,	1714,	2057

mod p^κ	$\xi = 49$										
	y	y	y	y	y	y	y	y	y	y	y
7^3	8,	15,	22,	29,	36,	43,	57,	64,	71,	78,	85
	92,	106,	113,	120,	127,	134,	141,	155,	162,	169,	176
	183,	190,	204,	211,	218,	225,	232,	239,	253,	260,	267
	274,	281,	288,	302,	309,	316,	323,	330,	337		
7^4	50,	99,	148,	197,	246,	295,	393,	442,	491,	540,	589
	638,	736,	785,	834,	883,	932,	981,	1079,	1128,	1177,	1226
	1275,	1324,	1422,	1471,	1520,	1569,	1618,	1667,	1765,	1814,	1863
	1912,	1961,	2010,	2108,	2157,	2206,	2255,	2304,	2353		

mod p^κ	$\frac{1}{2}\xi = 49$										
	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'
7^3	6,	13,	20,	27,	34,	41,	55,	62,	69,	76,	83
	90,	104,	111,	118,	125,	132,	139,	153,	160,	167,	174
	181,	188,	202,	209,	216,	223,	230,	237,	251,	258,	265
	272,	279,	286,	300,	307,	314,	321,	328,	335		
7^4	48,	97,	146,	195,	244,	293,	391,	440,	489,	538,	587
	636,	734,	783,	832,	881,	930,	979,	1077,	1126,	1175,	1224
	1273,	1322,	1420,	1469,	1518,	1567,	1616,	1665,	1763,	1812,	1861
	1910,	1959,	2008,	2106,	2155,	2204,	2253,	2302,	2351		

Tables of $y^{p^a} \equiv \pm 1 \pmod{p^\kappa}$.

TAB. IV.

mod p^κ	$\xi=11$ y	y	y	y	y	y	y	y	y	y
11^2	12,	23,	34,	45,	56,	67,	78,	89,	100,	111
11^3	122,	243,	364,	485,	606,	727,	848,	969,	1090,	1211

mod p^κ	$\frac{1}{2}\xi=11$ y'	y'	y'	y'	y'	y'	y'	y'	y'	y'
11^2	10,	21,	32,	43,	54,	65,	76,	87,	98,	109
11^3	120,	241,	362,	483,	604,	725,	846,	967,	1087,	1209

mod p^κ	$\xi=121$ y	y	y	y	y	y	y	y	y	y
	12,	23,	34,	45,	56,	67,	78,	89,	100,	111
	133,	144,	155,	166,	177,	188,	199,	210,	221,	232
	254,	265,	276,	287,	298,	309,	320,	331,	342,	353
	375,	386,	397,	408,	419,	430,	441,	452,	463,	474
	496,	507,	518,	529,	540,	551,	562,	573,	584,	595
11^3	617,	628,	639,	650,	661,	672,	683,	694,	705,	716
	738,	749,	760,	771,	782,	793,	804,	815,	826,	837
	859,	870,	881,	892,	903,	914,	925,	936,	947,	958
	980,	991,	1002,	1013,	1024,	1035,	1046,	1057,	1068,	1079
	1101,	1112,	1123,	1134,	1145,	1156,	1167,	1178,	1189,	1200
	1222,	1233,	1244,	1255,	1266,	1277,	1288,	1299,	1310,	1321

mod p^κ	$\frac{1}{2}\xi=121$ y'	y'	y'	y'	y'	y'	y'	y'	y'	y'
11^3	110 *values* $y' =$ *above* $y-2$.									

Tables of $y^p \equiv \pm 1$ (mod p^κ).

TAB. V.

mod p^κ	$\xi = 13$											
	y	y	y	y	y	y	y	y	y	y	y	y
13^2	14,	27,	40,	53,	66,	79,	92,	105,	118,	131,	144,	157
13^3	170,	339,	508,	677,	846,	1015,	1184,	1353,	1522,	1691,	1860,	2029

mod p^κ	$\frac{1}{2}\xi = 13$											
	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'
13^2	12,	25,	38,	51,	64,	77,	90,	103,	116,	129,	142,	155
13^3	168,	337,	506,	675,	844,	1013,	1182,	1351,	1520,	1689,	1858,	2027

mod p^κ	$\xi = 17$							
	y	y	y	y	y	y	y	y
17^2	18,	35,	52,	69,	86,	103,	120,	137
	154,	171,	188,	205,	222,	239,	256,	273
17^3	290,	579,	868,	1157,	1446,	1735,	2024,	2313
	2602,	2891,	3180,	3469,	3758,	4047,	4336,	4625

mod p^κ	$\frac{1}{2}\xi = 17$							
	y'	y'	y'	y'	y'	y'	y'	y'
17^2	16,	33,	50,	67,	84,	101,	118,	135
	152,	169,	186,	203,	220,	237,	254,	271
17^3	288,	577,	866,	1155,	1444,	1733,	2022,	2311
	2600,	2889,	3178,	3467,	3756,	4045,	4334,	4623

mod p^κ	$\xi = 19$								
	y	y	y	y	y	y	y	y	y
19^2	20,	39,	58,	77,	96,	115,	134,	153,	172
	191,	210,	229,	248,	267,	286,	305,	324,	343
19^3	362,	723,	1084,	1445,	1806,	2167,	2528,	2889,	3250
	3611,	3972,	4333,	4694,	5055,	5416,	5777,	6138,	6499

mod p^κ	$\frac{1}{2}\xi = 19$								
	y'	y'	y'	y'	y'	y'	y'	y'	y'
19^2	18,	37,	56,	75,	94,	113,	132,	151,	170
	189,	208,	227,	246,	265,	284,	303,	322,	341
19^3	360,	721,	1082,	1443,	1804,	2165,	2526,	2887,	3248
	3609,	3970,	4331,	4692,	5053,	5414,	5775,	6136,	6497

Tables of $y^{qp^a} \equiv \pm 1$ (mod p^κ).

TAB. VI.

mod p^κ	$\grave{y}$	$\grave{y}$	$\grave{y}$	$\grave{y}$	$\frac{1}{2}\xi = 2.5$ $\grave{y}$	$\grave{y}$	$\grave{y}$	$\grave{y}$	$\grave{y}$	$\grave{y}$
5	2,	3,	. ,	8,	12,	13,	17,	. ,	22,	23
5	7,	18,	32,	43,	. ,	. ,	82,	93,	107,	118
5	57,	68,	. ,	193,	307,	318,	432,	. ,	557,	568
5	182,	443,	807,	. ,	1432,	1693,	. ,	2318,	2682,	2943

mod p^κ	$\grave{y}$	$\grave{y}$	$\grave{y}$	$\grave{y}$	$\frac{1}{2}\xi = 2.25$ $\grave{y}$	$\grave{y}$	$\grave{y}$	$\grave{y}$	$\grave{y}$	$\grave{y}$
5^3	2,	3,	. ,	8,	12,	13,	17,	. ,	22,	23
	27,	28,	. ,	33.	37,	38,	42,	. ,	47,	48
	52,	53,	. ,	58,	62,	63,	67,	. ,	72,	73
	77,	78,	, ,	83,	87,	88,	92,	. ,	97,	98
	102,	103,	. ,	108,	112,	113,	117,	. ,	122,	123
5^4	7,	18,	32,	43,	. ,	. ,	82,	93,	107,	118
	132,	143,	157,	168,	. ,	. ,	207,	218,	232,	243
	257,	268,	282,	293,	. ,	. ,	332,	343,	357,	368
	382,	393,	407,	418,	. ,	. ,	457,	468,	482,	493
	507,	518,	532,	543,	. ,	. ,	582,	593,	607,	618
5^5	57,	68,	. ,	193,	307,	318,	432,	. ,	557,	568
	682,	693,	. ,	818,	932,	943,	1057,	. ,	1182,	1193
	1307,	1318,	. ,	1443,	1557,	1568,	1682,	. ,	1807,	1818
	1932,	1943,	. ,	2068,	2182,	2193,	2307,	. ,	2432,	2443
	2557,	2568,	. ,	2693,	2807,	2818,	2932,	. ,	3057,	3068

Tables of $y^{qp} \equiv \pm 1$ (mod p^{κ}).

TAB. VII.

mod p^{κ}	$\xi = 3.7$													
	y	y	y	y	y	y	y	y	y	y	y	y	y	y
7^2	2,	4,	9,	11,	16,	. ,	23,	25,	. ,	32,	37,	39,	44,	46
7^3	. ,	30,	67,	79,	116,	128,	165,	177,	214,	226,	263,	275,	312,	.
7^4	18,	324,	361,	667,	704,	1010,	. ,	. ,	1390,	1696,	1733,	2039,	2076,	2382

mod p^{κ}	$\frac{1}{2}\xi = 3.7$													
	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'
7^2	3,	5,	10,	12,	17,	. ,	24,	26,	. ,	33,	38,	40,	45,	47
7^3	. ,	31,	68,	80,	117,	129,	166,	178,	215,	227,	264,	276,	313,	.
7^4	19,	325,	362,	668,	705,	1011,	. ,	. ,	1391,	1697,	1734,	2040,	2077,	2383

mod p^{κ}	$\xi = 5.11$										
	y	y	y	y	y	y	y	y	y	y	y
11^2	. ,	14,	25,	36,	47,	58,	69,	80,	91,	102,	113
	4,	15,	26,	37,	48,	59,	70,	. ,	92,	103,	114
	5,	16,	. ,	38,	49,	60,	71,	82,	93,	104,	115
	. ,	20,	31,	42,	53,	64,	75,	86,	97,	108,	119
11^3	3,	. ,	245,	366,	487,	608,	729,	850,	971,	1092,	1213
	9,	130,	251,	372,	493,	614,	. ,	856,	977,	1098,	1219
	27,	148,	269,	390,	511,	. ,	753,	874,	995,	1116,	1237
	81,	202,	323,	444,	565,	686,	807,	928,	1049,	. ,	1291

mod p^{κ}	$\frac{1}{2}\xi = 5.11$										
	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'	y'
11^2	2,	13,	24,	35,	46,	57,	68,	79,	90,	101,	.
	6,	17,	28,	39,	50,	61,	72,	83,	. ,	105,	116
	7,	18,	29,	. ,	51,	62,	73,	84,	95,	106,	117
	8,	19,	30,	41,	52,	63,	74,	85,	96,	107,	.
11^3	40,	. ,	282,	403,	524,	645,	766,	887,	1008,	1129,	1250
	94,	215,	336,	457,	578,	. ,	820,	941,	1062,	1183,	1304
	112,	233,	354,	475,	. ,	717,	838,	959,	1080,	1201,	1322
	118,	239,	360,	481,	602,	723,	844,	965,	1086,	. ,	1328

Tables of $y^{qp} \equiv \pm 1 \pmod{p^\kappa}$.

