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J. W. L. GLAISHER, Sc.D., F.R.S.,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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MESSENGER OF MATHEMATICS.

THE PROBLEM OF THE SQUARE PYRAMID.

By *G. N. Watson*.

THE following problem was proposed by Lucas* in 1875: *The number of cannon balls piled in a pyramid on a square base is a perfect square. Shew that the number of balls in a side of the base is* 24.

An imperfect solution of the problem was given by Moret-Blanc.† After pointing out‡ the defect of this solution, Lucas gave a demonstration§ of a more satisfactory character in 1877. The latter investigation is made by Lucas to depend on a number of results scattered through a series of papers by himself and by Gerono. It is consequently a very troublesome matter to follow the analysis, and moreover it appears that there is a serious flaw which vitiates the argument in one of the earlier papers by Lucas. It consequently seems desirable to collect into a single paper the analysis contained in the various papers to which reference has been made, and to give a sound proof of the theorem of which an incomplete investigation was given by Lucas. This proof is of a more complex nature than I anticipated,‖ and, in fact, it seems impracticable to set it out briefly without making use of properties of Lemniscate functions, *i.e.* elliptic functions of modulus $1/\sqrt{2}$.

Like many other arithmetical problems, the problem of the square pyramid is discussed by Dudeney, *Amusements in Mathematics* (1917), pp. 26, 167; the author states that he is unaware of any complete solution of the problem.

2. If the number of cannon balls in a side of the base of the pyramid is n, then the total number of balls in the pyramid is

$$1^2+2^2+3^2+\ldots+n^2=\tfrac{1}{6}n(n+1)(2n+1);$$

* *Nouvelles Annales de Math.* (2), xiv. (1875), p. 336, problem 1180.

† *Ibid.* (2), xv., pp 46–48.

‡ *Ibid.* (2), xv., p. 528.

§ *Ibid.* (2), xvi., pp. 429–432. The papers by Lucas and Gerono, to which reference is made in § 5, should also be consulted.

‖ With the exception of this proof, which will be found in §§ 6–8, this paper contains nothing which is not given in the writings of Lucas and Gerono.

and, if this number be a perfect square, let it be m^2. Our object is therefore to shew that the equation

$$m^2 = \tfrac{1}{6} n(n+1)(2n+1) \text{.................} (1)$$

has no solutions in positive integers other than

(i) $n = m = 1$,*

(ii) $n = 24$, $m = 70$.

The symbols which will be used are supposed to denote positive integers, except in §§ 6–8, where it will usually be evident whether a symbol denotes a positive integer or not. Also the three following theorems will be constantly used:

(i) Every square number is of the form $3N$ or $3N + 1$.

(ii) Every square number is of the form $4N$ or $8N + 1$.

(iii) If p, q, r are coprime, and if

$$p^2 + q^2 = r^2,$$

then p, q, r are expressible in the forms†

$$p \text{ or } q = M^2 - N^2, \quad q \text{ or } p = 2MN, \quad r = M^2 + N^2,$$

where one (but not both) of the integers M, N is even.

3. We now consider the equation (1). It is evident that 3 must divide one of the numbers n, $n+1$, $2n+1$, and, since $2n+1$ is odd, 2 must divide one of the numbers n, $n+1$. Further, since n, $n+1$, $2n+1$ are coprime (unless $n = 1$), the remaining factors must be perfect squares if equation (1) is to be satisfied. We consequently obtain the following scheme‡ of values for n, $n+1$, $2n+1$:

	n	$n+1$	$2n+1$
(I)	$3a^2$	$2b^2$	c^2
(II)	$2a^2$	$3b^2$	c^2
(III)	$2a^2$	b^2	$3c^2$
(IV)	a^2	$6b^2$	c^2
(V)	$6a^2$	b^2	c^2
(VI)	a^2	$2b^2$	$3c^2$

* In this case the 'pyramid' consists of a single cannon ball.

† Chrystal, *Algebra*, ii, p. 531.

‡ Lucas gives these cases in a different order.

4. It can be shewn at once in the following manner that cases (I)–(IV) in the scheme give rise to no solutions:

(I) Since $2b^2 \equiv 0$ or $2 \pmod 3$, the equation $2b^2 - 3a^2 = 1$ leads to an impossible congruence (mod 3).

Similarly, in cases (II), (III), (IV), we obtain the following impossible congruences (mod 3) respectively:

(II) $6b^2 - c^2 \equiv 1$.

(III) $3c^2 - 4a^2 \equiv 1$.

(IV) $6b^2 - a^2 \equiv 1$.

Hence the only cases to be considered are:

(V) $n = 6a^2$, $n + 1 = b^2$, $2n + 1 = c^2$. And we shall shew (§ 5) that the only solution of this system in positive integers (a not being zero) is

$$a = 2, \quad b = 5, \quad c = 7, \quad n = 24.$$

(VI) $n = a^2$, $n + 1 = 2b^2$, $2n + 1 = 3c^2$. And we shall shew (§§ 6–8) that the only solution of this system in positive integers is

$$a = b = c = n = 1.$$

The flaw in Lucas' analysis* occurs in case (VI). He is considering an equation of the type

$$9x^4 - 12x^2y^2 - 4y^4 = z^2,$$

and he proves that it has no solution in integers except when $y = 0$. He has, however, failed to observe the fact that y may be a multiple of 3; and, in fact, the equation has an infinite number of solutions of which the simplest is

$$x = 5, \quad y = 3, \quad z = 51.$$

The oversight is the more remarkable because y is originally given as a factor of a number which is divisible by 3.

5. In case (V) the system to be resolved is

$$n = 6a^2, \quad n + 1 = b^2, \quad 2n + 1 = c^2 \ldots\ldots\ldots\ldots (2).$$

It is obvious that c is odd and that

$$(c - 1)(c + 1) = 12a^2.$$

Now $c - 1$, $c + 1$ are even and have no common factor except 2.

* *Nouv. Ann. de Math.* (2), xvi, pp. 409 *et seq.*

Hence we must have one of the two following resolutions,

either $\frac{1}{2}(c-1) = d^2,\ \frac{1}{2}(c+1) = 3e^2,\ a = de,$

or $\frac{1}{2}(c-1) = 3d^2,\ \frac{1}{2}(c+1) = e^2,\ a = de.$

The former gives rise to the impossible congruence (mod 3), $3e^2 - d^2 \equiv 1$, *and so we must have*

$$c - 1 = 6d^2,\quad c + 1 = 2e^2.$$

Hence $$c + 1 = 2e^2,\quad c^2 + 1 = 2b^2 \quad \ldots\ldots\ldots\ldots\ldots(3).$$

But it has been shewn by Gerono* that the only solutions of (3) in integers are obtained by taking c equal to 1 or to 7. His proof is as follows:

By eliminating c from equations (3), we have

$$e^4 + (e^2 - 1)^2 = b^2,$$

and so (§ 2)

either $e^2 = f^2 - g^2,\ e^2 - 1 = 2fg\ ,\ b = f^2 + g^2 \ldots\ldots(4a),$

or $e^2 = 2fg,\quad e^2 - 1 = f^2 - g^2,\ b = f^2 + g^2 \ldots\ldots(4b).$

First, if equation $(4a)$ holds, e is odd, since its square is $2fg + 1$, and so from the equation $g^2 + e^2 = f^2$ we get

$$g = 2hk,\quad e = h^2 - k^2,\quad f = h^2 + k^2,$$

whence $$(h^2 - k^2)^2 - 1 = 4hk(h^2 + k^2).$$

Writing $h - k = \alpha$, $h + k = \beta$ (so that α, β are both odd), we get

$$2\alpha^2\beta^2 - 2 = \beta^4 - \alpha^4,$$

i.e. $$\alpha^2 + \beta^2 = \sqrt{\{2(\beta^4 + 1)\}}.$$

Consequently $2(\beta^4 + 1)$ must be an even square, and so we have

$$\beta^4 + 1 = 2\gamma^2,$$

i.e. $$\gamma^4 - \beta^4 = (\tfrac{1}{2}\beta^4 - \tfrac{1}{2})^2.$$

But it has been shewn by Legendre† that the only solutions in positive integers of the equation

$$\gamma^4 - \beta^4 = \delta^2$$

* *Nouv. Ann.* (2), xvi., p. 231. See also Lucas on the system of equations

$$b^2 + 6a^2 = c^2,\quad b^2 - 6a^2 = d^2;$$

ibid. (2), xv., p. 466.

† *Théorie des Nombres*, II., pp. 4, 5.

are obtained by taking $\gamma=\beta$ or $\beta=0$; and therefore we have $\gamma=\beta=1$ and so $e=1$, $a=n=0$ and $b=c=1$.

To prove Legendre's theorem just quoted, consider the equation

$$X^4+Y^2=Z^4 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

in which X and Z are supposed to be coprime.

When X is odd, we have the resolution

$$X^2=p^2-q^2,\quad Y=2pq,\quad Z^2=p^2+q^2.$$

From the first of these, q is even and p odd, and

$$Z^2-X^2=2q^2,$$

whence $$Z\pm X=2y^2,\quad Z\mp X=4s^2,\quad q=2ys$$

(where y, s are coprime and y is odd, for otherwise X and Z would both be even), and

$$p^2=y^4+4s^4,$$

whence $$y^2=t^2-u^2,\quad s^2=tu,\quad p=t^2+u^2.$$

From the first of these, u is even, since y is odd, and from the second, $t=z^2$, $u=x^2$ (for if t, u were not each perfect squares they would have a common factor, and y, s, p would not be coprime), and therefore

$$y^2=z^4-x^4;$$

also $$XY\geq Y=4ys(x^4+z^4)\geq 4xy,$$

and x is even.

The equation is thus reduced to another of the same type in which the product of the first two variables is less than in the original equation, and the new first variable is even.

Leaving this case for the moment, we take *the case in which X is even*, when we have the resolution

$$X^2=2pq,\quad Y=p^2-q^2,\quad Z^2=p^2+q^2;$$

from the first of these, p or $q=y^2$, q or $p=2s^2$, whence

$$Z^2=y^4+4s^4,$$

which gives $$y^2=t^2-u^2,\quad s^2=tu,$$

and therefore $$t=z^2,\quad u=x^2,$$

and $$y^2=z^4-x^4,$$

where $$XY=2sy\geq 2xy.$$

Hence, *whether X be odd or even*, the equation $X^4 + Y^2 = Z^4$ can be reduced to an equation of the same type in which the product of the first two variables is less than half of its original value; this process can be carried on until the product is less than 2, and, since it is even, the product must be zero; but in either case the equation $xy = 0$ leads back to the equation $XY = 0$, and so the equation has no solution for which XY is not zero, and Legendre's theorem is proved.

The investigation of equation ($4a$) is now complete, and the only solution of the original system derived from it is one for which $a = n = 0$.

We next have to consider the alternative supplied by ($4b$), namely,

$$e^2 = 2fg, \quad e^2 - 1 = f^2 - g^2, \quad b = f^2 + g^2.$$

It is evident that e is even, so $f^2 - g^2 \equiv -1 \pmod 4$, and hence f must be even and g odd. From the first equation we obtain the resolution

$$f = 2h^2, \quad g = k^2, \quad e = 2hk,$$

which gives
$$4h^4 - k^4 - 4h^2k^2 = -1,$$

and so
$$k^2 + 2h^2 = \sqrt{(8h^4 + 1)}.$$

Hence $8h^4 + 1$ must be an odd square, say $(2p+1)^2$ and therefore $h^4 = \frac{1}{2}p(p+1)$.

Now Legendre* has shewn that the only solutions of this equation in positive integers (zero included) are

$$h = p = 0, \quad h = p = 1.$$

For, since p, $p+1$ are coprime, we have one of two resolutions, *either*

$$p = q^4, \quad p + 1 = 2r^4,$$

or
$$p = 2q^4, \quad p + 1 = r^4.$$

The former gives
$$q^4 + 1 = 2r^4,$$

so that
$$r^8 - q^4 = (q^4 - 1)^2,$$

an equation of the type $Z^4 - X^4 = Y^2$, so that $q = 0$ or 1 by previous analysis, and in the latter case $p = 1$ (the former gives no solution); and if $p = 1$, then $e = 2$, $c = 7$, and we have the solution $a = 2$, $b = 5$, $c = 7$, $n = 24$.

In the second resolution $2q^4 + 1 = r^4$, an equation of the type $X^4 + 2Y^4 = Z^2$, in which X is odd and Z is a square.

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* *Théorie des Nombres*, II, p. 7.

Hence $Z \pm X^2 = 2t^4$, $Z \mp X^2 = 16s^4$, $Y = 2ts$,

and so $$\pm X^2 = t^4 - 8s^4.$$

The lower sign gives an impossible congruence (mod 4), since t must be odd, and so $t^4 - X^2 = 8s^4$, whence

$$t^2 \pm X = 2x^4, \quad t^2 \mp X = 4y^4, \quad s = xy,$$

where x is odd, and $x^4 + 2y^4 = t^2$, an equation of the type with which we started, such that $Y = 2txy > 2y$.

Hence the equation can be reduced to one in which the second variable is less than 1, and so it must be zero; and, if $y = 0$, we are led back to the equation $Y = 0$. Hence q must be zero and $r = 1$, which gives $e = 0$, and $a = 0$, $b = c = 1$, $n = 0$. Hence $a = 2$, $b = 5$, $c = 7$, $n = 24$ is the only solution in positive integers arising from case (V).

6. We now have to discuss case (VI), namely,

$$n = a^2, \quad n + 1 = 2b^2, \quad 2n + 1 = 3c^2.$$

This system is obviously equivalent to

$$2b^2 - a^2 = 1, \quad 2b^2 + a^2 = 3c^2.$$

Our procedure will be to resolve the more general system

$$2Y^2 - X^2 = Z^2, \quad 2Y^2 + X^2 = 3W^2 \quad \text{.........(6)},$$

and we shall shew that there is only one solution of this system in positive integers for which $Z = 1$.

In equations (6) it is supposed that X, Y have no common factor, and then X, Y, Z, W are all coprime; and as X, Y, Z, W only enter by their squares, we shall suppose that they have either sign; it is, however, convenient to take W positive.

From the second member of (6) we see that neither X nor Y is divisible by 3 (for, if either were, the other would be), and so either $2Y + X$ or $2Y - X$ is divisible by 3; we fix the sign of $X : Y$ so that $2Y + X$ is divisible by 3, and then also $Y - X$ is divisible by 3. We then have

$$W^2 - (\tfrac{2}{3}Y + \tfrac{1}{3}X)^2 = 2(\tfrac{1}{3}Y - \tfrac{1}{3}X)^2 \quad \text{........(7)},$$

and the resolution of this equation gives

$$W \pm (\tfrac{2}{3}Y + \tfrac{1}{3}X) = 2r^2, \quad W \mp (\tfrac{2}{3}Y + \tfrac{1}{3}X) = 4t^2, \quad \pm(\tfrac{1}{3}Y - \tfrac{1}{3}X) = 2rt,$$

and so

$$\pm(\tfrac{2}{3}Y + \tfrac{1}{3}X) = r^2 - 2t^2, \quad \pm(\tfrac{1}{3}Y - \tfrac{1}{3}X) = 2rt, \quad W = r^2 + 2t^2,$$

where r is odd; a change of sign of t (if necessary) makes it possible to take the same sign in the last two ambiguities without loss of generality. We thus get

$$\pm Y = r^2 + 2rt - 2t^2, \quad \pm X = r^2 - 4rt - 2t^2, \quad W = r^2 + 2t^2;$$

and, on substituting into the first member of (6), we find that

$$Z^2 = (r^2 + 8rt - 2t^2)^2 - 72r^2t^2.$$

Resolving this equation in the same way as (7), we get

$$r^2 + 8rt - 2t^2 = \pm(d^2 + 2e^2), \quad 3rt = de, \quad d^2 - 2e^2 = Z,$$

where d is odd and the sign of Z is determined by the last equation.

If t is even so is e, and $8rt - 2t^2 \mp 2e^2 \equiv 0 \pmod 8$; hence $r^2 \mp d^2 \equiv 0 \pmod 8$, and so the upper sign must be taken; whereas if t is odd so is e, and

$$8rt - 2t^2 - 2e^2 \equiv 4 \pmod 8, \quad r^2 - d^2 \equiv 0 \pmod 8,$$

$$8rt - 2t^2 + 2e^2 \equiv 0 \pmod 8, \quad r^2 + d^2 \equiv 2 \pmod 8,$$

and whether upper or lower sign be taken, we are led to an impossibility, since neither 4 nor 2 is congruent to 0 (mod 8). Hence t is even, and the upper sign must betaken, so that

$$r^2 + 8rt - 2t^2 = d^2 + 2e^2, \quad 3rt = de, \quad r^2 - 2t^2 = Z.$$

We now put $e = 3Ar$, $t = Ad$, where A is a *rational fraction*, and then we get

$$A^2(2d^2 + 18r^2) - 8Adr - r^2 + d^2 = 0,$$

and since this quadratic in A has a rational root, $18r^4 - 2d^4$ must be a square.

When the fraction r^4/d^4 is in its lowest terms, let it be w^4/z^4; and then $18w^4 - 2z^4$ must be a square, and it must also be a multiple of 16, since w and z are both odd. Hence

$$9w^4 - z^4 = 8N^2,$$

and this gives, on resolution,

$$3w^2 \pm z^2 = 4y^2, \quad 3w^2 \mp z^2 = 2x^2, \quad N = \pm xy.$$

The lower signs in the first two equations are impossible by a congruence (mod 4), and so the upper signs have to be taken, and we get

$$2y^2 - x^2 = z^2 \quad 2y^4 + x^2 = 3w^2,$$

a system of the same type as the original system.

Now
$$A = \frac{2(zw \pm xy)}{z^2 + 9w^2}$$
(in which the denominator is even and not divisible by 4) and
$$Z : W = z^2 - 18A^2w^2 : w^2 + 2A^2z^2,$$
so, since Z and W are coprime integers, we must have
$$Z = T^2\{z^2 - 18A^2w^2\}, \quad W = T^2\{w^2 + 2A^2z^2\},$$
where T is the denominator of the fraction A when in its lowest terms (this denominator has no factors in common with $3w$ or z or 2 because these numbers are coprime).

It is therefore evident that $W \geq w^2$. Now define α, u by the equations
$$\operatorname{sn}\alpha = \frac{\sqrt{2}}{\sqrt{3}}, \quad \operatorname{cn}\alpha = \frac{1}{\sqrt{3}}, \quad \operatorname{dn}\alpha = \frac{\sqrt{2}}{\sqrt{3}}, \quad \operatorname{cn}u = \frac{z}{w\sqrt{3}},$$
the modulus of the elliptic functions being in each case $1/\sqrt{2}$. It follows that
$$A = \frac{4\sqrt{3}\operatorname{cn}u \pm 6\operatorname{sn}u\operatorname{dn}u}{24(1 - \frac{1}{4}\operatorname{sn}^2u)}$$
$$= \frac{\operatorname{sn}\alpha\operatorname{dn}\alpha\operatorname{cn}u \pm \operatorname{sn}u\operatorname{dn}u\operatorname{cn}\alpha}{2\sqrt{3}\{\operatorname{dn}^2\alpha - k^2\operatorname{sn}^2u\operatorname{cn}^2\alpha\}}$$
$$= \frac{\operatorname{sd}(\alpha \pm u)}{2\sqrt{3}}$$
(Cf. Cayley, *Elliptic Functions*, p. 64). Hence we have
$$\frac{Z}{W} = \frac{3\operatorname{cn}^2u - \frac{3}{2}\operatorname{sd}^2(\alpha \pm u)}{1 + \frac{1}{2}\operatorname{cn}^2u\operatorname{sd}^2(\alpha \pm u)}$$
$$= \frac{3\{1 - \operatorname{sn}^2u - \operatorname{sn}^2(\alpha \pm u) + \frac{1}{2}\operatorname{sn}^2u\operatorname{sn}^2(\alpha \pm u)\}}{1 - \frac{1}{2}\operatorname{sn}^2u\operatorname{sn}^2(\alpha \pm u)}$$
$$= 3\operatorname{cn}(u + \alpha \pm u)\operatorname{cn}(u - \alpha \mp u)$$
$$= \sqrt{3}\operatorname{cn}(2u \pm \alpha)$$
$$= \sqrt{3}\operatorname{cn}U,$$
where $U = 2u \pm \alpha$.

Now, since $W \geq w^2$, we can continue the reduction of the equations until we get a system in which $w = 1$, and then $z = 1$, so that $u = \alpha$; reversing the process of reduction we see that, corresponding to this value of u, we get $U = \alpha$ or 3α;

now $u=\alpha$ is the case already considered, and $u=3\alpha$ gives $U=5\alpha$ or 7α; $u=5\alpha$ gives $U=9\alpha$ or 11α, while $u=7\alpha$ gives $U=13\alpha$ or 15α; and so on. Hence *all* solutions of the system (7) are such that

$$\frac{Z}{W}=\sqrt{3}\,\mathrm{cn}\,\{(2r+1)\alpha\},$$

where r is zero or a positive integer, and conversely, if we express $\sqrt{3}\,\mathrm{cn}\,\{(2r+1)\alpha\}$ as a rational fraction in its lowest terms, we obtain a solution of the system of equations, in which

$$X:Y:Z:W=\pm\frac{\sqrt{3}}{\sqrt{2}}\,\mathrm{sn}\,\{(2r+1)\alpha\}:\pm\frac{\sqrt{3}}{\sqrt{2}}\,\mathrm{dn}\,\{(2r+1)\alpha\}$$
$$:\sqrt{3}\,\mathrm{cn}\,\{(2r+1)\alpha\}:1.$$

It is, however, a little more convenient to work subsequently* with the complement of α; if we put $\beta=K-\alpha$, we get

$$\mathrm{sn}\,\beta=\frac{1}{\sqrt{2}},\quad \mathrm{cn}\,\beta=\frac{1}{\sqrt{2}},\quad \mathrm{dn}\,\beta=\frac{\sqrt{3}}{2},$$

and
$$\frac{Z}{W}=(-)^r\,\mathrm{sd}\,\{(2r+1)\beta\}\sqrt{\tfrac{3}{2}}.$$

7. We shall now examine closely the numerators and denominators of $\mathrm{sn}\,2r\beta$, $\mathrm{cn}\,2r\beta$, $\mathrm{dn}\,2r\beta$ and $\mathrm{sn}\,(2r+1)\beta$, $\mathrm{cn}\,(2r+1)\beta$, $\mathrm{dn}\,(2r+1)\beta$; and we shall give a method of constructing these expressions as rational fractions (save for possible factors of the forms $\sqrt{2}$, $\sqrt{3}$) in their lowest terms from which it will appear that the numerator of $\mathrm{sn}\,(2r+1)\beta$ is never numerically equal to unity except when r is zero.

The method of constructing the elliptic functions is that indicated by Cayley, *Elliptic Functions* (1895), p. 79, except that k^2 is treated as a fraction $(=\frac{1}{2})$, whereas Cayley's formulæ are integral in k^2; it consists in applying the ordinary addition formulæ to multiples of β whose sum is $2r\beta$ or $(2r+1)\beta$ (as the case may be), and whose difference is as small as possible. Thus the functions of $2r\beta$ are obtained by applying duplication formulæ to functions of $r\beta$, and the functions of $(2r+1)\beta$ are obtained by applying the addition formulæ to functions of $r\beta$ and $(r+1)\beta$.

* The analysis in evaluating A and Z/W is simpler when we work with α than when we work with β.

It is evident from the general formulæ quoted by Cayley (*loc. cit.*, pp. 86–87) that we may write

$$\operatorname{sn} 2r\beta = \frac{4S_{2r}\sqrt{3}}{N_{2r}}, \quad \operatorname{cn} 2r\beta = \frac{C_{2r}}{N_{2r}}, \quad \operatorname{dn} 2r\beta = \frac{D_{2r}}{N_{2r}},$$

$$\operatorname{sn}(2r+1)\beta = \frac{S_{2r+1}}{N_{2r+1}\sqrt{2}}, \quad \operatorname{cn}(2r+1)\beta = \frac{C_{2r+1}}{N_{2r+1}\sqrt{2}},$$

$$\operatorname{dn}(2r+1)\beta = \frac{D_{2r+1}\sqrt{3}}{2N_{2r+1}},$$

and we shall shew by induction that if the expressions S_r, C_r, D_r, N_r are formed by the rules indicated above, then (*a*) C_{2r}, D_{2r}, N_{2r} are odd and not divisible by 3; (*b*) S_{2r+1}, C_{2r+1}, D_{2r+1}, N_{2r+1} are odd and (except D_{2r+1}) not divisible by 3; (*c*) S_{2r}, C_{2r}, D_{2r}, N_{2r} have no common factor; (*d*) S_{2r+1}, C_{2r+1}, D_{2r+1}, N_{2r+1} have no common factor.

To obtain these results we assume the above expressions for functions of $2r\beta$, $(2r+1)\beta$, and write down the expressions for functions of $(4r-1)\beta$, $4r\beta$, $(4r+1)\beta$, $(4r+2)\beta$.

We get without difficulty

$$\text{(I)}\quad \begin{cases} S_{4r} = 2S_{2r}C_{2r}D_{2r}N_{2r} \\ C_{4r} = C^2_{2r}N^2_{2r} - 48S^2_{2r}D^2_{2r} \\ D_{4r} = D^2_{2r}N^2_{2r} - 24S^2_{2r}C^2_{2r} \\ N_{4r} = N^4_{2r} - 1152S^4_{2r} = 2D^4_{2r} - C^4_{2r} \end{cases}$$

$$\text{(II)}\quad \begin{cases} S_{4r+2} = S_{2r+1}C_{2r+1}D_{2r+1}N_{2r+1} \\ C_{4r+2} = 4C^2_{2r+1}N^2_{2r+1} - 3S^2_{2r+1}D^2_{2r+1} \\ D_{4r+2} = 6D^2_{2r+1}N^2_{2r+1} - S^2_{2r+1}C^2_{2r+1} \\ N_{4r+2} = 8N^4_{2r+1} - S^4_{2r+1} = 9D^4_{2r+1} - 2C^4_{2r+1} \end{cases}$$

$$\text{(III)}\quad \begin{cases} S_{4r+1} = 6S_{2r}N_{2r}C_{2r+1}D_{2r+1} + S_{2r+1}N_{2r+1}C_{2r}D_{2r} \\ C_{4r+1} = C_{2r}N_{2r}C_{2r+1}N_{2r+1} - 6S_{2r}D_{2r}S_{2r+1}D_{2r+1} \\ D_{4r+1} = D_{2r}N_{2r}D_{2r+1}N_{2r+1} - 2S_{2r}C_{2r}S_{2r+1}C_{2r+1} \\ N_{4r+1} = N^2_{2r}N^2_{2r+1} - 12S^2_{2r}S^2_{2r+1} = \tfrac{1}{2}(3D^2_{2r}D^2_{2r+1} - C^2_{2r}C^2_{2r+1}) \end{cases}$$

and the expressions for S_{4r-1}, C_{4r-1}, D_{4r-1}, N_{4r-1} are obtained by leaving the functions involving $2r$ unchanged and replacing the functions of $2r+1$ in the last four equations by functions of $2r-1$.

From (I), (II), (III) we see that, if C_{2r}, D_{2r}, N_{2r}, $S_{2r\pm1}$, $C_{2r\pm1}$, $D_{2r\pm1}$, $N_{2r\pm1}$ are assumed to be all odd, it follows that C_{4r}, D_{4r}, N_{4r}, C_{4r+2}, D_{4r+2}, N_{4r+2}, $S_{4r\pm1}$, $C_{4r\pm1}$, $D_{4r\pm1}$, $N_{4r\pm1}$ all odd; and since the numbers are odd, in the case of 0β and 1β, they are odd for all values of r, by induction.

Similarly, if C_{2r}, D_{2r}, N_{2r}, S_{2r+1}, C_{2r+1}, N_{2r+1} are not divisible by 3, it follows that C_{4r}, D_{4r}, N_{4r}, C_{4r+2}, D_{4r+2}, N_{4r+2}, $S_{4r\pm1}$, $C_{4r\pm1}$, $N_{4r\pm1}$ are not divisible by 3, and the induction holds as in the previous case.

We have now proved the results (a) and (b). The results (c) and (d) are slightly more difficult; we shall prove them by a process of induction. The process of induction which we shall establish is the following: We take a set of eight numbers S_p, C_p, D_p, N_p, S_{p+1}, C_{p+1}, D_{p+1}, N_{p+1}, and we shall shew that if they satisfy any one of the following five conditions:

(i) Any pair of the four S_p, C_p, D_p, N_p have a common factor σ (or any pair of the four S_{p+1}, C_{p+1}, D_{p+1}, N_{p+1} have a common factor σ).

(ii) S_p and N_{p+1} have a common factor σ.

(iii) S_{p+1} and N_p have a common factor σ.

(iv) C_p and D_{p+1} have a common factor σ.

(v) C_{p+1} and D_p have a common factor σ.

Then there is another similar set of eight numbers *for which p has a smaller value*, which also satisfies one of the five conditions.

Since $S_1 = C_1 = D_1 = N_1 = 1$, $S_2 = C_2 = 1$, $D_2 = 5$, $N_2 = 7$, it follows by a *reductio ad absurdum* that none of the five conditions is ever satisfied.

In dealing with these conditions, if the common factor is a composite number, we take σ to be a prime factor of this number; the numbers 2 and 3 are exceptional values of σ for which the following investigation fails; however, there is only one of the eight numbers S_p, C_p, D_p, N_p, S_{p+1}, C_{p+1}, D_{p+1}, N_{p+1} which can be divisible by 2; and it has already been shewn that S_{2r+1} is not divisible by 3, so that while the investigation which we are about to give shews that there is no prime number σ (other than 2 or 3) which is a common factor of S_{2r+1} and D_{2r+1}, it has been shewn by other considerations that neither 2 nor 3 is a common factor.

We first observe that

$$48 S^2_{2r} + C^2_{2r} = N^2_{2r}, \quad 24 S^2_{2r} + D^2_{2r} = N^2_{2r}, \quad 24 S^2_{2r} + C^2_{2r} = D^2_{2r}.$$

and that

$$S^2_{2r+1} + C^2_{2r+1} = 2N^2_{2r+1}, \quad S^2_{2r+1} + 3D^2_{2r+1} = 4N^2_{2r+1},$$
$$S^2_{2r+1} + 2C^2_{2r+1} = 3D^2_{2r+1},$$

and hence, if σ is a factor of any pair of the numbers S_p, C_p, D_p, N_p, it is also a factor of each of the other pair.

We now take conditions (i)–(v) in turn.

Condition (i). First let a pair (and therefore all four) of S_{4r}, C_{4r}, D_{4r}, N_{4r} have a common factor σ; then, from (I), σ is a factor of one of the four numbers S_{2r}, C_{2r}, D_{2r}, N_{2r}; if σ is a factor of S_{2r}, it follows from the equation

$$N_{4r} = N^4_{2r} - 1152\,S^4_{2r}$$

that it is also a factor of N_{2r}; similarly, if it is a factor of N_{2r}, it must be a factor of S_{2r}; and from the equation

$$N_{4r} = 2D^4_{2r} - C^4_{2r}$$

it follows that if σ is a factor of either of the numbers C_{2r}, D_{2r}, it must be a factor of the other.

Therefore if condition (i) is satisfied for $p = 4r$, it is also satisfied for $p = 2r$.

Similarly if condition (i) is satisfied for $p = 4r + 2$, it is also satisfied for $p = 2r + 1$.

Next let a pair (and therefore all four) of* $S_{4r\pm1}$, $C_{4r\pm1}$, $D_{4r\pm1}$, $N_{4r\pm1}$ have a common factor σ.

Since $(2r\pm1) - 2r = \pm1$ and since $\operatorname{sn}\beta = 1/\sqrt{2}$, $\operatorname{cn}\beta = 1/\sqrt{2}$, $\operatorname{dn}\beta = \frac{1}{2}\sqrt{3}$, we see that

$$\begin{aligned}
&\pm\{S_{2r\pm1}N_{2r\pm1}C_{2r}D_{2r} - 6S_{2r}N_{2r}C_{2r\pm1}D_{2r\pm1}\}\\
&= C_{2r}N_{2r}C_{2r\pm1}N_{2r\pm1} + 6S_{2r}D_{2r}S_{2r\pm1}D_{2r\pm1}\\
&= D_{2r}N_{2r}D_{2r\pm1}N_{2r\pm1} + 2S_{2r}C_{2r}S_{2r\pm1}C_{2r\pm1}\\
&= N^2_{2r}N^2_{2r\pm1} - 12S^2_{2r}S^2_{2r\pm1}\\
&= N_{4r\pm1}.
\end{aligned}$$

We thus get

$$\text{(IV)} \quad \begin{cases} \pm S_{4r\pm1} + N_{4r\pm1} = 2S_{2r\pm1}N_{2r\pm1}C_{2r}D_{2r}, \\ \pm S_{4r\pm1} - N_{4r\pm1} = 12S_{2r}N_{2r}C_{2r\pm1}D_{2r\pm1}. \end{cases}$$

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* All the upper signs are to be taken in this investigation or else all the lower signs.

$$\text{(V)} \qquad \begin{cases} N_{4r\pm1} + C_{4r\pm1} = 2\, C_{2r} N_{2r} C_{2r\pm1} N_{2r\pm1}, \\ N_{4r\pm1} - C_{4r\pm1} = 12\, S_{2r} D_{2r} S_{2r\pm1} D_{2r\pm1}. \end{cases}$$

$$\text{(VI)} \qquad \begin{cases} N_{4r\pm1} + D_{4r\pm1} = 2\, D_{2r} N_{2r} D_{2r\pm1} N_{2r\pm1}, \\ N_{4r\pm1} - D_{4r\pm1} = 4\, S_{2r} C_{2r} S_{2r\pm1} C_{2r\pm1}. \end{cases}$$

Since $S_{4r\pm1}$, $C_{4r\pm1}$, $D_{4r\pm1}$, $N_{4r\pm1}$ have σ as a factor, each of the expressions on the right in (IV), (V) and (VI) have σ as a factor; and so, in particular, σ is a factor of one of $S_{2r\pm1}$, $N_{2r\pm1}$, C_{2r}, D_{2r}.

First let σ be a factor of $S_{2r\pm1}$; then from the second equation in (IV) σ is a factor of $C_{2r\pm1}$ or of $D_{2r\pm1}$ [in which case condition (i) is satisfied for $p = 2r \pm 1$] or else σ is a factor of $S_{2r} N_{2r}$ and also [from (V) and (VI)] of $C_{2r} N_{2r}$ and of $D_{2r} N_{2r}$; hence σ is a factor of each of S_{2r}, C_{2r}, D_{2r} [in which case condition (i) is satisfied for $p = 2r$] or else σ is a factor of both $S_{2r\pm1}$ and N_{2r}.

Similarly, if σ is a factor of $N_{2r\pm1}$ or C_{2r} or D_{2r}, we find that *either* condition (i) is satisfied for $p = 2r$ or $2r \pm 1$ *or else* that σ is a factor of S_{2r}, $D_{2r\pm1}$, $C_{2r\pm1}$ respectively. Therefore if condition (i) is satisfied for $p = 4r + 1$ it is also satisfied for $p = 2r$ or $2r + 1$, or else one of the conditions (ii)–(v) is satisfied for $p = 2r$; while if condition (i) is satisfied for $p = 4r - 1$, it is also satisfied for $p = 2r - 1$ or $2r$, or else one of the conditions (ii)–(v) is satisfied for $p = 2r - 1$.

This completes the reduction of condition (i).

Conditions (ii) *and* (iii). First take the case in which S_{4r}, $N_{4r\pm1}$ have the factor σ, so that $S_{2r} C_{2r} D_{2r} N_{2r}$ has σ as a factor.

Since

$$N_{4r\pm1} = N^2_{2r} N^2_{2r\pm1} - 12\, S^2_{2r} S^2_{2r\pm1} = \tfrac{3}{2} D^2_{2r} D^2_{2r\pm2} - \tfrac{1}{2} C^2_{2r} C^2_{2r\pm1},$$

it follows that if σ is a factor of S_{2r}, C_{2r}, D_{2r} or N_{2r} respectively, then *either* condition (i) is satisfied for $p = 2r$ *or else* σ is a factor of $N_{2r\pm1}$, $D_{2r\pm1}$, $C_{2r\pm1}$, or $S_{2r\pm1}$ respectively, so that one of the conditions (ii)–(v) is satisfied for $p = 2r$ when the upper signs are taken, and for $p = 2r - 1$ when the lower signs are taken.

Similarly, if $S_{4r\pm2}$, $N_{4r+2\pm1}$ have the factor σ, we find that *either* condition (i) is satisfied for $p = 2r + 1$ *or else* one of the conditions (ii)–(v) is satisfied for $p = 2r + 1$ when the upper signs are taken, and for $p = 2r$ when the lower signs are taken.

Next take the case in which N_{4r}, $S_{4r\pm1}$ have the factor σ; it is easy to see that

$$\pm S_{4r\pm1}N_{4r\pm1} = S_{2r\pm1}^2 N_{2r\pm1}^2 C_{2r}^2 D_{2r}^2 - 36 S_{2r}^2 N_{2r}^2 C_{2r\pm1}^2 D_{2r\pm1}^2$$

$$= S_{2r\pm1}^2 N_{2r\pm1}^2 (N_{2r}^2 - 48 S_{2r}^2)(N_{2r}^2 - 24 S_{2r}^2)$$

$$- 12 S_{2r}^2 N_{2r}^2 (2 N_{2r\pm1}^2 - S_{2r\pm1}^2)(4 N_{2r\pm1}^2 - S_{2r\pm1}^2)$$

$$= (S_{2r\pm1}^2 N_{2r}^2 - 96 S_{2r}^2 N_{2r\pm1}^2)(N_{2r}^2 N_{2r\pm1}^2 - 12 S_{2r}^2 S_{2r\pm1}^2),$$

and so $$\pm S_{4r\pm1} = S_{2r\pm1}^2 N_{2r}^2 - 96 S_{2r}^2 N_{2r\pm1}^2 \quad \ldots\ldots\ldots\ldots (8).$$

Consequently

$$\pm S_{4r\pm1} N_{2r}^2 - S_{2r\pm1}^2 N_{4r} = N_{2r}^2 (S_{2r\pm1}^2 N_{2r}^2 - 96 S_{2r}^2 N_{2r\pm1}^2)$$

$$- S_{2r\pm1}^2 (N_{2r}^4 - 1152 S_{2r}^4)$$

$$= - 96 S_{2r}^2 N_{4r\pm1};$$

and so, if σ is a factor of $S_{4r\pm1}$ and N_{4r}, it is also a factor of S_{2r} or of $N_{4r\pm1}$.

But, if σ is a factor of S_{2r} and N_{4r}, it is also a factor of N_{2r} from (I); and, if not, σ is a factor of both $S_{4r\pm1}$ and $N_{4r\pm1}$. Hence, if σ is a factor of N_{4r} and $S_{4r\pm1}$, *either* condition (i) is satisfied for $p = 2r$ *or else* condition (i) is satisfied for $p = 4r \pm 1$, which is a case already reduced.

Lastly, take the case in which N_{4r+2}, $S_{4r+2\pm1}$ have the factor σ; we find from (8) that

$$\begin{cases} S_{4r+1} = S_{2r+1}^2 N_{2r}^2 - 96 S_{2r}^2 N_{2r+1}^2, \\ - S_{4r+3} = S_{2r+1}^2 N_{2r+2}^2 - 96 S_{2r+2}^2 N_{2r+1}^2. \end{cases}$$

and so

$$\begin{cases} S_{4r+1} N_{2r+1}^2 + 12 N_{4r+2} S_{2r}^2 = S_{2r+1}^2 N_{4r+1}, \\ S_{4r+3} N_{2r+1}^2 - 12 N_{4r+2} S_{2r+2}^2 = - S_{2r+1}^2 N_{4r+3}. \end{cases}$$

Hence, if σ is a factor of N_{4r+2} and S_{4r+1}, it is *either* a factor of S_{2r+1}, and consequently also of N_{2r+1} from the equation $N_{4r+2} = 8 N_{2r+1}^4 - S_{2r+1}^4$, *or else* it is a factor of N_{4r+1} as well as of S_{4r+1}. Therefore, either condition (i) is satisfied for $p = 2r+1$, or else condition (i) is satisfied for $p = 4r + 1$, which is a case already reduced.

Similarly, if σ is a factor of N_{4r+2} and S_{4r+3}, condition (i) is satisfied for $p = 2r + 1$, or else condition (i) is satisfied for $p = 4r + 3$, which is a case already reduced.

This completes the reduction of conditions (ii) and (iii).

Conditions (iv) *and* (v). First take the case in which C_{4r} and $D_{4r\pm1}$ have the factor σ. It is easy to verify by (V) and (VI) that

$$(N_{4r\pm1}+C_{4r\pm1})(N_{4r\pm1}+D_{4r\pm1})-(N_{4r\pm1}-C_{4r\pm1})(N_{4r\pm1}-D_{4r\pm1})$$
$$=4C_{2r}D_{2r}C_{2r\pm1}D_{2r\pm1}\{N_{2r}^{2}N_{2r\pm1}^{2}-12S_{2r}^{2}S_{2r\pm1}^{2}\},$$

and so $$C_{4r\pm1}+D_{4r\pm1}=2C_{2r}D_{2r}C_{2r\pm1}D_{2r\pm1}.$$

In like manner

$$(N_{4r\pm1}-C_{4r\pm1})(N_{4r\pm1}+D_{4r\pm1})-(N_{4r\pm1}+C_{4r\pm1})(N_{4r\pm1}-D_{4r\pm1})$$
$$=16S_{2r}N_{2r}S_{2r\pm1}N_{2r\pm1}\{\tfrac{3}{2}D_{2r}^{2}D_{2r\pm1}^{2}-\tfrac{1}{2}C_{2r}^{2}C_{2r\pm1}^{2}\},$$

so that $$D_{4r\pm1}-C_{4r\pm1}=8S_{2r}N_{2r}S_{2r\pm1}N_{2r\pm1},$$

and consequently

$$D_{4r\pm1}=C_{2r}D_{2r}C_{2r\pm1}D_{2r\pm1}+4S_{2r}N_{2r}S_{2r\pm1}N_{2r\pm1}\ldots\ldots(9).$$

Further, it is easy to deduce from (I) that

$$C_{4r}=C_{2r}^{2}D_{2r}^{2}-24S_{2r}^{2}N_{2r}^{2}\ldots\ldots\ldots\ldots\ldots(10),$$

and so

$$D_{4r\pm1}C_{2r}D_{2r}-C_{4r}C_{2r\pm1}D_{2r\pm1}=4S_{2r}N_{2r}\{6S_{2r}N_{2r}C_{2r\pm1}D_{2r\pm1}$$
$$+C_{2r}D_{2r}S_{2r\pm1}N_{2r\pm1}\},$$

whence we see that $S_{2r}N_{2r}S_{4r\pm1}$ has σ as a factor. Hence either condition (i) is satisfied for $p=4r\pm1$ (which is a case already reduced) or else from (10) condition (i) is satisfied for $p=2r$.

Next take the case in which C_{4r+2} and $D_{4r+2\pm1}$ have the factor σ; we have, as in (9),

$$\begin{cases} D_{4r+1}=C_{2r}D_{2r}C_{2r+1}D_{2r+1}+4S_{2r}N_{2r}S_{2r+1}N_{2r+1}, \\ D_{4r+3}=C_{2r+2}D_{2r+2}C_{2r+1}D_{2r+1}+4S_{2r+2}N_{2r+2}S_{2r+1}N_{2r+1}, \end{cases}$$

while $$C_{4r+2}=3C_{2r+1}^{2}D_{2r+1}^{2}-2S^{2}_{2r+1}N_{2r+1}^{2},$$

and therefore

$$\begin{cases} 3D_{4r+1}C_{2r+1}D_{2r+1}-C_{4r+2}C_{2r}D_{2r}=2S_{2r+1}N_{2r+1}S_{4r+1}, \\ 3D_{4r+3}C_{2r+1}D_{2r+1}-C_{4r+2}C_{2r+2}D_{2r+2}=2S_{2r+1}N_{2r+1}S_{4r+3}. \end{cases}$$

In the first case, when C_{4r+2} and D_{4r+1} have the factor σ, it follows that $S_{2r+1}N_{2r+1}S_{4r+1}$ has the factor σ, and so (as in the case immediately preceding) condition (i) is satisfied either for $p=4r+1$ (a case already reduced) or for $p=2r+1$; and in the second case condition (i) is satisfied either for $p=4r+3$ or for $p=2r+1$.

Lastly, we have to take the cases in which C has the odd suffix and D has the even suffix.

We get from the equations by which (9) was deduced

$$C_{4r\pm 1} = C_{2r} D_{2r} C_{2r\pm 1} D_{2r\pm 1} - 4 S_{2r} N_{2r} S_{2r\pm 1} N_{2r\pm 1},$$

and
$$D_{4r} = C_{2r}^2 D_{2r}^2 + 24 S_{2r}^2 N_{2r}^2,$$

so that
$$D_{4r} C_{2r\pm 1} D_{2r\pm 1} - C_{4r\pm 1} C_{2r} D_{2r} = 4 S_{2r} N_{2r} S_{4r\pm 1}.$$

Hence if D_{4r} and $C_{4r\pm 1}$ have the factor σ, we deduce in the usual manner that condition (i) is satisfied either for $p = 4r \pm 1$ or for $p = 2r$.

Next, from the equations

$$\begin{cases} C_{4r+1} = C_{2r} D_{2r} C_{2r+1} D_{2r+1} - 4 S_{2r} N_{2r} S_{2r+1} N_{2r+1}, \\ C_{4r+3} = C_{2r+2} D_{2r+2} C_{2r+1} D_{2r+1} - 4 S_{2r+2} N_{2r+2} S_{2r+1} N_{2r+1}, \end{cases}$$

combined with

$$D_{4r+2} = 3 C_{2r+1}^2 D_{2r+1}^2 + 2 S_{2r+1}^2 N_{2r+1}^2,$$

we find that, if C_{4r+1} and D_{4r+2} have the factor σ, then condition (i) is satisfied either for $p = 4r + 2$ or for $p = 2r + 1$, while if C_{4r+3} and D_{4r+2} have the factor σ, then condition (i) is satisfied either for $p = 4r + 3$ or for $p = 2r + 1$.

This completes the reduction of conditions (iv) and (v). We have therefore proved that if any one of conditions (i) to (v) is satisfied for any value of p, then also some one of the conditions must be satisfied for a smaller value of p. Hence, since none of the conditions is satisfied for $p = 1$, the conditions are never satisfied.

In particular we have therefore shewn that S_{2r+1} and D_{2r+1} have no common factor, which is the result required. It remains to be shewn that S_{2r+1} is never numerically equal to unity.

8. When we compare the construction of S_p, C_p, D_p, N_p in § 7 with the mode of formation of the general expressions sn pu, cn pu, dn pu (for any argument u and any modulus k) in terms of sn u, given by Cayley,* it becomes evident that the only difference between the fractions at the beginning of § 7 and the functions given by Cayley is that Cayley's numerators and denominators are integral† expressions in $\text{sn}^2 u$ and k^2 (with no algebraic common factors), whereas in forming S_p, C_p, D_p, N_p the values of $\text{sn}^2\beta$ and k^2 have been regarded as fractions (each being equal to $\frac{1}{2}$), and so the numerators

* Cayley, *Elliptic Functions*, pp. 78–92.

† Apart from the factors sn u, dn u, which multiply some of them.

and denominators of § 7 are obtained from Cayley's numerators and denominators by multiplication by a suitable power of 2; and, in particular, Cayley's expression for the numerator of $\operatorname{sn}(2r+1)\beta$ is

$$(\operatorname{sn}\beta)^{4r(r+1)}\,k^{2r(r-1)}\,S_{2r+1}.$$

Hence, from Cayley's general formula,

$$(-)^r S_{2r+1} = 1 - \frac{a^{(2)}_{2r+1,1}}{k^2\operatorname{sn}^4\beta} + \left\{\frac{a^{(3)}_{2r+1,1}}{k^2} + \frac{a^{(3)}_{2r+1,2}}{k^4}\right\}\frac{1}{\operatorname{sn}^6\beta}$$
$$- \left\{\frac{a^{(4)}_{2r+1,1}}{k^2} + \frac{a^{(4)}_{2r+1,2}}{k^4} + \frac{a^{(4)}_{2r+1,3}}{k^6}\right\}\frac{1}{\operatorname{sn}^8\beta} + \dots,$$

where the coefficients $a^{(q)}_{2r+1,t}$ are integers, the coefficient of $1/\operatorname{sn}^{2q}\beta$ is a polynomial in $1/k^2$ containing no term independent of k^2, and in evaluating S_{2r+1} we have to take k^2 and $\operatorname{sn}^2\beta$, each equal to $\frac{1}{2}$.

Moreover, Cayley's general expression for the denominator of $\operatorname{sn}pu$ (whether p be odd or even) is

$$1 - a^{(2)}_{p,1}k^2\operatorname{sn}^4 u + \{a^{(3)}_{p,1}k^4 + a^{(3)}_{p,2}k^2\}\operatorname{sn}^6 u$$
$$+ \{a^{(4)}_{p,1}k^6 + a^{(4)}_{p,2}k^4 + a^{(4)}_{p,3}k^2\}\operatorname{sn}^8 u - \dots,$$

where $a^{(q)}_{p,t}$ is an integer and $a^{(q)}_{p,t} = 0$ when $t = 0$ or $t \geq q$ (except when $q = 0$ or 1). In particular*

$$a^{(2)}_{p,1} = \tfrac{1}{12}p^2(p^2-1).$$

We shall now obtain some general results concerning a minimum power of 2 which divides certain of the coefficients $a^{(q)}_{p,t}$. As the results in the case in which p is odd may be derived from the case in which p is even, we first consider the case in which p is even. Then p may be written in the form $p = 2^N M$, where M is odd and N is equal to or greater than 1. It is evident that $a^{(2)}_{p,1}$ is divisible by 2^{2N-2}, but not by any higher power of 2. We shall shew by induction (for increasing values of N) that, when q has the values 3, 4, ..., $2N$, then $a^{(q)}_{p,t}$ is divisible by 2^{2N-q+1} at least.

We have

$$\operatorname{sn}2Mu = \frac{2\operatorname{sn}Mu\operatorname{cn}Mu\operatorname{dn}Mu}{1-k^2\operatorname{sn}^4 Mu}.$$

* Cayley, *Collected Papers* I., p. 299.

and so the denominator of $\operatorname{sn} 2Mu$ can be written in the form*

$$\{1 - a^{(2)}_{M,1} k^2 \operatorname{sn}^4 u + \ldots\}^4 - k^2 \operatorname{sn}^4 u \,.\, P_1 (\operatorname{sn}^2 u,\ k^2),$$

and the numerator is $2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \,.\, P_2 (\operatorname{sn}^2 u,\ k^2)$.

Next we have

$$\operatorname{sn} 4Mu = \frac{2 \operatorname{sn} 2Mu \operatorname{cn} 2Mu \operatorname{dn} 2Mu}{1 - k^2 \operatorname{sn}^4 2Mu},$$

and so the denominator of $\operatorname{sn} 4Mu$ is

$$\{1 - a^{(2)}_{2M,1} k^2 \operatorname{sn}^4 u + \ldots\}^4 - 16k^2 \operatorname{sn}^4 u \,.\, P_3 (\operatorname{sn}^2 u,\ k^2),$$

while the numerator is $4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \,.\, P_4 (\operatorname{sn}^2 u,\ k^2)$.

Since we have to prove that $a^q_{4M,t}$ is divisible by 2^{5-q} $(q = 3, 4)$, and $2^{5-q} < 16$, it is sufficient to prove the divisibility of the corresponding coefficients in

$$\{1 - a^{(2)}_{2M,1} k^2 \operatorname{sn}^4 u + \ldots\}^4,$$

which we shall call $(1 - U_{2M})^4$.

Now $\quad (1 - U_{2M})^4 = 1 - 4U_{2M} + 6U^2_{2M} - 4U^3_{2M} + U^4_{2M}.$

The only terms in $\operatorname{sn}^6 u$ on the right occur in the term $-4U_{2M}$, and so these coefficients of all the terms involving $\operatorname{sn}^6 u$ (multiplied by powers of k^2) are divisible by 4; while the only terms in $\operatorname{sn}^8 u$ occur in $-4U_{2M} + 6U^2_{2M}$, and so the coefficients of k^2 all the terms in $\operatorname{sn}^8 u$ (multiplied by powers of k^2) are divisible by 2.

The result is therefore true for $N = 2$.

Assuming the result for any particular value of N, and writing $2^N M = p$, we have

$$\operatorname{sn} 2pu = \frac{2 \operatorname{sn} pu \operatorname{cn} pu \operatorname{dn} pu}{1 - k^2 \operatorname{sn}^4 pu},$$

and, by an obvious induction, the numerator of $\operatorname{sn} pu$ is

$$2^N \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \, P_5 (\operatorname{sn}^2 u,\ k^2).$$

Hence the denominator of $\operatorname{sn} 2pu$ is

$$\{1 - a^{(2)}_{p,1} k^2 \operatorname{sn}^4 u + \ldots\}^4 - 2^{4N} k^2 \operatorname{sn}^4 u \, P_6 (\operatorname{sn}^2 u,\ k^2).$$

* We use the symbol P to denote a polynomial in two variables with integral coefficients.

Since $4N > 2(N+1) - q + 1$ when $q = 3, 4, \ldots, 2(N+1)$, it is sufficient to prove the divisibility of the coefficients in

$$\{1 - a^{(2)}_{p,1} k^2 \operatorname{sn}^4 u + \ldots\}^4,$$

which we shall call $(1 - U)^4$.

Now $a^{(q)}_{p,t}$ is divisible by 2^{2N-q} when $q = 2, 3, \ldots, 2N$.

Next consider how the coefficients of the terms in $k^{2(q-1)} \operatorname{sn}^{2q} u$ (where $q = 3, 4, \ldots, 2N+2$) can arise from

$$(1 - U)^4 \equiv 1 - 4U + 6U^2 - 4U^3 + U^4.$$

The parts of the coefficients arising from $4U$ are divisible by $2^{2N-q+1+2}$ when $q = 3, 4, \ldots, 2N$, by 2 when $q = 2N+1$, and by 2 when $q = 2N+2$. Hence they are divisible by

$$2^{2(N+1)-q+1} \ (q = 3, 4, \ldots, 2N+2).$$

The parts of the coefficients arising from $6U^2$ are composed of a number of terms, each of which is divisible by the power of 2 whose index is

$$1 + (2N - q_1) + (2N - q_2)$$

where $$q_1 + q_2 = q, \ 2 \leq q_1 \leq 2N, \ 2 \leq q_2 \leq 2N,$$

so that q has all values from 4 up to* $2N + 2$. But

$$1 + (2N - q_1) + (2N - q_2) > 2(N+1) - q + 1.$$

Similarly the parts of the coefficients arising from $-4U^3$ are composed of a number of terms each of which is divisible by the power of 2 whose index is

$$1 + (2N - q_1) + (2N - q_2) + (2N - q_3) > 2(N+1) - q + 1,$$

where $q_1 + q_2 + q_3 = q$; and finally the parts of the coefficients arising from U^4 are composed of a number of terms each of which is divisible by

$$1 + (2N - q_1) + (2N - q_2) + (2N - q_3) + (2N - q_4) > 2(N+1) - q + 1.$$

Hence finally $a^q_{2p,t}$ is the sum of a number of terms each of which is divisible by $2^{2(N+1)-q+1}$ when q has the values $3, 4, \ldots, 2N+2$; and the induction is complete.

Next we take the case in which p is odd. Every odd number differs from a multiple of 4 by ± 1, so we may write p

* Terms in $\operatorname{sn}^{2N+2} u$ in U^2 do not arise from any term for which q_1 or q_2 exceeds $2N$; and there are *no* terms of the form $\operatorname{sn}^4 u$, $\operatorname{sn}^6 u$ in U^2.

either in the form $2^N M+1$ or else in the form $2^N M-1$, where M is odd and N is at least equal to 2.

It is evident that $a_{p,1}^{(2)}$ is now divisible by 2^{N-1}, but not by any higher power of 2. We shall shew by induction (for increasing values of N) that, when q has the values $3,4,\ldots,N+1$, then $a_{p,t}^{(q)}$ is divisible by 2^{N-q+2} at least.

We have $2^N M\pm 1=p$, so that

$$2^{N+1}M\pm 1=2p\mp 1=p+(p\mp 1).$$

Hence

$$\operatorname{sn}(2^{N+1}M\pm 1)u=\frac{\operatorname{sn}pu\operatorname{cn}(p\pm 1)u\operatorname{dn}(p\mp 1)u+\operatorname{sn}(p\mp 1)u\operatorname{cn}pu\operatorname{dn}pu}{1-k^2\operatorname{sn}^2pu\operatorname{sn}^2(p\mp 1)u}$$

In particular the denominator of $\operatorname{sn}^2(4M\pm 1)u$ is

$$\{1-a_{2M,1}^{(2)}k^2\operatorname{sn}^4u+\ldots\}^2\{1-a_{2M\pm 1,1}^{(2)}k^2\operatorname{sn}^4u+\ldots\}^2-4P_7(\operatorname{sn}^2u,k^2),$$

the factor 4 multiplying the polynomial P_7 because the numerator of $\operatorname{sn}2Mu$ is divisible by 2; and it is obvious that the coefficient of all the terms in sn^6u in the denominator of $\operatorname{sn}(4M\pm 1)u$ are divisible by 2. Hence the result stated is true for $N=2$.

Assuming the result for any particular value of N, we see that the denominator of $\operatorname{sn}(2^{N+1}M\pm 1)u$ is

$$\{1-a^{p,1}k^2\operatorname{sn}^4u+\ldots\}^2\{1-a^{p\pm 1,1}k^2\operatorname{sn}^4u+\ldots\}^2-2^{2N}k^2\operatorname{sn}^4uP_8(\operatorname{sn}^2u,k^2),$$

the factor 2^{2N} occurring since the numerator of $\operatorname{sn}(p\mp 1)u$ is divisible by 2^N.

We write this in the form

$$(1-V)^2(1-U)^2-2^{2N}k^2\operatorname{sn}^4uP_8(\operatorname{sn}^2u,k^2),$$

and it is obviously sufficient to prove that, when $q=3,4,\ldots,N+2$, the coefficient of $\operatorname{sn}^{2q}u$ is a polynomial in k^2 with integral coefficients, these integers being divisible by $2^{N+1-q+2}$ when $q=3,4,\ldots,N+1$. Now

$$(1-V)^2(1-U)^2=1-2U-2V+V^2$$
$$+4UV+V^2-2UV^2-2U^2V+U^2V^2.$$

Now for the values of q under consideration the integers may be expressed as a sum of integers arising from the individual terms $-2U$, $-2V$, ..., U^2V^2, and the integers composing these sums are respectively divisible by the powers of 2 whose indices are

$$2N-q+1,\quad 1+N-q+1,\quad (2N-q_1+1)+(2N-q_2+1),$$
$$2+(2N-q_1+1)+(N-q_2+1),\quad (N-q_1+1)+(N-q_2+1),$$
$$1+(2N-q_1+1)+(N-q_2+1)+(N-q_3+1),$$
$$1+(2N-q_1+1)+(2N-q_2+1)+(N-q_3+1),$$
$$(2N-q_1+1)+(2N-q_2+1)+(N-q_3+1)+(N-q_4+1),$$

where, in each sum which occurs, $\Sigma q_r = q$.

Now each of these sums is at least equal to $N-q+2$, and the induction follows, so that when $q=3, 4, \ldots, N$, we have proved that $a^{(q)}_{p,t}$ is divisible by 2^{N-q+1}.

The proof that S_{2r+1} is not equal to ± 1 except when $r=0$ is now immediate; for taking $2r+1$ to be equal to $2^N M \pm 1$ where $N \geq 2$, we may write

$$(-)^r S_{2r+1} = 1 - 2^{N+2} M' + 2^{4+N-1} M'' + M''',$$

where $2^{N+2} M' = -8a^{(2)}_{2r+1,1}$, so that M' is odd, and M'' is an integer; while M''' arises from the terms involving $a^{(q)}_{2r+1,p}$ where $q \geq N+2$ and these terms consist of integral coefficients multiplied by $1/\{k^2 \operatorname{sn}^{2q}\beta\}$, or higher negative powers of k, so that M''' is divisible by 2^{N+3} at least.

Hence, when $r > 0$,

$$(-)^r S_{2r+1} = 1 + 2^{N+2} + M^{(iv)} . 2^{N+3},$$

where $N \geq 2$ and $M^{(iv)}$ is an integer; and this obviously cannot be equal to ± 1.

Returning to the end of § 6 we see that Z is numerically equal to unity *only* when $r=0$, and then

$$X^2 = Y^2 = Z^2 = W^2 = 1.$$

Hence the only solution in integers of the system of equations

$$2b^2 - a^2 = 1,\quad 2b^2 + a^2 = 3c^2,$$

given at the beginning of § 6 is

$$a = b = c = 1,$$

and this gives $n=1$, $m=1$.

Thus the only solution of the system (1) in Case (VI) is given by $n=1$.

NOTE ON THE REPRESENTATION OF THE EXPANSION OF A BORDERED DETERMINANT.

By *Sir Thomas Muir, LL.D.*

1. The usual expansion of a bordered determinant is a series of terms each consisting of three factors, namely, a primary minor of one of the bordering arrays, a minor of the unbordered square array, and a primary minor of the other bordering array, the first and last being of coordinate importance. For almost all purposes the best mode of representing this expansion is in the form of a bilinear (or bipartite) function. For example,

$$\begin{vmatrix} a_1 & a_2 & a_3 & x_1 \\ b_1 & b_2 & b_3 & x_2 \\ c_1 & c_2 & c_3 & x_3 \\ y_1 & y_2 & y_3 & \cdot \end{vmatrix} = -\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline |b_2\, c_3| & -|a_2\, c_3| & |a_2\, b_3| & y_1 \\ -|b_1\, c_3| & |a_1\, c_3| & -|a_1\, b_3| & y_2 \\ |b_1\, c_2| & -|a_1\, c_2| & |a_1\, b_2| & y_3 \end{array}$$

$$= -x_1 \cdot |b_2\, c_3| \cdot y_1 + \ldots - x_3 \cdot |a_1\, b_2| \cdot y_3 ;$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & x_1 & x_2 \\ b_1 & b_2 & b_3 & y_1 & y_2 \\ c_1 & c_2 & c_3 & z_1 & z_2 \\ \xi_1 & \eta_1 & \zeta_1 & \cdot & \cdot \\ \xi_2 & \eta_2 & \zeta_2 & \cdot & \cdot \end{vmatrix} = \begin{array}{ccc|c} |x_1\, y_2| & |x_1\, z_2| & |y_1\, z_2| & \\ \hline c_3 & -b_3 & a_3 & |\xi_1\, \eta_2| \\ -c_2 & b_2 & -a_2 & |\xi_1\, \zeta_2| \\ c_1 & -b_1 & a_1 & |\eta_1\, \zeta_2| \end{array}$$

$$= |x_1\, y_2| \cdot c_3 \cdot |\xi_1\, \eta_2| - \ldots + |y_1\, z_2| \cdot a_1 \cdot |\eta_1\, \zeta_2| ;$$

the number of terms in the expansion being 3^2 in each case.

2. To make clear the law of formation for a bordered determinant of the $(m+r)^{\text{th}}$ order, r being the breadth of the border, it suffices to consider the case where $m, r = 5, 3$, say the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{15} & \theta_{11} & \theta_{12} & \theta_{13} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{51} & a_{52} & \cdots & a_{55} & \theta_{51} & \theta_{52} & \theta_{53} \\ \phi_{11} & \phi_{21} & \cdots & \phi_{51} & \cdot & \cdot & \cdot \\ \phi_{12} & \phi_{22} & \cdots & \phi_{52} & \cdot & \cdot & \cdot \\ \phi_{13} & \phi_{23} & \cdots & \phi_{53} & \cdot & \cdot & \cdot \end{vmatrix} .$$

As is known the procedure consists simply in the repeated use of Laplace's expansion-theorem. In the first place, expansion is taken according to minors of the last 3 columns, the number of product-terms being $C_{5,3}$, and the second factor in each being of the 5^{th} order with its last 3 rows taken from the array of ϕ's. In the next place, each of these second factors is expanded according to minors of its last 3 rows, the number of product-terms being again $C_{5,3}$, and the second factor in each a 2-line minor of the array of a's. The full result thus is $(C_{5,3})^2$ terms, each of three factors, namely, a 3-line minor of θ's, a 3-line minor of ϕ's, and a 2-line minor of a's. Further, when the first two are known, the third can readily be specified: for the rows and columns of the 8-line determinant which have been deleted to obtain the said third factor are exactly the same rows and columns of the a determinant. It is thus seen that in finding the third factors we are simply finding *the elements of the second compound of* $|a_{15}|$, and that the representation above suggested consists of the matrix of this compound with the primary minors of the θ array for the one lateral, and the primary minors of the ϕ array for the other. Consequently, as the most convenient notation for a primary minor of an oblong array consists of the numbers specifying the places which its columns occupy in the array, *e.g.*

$$(123)_\theta \text{ standing for } |\theta_{11}\ \theta_{22}\ \theta_{33}|,$$

and

$$(245)_\phi \text{ standing for } |\phi_{21}\ \phi_{42}\ \phi_{53}|,$$

the bilinear representation of our 8-line determinant takes the form

$$
-\begin{array}{cccc|c}
(123)_\theta & (124)_\theta & \cdots & (345)_\theta & \\
\hline
\begin{vmatrix}45\\45\end{vmatrix} & -\begin{vmatrix}35\\45\end{vmatrix} & \cdots & \begin{vmatrix}12\\45\end{vmatrix} & (123)_\phi \\
-\begin{vmatrix}45\\35\end{vmatrix} & \begin{vmatrix}35\\35\end{vmatrix} & \cdots & -\begin{vmatrix}12\\35\end{vmatrix} & (124)_\phi \\
\cdots & \cdots & \cdots & \cdots & \\
\begin{vmatrix}45\\12\end{vmatrix} & -\begin{vmatrix}35\\12\end{vmatrix} & \cdots & \begin{vmatrix}12\\12\end{vmatrix} & (345)_\phi
\end{array}
$$

where the row-numbers and column-numbers of any one of the 2-line minors, $\begin{vmatrix}45\\12\end{vmatrix}$ say, are known from its cofactors $(123)_\theta$, $(345)_\phi$. Further, as the briefest suggestive notation for the second compound of $|a_{15}|$ is $\|\,|a_{15}|\,\|_2$ we may use even the more concise form

$$-\begin{array}{c|c} (123)_\theta \quad (124)_\theta \quad \ldots \quad (345)_\theta & \\ \hline & (123)_\phi \\ \|\,|a_{15}|\,\|_2 & (124)_\theta \\ & \ldots \end{array}$$

3. An additional advantage of the proposed representation is connected with the question of sign. This is seen very readily if in obtaining any term of the expansion from the given determinant we take out the factors in the opposite order from that followed in the preceding, that is to say, if we begin with the factor obtainable from $|a_{15}|$. For example, such a factor being $|a_{12}\, a_{35}|$, we have the term

$$-\begin{vmatrix} a_{12} & a_{15} \\ a_{32} & a_{35} \end{vmatrix} \cdot \begin{vmatrix} a_{21} & a_{23} & a_{24} & \theta_{21} & \theta_{22} & \theta_{23} \\ a_{41} & a_{43} & a_{44} & \theta_{41} & \theta_{42} & \theta_{43} \\ a_{51} & a_{53} & a_{54} & \theta_{51} & \theta_{52} & \theta_{53} \\ \phi_{11} & \phi_{31} & \phi_{51} & \cdot & \cdot & \cdot \\ \phi_{12} & \phi_{32} & \phi_{52} & \cdot & \cdot & \cdot \\ \phi_{13} & \phi_{33} & \phi_{53} & \cdot & \cdot & \cdot \end{vmatrix}$$

where the minus sign, though obtained as the sign pertaining to $|a_{12}\, a_{35}|$ in the 8-line determinant is also its proper sign in $|a_{15}|$, and the sign to be prefixed to the product

$$|\theta_{21}\, \theta_{42}\, \theta_{53}| \cdot |\phi_{11}\, \phi_{32}\, \phi_{53}|$$

is seen to be $$(-1)^{(1+2+3)+(4+5+6)},$$

and seen also not to require alteration for any other like product. This means that the sign to precede the bilinear representation is

$$(-1)^{1+2+\ldots+2r} \ \ i.e.\ (-1)^{r(2r+1)} \ \ i.e.\ (-1)^r.$$

4. An easy deduction that may be noted in passing is that *If in an n-line determinant the elements of an r-line minor be made all zero, the number of terms in the resulting determinant is $n!\, n!/(n-r)!$*

5. The special case where the unbordered determinant has units for the elements in its diagonal and zeros elsewhere is

also worth a little attention. Any compound of such an unbordered determinant is a determinant of the same kind: consequently the expansion of the bordered determinant has then only the terms, $C_{n,r}$ in number, in which the column-numbers of the one factor are the same as those of the other. It is thus representable simply as the so-called product of the two bordering arrays, that is, when $m, r = 5, 3$ the product

$$- \left\| \begin{matrix} \theta_{11} & \theta_{21} & \cdots & \theta_{51} \\ \theta_{12} & \theta_{22} & \cdots & \theta_{52} \\ \theta_{13} & \theta_{23} & \cdots & \theta_{53} \end{matrix} \right\| \cdot \left\| \begin{matrix} \phi_{11} & \phi_{21} & \cdots & \phi_{51} \\ \phi_{12} & \phi_{22} & \cdots & \phi_{52} \\ \phi_{13} & \phi_{23} & \cdots & \phi_{53} \end{matrix} \right\| ,$$

—a fact recognized, apparently, by Sylvester as early as 1852 (*Hist.* II., p. 199).

In connection with this special case note should be taken that if we change each diagonal element from 1 into u, we are not really reaching a more general form, for by multiplication of columns and subsequent division of rows it becomes at once clear that the new determinant is simply a multiple of the former by u^{m-r}.

6. The main interest of the bordered unit-matrix referred to in the preceding paragraph lies in the fact that it is, as it were, an elemental form in terms of which other forms can be expressed. A fresh instance of this is the determinant

$$\begin{vmatrix} U & . & . & . & . & \theta_{11} & \theta_{12} & \theta_{13} \\ . & U & . & . & . & \theta_{21} & \theta_{22} & \theta_{23} \\ . & . & U & . & . & \theta_{31} & \theta_{32} & \theta_{33} \\ . & . & . & U & . & \theta_{41} & \theta_{42} & \theta_{43} \\ . & . & . & . & U & \theta_{51} & \theta_{52} & \theta_{53} \\ \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} & \phi_{51} & V & . & . \\ \phi_{12} & \phi_{22} & \phi_{32} & \phi_{42} & \phi_{52} & . & V & . \\ \phi_{13} & \phi_{23} & \phi_{33} & \phi_{43} & \phi_{53} & . & . & V \end{vmatrix}$$

a variant of which is got by multiplying each of the last three columns by U and then dividing the first five rows by the same, and another by multiplying each of the first five rows by V and then dividing the last three columns by the same. If we expand the determinant according to descending powers of U it is clear that we must obtain one term in U^5, five in

U^4, ten in U^3, and so on. Now since the cofactors of each of these powers of U is independent of U the places of the U's in them must be occupied by zeros, and they must thus be of the type of § 5. The expansion, in fact, is

$$U^5V^3 + U^4\Sigma \begin{vmatrix} \cdot & \theta_{51} & \theta_{52} & \theta_{53} \\ \phi_{51} & V & \cdot & \cdot \\ \phi_{52} & \cdot & V & \cdot \\ \phi_{53} & \cdot & \cdot & V \end{vmatrix} + U^3\Sigma \begin{vmatrix} \cdot & \cdot & \theta_{41} & \theta_{42} & \theta_{43} \\ \cdot & \cdot & \theta_{51} & \theta_{52} & \theta_{53} \\ \phi_{41} & \phi_{51} & V & \cdot & \cdot \\ \phi_{42} & \phi_{52} & \cdot & V & \cdot \\ \phi_{43} & \phi_{53} & \cdot & \cdot & V \end{vmatrix}$$

$$+ U^2\Sigma \begin{vmatrix} \cdot & \cdot & \cdot & \theta_{31} & \theta_{32} & \theta_{33} \\ \cdot & \cdot & \cdot & \theta_{41} & \theta_{42} & \theta_{43} \\ \cdot & \cdot & \cdot & \theta_{51} & \theta_{52} & \theta_{53} \\ \phi_{31} & \phi_{41} & \phi_{51} & V & \cdot & \cdot \\ \phi_{32} & \phi_{42} & \phi_{52} & \cdot & V & \cdot \\ \phi_{33} & \phi_{43} & \phi_{53} & \cdot & \cdot & V \end{vmatrix} + O + O,$$

the V's in the determinants being replaceable by I's on annexing V^2, V^1, V^0 to U^4, U^3, U^2 respectively. On using the result of § 5 we consequently have

$$U^5V^3 + U^4V^2\Sigma(\theta_{11}\theta_{12}\theta_{13})(\phi_{11}\phi_{12}\phi_{13}) + U^3V\Sigma \begin{vmatrix} \theta_{11}\theta_{12}\theta_{13} \\ \theta_{21}\theta_{22}\theta_{23} \end{vmatrix} \begin{vmatrix} \phi_{11}\phi_{12}\phi_{13} \\ \phi_{21}\phi_{22}\phi_{23} \end{vmatrix}$$

$$+ U^2\Sigma \begin{vmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{vmatrix} \begin{vmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{vmatrix}.$$

In like manner we may expand the given determinant according to descending powers of V, the result being

$$V^3U^5 + V^2\Sigma \begin{vmatrix} U & \cdot & \cdot & \cdot & \cdot & \theta_{11} \\ \cdot & U & \cdot & \cdot & \cdot & \theta_{21} \\ \cdot & \cdot & U & \cdot & \cdot & \theta_{31} \\ \cdot & \cdot & \cdot & U & \cdot & \theta_{41} \\ \cdot & \cdot & \cdot & \cdot & U & \theta_{51} \\ \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} & \phi_{51} & \cdot \end{vmatrix} + \ldots .$$

The existence of the two expansions leads of course to the comparison of the coefficients of like powers and thereby to such results as

$$\Sigma(\theta_{11}\ \theta_{12}\ \theta_{13})(\phi_{11}\ \phi_{12}\ \phi_{13}) = \Sigma(\theta_{11}\ \theta_{21}\ \cdots\ \theta_{51})(\phi_{11}\ \phi_{21}\ \cdots\ \phi_{51}),$$

$$\Sigma\begin{vmatrix}\theta_{11} & \theta_{12} & \theta_{13}\\ \theta_{21} & \theta_{22} & \theta_{23}\end{vmatrix}\cdot\begin{vmatrix}\phi_{11} & \phi_{12} & \phi_{13}\\ \phi_{21} & \phi_{22} & \phi_{23}\end{vmatrix} = \Sigma\begin{vmatrix}\theta_{11} & \theta_{21} & \cdots & \theta_{51}\\ \theta_{12} & \theta_{22} & \cdots & \theta_{52}\end{vmatrix}\cdot\begin{vmatrix}\phi_{11} & \phi_{21} & \cdots & \phi_{51}\\ \phi_{12} & \phi_{22} & \cdots & \phi_{52}\end{vmatrix}.$$

....

7. We come now naturally to the use made of bordered determinants by Arnaldi* in obtaining an expansion for *any determinant whatever*. His result is perhaps best described as *an aggregate of binary products, the first factor of each of which is a minor of a fixed coaxial minor of the given determinant, and the second factor a determinant got by bordering the complementary of the said fixed minor*.

For example, the given determinant being $|a_1\ b_2\ c_3\ d_4\ e_5|$, the fixed coaxial minor $|a_1\ b_2\ c_3|$, and consequently the minor to be bordered $|d_4\ e_5|$, we have

$$\begin{aligned}|a_1\ b_2\ c_3\ d_4\ e_5| = {} & |a_1\ b_2\ c_3|\,|d_4\ e_5|\\ & + \left\{|a_1\ b_2|\begin{vmatrix}\cdot & c_4 & c_5\\ d_3 & d_4 & d_5\\ e_3 & e_4 & e_5\end{vmatrix} - \ldots + |b_2\ c_3|\begin{vmatrix}\cdot & a_4 & a_5\\ d_1 & d_4 & d_5\\ e_1 & e_4 & e_5\end{vmatrix}\right\}\\ & + \left\{a_1\begin{vmatrix}\cdot & \cdot & b_4 & b_5\\ \cdot & \cdot & c_4 & c_5\\ d_2 & d_3 & d_4 & d_5\\ e_2 & e_3 & e_4 & e_5\end{vmatrix} - \ldots + c_3\begin{vmatrix}\cdot & \cdot & a_4 & a_5\\ \cdot & \cdot & b_4 & b_5\\ d_1 & d_2 & d_4 & d_5\\ e_1 & e_2 & e_4 & e_5\end{vmatrix}\right\},\end{aligned}$$

where it will be observed (1) that we begin with the highest minor of $|a_1\ b_2\ c_3|$, namely, $|a_1\ b_2\ c_3|$ itself, then give an assemblage of terms in which the first factors are the 2-line minors of $|a_1\ b_2\ c_3|$, and lastly an assemblage in which the first factors are the 1-line minors, *i.e.* the elements, (2) that we begin with a border of no breadth for $|d_4\ e_5|$, then affix a 1-line border, and lastly a 2-line border.

8. Any Arnaldi expansion is written with the greatest ease if what we have called the first factor be taken first, then

* Arnaldi, M. Sui determinanti orlati, e sullo sviluppo di un determinante per determinanti orlati. *Giornale di Mat.* xxxiv., pp. 209–214.

there be joined to it its complementary minor in the given determinant, and lastly the said complementary be altered so as to be of the bordered type required. For example, in the case used in the preceding paragraph, knowing that $|b_1\,c_3|$ is a first factor we take its cofactor $-|a_2\,d_4\,e_5|$, alter the latter so that the complementary of $|d_4\,e_5|$ in it has only zero elements, and we have the term

$$-|b_1\,c_3|\begin{vmatrix} \cdot & a_4 & a_5 \\ d_2 & d_4 & d_5 \\ e_2 & e_4 & e_5 \end{vmatrix}.$$

As a consequence of this observation we may formulate Arnaldi's equality as a theorem in words, namely, *If in any determinant P and Q be complementary coaxial minors, and if every possible minor of P be taken and multiplied by its complementary in the given determinant, the said complementary being so altered that the complementary of Q in it has only zero elements, the aggregate of the products so obtained is equal to the given determinant.*

9. Of course the roles played by the two coaxial minors may be reversed, and an alternative expansion obtained. For example, besides the equality of § 7, we have

$$|a_1b_2c_3d_4e_5| = |d_4e_5|\,|a_1b_2c_3| + \left\{ d_4\begin{vmatrix} a_1 & a_2 & a_3 & a_5 \\ b_1 & b_2 & b_3 & b_5 \\ c_1 & c_2 & c_3 & c_5 \\ e_1 & e_2 & e_3 & \cdot \end{vmatrix} - \ldots + e_5\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & \cdot \end{vmatrix}\right\}$$

$$+\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & \cdot & \cdot \\ e_1 & e_2 & e_3 & \cdot & \cdot \end{vmatrix},$$

where it will be observed that among the first factors we have to take cognisance of a minor of the 0^{th} degree, just as in the previous expansion we had among the second factors to reckon with a border of no breadth.

10. An interesting point in connection with these two expansions is the fact that not only are they equal as wholes, but the aggregate of each group of terms in the one is equal to the aggregate of the corresponding group in the other. In the case of the first group, consisting of one term, the equality is manifest: in the case of the last group it is almost so: we cannot, however, so readily see that

$$|a_1\ b_2| \begin{vmatrix} \cdot & c_4 & c_5 \\ d_3 & d_4 & d_5 \\ e_3 & e_4 & e_5 \end{vmatrix} - \ldots + |b_2\ c_3| \begin{vmatrix} \cdot & a_4 & a_5 \\ d_1 & d_4 & d_5 \\ e_1 & e_4 & e_5 \end{vmatrix}$$

$$= d_4 \begin{vmatrix} a_1 & a_2 & a_3 & a_5 \\ b_1 & b_2 & b_3 & b_5 \\ c_1 & c_2 & c_3 & c_5 \\ e_1 & e_2 & e_3 & \cdot \end{vmatrix} - \ldots + e_5 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & \cdot \end{vmatrix} .$$

Verification is effected on comparing the cofactor of such a minor as $|a_1\ b_2|$ on the one side with the partitioned cofactor of it on the other; or by expanding all the bordered determinants in the equality, when there arise 9×4 terms on the one side and 4×9 terms equal to them on the other.