Tab. VIII.

mod p^κ	$\frac{1}{2}\xi = 2.13$												
	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$
13^2	5,	18,	31,	44,	57,	. ,	83,	96,	109,	122,	135,	148,	161
	8,	21,	34,	47,	60,	73,	86,	. ,	112,	125,	138,	151,	164
13^3	70,	. ,	408,	577,	746,	915,	1084,	1253,	1422,	1591,	1760,	1929,	2098
	99,	268,	437,	606,	775,	944,	1113,	1282,	1451,	1620,	1789,	. ,	2127

mod p^κ	$\xi = 3.13$												
	y	y	y	y	y	y	y	y	y	y	y	y	y
13^2	3,	16,	29,	42,	55,	68,	81,	94,	107,	120,	133,	. ,	159
	9,	. ,	35,	48,	61,	74,	87,	100,	113,	126,	139,	152,	165
13^3	22,	191,	360,	529,	698,	867,	. ,	1205,	1374,	1543,	1712,	1881,	2050
	146,	315,	484,	653,	822,	991,	. ,	1329,	1498,	1667,	1836,	1205,	2174

mod p^κ	$\frac{1}{2}\xi = 3.13$												
	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$
13^2	4,	17,	30,	43,	56,	69,	82,	95,	108,	121,	134,	. ,	160
	10,	. ,	36,	49,	62,	75,	88,	101,	114,	127,	140,	148,	166
13^3	23,	192,	361,	530,	699,	868,	. ,	1206,	1375,	1544,	1713,	1882,	2051
	147,	316,	485,	654,	823,	992,	. ,	1331,	1449,	1668,	1837,	1206,	2175

mod p^κ	$\frac{1}{2}\xi = 2.3.13$												
	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$	$y^{\backprime}$
13^2	2,	15,	28,	41,	54,	67,	. ,	93,	106,	119,	132,	145,	158
	6,	. ,	32,	45,	58,	71,	84,	97,	110,	123,	136,	149,	162
	7,	20,	33,	46,	59,	72,	85,	98,	111,	124,	137,	. ,	163
	11,	24,	37,	50,	63,	76,	. ,	102,	115,	128,	141,	154.	167
13^3	19,	188,	357,	526,	695,	864,	1033,	1203,	1371,	. ,	1709,	1878,	2047
	80,	249,	. ,	587,	756,	925,	1094,	1263,	1432,	1601,	1770,	1939,	2108
	89,	258,	427,	596,	765,	934,	1103,	1272,	1441,	1610,	. ,	1948,	2117
	150,	319,	488,	. ,	826,	995,	1164,	1333,	1502,	1671,	1840,	2009,	2178

A GENERAL RESULT IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS.

By *H. Bateman.*

1. Considerable progress has been made recently in the theory of a partial differential equation of the type*

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) u = 0,$$

where f is a homogeneous polynomial of the n^{th} degree in its three arguments.

The theory is closely connected with that of the algebraic curve $f(x_1, x_2, x_3) = 0$, where x_1, x_2, x_3 are homogeneous co-ordinates: the solutions of the characteristic equation

$$f\left(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}\right) = 0$$

plays a very important part.

When we pass on to the study of a partial differential equation of type

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right) u = 0 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

some new problems present themselves. We shall consider in this note the general problem of finding solutions of the form

$$u = \gamma\, \phi(\alpha, \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where α, β, γ are certain functions of x, y, z, t, and ϕ is an arbitrary function which possesses a suitable number of derivates.

In order that solutions of this type may exist, the equation $f(x_1, x_2, x_3, x_4) = 0$ must represent either a ruled surface or a surface containing at least one straight line, the case in which the surface in ruled is, however, of chief interest; we suppose as before that f is a homogeneous polynomial of the n^{th} degree in its arguments.

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* See, for instance, A. R. Forsyth, *Mess. of Math.*, vol. xxvii. (1898), p. 99. *Phil. Trans*, A, vol. cxci. (1898), p. 1. Ivar Fredholm, *Comptes Rendus*, t. cxxiv. (1899), p. 32. *Acta Math.*, t. xxiii. (1900). *Rend. Palermo*, t. xxv. (1908). J. le Roux, *Liouville's Journal* (5), t. vi. (1900). *Comptes Rendus*, December 28th, 1903. E. T. Whittaker, *Monthly Notices of the Royal Astronomical Society*, vol. lxii. (1902). *Math. Ann.* (1903). H. Bateman, *Proc. London Math. Soc.* (2), vol. i. (1904). N. Zeilon, *Arkiv. f. Mat. Astr. o. Fys.*, Stockholm (1911), (1913).

To obtain a particular solution of the required type, we choose two sets of constants l_1, l_2, l_3, l_4 and m_1, m_2, m_3, m_4 such that, when λ and μ are arbitrary, the equation

$$f(\lambda l_1 + \mu m_1,\ \lambda l_2 + \mu m_2,\ \lambda l_3 + \mu m_3,\ \lambda l_4 + \mu m_4) = 0 \ldots (3)$$

is satisfied identically. This may be done by taking the homogeneous coordinates of two points on the same generator for the l's and m's respectively. Now write $\gamma = 1$ and

$$\left.\begin{array}{l} \alpha = \ l_1 x + \ l_2 y + \ l_3 z + \ l_4 t + a \\ \beta = m_1 x + m_2 y + m_3 z + m_4 t + b \end{array}\right\} \ldots\ldots\ldots\ldots (4),$$

where a and b are arbitrary constants, then it is easy to verify that the expression (2) satisfies the partial differential equation (1).

To generalise this solution, we regard l_1, l_2, l_3, l_4, a and m_1, m_2, m_3, m_4, b as functions of two parameters α_0, β_0, and consider the double integral

$$u = \iint \Phi(\alpha_0, \beta_0) \frac{d\alpha_0\, d\beta_0}{\alpha\beta} \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

taken over some domain of the complex variables α_0, β_0. Since each element of the double integral satisfies the equation (1), it follows that u is generally a solution of (1) provided the domain of integration does not depend directly on x, y, z, t.

With a suitable choice of a domain of integration the double integral can be evaluated with the aid of Poincaré's theory of the residues of double integrals.* Let us suppose that α and β both vanish when $\alpha = \alpha_1$, $\beta = \beta_1$, then the term in u which depends on the residue at α_1, β_1 is

$$\frac{(2\pi i)^2}{J} \Phi(\alpha_1, \beta_1),$$

where J is the value of the Jacobian $\dfrac{\partial(\alpha, \beta)}{\partial(\alpha_0, \beta_0)}$ for $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$. It is natural to expect that each such term will itself be a solution of the differential equation (1), and so we are led to enunciate the following general theorem:—

Let the equations $\alpha = 0$, $\beta = 0$ *be solved for* α_0, β_0, *giving*

$$\alpha_0 = \alpha_1(x, y, z, t), \quad \beta_0 = \beta_1(x, y, z, t) \ldots\ldots\ldots (6),$$

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* *Comptes Rendus*, t. cii. (1886), p. 202. *Acta Math.*, t. ix. (1887), p. 321. See also E. Picard and G. Simart, *Théorie des fonctions algébriques de deux variables indépendantes*, t. i., Paris (1897). E. Picard, *Traité d'analyse*, t. ii. (1905), p. 276.

then if J is the value of the Jacobian $\frac{\partial(\alpha, \beta)}{\partial(\alpha_0, \beta_0)}$ after these expressions have been substituted for α_0, β_0, the function

$$u = \frac{1}{J}\Phi(\alpha_1, \beta_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

is a solution of the differential equation (1).

This result is difficult to prove directly in the general case, but a simple verification can be given when the differential equation (1) is the equation of wave motion

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2} \quad \ldots\ldots\ldots\ldots (8).$$

In the general case it can be verified by differentiation that the functions α_1, β_1 are such that an arbitrary function $\Phi(\alpha_1, \beta_1)$ satisfies the partial differential equation of the characteristics, viz.,

$$f\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}, \frac{\partial \phi}{\partial t}\right) = 0 \quad \ldots\ldots\ldots\ldots (9).$$

In the present case this implies that α_1 and β_1 satisfy the equations

$$\left.\begin{aligned} \left(\frac{\partial \alpha_1}{\partial x}\right)^2 + \left(\frac{\partial \alpha_1}{\partial y}\right)^2 + \left(\frac{\partial \alpha_1}{\partial z}\right)^2 &= \left(\frac{\partial \alpha_1}{\partial t}\right)^2 \\ \frac{\partial \alpha_1}{\partial x}\frac{\partial \beta_1}{\partial x} + \frac{\partial \alpha_1}{\partial y}\frac{\partial \beta_1}{\partial y} + \frac{\partial \alpha_1}{\partial z}\frac{\partial \beta_1}{\partial z} &= \frac{\partial \alpha_1}{\partial t}\frac{\partial \beta_1}{\partial t} \\ \left(\frac{\partial \beta_1}{\partial x}\right)^2 + \left(\frac{\partial \beta_1}{\partial y}\right)^2 + \left(\frac{\partial \beta_1}{\partial z}\right)^2 &= \left(\frac{\partial \beta_1}{\partial t}\right)^2 \end{aligned}\right\} \ldots (10).$$

These equations may be replaced by three equations of type

$$\frac{\partial(\alpha_1, \beta_1)}{\partial(y, z)} = i\,\frac{\partial(\alpha_1, \beta_1)}{\partial(x, t)} \quad \ldots\ldots\ldots\ldots (11).$$

Writing $M_x . F(\alpha_1, \beta_1)$ for the quantity which occurs on the left-hand side and $M_y . F(\alpha_1, \beta_1)$, $M_z . F(\alpha_1, \beta_1)$ for analogous quantities occurring in the other two equations, we can easily verify that the following equations are satisfied:—

$$\left.\begin{aligned} \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} + i\frac{\partial M_x}{\partial t} = 0, &\quad \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x} + i\frac{\partial M_y}{\partial t} = 0 \\ \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} + i\frac{\partial M_z}{\partial t} = 0, &\quad \frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} + \frac{\partial M_z}{\partial z} = 0 \end{aligned}\right\} \ldots (12).$$

Now it follows from these equations that M_x, M_y, M_z are all solutions of equation (8). Hence, since the function $F(\alpha_1, \beta_1)$ is arbitrary, we may conclude that if $\Psi(\alpha_1, \beta_1)$ is an arbitrary function,

$$\frac{\partial(\alpha_1, \beta_1)}{\partial(y, z)} \Psi(\alpha_1, \beta_1)$$

is a solution of equation (8). On calculating the first factor, we find that

$$\frac{\partial(\alpha_1, \beta_1)}{\partial(y, z)} J = l_2 m_3 - l_3 m_2 :$$

now the quantity on the right-hand side depends only on α_1 and β_1, hence we have the result that if Φ is an arbitrary function an expression of type (7) satisfies the differential equation (8).

2. The theorem can be generalised in several ways. First of all we can introduce two sets of variables x, y, z, t; x_1, y_1, z_1, t_1 and write in place of (4)

$$\left.\begin{aligned} \alpha &= l_1 x + l_2 y + l_3 z + l_4 t + a \\ \beta &= m_1 x_1 + m_2 y_1 + m_3 z_1 + m_4 t_1 + b \end{aligned}\right\} \quad \text{..........(13).}$$

When the l's and m's are constants connected by equation (3), it is easy to verify that $\phi(\alpha, \beta)$ satisfies the partial differential equation

$$f\left(\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial x_1},\ \lambda \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial y_1},\right.$$
$$\left.\lambda \frac{\partial}{\partial z} + \mu \frac{\partial}{\partial z_1},\ \lambda \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial t_1}\right) u = 0 \quad \text{...(14)}$$

for all values of the constants λ, μ. Generalising this solution as before, we regard the l's and m's as functions of two parameters α_0, β_0 and form the double integral (5). The equations $\alpha = 0$, $\beta = 0$ now give
$\alpha_0 = \alpha_1(x, y, z, t;\ x_1, y_1, z_1, t_1)$, $\beta_0 = \beta_1(x, y, z, t;\ x_1, y_1, z_1, t_1)$ and formula (7) provides us with a solution of all the equations of type (14). Putting $\mu = 0$ we see that equation (1) is satisfied, while if we put $\lambda = 0$ we find that the same function also satisfies the equation

$$f\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial t_1}\right) u = 0.$$

Hence we have a solution of (1) which depends on the two sets of variables x, y, z, t; x_1, y_1, z_1, t_1, and remains a solution when we put $x_1 = x$, $y_1 = y$, $z_1 = z$, $t_1 = t$.

In the case of the equation of wave motion (8) the equation (14) implies that the three equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial^2 u}{\partial x \partial x_1} + \frac{\partial^2 u}{\partial y \partial y_1} + \frac{\partial^2 u}{\partial z \partial z_1} = \frac{\partial^2 u}{\partial t \partial t_1},$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial z_1^2} = \frac{\partial^2 u}{\partial t_1^2}$$

are satisfied. With the aid of a function u of this type we can obtain some interesting expressions for the components of the electric and magnetic forces E, H in a type of electromagnetic field in free aether. If the units are chosen so that the velocity of light is represented by unity, the electromagnetic potentials

$$A_x = \frac{\partial u}{\partial x_1}, \quad A_y = \frac{\partial u}{\partial y_1}, \quad A_z = \frac{\partial u}{\partial z_1}, \quad \Phi = -\frac{\partial u}{\partial t_1}$$

satisfy the relation

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{\partial \Phi}{\partial t} = 0,$$

and are all solutions of the wave equation; consequently the equations

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t} = \frac{\partial^2 u}{\partial x \partial t_1} - \frac{\partial^2 u}{\partial x_1 \partial t},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$H_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial^2 u}{\partial y \partial z_1} - \frac{\partial^2 u}{\partial y_1 \partial z}$$

may be used to specify the vectors E, H in a type of electromagnetic field in free aether. It is interesting to note that when x_1, y_1, z_1, t_1 are regarded as the variables, E and H are the electric and magnetic forces in a type of electromagnetic field for which the potentials are

$$A_x' = -\frac{\partial u}{\partial x}, \quad A_y' = -\frac{\partial u}{\partial y}, \quad A_z' = -\frac{\partial u}{\partial z}, \quad \Phi' = \frac{\partial u}{\partial t},$$

3. Returning to the general theory of § 1 let us write

$$p_{23}=\frac{\partial(\alpha_1, \beta_1)}{\partial(y, z)}, \quad p_{31}=\frac{\partial(\alpha_1, \beta_1)}{\partial(z, x)}, \quad p_{12}=\frac{\partial(\alpha_1, \beta_1)}{\partial(x, y)},$$

$$p_{14}=\frac{\partial(\alpha_1, \beta_1)}{\partial(x, t)}, \quad p_{24}=\frac{\partial(\alpha_1, \beta_1)}{\partial(y, t)}, \quad p_{34}=\frac{\partial(\alpha_1, \beta_1)}{\partial(x, t)};$$

and regard the p's as the six coordinates of a line. It is easy to see that if an arbitrary function of α_1 and β_1 satisfies equation (9) the functions α_1, β_1 will satisfy three partial differential equations of type

$$\left.\begin{aligned} G_1(p_{23}, p_{31}, p_{12}, p_{14}, p_{24}, p_{34}) = 0 \\ G_2(p_{23}, p_{31}, p_{12}, p_{14}, p_{24}, p_{34}) = 0 \\ G_3(p_{23}, p_{31}, p_{12}, p_{14}, p_{24}, p_{34}) = 0 \end{aligned}\right\} \text{.........} (15),$$

where $G_1 = 0$, $G_2 = 0$, $G_3 = 0$ are the line equations of the ruled surface $f(x_1, x_2, x_3, x_4) = 0$. Conversely, if we are given three partial differential equations of type (15) for two functions α_1, β_1 we can obtain a solution by finding the ruled surface $f = 0$ common to the three complexes $G_1 = 0$, $G_2 = 0$, $G_3 = 0$, and solving a differential equation of type (9).

4. The general theory of § 1 can evidently be extended to the case in which there are $2n$ independent variables instead of four; it is necessary, of course, to use the theory of the residues of multiple integrals. The verification used for the case of the wave equation can also be extended to the case of the equation

$$\sum_{m=1}^{2n} \frac{\partial^2 u}{\partial x_m^{\ 2}} = 0 \text{ ....} (16),$$

the system of equations (11) being now replaced by a system of equations of type

$$\frac{\partial(\alpha_1, \alpha_2, \ldots, \alpha_n)}{\partial(x_\lambda, x_\mu, \ldots)} = \pm \frac{\partial(\alpha_1, \alpha_2, \ldots, \alpha_n)}{\partial(x_\xi, x_\eta, \ldots)} \text{ ....} (17),$$

where the indices λ, μ, ..., ξ, η, ... are all different. The system of equations (12) is replaced by a more general system of linear equations which are of the types considered by Volterra;* it follows from his results that the functions M

* "Sulle funzione conjugate," *Rend. Lincei* (4), t. v. (1889), pp. 599, 630.

which occur in these equations are all solutions of (16). Hence one of the Jacobians in (17), when multiplied by an arbitrary function of $\alpha_1, \ldots, \alpha_n$, represents a solution of (16). On calculating the Jacobian, we obtain the desired result.

To make things clear, let us consider the case of six independent variables. We require, first of all, three functions α_1, α_2, α_3 (which we shall call α, β, γ) which satisfy the equations

$$\sum_1^6 \left(\frac{\partial \alpha}{\partial x_r}\right)^2 = 0, \quad \sum_1^6 \left(\frac{\partial \beta}{\partial x_r}\right)^2 = 0, \quad \sum_1^6 \left(\frac{\partial \gamma}{\partial x_r}\right)^2 = 0,$$

$$\sum_1^6 \frac{\partial \beta}{\partial x_r}\frac{\partial \gamma}{\partial x_r} = 0, \quad \sum_1^6 \frac{\partial \gamma}{\partial x_r}\frac{\partial \alpha}{\partial x_r} = 0, \quad \sum_1^6 \frac{\partial \alpha}{\partial x_r}\frac{\partial \beta}{\partial x_r} = 0,$$

Using M_{123} to denote the determinant $\dfrac{\partial(\alpha, \beta, \gamma)}{\partial(x_1, x_2, x_3)}$, we may replace the preceding equations by the following set of equations:

$$\left.\begin{array}{ll} \pm M_{123} = M_{456}, & \pm M_{124} = M_{536} \\ \pm M_{234} = M_{516}, & \pm M_{341} = M_{256} \\ \pm M_{125} = M_{346}, & \pm M_{126} = M_{354} \\ \pm M_{235} = M_{146}, & \pm M_{236} = M_{154} \\ \pm M_{246} = M_{135}, & \pm M_{136} = M_{245} \end{array}\right\} \quad \ldots\ldots\ldots(18),$$

wherein either the upper or the lower sign is taken in each case.