11. Instead of proceeding to generalize this equality it is of more importance to inquire what is the common characteristic of the 72 terms of $|a_1\ b_2\ c_3\ d_4\ e_5|$ which each member of the equality represents. A little examination of either shows that every term involved in it contains only 2 elements belonging to $|a_1\ b_2\ c_3|$ and only one belonging to $|d_4\ e_5|$. Further, a glance at the full expansion of $|a_1\ b_2\ c_3\ d_4\ e_5|$ as given either in § 7 or § 9 suffices to make clear that none of the remaining terms resemble them in this. Not only so, but we also see that the said remaining terms contain either 3 elements from $|a_1\ b_2\ c_3|$ and 2 from $|d_4\ e_5|$, or else 1 from $|a_1\ b_2\ c_3|$ and 0 from $|d_4\ e_5|$. We are thus led to take a little wider survey, and so reach the general theorem: *If P be a p-line coaxial minor of $|a_{1n}|$ and Q be its complementary, then any term of $|a_{1n}|$ that has only $p-r$ elements belonging to P must have only $n-p-r$ elements belonging to Q.* For a term which has only $p-r$ elements belonging to P must have drawn for its supply upon $p-r$ rows of P and have still r prolongations of rows of P to draw upon. But if r elements be selected from these prolongations they must also be from

prolongations of r of the $n-p$ columns of Q; and thus from inside Q we are restricted to take $n-p-r$ elements.

For example, if p be taken equal to 5, the 40320 terms of an 8-line determinant are classifiable into

720	with	5	elements from P	and	3	from Q
10800	,,	4	,,	,,	2	,,
21600	,,	3	,,	,,	1	,,
7200	,,	2	,,	,,	0	,,

all other pairs of numbers being impossible.

12. In connection with this mode of classifying the terms of a determinant the general census theorem is: *The number of terms of an n-line determinant that contain only $p-r$ elements taken from a fixed p-line coaxial minor is*

$$p!\,(n-p)!\;C_{p,r}C_{n-p,r}.$$

As for proof, it will be sufficient guide to say that the number is got first in the form

$$(C_{p,r})^2\times(p-r)!\times C_{n-p,r}\,.\,r!\,(n-p)!$$

We note that the symmetry of the result in regard to p and $n-p$ is in keeping with the equality of § 10. Also, on giving all possible values to r and performing addition, the sum $n!$ ought to be obtained, and this will be found to be the case if we make use of the equality

$$(p+q)_s=(p_s)+(p_{s-1})(q_1)+(p_{s-2})(q_2)+\cdots.$$

13. As the second factor in every term of an Arnaldi expansion is a bordered determinant, it is natural to inquire what form the expansion would finally assume if for each bordered determinant there were substituted the equivalent expression given by the theorem dealt with in §§ 1—3.

For one thing it is clear that each term of this dilated expansion will consist of four factors, namely, one from each of the coaxial minors and one from each of the mutually complementary arrays. We thus learn that *a determinant can be expressed in terms of minors drawn from four mutually exclusive arrays, two of which are coaxial and complementary to one another.*

In the second place it is readily seen on examining a term of an Arnaldi expansion that if the coaxial minors, P and Q

say, be of the p^{th} and q^{th} orders respectively, *a minor of the* $(p-r)^{\text{th}}$ *order taken from P can only have from Q a companion of the* $(q-r)^{\text{th}}$ *order:* in fact, that the orders of the four factors of the terms in the first, second, third, ... groups are

$$\begin{array}{cccc} p, & q, & 0, & 0, \\ p-1, & q-1, & 1, & 1, \\ p-2, & q-2, & 2, & 2, \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

Further, the number of terms in the Arnaldi expansion being

$$1 + (C_{p,1})^2 + (C_{p,2})^2 + \cdots,$$

the number in that now reached is

$$1 + (C_{p,1})^2 (C_{q,1})^2 + (C_{p,2})^2 (C_{q,2})^2 + \ldots .$$

Lastly, a rule for the formation of any individual term can readily be framed on the model of that of § 8. Having fixed on any minor of P and any minor of Q we have only got to delete from $|a_{1n}|$ the rows and columns to which these minors belong, and in the minors of $|a_{1n}|$ thus resulting to change all the P and Q elements to zeros. For example, if the given determinant be $|a_1\ b_2\ c_3\ d_4\ e_5\ f_6\ g_7\ h_8|$, P be $|a_1\ b_2\ c_3\ d_4\ e_5|$, and the minors chosen from P and Q be $|b_3\ c_4\ d_5|$ and g_8 respectively, the prescribed deletion of rows and columns produces the minor $|a_1\ e_2\ f_6\ h_7|$, and the nullification of elements produces

$$\begin{vmatrix} \cdot & \cdot & a_6 & a_7 \\ \cdot & \cdot & e_6 & e_7 \\ f_1 & f_2 & \cdot & \cdot \\ h_1 & h_2 & \cdot & \cdot \end{vmatrix},$$

whence the term

$$|b_3\ c_4\ d_5| \cdot g_8 \cdot |a_6\ e_7| \cdot |f_1 h_2|.$$

Rondebosch, South Africa.

ON A PLANE CONFIGURATION OF POINTS AND LINES CONNECTED WITH THE GROUP OF 168 PLANE COLLINEATIONS.

By *Prof. W. Burnside.*

THE configuration in question is one of 21 points and 21 lines such that 4 of the points lie on each line and 4 of the lines pass through each point.

Calling the points $P_i Q_i R_i$ and the lines $p_i q_i r_i$ $(i=0,1,2,\ldots,6)$, the configuration is defined by the tables

	0	1	2	3	4	5	6
P_i	$1, 0, 0$	$0, 1, 1$	$0, 1, -1$	$1, 0, 1$	$-1, 0, 1$	$1, 1, 0$	$1, -1, 0$
Q_i	$0, 1, 0$	$\alpha, 1, -1$	$\alpha, 1, 1$	$-1, \alpha, 1$	$1, \alpha, 1$	$1, -1, \alpha$	$1, 1, \alpha$
R_i	$0, 0, 1$	$\alpha, -1, 1$	$\alpha, -1, -1$	$1, \alpha, -1$	$-1, \alpha, -1$	$-1, 1, \alpha$	$-1, -1, \alpha$
p_i	$x=0$	$y+z=0$	$y-z=0$	$z+x=0$	$z-x=0$	$x+y=0$	$x-y=0$
q_i	$y=0$	$2x+\alpha(y-z)=0$	$2x+\alpha(y+z)=0$	$2y+\alpha(z-x)=0$	$2y+\alpha(z+x)=0$	$2z+\alpha(x-y)=0$	$2z+\alpha(x+y)=0$
r_i	$z=0$	$2x-\alpha(y-z)=0$	$2x-\alpha(y+z)=0$	$2y-\alpha(z-x)=0$	$2y-\alpha(z+x)=0$	$2z-\alpha(x-y)=0$	$2z-\alpha(x+y)=0$

where α is one of the roots of the equation

$$\alpha^2 + \alpha + 2 = 0.$$

When this condition is satisfied, it is easily verified that the distribution of the points on the lines is given by

p_i	$Q_0R_0P_1P_2$	$Q_1R_1P_2P_0$	$Q_2R_2P_0P_1$	$Q_3R_3Q_0P_4$	$Q_4R_4Q_0P_3$	$Q_5R_5R_0P_6$	$Q_6R_6R_0P_5$
q_i	$R_0P_0P_3P_4$	$R_1P_1R_3P_6$	$R_2P_2Q_4Q_6$	$R_3P_3R_2R_5$	$R_4P_4Q_2Q_6$	$R_5P_5R_1R_4$	$R_6P_6Q_2Q_4$
r_i	$P_0Q_0P_5P_6$	$P_1Q_1R_4Q_5$	$P_2Q_2Q_3R_5$	$P_3Q_3Q_1R_6$	$P_4Q_4R_1Q_5$	$P_5Q_5R_2Q_3$	$P_6Q_6Q_1R_3$

The configuration is invariant for the 24 collineations which arise from

$$\begin{aligned} x' &= x; & x' &= -x; & x' &= y; \\ y' &= z; & y' &= z; & y' &= z; \\ z' &= -y; & z' &= y; & z' &= x; \end{aligned}$$

whatever α may be, the points P_0, Q_0, R_0; P_1, P_2, P_3, P_4, P_5, P_6; and the remaining twelve being separately permuted among themselves.

If S is the substitution whose coefficients are

$$\begin{array}{ccc} \frac{\alpha}{2} & \frac{\alpha}{2} & \cdot \\[6pt] -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\alpha} \\[6pt] -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{\alpha} \end{array}$$

and α is determined as above, then the powers of S are given by

$$S^2 \qquad\qquad\qquad S^3 \qquad\qquad\qquad S^4$$

$$\begin{array}{ccc} -\frac{1}{2} & -\frac{1}{\alpha} & -\frac{1}{2} \\[6pt] \frac{1}{2} & -\frac{1}{\alpha} & \frac{1}{2} \\[6pt] -\frac{\alpha}{2} & \cdot & \frac{\alpha}{2} \end{array} \qquad \begin{array}{ccc} -\frac{\alpha}{2} & \cdot & -\frac{\alpha}{2} \\[6pt] -\frac{1}{2} & \frac{1}{\alpha} & \frac{1}{2} \\[6pt] \frac{1}{2} & \frac{1}{\alpha} & -\frac{1}{2} \end{array} \qquad \begin{array}{ccc} -\frac{1}{\alpha} & -\frac{1}{2} & \frac{1}{2} \\[6pt] \cdot & \frac{\alpha}{2} & \frac{\alpha}{2} \\[6pt] -\frac{1}{\alpha} & \frac{1}{2} & -\frac{1}{2} \end{array}$$

$$S^5 \qquad\qquad\qquad S^6 \qquad\qquad\qquad S^7$$

$$\begin{array}{ccc} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\alpha} \\[6pt] -\frac{\alpha}{2} & -\frac{\alpha}{2} & \cdot \\[6pt] -\frac{1}{2} & \frac{1}{2} & \frac{1}{\alpha} \end{array} \qquad \begin{array}{ccc} \frac{1}{\alpha} & -\frac{1}{2} & -\frac{1}{2} \\[6pt] -\frac{1}{\alpha} & -\frac{1}{2} & -\frac{1}{2} \\[6pt] \cdot & \frac{\alpha}{2} & -\frac{\alpha}{2} \end{array} \qquad \begin{array}{ccc} 1 & \cdot & \cdot \\[6pt] \cdot & 1 & \cdot \\[6pt] \cdot & \cdot & 1 \end{array}.$$

The 21 points are permuted among themselves by S, the permutation being

$$(P_0 R_2 Q_5 Q_1 P_3 Q_4 P_6)\,(Q_0 Q_2 P_5 P_1 Q_3 R_4 R_6)\,(R_0 P_2 R_5 R_1 R_3 P_4 Q_6).$$

This collineation of order 7, taken with the preceding group of 24 collineations, gives rise to a group of 168 collineations. It is almost certain that this well-known group cannot be represented in a simpler form, since there are only eight distinct coefficients, viz. ± 1, $\pm\frac{1}{2}$, $\pm\frac{\alpha}{2}$ and $\pm\frac{1}{\alpha}$.

Among the collineations are included the 21 perspectives of order 2, for which the fixed points and lines are P_i, p_i; Q_i, q_i; R_i, r_i.

THE CONJOINT METHOD OF FACTORIZATION OF $N=xy$, BY ASCENT AND DESCENT UPON x, THE PROCESS BEING, IN EACH PART, ALIKE IN PRINCIPLE.*

By *D. Biddle, M.R.C.S.*

$N=xy$, where x is always prime and $< N^{\frac{1}{2}}$, whilst $y > N^{\frac{1}{2}}$ and either prime or composite, this being a matter for subsequent determination. Also $y - N^{\frac{1}{2}} > N^{\frac{1}{2}} - x$.

Since 2, 3 and 5 can readily be detected as divisors, let N be regarded as divested of these before the conjoint method is applied. N will then be an odd number, of form $6n \pm 1$, indistinguishable at first sight from a prime; and its factors will be of this form also. Sometimes the factors of N are further known to be of form $2\Delta m + 1$, where $\Delta > 3$. In other cases we also know that x lies above (or below) a certain limit, to which trial has already been carried. In all cases we are entitled to begin wherever our previous knowledge and a regard for safety allow.

The Method by Ascent. Let p_1 be a prime known to be not above x. It will be of form $6n \pm 1$, whether N be of form $6n+1$ or $6n-1$; it may also have a right to be of form $2\Delta m + 1$, but this must be determined beforehand according to known laws. Dividing N by p_1 we obtain Q_1, which may stand alone or with r_1 as remainder. If alone, $Q_1 = y$, and $N = p_1 Q_1$. Otherwise, $N = p_1 Q_1 + r_1$. We then deal with Q_1 and r_1 by a regard to the difference between p_1 and p_2 (the next higher possible factor). For we have, as our final objective, $(p_n - p_1) Q_1 - r_1 = p_n Q_n$, where $p_n = x$ and $Q_1 - Q_n = y$. But, if there be a remainder, as when $(p_2 - p_1) Q_1 - r_1 = p_2 Q_2 + r_2$, we go on without even entering Q_2 or r_2, but add $(p_3 - p_2) Q_1$ to $(p_2 - p_1) Q_1 - r_1$, already entered, giving as our next entry $(p_3 - p_1) Q_1 - r_1$; and thus we get on step by step to the final objective. Also, $p_n = x$, $Q_1 - Q_n = y$.

In the absence of a table of primes, it will be found advisable to proceed by a regard to known forms only, namely, $6n \pm 1$ or $2\Delta m + 1$. In the former the differences are 2 and 4 alternately, in the latter 2Δ uniformly.

The Method by Descent. Let P_1 equal the prime (of form $6n \pm 1$ or haply $2\Delta m + 1$) next below $N^{\frac{1}{2}}$. Dividing N by P_1,

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* The part by ascent originated in the present writer's Question 13,815, which was proposed in *The Educational Times* of March, 1898.

we obtain q_1, with or without remainder, so that $N=P_1q_1+r_1$, where r_1' may be zero; and we proceed as before, except that we go down instead of up with our divisors, and *add* r_1' in the first step, instead of subtracting it as before, because our formula is now $(P_1-P_n)q_1+r_1'=P_nq_n$, whence $P_n=x$, $q_1+q_n=y$.

There is no need to proceed to x by both methods. If there were any direct means of knowing that x was in the lower half, or the upper half, of the series of primes below $N^{\frac{1}{2}}$, the one particular method would suffice. But where the methods are used conjointly, two persons should be engaged, as errors are likely to arise when one person has to go up and down alternately. But the finder of x must at once announce the fact, in order to prevent waste of labour on the part of his fellow-worker, and so that the factorization of N may be completed by attention to y, treated as a subsidiary N', unless known to be a prime.

As an example of the conjoint method, let $N = 1843$.

$$p_1 = 7)\underline{1843} = N \qquad P_1 = 41)\underline{1843} = N$$
$$Q_1 = 263,\ 2 = r_1 \qquad q_1 = 44,\ 39 = r_1'$$

$4.263 = 1052$	4	4	$4.44 = 176$
$2.263 = 526$	$11)1050$	$37)\ 215$	$2.44 = 88$
	$13)1576$	$31)\ 479$	
	$17)2628$	$29)\ 567$	
	$p_n = 19)3154$	$23)\ 831$	
	$Q_n = 166$	$P_n = p_n = 19)1007$	
		$53 = q_n$	

$$N = 19\,(263 - 166) = 19\,(44 + 53) = 19 . 97.$$

N.B.—In the first method here, as N' becomes big, we might start afresh by dividing N by $17 = 108 + \underline{7}$, then multiplying by 2, $= 216 - \underline{7} = 209 = 19 . 11$, $N = 19\,(108 - 11)$.

It now behoves us to garner the facts obtainable in the foregoing conjoint processes:

(1) $P_n = p_n = x$.

(2) $Q_1 - Q_n = q_1 + q_n = y$.

(3) $p_1Q_1 + r_1 = P_1q_1 + r_1' = N$.

(4) $(p_n - p_1)\,Q_1 - r_1 = p_nQ_n$.

(5) $(P_1 - P_n)\,q_1 + r_1' = P_nq_n$.

(6) $Q_1 - y = Q_n$.

(7) $y - q_1 = q_n$.

The known quantities (that is to say, easily found) are p_1, Q_1, r_1; P_1, q_1, r_1'. The unknown quantities are p_n, Q_n; P_n, q_n. But, since $P_n = p_n = x$, there are in reality only three unknown quantities, namely, p_n, Q_n, q_n, and two of these may be eliminated as follows:

$$(8)\quad Q_n = Q_1 - N/p_n, \qquad (9)\quad q_n = N/p_n - q_1.$$

Therefore $$(10)\quad Q_1 > N/p_n > q_1,$$

but $$(11)\quad p_n(q_1 + q_n) = N,$$

whence $$(12)\quad p_n Q_1 > N > p_n q_n,$$

as well as $p_n q_1$.

Consequently, the *second* of the two methods is preferable, when *one only* is adopted, to be pursued to a successful termination, since the dividend here never exceeds N. In fact, the dividend cannot reach N after the start has been made. And, not only has the second method this advantage, but the divisor becomes continuously smaller as we proceed. Nevertheless, a combination of the two methods and their formulæ might be expected to lead to a direct method of factorization—one divested of the necessity for prolonged trial. A few further remarks may tend in this direction.

Let $p_n - p_1 = k$ and $P_1 - p_n = K$. Then K and k are both even, each being the difference between two primes (or at least odd numbers). Thus we can take

$$\tfrac{1}{2}(K + k) = \tfrac{1}{2}(P_1 - p_1) = \kappa \;\ldots\ldots\ldots\ldots\ (13)$$

and $$\tfrac{1}{2}(K - k) = \tfrac{1}{2}(P_1 + p_1) - p_n = \pm\lambda \ldots\ldots (14).$$

Now κ is a fixed value and easily found, but λ, though partly known, depends for its value upon an unknown. But we have

$$kQ_1 - r_1 = (k + p_1)Q_n, \quad Kq_1 + r_1' = (P_1 - K)q_n \ldots (15), (16),$$

whence

$$k(Q_1 - Q_n) = p_1 Q_n + r_1, \quad K(q_1 + q_n) = P_1 q_n - r_1' \ldots (17), (18).$$

By (2), $Q_1 - Q_n = q_1 + q_n$. Therefore, multiplying both sides of (17) by K, and both sides of (18) by k, we make the left sides of the two equal, and have the right sides also equal.

$$p_1 K Q_n + K r_1 = P_1 k q_n - k r_1' \ldots\ldots\ldots\ldots (19).$$

Reducing, by aid of (8) and (9), we obtain

$$(K + k)p_n = p_1 K + P_1 k \;\ldots\ldots\ldots\ldots (20),$$

and, by (13), (14),

$$2\kappa p_n = p(\kappa \pm \lambda) + P_1(\kappa \mp \lambda) \ldots\ldots\ldots\ldots (21),$$

where P_1, p_1, and κ are known. But neither (20) nor (21) fixes the value of p_n, since the former results in $p_n = p_1 + k$, and the latter results in $p_n = \frac{1}{2}(P_1 + p_1) \mp \lambda$, both which were already known from (14) and the remarks above (13). Yet, at first sight, (20) seemed to promise much, seeing that P_1 and p_1 are known primes, and K and k even numbers at an equal distance above and below κ, which is also known, being $\frac{1}{2}(P_1 - p_1)$.

Let $\rho = K - p_n$; then, having $p_n - k = p_1$, we also have

$$\rho + p_1 = K - k = 2\lambda \ldots\ldots\ldots\ldots (22).$$

This shows what a useful quantity, to search out, ρ is; so that a few of its characteristics here follow.

Since $K + p_n = P_1$, and since K is always even, but p_n always odd, K and p_n can never be equal, nor can ρ be otherwise than odd, $\pm(1, 3, 5, \&c.)$. The average meeting-place of K and p_n may be said to be at $\frac{1}{2}(P_1 \pm 1)$, and ρ is the actual distance apart of K and p_n, which is never less than 1 (taken neutrally) nor ever greater than $P_1 - 2p_1$. When $\rho = +1$, $K = \frac{1}{2}(P_1 + 1)$, and $p_n = \frac{1}{2}(P_1 - 1)$; when $\rho = -1$, $K = \frac{1}{2}(P_1 - 1)$, and $p_n = \frac{1}{2}(P + 1)$; when $\rho = \pm 3$, $K = \frac{1}{2}(P_1 \pm 3)$, and $p_n = \frac{1}{2}(P_1 \mp 3)$; and so on. The alternative sign is governed by the necessity of making K even and p_n odd. This is a very important matter. There are other important considerations. Thus, it is not enough that K be even and p_n odd, but p_n must also be a prime of form $6n \pm 1$ (and mayhap also of form $2\Delta m + 1$). Moreover, the form of P_1, as $4m \pm 1$, determines on which sides of $\frac{1}{2}P_1$ respectively the odd and even numbers alternately lie, or, in other words, which of the alternate neutral values of ρ that are possible in the given instance are positive and which negative. Thus, when P_1 is of form $4m + 1$, as in our example, we know that ρ belongs to the series -1, $+3$, -5, $+7$, &c. On the contrary, when P_1 is of form $4m - 1$, ρ belongs to the series $+1$, -3, $+5$, -7, &c. That is to say, the neutral values of ρ that are of the same form as P_1 are always negative, those intervening being positive. We also have

$$2p_n = P_1 - \rho \ldots\ldots\ldots\ldots (23),$$

$$2K = P_1 + \rho \ldots\ldots\ldots\ldots (24),$$

$$2k = P_1 - 2p_1 - \rho \ldots\ldots\ldots\ldots (25).$$

Moreover, $\quad K - k = 2\lambda = p_1 + \rho$, as given in (22).

$$p_n + \lambda = \frac{1}{2}(P_1 + p_1) \ldots\ldots\ldots\ldots (26),$$

$$p_n - \lambda = \frac{1}{2}(P_1 - p_1) - \rho \ldots\ldots\ldots\ldots (27).$$

Now, not only are K and k unknown separately, though $K+k=P_1-p_1$, but we also have Q_n and q_n unknown separately, although, by (2), $Q_n+q_n=Q_1-q_1$.

Reverting to (15), (16), and observing that p_1+k in the former equals P_1-K in the latter, let us take

$$(Kq_1+r_1')\,Q_n=(kQ_1-r_1)\,q_n \ldots\ldots\ldots\ldots(28),$$

in which all four of these unknown quantities appear. This results in

$$Q_1k-r_1=Q_n(p_1+k) \ldots\ldots\ldots\ldots\ldots(29),$$

which enunciates the very law on which the first of the conjoined methods is based. Reversing the transforming process, we obtain, from (28),

$$Kq_1+r_1'=q_n(P_1-K) \ldots\ldots\ldots\ldots\ldots(30),$$

which gives the law of the second method.

In order to test the possibility of discovering the separate values of the four quantities, K, k, Q_n, q_n, which at present are only known in pairs, let us treat them as follows: We have

$$K+k=P_1-p_1=a,\quad Q_n+q_n=Q_1-q_1=b,$$

whence

$$\underset{\text{(i)}}{KQ_n}+\underset{\text{(ii)}}{kQ_n}+\underset{\text{(iii)}}{Kq_n}+\underset{\text{(iv)}}{kq_n}=(P_1-p_1)(Q_1-q_1)=ab=M \ldots(31).$$

Bracketing together (i) and (iv) in one set, and (ii) and (iii) in another, we have reason to believe that on the average the two sums will be nearly equal, sometimes one, and sometimes the other, being the greater. But the difference between the sets will invariably be

$$\{\text{(i)}+\text{(iv)}\}\sim\{\text{(ii)}+\text{(iii)}\}=(K\sim k)(Q_n-q_n) \ldots(32).$$

We can find the difference between the terms in the first set as follows: (i) + (iii) $=bK$, (iii) + (iv) $=aq_n$. Therefore

$$\text{(i)}-\text{(iv)}=KQ_n-kq_n=bK-aq_n \ldots\ldots\ldots(33).$$

Similarly, the difference between the terms in the second set is found as follows: (ii) + (iv) $=bk$, (iii) + (iv) $=aq_n$. Thus

$$\text{(ii)}-\text{(iii)}=kQ_n-Kq_n=bk-aq_n \ldots\ldots\ldots(34).$$

Consequently

$$\{\text{(i)}-\text{(iv)}\}\sim\{\text{(ii)}-\text{(iii)}\}=b(K\sim k) \ldots\ldots(35),$$

and

$$\{\text{(i)}-\text{(iv)}\}+\{\text{(ii)}-\text{(iii)}\}=a(Q_n-q_n)=a(b-2q_n) \ldots(36),$$

which is the difference between the two first and the two last terms in (31).

Let us next call in the aid of the known sums, as follows:

$$(P_1 - p_1)(Q_n + q_n) = (Q_1 - q_1)(K + k) = ab \ldots\ldots (37),$$

or $$P_1 Q_n + P_1 q_n - p_1 Q_n - p_1 q_n = Q_1 K + Q_1 k - q_1 K - q_1 k \ldots (38).$$

The right of this equation begins much larger than the left, but is soon balanced. Thus, putting $(a-k)$ for K, and $(b-q_n)$ for Q_n, we have

$$Q_1 K - P_1 Q_n = Q_1 (a-k) - P_1 (b - q_n) = Q_1 a + P_1 q_n - Q_1 k - P_1 b.$$

Here $Q_1 a - P_1 b$ reduces to $r_1 - r_1'$, which may or may not be positive. In our example it is negative, showing that the advantage, midway on either side, is no longer with the right side, for $P_1 q_n$ and $Q_1 k$, the respective second terms of (38), are now disposed of, as well as the first terms, of which the difference has been taken. Thus we have

$$P_1 Q_n + P_1 q_n = Q_1 K + Q_1 k - (r_1 - r_1') \ldots\ldots\ldots (39),$$

and $$p_1 Q_n + p_1 q_n = q_1 K + q_1 k - (r_1 - r_1') \ldots\ldots\ldots (40).$$

Now, (35) and (36) contain the essential ingredients for a direct method of factorization, especially as the coefficients of the differences of each unknown pair are known, as well as the sum of each pair.

A mode of partitioning the known $ab\,(=M)$ would greatly help, observing that the product of (i) and (iv) equals the product of (ii) and (iii); also that the difference of the products of (i) and (iii), and of (ii) and (iv), equals $(K^2 - k^2)\,Q_n q_n$, of which $K + k$ is a known factor; again, (i) and (ii) have a common factor, which is large, Q_n; whilst (iii) and (iv) have a common factor not so large, q_n; (i) + (iii) is a small multiple of a known quantity, $Q_1 - q_1$; and (ii) + (iv) is a still smaller multiple of the same known quantity. Taking our original example

$$M = 34 \,.\, 219 = 3652 + 1992 + 1166 + 636 = 7446.$$

$$[(i) + (iii)] - [(ii) + (iv)] = 4818 - 2628 = 2190$$

$$= (Q_n + q_n)\,(K - k) = (219)\,(10).$$

N.B. Taking our original example as an illustration of the partition of M, we make a double distribution. Thus

$$34 \,.\, 219 = 22 \,.\, 219 + 12 \,.\, 219 = 34 \,.\, 166 + 34 \,.\, 53,$$

$$\left\{\begin{matrix} (i) + (iii) \\ + \quad + \\ (ii) + (iv) \end{matrix}\right\} \begin{matrix} (i) + (iii) \quad (ii) + (iv) \quad (i) + (ii) \quad (iii) + (iv) \\ \\ (i) = 22 \,.\, 166, \ (ii) = 12 \,.\, 166, \ (iii) = 22 \,.\, 53, \ (iv) = 12 \,.\, 53. \end{matrix}$$

The accompanying diagram sets forth some of the relations of the principal quantities in a known instance, namely, our original example.

Let $AD = P_1$, $AE = k$, $EB = K$, $BG = p_1$. Draw EF parallel to AD and GH to AB. Also join AC and EC. Through O, where GH and EC intersect, draw IJ parallel to EF, also DT through the same point. Moreover, through R, the intersection of IJ and AC, draw UV parallel to AB. Then we have the rectangles $FI = HB$, $JA = FA + HB = UB$. This visibly sets forth (20), and $IR = p_n = x$. Thus, since $OI = p_1$, we have $OR = AE = k$. Then $P_1 - p_1 : AI = P_1 : k + ET$ and $p_n : AI = P_1 : P_1 - p_1 = SE : K$, whence $k + ET = p_n = k + p_1$, $ET = p_1$, and $SE = P_1 K/(P_1 - p_1)$. This value of SE is that of the second term to the right in (20) after division by $K + k$. Consequently SZ similarly represents the first term (so divided) or $SZ = p_1 K/(P_1 - p_1)$. Therefore $LM = p_1$. Now, whenever O lies on the parallel $G\gamma$, $E'T'$, marked off by CE' and DT', will always equal p_1, but the relative lengths of EI and IT will vary. For $P_1 : K = P_1 - p_1 : BI\,(=GO)$, whence $EI = p_1 K/P_1$ and $IT = p_1(P_1 - K)/P_1$. Therefore $EI : IT = K : p_n = EB : AT$.

Also the rectangles αV, αB are the identical values of $p_1 K$ and $P_1 k$ in (20). Again, drawing γC, we have the common diagonal of three squares, K^2, k^2 and $(K + k)^2$, the two additional positions $UF = ZG = Kk$ being also clearly defined. The join TU passes through H, and intersects γC at right angles.

We next come to a law of universal application.

If p_1, p_2, p_3, &c., be successive primes from 7 onwards, and N, a composite, be divided by each leaving remainders r_1, r_2, r_3, &c., the quotients being Q_1, Q_2, Q_3, &c., and $d_1 = p_2 - p_1$, $d_2 = p_3 - p_2$, &c, then

$$p_1(Q_1 - Q_2) = d_1 Q_2 + r_2 - r_1 \ldots\ldots\ldots\ldots\ldots (41),$$

$$p_2(Q_1 - Q_2) = d_1 Q_1 + r_2 - r_1 \ldots\ldots\ldots\ldots\ldots (42).$$

Similarly $$p_2(Q_2 - Q_3) = d_2 Q_3 + r_3 - r_2 \ldots\ldots\ldots\ldots\ldots (43),$$

and $$p_3(Q_2 - Q_3) = d_2 Q_2 + r_3 - r_2 \ldots\ldots\ldots\ldots\ldots (44).$$

Moreover, if D be the difference between primes more remote, say p_1 and p_3, we have

$$p_1(Q_1 - Q_3) = D.Q_3 + r_3 - r_1 \ldots\ldots\ldots\ldots\ldots (45),$$

and $$p_3(Q_1 - Q_3) = D.Q_1 + r_3 - r_1 \ldots\ldots\ldots\ldots\ldots (46).$$

Consequently, taking $D = k$ and $N = p_n y$, we have

$$p_1(Q_1 - y) = ky - r_1 \ldots\ldots\ldots\ldots\ldots\ldots (47),$$

$$p_n(Q_1 - y) = kQ_1 - r_1 \ldots\ldots\ldots\ldots\ldots\ldots (48),$$

where $Q_1 - y = Q_n$.

By transforming (41), (42) somewhat, we obtain

$$kQ_1 - r_1 \equiv 0 \pmod{p_n} \quad \ldots\ldots\ldots\ldots\ldots (49),$$

$$(k - d_1)Q_2 - r_2 \equiv 0 \pmod{p_n} \quad \ldots\ldots\ldots\ldots\ldots (50),$$

both these being established by our first method.

Now, if we denote $(k - d_1)$ by k', k becomes $k' + d_1$; and we can take (50) from (49), and (49) from (50), the former in terms of k, the latter in terms of k', getting, as added quantities, those on the right in (41) and (42) respectively. Thus

$$k(Q_1 - Q_2) + d_1 Q_2 + r_2 - r_1 \equiv 0 \pmod{p_n} \ldots\ldots (51),$$

$$k'(Q_1 - Q_2) + d_1 Q_1 + r_2 - r_1 \equiv 0 \pmod{p_n} \ldots\ldots (52).$$

Dividing by $(Q_1 - Q_2)$, we obtain, by (41), (42), what are the actual values of p_n, namely, $k + p_1 = k' + p_2$, which are identical. But, taking (51), let us suppose that $Q_1 - Q_2$ is a composite quantity, as is frequently, not to say generally, the case. Then it is possible to divide by a prime factor of $Q_1 - Q_2$, leaving the congruence still holding good. It is also possible to augment the multiple of $k + p_1$, which forms the left of the congruence, by another small multiple of the same, until the coefficient of k is itself a multiple of $k + p_1$, that is, of p_n; and such final multiple being numerically divided by p_n, shall leave a quotient, consisting of a small multiple of $k + p_1$ of the identical value of the coefficient of k in the "final multiple" above mentioned.

Take $N = 6049$ as an example, where

$N = 7.864 + 1 = 11.549 + 10$, $Q_1 - Q_2 = 864 - 549 = 315$, $d_1 = 4$, and $d_1 Q_2 + r_2 - r_1 = 2205$. Thus we have

$$315k + 2205 \equiv 0 \pmod{p_n}.$$

Dividing by 5, $\quad 63k + \ 441 \equiv 0 \pmod{p_n}$.

Adding $6(k + 7)$, $\quad 69k + \ 483 \equiv 0 \pmod{p_n}$.

Dividing by 23, $\quad 3k + \ 21 \equiv 0 \pmod{p_n}$.

Taking $k = 16$, $\quad 3k + \ 21 = 69 = 3p_n = 3(16 + 7)$.

Therefore $N = 23.263$. The values fit in like a Chinese puzzle.

Following is a table giving the values of the similar quantities in five other examples:

N.B. $p_1 = 7$, $d_1 = 4$, in all. f is the factor (like 5), m is the added multiple.

N	Q_1	Q_2	r_1	r_2	f	m	k	p_n
559	79	50	6	9	—	10	6	13
589	84	53	1	6	—	7	12	19
1843	263	167	2	6	3	6	12	12
11771	1681	1070	4	1	13	32	72	79
150809	21544	13709	1	10	5	106	232	239

Unfortunately, the process just described fails to determine either k or p_n, since suitable pairs of values are yielded by other added multiples of $(k+7)$. Thus, if 2 instead of 6 were the multiple, in the instance above given at length, we should get as the result $65k+455$, and 13 might be taken as the dividing p_n, yielding $5k+35=65$, and $k=6$, which, added to 7, gives 13 also.

It is also clear that we should fare no better by utilizing (52) instead of (51). Moreover, success is not achieved by taking either their difference or their sum. The same stultifying episode, with but slight variations, marks every such effort.

By reverting, however, to (41), (42), we find a very remarkable coincidence in reference to our second example of $N=6049$. Here $Q_1=864$, $r_1=1$, $Q_2=549$, $r_2=10$, and $d_1=4$. Taking the right hand of (41) and (42), we have respectively 2205 and 3465. Dividing these by 5, and subtracting the smaller from the greater, we have 252, $\equiv -r_1$ (mod 23, $=x$), and $\equiv -(r_1+r_2)$ (mod 263, $=y$).

Taking the other examples we generally find the same kind of coincidence with a slight variation, and, perhaps, applying to only one of the factors of N.

Thus, in $N=559$, we arrive at the difference 116; dividing by 2 we have 58, $\equiv r_1$ (mod $13=x$), $\equiv (r_1+r_2)$ (mod 43, $=y$).

In $N=589$ we arrive at 124, $\equiv 0$ (mod 31, $=y$), but find nothing pointing directly to x.

In $N=1843$ we arrive at 384, $\equiv (r_2-r_1)$ (mod 19, $=x$), and $\equiv -(r_2-r_1)$ (mod 97, $=y$).

In $N=11771$ we arrive at 2444, $\equiv -(r_1+r_2)$ (mod 79, $=x$), but nothing apparently of value as regards y.

In $N=150809$ we arrive at 31340, which yields little that is satisfactory, except that $31340\equiv(3r_2+r_1)$ (mod 239, $=x$).

On the whole, therefore, I am inclined to recommend an adherence to the conjoint method as originally devised. It is exceedingly simple, and not very laborious, except in regard to numbers beyond needed calculation.

N.B. Whenever a considerable number of calculators are at hand to conduct the factorization of a large number, the range between p_1 and P_1 can be divided into an equal number of parts of nearly equal range, one for each calculator; and a number, of form $6n \pm 1$ (or $2\Delta m + 1$, where known), should mark the starting-point in each case, the advance being upward or downward according to the position of the particular part-range, in reference to $N^{\frac{1}{2}}$, as belonging to the first or the second method.

There is, however, one suggestion that may be made for lightening the burden of the increasing dividend in the first method. It is that division of it should be made according to the following series of formulæ:

$$(p_2 - p_1) Q_1 - r_1 = p_2 Q_2' + r_2' \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)},$$
$$p_2 (Q_1 - Q_2') - r_2' = p_1 Q_1 + r_1 = N,$$
$$(p_3 - p_2)(Q_1 - Q_2') + r_2' = p_3 Q_3' + r_3' \quad \ldots\ldots\ldots \text{(ii)},$$
$$p_3 (Q_1 - Q_2' - Q_3') - r_3' = N,$$
$$\&c. \qquad\qquad \&c.$$

We begin as before by dividing N by p_1, and multiplying the quotient by $(p_2 - p_1)$, also subtracting r_1. Next, however, we divide the amount, so arrived at, by p_2, entering both quotient and remainder, namely, Q_2' and r_2'. The lines next after (i) and next after (ii) in the above statement are merely explanatory. But they show how the quantity dealt with is reduced by quotient after quotient, whilst the multiplier increases. Moreover, the remainder, after the first step, is added, as in (ii), and not again subtracted.

When no remainder follows division, the process is at an end, p_n being arrived at as the last divisor, and y is given as follows

$$(Q_1 - Q_2' - \ldots - Q_n') = y,$$

where Q_n' is the quotient without remainder.

A few examples will show the process better than any explanation. The gradually diminishing quantity, formed by subtraction of subsidiary quotients, is indicated by brackets { }, forming a "cap."

$p_1 = 7$) $559 = N_1$	$p_1 = 7$) $589 = N_2$	$p_1 = 7$) $6049 = N_3$
$Q_1 = 79 + 6 = r_1$	$Q_1 = 84 + 1 = r_1$	$Q_1 = 864 + 1 = r_1$
$4 = p_2 - p_1$	$4 = p_2 - p_1$	$4 = p_2 - p_1$
$p_2 = 11$) 310	$p_2 = 11$) 335	$p_2 = 11$) 3455
$Q_2' = 28 + 2 = r_2'$	$Q_2' = 30 + 5 = r_2'$	$Q_2' = 314 + 1 = r_2'$

$$\begin{array}{r}
\{51\} \\
2 = p_n - p_2 \\
p_n = 13\ \overline{)\ 104} \\
Q_n' = 8 \\
\{43\} \\
N_1 = 13.43
\end{array}$$

$$\begin{array}{r}
\{54\} \\
2 = p_3 - p_2 \\
p_3 = 13\ \overline{)\ 113} \\
Q_3' = 8 + \underline{9} = r_3' \\
\{46\} \\
4 = p_4 - p_3 \\
p_4 = 17\ \overline{)\ 193} \\
Q_4' = 11 + \underline{6} = r_4' \\
\{35\} \\
2 = p_n - p_4 \\
p_n = 19\ \overline{)\ 76} \\
Q_n'\ 4 \\
\{31\} \\
N_2 = 19.31
\end{array}$$

$$\begin{array}{r}
\{550\} \\
2 = p_3 - p_2 \\
p_3 = 13\ \overline{)\ 1101} \\
Q_3' = 84 + \underline{9} = r_3' \\
\{466\} \\
4 = p_4 - p_3 \\
p_4 = 17\ \overline{)\ 1873} \\
110 + \underline{3} = r_4' \\
\{356\} \\
2 = p_5 - p_4 \\
p_5 = 19\ \overline{)\ 715} \\
37 + \underline{12} = r_5 \\
\{319\} \\
4 = p_n - p_5 \\
p_n = 23\ \overline{)\ 1288} \\
Q_n = 56 \\
\{263\} \\
N_3 = 23.263
\end{array}$$

By a similar, though different, principle, it is possible to transform also the method of descent. Thus we have

$$(P_1 - P_2)\,q_1 + r_1' = P_2 q_2' + r_2'' \ldots\ldots\ldots\ldots\ldots\text{(iii)},$$
$$P_2(q_1 + q_2') + r_2'' = P_1 q_1 + r_1' = N,$$
$$(P_2 - P_3)(q_1 + q_2') + r_2'' = P_3 q_3' + r_3'' \ldots\ldots\text{(iv)},$$
$$P_3(q_1 + q_2 + q_3) + r_3'' = N,$$
&c. &c.

We begin, as before, by dividing N by P_1, and multiplying the quotient by $(P_1 - P_2)$, also adding r_1'. Next, we divide the said amount by P_2, entering both quotient and remainder. But, instead of subtracting, we now add q_2' to q_1, putting the cap over the sum instead of the difference. Moreover, we continue to add the previous remainder to the next product, as shown in (iv). When no remainder follows division, the process, as before, is at an end, P_n being arrived at as the last divisor, and q_n being added to the previous sum of quotients, revealing y without division of N, for

$$(q_1 + q_2 + q_3 + \ldots + q_n) = y.$$

In some cases these transformed conjoint methods may be preferred to the original ones, for though more lines are used, the number of figures is often less, and the mind being more engaged, and there being less copying of the same figures, there is less fatigue and less liability to error. Each computer must choose. Two or three examples of the transformed method of descent will doubtless be welcome with the rest:

$$\begin{array}{rl}
P_1 = 23\ \underline{)\ 559} = N_1 & \\
q_1 = 24 & 7 = r_1' \\
\underline{P_1 - P_2 = 4} & \\
P_2 = 19\ \underline{)\ 103} & \\
q_2 = 5 & 8 = r_2'' \\
q_1 + q_2 = \{29\} & \\
\underline{P_2 - P_3 = 2} & \\
P_3 = 17\ \underline{)\ 66} & \\
q_3 = 3 & 15 = r_3'' \\
q_1 + \ldots + q_3 = \{32\} & \\
\underline{P_3 - P_n = 4} & \\
P_n = 13\ \underline{)\ 143} & \\
q_n = 11 & \\
q_1 + \ldots + q_n = \{43\} & \\
 & \\
N_1 = 13.43 &
\end{array}$$

$$\begin{array}{rl}
P_1 = 89\ \underline{)\ 9047} = N_2 & \\
q_1 = 101 & 58 = r_1' \\
\underline{P_1 - P_2 = 6} & \\
P_2 = 83\ \underline{)\ 664} & \\
q_n = 8 & \\
q_1 + q_n = \{109\} & \\
N_2 = 83.109^{*} &
\end{array}$$

$$\begin{array}{rl}
P_1 = 107\ \underline{)\ 11771} = N_3 & \\
q_1 = 110 & 1 = r_1' \\
\underline{P_1 - P_2 = 4} & \\
P_2 = 103\ \underline{)\ 441} & \\
q_2 = 4 & 29 = r_2'' \\
q_1 + q_2 = \{114\} & \\
\underline{P_2 - P_3 = 2} & \\
P_3 = 101\ \underline{)\ 257} & \\
q_3 = 2 & 55 = r_3'' \\
q_1 + \ldots + q_3 = \{116\} & \\
\underline{P_3 - P_4 = 4} & \\
P_n = 97\ \underline{)\ 519} & \\
q_n = 5 & 34 = r_4'' \\
q_1 + \ldots + q_n = \{121\} & \\
\underline{P_4 - P_5 = 8} & \\
P_5 = 89\ \underline{)\ 1002} & \\
q_5 = 11 & 23 = r_5'' \\
q_1 + \ldots + q_5 = \{132\} & \\
\underline{P_5 - P_6 = 6} & \\
P_6 = 83\ \underline{)\ 815} & \\
q_6 = 9 & 68 = r_6'' \\
q_1 + \ldots + q_6 = \{141\} & \\
\underline{P_6 - P_n = 4} & \\
P_n = 79\ \underline{)\ 632} & \\
q_n = 8 & \\
q_1 + \ldots + q_n = \{149\} & \\
N_3 = 79.149 &
\end{array}$$

In actual practice it will be found possible to avoid all symbols, indicative of values, except the caps.

* N_2 is really resolved by the original method of descent; but this could not be foreseen at the start.

ON THE PROBABLE REGULARITY OF A RANDOM DISTRIBUTION OF POINTS.

By *Prof. W. Burnside.*

THE problem it is proposed to consider is the following: A number of points are distributed at random on the perimeter of a circle (or any other closed curve): what is the probability that no gap (arcual distance) between two consecutive points shall exceed a given length l?

The number of points is represented by n, and the perimeter of the curve is taken to be unity. It is obvious, by considering the regular distribution of the points, that some gap must be equal to or greater than $1/n$; or, which is the same thing, that the whole of the points must lie on some continuous portion of the perimeter whose length does not exceed $1-1/n$.

The probability will first be determined that the whole of the points lie on a continuous portion of the perimeter of length $1-x$, so that there is at least one gap whose length is equal to or greater than x.

The probability that all the points lie on a continuous arc of the perimeter of length $1-x$, starting from the r^{th} point and taken clockwise round the perimeter, is $(1-x)^{n-1}$. If $1-x<1/2$, this distribution and the corresponding one starting from any other one of the points are clearly mutually exclusive. Moreover, they cover all possibilities. Hence if $x>1/2$, the probability that there is at least one gap equal to or exceeding x is $n(1-x)^{n-1}$.

If $x<\frac{1}{2}$, the two distributions in which all the points lie on an arc $1-x$, reckoned clockwise from the r^{th}, and on an arc length $1-x$, reckoned clockwise from the s^{th}, are not all distinct. If $1/2>x>1/3$, these two arcs necessarily overlap, and may have in common either one continuous piece of arc or two. When they have one common part the distributions are distinct as before. When they have two parts in common, and all the points lie in these two parts, the distribution has been counted (in the previous formula) both as arising from the r^{th} point and from the s^{th}. Now when the two arcs have two parts in common, the sum of the lengths of the parts is $2(1-x)-1=1-2x$. Assigning the r^{th} point, the probability that the remaining $n-1$ points lie in a length $1-2x$ is $(1-2x)^{n-1}$. This in the previous formula has been taken twice over, and the same applies to each pair of the n points.

Hence when $1/2 > x > 1/3$, the probability that there is at least one gap equal to or exceeding x is

$$n(1-x)^{n-1} - \frac{n(n-1)}{1.2}(1-2x)^{n-1}.$$

If $1/3 > x > 1/4$, the clockwise arcs starting from the r^{th}, s^{th} and t^{th} points may overlap in three separate continuous pieces, and when they do the sum of the lengths of these pieces is $3(1-x) - 2 = 1 - 3x$. Assigning the r^{th} point, the chance that the remaining points lie in these overlapping pieces is $(1-3x)^{n-1}$. In the previous formula this has been reckoned three times positively in the first term and three times negatively in the second. Hence for each set of three points such a term must be added, in this case, to the previous formula. This process may be continued, and the general result is as follows: When n points are distributed at random on the perimeter (of unit length) of a closed curve, the probability that the arcual distance between some two consecutive points shall be equal to or exceed x, where $\{1/m > x > 1/(m+1)\}$, is

$$n(1-x)^{n-1} - \frac{n(n-1)}{1.2}(1-2x)^{n-1} + \ldots$$
$$+ (-1)^{m+1}\frac{n(n-1)\ldots(n-m+1)}{m!}(1-mx)^{n-1}.$$

When x is less than $1/n$, the probability is unity. Assuming n large, the sum of the above series is very nearly unity, so long as x is a moderate multiple of $1/n$. When x is equal to $\log n/n$, the sum is approximately $1 - e^{-1}$ $(= .632)$, and for larger values of x the sum diminishes rapidly towards zero. These results indicate the degree of regularity that may be expected in the distribution of the points.

ON NIELSEN'S FUNCTIONAL EQUATIONS.

By *G. N. Watson.*

1. THE system of simultaneous equations

$$\begin{cases} F_{\nu-1}(z) - F_{\nu+1}(z) - 2\dfrac{dF_\nu(z)}{dz} = \dfrac{2}{z} f_\nu(z) \quad \ldots\ldots\ldots(1), \\ F_{\nu-1}(z) + F_{\nu+1}(z) - \dfrac{2\nu}{z} F_\nu(z) = \dfrac{2}{z} g_\nu(z) \quad \ldots\ldots\ldots(2), \end{cases}$$

(which are a generalisation of the recurrence formulæ satisfied by Bessel functions), where $f_\nu(z)$ and $g_\nu(z)$ are arbitrary given functions of the two variables z and ν, has been studied by Nielsen.* It has been shewn that, if the pair of equations are consistent, it is necessary that $f_\nu(z)$ and $g_\nu(z)$ should not be completely arbitrary, but they must be connected by the relation

$$g_{\nu-1}(z) - g_{\nu-1}(z) - 2g_\nu'(z) = f_{\nu-1}(z) + f_{\nu+1}(z) - (2\nu/z) f_\nu(z),$$

the prime denoting a differentiation with respect to z.

When this necessary condition is satisfied, Nielsen does not succeed in constructing the solution of the given system. I propose to shew that, when the condition is satisfied, it is possible to reduce the system to a pair of independent linear difference equations of the first order; these equations are of the simplest type and, if a certain convergence-condition is satisfied, it is possible to express their solutions in terms of integrals.

2. If we write

$$f_\nu(z) + g_\nu(z) \equiv \alpha_\nu(z), \quad f_\nu(z) - g_\nu(z) \equiv \beta_\nu(z),$$

and if we denote the operator $z(d/dz)$ by the symbol ϑ, it is evident that the given system is precisely equivalent to the system

$$\begin{cases} (\vartheta + \nu) F_\nu(z) = \quad zF_{\nu-1}(z) - \alpha_\nu(z) \quad \ldots\ldots\ldots\ldots(3), \\ (\vartheta - \nu) F_\nu(z) = -zF_{\nu+1}(z) - \beta_\nu(z) \quad \ldots\ldots\ldots\ldots(4). \end{cases}$$

* *Ann. di Mat.* (3), vi. (1901), pp. 51–59.

We proceed to investigate this system of equations.
It is evident that

$$(\vartheta^2-\nu^2) F_\nu(z) = (\vartheta-\nu)\left[zF_{\nu-1}(z)-\alpha_\nu(z)\right]$$
$$= z(\vartheta-\nu+1) F_{\nu-1}(z) - (\vartheta-\nu)\alpha_\nu(z)$$
$$= z\{-zF_\nu(z)-\beta_{\nu-1}(z)\} - (\vartheta-\nu)\alpha_\nu(z),$$

by using first (3) and then (4).

This result may be written in the form

$$z^2\frac{d^2F_\nu(z)}{dz^2}+z\frac{dF_\nu(z)}{dz}+(z^2-\nu^2)F_\nu(z)=z\varpi_\nu(z),$$

where $\qquad \varpi_\nu(z) + (\nu/z)\alpha_\nu(z) - \alpha_\nu'(z) - \beta_{\nu-1}(z) \quad \ldots\ldots\ldots(5).$

Also, by using first (4) and then (3), we have

$$(\vartheta^2-\nu^2) F_\nu(z) = (\vartheta+\nu)\left[-zF_{\nu+1}(z)-\beta_\nu(z)\right]$$
$$= -z(\vartheta+\nu+1) F_{\nu+1}(z) - (\vartheta+\nu)\beta_\nu(z)$$
$$= -z\{zF_\nu(z)-\alpha_{\nu+1}(z)\} - (\vartheta+\nu)\beta_\nu(z),$$

whence we see that $\varpi_\nu(z)$ must also be defined by the equation

$$\varpi_\nu(z) = -(\nu/z)\beta_\nu(z) - \beta_\nu'(z) + \alpha_{\nu+1}(z) \ldots\ldots\ldots(6).$$

A comparison of the values of $\varpi_\nu(z)$ given by (5) and (6) shews that

$$(\nu/z)\{\alpha_\nu(z)+\beta_\nu(z)\} - \{\alpha_\nu'(z)-\beta_\nu'(z)\} - \beta_{\nu-1}(z) - \alpha_{\nu+1}(z) = 0,$$

that is to say

$$2(\nu/z)f_\nu(z) - 2g_\nu'(z) - \{f_{\nu-1}(z) - g_{\nu-1}(z)\}$$
$$- \{f_{\nu+1}(z) + g_{\nu+1}(z)\} = 0 \ldots(7),$$

and this is the necessary condition of consistency obtained by Nielsen.

We now solve the equation

$$z^2\frac{d^2F_\nu(z)}{dz^2}+z\frac{dF_\nu(z)}{dz}+(z^2-\nu^2)F_\nu(z)=z\varpi_\nu(z) \ldots.(8)$$

by the method of variation of parameters.

The general solution is

$$F_\nu(z)=J_\nu(z)\left[c_\nu-\tfrac{1}{2}\pi\int_a^z Y_\nu(t)\,\varpi_\nu(t)\,dt\right]$$
$$+Y_\nu(z)\left[d_\nu+\tfrac{1}{2}\pi\int_b^z J_\nu(t)\,\varpi_\nu(t)\,dt\right]\text{......}(9),$$

where a and b are arbitrary, c_ν and d_ν are independent of z, though they may depend on ν, and $Y_\nu(z)$ is the Bessel function of the second kind which is regarded as the canonical function in Nielsen's *Handbuch der Cylinderfunktionen*; it is defined by the equation

$$Y_\nu(z)=\frac{J_\nu(z)\cos\nu\pi-J_{-\nu}(z)}{\sin\nu\pi},$$

or by the limiting form of this equation when ν is an integer.

3. We now have to determine the circumstances in which the general solution of (8) is a solution of (3) and (4), and this is the point at which Nielsen's analysis appears to fail.

If the value of $F_\nu(z)$ given by (9) is a solution of (3), we find, on substitution, that

$$zJ_{\nu-1}(z)\left[c_\nu-\tfrac{1}{2}\pi\int_a^z Y_\nu(t)\,\varpi_\nu(t)\,dt\right]-\tfrac{1}{2}\pi zJ_\nu(z)\,Y_\nu(z)\,\varpi_\nu(z)$$
$$+zY_{\nu-1}(z)\left[d_\nu+\tfrac{1}{2}\pi\int_b^z J_\nu(t)\,\varpi_\nu(t)\,dt\right]+\tfrac{1}{2}\pi zY_\nu(z)\,J_\nu(z)\,\varpi_\nu(z)$$
$$=zJ_{\nu-1}(z)\left[c_{\nu-1}-\tfrac{1}{2}\pi\int_a^z Y_{\nu-1}(t)\,\varpi_{\nu-1}(t)\,dt\right]$$
$$+zY_{\nu-1}(z)\left[d_{\nu-1}+\tfrac{1}{2}\pi\int_b^z J_{\nu-1}(t)\,\varpi_{\nu-1}(t)\,dt\right]-\alpha_\nu(z),$$

that is to say

$$zJ_{\nu-1}(z)\left[c_\nu-c_{\nu-1}-\tfrac{1}{2}\pi\int_a^z\{Y_\nu(t)\,\varpi_\nu(t)-Y_{\nu-1}(t)\,\varpi_{\nu-1}(t)\}\,dt\right]$$
$$+zY_{\nu-1}(z)\left[d_\nu-d_{\nu-1}+\tfrac{1}{2}\pi\int_b^z\{J_\nu(t)\,\varpi_\nu(t)\right.$$
$$\left.-J_{\nu-1}(t)\,\varpi_{\nu-1}(t)\}\,dt\right]+\alpha_\nu(z)=0\text{...}(10),$$

and, similarly, if the value of $F_\nu(z)$ given by (9) is a solution of (4), we find that

$$zJ_{\nu+1}(z)\left[c_\nu - c_{\nu+1} - \tfrac{1}{2}\pi\int_a^z \{Y_\nu(t)\,\varpi_\nu(t) - Y_{\nu+1}(t)\,\varpi_{\nu+1}(t)\}\,dt\right]$$

$$+ zY_{\nu+1}(z)\left[d_\nu - d_{\nu+1} + \tfrac{1}{2}\pi\int_b^z \{J_\nu(t)\,\varpi_\nu(t) - J_{\nu+1}(t)\,\varpi_{\nu+1}(t)\}\,dt\right]$$

$$-\beta_\nu(z) = 0 \quad \ldots(11).$$

We now transform the equations (10) and (11). In the first place we find, on using recurrence formulæ, that

$$(d/dz)\left[J_{\nu-1}(z)\,\beta_{\nu-1}(z) - J_\nu(z)\,\alpha_\nu(z)\right]$$

$$= J'_{\nu-1}(z)\,\beta_{\nu-1}(z) + J_{\nu-1}(z)\,\beta'_{\nu-1}(z) - J_\nu'(z)\,\alpha_\nu(z) - J_\nu(z)\,\alpha_\nu'(z)$$

$$= \left[\{(\nu-1)/z\}\,J_{\nu-1}(z) - J_\nu(z)\right]\beta_{\nu-1}(z) + J_{\nu-1}(z)\,\beta'_{\nu-1}(z)$$

$$+ \left[\nu/z)\,J_\nu(z) - J_{\nu-1}(z)\right]\alpha_\nu(z) - J_\nu(z)\,\alpha_\nu'(z)$$

$$= \varpi_\nu(z)\,J_\nu(z) - \varpi_{\nu-1}(z)\,J_{\nu-1}(z) \quad \ldots\ldots\ldots\ldots(12),$$

and in this result the functions of the first kind may be replaced throughout by functions of the second kind. It follows that (10) is equivalent to

$$zJ_{\nu-1}(z)\left\{c_\nu - c_{\nu-1} - \tfrac{1}{2}\pi\left[Y_{\nu-1}(z)\,\beta_{\nu-1}(z) - Y_\nu(z)\,\alpha_\nu(z)\right]_a^z\right\}$$

$$+ zY_{\nu-1}(z)\left\{d_\nu - d_{\nu-1} + \tfrac{1}{2}\pi\left[J_{\nu-1}(z)\,\beta_{\nu-1}(z) - Y_\nu(z)\,\alpha_\nu(z)\right]_b^z\right\}$$

$$+ \alpha_\nu(z) = 0,$$

that is to say

$$zJ_{\nu-1}(z)\left\{c_\nu - c_{\nu-1} + \tfrac{1}{2}\pi\left[Y_{\nu-1}(a)\,\beta_{\nu-1}(a) - Y_\nu(a)\,\alpha_\nu(a)\right]\right\}$$

$$+ zY_{\nu-1}(z)\left\{d_\nu - d_{\nu-1} - \tfrac{1}{2}\pi\left[J_{\nu-1}(b)\,\beta_{\nu-1}(b) - J_\nu(b)\,\alpha_\nu(b)\right]\right\} = 0.$$

This result can be true for general values of z if, and only if,

$$\begin{cases} c_\nu - c_{\nu-1} = -\tfrac{1}{2}\pi\,\{Y_{\nu-1}(a)\,\beta_{\nu-1}(a) - Y_\nu(a)\,\alpha_\nu(a)\} \ldots\ldots(13), \\ d_\nu - d_{\nu-1} = \tfrac{1}{2}\pi\,\{J_{\nu-1}(b)\,\beta_{\nu-1}(b) - J_\nu(b)\,\alpha_\nu(b)\} \ldots\ldots(14). \end{cases}$$

These are the difference equations of the first order,* to which reference was made in § 1.

* There are, of course, numerous existence theorems concerning equations of this type.

If the integrals involved are convergent, the solutions of them are

$$c_\nu = \Omega_1(\nu) + \tfrac{1}{2}\pi \int_a^\infty \varpi_\nu(t)\, Y_\nu(t)\, dt,$$

$$d_\nu = \Omega_2(\nu) - \tfrac{1}{2}\pi \int_b^\infty \varpi_\nu(t)\, J_\nu(t)\, dt,$$

where $\Omega_1(\nu)$ and $\Omega_2(\nu)$ are arbitrary periodic functions of ν of period unity. A sufficient condition for convergence is that $\varpi_\nu(t)$ should be analytic near $t=\infty$.

If $\varpi_\nu(t)$ is expressible as a sum $\varpi_\nu(t, 1) + \varpi_\nu(t, 2)$, of which the former part is analytic near $t=\infty$ and the latter part near $t=0$, the solutions when* $-1 < R(\nu) < 1$ are

$$c_\nu = \Omega_1(\nu) + \tfrac{1}{2}\pi \int_a^\infty \varpi_\nu(t, 1)\, Y_\nu(t)\, dt - \tfrac{1}{2}\pi \int_0^a \varpi_\nu(t, 2)\, Y_\nu(t)\, dt,$$

$$d_\nu = \Omega_2(\nu) - \tfrac{1}{2}\pi \int_b^\infty \varpi_\nu(t, 1)\, J_\nu(t)\, dt + \tfrac{1}{2}\pi \int_0^b \varpi_\nu(t, 2)\, J_\nu(t)\, dt.$$

Hence, subject to the general condition of convergence, Nielsen's condition is sufficient as well as necessary for the consistency of the equations.

If it is possible to find numbers c and d such that

$$Y_{\nu-1}(c)\, \beta_{\nu-1}(c) - Y_\nu(c)\, \alpha_\nu(c) = 0 \quad \ldots\ldots\ldots(15),$$

$$J_{\nu-1}(d)\, \beta_{\nu-1}(d) - J_\nu(d)\, \alpha_\nu(d) = 0 \quad \ldots\ldots\ldots(16),$$

then the general solution of (3) and (4) may be written in the form

$$F_\nu(z) = J_\nu(z)\left[\Omega_1(\nu) - \tfrac{1}{2}\pi \int_c^z Y_\nu(t)\, \varpi_\nu(t)\, dt\right]$$

$$+ Y_\nu(z)\left[\Omega_2(\nu) + \tfrac{1}{2}\pi \int_d^z J_\nu(t)\, \varpi_\nu(t)\, dt\right].$$

It is possible that Nielsen had this result in mind (though he does not give it in his memoir) in the special case in which $c=d=0$, when he states that (*loc. cit.*, p. 56) the functions $f_\nu(z)$, $g_\nu(z)$ must, in certain circumstances, behave in a very particular manner near the origin. But this seems unlikely, since his conclusion on p. 57 is that "Après une discussion avec M. J.-P.- Gram je désespère d'un résultat favorable de ces recherches prises dans toute leur généralité".

* Solutions for other values of ν may be derived by using (13) and (14) a finite number of times.

ON A SIMPLE SUMMATION OF THE SERIES $\sum\limits_{s=0}^{n-1} e^{2s^2\pi i/n}$.

By *L. J. Mordell*, Birkbeck College, London.

KRONECKER* has found the sum of the series

$$S = \sum_{s=0}^{n-1} e^{2s^2\pi i/n}$$

in a very simple and elegant manner by considering the integral

$$\int \frac{e^{2\pi i z^2/n}\,dz}{e^{2\pi i z} - 1}$$

taken around a contour consisting of an infinite rectangle, two of whose sides are the lines $x = 0$ and $x = \frac{1}{2}n$ indented at the points where they cut the x axis. This integral however is only a particular case of the integral

$$\int_{-\infty}^{\infty} \frac{e^{\pi i \omega z^2 - 2\pi z x}\,dz}{e^{2\pi z} - 1},$$

wherein $I(\omega) > 0$ and the path of integration is the real axis indented by the lower half of a small circle described about the origin as centre, and may be deformed into a line parallel to the real axis of z and below it at a distance not greater than unity, or into a line cutting the imaginary axis of z between $z = 0$ and $z = -i$ and the real axis at an acute angle a, which is such that $I(\omega e^{2ia}) > 0$. I have shown that the value† of this integral can be expressed in terms of what can be considered as known functions, and that in the particular case, when ω is a rational quantity, the value can be expressed in terms of elementary functions. For example,

$$(e^{2n\pi i x} - 1)\int_{-\infty}^{\infty} \frac{e^{-2\pi i z^2/n - 2\pi z x}\,dz}{e^{2\pi z} - 1}$$

$$= i\sum_{s=1}^{n} e^{s\pi i(2x + 2s/n)} - i\sqrt{(in/2)}\,(e^{-\pi i n x^2/2} + e^{\mp\pi i n(x-1)^2/2}),$$

* See, for example, *The Theory of Numbers*, by G. B. Matthews, pp. 202–205.

† Similar results hold if -1 in the denominator of the integral is replaced by any constant. See my paper, "The value of the definite integral $\int_{-\infty}^{\infty} \frac{e^{at^2+bt}}{e^{ct}+d}\,dt$", which will, I hope, appear shortly in the *Quarterly Journal*.

where $\sqrt{(i)}$ is taken with a positive real part and the path of integration is as before. When the path of integration is the real axis the integral exists only if x lies between 0 and -1, but by considering the inclined path of integration it is clear that the integral is an integral function of x.

Putting $x=0$, we have

$$S=\sqrt{(in/2)}\,(1+i^{-n}),$$

which reduces to the well-known result

$$S=\sqrt{(n)}\left(\frac{i+i^{1-n}}{1+i}\right).$$

The above however suggested to me the evaluation of the series by the following method, which is rather simpler than Kronecker's.

Let a function of z be defined by the equation

$$f(z)\,(e^{2\pi iz}-1)=e^{2\pi iz^2/n}+e^{2\pi i(z+1)^2/n}+e^{2\pi i(z+2)^2/n}+\ldots$$
$$\ldots+e^{2\pi i(z+n-1)^2/n}.$$

Then

$$[f(z+1)-f(z)]\,(e^{2\pi iz}-1)=e^{2\pi i(z+n)^2/n}-e^{2\pi iz^2/n},$$

so that $$f(z+1)-f(z)=e^{2\pi iz^2/n}\,(e^{2\pi iz}+1).$$

Consider now the integral $\int f(z)\,dz$ taken around an infinite parallelogram $ABCD$ of which the parallel sides AB and CD are inclined to the axis of x at an angle lying between 0 and $\pi/2$ and cutting it at the points $z=-\frac{1}{2}$ and $z=\frac{1}{2}$ respectively, while the sides BC and DA are parallel to the axis of x and at an infinite distance above it and below it respectively.

The only pole of $f(z)$ inside the parallelogram is at the origin and the residue there is $2\pi iS$. The integral taken along each of the sides BC and DA vanishes, while the integral along the sides AB and CD reduces to

$$\int_A^B [f(z)-f(z+1)]\,dz=-\int_A^B e^{2\pi iz^2/n}\,(e^{2\pi iz}+1)\,dz.$$

Hence, by Cauchy's theorem,

$$S=\int_A^B e^{2\pi iz^2/n}\,(e^{2\pi iz}+1)\,dz.$$

The value of the integral on the right-hand side is well known since we can deform the path of integration into the real axis from $-\infty$ to ∞. There is no need however to make use of the value of the integral, for if we write $z-\frac{1}{2}n$ for z in the first part of the integral and then put $z=y\sqrt{(n)}$ in the new integral for S, we have at once

$$S=\sqrt{(n)}\,(1+e^{-\pi in/2})\,k,$$

where k is a constant independent of n and $\sqrt{(n)}$ is taken positively. If we put $n=1$, we find as before

$$S=\sqrt{(n)}\left(\frac{i+i^{1-n}}{1+i}\right).$$

It may be noted that we find in exactly the same way

$$\sum_{s=0}^{n-1} e^{\pi is^2/n}=\frac{1+i}{\sqrt{(2)}}\sqrt{(n)}=\sqrt{(in)}$$

if n is even. When n is odd, by grouping the terms in pairs, it is seen that the sum is unity.

RADIATION FROM A MOVING MAGNETON.

By *Dr. H. Bateman.*

§ 1. The rate of radiation of energy from a ring of electrons revolving in a circular orbit and from various other distributions of moving electric charges and magnetic poles has been calculated by G. A. Schott*, who finds that the rate of radiation of energy is almost invariably positive. A steady distribution, such as a Parson magneton† which consists of a complete ring of electric charges following one another round the ring with constant speed, will evidently give no radiation when the ring is stationary as a whole, but as Schott remarks the ring may be expected to radiate energy when its centre has an acceleration. We shall endeavour to prove this by calculating the rate of radiation of energy from

* *Electromagnetic radiation*, Camb. Univ. Press (1912). *Phil. Mag.* (6), xxxvi. (1918), p. 243.

† 'A magneton theory of the structure of the atom,' *Smithsonian Miscellaneous Collections* (Nov. 29, 1915).

an electric pole and magnetic doublet which move together, and shall develop a method by which the rate of radiation of energy from more complicated distributions of moving poles and doublets may be calculated without an accessive amount of labour.

§ 2. Starting with the case of an *electromagnetic doublet*, *i.e.*, an electric doublet and magnetic doublet which move together, we determine the electric force $\mathbf{E}$ and the magnetic force $\mathbf{H}$ from the equations

$$\mathbf{M} \equiv \mathbf{H} + i\mathbf{E} = i \operatorname{rot} \mathbf{L},$$

$$\mathbf{L} = \frac{\partial \mathbf{G}}{\partial t} + ic \operatorname{rot} \mathbf{G},$$

$$\mathbf{G} = \frac{1}{\nu}\, \mathbf{g}(\tau), \quad 4\pi \mathbf{g} = \mathbf{q} + i\mathbf{p} - \frac{1}{c}(\mathbf{v} \times \mathbf{p}) + \frac{1}{c}(\mathbf{v} \times \mathbf{q}),$$

$$\nu = \xi'(\tau)\,[x - \xi(\tau)] + \eta'(\tau)\,[y - \eta(\tau)]$$
$$+ \zeta'(\tau)\,[z - \zeta(\tau)] - c^2(t - \tau) \equiv r\,[(\mathbf{v}\,.\,\mathbf{s}) - c],$$

$$r^2 \equiv [x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \zeta(\tau)]^2 = c^2(t - \tau)^2, \quad t \geq \tau.$$

In these equations $\mathbf{v}$ denotes the velocity at time τ of the moving point P, whose coordinates at this instant are $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$; x, y, z are the coordinates of an arbitrary point Q, $\mathbf{s}$ is a unit vector in the direction of the line PQ, $\mathbf{p}$ and $\mathbf{q}$ are vectors representing the electric and magnetic moments at time τ, t is the time and c the velocity of light which is supposed to be constant. The symbol $\mathbf{v} \times \mathbf{p}$ denotes the vector product of the two vectors $\mathbf{v}$ and $\mathbf{p}$, while $\mathbf{v}\,.\,\mathbf{p}$ denotes their scalar product.

In calculating the rate of radiation we need only retain terms of order $\frac{1}{r}$ in the expressions for the electric and magnetic forces. With this understanding we may write for the three components of $\mathbf{M}$

$$\mathbf{M}_x = \overset{*}{\mathbf{g}}\,.\,\mathbf{K}_x, \quad \mathbf{M}_y = \overset{*}{\mathbf{g}}\,.\,\mathbf{K}_y, \quad \mathbf{M}_z = \overset{*}{\mathbf{g}}\,.\,\mathbf{K}_z \ldots\ldots (1),$$

where $\overset{*}{\mathbf{g}}$ denotes the vector

$$\overset{*}{\mathbf{g}} = \frac{1}{\nu}\, \mathbf{g}''(\tau) - \frac{3\lambda}{\nu^2}\, \mathbf{g}'(\tau) + \left(\frac{3\lambda^2}{\nu^3} - \frac{\mu}{\nu^2}\right) \mathbf{g}(\tau) \ldots\ldots\ldots (2),$$

$$\lambda = r\,(\mathbf{s}\,.\,\mathbf{v}'), \quad \mu = r\,(\mathbf{s}\,.\,\mathbf{v}''),$$

and primes denote differentiations with respect to τ so that the vector $\mathbf{V}'$ represents the acceleration of the moving point P.

The vectors $\mathbf{K}_x$, $\mathbf{K}_y$, $\mathbf{K}_z$ have the following components

$$\mathbf{K}_x \quad c\left[\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2\right], \quad -c\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial y}-i\frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial z}, \quad -c\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial z}+i\frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial y},$$

$$\mathbf{K}_y \quad -c\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial y}+i\frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial z}, \quad c\left[\left(\frac{\partial\tau}{\partial z}\right)^2+\left(\frac{\partial\tau}{\partial x}\right)^2\right], \quad -c\frac{\partial\tau}{\partial y}\frac{\partial\tau}{\partial z}-i\frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial x},$$

$$\mathbf{K}_z \quad -c\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial z}-i\frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial y}, \quad -c\frac{\partial\tau}{\partial y}\frac{\partial\tau}{\partial z}+i\frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial x}, \quad c\left[\left(\frac{\partial\tau}{\partial x}\right)^2+\left(\frac{\partial\tau}{\partial y}\right)^2\right].$$

It is important to notice that the vectors $\mathbf{K}_x$, $\mathbf{K}_y$, $\mathbf{K}_z$ are *all in the same direction*, hence we may write

$$\mathbf{K}_x=c\left[\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2\right]\mathbf{k}, \quad \mathbf{K}_y=\left[i\frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial z}-c\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial y}\right]\mathbf{k},$$

$$\mathbf{K}_z=-\left[i\frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial y}+c\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial z}\right]\mathbf{k},$$

where $\mathbf{k}$ is a complex vector which may be defined by the first of these equations.

We thus have

$$c\left[\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2\right]\mathbf{M}_x=c\left[\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2\right](\overset{*}{\mathbf{g}}\cdot\mathbf{K}_x),$$

$$c\left[\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2\right]\mathbf{M}_y=\left[i\frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial z}-c\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial y}\right](\overset{*}{\mathbf{g}}\cdot\mathbf{K}_x),$$

$$c\left[\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2\right]\mathbf{M}_z=-\left[i\frac{\partial\tau}{\partial y}\frac{\partial\tau}{\partial t}+c\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial z}\right](\overset{*}{\mathbf{g}}\cdot\mathbf{K}_x).$$

Now $$2i(\mathbf{E}\times\mathbf{H})=\mathbf{M}\times\overline{\mathbf{M}},$$

where $\overline{\mathbf{M}}$ is the complex vector conjugate to $\mathbf{M}$, hence

$$\left[\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2\right]\left(\mathbf{E}\times\mathbf{H}\right)=-\frac{1}{c}\frac{\partial\tau}{\partial t}\nabla\tau\,|\overset{*}{\mathbf{g}}.\mathbf{K}_x|^2,$$

where $\nabla\tau$ denotes the vector whose components are

$$\frac{\partial\tau}{\partial x},\ \frac{\partial\tau}{\partial y},\ \frac{\partial\tau}{\partial z}.$$

If $\overset{*}{\mathbf{g}}_0$ denotes the complex vector conjugate to $\overset{*}{\mathbf{g}}$, the above expression may be reduced to the following form

$$\mathbf{E}\times\mathbf{H}=c^2\frac{r^4}{\nu^4}\mathbf{s}\left[\overset{*}{\mathbf{g}}.\overset{*}{\mathbf{g}}_0-(\mathbf{s}.\overset{*}{\mathbf{g}})(\mathbf{s}.\overset{*}{\mathbf{g}}_0)-i\{\mathbf{s}.(\overset{*}{\mathbf{g}}\times\overset{*}{\mathbf{g}}_0)\}\right]\dots(3).$$

This expression for Poynting's vector is of wide generality, as it is valid whenever the components of the vector $\mathbf{M}$ can be expressed in the form (1), where $\overset{*}{\mathbf{g}}$ is some complex vector which is not necessarily given by the formula (2).

§ 3. In the case of an *electromagnetic pole*, the appropriate expression for $\overset{*}{\mathbf{g}}$ is

$$\overset{*}{\mathbf{g}}=\frac{m}{\nu}\left[\mathbf{v}'+\frac{i}{c}(\mathbf{v}\times\mathbf{v}')\right]\dots\dots\dots\dots(4),$$

where $4\pi e$ and $4\pi h$ are the electric and magnetic charges associated with the pole and $m=h+ie$.