Now the quantities M evidently satisfy the following equations:

$$\left.\begin{array}{l} \dfrac{\partial M_{123}}{\partial x_4} + \dfrac{\partial M_{324}}{\partial x_1} + \dfrac{\partial M_{134}}{\partial x_2} + \dfrac{\partial M_{214}}{\partial x_3} = 0 \\ \dfrac{\partial M_{123}}{\partial x_5} + \dfrac{\partial M_{325}}{\partial x_1} + \dfrac{\partial M_{135}}{\partial x_2} + \dfrac{\partial M_{215}}{\partial x_3} = 0 \\ \dfrac{\partial M_{123}}{\partial x_6} + \dfrac{\partial M_{326}}{\partial x_1} + \dfrac{\partial M_{136}}{\partial x_2} + \dfrac{\partial M_{216}}{\partial x_3} = 0 \\ \dfrac{\partial M_{456}}{\partial x_1} + \dfrac{\partial M_{516}}{\partial x_4} + \dfrac{\partial M_{461}}{\partial x_5} + \dfrac{\partial M_{415}}{\partial x_6} = 0 \\ \dfrac{\partial M_{456}}{\partial x_2} + \dfrac{\partial M_{526}}{\partial x_4} + \dfrac{\partial M_{462}}{\partial x_5} + \dfrac{\partial M_{425}}{\partial x_6} = 0 \\ \dfrac{\partial M_{456}}{\partial x_3} + \dfrac{\partial M_{536}}{\partial x_4} + \dfrac{\partial M_{463}}{\partial x_5} + \dfrac{\partial M_{435}}{\partial x_6} = 0 \end{array}\right\} \quad \ldots\ldots\ldots(19).$$

Differentiating these with regard to x_4, x_5, x_6, x_1, x_2, x_3 respectively and making use of the equations (18), we get

$$\sum_{r=1}^{6} \frac{\partial^2 M_{123}}{\partial x_r^{\,2}} = 0.$$

In a similar way it can be shown that the other quantities M satisfy this equation. If, moreover, we multiply all the quantities M by the same function $\phi(\alpha, \beta, \gamma)$ the relations (19) will still be satisfied, and so we may conclude that a function of type

$$u = \frac{\partial(\alpha, \beta, \gamma)}{\partial(x_1, x_2, x_3)} \phi(\alpha, \beta, \gamma)$$

satisfies the partial differential equation

$$\sum_{r=1}^{6} \frac{\partial^2 u}{\partial x_r^{\,2}} = 0,$$

ϕ being an arbitrary function.

ON CENTRO-SYMMETRIC AND SKEW-CENTRO-SYMMETRIC DETERMINANTS.

By *W. H. Metzler.*

1. Certain properties of these determinants have been given by Muir*, who shows that they may be expressed as the product of two factors D and D', and for a centro-symmetric determinant Δ of order $2m$ he shows that it may be expressed as the difference of two squares. Thus

$$\Delta = D . D' \equiv \tfrac{1}{4}\{(D + D')^2 - (D - D')^2\},$$

but this being an identity is independent of what D and D' are. In this paper it is shown that D is the sum of two sets of minors of order m formed from the first m rows of Δ, and that D' is the difference of the same two sets of minors, and in this way their product is the difference of two squares. The same thing is shown for skew-centro-symmetric determinants, and other interesting results are given in articles 9, 10, 11, 12, 13.

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* "On skew determinants," *Philosophical Magazine* (1881).

2. Two constituents are said to be *conjugate* with respect to the centre of a determinant when they lie on a line through the centre and are equally distant from it.

A determinant is *centro-symmetric* when every constituent is equal to its conjugate with respect to the centre.

A determinant is *skew-centro-symmetric* when every constituent is the negative of its conjugate with respect to the centre.

It follows from this definition that a skew-centro-symmetric determinant of odd order has its centre constituent zero.

3. *Combinations and minors.* Let $\alpha \equiv (\alpha_1 \alpha_2 \ldots \alpha_m)$ be a combination, m at a time, of the numbers 1, 2, 3, ..., $2m$, such that $\alpha_h + \alpha_k \neq 2m+1$ for all values of h and k from l to m. There are 2^m such combinations, for they evidently may be formed by writing the numbers 1, 2, ..., $2m$ in m pairs, the sum of each pair being $2m+1$, and taking one number from each pair.

Let $\beta \equiv (\beta_1 \beta_2 \ldots \beta_m)$ be the complementary combination of α, then β is also the *reflex-combination* of α, that is β_k is the defect from $2m+1$ for some one of the numbers in α for each value of k from l to m. For by hypothesis the defect of α_k from $2m+1$ is not found in α and therefore must be in β. It follows therefore that of the $(2m)_m$ combinations of the numbers 1, 2, ..., $2m$ taken m at a time there are 2^m, the complementary and reflex of each of which are alike.

Two minors of a determinant may be called the *reflex* of each other when the rows and columns of one are the reflex combinations of the rows and columns respectively of the other.

Two minors are said to be *trans-reflex* of each other when the row numbers of the two are the same and the column numbers of the two are reflex combinations.

Two minors are said to be *sub-reflex* when the column numbers of the two are the same and the row numbers are reflex combinations.

4. *Every centro-symmetric determinant Δ of even order is expressible as the difference of two squares.*

For if we perform the following operations:

(*a*) $$r_1 + r_{2m},\ r_2 + r_{2m-1},\ \ldots,\ r_m + r_{m+1},$$

and

(*b*) $$c_1 - c_{2m},\ c_2 - c_{2m-1},\ \ldots,\ c_m - c_{m+1},$$

the resulting determinant has a square of m^2 zeros in the upper left-hand corner and therefore breaks up into the product of two determinants with binomial elements, D and D'.

We may write

$$D = |a_{rs} + a_{rt}|,$$

$$D' = |a_{rs} - a_{rt}|,$$

where
$$r = 1, 2, 3, \ldots, m,$$

$$s = 1, 2, 3, \ldots, m,$$

$$t = 2m, 2m-1, \ldots, m+1.$$

The determinant D may be written as the sum of 2^m determinants with monomial elements, concerning which it may be observed that:

(1) *For every determinant*

$$M_\alpha \equiv \begin{vmatrix} 1, & 2, & \ldots, & m \\ \alpha_1, & \alpha_2, & \ldots, & \alpha_m \end{vmatrix},$$

there is another

$$M_\beta \equiv \begin{vmatrix} 1, & 2, & \ldots, & m \\ \beta_1, & \beta_2, & \ldots, & \beta_m \end{vmatrix},$$

where $\alpha_k + \beta_k = 2m + 1 \quad (k = 1, 2, \ldots, m).$

That is M_α and M_β are trans-reflex minors.

(2) *The signs of M_α and M_β, when the columns are arranged in their natural order, are the same or opposite according as $\frac{1}{2}\{m(m-1)\}$ is even or odd.*

For if there are g_k numbers following α_k smaller than α_k, there are g_k numbers following β_k larger than β_k. Therefore g_k is the number of inversions due to the position of α_k in M_α, and $m-k-g_k$ is the number of inversions due to the position β_k in M_β. The sign factor, therefore, for M_α when the column numbers are written in their natural order is $(-1)^{g_1+g_2+\ldots+g_m}$ and that for M_β under similar circumstances is $(-1)^{m^2-\frac{1}{2}\{m(m+1)\}-(g_1+\ldots+g_m)}$ or $(-1)^{\frac{1}{2}\{m(m-1)\}-(g_1+g_2+\ldots+g_m)}$. Since the exponents differ from $\frac{1}{2}\{m(m-1)\}$ by an even number the truth of the theorem appears.

(3) *There are as many positive as negative terms in the series of terms.*

Considering two consecutive cases, say when $m = k$ and $m = k+1$, we see that for every term

$$\begin{vmatrix} 1, & 2, & \ldots, & k \\ \alpha_1, & \alpha_2, & \ldots, & \alpha_k \end{vmatrix} \equiv M,$$

when $m=k$, there are two terms

$$\begin{vmatrix} 1, & 2, & 3, & \ldots, & k+1 \\ 1, & \alpha_1+1, & \alpha_2+1, & \ldots, & \alpha_k+1 \end{vmatrix} \equiv M',$$

and

$$\begin{vmatrix} 1, & 2, & 3, & \ldots, & k+1 \\ 2k+2, & \alpha_1+1, & \alpha_2+1, & \ldots, & \alpha_k+1 \end{vmatrix} \equiv M'',$$

when $m=k+1$.

The term M' will obviously have the same sign as M, and M'' will have the same or opposite sign according as k is even or odd. It follows, therefore, that if there are as many positive as negative terms when $m=k$, there will be as many positive as negative terms when $m=k+1$, and since it is true when $m=2$ and $m=3$, it is true in general.

5. In the case of D' it is obvious from the method of formation that the same 2^m determinants occur as in D, and the signs of the various terms will be the same as in D except that whenever there is an odd number of columns with negative elements the sign will be changed. If k be the number of such columns taken the sign factor will be multiplied by $(-1)^k$, and there are m_k such determinants. The number of terms changing sign on account of negative elements would therefore be $m_1+m_3+m_5 \ldots + m_{2h+1} + \ldots = 2^{m-1}$, which is just half of the whole number of terms.

6. It follows from the foregoing that D is the sum of two sets of minors of order m, and that D' is the difference of the same two sets of minors, and therefore Δ, which is their product, may be expressed as the difference of two squares.

7. If Δ is of odd order $2m+1$, it still breaks up into two factors, the one factor being the sum, with proper sign, of those minors of order m, formed from the first m rows, which have for their column numbers the 2^m combinations the complementary and reflex of each of which are alike. The other factor consists of the sum, with proper sign, of those same minors each bordered with elements from the first $m+1$ elements of the $(m+1)$st column and the first m elements of the $(m+1)$st row.

8. *Every skew-centro-symmetric determinant of even order is expressible as the difference of two squares.*

For if we perform the operations:

(*a*) $\qquad r_1 + r_{2m},\ r_2 + r_{2m-1},\ \ldots,\ r_m + r_{m+1},$

and

(*b*) $\qquad c_1 + c_{2m},\ c_2 + c_{2m-1},\ \ldots,\ c_m + c_{m+1},$

the result will be seen to break up into two determinants D and D' with binomial elements. Here as in the case of centro-symmetric determinants if the element in the r^{th} row and s^{th} column of D is $x-y$, then the element in the same position of D' (or $-D'$ if m is odd) is $x-y$, and hence the theorem follows as for centro-symmetric determinants.

9. *Every skew-centro-symmetric determinant of odd order vanishes.*

For, performing the operations

$$(a) \qquad r_1+r_{2m+1},\ r_2+r_{2m},\ \ldots,\ r_m+r_{m+2},$$

and

$$(b) \qquad c_1+c_{2m+1},\ c_2+c_{2m},\ \ldots,\ c_m+c_{m+2},$$

the result will be a determinant with a square of $(m+1)^2$ zeros and therefore vanishes.

10. *The sum of the coaxial minors of odd order of a skew-centro-symmetric determinant is zero.*

For all those which are *bicoaxial*, that is coaxial with respect to both principal and secondary axis, are determinants of the same type as the original and therefore vanish. Those which are not bicoaxial go in pairs which are the negative of each other, and therefore the whole sum vanishes.

11. It follows from the preceding article that *the determinantal equation of a skew-centro-symmetric determinant contains either only even or only odd powers of the variable.*

12. *Vanishing aggregates for skew-centro-symmetric determinants.* Since every minor of order m of a skew-centro-symmetric determinant Δ of order $2m$ is equal to the reflex when m is even, and to the negative of the reflex when m is odd, it is evident that the known aggregate* for centro-symmetric determinants takes the following form:

$$\sum_{1}^{m_k} i \left| \begin{matrix} \overline{(2m\,|\,\underset{a}{m}\,|\,\underset{i}{k})}\ (2m\,|\,\underset{a}{\overline{m}}\,|\,\underset{i}{k}) \\ (2m\,|\,\underset{a}{m}) \end{matrix} \right|$$

$$= (-1)^m \sum_{1}^{m_k} i \left| \begin{matrix} (2m\,|\,\underset{a}{m}) \\ \overline{(2m\,|\,\underset{a}{m}\,|\,\underset{i}{k})}\ (2m\,|\,\underset{a}{\overline{m}}\,|\,\underset{i}{k}) \end{matrix} \right| \ldots\ldots(\text{A}),$$

where $\overline{(2m\,|\,\underset{a}{m}\,|\,\underset{i}{k})}$ denotes the reflex of $(2m\,|\,\underset{a}{m}\,|\,\underset{i}{k})$.

* Metzler, "On certain aggregates of determinant minors," *Trans. Amer. Math. Soc.*, vol. ii., p. 4.

13. It may be observed that this aggregate could be stated in a somewhat more general form and not be confined to minors of order m of a determinant of order $2m$.

We might write our aggregate for a skew-centro-symmetric determinant as follows:

$$\sum_{1}^{m_k} i \left| \begin{matrix} \overline{(n|m|k)}_{a\;\;i} \; (n|\overline{m}|k)_{a\;\;i} \\ (n|m)_{a} \end{matrix} \right|$$

$$= (-1)^m \sum_{1}^{m_k} i \left| \begin{matrix} (n|m)_{a} \\ \overline{(n|m|k)}_{a\;\;i} \; (n|\overline{m}|k)_{a\;\;i} \end{matrix} \right| \ldots\ldots(B),$$

where, if m is greater than $\frac{1}{2}(n)$, the aggregate may be either a trivial identity (consisting of certain terms and their negatives) or the extensional of an aggregate of lower order.

Thus, if $n=6$, $m=4$, $k=2$, and $(n|m)_a \equiv 1234$, eight of the twelve terms vanish on account of identical rows or columns and the remaining four gives the trivial identity

$$\begin{vmatrix} 1243 \\ 1234 \end{vmatrix} + \begin{vmatrix} 6534 \\ 1234 \end{vmatrix} - \begin{vmatrix} 1234 \\ 1243 \end{vmatrix} - \begin{vmatrix} 1234 \\ 6534 \end{vmatrix} \equiv 0.$$

If we take $n=6$, $m=4$, $k=1$, and $(n|m)_a \equiv 1234$, then four of the eight terms vanish on account of identical rows or columns and the remaining four gives the identity

$$\begin{vmatrix} 1534 \\ 1234 \end{vmatrix} + \begin{vmatrix} 6234 \\ 1234 \end{vmatrix} - \begin{vmatrix} 1234 \\ 1534 \end{vmatrix} - \begin{vmatrix} 1234 \\ 6234 \end{vmatrix} = 0,$$

which is the extensional of

$$\begin{vmatrix} 15 \\ 12 \end{vmatrix} + \begin{vmatrix} 62 \\ 12 \end{vmatrix} - \begin{vmatrix} 12 \\ 15 \end{vmatrix} - \begin{vmatrix} 12 \\ 62 \end{vmatrix} = 0$$

The corresponding aggregate for centro-symmetric determinants would be the same as (B) except the sign factor $(-1)^m$ would be omitted.

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NOTE ON THE SUM OF EQUIGRADE MINORS OF A DETERMINANT.

By *Thomas Muir, LL.D.*

1. An expression for the sum of the co-factors of the elements in any determinant is obtained by bordering the determinant with the row

$$0,\ 1,\ 1,\ 1,\ \ldots,$$

and the column

$$0,\ -1,\ -1,\ -1,\ \ldots;$$

for example,

$$\begin{vmatrix} . & 1 & 1 & 1 \\ -1 & a_1 & a_2 & a_3 \\ -1 & b_1 & b_2 & b_3 \\ -1 & c_1 & c_2 & c_3 \end{vmatrix} = A_1 + A_2 + A_3 + B_1 + \ldots + C_3.$$

If instead of the co-factors of the elements we wish to have the sum of the unsigned primary minors, we border the row

$$0,\ 1,\ -1,\ 1,\ -1,\ \ldots,$$

and the column

$$0,\ -1,\ 1,\ -1,\ 1,\ \ldots$$

2. The obtaining of a like expression for the sum of the secondary minors depends on the possibility of finding values for the x's and y's, which will satisfy the equations

$$\left\| \begin{matrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{matrix} \right\| = 1.$$

When n is 3 a solution with two disposable quantities is readily obtained, namely,

$$\left\| \begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix} \right\| = \left\| \begin{matrix} x_1 & x_1y_2 - 1 & -x_1y_2 - x_1 + 1 \\ 1 & y_2 & -y_2 - 1 \end{matrix} \right\|.$$

For example, putting $x_1 = 1$, $y_2 = 2$, we have

$$\begin{vmatrix} . & . & 1 & 1 & -2 \\ . & . & 1 & 2 & -3 \\ 1 & 1 & a_1 & a_2 & a_3 \\ 1 & 2 & b_1 & b_2 & b_3 \\ -2 & -3 & c_1 & c_2 & c_3 \end{vmatrix} = a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + c_1 + c_2 + c_3.$$

The said solution, however, is in a sense not more general than

$$\left\|\begin{matrix} . & -1 & 1 \\ 1 & y_2 & -y_2 - 1 \end{matrix}\right\|,$$

or even than

$$\left\|\begin{matrix} . & -1 & 1 \\ 1 & . & -1 \end{matrix}\right\|.$$

3. In view of the multiplicity of solutions when n is 3, it is somewhat curious that when n is 4 there is no solution at all. There are then six equations to be satisfied, namely,

$$x_1 y_2 - x_2 y_1 = \quad 1 = x_2 y_3 - x_3 y_2,$$

$$x_1 y_3 - x_3 y_1 = -1 = x_2 y_4 - x_4 y_2,$$

$$x_1 y_4 - x_4 y_1 = \quad 1 = x_3 y_4 - x_4 y_3.$$

Taking in pairs the three on the left we have

$$x_1 (y_2 + y_3) = y_1 (x_2 + x_3),$$

$$x_1 (y_2 - y_4) = y_1 (x_2 - x_4),$$

$$x_1 (y_3 + y_4) = y_1 (x_3 + x_4),$$

whence, on multiplying by $-x_4$, $-x_3$, x_2 respectively and adding, there is obtained

$$x_1 \{-x_4 (y_2 + y_3) - x_3 (y_2 - y_4) + x_2 (y_3 + y_4)\} = 0.$$

On the other hand, if we take the remaining three, we obtain directly by addition

$$-x_4 (y_2 + y_3) - x_3 (y_2 - y_4) + x_2 (y_3 + y_4) = 1.$$

We thus have a manifest contradiction unless x_1 be 0. But from the three equations just used we also have

$$\left.\begin{array}{r} x_2y_1y_3 - x_3y_1y_2 = \ \ y_1 \\ x_2y_1y_4 - x_4y_1y_2 = -y_1 \\ x_3y_1y_4 - x_4y_1y_3 = \ \ y_1 \end{array}\right\},$$

whence, if x_1 be 0, we obtain, with the help of the first three,

$$\left.\begin{array}{r} -y_3 - y_2 = \ \ y_1 \\ -y_4 + y_2 = -y_1 \\ y_4 + y_3 = \ \ y_1 \end{array}\right\}$$

and therefore by addition

$$0 = y_1$$

—a result which again involves a contradiction. There is therefore no solution.