To prove this let us first consider the case of an electric pole at the moving point P. The electric and magnetic forces are determined by the equations

$$\mathbf{H}=\operatorname{rot}\mathbf{A},\quad \mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}-\nabla\Phi,$$

$$\mathbf{A}=-\frac{e}{\nu}\mathbf{v},\quad \Phi=-\frac{ec}{\nu}.$$

Retaining only terms of order $\frac{1}{r}$, we find that

$$\mathbf{M}\equiv\mathbf{H}+i\mathbf{E}=\nabla\tau\times\left[\frac{e\lambda}{\nu^2}\mathbf{v}-\frac{e}{\nu}\mathbf{v}'\right]$$

$$-\frac{i}{c}\frac{\partial\tau}{\partial t}\left[\frac{e\lambda}{\nu^2}\mathbf{v}-\frac{e}{\nu}\mathbf{v}'\right]-\frac{iec\lambda}{\nu^2}\nabla\tau.$$

Now let $$\frac{\partial \tau}{\partial t}\left(\frac{e\lambda}{\nu^2}\mathbf{v}-\frac{e}{\nu}\mathbf{v}'\right)+\frac{ec^2\lambda}{\nu^2}\nabla\tau=\nabla\tau\times\mathbf{B}.$$

The vector **B** is not completely defined by this equation, so we may impose the further condition $\mathbf{B}.\nabla\tau=0$, we then find that

$$-\mathbf{B}\left[\left(\frac{\partial\tau}{\partial x}\right)^2+\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2\right]=\nabla\tau\times\left(\frac{e\lambda}{\nu^2}\mathbf{v}-\frac{e}{\nu}\mathbf{v}'\right)\frac{\partial\tau}{\partial t}.$$

This equation may be simplified with the aid of the relation

$$\left(\frac{\partial\tau}{\partial x}\right)^2+\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2=\frac{1}{c^2}\left(\frac{\partial\tau}{\partial t}\right)^2,$$

and we finally find that

$$\mathbf{M}=\nabla\tau\times\left\{\frac{e\lambda}{\nu^2}\mathbf{v}-\frac{e}{\nu}\mathbf{v}'+\frac{i\nabla\tau\times\left(\frac{e\lambda}{\nu^2}\mathbf{v}-\frac{e}{\nu}\mathbf{v}'\right)}{\frac{1}{c}\frac{\partial\tau}{\partial t}}\right\}.$$

It is now easy to see that **M** can be thrown into the form

$$M_x=\overset{*}{\mathbf{g}}.\mathbf{K}_x,\quad M_y=\overset{*}{\mathbf{g}}.\mathbf{K}_y,\quad M_z=\overset{*}{\mathbf{g}}.\mathbf{K}_z,$$

where $$\overset{*}{\mathbf{g}}=-\frac{i}{c}\left[\frac{e\lambda}{\nu^2}\mathbf{v}-\frac{e}{\nu}\mathbf{v}'\right]\frac{c}{\frac{\partial\tau}{\partial t}}=\frac{ie\lambda}{c\nu r}\mathbf{v}-\frac{ie}{cr}\mathbf{v}'.$$

Writing $m\equiv h+ie$ instead of ie we obtain the expression

$$\overset{*}{\mathbf{g}}=\frac{m}{cr}\left[\frac{\lambda}{\nu}\mathbf{v}-\mathbf{v}'\right].$$

It is important to notice that the same results are obtained by using the expression

$$\overset{*}{\mathbf{g}}=\frac{m}{\nu}\left[\mathbf{v}'+\frac{i}{c}(\mathbf{v}\times\mathbf{v}')\right] \quad\ldots\ldots\ldots\ldots\ldots(4).$$

This result is a consequence of the fact that the expression

$$\mathbf{s}\times[\mathbf{V}-i(\mathbf{s}\times\mathbf{V})]$$

vanishes when the vector **V** has the value

$$\mathbf{V}=\mathbf{v}(\mathbf{s}.\mathbf{v}')-\mathbf{v}'(\mathbf{s}.\mathbf{v})-i(\mathbf{v}\times\mathbf{v}').$$

This follows immediately from the identity

$$(\mathbf{s}.\mathbf{v})(\mathbf{s}\times\mathbf{v}')-(\mathbf{s}.\mathbf{v}')(\mathbf{s}\times\mathbf{v})=(\mathbf{v}\times\mathbf{v}')-\mathbf{s}.\{\mathbf{s}.(\mathbf{v}\times\mathbf{v}')\}.$$

Let us now write

$$\mathbf{b}=\mathbf{g}''+m\mathbf{v}'+i\frac{m}{c}(\mathbf{v}\times\mathbf{v}'),$$

then an appropriate expression for $\overset{*}{\mathbf{g}}$ for an electromagnetic pole and electromagnetic doublet which move together is

$$\overset{*}{\mathbf{g}}=\frac{1}{\nu}\mathbf{b}-3\frac{r}{\nu^2}(\mathbf{s}.\mathbf{v}')\mathbf{g}'+\frac{3r^2}{\nu^3}(\mathbf{s}.\mathbf{v}')^2\mathbf{g}-\frac{r}{\nu^2}(\mathbf{s}.\mathbf{v}'')\mathbf{g}\ldots(5).$$

§ 4. Let us now endeavour to calculate the rate at which energy flows across a very large sphere of radius r, which has its centre at the moving point P, and consequently moves with P. According to the usual theory, the flow of energy at a point Q is specified by the vector $c(\mathbf{E}\times\mathbf{H})$ which, in the present case, is in the direction of the unit vector $\mathbf{s}$. An amount of energy represented by $\{\mathbf{s}.(\mathbf{E}\times\mathbf{H})\}$ may, in fact, be supposed to move with velocity c in the direction of $\mathbf{s}$. On account of the motion of the sphere the velocity of this energy relative to a surface element at Q is represented by the vector $c\mathbf{s}-\mathbf{v}$, whose component in the direction of $\mathbf{s}$ is $c-(\mathbf{v}.\mathbf{s})$. Hence the rate at which energy flows across the sphere is

$$I=\int_0^{2\pi}d\phi\int_0^{\pi}r^2\sin\theta\,d\theta\,d\phi\,\{\mathbf{s}.(\mathbf{E}\times\mathbf{H})\}\,\{c-(\mathbf{v}.\mathbf{s})\}.$$

Writing $\sigma^{-1}=c-(\mathbf{s}.\mathbf{v})$ and introducing the notation

$$I_n^m=\int_0^{\pi}\sigma^n\cos^m\theta\sin\theta\,d\theta,$$

we can express the integral I in terms of the integrals I_n^m by choosing the spherical polar coordinates, so that

$$(\mathbf{v}.\mathbf{s})=v\cos\theta.$$

Using the symbol $\mathbf{b}_0$ to denote the complex vector conjugate to $\mathbf{b}$, we find that I is represented by the real part of the following expression

$$\pi c^2\Big[2(\mathbf{b}.\mathbf{b}_0)I_5^0+\frac{12}{v}(\mathbf{v}.\mathbf{v}')(\mathbf{b}.\mathbf{g}_0')I_6^1$$

$$+\frac{9}{v^2}\{v^2v'^2-(\mathbf{v}.\mathbf{v}')^2\}(\mathbf{g}'.\mathbf{g}_0')I_7^0+\frac{9}{v^2}\{3(\mathbf{v}.\mathbf{v}')^2-v^2v'^2\}(\mathbf{g}'.\mathbf{g}_0')I_7^2$$

$$+\frac{6}{v^2}\{v^2v'^2-(\mathbf{v}\cdot\mathbf{v}')^2\}(\mathbf{b}\cdot\mathbf{g}_0)I_7^0+\frac{6}{v^2}\{3(\mathbf{v}\cdot\mathbf{v}')^2-v^2v'^2\}(\mathbf{b}\cdot\mathbf{g}_0)I_7^2$$

$$+\frac{4}{v}(\mathbf{v}\cdot\mathbf{v}'')(\mathbf{b}\cdot\mathbf{g}_0)I_6^1+\frac{54}{v^3}(\mathbf{v}\cdot\mathbf{v}')\{v^2v'^2-(\mathbf{v}\cdot\mathbf{v}')^2\}(\mathbf{g}'\cdot\mathbf{g}_0)I_8^1$$

$$+\frac{18}{v^3}(\mathbf{v}.\mathbf{v}')\{5(\mathbf{v}.\mathbf{v}')^2-3v^2v'^2\}(\mathbf{g}'.\mathbf{g}_0)I_8^3$$

$$+\frac{6}{v^2}\{v^2(\mathbf{v}'.\mathbf{v}'')-(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{v}'')\}(\mathbf{g}'.\mathbf{g}_0)I_7^0$$

$$+\frac{6}{v^2}\{3(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{v}'')-v^2(\mathbf{v}'.\mathbf{v}'')\}(\mathbf{g}'.\mathbf{g}_0)I_7^2$$

$$+\frac{27}{4v^4}\{v^2v'^2-(\mathbf{v}.\mathbf{v}')^2\}(\mathbf{g}\cdot\mathbf{g}_0)I_9^0$$

$$+\frac{27}{2v^4}\{5(\mathbf{v}.\mathbf{v}')^2-v^2v'^2\}\{v^2v'^2-(\mathbf{v}.\mathbf{v}')^2\}(\mathbf{g}\cdot\mathbf{g}_0)I_9^2$$

$$+\frac{9}{4v^4}\{3v^4v'^4-30v^2v'^2(\mathbf{v}.\mathbf{v}')^2+35(\mathbf{v}.\mathbf{v}')^4\}(\mathbf{g}\cdot\mathbf{g}_0)I_9^4$$

$$+\frac{6}{v^3}\{2v^2(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{v}'')+(\mathbf{v}.\mathbf{v}'')[v^2v'^2-3(\mathbf{v}.\mathbf{v}')^2]\}(\mathbf{g}\cdot\mathbf{g}_0)I_8^1$$

$$+\frac{6}{v^3}\{(\mathbf{v}.\mathbf{v}'')[5(\mathbf{v}.\mathbf{v}')^2-v^2v'^2]-2v^2(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{v}'')\}(\mathbf{g}\cdot\mathbf{g}_0)I_8^3$$

$$+\frac{1}{v^2}\{v^2v''^2-(\mathbf{v}.\mathbf{v}'')^2\}(\mathbf{g}\cdot\mathbf{g}_0)I_7^0+\frac{1}{v^2}\{3(\mathbf{v}.\mathbf{v}'')^2-v^2v''^2\}(\mathbf{g}\cdot\mathbf{g}_0)I_7^2$$

$$-\frac{1}{v^2}\{v^2(\mathbf{b}\cdot\mathbf{b}_0)-(\mathbf{v}.\mathbf{b})(\mathbf{v}\cdot\mathbf{b}_0)\}I_5^0-\frac{1}{v^2}\{3(\mathbf{v}.\mathbf{b})(\mathbf{v}.\mathbf{b}_0)-v^2(\mathbf{b}.\mathbf{b}_0)\}I_5^2$$

$$-\frac{6}{v^3}\{v^2[(\mathbf{v}.\mathbf{v}')(\mathbf{b}\cdot\mathbf{g}_0')+(\mathbf{v}.\mathbf{b})(\mathbf{v}'\cdot\mathbf{g}_0')+(\mathbf{v}\cdot\mathbf{g}_0')(\mathbf{v}'.\mathbf{b})]$$
$$-3(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{b})(\mathbf{v}\cdot\mathbf{g}_0')\}I_6^1$$

$$-\frac{6}{v^3}\{5(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{b})(\mathbf{v}\cdot\mathbf{g}_0')-v^2[(\mathbf{v}.\mathbf{v}')(\mathbf{b}\cdot\mathbf{g}_0')$$
$$+(\mathbf{v}.\mathbf{b})(\mathbf{v}'\cdot\mathbf{g}_0')+(\mathbf{v}\cdot\mathbf{g}_0')(\mathbf{v}'.\mathbf{b})]\}I_6^3$$

$$-\frac{1}{4v^4}\{v^4A-v^2B+3C\}I_7^0-\frac{1}{2v^4}\{3v^2B-v^4A-15C\}I_7^2$$

$$-\frac{1}{4v^4}\{v^4A-5v^2B+35C\}I_7^4$$

$$- \frac{2}{v^3}\{v^2[(\mathrm{v}.\mathrm{v}'')(\mathrm{b}.\mathrm{g}_0) + (\mathrm{v}.\mathrm{b})(\mathrm{v}''.\mathrm{g}_0) + (\mathrm{v}.\mathrm{g}_0)(\mathrm{v}''.\mathrm{b})]$$

$$- 3(\mathrm{v}.\mathrm{v}'')(\mathrm{v}.\mathrm{b})(\mathrm{v}.\mathrm{g}_0)\} I_6^1$$

$$- \frac{2}{v^3}\{5(\mathrm{v}.\mathrm{v}'')(\mathrm{v}.\mathrm{b})(\mathrm{v}.\mathrm{g}_0) - v^2[(\mathrm{v}.\mathrm{v}'')(\mathrm{b}.\mathrm{g}_0)$$

$$+ (\mathrm{v}.\mathrm{b})(\mathrm{v}'.\mathrm{g}_0) + (\mathrm{v}.\mathrm{g}_0)(\mathrm{v}''.\mathrm{b})]\} I_6^3$$

$$- \frac{1}{v^5}\{X I_8^1 + (Y - 2X) I_8^3 + (2Z - Y + X) I_8^5\}$$

$$- \frac{9}{v^6}\{U I_9^0 + (V - 3U) I_9^2 + (W - 2V + 3U) I_9^4 + (T - W + V - U) I_9^6\}$$

$$- \frac{2i}{v}\{\mathrm{v}.(\mathrm{b} \times \mathrm{b}_0)\} I_5^1 - \frac{6i}{v^2}\{v^2[\mathrm{v}'.(\mathrm{b} \times \mathrm{g}_0')] - (\mathrm{v}.\mathrm{v}')[\mathrm{v}.(\mathrm{b} \times \mathrm{g}_0')]\} I_6^0$$

$$- \frac{6i}{v^2}\{3(\mathrm{v}.\mathrm{v}')[\mathrm{v}.(\mathrm{b} \times \mathrm{g}_0')] - v^2[\mathrm{v}'.(\mathrm{b} \times \mathrm{g}_0')]\} I_6^2$$

$$- \frac{i}{v^3}\{v^2 L - 3N\} I_7^1 - \frac{i}{v^3}\{5N - v^2 L\} I_7^3$$

$$- \frac{2i}{v^2}\{v^2[\mathrm{v}''.(\mathrm{b} \times \mathrm{g}_0)] - (\mathrm{v}.\mathrm{v}'')[\mathrm{v}.(\mathrm{b} \times \mathrm{g}_0)]\} I_6^0$$

$$- \frac{2i}{v^2}\{3(\mathrm{v}.\mathrm{v}'')[\mathrm{v}.(\mathrm{b} \times \mathrm{g}_0)] - v^2[\mathrm{v}''.(\mathrm{b} \times \mathrm{g}_0)]\} I_6^2$$

$$- \frac{i}{4v^4}\{v^4 P - v^2 Q + 3R\} I_8^0 - \frac{i}{2v^4}\{3v^2 Q - v^4 P - 15R\} I^2$$

$$- \frac{i}{4v^4}\{v^4 P - 5v^2 Q + 35R\} I_8^4$$

$$- \frac{9i}{v^5}\{P_1 I_9^1 + (Q_1 - 2P_1) I_9^3 + (2R_1 - Q_1 + P_1) I_9^5\}\Bigg].$$

In this expression the symbols used have the following values:

$$A = 9[v'^2(\mathrm{g}'.\mathrm{g}_0') + 2(\mathrm{v}'.\mathrm{g}')(\mathrm{v}'.\mathrm{g}_0')]$$

$$+ 6[v'^2(\mathrm{b}.\mathrm{g}_0) + 2(\mathrm{v}'.\mathrm{b})(\mathrm{v}'.\mathrm{g}_0)]$$

$$+ 6[(\mathrm{v}'.\mathrm{v}'')(\mathrm{g}'.\mathrm{g}_0) + (\mathrm{v}'.\mathrm{g}')(\mathrm{v}''.\mathrm{g}_0) + (\mathrm{v}''.\mathrm{g}')(\mathrm{v}'.\mathrm{g}_0)]$$

$$+ [v''^2(\mathrm{g}.\mathrm{g}_0) + 2(\mathrm{v}''.\mathrm{g})(\mathrm{v}''.\mathrm{g}_0)],$$

$$B = 9\,[(\mathbf{v}.\mathbf{v}')^2(\mathbf{g}'.\mathbf{g}_0') + 2(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0')$$
$$+ 2(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0') + v'^2(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0')]$$
$$+ 6\,[(\mathbf{v}.\mathbf{v}')^2(\mathbf{b}.\mathbf{g}_0) + 2(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{b})(\mathbf{v}'.\mathbf{g}_0)$$
$$+ 2(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{b})(\mathbf{v}.\mathbf{g}_0) + v'^2(\mathbf{v}.\mathbf{b})(\mathbf{v}.\mathbf{g}_0)]$$
$$+ 6\,[(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{v}'')(\mathbf{g}'.\mathbf{g}_0) + (\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{v}'')$$
$$+ (\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}')(\mathbf{v}''.\mathbf{g}_0) + (\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}')$$
$$+ (\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0) + (\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}''.\mathbf{g}')]$$
$$+ (\mathbf{v}.\mathbf{v}'')^2(\mathbf{g}.\mathbf{g}_0) + 2(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}''.\mathbf{g}_0)$$
$$+ 2(\mathbf{v}.\mathbf{v}'')(\mathbf{v}''.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) + v''^2(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0),$$

$$C = 9(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0') + 6(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{b})(\mathbf{v}.\mathbf{g}_0)$$
$$+ 6(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0) + (\mathbf{v}.\mathbf{v}'')^2(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0),$$

$$X = \tfrac{9}{2}\,[v^2v'^2 - (\mathbf{v}.\mathbf{v}')^2]\,v^2\,[(\mathbf{v}.\mathbf{v}')(\mathbf{g}'.\mathbf{g}_0)$$
$$+ (\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0) + (\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}')]$$
$$- \tfrac{27}{2}\,[v^2v'^2 - (\mathbf{v}.\mathbf{v}')^2](\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0)$$
$$+ 9v^4\,[(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0)$$
$$+ v'^2(\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0) + v'^2(\mathbf{v}'.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0)]$$
$$- 27v^2(\mathbf{v}.\mathbf{v}')\,[(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0) + (\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}')$$
$$+ v'^2(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0)] + 54(\mathbf{v}.\mathbf{v}')^3(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0)$$
$$+ 9v^4(\mathbf{v}.\mathbf{v}')\,[v'^2(\mathbf{g}'.\mathbf{g}_0) + 2(\mathbf{v}'.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0)]$$
$$- 9v^2(\mathbf{v}.\mathbf{v}')^2\,[(\mathbf{v}.\mathbf{v}')(\mathbf{g}'.\mathbf{g}_0) + (\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0) + (\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}')]$$
$$+ \tfrac{3}{2}\,[v^2v'^2 - (\mathbf{v}.\mathbf{v}')^2]\,[(\mathbf{v}.\mathbf{v}'')(\mathbf{g}.\mathbf{g}_0)$$
$$+ (\mathbf{v}.\mathbf{g})(\mathbf{v}''.\mathbf{g}_0)(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}''.\mathbf{g})]\,v^2$$
$$- \tfrac{9}{2}\,[v^2v'^2 - (\mathbf{v}.\mathbf{v}')^2](\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0)$$
$$+ 3v^4\,[(\mathbf{v}\,\mathbf{v}'')(\mathbf{v}'.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0) + (\mathbf{v}'.\mathbf{v}'')(\mathbf{v}'.\mathbf{g}_0)(\mathbf{v}.\mathbf{g})$$
$$+ (\mathbf{v}'.\mathbf{v}'')(\mathbf{v}'.\mathbf{g})(\mathbf{v}.\mathbf{g}_0)] + 18(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0)$$
$$- 9v^2(\mathbf{v}.\mathbf{v}')\,[(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0) + (\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{v}'')$$
$$+ (\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g})] + 3v^4(\mathbf{v}.\mathbf{v}')\,[(\mathbf{v}'.\mathbf{v}'')(\mathbf{g}.\mathbf{g}_0)$$
$$+ (\mathbf{v}'.\mathbf{g})(\mathbf{v}''.\mathbf{g}_0) + (\mathbf{v}'.\mathbf{g}_0)(\mathbf{v}''.\mathbf{g})]$$
$$- 3v^2(\mathbf{v}.\mathbf{v}')^2\,[(\mathbf{v}.\mathbf{v}'')(\mathbf{g}\,\mathbf{g}_0) + (\mathbf{v}.\mathbf{g})(\mathbf{v}''.\mathbf{g}_0) + (\mathbf{v}.\mathbf{g}_0)(\mathbf{v}''.\mathbf{g})],$$

$$\begin{aligned}
Y = {} & 18v^2 (\mathbf{v}.\mathbf{v}')^2 [(\mathbf{v}.\mathbf{v}')(\mathbf{g}'.\mathbf{g}_0) + (\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0) + (\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}')] \\
& - 180 (\mathbf{v}.\mathbf{v}')^3 (\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0) + 18 v^2 v'^2 (\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0) \\
& + 36 v^2 (\mathbf{v}.\mathbf{v}')[(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0) + v'^2 (\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0) \\
& + (\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}')] + 6 (\mathbf{v}.\mathbf{v}')^2 v^2 [(\mathbf{v}.\mathbf{v}'')(\mathbf{g}.\mathbf{g}_0) \\
& + (\mathbf{v}.\mathbf{g})(\mathbf{v}''.\mathbf{g}_0) + (\mathbf{v}.\mathbf{g}_0)(\mathbf{v}''.\mathbf{g})] \\
& - 60 (\mathbf{v}.\mathbf{v}')^2 (\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) + 6 v^2 v'^2 (\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) \\
& + 12 v^2 (\mathbf{v}.\mathbf{v}')\{(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0) + (\mathbf{v}.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0)(\mathbf{v}'.\mathbf{v}'') \\
& + (\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g})\},
\end{aligned}$$

$$Z = 18 (\mathbf{v}.\mathbf{v}')^3 (\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0) + 6 (\mathbf{v}.\mathbf{v}')^2 (\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0),$$

$$\begin{aligned}
U = {} & \tfrac{1}{8} [v^2 v'^2 - (\mathbf{v}.\mathbf{v}')^2]^2 [v^2 (\mathbf{g}.\mathbf{g}_0) - (\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0)] \\
& + \tfrac{1}{2} [v^2 v'^2 - (\mathbf{v}.\mathbf{v}')^2][v^2 (\mathbf{v}'.\mathbf{g}) - (\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g})][v^2 (\mathbf{v}'.\mathbf{g}_0) \\
& - (\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g})],
\end{aligned}$$

$$\begin{aligned}
V = {} & [3v'^2 (\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0) + 3v'^2 (\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}) \\
& + 3 (\mathbf{v}.\mathbf{v}')^2 (\mathbf{v}'.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0)] v^4 \\
& - 6v^2 (\mathbf{v}.\mathbf{v}')^3 (\mathbf{v}.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0) - 6v^2 (\mathbf{v}.\mathbf{v}')^3 (\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}) \\
& - 9 v^2 v'^2 (\mathbf{v}.\mathbf{v}')^2 (\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) \\
& + \tfrac{3}{4} v^4 v'^4 (\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) + \tfrac{3}{2} v^4 (\mathbf{v}.\mathbf{v}') v'^2 (\mathbf{g}.\mathbf{g}_0) \\
& - \tfrac{3}{2} v^2 (\mathbf{v}.\mathbf{v}')^4 (\mathbf{g}.\mathbf{g}_0) + \tfrac{45}{4} (\mathbf{v}.\mathbf{v}')^4 (\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0),
\end{aligned}$$

$$\begin{aligned}
W = {} & v^2 (\mathbf{v}.\mathbf{v}')^4 (\mathbf{g}.\mathbf{g}_0) + 4v^2 (\mathbf{v}.\mathbf{v}')^3 (\mathbf{v}.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0) \\
& + 4v^2 (\mathbf{v}.\mathbf{v}')^3 (\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}) \\
& + 6 v^2 v'^2 (\mathbf{v}.\mathbf{v}')^2 (\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) - 15 (\mathbf{v}.\mathbf{v}')^4 (\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0),
\end{aligned}$$

$$T = 2 (\mathbf{v}.\mathbf{v}')^4 (\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0),$$

$$\begin{aligned}
L = {} & 9 [2 (\mathbf{v}.\mathbf{v}')\{\mathbf{v}'.(\mathbf{g}' \times \mathbf{g}_0')\} + v'^2 \{\mathbf{v}.(\mathbf{g}' \times \mathbf{g}_0')\}] \\
& + 6 [2 (\mathbf{v}.\mathbf{v}')\{\mathbf{v}'.(b \times \mathbf{g}_0)\} + v'^2 \{\mathbf{v}.(b \times \mathbf{g}_0)\}] \\
& + 6 [(\mathbf{v}.\mathbf{v}')\{\mathbf{v}''.(\mathbf{g}' \times \mathbf{g}_0)\} + (\mathbf{v}.\mathbf{v}'')\{\mathbf{v}'.(\mathbf{g}' \times \mathbf{g}_0)\} \\
& + (\mathbf{v}'.\mathbf{v}'')\{\mathbf{v}.(\mathbf{g}' \times \mathbf{g}_0)\}] \\
& + [2 (\mathbf{v}'.\mathbf{v}'')\{\mathbf{v}''.(\mathbf{g} \times \mathbf{g}_0)\} + v''^2 \{\mathbf{v}.(\mathbf{g} \times \mathbf{g}_0)\}],
\end{aligned}$$

$$\begin{aligned}
N = {} & 9 (\mathbf{v}.\mathbf{v}')^2 \{\mathbf{v}.(\mathbf{g}' \times \mathbf{g}_0')\} + 6 (\mathbf{v}.\mathbf{v}')^2 \{\mathbf{v}.(b \times \mathbf{g}_0)\} \\
& + 6 (\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{v}'')\{\mathbf{v}.(\mathbf{g}' \times \mathbf{g}_0)\} + (\mathbf{v}.\mathbf{v}'')^2 \{\mathbf{v}.(\mathbf{g} \times \mathbf{g}_0)\},
\end{aligned}$$

$$P = 54v'^2\{\mathbf{v}'.(\mathbf{g}'\times\mathbf{g}_0)\} + 6\,[2\,(\mathbf{v}'.\mathbf{v}'')\{\mathbf{v}'.(\mathbf{g}\times\mathbf{g}_0)\}$$
$$+ v'^2\{\mathbf{v}''.(\mathbf{g}\times\mathbf{g}_0)\}],$$

$$Q = 54\,[(\mathbf{v}.\mathbf{v}')^2\{\mathbf{v}'.(\mathbf{g}'\times\mathbf{g}_0)\} + (\mathbf{v}.\mathbf{v}')\,v'^2\{\mathbf{v}.(\mathbf{g}'\times\mathbf{g}_0)\}]$$
$$+ 6\,[(\mathbf{v}.\mathbf{v}')^2\{\mathbf{v}''.(\mathbf{g}\times\mathbf{g}_0) + 2\,(\mathbf{v}\,\mathbf{v}')(\mathbf{v}.\mathbf{v}'')\{\mathbf{v}'.(\mathbf{g}\times\mathbf{g}_0)\}$$
$$+ 2\,(\mathbf{v}.\mathbf{v}')\,(\mathbf{v}'.\mathbf{v}'')\{\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)\} + v'^2\,(\mathbf{v}.\mathbf{v}'')\{\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)\},$$

$$R = 18\,(\mathbf{v}.\mathbf{v}')^3\{\mathbf{v}.(\mathbf{g}'\times\mathbf{g}_0)\} + 6\,(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{v}'')\{\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)\},$$

$$P_1 = 3v^2\,[v^2v'^2 - (\mathbf{v}.\mathbf{v}')^2]\,(\mathbf{v}.\mathbf{v}')\{\mathbf{v}'.(\mathbf{g}\times\mathbf{g}_0)\}$$
$$+ \tfrac{3}{4}\,[v^4v'^4 - 6v^2v'^2(\mathbf{v}.\mathbf{v}')^2 + 5\,(\mathbf{v}.\mathbf{v}')^4]\,\{\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)\},$$

$$Q_1 = 4v^2\,(\mathbf{v}.\mathbf{v}')^3\{\mathbf{v}'.(\mathbf{g}\times\mathbf{g}_0)\}$$
$$+ [6v^2v'^2\,(\mathbf{v}.\mathbf{v}')^2 - 10\,(\mathbf{v}.\mathbf{v}')^4]\,\{\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)\},$$

$$R_1 = (\mathbf{v}.\mathbf{v}')^4\{\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)\}.$$

The values of the different integrals I_n^m are as follows:

$$I_5^0 = \frac{2c\,(c^2+v^2)}{(c^2-v^2)^4}, \quad I_5^1 = \frac{2v\,(5c^2+v^2)}{3\,(c^2-v^2)^4}, \quad I_5^2 = \frac{2c\,(c^2+5v^2)}{3\,(c^2-v^2)^4},$$

$$I_6^0 = \frac{2}{5}\,\frac{5c^4+10c^2v^2+v^4}{(c^2-v^2)^5}, \quad I_6^1 = \frac{4}{5}\,\frac{cv\,(5c^2+3v^2)}{(c^2-v^2)^5},$$

$$I_6^2 = \frac{2}{15}\,\frac{5c^4+38c^2v^2+5v^4}{(c^2-v^2)^5}, \quad I_6^3 = \frac{4}{5}\,\frac{cv\,(3c^2+5v^2)}{(c^2-v^2)^5},$$

$$I_7^0 = \frac{2}{3}\,\frac{c\,(3c^4+10c^2v^2+3v^4)}{(c^2-v^2)^6}, \quad I_7^1 = \frac{1}{15}\,\frac{v\,(70c^4+84c^2v^2+6v^4)}{(c^2-v^2)^6},$$

$$I_7^2 = \frac{2}{15}\,\frac{c\,(5c^4+54c^2v^2+21v^4)}{(c^2-v^2)^6}, \quad I_7^3 = \frac{2}{15}\,\frac{v\,(21c^4+54c^2v^2+5v^4)}{(c^2-v^2)^6},$$

$$I_7^4 = \frac{2}{15}\,\frac{c\,(3c^4+42c^2v^2+35v^4)}{(c^2-v^2)^6},$$

$$I_8^0 = \frac{2}{7}\,\frac{7c^6+35c^4v^2+21c^2v^4+v^6}{(c^2-v^2)^7}, \quad I_8^1 = \frac{16}{21}\,\frac{cv\,(7c^4+14c^2v^2+3v^4)}{(c^2-v^2)^7},$$

$$I_8^2 = \frac{2}{105}\,\frac{35c^6+511c^4v^2+393c^2v^4+21v^6}{(c^2-v^2)^7},$$

$$I_8^3 = \frac{16}{35}\,\frac{cv\,(7c^4+26c^2v^2+7c^4)}{(c^2-v^2)^7},$$

$$I_8^4 = \frac{2}{105}\,\frac{21c^6+393c^4v^2+511c^2v^4+35v^6}{(c^2-v^2)^7},$$

$$I_8^5 = \frac{16}{21}\frac{cv(3c^4 + 14c^2v^2 + 7v^4)}{(c^2 - v^2)^7},$$

$$I_9^0 = \frac{2c(c^6 + 7c^4v^2 + 7c^2v^4 + v^6)}{(c^2 - v^2)^8}, \quad I_9^1 = \frac{2v}{7}\,\frac{21c^6 + 63c^4v^2 + 27c^2v^4 + v^6}{(c^2 - v^2)^8},$$

$$I_9^2 = \frac{2c}{21}\,\frac{7c^6 + 133c^4v^2 + 169c^2v^4 + 27v^6}{(c^2 - v^2)^8},$$

$$I_9^3 = \frac{2v}{35}\,\frac{63c^6 + 321c^4v^2 + 169c^2v^4 + 7v^6}{(c^2 - v^2)^8},$$

$$I_9^4 = \frac{2c}{35}\,\frac{7c^6 + 169c^4v^2 + 321c^2v^4 + 63v^6}{(c^2 - v^2)^8},$$

$$I_9^5 = \frac{2v}{21}\,\frac{27c^6 + 169c^4v^2 + 133c^2v^4 + 7v^6}{(c^2 - v^2)^8},$$

$$I_9^6 = \frac{2c}{7}\,\frac{c^6 + 27c^4v^2 + 63c^2v^4 + 21v^6}{(c^2 - v^2)^8}.$$

Substituting these values in the above expression and collecting like terms together we find eventually that I is equal to the real part of the following expression:

$$\pi c^3 \Big[\tfrac{8}{3}\frac{c^2 + 2v^2}{(c^2 - v^2)^4}(\mathbf{b}.\mathbf{b}_0) - \frac{8}{(c^2 - v^2)^4}(\mathbf{v}.\mathbf{b})(\mathbf{v}.\mathbf{b}_0)$$

$$+ \tfrac{16}{5}\frac{c^2 + v^2}{(c^2 - v^2)^5}\{12(\mathbf{v}.\mathbf{v}')(\mathbf{b}\,\mathbf{g}_0') + 3v'^2(\mathbf{g}'.\mathbf{g}_0')$$

$$+ 4(\mathbf{v}.\mathbf{v}'')(\mathbf{b}.\mathbf{g}_0) + 2v'^2(\mathbf{b}.\mathbf{g}_0) + 2(\mathbf{v}'.\mathbf{v}'')(\mathbf{g}'.\mathbf{g}_0) + \tfrac{1}{3}v''^2(\mathbf{g}.\mathbf{g}_0)\}$$

$$+ \tfrac{8}{5}\frac{3c^2 + 2v^2}{(c^2 - v^2)^6}\{24(\mathbf{v}.\mathbf{v}')^2(\mathbf{g}'.\mathbf{g}_0')$$

$$+ 16(\mathbf{v}.\mathbf{v}')^2(\mathbf{b}.\mathbf{g}_0) + \tfrac{144}{7}v'^2(\mathbf{v}.\mathbf{v}')(\mathbf{g}'.\mathbf{g}_0)$$

$$+ 16(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{v}'')(\mathbf{g}'.\mathbf{g}_0) + \tfrac{9}{7}v'^4(\mathbf{g}.\mathbf{g}_0)$$

$$+ \tfrac{16}{7}(\mathbf{v}.\mathbf{v}'')v'^2(\mathbf{g}.\mathbf{g}_0) + \tfrac{32}{7}(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{v}'')(\mathbf{g}.\mathbf{g}_0) + \tfrac{8}{3}(\mathbf{v}.\mathbf{v}'')^2(\mathbf{g}.\mathbf{g}_0)\}$$

$$+ \tfrac{64}{7}\frac{2c^2 + v^2}{(c^2 - v^2)^7}\{24(\mathbf{v}.\mathbf{v}')^3(\mathbf{g}'.\mathbf{g}_0)$$

$$+ 9v'^2(\mathbf{v}.\mathbf{v}')^2(\mathbf{g}.\mathbf{g}_0) + 8(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{v}')^2(\mathbf{g}.\mathbf{g}_0)\}$$

$$- \tfrac{8}{5}\frac{1}{(c^2 - v^2)^4}\{6(\mathbf{v}.\mathbf{b})(\mathbf{v}'.\mathbf{g}_0') + 6(\mathbf{v}.\mathbf{g}_0')(\mathbf{v}'.\mathbf{b})$$

$$+ 3(\mathbf{v}'.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0') + 2(\mathbf{v}'.\mathbf{b})(\mathbf{v}'.\mathbf{g}_0)$$

$$+ (\mathbf{v}'.\mathbf{v}'')(\mathbf{g}'.\mathbf{g}_0) + (\mathbf{v}''.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0) + (\mathbf{v}'.\mathbf{g}')(\mathbf{v}''.\mathbf{g}_0) + \tfrac{1}{3}(\mathbf{v}''.\mathbf{g})(\mathbf{v}''.\mathbf{g}_0)$$

$$+ 2(\mathbf{v}.\mathbf{b})(\mathbf{v}''.\mathbf{g}_0) + 2(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}''.\mathbf{b})\}$$

$$- \tfrac{32}{5}\frac{1}{(c^2 - v^2)^5}\{12(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}_0')(\mathbf{v}.\mathbf{b})$$

$$+ 6(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0') + 6(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0')$$

$$+ 3v'^2(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0') + 2v'^2(\mathbf{v}.\mathbf{b})(\mathbf{v}.\mathbf{g}_0)$$

$$+ 4(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{b})(\mathbf{v}'.\mathbf{g}_0) + 4(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{b})(\mathbf{v}.\mathbf{g}_0)$$

$$+ 2(\mathbf{v}'.\mathbf{v}'')(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0) + 2(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}')(\mathbf{v}''.\mathbf{g}_0)$$

$$+ 2(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}') + 2(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0)$$

$$+ 2(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}''.\mathbf{g}') + \tfrac{4}{3}(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}''.\mathbf{g}_0)$$

$$+ \tfrac{1}{3}\mathbf{v}''^2(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) + \tfrac{18}{7}v'^2(\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0) + \tfrac{18}{7}v'^2(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}')$$

$$+ \tfrac{4}{7}(\mathbf{v}.\mathbf{v}'')(\mathbf{v}'.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0) + \tfrac{4}{7}(\mathbf{v}'.\mathbf{v}'')(\mathbf{v}'.\mathbf{g}_0)(\mathbf{v}.\mathbf{g})$$

$$+ \tfrac{4}{7}(\mathbf{v}'.\mathbf{v}'')(\mathbf{v}'.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) + \tfrac{4}{7}(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{g})(\mathbf{v}''.\mathbf{g}_0)$$

$$+ \tfrac{4}{7}(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{g}_0)(\mathbf{v}''.\mathbf{g}) + 4(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{b})(\mathbf{v}.\mathbf{g}_0)$$

$$+ \tfrac{36}{7}(\mathbf{v}.\mathbf{v}')(\mathbf{v}'.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0) + \tfrac{4}{7}(\mathbf{v}.\mathbf{g})(\mathbf{v}''.\mathbf{g}_0)v'^2$$

$$+ \tfrac{9}{14}v'^2(\mathbf{v}'.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0)\} - 64\frac{1}{(c^2 - v^2)^6}\{3(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0')$$

$$+ 2(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{b})(\mathbf{v}.\mathbf{g}_0) + 2(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0)$$

$$+ \tfrac{1}{3}(\mathbf{v}.\mathbf{v}'')^2(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) + \tfrac{18}{7}(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{g}')(\mathbf{v}'.\mathbf{g}_0)$$

$$+ \tfrac{18}{7}(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g}') + \tfrac{4}{7}(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0)$$

$$+ \tfrac{4}{7}(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{v}'') + \tfrac{4}{7}(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g}_0)(\mathbf{v}'.\mathbf{g})$$

$$+ \tfrac{18}{7}v'^2(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0)$$

$$+ \tfrac{4}{7}(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{g})(\mathbf{v}''.\mathbf{g}_0) + \tfrac{2}{7}v'^2(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0)$$

$$+ \tfrac{9}{56}v'^4(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) + \tfrac{9}{14}(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}'.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0)$$

$$+ \tfrac{9}{7}v'^2(\mathbf{v}.\mathbf{v}')(\mathbf{v}.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0) + \tfrac{192}{7}\{24(\mathbf{v}.\mathbf{v}')^3(\mathbf{v}.\mathbf{g}')(\mathbf{v}.\mathbf{g}_0)$$

$$+ 8(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{v}'')(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0) + 9v'^2(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0)$$

$$+ 12(\mathbf{v}.\mathbf{v}')^3(\mathbf{v}.\mathbf{g})(\mathbf{v}'.\mathbf{g}_0)\} - 576\frac{1}{(c^2 - v^2)^8}(\mathbf{v}.\mathbf{v}')^4(\mathbf{v}.\mathbf{g})(\mathbf{v}.\mathbf{g}_0)\Big]$$

$$- i\pi c^2\Big[\tfrac{4}{15}\frac{5c^2 + v^2}{(c^2 - v^2)^4}\{5[\mathbf{v}.(\mathbf{b}\times\mathbf{b}_0)] + 6[\mathbf{v}'.(\mathbf{b}\times\mathbf{g}_0')]$$

$$+ 2[\mathbf{v}''.(\mathbf{b}\times\mathbf{g}_0)]\} + \tfrac{4}{5}\frac{7c^2 + v^2}{(c^2 - v^2)^5}\{4(\mathbf{v}.\mathbf{v}'')[\mathbf{v}.(\mathbf{b}\times\mathbf{g}_0)]$$

$$+ 12\,(\mathbf{v}.\mathbf{v}')\,[\mathbf{v}.(\mathbf{b}\times\mathbf{g}_0')] + 6\,(\mathbf{v}.\mathbf{v}')\,[\mathbf{v}'.(\mathbf{g}'\times\mathbf{g}_0')]$$
$$+ 3v'^2\,[\mathbf{v}.(\mathbf{g}'\times\mathbf{g}_0')] + 4\,(\mathbf{v}.\mathbf{v}')\,[\mathbf{v}'.(\mathbf{b}\times\mathbf{g}_0)]$$
$$+ 2v'^2\,[\mathbf{v}.(\mathbf{b}\times\mathbf{g}_0)] + 2\,(\mathbf{v}.\mathbf{v}')\,[\mathbf{v}''.(\mathbf{g}'\times\mathbf{g}_0)]$$
$$+ 2\,(\mathbf{v}.\mathbf{v}'')\,[\mathbf{v}'.(\mathbf{g}'\times\mathbf{g}_0)] + 2\,(\mathbf{v}'.\mathbf{v}'')\,[\mathbf{v}.(\mathbf{g}'\times\mathbf{g}_0)]$$
$$+ \tfrac{2}{3}\,(\mathbf{v}.\mathbf{v}'')\,[\mathbf{v}''.(\mathbf{g}\times\mathbf{g}_0)] + \tfrac{1}{3}\,v''^2\,[\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)]$$
$$+ \tfrac{18}{7}\,v'^2\,[\mathbf{v}'.(\mathbf{g}'\times\mathbf{g}_0)] + \tfrac{4}{7}\,(\mathbf{v}'.\mathbf{v}'')\,[\mathbf{v}'.(\mathbf{g}\times\mathbf{g}_0)] + \tfrac{2}{7}\,v'^2\,[\mathbf{v}''.(\mathbf{g}\times\mathbf{g}_0)]\}$$
$$+ \tfrac{16}{5}\,\frac{9c^2+v^2}{(c^2-v^2)^6}\,\{6\,(\mathbf{v}.\mathbf{v}')^2\,[\mathbf{v}.(\mathbf{g}'\times\mathbf{g}_0')] + 4\,(\mathbf{v}.\mathbf{v}')^2\,[\mathbf{v}.(\mathbf{b}\times\mathbf{g}_0)]$$
$$+ 4\,(\mathbf{v}.\mathbf{v}')\,(\mathbf{v}.\mathbf{v}'')\,[\mathbf{v}.(\mathbf{g}'\times\mathbf{g}_0)] + \tfrac{2}{3}\,(\mathbf{v}.\mathbf{v}'')^2\,[\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)]$$
$$+ \tfrac{9}{7}\,v'^2\,(\mathbf{v}.\mathbf{v}')\,[\mathbf{v}'.(\mathbf{g}\times\mathbf{g}_0)] + \tfrac{9}{28}\,v'^4\,[\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)]$$
$$+ \tfrac{4}{7}\,(\mathbf{v}'.\mathbf{v}')^2\,[\mathbf{v}''.(\mathbf{g}\times\mathbf{g}_0)] + \tfrac{8}{7}\,(\mathbf{v}.\mathbf{v}')\,(\mathbf{v}.\mathbf{v}'')\,[\mathbf{v}'.(\mathbf{g}\times\mathbf{g}_0)]$$
$$+ \tfrac{8}{7}\,(\mathbf{v}.\mathbf{v}')\,(\mathbf{v}'.\mathbf{v}'')\,[\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)] + \tfrac{4}{7}\,v'^2\,(\mathbf{v}.\mathbf{v}'')\,[\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)]\}$$
$$+ \tfrac{16}{7}\,\frac{11c^2+v^2}{(c^2-v^2)^7}\,\{24\,(\mathbf{v}.\mathbf{v}')^3\,[\mathbf{v}.(\mathbf{g}'\times\mathbf{g}_0)] + 12\,(\mathbf{v}.\mathbf{v}')^2(\mathbf{v}.\mathbf{v}''\,[\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)]$$
$$+ 6\,(\mathbf{v}.\mathbf{v}')^3\,[\mathbf{v}'.(\mathbf{g}\times\mathbf{g}_0)] + 9v'^2\,(\mathbf{v}.\mathbf{v}')^2\,[\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)]\}$$
$$+ \tfrac{288}{7}\,\frac{13c^2+v^2}{(c^2-v^2)^8}\,(\mathbf{v}.\mathbf{v}')^4\,[\mathbf{v}.(\mathbf{g}\times\mathbf{g}_0)]\Big].$$

As a check on this result the integral representing I has been calculated directly in the case when $\mathbf{v}$ is instantaneously zero and $\mathbf{v}'$, $\mathbf{v}''$, different from zero. The result obtained agrees with that given by the above formula.

§ 5. Let us now consider the special case of a magneton of invariable moment describing a circular orbit at a constant speed and in such a way that the axis of the magneton is always perpendicular to the plane of the orbit. In this case we may write

$$\mathbf{q} + i\mathbf{p} = 4\pi\,\frac{k}{vv'}\,(\mathbf{v}\times\mathbf{v}') = 4\pi k\mathbf{w}, \text{ say,}$$

where κ is a real constant. We then have

$$\mathbf{g} = \kappa\left[\mathbf{w} - \frac{i}{c}\,\frac{v}{v'}\,\mathbf{v}'\right],$$

$$\mathbf{g}' = -\frac{i\kappa}{c}\,\frac{v}{v'}\,\mathbf{v}'' = -\frac{i\kappa}{c}\,\frac{v'}{v}\,\mathbf{v},$$

$$\mathbf{g}'' = \frac{i\kappa}{c}\,\frac{v'}{v}\,\mathbf{v}', \quad \mathbf{b} = \left(m + \frac{i\kappa}{c}\,\frac{v'}{v}\right)\mathbf{v}' + i\,\frac{m}{c}\,vv'\mathbf{w}.$$

Writing $m = ie$ we find from our general formula that the rate of radiation of energy is represented by

$$\tfrac{8}{3}\pi c e^2 \frac{v'^2}{(c^2 - v^2)^2} + \tfrac{8}{3}\pi c^2 e k \frac{v'^3}{v\,(c^2 - v^2)^3} + \tfrac{16}{15}\pi c k^2 \frac{v'^4}{v^2} \frac{c^2 + \frac{3}{2}v^2}{(c^2 - v^2)^4}\,.$$

In this expression it must be remembered that $4\pi e$ is the electric charge and $4\pi k$ the magnetic moment of the magneton. The units are the same as those used in Lorentz's *Theory of Electrons*. To reduce to electrostatic units we must divide the above expression by 4π and interpret e as the electric charge, k as the magnetic moment.

Some years ago Dr. W. F. G. Swann suggested to me the desirability of calculating the rate of radiation of energy from an electron which rotates round its axis like a planet while describing a circular orbit. If such an electron can be treated to a first approximation as a magneton whose axis is perpendicular to the plane of the orbit, it appears that the rate of radiation is always positive. If e, v, and v' are given, the minimum rate is obtained when k is defined by the equation

$$ce\,(c^2 - v^2)\,v + \tfrac{4}{5}v'\,(c^2 + \tfrac{3}{2}v^2)\,k = 0,$$

and this minimum rate is found to be

$$\frac{2\pi}{3} \frac{ce^2 v'^2}{(c^2 - v^2)^2} \frac{c^2 + 4v^2}{2c^2 + 3v^2}\,.$$

If v is small compared with c, this rate of radiation is roughly about $\frac{1}{8}$ of the rate of radiation from the electric charge alone.

§ 6. Let us next consider the case of a magneton of invariable moment describing a circular orbit at a constant speed and in such a way that the axis of the magneton is always *tangential* to its path. In this case we may write

$$\mathbf{q} + i\mathbf{p} = 4\pi k\mathbf{v}, \quad \mathbf{g} = k\mathbf{v}.$$

The rate of radiation of energy is in this case found to be

$$\frac{8\pi}{3} ce^2 \frac{v'^2}{(c^2 - v^2)^2} + \pi k^2 \frac{v'^4}{15v^2} \frac{40c^4 + 168c^2v^2 + 72v^4}{(c^2 + v^2)^5}\,,$$

and is always greater than the rate of radiation from the electric charge alone.

§ 7. It should be noticed that if $\mathbf{v}' = \mathbf{v}'' = \mathbf{g}'' = 0$ the rate of radiation of energy vanishes. Hence when an electromagnetic pole and an electromagnetic doublet move together with constant velocity along a rectilinear path, the moment of the doublet can change at a constant rate without there being any radiation of energy.

§ 8. Formula (3) may be used to calculate the rate of radiation from a complex arrangement of poles and doublets concentrated round a moving point P. To illustrate this, let us calculate the vector $\overset{*}{\mathrm{g}}$ when the vector $\mathbf{G}$ is defined by the equation

$$\mathbf{G}=\frac{\partial}{\partial x}\left[\frac{1}{\nu}\,\mathrm{g}_1(\tau)\right]+\frac{\partial}{\partial y}\left[\frac{1}{\nu}\,\mathrm{g}_2(\tau)\right]+\frac{\partial}{\partial z}\left[\frac{1}{\nu}\,\mathrm{g}_3(\tau)\right].$$

We have now to collect the terms of order $\frac{1}{r}$ in expressions of type

$$M_x=\frac{\partial}{\partial x}[\overset{*}{\mathrm{g}}_1\,.\,K_x]+\frac{\partial}{\partial y}[\overset{*}{\mathrm{g}}_2\,.\,K_x]+\frac{\partial}{\partial z}[\overset{*}{\mathrm{g}}_3\,.\,K_x].$$

In performing the differentiations we can replace

$$\frac{\partial^2\tau}{\partial x^2},\ \frac{\partial^2\tau}{\partial x\,\partial y},\ \dots\ \text{by}\ \frac{\lambda}{\nu}\left(\frac{\partial\tau}{\partial x}\right)^2,\ \frac{\lambda}{\nu}\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial y},\ \dots,$$

respectively, since the terms which are neglected in the approximations do not affect the terms of order $\frac{1}{r}$. We may thus write

$$\frac{\partial}{\partial x}\,\mathbf{K}_x=2\,\frac{\lambda}{\nu}\,\frac{\partial\tau}{\partial x}\,\mathbf{K}_x,$$

$$\frac{\partial}{\partial x}\overset{*}{\mathrm{g}}_1=\left[\frac{1}{\nu}\,\mathrm{g}_1'''(\tau)-\frac{4\lambda}{\nu^2}\,\mathrm{g}_1''(\tau)+\left(\frac{9\lambda^2}{\nu^3}-\frac{4\mu}{\nu^2}\right)\mathrm{g}_1'(\tau)\right.$$
$$\left.+\left(\frac{8\lambda\mu}{\nu^3}-\frac{9\lambda^3}{\nu^4}-\frac{\rho}{\nu^2}\right)\mathrm{g}_1(\tau)\right]\frac{\partial\tau}{\partial x},$$

where $\rho=r\,(\mathbf{s}\,.\,\mathbf{v}''')$. Hence we may also write

$$\frac{\partial}{\partial x}[\overset{*}{\mathrm{g}}_1\,.\,\mathbf{K}_x]=\frac{\partial\tau}{\partial x}\,(\overset{**}{\mathrm{g}}_1\,.\,\mathbf{K}_x),$$

where

$$\overset{**}{\mathrm{g}}_1=\frac{1}{\nu}\,\mathrm{g}_1'''(\tau)-\frac{2\lambda}{\nu^2}\,\mathrm{g}_1''(\tau)+\left(\frac{3\lambda^2}{\nu^3}-\frac{4\mu}{\nu^2}\right)\mathrm{g}_1'(\tau)$$
$$+\left(\frac{6\lambda\mu}{\nu^3}-\frac{3\lambda^3}{\nu^4}-\frac{\rho}{\nu^2}\right)\mathrm{g}_1(\tau).$$

Finally writing

$$\mathrm{g}_1\ (\tau)\frac{\partial\tau}{\partial x}+\mathrm{g}_2\ (\tau)\frac{\partial\tau}{\partial y}+\mathrm{g}_3\ (\tau)\frac{\partial\tau}{\partial z}=\mathrm{a}\ =\frac{r}{\nu}\,(\mathrm{s.g}),$$

$$\mathrm{g}_1'\ (\tau)\frac{\partial\tau}{\partial x}+\mathrm{g}_2'\ (\tau)\frac{\partial\tau}{\partial y}+\mathrm{g}_3'\ (\tau)\frac{\partial\tau}{\partial z}=\mathrm{a}'\ =\frac{r}{\nu}\,(\mathrm{s.g}'),$$

$$\mathrm{g}_1''\ (\tau)\frac{\partial\tau}{\partial x}+\mathrm{g}_2''\ (\tau)\frac{\partial\tau}{\partial y}+\mathrm{g}_3''\ (\tau)\frac{\partial\tau}{\partial z}=\mathrm{a}''=\frac{r}{\nu}\,(\mathrm{s.g}''),$$

$$\mathrm{g}_1'''(\tau)\frac{\partial\tau}{\partial x}+\mathrm{g}_2'''(\tau)\frac{\partial\tau}{\partial y}+\mathrm{g}_3'''(\tau)\frac{\partial\tau}{\partial z}=\mathrm{a}'''=\frac{r}{\nu}\,(\mathrm{s.g}'''),$$

we find that $$M_x=\overset{*}{\mathrm{g}}\,.\,\mathrm{K}_x,$$
where

$$\overset{*}{\mathrm{g}}=\frac{1}{\nu}\,\mathrm{a}'''-\frac{2\lambda}{\nu^2}\,\mathrm{a}''+\left(\frac{3\lambda^2}{\nu^3}-\frac{4\mu}{\nu^2}\right)\mathrm{a}'+\left(\frac{6\lambda\mu}{\nu^3}-\frac{3\lambda^3}{\nu^4}-\frac{\rho}{\nu^2}\right)\mathrm{a}.$$

It should be noticed that in the above expressions the symbol g stands for a dyad with the 9 components

$$\begin{matrix} g_{1x} & g_{1y} & g_{1z}, \\ g_{2x} & g_{2y} & g_{2z}, \\ g_{3x} & g_{3y} & g_{3z}. \end{matrix}$$

The expression (s.g) thus represents a vector whose x-component is obtained by taking the 3 components of g with suffix x, thus

$$(\mathrm{s}.\mathrm{g})_x=s_x g_{1x}+s_y g_{2x}+s_z g_{3x}.$$

The calculation of the radiation will in this case be exceedingly laborious.

§ 9. In Bohr's theory of the structure of the hydrogen atom* it is assumed that there are certain orbits in which an electron can revolve round a positive nucleus without losing energy by radiation. This assumption is incompatible with the usual postulates of electromagnetic theory if the electron be treated as an electric pole for the expressions given by Larmor† and Liénard,‡ for the radiation from a moving electric pole indicates that there is a loss of energy by radiation except in the case of uniform rectilinear motion.

* N. Bohr, *Phil. Mag.*, 26 (1913), pp. 1, 476, 857; 30 (1915), p. 394.
† J. Larmor, *Phil. Mag.* (1897), p. 512; *Aether and Matter*, p. 227.
‡ A. Liénard, *L'Eclairage Electrique*, t. 16 (1898), pp. 5, 53 and 106.

If the electron is supposed to rotate about an axis, or is treated as a magneton, the situation is slightly improved, because the rate of radiation may be reduced to about one-eighth of its former value. This, however, is not sufficient to secure a long life for the Bohr atom, and so it seems that we must either reject Bohr's idea of non-radiating orbits, or endeavour to modify the fundamental postulates of electro-magnetic theory.

As an alternative to the idea of electrons describing orbits round a positive nucleus, we have the conception of the static atom in which the electrons are practically in fixed positions relative to the positive nucleus or nuclei, and magnetic properties are attributed to the magnetic moment of the generalised electron. This idea has been favoured by many American physical chemists,§ and is certainly very fascinating.

Many scientists, however, are unwilling to give up Larmor's idea of electrons revolving in orbits. The rate of radiation may not be excessive when there are several electrons in a ring, and the idea of circular orbits has been adopted with considerable success in the beautiful theory of atomic structure developed by J. W. Nicholson.

If an attempt is to be made to modify the postulates of electromagnetic theory, the following possibility is one which ought to be considered.

Writing the electromagnetic equations in the form

$$c \operatorname{rot} \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t} + \rho \mathbf{v} \quad \operatorname{div} \mathbf{E} = \rho,$$

$$c \operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} \quad \operatorname{div} \mathbf{H} = 0,$$

let us consider the consequences of assuming that

$$\rho = -\frac{1}{c}\frac{\partial \psi}{\partial t}, \quad \rho \mathbf{v} = c\nabla\psi \quad \ldots\ldots\ldots\ldots\ldots (6),$$

where ψ is a *positive* quantity which is constant in the æther and variable where there are electric charges.

As we have seen elsewhere, if we assume that the total energy density is $\frac{1}{2}(E^2 + H^2 + \psi^2)$, and that the total flow of energy is specified by the vector $c(\mathbf{E} \times \mathbf{H}) + c\psi\mathbf{E}$, the career

§ A. L. Parson, *loc. cit.*, D. L. Webster, *Proc. Amer. Acad. of Arts and Sci.*, t. 50 (1915), p. 131; *Phys. Review*, t. 9 (1917), p. 484; *Amer. Chem. Journ.*, Feb. (1918); C. Davisson, *Phys. Review*, t. 9 (1917), p. 484; A. H. Compton, *Journ. Washington Acad. Sci.*, 8 (1918), p. 1; *Phys. Review*, t. 11 (1918), p. 330; G. N. Lewis, *Amer. Chem. Journ.*, April (1916), vol. 38, p. 762; I. Langmuir, *Journal of the Franklin Institute*, vol. 187, March (1919), p. 359.

of the energy of electromagnetic type that is transformed into work is not lost sight of. The additional term $c\psi\mathbf{E}$ in the energy flux indicates that there is a continual flow of energy from the positive to the negative charges. This energy deserves the name of *concealed energy*, because the rate at which a charge receives or loses energy is simply proportional to the charge, and so remains constant when the charge remains constant. This flow of energy would presumably remain undetected on account of its invariability.

It may be asked, however, whether the electric charge associated with an electron or positive nucleus really does remain constant. If we regard the field of an electric pole as a limiting case of two superposed simple radiant fields whose line charges cancel one another, it is easy to see that if the line charges do not exactly cancel it is possible to obtain a type of electromagnetic field in which the charge associated with a point singularity varies owing to the emission of electrified particles which travel with the velocity of light.

Now the phenomenon of gravitation provides some slight evidence for the existence of a slight fluctuation of the charge on an electron or positive nucleus.

Let us suppose that such fluctuations are exceedingly rapid and more or less spasmodic. We shall suppose in fact that during a short interval of time T the charge on a particle A oscillates slightly, and that all the positive and negative charges that are emitted travel with the velocity of light along rectilinear paths to another particle B. We shall suppose, moreover, that T is so small that a similar transfer of charge takes place between A and other particles in the universe in such a way that the events are practically independent, being separated by intervals of time.

Now let the charge on A at time τ be $e-\kappa\sin p\tau$ and the charge on B at time $\tau+\frac{1}{c}(AB)$ be $e'+\kappa\sin p\tau$, then if the force exerted by A on B depends on A's charge at time τ and B's charge at time $\tau+\frac{1}{c}(AB)$, the force will be of type

$$\frac{1}{r^2}(e-\kappa\sin p\tau)(e'+\kappa\sin p\tau).$$

Averaging over a period $\frac{2\pi}{p}$ we find that the average force is

$$\frac{ee'}{r^2}-\frac{\kappa^2}{2r^2}.$$

If now the time during which A sends charged particles to B is a fraction $\frac{1}{n}$ of the whole of any time interval, there will be an attraction between A and B of order $\frac{\kappa^2}{2nr^2}$. In order to explain gravitation $\frac{\kappa^2}{2n}$ need only be about 10^{-42} of ee', hence, even if κ were $10^{-10}e$, the number n could be very large, say 10^{22}.

A slight fluctuation of the charge on an electron seems reasonable when we remember that electrons are very much alike and that consequently some kind of balance has been set up, most likely by a transfer of charge from one to another until a steady state has been very nearly attained.

The question now arises whether an atom in which there are 'non-radiating orbits' is really neutral. If the negative charge slightly exceeded the positive charge the rate of radiation specified by Poynting's vector might be just balanced by the gain of energy depending on the flow specified by $c\psi\mathbf{E}$.

On this hypothesis a re-adjustment of charge would take place in the transition from one of Bohr's stationary orbits to another, and so a concentrated type of radiation would be emitted in addition to the usual continuous type of radiation. This concentrated type of radiation may produce the photoelectric effect and similar phenomena. In attempting to put this hypothesis to the test many questions arise which need to be answered. In the first place, if there is a readjustment of charge during the emission of light, there may be a slight effect of the emission of light and heat on gravitation. It is well known that P. E. Shaw* has obtained experimental evidence of an influence of temperature on gravitation.

Unfortunately we can only guess at the value of ψ, and so a numerical test is at present out of the question. It is easily seen, however, that ψ must be enormously large,† because ρ is very large. This requires a gigantic density of concealed energy in the æther,‡ and this requirement may, perhaps, be used as an argument against the hypothesis (6), which is analogous to the hypothesis of irrotational motion in hydrodynamics.

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* *Phil. Trans.*, A, vol. ccxvi. (1916), pp. 349–392; *Proc. Phys. Soc.*, vol. xxix. (1917), pp. 163–169

† The value of ψ for a positive charge may be as high as 10^{50} in C.G.S. units.

‡ The view that the total energy per c.c. in the æther is enormously large is adopted by Sir Oliver Lodge, *British Association Report* (1907), p. 453, who gives 10^{33} ergs as an estimate The present hypothesis seems to demand a still larger value. In his book, *The New Science of the Fundamental Physics* (1918), W. W. Strong expresses the view that the density of energy in the æther is so large that it is comparatively unchanged by the increments arising from transformed kinetic energy.

An objection which may, perhaps, be raised against the present hypothesis is that it implies that the world is progressing continually towards a state in which ψ is uniform, and so its life would be finite. It may be replied, however, that it is by no means certain that this rate of progress would remain constant; if ρ or $\frac{\partial \psi}{\partial t}$ diminishes in magnitude as the differences in values of ψ become less, the progress in one direction could continue indefinitely.

The one great advantage of the hypothesis is that it gives a definite physical meaning to the lines of electric force; they are, in fact, the lines of flow of the concealed energy.

It should be remarked that the total energy of an electron or positive nucleus of an atom is on the present view enormously large, and is not represented by mc^2, where m is the mass. It is possible, however, that it may be equal to Mc^2, where M is the sum of the masses of all the minute electric charges, positive and negative, which make up the electron. It is easy to see that M can be very much greater than m.

It is probable that if an electron is made up of minute electric charges, these are extremely small, and perhaps as numerous as the electrons in the universe. It is generally thought that such charges would fly apart under their mutual repulsion, but even if we exclude the attractive influence of the minute positive charges, it is still possible to secure permanence if the masses of the minute charges are enormously large in comparison with the forces which act on them, a condition that may be satisfied by making the size of the minute charges sufficiently small. The idea that these minute charges are as numerous as the electrons in the universe is akin to the idea that the surface of an electron is a sphere of inversion, and that the interior of the electron is an inverse copy of the external universe. This simple correspondence may not apply exactly when the motion of electric charges is taken into account, but there may be some analogous correspondence which holds in this case.

ON A DIOPHANTINE PROBLEM.

By *H. Holden.*

1. THE equations

$$\alpha a^2+\beta b^2-\gamma c^2=z^2,$$
$$\alpha b^2+\beta c^2-\gamma a^2=x^2,$$
$$\alpha c^2+\beta a^2-\gamma b^2=y^2,$$

where α, β, γ are given quantities, can usually be solved, and a, b, c, x, y, z expressed in terms of one rational parameter whenever the auxiliary equation $\beta\gamma p^2+\gamma\alpha q^2=\alpha\beta r^2$ can be similarly solved.

For if p, q, r, expressed in terms of the rational parameter k, satisfy the auxiliary equation, it will be seen that the linear relation $p_1a+q_1b+r_1c=0$ makes $\alpha a^2+\beta b^2-\gamma c^2$ a perfect square. Similarly $r_2a+p_2b+q_2c=0$, where p_2, q_2, r_2 are expressed in terms of m, makes $\alpha b^2+\beta c^2-\gamma a^2$ a perfect square.

On solving the two equations

$$p_1a+q_1b+r_1c=0,$$
$$r_2a+p_2b+q_2c=0,$$

values of a, b, c, expressed as quadratic functions of m and k, are obtained, satisfying the first two original equations.

Substituting these values of a, b, c in the third expression $\alpha c^2+\beta a^2-\gamma b^2$, an expression of the fourth degree in m or k is obtained, with the condition that this expression must be a square. Arranging in powers of m it will be seen that the first and last coefficients (functions of k) are perfect squares,* and so by Fermat's method we find two suitable values of m in terms of k, from which others may, in general, be obtained.

Substituting either of these values of m, a, b, c, and hence x, y, z are got in terms of the arbitrary rational parameter k. By altering the sign of b in one of the two linear equations, their solution will yield a different biquadratic expression, and two more sets of values for a, b, c, x, y, z.

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* This statement holds for the examples given below, but in some cases it is only true for special values of k.

The above equations are equivalent to others of the usual Diophantine type: for example, they may be regarded as furnishing solutions of

$$x^2+2\gamma a^2=y^2+2\gamma b^2=z^2+2\gamma c^2=\alpha a^2+\beta b^2+\gamma c^2$$

and $$(\alpha+\beta-\gamma)(a^2+b^2+c^2)=x^2+y^2+z^2.$$

2. The equations

$$2a^2+2b^2-c^2=z^2,$$
$$2b^2+2c^2-a^2=x^2,$$
$$2c^2+2a^2-b^2=y^2,$$

can be solved by the above method.

For the auxiliary equation is $p^2+q^2=2r^2$ or

$$(k^2-2k-1)^2+(k^2+2k-1)^2=2(k^2+1)^2,$$

and so the first equation is satisfied by

$$(k^2-2k-1)\,a+(k^2+2k-1)\,b+(k^2+1)\,c=0,$$

and the second equation by

$$(m^2+1)\,a+(m^2-2m-1)\,b+(m^2+2m-1)\,c=0.$$

Solving these two linear equations in a, b, c, we may write, since the signs of a, b, c are immaterial,

$$a=m^2(k-1)+2mk(k+1)-(k-1),$$
$$b=m^2(k+1)-m(k^2-2k-1)+k(k-1),$$
$$c=2km^2+m(k^2-2k-1)+(k^2-1).$$

Substituting in $2c^2+2a^2-b^2$ we get the condition that

$$m^4(3k-1)^2+2m^3(9k^3-9k^2-11k-1)+3m^2(3k^4+6k^3+2k^2+2k-1)$$
$$+2m(3k^4-11k^3-3k^2+9k+2)+(k^2+k-2)^2$$

is a square. This is satisfied by

$$m=-\frac{(12k^4+23k^3+12k^2+3k+4)}{(k+2)(9k^2+8k+1)}$$

and by $$m=\frac{(k-1)(3k^2+4k+2)}{(k+2)(-k^2+2k+1)}.$$

Putting $k=2$ in the second relation, we get $m=\frac{11}{2}$ and

$$a=127,\qquad x=261,$$
$$b=131,\qquad y=255,$$
$$c=158,\qquad z=204.$$

These values give solutions of

$$x^2 + 3a^2 = y^2 + 3b^2 = z^2 + 3c^2 = 2(a^2 + b^2 + c^2),$$
$$3(a^2 + b^2 + c^2) = x^2 + y^2 + z^2,$$
$$9(a^4 + b^4 + c^4) = x^4 + y^4 + z^4,$$

and hence of

$$(3bc)^2 + (3ca)^2 + (3ab)^2 = (yz)^2 + (zx)^2 + (xy)^2,$$

and so the problem of finding six numbers, such that the sum of the squares of the first three is equal to three times the sum of the squares of the second three, and the sum of the fourth powers of the first three is equal to nine times the sum of the fourth powers of the second three, may be attacked by the present method.

Again, if a, b, c be regarded as the sides of a triangle ABC, x, y, z would represent double its medians, and so the problem of finding triangles whose sides and medians are rational or integral has been solved. From this point of view, since a triangle XYZ, whose sides are double the medians of ABC, will have medians, which when doubled will be three times the sides of ABC, we may infer that the above values have the companion-set

$$a = 261, \qquad x = 381,$$
$$b = 255, \qquad y = 393,$$
$$c = 204, \qquad z = 474.$$

These results reduce to

$$a = 87, \qquad x = 127,$$
$$b = 85, \qquad y = 131,$$
$$c = 68, \qquad z = 158.$$

They are got, in a different order, by using $k = 4$, $m = -\frac{33}{7}$.

Using again the last value of m, and putting $k = -3$, $m = -\frac{34}{7}$, we have

$$a = 607, \qquad x = 1011,$$
$$b = 134, \qquad y = 1440,$$
$$c = 823, \qquad z = 309,$$

which show that values of a, b, c may be found for which no triangle is possible. From these values may be deduced

$$a = 337, \qquad x = 607,$$
$$b = 480, \qquad y = 134,$$
$$c = 103, \qquad z = 823.$$

Substituting for m by the second value, discarding the common factor $(k-1)k(k+1)$, and clearing fractions, we have

$$a = 6k^5 - 24k^3 - 53k^2 - 23k + 4,$$

$$b = 4k^5 + 3k^4 - 13k^3 + 24k^2 + 40k + 14,$$

$$c = 2k^5 - 23k^4 - 21k^3 + 23k^2 + 27k + 10,$$

$$x = 2k^5 - 34k^4 - 42k^3 + 31k^2 + 73k + 24,$$

$$y = 8k^5 - 13k^4 + 15k^3 + 70k^2 + 34k - 6,$$

$$z = 10k^5 + 7k^4 - 63k^3 - 61k^2 - 37k - 18.$$

Thus if $k = -\frac{2}{3}$, we get, after clearing fractions,

$$a = 255, \qquad x = 659,$$

$$b = 233, \qquad y = 683,$$

$$c = 442, \qquad z = 208.$$

If the linear equations had been written

$$(k^2 - 2k - 1)\,a + (k^2 + 2k - 1)\,b + (k^2 + 1)\,c = 0,$$

$$(m^2 + 1)\,a - (m^2 - 2m - 1)\,b + (m^2 + 2m - 1)\,c = 0,$$

and the above procedure followed, two different sets of values of a, b, c would have been obtained.

3. The equations in the preceding section may be regarded as a special case of

$$(s^2 + st)\,a^2 + (st + t^2)\,b^2 - st\,c^2 = z^2,$$

$$(s^2 + st)\,b^2 + (st + t^2)\,c^2 - st\,a^2 = x^2,$$

$$(s^2 + st)\,c^2 + (st + t^2)\,a^2 - st\,b^2 = y^2,$$

for which the auxiliary equation is $sp^2 + tq^2 = (s + t)\,r^2$ or

$$s\,(sm^2 - 2tm - t)^2 + t\,(sm^2 + 2sm - t)^2 = (s + t)\,(sm^2 + t)^2.$$

Hence the above equations can be solved, and also yield solutions of the simultaneous equations

$$(s^2 + st + t^2)\;(a^2 + b^2 + c^2) = x^2 + y^2 + z^2,$$

$$(s^2 + st + t^2)^2\,(a^4 + b^4 + c^4) = x^4 + y^4 + z^4;$$

and by putting $$s = S^2 - T^2,$$

$$t = 2ST + T^2,$$

we get solutions of

$$A^2+B^2+C^2=x^2+y^2+z^2,$$
$$A^4+B^4+C^4=x^4+y^4+z^4.$$

When both the linear equations in a, b and c have literal coefficients, the work involved is somewhat laborious, but, in general, any number of solutions may be obtained by using numerical coefficients in one equation.

Thus, taking the equations

$$6a^2+3b^2-2c^2=z^2,$$
$$6b^2+3c^2-2a^2=x^2,$$
$$6c^2+3a^2-2b^2=y^2,$$

for which the auxiliary equation is $p^2+2q^2=3r^2$, the first equation is satisfied by $5a+b+3c=0$, and the second equation by $(m^2+2)a-(m^2-4m-2)b+(m^2+2m-2)c=0$, which give

$$a=2m^2-5m-4,$$
$$b=m^2+5m-8,$$
$$c=3m^2-10m-4.$$

Substituting in the third equation, the expression

$$64m^4-440m^3+465m^2+760m+16$$

is a square. This is satisfied by $m=\frac{80}{29}$ or $\frac{240}{101}$,* and so

$$a=541, \qquad x=7565,$$
$$b=2818, \qquad y=2309,$$
$$c=1841, \qquad z=4336.$$

In this case a, b, c do not form a triangle, but when they do, if each side, taken in order, be produced by its own length, x, y, z would be the lines joining the ends. Or if each side of the triangle be divided in the ratio of 2 to 1, x, y, z would be 3 times the lines joining the points of division to the opposite angles.

4. Legendre's equations

$$a^2+b^2-c^2=z^2,$$
$$b^2+c^2-a^2=x^2,$$
$$c^2+a^2-b^2=y^2$$

may be solved.

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* Also by $m=-\frac{22}{5}$, which gives

$$a=709, \qquad x=1900,$$
$$b=133, \qquad y=3239,$$
$$c=1226, \qquad z=251.$$

The auxiliary equation is $p^2+q^2=r^2$.
The first equation is satisfied by

$$(k^2-1)\,a+2kb+(k^2+1)\,c=0,$$

and the second by

$$(m^2+1)\,a+(m^2-1)\,b+2mc=0,$$

or by
$$a=-m^2(k^2+1)+4km+k^2+1,$$
$$b=\quad m^2(k^2+1)-2m(k^2-1)+k^2+1,$$
$$c=\quad m^2(k^2-2k-1)-(k^2+2k-1),$$

and we get the condition that

$$m^4(k^2-2k-1)^2+4m^3(k^2+1)(k^2-2k-1)-2m^2(5k^4-14k^2+5)$$
$$+4m(k^2+1)(k^2+2k-1)+(k^2+2k-1)^2$$

is a square.* This is true if

$$m=\frac{3k^4-2k^2+3}{2(k^2+1)(k^2-2k-1)}\text{ or }\frac{2(k^2+1)(k^2+2k-1)}{3k^4-2k^2+3},$$

so that a, b, c, x, y, z may be expressed in terms of k. These results solve

$$x^2+2a^2=y^2+2b^2=z^2+2c^2=a^2+b^2+c^2$$

and
$$a^2+b^2+c^2=x^2+y^2+z^2.$$

5. The equations in §§ 2 and 4 are special cases of

$$(s^2+1)\,b^2+(s^2+1)\,c^2-a^2=x^2,$$
$$(s^2+1)\,c^2+(s^2+1)\,a^2-b^2=y^2,$$
$$(s^2+1)\,a^2+(s^2+1)\,b^2-c^2=z^2,$$

where s is a given rational quantity.
The auxiliary equation is $p^2+q^2=(s^2+1)\,r^2$ or

$$(m^2-2sm-1)^2+(sm^2+2m-s)^2=(s^2+1)(m^2+1)^2.$$

Hence the method is applicable to this type of equation, and thus solves

$$(2s^2+1)(a^2+b^2+c^2)=x^2+y^2+z^2.$$

6. Certain simpler, but less general, solutions of the above types of equations may be found without using the biquadratic

* In this case it is noteworthy that if the expression were arranged in powers of k, neither the coefficient of k^4, nor the term independent of k, would be a perfect square.

expression in m. These are got by using the trivial solutions of the auxiliary equation. Thus for the equations

$$(s^2+1)\,a^2+(s^2+1)\,b^2-c^2=z^2, \text{ \&c.},$$

the trivial solution of $p^2+q^2=(s^2+1)\,r^2$ is $1, s, 1$, and it will be found that the relation $a-sb-c=0$ satisfies two of the equations. Hence, combining this with

$$(m^2-2sm-1)\,a-(m^2+1)\,b-(sm^2+2m-s)\,c=0,$$

which satisfies the third equation, we get the solutions

$$a=m^2(s^2-1)+2sm-(s^2+1),$$
$$b=m^2(s-1)+2m\,(s+1)-(s-1),$$
$$c=m^2(s-1)-2s^2m-(s+1),$$

and another set of values by combining $a-sb-c=0$ and

$$(m^2-2sm-1)\,a+(m^2+1)\,b-(sm^2+2m-s)\,c=0.$$

Thus for the equations $5a^2+5b^2-c^2=z^2$, etc., we get

$$a=3m^2+4m-5, \qquad x=m^2-22m-5,$$
$$b=m^2+6m-1, \qquad y=7m^2+2m+13,$$
$$c=m^2-8m-3, \qquad z=7m^2+14m-11,$$

and

$$a=3m^2+4m-5, \qquad x=9m^2-18m-5,$$
$$b=3m^2-2m-3, \qquad y=9m^2-6m+11,$$
$$c=3m^2-8m-1, \qquad z=9m^2+6m-13.$$

For equations of the type

$$(s^2+st)\,a^2+(st+t^2)\,b^2-st\,c^2=z^2$$

the trivial solutions of the auxiliary equation are $1, 1, 1$, and it will be found that the relation $c=a+b$ satisfies each of the three equations. This would give

$$x=ta+(s+t)\,b,$$
$$y=(s+t)\,a+sb,$$
$$z=sa-tb.$$

In this case the triangle ABC becomes a straight line, and so the lines got by producing the sides in the opposite order would also be rational, and hence

$$\{ta+(s+t)\,b\}^2+\{(s+t)\,a+sb\}^2+(sa-tb)^2$$
$$=\{sa+(s+t)\,b\}^2+\{(s+t)\,a+tb\}^2+(ta-sb)^2$$
$$=(s^2+st+t^2)\,\{a^2+b^2+(a+b^2)\}$$

and $$\{ta+(s+t)b\}^4+\{(s+t)a+sb\}^4+(sa-tb)^4$$
$$=\{sa+(s+t)b\}^4+\{(s+t)a+tb\}^4+(ta-sb)^4$$
$$=(s^2+st+t^2)^2\{a^4+b^4+(a+b)^4\}.$$

By putting $$s=S^2-T^2,$$
$$t=2ST+T^2,$$

s^2+st+t^2 becomes a square, and so we get integral solutions of

$$A^2+B^2+C^2=x_1^2+y_1^2+z_1^2=x_2^2+y_2^2+z_2^2,$$
$$A^4+B^4+C^4=x_1^4+y_1^4+z_1^4=x_2^4+y_2^4+z_2^4.$$

Thus if $$s=5,\quad t=3,\quad a=3,\quad b=1,$$
$$17^2+29^2+12^2=23^2+27^2+4^2=7^2+21^2+28^2,$$
$$17^4+29^4+12^4=23^4+27^4+4^4=7^4+21^4+28^4.$$

7. The method may be used for equations of a somewhat different type to that originally mentioned. Thus for

$$bc+ca-ab=z^2,$$
$$ca+ab-bc=x^2,$$
$$ab+bc-ca=y^2,$$

the linear relation for the first equation is

$$k^2a+b-(k+1)^2c=0,$$

and using for the second equation $4b+c-9a=0$ we get the condition that $48m^4+176m^3+272m^2+252m+81$ is a square. Thus $m=-\frac{20}{47}$ and

$$a=9605,\qquad x=11733,$$
$$b=8541,\qquad y=5139,$$
$$c=52281,\qquad z=29439.$$

Similarly the equations

$$a^2\pm bc=x^2,\quad b^2\pm ca=y^2,\quad c^2\pm ab=z^2$$

may be solved by using $2ka+k^2b\mp c=0$ for the first equation and proceeding as before.

Finally, chains of simultaneous equations of the same quadratic form may be solved by using the known solution of one of them to form the linear relation. Thus

$$a^2+ab+b^2=z^2,\quad b^2+bc+c^2=x^2,\quad c^2+ca+a^2=y^2$$

have as linear relation $a(k^2-1)-b(2k+1)=0$. Using this with $3b-5c=0$ we get

$$a=-80, \quad x=147,$$
$$b=105, \quad y=73,$$
$$c=63, \quad z=95.$$

8. The same idea avails for Legendre's equations, which may be written

$$x^2+y^2=2c^2, \quad y^2+z^2=2a^2, \quad z^2+x^2=2b^2,$$

and so $$(k^2-2k-1)x-(k^2+2k-1)y=0$$
$$(m^2-2m-1)y-(m^2+2m-1)z=0.$$

Hence $$x=(k^2+2k-1)(m^2+2m-1),$$
$$y=(k^2-2k-1)(m^2+2m-1),$$
$$z=(k^2-2k-1)(m^2-2m-1),$$

with the condition that

$$m^4(k^2+1)^2+16m^3k(k^2-1)+2m^2(k^2+1)^2-16mk(k^2-1)+(k^2+1)^2$$

is a square. This is satisfied by $m=-\dfrac{(k^2+1)^2}{2k(k^2-1)}$ with

$$a=(k^2-2k-1)(m^2+1),$$
$$b=(k^2+1)(m^2-3),$$
$$c=(k^2+1)(m^2+2m-1).$$

If $k=2$, $m=-\frac{25}{12}$, and

$$a=769, \quad x=833,$$
$$b=965, \quad y=119,$$
$$c=595, \quad z=1081.$$

Again, writing the linear relations as

$$x(k^2-2kl-l^2)-(k^2+2kl-l^2)y=0,$$
$$y(m^2-2mn-n^2)-(m^2+2mn-n^2)z=0,$$

k, l, m, n being now integers, it will be seen that simpler results would be obtained if $k^2-2kl-l^2$ and $m^2+2mn-n^2$ could be made equal. For then

$$x=k^2+2kl-l^2 \quad a=m^2+n^2,$$
$$y=k^2-2kl-l^2=m^2+2mn-n^2,$$
$$z=m^2-2mn-n^2 \quad c=k^2+l^2,$$

with the condition that

$$(m^2 - 2mn - n^2)^2 + (k^2 + 2kl - l^2)^2 = 2b^2.$$

Adding $\quad (m^2 + 2mn - n^2)^2 - (k^2 - 2kl - l^2)^2 = 0,$

it follows that $\quad (m^2 + n^2)^2 + 4kl(k^2 - l^2)$

is a square. This will be satisfied if $2(m^2 + n^2)$ is equal to the difference of any two factors of the last term in the expression. Thus if

$$2(m^2 + n^2) = k^2 - 4kl - l^2$$

and also $\quad m^2 + 2mn - n^2 = k^2 - 2kl - l^2$

we get $\quad 4m(m+n) = (3k + l)(k - 3l)$

$$4n(n - m) = (l + k)(l - k).$$

Let $\quad 2m = 3k + l$

$$2(m+n) = k - 3l,$$

then $\quad 2n = -2(2l + k)$

$$2(n - m) = -5(l + k),$$

and so $\quad 10(l + k)(2l + k) = (l + k)(l - k),$

which gives $\quad k = 19, \quad l = -11, \quad m = 23, \quad n = 3.$

and $\quad a = 269, \quad x = 89,$

$$b = 149, \quad y = 329,$$

$$c = 241, \quad z = 191,$$

which are the values obtained by Legendre. The equations

$$px^2 + qy^2 = (p + q)c^2,$$

$$py^2 + qz^2 = (p + q)a^2,$$

$$pz^2 + qx^2 = (p + q)b^2,$$

may be treated in a similar way, and so equations of the type

$$q^2b^2 + p^2c^2 - pqa^2 = (p^2 - pq + q^2)x^2$$

may be solved.

9. The equations $2a^2 + b^2 - c^2 = 2z^2$, etc., for which the linear relation is $4mb + (m^2 - 2)c + (m^2 - 2)a = 0$, give less general results, as the coefficient of m^4 is a square for certain

values of k only. Thus for $k=1$, $m=\frac{8}{5}$, we get

$$a=19, \quad x=19,$$
$$b=11, \quad y=31,$$
$$c=29, \quad z=1.$$

Similarly for the equations $a^2+2b^2-2c^2=z^2$, for which the linear relation is $2ma+(m^2-2)b+(m^2+2)c=0$. A solution is

$$a=97, \quad x=67,$$
$$b=107, \quad y=43,$$
$$c=77, \quad z=143.$$

Solutions of the equations

$$a^2+b^2-c^2=z^2,$$
$$b^2+c^2-d^2=t^2,$$
$$c^2+d^2-a^2=x^2,$$
$$d^2+a^2-b^2=y^2$$

may be obtained.

A SIMPLE PROOF OF THE FUNDAMENTAL EQUALITY IN THE THEORY OF THE GAMMA FUNCTION.

By *S. Pollard*.

THE equality with which we are concerned is

$$(1) \qquad \lim_{n\to\infty} \frac{n^z\, n!}{z(z+1)\ldots(z+n)} = \int_0^\infty t^{z-1} e^{-t}\, dt;$$

and we shall shew that it is true whenever the right-hand side exists. It is fundamental because by its means we reconcile the two primary definitions of the Gamma-function, Gauss's (which is equivalent to Weierstrass's) and Euler's. For the former expresses $\Gamma(z)$ as the left-hand member of the equality and the latter expresses it as the right.