4. Further, as the equations which have just been proved to be inconsistent when n is 4 make their appearance in every higher case, our conclusion is that the equations

$$\left\| \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{array} \right\| = 1$$

are not soluble when n is greater than 3.

5. Before considering arrays of *three* rows, it is desirable to call attention to the general theorem that *if all the primary minors of an* $(n-1)$*-by-*n *array be equal, the sum of each row of the array vanishes.* This is readily seen to be true on using in order the multipliers $x_n, x_{n-1}, \cdots, x_1$ with the given equations and then performing addition; for thereby we obtain when n is 4 and c is the common value of the minors

$$\left| \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ x_1 & x_2 & x_3 & x_4 \end{array} \right| = c\,(x_4 + x_3 + x_2 + x_1),$$

i.e.,

$$0 = c\,.\,\Sigma x,$$

and therefore

$$0 = \Sigma x.$$

Were we to use the y's as multipliers instead of the x's we should similarly find the fourth row to be

$$y_1 \; y_2 \; y_3 \; y_4,$$

the right-hand side to be

$$c\,(y_4+y_3+y_2+y_1),$$

and the result to be

$$0=\Sigma y.$$

6. In connection with this it is also worth noting that the data can be expressed as a set of linear equations in the elements of any row of the array, and that the determinant of the set is skew with zero diagonal. Thus, if

$$\left\|\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix}\right\| = c,$$

and we put, for shortness' sake,

$$\left\|\begin{matrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix}\right\| = A,\ B,\ C,\ D,\ E,\ F,$$

the set of linear equations in x_1, x_2, x_3, x_4 is

$$\left.\begin{array}{rrrr} & x_3A - x_2B & + x_1D & = c \\ -x_4A & & + x_2C - x_1E & = c \\ & x_4B - x_3C & + x_1F & = c \\ & -x_4D + x_3E - x_2F & & = c \end{array}\right\},$$

7. Even if A, B, C, ... in these equations were unconditioned, it would follow on multiplying by x_4, x_3, x_2, x_1 and adding that

$$x_4+x_3+x_2+x_1=0.$$

On solving we should also have

$$\left.\begin{array}{l} x_1=c(\qquad\;\; A+B+C)\div(AF-BE+CD) \\ x_2=c(-A\qquad +D+E)\div(AF-BE+CD) \\ x_3=c(-B-D\qquad +F)\div(AF-BE+CD) \\ x_4=c(-C-E-F\qquad\;\;)\div(AF-BE+CD) \end{array}\right\},$$

from which the same fact is evident.

8. As things stand, however, A, B, ... are such that

$$AF - BE + CD = 0,$$

and consequently the determinant of the set, being the square of $AF - BE + CD$, vanishes. In other like cases, namely, where n is even, the same occurs, that is to say, the pfaffian, which is the square root of the determinant, vanishes because of a relation between its elements. In the cases where n is odd, the determinant vanishes altogether from any relation between the elements on one side of the diagonal.

9. Passing now to arrays of three rows let us try to find a solution of the set of equations

$$\left\|\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix}\right\| = 1.$$

In the first place it is clear that a three-line determinant equal to 1 can readily be constructed by utilizing the result of § 2. We have only got to take for two of its columns such a 2-by-3 array as is there given, for example,

$$\begin{matrix} u & 1 \\ uv - 1 & v \\ -uv - u + 1 & -v - 1 \end{matrix}$$

and then prefix or annex another column whose sum is 1, for example,

$$\begin{matrix} r \\ -r - s + 1 \\ s \end{matrix}$$

By both prefixing and annexing such a column we should have the 3-by-4 array

$$\left\|\begin{matrix} x_1 & x_2 & 1 & -x_4 \\ -x_1 - z_1 + 1 & x_2 y_3 - 1 & y_3 & -z_4 \\ z_1 & -x_2 y_3 - x_2 + 1 & -y_3 - 1 & x_4 + z_4 - 1 \end{matrix}\right\|,$$

with two of its four primary minors already satisfying the required condition. This, however, can, by legitimate operations on rows, be changed into

$$\left\|\begin{array}{cccc} x_1 & x_2 & 1 & -x_4 \\ -x_1-z_1+1 & x_2y_3-1 & y_3 & -z_4 \\ z_1 & . & . & -1 \end{array}\right\|$$

To secure the fulfilment of the two remaining conditions we might formulate them and solve for x_4 and z_4; but it is more promptly effective to use the fact (§ 5) that the sum of each row must vanish. This gives us at once the result

$$\left\|\begin{array}{cccc} x_1 & x_2 & 1 & -x_1-x_2-1 \\ -x_1-z_1+1 & x_2y_3-1 & y_3 & -y_3(x_2+1)+x_1+z_1 \\ 1 & . & . & -1 \end{array}\right\| = 1.$$

By operating on rows, however, it may be simplified into

$$\left\|\begin{array}{cccc} . & x_2 & 1 & -x_2-1 \\ 1 & x_2y_3-1 & y_3 & -y_3(x_2+1) \\ 1 & . & . & -1 \end{array}\right\| = 1,$$

and thence into

$$\left\|\begin{array}{cccc} . & x_2 & 1 & -x_2-1 \\ 1 & -1 & . & . \\ 1 & . & . & -1 \end{array}\right\| = 1,$$

and finally into

$$\left\|\begin{array}{cccc} . & . & 1 & -1 \\ 1 & -1 & . & . \\ 1 & . & . & -1 \end{array}\right\| = 1,$$

which is also what we should have got, before operating on the rows, by merely putting

$$x_1,\ x_2,\ y_3,\ z_1 = 0,\ 0,\ 0,\ 0.$$

10. We have now to note that in the case of the 3-by-5 array, exactly as in the case of the 2-by-4 array, there is no

solution at all. In order that this may be seen, we first remark that included in

$$\left\|\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{matrix}\right\| = 1$$

there are ten equations, namely, the four included in

$$\left\|\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix}\right\| = 1$$

and other six which all involve the fifth column. But by reason of the first four we must have

$$x_4 = -(x_1 + x_2 + x_3),$$
$$y_4 = -(y_1 + y_2 + y_3),$$
$$z_4 = -(z_1 + z_2 + z_3),$$

and three of the six being

$$|x_1 y_2 z_5| = 1, \quad |x_1 y_3 z_5| = -1, \quad |x_2 y_3 z_5| = 1,$$

there is obtained thence by addition or subtraction

$$|x_1 (y_2 + y_3) z_5| = 0, \quad |(x_1 + x_2) y_3 z_5| = 0, \quad |(x_1 + x_3) y_2 z_5| = 0,$$

and therefore by operating on columns and substituting

$$-|x_1 y_4 z_5| = 0, \quad -|x_4 y_3 z_5| = 0, \quad -|x_4 y_2 z_5| = 0,$$

—results which are at variance with the other three of the six equations, namely,

$$|x_1 y_4 z_5| = 1, \quad |x_3 y_4 z_5| = 1, \quad |x_2 y_4 z_5| = -1.$$

There is thus no solution for

$$\left\|\begin{matrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{matrix}\right\| = 1$$

when n is greater than 4.

11. When the number of rows is greater than 3, the like results are obtained in similar fashion, there being no solution of

$$\begin{Vmatrix} x_{11} & x_{12} & \cdots & x_{1,n+h} \\ x_{21} & x_{22} & \cdots & x_{2,n+h} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{n,n+h} \end{Vmatrix} = 1$$

when $h > 1$; and the simplest solution when $h = 1$ being got by taking equal to 1 each element of the diagonal passing through x_{11}, each element of the last column equal to 1, and every other element equal to 0.

Capetown, S.A.
January 28th, 1914.

THE DETERMINANT OF THE SUM OF A SQUARE MATRIX AND ITS CONJUGATE.

By *Thomas Muir, LL.D.*

1. If the matrix of any determinant D be increased by the matrix of the conjugate determinant D', the determinant of the matrix thus produced may be conveniently called the *duplicant* of D or of D'. Thus the duplicant of $|a_1 b_2|$ is

$$\begin{vmatrix} 2a_1 & a_2 + b_1 \\ b_1 + a_2 & 2b_2 \end{vmatrix},$$

and is equal to

$$2\,|a_1 b_2| + 2a_1 b_2 - (a_2^2 + b_1^2),$$

$$4\,|a_1 b_2| - (a_2 - b_1)^2.$$

2. A duplicant is necessarily axisymmetric. The duplicant of an axisymmetric determinant of the n^{th} order is 2^n times the original. The duplicant of a skew determinant is 2^n times the diagonal term of the original, and therefore is 0 when the original is zero-axial.

3. A duplicant on account of having all its elements binomials is expressible as the sum of 2^n determinants with monomial elements; and as each determinant of the sum has

every one of its columns taken either from D or from D', a convenient notation for it is obtained by simply specifying in some short way the columns of which it is composed. Thus, if we denote the columns of D by 1, 2, 3, ... and those of D' by $1'$, $2'$, $3'$, ..., one of the determinants of the sum would be readily recognised from the notation

$$|12'34'5'\dots| \text{ or } 12'34'5'\dots .$$

4. Taking the case of the 3rd order we have

$$\begin{vmatrix} 2a_1 & a_2+b_1 & a_3+c_1 \\ b_1+a_2 & 2b_2 & b_3+c_2 \\ c_1+a_3 & c_2+b_3 & 2c_3 \end{vmatrix} = \begin{aligned} &(123+123')+(12'3+12'3') \\ &+(1'23+1'23')+(1'2'3+1'2'3'). \end{aligned}$$

The development is however more suitably arranged in the form

$$\begin{aligned} 123 &+ (123'+12'3+1'23) \\ &+ (12'3'+1'23'+1'2'3)+1'2'3'; \end{aligned}$$

for since $$123 = 1'2'3'$$

and[*] $$123'+12'3+1'23 = 12'3'+1'23'+1'2'3,$$

it thus assumes the alternative forms

$$2\left\{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & c_1 \\ b_1 & b_2 & c_2 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & b_2 & b_3 \\ a_3 & c_2 & c_3 \end{vmatrix}\right\} \quad \text{(I)},$$

$$2\left\{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & a_3 \\ a_2 & b_2 & b_3 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ b_1 & b_2 & c_2 \\ c_1 & b_3 & c_3 \end{vmatrix}\right\} \quad \text{(I')}.$$

As another possible mode of viewing the construction of these expressions we may note that the first determinant in (I) is D, and that each of the last three determinants is a coaxial two-line minor of D bordered by a row of D: and that D' forms a like basis for (I′).