The proof falls into two parts, which we will summarise in order. The first consists in shewing that the points at which the left-hand member exists form a connected region in which it is one-valued and analytic, and that those at which the right-hand member exists are either interior or boundary points of another connected region in which it also is one-valued and analytic. By the theory of analytic continuation it follows that, in order to establish the equality at all points z where its members exist, it is sufficient to establish it at all points of a portion of a curve within the connected region common to both the regions above. The second part consists in doing this for the particular portion of a curve given by real values of z between 0 and 1.

For the first part it is sufficient to state the following results. The left-hand member of (1) is one-valued and analytic in the region consisting of the whole plane minus the points $0, -1, -2, \ldots$, and does not exist anywhere else*. And the right-hand member is one-valued and analytic in the region $R(z) > 0$, and does not exist where $R(z) < 0$; *i.e.* it exists only in $R(z) > 0$, and possibly at points which are on its boundary $R(z) = 0$.†

For the second part of the proof, we notice that the common region is $R(z) > 0$, which contains the portion of the real axis given by $0 < R(z) < 1$; and so it is sufficient for our purpose to prove that

$$\lim_{n \to \infty} \frac{n^x \, n!}{x(x+1)\ldots(x+n)} = \int_0^\infty t^{x-1} e^{-t} \, dt$$

for $x > 0$. Now by means of repeated integrations by parts we can shew that the respective members of this equation are identical with

$$\lim_{n \to \infty} \frac{1}{x} \int_0^n t^x \left(1 - \frac{t}{n}\right)^n dt, \qquad \frac{1}{x} \int_0^\infty t^x e^{-t} \, dt,$$

respectively. We are thus left with the problem of shewing that

$$(2) \qquad \lim_{n \to \infty} \int_0^n t^x \left(1 - \frac{t}{n}\right)^n dt = \int_0^\infty t^x e^{-t} \, dt \quad (x > 0).$$

Various methods of doing this will be found referred to in a paper by Dr. G. N. Watson.‡ All these methods involve the use of one or more inequalities which are not obvious and which require a fair amount of analysis to obtain. It is, however, possible to get a simple proof of (2) by the application of a result in the theory of double series. This result, which is fundamental in the theory, is as follows:

* Whittaker and Watson, *Modern Analysis*, 2nd ed., pp. 230 and 231.

† Goursat, *Cours d'Analyse Mathématique*, 2nd ed., vol. ii., p. 269.

‡ *Messenger of Mathematics*, vol. xlv., p. 28, June, 1915.

*A double series of positive terms which possesses a repeated sum also possesses a double sum, and so all sums which take into account every term exist and are equal.**

The theorem evidently holds good for an increasing double sequence as well as for a double series of positive terms.

Define a sequence $S_{m,n}$ by the equations

$$S_{m,n} = \int_0^m t^x \left(1 - \frac{t}{n}\right)^n dt \quad (m \leq n)$$

$$= \int_0^n t^x \left(1 - \frac{t}{n}\right)^n dt \quad (m > n).\dagger$$

Then the integrand is positive, and so $S_{m,n}$ increases with m. Again, t^x is positive, and $1-(t/n)^n$ increases with n for all values of t within the limits of integration.‡ Thus $S_{m,n}$ increases with m or n, and we can apply the given result.

Now, for $x > 0$, $S_{m,\infty}$ exists and is equal to $\int_0^m t^x e^{-t} dt$, because $\left(1 - \frac{t}{n}\right)^n$ tends uniformly to e^{-t} for $0 \leq t \leq m$, and $\int_0^m |t^x| dt$ is convergent for $x > 0$.§

Again, from the convergence of $\int_0^\infty t^x e^{-t} dt$, it follows that $\lim_{m \to \infty} S_{m,\infty}$ exists. Thus a repeated limit exists, and therefore, by the theorem quoted, the double limit and all other limits in which both m and n tend to ∞ exist and are equal. In particular we have

$$\lim_{n \to \infty} S_{n,n} = \int_0^\infty t^x e^{-t} dt,$$

and this is (2).

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* See Bromwich, *Infinite Series*, pp. 76—78.

† The point of this artifice is that we secure $0 < t < n$ for all values of t between the limits of integration.

‡ For

$$\frac{d}{d\nu} \log \left(1 - \frac{t}{\nu}\right)^\nu = \log\left(1 - \frac{t}{\nu}\right) + \frac{\nu}{1-(t/\nu)} \frac{t}{\nu^2}$$

$$= \log\left(1 - \frac{t}{\nu}\right) + \frac{t}{\nu}\left(1 - \frac{t}{\nu}\right)^{-1}$$

$$= \tfrac{1}{2}\frac{t^2}{\nu^2} + \tfrac{2}{3}\frac{t^3}{\nu^3} + \dots,$$

if $0 < t < \nu$, and is positive.

§ See Bromwich, *Infinite Series*, p. 448, where the analogous result for series is given.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

L.

On the integral of Stieltjes and the formula for integration by parts.

1. In the first note of this series* I gave a very direct and simple proof that

$$(1) \qquad \int_a^b f\,dx \int_a^x g\,dt + \int_a^b g\,dx \int_a^x f\,dt = \int_a^b f\,dx \int_a^b g\,dx$$

whenever f and g are bounded and integrable in Riemann's sense. I afterwards found that this form of the theorem of integration by parts was not new, and is really due to Thomae†; Thomae's proof, which is in all essentials the same as mine, is reproduced in his *Vorlesungen über bestimmte Integrale und die Fourierschen Reihen* (second edition, 1908). It is remarkable that the theorem has not found a place in any of the standard works. Thus in Hobson's *Theory of functions of a real variable* (1907) a less simple and comprehensive result, due to du Bois-Reymond, is given instead‡.

The theorem, like all theorems of this character, has lost a good deal of its interest owing to the general supersession of Riemann's concept of an integral by Lebesgue's. The theorem is true for all summable functions, bounded or unbounded. But, when the sign of integration is interpreted in the sense of Lebesgue, the formula (1) ceases to be more general than the ordinary form of the theorem of integration by parts, viz.

$$(2) \qquad \int_a^b F'G\,dx = F(b)\,G(b) - \int_a^b FG'\,dx,$$

where
$$F = \int_a^x f\,dt, \qquad G = \int_a^x g\,dt.$$

For F and G, being integrals, possess derivatives $F' = f$ and $G' = g$ for almost all values of x, and (1) and (2) are exactly equivalent§.

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* *Messenger*, vol. xxx. (1901), pp 185–187.

† J. Thomae, 'Die partielle Integration', *Zeitschrift für Mathematik*, vol. xx. (1875), pp. 475–478.

‡ *l.c.*, pp. 407–409. The theorem may be deduced from the theorems concerning repeated integration, due to Du Bois Reymond, Harnack, Arzelà, Jordan, and Pringsheim, given by Hobson on pp. 421–430.

§ See de la Vallée-Poussin, *Cours d'Analyse*, vol. i. (ed. 3, 1914), pp. 268 *et seq.*

It does not seem possible to give a proof of the theorem, even for *bounded* summable functions, of quite so elementary a character as that of the more special theorem given by Thomae and by myself—a proof, that is to say, which depends merely on a simple passage to the limit from 'Abel's Lemma' or some similar algebraical identity*.

The restricted theorem and its proof therefore still retain a certain interest. This interest is considerably increased by the fact that the theorem may easily be generalised in another direction, when the integrals considered are the 'Stieltjes' integrals' which have recently attracted so much attention, that is to say integrals of the type

$$\int f\, d\phi,$$

where ϕ is a bounded and monotonic (but not necessarily continuous) function, or, more generally, any function of bounded variation. We have, in fact, the following theorem.

THEOREM. *Suppose that ϕ and ψ are functions of x, of bounded variation in the interval (a, b), and not simultaneously discontinuous; and that f is bounded and possesses an integral with respect to ϕ, and g is bounded and possesses an integral with respect to ψ. Then*

$$(3) \quad \int_a^b f\, d\phi \int_a^x g\, d\psi + \int_a^b g\, d\psi \int_a^x f\, d\phi = \int_a^b f\, d\phi \int_a^b g\, d\psi.$$

2. It will be necessary to begin by a short sketch of some fundamental propositions concerning Stieltjes' integrals.†

* The theorem, in its most general form, is a simple corollary of the theorem that a Lebesgne double integral may be evaluated by repeated integration in either of the two possible orders: but such a proof appeals to considerations of a distinctly more difficult character.

† In the account of the theory which follows I am much indebted to Prof. W. H. Young's very valuable paper, 'On integration with respect to a function of bounded variation'. *Proc. London Math. Soc.*, ser. 2, vol. xiii. (1913), pp. 109–150. In this paper Prof. Young develops a theory of integration which includes those of both Stieltjes and Lebesgue. In this note I am only concerned with what he calls 'Riemann integrals with respect to a function of bounded variation', that is to say integrals defined in accordance with the ideas of Stieltjes himself (see pp. 130 *et seq.* of Prof. Young's paper). I develop the theory in a quite different manner from that adopted by Prof. Young, my object being to assimilate it as far as possible to de la Vallée-Poussin's treatment (*l.c.*, pp. 250 *et seq.*) of the theory of the ordinary Riemann integral. This has led me to begin by a separate treatment of the case in which ϕ is *continuous*, a course for which there is a good deal to be said in any event. In this case the theory runs on almost exactly the same lines as the Riemann-Darboux theory, so much so that it is often unnecessary to state proofs at length; and, as the discontinuities of ϕ are in any case enumerable, the transition to the general case is (as will be seen in § 7) a simple matter.

I should also refer, in connection with the integral of Stieltjes and other generalisations of the notion of integration, to

G. A. Bliss, 'Integrals of Lebesgue', *Bulletin of the American Mathematical Society*, vol. xxiv., pp. 1–46 (pp. 12–17, 28–30):

T. H. Hildebrandt, 'On integrals related to and extensions of Lebesgue integrals', *ibid.*, pp. 113–144, 177–202).

The Stieltjes' integral

$$\int_a^b f\,d\phi, \tag{4}$$

where ϕ is a function of x of bounded variation in (a, b), may be defined as being the limit, if it exists, of

$$\sum_0^{n-1} f(\xi_i)\,\{\phi(x_{i+1}) - \phi(x_i)\}, \tag{5}$$

where $$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

and $x_i \leq \xi_i \leq x_{i+1}$, when the intervals (x_i, x_{i+1}) tend to zero. We may obviously suppose, without loss of generality, that ϕ is *monotonic*, say increasing. In this form the definition is that of Stieltjes himself. When $\phi = x$, the definition reduces to that of Riemann.

I shall adopt here a different definition. The points of discontinuity of ϕ form at most an enumerable set. I find it convenient to suppose, in the first instance, that ϕ is *continuous*. Suppose then that ϕ is steadily increasing and continuous, and write

$$\delta_i = x_{i+1} - x_i, \quad \eta_i = \phi(x_{i+1}) - \phi(x_i);$$

and let M_i, m_i, and $O_i = M_i - m_i$ denote the upper and lower bounds and the oscillation of f in δ_i. Further, let

$$s = \Sigma m_i \eta_i, \quad S = \Sigma M_i \eta_i,$$

so that s and S are bounded and $s \leq S$. Then we define the lower and upper integrals of f with respect to ϕ, viz.

$$L = \overline{\int_a^b} f\,d\phi, \quad U = \underline{\int_a^b} f\,d\phi,$$

as being respectively the upper bounds of the sums s and the lower bounds of the sums S. It is obvious that $L \leq U$.

When the intervals δ_i tend to zero, s tends to its upper bound L and S to its lower bound U. This is the analogue of Darboux's theorem concerning lower and upper Riemann integrals, and may be proved in the same manner; in fact the modifications necessary in de la Vallée-Poussin's proof of Darboux's theorem* are of so trivial a character that they may be left to the reader.

If $L = U$, f is integrable with respect to ϕ, and the Stieltjes' integral (4) is, by definition, their common value.

* *l.c.*, pp. 250–251. The truth of this remark will be found to depend essentially on the fact that ϕ is continuous.

The necessary and sufficient condition for integrability is that

$$\lim O_i \eta_i = 0:$$

in this case the sum (5) tends to the limit (4), so that the definition agrees with that of Stieltjes. The sum or product of two integrable functions is integrable, and so forth.

Another form of the condition of integrability is

$$\overline{\int_a^b} Of\, d\phi = 0, \tag{6}$$

where Of is the oscillation of f at the point x, that is to say the limit of the oscillation of f in a small interval including x. In particular any continuous function (for which $Of = 0$) is integrable.

All these propositions are easily proved by trifling modifications of the arguments used by de la Vallée-Poussin.*

3. Before proceeding further, we introduce the ideas of the *increment* and the *variation* of ϕ in a set of points S. Of these ideas the first is a generalisation of Cantor's concept of the *content* of S (to which it reduces when $\phi = x$). The second is a similar generalisation of the concept of *measure*. The increment of ϕ in S is *the lower bound of the sum of the increments of ϕ in a finite set of intervals containing S*; the variation of ϕ in S is the lower bound of the sum of the increments in a *finite or infinite* set. It is obvious from the definition that the variation cannot exceed the increment. The increment may be expressed in the form

$$\overline{\int} e\, d\phi,$$

where e is the 'characteristic function' of the set S, *i.e.* the function which is unity in points of S and zero elsewhere.

If S is closed, the increment and the variation are the same, and each is equal to

$$\omega = \phi(b) - \phi(a) - \Sigma \Delta\phi_\beta,$$

where $\Delta\phi_\beta = \phi(\xi') - \phi(\xi)$ is the increment of ϕ in a 'black interval' $\beta = (\xi, \xi')$ of S.†

* It should however be remarked that de la Vallée-Poussin's deduction of the last form of the condition of integrability requires some expansion: see a note by S. Pollard, 'Note on a proof in de la Vallée-Poussin's *Cours d'Analyse*', *Messenger*, vol. xlix. (1918), pp. 141–144.

† Following Prof. Young, I use the phrase 'black intervals of S' to denote the intervals whose internal points form the set complementary to S.

(i) Enclose S in any manner in a system of intervals j. From these, together with the black intervals, we can select a finite subset covering up the whole interval (a, b); and the sum of the increments of ϕ in this subset of intervals is at least $g(b)-g(a)$. The sum of the increments in the intervals j is consequently not less than ω; and so the variation of ϕ in S is not less than ω.

(ii) Choose a finite number of black intervals β' so that

$$\Sigma\Delta\phi_{\beta'} > \Sigma\Delta\phi_{\beta} - \epsilon.$$

These leave over in (a, b) a finite system of intervals including S; and the sum of the increments of ϕ in this system is

$$\phi(b) - \phi(a) - \Sigma\Delta\phi_{\beta'} < \omega + \epsilon.$$

As ϵ is arbitrary, the increment of ϕ in S is not greater than ω.

Since the variation cannot exceed the increment, it follows from (i) and (ii) that the two are equal and that their common value is ω.

Variation possesses the characteristic properties of measure. In particular, the variation of ϕ in a sum of sets S_1, S_2, S_3, ..., which have no point in common, is equal to the sum of the variations of ϕ in the individual sets.

This is almost obvious if ϕ has no stretches of invariability; for, if we write $y = \phi(x)$, the variation of ϕ in S is the same thing as the measure of the corresponding set T of values of y. If ϕ has stretches of invariability, $\psi = \phi + \alpha x$, where $\alpha > 0$, will have none. The variation of ϕ is less than that of ψ by α times the measure of S; so that our proposition is true in any case.

4. We can now express the condition of integrability in terms of the variation of ϕ in the set of discontinuities of f.

(i) *In order that f should be integrable with respect to ϕ, it is necessary and sufficient that the increment of ϕ, in the set of points S_ϵ at which the oscillation* of f is not less than ϵ, should be zero for all values of ϵ.*

Since S_ϵ is closed, it is indifferent whether we consider the increment or the variation of ϕ. The proof of the proposition is practically a reproduction of de la Vallée-Poussin's proof† of the corresponding proposition for Riemann integrals.

* Following de la Vallée-Poussin, I use 'oscillation' as equivalent to 'measure of discontinuity', even when the discontinuity is of what is often described as an 'ordinary', as opposed to an 'oscillatory', type.

† *l. c.*, p. 256.

From (i) and the last paragraph of § 3 we can at once deduce an important theorem due to Young.

(ii) *In order that f should be integrable with respect to ϕ, it is necessary and sufficient that the variation of ϕ in the set S of discontinuities of f should be zero.*

If f is integrable, the variation of ϕ in S_ϵ, and therefore in S, is zero. And if the variation of ϕ in S is zero, that in S_ϵ is zero *a fortiori*, so that f is integrable.

As a corollary it may be observed that f is certainly integrable if its discontinuities are enumerable. In particular any monotonic function, or any function of bounded variation, is integrable.

5. From among the elementary properties of the Stieltjes' integral, as I have defined it, I select those which are required for my immediate purpose. In the first place, the theorem of the mean plainly holds in the form

$$m\{\phi(b)-\phi(a)\} \leq \int_a^b f d\phi \leq M\{\phi(b)-\phi(a)\}.$$

From this it is an immediate deduction that the Stieltjes' integral, with respect to a continuous ϕ, is a continuous function of its upper limit. The product of two integrable functions, and in particular of a continuous function and an integrable function, is integrable. This (as I have already stated) may be proved directly; it is also an immediate corollary of Young's theorem proved in § 4 (ii). Finally, *if f is integrable with respect to ϕ in (a, b), it is uniformly integrable with respect to ϕ in (a, x), where $a \leq x \leq b$.* For the difference between

$$\int_a^x f d\phi,$$

and one of the sums (s or S) by means of which it is defined, is less than $\Sigma O_i \delta_i$, where the intervals δ_i define a sub-division of (a, x), and *a fortiori* less than the corresponding sum extended over intervals defining a sub-division of the complete interval (a, b).

6. We can now prove the theorem of integration by parts, stated in § 1, on the assumption that ϕ and ψ are continuous.

We write

$$\eta_i = \phi(x_{i+1}) - \phi(x_i), \qquad \zeta_i = \psi(x_{i+1}) - \psi(x_i),$$
$$f_i = f(x_{i+1}), \qquad g_i = g(x_{i+1}),$$
$$F_i = F(x_{i+1}) = \int_a^{x_{i+1}} f d\phi, \quad G_i = G(x_{i+1}) = \int_a^{x_{i+1}} g d\psi;$$

and we have identically

$$(7) \qquad \sum_0^{n-1} f_i\eta_i \sum_0^{i} g_j\zeta_j + \sum_1^{n-1} g_i\zeta_i \sum_0^{i-1} f_j\eta_j = \sum_0^{n-1} f_i\eta_i \sum_0^{n-1} g_i\zeta_i.$$

Given ϵ, we shall have

$$\left| \sum_0^{i} f_j\eta_j - F_i \right| < \epsilon \quad (i = 0,\ 1,\ 2,\ \ldots,\ n-1),$$

$$\left| \sum_0^{i} g_j\zeta_j - G_i \right| < \epsilon \quad (i = 0,\ 1,\ 2,\ \ldots,\ n-1),$$

provided only the intervals δ_j are sufficiently small. Thus the left-hand side of (7) differs by as little as we please from

$$\sum_0^{n-1} f_i G_i \eta_i + \sum_1^{n-1} g_i F_i \zeta_i;$$

and so, since fG and gF are integrable with respect to ϕ and ψ respectively, its limit is

$$\int_a^b f d\phi \int_a^x g d\psi + \int_a^b g d\psi \int_a^x f d\phi;$$

while the limit of the right-hand side is plainly

$$\int_a^b f d\phi \int_a^b g d\psi.$$

7. So far we have supposed ϕ continuous as well as monotonic: we must now consider the general case.

The discontinuities of ϕ form an enumerable set, and we may denote one of them by γ_ν. If we write

$$\alpha_\nu' = \phi(\gamma_\nu) - \phi(\gamma_\nu - 0), \quad \alpha_\nu'' = \phi(\gamma_\nu + 0) - \phi(\gamma_\nu),$$

then
$$\alpha_\nu = \alpha_\nu' + \alpha_\nu''$$

is the oscillation at γ_ν. It is convenient to regard α_ν' as zero when $\gamma_\nu = a$, and α_ν'' as zero when $\gamma_\nu = b$.

Let

$$(8) \qquad \Phi(x) = \sum_{\gamma_\nu \leq x} \alpha_\nu,$$

it being understood that α_ν is replaced by α_ν' if $\gamma_\nu = x$. It is plain that Φ is a steadily increasing function with the same discontinuities as ϕ, and constant in any interval in which it is continuous. And

$$(9) \qquad \phi = \bar{\phi} + \Phi,$$

where $\bar{\phi}$ is steadily increasing and continuous.

We now define the integrals of f with respect to Φ and ϕ by the equations

$$(10) \qquad \int_a^b f d\Phi = \sum_{\gamma_\nu \leq x} f(\gamma_\nu)\, \alpha_\nu,$$

$$(11) \qquad \int_a^b f d\phi = \int_a^b f d\bar{\phi} + \int_a^b f d\Phi.$$

It is convenient however, in order to ensure the agreement of our definition with that of Stieltjes, to suppose that *f is continuous at every point of discontinuity of ϕ**. We can then easily verify that *our definition is equivalent to that of Stieltjes, and the necessary and sufficient condition for integrability may still be stated as in* § 4 (ii).

In fact our definition is

$$\int_a^b f d\phi = \Sigma f(\gamma_\nu)\, \alpha_\nu + \lim \Sigma f(\xi_i)\, (\bar{\phi}_{i+1} - \bar{\phi}_i),$$

while that of Stieltjes is

$$\int_a^b f d\phi = \lim \Sigma f(\xi_i)\, (\phi_{i+1} - \phi_i).$$

Now $$\phi_{i+1} - \phi_i = \bar{\phi}_{i+1} - \bar{\phi}_i + \Sigma_i \alpha_\nu,$$

where the summation applies to all points of discontinuity which lie in δ_i, and it is understood that α_ν is replaced by α_ν'' if $\xi_\nu = x_i$ and by α'_ν if $\xi_\nu = x_{i+1}$. The difference between the defining sums is therefore

$$\Sigma f(\gamma_\nu)\, \alpha_\nu - \sum_{(i)} f(\xi_i)\, \Sigma_i \alpha_\nu = \Sigma f(\gamma_\nu)\, \alpha_\nu - \sigma,$$

say.

Choose k so that

$$(12) \qquad \sum_{k+1}^{\infty} \alpha_\nu < \epsilon,$$

and write $$\sigma = \sigma_k + \rho,$$

where σ_k and ρ are derived from σ by omitting all α's whose suffixes do and do not exceed k. It is plain that $|\sigma_k|$ and

$$\left| \sum_{k+1}^{\infty} f(\gamma_\nu)\, \alpha_\nu \right|$$

* Without this limitation, our definition would be somewhat more general than that of Stieltjes.

are each less than a constant multiple of ϵ. On the other hand

$$\sum_1^k f(\gamma_\nu)\,\alpha_\nu - \sigma_k = \sum_1^k \{f(\gamma_\nu) - f(\xi_i)\}\,\alpha_\nu,$$

where ξ_i is a point in the interval δ_i which includes γ_ν*. Since f is continuous at γ_ν, this sum tends to zero. Thus

$$\sigma \rightarrow \Sigma f(\gamma_\nu)\,\alpha_\nu,$$

which establishes the equivalence of the two definitions.

That the necessary and sufficient condition for integrability has the same form as when ϕ is continuous is almost obvious. For to say that the variation of ϕ in the set of discontinuities of f is zero is the same thing as to say, first that ϕ is continuous where f is discontinuous, and secondly that the variation of $\bar{\phi}$ in the set of discontinuities of f is zero. It follows that the product of two integrable functions is integrable. Again any bounded function whose discontinuities are enumerable, and all different from those of ϕ, is integrable. In particular any continuous function, or any monotonic function continuous at all points of discontinuity of ϕ, is integrable; and so is the product of any such function by any other integrable function.

The integral

$$\int_a^x f d\phi$$

is a continuous function of x at any point of continuity of ϕ. At a point of discontinuity of ϕ the integral has also a discontinuity, the oscillation being the product of the value of f by the oscillation of ϕ.

8. We are now in a position to prove the theorem of integration by parts in the general form in which it was stated in § 1. There is obviously no loss of generality in supposing ϕ and ψ to be increasing functions.

Suppose first that one of the functions, say ψ, is continuous, and that $\phi = \bar{\phi} + \Phi$, as in § 7. Then

$$\int_a^b f d\phi \int_a^x g d\psi = \Sigma f_\nu G_\nu \alpha_\nu + \int_a^b f d\bar{\phi} \int_a^x g d\psi,$$

---

* If γ_ν coincides with an x_i, the corresponding α_ν must be regarded as composed of the two parts α_ν' and α_ν'', which occur in the summation on the right in the form

$$\{f(\gamma_\nu) - f(\xi_{i-1})\}\,\alpha_\nu' + \{f(\gamma_\nu) - f(\xi_i)\}\,\alpha_\nu''.$$

The argument is not affected.

where $f_\nu = f(\gamma_\nu)$ and $G_\nu = G(\gamma_\nu)$;

$$\int_a^b g d\psi \int_a^x f d\phi = \int_a^b (\sum_{\gamma_\nu \leq x} f_\nu \alpha_\nu) g d\psi + \int_a^b g d\psi \int_a^x f d\bar{\phi};$$

and $$\int_a^b f d\phi \int_a^b g d\psi = \left(\Sigma f_\nu \alpha_\nu + \int_a^b f d\bar{\phi}\right) \int_a^b g d\psi.$$

The theorem having been proved already when both ϕ and ψ are continuous, what we have to verify is that

$$(13) \quad \Sigma f_\nu G_\nu \alpha_\nu + \int_a^b (\sum_{\gamma_\nu \leq x} f_\nu \alpha_\nu) g d\psi = \Sigma f_\nu \alpha_\nu \int_a^b g d\psi.$$

Suppose first that the number of discontinuities is finite. Then the coefficient of α_ν on the left-hand side of (13) is

$$f_\nu G_\nu + f_\nu \int_{\gamma_\nu}^b g d\psi = f_\nu \int_a^b g d\psi.$$

so that (13) holds in this case. In the general case we can determine k as in (12). Each term in (13) then splits up into two parts, corresponding respectively to values of ν which do not or do exceed k. Each of the latter parts is plainly less in absolute value than a constant multiple of ϵ, so that the truth of (13) follows by a passage to the limit. Our theorem is therefore proved when either of the two functions ϕ and ψ is continuous.

9. Finally, suppose that ψ as well as ϕ is discontinuous, that its discontinuities occur for $x = \eta_\mu$, and that its oscillation at η_μ is β_μ. *Ex hypothesi* η_μ and γ_ν are different, for all values of μ and ν.

We have

$$\int_a^b f d\phi \int_a^x g d\psi = \int_a^b (\sum_{\eta_\mu \leq x} g_\mu \beta_\mu) f d\phi + \int_a^b f d\phi \int_a^x g d\bar{\psi},$$

$$\int_a^b g d\psi \int_a^x f d\phi = \Sigma g_\mu F_\mu \beta_\mu + \int_a^b g d\bar{\psi} \int_a^x f d\phi,$$

and

$$\int_a^b f d\phi \int_a^b g d\psi = \int_a^b f d\phi \left(\Sigma g_\mu \beta_\mu + \int_a^b g d\bar{\psi}\right).$$

What we have to verify is that

$$(14) \quad \int_a^b (\sum_{\eta_\mu \leq x} g_\mu \beta_\mu) f d\phi + \Sigma g_\mu F_\mu \beta_\mu = \int_a^b f d\phi (\Sigma g_\mu \beta_\mu).$$

The verification of this identity is substantially the same as that of (13), the only additional remark that is necessary being that, since every η_μ is a point of continuity of ϕ,

$$F = \int_a^x f d\phi$$

is continuous for $x = \eta_\mu$.

Our theorem is thus proved in all cases. If we suppose in particular that

$$f = 1, \quad g = 1, \quad F = \int_a^x d\phi = \phi, \quad G = \int_a^x d\psi = \psi,$$

we obtain

$$\int_a^b \psi d\phi + \int_a^b \phi d\psi = \phi(b)\,\psi(b);$$

the form in which the theorem of integration by parts for Stieltjes' integrals is usually stated*.

The condition that ϕ and ψ should not be simultaneously discontinuous is essential to the truth of the theorem. Suppose, for example, that f and g are equal to 1 for $0 \leq x \leq 1$, and that ϕ and ψ are equal to 1 if $x = 1$ and to 0 otherwise. Then

$$\int_0^1 f(x)\,G(x)\,d\phi = f(1)\,G(1) = 1,$$

$$\int_0^1 g(x)\,F(x)\,d\psi = g(1)\,F(1) = 1,$$

$$\int_0^1 f(x)\,d\phi = \int_0^1 g(x)\,d\psi = 1;$$

and the formula is false.

METHOD OF EXPRESSING THE CROSS-RATIOS OF THE RANGE GIVEN BY THE ROOTS OF A BIQUADRATIC EQUATION IN TERMS OF AN AUXILIARY ANGLE CONNECTED WITH THE ROOTS OF THE REDUCING CUBIC OF THE BIQUADRATIC.

By *Alfred Lodge*.

THE six cross-ratios of a given range of four points (or pencil of four lines) are, if *any one* of them is denoted by x,

$$x, \quad \frac{1}{1-x}, \quad \frac{x-1}{x}; \text{ and } 1-x, \quad \frac{x}{x-1}, \quad \frac{1}{x}.$$

---

* See Young, *l.c.*, pp. 136–137.

If the first three are denoted by x_1, x_2, x_3, and the second three by x_4, x_5, x_6, we have

$$x_2 = \frac{1}{1-x_1}, \quad x_3 = \frac{1}{1-x_2}, \quad x_1 = \frac{1}{1-x_3} \text{ ....(1)},$$

with similar relations between x_4, x_5, x_6, but in different order. There are also slant connections between the two triplets, *e.g.*

$$x_2x_4 = x_3x_5 = x_1x_6 = 1 \text{ ....(2)},$$

and

$$x_3x_4 + x_1x_5 + x_2x_6 = 3 \text{ ....(3)}.$$

The products x_1x_4, x_2x_5, x_3x_6 will be considered later.

Further,

$$x_4 = 1 - x_1,$$

$$x_5 = 1 - x_2,$$

$$x_6 = 1 - x_3,$$

therefore the sum of all the ratios $= 3$.

Let

$$y = x_1 + x_2 + x_3 - \tfrac{3}{2},$$

whence

$$-y = x_4 + x_5 + x_6 - \tfrac{3}{2}.$$

From (1) we have

$$x_1x_2 = x_2 - 1 = -\frac{1}{x_3},$$

$$x_2x_3 = x_3 - 1,$$

$$x_3x_1 = x_1 - 1,$$

therefore

$$x_1x_2 + x_2x_3 + x_3x_1 = y - \tfrac{3}{2},$$

and

$$x_1x_2x_3 = -1,$$

therefore x_1, x_2, x_3 are the roots of the cubic equation

$$x^3 - (y + \tfrac{3}{2})x^2 + (y - \tfrac{3}{2})x + 1 = 0 \text{ ....(4)},$$

and the cubic for the other ratios is obtained by changing the sign of y.

To obtain the values of x when y is known, an auxiliary angle is available as follows:

Let

$$x = z + \tfrac{1}{2},$$

then

$$z^3 - yz^2 - \frac{9z}{4} + \frac{y}{4} = 0,$$

i.e.
$$y=\frac{9z-4z^3}{1-4z^2}.$$

Let
$$z=k\cot\theta,$$

therefore
$$y=3k\,.\,\frac{3\cot\theta-\frac{1}{3}(4k^2)\cot^3\theta}{1-4k^2\cot^2\theta},$$

which, if $k=\frac{1}{2}\sqrt{3}$, reduces to

$$y=\frac{3\sqrt{3}}{2}\cdot\frac{3\cot\theta-\cot^3\theta}{1-3\cot^2\theta},$$

i.e.
$$y=\tfrac{1}{2}(3\sqrt{3})\cot 3\theta \quad\text{......................(5)},$$

and
$$x=\tfrac{1}{2}+\tfrac{1}{2}(\sqrt{3})\cot\theta$$
$$=\cos 60^\circ+\sin 60^\circ\cot\theta$$
$$=\frac{\sin(\theta+60^\circ)}{\sin\theta} \quad\text{........................(6)}.$$

By taking all the needful values of θ given by (5) the values of x_1, x_2, x_3 are given by (6). Also, by changing the sign of y, *i.e.* of θ, the values of x_4, x_5, x_6 are given by

$$\frac{\sin(\theta-60^\circ)}{\sin\theta} \quad\text{.....................(7)},$$

or, taking one value of θ, we can write

$$x_1=\frac{\sin(\theta+60^\circ)}{\sin\theta},\qquad x_4=\frac{\sin(\theta-60^\circ)}{\sin\theta},$$

$$x_2=\frac{\sin\theta}{\sin(\theta-60^\circ)},\qquad x_5=\frac{\sin(\theta-120^\circ)}{\sin(\theta-60^\circ)},$$

$$x_3=\frac{\sin(\theta-60^\circ)}{\sin(\theta-120^\circ)},\qquad x_6=\frac{\sin(\theta-180^\circ)}{\sin(\theta-120^\circ)}=\frac{\sin\theta}{\sin(\theta+60^\circ)}.$$

The cubic (4) gives only three of the cross-ratios, the other three requiring the sign of y to be changed. To obtain an equation giving *all* the cross-ratios, it is necessary to work with y^2. Thus

$$y^2=-(x_1+x_2+x_3-\tfrac{3}{2})(x_4+x_5+x_6-\tfrac{3}{2})$$
$$=-\Sigma x_1x_4-\Sigma x_1x_5-\Sigma x_1x_6+\tfrac{3}{2}\Sigma x-\tfrac{9}{4}$$
$$=-\Sigma x_1x_4-3\qquad -3\qquad +\tfrac{9}{2}\qquad -\tfrac{9}{4},$$

therefore

$$\left.\begin{aligned} y^2 + \tfrac{27}{4} &= (1 - x_1x_4) + (1 - x_2x_5) + (1 - x_3x_6) \\ &= (1 - x_1 + x_1^2) + (1 - x_2 + x_2^2) + (1 - x_3 + x_3^2) \end{aligned}\right\} \text{....(8)},$$

the quantities in brackets having the curious property that their product equals their sum, which is easily proved by actual multiplication of $(1-x_1x_4)(1-x_2x_5)(1-x_3x_6)$ and making use of the identities (2) and (3). Further, the sum of their products taken in pairs equals twice their sum. The proof may be given in full, thus

$$\begin{aligned} &\Sigma (1 - x_1x_4)(1 - x_2x_5) \\ &= 3 - 2\Sigma x_1x_4 + (x_1x_4x_2x_5 + x_1x_4x_3x_6 + x_2x_5x_3x_6) \\ &= 3 - 2\Sigma x_1x_4 + (x_1x_5 + x_4x_3 + x_2x_6), \text{ from } (2) \\ &= 6 - 2\Sigma x_1x_4, \text{ from } (3) \\ &= \text{twice the sum of the quantities.} \end{aligned}$$

Hence, denoting any one of these quantities by z,

$$z^3 - (y^2 + \tfrac{27}{4})(z^2 - 2z + 1) = 0,$$

therefore $$y^2 + \tfrac{27}{4} = \frac{z^3}{(z-1)^2} = \frac{(1 - x + x^2)^3}{x^2(1-x)^2} \text{ ....(9)},$$

where x is any one of the six cross-ratios.

Also, since $$y = \tfrac{1}{2}(3\sqrt{3}) \cot 3\theta,$$

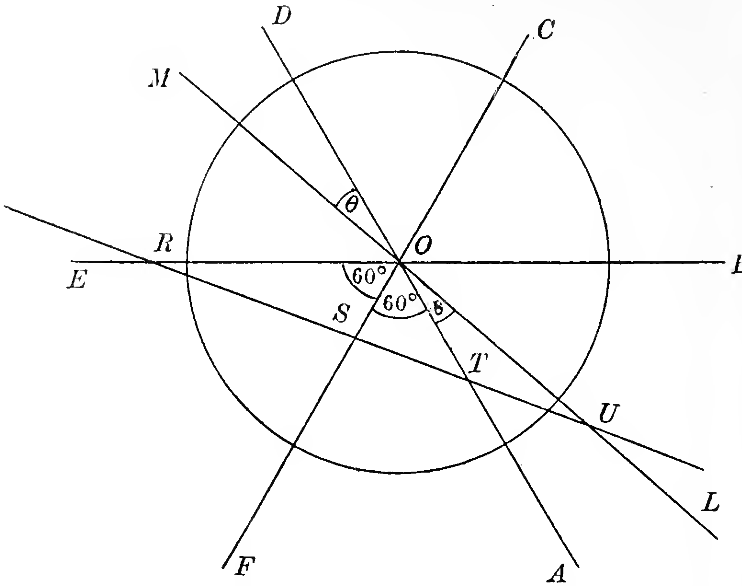
$$y^2 + \tfrac{27}{4} = \tfrac{27}{4} \operatorname{cosec}^2 3\theta \text{.....................(10)},$$

and the roots of (9) when y is known are given by

$$x = \frac{\sin(\theta \pm 60^\circ)}{\sin\theta}$$

for the values of θ given by (10). In fact, however, since (10) gives six different effective values for θ, there is no need for the ambiguity in sign in the value of x, though in practice it is probably convenient.

A fairly obvious geometrical meaning can be assigned to θ in connection with any given range of four points by taking a pencil of rays through the four points so that two contiguous sections of the range subtend angles of 60° at the vertex, then the angle subtended by anyone of the the other sections of the range may be taken as θ. Thus, in



the diagram, if $AOL = \theta$, and we take any one of the cross-ratios at random, *e.g.* $\frac{RS}{ST} : \frac{RU}{UT} = \frac{RS.UT}{ST.RU}$, it is equal to

$$\frac{\sin ROS . \sin UOT}{\sin SOT . \sin ROU} = \frac{\sin 60° . \sin(-\theta)}{\sin 60° . \sin(120° + \theta)} = \frac{\sin\theta}{\sin(\theta - 60°)} = x_2.$$

If $SOU = \theta$, the same cross-ratio is

$$\frac{\sin(60° - \theta)}{\sin(60° + \theta)} = \frac{\sin(\theta - 60°)}{\sin(\theta - 120°)} = x_3,$$

and, if $ROU = \theta$, it becomes

$$\frac{\sin(120° - \theta)}{\sin\theta} = \frac{\sin(\theta + 60°)}{\sin\theta} = x_1;$$

while, if we take $LOB = \theta$, the same ratio becomes x_4, and so on. Or, we may keep to any *one* of the possible values of θ, and vary the cross-ratios. Any other transversal would have done equally well.

If the points on the range are given by the roots of a biquadratic equation, whose reducing cubic is of the form

$$4s^3 - g_2 s - g_3 = 0,$$

which in Burnside and Pantin's *Theory of Equations* is written

$$4(a\theta)^3 - I(a\theta) + J = 0,$$

we find, on p. 147 of the second edition, that

$$\frac{(1-x+x^2)^3}{x^2(1-x)^2} = \frac{27}{4} \cdot \frac{g_2^3}{\Delta},$$

where
$$\Delta \equiv g_2^3 - 27g_3^2,$$

and x denotes any one of the cross-ratios.

Hence
$$\frac{g_2^3}{\Delta} = \operatorname{cosec}^2 3\theta, \quad \frac{27g_3^2}{\Delta} = \cot^2 3\theta,$$

$$\frac{\Delta}{g_2^3} = \sin^2 3\theta, \quad \frac{27g_3^2}{g_2^3} = \cos^2 3\theta.$$

Further, comparing the reducing cubic with the identity

$$n^3(4\cos^3\theta - 3\cos\theta - \cos 3\theta) = 0,$$

we see that
$$s = n\cos\theta$$

if
$$g_2 = 3n^2,$$

and
$$g_3 = n^3 \cos 3\theta,$$

which makes
$$\cos^2 3\theta = \frac{27g_3^2}{g_2^3},$$

therefore it is the same θ as above. It may be noted that θ is real if $g_2^3 > 27g_3^2$, *i.e.* if Δ is positive, which is the case whenever the range given by the biquadratic is real.

It is evident also that θ is real and the cross-ratios are real so long as Δ is a positive real number, even though the roots of the biquadratic are all imaginary; so that in this case we have a pencil of imaginary lines making real angles with each other, and cutting real transversals in imaginary points having real cross-ratios. In fact, if $kx + iy = 0$ is one of the lines, the lines making angles of $\pm 60°$ with it are

$$kx + iy \pm \sqrt{3}\,(ky - ix) = 0,$$

and the line making an angle θ with it is

$$kx + iy + (ky - ix)\tan\theta = 0$$

for all values of k except 0, ∞, and ± 1. If $k = 0$ or ∞, all the lines are real, and if $k = \pm 1$, all the gradients assume the indeterminate form $(0/0)$.

For all other values of k the pencil has the necessary analytical properties, and the cross-ratios of the intercepts made by any real line have the values given in this paper in terms of θ.

Addendum. The auxiliary angle for the roots of the cubic equation (4) was found by putting $x = k\cot\theta + \frac{1}{2}$ instead of by the usual method of getting rid of the second degree term and using a sine or cosine substitution. The general principle on which this substitution is based will probably be of interest. It is immaterial whether $\tan\theta$ or $\cot\theta$ is used, since, whether $f(\theta) \equiv \tan\theta$ or $\cot\theta$,

$$f(3\theta) = \frac{3f - f^3}{1 - 3f^2}.$$

Let the equation be

$$ax^3 + bx^2 + cx + d = 0,$$

then, on putting $x = z + h$, it becomes

$$az^3 + Bz^2 + Cz + D = 0,$$

where

$$B = 3ah + b,$$
$$C = 3ah^2 + 2bh + c,$$
$$D = ah^3 + bh^2 + ch + d.$$

Now put $z = k\tan\theta$, and consider the fraction

$$\frac{Cz + az^3}{D + Bz^2}, \quad i.e. \quad \frac{Ck\tan\theta + ak^3\tan^3\theta}{D + Bk^2\tan^2\theta},$$

which we want to be some multiple of

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta},$$

therefore

$$Ck = -3ak^3,$$
$$3D = -Bk^2,$$

therefore

$$\frac{C}{3a} = \frac{3D}{B} = -k^2.$$

Hence we must have $BC = 9aD$, *i.e.*

$$9a^2h^3 + 9abh^2 = (3ac + 2b^2)h + bc = 9a^2h^3 + 9abh^2 + 9ach + 9ad,$$

therefore

$$h = \frac{9ad - bc}{2(b^2 - 3ac)},$$

therefore h can always be found, and uniquely, unless $b^2 = 3ac$.

[If the original equation were written in the form

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

the value of h would assume the neater form $\dfrac{ad - bc}{2(b^2 - ac)}$, but for numerical work the binomial coefficients are not very convenient.]

In the case of equation (4), where

$$a=1,\quad b=-(9+\tfrac{3}{2}),\quad c=y-\tfrac{3}{2},\quad d=1,$$

$$h=\frac{9+y^2-\frac{9}{4}}{2(y+\frac{3}{2})^2-6(y-\frac{3}{2})}=\frac{y^2+\frac{27}{4}}{2y^2+\frac{27}{2}}=\tfrac{1}{2},$$

which is the value used. Moreover,

$$C=\tfrac{3}{4}-(y+\tfrac{3}{2})+(y-\tfrac{3}{2})=-\tfrac{9}{4},$$

therefore $k^2=\frac{3}{4}$, agreeing with the value previously found.

[*Note.* If the roots of the cubic were not all real, k^2 would be negative, and we should have to use tanh or coth, instead of tan or cot.]

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

LI.

On Hilbert's double-series theorem, and some connected theorems concerning the convergence of infinite series and integrals.

1. It was first proved by Hilbert that *the series*

$$\Sigma\Sigma\frac{a_m a_n}{m+n}\quad (a_n>0)$$

is convergent whenever Σa_n^2 *is convergent*; Hilbert's proof, which is based on considerations drawn from the theory of Fourier series, is sketched by Weyl on p. 83 of his *Inaugural-Dissertation* ('Singuläre Integralgleichungen', Göttingen, 1908). The theorem is very beautiful, and has attracted a good deal of attention; and at last three independent proofs have been published later. The first of these is due to Wiener*, and is genuinely elementary, but decidedly artificial; the second and third are due to Schur†. Of Schur's two proofs the first depends upon the general theory of quadratic and bilinear forms in an infinity of variables. The second is the simplest and most elegant that has yet been given; but, inasmuch as it depends upon a change of variables in a double integral, it cannot be regarded as elementary‡.

* F. Wiener, 'Elementarer Beweis eines Reihensatzes von Herrn Hilbert', *Math. Annalen*, vol. lxviii., 1910, pp. 361—366.

† J. Schur, 'Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen', *Journal für Math.*, vol. cxl., 1912, pp. 1—28.

‡ It is possible to remove the reference to integrals: this is done, for example, in the note of my own quoted below; but then the proof loses a good deal of its quality.

2. In Note XLI. of this series I stated another theorem which I showed to be equivalent to Hilbert's, viz.:

THEOREM A. *If* Σa_n^2 *is convergent then*

$$\Sigma \left(\frac{A_n}{n}\right)^2,$$

where $A_n = a_1 + a_2 + \ldots + a_n$, *is convergent.*

I showed that Theorem A and Hilbert's theorem may be deduced, either from the other, by arguments of an entirely simple and elementary kind. But I was still unable to find a proof of either theorem simpler than those which other writers had published before. It was only very recently that, returning to the subject once more, I have been able to prove Theorem A, and so Hilbert's theorem, by a method which really seems to me to lack nothing in directness and simplicity.

3. (a) *Proof of Theorem A.* It is evident that

$$\left(\frac{A_n}{n}\right)^2 = \left(a_n + \frac{A_n}{n} - a_n\right)^2 \leq 2a_n^2 + 2\left(\frac{A_n}{n} - a_n\right)^2$$

$$= 4a_n^2 + 2\left(\frac{A_n}{n}\right)^2 - 4\frac{a_n A_n}{n};$$

and so

$$(1) \qquad \sum_1^\nu \left(\frac{A_n}{n}\right)^2 \leq 4\sum_1^\nu a_n^2 + 2\sum_1^\nu \left(\frac{A_n}{n}\right)^2 - 4\sum_1^\nu \frac{a_n A_n}{n}.$$

But $\qquad -2a_n A_n = -(A_n^2 - A_{n-1}^2) - a_n^2 \leq -(A_n^2 - A_{n-1}^2);$

and therefore

$$-2\sum_1^\nu \frac{a_n A_n}{n} \leq -\sum_1 \frac{A_n^2 - A_{n-1}^2}{n}$$

$$= -\frac{A_1^2}{1.2} - \frac{A_2^2}{2.3} - \ldots - \frac{A_{\nu-1}^2}{(\nu-1)\nu} - \frac{A_\nu^2}{\nu}$$

$$\leq -\sum_1^\nu \frac{A_n^2}{n(n+1)}.$$

Substituting into (1), we obtain

$$\sum_1^\nu \left(\frac{A_n}{n}\right)^2 \leq 4\sum_1^\nu a_n^2 + 2\sum_1^\nu \left(\frac{A_n}{n}\right)^2 - 2\sum_1^\nu \frac{A_n^2}{n(n+1)}$$

$$= 4\sum_1^\nu a_n^2 + 2\sum_1^\nu \frac{A_n^2}{n^2(n+1)};$$

or

(2) $$\sum_1^\nu \left(1 - \frac{2}{n+1}\right) \left(\frac{A_n}{n}\right)^2 \leq 4 \sum_1^\nu a_n^2;$$

and it is obvious that this inequality contains a proof of the theorem.

(*b*) *Proof of Hilbert's theorem.* The convergence of Σa_n^2 and $\Sigma (A_n/n)^2$ plainly involves that of

$$\Sigma \frac{a_n A_n}{n},$$

and *a fortiori* of

$$\sum_n a_n \sum_{m=1}^n \frac{a_m}{m+n}$$

or of
$$\sum_{(m \leq n)} \sum \frac{a_m a_n}{m+n},$$

and therefore of Hilbert's series.

4. There is no substantial difference, so far as theorems of this character are concerned, between infinite series and integrals with an infinite upper limit; and the arguments required are slightly more compact when integrals are considered. I shall therefore state the theorem which follows, which is a more complete form of one which I stated in my former note, in terms of integrals of this type.

THEOREM B. *Suppose that a is positive, and f (x) positive and summable, and that*

$$F(x) = \int_a^x f(t)\, dt, \quad \phi(x) = \int_x^\infty \frac{f(t)}{t}\, dt.$$

Then the convergence of any one of the five integrals

$$(1) \int_a^\infty \left(\frac{F}{x}\right)^2 dx, \quad (2) \int_a^\infty \frac{fF}{x}\, dx, \quad (3) \int_a^\infty \phi^2\, dx, \quad (4) \int_a^\infty f\phi\, dx,$$

$$(5) \int_a^\infty \int_a^\infty \frac{f(x) f(y)}{x+y}\, dx\, dy$$

involves that of all the rest. In particular, all of them are convergent whenever

$$(6) \int_a^\infty f^2\, dx$$

is convergent.

This was proved in my former note, so far as it concerns the integrals (1), (2), and (5).

If (1) is convergent, $F = o(\sqrt{x})$, and

$$\int_a^x \frac{f}{t}\,dt = \frac{F(x)}{x} + \int_a^x \frac{F}{t^2}\,dt,$$

so that the integral which defines ϕ is convergent. Also

$$\phi(x) = \int_x^\infty \frac{f}{t}\,dt = -\frac{F(x)}{x} + \int_x^\infty \frac{F}{t^2}\,dt$$

$$= o\left(\frac{1}{\sqrt{x}}\right) + o\left(\int_x^\infty \frac{dt}{t\sqrt{t}}\right) = o\left(\frac{1}{\sqrt{x}}\right).$$

But
$$\int_a^x f\phi\,dt = F(x)\,\phi(x) + \int_a^x \frac{fF}{t}\,dt$$

$$= o(1) + \int_a^x \frac{fF}{t}\,dt.$$

Hence the convergence of (1), involving as it does that of (2), also involves that of (4). Conversely, since $F\phi > 0$, the convergence of (4) involves that of (2), and so that of (1).

Again

$$\int_a^x \phi^2 dt = x\,\{\phi(x)\}^2 - a\,\{\phi(a)\}^2 + 2\int_a^x f\phi\,dt.$$

If (3) is convergent

$$x\,\{\phi(x)\}^2 < 2\int_{\frac{1}{2}x}^x \phi^2 dt = o(1),$$

so that (4) is convergent. On the other hand, if (4) is convergent, (1) also is convergent; in which case we have seen already that $x\phi^2 = o(1)$. Thus the convergence of (4) involves that of (3). This completes the proof of the theorem.

It should be observed that we have proved incidentally that

$$\int_a^\infty \left(\frac{F}{t}\right)^2 dt = 2\int_a^\infty \frac{fF}{t}\,dt = 2\int_a^\infty f\phi\,dt = \int_a^\infty \phi^2 dt - a\,\{\phi(a)\}^2.$$

The argument used in § 3 shows that

$$\int_a^\infty \left(\frac{F}{t}\right)^2 dt \leq 4\int_a^\infty f^2 dt$$

whenever the last integral is convergent.

5. THEOREM C. *Suppose that a is positive and $f(x)$ positive and summable, and that*

$$F(x)=\int_0^x f(t)\,dt,\quad \phi(x)=\int_x^a \frac{f(t)}{t}\,dt.$$

Then the convergence of any one of the five integrals

$$(1)\int_0^a\left(\frac{F}{x}\right)^2 dx,\quad (2)\int_0^a \frac{fF}{x}\,dx,\quad (3)\int_0^a \phi^2 dx,\quad (4)\int_0^a f\phi\,dx,$$

$$(5)\int_0^a\int_0^a \frac{f(x)f(y)}{x+y}\,dx\,dy$$

involves that of all the rest. In particular all of them are convergent whenever

$$(6)\int_0^a f^2 dx$$

is convergent.

This theorem is immediately deducible from Theorem B. Write $x=1/y$, $a=1/b$, and $f(x)=y\,g(y)$. Then

$$F(x)=\int_y^\infty \frac{g(u)}{u}\,du=\psi(y)$$

say, and

$$\int_0^a\left(\frac{F}{x}\right)^2 dx=\int_b^\infty \psi^2\,dy,$$

whenever either integral is convergent. It will be found that the five integrals of Theorem C transform in this manner into the five of Theorem B.

6. Similarly it may be shown that, if $a_n \geq 0$ and

$$A_n=a_1+a_2+\ldots+a_n,\quad \mathrm{A}_n=\frac{a_{n+1}}{n+1}+\frac{a_{n+2}}{n+2}+\ldots,$$

the five series

$$(1)\ \Sigma\left(\frac{A_n}{n}\right)^2,\ (2)\ \Sigma\frac{a_nA_n}{n},\ (3)\ \Sigma\mathrm{A}_n^2,\ (4)\ \Sigma a_n\mathrm{A}_n,\ (5)\ \Sigma\frac{a_ma_n}{m+n}$$

converge or diverge together; and that all are convergent whenever Σa_n^2 is convergent.

7. There is a very interesting reciprocity between the theorems for series and for integrals, which is brought out by the theory of Fourier series.

Suppose that Σa_n^2 is convergent, so that

$$f(\theta) \sim \Sigma a_n \cos n\theta$$

is an even function whose square is summable. Then an easy calculation shows that

$$a_0 + a_1 + \ldots + a_{n-1} + \tfrac{1}{2}a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cot \tfrac{1}{2}\theta \sin n\theta \, d\theta.$$

And if we write

$$g(\theta) = \tfrac{1}{2} P \int_{\theta}^{\pi} f(u) \cot \tfrac{1}{2}u \, du,$$

where P is the symbol of Cauchy's principal value, we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos n\theta \, d\theta = \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(\theta) \cot \tfrac{1}{2}\theta \sin n\theta \, d\theta.$$

Thus $g(\theta)$ is an even function whose typical Fourier constant is

$$\alpha_n = \frac{a_0 + a_1 + \ldots + a_{n-1} + \tfrac{1}{2}a_n}{n}.$$

But it follows from Theorem C that the square of

$$h(\theta) = P \int_{\theta}^{\pi} \frac{f(u)}{u} \, du$$

is summable, and therefore that the square of $g(\theta)$ is summable; and so that $\Sigma \alpha_n^2$ is convergent. And from this we can deduce immediately that

$$\Sigma \left(\frac{A_n}{n}\right)^2$$

is convergent. Similarly, by considering the function

$$k(\theta) = \frac{1}{\theta} \int_0^{\theta} f(u) \, du,$$

we can deduce the convergence of ΣA_n^2.

It will be observed that the reciprocal relations indicated in this paragraph and § 5 only become clear when the theorems are stated in the complete form given in this note.

GENERAL PENTASPHERICAL CO-ORDINATES.

By *T. C. Lewis, M.A.*

1. If five points be taken which are not co-spherical, and no four co-planar, there is a complex of five spheres, having their centres at the five points, which may be taken as the spheres of reference for a system of pentaspherical co-ordinates. The co-ordinate of any point with reference to a sphere is defined as the power of the point divided by the radius of the sphere.

If the five spheres cut one another orthogonally, each of the five points is the orthocentre of the tetrahedron whose vertices coincide with the four other points. If the five points are not so related, the intersections of the spheres are not all orthogonal. Since a tetrahedon is in general not orthocentric, problems may arise in which it might be an advantage to use a complex of spheres associated with such a tetrahedron, a suitable fifth point being selected. The investigation of such a non-orthogonal system may therefore be of practical as well as of theoretic interest.

A still more general system of co-ordinates would be one rèferred not to a complex of spheres arising out of the geometrical relations of their centres, but having radii independent of them.

The method adopted is applicable to space of any dimensions, the number of centres and of co-ordinates of a point in n-space being $n+2$.

2. Let $p_1, p_2, \ldots, p_{n+2}$ be the powers of $n+2$ vertices $A_1, A_2, \ldots, A_{n+2}$ respectively with regard to the circle, sphere, hypersphere, &c., which passes through the remaining $n+1$ vertices.

Let the $(n-1)$-space containing the n vertices $A_3 \ldots A_{n+2}$ be intersected by the straight line A_1A_2 in the point B.

Let A_1A_2 cut the circle, sphere, &c., through $A_2 \ldots A_{n+2}$ in C,

and „ „ „ „ $A_1A_3 \ldots A_{n+2}$ in D.

Then the point B, being on the common $(n-1)$-space of these two circles, spheres, &c., has the same power with regard to each of them.

Therefore $$BA_2.BC = BA_1.BD.$$

Add $BA_1.BA_2$ to each side, then

$$BA_2.A_1C = BA_1.A_2D.$$

But $$p_1 : p_2 :: A_1C : A_2D$$
$$:: A_1B : BA_2 \quad \text{..................(1)}.$$

Therefore B is the mean centre for multiples $1/p_1$, $1/p_2$ at A_1, A_2 respectively. And similarly for the corresponding multiples at all the vertices. Therefore any final vertex A_{n+2} is the mean centre for a system of multiples inversely proportional to their proper powers at all the $n+1$ other points. Add the corresponding multiple $1/p_{n+2}$ at A_{n+2}, and this point is still the mean centre for the complete system of multiples at the $n+2$ points. So also is any other of the vertices. But there is only one mean centre for a given system of multiples at given points, unless their sum is zero.

Therefore in space of any dimensions $\Sigma(1/p)=0$; that is:

The sum of the reciprocals of the powers of each point with regard to the circle, sphere, &c., through the remaining $n+1$ points is zero.

Also the mean centres of any two complementary groups of vertices, with the given system of multiples, are identical.

The relation (1) may also be expressed in the form

$$p_1\Delta_1 = p_2\Delta_2 = p_k\Delta_k = H, \text{ a constant for all values of } k,$$

where Δ_k is the area of the triangle, volume of tetrahedron, &c., whose vertices are all the given vertices except A_k. Hence if Δ be considered positive when p is positive, it must be taken as negative when p is negative, and $\Sigma\Delta_k = H\Sigma(1/p)=0$; and the whole area, volume, &c., is covered twice over, once positively and once negatively.

Also $$\Sigma p_k\Delta_k^2 = H^2\Sigma(1/p_k)$$
$$= 0.$$

Therefore the sum of the products of each power into the square of the area, volume, &c., of the triangle, tetrahedron, &c., whose vertices are those appertaining to the remaining powers is zero.

3. It is known that if there are masses m_1, m_2, ..., at points A_1, A_2, ... whose distances from their mean centre are d_1, d_2, ..., and if l_{12} is the join of A_1A_2, and similarly for other pairs of points, then

$$\Sigma m_k \, \Sigma m_k d_k^2 = \Sigma m_h m_k l_{hk}^2.$$

If $\Sigma m_k = 0$, any point is the mean centre.

Therefore $\Sigma m_k d_k^2$ is constant, the distances d being measured from any point whatsoever.

To determine this constant when $m_1 = \frac{1}{p_1}$, $m_2 = \frac{1}{p_2}$, &c., let the distances d be measured from the centre of the circle, sphere, &c., passing through the $n+1$ points $A_1, \ldots, A_{n+1}$, whose radius is R, and centre at O. Then $\Sigma m_k d_k^2$ becomes

$$\left(\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_{n+1}}\right) R^2 + \frac{OA_{n+2}^{\ 2}}{p_{n+2}} = \frac{1}{p_{n+2}} (OA_{n+2}^{\ 2} - R^2) = 1.$$

Therefore $$\Sigma \frac{1}{p_k} \cdot d_k^2 = 1.$$

4. The powers p_1, &c., may now be expressed in terms of the joins of the vertices of the figure.

Let d_{12} be the square of the line joining $A_1 A_2$, and similarly for other suffixes.

Taking each vertex in turn as mean centre of the system, we have

$$\left.\begin{array}{l} \phantom{\frac{d_{12}}{p_1}} \quad \frac{d_{12}}{p_2} + \frac{d_{13}}{p_3} + \ldots \frac{d_{1,n+2}}{p_{n+2}} = 1 \\ \frac{d_{12}}{p_1} \quad\quad\quad + \frac{d_{23}}{p_3} + \ldots \frac{d_{2,n+2}}{p_{n+2}} = 1 \\ \quad \&c. \qquad \&c. \qquad \&c. \\ \frac{d_{1,n+2}}{p_1} + \frac{d_{2,n+2}}{p_2} + \ldots \qquad\quad = 1 \end{array}\right\}.$$

Therefore

$$\frac{1}{p_1} = \begin{vmatrix} 1 & d_{12} & d_{13} & \ldots & d_{1,n+2} \\ 1 & 0 & d_{23} & \ldots & d_{2,n+2} \\ \&c. & & \&c. & & \&c. \\ 1 & d_{2,n+2} & d_{3,n+2} & \ldots & 0 \end{vmatrix} \div \begin{vmatrix} 0 & d_{12} & d_{13} & \ldots & d_{1,n+2} \\ d_{12} & 0 & d_{23} & \ldots & d_{2,n+2} \\ \&c. & & \&c. & & \&c. \\ d_{1,n+2} & d_{2,n+2} & d_{3,n+2} & \ldots & 0 \end{vmatrix} \ldots (2),$$

and $p_2 \ldots p_{n+2}$ are similarly determined.

To the equations in p_1, &c., we may add the further equation

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_{n+2}} = 0.$$

Therefore, if the $n+2$ points lie in space of n dimensions, we obtain Cayley's condition

$$\begin{vmatrix} 0 & d_{12} & d_{13} & \dots & d_{1,n+2} & 1 \\ d_{12} & 0 & d_{23} & \dots & d_{2,n+2} & 1 \\ & \&c. & \&c. & & \&c. & \\ d_{1,n+2} & d_{2,n+2} & d_{3,n+2} & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{vmatrix} = 0.$$

Moreover, if the $n+2$ points all lie in a space of $n-1$ dimensions, the problem of determining $\frac{1}{p_1}$, &c., consistently with $\Sigma \frac{1}{p_k} = 0$ is indeterminate.

Therefore, if $n+3$ points lie in a space of n dimensions, we obtain the condition

$$\begin{vmatrix} 0 & d_{12} & d_{13} & \dots & d_{1,n+3} \\ d_{12} & 0 & d_{23} & \dots & d_{2,n+3} \\ & \&c. & \&c. & & \&c. \\ d_{1,n+3} & d_{2,n+3} & d_{3,n+3} & \dots & 0 \end{vmatrix} = 0 \dots\dots\dots\dots (3),$$

and the determinant also vanishes if the elements in any column are replaced by unity.

We also conclude from (2) that, if the $n+2$ points lie on an n-sphere,

$$\begin{vmatrix} 0 & d_{12} & d_{13} & \dots & d_{1,n+2} \\ d_{12} & 0 & d_{23} & \dots & d_{2,n+2} \\ \dots & \dots & \dots & \dots & \dots \\ d_{1,n+2} & d_{2,n+2} & \dots & \dots & 0 \end{vmatrix} = 0 \dots\dots\dots\dots (4),$$

a result which has been given by Cayley as regards 3-space.

Similarly if there be, in the n-space considered, another point, viz. the $(n+4)^{\text{th}}$, the distance of which from the $(n+3)^{\text{th}}$ point is ρ,

$$\begin{vmatrix} 0 & d_{12} & \dots & d_{1,n+4} & \\ d_{12} & 0 & \dots & d_{2,n+4} & \\ \dots & \dots & \dots & \dots & \\ \dots & \dots & 0 & \rho^2 & \\ d_{1,n+4} & d_{2,n+4} & \dots & \rho^2 & 0 \end{vmatrix} = 0.$$

Since there can be only one value of ρ^2, this equation in ρ^2 must have equal roots. (This may be proved directly). It may be written

$$\rho^4 \begin{vmatrix} 0 & d_{12} & \ldots & d_{1,n+2} \\ d_{12} & 0 & \ldots & d_{2,n+2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ d_{1,n+2} & \ldots & \ldots & 0 \end{vmatrix} + 2\rho^2 \begin{vmatrix} 0 & d_{12} & \ldots & d_{1,n+3} \\ d_{12} & 0 & \ldots & d_{2,n+3} \\ \cdots & \cdots & \cdots & \cdots \\ d_{1,n+2} & \ldots & \ldots & \\ d_{1,n+4} & \ldots & \ldots & 0 \end{vmatrix} + \&c. = 0,$$

and therefore, since the left-hand side is a perfect square,

$$\rho^2 \begin{vmatrix} 0 & d_{12} & \ldots & d_{1,n+2} \\ \cdots & \cdots & \cdots & \cdots \\ d_{1,n+2} & \ldots & \ldots & 0 \end{vmatrix} + \begin{vmatrix} 0 & d_{12} & \ldots & d_{1,n+3} \\ \cdots & \cdots & \cdots & \cdots \\ d_{1,n+2} & \ldots & \ldots & \ldots \\ d_{1,n+4} & \ldots & \ldots & 0 \end{vmatrix} = 0,$$

and ρ^2 is therefore determined.

In 3-space, if a, b, c be the sides of the base of the tetrahedron of reference; d, e, f the edges joining the vertex to the angular points of the base; g, h, i, j the distances of a fifth point from the first four; g_1, h_1, i_1, j_1, k_1 the distances of a sixth point from the first five; g_2, h_2, i_2, j_2, k_2 the distances of a seventh point from the first five; then ρ, the distance between these last two points is given by

$$\rho^2 \begin{vmatrix} 0 & c^2 & b^2 & d^2 & g^2 \\ c^2 & 0 & a^2 & e^2 & h^2 \\ b^2 & a^2 & 0 & f^2 & i^2 \\ d^2 & e^2 & f^2 & 0 & j^2 \\ g^2 & h^2 & i^2 & j^2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & c^2 & b^2 & d^2 & g^2 & g_1^2 \\ c^2 & 0 & a^2 & e^2 & h^2 & h_1^2 \\ b^2 & a^2 & 0 & f^2 & i^2 & i_1^2 \\ d^2 & e^2 & f^2 & 0 & j^2 & j_1^2 \\ g^2 & h^2 & i^2 & j^2 & 0 & k_1^2 \\ g_2^2 & h_2^2 & i_2^2 & j_2^2 & k_2^2 & 0 \end{vmatrix} = 0 \ldots (5),$$

or $D_1\rho^2 + D_2 = 0$, and

$$D_2^2 = D_1 \begin{vmatrix} 0 & c^2 & \ldots & g^2 & g_1^2 & g_2^2 \\ c^2 & 0 & \ldots & h^2 & h_1^2 & h_2^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g^2 & h^2 & \ldots & 0 & k_1^2 & k_2^2 \\ g_1^2 & h_1^2 & \ldots & k_1^2 & 0 & 0 \\ g_2^2 & h_2^2 & \ldots & k_2^2 & 0 & 0 \end{vmatrix}$$

Here $$D_1 = 288 p_5^2 V_5^2 = 288 H^2 \text{..............} (6).$$

Use will be made of the following property of determinants. If Δ be any determinant of order n, viz.

$$\begin{vmatrix} a & \dots & b \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ c & \dots & d \end{vmatrix}$$

and A, B, C, D the minors of the corner elements a, b, c, d; and N the determinant without the outside rows and columns, *i.e.* the complementary minor of

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix};$$

then $$N\Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix}.$$

If the determinant Δ be symmetrical about the leading diagonal, $B = C$ and $N\Delta = AD - B^2$.

If, moreover, Δ vanishes, $B^2 = AD$; while if A or D vanishes, $B^2 = -N\Delta$.

5. In 2-space, if a, b, c are the sides of the triangle having its vertices at three of the given points, and d, e, f the distances of the fourth point from those vertices, we have

$$\frac{1}{p_1} = \begin{vmatrix} c^2 & b^2 & d^2 & 1 \\ 0 & a^2 & e^2 & 1 \\ a^2 & 0 & f^2 & 1 \\ e^2 & f^2 & 0 & 1 \end{vmatrix} \div \begin{vmatrix} 0 & c^2 & b^2 & d^2 \\ c^2 & 0 & a^2 & e^2 \\ b^2 & a^2 & 0 & f^2 \\ d^2 & e^2 & f^2 & 0 \end{vmatrix} (-1).$$

The denominator, common to the reciprocals of all the four powers, is

$$-a^4d^4 - b^4e^4 - c^4f^4 + 2b^2c^2e^2f^2 + 2c^2a^2f^2d^2 + 2a^2b^2d^2e^2;$$

and if we write

$$ad = hx,$$
$$be = hy,$$
$$cf = hz,$$

this becomes $16h^4\Delta_h^2$, where Δ_h is the area of the triangle whose sides are x, y, z. And the numerators are

$$\text{for } \frac{1}{p_1},\quad a^2d^2(e^2+f^2-a^2)+b^2e^2(f^2+a^2-e^2) + c^2f^2(a^2+e^2-f^2)-2a^2e^2f^2;$$

$$\text{for } \frac{1}{p_2},\quad a^2d^2(b^2+f^2-d^2)+b^2e^2(f^2+d^2-b^2) + c^2f^2(d^2+b^2-f^2)-2b^2d^2f^2;$$

$$\text{for } \frac{1}{p_3},\quad a^2d^2(e^2+c^2-d^2)+b^2e^2(c^2+d^2-e^2) + c^2f^2(d^2+e^2-c^2)-2c^2d^2e^2;$$

$$\text{for } \frac{1}{p_4},\quad a^2d^2(b^2+c^2-a^2)+b^2e^2(c^2+a^2-b^2) + c^2f^2(a^2+b^2-c^2)-2a^2b^2c^2.$$

The relation $\Sigma\frac{1}{p}=0$ gives the relation between the lines joining four points in a plane in the form

$$a^2d^2(b^2+e^2+c^2+f^2-a^2-d^2)+b^2e^2(c^2+f^2+a^2+d^2-b^2-e^2) + c^2f^2(a^2+d^2+b^2+e^2-c^2-f^2)=a^2e^2f^2+b^2f^2d^2+c^2d^2e^2+a^2b^2c^2,$$

which is identical with Cayley's relation.

Moreover,

$$\{a^2d^2(e^2+f^2-a^2)+b^2e^2(f^2+a^2-e^2)+c^2f^2(a^2+e^2-f^2)-2a^2e^2f^2\}^2$$
$$-(-a^4d^4-b^4e^4-c^4f^4+2b^2e^2c^2f^2+\ldots)(-a^4-e^4-f^4+2e^2f^2+\ldots)$$
$$=-4a^2e^2f^2\{a^2d^2(b^2+e^2+c^2+f^2-a^2-d^2)+\ldots-a^2e^2f^2-\&\text{c.}\}=0.$$

Therefore
$$\frac{1}{p_1}=\frac{16\Delta_1^2}{a^2d^2(e^2+f^2-a^2)+\ldots-2a^2e^2f^2},$$

therefore

$$16p_1\Delta_1^2=a^2d^2(e^2+f^2-a^2)+b^2e^2(f^2+a^2-e^2) + c^2f^2(a^2+e^2-f^2)-2a^2e^2f^2,$$

and similarly for $p_2\Delta_2^2$, &c., whence the relation $\Sigma p_k\Delta_k^2=0$ is verified. Also

$$p_1^2\Delta_1^2=\frac{\{a^2d^2(e^2+f^2-a^2)+\ldots-2a^2e^2f^2\}^2}{256\Delta_1^2}=h^4\Delta_h^2.$$

Therefore
$$H=h^2\Delta_h,$$

H being the constant so named in para. 2.

The values of p_1, &c., assume simpler forms for particular positions of A_4 relatively to the triangle $A_1A_2A_3$.

(1) If A_4 be the orthocentre of the triangle

$$p_1 = b^2 + c^2 - a^2, \text{ \&c., \&c.}$$

(2) If A_4 be the circumcentre

$$p_1 = \frac{b^2c^2}{b^2+c^2-a^2}, \text{ \&c.,}$$

and

$$p_4 = -R^2.$$

(3) If A_4 be the centre of gravity of the triangle

$$p_1 = p_2 = p_3 = (d^2+e^2+f^2) = \tfrac{1}{3}(a^2+b^2+c^2) = -3p_4.$$

6. In 3-space, if a, b, c be the sides of the triangle $A_1A_2A_3$; d, e, f the distances of A_4 from A_1, A_2, A_3 respectively; and g, h, i, j the distances of A_5 from the other vertices taken in order; we have

$$\left.\begin{array}{l} \dfrac{c^2}{p_2} + \dfrac{b^2}{p_3} + \dfrac{d^2}{p_4} + \dfrac{g^2}{p_5} = 1 \\[2ex] \dfrac{c^2}{p_1} + \dfrac{a^2}{p_3} + \dfrac{e^2}{p_4} + \dfrac{h^2}{p_5} = 1 \\[2ex] \dfrac{b^2}{p_1} + \dfrac{a^2}{p_2} + \dfrac{f^2}{p_4} + \dfrac{i^2}{p_5} = 1 \\[2ex] \dfrac{d^2}{p_1} + \dfrac{e^2}{p_2} + \dfrac{f^2}{p_3} + \dfrac{j^2}{p_5} = 1 \\[2ex] \dfrac{g^2}{p_1} + \dfrac{h^2}{p_2} + \dfrac{i^2}{p_3} + \dfrac{j^2}{p_4} = 1 \end{array}\right\}.$$

Therefore

$$1/p_1 = \begin{vmatrix} 1 & c^2 & b^2 & d^2 & g^2 \\ 1 & 0 & a^2 & e^2 & h^2 \\ 1 & a^2 & 0 & f^2 & i^2 \\ 1 & e^2 & f^2 & 0 & j^2 \\ 1 & h^2 & i^2 & j^2 & 0 \end{vmatrix} \div \begin{vmatrix} 0 & c^2 & b^2 & d^2 & g^2 \\ c^2 & 0 & a^2 & e^2 & h^2 \\ b^2 & a^2 & 0 & f^2 & i^2 \\ d^2 & e^2 & f^2 & 0 & j^2 \\ g^2 & h^2 & i^2 & j^2 & 0 \end{vmatrix}.$$

Also, since $\Sigma(1/p_k) = 0$,

$$\begin{vmatrix} 0 & c^2 & b^2 & d^2 & g^2 & 1 \\ c^2 & 0 & a^2 & e^2 & h^2 & 1 \\ \vdots & & \vdots & & & \vdots \\ g^2 & h^2 & i^2 & j^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0,$$

and therefore the square of

$$\begin{vmatrix} 1 & c^2 & b^2 & d^2 & g^2 \\ 1 & 0 & a^2 & e^2 & h^2 \\ \vdots & & \vdots & & \vdots \\ 1 & h^2 & i^2 & j^2 & 0 \end{vmatrix}$$

is equal to the product of

$$\begin{vmatrix} 0 & c^2 & b^2 & d^2 & g^2 \\ c^2 & 0 & a^2 & e^2 & h^2 \\ \vdots & & \vdots & & \vdots \\ g^2 & h^2 & i^2 & j^2 & 0 \end{vmatrix} \text{ and } \begin{vmatrix} 0 & a^2 & e^2 & h^2 & 1 \\ a^2 & 0 & f^2 & i^2 & 1 \\ \vdots & & \vdots & & \vdots \\ h^2 & i^2 & j^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}.$$

Therefore

$$1/p_1 = \begin{vmatrix} 0 & a^2 & e^2 & h^2 & 1 \\ a^2 & 0 & f^2 & i^2 & 1 \\ \vdots & & \vdots & & \vdots \\ h^2 & i^2 & j^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \div \begin{vmatrix} 1 & c^2 & b^2 & d^2 & g^2 \\ 1 & 0 & a^2 & e^2 & h^2 \\ \vdots & & \vdots & & \vdots \\ 1 & h^2 & i^2 & j^2 & 0 \end{vmatrix},$$

therefore $288\,V_1^{\,2} p_1 = \begin{vmatrix} 1 & c^2 & b^2 & d^2 & g^2 \\ 1 & 0 & a^2 & e^2 & h^2 \\ \vdots & & \vdots & & \vdots \\ 1 & h^2 & i^2 & j^2 & 0 \end{vmatrix}$,

where V_1 is the volume of the tetrahedron $A_2A_3A_4A_5$;

and
$$288\,V_1^{\,2} p_1^{\,2} = \begin{vmatrix} 0 & c^2 & b^2 & d^2 & g^2 \\ c^2 & 0 & a^2 & e^2 & h^2 \\ b^2 & a^2 & 0 & f^2 & i^2 \\ d^2 & e^2 & f^2 & 0 & j^2 \\ g^2 & h^2 & i^2 & j^2 & 0 \end{vmatrix}.$$

If we write $a^3g^2j^2 = k^2u^3hi$; $d^3h^2i^2 = k^2x^3gj$;

$b^3h^2j^2 = k^2v^3gi$; $e^3g^2i^2 = k^2y^3hj$;

$c^3i^2j^2 = k^2w^3gh$; $f^3g^2h^2 = k^2z^3ij$;

the last-written determinant becomes $288k^4V_k^2$, where V_k is the volume of the tetrahedron whose edges are u, v, w, x, y, z, corresponding to a, b, c, d, e, f. Therefore

$$p_1^2V_1^2 = p_2^2V_2^2 = \&c. = k^4V_k^2$$

and
$$p_1V_1 = p_2V_2 \ldots \quad = k^2V_k.$$

This is, therefore, the constant H of para. 2.

For definite positions of A_5 with respect to the tetrahedron $A_1 \ldots A_4$, the expressions for the powers may be much simplified.

(1) If the tetrahedron be orthocentric and A_5 the orthocentre,

$$p_1 = b^2 + c^2 - a^2.$$

(2) If A_5 be the circumcentre,

$$p_1 = \tfrac{1}{2}\,\frac{-a^4d^4 - b^4e^4 - c^4f^4 + 2b^2c^2e^2f^2 + 2c^2a^2f^2d^2 + 2a^2b^2d^2e^2}{a^2d^2(e^2+f^2-a^2) + b^2e^2(f^2+a^2-e^2) + c^2f^2(a^2+e^2-f^2) - 2a^2e^2f^2},$$

$$p_5 = -R^2.$$

(3) If A_5 be the centre of gravity,

$$\begin{aligned} p_1 = p_2 = p_3 = p_4 &= g^2 + h^2 + i^2 + j^2 \\ &= \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) \\ &= -4p_5. \end{aligned}$$

(4) If A_5 be the centre of the hyperboloid of which the four altitudes of the tetrahedron are generators, we have

$$g^2 = \tfrac{1}{4}(b^2 + c^2 + d^2) - \tfrac{1}{4}(a^2 + e^2 + f^2) + R^2, \ \&c.,$$

$$g^2 + h^2 + i^2 + j^2 = 4R^2,$$

$$p_5 = 3R^2 - \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2),$$

and $p_1, \ldots, p_4$ may be expressed in terms of a, b, c, d, e, f.

The co-ordinates.

7. In n-space describe about the vertices $A_1 \ldots A_{n+2}$ as centres, circles, spheres, &c., with radii $\rho_1, \rho_2, \ldots, \rho_{n+2}$ respectively, such that

$$n\rho_1^2 = p_1,$$

and similarly for other suffixes from 1 to $n+2$.

Let a co-ordinate of any point be the power of the point with regard to any one of these circles, spheres, &c., divided by the radius thereof. Let these $n+2$ co-ordinates be $x_1, x_2, \ldots x_{n+2}$.

The co-ordinates are not all independent of one another. They must be connected by two relations, which may be at once determined.

The square of the distance of any point P from any vertex A_r is

$$\rho_r^2 + \rho_r x_r.$$

Therefore, by para. 3,

$$\Sigma \frac{\rho_r^2 + \rho_r x_r}{\rho_r^2} = n,$$

therefore
$$\Sigma \frac{x_r}{\rho_r} = -2.$$

This is the first relation. The second is given by (3), writing $\rho_r^2 + \rho_r x_r$ for $d_{r,n+3}$, making use of the first relation, and writing σ for $\frac{1}{2}\left(\frac{x_1}{\rho_1} + \frac{x_2}{\rho_2} + \ldots + \frac{x_{n+2}}{\rho_{n+2}}\right)$, whose value is -1. We have

$$\begin{vmatrix} 0 & d_{12} & d_{13} & \cdots & \rho_1^2\left(\sigma - \frac{x_1}{\rho_1}\right) \\ d_{12} & 0 & d_{23} & \cdots & \rho_2^2\left(\sigma - \frac{x_2}{\rho_2}\right) \\ \vdots & \vdots & & & \vdots \\ \rho_1^2\left(\sigma - \frac{x_1}{\rho_1}\right) & \cdots & \cdots & \cdots & 0 \end{vmatrix} = 0.$$

This is homogeneous of the second degree in the co-ordinates. Here the radii of the spheres need not be related to one another in any way, provided the form of σ is adapted to the general case, as in para. 9, viz., $-\frac{1}{k'}\Sigma\frac{k_r x_r}{\rho_r}$.