---

* The theorem here used, although first published in 1888, has unfortunately not become well known. It is to the effect that "If any two determinants A and B of the nth order be taken, and from these two sets of determinants be formed, namely, first, a set of $C_{n,r}$ determinants each of which is in r rows identical with A and in the remaining rows with B, and, secondly, a set of the same number of determinants each of which is in r columns identical with A and in the remaining columns with B, then the sum of the first set of determinants is equal to the sum of the second set." *Proc. Roy. Soc. Edinburgh*, vol. xv., p. 103.

5. If we express each of the last three determinants in (I) in terms of the elements of the columns taken from D' and their co-factors, we obtain the development

$$2\left\{\begin{array}{l}|a_1b_2c_3| + c_3C_3 + c_2B_3 + c_1A_3 + b_3C_2 + b_2B_2 \\ \qquad\qquad\qquad + b_1A_2 + a_3C_1 + a_2B_1 + a_1A_1\end{array}\right\},$$

which is best written in the form

$$2\left\{\begin{array}{l}|a_1b_2c_3| + (a_1,\ a_2,\ a_3\between A_1,\ B_1,\ C_1) \\ \qquad\qquad + (b_1,\ b_2,\ b_3\between A_2,\ B_2,\ C_2) \\ \qquad\qquad + (c_1,\ c_2,\ c_3\between A_3,\ B_3,\ C_3)\end{array}\right\} \qquad \text{(II)}.$$

The same is obtained from (I′).

6. From this is derived an interesting theorem regarding the duplicant of the adjugate of $|a_1b_2c_3|$. For, by (II), we have

$$\begin{vmatrix} 2A_1 & A_2+B_1 & A_3+C_1 \\ B_1+A_2 & 2B_2 & B_3+C_2 \\ C_1+A_3 & C_2+B_3 & 2C_3 \end{vmatrix} = 2\left\{\begin{array}{l}|A_1B_2C_3| + (A_1,\ldots\between|B_2C_3|,\ldots) \\ \qquad + \ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\},$$

and as in the right-hand member we can put

$$|A_1B_2C_3| = |a_1b_2c_3|^2,$$
$$|B_2C_3| = a_1|a_1b_2c_3|,\ \ldots,$$

it follows that that member has $|a_1b_2c_3|$ for a factor, and that the co-factor is

$$2\left\{\begin{array}{l}|a_1b_2c_3| + (A_1,\ A_2,\ A_3\between a_1,\ b_1,\ c_1) \\ \qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\}.$$

This, however, being identical with (II), we have the proposition that *the duplicant of the adjugate of* $|a_1b_2c_3|$ *is the product of* $|a_1b_3c_3|$ *into the duplicant of* $|a_1b_2c_3|$. (III).

7. In the case of the 4th order, the initial form of development is

$$\begin{array}{l}1234 + (1234' + 123'4 + 12'34 + 1'234) \\ \quad + (123'4' + 12'34' + 1'234' + 12'3'4 + 1'23'4 + 1'2'34) \\ \quad + (12'3'4' + 1'23'4' + 1'2'34' + 1'2'3'4) \\ \quad + 1'2'3'4'.\end{array}$$

And since, as before, we have

$$1234 = 1'2'3'4', \quad \Sigma 1234' = \Sigma 1'2'3'4.$$

it follows that the four-line duplicant is equal to

$$2\{1234 + \Sigma 1234'\} + \Sigma 123'4'. \qquad \text{(IV)}.$$

It would thus seem that in this case 2 is not a factor: it remains however to examine more closely

$$\Sigma\, 123'4', \quad i.e., \quad \Sigma \begin{vmatrix} a_1 & a_2 & c_1 & d_1 \\ b_1 & b_2 & c_2 & d_2 \\ c_1 & c_2 & c_3 & d_3 \\ d_1 & d_2 & c_4 & d_4 \end{vmatrix}.$$

Taking each of the six determinants included in it, and expanding in terms of the two-line minors formed from the columns of D and their co-factors, we obtain an expression of thirty-six products, namely,

$$\begin{array}{llllll}
|a_1b_2|\cdot|c_3d_4| & -\,|a_1c_2|\cdot|c_2d_4| & +\,|a_1d_2|\cdot|c_2d_3| & +\,|b_1c_2|\cdot|c_1d_4| & -\,|b_1d_2|\cdot|c_1d_3| & +\quad |c_1d_2|^2 \\
|a_1b_3|\cdot|b_3d_4| & +\,|a_1c_3|\cdot|b_2d_4| & -\,|a_1d_3|\cdot|b_2d_3| & -\,|b_1c_3|\cdot|b_1d_4| & +\quad |b_1d_3|^2 & -\,|c_1d_3|\cdot|b_1d_2| \\
|a_1b_4|\cdot|b_3c_4| & -\,|a_1c_4|\cdot|b_2c_4| & +\,|a_1d_4|\cdot|b_2c_3| & +\quad |b_1c_4|^2 & -\,|b_1d_4|\cdot|b_1c_3| & +\,|c_1d_4|\cdot|b_1c_2| \\
|a_2b_3|\cdot|a_3d_4| & -\,|a_2c_3|\cdot|a_2d_4| & +\quad |a_2d_3|^2 & +\,|b_2c_3|\cdot|a_1d_4| & -\,|b_2d_3|\cdot|a_1d_3| & +\,|c_2d_3|\cdot|a_1d_2| \\
|a_2b_4|\cdot|a_3c_4| & +\quad |a_2c_4|^2 & -\,|a_2d_4|\cdot|a_2c_3| & -\,|b_2c_4|\,|a_1c_4| & +\,|b_2d_4|\cdot|a_1c_3| & -\,|c_2d_4|\cdot|a_1c_2| \\
\quad |a_3b_4|^2 & -\,|a_3c_4|\cdot|a_2b_4| & +\,|a_3d_4|\cdot|a_2b_3| & +\,|b_3c_4|\cdot|a_1b_4| & -\,|b_3d_4|\cdot|a_1b_3| & -\,|c_3d_4|\,|a_1b_2|
\end{array}$$

where necessarily the first factor of every product is a two-line minor of D and has joined to it the two-line minor which is the conjugate of its complementary. Thus, taking the second product in the first line, we note that the first factor $|a_1c_2|$ has in D the complementary minor $|b_3d_4|$; and, since the minor conjugately situated to $|b_3d_4|$ in D (and in D') is $|c_2d_4|$, the correctness of the product is verified. When the first factor is a coaxial minor of D, its complementary is not different from the conjugate of its complementary: hence the peculiarity of the products occurring in the primary diagonal above. When the first factor has none of its elements in the primary diagonal of D, its complementary is the same as its conjugate, and the conjugate of its complementary is thus itself: hence the peculiarity of the products occupying the secondary diagonal. Further, the assemblage of first factors necessarily contains *all* the two-line minors of D: and the assemblage of second factors being obtained as two-line minors of D' is not a different assemblage but merely a different arrangement of the members of the first assemblage. In fact, the second assemblage is the first rotated 180° round its secondary diagonal. On this account the quadriform collection of products is symmetric with respect to its secondary diagonal, and thus save for the portion

$$|a_3b_4|^2+|a_2c_4|^2+|a_2b_3|^2+|b_1c_4|^2+|b_1d_3|^2+|c_1d_2|^2,$$

contains 2 as a factor.

These six non-repeated terms correspond to the two $-a_2^2-b_1^2$ in the case of the duplicant of the second order (§1).

8. The development of the duplicant of the 4^{th} order thus is

$$2\left\{\begin{array}{l}|a_1b_2c_3d_4|+\Sigma(A_1,\ B_1,\ C_1,\ D_1)(a_1,\ a_2,\ a_3,\ a_4)\\ \qquad\qquad+\Sigma(|a_1b_2|.\text{conj comp}|a_1b_2|)\end{array}\right\} \quad (\text{IV}');$$

and therefore, if we pass from $|a_1b_2c_3d_4|$ to its adjugate $|A_1B_2C_3D_4|$, we come on a state of affairs exactly similar to that encountered in §6, save that the common factor found is now not the first, but the second power of the original determinant. We thus have the theorem that *the duplicant of the adjugate of* $|a_1b_2c_3d_4|$ *is equal to the product of* $|a_1b_2c_3d_4|^2$ *into the duplicant of* $|a_1b_2c_3d_4|$. (V).

9. Proceeding to the consideration of the 5^{th} order, we find, in the same manner as before, the initial development of the duplicant to be

$$12345+\Sigma 12345'+\Sigma 1234'5'+\ldots,$$

the grouping of the thirty-two determinants with monomial elements being

$$1+5+10+10+5+1;$$

and since we know that

$$12345=1'2'3'4'5',$$

$$\Sigma 12345'=\Sigma 12'3'4'5',$$

$$\Sigma 1234'5'=\Sigma 123'4'5',$$

it follows without further examination that 2 is a factor. (VI).