If we write $d_{rs} = \rho_r^2 + \rho_s^2 - 2\rho_r\rho_s c_{rs}$,

so that c_{rs} is the cosine of the angle of intersection of the spheres, &c., having radii ρ_r, ρ_s, this becomes

$$\begin{vmatrix} 1 & c_{12} & c_{13} & \cdots & c_{1,n+2} & x_1 \\ c_{12} & 1 & c_{23} & \cdots & c_{2,n+2} & x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{1,n+2} & c_{2,n+2} & \cdots & \cdots & 1 & x_{n+2} \\ x_1 & x_2 & \cdots & \cdots & x_{n+2} & 0 \end{vmatrix} = 0 \ldots\ldots\ldots (7).$$

For an orthogonal system this reduces to $\Sigma x_r^2 = 0$.

8. Also in 3-space, from (5) and (6), the distance ρ between two points $(x_1', x_2', \ldots)$ and $(x_1, x_2, \ldots)$ is given by

$$72 V_5^2 \rho_5^2 \rho^2 + \rho_1^2 \rho_2^2 \rho_3^2 \rho_4^2 \begin{vmatrix} 1 & c_{12} & \ldots & c_{15} & x_1 \\ c_{12} & 1 & \ldots & c_{25} & x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{15} & c_{25} & \ldots & 1 & x_5 \\ x_1' & x_2' & \ldots & x_5' & 0 \end{vmatrix} = 0 \ldots (8),$$

and this, by means of (7), may be written

$$144 V_5^2 \rho_5^2 \rho^2 = \rho_1^2 \rho_2^2 \rho_3^2 \rho_4^2 \begin{vmatrix} 1 & c_{12} & \ldots & c_{15} & x_1 - x_1' \\ c_{12} & 1 & \ldots & c_{25} & x_2 - x_2' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{15} & c_{25} & \ldots & 1 & x_5 - x_5' \\ x_1 - x_1' & \ldots & \ldots & x_5 - x_5' & 0 \end{vmatrix},$$

thus the square of an element of length is determined in the form $(ds)^2 = \Sigma\Sigma g_{rs} dx_r dx_s$.

For an orthogonal system, since

$$36 V_5^2 \rho_5^2 + \rho_1^2 \rho_2^2 \rho_3^2 \rho_4^2 = 0,$$

these formulæ become

$$2\rho^2 + \Sigma x_r x_r' = 0,$$

and $$4\rho^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2 + \ldots + (x_5 - x_5')^2.$$

Thus the quadratic relation between the co-ordinates simply expresses the fact that the distance of a point from itself is zero.

The relation (7) being true for the point at infinity, it holds when $1/\rho_1, 1/\rho_2, \ldots, 1/\rho_{n+2}$ are substituted for $x_1, x_2, \ldots, x_{n+2}$. This result may be derived immediately from Cayley's relation.

Taking the quadratic relation between the co-ordinates of a point, consider a sphere of radius ρ about that point as centre, and making with the spheres of reference angles whose cosines are c_1, c_2, &c. It follows that

$$\begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & c_{15} & c_1 \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} & c_2 \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} & c_3 \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} & c_4 \\ c_{15} & c_{25} & c_{35} & c_{45} & 1 & c_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 1 \end{vmatrix} = 0.$$

9. In the most general pentaspherical system of co-ordinates the radii of the five spheres of reference are independent of one another and of the configuration of their centres.

Let ρ_1^2 be $k_1 \times \frac{1}{3}$ of the power of A_1 with regard to the sphere $A_2 \dots A_5$, *i.e.*, k_1 times the square of the radius for the complex of spheres associated with the five points; and similarly for the radii of the other spheres of reference. Then we have

$$\Sigma \frac{k_r}{\rho_r^2} = 0, \qquad \Sigma \frac{k_r x_r}{\rho_r} = 3 - \Sigma k_r = k';$$

and the quadratic relation (7) between the co-ordinates holds without alteration.

The equation to determine ρ^2, the square of the distance between two points $(x_1 \dots)$ and $(x_1' \dots)$, is

$$288 \times 9\rho_5^4 \times \frac{1}{k_5^2} V_5^2 \rho^2 + \begin{vmatrix} 0 & c^2 & b^2 & d^2 & g^2 & \rho_1^2 + \rho_1 x_1 \\ c^2 & 0 & a^2 & e^2 & h^2 & \rho_2^2 + \rho_2 x_2 \\ b^2 & a^2 & 0 & \&c. & \dots & \rho_3^2 + \rho_3 x_3 \\ d^2 & e^2 & \&c. & \dots & & \rho_4^2 + \rho_4 z_4 \\ g^2 & \&c. & \dots & & & \rho_5^2 + \rho_5 x_5 \\ \rho_1^2 + \rho_1 x_1' & \&c. & \dots & & \rho_5^2 + \rho_5 x_5' & 0 \end{vmatrix} = 0,$$

which reduces to

$$18k'^2 V_5^2 \rho_5^2 \rho^2 + \rho_1^2 \rho_2^2 \rho_3^2 \rho_4^2 \begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & c_{15} & x_1 \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} & x_2 \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} & x_3 \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} & x_4 \\ c_{15} & c_{25} & c_{35} & c_{45} & 1 & x_5 \\ x_1' & x_2' & x_3' & x_4' & x_5' & 0 \end{vmatrix} = 0.$$

This agrees with (8) since in the case there considered $k' = -2$.

It may be written in the form

$$36k'^2 V_5^2 \rho_5^2 \rho^2 = \rho_1^2 \rho_2^2 \rho_3^2 \rho_4^2 \begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & c_{15} & x_1 - x_1' \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} & x_2 - x_2' \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} & x_3 - x_3' \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} & x_4 - x_4' \\ c_{15} & c_{25} & c_{35} & c_{45} & 1 & x_5 - x_5' \\ x_1 - x_1' & \&c. & & & & 0 \end{vmatrix},$$

which gives for the square of an element of length the form

$$(ds)^2 = \Sigma\Sigma g_{rs} dx_r dx_s,$$

r and s having values 1, 2, 3, 4, 5; and the coefficients g_{rs} being determined.

It is easily proved that

$$2\Sigma \frac{k_r c_{1r}}{\rho_r} = -\frac{1}{\rho_1}(2k_1 + k'),$$

r having all values except 1. This has been used in reducing the determinant given above to its final form.

10. If $S_1, S_2, \ldots, S_{n+2}$ are the powers of a point with regard to the $n+2$ spheres of reference, the equation to any n-sphere may be written

$$\frac{S}{\rho} \equiv \alpha_1 \frac{S_1}{\rho_1} + \alpha_2 \frac{S_2}{\rho_2} + \ldots + \alpha_{n+2} \frac{S_{n+2}}{\rho_{n+2}} = 0 \ \ldots\ldots (9),$$

where S is the power of any point with regard to it, and ρ the radius. This is the same thing as

$$\frac{S}{\rho} \equiv \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_{n+2} x_{n+2} = 0.$$

Let K be the centre of S, c_k the cosine of the angle of intersection of S and S_k, c_{hk} the cosine of the angle of intersection of S_h and S_k. Then

$$A_h K^2 = \rho^2 + \rho_h^2 - 2\rho\rho_h c_h,$$

therefore $$\frac{A_h K^2 - \rho_h^2}{\rho_h^2} = \frac{\rho^2}{\rho_h^2} - 2\rho \frac{c_h}{\rho_h},$$

therefore $$\Sigma \frac{A_h K^2 - \rho_h^2}{\rho_h^2} = -2\rho \Sigma \frac{c_h}{\rho_h},$$

therefore $$\frac{1}{\rho} = \Sigma \frac{c_h}{\rho_h}.$$

In the identity (9) substitute the co-ordinates of the $n+2$ vertices in succession. Therefore

$$\frac{\rho_1^2}{\rho} - 2c_1\rho_1 = -\alpha_1\rho_1 + \alpha_2\left(\frac{\rho_1^2}{\rho_2} - 2c_{12}\rho_1\right) + \ldots,$$

therefore

$$\frac{1}{\rho} - 2\frac{c_1}{\rho_1} = -\frac{\alpha_1}{\rho_1} + \frac{\alpha_2}{\rho_2} + \frac{\alpha_3}{\rho_3} + \ldots - 2\alpha_2 \frac{c_{12}}{\rho_1} - 2\alpha_3 \frac{c_{13}}{\rho_1} - \ldots \ \ldots (10).$$

Taking the sum of the $n+2$ such equations, we have

$$\frac{n+2}{\rho} - 2\Sigma \frac{c_h}{\rho_h} = n\Sigma \frac{\alpha_h}{\rho_h} - 2\alpha_1 \Sigma \frac{c_{1h}}{\rho_h} - 2\alpha_2 \Sigma \frac{c_{2h}}{\rho_h} - \&\text{c}.$$

Here

$$\Sigma \frac{c_{1h}}{\rho_h} = \tfrac{1}{2} \cdot \frac{1}{\rho_1} \Sigma \frac{2\rho_1\rho_h c_{1h}}{\rho_h^2} = \tfrac{1}{2} \cdot \frac{1}{\rho_1} \Sigma \frac{\rho_1^2 + \rho_h^2 - d_{1h}}{\rho_h^2} = 0.$$

Therefore $$\frac{1}{\rho} = \Sigma \frac{\alpha_h}{\rho_h}.$$

11. If we substitute the co-ordinates of the centre of S in the same identity, we have

$$-\rho = \alpha_1 \left(\frac{\rho^2}{\rho_1} - 2c_1\rho\right) + \ldots = \rho^2 \Sigma \frac{\alpha_h}{\rho_h} - 2\rho\Sigma\alpha_h c_h,$$

therefore $$\Sigma\alpha_h c_h = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Also by (10) $$\frac{1}{\rho} - 2\frac{c_1}{\rho_1} = -\frac{2\alpha_1}{\rho_1} + \frac{1}{\rho} - 2\Sigma\frac{\alpha_h c_{1h}}{\rho_1}.$$

Therefore $$c_1 = \alpha_1 + \Sigma\alpha_h c_{1h}.$$

Here h is different from 1. Therefore (11) becomes

$$\Sigma\alpha_h^2 + 2\Sigma\alpha_h\alpha_k c_{hk} = 1.$$

12. Let $$\frac{S}{\rho} \equiv \Sigma\alpha_h x_h = 0,$$

and $$\frac{S'}{\rho'} \equiv \Sigma\alpha_h' x_h = 0,$$

be two n-spheres; c the cosine of their angle of intersection. Substitute the co-ordinates of the centre of the first n-sphere in the second of the above identities. Then

$$\frac{\rho^2}{\rho'} - 2c\rho = \Sigma\alpha_h' \left(\frac{\rho^2}{\rho_h} - 2c_h\rho\right) = \frac{\rho^2}{\rho'} - 2\rho\Sigma\alpha_h' c_h,$$

therefore $$c = \Sigma\alpha_h' (\alpha_1 c_{1h} + \alpha_2 c_{2h} + \ldots) = \Sigma\alpha_h\alpha_k' c_{hk},$$

where h and k may have all values from 1 to $n+2$, including equal values.

13. In 3-space the co-ordinates of the vertex A_1 are

$$-\rho_1, \quad \frac{c^2 - \rho_2^2}{\rho_2}, \quad \frac{b^2 - \rho_3^2}{\rho_3}, \quad \frac{d^2 - \rho_4^2}{\rho_4}, \quad \frac{g^2 - \rho_5^2}{\rho_5};$$

and similarly for the other vertices. Hence at A_1

$$-\rho_1 x_1 = \rho_2 x_2 + 2\rho_1\rho_2 c_{12} = \rho_3 x_3 + 2\rho_1\rho_3 c_{13} = \&c. = \rho_5 x_5 + 2\rho_1\rho_5 c_{15}.$$

The equation to the plane $A_1A_2A_3$ is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ -\rho_1 & \dfrac{c^2-\rho_2^2}{\rho_2} & \dfrac{b^2-\rho_3^2}{\rho_3} & \dfrac{d^2-\rho_4^2}{\rho_4} & \dfrac{g^2-\rho_5^2}{\rho_5} \\ \dfrac{c^2-\rho_1^2}{\rho_1} & -\rho_2 & \dfrac{a^2-\rho_3^2}{\rho_3} & \dfrac{e^2-\rho_4^2}{\rho_4} & \dfrac{h^2-\rho_5^2}{\rho_5} \\ \dfrac{b^2-\rho_1^2}{\rho_1} & \dfrac{a^2-\rho_2^2}{\rho_2} & -\rho_3 & \dfrac{f^2-\rho_4^2}{\rho_4} & \dfrac{i^2-\rho_5^2}{\rho_5} \\ \dfrac{1}{\rho_1} & \dfrac{1}{\rho_2} & \dfrac{1}{\rho_3} & \dfrac{1}{\rho_4} & \dfrac{1}{\rho_5} \end{vmatrix} = 0,$$

which reduces to

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} \\ \dfrac{1}{\rho_1} & \dfrac{1}{\rho_2} & \dfrac{1}{\rho_3} & \dfrac{1}{\rho_4} & \dfrac{1}{\rho_5} \end{vmatrix} = 0;$$

and for an orthogonal system this becomes

$$\rho_4x_4 = \rho_5x_5.$$

14. The equations to the sphere on A_1A_2 as diameter is

$$\rho_1x_1 + \rho_1^2 + \rho_2x_2 + \rho_2^2 = c^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2c_{12},$$

i.e.,

$$\rho_1x_1 + \rho_2x_2 + 2\rho_1\rho_2c_{12} = 0,$$

which may be rendered homogeneous by means of the relation $\Sigma\dfrac{x_r}{\rho_r} = -2$.

15. The equation to the circumscribing sphere $A_1A_2A_3A_4$ is

$$\begin{vmatrix} \rho_1x_1 & \rho_2x_2 & \rho_3x_3 & \rho_4x_4 & \rho_5x_5 \\ -\rho_1^2 & \rho_1^2-2\rho_1\rho_2c_{12} & \rho_1^2-2\rho_1\rho_3c_{13} & \cdots & \rho_1^2-2\rho_1\rho_5c_{15} \\ \rho_2^2-2\rho_1\rho_2c_{12} & -\rho_2^2 & \rho_2^2-2\rho_2\rho_3c_{23} & \cdots & \rho_2^2-2\rho_2\rho_5c_{25} \\ \rho_3^2-2\rho_1\rho_3c_{13} & \rho_3^2-2\rho_2\rho_3c_{23} & -\rho_3^2 & \cdots & \rho_3^2-2\rho_3\rho_5c_{35} \\ \rho_4^2-2\rho_1\rho_4c_{14} & \rho_4^2-2\rho_2\rho_4c_{24} & \rho_4^2-2\rho_3\rho_4c_{34} & -\rho_4^2 & \rho_4^2-2\rho_4\rho_5c_{45} \end{vmatrix}$$

$= 0,$

which, for an orthogonal system, becomes

$$\rho_1x_1 + \rho_2x_2 + \rho_3x_3 + \rho_4x_4 - 2\rho_5x_5 = 0.$$

NOTE ON THE DEFLECTION OF BEAMS.

By *W. H. Macaulay, M.A.*

In the problem of a loaded beam, supported in any manner, it is assumed that the bending moment at any point of it is $EI\frac{d^2y}{dx^2}$, where x is the coordinate of the point measured from one end of the beam, y is the deflection, E is Young's modulus, and I depends on the section of the beam and may be either constant or a function of x. The equation to be solved is that given by equating the bending moment at the point x to the sum of the moments of the forces acting on the beam between this point and the origin. The beam is divided into segments by the points at which a support or a clamp or a single load occurs, or at which a continuous loading, whether uniform or a function of x, begins or ends. Each of these segments has its own differential equation, and each of these equations, if solved separately, introduces two constants of integration.

The usual way of avoiding this complication is by means of the theorem of three moments, which has various forms applicable to different cases, and by employing devices depending on the superposition of loads. It should however be noticed that by the use of a suitable notation the equations can be greatly simplified; for it becomes possible, in all cases, to write down a single differential equation for the whole length of the beam, and to obtain a single solution of it, introducing only two constants of integration. This seems to be generally the simplest, and sometimes the most expeditious procedure for dealing with the problem.

Let us denote by $\{f(x)\}_a$ a function of x which is zero when x is less than a, and equal to $f(x)$ when x is equal to or greater than a. Then, to take a simple example, the equation for the deflection of a beam, with uniform loading w per unit length, and with isolated loads or supports at successive points whose coordinates are a, b, ... is

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{2}wx^2 + \{P(x-a)\}_a + \{Q(x-b)\}_b + \dots,$$

and, if I is constant, the integrals of this may be written

$$EI\frac{dy}{dx} = \tfrac{1}{6}wx^3 + \{\tfrac{1}{2}P(x-a)^2\}_a + \{\tfrac{1}{2}Q(x-b)^2\}_b + \ldots + A,$$

$$EIy = \tfrac{1}{24}wx^4 + \{\tfrac{1}{6}P(x-a)^3\}_a + \{\tfrac{1}{6}Q(x-b)^3\}_b + \ldots + Ax + B.$$

Here the first term is integrated between the limits 0 and x, the second between the limits a and x, the third between the limits b and x, and so on. The beam is divided into segments by the points a, b, ..., and when x reaches any one of these values a new term, in each of the three equations, comes in with the value zero. Accordingly the constants of integration, A and B, are the same for all segments, for this is what is required to secure the continuity of $\frac{dy}{dx}$ and y. If l is the length of the beam

$$wl + P + Q + \ldots = 0,$$

$$\tfrac{1}{2}wl^2 + aP + bQ + \ldots = 0,$$

also if, for example, $x = b$ is a point at which there is a support and the deflection is zero,

$$\tfrac{1}{24}wx^4 + \tfrac{1}{6}P(b-a)^3 + Ab + B = 0,$$

accordingly there are enough equations to determine A and B and the pressure at each support.

If the beam is clamped at some point an unknown couple is introduced, and the given slope at that point provides an additional equation to determine it.

If w and I are functions of x the same procedure is applicable, the term involving P in the first integral being $\left\{\int_a^x \frac{P(x-a)}{I}\,dx\right\}_a$. If a continuous loading w is confined to a limited portion of the beam, say between $x = \alpha$ and $x = \beta$, the terms representing this are

$$\left\{\int_\alpha^x \frac{w}{I}(x-z)\,dz\right\}_\alpha - \left\{\int_\beta^x \frac{w}{I}(x-z)\,dz\right\}_\beta$$

in the differential equation, and

$$\left\{\int_\alpha^x\int_\alpha^x \frac{w}{I}(x-z)\,dz\,dx\right\}_\alpha - \left\{\int_\beta^x\int_\beta^x \frac{w}{I}(x-z)\,dz\,dx\right\}_\beta$$

in the first integral.

LAWS OF FACILITY OF ERROR.

By *Professor A. R. Forsyth.*

1. Let n independent and equally trustworthy measurements $x_1, \ldots, x_n$ of an unknown quantity z be supposed known, with the customary result of experience that they are not entirely concordant. Let u_r denote $z - x_r$, for $r = 1, \ldots, n$; and let the quantities u be called the errors.

On the assumption that the law of facility of an error u is a function, say $\phi(u)$, of the error and solely of the error, the combined probability of all the deviations of the measurements is proportional to

$$\phi(u_1)\ldots\phi(u_n).$$

The most probable value of z is defined to be the value which makes this combined probability the greatest. Hence the value of z is given by

$$\frac{\phi'(u_1)}{\phi(u_1)} + \ldots + \frac{\phi'(u_n)}{\phi(u_n)} = 0,$$

say by

$$\psi(u_1) + \ldots + \psi(u_n) = 0,$$

an equation which gives a maximum or minimum to the combined probability; and the deduced value of z must, in order to secure a maximum, make

$$\psi'(u_1) + \ldots + \psi'(u_n)$$

negative.

Now let

$$t_0 = \frac{1}{n}(x_1 + \ldots + x_n),$$

$$z - t_0 = v,$$

so that

$$z - x_r = v - \frac{1}{n}(x_r - x_1) - \frac{1}{n}(x_r - x_2) - \ldots - \frac{1}{n}(x_r - x_n);$$

then the value of v, and therefore the value of z, is given by the equation

$$\sum_{r=1}^{n} \psi\left\{v - \frac{1}{n}(x_r - x_1) - \ldots \frac{1}{n}(x_r - x_n)\right\} = 0.$$

The left-hand side of this equation is unaltered by any and every interchange between the measurements x; and therefore the quantity v is a symmetric function of the differences between these n measurements. Further, this symmetric function, being equal to $z - t_0$, is the difference between the most probable value z and the arithmetic mean of the n measurements.

Any symmetric function of the differences, whatever it be, will satisfy the requirement. When assumed, it will serve to determine the form of the function ψ and therefore the form of the function ϕ. But there is nothing in the statement of the assumed law of facility of error, nor is there anything in the statement of the assigned definition of the most probable value of z, which will serve to determine the form of the function. It is only by further assumption as to the form of the symmetric function of the differences that the form of the law of error can be deduced; and conversely.

2. The simplest case arises when the symmetric function of the differences of the measurements is supposed to vanish. On that assumption, we have

$$z - t_0 = 0,$$

which gives the arithmetic mean as the most probable value of z. Then we have

$$u_1 + \ldots + u_n = 0$$

as a relation among the quantities u; and it is the only relation among them. Moreover, we always have the equation

$$\psi(u_1) + \ldots + \psi(u_n) = 0;$$

consequently, we have

$$\psi(u_r) = \lambda u_r,$$

where λ is independent of the quantities u. Thus

$$\psi'(u_r) = \lambda;$$

and the quantity $\psi'(u_1) + \ldots + \psi'(u_n)$

must be negative. Hence λ must be negative; so we write

$$\lambda = -2h^2,$$

where h is real. Then

$$\frac{\phi'(u)}{\phi(u)} = \psi(u) = -2h^2 u,$$

and therefore $\phi(u) = Ae^{-h^2u^2}$.

The condition that certainty must be achieved, viz.

$$\int_{-\infty}^{\infty} \phi(u)\,du = 1,$$

gives
$$A = \frac{h}{\sqrt{\pi}},$$

so that
$$\phi(u) = \frac{h}{\sqrt{\pi}} e^{-h^2u^2},$$

which is the ordinary Gauss normal law of error. Its consequences are developed in many treatises and memoirs concerned with the combination of observations.

3. We proceed now to the hypothesis that the symmetric function of the differences of the measurements does not vanish identically.

One general mathematical inference can be made at once. Let $x_1, \ldots, x_n$ be the roots of the equation

$$x^n + na_1x^{n-1} + \frac{n(n-1)}{2!} a_2x^{n-2} + \ldots + a_n = 0,$$

or, with the usual notation,

$$(1, a_1, a_2, \ldots, a_n \between x, 1)^n = 0.$$

It is a known theorem that any rational integral symmetric function of the differences of the roots of this equation is a seminvariant (the leading coefficient of a covariant) of the binary quantic $(1, a_1, a_2, \ldots, a_n \between x, 1)^n$.

The two simplest covariants are the Hessian and the cubicovariant. Denoting their leading coefficients by h_2 and ϕ_3 respectively, we have

$$\begin{aligned} h_2 &= a_2 - a_1^2 \\ &= -\frac{1}{n^2(n-1)} \Sigma (x_r - x_s)^2, \\ \phi_3 &= a_3 - 3a_1a_2 + 2a_1^3 \\ &= -\frac{2}{n^3(n-1)(n-2)} \Sigma \{(x_r - x_s)^2 (x_r - x_t)\}. \end{aligned}$$

These expressions for symmetric functions of differences arise in connection with other possible laws of facility of error. They might be calculated from their values in terms of the coefficients of the quantic; but they can be calculated more easily from the sums of the powers of the quantities x, as given by

$$h_2 = \frac{1}{n^2(n-1)}(s_1^2 - ns_2),$$

$$\phi_3 = -\frac{2}{n^3(n-1)(n-2)}(2s_1^3 - 3ns_1s_2 + n^2s_3),$$

where $s_1 = \Sigma x_r$, $s_2 = \Sigma x_r^2$, $s_3 = \Sigma x_r^3$.

4. We shall consider two possible laws of error, both differing from the customary normal law, and both satisfying the general condition

$$\Sigma \psi(u_r) = 0;$$

and, in both of them, we have to determine the proper value of z to be extracted (under the hypotheses adopted) from the assigned measurements.

5. The formally simplest law of error that is permissible, and is distinct from the Gauss normal law, is

$$\phi(u) = Ce^{-cu^4},$$

where c is necessarily positive*. To determine C, we use the relation for certainty, viz.

$$\int_{-\infty}^{\infty} \phi(u)\,du = 1,$$

so that

$$C\int_{-\infty}^{\infty} e^{-cu^4}\,du = 1,$$

that is,

$$2C\int_{0}^{\infty} e^{-cu^4}\,du = 1.$$

Writing

$$cu^4 = v,$$

we have

$$\int_0^\infty e^{-cu^4}\,du = \frac{1}{4c^{\frac{1}{4}}}\int_0^\infty e^{-v}v^{-\frac{3}{4}}\,dv = \frac{1}{4c^{\frac{1}{4}}}\Gamma(\tfrac{1}{4});$$

and therefore

$$C = \frac{2c^{\frac{1}{4}}}{\Gamma(\frac{1}{4})},$$

* It seems useless to consider laws such as

$$\phi(u) = B'e^{-au^2-bu^3}, \quad \phi(u) = C'e^{-bu^3};$$

for they make the chance of a large negative error much greater than that of a large positive error when b is positive, and conversely if b is negative making the law of error to be

$$\frac{2c^{\frac{1}{4}}}{\Gamma\left(\frac{1}{4}\right)} e^{-cu^4}.$$

As regards the value of z to be taken under this law, the quantity

$$\Pi e^{-c(z-x_r)^4}$$

must be a maximum; and therefore

$$\Sigma (z - x_r)^3 = 0,$$

the maximum being secured because c is assumed positive. Now

$$\begin{aligned}\Sigma (z - x_r)^3 &= nz^3 - 3z^2 . nt_0 + 3z\, \Sigma x_r^2 - \Sigma x_r^3 \\ &= n (z - t_0)^3 + 3 (z - t_0) (\Sigma x_r^2 - nt_0^2) + U,\end{aligned}$$

where $$U = 3t_0\, \Sigma x_r^2 - \Sigma x_r^3 - 2nt_0^3.$$

We have $$\Sigma x_r^2 - nt_0^2 = \frac{1}{n} \Sigma (x_r - x_s)^2 = nA,$$

where A, an obviously positive quantity, is given by

$$A = -(n - 1) h_2.$$

Also $$U = -\frac{1}{n^2} \Sigma (x_r - x_s)^2 (x_r - x_t) = -nB,$$

where $$B = \tfrac{1}{2} (n - 1) (n - 2) \phi_3.$$

Thus the equation for z becomes the cubic

$$(z - t_0)^3 + 3 (z - t_0) A - B = 0.$$

As A is positive, the quantity $B^2 + 4A^3$ is positive; and therefore the cubic possesses only one real root, which is given by

$$z - t_0 = \lambda + \mu,$$

where

$$\lambda^3 = \tfrac{1}{2} B + \tfrac{1}{2} (B^2 + 4A^3)^{\frac{1}{2}}, \quad \mu^3 = \tfrac{1}{2} B - \tfrac{1}{2} (B^2 + 4A^3)^{\frac{1}{2}},$$

and λ and μ are the real cube roots of λ^3 and μ^3. The magnitude $B^2 + 4A^3$ is a seminvariant of the quantic

$$(1, a_1, a_2, \ldots \between x, 1)^n;$$

it can be expressed in either of the two forms

$$\frac{1}{n^6} [\{\Sigma (x_r - x_s)^2 (x_r - x_t)\}^2 + 4 \{\Sigma (x_r - x_s)^2\}^3],$$

$$\frac{1}{n^4} \{n^2 s_3^2 - 6ns_1 s_2 s_3 + 4ns_2^3 - 3s_1^2 s_2^2 + 4s_1^3 s_3\},$$

the latter of which is the more useful for purposes of calculation. The result is to give z in the form

$$z = t_0 + (\lambda + \mu),$$

where $\lambda + \mu$ is a symmetric function of the differences.

6. If the suggested law were to be tried upon any set of measurements, the constant c must be determinable from them. Moreover, there is the necessity for having some criterion as to the suitability of the law. We take these in turn.

Writing $c = h^4$, we have the law in the form

$$\frac{2h}{\Gamma\left(\frac{1}{4}\right)} e^{-h^4u^4}.$$

Let ϵ_2 denote the mean square deviation from the adopted measure, defined by the relation

$$\epsilon_2^{\,2} = \frac{1}{n} \Sigma (z - x_r)^2;$$

and let ϵ_4 denote the mean quartic deviation from the adopted measure, defined by the relation

$$\epsilon_4^{\,4} = \frac{1}{n} \Sigma (z - x_r)^4.$$

We obtain the suitable approximation for ϵ_2 by the equation

$$\epsilon_2^{\,2} = \frac{2h}{\Gamma\left(\frac{1}{4}\right)} \int_{-\infty}^{\infty} u^2 e^{-h^4u^4} du = \frac{4h}{\Gamma\left(\frac{1}{4}\right)} \int_0^{\infty} u^2 e^{-h^4u^4} du.$$

Now, substituting $h^4u^4 = t$, we have

$$\int_0^{\infty} u^2 e^{-h^4u^4} du = \frac{1}{4h^3} \int_0^{\infty} t^{-\frac{1}{4}} e^{-t} dt = \frac{1}{4h^3} \Gamma\left(\tfrac{3}{4}\right),$$

so that

$$\epsilon_2^{\,2} = \frac{1}{h^2} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} = \frac{\pi\sqrt{2}}{h^2 \Gamma^2\left(\frac{1}{4}\right)}.$$

Similarly we obtain the suitable approximation for ϵ_4 by the equation

$$\epsilon_4^{\,4} = \frac{2h}{\Gamma\left(\frac{1}{4}\right)} \int_{-\infty}^{\infty} u^4 e^{-h^4u^4} du = \frac{2h}{\Gamma\left(\frac{1}{4}\right)} \frac{2}{4h^5} \Gamma\left(\tfrac{5}{4}\right) = \frac{1}{4h^4}.$$

Consequently

$$\frac{\epsilon_2^{\,4}}{\epsilon_4^{\,4}} = \frac{8\pi^2}{\Gamma^4\left(\frac{1}{4}\right)};$$

where

$$\epsilon_2^{\,2} = \frac{1}{n} \Sigma (z - x_r)^2, \quad \epsilon_4^{\,4} = \frac{1}{n} \Sigma (z - x_r)^4.$$

If the law is to be fairly admissible, the last relation, viz.

$$\frac{\{\Sigma (z-x_r)^2\}^2}{\Sigma (z-x_r)^4} = \frac{8n\pi^2}{\Gamma^4(\frac{1}{4})}$$

must be approximately satisfied. Should the relation be approximately satisfied, the value of h is given by

$$h = \frac{1}{\epsilon_4 \sqrt{2}} .$$

7. In his attempt to establish the law of error in the form $Ae^{-h^2u^2}$ without introducing the assumption of the arithmetic mean, Laplace obtained an expression

$$\phi(u) = Ae^{-au^2 - cu^4},$$

where c has a factor $(1/n)$ and vanishes only when the number of measurements becomes infinite. It is therefore worth while noting some of the consequences of this law.

We shall assume that a and c are positive; although, when c is not zero, it might be sufficient to assume that c alone is positive. The constant A is determined by the condition for certainty, which is

$$\int_{-\infty}^{\infty} \phi(u)\, du = 1,$$

so that

$$AI = 1,$$

where

$$I = \int_{-\infty}^{\infty} e^{-au^2 - cu^4} du.$$

8. Before obtaining the quantity z, to be adopted as the best representative of the n measurements, we shall discuss some properties of this integral I and of cognate integrals.

We have

$$\tfrac{1}{2}I = \int_0^{\infty} e^{-au^2 - cu^4} du$$

$$= \Sigma \int_0^{\infty} (-1)^m \frac{a^m}{m!} u^{2m} e^{-cu^4} du.$$

Now

$$\int_0^{\infty} u^{2m} e^{-cu^4} du = \int_0^{\infty} \frac{v^{\frac{1}{4}(2n-3)}}{4c^{\frac{1}{4}(2n+1)}} e^{-v} dv = \frac{1}{4c^{\frac{1}{4}(2n+1)}} \Gamma\left(\frac{2n+1}{4}\right);$$

consequently

$$2Ic^{\frac{1}{4}} = \Sigma \frac{(-1)^m}{m!} \left(\frac{a}{c^{\frac{1}{2}}}\right)^m \Gamma\left(\tfrac{1}{2}m + \tfrac{1}{4}\right).$$

Further, integrating I by parts, we have

$$I = \left[u e^{-au^2 - cu^4} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2au^2 + 4cu^4)\, e^{-au^2 - cu^4}\, du$$

$$= -2a \frac{\partial I}{\partial a} - 4c \frac{\partial I}{\partial c},$$

a partial differential equation of the first order, of which the most general integral is

$$I a^{\frac{1}{2}} = f\left(\frac{a^2}{c}\right),$$

where f is any function. Again, we have

$$\frac{\partial^2 I}{\partial a^2} + \frac{\partial I}{\partial c} = 0,$$

manifestly a different equation. When the value of I just obtained is substituted in this equation, we find

$$\tfrac{3}{4} \frac{1}{a^4} f - \frac{1}{c^2} f' + \frac{4}{c^2} f'' = 0.$$

Let

$$t = \frac{a^2}{c};$$

the equation for f becomes

$$4 \frac{d^2 f}{dt^2} - \frac{df}{dt} + \tfrac{3}{4} \frac{f}{t^2} = 0,$$

a Fuchsian equation for expansions in ascending powers of t. When it is solved by the Frobenius method, and when the arbitrary constants are determined so as to make the first two terms in $a^{-\frac{1}{2}} f(t)$ the same as

$$\int_{-\infty}^{\infty} (1 - au^2)\, e^{-cu^4}\, du,$$

which are the first two terms in I when e^{-au^2} is expanded, we obtain the same result as before, viz.

$$I = \frac{1}{2c^{\frac{1}{4}}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{a}{c^{\frac{1}{2}}}\right)^m \Gamma\left(\tfrac{1}{2}m + \tfrac{1}{4}\right),$$

which seems the simplest expression for I free from quadratures.

Another expression for I, curious but not at all useful, can be obtained as follows. Writing

$$I = \int_{-\infty}^{\infty} e^{-ax^2 - cx^4}\, dx = \int_{-\infty}^{\infty} e^{-ay^2 - cy^4}\, dy,$$

we have

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)-c(x^4+y^4)}\,dx\,dy.$$

Changing to polar coordinates, we have

$$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-ar^2-cr^4(\cos^4\theta+\sin^4\theta)}\,r\,dr\,d\theta$$

$$= \tfrac{1}{2}\int_0^{\infty}\int_0^{2\pi} e^{-at-ct^2(\cos^4\theta+\sin^4\theta)}\,dt\,d\theta$$

$$= 2\int_0^{\infty}\int_0^{\frac{1}{2}\pi} e^{-at-ct^2(\cos^4\theta+\sin^4\theta)}\,dt\,d\theta$$

$$= 2\int_0^{\infty} e^{-at-\frac{3}{4}ct^2}\,dt\int_0^{\frac{1}{2}\pi} e^{-\frac{1}{4}ct^2\cos 4\theta}\,d\theta.$$

Let $\qquad -\tfrac{1}{4}ct^2 = \mu;$

then, as $\qquad \int_0^{\frac{1}{2}\pi} e^{\mu\cos 4\theta}\,d\theta = \tfrac{1}{4}\int_0^{2\pi} e^{\mu\cos\phi}\,d\phi,$

we have $\qquad I^2 = \tfrac{1}{2}\int_0^{\infty} e^{-at-\frac{3}{4}ct^2}\,dt\int_0^{2\pi} e^{\mu\cos\phi}\,d\phi.$

But

$$e^{\frac{1}{2}a'\{z-(1/z)\}} = J_0(a') + \left(z-\frac{1}{z}\right)J_1(a') + \left(z^2+\frac{1}{z^2}\right)J_2(a') + \ldots;$$

so, taking $\qquad z = ie^{i\phi},\quad \dfrac{1}{z} = -ie^{-i\phi},$

we have

$$e^{a'i\cos\phi} = J_0(a') + 2i\cos\phi\,.\,J_1(a') - 2\cos 2\phi\,.\,J_2(a') + \ldots,$$

and therefore $\qquad \int_0^{2\pi} e^{a'i\cos\phi}\,d\phi = 2\pi J_0(a').$

Thus $\qquad I^2 = \pi\int_0^{\infty} e^{-at-\frac{3}{4}ct^2} J_0(-\tfrac{1}{4}ict^2)\,dt$

$$= \pi\int_0^{\infty} e^{-at-\frac{3}{4}ct^2} J_0(\tfrac{1}{4}ict^2)\,dt,$$

because J_0 is an even function of its argument. This verifies to the customary value when $c=0$.

9. The quantity z is selected as the best representative of the unknown quantity by making

$$\phi(u_1).\phi(u_2)\ldots\phi(u_n)$$

a maximum; and therefore the quantity

$$a\Sigma u_r^2 + c\Sigma u_r^4$$

is a minimum. Hence

$$a\Sigma(z - x_r) + 2c\Sigma(z - x_r)^3 = 0;$$

and the further condition, that

$$a + 6c\Sigma(z - x_r)^2$$

shall be positive, is satisfied because a and c are assumed to be positive. Thus

$$a(nz - \Sigma x_r) + 2c\Sigma(z - x_r)^3 = 0,$$

or, if

$$\frac{a}{2c} = \mu',$$

we have

$$z - t_0 = -\frac{1}{n\mu'}\Sigma(z - x_r)^3.$$

But

$$\begin{aligned}\Sigma(z - x_r)^3 &= nz^3 - 3z^2 . nt_0 + 3z\Sigma x_r^2 - \Sigma x_r^3 \\ &= n(z - t_0)^3 + 3(z - t_0)(\Sigma x_r^2 - nt_0^2) + U,\end{aligned}$$

where

$$\begin{aligned}U &= 3t_0\Sigma x_r^2 - \Sigma x_r^3 - 2nt_0^3 \\ &= -\frac{1}{n^2}\Sigma(x_r - x_s)^2(x_r - x_t) \\ &= -nB,\end{aligned}$$

as before, and

$$\Sigma x_r^2 - nt_0^2 = \frac{1}{n}\Sigma(x_r - x_s)^2 = nA,$$

also as before. Hence the equation for z becomes

$$(z - t_0)^3 + (3A + \mu')(z - t_0) - B = 0.$$

Writing

$$3A + \mu' = 3H,$$

we note that H is an obviously positive quantity; and therefore $B^2 + 4H^3$ is positive. Thus the cubic has one real root; it is given by

$$z - t_0 = \kappa + \lambda,$$

where κ and λ are the real cube roots of κ^3 and λ^3, defined as

$$\kappa^3 = \tfrac{1}{2}B + \tfrac{1}{2}(B^2 + 4H^3)^{\frac{1}{2}}, \quad \lambda^3 = \tfrac{1}{2}B - \tfrac{1}{2}(B^2 + 4H^3)^{\frac{1}{2}}.$$

The result becomes the result for the earlier case when $\mu' = 0$, that is, when $a = 0$.

10. To obtain relations for the determination of the constants a and c, we proceed as follows: Let m_p denote the

mean of the p^{th} power of the deviations of z from the given measurements, so that

$$m_p = \frac{1}{n} \Sigma (z - x_r)^p,$$

taken from the point of view of the calculations. From the point of view of the law, we have

$$m_p = \int_{-\infty}^{\infty} u^p \phi(u)\, du,$$

and therefore

$$Im_p = \int_{-\infty}^{\infty} u^p e^{-au^2-cu^4} du\,;$$

and we take these for the successive even values of p.

In the first place,

$$\begin{aligned} I &= \int_{-\infty}^{\infty} e^{-au^2-cu^4} du \\ &= \Big[ue^{-au^2-cu^4}\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2au^2 + 4cu^4)\, e^{-au^2-cu^4} du \\ &= 2am_2 I + 4cm_4 I. \end{aligned}$$

Next,

$$\begin{aligned} m_2 I &= \int_{-\infty}^{\infty} u^2 e^{-au^2-cu^4} du \\ &= \Big[\tfrac{1}{3} u^3 e^{-au^2-cu^4}\Big]_{-\infty}^{\infty} + \tfrac{1}{3}\int_{-\infty}^{\infty} (2au^4 + 4cu^6)\, e^{-au^2-cu^4} du \\ &= \tfrac{2}{3} am_4 I + \tfrac{4}{3} cm_6 I. \end{aligned}$$

Similarly $\qquad m_4 I = \tfrac{2}{5} am_6 I + \tfrac{4}{5} cm_8 I,$

and so on. Consequently,

$$\begin{aligned} 2am_2 + 4cm_4 &= 1, \\ 2am_4 + 4cm_6 &= 3m_2, \\ 2am_6 + 4cm_8 &= 5m_4, \end{aligned}$$

and generally

$$2am_{2p} + 4cm_{2p+2} = (2p-1)\, m_{2p-2}.$$

The first two of these will serve to determine a and c; so that, if the law may be used, its expression can be regarded as completed. In order that the law may be used, the relation

$$\begin{vmatrix} 1\ , & m_2, & m_4 \\ 3m_2, & m_4, & m_6 \\ 5m_4, & m_6, & m_8 \end{vmatrix} = 0$$

must be approximately satisfied by the quantities m as deduced from the original measurements and the selected value of z.

11. The two preceding possible laws of error seem to be the simplest which are distinct from the Gauss law and which are subject to a natural convention that a very large positive error is neither more likely nor less likely to occur than a very large negative error—a convention which excludes odd powers of u from the index of the exponential. Both of the laws lead to expressions for the selected value of z which, while differing slightly from the arithmetic mean, lack the simplicity of that mean; but the numerical calculations need not prove so heavy as the mere mathematical form suggests. Probably labour will best be saved by using sums of powers of the numerical measurements made; the preliminary formulæ have been stated.

12. A question arises as to whether such laws of error, as yet untried for practical purposes (or so I believe), can have any relation to Pearson's frequency curves, or any influence in suggesting new types of such curves. These all occur in connection with a general characteristic differential equation

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x+a}{b_0 + b_1 x + b_2 x^2},$$

which can be established by various methods involving approximations and generalisations. The following method is based on inferences derived from the graphical grouping of statistics. The proof cannot be regarded as more than suggestive; in any case, the assumptions made are sufficiently obvious.

When regard is paid to the mathematical establishment of the preceding laws of error, whereby an equation

$$\Sigma \frac{\phi'(u_r)}{\phi(u_r)} = 0$$

was needed, it is reasonable to begin with the quantity

$$\frac{1}{y}\frac{dy}{dx}$$

as a magnitude to be obtained. An expression is obtained by using the following inferences from the general appearance of any smooth curve, taken to represent any group of statistics:

(i) for the smallest values of x, the frequency y is small or is zero, while (dy/dx) may be zero and otherwise has a positive value;

(ii) for the largest values of x, the frequency y is small or is zero, while (dy/dx) may be zero and otherwise has a negative value;

(iii) for some one value of x, and (in the simplest sets of statistics selected) for only a single value of x, the frequency y is a maximum;

(iv) usually the slope of increase to the maximum is not the same as the slope of decrease from the maximum.

Suppose that we can take

$$\frac{1}{y}\frac{dy}{dx} = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials in x. There is a single value of x for which (dy/dx) vanishes, say $x = -a$; then

$$P(x) = (x + a) N(x),$$

where $N(x)$ does not vanish for real values of x and must therefore be a polynomial of even degree, if it is not a constant.

At the utmost, in non-periodic statistics, there are two values of x for which y vanishes in the range; then, if we suppose them given by

$$b_0 + b_1 x + b_2 x^2 = 0,$$

where we shall assume b_2 not negative (it might be zero),

$$Q(x) = (b_0 + b_1 x + b_2 x^2) D(x),$$

where $D(x)$ does not vanish for real values of x, and must therefore be a polynomial of even degree, if it is not a constant. Thus

$$\frac{1}{y}\frac{dy}{dx} = \frac{N(x)}{D(x)} \frac{x + a}{b_0 + b_1 x + b_2 x^2}.$$

Next, for large positive values of x, the value of (dy/dx) is negative; that is,

$$\frac{N(x)}{D(x)} \frac{1}{b_2 x}$$

is positive as $x \rightarrow +\infty$; thus

$$\frac{N(x)}{D(x)} \frac{1}{b_2}$$

is negative for large positive values of x. Again, for large negative values of x—the smallest values of x can be supposed negative, because no origin has yet been assigned—the value of (dy/dx) is positive; that is,

$$\frac{N(x)}{D(x)} \frac{1}{b_2 x}$$

is positive as $x \to -\infty$; thus

$$\frac{N(x)}{D(x)} \frac{1}{b_2}$$

is negative for large negative values of x. Further, both $N(x)$ and $D(x)$ are polynomials of even degree. Consequently, the sign of $N(x)$ is the same for large values of x, whether these are positive or negative; and likewise for the sign of $D(x)$. We have assumed that the sign of b_2 is positive, when b_2 is not zero; thus the sign of $N(x)/D(x)$ is negative, as x tends to large values.

13. There are simple cases, which arise by assuming special forms for $N(x)$ and $D(x)$.

(*a*) Let both $N(x)$ and $D(x)$ be constants; their ratio is a constant, which must be negative. The constant, being absorbable into the constants b_0, b_1, b_2, can be taken as equal to -1; so, in this case, we have

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x+a}{b_0 + b_1 x + b_2 x^2}.$$

(*b*) Let $D(x)$ be a constant, and $N(x)$ the simplest polynomial of even degree not vanishing for real values of x, say

$$b + c(x+a)^2,$$

where b and c have the same sign, say positive. As $N(x)/D(x)$ is to be negative, and as $D(x)$ is a constant, the latter can (as before) be taken as equal to -1; and so we have

$$\frac{1}{y}\frac{dy}{dx} = -\frac{(x+a)\{b + c(x+a)^2\}}{b_0 + b_1 x + b_2 x^2}$$

as a possible form, where c can be made unity when it is not zero.

The very special case when $b_1 = 0$, $b_2 = 0$, leads to the form of the possible law of error stated in § 7; and the further limitation, $b = 0$, leads to the form of the possible law of error stated in § 4.

14. Other forms arise by taking special forms for $N(x)$ and $D(x)$ simultaneously as polynomials of even degree that do not vanish for real values of x. Thus we might have

$$\frac{1}{y}\frac{dy}{dx} = -\frac{b + c(x+a)^2}{b' + c'(x+a)^2}\frac{x+a}{b_0 + b_1 x + b_2 x^2},$$

where b, c, b', c' are positive constants; but I have no means of judging whether such a form, even if much specialised, is likely to be useful for a practical purpose.

ON NAPIER'S CIRCULAR PARTS.

By *W. Woolsey Johnson.*

1. Let us consider five quantities known, by virtue of their geometric connection, to be such that, arbitrary values being assigned to two of them, the remaining three have their values fixed. It follows that there must exist a relation between the members of each of the ten triads which can be formed from the five quantities. Suppose further that the five quantities form a reversible cycle as do, for example, the parts (omitting the right angle) of a right spherical triangle; so that there is one unique part, and the others form two pairs, such that the members of either pair may be interchanged provided those of the other pair are also interchanged: then there will be but six distinct relations to be established between the members of a triad, the other four relations following from a proper interchange of the letters denoting the quantities.

2. But if in such a reversible cycle, $x_1x_2x_3x_4x_5$ (which we may suppose placed at the vertices of a pentagon), there is no unique part, so that cyclic interchanges can take place, the relation between the members of a triad will depend only upon the *collocation* of the parts. Of these collocations there exist but two, namely, the case in which the three parts are adjacent, for example, $x_1x_2x_3$, and that in which one part is detached from the other two, as for example $x_1x_2x_4$. There will now be only two distinct relations to be established, the remaining eight following from cyclic interchanges of the letters.

Such a set of quantities may be called a set of "Circular Parts".

3. Napier's work in connection with right spherical triangles may be regarded as consisting of two parts; namely, first the introduction of the idea of Circular Parts as above defined; secondly, the "invention", to use his own term, of a set of Circular Parts closely connected with the parts of a spherical right triangle, whereby the six relations which constitute the doctrine of right spherical trigonometry are virtually reduced to two.

4. The general case of Circular Parts does not seem to have attracted the attention it merits, and the writer hopes, in another paper, to make a contribution to the subject. It is the object of the present paper to trace the history of Napier's own Circular Parts in connection with that of spherical trigonometry.

5. Napier's Rules of the Circular Parts of a right-angled spherical triangle were published in 1614 in the *Mirifici Logarithmorum Canonis Descriptio.* Immediately upon its appearance, Edward Wright, the Navigator, who received it with great enthusiasm, translated it into English and sent his version to the author, who found it "most exact and precisely conformable to his mind and the original". The translation was returned to Wright shortly before his death in 1615, and during the next year was seen through the press by his son*. The quotations from Napier in this paper will be made from Wright's version.

6. All of the six formulæ of right triangles were well known in Napier's time, and are quoted by him from Regiomontanus and others; but he himself gave no proofs. The current demonstrations made use of constructions "within the sphere" and others on the surface of the sphere. The latter reduce themselves essentially to a single one; namely, the construction of the "complementary triangle" formed by producing through the vertex of one of the oblique angles of a given right triangle the containing side and hypothenuse, and then intersecting them by the great-circle, whose pole is the vertex of the other oblique angle. All of the parts of the triangle thus formed (except the equal vertical angles) are complements of parts of the original triangles.

7. This construction was much used in demonstration until comparatively recent times. For example, in the texts of Playfair and of Robert Simson, the authors, after demonstrating by constructions within the sphere (that is by intersecting the triedral angle by a plane) the theorem $\sin A = \frac{\sin a}{\sin c}$, apply it to each angle of the complementary triangle; each result is then stated in terms of the parts of the original triangle and

* *National Dictionary of Biography, Art: Napier.* The Latin original was republished by Maseres in his *Scriptores Logarithmici,* vol. vi., p. 475.

gives a new theorem. In like manner, $\tan A = \frac{\tan a}{\sin b}$, demonstrated by another construction within the sphere, gives rise to two other theorems completing the six, and incidentally separating them into two groups of three.

8. The necessity of using separate constructions in demonstrating the two main formulæ quoted above was due to the fact that the line-definitions of the trigonometric functions were then in vogue. The complementary triangle was for this reason particularly useful at that time. But writers who now derive the formulæ of right triangles independently of the general case obtain four of them directly from the triedral angle figure, and the remaining two by elimination; thus having no occasion to use the complementary triangle.

The latest writer I know of to use this method is Elias Loomis, in 1848 (who only at the last, and reluctantly, gave up the line-definitions).

9. Delambre, in discussing the *Opus Palatinum* of Rheticus, 1596*, notes his employment of the figure of the complementary triangle. Delambre shows how by the help of this figure Rheticus might have derived all the other five theorems from a single one (the initial theorem for the second group being derived by elimination from two of the first group). He adds, however, that Rheticus did not thus carry formulæ into the complementary triangle, but that "these complementary triangles give him rectilinear triangles which he compares in all possible manners".

Braunmühl, in his *Vorlesungen über Geschichte der Trigonometrie*, says that the earliest use of the figure occurs in Ptolemy's *Almagest*.

10. To return to Napier, the Circular Parts are introduced in Chapter IV. of the second book of the *Descriptio*, which is entitled "Of single quadrantals". In this term he includes triangles having either a side or an angle equal to 90°. After defining the five parts, he says: "Of these five parts which are not quadrants, those which are furthest removed from the right angle or the side which is a quadrant we turn into their complements, and retaining the old order, we bring all five into a quinquangular situation and call them Circulars". The examples used in illustration are the right triangle *PBS* and

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* *Astronomie Moderne*, vol. ii., p. 5.

the astronomical triangle *PSZ* with the sun on the horizon in summer, so that *BS* is less than a quadrant. The right and quadrantal triangles thus related may be called complemental, having corresponding parts either equal or complementary; the obtuse angle subtended by the quadrant is virtually regarded by Napier as equal to its supplement, having the same complement which he calls "the difference ascensional, that is the difference in the Sunne's rising or setting from sixe a clock".

11. Napier now shows that these triangles have the same circular parts, and concludes that there are many triangles which differ in their actual parts, but have the same circular parts. He continues: "This uniformity of the circular parts most manifestly appeareth in right triangles made on the superficies of a globe of five great circles the first whereof cutteth the second, the second the third, the third the fourth, the fourth the fifth, and lastly the fifth the first, at right angles, but the other sections shall be made at oblique angles". The figure illustrating this is an enlargement of that already used, the five great circles being the meridian, the horizon, the circle of which the sun is at the instant the pole, the hour circle of the sun, and the equator. It constitutes a crossed pentagon, of which the inner convex pentagon is self-polar. The five quadrantal triangles are cut off from this pentagon by its diagonals, and the five right triangles are the stellations formed by producing the sides.

Gauss was much attracted by this figure, which he named the *Pentagramma Mirificum*, and amplified further by producing the great circles to their other intersections, thus forming five trirectangular quadrilaterals, another set of figures subsidiary to the pentagon, and so possessing the same circular parts.

12. Napier next remarks that, of the three parts which enter any formula, one must be a middle part and the two extremes either adjacent or opposite. He then gives his two rules in the logarithmic* form appropriate to the main purpose of the *Descriptio*. He then says: "The theorem is proved by induction of all the three parts or triplicities that can be made or come in question of the five circular parts of the right-angled quadrantal *BPS*". Enumerating the five cases of adjacent parts in this triangle, he continues:

* Logarithmus intermediae aequatur differentialibus circumpositarum extremarum seu antilogarithmis oppositarum extremarum.

"In all these cases, the tangent of either extreme is to the sine of the intermediate as the whole sine is to the tangent of the other extreme, as appears from the ordinary demonstrations of trigonometry".

13. After the appearance of Napier's Rules it was natural that their aspect as mnemonics should have been made prominent by the writers of text books. After demonstration of the six theorems, for the most part, the Rules were given, and proved by "induction", just as by Napier. The Rules of course recognized the connection of the members of the two groups of the formulæ; but the pentagram which explains it was lost sight of, so that this connection appeared to be the result of a number of curious coincidences discovered by Napier. Thus Wilson, in 1831, says: "For common purposes a technical memory has been invented under the title 'Napier's rules for Circular Parts'". The earliest expression of this view which I have met with occurs in Hodgson's *System of Mathematics*, 1723; after giving the six formulæ as corollaries to certain theorems, Hodgson says: "From a diligent consideration of the preceding Corollaries, the Lord Napier, the first inventor of the Logarithms, contrived two General Rules easy to be retained in the Memory". Many writers admire the Rules "as one of the happiest examples of artificial memory that can anywhere be found". Montucla expressed his regret that they were not more familiar to French writers.

14. On the other hand, Delambre in his *Astronomie Theoretique et Pratique*, vol. I., p. 203, says that "Neper . . . a tenté de reduire toutes ces regles . . . à deux", but goes on to say that he has never used them nor found any difficulty in remembering the six formulæ. "It is extremely inconvenient", he adds, "to substitute the complements, to examine which is the middle part, etc." De Morgan, in his *Spherical Trigonometry*, refuses to give the Rules, on the ground that "they only create confusion instead of assisting the memory". They were excluded for the same reason from Dr. Hutton's *Course of Mathematics*.

The fact that in astronomical applications the actual parts of the triangle are frequently themselves complements of the quantities that are to appear in the final formula increases the confusion alluded to by Delambre and De Morgan; so that we are not surprised at their objection to the indirectness of the Rules. For the same reason Dr. Glaisher, in the article

"Napier", *Enc. Brit.*, 11th edition, speaks of them as "of very doubtful utility, as the formulæ are best remembered by the practical computer in their unconnected form".

15. James Ivory, in 1821, in a paper published in *Tillock's Phil. Mag.*, vol. lviii., p. 255, controverts the view that Napier's Rules "are so contrived that by a particular classification and nomenclature of the parts of a triangle they include all the propositions necessary for solving each case". He represents his opponents as making the curious claim that "Rules entirely dependent on dexterity of arrangement cannot admit of a separate demonstration".

While admitting that Napier *does* prove the Rules by induction, Ivory says that, because he was not writing "an express treatise on trigonometry, it became necessary to show the agreement of the rules with the writings of others. At the close of this demonstration he immediately indicates another and a more general one". This looks as if Ivory held that Napier might have given a complete proof independent of the work of his predecessors, which is of course untrue. But, after all, Ivory's summary speaks of two theorems as precedent to what he calls Napier's one "proposition". He then contrasts Napier's method with the use made by Playfair and Simson of the complementary triangle. In fact, it seems remarkable that these authors should have given Napier's Rules without alluding to their connection with the complementary triangle.

16. The "one proposition" is, in fact, the existence of the central pentagon (with its five-fold instead of two-fold "symmetry"), which Napier evolved out of one of its subsidiary figures, the right triangle. Had our interest been primarily concerned with this figure there would still have been ten relations between the five parts, say the sides of the pentagon (as in Mauduit's version of the Rules), to be considered, but only two distinct ones to be derived from constructions within the sphere; the remaining eight would be obvious by "symmetry", that is, by mere interchange of letters.

17. The same language about a "proof of Napier's Rules" has been used in connection with the rediscoveries of Napier's method, of which the best known is that of R. L. Ellis. Not, however, by Ellis himself, who notes explicitly that two of the six formulæ must be independently proved, just as Napier

does in the final paragraph quoted by Ivory*; but by Goodwin, who was led by Ellis' paper to look up the *Descriptio*, and expressed his astonishment at its long neglect on the part of writers on trigonometry and indignation against Airy and others, who state "there is no other proof, etc.". Also by Todhunter, who gave an account of Ellis' work sometime before the appearance of Walton's memorial volume.

One of the recent rediscoverers of the pentagram believes that a proof of Napier's Rules "appears here for the first time in an elementary text book".

18. The separation of the formulæ into two groups, though not due to Napier, is made so prominent by the rules that it had come to be regarded as their principal part. Of the rest Ivory makes light, saying: "The invention of the circular parts merely enables the author to enunciate the two theorems with reference to the given triangle alone instead of the five associated triangles".

19. The same view of the relative importance of the parts of Napier's work appears in a paper by De Morgan in 1843, "On the invention of the Circular Parts", *Phil. Mag.*, 3rd Series, vol. xxii., p. 350. In this paper he credits Nathaniel Torporley (1564–1633) with anticipating Napier "by 12 years in a very material portion of Napier's rule of circular parts". After showing how Torporley separates the formulæ into two groups, De Morgan says: "Here is the reduction of the six cases to two". He also says Torporley "discovers the necessity of using the complements", as though he were the first discoverer of the complementary triangle.

20. I have not been able to see a copy of Torporley's work of which the title is given by De Morgan as "Diclides coelometricae, seu valvae astronomicae universales, omnia artis totius munera psephophoretica in sat modicis finibus duarum tabularum methodo nova generali et facillima continentes, London, 1602". An account of the work is also given by Delambre in the *Histoire de l'Astronomie Moderne*, vol. ii, p. 36.

21. The "diclides" and "valvae" of the title appear to be equivalent terms referring to the doors of knowledge.

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* Praeter hanc probationem per inductionem omnium casuum, qui occurrere possunt, potest idem theorema lucide perspici ex 19 et 20, praecedetibus, in quorum schemate, homologa circularium partium constitutio earundum analogiae similitudinem arguit: ita ut quod de una intermediâ et suis extremis circumpositis, aut oppositis vere enuntiatur, de caeteris quatuor intermediis et suis extremis respective circumpositis, aut oppositis, negari non possit.

But the term valvae in the text is applied to the six "triplicities" corresponding to the six formulæ. These "valves" receive fanciful names, evidently suggested by the diagrams formed by heavily shading, in the figure of a right triangle, the three parts which occur in the triplicity. For examples, the valve of the three sides is called "carcer", that of the hypothenuse and the angles (a shaft with two spear points) is called "hasta". These valves are, as De Morgan puts it, "mounted on two 'mitres' three on each". The figure of the mitre is given by Delambre; it consists of a triangle with its two complementary ones, and the connection between the valves of each set is this: if, in the central triangle, we mark the parts concerned in one of the formulae as given, three parts become given in each of the lateral triangles. The central valve in each mitre is called the "mother" and the others the "daughters". The formulae corresponding to the mothers are those given in Sec. 7. This choice of the central valves is such that the lateral valves are distinct.

22. It appears, from De Morgan's account, that Torporley gives rules in the form of mnemonic verses, "for the reduction of each daughter to the mother". The whole mnemonic scheme De Morgan characterises as ridiculous. Yet he speaks of Torporley's two *formulæ* which "resemble of course those of Napier in their structure", and in conclusion says: "The reduction of all six cases to two, and the first exhibition of an organized mechanical mode of reducing each of the six cases to its primitive belongs to him. Napier afterward did the latter in a better manner, without the necessity of mnemonical verses".

De Morgan gives specimens of one of the two tables mentioned in Torporley's title: these were, no doubt, made available each for the three members of one group of triplicities.

23. De Morgan is of opinion that Napier must have seen Torporley's book. His arguments are two: first that Torporley's figure is included in Napier's. To this it may be answered that the connection of each adjacent pair of triangles in either figure had long been known, and even the figure of *three* triangles had been given by Brescius and probably many others. Thus Napier needed no further suggestion in discovering the chain of five associated triangles. The second "suspicious circumstance", as De Morgan calls it, is that

both authors use the word "triplicitas" for a group of three. This word, being a term of judicial astrology, he thinks was naturally used by Torporley, who was an astrologer, but that Napier, a mathematician, would be expected to use "Ternio" or "Trias". To this it may be replied, that neither of the latter words being in general use, it should be regarded as merely a coincidence that the same selection out of a very limited number should have been made by both writers.

Against the idea that Napier derived anything from Torporley without acknowledgment is to be set his character and the cordial relations he sustained with his contemporaries.

24. Braunmühl, in his account of Torporley, says that he has unfortunately been unable to see Torporley's book, and therefore must rely upon De Morgan and Delambre. He accepts the argument from the use of "Triplicitas" without question. He recalls the fact that Torporley was, for a period, secretary to Vieta, and says his writings are characterised by a predilection for new demonstrations devised by his master carried sometimes to a ridiculous extent. He sums up as follows: "In a history of trigonometry we should hardly have had occasion to mention Torporley, if he had not had the happy thought (perhaps also due to Vieta) to unite the six cases, already separated into two groups, by means of a single figure. To this extent has even this crazy book contributed to the simplification of trigonometric doctrine".

THEOREMS IN THE EXPANSION OF POLYNOMIALS, OBTAINED BY AN APPLICATION OF THE CALCULUS OF RESIDUES.

By *E. A. Milne, B.A.*, Scholar of Trinity College, Cambridge.

§ 1. In this paper a simple application of the calculus of residues is made to deduce certain results concerning expansions of polynomials. If from the results obtained it is attempted to deduce further results by the method of "equating coefficients", the further results are merely cases of the binomial theorem or simple deductions from it. It follows that the expansions could all be built up by elementary algebraic methods, though they seem to be suggested directly only by the method of residues.

§ 2. Let n be an integer, $f(z)$ a polynomial of degree not greater than n, and consider the rational function $F(z)$ given by

$$F(z) = \frac{f(z)}{(z-a)^n \{h - z(z-a)\}}.$$

Let α, β be the roots of the quadratic

$$z(z-a) = h \quad\text{.......................}(1),$$

and suppose that α, β, a are all unequal.

The degree of the denominator of $F(z)$ exceeds that of the numerator by at least 2, and therefore the residue of $F(z)$ at infinity is zero. Hence the sum of the residues of $F(z)$ at its poles must be zero. These poles occur at the points α, β, a. Writing $F(z)$ in the form

$$\frac{f(z)}{(z-a)^n} \cdot \frac{1}{\beta-\alpha}\left(\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right),$$

we see that the residues of $F(z)$ at α, β are

$$\frac{1}{\beta-\alpha} \cdot \frac{f(\alpha)}{(\alpha-a)^n}, \quad \frac{1}{\alpha-\beta} \cdot \frac{f(\beta)}{(\beta-a)^n};$$

or, since α, β each satisfy (1),

$$\frac{1}{\beta-\alpha} \cdot \frac{\alpha^n f(\alpha)}{h^n}, \quad \frac{1}{\alpha-\beta} \cdot \frac{\beta^n f(\beta)}{h^n};$$

the sum of which is

$$-\frac{\alpha^n f(\alpha) - \beta^n f(\beta)}{h^n(\alpha-\beta)}.$$

To determine the residue at the point a, we expand $\{h - z(z-a)\}^{-1}$ in powers of $z-a$. Let C be a circle of radius ρ and with centre at the point a. Choose ρ so small that the points α, β lie outside C, and that $\{h - z(z-a)\}^{-1}$ can be expanded on C in the uniformly convergent series

$$\frac{1}{h}\left[1 + \sum_{r=1}^{\infty}\left(\frac{z(z-a)}{h}\right)^r\right].$$

Then the residue of $F(z)$ at a is equal to

$$\frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^n} \frac{1}{h}\left[1 + \sum_{r=1}^{\infty}\left(\frac{z(z-a)}{h}\right)^r\right] dz.$$

This may be integrated term by term. The terms for which r exceeds $n-1$ contribute zero, and the rest give

$$\frac{1}{h}\left[\frac{1}{h^{n-1}}\cdot a^{n-1}f(a)+\frac{1}{1\,!}\cdot\frac{1}{h^{n-2}}\cdot\frac{d}{da}\{a^{n-2}f(a)\}\right.$$
$$\left.+\frac{1}{2\,!}\cdot\frac{1}{h^{n-3}}\cdot\frac{d}{da}\{a^{n-3}f(a)\}+\ldots+\frac{1}{n-1\,!}\cdot\frac{d^{n-1}}{da^{n-1}}\{f(a)\}\right].$$

Equating the sum of the residues of $F(z)$ to zero, we have finally the expansion

$$\frac{\alpha^n f(\alpha)-\beta^n f(\beta)}{\alpha-\beta}=a^{n-1}f(a)+\frac{h}{1\,!}\frac{d}{da}\{a^{n-2}f(a)\}$$
$$+\frac{h^2}{2\,!}\cdot\frac{d^2}{da^2}\{a^{n-3}f(a)\}+\ldots+\frac{h^{n-1}}{n-1\,!}\cdot\frac{d^{n-1}}{da^{n-1}}\{f(a)\}\ldots(2).$$

In this expansion

$$a=\alpha+\beta,\quad h=-\alpha\beta,$$

and the expansion is valid provided the degree of f is not greater than n. As particular cases, put $\beta=1/\alpha$ and $\beta=-1/\alpha$. We obtain

$$\frac{\alpha^n f(\alpha)-\alpha^{-n}f(\alpha^{-1})}{\alpha-\alpha^{-1}}=a^{n-1}f(a)-\frac{1}{1\,!}\cdot\frac{d}{da}\{a^{n-2}f(a)\}$$
$$+\ldots+\frac{(-1)^{n-1}}{n-1\,!}\cdot\frac{d^{n-1}}{da^{n-1}}\{f(a)\}\ldots(3)$$
$$(a=\alpha+\alpha^{-1}),$$

$$\frac{\alpha^n f(\alpha)-(-1)^n\alpha^{-n}f(\alpha^{-1})}{\alpha+\alpha^{-1}}=a^{n-1}f(a)+\frac{1}{1\,!}\cdot\frac{d}{da}\{a^{n-2}f(a)\}$$
$$+\ldots+\frac{1}{n-1\,!}\cdot\frac{d^{n-1}}{da^{n-1}}\{f(a)\}\ldots(4)$$
$$(a=\alpha-\alpha^{-1}).$$

Putting $f(\alpha)=1$ in these, we obtain the known expansions

$$\frac{\alpha^n-\alpha^{-n}}{\alpha-\alpha^{-1}}=(\alpha+\alpha^{-1})^{n-1}-\frac{n-2}{1\,!}(\alpha+\alpha^{-1})^{n-3}$$
$$+\frac{(n-3)(n-4)}{2\,!}(\alpha+\alpha^{-1})^{n-5}-\ldots\ldots\ldots(5),$$

$$\frac{\alpha^n-(-1)^n\alpha^{-n}}{\alpha+\alpha^{-1}}=(\alpha-\alpha^{-1})^{n-1}+\frac{n-2}{1\,!}(\alpha-\alpha^{-1})^{n-3}$$
$$+\frac{(n-3)(n-4)}{2\,!}(\alpha-\alpha^{-1})^{n-5}+\ldots\ldots\ldots\ldots(6),$$

each series terminating at the first term which vanishes identically.

§ 3. If the same method is applied to the rational function

$$\frac{f(z)}{(z-a)^n\{z^2-h(z-a)\}},$$

we obtain the result

$$\frac{1}{\beta-\alpha}\left[\frac{f(\alpha)}{\alpha^{2n}}-\frac{f(\beta)}{\beta^{2n}}\right]=\frac{1}{h}\left[\frac{f(a)}{a^{2n}}+\frac{1}{1!}\cdot\frac{1}{h}\cdot\frac{d}{da}\frac{f(a)}{a^{2n-2}}\right.$$

$$\left.+\frac{1}{2!}\cdot\frac{1}{h^2}\cdot\frac{d^2}{da^2}\frac{f(a)}{a^{2n-4}}+\ldots+\frac{1}{n-1!}\cdot\frac{1}{h^{n-1}}\cdot\frac{d^{n-1}}{da^{n-1}}\frac{f(a)}{a^2}\right]\ldots(7),$$

where now $$h=\alpha+\beta,$$

$$a^{-1}=\alpha^{-1}+\beta^{-1},$$

and $f(z)$ is a polynomial of degree not greater than n. If in this we put $f(z)=z\phi(z)$, where $\phi(z)$ is a polynomial of degree not greater than $n-1$, and if we then replace ϕ by f, we obtain

$$\frac{1}{\beta-\alpha}\left[\frac{f(\alpha)}{\alpha^{2n-1}}-\frac{f(\beta)}{\beta^{2n-1}}\right]=\frac{1}{h}\left[\frac{f(a)}{a^{2n-1}}+\frac{1}{1!}\cdot\frac{1}{h}\cdot\frac{d}{da}\frac{f(a)}{a^{2n-3}}+\ldots\right.$$

$$\left.+\frac{1}{n-1}\cdot\frac{1}{h^{n-1}}\cdot\frac{d^{n-1}}{da^{n-1}}\frac{f(a)}{a}\right]\ldots\ldots\ldots\ldots(8),$$

where the degree of f is not greater than $n-1$.

As examples, put $\beta=1/\alpha$, $\beta=-1/\alpha$. We obtain

$$\frac{\alpha^{2n}f(1/\alpha)-\alpha^{-2n}f(\alpha)}{\alpha-\alpha^{-1}}=\frac{af(a)}{a^{2n}}+\frac{a^2}{1!}\cdot\frac{d}{da}\frac{f(a)}{a^{2n-2}}+\ldots$$

$$+\frac{a^n}{n-1!}\cdot\frac{d^{n-1}}{da^{n-1}}\frac{f(a)}{a^2}\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

$$\{a=\alpha/(1+\alpha^2)\},$$

$$\frac{\alpha^{2n}f(-1/\alpha)-\alpha^{-2n}f(\alpha)}{\alpha+\alpha^{-1}}=-\frac{af(a)}{a^{2n}}+\frac{a^2}{1!}\cdot\frac{d}{da}\frac{f(a)}{a^{2n-2}}-\ldots$$

$$+\frac{(-a)^n}{n-1!}\cdot\frac{d^{n-1}}{da^{n-1}}\frac{f(a)}{a^2}\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

$$\{a=\alpha/(1-\alpha^2)\},$$

with similar expansions for an odd exponent.

§ 4. In the result (2) let β tend to α. We obtain

$$\frac{d}{d\alpha}\{\alpha^n f(\alpha)\} = (2\alpha)^{n-1} f(2\alpha) - \frac{\alpha^2}{1!} \cdot \frac{d}{d(2\alpha)}\{(2\alpha)^{n-2} f(2\alpha)\} + \dots$$

$$+ \frac{(-1)^{n-1}\alpha^{2n-2}}{n-1!} \cdot \frac{d^{n-1}}{d(2\alpha)^{n-1}}\{f(2\alpha)\}$$

or

$$\frac{d}{d\alpha}\{\alpha^n f(\alpha)\} = 2^{n-1}\alpha^{n-1} f(2\alpha) - \frac{2^{n-3}\alpha^2}{1!} \cdot \frac{d}{d\alpha}\{\alpha^{n-2} f(2\alpha)\}$$

$$+ \frac{2^{n-5}\alpha^4}{2!} \cdot \frac{d^2}{d\alpha^2}\{\alpha^{n-3} f(2\alpha)\} - \dots + \frac{2^{-n+1}\alpha^{2n-2}}{n-1!} \cdot \frac{d^{n-1}}{d\alpha^{n-1}}\{f(2\alpha)\} \dots. (11).$$

Similarly, from (7) and (8), we obtain

$$-\frac{d}{d\alpha}\left\{\frac{f(\alpha)}{\alpha^{2n}}\right\} = \frac{2^{2n-1}}{\alpha} \cdot \frac{f(\frac{1}{2}\alpha)}{\alpha^{2n}} + \frac{2^{2n-3}}{1!\,\alpha^2} \cdot \frac{d}{d\alpha} \frac{f(\frac{1}{2}\alpha)}{\alpha^{2n-2}}$$

$$+ \frac{2^{2n-5}}{2!\,\alpha^3} \cdot \frac{d^2}{d\alpha^2} \frac{f(\frac{1}{2}\alpha)}{\alpha^{2n-4}} + \dots + \frac{2}{n-1!\,\alpha^n} \cdot \frac{d^{n-1}}{d\alpha^{n-1}} \frac{f(\frac{1}{2}\alpha)}{\alpha^2} \dots (12),$$

and a similar expansion when the exponent on the left-hand side is odd.

§ 5. If we apply the method to the rational function

$$\frac{f(z)}{(z-a)^n\{hz-(z-a)\}},$$

we obtain the result

$$\frac{f(\alpha)}{\alpha^n} = (1-h)\left[\frac{f(a)}{a^n} + \frac{h}{1!} \cdot \frac{d}{da} \frac{f(a)}{a^{n-1}}\right.$$

$$\left. + \frac{h^2}{2!} \cdot \frac{d^2}{da^2} \frac{f(a)}{a^{n-2}} + \dots + \frac{h^{n-1}}{n-1!} \cdot \frac{d^{n-1}}{da^{n-1}} \frac{f(a)}{a}\right] \dots (13),$$

where $$a = \alpha(1-h),$$

and $f(z)$ is a polynomial of degree not greater than $n-1$.

§ 6. Consider the rational function

$$\frac{f(z)}{(z-a)^m (z-b)^n [A(z-a)-B(z-b)]},$$

where $f(z)$ is a polynomial of degree not greater than $m+n-1$. The residue of this function at infinity is zero, and hence the sum of the residues at its poles is zero. Put

$$\alpha = (Aa - Bb)/(A-B),$$

and suppose that a and b are unequal. The residue at the pole $z=\alpha$ is

$$\frac{f(\alpha)}{(\alpha-a)^m(\alpha-b)^n(A-B)},$$

which is equal to

$$\frac{f(\alpha)(A-B)^{m+n-1}}{(a-b)^{m+n}A^nB^m}.$$

The residues at the points a, b are obtained by expanding $\{A(z-a)-B(z-b)\}^{-1}$ in power series in $z-a$, $z-b$ in the neighbourhoods of these points. We have finally

$$\frac{f(\alpha)(A-B)^{m+n-1}}{(a-b)^{m+n}A^nB^m}=\frac{1}{A}\left[\left(\frac{B}{A}\right)^{n-1}\frac{f(b)}{(b-a)^{m+n}}\right.$$
$$+\frac{1}{1!}\cdot\left(\frac{B}{A}\right)^{n-2}\frac{d}{db}\frac{f(b)}{(b-a)^{m+n-1}}+\ldots+\frac{1}{n-1!}\cdot\frac{d^{n-1}}{db^{n-1}}\left.\frac{f(b)}{(b-a)^{m-n}}\right]$$
$$-\frac{1}{B}\left[\left(\frac{A}{B}\right)^{m-1}\frac{f(a)}{(a-b)^{m+n}}+\frac{1}{1!}\cdot\left(\frac{A}{B}\right)^{m-2}\frac{d}{da}\frac{f(a)}{(a-b)^{m+n-1}}+\ldots\right.$$
$$\left.+\frac{1}{m-1!}\cdot\frac{d^{m-1}}{da^{m-1}}\frac{f(a)}{(a-b)^{n+1}}\right]\ldots(14).$$

In this put $\alpha=0$, *i.e.* $A/b=B/a$. Then

$$\frac{(-1)^{m+n}f(0)}{(b-a)a^nb^m}=\frac{1}{b}\left[\left(\frac{a}{b}\right)^{n-1}\frac{f(b)}{(b-a)^{m+n}}+\frac{1}{1!}\cdot\left(\frac{a}{b}\right)^{n-2}\frac{d}{db}\frac{f(b)}{(b-a)^{m+n-1}}+\ldots\right.$$
$$\left.+\frac{1}{n-1!}\cdot\frac{d^{n-1}}{db^{n-1}}\frac{f(b)}{(b-a)^{m+1}}\right]-\frac{1}{a}\left[\left(\frac{b}{a}\right)^{m-1}\frac{f(a)}{(a-b)^{m+n}}\right.$$
$$\left.+\frac{1}{1!}\cdot\left(\frac{b}{a}\right)^{m-2}\frac{d}{da}\frac{f(a)}{(a-b)^{m+n-1}}+\ldots+\frac{1}{m-1!}\cdot\frac{d^{m-1}}{da^{m-1}}\frac{f(a)}{(a-b)^{n+1}}\right]$$
$$\ldots\ldots(15),$$

which may be written

$$\frac{(-1)^{m+n}f(0)}{(b-a)}=\left[a^{m+n-1}\frac{f(b)}{(b-a)^{m+n}}+\frac{a^{m+n-2}b}{1!}\cdot\frac{d}{db}\frac{f(b)}{(b-a)^{m+n-1}}+\ldots\right.$$
$$\left.+\frac{a^mb^{n-1}}{n-1!}\cdot\frac{d^{n-1}}{db^{n-1}}\frac{f(b)}{(b-a)^{m+1}}\right]-\left[b^{m+n-1}\frac{f(a)}{(a-b)^{m+n}}\right.$$
$$\left.+\frac{b^{m+n-2}a}{1!}\cdot\frac{d}{da}\frac{f(a)}{(a-b)^{m+n-1}}+\ldots+\frac{b^na^{m-1}}{m-1!}\cdot\frac{d^{m-1}}{db^{m-1}}\frac{f(b)}{(a-b)^{n+1}}\right]\ldots(16).$$

Now suppose m greater than n. Interchange m and n in (16), and subtract from (16) the result so obtained. We get the curious result,

$$\frac{1}{n!}\left[a^{m-1}b^{n}\frac{d^{n}}{db^{n}}\quad\frac{f(b)}{(b-a)^{m}}+b^{m-1}a^{n}\frac{d^{n}}{da^{n}}\quad\frac{f(a)}{(a-b)^{m}}\right]$$

$$+\frac{1}{n+1!}\left[a^{m-2}b^{n+1}\frac{d^{n+1}}{db^{n+1}}\quad\frac{f(b)}{(b-a)^{m-1}}+b^{m-2}a^{n+1}\frac{d^{n+1}}{da^{n+1}}\quad\frac{f(a)}{(a-b)^{n-1}}\right]$$

$$+\ \ldots\ldots$$

$$+\frac{1}{m-1!}\left[a^{n}b^{m-1}\frac{d^{m-1}}{db^{m-1}}\quad\frac{f(b)}{(b-a)^{n-1}}+b^{n}a^{m-1}\frac{d^{m-1}}{da^{m-1}}\quad\frac{f(a)}{(a-b)^{n+1}}\right]=0$$

......(17).

This holds provided the degree of f is not greater than $m+n-1$.

§ 7. The foregoing results can be extended to rational functions by taking into account the residues of the function $f(z)$ at its own poles.

THE DISSECTION OF RECTILINEAL FIGURES.

By *W. H. Macaulay, M.A.*

The problem of dissecting two rectilineal figures of equal area, by straight lines, so that the parts of either figure fit on the other, is one with regard to which a certain number of results have been published.* The problem can obviously be solved in a variety of ways, but it becomes interesting when the object is to discover how, and in how many different ways, a dissection can be made which gives the smallest possible number of parts, for a pair of figures which are independent of one another, in the sense of the dimensions of one figure having no relation to those of the other beyond the assumed equality of areas. It will be assumed that a pair of figures for which a type of dissection is sought are independent, if nothing is said to the contrary.

It is necessary to give some explanation of terminology. A pair of parallelograms is said to have a three-part dissection

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* Dudeney's *Canterbury Puzzles*, p. 143, the *Messenger of Mathematics*, vol. xxxv., p. 81, and the *Mathematical Gazette*, vol. vii., p. 381, and vol. viii., p. 72, continued on p. 109.

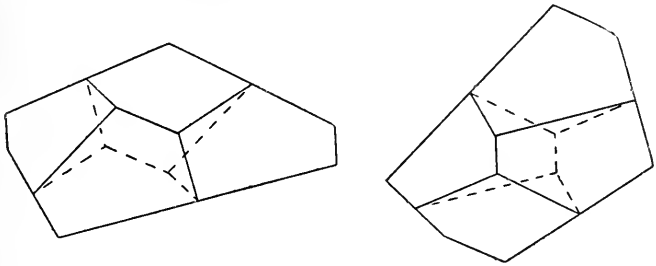
(each being divided by two lines equal and parallel to sides of the other), although this dissection actually gives three parts only when the dimensions of the second parallelogram lie within a certain range with reference to those of the first parallelogram. When they are not within this range the same type of dissection is still applicable, but one or both of the dividing lines must be replaced by what are called "broken lines", each consisting of a series of parallel lines; and the dissection has a greater number of parts. This may be expressed otherwise by saying that every pair of parallelograms has a certain dissection, which has three-parts in the "fundamental" case, and which can be extended to other cases by means of broken lines. When a broken line is used it has the same total length as it would have, according to the theory of the dissection, if unbroken. The same terminology is applied to other dissections.

There is one three-part type of dissection of independent figures, namely that of a pair of parallelograms. Various "derived" dissections may be obtained from this by means of additional cuts in each figure. Two of these are four-part dissections, namely, those of a parallelogram and a triangle, and of a parallelogram and a quadrilateral with two sides parallel. The derived dissections of a pair of triangles, and of a pair of quadrilaterals with two sides parallel, have each five parts in their fundamental case. A dissection which is not derived from another dissection of independent figures by means of additional cuts will be called a "radical" dissection. Thus the three-part dissection of a pair of parallelograms is a radical dissection.

My present object is mainly to call attention to the fact that all the radical four-part dissections of independent figures which are known to me, perhaps all that exist, are particular cases of a four-part dissection of a pair of hexagons, each with a pair of opposite sides equal and parallel; these hexagons not being independent of one another, but related in a way which I will explain. A hexagon has a pair of "parallel sides" and four "inclined sides", the latter meeting at two "vertices". In each hexagon, join the middle points of the inclined sides so as to form a parallelogram of area equal to half that of a hexagon; for our present purpose I will call this the "core" of the hexagon. Then the relation between the hexagons is that they have identical cores, the side of the core which is opposite a vertex in one hexagon being equal to a side of the core which is opposite one of the parallel sides in the other hexagon.

Each hexagon must be restricted to being such that, when the perimeter is traversed, the two parallel sides are traversed in opposite directions; also to having a single area, no boundary lines crossing. And to begin with, dealing with fundamental cases, it will be assumed that there are no re-entrant angles.

The dissection of the pair of hexagons follows very obviously from the fact that the core of a hexagon can be dissected into four parts, which are identical with the portions of the hexagon which lie outside the core, namely, two triangles and two quadrilaterals. This dissection of the core is made by drawing (in either of the two possible ways) the two triangles with their bases on two sides of the core, and then joining their vertices so as to complete the two quadrilaterals. Having dissected the core of the first hexagon in this way, let us apply this set of dissecting lines to the core of the second hexagon. This hexagon is thus divided into



four pentagons, each of which is made up of a triangle from one hexagon and a quadrilateral from the other. Accordingly we have obtained a four-part dissection of the pair of hexagons. Moreover we have two distinct dissections, because a triangle belonging to the first hexagon can be combined with either of the quadrilaterals belonging to the second. If the first hexagon alone is given, the second has four degrees of freedom, that is to say, is determined by the choice of four quantities; for in order to construct it we have to choose a pair of points within the given hexagon and join them to the middle points of the inclined sides. There are two ways in which the pair of points may be taken in order to produce a given second hexagon. The two dissections, one of them drawn with dotted lines, are shewn in the figure.

This very symmetrical dissection of a pair of related figures does not appear to be especially interesting till it is perceived that there are a number of particular cases in which the relation is eliminated, so that we get a pair of independent figures. All these cases can easily be found by examination of a diagram. They include, I think, all the known radical types of four-part dissection of independent figures, and at least one which perhaps has not been noticed before. It is remarkable that these should all be examples of one geometrical construction. Particular cases are obtained by making two points coincide, or bringing two lines into one straight line, either making an angle zero, or straightening it so that it is two right angles. In each case the core must depend on the two figures jointly, one dimension of it being derived from our figure and another from the other figure.

The following are the principal varieties of four-part dissections of independent figures obtained in this way:

(1) The most obvious way of giving the necessary freedom to the core is simply to straighten, in each hexagon, one angle between adjacent inclined sides; we thus get a pair of pentagons, each with two sides equal and parallel.

(2) If we also make the parallel sides zero we get a pair of triangles.

(3) By straightening both the angles between adjacent inclined sides, in both hexagons, we get a pair of parallelograms.

(4) We get a triangle and a parallelogram (or square) by making one hexagon into a triangle, and the other into a parallelogram, as above. This, as well as the previous case, gives a continuous series of dissections.

(5) We get a pair of quadrilaterals, each with two sides parallel, as follows: in the first hexagon make one parallel side and the two inclined sides adjacent to it into one straight line, and in the second hexagon straighten two opposite angles between a parallel side and an inclined side.

(6) From (5) we get another dissection of a triangle and a parallelogram (or square), quite different in character from (4), by making the parallel sides zero in the first hexagon, and making the opposite sides equal in the quadrilateral given by the second hexagon.

We obtain in this way the fundamental case of each type of dissection, which can be extended to other cases by means of broken lines. Some of these cases arise from angles being made zero, instead of being straightened.

Let us also apply the principle of the dissection of a core to a pair of hexagons, each with three pairs of opposite sides equal and parallel, and with a common core consisting of the triangle formed by joining alternate angular points of each hexagon, the term core being used in the sense of a figure which has the property corresponding to that of the parallelogram in the previous case. In each hexagon the core can be dissected into three triangles indentical with the portions of the hexagon outside the core. Thus by interchanging, as before, the dissections of the cores, we get a three-part dissection of the pair of hexagons in which each part is a quadrilateral. By straightening a pair of opposite angles of each hexagon we get the three-part dissection of a pair of independent parallelograms which has already been referred to.

There are various six-part dissections; but the next problem that seems to have much interest is that of the dissection of a pair of quadrilaterals. This appears to require at least eight parts. There are at least three types of eight-part dissection. Two of these are derived dissections, obtained from the four-part dissection of a pair of pentagons with two sides equal and parallel. The other is a radical dissection obtained from a parallelogram of the same area, whose sides are equal to diagonals of the two quadrilaterals, one diagonal being selected from each. There may be other types, but it seems unlikely that one will be found which gives less than eight parts.

So far I have dealt only with "fundamental cases" of dissections, the object being the classification of types. I propose to consider now how the theory of the four-part dissection of a pair of hexagons, with two opposite sides equal and parallel, and with a common core, may be completed by the introduction of broken lines. In the description of this dissection, given above, it is assumed that the hexagons have no re-entrant angles. When there are re-entrant angles the description needs some modification, and two dissections with four parts do not always exist. It will be found that the use of broken lines, each considered to be equivalent to an unbroken line of the same length, abolishes exceptional cases; and that a pair of hexagons, with or without re-entrant angles, has always exactly four distinct dissections of the type in question, though not more than two can actually have four parts. The four constructions are all made on the same plan, and no distinction of universal application can be made between them; so they must be classed together as one type of dissection. It is called a four-part dissection because it has four parts in the fundamental case.

The scheme of four dissections can easily be understood by placing two hexagons side by side, with their cores similarly situated, and with two dissections with four parts drawn, and attempting to draw the other two in the same way. From each centre of inclined sides of the first hexagon we have already drawn two lines equal and parallel to two inclined half sides of the second hexagon, each set giving a dissection when the points of intersection are joined. Let us now draw from each centre two lines equal and parallel to the other two inclined half sides of the second hexagon, and see how by the use of broken lines each set of these can be made to give a dissection. In the course of this construction, when a line which is being drawn meets a boundary of the hexagon it must be continued by a parallel line. If the boundary which is encountered is an inclined side, the continuation starts from a point on that side, such that the centre of the side is half-way between the first part of the line and its continuation; and if the boundary which is encountered is a parallel side the continuation starts from the corresponding point in the opposite parallel side. Each set of lines drawn in this way gives, as before, two points of intersection, and the dissections are completed by lines, either broken or unbroken, drawn to join these. In this way we get altogether four dissections, which correspond to dissections similarly drawn for the second hexagon. In the general case, when the hexagons are permitted to have re-entrant angles, all the four dissections may require broken lines, and none be distinguished from the rest by any special characteristic.

The two additional dissections might have been obtained in the following way. In the second hexagon draw a set of lines which dissect the core into parts identical with the parts of the hexagon outside the core. Now construct the first hexagon from the second by means of one of the dissections with four parts; then the lines which have been drawn form one of the additional dissections of the first hexagon. The other is found from the other dissection of the core. This procedure is unsymmetrical; but it provides an explanation of the rule for drawing broken lines, by shewing the displacement of parts by means of which a broken line is obtained from a continuous one.

It cannot, I think, be doubted that, for actual hexagons, the number of distinct dissections of this type is exactly four. When, however, we consider particular cases in which the figures are no longer actual hexagons, but may be quadrilaterals or triangles, another consideration has to be taken into account. There may then be a greater number of dissections of the type in question, because there may be several different ways in which the pair of figures can be interpreted as a pair of hexagons. Let us consider the case of a pair of triangles. We shall find that a pair of independent triangles may have as many as 72 dissections of the four-part type.

For the purpose of counting the dissections of a pair of triangles, it will be assumed that there is no accidental coincidence of their dimensions which would affect the number. To secure this we must suppose that the six sides and six perpendiculars from the angles to the opposite sides are all different. A pair of quadrilaterals with a common core is obtained from the hexagons by making the parallel sides zero, and the number of dissections is unaltered, being still exactly four, provided that opposite sides are not parallel. A pair of triangles, so far as their areas are concerned, may be regarded, in any number of different ways, as a pair of quadrilaterals, each with one angle either zero or equal to two right angles. Accordingly, if we count the number of different ways in which this pair of quadrilaterals can be chosen so as to have a common core, and multiply this by four, we shall have obtained the number of distinct dissections of the type in question for the pair of triangles. It is clear that each of these cores is a parallelogram with one side equal to a half side of one triangle, and another equal to a half side of the other triangle, and with area equal to half the area of a triangle; also that each possible core gives one pair of quadrilaterals. Now, if a, b, c, a', b', c' are the sides of the triangles, the number of products aa', ab', ..., which are greater than twice the area of a triangle is either 6 or 7 or 8 or 9. This is fairly obvious, and Mr. H. W. Richmond has given me a proof of it. So the number of possible cores is either 12 or 14 or 16 or 18, and the number of distinct dissections is either 48 or 56 or 64 or 72. I have proved elsewhere that there may be 12, but cannot be more, which actually have four parts; two occur when each quadrilateral has an angle equal to two right angles. The use of quadrilaterals gives the most convenient systematic way of drawing the dissections, and it is easy to draw enough of them to verify the count. Note that a useful reciprocity governs the selection of the lines to be drawn from each centre. The type of dissection may be defined as one in which each triangle is divided by three lines equal and parallel to half sides of the other triangle.

A triangle can be interpreted as a hexagon in another way, one side being made up of two inclined and two parallel sides of a hexagon; but this gives the same result, with a little needless complication.

ON A DIOPHANTINE PROBLEM.

(Second Paper.)

By *H. Holden.*

1. The method used in the previous paper* for the solution in integers of Diophantine systems containing three equations may be used to solve certain systems of four equations.

Consider the system

$$a^2+p^2b^2-c^2=z^2,$$
$$b^2+p^2c^2-d^2=t^2,$$
$$c^2+p^2d^2-a^2=x^2,$$
$$d^2+p^2a^2-b^2=y^2,$$

where p is a given rational quantity. Values of a, b, c, d (and hence of x, y, z, t) expressed in terms of two arbitrary rational parameters can be found, which satisfy these equations.

It may be noticed that the above system is equivalent to

$$(p^2+1)a^2+p^2b^2+(p^2-1)c^2=y^2+z^2+t^2,$$
$$(p^2+1)b^2+p^2c^2+(p^2-1)d^2=z^2+t^2+x^2,$$
$$(p^2+1)c^2+p^2d^2+(p^2-1)a^2=t^2+x^2+y^2,$$
$$(p^2+1)d^2+p^2a^2+(p^2-1)b^2=x^2+y^2+z^2.$$

Returning to the first system, it will be seen that the first three equations will be satisfied if

$$(l^2-1)a+2plb-(l^2+1)c=0,$$
$$(k^2-1)b+2pkc-(k^2+1)d=0,$$
$$(m^2-1)c+2pmd-(m^2+1)a=0.$$

Solving these linear relations for a, b, c, d, we get

$$b=2(l^2+1)(l^2+m^2)k^2+\ldots+2(l^2+1)(l^2+m^2),$$
$$d=2(l^2+1)(l^2+m^2)k^2+\ldots-2(l^2+1)(l^2+m^2).$$

On substituting the values so found in $d^2+p^2a^2-b^2=y^2$, the left-hand side becomes a biquadratic expression in l, k, or m, and if this expression be arranged in powers of k it is obvious that the coefficients of k^4 and of k^0 will be the same as the corresponding coefficients in p^2a^2, and hence are squares.

Hence, in general, one or more suitable values of k, expressed in terms of l, m, and p, may be found which make

* pp. 77-87 of the present volume.

the biquadratic expression a square, and so values of a, b, c, d, similarly expressed, are obtained, satisfying the four equations of either system.

Instead of working with the general linear relations given above, we may, of course, get special solutions by substituting, at the outset, numerical values for l and m. These values should not be equal.

2. As an example, take $p=2$, when the equations become

$$\begin{array}{lll} a^2+4b^2-c^2=z^2, & & 5a^2+4b^2+3c^2=y^2+z^2+t^2, \\ b^2+4c^2-d^2=t^2, & \text{or} & 5b^2+4c^2+3d^2=z^2+t^2+x^2, \\ c^2+4d^2-a^2=x^2, & & 5c^2+4d^2+3a^2=t^2+x^2+y^2, \\ d^2+4a^2-b^2=y^2, & & 5d^2+4a^2+3b^2=x^2+y^2+z^2. \end{array}$$

If, to obtain a special solution, we take $l=2$, the linear relations satisfying the first three equations of the first system are

$$3a+8b-5c=0,$$
$$(k^2-1)b+4kc-(k^2+1)d=0,$$
$$(m^2-1)c+4md-(m^2+1)a=0,$$

which yield, after reduction,

$$a=k^2(4m^2+10m-4)+64mk+4m^2-10m-4,$$
$$b=k^2(m^2+4)-24mk+m^2+4,$$
$$c=k^2(4m^2+6m+4)+4m^2-6m+4,$$
$$d=k^2(m^2+4)+k(16m^2+16)-(m^2+4).$$

We might now substitute these values in $d^2+4a^2-b^2$, and so obtain solutions in terms of m, but to shorten the work take $m=\frac{2}{3}$. This gives, after reduction,

$$a=5k^2+48k-10,$$
$$b=5k^2-18k+5,$$
$$c=11k^2+2,$$
$$d=5k^2+26k-5,$$

and

$$\begin{aligned} d^2+4a^2-b^2 &= 4(25k^4+590k^3+2267k^2-980k+100) \\ &= 4(5k^2-49k+10)^2 \text{ if } k=\tfrac{13}{60} \\ &= 4(5k^2+49k-10)^2 \text{ if } k=\tfrac{17}{50}, \end{aligned}$$

and the first value of k gives

$$a = 2285, \quad x = 10766,$$
$$b = 4805, \quad y = 2750,$$
$$c = 9059, \quad z = 3938,$$
$$d = 3125, \quad t = 18482.$$

For the similar system of three equations

$$\begin{aligned} a^2 + 4b^2 - c^2 &= z^2, & 5a^2 + 3b^2 &= y^2 + z^2, \\ b^2 + 4c^2 - a^2 &= x^2, \quad \text{or} & 5b^2 + 3c^2 &= z^2 + x^2, \\ c^2 + 4a^2 - b^2 &= y^2, & 5c^2 + 3a^2 &= x^2 + y^2, \end{aligned}$$

two solutions are

$$a = 313, \quad x = 434,$$
$$b = 263, \quad y = 614,$$
$$c = 233, \quad z = 566,$$

and

$$a = 347, \quad x = 434,$$
$$b = 43, \quad y = 746,$$
$$c = 277, \quad z = 226.$$

3. Perhaps the most interesting case is the system

$$\begin{aligned} a^2 + b^2 - c^2 &= z^2, & 2a^2 + b^2 &= y^2 + z^2 + t^2, \\ b^2 + c^2 - d^2 &= t^2, \quad \text{or} & 2b^2 + c^2 &= z^2 + t^2 + x^2, \\ c^2 + d^2 - a^2 &= x^2, & 2c^2 + d^2 &= t^2 + x^2 + y^2, \\ d^2 + a^2 - b^2 &= y^2, & 2d^2 + a^2 &= x^2 + y^2 + z^2. \end{aligned}$$

As before stated we might work with general values of l, k and m, but to lessen the labour involved write the linear relations as

$$3a + 4b - 5c = 0,$$
$$(k^2 - 1)\, b + 2kc - (k^2 + 1)\, d = 0,$$
$$(m^2 - 1)\, c + 2md - (m^2 + 1)\, a = 0,$$

which yield, after reduction,

$$\begin{aligned} a &= k^2 (2m^2 + 5m - 2) + 8km + 2m^2 - 5m - 2, \\ b &= k^2 (m^2 + 4) - 6km + m^2 + 4, \\ c &= k^2 (2m^2 + 3m + 2) + 2m^2 - 3m + 2, \\ d &= k^2 (m^2 + 4) + k (4m^2 + 4) - (m^2 + 4), \end{aligned}$$

and $$d^2+a^2-b^2$$
$$=k^4(2m^2+5m-2)^2+2k^3(4m^4+22m^3+60m^2+8m+16)$$
$$+k^2(20m^4-38m^2-40)-2k(4m^4-22m^3+60m^2-8m+16)$$
$$+(2m^2-5m-2)^2,$$
which might be used to express a, b, c, d in terms of the arbitrary parameter m.

To shorten the work once more put $m=\frac{2}{3}$, which gives, after reduction,

$$a=5k^2+12k-10,$$
$$b=10k^2-9k+10,$$
$$c=11k^2+2,$$
$$d=10k^2+13k-10,$$

and $$d^2+a^2-b^2=25k^4+560k^3-268k^2-320k+100$$
$$=(5k^2-16k+10)^2 \quad (\text{if } k=\tfrac{13}{15})$$
$$=(5k^2+56k+10)^2 \quad (\text{if } k=-\tfrac{30}{73})$$
$$=(5k^2+16k-10)^2 \quad (\text{if } k=\tfrac{53}{50}).$$

Other values of k may be got, but they are much more cumbrous than the above.

For $k=\frac{13}{15}$,

$$a=935,\quad x=2891,$$
$$b=2185,\quad y=25,$$
$$c=2309,\quad z=563,$$
$$d=1975,\quad t=2491.$$

For $k=-\frac{30}{73}$,

$$a=37535,\quad x=4504,$$
$$b=41000,\quad y=32425,$$
$$c=10279,\quad z=54628,$$
$$d=36380,\quad t=21521.$$

For $k=\frac{53}{50}$,

$$a=20845,\quad x=47576,$$
$$b=29240,\quad y=31445,$$
$$c=35899,\quad z=868,$$
$$d=37540,\quad t=27101.$$

4. It may be of service to point out that some arrangements of the linear relations or of the biquadratic expression are more suitable than others, and that, if we are using the

linear relations in their general form, or if we are dealing with systems containing a large number of equations, the difficulties caused by an unsuitable arrangement will not be easily surmounted.

To illustrate, arrange the values of a, b, c, d obtained from the linear relations in the preceding section in powers of m and we get

$$\begin{aligned} a &= m^2(2k^2+2)+m(5k^2+8k-5)-(2k^2+2), \\ b &= m^2(k^2+1)-6km+4k^2+4, \\ c &= m^2(2k^2+2)+m(3k^2-3)+(2k^2+2), \\ d &= m^2(k^2+4k-1)+4k^2+4k-4, \end{aligned}$$

and on substituting these values in $d^2+a^2-b^2$ we find that the coefficient of m^4 is $4(k^4+2k^3+5k^2-2k+1)$; and the fact that neither this, nor the coefficient of m^0, is a perfect square would prevent us from expressing a, b, c, d in general terms.

We may, of course, find special values of k, such as $k=2$, and get

$$\begin{aligned} a &= 10m^2+31m-10, \\ b &= 5m^2-12m+20, \\ c &= 10m^2+9m+10, \\ d &= 11m^2+20, \end{aligned}$$

and

$$\begin{aligned} d^2+a^2-b^2 &= 196m^4+740m^3+857m^2-140m+100 \\ &= (14m^2-7m+10)^2 \quad \text{if} \quad m=-\tfrac{22}{39} \\ &= \left(14m^2+\frac{185m}{7}-10\right)^2 \quad \text{if} \quad m=-\tfrac{70}{79} \\ &= (\tfrac{202}{5}m^2-7m+10)^2 \quad \text{if} \quad m=\tfrac{10}{11}. \end{aligned}$$

The first two values of m give, after reduction, the same solution

$$\begin{aligned} a &= 4621, & x &= 992, \\ b &= 5392, & y &= 3499, \\ c &= 1541, & z &= 6932, \\ d &= 4468, & t &= 3389, \end{aligned}$$

whilst $m=\frac{10}{11}$ gives

$$\begin{aligned} a &= 10, & x &= 11, \\ b &= 5, & y &= 14, \\ c &= 10, & z &= 5, \\ d &= 11, & t &= 2, \end{aligned}$$

which are marred by the fact that $a=c$.

With the above arrangement we can, without tentative methods, get special but not general solutions.

If, however, we had written the linear relations as

$$(k^2-1)a+2kb-(k^2+1)c=0,$$
$$3b+4c-5d=0,$$
$$(m^2-1)c+2md-(m^2+1)a=0,$$

we should have

$$a=5km^2+m(3k^2+8k+3)-5k,$$
$$b=5m^2-m(4k^2-4)+5k^2,$$
$$d=(4k+3)m^2+3k^2+4k,$$

and

$$d^2+a^2-b^2=m^4(41k^2+24k-16)+\ldots+(-16k^4+24k^3+41k^2),$$

so that suitable values of k can only be found by tentative methods. It may be noticed that the value $k=\frac{1}{2}$ would lead to the same values of a, b, c, d as those obtained above by putting $m=-\frac{22}{39}$ or $-\frac{70}{79}$. Lastly, if we had arranged $d^2+a^2-b^2$ in powers of k, it would have been found equal to

$$k^4(-7m^2+40m-16)+\ldots+(-16m^4-40m^3-7m^2),$$

which is a still more unsuitable form.

Another very curious difficulty may be illustrated by using the three equations

$$a^2+b^2-c^2=z^2,$$
$$b^2+c^2-a^2=x^2,$$
$$c^2+a^2-b^2=y^2.$$

The relation

$$(k^2-1)a+2kb-(k^2+1)c=0,$$

or say $3a+4b-5c=0$ satisfies the first equation. But so also does $2ka+(k^2-1)b-(k^2+1)c=0$ equally well.

If now we write the two linear relations as

$$3a+4b-5c=0,$$
$$(m^2-1)b+2mc-(m^2+1)a=0,$$

we get the biquadratic expression $49m^4+140m^3-58m^2-20m+1$, which is quite suitable. But if we write the linear relations as

$$3a+4b-5c=0,$$
$$2mb+(m^2-1)c-(m^2+1)a=0,$$

we get $28m^4+128m^3+104m^2-32m-32$, which only yields to tentative methods.

It might be argued that a want of symmetry in the two linear relations causes the difficulty: to test this use

$$4a+3b-5c=0,$$

$$2mb+(m^2-1)c-(m^2+1)a=0,$$

which will be found to lead to

$$17m^4+108m^3+146m^2-12m-63.$$

It seems very difficult to say why one arrangement should be so much more satisfactory than the other, unless it implies that it is possible to devise a method for dealing with expressions like $17m^4+108m^3+146m^2-12m-63$ as easily and directly as if the first and last coefficients had been squares.

At all events, even if solutions of systems containing more than four equations exist, the above considerations considerably increase the difficulty of obtaining them.

5. A general solution of the equations

$$(a+b)^2+c^2=z^2,$$

$$(b+c)^2+a^2=x^2,$$

$$(c+a)^2+b^2=y^2,$$

is possible. We have

$$2ma+2mb+(m^2-1)c=0,$$

$$2kb+2kc+(m^2-1)a=0,$$

which give

$$a=-2k(m^2-1)+4km,$$

$$b=(k^2-1)m^2-4km-(k^2-1),$$

$$c=-2m(k^2-2k-1),$$

and

$$(c+a)^2+b^2=m^4(k^2+1)^2-32m^3k^3$$
$$+2m^2(k^4-16k^3+34k^2+16k+1)+32mk^3+(k^2+1)^2,$$

which yields

$$m=\frac{2k(k^2+1)^2}{(k^2-1)(k^4-2k^3+2k^2+2k+1)}.$$

For $k=2$, $m=\frac{100}{39}$, and

$$a=-2716, \qquad x=3395,$$
$$b=-5763, \qquad y=7685,$$
$$c=7800, \qquad z=11521.$$

6. In the previous paper the biquadratic expression resulting from the equations

$$bc+ca-ab=z^2,$$
$$ca+ab-bc=x^2,$$
$$ab+bc-ca=y^2,$$

was $$48m^4+176m^3+272m^2+252m+81.$$

By Fermat's method this yields $m=-\frac{20}{47}$ and very large values of a, b, c.

By writing the expression as

$$(14m+9)^2+4m^2(6m+19)(2m+1),$$

we get the value $m=-\frac{19}{6}$, which gives

$$a=89, \qquad x=23,$$
$$b=145, \qquad y=159,$$
$$c=221, \qquad z=197;$$

$m=-\frac{1}{2}$ gives trivial values for a, b, c.

As already noted, some uncertainty may exist as to the best way of arranging the first two coefficients in the linear equations, or as to the most suitable parameter, in powers of which to express the unknown quantities. The following considerations will help to remedy this defect, and also enable us to predict the existence of a solution. As an example, take the system

$$a^2+b^2-c^2=z^2,$$
$$b^2+c^2-d^2=t^2,$$
$$c^2+d^2-a^2=x^2,$$
$$d^2+a^2-b^2=y^2,$$

and write the linear relations for the first three equations as

$$(l^2-1)a+2lb-(l^2+1)c=0,$$
$$(k^2-1)b+2kc-(k^2+1)d=0,$$
$$(m^2-1)c+2md-(m^2+1)a=0.$$

On solving these three equations, and substituting the values of a, b, d thus obtained in the expression $d^2+a^2-b^2$, the result, arranged in powers of one of the parameters, will be of the fourth degree, and we can, in general, get a solution if the first or last coefficient of this biquadratic expression is a square.

Suppose the values of a, b, c, d, obtained from these equations, be arranged in powers of k, and let

$$a=p_1k^2+q_1k+r_1,$$

with similar expressions for b, c, d.

On substituting in the second linear relation we get

$$k^2[(p_2-p_4)k^2+(q_2-q_4)k+(r_2-r_4)]+2k(p_3k^2+q_3k+r_3)$$
$$-(p_2+p_4)k^2-(q_2+q_4)k-(r_2+r_4)=0,$$

and as this is true for all values of k, we have $p_2=p_4$, and $r_2=-r_4$, and so

$$d^2+a^2-b^2=p_1^2k^4+\ldots+r_1^2,$$

and hence, in general, a suitable value of k, expressed as a function of l and m, can be found, and so a, b, c, d can be similarly expressed.

It is clear that a similar result would have been got if we had interchanged the first two coefficients of either or both of the first and third relations, but not if we had altered the second one, which may be called the key relation. Nor should we have obtained a similar result if we had expressed a, b, c, d in powers of l and m.

Of course, these other arrangements might lead to a solution, as the essential condition is not that $p_2=p_4$ or $r_2=-r_4$, but only that $p_4^2+p_1^2-p_2^2$ or $r_4^2+r_1^2-r_2^2$ should be a square.

7. The above remarks apply equally well to the system

$$\begin{aligned} pa^2+q^2b^2-pc^2&=z^2,\\ pb^2+q^2c^2-pd^2&=t^2,\\ pc^2+q^2d^2-pa^2&=x^2,\\ pd^2+q^2a^2-pb^2&=y^2,\end{aligned}$$

p and q being any given rational quantities, and so this system, and also the equivalent one

$$\begin{aligned}(q^2+p)a^2+q^2b^2+(q^2-p)c^2&=y^2+z^2+t^2,\\ (q^2+p)b^2+q^2c^2+(q^2-p)d^2&=z^2+t^2+x^2,\\ (q^2+p)c^2+q^2d^2+(q^2-p)a^2&=t^2+x^2+y^2,\\ (q^2+p)d^2+q^2a^2+(q^2-p)b^2&=x^2+y^2+z^2,\end{aligned}$$

may be solved, and a, b, c, d expressed in terms of two arbitrary rational parameters.

As an example, in which l and m are given numerical values, take the system

$$\begin{aligned} 2a^2 + q^2b^2 - 2c^2 &= z^2, \\ 2b^2 + q^2c^2 - 2d^2 &= t^2, \\ 2c^2 + q^2d^2 - 2a^2 &= x^2, \\ 2d^2 + q^2a^2 - 2b^2 &= y^2, \end{aligned}$$

and for the linear relations use

$$\begin{aligned} -a + 2qb - 3c &= 0, \\ (k^2 - 2)\, b + 2qkc - (k^2 + 2)\, d &= 0, \\ 7c + 6qd - 11a &= 0, \end{aligned}$$

from which, after reduction, we get

$$\begin{aligned} a &= 8qk^2 + 6q^3k - 2q, \\ b &= 10k^2 + 3q^2k + 20, \\ c &= 4qk^2 + 14q, \\ d &= 10k^2 + 11q^2k - 20. \end{aligned}$$

It will then be found in the usual way that $2d^2 + q^2a^2 - 2b^2$ is a square if $k = -\dfrac{3q^2}{5}$, which gives

$$\begin{aligned} a &= 18q^5 + 50q, & x &= 87q^5 + 700q, \\ b &= 45q^4 + 500, & y &= 18q^6 + 250q^2, \\ c &= 36q^5 + 350q, & z &= 9q^5 - 100q, \\ d &= 75q^4 + 500, & t &= 36q^6 + 250q^2. \end{aligned}$$

Or, taking the equations

$$\begin{aligned} 3a^2 + q^2b^2 - 3c^2 &= z^2, \\ 3b^2 + q^2c^2 - 3d^2 &= t^2, \\ 3c^2 + q^2d^2 - 3a^2 &= x^2, \\ 3d^2 + q^2a^2 - 3b^2 &= y^2, \end{aligned}$$

and using the linear relations

$$\begin{aligned} -a + qb - 2c &= 0, \\ (k^2 - 3)\, b + 2qkc - (k^2 + 3)\, d &= 0, \\ c + 4qd - 7a &= 0, \end{aligned}$$

we get
$$a = 9qk^2 + 8q^3k - 21q,$$
$$b = 15k^3 + 8q^2k + 45,$$
$$c = 3qk^2 + 33q,$$
$$d = 15k^3 + 14q^2k - 45,$$

and it will be found that $3d^2 + q^2a^2 - 3b^2$ is a square if $k = -\frac{4q^2}{5}$, and hence

$$a = 16q^5 + 525q, \qquad x = 88q^5 + 1575q,$$
$$b = 80q^4 + 1125, \qquad y = 16q^6 + 75q^2,$$
$$c = 48q^5 + 825q, \qquad z = 16q^5 - 225q,$$
$$d = 40q^4 + 1125, \qquad t = 48q^6 + 975q^2.$$

Hitherto, in the examples given, the right-hand side of each equation has been a square, but as p may be any rational quantity this restriction is only apparent. Hence a system of four equations of the type $pa^2 + qb^2 - pc^2 = qz^2$ or the equivalent system of the type

$$(q+p)\, a^2 + qb^2 + (q-p)\, c^2 = q\,(y^2 + z^2 + t^2)$$

may be solved, p and q being integers.

As an example, take

$$3a^2 + 2b^2 - 3c^2 = 2z^2,$$
$$3b^2 + 2c^2 - 3d^2 = 2t^2,$$
$$3c^2 + 2d^2 - 3a^2 = 2x^2,$$
$$3d^2 + 2a^2 - 3b^2 = 2y^2,$$

and write
$$-a + 4b - 5c = 0,$$
$$(2k^2 - 3)\, b + 4kc - (2k^2 + 3)\, d = 0,$$
$$5c + 4d - 7a = 0,$$

which give
$$a = 10k^2 + 8k,$$
$$b = 10k^2 + 2k + 15,$$
$$c = 6k^2 + 12,$$
$$d = 10k^2 + 14k - 15,$$

and
$$3d^2 + 2a^2 - 3b^2 = 2\,(100k^4 + 520k^3 - 548k^2 - 720k)$$
$$= 2\,(10k^2 + 26k)^2 \text{ if } k = -\tfrac{10}{17},$$

and hence, after reduction, and neglecting signs,

$$a = 40, \quad x = 841,$$
$$b = 555, \quad y = 380,$$
$$c = 452, \quad z = 63,$$
$$d = 635, \quad t = 248.$$

Or taking the equations

$$q^2a^2 + 2b^2 - q^2c^2 = 2z^2,$$
$$q^2b^2 + 2c^2 - q^2d^2 = 2t^2,$$
$$q^2c^2 + 2d^2 - q^2a^2 = 2x^2,$$
$$q^2d^2 + 2a^2 - q^2b^2 = 2y^2,$$

and using $$-qa + 4b - 3qc = 0,$$
$$q(k^2 - 2)b + 4kc - q(k^2 + 2)d = 0,$$
$$7qc + 12d - 11qa = 0,$$

which give, after reduction,

$$a = 8q^2k^2 + 24k - 2q^2,$$
$$b = 5q^3k^2 + 6qk + 10q^3,$$
$$c = 4q^2k^2 + 14q^2,$$
$$d = 5q^3 + 22q - 10q^3,$$

and it will be found that $q^2d^2 + 2a^2 - q^2b^2$ is double a square if $k = -\dfrac{12}{5q^2}$, so that

$$a = 25q^4 + 144, \quad x = 175q^5 + 348q,$$
$$b = 125q^5 + 180q, \quad y = 125q^4 + 144,$$
$$c = 175q^4 + 288, \quad z = 25q^5 - 36q,$$
$$d = 125q^5 + 300q, \quad t = 125q^4 + 288.$$

Or, lastly, taking the system

$$q^2a^2 + 3b^2 - q^2c^2 = 3z^2,$$
$$q^2b^2 + 3c^2 - q^2d^2 = 3t^2,$$
$$q^2c^2 + 3d^2 - q^2a^2 = 3x^2,$$
$$q^2d^2 + 3a^2 - q^2b^2 = 3y^2,$$

and using the relations

$$-qa + 3b - 2qc = 0,$$
$$q(k^2 - 3)b + 6kc - q(k^2 + 3)d = 0,$$
$$qc + 12d - 7qa = 0,$$

we get $$a = 9q^2k^2 + 72k - 21q^2,$$
$$b = 5q^3k^2 + 24qk + 15q^3,$$
$$c = 3q^2k^2 + 33q^2,$$
$$d = 5q^3k^2 + 42qk - 15q^3.$$

A suitable value of k is $-\frac{36}{5q^2}$, so that

$$\begin{aligned} a &= 175q^4 + 432, & x &= 175q^5 + 792q, \\ b &= 125q^5 + 720q, & y &= 625q^4 + 432, \\ c &= 275q^4 + 1296, & z &= 25q^5 - 144q, \\ d &= 125q^5 + 360q, & t &= 325q^4 + 1296. \end{aligned}$$

8. Consider a system of five equations of the type

$$a^2 + b^2 - c^2 = z^2.$$

Two key relations—the second and fourth—must be used, and should be written

$$(k^2 - 1)\, b + 2kc - (k^2 + 1)\, d = 0,$$
$$(m^2 - 1)\, d + 2me - (m^2 + 1)\, a = 0,$$

whilst in the first and third relations numerical values may be used, and the first two coefficients interchanged.

Suppose now that a, b, c, d, e are expressed in powers of k, with the coefficients arranged in powers of m, we may put $a = k^2 (p_1 m^2 + q_1 m + r_1) + \ldots$, with similar expressions for b, c, d, e.

On substituting in the first key relation, we have

$$k^4 \{(p_2 - p_4)\, m^2 + (q_2 - q_4)\, m + r_2 - r_4\} + \ldots = 0,$$

and, as these relations hold for all values of k and m, we have $p_2 = p_4$.

Again, arranging the values in powers of m, we have

$$a = m^2 (p_1 k^2 + q_1' k + r_1') + \ldots,$$

and on substituting in the second key relation, we have $p_4 = p_1$.

Hence $p_1 = p_2$ and so $e^2 + a^2 - b^2 = k^4 (p_5^2 m^4 + \ldots) + \ldots$, so that a value of m may be found which makes the coefficient of k^4 a square, and then k may be found to make $e^2 + a^2 - b^2$ a square. This double process, however, makes the solution very laborious, and leads to large values of a, b, c, d, e.

The above arrangement is effective only when, in the original equations, the second and third squares have the same numerical coefficients, and so we cannot definitely say that systems of five equations of the type $pa^2 + q^2 b^2 - pc^2 = z^2$ can be solved. With systems of six equations the two key relations would again be the second and fourth, and should be written

$$(k^2 - 1)\, b + 2kc - (k^2 + 1)\, d = 0,$$
$$(m^2 - 1)\, d + 2me - (m^2 + 1)\, f = 0,$$

and, reasoning as before, we find $p_2 = p_4$ and $p_4 = p_6$. Hence $p_2 = p_6$ and $f^2 + a^2 - b^2 = k^4 (p_1^2 m^4 + \ldots) + \ldots$. This result holds for equations of the type

$$pa^2 + q^2b^2 - pc^2 = z^2,$$

and hence systems of six equations of this kind can be solved.

With systems of seven and eight equations, three key relations would be required, and the work of solution becomes increasingly laborious, and the values obtained extremely large.

The reasoning used above seems to apply generally, and so it may be concluded that systems containing an even number of equations of the type $pa^2 + q^2b^2 - pc^2 = z^2$ can be solved, and that, if $p = q = 1$, solutions may be found, whatever be the number of equations in the system.

To solve systems containing an odd number of equations of the type $pa^2 + q^2b^2 - pc^2 = z^2$ all the linear relations should be used with literal coefficients.

9. In the first paper, the solution of

$$6a^2 + 3b^2 - 2c^2 = z^2,$$
$$6b^2 + 3c^2 - 2a^2 = x^2,$$
$$6c^2 + 3a^2 - 2b^2 = y^2$$

gave the condition that $64m^4 - 440m^3 + 465m^2 + 760m + 16$ is a square. This is satisfied by $m = 11$, which gives

$$a = 61, \quad x = 179,$$
$$b = 56, \quad y = 215,$$
$$c = 83, \quad z = 134.$$

SUR QUELQUES INTÉGRALES DÉFINIES.

Par *S. P. Sheusen.*

Considérons l'intégrale d'Euler

$$F_0 = \int_0^{\frac{1}{2}\pi} \sin^{2x-1}\phi \cos^{2y-1}\phi \, d\phi = \tfrac{1}{2}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad R(x) > 0, \; R(y) > 0,$$

et dérivons sous la signe d'intégrale par rapport à x; nous avons

$$F_1 = 2\int_0^{\frac{1}{2}\pi} \sin^{2x-1}\phi \, . \cos^{2y-1}\phi \, . \log\sin\phi \, d\phi$$
$$= \tfrac{1}{2}\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \{\psi(x) - \psi(x+y),$$

en appelant $\partial_x \log\Gamma(x) = \psi(x)$.

Dérivons encore une fois:

$$F_2 = 4\int_0^{\frac{1}{2}\pi} \sin^{2x-1}\phi.\cos^{2y-1}\phi.\log^2\sin\phi\, d\phi$$

$$= \tfrac{1}{2}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}[\{\psi(x)-\psi(x+y)\}^2+\psi'(x)-\psi'(x+y)],$$

en posant $\partial_x\psi(x) = \psi'(x)$.

Posons pour abréger

$$\psi(x)-\psi(x+y) = a, \quad \tfrac{1}{2}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = A.$$

Par ces significations les équations précedentes deviennent

$$F_0 = A, \quad F_1 = Aa, \quad F_2 = A(a^2+a').$$

Supposons que l'intégrale

$$F_q = 2^q\int_0^{\frac{1}{2}\pi} \sin^{2x-1}\phi.\cos^{2y-1}\phi.\log^q\sin\phi\, d\phi = A.\phi_q(a),$$

$\phi_q(a)$ désignant une fonction entier avec des coéfficients numériques entières, et supposons que la fonction $\phi_q(a)$ soit homogène de l'ordre q de $a^{(r)}$, en supposant que $a^{(r)}$ soit de l'ordre $(r+1)$.

On voit immédiatement que la supposition est correcte pour des valeurs petites de q, et il nous reste de démontrer, qu'elle est aussi correcte, quand on remplace q par $q+1$. À cette occasion on trouve directement

$$F_{q+1} = A\{a\phi_q(a)+\phi_q'(a)\},$$

et en outre c'est clair, que $\phi_q(a)$ ne peut contenir que des termes donc les facteurs seules sont des puissances de a, de $a^{(r)}$, et enfin des termes, qui contiennent puissances de a aussi bien que de $a^{(r)}$.

Par differentiation de chaque de ces termes la justesse de la proposition résulte immédiatement.

De cette proposition nous allons déduire quelques autres. Posons $x = y = \frac{1}{3}$, et remarquons que

$$\psi^{(r)}(\tfrac{1}{3}) - \psi^{(r)}(\tfrac{2}{3}) = (-1)^{r+1}r!\,3^{r+1}H_{r+1},$$

quand $$H_n = \sum_{s=0}^{s=\infty}\left\{\frac{1}{(3s+1)^n} - \frac{1}{(3s+2)^n}\right\},$$

nous avons, tandis que

$$\int_0^1 \frac{dt}{\sqrt{(1-t^3)}} = F(1,\,3) = \frac{\Gamma^3(\frac{1}{3})}{\pi\sqrt{(3)}.2^{\frac{4}{3}}},$$

$$\int_0^{\frac{1}{2}\pi}(\sin 2\phi)^{-\frac{1}{3}}\log^n\sin\phi\, d\phi = \frac{3F(1,\,3)}{2^{n+1}}F_n(H)\ldots\ldots(1)$$

$F_n(H)$ désignant une fonction entier avec des coéfficients numériques entières, et cette fonction est en outre homogène de l'ordre n, quand on suppose H_s soit de l'ordre s.

Posons encore $x = y = \frac{2}{3}$, nous avons d'une manière analogue, tandis que $H_r' = 1 - H_r$

$$\int_0^{\frac{1}{4}\pi} (\sin 2\phi)^{\frac{1}{3}} \log^n \sin\phi \, d\phi = \frac{\pi}{2^n F(1,\ 3)\sqrt{3}} F_n(H') \ldots\ldots (2).$$

Puis, par multiplication des deux formules (1) et (2), nous obtenons la formule intéressante

$$\int_0^{\frac{1}{4}\pi} (\sin 2\phi)^{\frac{1}{3}} \log^n \sin\phi \, d\phi \,.\int_0^{\frac{1}{4}\pi} (\sin 2\phi)^{-\frac{1}{3}} \log^n \sin\phi \, d\phi$$
$$= \frac{\pi\sqrt{3}}{2^{2n+1}} F_n(H) \,.\, F_n(1-H).$$

Transformons ensuite l'intégrale de la formule (1) en posant $\sin\phi = e^{-\theta}$; nous obtenons

$$\int_0^{\infty} \frac{\theta^n \, d\theta}{(\sinh\theta)^{\frac{2}{3}}} = (-1)^n \frac{3 \,.\, F(1,\ 3)}{2^n} F_n(H),$$

quand $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

Remplaçant en (1) ϕ par 2ϕ, il suit

$$\int_0^{\frac{1}{4}\pi} (\sin^4\phi)^{-\frac{1}{3}} \log^n \sin\phi \, d\phi = \int_0^{\frac{1}{4}\pi} \frac{(\log 2 + \log\sin\phi + \log\cos\phi)^n}{(\sin^4\phi)^{\frac{1}{3}}} d\phi\,;$$

d'où par application de la formule du binome

$$\int_0^{\frac{1}{4}\pi} \frac{(\log\sin\phi + \log\cos\phi)^{n+1}}{(\sin^4\phi)^{\frac{1}{3}}} d\phi$$
$$= -\sum_{r=1}^{r=n} \frac{n!\,\sigma_1^{\,n-r+1}}{(n-r)!\,(r-1)!} \int_0^{\frac{1}{4}\pi} \frac{(\log\sin\phi + \log\cos\phi)^r}{(\sin^4\phi)^{\frac{1}{3}}} d\phi$$
$$+ \frac{3F(1,\ 3)}{2^{n+2}} \{\tfrac{1}{2} F_{n+1}(H) \div \sigma_1 F_n(H)\},$$

en posant $\log 2 = \sigma_1$.

On voit donc, que les intégrales

$$\int_0^{\frac{1}{4}\pi} \frac{(\log\sin\phi + \log\cos\phi)^q}{(\sin^4\phi)^{\frac{1}{3}}} d\phi,$$

se déterminent par la fonction $F_s(H)$, multiplié par une intégrale élliptique, car on a immédiatement

$$\int_0^{\frac{1}{4}\pi} \frac{d\phi}{(\sin^4\phi)^{\frac{1}{3}}} = \tfrac{3}{4} F(1,\ 3),$$

et $$\int_0^{\frac{1}{4}\pi} \frac{\log\sin\phi + \log\cos\phi}{(\sin^4\phi)^{\frac{1}{3}}} d\phi = -\tfrac{3}{4} F(1,\ 3)(\tfrac{3}{2} H_1 + \sigma_1).$$

Nous allons considérer l'intégrale

$$\int_0^{\frac{1}{4}\pi} \frac{(\log\sin\phi + \log\cos\phi)^n}{(\sin^4\phi)^{\frac{1}{3}}} d\phi,$$

et remplaçer ϕ par $\frac{1}{2}\pi - \phi$; nous obtenons

$$\int_0^{\frac{1}{4}\pi} \frac{(\log\sin\phi + \log\cos\phi)^n}{(\sin^4\phi)^{\frac{1}{3}}} d\phi = \div \int_0^{\frac{1}{4}\pi} \frac{(\log\sin\phi + \log\cos\phi)^n}{(\sin^4\phi)^{\frac{1}{3}}} d\phi,$$

ou, ce qui est le même,

$$\int_0^{\frac{1}{2}\pi} \frac{(\log\sin\phi + \log\cos\phi)^n}{(\sin^4\phi)^{\frac{1}{3}}} d\phi = 0,$$

d'où l'on tire par application de la formule du binome et des transformations simples, quand $n = 2q$

$$\int_0^{\frac{1}{2}\pi} \frac{\log^q\sin\phi \log^q\cos\phi}{(\sin^4\phi)^{\frac{1}{3}}} d\phi = 0.$$

Spécialment on trouve de la formule (1) pour $n = 2$,

$$\int_0^{\frac{1}{2}\pi} \frac{\log^2\sin\phi}{(\sin 2\phi)^{\frac{1}{3}}} d\phi = \frac{27}{8} F(1, 3)(H_1^2 + H_2).$$

Nous remarquons encore, que le procédé indiqué s'applique sur beaucoup d'autres expressions, et on voit ainsi, q'une foule d'intégrales définies sous forme finie se détermine par les constantes H_r.

Nous allons traiter l'autre type d'intégrale:

$$F_n' = \int_0^{\frac{1}{2}\pi} (\sin 2\phi)^{\frac{1}{3}} \log^n\sin\phi \, d\phi = \frac{\pi}{2^n F(1, 3)\sqrt{3}} F_n(H') \ldots (3).$$

En posant $\sin\phi = 2^{-\theta}$, nous obtenons après une réduction simple

$$\int_0^{\infty} \frac{e^{-\theta}\,\theta^n\, d\theta}{(\sinh\theta)^{\frac{1}{3}}} = \frac{(-1)^n \pi}{2^n F(1, 3)\sqrt{3}} F_n(H').$$

Remplaçant dans la formule (3) ϕ par 2ϕ, il suit

$$F_n' = 2\int_0^{\frac{1}{4}\pi} (\sin^4\phi)^{\frac{1}{3}} \log^n\sin 2\phi \, d\phi \ldots\ldots \ldots\ldots (4),$$

d'où en remplaçant ϕ par $\frac{1}{2}\pi - \phi$

$$F_n' = -2\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (\sin^4\phi)^{\frac{1}{3}} \log^n\sin 2\phi \, d\phi,$$

ou

$$\int_0^{\frac{1}{2}\pi} (\sin^4\phi)^{\frac{1}{3}} \log^n\sin 2\phi \, d\phi = 0.$$

Par l'application de la formule du binome on trouve par le même procédé, que nous avons indiqué plus haut

$$\int_0^{\frac{1}{2}\pi} (\sin^4\phi)^{\frac{1}{3}} \log^q \sin\phi . \log^q \cos\phi \, d\phi = 0.$$

Revenons à la formule (4). Si l'on applique notre procédé ordinaire, on trouve aisement la formule de récursion:

$$\int_0^{\frac{1}{4}\pi} (\sin^4\phi)^{\frac{1}{3}} (\log\sin\phi + \log\cos\phi)^{n+1} d\phi$$

$$= -\sum_{r=1}^{r=n} \frac{n!\,\sigma_1^{\,n-r+1}}{(n-r)!\,(r-1)!} \int_0^{\frac{1}{4}\pi} (\sin^4\phi)^{\frac{1}{3}} (\log\sin\phi + \log\cos\phi)^r \, d\phi$$

$$+ \frac{\pi}{2^n F(1, 3)\sqrt{3}} \{\tfrac{1}{2} F_{n+1}(H') - \sigma_1 F_n(H')\}.$$

Par application de cette formule les intégrales

$$\int_0^{\frac{1}{4}\pi} (\sin^4\phi)^{\frac{1}{3}} (\log\sin\phi + \log\cos\phi)^q \, d\phi$$

se déterminent immédiatement, car on a

$$\int_0^{\frac{1}{4}\pi} (\sin^4\phi)^{\frac{1}{3}} d\phi = \frac{\pi}{F(1, 3)\sqrt[3]{2}.\sqrt{3}}$$

et $\displaystyle\int_0^{\frac{1}{4}\pi} (\sin^4\phi)^{\frac{1}{3}} (\log\sin\phi + \log\cos\phi)\, d\phi$

$$= \frac{\pi}{2\sqrt{3}.F(1, 3)} \left(\frac{\pi}{2\sqrt{3}} - \frac{3}{2} - \sigma_1 \sqrt[3]{4}\right).$$

On trouve ensuite

$$\int_0^{\frac{1}{2}\pi} (\sin 2\phi)^{-\frac{1}{3}} \log^n \sin\phi \, d\phi . \int_0^{\frac{1}{2}\pi} (\sin 2\phi)^{\frac{1}{3}} \log^n \sin\phi \, d\phi$$

$$= \frac{\pi\sqrt{3}}{2^{2n+1}} F_n(H) . F_n(1 \div H).$$

Plus haut nous avons regardé quelques intégrales, qui sous forme finie s'expressent par les séries

$$H_r = \sum_{s=0}^{s=\infty} \left\{\frac{1}{(3s+1)^r} - \frac{1}{(3s+2)^r}\right\}.$$

Quant à la nature des nombres H_r, on démontre sans difficulté que H_{2r-1} sous forme finie s'expressent par π et des nombres du Bernoulli, ainsi que les nombres H_{2r+1} sont trancendentes. La sommation des séries

$$\sum_{s=0}^{s=\infty} \left\{\frac{1}{(3s+1)^{2q}} - \frac{1}{(3s+2)^{\,q}}\right\}$$

est cependant plus difficile, et il me semble, q'on trouve ici les mêmes difficultés, q'on trouve par la sommation des séries

$$\sum_{s=0}^{s=\infty}\left\{\frac{1}{(4s+1)^{2q}}-\frac{1}{(4s+3)^{2q}}\right\};$$

mais je me réserve de revenir à cette question dans une autre occasion.

ON n-POLED CASSINOIDS.

By *Harold Hilton.*

§ 1. SUPPOSE that a curve of degree $2n$ has a multiple point of order n at each circular point at infinity ω and ω', such that each tangent at ω or ω' has $(n+1)$-point contact, and therefore does not meet the curve at any finite point. Then the curve is the locus of a point P, such that

$$A_1P.A_2P.\dots.A_nP=a^n\dots\dots\dots\dots\dots\text{(i)},$$

where a is a constant and A_1, A_2, ..., A_n are the real singular foci of the curve, *i.e.*, the real intersections of tangents at ω and ω'.

If also the tangents from ω (and ω') are each tangents of n-point contact, so that they meet the curve only at ω (or ω') and at their point of contact, then the polar equation of the curve can be put in the form

$$r^{2n}-2r^nc^n\cos n\theta+c^{2n}=a^{2n}\dots\dots\dots\dots\text{(ii)},$$

and the singular foci A_1, A_2, ..., A_n are the vertices of a regular polygon, whose centre is the pole O and whose vertices are

$$(c,\ 2k\pi/n),\ \text{where } k=1,\ 2,\ \dots,\ n.$$

The curve has the symmetry of the regular polygon. It has been called the *n-poled Cassinoid.* If $n=1$, it is a circle. If $n=2$, it is the Cassinian curve*, and is the most general quartic with biflecnodes at ω and ω'.

We give here some of the more interesting properties of the curve with a brief indication of their proof†.

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* Also called "Cassinian Ellipse" or "Cassinian Oval". The curve for which $n=2$, $c=a$ is the "Lemniscate of Bernouilli".

† The reader may consult Darboux, *Sur une classe remarquable de courbes et de surfaces algébriques*, Paris (1873), pp. 66–75; La Goupillière, *Journal de l'École Polytechnique*, 38 (1861), pp. 15–112 (a diagram is given for $n=4$); Serret, *Liouville*, 8 (1843), pp. 145, 495; Roberts, *Liouville*, 13 (1848), pp. 38, 209.

The n-poled Cassinoids are of three types.

TYPE I. The n-circuited type for which $c > a$.

The curve consists of n convex ovals. If we write

$$d^n c^n = c^{2n} - a^{2n},$$

the equation (ii) becomes

$$r^{2n} - 2r^n c^n \cos n\theta + c^n d^n = 0 \text{.............(iii).}$$

The real foci $B_1, B_2, \ldots, B_n$ form the vertices of a regular polygon with centre O

$$(d, 2k\pi/n), \text{ where } k = 1, 2, \ldots, n.$$

We called these points (ordinary) "foci" as distinguished from the singular foci $A_1, A_2, \ldots, A_n$. But each is really a multiple focus formed of $n-1$ real foci, since the tangents from ω and ω' have all n-point contact.

Each of the n ovals is its own inverse with respect to the circle with centre O and radius $\sqrt{(cd)}$, the points A_k and B_k being inverse points for this circle. Each oval contains a singular focus and an ordinary focus collinear with O, the latter being the nearer to O.

The tangents from O to the ovals are the lines

$$\sin n\theta = \pm (a/c)^n.$$

TYPE II. The one-circuited type for which $c < a$.

The curve consists of a single oval. If we write

$$d^n c^n = a^{2n} - c^{2n},$$

the equation (ii) becomes

$$r^{2n} - 2r^n c^n \cos n\theta - c^n d^n = 0 \text{.............(iv).}$$

The foci $B_1, B_2, \ldots, B_n$ form the vertices of a regular polygon with centre O

$$[d, (2k+1)\pi/n], \text{ where } k = 1, 2, \ldots, n.$$

They lie outside the oval, and the singular foci lie inside the oval. The oval has $2n$ real inflections, if

$$n^{1/n} c > a > c.$$

If $a = n^{1/n} c$, the oval has n points of undulation. If $a > n^{1/n} c$, the oval is convex.

TYPE III. The n-looped type for which $c = a$.

The equation (ii) becomes

$$r^n = 2c^n \cos n\theta,$$

which represents a curve with a multiple point of order n at O and n loops. It has no ordinary foci. Its properties are very well known.

The class m of Types I. and II. is $2n^2$; the number ι of inflexions is $6n(n-1)$. The tangents at $2n(2n-3)$ of these inflexions pass through ω or ω'. For Type III.

$$m = n(n+1), \quad \iota = 3n(n-1).$$

For Types I. and II. the equations (iii) and (iv) may be written

$$r^{2n} \mp 2r^n d^n \cos n\theta + d^{2n} = (ar/c)^{2n} \text{..........(v)}.$$

Therefore $$c^n . B_1P . B_2P . \ldots . B_nP = a^n . OP^n \text{........(vi)}.$$

We may obtain this result also by inverting (i) with respect to O.

The directrix corresponding to the focus B_k (the chord of contact of $B_k\omega$ and $B_k\omega'$) is the perpendicular bisector of OB_k.

Writing (ii) in the form

$$(r^n - c^n)^2 + 2r^n c^n (1 - \cos n\theta) = a^{2n},$$

we see that the curves

$$r^{\frac{1}{2}n} \sin \tfrac{1}{2} n\theta = \pm a^n / 2c^{\frac{1}{2}n} \text{................(vii)}$$

touch the Cassinoid at each intersection with the circle $A_1 A_2 \ldots A_n$.

§ 2. If the line joining $P(r, \theta)$ to $Q_k(h, 2k\pi/n + \alpha)$ makes an angle η_k with $\theta = 0$,

$$e^{2i\eta_k} = \{re^{\theta i} - he^{(2k\pi/n+\alpha)i}\} \div (re^{-\theta i} - he^{-(2k\pi/n+\alpha)i});$$

hence

$$e^{2i(\eta_1+\eta_2+\ldots+\eta_n)} = (r^n e^{n\theta i} - h^n e^{n\alpha i}) \div (r^n e^{-n\theta i} - h^n e^{-n\alpha i}),$$

or

$$\tan(\eta_1 + \eta_2 + \ldots + \eta_n)$$
$$= (r^n \sin n\theta - h^n \sin n\alpha) \div (r^n \cos n\theta - h^n \cos n\alpha) \text{...(viii)}.$$

If the line OQ_k subtends an angle ϵ_k at P, $\epsilon_k = \eta_k - \theta$, so that

$$\tan(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n) = h^n \sin n(\theta - \alpha) \div \{r^n - h^n \cos n(\theta - \alpha)\} \text{......(ix)}.$$

The product

$$PQ_1 . PQ_2 . \ldots . PQ_n = r^{2n} - 2r^n h^n \cos n(\theta - \alpha) + h^{2n} \text{...(x)}.$$

If ϕ is the angle between OP and the tangent at P to (ii),

$$\tan\phi = r\frac{d\theta}{dr} = (c^n \cos n\theta - r^n) \div c^n \sin n\theta$$
$$= (\pm d^n - r^n \cos n\theta) \div r^n \sin n\theta;$$
$$\cos\phi = \pm c^n \sin n\theta / a^n \text{ ....(xi)}.$$

Putting in (ix) $\alpha = 0$, $h = c$, we have

The angle between OP and the normal at P differs by a multiple of π from the algebraic sum of the angles subtended at P by OA_1, OA_2, ..., OA_n.

Putting in (viii) $\alpha = 0$, $h = d$ or $\alpha = \pi/n$, $h = d$, we have

The angle between OP and the normal at P differs by a multiple of π from the algebraic sum of the angles made by PB_1, PB_2, ..., PB_n with OA_1.

Similarly

The ratio of the sines of (1) *the angle between OP and the normal at P,* (2) *the sum of the angles made by $PA_1, PA_2, \ldots, PA_n$ with OA_1, is $\pm(OA_1/OP)^n$.*

The ratio of the sines of (1) *the angle between OP and the normal at P,* (2) *the sum of the angles subtended at P by OB_1, OB_2, ..., OB_n, is $\pm(OP/OB_1)^n$.*

§ 3. Suppose we have two regular polygons with centre O, namely, $L_1L_2\ldots L_n$ and $M_1M_2\ldots M_n$, whose vertices have respectively the polar coordinates

$$(l,\ 2k\pi/n + \lambda) \quad \text{and} \quad (m,\ 2k\pi/n + \mu),$$

where $k = 1, 2, \ldots, n$, and $2\pi/n > \mu \geq \lambda \geq 0$, $l > 0$, $m > 0$.

Then the locus of a point P, such that

$$PL_1 . PL_2 . \ldots . PL_n = p . PM_1 . PM_2 . \ldots . PM_n,$$

where p is constant, is by (x)

$$\{r^{2n} - 2r^n l^n \cos n(\theta - \lambda) + l^{2n}\}$$
$$= p^2 \{r^{2n} - 2r^n m^n \cos n(\theta - \mu) + m^{2n}\} \ldots\ldots \text{(xii)},$$

which is an n-poled Cassinoid.

It is identical with the given Cassinoid (ii), if

$$(1 - p^2)\, c^n = l^n \cos n\lambda - p^2 m^n \cos n\mu, \quad l^n \sin n\lambda = p^2 m^n \sin n\mu,$$
$$(1 - p^2)(c^{2n} - a^{2n}) = l^{2n} - p^2 m^{2n} \ldots\ldots\ldots\ldots \text{(xiii)}.$$

(1) If $\lambda = \mu = 0$, these equations are satisfied by

$$(c^n - l^n)(c^n - m^n) = a^{2n}, \quad p^2(c^n - m^n) = c^n - l^n.$$

(2) If $\lambda = 0$, $\mu = \pi/n$, they are satisfied by

$$(c^n - l^n)(c^n + m^n) = a^{2n}, \quad p^2(c^n + m^n) = c^n - l^n.$$

(3) If $\lambda = \mu = \pi/n$, we have

$$(c^n + l^n)(c^n + m^n) = a^{2n}, \quad p^2(c^n + m^n) = c^n + l^n.$$

In each case there are an infinite number of possible pairs of values for l and m, but the third alternative is only possible if $a > c$.

If λ and μ have other given unequal values, there is only one position of the polygons $L_1L_2\ldots L_n$ and $M_1M_2\ldots M_n$ which makes (xii) identical with (ii). In fact, we have from (xiii)

$$l^n = (1-p^2)c^n \sin n\mu \operatorname{cosec} n(\mu-\lambda),$$
$$p^2m^n = (1-p^2)c^n \sin n\lambda \operatorname{cosec} n(\mu-\lambda),$$
$$p^4 \sin^2 n\mu - p^2\{(a/c)^{2n} \sin^2 n(\mu-\lambda) + 2\cos n(\mu-\lambda) \sin n\lambda \sin n\mu\} + \sin^2 n\lambda = 0,$$

giving one value for l, m, p^2, provided

$$(a/c)^{2n} > \sin n\lambda \sin n\mu \sec^2 \tfrac{1}{2}n(\mu-\lambda).$$

We have thus obtained every method of defining a given Cassinoid as the locus of a point, the products of whose distances from the vertices of two concentric regular polygons have a constant ratio.

§ 4. Consider now the locus of a point P, such that the algebraic sum of the angles made with a fixed direction by the lines PL_1, PL_2, ..., PL_n less the sum of the angles made with the direction by PM_1, PM_2, ..., PM_n is constant ($=\sigma$, say). This is the same thing as saying that $L_1M_1, L_2M_2, \ldots, L_nM_n$ subtend at P angles with the constant algebraic sum σ.

Then by (ix)

$$r^{2n} - r^n\{l^n \cos n(\theta-\lambda) + m^n \cos n(\theta-\mu)\} + l^nm^n \cos n(\lambda-\mu)$$
$$= \cot\sigma \, . \, r^n\{l^n \sin n(\theta-\lambda) - m^n \sin n(\theta-\mu)\} + \cot\sigma \, . \, l^nm^n \sin n(\lambda-\mu) \ldots\ldots \text{(xiv)}.$$

This is an n-poled Cassinoid passing through the vertices of the regular polygons $L_1L_2\ldots L_n$ and $M_1M_2\ldots M_n$.

It is identical with (ii), if

$$l^n \sin(\sigma - n\lambda) + m^n \sin(\sigma + n\mu) = 2c^n \sin\sigma,$$
$$l^n \cos(\sigma - n\lambda) = m^n \cos(\sigma + n\mu),$$
$$l^nm^n \sin(\sigma - n\lambda + n\mu) = (c^{2n} - a^{2n}) \sin\sigma.$$

These equations may be written

$$l^n \sin(2\sigma - n\lambda + n\mu) = 2c^n \sin\sigma \, . \cos(\sigma + n\mu),$$
$$m^n \sin(2\sigma - n\lambda + n\mu) = 2c^n \sin\sigma \, . \cos(\sigma - n\lambda),$$
$$(a/c)^{2n} \sin^2(2\sigma - n\lambda + n\mu) = \sin^2 n\lambda + \sin^2 n\mu - 2\sin n\lambda \sin n\mu \cos(2\sigma - n\lambda + n\mu),$$

which give l, m, σ, when λ and μ are known; $(a/c)^{2n}$ must be greater than $\sin^2 n\lambda$ and $\sin^2 n\mu$.

§ 5. Suppose any line through O meets an oval of the Cassinoid (iii) in P and P'. Then a circle touches the Cassinoid at P and P' which is orthogonal to the fixed circle with centre O and radius $\sqrt{(cd)}$. The locus of the centre of the circle is the envelope of the perpendicular bisector of PP'. This envelope is the first negative pedal of the locus of the middle point of PP', which is

$$2r/c = \{\cos n\theta + (\cos^2 n\theta - d^n/c^n)^{\frac{1}{2}}\}^{1/n} + \{\cos n\theta - (\cos^2 n\theta - d^n/c^n)^{\frac{1}{2}}\}^{1/n},$$

i.e.

$$2\cos n\theta = (2r/c)^n - ({}^{n-1}C_1 + 1)(2r/c)^{n-2}(d/c) + ({}^{n-2}C_2 + {}^{n-3}C_1)(2r/c)^{n-4}(d/c)^2 - \dots .$$

The envelope is a curve of the n-th class with foci at the singular foci of the Cassinoid. It belongs to the type of curves of the n-th class, which are such that the product of the perpendiculars on any tangent from the vertices of an n-sided regular polygon is proportional to the cosine of n times the angle which these perpendiculars make with the line joining any vertex of the polygon to its centre.*

§ 6. We now consider the family of Cassinoids with given singular foci.

In the equation (iii) c^n is kept fixed and d^n varies, the singular foci being the points $(c, 2k\pi/n)$. The curve is of Type I., II., or III., according as $d^n > 0$, < 0, or $= 0$.

The orthogonal trajectories of the family are

$$r^n \cos n(\theta - \alpha) = c^n \cos n\alpha \quad \dots\dots\dots\dots (xv).$$

These curves are inverses of n-poled Cassinoids of Type III. with respect to their centre. [Compare equation (vii).] They all pass through the singular foci of the Cassinoids and have n asymptotes through O.

The pedal equation of the Cassinoid (ii) is

$$\pm 2c^n p r^{n-1} = r^{2n} + a^{2n} - c^{2n} \quad \dots\dots\dots\dots (xvi).$$

It follows that the radius of curvature at any point is

$$2a^n r^{n+1} \div \{(n+1) r^{2n} + (n-1)(c^{2n} - a^{2n})\},$$

and that the locus of the inflexions of those members of the family, for which $a > c$, is

$$r^n = (n-1) c^n \cos n(\theta + \pi/n) \quad \dots\dots\dots\dots (xvii),$$

which is a Cassinoid of Type III.

* The results of §§ 3, 4, 5 are given by Darboux, *loc. cit.*, for the case $n=2$.

The locus of the point of contact of tangents from O to the family is

$$r^n = c^n \cos n\theta,$$

which is also a Cassinoid of Type III.

From (xi) we see that the circle through the singular foci meets the family at points where $\frac{1}{2}n$ times the vectorial angle plus the angle between tangent and radius vector is a multiple of π.

From (xi) it also follows that, if we are given two concentric regular polygons $L_1L_2 \ldots L_n$ and $M_1M_2 \ldots M_n$, the locus of the intersection of a Cassinoid with singular foci $L_1, L_2, \ldots, L_n$ and a Cassinoid with singular foci $M_1, M_2, \ldots, M_n$, which cut at a constant angle, is a concentric Cassinoid.

In figure 1 we show the curves of the family for the case $n = 3$ and $a/c = \cdot 9$, 1, 1·1, 1·5. The singular foci are denoted by dots and the ordinary foci by crosses.

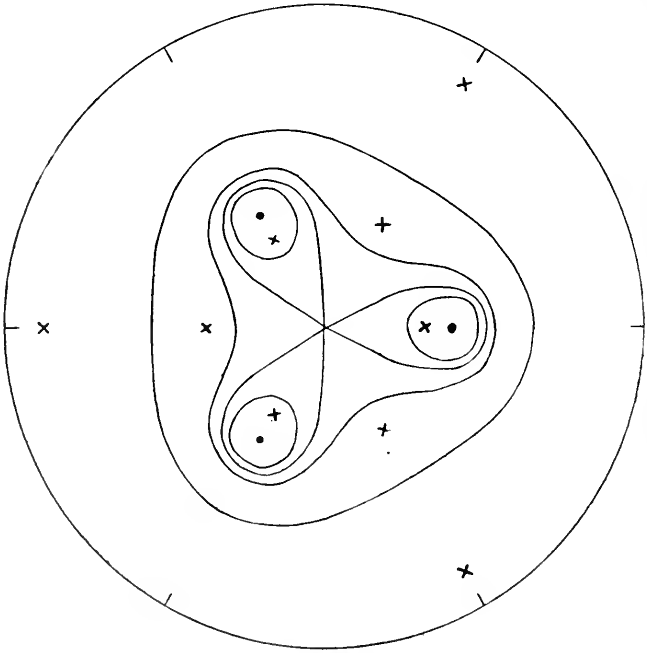

FIGURE 1.

§ 7. Now consider the family of Cassinoids with given ordinary foci.

In equation (iii) d^n is kept fixed (and positive), while c^n

varies, the foci being the points $(d,\ 2k\pi/n)$. The curve is of the Type I., II., III., according as $c^n > 0,\ < 0,\ = 0$. The family is the inverse, with respect to O, of the family of § 6. It is shown in figure 2.

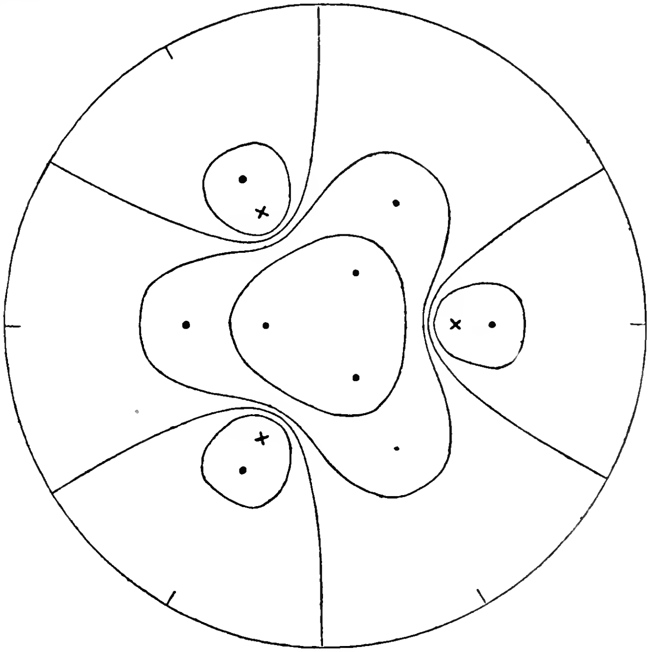

FIGURE 2.

The orthogonal trajectories of the family are

$$r^n = d^n \sec n\alpha . \cos n\,(\theta - \alpha) \qquad \text{(xviii)},$$

which are Cassinoids of Type III. passing through the given foci. The locus of the inflexions of the family is

$$r^n \cos n\,(\theta + \pi/n) = d/(n+1) \qquad \text{(xix)},$$

which is of the same type as the curve (xv).

The locus of the points of contact of tangents from O is $r^n \cos n\theta = d^n$, which is also of the same type as (xv).

The circle through the foci meets the family at points where $\frac{1}{2}n$ times the vectorial angle differs from the angle between tangent and radius vector by a multiple of π.

The locus of the intersection of two Cassinoids, whose foci are the vertices of two given regular concentric polygons and which cut at a constant angle, is a concentric Cassinoid.

If we are given a regular n-sided polygon $F_1F_2 \ldots F_n$ with centre O, and consider (1) the Cassinoids with $F_1,\ F_2,\ \ldots,\ F_n$

as singular foci, (2) the Cassinoids with $F_1, F_2, \ldots, F_n$ as ordinary foci, then the locus of a point P of the first family, such that the angle between OP and the tangent at P is constant, is an orthogonal trajectory of the second family, and *vice versâ.*

Two Cassinoids, such that the singular foci of one are the ordinary foci of the other and *vice versâ*, are necessarily of Type II. Their equations may be put in the form

$$\left.\begin{aligned} r^{2n} - 2r^n b^n \cos^2\beta \cos n\theta = b^{2n} \cos^2\beta \sin^2\beta \\ r^{2n} + 2r^n b^n \sin^2\beta \cos n\theta = b^{2n} \cos^2\beta \sin^2\beta \end{aligned}\right\}.$$

They meet on the bisectors of the angles between the lines joining O to the $2n$ foci, and they cut orthogonally.

They are superposable if $\beta = \pi/4$, when $a^{2n} = 2c^{2n}$.

§ 8. Any polar curve of the centre O with respect to all Cassinoids having given foci is the same. It is a curve of the type (xv).

Any polar curve of a point at infinity with respect to all Cassinoids having given singular foci is the same.

The first polar curve of the point at infinity in the direction $\theta = \alpha$ with respect to (iii) is

$$r^n \cos(\theta - \alpha) = c^n \cos(n\theta - \theta + \alpha) \quad \ldots\ldots\ldots \text{(xx)}.$$

It is the locus of the point of contact of tangents drawn in the direction $\theta = \alpha$ to the family of Cassinoids with given singular foci $(c, 2k\pi/n)$. It is of degree $2n - 1$, and has superlinear branches of order $n - 1$ at each circular point, O being both the singular focus and a multiple point of order $n - 1$. It has a real asymptote perpendicular to the tangents.

Eliminating c^n from (iii) and (xx), we have

$$r^n \cos(n\theta + \theta - \alpha) = d^n \cos(\theta - \alpha),$$

as the locus of the point of contact of tangents drawn in the direction $\theta = \alpha$ to Cassinoids with given ordinary foci. It is a curve of degree $n + 1$ passing through O.

The locus of a point whose polar conic with respect to the Cassinoid (ii) has eccentricity e is

$$(n-1)^2 (2 - e^2)^2 (r^{2n} - 2r^n c^n \cos n\theta + c^{2n}) = n^2 e^4 r^{2n}.$$

It is the same for all Cassinoids having the same singular foci; and it is a Cassinoid having these singular foci as ordinary foci.

If the product of OP^{n-1} and the perpendicular from P on the polar line of P with respect to a Cassinoid is constant, the locus of P is a pair of Cassinoids with the same singular foci.

END OF VOL. XLVIII.

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J. W. L. GLAISHER, Sc.D., F.R.S.,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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MESSENGER OF MATHEMATICS.

FACTORISATION OF N & $N' = (x^n \mp y^n) \div (x \mp y)$, &c. [when $x - y = 1$].

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's College, London.

[The author is indebted to Mr. H. J. Woodall, A.R.C.Sc., for help in reading the proof-sheets of this Paper].

1. *Introduction.* Using the abbreviation—

M.A.P.F. meaning *Max. Algebraic Prime Factor*..............(1),

and notation—

$\mathbf{N}_n$ = M.A.P.F. of $(x^n - y^n)$, $\mathbf{N}_n'$ = M.A.P.F. of $(x^n + y^n)$, [x, y unrestricted].(2a),

N_n = M.A.P.F. of $(x^n - y^n)$, N_n' = M.A.P.F. of $(x^n + y^n)$, [$x - y = 1$].........(2b),

H_n = M.A.P.F. of $(y^n - 1)$, H_n' = M.A.P.F. of $(y^n + 1)$, [$x = 1$]..............(2c),

it is proposed in this Memoir to study the properties of the two quantities N_n, N_n' above defined, with a special view to their factorisation.

The six quantities are all included under the generic title of "n-ans," whilst they will be distinguished as—

General n-ans $\mathbf{N}_n$, $\mathbf{N}_n'$; *Sub-Simple n-ans* N_n, N_n'; *Simple n-ans* H_n, H_n'.

The subscript n indicates the exponent of the algebraic form: it will be *omitted* (when not necessary to specify it), so as to simplify the notation.

It will be seen that N, N', H, H' are only special forms of the General n-ans $\mathbf{N}$, $\mathbf{N}'$. The forms H, H' are the only ones that have been as yet much studied. It will be shown that the four forms N, N', H, H' are closely related and in an interesting manner.

The main divisions of the subject are:—

General, Art. 1–5.
Congruence Roots, Art. 6–16.
Chains & Aurifeuillians, Art. 17–22.
Cubans, Art. 23–31.
n-ans, Art. 32.
Congruence Tables, C3–C15.
Factorisation Tables, F3–F17.

1a. *Notation.* All symbols here denote integers—

p, a, b, c denote *odd primes* [a≠b≠c]; I means an integer.
ω, Ω denote *odd* numbers; ϵ, e, E denote *even* numbers.

M.A.P.F.; **N**, **N`**, $N$, $N`$, H, $H`$ are explained in Art. 1.

x, y; x', y' are styled *roots* of $N \equiv 0$, $N` \equiv 0$ (mod p & p^κ).

η, $\eta`$ are often used (instead of $y$) in the forms $H$, $H`$; and are styled *roots* of $H \equiv 0$, $H` \equiv 0$ (mod p & p^κ).

The accented letters x', y', η' belong especially to $N`$, $N`$, $H`$; but the accents will be omitted when there is no risk of mistake.

$v = xy$, or $x'y'$ (a contraction for shortness' sake).

(x, y) means N; $\{x', y'\}$ means $N`$; $[\eta', 1]$ means $H`$: these exhibit the the elements x, y; x', y', η' of N, $N`$, $H`$ when required.

m, M mean *Multiples of.*

$\tau(n)$ means the Totient of n; $\tau(\mathrm{a}) = \mathrm{a} - 1$, $\tau(\mathrm{a}^2) = \mathrm{a}(\mathrm{a} - 1)$, $\tau(\mathrm{ab}) = \tau(\mathrm{a}).\tau(\mathrm{b})$, &c.

1b. *Working condition.* To avoid unnecessary factors in **N**, **N`**, &c., it is assumed throughout that

x and y, x' and y' have *no common factor*..................(3).

2. *Simpler forms of* N_n, $N`_n$. The Sub-simple n-ans (2*b*) take the following simple forms for the simpler cases of $n = \mathrm{a}, \mathrm{a}^2, \mathrm{ab}, 2\omega$,

1°. $n = \mathrm{a}$; $N = (x^n - y^n) \div (x - y)$; $N` = (x^n + y^n) \div (x + y)$ ....(4*a*),

2°. $n = \mathrm{a}^2$; $N = (x^n - y^n) \div (x^{\mathrm{a}} - y^{\mathrm{a}})$; $N` = (x^n + y^n) \div (x^{\mathrm{a}} + y^{\mathrm{a}})$.......(4*b*),

3°. $n = \mathrm{ab}$; $N = \dfrac{(x^n - y^n)(x - y)}{(x^{\mathrm{a}} - y^{\mathrm{a}})(x^{\mathrm{b}} - y^{\mathrm{b}})}$; $N` = \dfrac{(x^n + y^n)(x + y)}{(x^{\mathrm{a}} + y^{\mathrm{a}})(x^{\mathrm{b}} + y^{\mathrm{b}})}$.........(4*c*),

4°. $n = 2\omega$; $N =$ M.A.P.F. of $(x^n - y^n) =$ M.A.P.F. of $(x^{\frac{1}{2}n} + y^{\frac{1}{2}n})$.......(4*d*),

and the N of Case 1° above may also be written in the (apparently) yet simpler form

$N = x^n - y^n$ ....(4*a'*).