Carrying the development a stage farther, as in the previous cases, we have

$$2\left\{\begin{array}{l}|a_1b_2c_3d_4e_5|+\Sigma(|a_1b_2c_3d_4|.\text{conj comp}|a_1b_2c_3d_4|)\\ \qquad\qquad+\Sigma(|a_1b_2c_3|.\text{conj comp}|a_1b_2c_3|)\end{array}\right\};$$

and applying this to $|A_1B_2C_3D_4E_5|$ we find that $|a_1b_2c_3d_4e_5|^3$ is a factor of it, and that the co-factor is the duplicant of $|a_1b_2c_3d_4e_5|$. (VII).

10. The consideration of further cases is seen to be unnecessary, it being clear that 2 is a factor of the duplicant only when the order is odd, but that the theorem regarding the adjugate holds also when the order is even. The latter matter, be it noted, is quite unconnected with the former: to deal with it we only need to change the initial form of development

$$123456 + \Sigma 123456' + \ldots$$

into the more advanced form

$$|a_1b_2c_3d_4e_5f_6| + \Sigma\,(|a_1b_2c_3d_4e_5|.\text{conj comp}) \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

and then use the familiar theorems regarding the adjugate and its minors.

11. The matter of next importance concerns the determinants which arise from bordering a duplicant by two rows or two columns of the original determinant. In dealing with it let us at once take the fourth order, and in the first place use the same row for forming both borders, the subject of consideration thus being

$$\begin{vmatrix} . & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_1+a_1 & a_2+b_1 & a_3+c_1 & a_4+d_1 \\ a_2 & b_1+a_2 & b_2+b_2 & b_3+c_2 & b_4+d_2 \\ a_3 & c_1+a_3 & c_2+b_3 & c_3+c_3 & c_4+d_3 \\ a_4 & d_1+a_4 & d_2+b_4 & d_3+c_4 & d_4+d_4 \end{vmatrix}.$$

The performance of the operations

$$\text{row}_2 - \text{row}_1, \quad \text{col}_2 - \text{col}_1$$

leads to

$$\begin{vmatrix} . & a_1 & a_2 & a_3 & a_4 \\ a_1 & . & b_1 & c_1 & d_1 \\ a_2 & b_1 & b_2+b_2 & b_3+c_2 & b_4+d_2 \\ a_3 & c_1 & c_2+b_3 & c_3+c_3 & c_4+d_3 \\ a_4 & d_1 & d_2+b_4 & d_3+c_4 & d_4+d_4 \end{vmatrix};$$

and if we multiply the last three rows of this by a_1 and then perform the operations

$$\text{row}_3 - \frac{b_1}{a_1}\text{row}_1\,, \quad \text{row}_4 - \frac{c_1}{a_1}\text{row}_1\,, \quad \text{row}_5 - \frac{d_1}{a_1}\text{row}_1\,,$$

followed by the operations

$$\text{col}_3 - \frac{b_1}{a_1}\text{col}_1\,, \quad \text{col}_4 - \frac{c_1}{a_1}\text{col}_1\,, \quad \text{col}_5 - \frac{d_1}{a_1}\text{col}_1\,,$$

we find our determinant

$$= \frac{1}{a_1{}^3}\begin{vmatrix} \cdot & a_1 & a_2 & a_3 & a_3 \\ a_1 & \cdot & \cdot & \cdot & \cdot \\ a_2 & \cdot & 2\,|a_1b_2| & |a_1b_3| + |a_1c_2| & |a_1b_4| + |a_1d_2| \\ a_3 & \cdot & |a_1c_2| + |a_1b_3| & 2\,|a_1c_3| & |a_1c_4| + |a_1d_3| \\ a_4 & \cdot & |a_1d_2| + |a_1b_4| & |a_1d_3| + |a_1c_4| & 2\,|a_1d_4| \end{vmatrix}$$

$$= -\frac{1}{a_1}\,.\,\text{duplicant of} \begin{Vmatrix} |a_1b_2| & |a_1b_3| & |a_1b_4| \\ |a_1c_2| & |a_1c_3| & |a_1c_4| \\ |a_1d_2| & |a_1d_3| & |a_1d_4| \end{Vmatrix} \qquad \text{(VIII)},$$

where instead of $-1/a_1$ we should have had the co-factor $-1/a_1{}^{n-3}$ if the basic determinant had been of the n^{th} order.

Similar results follow when the bordering is done with any other row. In fact, these results are all included in this one, because, the duplicant being axisymmetric, any one of its rows can be made the first.

12. *The result of the double bordering of any duplicant by one of the rows of the original is the same as the result of the like bordering with the corresponding column.* (IX).

For the determinant obtained after the performance of the first operation in § 11 needs only to have its first two rows interchanged and then its first two columns in order to become

$$\begin{vmatrix} \cdot & a_1 & b_1 & c_1 & d_1 \\ a_1 & \cdot & a_2 & a_3 & a_4 \\ b_1 & a_2 & b_2 + b_2 & b_3 + c_2 & b_4 + d_2 \\ c_1 & a_3 & c_2 + b_3 & c_3 + c_3 & c_4 + d_3 \\ d_1 & a_4 & d_2 + b_4 & d_3 + c_4 & d_4 + d_4 \end{vmatrix}.$$

which by the operation

$$\text{row}_2 + \text{row}_1, \quad \text{col}_2 + \text{col}_1$$

gives the bordering of the duplicant by

$$a_1,\ b_1,\ c_1,\ d_1.$$

13. The theorem corresponding to that of § 11 when we border with two different rows instead of using the same row twice is sufficiently indicated by giving the example in which the bordering rows are

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4. \end{matrix}$$

The result then is

$$a_2 b_1 \begin{vmatrix} \dfrac{|a_2b_1|}{a_2} + \dfrac{|b_1a_2|}{b_1} & \dfrac{|a_2b_3|}{a_2} + \dfrac{|b_1c_2|}{b_1} & \dfrac{|a_2b_4|}{a_2} + \dfrac{|b_1d_2|}{b_1} \\ \dfrac{|a_2c_1|}{a_2} + \dfrac{|b_1a_3|}{b_1} & \dfrac{|a_2c_3|}{a_2} + \dfrac{|b_1c_3|}{b_1} & \dfrac{|a_2c_4|}{a_2} + \dfrac{|b_1d_3|}{b_1} \\ \dfrac{|a_2d_1|}{a_2} + \dfrac{|b_1a_4|}{b_1} & \dfrac{|a_2d_3|}{a_2} + \dfrac{|b_1c_4|}{b_1} & \dfrac{|a_2d_4|}{a_2} + \dfrac{|b_1d_4|}{b_1} \end{vmatrix} \quad \text{(X)},$$

where, it will be seen, the determinant is no longer a duplicant.

Had we bordered with the corresponding *columns*, namely,

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2, \end{matrix}$$

we should have obtained the same result. (XI).

14. The continuation of the investigation of § 11 naturally splits up into two branches, the one dealing with the duplicants of odd order and the other with those of even order.

In the case there taken, namely, where the duplicant which is bordered is of the 4[th] order, we use the result (II) of § 5 on the co-factor of $-1/a_1$ in (VIII), and the outcome is readily seen to be

$$-2\left\{\begin{aligned} a_1|a_1b_2c_3d_4| &+ (\ \ |a_1b_2|, -|a_1b_3|, \ \ |a_1b_4|\mathbin{)(}|a_1c_3d_4|, |a_1b_3d_4|, |a_1b_3c_4|) \\ &+ (-|a_1c_2|, \ \ |a_1c_3|, -|a_1c_4|\mathbin{)(}|a_1c_2d_4|, |a_1b_2d_4|, |a_1b_2c_4|) \\ &+ (\ \ |a_1d_2|, -|a_1d_3|, \ \ |a_1d_4|\mathbin{)(}|a_1c_2d_3|, |a_1b_2d_3|, |a_1b_2c_3|) \end{aligned}\right\}$$

In the case where the duplicant which is bordered is of the 3rd order we have from § 11

$$\begin{vmatrix} \cdot & a_1 & a_2 & a_3 \\ a_1 & 2a_1 & a_2+b_1 & a_3+c_1 \\ a_2 & a_2+b_1 & 2b_2 & b_3+c_2 \\ a_3 & a_3+c_1 & b_3+c_2 & 2c_3 \end{vmatrix} = -\begin{vmatrix} 2\,|a_1b_2| & |a_1b_3|+|a_1c_2| \\ |a_1b_3|+|a_1c_2| & 2\,|a_1c_3| \end{vmatrix}$$

$$= -4a_1\,|a_1b_2c_3| + (|a_1b_3| - |a_1c_2|)^2,$$

and from § 13

$$\begin{vmatrix} \cdot & a_1 & a_2 & a_3 \\ b_1 & 2a_1 & a_2+b_1 & a_3+c_1 \\ b_2 & a_2+b_1 & 2b_2 & b_3+c_2 \\ b_3 & a_3+c_1 & b_3+c_2 & 2c_3 \end{vmatrix}$$

$$= -a_2b_1\begin{vmatrix} \dfrac{|a_2b_1|}{a_2} + \dfrac{|b_1a_2|}{b_1} & \dfrac{|a_2b_3|}{a_2} + \dfrac{|b_1c_2|}{b_1} \\[2ex] \dfrac{|a_2c_1|}{a_2} + \dfrac{|b_1a_3|}{b_1} & \dfrac{|a_2c_3|}{a_2} + \dfrac{|b_1c_3|}{b_1} \end{vmatrix}$$

$$= -2\,(a_2+b_1)\,|a_1b_2c_3| + (|a_1b_3| - |a_1c_2|)\,.\,(|a_2b_3| - |b_1c_2|).$$

The last two identities may be interestingly applied in connection with a result of Cayley's on geometrical reciprocity (*Collected Math. Papers*, vol. i., pp. 377–382).

Capetown, S.A.,
February 20*th*, 1914.

END OF VOL. XLIII.

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