The four Cases above are the only ones dealt with in this Memoir. To have treated more complicated Cases (*e.g.* $n = \mathrm{a}^3$, a^4, &c.; abc, &c.; $n = \epsilon$, &c.) would have needed a Memoir of great length.

3. *N & $N`$ as functions of $xy$.* It will now be shown that $N$ and $N`$ can always be expressed as functions of xy.

It is easily seen that N, $N`$ are *symmetric functions* of x, y, of *even* degree which can be arranged as a sum of pairs of terms of form

N & $N` = \Sigma_r A_r (xy)^{\alpha_r} . (x^{e_r \beta_r} + y^{e_r \beta_r})$.....................(5),

where $e_r = 2^{\kappa_r}$, $\beta_r = \omega$..................................(5*a*),

a form which sufficiently exhibits the *symmetry* (in x, y).

And, it will suffice to show that—(under the condition $x - y = 1$)—the quantity $(x^{e_r \beta_r} + y^{e_r \beta_r})$ is always expressible as a function of xy. This may be shown by taking $e_r = 2, 2^2, 2^3$, &c.; $\beta_r = 1, 3, 5$, &c., in succession. In what follows, $xy = v$ is written for shortness.

$$x^2 + y^2 = (x-y)^2 + 2xy = 1 + 2v \quad \text{.....(6a)},$$

$$x^4 + y^4 = 1 + 4v + 2v^2 \quad \text{.....(6b)},$$

$$x^8 + y^8 = (1 + 4v + 2v^2)^2 - 2v^4 \quad \text{.....(6c)},$$

&c. = &c.

$$x^6 + y^6 = (x^3 - y^3)^2 + 2(xy)^3$$
$$= (x-y)^2 . (x^2 + xy + y^2)^2 + 2(xy)^3$$
$$= (1 + 3v)^2 + 2v^3 \quad \text{.....(6d)},$$

$$x^{12} + y^{12} = (x^6 + y^6)^2 - 2(xy)^6$$
$$= \{(1 + 3v)^2 + 2v^3\}^2 - 2v^6 \quad \text{.....(6e)},$$

&c. = &c.

$$x^{10} + y^{10} = (x^5 - y^5)^2 + 2(xy)^5$$
$$= (x-y)^2 . (x^4 + x^3y + x^2y^2 + xy^3 + y^4)^2 + 2(xy)^5$$
$$= (1 + 5v + 5v^2)^2 + 2v^5 \quad \text{.....(6f)},$$

&c. = &c.

The mode in which these are successively formed suffices to show that

$(x^{e_r\beta_r} + y^{e_r\beta_r})$ is always a function of xy .....(7),

whence it follows, from (5) that—

N & N' are always expressible as functions of xy.....(8).

3a. *Linear forms of* N, N'. It is known that—using the symbols m, M to mean *multiples of*—

$\mathbf{N}_n = 1 + M.n$, *always* .....(9a),

$\mathbf{N}_n' = 1 + M.n$, when $x + y \neq m.n$ .....(9b),

$\frac{1}{n}.\mathbf{N}_n' = 1 + M.n$, when $x + y = m.n$ .....(9c).

Next, writing $x = y + 1$, and $y = x - 1$, alternately in both **N**, **N'**, it follows at once that—

N_n & N_n' both $= 1 + M(xy)$ .....(10a),

and that hence

$N_n = 1 + nxy . f_1(x, y)$, $N_n' = 1 + xy . f_2(x, y)$ *always*.....(10b).

3b. *N & N' as functions of xy (continued).* From Art. 3*a* it follows that the quantities $f_1(x, y)$, $f_2(x, y)$ of last Article must themselves be functions of xy. It does not seem possible to exhibit these functions of xy in a general formula for all values of n. The following Table, however, shows the quantities N, N' expressed as functions of $v = xy$ for all *odd* values of $n \not> 15$.

n	N_n	N_n'
3	$1+3v$	$1+v$
5	$1+5v(1+v)$	$1+v(3+v)$
7	$1+7v(1+2v+v^2)$	$1+v(1+v)(5+v)$
9	$1+3v(1+v)(2+v)$	$1+v(6+9v+v^2)$
11	$1+11v(1+v)(1+3v+4v^2+v^3)$	$1+v(9+28v+35v^2+15v^3+v^4)$
13	$1+13v(1+v)(1+v)(1+3v+5v^2+v^3)$	$1+v(1+v)(11+34v+50v^2+20v^3+v^4)$
15	$1+v(1+v)(7+7v+v^2)$	$1+v(9+26v+24v^2+v^3)$

The following inferences seem evident from the Table (though not easily proved in a general manner)—

$N = 1 + v \,.\, F_1(v), \quad N' = 1 + v \,.\, F_2(v)$ .... (11*a*),

$N = 1 + nv(1+v) \,.\, \{1 + v\phi_1(v)\}$, when n = prime > 3 .... (11*b*),

$N' = 1 + v \,.\, \{n - 2 + v \,.\, \phi_2(v)\}$, when n = prime > 3 .... (11*c*),

$N = 1 + nv(1+v)^2 \,.\, \{1 + v \,.\, \phi_1(v)\}$, when n = prime $= 6\varpi + 1$.......... (11*d*).

$N' = 1 + v \,.\, (1+v) \,.\, \{n - 2 + v \,.\, \phi_2(v)\}$, when n = prime $= 6\varpi + 1$ ...... (11*e*).

4. *Quadratic forms.* The numbers **N**, **N'** are known to be expressible in the following 2^ic forms when n = a or ab—

$n = 4k+1$ has $\mathbf{N}_n = X^2 - nY^2$, $\quad \mathbf{N}_n' = X'^2 - nY'^2$ .... (12*a*),

$\mathbf{N}_n = T^2 - nxyU^2$, $\quad \mathbf{N}_n' = T'^2 + nxyU'^2$ .... (12*b*),

$n = 4k-1$ has $\mathbf{N}_n = X^2 + nY^2$, $\quad \mathbf{N}_n' = X'^2 + nY'^2$ .... (12*c*),

$\mathbf{N}_n = T^2 + nxyU^2$, $\quad \mathbf{N}_n' = T'^2 - nxyU'^2$ .... (12*d*).

The forms of X, Y, X', Y', T, U, T', U' for N_n, N_n' are the same as for the general n-ans ($\mathbf{N}_n$, $\mathbf{N}_n'$); so need not be detailed here.

It will suffice to say that the 2^ic parts (X, Y, &c.) of N_n, N_n' are not in general expressible as functions of xy.

5. *Factorisation by the Factor-Tables.* The factorisation of N, N' may be effected to a certain extent—(up to the limit N & $N' \not> 10017000$)—by the large* Factor-Tables: but this can be done only to the very limited extent shown below:—

	n =	3	5	7	9	11	13	15
Limit of x	in N	1827	38	11	12	4	3	7
	in N'	3165	56	15	15	5	3	7

so that, to push it further, other means—(explained in next Article)—must be sought.

* *Factor-Tables for the first ten millions*, by D. N. Lehmer, Washington, 1909: these extend to 10017000.

6. *Congruence-Tables.* Solutions (x, y, x', y') of the Congruences

$$N_n \equiv 0 \;\&\; N_n' \equiv 0 \pmod{p \;\&\; p^\kappa} \quad \ldots\ldots\ldots(13),$$

for all primes (p) capable of acting as divisors of the forms N, N' would evidently supply divisors of N, N'.

It will now be shown how to find solutions (x, y, x', y') or—(as they are often called)—*Roots* of these congruences. And, since here $x - y = 1$ and $x' - y' = 1$, it will evidently suffice to record *one* (say x and x') of each pair in Tables of solutions. It will be shown that they are intimately connected with the roots of η, η' of the Associate Congruences

$$H \equiv 0,\; H' \equiv 0 \pmod{p \;\&\; p^\kappa} \quad \ldots\ldots\ldots(14).$$

7a. *Special divisor n.* The Theory of Numbers shows that

n is a non-divisor of every N ....(15*a*),

a is a divisor of every N' when—(and only when)—$x' + y' = m \cdot \text{a}$, and $n = \text{a}, \text{a}^2, \text{a}^3$, &c.; hence $x' = \frac{1}{2}(m \cdot \text{a} + 1)$, $y' = \frac{1}{2}(m \cdot \text{a} - 1)$ ......(15*b*),

n^2 is a non-divisor of every N' ....(15*c*).

7b. *Form of divisors (p).* *Excluding* the exceptional divisor n—see Art. 7*a*—it is known that

Every divisor of N & N' must be of form $p = 2m.n + 1$ ......(16).

7c. *Number of roots* (x, x') *of Congruences.* It is known that

The *number* of incongruous roots (x, x')—all $< p$ or p^κ—of each of the Congruences (13) is $\tau(n)$, when n is *odd* ....(17),

where $\tau(n)$ means the *Totient* of n.

8. *Roots* (x, x') *from Factorisations.* Every actually factorised number N, N' evidently supplies one root, x or x', for every prime (p) contained in N or N': so that a few roots (x, x') of the Congruences (13) will be supplied by the factorisations found from the Factor-Tables (Art. 5): but this number is evidently very limited when $n > 3$ (*see* Art. 5).

9. *Connexion of roots* x, x' *with* η, η':

$$N \equiv 0 \pmod p \text{ gives } x^n \equiv y^n \pmod p.$$

$$\text{Hence } (y/x)^n \equiv +1 \pmod p: \quad \text{but } \eta^n \equiv +1 \pmod p.$$

$$\text{Therefore } (y/x)^n \equiv \eta^n, \quad \text{whence } (y/x) \equiv \eta \pmod p.$$

$$\text{Hence } (x-1)/x \equiv \eta, \quad \text{and } x \equiv -1/(\eta - 1) \pmod p.$$

In this way the four results below are obtained

$$x = \frac{-1}{\eta - 1}, \quad x \equiv \frac{+1}{\eta' + 1}; \quad x' \equiv \frac{-1}{\eta' - 1}, \quad x' = \frac{+1}{\eta + 1} \pmod{p} \ldots\ldots(18).$$

Similar results may be found for the modulus p^κ.

Now extensive Tables* exist of the roots η, η' modulo p & p^κ. Thus the roots x, x' may be computed from the known roots η, η' by means of the Congruences (18): and this affords one of the readiest means of computing x, x'.

10. *Conjugate Roots* (x_r, x_s), (x_r', x_s'). Let η_r, η_s be roots of $H \equiv 0$.

Take two (different) numbers r, s (both prime to n). Let η_r, η_s be the Least + Residues of η_1^r, η_1^s modulo p.

Then η_r, η_s are hereby new roots of $H \equiv 0 \pmod{p}$.

Now choose r, s so that $r + s = n$ ....(19a).

This involves $\eta_r . \eta_s \equiv +1 \pmod{p}$ ....(19b).

Let x_r, x_s be the roots of $X \equiv 0$ arising from η_r, η_s.

Then, by Result (18)

$$x_r + x_s \equiv \frac{-1}{\eta_r - 1} + \frac{-1}{\eta_s - 1} = \frac{-(\eta_r + \eta_s) + 2}{\eta_r \eta_s - (\eta_r + \eta_s) + 1} \equiv +1 \pmod{p} \ldots(20).$$

As x_r, x_s are both $< p$, this involves

$$x_r + x_s = p + 1 \ldots\ldots(20a).$$

Similarly it may be shown that

$$x_r' + x_s' = p + 1 \ldots\ldots(20b).$$

Similar results may be proved in the same way for the prime-power modulus p^κ.

These results are important as they show it *suffices to compute one-half* of the complete set of $\tau(n)$ roots of each kind (x, x'); the other half being obtained by simple subtraction from $(p + 1)$.

Note that the *conjugate* roots x_r, x_s occur *in the same set* of x, and arise from the *reciprocal* associate roots η_r, η_s of the same set of η. Similar results hold with x_r', x_s', η_r', η_s'. Contrast this with conjugate roots η, η', which have $\eta + \eta' = p$ (not $p + 1$), and occur in pairs, one member from each set η, η'.

11. *Pairs of roots* (x_r, x_r'). Let η_r, η_r' be *conjugate* roots of $H \equiv 0$, $H' \equiv 0$ (mod p or p^κ), connected by the relation

$$\eta_r + \eta_r' \equiv p, \text{ or } p^\kappa \ldots\ldots(21).$$

* The Author has prepared Tables of η, η' for *all exponents* $n \not> 15$ for all primes and prime-powers p & $p^\kappa \not> 100000$: most of these are in type, and printed off, in a series of volumes styled *Binomial Factorisations*. Mr. T. G. Creak has prepared similar Tables for $n > 15$ up to 49 for $p \not> 10000$, and in some cases up to 10000, or even 100000.

And, let x_r, x_r' be the roots of $N \equiv 0$, $N' \equiv 0$ (mod p or p^κ) arising from η_r, η_r'. Then, noting that η_r is derived from

$$\eta_r = \text{Least} + \text{Residue of } \eta_1{}^r \text{ modulo } p \text{ or } p^\kappa,$$

it follows from (18) that

$$x_r + x_r' \equiv \frac{-1}{\eta_r - 1} + \frac{1}{\eta_r + 1} = \frac{-2}{\eta_r{}^2 - 1} \equiv \frac{-2}{\eta_{2r} - 1}, \text{ (mod } p)$$

$$\equiv 2x_{2r} \text{ (mod } p) \text{ .... (22a)}.$$

And $$x_r x_r' \equiv \frac{-1}{\eta_r - 1} \cdot \frac{1}{\eta_r + 1} \equiv \frac{-1}{\eta_r{}^2 - 1} \equiv x_{2r}, \text{ (mod } p) \text{ .... (22b)}.$$

Hence also $$\frac{1}{x_r} + \frac{1}{x_r'} \equiv 2, \text{ (mod } p) \text{ .... (22c)}.$$

And similar results may be proved for the modulus p^κ.

The above *three* Results (22*a*–*c*) are true for all values of r (prime to n), and may be used as *succession-formulæ* for computing the complete sets of roots x, x' of any prime or prime-power (p, p^κ) from one given root x_1 or x_1'.

Thus x_1', or x_1, may be computed from a given x_1, or x_1', by

$$x_1' \equiv \frac{x_1}{2x_1 - 1}, \quad x_1 \equiv \frac{x_1'}{2x_1' - 1}, \text{ (mod } p \text{ or } p^\kappa) \text{ .... (23)},$$

and x_2 may be computed from

$$x_2 \equiv x_1 x_1', \text{ (mod } p \text{ or } p^\kappa) \text{ .... (23a)},$$

and the formula

$$x_2 = \tfrac{1}{2}(x_1 + x_1'), \text{ (mod } p \text{ or } p^\kappa) \text{ .... (23b)},$$

may be used as a check on the work, and so on.

12. *Simple formula for* x', [n prime]. Take

$$X' = \eta' + \eta'^3 + \eta'^5 + \ldots + \eta'^{n-2}, \text{ (mod } p) \text{ .... (24)},$$

and let η, η' be *conjugate* roots of $H \equiv 0$, $H' \equiv 0$, (mod p).

Hence $$X' \equiv -(\eta + \eta^3 + \eta^5 + \ldots + \eta^{n-2}), \text{ (mod } p),$$

therefore $$X' . (\eta + 1) \equiv -(\eta + \eta^2 + \eta^3 + \ldots + \eta^{n-2} + \eta^{n-1}), \text{ (mod } p),$$

$$\equiv +1 \text{ (mod } p), \quad [n \text{ being prime}].$$

But, by (18), $x' . (\eta + 1) \equiv +1$, (mod p),

therefore $$x' \equiv X', \text{ (mod } p) \text{ .... (24a)}.$$

This formula is really much easier to use for computing x' (when n is prime), especially when n is small, in which case it takes the following very simple forms—

$n =$	3	5	7	11	
$x' \equiv$	η_1'	$\eta_1' + \eta_3'$	$\eta_1' + \eta_3' + \eta_5'$	$\eta_1' + \eta_3' + \eta_5' + \eta_7' + \eta_9'$	...(24*b*).

[To use these formulæ conveniently, it is necessary to have Tables of the roots η or η' arranged* so as to show each root η_r, or η_r', *along with the index* r, showing thereby its derivation from η_1, η_1'. Any root may then be taken for η_1, or η_1', and the *corresponding* η_r, η_r' can be pretty easily picked out].

There appears to be no simple formula for the roots x similar to the above for x'.

13. *Sum of roots.* It is evident from results (20–20*b*) that

$$\Sigma(x) = \Sigma(x') = \tfrac{1}{2}\tau(n).(p+1), \text{ [modulus } = p] \quad\ldots\ldots\ldots\ldots(25a),$$

$$= \tfrac{1}{2}\tau(n).(p^\kappa+1), \text{ [modulus } = p^\kappa] \quad\ldots\ldots\ldots(25b).$$

14. *Product of roots.* The Residues (modulo p & p^κ) of the continued product of the whole set of roots (x), and also of the whole set of roots x', are as shown in the scheme below:—

$$n = a; \quad \Pi(x) \equiv \dagger -\tfrac{1}{n}(p-1), \quad \Pi(x') \equiv +1 \quad\ldots\ldots\ldots\ldots(26a),$$

$$n = a^2; \quad \Pi(x) \equiv \dagger -\tfrac{1}{a}(p-1), \quad \Pi(x') \equiv +1 \quad\ldots\ldots\ldots\ldots(26b),$$

$$n = ab; \quad \Pi(x) \equiv \dagger +1, \quad \Pi(x') \equiv +1 \quad\ldots\ldots\ldots\ldots(26c).$$

The results for the $\pi(x')$ are proved as follows:—
By Result (22*a*) it follows that

$$\pi(x_r).\pi(x_r') \equiv \pi(x_{2r}) \text{ (mod } p \text{ \& } p^\kappa) \textit{ generally} \ldots\ldots\ldots\ldots(27).$$

Now by giving r all values $< \frac{1}{2}n$, the partial product $\pi(x'_{2r})$ includes *all* the x_{2r} with *even suffix*: and, by then giving r all the values $> \frac{1}{2}n$ up to n, each x_{2r} becomes effectively x_{2r-n} (wherein $2r - n = \omega > n$), so that the partial product $\pi(x_{2r})$ here includes *all* the $x_{2r} = x_{2r-n}$ with the *odd* suffix $(2r-n)$, and

$$\text{The complete product } \pi(x_{2r}) \equiv \pi(x_r) \ldots\ldots\ldots\ldots\ldots\ldots(27a),$$

and this cancels out of (27), which thus reduces to

$$\pi(x') \equiv +1 \text{ (mod } p \text{ \& } p^\kappa) \ldots\ldots\ldots\ldots\ldots\ldots(27b).$$

[These results (26*a*–*c*) may be used as a Test of the accuracy of the sets of roots x, x'].

15. *Congruence-Tables.* The Tables‡ C_3, C_5, C_7, C_9, C_{11}, &c.—at end of this Memoir—give the complete set of $\tau(n)$ roots (x, x') of both Congruences $N_n \equiv 0$, $N_n' \equiv 0$ (mod p & p^κ),

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* The Tables I. for each value of n in Reuschle's *Tafeln Complexer Primzahlen*, &c. give the roots of $y^n - 1 \equiv 0$, or of M.A.P.F. of $(y^n - 1) \equiv 0$ (mod p) arranged in this way for most values of $n \not> 105$ for all suitable primes $\not> 1000$. The present Author's Tables of η, η', quoted in footnote of Art 9, are arranged in the *numerical order* of the roots η, η'; so are not convenient for the purpose of this Article.

† The results for $\pi(x)$ are (at present) only empirical (no general proof as yet known).

‡ These Tables were computed partly by the present Author, partly by an Assistant (Mr. R. F. Woodward) under the Author's superintendence. Every root (x, x') was checked by one of the Rules in Art. 11.

for all *odd* values of $n \not> 15$—except 13—for all primes and prime-powers (p & p^κ) proper to each index n, up to p & $p^\kappa \not> 1000$.

[It will be noted that $x' = y'$ when $n = 3$].

16. *Factorisation-Tables.* The Tables F_3, F_5, F_7, F_9, F_{11}, F_{13}, F_{15}, F_{17}—at end of this Memoir—give the factorisation into prime factors—(as far as found possible with the means available)—of both N & N' up to following limits of x, x'.

$n =$	3	5	7	9	11	13	15	17
x & $x' \not>$	100,	100,	50,	50,	40,	11,	40,	8

The aids (to factorisation) used were :—

1°. The *Congruence-Tables* quoted in Art. 15: these have enabled *all* divisors $\not> 1000$ to be cast out, (*i.e.* except in F_{13} and F_{17}).

2°. Certain *Numerical* MS. Canons* (2^{ary}, 3^{ary}, 5^{ary}, ..., 11^{ary}) which give the residues (R, both $\pm$) of x^r (mod p & p^κ) for each of the Bases $x = 2$, 3, 5, ..., 11, up to p & $p^\kappa \not> 10000$, and up to the limit $r = 100$ for Base 2, and $r = 30$ for the other Bases.

These Canons have enabled all divisors $\not> 10000$ to be cast out when $x \not> 11$.

16a. *Explanation of signs* (., ;, :, ?, §) *in the Tables.*

1°. A semi-colon (;) on the exteme right shows that the factorisation (into prime factors) is complete.

2°. A full stop (.) on the extreme right shows the presence of other (unknown) factors (each > 1000).

3°. A colon (:) in the middle is used to separate two important algebraic factors, *e.g.* Chain-Factors, or Aurifeuillian Factors (see Art. 17, 18).

4°. A query (?) on the right of a large factor ($> 10^7$) indicates that the composition of this factor is unknown (but contains no factor < 1000).

5°. The sign (‡) on the extreme right shows that all factors $< 10^4$ have been cast out.

17. *Numerical Chains.* Let N_1, N_2, N_3,† &c., be a series of *composite* numbers, each *formed in the same way* from a pair of elements (x, y), so that

$$N_1 = f(x_1, y_1) = L_1 M_1,\ \ N_2 = f(x_2, y_2) = L_2 M_2, \ldots, N_r = f(x_r, y_r) = L_r M_r \ldots (28),$$

when the functional operator is the *same throughout.* When the factors (L, M) of every three successive numbers (N_{r-1}, N_r, N_{r+1}) are so connected that

$$M_{r-1} = L_r,\ M_r = L_{r+1} \text{ (for all values of } r) \ldots\ldots (29),$$

then the numbers (N_r) are said to be *in chain*‡; the series is styled a *Chain-series*,‡ and the factors (L, M) are styled *Chain-factors.*‡

* These Canons are still in MS. They await funds for publication! The 2^{ary} and 10^{ary} Canons were computed by Mr. H. J. Woodall, A.R.C.Sc., and by an Assistant (Miss A. Woodward) under the present Author's superintendence; the two worked independently. The rest were computed by Miss A. Woodward under same superintendence.

† The N_1, N_2, N_3, &c., of this Article are not necessarily of the type N, N' of this Memoir, but are conditioned *only as here stated.*

‡ These terms were introduced by the present Author.

The most salient properties of such Chains are

$$\frac{N_2N_4N_6\ldots N_{2r}}{N_1N_3N_5\ldots N_{2r-1}}=\frac{M_{2r}}{L_1} \quad\ldots\ldots(30a),$$

$$\frac{N_1N_3N_5\ldots N_{2r-1}N_{2r+1}}{N_2N_4N_6\ldots N_{2r}}=L_1M_{2r+1}\quad\ldots\ldots(30b),$$

$$N_1N_2N_3N_4\ldots N_r=L_1(L_2L_3\ldots L_r)^2.\,M_r$$
$$=L_1(M_1M_2\ldots M_{r-1})^2.\,M_r\ldots\ldots(30c).$$

Examples will be found in Art. 25*a*, 26, 29, 30, 32 (see the Tables thereof).

[*Notation.* In numerical work the chain-factors (L_r, M_r) of a number N_r of a Chain are separated by a *colon* (:), thus $N=91=7:13$; thus, this colon is a *sign of multiplication* between two chain-factors].

18. *Aurifeuillians, Ant-Aurifeuillians* (**A**, **A'**). It is known that the general *n*-an $\mathbf{N}_n$ or $\mathbf{N}_n'$ may always be expressed in one of the *impure* 2ic forms shown below—the determinant D of the form ($D=\pm nxy$, $\pm \mathrm{a}xy$, &c.) depending on the form of n—

Form of n	N	N'	Form of n	N	N'
$n=4a+1\neq\square$	P^2-nxyK^2	$P'^2+nxyK'^2$	$n=4a-1$	$P'^2+nxyK'^2$	P^2-nxyK^2..(31*a*),
$n=\mathrm{a}^2$, $\mathrm{a}=4a+1$	$P^2-\mathrm{a}xyK^2$	$P'^2+\mathrm{a}xyK'^2$	$n=\mathrm{a}^2$, $\mathrm{a}=4a-1$	$P'^2+\mathrm{a}xyK'^2$	$P^2-\mathrm{a}xyK^2$..(31*b*),
$n=\mathrm{ab}$, $\mathrm{a}=4a+1$	$P^2-\mathrm{a}xyK^2$	$P'^2+\mathrm{a}xyK'^2$	$n=\mathrm{ab}$, $\mathrm{a}=4a-1$	$P'^2+\mathrm{a}xyK'^2$	$P^2-\mathrm{a}xyK^2$..(31*c*).

Here, introducing the condition

$$nxy=\square=(n\tau v)^2,\ \text{or}\ \mathrm{a}xy=\square=(\mathrm{a}\tau v)^2\ldots\ldots(32),$$

N and **N'** become either a *sum* or a *difference of squares*, viz.

$$\mathbf{N}=P^2-Q^2,\ \text{or}\ P'^2+Q'^2;\quad \mathbf{N}'=P'^2+Q'^2,\ \text{or}\ P^2-Q^2\ldots\ldots(33),$$

where $\quad Q^2=nxy.K^2$, or $\mathrm{a}xyK^2$; $\quad Q'^2=nxyK'^2$ or $\mathrm{a}xyK'^2$....(33*a*).

When $\mathbf{N}_1$ or $\mathbf{N}_1'=P^2-Q^2$, it is styled an *Aurifeuillian*,* denoted by **A**;

When **N**, or **N'**, $=P'^2+Q'^2$, it is styled an *Ant-Aurifeuillian**, denoted by **A'**;

And the condition (32) producing an **A**, or **A'** is styled the *Aurifeuillian Condition.*

And, since x, y are supposed *mutually prime* (see Art. 1*b*), this condition requires that x, y should be of the following forms

$$x=\tau^2,\ \text{or}\ nv^2;\ \text{or}\ =\tau^2,\ \text{or}\ \mathrm{a}v^2\ldots\ldots(34a),$$

$$y=nv^2,\ \text{or}\ \tau^2;\ \text{or}\ =\mathrm{a}v^2,\ \text{or}\ \tau^2\ldots\ldots(34b),$$

* These terms are due to the present Author, see his Memoir *On Aurifeuillians* in *Proc. Lond. Math. Soc.*, vol. xxix., 1898. This term commemorates the first considerable use of this function in the process of factorisation by the late M. Aurifeuille, of Toulouse.

and, when $n = \text{ab}$, b may be substituted for a in the Results (31*c*, 33*a*, 34*a*, 34*b*).

Introducing now the condition $x - y = 1$ of this Memoir—whereby **N** and **N'** become N and N')—the above forms (33) of x, y require the following Pellian equations to be satisfied—(depending on the form of n)—

Form of n	*Pellian Equatⁿ.*	*Form of n*	*Pellian Equation*
$n = 4a + 1 \neq \square$	$\tau^2 - nv^2 = \pm 1$	$n = 4a - 1$	$\tau^2 - nv^2 = +1$ .....(35*a*),
$n = \text{a}^2$, $\text{a} = 4a + 1$	$\tau^2 - \text{a}v^2 = \pm 1$	$n = \text{a}^2$, $\text{a} = 4a - 1$	$\tau^2 - \text{a}v^2 = +1$......(35*b*),
$n = \text{ab}$, $\text{a} = 4a + 1$	$\tau^2 - \text{a}v^2 = \pm 1$	$n = \text{ab}$, $\text{a} = 4a - 1$	$\tau^2 - \text{a}v^2 = +1$ .....(35*c*).

Every solution (τ, v) of the above Pellian equations gives *one* (and only one)—form **A**, or **A'**, of N, and *one* (and only one)—form **A'**, or **A**, of N'. For any one value of n the number of values of N of form **A** or **A'**, and also the number of values of N' of form **A'** or **A**, is therefore *infinite*; but the number practically useful is very limited, because only a few of the x, y arising from the Pellian equations are small enough to be useful.

19. *Aurifeuillians*, **A.** These are the more interesting of the two functions (**A**, **A'**) introduced in Art. 18; as N, or N', being then (algebraically) expressible as a *difference of squares*, is hereby always (algebraically) *factorisable into two factors* (say L, M): thus

$$N \text{ or } N' = \mathbf{A} = P^2 - Q^2 = L.M, \quad [Q = n\tau v.K, \text{ or } \text{a}\tau v.K] \text{.........(36)};$$

$$L = P - Q, \quad M = P + Q \text{ ....(36}a\text{)}.$$

The two factors L, M are styled the *Aurifeuillian Factors* of **A**, and have the peculiarities

L or M are both (algebraically) expressible in the same *pure* 2^{ic} forms as their product **A** ....(37).

20. *Quotient-Aurifeuillians.* If $\mathbf{A} = \mathbf{L}.\mathbf{M}$ and $A_1 = L_1.M_1$ be two Aurifeuillians of same order (n), such that **A** contains A_1 as a factor, then

$$\text{The Quotient } \frac{\mathbf{A}}{A_1} = \frac{\mathbf{L}.\mathbf{M}}{L_1.M_1} = A_2 \text{ is also an Aurifeuillian of same order.(38)},$$

so that
$$A_2 = L_2.M_2 \text{ ....(38}a\text{)},$$

and
$$L_2 = \frac{\mathbf{L}}{L_1} \text{ or } \frac{\mathbf{L}}{M_1}; \quad M_2 = \frac{\mathbf{M}}{M_1} \text{ or } \frac{\mathbf{M}}{L_1} \text{ ....(38}b\text{)}.$$

Hereby it is possible to compute the Aurifeuillian Factors L_2, M_2 of A_2 *algebrabically* from the above quotient-forms; but in numerical work, it is often much easier to compute the numerical values of **L**, **M**, L_1, M_1, and thence to compute L_2, M_2 as the numerical quotients.

21. *Ant-Aurifeuillians*, **A`**. These are of interest chiefly as giving the algebraic expression of **N** or **N`** as a *sum of Squares*.

N or $N` = P'^2 + Q'^2$, $[Q' = n\tau v . K'$, or $a\tau v . K']$............(39).

The following general relations between P and P', and Q and Q' apply to the general case of **N**, **N`**,

If $P = \phi(x, y)$, and $Q = \psi(x, y)$;

then $P' = \phi(x, -y)$, $Q' = \psi(x, -y)$, and *vice-versâ*.......(40).

22. *Formulæ for P, Q, P', Q'.* The formulæ for the four quantities P, Q, P', Q' which enter into the formulæ (33) for **A**, **A`** may be adapted to the *present case* of $N$, $N`$ by writing $x - y = 1$ in the known* formulæ of these four quantities in the *general case* of **N**, **N`**. The results are somewhat lengthy when $n > 7$ (increasing in length as n increases): they become more concise by expression in terms of two new quantities

$s = x + y$, $v = xy$ ....(41).

The Table below shows the new formulæ for P, K, P', K' for all values of $n \not> 17$.

[It will be seen that the formulæ are *unsymmetrical*, whereas the formulæ (in terms of x, y) in the general case of **N**, **N`** are symmetrical: this loss of symmetry is due to the disappearance of the term $(x - y)$ by the substitution of $x - y = 1$. Also—for the same reason—the general relation (40) between $P$ & $P'$, and $Q$ & $Q'$, does not obtain in the case of $N$, $N`$]. Note that—

One of P, Q is a function of both† s and v......................(42*a*);

The other (Q or P) is a function of v only. ....(42*b*);

Both P', Q' are functions of v only ...,...........................(42*c*).

n, a	N, N`	N or $N` = \mathbf{A} = P^2 - Q^2 = L . M$; $Q = \mu K$		
		P	μ	K
3, 3	N`	s	$3\tau v$	1
5, 5	N	$1 + 5v$	$5\tau v$	s
7, 7	N`	s^3	$7\tau v$	$1 + 3v$
9, 3	N`	$s(1 + v)$	$3\tau v$	v
11, 11	N`	$s^5 - 11v^2s$	$11\tau v$	$1 + 5v + 3v^2$
13, 13	N	$1 + 13v(1 + 4v + 5v^2)$	$13\tau v$	$s(s^2 - v)^2$
15, 3	N`	$1 + 6v + 7v^2$	$3\tau v$	$s(1 + 2v)$
15, 5	N	$(1 + 3v)^2$	$5\tau v$	$s(1 + 2v)$
15, 15	N`	$1 + 12v + 31v^2$	$15\tau v$	s^3
17, 17	N	$1 + 17v(1 + 5v + 8v^2 + v^3)$	$17\tau v$	$s(s^6 - 4vs^4 + 2v^3)$

* See the author's Paper on "Factorisation of $(Y^Y \mp 1)$, &c.", Art. 13, in vol. xlv. of this Journal, 1915.

† Since $s^2 = (1 + 4v)$, even powers of s can be replaced by powers of $(1 + 4v)$; but a single s cannot be removed.

n, a	N, N'	N or N' = A' = $P'^2 + Q'^2$; $Q' = \mu' K'$		
		P'	μ'	K'
3, 3	N	1	$3\tau v$	1
5, 5	N'	$1 - v$	$5\tau v$	1
7, 7	N	1	$7\tau v$	$1 + v$
9, 3	N	$1 + 3v$	$3\tau v$	v
11, 11	N	$1 - 11v^2$	$11\tau v$	$1 + 3v - v^2$
13, 13	N'	$1 - v(1 + 4v^2 + v^3)$	$13\tau v$	$1 + 2v + v^2$
15, 3	N	$1 + 2v - v^2$	$3\tau v$	$1 + 2v$
15, 5	N'	$(1 + v)^2$	$5\tau v$	$1 + 2v$
15, 15	N	$1 - 4v - v^2$	$15\tau v$	1
17, 17	N'	$1 - v(1 + 23v + 16v^2 - v^3)$	$15\tau v$	$1 + 4v - 2v^3$

23. *Properties of Sub-simple Cubans* N_{iii}, N_{iii}'. The following Articles (24—31) are devoted to developing the properties of Sub-simple Cubans N_{iii}, N_{iii}'. These are so numerous—[compared with those for other indices ($n > 3$)]—as to require separate treatment.

They are dealt with as follows:—

General properties, Art. 24, 24*a*.	Perfect squares, Art. 27.
Equalities, Art. 25, 25*a*.	Aurifeuillians, Art. 28—30.
Product properties, Art. 26.	Powers of small bases, Art. 31.

These properties arise chiefly from the following general theorem.

24. *Cuban Identity.* Every general Cuban $\mathbf{N}_{iii}$, $\mathbf{N}_{iii}'$ can be expressed in *three* equivalent forms:—(one of $\mathbf{N}_{iii}$, two of $\mathbf{N}_{iii}'$)—

$$\mathbf{N}_{iii} = \frac{x^3 - y^3}{x - y} = \frac{z^3 + x^3}{z + x} = \frac{z^3 + y^3}{z + y} = \mathbf{N}_{iii}', \textit{ identically}, \ [z = x + y] \ldots (43);$$

Under the restriction $x - y = 1$ of Sub-simple Cubans, these take the form

$$N_{iii} = x^3 - y^3 = \frac{(2x-1)^3 + x^3}{(2x-1) + x} = \frac{(2y+1)^3 + y^3}{(2y+1) + y}, \ = \mathbf{N}_{iii}' \text{ (in two ways)} \ldots (43a),$$

$$N_{iii}' = \frac{x^3 + y^3}{x + y} = \frac{x^3 + 1}{x + 1} = \frac{y^3 - 1}{y - 1} = H' = H \ldots (43b).$$

Properties latent in N_{iii}, N_{iii}' are in many cases obvious in one or other of these equivalent forms. This leads to *many* properties peculiar to Sub-simple Cubans, which obtain in no other Sub-simple n-ans.

24a. *Various formulæ for* N_{iii}, N_{iii}', H_3, H_3'.

$$N = \frac{x^3 - y^3}{x - y} = x^3 - y^3 = x^2 + xy + y^2 = 3x^2 - 3x + 1 = 3y^2 + 3y + 1 \ldots (44a),$$

$$N' = \frac{x'^3 + y'^3}{x' + y'} = x'^2 - x'y' + y'^2 = x'^2 - x' + 1 = y'^2 + y' + 1 \ldots (44b),$$

$$= (x' + y')^2 - 3x'y' = (x' + 1)^2 - 3x' = (y' - 1)^2 + 3y' \ldots (44c).$$

25. *Equality of* $N_{iii}^{\backprime}$, H_{iii}, $H_{iii}^{\backprime}$. The formulæ (44 b, c) show at sight that

$$H = H^{\backprime}, \text{ when } \eta' - \eta = 1 \quad\text{.......... (45a)},$$

$$N^{\backprime} = H, \text{ when } y' = \eta, \; N^{\backprime} = H^{\backprime} \text{ when } x' = \eta' \quad\text{.......... (45b)},$$

As $y' = \eta$, $x' = \eta'$ involve $\eta' - \eta = 1$, this shows that the equality

$$N^{\backprime} = H = H^{\backprime} \text{ obtains for every number } N_{iii}^{\backprime} \quad\text{...... (45c)}.$$

25a. *Equality of* N_{iii}, $N_{iii}^{\backprime}$. The formulæ (44 a—c) shows that

$$N = N^{\backprime} \text{ requires } 3x^2 - 3x + 1 = x'^2 - x' + 1 \quad\text{.......... (46)},$$

whence $$(2x' - 1)^2 - 3(2x - 1)^2 = -2,$$

and, writing $$x' = \tfrac{1}{2}(t + 1), \quad x = \tfrac{1}{2}(u + 1),$$

this gives $$t^2 - 3u^2 = -2 \quad\text{.......... (47)}.$$

The solutions (t, u) of these Equations may be obtained from the known solutions (τ, υ) of the Pellian equation

$$\tau^2 - 3\upsilon^2 = +1 \quad\text{.......... (48)}.$$

The Table below shows the corresponding solutions (τ_r, υ_r) (t_r, u_r), $(x_r^{\backprime}, x_r)$, and the final values of $N_r = N_r^{\backprime}$.

$r =$	1	2	3	4	5	6	7	8
$\tau, \upsilon =$	1, 0	2, 1	7, 4	26, 15	97, 56	362, 209	1351, 780	5042, 2911
$t, u =$	.	1, 1	5, 3	19, 11	71, 41	265, 153	989, 571	3691, 2131
$x', x =$	.	1, 0	3, 2	10, 6	36, 21	133, 77	495, 286	1846, 1066
$N = N^{\backprime} =$	.	1;	1:7;	7:13;	13:97;	97:181;	181:7.193;	7.193:2521;

The (t, u), (x, y), (x', y') may be found from the known (τ, υ) by the formulæ

$$t_r = \tau_{r+1} - \tau_r, \quad u_r = \upsilon_{r+1} - \upsilon_r \quad\text{.......... (49a)},$$

$$x' = \tfrac{1}{2}(t + 1), \; y' = \tfrac{1}{2}(t - 1); \quad x = \tfrac{1}{2}(u + 1), \; y = \tfrac{1}{2}(u - 1) \quad\text{.......... (49b)}.$$

There are also the succession-formulæ

$$t_{r+1} = 4t_r - t_{r-1}, \quad u_{r+1} = 4u_r - u_{r-1} \quad\text{.......... (50a)},$$

$$x'_{r+1} = 4x'_r - x'_{r-1} - 1, \quad x_{r+1} = 4x_r - x_{r-1} - 1 \quad\text{.......... (50b)}.$$

It will be seen that the series of $N = N^{\backprime}$ is a *Chain-series*: the chain-factors are shown separated by a *colon* (:).

26. *Products of two adjacent* N_{iii}, $N_{iii}^{\backprime}$.

Let N_1, N_2, be a pair of adjacent N_{iii}.

Let $N_1^{\backprime}$, $N_2^{\backprime}$ be a pair of adjacent $N_{iii}^{\backprime}$.

It will now be shown that—

1°. Every $N_1 . N_2 =$ a certain $N^{\backprime} = H^{\backprime}$ .... (51a).

2°. Every $N_1^{\backprime} . N_2^{\backprime} =$ a certain $N^{\backprime} = H$ .... (51b).

3°. Every $\tfrac{1}{3} N_1^{\backprime} . N_2^{\backprime} =$ a certain N (when $N_1^{\backprime} . N_2^{\backprime} = 3I$) .... (51c).

4°. The two series of products in Props. 1° & 2° are both *Chain-Series* ...(51d).

PROP. 1°. Here N_1, N_2 are of forms

$$N_1 = \frac{x^3-(x-1)^3}{x-(x-1)} = 3x^2-3x+1, \quad N_2 = \frac{(x+1)^3-x^3}{(x+1)-x} = 3x^2+3x+1 \text{......(52)},$$

whence $N_1 . N_2 = (3x^2-3x+1)(3x^2+3x+1) = 9x^4-3x^2+1$.........(52*a*),

$$= \frac{(3x^2)^3+1}{3x^2+1}, \text{ which is of form } H' \text{..................(52}b\text{)},$$

$$= \frac{(3x^2-1)^3+(3x^2)^3}{(3x^2-1)+3x^2}, \text{ which is of form } N' \text{..........(52}c\text{)}.$$

PROP. 2°. Here N_1', N_2' are of forms

$$N_1' = \frac{x^3+(x-1)^3}{x+(x-1)} = x^2-x+1, \quad N_2 = \frac{(x+1)^3+x^3}{(x+1)+x} = x^2+x+1 \text{.............(53)},$$

whence $N_1' . N_2' = (x^2-x+1)(x^2+x+1) = x^4+x^2+1$..............(53*a*),

$$= \frac{(x^2)^3-1^3}{x^2-1^3}, \text{ which is of form } H \text{..................(53}b\text{)},$$

$$= \frac{(x^2+1)^3+(x^2)^3}{(x^2+1)+x^2}, \text{ which is of form } N' \text{..........(53}c\text{)}.$$

PROP. 3°. From the detail in Prop. 2°, it follows that

$\frac{1}{3} . N_1' . N_2' = \frac{1}{3}(x^4+x^2+1)$; [$x$ must be of form $x = 3a \pm 1$, in order that $\frac{1}{3} N_1' . N_2' = i$]......(54).

Now, take $X = \frac{1}{3}(x^2+2)$, $Y = \frac{1}{3}(x^2-1)$ [which are both integers]...(54*a*).

Then $\frac{X^3-Y^3}{X-Y}$ is of form N [since $X - Y = 1$]...........(54*b*).

On reduction it will be found that this $N = \frac{1}{3}(x^4+x^2+1)$............(54*c*),

This shows that $\frac{1}{3} N_1' . N_2' = N$ ....(54*d*).

PROP. 4°. Let (N_1, N_2, N_3, &c.), (N_1', N_2', N_3', &c.) be successive numbers of the types N_{iii}, N_{ii}' respectively: the products of adjacent pairs are all of type N_{iii}' (by Prop. 1°, 2° above) and may be written $N_{1,2}'$, $N_{2,3}'$, $N_{3,4}'$, &c., where

In Prop. 1°; $N_{1,2}' = N_1 N_2$, $N_{2,3}' = N_2 N_3$, $N_{3,4}' = N_3 N_4$, &c. ....(55*a*);

In Prop. 2°; $N_{1,2}' = N_1' N_2'$, $N_{2,3}' = N_2 N_3$, $N_{3,4}' = N_3 N_4$, &c. ....(55*b*).

From the mode of formation it is now seen at once that both series of $N_{r,r+1}'$ are *Chain-series*, the chain-factors being the original (N_2, N_3, &c.) and (N_2', N_3', &c.).

Examples. The Table below shows *eight* examples—(taking $x = 1$ to 8)—of each of the Props 1°, 2°, 3°. The L, M column of the examples of Props 1°, 2° show the chain property of Prop. 4°. Here (x, y) means N; $\{x', y'\}$ means N'; $[\eta', 1]$ means H'; this notation exhibits the elements x, y, x', y', η'.

1°.	$N_r . N_{r+1} = N_{r,\,r+1}' = H_{r,\,r+1}'$		
$N_r \,.\, N_{r+1}$	$L : M$	$N_{r,\,r+1}'$	$H_{r,\,r+1}'$
(1, 0).(2, 1)	1:7;	{3, 2}	[3, 1]
(2, 1).(3, 2)	7:19;	{12, 11}	[12, 1]
(3, 2).(4, 3)	19:37;	{27, 26}	[27, 1]
(4, 3).(5, 4)	37:61;	{48, 47}	[48, 1]
(5, 4).(6, 5)	61:7.13;	{75, 74}	[75, 1]
(6, 5).(7, 6)	7.13:127;	{108, 107}	[108, 1]
(7, 6).(8, 7)	127:13.13;	{147, 146}	[46, 1]
(8, 7).(9, 8)	13.13:7.31;	{192, 191}	[192, 1]

2°.	$N_r' . N_{r+1}' = N_{r,\,r+1}' = H_{r,\,r+1}'$		
$N_r' \,.\, N_{r+1}'$	$L : M$	$N_{r,\,r+1}'$	$H_{r,\,r+1}'$
{1, 0}.{2, 1}	1:3;	{2, 1}	[2, 1]
{2, 1}.{3, 2}	3:7;	{5, 4}	[5, 1]
{3, 2}.{4, 3}	7:13;	{10, 9}	[10, 1]
{4, 3}.{5, 4}	13:3.7;	{17, 16}	[17, 1]
{5, 4}.{6, 5}	3.7:31;	{26, 25}	[26, 1]
{6, 5}.{7. 6}	31:43;	{37, 36}	[37, 1]
{7, 6}.{8, 7}	43:3.19;	{50, 49}	[50, 1]
{8, 7}.{9, 8}	3.19:73;	{65, 64}	[65, 1]

3°.	$N_r' . N_{r+1}' = 3 N_{r,r+1}$	
$N_r' \,.\, N_{r+1}'$	$L : M$	$3N_{r,r+1}$
{1, 0} {2, 1}	1:3	3.(1, 0)
{2, 1}.{3, 2}	3:7	3.(2, 1)
· ·	·	·
{4, 3}.{5, 4}	13:3.7	3.(6, 5)
{5, 4}.{6, 5}	3.7:31	3.(9, 8)
· ·	·	·
{7, 6}.{8, 7}	43:3.19	3.(17, 16)
{8, 7}.{9, 8}	3.19:73	3.(22, 21)

27. N_{iii}, N_{iii}' *as perfect squares.* The formulæ (44*a*) show that

1°. $N = z^2$ involves $3x^2 - 3x + 1 = z^2$.........(56),

whence $(2z)^2 - 3(2x-1)^2 = +1$..............................(56*a*).

And, writing $z = \frac{1}{2}\tau$, $x = \frac{1}{2}(v+1)$, $y = \frac{1}{2}(v-1)$,

this gives $\tau^2 - 3v^2 = +1$ ....(57).

The known solutions (τ, v) of this Pellian equation—taking only those in which τ is *even*—give the following values for x, z.

$x = 0$, 8, 105, 1456, 20273, 282360,

$z = 1$, 13, 181, 2521, 35113, 489061.

2°. A similar process applied to $N_{iii}' = z^2$ shows that the *only* solution is

$x = 1$; giving $N' = 1^2$ ....(58).

28. *Aurifeuillian and Ant-Aurifeuillian* N_{iii} & N_{iii}'. It has been shown (Art. 18) that every solution (τ, v) of the Pellian equation

$$\tau^2 - 3v^2 = +1 \quad \text{.......(59)},$$

will yield One N of form A', and one N' of form A.............(60).

1°. $N = (x^3 - y^3) \div (x - y) = (x - y)^2 + 3xy.$

Taking $x = \tau^2$, and $y = 3v^2$, gives

$$N = 1^2 + (3\tau v)^2 = P'^2 + Q'^2 = \text{A}' \text{ [an Ant-Aurifeuillian] ....(61)}.$$

2°. $N' = (x^3 + y^3) \div (x + y) = (x + y)^2 - 3xy.$

Taking $x = \tau^2$, $y = 3v^2$ gives

$$N' = (x + y)^2 - (3\tau v)^2 = P^2 - Q^2 = \text{A} \text{ [an Aurifeuillian]..........(62)},$$

wherein $L = P - Q, \quad M = P + Q$ ....(62a).

Also, these N' are a *Chain-series*:

For, let $N_1' = L_1 M_1$, $N_2' = L_2 M_2$ be any two adjacent terms: then

$$M_1 = \tau_1^2 + 3v_1^2 + 3\tau_1 v_1, \qquad L_2 = \tau_2^2 + 3v_2^2 - 3\tau_2 v_2$$
$$= 3v_1(2v_1 + \tau_1) + 1, \qquad = 3v_2(2v_2 - \tau_2) + 1$$
$$= 3v_1 v_2 + 1, \qquad = 3v_2 v_1 + 1 \text{ ....(63)},$$

by the known properties of τ, v: hence $M_2 = L_1$.

This proves that the N' numbers are *in chain*.

Further, this $N' = \dfrac{x^3 + y^3}{x + y} = \dfrac{(x - y)^3 - y^3}{(x - y) - y}$ [see Art. 24].

$$= \frac{y^3 - 1}{y - 1} = y^2 + y + 1 = (y - 1)^2 + 3y$$
$$= (y - 1)^2 + (3v)^2 = P'^2 + Q'^2 = \text{A}' \text{ [an Ant-Aurifn]......(64)}.$$

Also, the above N, N' are thus connected

$$\tfrac{1}{3}(N + 2) = 1 + 3\tau^2 v^2 = 1 + 3v^2 + 9v^4 = 1 + y + y^2 = N' \text{.........(64a)}.$$

Examples. The Table below gives nine examples. In the first six—(numbered $r = 1$ to 6)—the data (τ, v) and (x, y) and the results thereof, viz. N with its P', Q'; and N' with its L, M and P', Q', are given, illustrating all the above formulæ.

r =	1	2	3	4	5	6
τ, v =	1, 0	2, 1	7, 4	26, 15	97, 56	262, 209
x, y =	1, 0	4, 3	49, 48	676, 675	9409, 9408	131044, 131043
N =	1	37;	7057;	1368901	265559617	51517196677
P', Q' =	1, 0	1, 6	1, 84	1, 1170	1, 16296	1, 226974
N' =	1	13;	2353	456301	88519873	17172398893
L, M =	1:1;	1:13;	13:181;	181:2521;	2521:35113;	35113:489061;
P', Q' =	1, 0	2, 3	47, 12	674, 45	9407, 168	131042, 627

r =	**7**	**8**	**9**
τ, υ =	1351, 780	5042, 2911	18817, 10864
x, y =	1825201, 1825200	25421764, 25421763	18817^2, 3.10864^2
L, M =	489061:6811741;	6811741:13.61.181.661;	13.61.181.661:1321442641?†

In the remaining three cases—(numbered $r=7, 8, 9$)—the data (τ, υ) and (x, y) are given, and the resolution of the resulting N into its Aurifeuillian factors (L, M).

Note that the P', Q' of N' are connected by the relation—

$$3P'_r - 2 = Q'_{r-1} \cdot Q'_{r+1} \quad\text{.............................}(65).$$

29. *Aurifeuillian property of adjacent* N'_{iii}. It was shown in Art. 26 that two *adjacent* N'_{iii} may be expressed—(by taking $x = 3r^2$)—in form

$$N'_1 = x^2 - x + 1 = (x+1)^2 - 3x \qquad N'_2 = x^2 + x + 1 = (x-1)^2 + 3x \text{......}(66),$$
$$= (x+1)^2 - (3r)^2 \qquad = (x-1)^2 + (3r)^2$$
$$= P^2 - Q^2 = \mathbf{A} \qquad = P'^2 + Q'^2 = \mathbf{A}' \text{ ....}(66a).$$
$$L = x + 1 - 3r, \quad M = x + 1 + 3r$$

Here the series of $N'_1 = \mathbf{A}$ is a *Chain-series*. For if $\mathbf{A}_1$, $\mathbf{A}_2$ be any pair of adjacent members thereof, then

$$M_1 = x_1 + 3r_1 + 1, \qquad L_2 = x_2 - 3r_2 + 1$$
$$= 3r_1^2 + 3r_1 + 1, \qquad = 3r_2^2 - 3r_2 + 1$$
$$= 3r_1r_2 + 1, \qquad = 3r_2r_1 + 1 \quad (\text{for } r_2 = r_1 + 1) \text{ ....}(66b).$$

This establishes the Chain-property.

Also, N'_1, N'_2 are connected by the relation

$$N'_2 - N'_1 = 6r^2 \text{ ....}(67).$$

r =	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
x, y =	3,2	12,11	27,26	48,47	75,74	108,107	147,146	192,191
N'_1 =	7;	133	703	2257	5551	11557	21463	36673
L, M =	1:7;	7:19;	19:37;	37:61;	61:7.13;	7.13:127;	127:13.13;	13.13:7.31
x, y =	4,3	13,12	28,27	49,48	76,75	109,108	148,147	193,192
N'_2 =	13	157	757	2353	5701	11773	21757	37057
P', Q' =	2,3	11,6	26,9	47,12	74,15	107,18	146,21	191,24

Examples. The Table shows the data r, x, y of eight examples—[$r=1$ to 8]—with the result N'_1 with its Aurifeuillian Factors L, M and N'_2 with its 2^{ic} parts P', Q', illustrating all the above formulæ.

Note that the P', Q' of N'_2 are connected by the relation (65)—(see above).

30. *Mixed Aurifeuillian-Chains* (of two N_{iii}). By the cuban identity (43)

$$N = \frac{x^3 - y^3}{x - y} = \frac{z^3 + y^3}{z + y} = (z+y)^2 - 3zy \quad [z = x + y] \text{.............}(68).$$

This becomes an Aurifeuillian **A**

$$N = (z+y)^2 - (3\tau\upsilon)^2 = \mathbf{A} \text{..............................}(69),$$

† Factors not known, none < 1000.

under *each of two sets* of conditions

1°. $z + y = 3v^2, \quad y = \tau^2,$

whence $3v^2 - 2\tau^2 = x - y = +1$ ....(70*a*),

2°. $z + y = \tau^2, \quad y = 3v^2,$

whence $\tau^2 - 6v^2 = x - y = +1$ ....(70*b*).

The Aurifeuillian Factors (L, M) have the same formula in each case, viz.

$$L = z + y - 3\tau v, \quad M = z + y + 3\tau v \quad \text{.....................(71)}.$$

Let the former set be written N_1, N_3, N_5, ..., and the latter set be written N_2, N_4, N_6. &c., ... : and let the two series be arranged into one series, thus N_1, N_2, N_3, N_4, ... (by taking members from each series alternately).

The combined series will be found to be a *Chain-series*. To prove this, it will suffice to show, taking *any* three adjacent links, say N_1, N_2, N_3, that $M_1 = L_2$ and $M_2 = L_3$. Now

$$M_1 = z_1 + y_1 + 3\tau_1 v_1 = 3\tau_1(\tau_1 + v_1) + 1 \quad \Big| \quad L_2 = z_2 + y_2 - 3\tau_2 v_2 = 3v_2(3v_2 - \tau_2) + 1 \quad \text{.........(72}a\text{)}.$$

Here $M_1 = L_2$; because $\tau_1 + v_1 = v_2$ and $\tau_1 = 3v_2 - \tau_2$ by the Pellian properties.

$$M_2 = z_2 + y_2 + 3\tau_2 v_2 = 3v_2(3v_2 + \tau_2) + 1 \quad \Big| \quad L_3 = z_3 + y_3 - 3\tau_3 v_3 = 3\tau_3(\tau_3 - v_3) + 1 \quad \text{.........(72}b\text{)}.$$

Here $M_2 = L_3$; because $v_2 = \tau_3 - v_3$, and $3v_2 + \tau_2 = \tau_3$ by the Pellian properties.

Hence, also the formulæ for the successive L, M may be written

$$M_1 = L_2 = 3\tau_1 v_2 + 1, \quad M_2 = L_3 = 3\tau_3 v_2 + 1, \quad M_3 = L_4 = 3\tau_3 v_4 + 1, \ \text{\&c ...(72}c\text{)}.$$

Examples. The Table shows—for the first four numbers **N** of each Series (1°, 2°) the data (τ, v), (x, y) and the resulting numbers **N** resolved into their Aurifeuillian Factors (L, M): thus illustrating all the above formulæ.

Series 1°.	r	**1**	**3**	**5**	**7**
	τ, v	1,1	11,9	109,89	1079,881
	x, y	2,1	122,121	11882,11881	1164242,1164241
	L, M	1:7;	67:661;	31.211:64747;	7.19.61.79:43.147547;
Series 2°.	r	**2**	**4**	**6**	**8**
	τ, v	5,2	49,20	485,198	4801,1960
	x, y	13,12	1201,1200	117613,117612	11524801,11524800
	L, M	7:67;	661:31.211;	64747:7.19.61.79;	43.147547:37.1697413;

31. *Powers of small bases* (q^a). By means of the cuban identity (43), the two following adjacent Sub-simple Cubans N_1', N_2' are transformed into H', H.

$$N_1' = \frac{(q^a)^3 + (q^a - 1)^3}{q^a + (q^a - 1)} = \frac{(q^a)^3 + 1}{q^a + 1} = H' \quad \text{....................(73}a\text{)},$$

$$N_2' = \frac{(q^a + 1)^3 + (q^a)^3}{(q^a + 1) + q^a} = \frac{(q^a)^3 - 1}{q^a - 1} = H \quad \text{....................(73}b\text{)}.$$

Now the complete or partial factorisation of $(q^a)^3 \mp 1$ is known up to very high limits when q is a *small* base. Hence, by taking $q=2$, 3, 4, &c., the more or less complete factorisation of the Special Cubans N' above can be obtained up to the same high limits.

Examples. Complete factorisation is known of the above Simple-Cubans H', H is known in the following cases:

$$H = 2^{117}-1,\ 3^{45}-1,\ 5^{27}-1,\ 10^{21}-1;$$
$$H' = 2^{135}+1,\ 3^{105}+1,\ 5^{27}+1,\ 10^{21}+1;$$

32. *Sub-simple n-ans* $(n>3)$. The only properties of Sub-simple n-ans of orders higher than $n=3$, which it seems at all easy to develop are

1°. Chain properties. 2°. Aurifeuillan properties.

These form the subjects of Art. 33, 34.

Chain Series of N_n, N_n', $n>3$. Chains do not appear to occur in orders >3 of the same interest as in the Cubans N_{iii}, N_{iii}'. But they may be readily found from any chosen N_n. N_n' as initial member.

Let C denote either N or N', so as to include the chain-formation of both N, N' in one investigation, and let C_1, C_2, C_3, ..., be the members of the Chain, where

$$C_1 = f(x_1, y_1) = L_1 M_1, \quad C_2 = f(x_2, y_2) = L_2 M_2, \quad C_3 = f(x_3, y_3) = L_3 M_3, \ \&c.$$

Thus, by the chain-property, Art. 17,

x_1, x_2 are two roots of $C_1 \equiv 0 \pmod{M_1 = L_2}$............(74*a*),

x_2, x_3 are two roots of $C_2 \equiv 0 \pmod{M_2 = L_3}$............(74*b*),

x_3, x_4 are two roots of $C_3 \equiv 0 \pmod{M_3 = L_4}$............(74*c*),

and so on.

Here, x_1 being assumed given, x_2 is given by (74*a*); and x_2 thus formed, gives x_3 by (74*b*); similarly x_3, x_4, &c., are found in succession.

As each Congruence $C_r \equiv 0 \pmod{M_r = L_{r+1}}$ has $\tau(n)$ or $\tau(a)$ roots (by Art. 7*c*), each root x_r found leaves a choice of $\{\tau(n)-1\}$ or $\{\tau(a)-1\}$ values for the next root x_{r+1}, so that many chains can be formed from every initial x_1.

The moduli $M_1 = L_2$, $M_2 = L_3$, &c., of the Congruences (13) are found as the chain-factors of C_1, C_2, C_3, &c., in succession. The successive links C_1, C_2, C_3, &c., of the chain are computed in succession from the roots x_1, x_2, x_3, ..., as they are found.

Numerical Examples. Although the mode of formation above is simple (in principle), it is not practicable to form

Chains of more than three or four links, as the numbers x rise too rapidly.

Examples. Several examples are given below of Chains of degrees $n = 5, 7, 9$. The Chains shown are of *three* types—

1°. All members of type N; 2°. All members of type N'; 3°. Members of types N', N alternating.

n	N or N'	$r =$	1	2	3	4
	All N	x, y L, M	4,3 1:11.71;	19,18 11.71:751;	316,315 751:M_3	
5	*All N'*	x, y L, M	2,1 1:11;	3,2 11:5;	8,7 5:661;	217,216 661:M_4
	N' & N	x, y L, M	2,1 1:11; N_1'	4,3 11:71; N_2	12,11 71:251; N_3'	39,38 251,43781; N_4
7	*All N'*	x, y L, M	2,1 1:43;	6,5 43:757;	13,12 757:5209;	
9	*All N'*	x, y L, M	2,1 3:19;	4,3 19:163	37,36 163:M_3	

33. *Aurifeuillian and Ant-Aurifeuillians* ($n > 3$). The mode of formation is fully explained in Art. 19—22. The values of x, y, x', y' rise so rapidly when $n > 3$ that the number of numerical examples which are practically useful is very limited.

Examples. The Table (Tab. A), shows the whole of the Aurifeuillians (**A**) and Ant-Aurifeuillians **A'** likely to be useful for the degrees $n = 5, 7, 11, 15$.

The top line in each set gives the solutions (τ', v') and (τ, v) of the two Pellian equations

$$\tau'^2 - av'^2 = -1, \quad \tau^2 - av^2 = +1.$$

Every solution (τ', v') and (τ, v) yields (see Art. 18) one Aurifeuillian form (**A**) of either N or N', and one Ant-Aurifeuillian form **A'** of either N' or N: the former (**A**) is shown by its Factors L, M; the latter (**A'**) by its 2ic parts P', Q'. The factorisation of the L, M has been made as complete as the means at disposal (see Art. 16) would allow. In each case the Pellian solutions (τ', v'), (τ, v) are carried one step beyond the limits of practical factorisability of L, M simply to show how rapidly the values of the elements (x, y), (x', y') rise.

Tab. A.

n, a			*N* or *N*' = **A** = *L*. *M*,	*N*' or *N* = **A**' = $P'^2 + Q'^2$.			
5, 5		τ', υ'	2,1	38,17	682, 305		
		x, *y*	5,4	1445,1444	$682^2, 682^2-1$		
	N	*L*, *M*	11:191	1101431:11.1796761;			
	N'	*P'*.*Q'*	19.10	2086579.3230			
		τ, υ	9,4	161,72	2889, 1292		
		x, *y*	81,80	25921,25920	$2889^2, 2889^2-1$		
	N	*L*, *M*	11.311:61381;	354657241†:11.41.41.344171;			
	N'	*P'*,*Q'*	6479.180	671872320.57960			
7, 7		τ, υ	8,3	127,48	2024,765		
		x, *y*	64,63	16129,16128	$2024^2, 2024^2-1$		
	N'	*L*, *M*	16087:4080679;	263250043729?:66864473313457?			
	N	*P'*.*Q'*	1,274244	1,11100203906736			
9, 3		τ, υ	2,1	7,4	26,15	97,56	
		x, *y*	4,3	49,48	676,675	9409,9408	
	N'	*L*, *M*	19:163;	37.829:19.73.307;	19.4346929:1150333651?		
	N	*P'*.*Q'*	37.72	6906,193368	1368901,533871000		
11,11		τ, υ	10,3	199,60			
		x, *y*	100,99	39601,39600			
	N'	*L*, *M*	67.7305607:23.463.18272321?				
	N	*P'*,*Q'*	1078109999,97980299				
13,13		τ', υ'	18,5	**13**, 13			649, 180
		x, *y*	325,324				$649^2, 649^2-1$
	N	*L*, *M*			*N*	*L*, *M*	
	N'	*P'*,*Q'*			*N*'	*P'*,*Q'*	
3		τ, υ	2,1	7,4	26,15		
		x, *y*	4,3	49,48	676,675		
	N'	*L*, *M*	31:2131;	401101:31.2486251			
	N	*P'*,*Q'*	119.150	5547199.38336340			
15, 5		τ, υ	2,1	38,17	τ, υ	9,4	161, 72
		x, *y*	5,4	1445,1444	*x*, *y*	81,80	$161^2, 161^2-1$
	N	*L*, *M*	31:7411;		*L*, *M*	31.75571:753562261?	
	N'	*P'*.*Q'*	441.410		*P'*,*Q'*	42003361,375609780	
15		τ, υ	4,1	31,8			
		x, *y*	16,15	961,960			
	N'	*L*, *M*	1021:3575941;				
	N	*P'*.*Q'*	58559.1787460				
27, 3		τ, υ	2,1	7,4			
		x, *y*	4,3	49,48			
	N'	*L*, *M*	157411:109.3727;				
	N	*P'* *Q'*	242641,124416				

Factorisation of $N = (x^3 \mp y^3) \div (x \mp y)$; $[x - y = 1]$.

TAB. F 3.

$N = x^3 - y^3$, $N' = (x^3 + y^3) \div (x + y)$.

x	N	N'	x	N	N'
1	1;	1;	51	7.1093;	2551;
2	7;	3;	52	73.109;	7.379;
3	19;	7;	53	8269;	3.919;
4	37;	13;	54	31.277;	7.409;
5	61;	3.7;	55	7 19.67;	2971;
6	7.13;	31;	56	9241;	3.13.79;
7	127;	43;	57	61.157;	31.103;
8	13.13;	3 19;	58	7.13.109;	3307;
9	7.31;	73;	59	10267;	3.7.163;
10	271;	7.13;	60	13.19.43;	3541;
11	331;	3.37;	61	79.139;	7.523;
12	397;	7.19;	62	7.1621;	3 13 97;
13	7.67;	157;	63	11719;	3907;
14	547;	3.61;	64	12097;	37.109;
15	631;	211;	65	7.1783;	3.19.73;
16	7.103;	241;	66	61.211;	7.613;
17	19.43;	3 7.13;	67	13267;	4423;
18	919;	307;	68	13669;	3.49.31;
19	13.79;	343;	69	7.2011;	13 19.19;
20	7.163;	3.127;	70	43.337;	4831;
21	13.97;	421;	71	13.31 37;	3.1657;
22	19 73;	463;	72	49.313;	5113;
23	49.31;	3.13.13;	73	13.1213;	7.751;
24	1657;	7.79;	74	19.853;	3.1801;
25	1801;	601;	75	16651;	61:7.13;
26	1951;	3.7.31;	76	49 349;	5701;
27	49.43;	19 37;	77	97.181;	3.1951;
28	2269;	757;	78	37.487;	6007;
29	2437;	3.271;	79	7.19.139;	6163;
30	7.373;	13.67;	80	67.283;	3.49.43;
31	2791;	49.19;	81	19441;	6481;
32	13.229;	3.331;	82	19927;	7.13.73;
33	3169;	7.151;	83	7.2917;	3.2269;
34	7.13.37;	1123;	84	13.1609;	19.367;
35	3571;	3.397;	85	31.691;	37.193;
36	19.199;	13 97;	86	7.13.241;	3.2437;
37	7.571;	31.43;	87	22447;	7.1069;
38	4219;	3.7.67;	88	103.223;	13.19.31;
39	4447;	1483;	89	23497;	3.7.373;
40	31.151;	7.223;	90	7.3433;	8011;
41	7.19.37;	3.547;	91	24571;	8191;
42	5167;	1723;	92	25117;	3 2791;
43	5419;	13.139;	93	7.19.193;	43.199;
44	7.811;	3 631;	94	26227;	7.1249;
45	13.457;	7.283;	95	73.367;	3 13 229;
46	6211;	19.109;	96	27361;	7.1303;
47	13.499;	3.7.103;	97	7.13.307;	67.139;
48	7.967;	37:61;	98	19.19.79;	3.3169;
49	7057;	13.181;	99	13.2239;	31.313;
50	7351;	3.19 43;	100	7.4243;	9901;

Factorisation of $N = (x^5 - y^5) \div (x - y)$; $[x - y = 1]$.

TAB. F 5.

x	N	x	N
1	1;	51	11.2956841;
2	31;	52	11.271.11801;
3	211;	53	31.41.71.421;
4	11.71;	54	181.461.491;
5	11:191;	55	
6	4651;	56	
7	11.821;	57	
8	11.1451;	58	
9	41.641;	59	11.1861.2861;
10	31.1321;	60	121.517981;
11	61051;	61	31.221.7691;
12	41.2141;	62	121.311.1901;
13	151.811;	63	11.6936701;
14	241.691;	64	31.131.20021;
15	121.1831;	65	
16	11.61 431;	66	251.366701;
17	371281;	67	
18	11.42701;	68	71.131.11161;
19	11.71.751;	69	
20	723901;	70	11.10606241;
21	331.2671;	71	11.41.273901;
22	31.34501;	72	31.4215751;
23	541.2371;	73	11.101.34351;
24	61.131.191;	74	11.41.323581;
25	1803001;	75	71.101.21481;
26	11.192341;	76	
27	11.101.2221;	77	61.2807521;
28	2861461;	78	181.996631;
29	11.101.2971;	79	
30	11.31.41.271;	80	
31	4329151;	81	11.311:61381;
32	4925281;	82	11.20055821†
33	31.41.4391;	83	241.881.1091;
34	881.7151;	84	11.31.712841;
35	7086451;	85	11.61.379931;
36	7944301;	86	
37	11.807071;	87	521.571.941;
38	11.61.14741;	88	
39	251.43781;	89	
40	11.1106891;	90	71.4518881;
41	11.31.39461;	91	41.8181011;
42	14835031?†	92	11.31.1027841;
43	151.108061;	93	11.33278951†
44	211.84871;	94	41.9320891;
45	19611901?†	95	11.31.1169411;
46	61.351391;	96	11.401.94291;
47	23382031?†	97	
48	11.601.3851;	98	
49	11.2515571;	99	61.7716271;
50	41.732311;	100	

TAB. F 5 (*cont.*)

Factorisation of $N' = (x^5 + y^5) \div (x + y)$, $[x - y = 1]$.

x	N'	x	N'
1	1;	51	6510151;
2	11;	52	31.227131;
3	5.11;	53	5.11.138251;
4	181;	54	11.41.18181;
5	461;	55	61.144751;
6	991;	56	31.31.41.241;
7	31.61;	57	11.331.2801;
8	5.661;	58	5.11.198901;
9	11.491;	59	
10	11.761;	60	71.176651;
11	31.401;	61	
12	71.251;	62	
13	5.121.41;	63	5.241.12671;
14	11.3061;	64	11.1479011;
15	41.1091;	65	11.1574371;
16	58321;	66	
17	131.571;	67	
18	5.18911;	68	5.11.41.61.151;
19	117991;	69	11.31.64601;
20	11.101.131;	70	41.431.1321;
21	11.31.521;	71	101.244711;
22	214831;	72	
23	5.51511;	73	5.31.151.1181;
24	11.27851;	74	271.107741;
25	11.31.1061;	75	11.2801741;
26	424451;	76	11.2955191;
27	41.12071;	77	
28	5.114761;	78	5.571.12641;
29	41.16141;	79	11.151.22871;
30	61.12451;	80	11.181.20071;
31	11.11.71.101;	81	
32	11.61.1471;	82	101.251.1741;
33	5.191.1171;	83	5.31.71.4211;
34	1051.1201;	84	151.322051;
35	11.129061;	85	211.24'711;
36	11.144671;	86	11.4859821;
37	1778221;	87	11.31.164231;
38	5.31.12781;	88	5.
39	2200771;	89	311.197311;
40	2438281;	90	11.5834921;
41	71.37951;	91	121.61.9091;
42	11.31.31.281;	92	
43	5.11.191.311;	93	5.61.240101;
44	401.8941;	94	
45	3926341;	95	41.1945651;
46	11.390101;	96	
47	11.425521;	97	11.41.41.4691;
48	5.1019261;	98	5.11.1643501;
49	211.26251;	99	811.116101;
50	6009851;	100	31.1621.1951;

Factorisation of $N=(x^7 \mp y^7) \div (x \mp y)$, $[x-y=1]$.

Tab. F 7.

$N=x^7-y^7$; $N' = (x^7+y^7) \div (x+y)$.

x	N	N'
1	1;	1;
2	127;	43;
3	29.71;	463;
4	14197;	7.379;
5	29 2129;	10501;
6	29.6959;	43.757;
7	543607;	29.2927;
8	1273609;	194713;
9	2685817;	404713;
10	5217031;	778051;
11	1499.6329;	7.29.6917;
12		29.197.421;
13	281.95789;	757.5209;
14	43.463.2143;	6228223;
15	43.113.13469;	127.75013;
16	71.1374311;	197.71933;
17		71.211.1373.
18	127.281.5657;	7.29.43.3347;
19		29.1403627;
20	43.911.9857;	113.493277;
21		547.137383;
22	71.1429.6833;	
23		29.281.16087;
24	29.43.43.22037;	
25	29.	7.
26		43.43.149899;
27	29.	
28		71.6134003;
29	43.197.659.673;	71.7596443;
30	43.	379.1749469;
31		
32	29.	7.
33		827.1432019;
34	29.	
35	29.617.660367;	
36		29.701.98869;
37	211.	
38	701.	43.
39		7.113.4131653;
40	113.701.335917;	29.
41		29.
42	127.	43.
43		71.127.655439;
44	743.	71.
45	197.	43.
46	113.	7.
47		29.
48	337.	29.
49		43.
50	71.239.6071479;	659.

Factorisation of $N = \dfrac{x^9 - y^9}{x^3 - y^3}$, $N' = \dfrac{x^9 + y^9}{x^3 + y^3}$, $[x - y = 1]$.

TAB. F 9.

x	N	N'
1	1;	1;
2	73;	3.19;
3	1009;	577;
4	6553;	19.163;
5	19.1459;	3.3907;
6	19.37.127;	35281;
7	19.12547;	90217;
8	199.2791;	3.68059;
9	1166833;	19.22123;
10	37.199.307;	802441;
11	4102561;	3.19.127.199;
12	1153.6121;	37.127.523;
13	19.611011;	4016377;
14	19.967617;	3.37.109 523;
15	19.37.40087;	
16		19 754939;
17		3.6930379;
18		19.37.41959;
19	577.209809;	
20		3.37.506071;
21		19.3983059;
22	109.163.16759;	
23	37.	3.19.181.12781;
24	19.73.365779;	
25	19.	73.
26	19.	3.37.2508391;
27		199.1760743;
28	37.	19.
29	109.523.28279;	3.
30		19.73.
31		73.
32	19.37 4178431;	3.
33	19.	109.
34	19.73.163.18793;	
35		3.19.
36	127.	109.
37	379.	163.
38		3.181.271.19009;
39	397.	
40	73.	19.883.227593;
41		3.
42		19.
43	19.37.	73.
44	19.	3.73.
45	19.	
46	523.	163.
47	37.163.	3.19.127.
48		
49		19.37.73.307.829;
50	73.271.	3.379.

$$\textit{Factorisation of } N = \frac{x^{11} - y^{11}}{x - y}, \quad N' = \frac{x^{11} + y^{11}}{x + y}, \ [x - y = 1].$$

TAB. F 11.

x	N	N'
1	1;	1;
2	23.89;	683;
3	23.23.331;	35839;
4	23.174659;	727.859;
5	6359.7019;	23.256147;
6	‡	11.23.353.419;
7	89.18140783;	23.419.18769;
8	‡	23.30629743;
9	23. ‡	67.35093059;
10	‡	23.89.3378013;
11	23. ‡	2333.7864627;
12	67.727.9396553;	
13	23.	
14		23.
15	23.67.	
16		23.
17	67.331.	11.23.
18		23.67.89.
19	353.	23.67.
20	23.	
21	23.	67.89.199.
22	23.	683.991.
23	991.	
24	89.	
25	23.67.	463.947.
26	23.	397.
27	23.	89.
28		11.199.
29		463.
30		23.419.89.
31	89.463.	199.
32	23.67.	
33	463.	67.
34	23.	617.
35		67.353.859.
36	23.67.419.	
37		23.
38	23.89.	
39	617.	11.
40		23.

$$N = \frac{x^{15} - y^{15}}{x^5 - y^5} \cdot \frac{x - y}{x^3 - y^3}, \quad N' = \frac{x^{15} + y^{15}}{x^5 + y^5} \cdot \frac{x + y}{x^3 + y^3}, \quad [x - y = 1].$$

TAB. F 15.

x	N	N'
1	1;	1;
2	151;	331;
3	3571;	31.241;
4	61.601;	31.2131;
5	31:7411;	362581;
6	1038811;	1481671;
7	3729391;	181.27271;
8	31.181.2011;	61.231661;
9	3271.9151;	691.52051;
10	151.631.751;	‡
11	61.2577511;	‡
12	31.10394851? †	
13	31.61.329431;	751.910771;
14		31.541.74101;
15		31.61.271.4231;
16		
17		31.192210511? †
18	211 42645661? †	31.61.5001481;
19	31.451684861? †	
20	31.751.914581;	61.363476161?†
21		
22	241.331.581041;	
23		
24	31.	181.
25		
26		211.
27	31.421.	
28	631.	31.271.
29	61.7197161401?	31.571.
30		
31		151.181.
32		
33	61.	
34		
35	271.7449595261?	
36	31.	
37		
38		
39	31.181.	331.541.751.36451;
40		751.

Factorisation of $N = \dfrac{x^{13} - y^{13}}{x - y}$, $N' = \dfrac{x^{13} + y^{13}}{x + y}$, $[x - y = 1]$.

TAB. F 13.

x	N	N'
1	1;	1;
2	8191;	3.2731;
3	53.29927;	79.4057;
4	131.500111;	313.31357;
5	79.14602459;	1483.96487;
6	†	79.157.104677;
7	53.79.20021093? †	13. †
8	157.3329.866477;	‡
9	6553 303999697? ‡	‡
10	2081.3583918391? ‡	53. ‡
11	937.26171517763? ‡	53.157.254792591? ‡

Factorisation of $N = \dfrac{x^{17} - y^{17}}{x - y}$, $N' = \dfrac{x^{17} + y^{17}}{x + y}$, $[x - y = 1]$.

TAB. F 17.

x	N	N'
1	1;	1;
2	131071;	43691;
3	‡	7039 3673;
4	‡	‡
5	1259.2381.248779;	‡
6	137.409.443.651169;	†
7	1021. †	1259. †
8	‡	137.1327.5101.178603;

Least Roots (x, x') *of* $(x^3 \mp y^3) \div (x \mp y) \equiv 0$, (*mod* p), $[x - y = 1]$.

TAB. C 3.

x x	p	x' x'	x x	p	x' x'
. .	3	. 2	78, 410	487	233, 255
2, 6	7	3, 5	47, 453	499	140, 360
6, 8	13	4, 10	195, 329	523	61, 463
3, 17	19	8, 12	224, 318	541	130, 412
9, 23	31	6, 26	14, 534	547	41, 507
4, 34	37	11, 27	37, 535	571	110, 462
17, 27	43	7, 37	264, 314	577	214, 364
5, 57	61	14, 48	209, 393	601	25, 577
13, 55	67	30, 38	273, 335	607	211, 397
22, 52	73	9, 65	183, 431	613	66, 548
19, 61	79	24, 56	409, 201	619	253, 367
21, 77	97	36, 62	15, 617	631	44, 588
16, 88	103	47, 57	274, 370	643	178, 466
52, 58	109	46, 64	122, 540	661	297, 365
7, 121	127	20, 108	310, 364	673	256, 418
61, 79	139	43, 97	85, 607	691	254, 438
40, 112	151	33, 119	161, 549	709	228, 482
57, 101	157	13, 145	149, 579	727	282, 446
20, 144	163	59, 105	103, 631	733	308, 426
77, 105	181	49, 133	140, 600	739	321, 419
93, 101	193	85, 109	275, 477	751	73, 679
36, 164	199	93, 107	262, 496	757	28, 730
66, 146	211	15, 197	377, 393	769	361, 409
88, 136	223	40, 184	127, 661	787	380, 408
32, 198	229	95, 135	44, 768	811	131, 681
86, 156	241	16, 226	333, 491	823	175, 649
10, 262	271	29, 243	235, 595	829	126, 704
54, 224	277	117, 161	74, 780	853	221, 633
80, 204	283	45, 239	260, 660	859	261, 599
97, 211	307	18, 290	387, 481	877	283, 595
72, 242	313	99, 215	113, 771	883	338, 546
11, 321	331	32, 300	431, 477	907	385, 523
70, 268	337	129, 209	18, 902	919	53, 867
76, 274	349	123, 227	108, 830	937	323, 615
95, 273	367	84, 284	48, 920	967	143, 825
30, 344	373	89, 285	293, 699	991	114, 878
144, 236	379	52, 328	102, 896	997	305, 693
12, 386	397	35, 363			
119, 291	409	54, 356		p^κ	
134, 288	421	21, 401	23, 27	49	19, 31
211, 223	433	199, 235	121, 223	343	19, 325
204, 236	439	172, 268	8, 162	169	23, 147
45, 413	457	134, 324	98, 264	361	69, 293
162, 302	463	22, 442	147, 815	961	440, 522

Least Roots (x, x') *of* $(x^5 \mp y^5) \div (x \mp y) \equiv 0 \pmod{p}$.

TAB. C5.

$[(x-y)=(x'-y')=1]$.

x	x	x	x	p	x'	x'	x'	x'
4,	5,	7,	8	11	2,	3,	9,	10
2,	10,	22,	30	31	7,	11,	21,	25
9,	12,	30,	33	41	13,	15,	27,	29
16,	24,	38,	46	61	7,	30,	32,	55
4,	19,	53,	68	71	12,	31,	41,	60
27,	29,	73,	75	101	20,	31,	71,	82
24,	64,	68,	108	131	17,	20,	112,	115
13,	43,	109,	139	151	68,	73,	79,	84
54,	78,	104,	128	181	4,	80,	102,	178
5,	24,	168,	187	191	33,	43,	149,	159
3,	44,	168,	209	211	49,	85,	127,	163
14,	83,	159,	228	241	56,	63,	179,	186
39,	66,	186,	213	251	12,	82,	170,	240
30,	52,	220,	242	271	74,	111,	161,	198
61,	119,	163,	221	281	42,	105,	177,	240
62,	81,	231,	250	311	43,	89,	223,	269
21,	148,	184,	311	331	57,	143,	189,	275
96,	148,	254,	306	401	11,	44,	358,	391
53,	131,	291,	369	421	186,	209,	213,	236
16,	189,	243,	416	431	70,	212,	220,	362
54,	103,	359,	408	461	5,	203,	259,	457
54,	116,	376,	438	491	9,	207,	285,	483
87,	153,	369,	435	521	21,	130,	392,	501
23,	144,	398,	519	541	189,	265,	277,	353
87,	251,	321,	485	571	17,	78,	494,	555
48,	252,	350,	554	601	146,	255,	347,	456
144,	164,	468,	488	631	124,	161,	471,	508
9,	231,	411,	633	641	189,	248,	394,	453
173,	309,	353,	489	661	8,	217,	445,	654
14,	212,	480,	678	691	239,	282,	410,	453
162,	231,	471,	540	701	148,	300,	402,	554
19,	316,	436,	733	751	102,	279,	473,	650
196,	361,	401,	566	761	10,	236,	526,	752
13,	124,	688,	799	811	99,	260,	552,	713
7,	313,	509,	815	821	190,	300,	522,	632
34,	83,	799,	848	881	132,	150,	732,	750
125,	253,	659,	787	911	381,	410,	502,	531
87,	338,	604,	855	941	165,	180,	762,	777
317,	441,	531,	655	971	103,	407,	565,	869
268,	451,	541,	724	991	6,	358,	634,	986
				p^κ				
15,	60,	62,	107	121	13,	31,	91,	109
165,	371,	591,	797	961	42,	56,	906,	920

Least Solutions (x, x') *of* $(x^7 \mp y^7) \div (x \mp y) \equiv 0$ (*mod* p), $[x - y = 1]$.

TAB. C7.

x	x	x	x	x	x	p	x'	x'	x'	x'	x'	x'
3,	5,	6,	24,	25,	27	29	7,	11,	12,	18,	19,	23
14,	15,	20,	24,	29,	30	43	2,	6,	18,	26,	38,	42
3,	16,	22,	50,	56,	69	71	17,	29,	28,	44,	43,	55
15,	40,	46,	68,	74,	99	113	20,	39,	52,	62,	75,	94
2,	18,	42,	86,	100,	126	127	15,	43,	51,	77,	85,	113
29,	45,	69,	129,	153,	169	197	12,	16,	80,	118,	182,	186
37,	65,	90,	122,	147,	175	211	17,	83,	97,	115,	119,	195
50,	53,	70,	170,	187,	190	239	71,	85,	87,	153,	155,	169
13,	18,	64,	218,	264,	269	281	23,	89,	137,	145,	193,	259
108,	153,	48,	290,	185,	230	337	75,	90,	140,	198,	248,	263
84,	107,	163,	217,	273,	296	379	4,	30,	61,	319,	350,	376
92,	166,	171,	251,	256,	330	421	12,	73,	161,	261,	349,	410
75,	132,	196,	254,	318,	375	449	105,	112,	190,	260,	338,	345
14,	93,	187,	277,	371,	450	463	3,	172,	214,	250,	292,	461
43,	89,	189,	303,	403,	449	491	203,	218,	220,	272,	274,	289
137,	205,	254,	274,	343,	411	547	21,	165,	273,	275,	383,	527
35,	134,	285,	333,	484,	583	617	86,	233,	264,	354,	385,	532
186,	197,	285,	347,	435,	446	631	87,	208,	259,	373,	424,	545
29,	61,	107,	553,	599,	631	659	50,	93,	153,	507,	567,	610
29,	163,	246,	428,	511,	645	673	80,	154,	220,	454,	520,	594
38,	40,	293,	309,	662,	664	701	36,	122,	230,	472,	580,	666
44,	287,	359,	385,	457,	700	743	214,	272,	313,	431,	472,	530
207,	249,	273,	485,	509,	551	757	6,	13,	165,	593,	745,	752
115,	316,	373,	455,	512,	713	827	33,	87,	177,	651,	741,	795
165,	243,	440,	444,	641,	719	883	111,	234,	239,	645,	650,	773
20,	114,	384,	528,	798,	892	911	153,	164,	301,	611,	748,	759
82,	217,	251,	703,	737,	872	953	88,	236,	439,	515,	718,	866
203,	414,	479,	489,	554,	765	967	149,	213,	339,	629,	955,	819
						p^κ						
90,	92,	111,	731,	750,	752	841	69,	273,	313,	529,	569,	773

Least Roots (x, x') *of* $\dfrac{x^9 \mp y^9}{x^3 \mp y^3} \equiv 0 \pmod{p}$, $[x - y = 1]$.

TAB. C9.

x	x	x	x	x	x	p	x'	x'	x'	x'	x'	x'
5,	6,	7,	13,	14,	15	19	2,	4,	9,	11,	16,	18
6,	10,	15,	23,	28,	32	37	12,	14,	18,	20,	24,	26
2,	24,	34,	40,	50,	72	73	25,	30,	31,	43,	44,	49
22,	29,	55,	57,	81,	88	109	14,	33,	36,	74,	77,	96
6,	36,	61,	67,	92,	122	127	11,	12,	47,	81,	116,	117
22,	34,	47,	117,	130,	142	163	4,	37,	46,	118,	127,	160
56,	82,	89,	93,	100,	126	181	23,	38,	86,	96,	144,	159
8,	10,	90,	110,	190,	192	199	11,	27,	95,	105,	173,	189
50,	59,	80,	192,	213,	222	271	38,	110,	114,	158,	162,	234
10,	117,	125,	183,	191,	298	307	49,	98,	109,	199,	210,	259
37,	123,	138,	242,	257,	343	379	50,	68,	108,	272,	312,	330
39,	57,	61,	337,	331,	359	397	53,	83,	134,	264,	315,	345
51,	91,	94,	340,	343,	383	433	156,	202,	195,	239,	232,	278
144,	169,	208,	280,	319,	344	487	58,	119,	216,	272,	369,	430
29,	46,	78,	446,	478,	495	523	12,	14,	184,	340,	510,	512
117,	128,	168,	374,	414,	425	541	79,	141,	170,	372,	401,	463
19,	116,	230,	348,	462,	559	577	3,	250,	267,	311,	328,	575
163,	196,	293,	321,	418,	451	613	93,	99,	191,	423,	515,	521
67,	176,	286,	346,	456,	565	631	153,	214,	236,	396,	418,	479
231,	297,	353,	387,	443,	509	739	165,	207,	265,	475,	533,	575
123,	285,	378,	380,	473,	635	757	121,	190,	228,	530,	568,	637
84,	106,	153,	659,	706,	728	811	59,	62,	389,	423,	750,	753
207,	368,	381,	449,	462,	623	829	49,	139,	287,	543,	691,	781
249,	258,	347,	537,	626,	635	883	40,	207,	436,	448,	617,	844
68,	382,	418,	502,	538,	852	919	89,	298,	367,	553,	622,	831
67,	152,	397,	541,	786,	871	937	78,	238,	285.	653,	700,	860
136,	335,	408,	584,	657,	856	991	159,	187,	313,	679,	805,	833
						p^κ						
71,	108,	110,	252,	254,	291	361	106,	113,	149,	213,	249,	256

Least Roots (x, x') of $\dfrac{x^{11} \mp y^{11}}{x \mp y} \equiv 0 \pmod{p}$, $[x - y = 1]$.

TAB. C11.

x	x	x	x	x	x	x	x	x	x	p	x'	x'	x'	x'	x'	x'	x'	x'	x'	x'
2	3	4	9	11	13	15	20	21	22	23	5	6	7	8	10	14	16	17	18	19
12	15	17	25	32	36	43	51	53	56	67	9	18	19	21	33	35	47	49	50	59
2	7	24	31	38	52	59	66	83	88	89	10	18	21	27	30	60	63	69	72	80
62	63	69	81	83	117	119	131	137	138	199	21	28	31	45	61	139	155	169	172	179
3	17	68	98	156	176	234	264	315	329	331	65	94	128	133	161	171	199	204	238	267
19	67	84	94	161	193	260	270	287	335	353	6	35	47	93	115	239	261	307	319	348
54	55	137	159	172	226	239	261	343	344	397	26	63	71	132	191	207	266	327	335	372
36	65	112	191	207	213	229	308	355	384	419	6	7	30	175	179	241	245	390	413	414
31	33	60	95	175	289	369	404	431	433	463	25	29	84	100	125	339	364	380	435	439
39	111	145	172	304	314	446	473	507	579	617	34	93	135	157	305	313	461	483	525	584
119	149	158	166	249	413	496	504	513	543	661	67	179	183	267	330	332	395	479	483	595
76	91	207	228	229	455	456	477	593	608	683	2	22	195	242	274	410	442	489	662	682
12	210	216	255	312	416	473	512	518	716	727	4	233	285	294	359	369	434	443	495	724
80	125	246	272	359	501	588	614	735	780	859	4	35	97	185	222	638	675	763	825	856
57	100	304	341	364	518	541	578	782	825	881	105	211	213	267	318	564	615	669	671	777
165	194	264	386	445	503	562	684	754	773	947	25	57	107	297	370	578	651	841	891	923
23	145	182	254	380	612	738	810	847	969	991	22	137	323	484	485	507	508	669	855	970
3	73	105	124	252	278	406	425	457	527	p^{κ} 529	31	48	99	129	212	318	401	431	489	499

Least Roots (x, x') *of* N & $N' = (x^{15} \mp y^{15})(x \mp y) \div (x^5 \mp y^5)(x^3 \mp y^3) \equiv 0$, $[x-y=1]$.

TAB. C15.

x	x	x	x	x	x	x	x	p	x'	x'	x'	x'	x'	x'	x'	x'
5	8	12	13	19	20	24	27	31	3	4	14	15	17	18	28	29
4	11	13	29	33	49	51	58	61	8	15	18	20	42	44	47	54
2	10	45	50	102	107	142	150	151	31	48	51	72	80	101	104	121
8	39	45	84	98	137	143	174	181	7	24	31	85	97	151	158	175
18	46	82	96	116	130	166	194	211	26	48	95	103	109	117	164	186
22	50	57	97	145	185	192	220	241	3	93	105	107	135	137	149	239
35	68	105	122	150	167	204	237	271	15	28	77	135	137	195	244	257
22	110	111	136	196	221	222	310	331	2	39	67	86	246	265	293	330
27	54	139	162	260	283	368	395	421	72	91	153	192	230	269	331	350
134	185	222	261	281	320	357	408	541	14	39	87	202	340	455	503	528
155	230	249	281	291	323	342	417	571	29	43	56	259	313	416	529	543
4	106	174	227	375	428	496	598	601	217	234	254	258	344	348	368	385
10	28	90	195	437	542	604	622	631	71	140	207	233	399	425	492	561
60	150	208	230	432	454	512	602	661	155	306	310	313	349	352	356	507
208	285	313	328	364	379	407	484	691	9	172	257	298	394	435	520	683
10	20	187	361	391	565	732	742	751	13	39	40	226	526	712	713	739
55	175	184	216	596	628	637	757	811	247	248	284	313	499	528	564	565
151	270	370	478	514	622	722	841	991	234	262	321	389	603	671	730	758
								p^{κ}								
44	58	423	442	520	539	904	918	961	59	122	327	452	510	635	840	903

THE ELIMINANT OF TWO BINARY QUANTICS WITH DETERMINANTAL COEFFICIENTS.

By *Sir Thomas Muir, LL.D.*

1. Two equations accidentally turned up, viz.

$$\left.\begin{array}{l} |123|x^3-|124|x^2+|134|x-|234|=0 \\ |234|x^3-|235|x^2+|245|x-|345|=0 \end{array}\right\},$$

where the coefficients are primary minors of the 3-by-5 array

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5, \end{array}$$

the requirement in connection with the equations being the elimination of x. On using the dialytic process it was found that the six-line eliminant so obtained had $|234|^2$ for a factor, and after some trouble the cofactor was transformed into a four-line determinant, the result thus reached being the curious equality

$$\begin{vmatrix} |123| & |124| & |134| & |234| & \cdot & \cdot \\ |234| & |235| & |245| & |345| & \cdot & \cdot \\ \cdot & |123| & |124| & |134| & |234| & \cdot \\ \cdot & |234| & |235| & |245| & |345| & \cdot \\ \cdot & \cdot & |123| & |124| & |134| & |234| \\ \cdot & \cdot & |234| & |235| & |245| & |345| \end{vmatrix}$$

$$=|234|^2 . \begin{vmatrix} |123| & |124| & |134| & |234| \\ |124| & |134|+|125| & |234|+|135| & |235| \\ |134| & |234|+|135| & |235|+|145| & |245| \\ |234| & |235| & |245| & |345| \end{vmatrix},$$

where the form of the four-line determinant on the right is as if it were constructed by superposing the persymmetric array

$$\begin{array}{cc} |125| & |135| \\ |135| & |145| \end{array}$$

on the persymmetric array

$$\begin{vmatrix} |123| & |124| & |134| & |234| \\ |124| & |134| & |234| & |235| \\ |134| & |234| & |235| & |245| \\ |234| & |235| & |245| & |345| \end{vmatrix},$$

and thus closely resembles Bezout's eliminant

$$\begin{vmatrix} |a_1b_2| & |a_1b_3| & |a_1b_4| & |a_1b_5| \\ |a_1b_3| & |a_1b_4|+|a_2b_3| & |a_1b_5|+|a_2b_4| & |a_2b_5| \\ |a_1b_4| & |a_1b_5|+|a_2b_4| & |a_2b_5|+|a_3b_4| & |a_3b_5| \\ |a_1b_5| & |a_2b_5| & |a_3b_5| & |a_4b_5| \end{vmatrix}$$

of the two quartics

$$\left.\begin{array}{l} a_1x^4+a_2x^3+a_3x^2+a_4x+a_5=0 \\ b_1x^4+b_2x^3+b_3x^2+b_4x+b_5=0 \end{array}\right\}.$$

2. In the next place, as the two initial cubics of § 1 may be written in the form

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} = 0 = \begin{vmatrix} 1 & x & x^2 & x^3 \\ a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \\ c_2 & c_3 & c_4 & c_5 \end{vmatrix},$$

our problem is seen to be essentially the same as a problem of Cayley's,* who obtained in a rather laboured manner the eliminant of

$$\begin{Vmatrix} x^4 & x^3y & x^2y^2 & xy^3 & y^4 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{Vmatrix} = 0$$

in the form

$$\begin{vmatrix} |1234| & |1246|-|1345| & |1456| \\ |1235| & |2354|-|2156| & |2564| \\ |1236| & |3165|-|3264| & |3645| \end{vmatrix},$$

* Cayley, A., "Solution of a problem of elimination", *Quart. Journ. of Math.*, viii., pp. 183–185; or *Collected Math. Papers*, vi., pp. 40–42.

where the column-numbers 1, 2, 3, 4, 5, 6 now refer to the more artificial array

$$\begin{matrix} a_1 & b_1 & c_1 & a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 & a_3 & b_3 & c_3 \\ a_3 & b_3 & c_3 & a_4 & b_4 & c_4 \\ a_4 & b_4 & c_4 & a_5 & b_5 & c_5, \end{matrix}$$

which though 4-by-6 in dimensions has only the same 15 different elements as before. Since this three-line compound determinant of Cayley's must be practically identical with our four-line compound determinant of § 1, another interesting subject of inquiry is thus suggested: and the interest in both cases is not lessened when it is borne in mind that what we are essentially dealing with is an equality not restricted to determinants of the low orders here appearing, but valid for all higher orders.

3. The generalization which helps to make the whole matter considerably simpler was suggested by multiplying Cayley's vanishing 4-by-5 array by y^5 in the form

$$\begin{vmatrix} y & -x & \cdot & \cdot & \cdot \\ \cdot & y & -x & \cdot & \cdot \\ \cdot & \cdot & y & -x & \cdot \\ \cdot & \cdot & \cdot & y & -x \\ \cdot & \cdot & \cdot & \cdot & y \end{vmatrix},$$

the result being

$$0 = \left\|\begin{matrix} \cdot & \cdot & \cdot & \cdot & y^5 \\ a_1y-a_2x & a_2y-a_3x & a_3y-a_4x & a_4y-a_5x & a_5y \\ b_1y-b_2x & b_2y-b_3x & b_3y-b_4x & b_4y-b_5x & b_5y \\ c_1y-c_2x & c_2y-c_3x & c_3y-c_4x & c_4y-c_5x & c_5y \end{matrix}\right\|,$$

that is,

$$\left\|\begin{matrix} a_1y-a_2x & a_2y-a_3x & a_3y-a_4x & a_4y-a_5x \\ b_1y-b_2x & b_2y-b_3x & b_3y-b_4x & b_4y-b_5x \\ c_1y-c_2x & c_2y-c_3x & c_3y-c_4x & c_4y-c_5x \end{matrix}\right\| = 0,$$

the four equations of which written separately enable us to eliminate y^3, y^2x, yx^2, x^3 and thereby obtain naturally and simply the four-line eliminant of § 1. As just stated, however, it is at once seen that the problem in this shape is not made more difficult by being made more general, namely, by replacing the array of coefficients of x in the said set of four equations by an entirely fresh array

$$\begin{matrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4, \end{matrix}$$

the generalized result thus obtained being *the eliminant of*

$$\left\| \begin{matrix} a_1y - l_1x & a_2y - l_2x & a_3y - l_3x & a_4y - l_4x \\ b_1y - m_1x & b_2y - m_2x & b_3y - m_3x & b_4y - m_4x \\ c_1y - n_1x & c_2y - n_2x & c_3y - n_3x & c_4y - c_4x \end{matrix} \right\| = 0$$

is

$$\left| \begin{matrix} |a_1b_2c_3| & |a_1b_2n_3| + |a_1m_2c_3| + |l_1b_2c_3| & |a_1m_2n_3| + |l_1b_2n_3| + |l_1m_2c_3| & |l_1m_2n_3| \\ |a_1b_2c_4| & |a_1b_2n_4| + |a_1m_2c_4| + |l_1b_2c_4| & |a_1m_2n_4| + |l_1b_2n_4| + |l_1m_2c_4| & |l_1m_2n_4| \\ |a_1b_3c_4| & |a_1b_3n_4| + |a_1m_3c_4| + |l_1b_3c_4| & |a_1m_3n_4| + |l_1b_3n_4| + |l_1m_3c_4| & |l_1m_3n_4| \\ |a_2b_3c_4| & |a_2b_3n_4| + |a_2m_3c_4| + |l_2b_3c_4| & |a_2m_3n_4| + |l_2b_3n_4| + |l_2m_3c_4| & |l_2m_3n_4| \end{matrix} \right|$$

the multiplication of which columnwise by

$$\left| \begin{matrix} -l_4 & -m_4 & -n_4 & -1 \\ l_3 & m_3 & n_3 & \cdot \\ -l_2 & -m_2 & -n_2 & \cdot \\ l_1 & m_1 & n_1 & \cdot \end{matrix} \right|$$

leads to the equivalent three-line form

$$\left| \begin{matrix} |a_1b_2c_3\,l_4| & |a_1b_2n_3\,l_4| + |a_1m_2c_3\,l_4| & |a_1m_2n_3\,l_4| \\ |a_1b_2c_3m_4| & |a_1b_2n_3m_4| + |l_1b_2c_3m_4| & |l_1b_2n_3m_4| \\ |a_1b_2c_3n_4| & |a_1m_2c_3n_4| + |l_1b_2c_3n_4| & |l_1m_2c_3n_4| \end{matrix} \right| .$$

The way in which the four-line determinant here simplifies for Cayley's special case is in itself quite interesting, the theorem on which the simplification depends being that of C. le Paige and J. Deruyts regarding the summation of determinants*.

* Paige, C. le, "Sur quelques points de la théorie des formes algébriques", *Mém. ... Soc. ... des Sci.* (Liège) (2) ix. no. 4, 23 pp. Deruyts, J., "Sur certaines sommes de déterminants", *Mém. ... Soc. ... des Sci.* (Liège) (2) x., 11 pp.

4. The origin of the external analogy between our four-line eliminant in Cayley's case and Bezout's eliminant of two binary quartics is evident on noting that Bezout's four equations can be written as vanishing determinants, namely,

$$\begin{vmatrix} a_1 & a_2x^3 + a_3x^2 + a_4x + a_5 \\ b_1 & b_2x^3 + b_3x^2 + b_4x + b_5 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_1x + a_2 & a_3x^2 + a_4x + a_5 \\ b_1x + b_2 & b_3x^2 + b_4x + b_5 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_1x^2 + a_2x + a_3 & a_4x + a_5 \\ b_1x^2 + b_2x + b_3 & b_4x + b_5 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_1x^3 + a_2x^2 + a_3x + a_4 & a_5 \\ b_1x^3 + b_2x^2 + b_3x + b_4 & b_5 \end{vmatrix} = 0,$$

and that these partition themselves into four-termed expressions in the same way as our set of vanishing determinants in Cayley's case.

Rondebosch, S.A.,
15th July, 1919.

ON CERTAIN PLANE CONFIGURATIONS OF POINTS AND LINES.

By *Prof. W. Burnside.*

By a configuration of points and lines in a plane is meant a set of m points and n lines such that r of the points lie on each line and s of the lines pass through each point. If the points are denoted by the first m integers, and the fact that the r points $m_1, m_2, \ldots, m_r$ (all distinct and none exceeding m) lie on a line is denoted by the multiple symbol $(m_1, m_2, \ldots, m_r)$, then the configuration is denoted by a set of n such multiple symbols, subject to the condition that every distinct m_i occurs the same number, s, times in the multiple symbols, so that $ms = nr$. There is no difficulty in forming such sets of multiple symbols, but it is seldom easy to determine directly whether there is an actual configuration corresponding to it.

Let p be a prime, and form all sets of 3 unequal integers i, j, k (less than p) such that

$$i + j + k \equiv 0 \quad (\text{mod. } p).$$

If i is 1, $i+j+k$ cannot exceed $2p-1$, and must therefore be equal to p. The sets are therefore $1, \alpha, p-\alpha-1$, for all values of α such that $1<\alpha<p-\alpha-1$, or $1<\alpha<\frac{1}{2}(p-1)$. There are therefore $\frac{1}{2}(p-5)$ such sets containing 1. Now if $1, \alpha, \beta$ is such a set, so also is $n, n\alpha, n\beta$ when $n\alpha$ and $n\beta$ are replaced by their least positive remainders (mod. p). There are therefore $\frac{1}{2}(p-5)$ such sets containing any given number between 0 and p. Since there are 3 numbers in each set; it follows that the number of sets is $\frac{1}{6}(p-5)(p-1)$. That corresponding to this set of $\frac{1}{6}(p-5)(p-1)$ multiple symbols formed from the first $p-1$ integers, there is a configuration of $p-1$ real points and $\frac{1}{6}(p-5)(p-1)$ real lines, such that 3 of the points lie on each line and $\frac{1}{2}(p-5)$ of the lines pass through each point may be verified immediately as follows.

Consider the cubic curve

$$y^2 = 4x^3 - g_2 x - g_3,$$

where g_2 and g_3 are real. The co-ordinates of any point on it are given by

$$x = \wp u, \quad y = \wp' u.$$

Let 2ϖ be that period of these elliptic functions such that $\wp t\varpi$ and $\wp' t\varpi$ are real when t is real. Then corresponding to the values

$$u = \frac{2i\varpi}{p} \quad (i = 1, 2, \ldots, p-1)$$

there are $p-1$ points P_i on the curve; and it is an immediate consequence of Abel's theorem that if

$$i+j+k \equiv 0 \quad (\text{mod. } p),$$

the points P_i, P_j, P_k lie on a line.

The same idea may be used to verify the existence of a configuration of p^2 points (p prime) and $\frac{1}{6}p^2(p^2-1)$ lines, such that 3 of the points lie on each line and $\frac{1}{2}(p^2-1)$ of the lines pass through each point, but in this case the points and lines will not be all real.

If 2ϖ and $2\varpi'$ are the two fundamental periods of the elliptic functions, consider the p^2 points $P_{i,j}$ on the cubic for which

$$u = \frac{2i\varpi + 2j\varpi'}{p} \quad (i, j = 0, 1, 2, \ldots, p-1).$$

Since $P_{i,j}$ and $P_{-i,-j}$ are distinct points, $\frac{1}{2}(p^2-1)$ lines can be drawn through $P_{0,0}$, each of which contains a pair of the remaining points. Further, since $P_{0,0}$, $P_{i,j}$, $P_{-i,-j}$ lie on a line, so also do $P_{\alpha,\beta}$, $P_{i-\alpha,j-\beta}$, $P_{-i,-j}$ for all values of i and j. Hence through each of the p^2 points $P_{i,j}$ there can be drawn $\frac{1}{2}(p^2-1)$ lines each containing a distinct pair of the remaining points; and this verifies the existence of the configuration. As is well known when $p=3$, the 9 points are the inflexions of the cubic and the configuration is the Hessian configuration often denoted by the symbol $(9_3, 12_4)$.

A PROPERTY OF GROUPS OF EVEN ORDER.

By *Prof. W. Burnside.*

Denote by A_1, A_2, ..., A_r the operations of order two contained in a group of order N. If $2r > N$, the operations of the two sets

$$E, \ A_1, \ A_2, \ldots, A_r,$$

and

$$A_1A_2, \ A_1A_3, \ldots, A_1A_r,$$

must have at least s in common, where

$$2r - s = N.$$

If

$$r > \tfrac{3}{4}N, \quad s > \tfrac{1}{2}N,$$

and, including itself, A_1 is permutable with $s+1$ operations of order two. The number of these operations being greater than half the order of the group, every operation of order two is a self-conjugate operation, and the group is an Abelian group of type (1, 1, 1, ...) and order a power of two, when $r > \frac{3}{4}N$.

The ratio of the number of operations of order two contained in a group to the order of the group cannot therefore exceed $\frac{3}{4}$ unless all the operations of the group, except identity, are of order two. That $\frac{3}{4}$ cannot be replaced by a smaller number is shown by considering the direct product of a dihedral group of order 8 (which has 5 operations of order 2) and a group of order 2^{n-3}, which has $2^{n-3}-1$. The number of operations of order 2 in the resulting group is $6.2^{n-3}-1$; so that for this group the ratio is $\frac{3}{4} - \frac{1}{2^n}$.

ON THE SOLUTION OF A CUBIC EQUATION.

By *Alfred Lodge*.

In the *Messenger* of November, 1918,* is an initial outline of a method of solving a cubic equation, with three real roots, which I find suggests a neat graphic solution.

I take the equation in the form

$$f(x) \equiv ax^3 + 3bx^2 + 3cx + d = 0 \qquad (1),$$

as the expressions involved are neater in form with these coefficients, and I use $\cot\theta$ as preferable to $\tan\theta$, which latter was used on p. 106 of the above paper.

Putting $x = z + h$, the equation becomes

$$az^3 + 3Bz^2 + 3Cz + D = 0 \qquad (2),$$

where

$$\left.\begin{aligned} B &\equiv ah + b \\ C &\equiv ah^2 + 2bh + c \\ D &= ah^3 + 3bh^2 + 3ch + d \end{aligned}\right\} \qquad (3),$$

with the identities

$$\left.\begin{aligned} C^2 - BD &\equiv (b^2 - ac)\,h^2 + (bc - ad)\,h + c^2 - bd \\ BC - aD &\equiv 2\,(b^2 - ac)\,h + (bc - ad) \\ B^2 - aC &\equiv b^2 - ac \end{aligned}\right\} \qquad (4).$$

From (2) we obtain

$$\frac{3Cz + az^3}{D + 3Bz^2} = -1,$$

which, on putting $z = k\cot\theta$, reduces to

$$\frac{3\cot\theta + (ak^2/C)\cot^3\theta}{1 + (3Bk^2/D)\cot^2\theta} = -\frac{D}{Ck};$$

therefore if

$$k^2 = -\frac{C}{a} = -\frac{D}{B} \qquad (5),$$

we shall have

$$\cot 3\theta = -\frac{D}{Ck} \qquad (6).$$

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* vol. xlviii., p. 100.

This requires $BC - aD = 0$; therefore from (4)

$$h = \frac{ad - bc}{2(b^2 - ac)} \quad \text{......................(7).}$$

Further

$$k^2 = \frac{C^2}{-aC} = \frac{-BD}{B^2} = \frac{C^2 - BD}{b^2 - ac} \quad \text{.........(8);}$$

therefore from (4)

$$k^2 = h^2 - 2h^2 + \frac{c^2 - bd}{b^2 - ac},$$

i.e.

$$h^2 + k^2 = \frac{c^2 - bd}{b^2 - ac} \quad \text{.....................(9).}$$

But the following relation between h, k is more useful, derived from (3), (4) and (5), viz.

$$\left(h + \frac{b}{a}\right)^2 + k^2 = \left(\frac{B}{a}\right)^2 - \frac{C}{a} = \frac{b^2 - ac}{a^2} \quad \text{.........(10).}$$

The condition for all the roots being real is that k^2 shall be positive; hence $b^2 - ac$ is positive and greater than $\left(h + \frac{b}{a}\right)^2$. This is the necessary and sufficient condition.

The equation for 3θ can be put into several alternative forms by reason of (5). Thus

$$\cot 3\theta = -\frac{D}{Ck} = -\frac{B}{ak} = \frac{Bk}{C}$$

of these, the simplest and most useful is the second, whence

$$k = -\frac{B}{a}\tan 3\theta = -\left(h + \frac{b}{a}\right)\tan 3\theta \quad \text{.........(11).}$$

The sign of k may be either + or −, but it will be sufficient to deal with the positive value only. Also, a will be considered to be positive.

With regard to 3θ, we shall take its principal value $(3\theta_1)$ as positive and less than 180°. The other values will differ from this by multiples of 360° to avoid ambiguity in sines and cosines. Therefore, from (10) and (11),

$$\sin 3\theta_1 = \frac{ak}{\sqrt{(b^2 - ac)}},$$

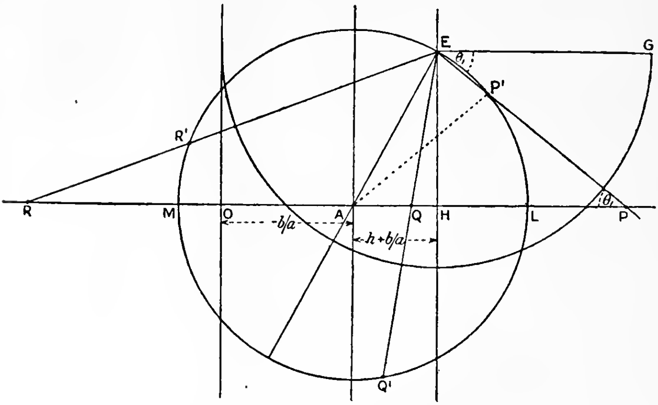
and

$$\cos 3\theta_1 = -\frac{ah + b}{\sqrt{(b^2 - ac)}}.$$

The working equations are therefore

$$\left.\begin{aligned} h &= \frac{ad - bc}{2(b^2 - ac)} \\ \cos 3\theta &= -\frac{ah + b}{\sqrt{(b^2 - ac)}} \\ x &= h + k \cot \theta \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (12).$$

On these equations the graphic solution is based.



Let O be the x-origin, and draw the ordinate $x = -b/a$, cutting the x-axis in A, which is the z-origin. This ordinate may be called the principal ordinate, as it passes through the inflexion point of the curve $y = f(x)$. Plot the point $(h, 0)$, denoting it by H. With centre A, radius $\sqrt{(b^2 - ac)} \div a$ describe a circle, cutting the ordinate of H in E (on the positive side). Join AE, and draw EG in the positive direction parallel to the x-axis. Then $AEG = 3\theta_1$, since $OH = h$, $OA = -b/a$, therefore $AH = h + b/a$, $AE = \sqrt{(b^2 - ac)} \div a$, and $HE = k$, therefore

$$\cos AEG = -\cos EAH = -\frac{ah + b}{\sqrt{(b^2 - ac)}}.$$

Now trisect the angle GEA by the line EP, cutting AH in P, so that $\angle GEP = \theta_1$, and $OP = OH + HP = h + k \cot \theta_1$, therefore OP is one of the roots. The other roots OQ, OR are obtained by taking $GEQ = \theta_1 + 60°$, which is the same line as $\theta_1 - 120°$, and $GER = \theta_1 + 120°$.

A good check on the accuracy of the constructions is the fact that if EP, EQ, ER cut the circle in P', Q', R', the lengths PP', QQ', RR' are each equal to the radius. For $2\theta_1 = AEP = AP'E = P'AP + P'PA = P'AP + \theta_1$, therefore the triangle $AP'P$ is isosceles, therefore $P'P$ = the radius of the circle, and similarly, though not quite so obviously, for $Q'Q$, $R'R$. (It may be noted that $P'Q'R'$ is an equilateral triangle).

Another obvious check is that $AP + AQ + AR = 0$.

For this construction the only quantities needed to be calculated from the coefficients are b/a, h, and $1/a\sqrt{(b^2 - ac)}$, where $h = \dfrac{ad - bc}{2(b^2 - ac)}$. It is worth noticing that, whatever may be the dimensions of a, all these quantities must represent lengths, for b/a, c/a, d/a are respectively of dimensions 1, 2, 3, to make the given equation homogeneous.

If the roots are to be obtained by calculation, it is desirable to modify equations (12) so as to avoid having to calculate k. This can be done by noting that $h + b/a = r \sin 3\theta$ and $k = -r\cos 3\theta$, where r denotes the radius $1/a\sqrt{(b^2 - ac)}$.

Therefore $$x = h + k\cot\theta$$

becomes
$$\begin{aligned} x &= -b/a + r(\sin 3\theta - \cos 3\theta \cot\theta) \\ &= -b/a + r\sin 2\theta \div \sin\theta \\ &= -b/a + 2r\cos\theta, \end{aligned}$$

which is also obtainable from the diagram; thus, for instance, $OP = OA + AP$, and

$$\frac{AP}{AE} = \frac{\sin AEP}{\sin APE} = \frac{\sin 2\theta_1}{\sin\theta_1} = 2\cos\theta_1,$$

therefore $$OP = -b/a + 2r\cos\theta_1,$$

therefore for calculation purposes the working equations are finally

$$\left.\begin{aligned} h &= \frac{ad - bc}{2(b^2 - ac)}, \quad \cos 3\theta = -\frac{ah + b}{\sqrt{(b^2 - ac)}} \\ x &= -b/a + 2\cos\theta\sqrt{\left(\frac{b^2 - ac}{a^2}\right)} \end{aligned}\right\}\ldots\ldots(13).$$

These equations are derivable at once by putting $x + b/a = z$ in the given equation, whence, after a little adjustment, the equation in z can be written

$$z^3 - 3r^2 z + 2r^2(h + b/a) = 0, \text{ where } r^2 = \frac{b^2 - ac}{a^2} \ldots(14),$$

i.e. $$4\left(\frac{z}{2r}\right)^3 - 3\left(\frac{z}{2r}\right) + \frac{h + b/a}{r} = 0,$$

which as usual is to be identified with

$$4\cos^3\theta - 3\cos\theta = \cos 3\theta,$$

so that θ is the usual auxiliary angle.

The object of the previous investigation was to introduce the diagram and so emphasize the importance of h and the circle, which are at once suggested by equations (12).

Further connections of the circle and point H with the graph of the function $f(x)$.

The curve representing the function being independent of the particular axes of reference, the connection can be established by moving the axes. The circle is independent of the position of the y-axis, since its centre is on the inflexion ordinate (where it cuts the x-axis), and its radius is invariable. If the y-axis cuts the circle in T, the ordinate $OT = \sqrt{(-c/a)}$ for

$$OT^2 = AT^2 - OA^2 = \frac{b^2 - ac}{a^2} - \frac{b^2}{a^2} = -c/a.$$

The gradient of the curve where it crosses $OT = 3c$, therefore the gradient of the curve $= -3a.OT^2$, and this holds for all points whose ordinates cut the circle. When O is outside the circle, let OT be the tangent, then $OT^2 = OA^2 - AT^2 = c/a$, therefore the gradient of the curve for such ordinates $= 3a.OT^2$.

If O is at either L or M where the circle cuts the x-axis, $OT = 0$, therefore the tangents at L, M pass through the points of zero gradient, the points of negative gradient lying between them, and those of positive gradient beyond them on either side.

The gradient at the inflexion point $= -3ar^2$, and its ordinate $= 2r^3(ah + b)$, as seen from the expression in equation (14).

With regard to H, its position on the x-axis is independent of the position of the y-axis, for, using the equation $h = \dfrac{ad - bc}{2(b^2 - ac)}$, when the y-ordinate goes through the inflexion point, the corresponding value of h, viz., AH, as obtained from (14), is $\dfrac{2r^2(h + b/a)}{2r^2} = h + b/a$, which shows that H does not change its position with the moving of the y-axis. When, however, the x-axis moves, H moves with it and also alters its abscissa. For if the x-axis moves through a positive distance y, the equation referred to the original axes becomes

$$ax^3 + 3bx^2 + 3cx + (d - y) = 0,$$

and the abscissa of the new H

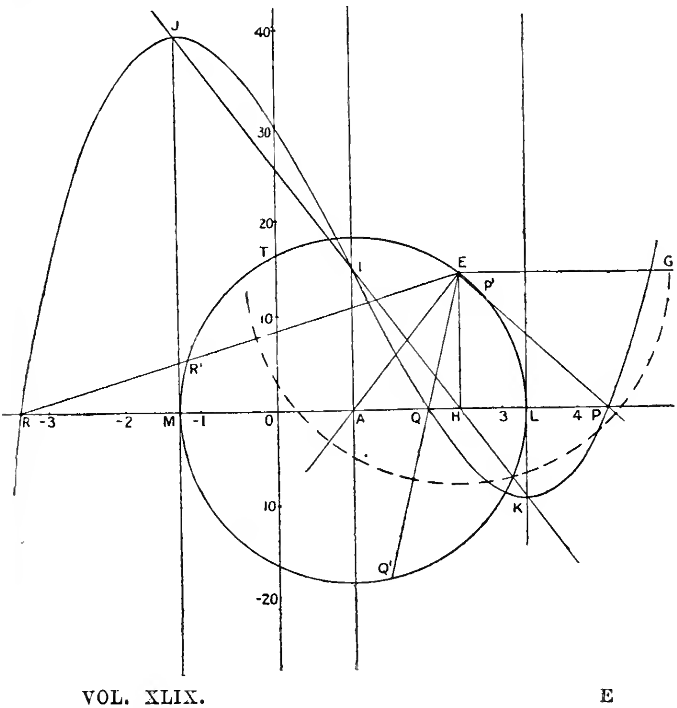
$$= \frac{a(d-y)-bc}{2(b^2-ac)},$$

i.e., H lies on the line

$$x = \frac{a(d-y)-bc}{2(b^2-ac)}, \text{ referred to the old axes......(15).}$$

By moving the y-axis it is seen that the above line cuts the curve on this axis whenever b or c is zero, for, then, $x=0$ when $y=d$. But, when $b=0$, the y-axis goes through the inflexion-point, and, when $c=0$, it goes through one of the points of zero gradient. Hence the locus of H is the straight line through these three points, so that it cuts the tangents at L, M of the circle in the points of zero gradient, and cuts the locus of the centre of the circle in the inflexion-point.

The gradient of the H-locus is $-2ar^2$, from (15), *i.e.* it is two-thirds of the inflexion gradient of the curve. This fact can be utilized in determining the limits between which one of the roots lies. For illustration see accompanying diagram.



Limits of roots. These are readily ascertained from equation (14), assisted by the diagram.

When $h=-b/a$, $z=0$, or $\pm r\sqrt{3}$;

when $h=-b/a+r$, $z=r$, r, $-2r$.

Between these values of h there are two positive values of z, one less than $h+b/a$ but greater than $\frac{2}{3}(h+b/a)$, the other between r and $r\sqrt{3}$, while the negative root lies between $-r\sqrt{3}$ and $-2r$.

When h is negative, the signs of z are reversed, but lie between the same numerical limits.

Two roots imaginary.

Even when r is real, if $h+b/a$ is numerically greater than r, two roots are imaginary. In this case the real value of z is numerically greater than $2r$, with its sign opposite to that of $h+b/a$. The gradient of $f(x)$ is given by $\pm 3a.OT^2$ as before.

When r is imaginary, there is only one real value of z, and it is always numerically less than $\frac{2}{3}(h+b/a)$, and of the same sign.

The gradient of $f(x)$, corresponding to any ordinate cutting the x-axis in O, is equal to $3a.OT^2$, where AT is drawn $=\sqrt{\left(\frac{ac-b^2}{a^2}\right)}$ at right angles to the x-axis. For

$$OT^2=OA^2+AT^2=\frac{b^2}{a^2}+\frac{ac-b^2}{a^2}.$$

For calculating the real root by an auxiliary angle the two cases are distinct.

(1) *b^2-ac positive but less than $(ah+b)^2$.*

The equation for z is

$$z^3-3r^2z+2r^2(h+b/a)=0,$$

i.e.

$$\left(\frac{z}{r}\right)^3-3\left(\frac{z}{r}\right)+\frac{2(h+b/a)}{r}=0.$$

If $\frac{z}{r}=\pm(p+p^{-1})$, p^3+p^{-3} will $=\mp\frac{2(h+b/a)}{r}$.

Let $p^3=\cot\theta$, then $p^3+p^{-3}=\cot\theta+\tan\theta=2\operatorname{cosec}2\theta$, therefore the working equations are

$$\left.\begin{aligned}\operatorname{cosec}2\theta&=\mp\frac{h+b/a}{r}\\ x&=-b/a\pm r(\cot^{\frac{1}{3}}\theta+\tan^{\frac{1}{3}}\theta)\end{aligned}\right\}\ldots\ldots\ldots\ldots(16).$$

Of the alternative signs it is better to choose the one which makes $\operatorname{cosec}2\theta$ positive.

The angle 2θ can be shown diagrammatically by drawing HE to touch the circle, centre A, radius r, in E, and drawing EG parallel to the x-axis as before. Then $HEG = 2\theta$. But it is not of much use, if any, as there seems no way of obtaining the root itself except by calculation.

(2) *$ac - b^2$ positive.*

Let
$$\frac{ac - b^2}{a^2} = s^2.$$

Then
$$z^3 + 3s^2 z = 2s^2 (h + b/a),$$

i.e.
$$\left(\frac{z}{s}\right)^3 + 3\left(\frac{z}{s}\right) = \frac{2(h + b/a)}{s}.$$

$$\left.\begin{array}{ll} \text{Putting} & \dfrac{z}{s} = \cot^{\frac{1}{3}}\theta - \tan^{\frac{1}{3}}\theta \\ \text{we obtain} & \cot\theta - \tan\theta = \dfrac{2(h + b/a)}{s} \\ \textit{i.e.} & \cot 2\theta = \dfrac{h + b/a}{s} \end{array}\right\} \quad \ldots\ldots\ldots\ldots(17).$$

If AT is drawn $= s$, perpendicular to the x-axis, and TG is parallel to this axis, the angle $HTG = 2\theta$, but as before the root seems to be obtainable only by calculation, viz.

$$x = -b/a + s(\cot^{\frac{1}{3}}\theta - \tan^{\frac{1}{3}}\theta).$$

Hence, in all cases, the only expressions required numerically are the linear quantities

$$\frac{b}{a}, \quad \frac{ad - bc}{2(b^2 - ac)} \quad \text{and} \quad \frac{1}{a}\sqrt{(b^2 - ac)} \quad \textit{or} \quad \frac{1}{a}\sqrt{(ac - b^2)},$$

and an auxiliary angle related to them in a way easily depicted in a diagram. The roots themselves can be given by the diagram when all are real, but not otherwise.

ON UNIFORM DIOPHANTINE APPROXIMATION.

By *H. T. J. Norton.*

Mr. Hardy and Mr. Littlewood in a paper in *Acta Mathematica** have discussed the approximate solution of the equation

$$q_n \vartheta - y - \alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

* Vol. xxxvii., pp. 155–190.

when q_n is a variable member of a given integral sequence and y is a variable integer. They have shewn among other things that if q_n is the value for argument n of a polynomial which has no absolute term, the left-hand side of (1) can be made arbitrarily small for any value of ϑ, provided α is an integer, and if α is not an integer it can be made arbitrarily small provided ϑ is irrational; further, that on the former supposition of α being integral, corresponding to any positive ϵ there exists a number $n_0\ \{= n_0(\epsilon)\}$, such that for all values of ϑ one of the numbers

$$|q_n\vartheta - y - \alpha| \quad (n = 1, 2, \ldots, n_0;\ y = 0, \pm 1, \pm 2, \ldots)$$

is less than ϵ; but if α is not an integer, there is no number $n_0(\epsilon)$ which possesses such a property for all irrational values of ϑ.

The purpose of the present note is, in the first place, to consider the conditions necessary for the existence of an n_0 and then to make some remarks on its magnitude in relation to ϵ when $q_n = n$ or n^2.

I start from equation (1), with α given arbitrarily, ϑ an arbitrary member of a given set E, y an integer and q_n a member of the integral sequence $q_1 \ldots q_n \ldots$. It may be supposed without loss of generality that $0 \leq \alpha < 1$, and that all the members of E are contained in the segment (0, 1). If, when any positive ϵ is chosen and ϑ is taken at random from E, values of n and y exist, such that $|q_n\vartheta - y - \alpha|$ is less than ϵ, α may be said to be a limiting point of $q_n\vartheta - y$ in respect of E; and if for any ϵ there exists a number $n_0 = n_0(\epsilon)$, such that for every ϑ contained in E one of the numbers

$$|q_n\vartheta - y - \alpha| \quad (n = 1, 2, \ldots, n_0;\ y = 0, \pm 1, \ldots)$$

is less than or equal to ϵ, then I shall say that $q_n\vartheta - y$ tends to α *uniformly* in respect of ϵ.

1. *If E is a closed set and α is a limiting point of $q_n\vartheta - y$ in respect of E, then $q_n\vartheta - y$ tends to α uniformly in respect of E.* This follows from a generalisation of the Heine-Borel theorem which is given by Professor Hobson*, viz., if (a, b) is a finite segment and if there is given "a closed set of points in (a, b) and a set of intervals such that each point is interior to at least one interval of the set, a finite number of intervals can be selected from the given set, which is also such that every point of the closed set is interior to at least

* *Theory of Functions of a Real Variable*, p. 90.

one of these intervals". For take any positive ϵ and consider the set of intervals whose end-points are

$$\left(\frac{y+\alpha-\epsilon}{q_n}, \frac{y+\alpha+\epsilon}{q_n}\right) \quad (n=1, 2, \ldots;\ y=0, \pm 1, \pm 2, \ldots).$$

If ϑ is an interior point of one of these intervals, values of n and y exist such that $|q_n\vartheta - y - \alpha|$ is less than ϵ; and conversely, if values of n and y exist such that $|q_n\vartheta - y - \alpha|$ is less than ϵ, then ϑ is an interior point of some interval of the set. Hence, since E is supposed to be closed, a finite number of intervals can be selected from the set which is also such that every point of E is contained in the interior of at least one of them. Let

$$\left(\frac{y_s+\alpha-\epsilon}{q_{n_t}}, \frac{y_s+\alpha+\epsilon}{q_{n_t}}\right) \quad (s=1, 2, \ldots, s_0;\ t=1, 2, \ldots, t_0)$$

be such a selection and n_0 the largest of $n_1, n_2, \ldots, n_{t_0}$; then it follows at once that for every ϑ contained in E, one of the numbers

$$|q_n\vartheta - y - \alpha| \quad (n=1, 2, \ldots, n_0;\ y=0, \pm 1, \pm 2, \ldots)$$

is less than ϵ.

Conversely, *if $q_n\vartheta - y$ tends uniformly to α in respect of E, then it also tends uniformly to α in respect of the set obtained by adding to E all its limiting points.* For, suppose that Θ is a limiting point of E, and $\vartheta_1 \ldots \vartheta_s \ldots$ a sequence contained in E which tends to Θ. By definition, if any ϵ is assigned, the inequalities

$$|q_n\vartheta_s - y - \alpha| \leq \epsilon,$$

$$n \leq n_0(\epsilon)$$

can be satisfied for any value of s. As there are only a finite number of n's which satisfy the second inequality, the first inequality must be satisfied by the same n for an infinite number of values of s.

If n_1 is such an n, and

$$|q_{n_1}\vartheta_s - y_s - \alpha| \leq \epsilon \quad (s=s_1, s_2, \ldots),$$

then, since ϑ_{s_n} tends to Θ, y_{s_n} is a constant, y_0, if n is sufficiently large, and $|q_{n_1}\Theta - y_0 - \alpha| \leq \epsilon$.

Thus, if E is a set of points between 0 and 1, and E' is the set obtained by adding to E all its limiting points, the limiting points of $q_n\vartheta - y$ in respect of E fall into two classes, those which are also limiting points in respect of E' and those which are not; and $q_n\vartheta - y$ approaches the members of the first class uniformly in respect of E (or E') and not uniformly the members of the second class. One can take a step further if one supposes that E is the set of *all* ϑ's in the segment (0, 1) for which α is a limiting point of $q_n\vartheta - y$. For Mr. Hardy and Mr. Littlewood have shewn this set is of measure one—provided, only, that the sequence $q_1 \ldots q_n \ldots$ tends to infinity*. E' is therefore the continuum, and therefore contains the point 0. But if ϑ is zero $q_n\vartheta - y$ is an integer, and therefore, if α is a limiting point of $q_n\vartheta - y$ in respect of E', α must be zero. We have thus proved the following proposition:—*If* $q_1, \ldots, q_n, \ldots$ *is a sequence of integers tending to infinity, if* $0 \leq \alpha < 1$, *and if* E *is the set of all points* ϑ *of the segment* (0, 1), *such that* α *is a limiting point of* $q_n\vartheta - y$, *then the necessary and sufficient conditions that* $q_n\vartheta - y$ *should tend uniformly to* α *in respect of* E *are that* α *is zero and* E *is the continuum.*

2. It follows from what has been said that if 0 is a limiting point of $|q_n\vartheta - y|$ for all points ϑ of the segment (0, 1), then $|q_n\vartheta - y|$ tends to zero uniformly in respect of that segment. This fact enables one to reduce the question whether 0 is a limiting point in respect of the segment to a form which does not involve any reference to irrational values of ϑ. One may suppose q_n to be positive. Take, now, a positive ϵ and an integer n_0 and consider the finite set of intervals

$$\left(\frac{y-\epsilon}{q_n}, \frac{y+\epsilon}{q_n}\right) \quad (n = 1, 2, \ldots, n_0;\ y = 0, 1, \ldots, q_n).$$

If ϵ is large enough, every point of the segment (0, 1) is an interior point of some interval of the set. If, n_0 being kept fixed, ϵ is allowed to decrease, then for small enough values of ϵ there are points of the segment which are external to every interval. And there is one value of ϵ which is such that every point is either an end-point or an internal point, but there is at least one point of the segment which is not an internal point.

Let ϵ_0 be this value of ϵ and ϑ_0 a point which is not an internal point. ϵ_0 is then the smallest value of ϵ, such that the inequalities

$$|q_n\vartheta - y| \leq \epsilon,$$
$$n \leq n_0$$

* *l.c.* Theorem 1, 40.

can be satisfied for every value of ϑ, and ϑ_0 is a point such that the least of the quantities

$$|q_n\vartheta - y| \quad (n=1, 2, \ldots, n_0;\ y=0, 1, 2, \ldots)$$

has the value ϵ_0.

Now, since ϑ_0 is not an internal point of any interval of the set, and the points in the immediate neighbourhood of ϑ_0, both to its right and to its left, are internal points or end-points, it follows at once that ϑ_0 is the right-hand end of one interval and the left-hand end of another; or, in other words, integers y_1, y_2, n_1, n_2 exist, such that

$$\vartheta_0 = \frac{y_1+\epsilon}{q_{n_1}} = \frac{y_2-\epsilon}{q_{n_2}} \quad (n_1 < n_2 \leq n_0),$$

that is, $$\vartheta_0 = \frac{y_1+y_2}{q_{n_1}+q_{n_2}}.$$

ϑ_0 is, therefore, a rational fraction with denominator $q_{n_1}+q_{n_2}$.

Hence, if a, n_1 and n_2 are integers and $1 \leq a \leq q_{n_1}+q_{n_2}-1$, and if $\epsilon_0(a, n_1, n_2)$ is the smallest of the numbers

$$\left|\frac{q_n a}{q_{n_1}+q_{n_2}} - y\right| \quad (n=1, 2, \ldots, n_0;\ y=0, 1, \ldots, q_n)$$

then the necessary and sufficient condition that 0 should be a limiting point of $|q_n\vartheta - y|$ in respect of the segment (0, 1) is that the largest of the numbers

$$\epsilon_0(a, n_1, n_2) \quad (n_1=1, 2, \ldots, n_0-1;\ n_2=n_1+1, \ldots, n_0;$$
$$a=1, 2, \ldots, q_{n_1}+q_{n_2}-1)$$

should tend to zero as n_0 tends to infinity.

I shall now apply this result to the cases in which $q_n=n$ or n^2. If $q_n=n$, the numbers $a/q_{n_1}+q_{n_2}$ of the last proposition are the rational fractions whose denominators are less than $2n_0$. If a/b is one of them, a being prime to b, and b is not greater than n_0, we can take $q_n=b$, and

$$\epsilon(a, n_1, n_2)=0.$$

If, on the other hand, $n_0 < b \leq 2n_0 - 1$, we can choose a q_n not greater than n_0, such that

$$q_n a \equiv \pm 1 \pmod{b},$$

and therefore $$\epsilon(a, n_1, n_2) = 1/b.$$

It follows from this that the largest of the numbers

$$\epsilon_0(a, n_1, n_2) \quad (n_1=1, 2, \ldots, n_0-1;\ n_2=n_1+1, \ldots, n_0;$$
$$a=1, 2, \ldots, q_{n_1}+q_{n_2}-1)$$

is $1/n_0+1$. This proves the familiar proposition: given any ϑ and any integer n_0, integers n and y exist such that

$$|n\vartheta - y| \leq 1/n_0+1,$$
$$n \leq n_0,$$

but these inequalities cannot in general be satisfied if any smaller number be substituted for $1/n_0+1$. Although this proposition is familiar, the proof just given seems to me of some interest, as it connects the result with one of the fundamental properties of whole numbers.

In the case in which $q_n = n^2$, Mr. Hardy and Mr. Littlewood showed in the memoir, already cited, that 0 is in fact a limiting point of $|q_n\vartheta - y|$ in respect of the segment (0, 1); and in an article of mine in the *Lond. Math. Soc. Jour.** it was shown that the inequalities

$$|n^2\vartheta - y| \leq 1/(\log n_0)^{1-\delta},$$
$$n \leq n_0$$

are soluble for every value of ϑ, δ being a positive quantity which tends to zero as n_0 tends to infinity; that is to say, in the notation of the present section, if $q_n = n^2$,

$$\epsilon_0 \leq 1/(\log n_0)^{1-\delta}.$$

This upper limit to ϵ_0 is, probably, much too large, but I have not succeeded in finding a smaller one. What, though, one can easily do by the methods of this note is to calculate the actual value of ϵ_0 for small values of n_0. The results I get are as follows:—

n_0	$\frac{1}{\epsilon_0}$	ϑ_0
2, 3, 4	2·5	$\frac{2}{5}$
5	3·4	$\frac{7}{17}$
6	$4.0\dot{6}$	$\frac{25}{61}$
7, 8, 9	5	$\frac{3}{10}$
10—16	$5{\cdot}\dot{6}$	$\frac{3}{17}$
(17, 18, 19	$\leq 5{\cdot}71\ldots$	$\frac{7}{40}$)

In this table the second column gives the largest number $1/\epsilon_0$, such that the inequalities

$$|n^2\vartheta - y| \leq \epsilon_0,$$
$$n \leq n_0$$

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can be satisfied for every ϑ, when n_0 is the number (or numbers) in the same row of the first column. The last column gives a value of ϑ for which the upper limit $1/\epsilon_0$ is actually attained.

The figures certainly suggest that ϵ_0 tends to zero very slowly as n_0 tends to infinity.

In conclusion, I propose to give a proof of a result which was stated, without proof, in my article cited above. It is to the effect that if $q_n = n^2$, then, as n_0 tends to infinity,

$$\overline{\lim}\, n_0 \epsilon_0 = \infty\ ;$$

this disproves a suggestion, made in Mr. Hardy and Mr. Littlewood's article, that ϵ_0 is not greater than Kn_0^{-1} where K is some constant.

To prove it, let $\pi_1, \pi_2, \ldots, \pi_n$ be the prime numbers in order of magnitude, and, n being given any fixed value, let P be the smallest prime in the arithmetic progression

$$1 + 4\lambda\pi_1\pi_2\ldots\pi_n.$$

Since $P \equiv 1 \pmod 8$,

$$\left(\frac{2}{P}\right) = 1,$$

$$\left(\frac{\pi_i}{P}\right) = \left(\frac{P}{\pi_i}\right) = 1 \quad (i = 2, 3, \ldots, n).$$

Thus all the numbers

$$\pm\pi_i \quad (i = 1, 2, \ldots, n)$$

are quadratic residues of P, and since the product of two residues is a residue, the numerically smallest non-residue is not less than π_{n+1}. Therefore, if π be any quadratic non-residue, the least of the quantities

$$\left| r^2 \frac{\pi}{P} - y \right| \quad (r = 1, 2, \ldots, P-1\ ;\ y = 0, \pm 1, \pm 2, \ldots)$$

is not less than $\dfrac{\pi_{n+1}}{P}$.

Hence $$\epsilon_{P-1}(P-1) \leq \pi_{n+1}(1 - 1/P),$$

which tends to infinity with n. *Q.E.D.*—When n is equal to 0 or 1, π/P can be taken equal to $\frac{2}{5}$ and $\frac{3}{17}$ respectively, and these, as will be seen from the table above, are the crucial values of ϑ for n_0 equal to $P-1$. But whether this is true for larger values of n_0 I do not know. If, by any chance, it were true for an infinity of n_0's, one could deduce a new and much smaller upper limit to ϵ_0.

STANDARD RELATION OF LEGENDRE'S FUNCTIONS.

By *R. Hargreaves, M.A.*

In the course of preparing short proofs of the properties of Legendre's functions, with special reference to expressions which do not require the order to be a positive integer, my attention was directed to the following question. Is it, or is it not, possible to deduce the whole scheme of relations connecting the functions and their first differential coefficients from any two of the fundamental formulæ, by process independent of the type of expression? The question was prompted by experience of the very different degrees of facility with which different formulæ are established, according as one or other type of expression is used for the function.

With respect to the seven relations taken below as fundamental I found it to be possible in all combinations. For each type of expression therefore we may direct attention to the choice of the two relations which are most readily proved; and in any general account of the functions a sensible economy may thus be realised. It is not necessary to dwell on the importance of such economy, in view of the task, incumbent on each generation of mathematicians, to mitigate the extra burden imposed on students by the accumulation of material.

I now set down the seven relations used; and make some comment on the reservations needed, if the sequence relation of P_n' is admitted to the group. The place of (ii) is suggested by the analogy with Bessel's functions.

§ 1. *Fundamental Relations.*

$$(n+1)P_{n+1} + nP_{n-1} = (2n+1)xP_n \quad\text{..........(1)},$$

$$(2n+1)(1-x^2)P_n' = n(n+1)(P_{n-1} - P_{n+1}) \quad\text{......(2)},$$

$$(2n+1)P_n = P'_{n+1} - P'_{n-1} \quad\text{..............(3)},$$

$$P'_{n+1} - xP_n' = (n+1)P_n \quad\text{.................(4)},$$

$$P'_{n-1} - xP_n' = -nP_n \quad\text{...............(5)},$$

$$(1-x^2)P_n' = (n+1)(xP_n - P_{n+1}) \quad\text{...........(6)},$$

$$(1-x^2)P_n' = n(P_{n-1} - xP_n) \quad\text{..............(7)}.$$

From any pair of these seven relations the rest are directly deducible (without use of special values), though not with the same facility in all cases. But if the relation

$$nP'_{n+1} + (n+1)P'_{n-1} = (2n+1)xP'_n \text{ .... } (8)$$

is used, then with respect to its combinations with (2), (6) or (7), cases where two P_n functions occur, recourse to special values seems to be needed to complete the argument. With

$$U_n \equiv (n+1)P_{n+1} + nP_{n-1} - (2n+1)xP_n,$$

the pair (8, 2) leads to $(2n-1)U_{n+1} = (2n+3)U_{n-1}$, or $U_n = C(2n+1)$;

,, (8, 6) ,, $nU_{n+1} = (n+1)xU_n$, or $U_n = Cnx^n$;

,, (8, 7) ,, $(n+1)U_{n-1} = nxU_n$, or $U_n = C(n+1)/x^n$.

As regards (6) if $(1-x^2)P_n'$ vanishes with $1-x$, then

$$P_{n+1}(1) = P_n(1) = P_{n-1}(1),$$

i.e. $U_n = 0$ for $x=1$ and so $C=0$; and (7) stands on the same footing. As regards (2), with the same assumption,

$$P_{n+1}(1) = P_{n-1}(1),$$

and $$U_n = (2n+1)\{P_{n+1}(1) - P_n(1)\},$$

or $$P_{n+1} - P_n(1) = C = P_n(1) - P_{n-1}(1),$$

or $$2P_{n+1}(1) = 2P_n(1);$$

that is again $C=0$. For these combinations some such supplementary argument as the one just given appears to be necessary.

§ 2. I proceed to the proofs of the relations selected as well adapted to the form of expression.

Laplace's Form

$$P_n = \frac{1}{\pi}\int_0^\pi \{x + \sqrt{(x^2-1)}\cos\phi\}^n d\phi = \frac{1}{\pi}\int_0^\pi X^n d\phi, \text{ say.}$$

Since

$$\sqrt{(x^2-1)}\sin\phi\,\frac{\partial X}{\partial\phi} = -(x^2-1)\sin^2\phi = X^2 - 2Xx + 1,$$

we get

$$\frac{\partial}{\partial\phi}\{X^n\sqrt{(x^2-1)}\sin\phi\} = (n+1)X^{n+1} + nX^{n-1} - (2n+1)xX^n;$$

from which the sequence equation follows by integration.

Also $$(x^2-1)\frac{\partial X^n}{\partial x}=n(xX^n-X^{n-1}),$$

and therefore (7) is true because the integrands on the two sides of the equation are equal.

Mehler's Form

$$P_n(\cos\theta)=\frac{2}{\pi}\int_0^\theta\frac{\cos(n+\frac{1}{2})\phi\,d\phi}{\sqrt{\{2(\cos\phi-\cos\theta)\}}}.$$

Here $$\frac{\partial}{\partial\phi}2\sin(n+\tfrac{1}{2})\phi\sqrt{(\cos\phi-\cos\theta)}$$

$$=[(2n+1)\cos(n+\tfrac{1}{2})\phi(\cos\phi-\cos\theta)$$
$$-\sin\phi\sin(n+\tfrac{1}{2})\phi]/\sqrt{(\cos\phi-\cos\theta)}$$

$$=[(n+\tfrac{1}{2})\{\cos(n-\tfrac{1}{2})\phi+\cos(n+\tfrac{3}{2})\phi\}-(2n+1)\cos\theta\cos(n+\tfrac{1}{2})\phi$$
$$-\tfrac{1}{2}\{\cos(n-\tfrac{1}{2})\phi-\cos(n+\tfrac{3}{2})\phi\}]/\sqrt{(\cos\phi-\cos\theta)}$$

$$=[(n+1)\cos(n+\tfrac{3}{2})\phi+n\cos(n-\tfrac{1}{2})\phi$$
$$-(2n+1)\cos\theta\cos(n+\tfrac{1}{2})\phi]/\sqrt{(\cos\phi-\cos\theta)};$$

the integration of which gives the sequence equation.

To deal with differential coefficients we may transform by using $\sin\frac{1}{2}\phi=\sin\frac{1}{2}\theta\sin\psi$, which gives

$$P_n=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\cos(n+\tfrac{1}{2})\phi\sec\tfrac{1}{2}\phi\,d\psi \quad \text{..........(9)},$$

and thence

$$P_{n-1}-P_{n+1}=\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\{\cos n\phi-\cos(n+1)\phi\}\,d\psi.$$

But $$\frac{\partial\phi}{\partial\theta}\cos\tfrac{1}{2}\phi=\cos\tfrac{1}{2}\theta\sin\psi,$$

therefore $$(1-x)\frac{\partial\phi}{\partial x}=-\tan\tfrac{1}{2}\phi,$$

and so

$$(1-x)(P'_{n-1}-P'_{n+1})$$

$$=\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\{n\sin n\phi-(n+1)\sin(n+1)\phi\}\tan\tfrac{1}{2}\phi\,d\psi$$

$$=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\{n\cos(n-\tfrac{1}{2})\phi+(n+1)\cos(n+\tfrac{3}{2})\phi-(2n+1)\cos(n+\tfrac{1}{2})\phi\}\sec\tfrac{1}{2}\phi\,d\psi$$

$$=nP_{n-1}+(n+1)P_{n+1}-(2n+1)P_n=-(2n+1)(1-x)P_n,$$

that is $(2n+1)P_n=P'_{n+1}-P'_{n-1}$, which is (3).

Schläfli's Form is the circuit integral

$$P_n(z) = \frac{1}{2\pi\iota}\int_C \frac{(t^2-1)^n dt}{2^n(t-z)^{n+1}}.$$

Here $$(z^2-1)P_n' = \frac{n+1}{2\pi\iota}\int_C \frac{(z^2-1)(t^2-1)^n dt}{2^n(t-z)^{n+2}},$$

or as $$z^2-1 = t^2-1-2z(t-z)-(t-z)^2,$$

$$(z^2-1)P_n' = \frac{n+1}{2\pi\iota}\int_C \left[\frac{(t^2-1)^{n+1}}{2^n(t-z)^{n+2}} - \frac{2z(t^2-1)^n}{2^n(t-z)^{n+1}} - \frac{(t^2-1)^n}{2^n(t-z)^n}\right] dt$$

$$= 2(n+1)(P_{n+1}-zP_n) - \frac{n+1}{2\pi\iota}\int_C \frac{(t^2-1)^n dt}{2^n(t-z)^n} \;\ldots(10).$$

But

$$\frac{\partial}{\partial t}\frac{(t^2-1)^{n+1}}{(n+1)2^{n+1}(t-z)^{n+1}} = \frac{\{(t-z)+z\}(t^2-1)^n}{2^n(t-z)^{n+1}} - \frac{(t^2-1)^{n+1}}{2^{n+1}(t-z)^{n+2}}$$

gives on integration

$$0 = \frac{1}{2\pi\iota}\int_C \frac{(t^2-1)^n dt}{2^n(t-z)^n} + zP_n - P_{n+1} \;\ldots\ldots\ldots(11),$$

which with (10) yields $(z^2-1)P_n' = (n+1)(P_{n+1}-zP_n)$, the relation (6).

Also differentiating (11) with regard to z, we have

$$0 = nP_n + \frac{d}{dz}(zP_n - P_{n+1}) \quad\text{or}\quad (n+1)P_n = P'_{n+1} - zP_n',$$

the relation (4).

§ 3. Schläfli's form is clearly suggested by that of Rodriguez, of which in effect it constitutes the generalisation when n is not integral. For the formula of Rodriguez equations (4) and (3) are readily proved, viz.

$$P'_{n+1} = \left(\frac{d}{dz}\right)^{n+2}\frac{(z^2-1)^{n+1}}{2^{n+1}(n+1)!} = \left(\frac{d}{dz}\right)^{n+1}\frac{z(z^2-1)^n}{2^n\, n!}$$

$$= z\left(\frac{d}{dz}\right)^{n+1}\frac{(z^2-1)^n}{2^n\, n!} + (n+1)\left(\frac{d}{dz}\right)^n\frac{(z^2-1)^n}{2^n\, n!}$$

$$= zP_n' + (n+1)P_n, \text{ which is (4).}$$

Also

$$P'_{n+1} = \left(\frac{d}{dz}\right)^n\left(\frac{d}{dz}\right)^2\frac{(z^2-1)^{n+1}}{2^{n+1}(n+1)!}$$

$$= \left(\frac{d}{dz}\right)^n\left[\frac{(2n+1)(z^2-1)^n}{2^n\, n!} + \frac{(z^2-1)^{n-1}}{2^{n-1}(n-1)!}\right] = (2n+1)P_n + P'_{n-1},$$

which is (3).

Lastly, the infinite series in powers of $\frac{1}{2}(z-1)$ is readily determined by the relations (2) and (3). For if

$$P_n = \sum_r {}_nA_r (z-1)^r 2^{-r},$$

and we compare the coefficients of the r^{th} power on the two sides of (2), writing $z^2 - 1 = 4 \times \frac{1}{2}(z-1)\{\frac{1}{2}(z-1)+1\}$, we get

$$2(2n+1)\{r_n A_r + (r-1)_n A_{r-1}\} = n(n+1)({}_{n+1}A_r - {}_{n-1}A_r) \ldots (12).$$

Also (3) gives

$$\tfrac{1}{2}r({}_{n+1}A_r - {}_{n-1}A_r) = (2n+1)\,{}_nA_{r-1} \;\ldots\ldots\ldots (13);$$

and on multiplication

$$r\{r_n A_r + (r-1)_n A_{r-1}\} = n(n+1)\,{}_nA_{r-1},$$

or

$$r^2{}_nA_r = (n-r+1)(n+r)\,{}_nA_{r-1} \;\ldots\ldots\ldots (14).$$

This with ${}_nA_0 = 1$, as representing $P_n(1) = 1$, gives the correct value, viz., ${}_nA_r = \binom{n}{r} \times \binom{n+r}{r}$, which is readily shown to agree with (13) and therefore with (12).

NOTE ON THE m^{th} COMPOUND OF A DETERMINANT OF THE $(2m)^{\text{th}}$ ORDER.

By *Sir Thomas Muir, LL.D.*

1. It is a familiar fact that the k^{th} compound of a determinant of the n^{th} order is an exact power of the latter, the index of the power being $C_{n-1,k-1}$: in symbols,

$$||a_{1n}||_k = |a_{1n}|^{C_{n-1,k-1}}.$$

The object of the present note is to give a generalisation of the special case of this where $n = 2m$ and $k = m$.

2. When m is 2 the determinant to be considered is

$$\begin{vmatrix} |a_1 b_2| & |a_1 b_3| & |a_1 b_4| & |a_2 b_3| & |a_2 b_4| & |a_3 b_4| \\ |a_1 c_2| & |a_1 c_3| & |a_1 c_4| & |a_2 c_3| & |a_2 c_4| & |a_3 c_4| \\ |b_1 c_2| & |b_1 c_3| & |b_1 c_4| & |b_2 c_3| & |b_2 c_4| & |b_3 c_4| \\ |d_1 e_2| & |d_1 e_3| & |d_1 e_4| & |d_2 e_3| & |d_2 e_4| & |d_3 e_4| \\ |f_1 g_2| & |f_1 g_3| & |f_1 g_4| & |f_2 g_3| & |f_2 g_4| & |f_3 g_4| \\ |h_1 k_2| & |h_1 k_3| & |h_1 k_4| & |h_2 k_3| & |h_2 k_4| & |h_3 k_4| \end{vmatrix}.$$

Calling this Q we readily see that

$$\left|\begin{array}{cccccc} |a_3\ b_4| & -|a_2\ b_4| & |a_2\ b_3| & |a_1\ b_4| & -|a_1\ b_3| & |a_1\ b_2| \\ |a_3\ c_4| & -|a_2\ c_4| & |a_2\ c_3| & |a_1\ c_4| & -|a_1\ c_3| & |a_1\ c_2| \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ |h_3\ k_4| & -|h_2\ k_4| & |h_2\ k_3| & |h_1\ k_4| & -|h_1\ k_3| & |h_1\ k_2| \end{array}\right| = -Q,$$

and that therefore by multiplication we have

$$-Q^2 = \left|\begin{array}{cccccc} & \cdot & \cdot & |a_1b_2d_3e_4| & |a_1b_2f_3g_4| & |a_1b_2h_3k_4| \\ & \cdot & \cdot & |a_1c_2d_3e_4| & |a_1c_2f_3g_4| & |a_1c_2h_3k_4| \\ & \cdot & \cdot & |b_1c_2d_3e_4| & |b_1c_2f_3g_4| & |b_1c_2h_3k_4| \\ |d_1e_2a_3b_4| & |d_1e_2a_3c_4| & |d_1e_2b_3c_4| & \cdot & |d_1e_2f_3g_4| & |d_1e_2h_3k_4| \\ |f_1g_2a_3b_4| & |f_1g_2a_3c_4| & |f_1g_2b_3c_4| & |f_1g_2d_3e_4| & \cdot & |f_1g_2h_3k_4| \\ |h_1k_2a_3b_4| & |h_1k_2a_3c_4| & |h_1k_2b_3c_4| & |h_1k_2d_3e_4| & |h_1k_2f_3g_4| & \cdot \end{array}\right|.$$

As, however, the determinant here on the right resolves itself into two factors that are conjugates one of the other, the next form of our equality is

$$-Q^2 = -\left|\begin{array}{ccc} |a_1\ b_2\ d_3\ e_4| & |a_1\ b_2\ f_3\ g_4| & |a_1\ b_2\ h_3\ k_4| \\ |a_1\ c_2\ d_3\ e_4| & |a_1\ c_2\ f_3\ g_4| & |a_1\ c_2\ h_3\ k_4| \\ |b_1\ c_2\ d_3\ e_4| & |b_1\ c_2\ f_3\ g_4| & |b_1\ c_3\ h_3\ k_4| \end{array}\right|^2,$$

and therefore finally the interesting result in condensation

$$\left|\begin{array}{cccccc} |a_1\ b_2| & |a_1\ b_3| & |a_1\ b_4| & |a_2\ b_3| & |a_2\ b_4| & |a_3\ b_4| \\ |a_1\ c_2| & \cdots & \cdots & \cdots & \cdots & \cdots \\ |b_1\ c_2| & \cdots & \cdots & \cdots & \cdots & \cdots \\ |d_1\ e_2| & \cdots & \cdots & \cdots & \cdots & \cdots \\ |f_1\ g_2| & \cdots & \cdots & \cdots & \cdots & \cdots \\ |h_1\ k_2| & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right|$$

$$= \left|\begin{array}{ccc} |a_1\ b_2\ d_3\ e_4| & |a_1\ b_2\ f_3\ g_4| & |a_1\ b_2\ h_3\ k_4| \\ |a_1\ c_2\ d_3\ e_4| & |a_1\ c_2\ f_3\ g_4| & |a_1\ c_2\ h_3\ k_4| \\ |b_1\ c_2\ d_3\ e_4| & |b_1\ c_2\ f_3\ g_4| & |b_1\ c_2\ h_3\ k_4| \end{array}\right|.$$

3. When in this we put $e = c$, the right-hand side becomes

$$-|a_1\ b_2\ c_3\ d_4| \cdot \left|\begin{array}{cc} |a_1\ c_2\ f_3\ g_4| & |a_1\ c_2\ h_3\ k_4| \\ |b_1\ c_2\ f_3\ g_4| & |b_1\ c_2\ h_3\ k_4| \end{array}\right|;$$

when in addition we put $g = b$ it becomes

$$-|a_1\ b_2\ c_3\ d_4| \cdot |a_1\ b_2\ c_3\ f_4| \cdot |b_1\ c_2\ h_3\ k_4|;$$

and when finally we put

$$f,\ h,\ k \equiv d,\ a,\ d,$$

it becomes $$-|a_1\ b_2\ c_3\ d_4|^3,$$

and we have reached the equality regarding $||a_1\ b_2\ c_3\ d_4||_2$, the second compound of $|a_1\ b_2\ c_3\ d_4|$, namely,

$$||a_1\ b_2|\ \ |a_1\ c_3|\ \ |a_1\ d_4|\ \ |b_1\ c_3|\ \ |b_2\ d_4|\ \ |c_3\ d_4|| = |a_1\ b_2\ c_3\ d_4|^3.$$

4. In reference to the higher cases there has first to be noted the mode of formation of the determinant which is the generalisation of the m^{th} compound. Taking the case where m is 3 and the given determinant is

$$|a_1\ b_2\ c_3\ d_4\ e_5\ f_6|,$$

we first form in order all the 3-line minors of the first 5-by-6 array of $|a_1\ b_2\ c_3\ d_4\ e_5\ f_6|$, namely,

$$\begin{array}{cccc} |a_1\ b_2\ c_3| & \cdots & \cdots & |a_4\ b_5\ c_6| \\ \cdots & \cdots & \cdots & \cdots \\ |c_1\ d_2\ e_3| & \cdots & \cdots & |c_4\ d_5\ e_6|, \end{array}$$

and then form the 3-line minors of any ten 3-by-6 arrays namely,

$$\begin{array}{cccc} |\alpha_1\ \beta_2\ \gamma_3| & \cdots & \cdots & |\alpha_4\ \beta_5\ \gamma_6| \\ \cdots & \cdots & \cdots & \cdots \\ |\xi_1\ \eta_2\ \zeta_3| & \cdots & \cdots & |\xi_4\ \eta_5\ \zeta_6|. \end{array}$$

The square array obtained by writing the latter oblong array of 3-line minors after the former is the array of the compound determinant required. The formation of the condensed equivalent of this is then

$$\left|\begin{array}{ccc} |a_1\ b_2\ c_3\ \alpha_4\ \beta_5\ \gamma_6| & \cdots & |a_1\ b_2\ c_3\ \xi_4\ \eta_5\ \zeta_6| \\ \cdots & \cdots & \cdots \\ |c_1\ d_2\ e_3\ \alpha_4\ \beta_5\ \gamma_6| & \cdots & |c_1\ d_2\ e_3\ \xi_4\ \eta_5\ \zeta_6| \end{array}\right|.$$

Rondebosch, S.A.

6 5

ON SOME SERIES WHOSE n^{th} TERM INVOLVES THE NUMBER OF CLASSES OF BINARY QUADRATICS OF DETERMINANT $-n$.

By *L. J. Mordell*, Birkbeck College, London.

LET $F(n)$ be the number of uneven classes of binary quadratics of determinant $-n$, and $G(n)$ the total number of classes, with the convention that the classes $(a, 0, a)$ and $(2a, a, 2a)$ are reckoned as $\frac{1}{2}$ and $\frac{1}{3}$ respectively, and that $F(0)=0$, $G(0)=-\frac{1}{12}$. For some time past it has seemed to me that the study of the arithmetic functions $F(n)$ and $G(n)$ would be facilitated by the study of the function

$$\chi(\omega)=\sum_1^\infty F(n)\,q^n,$$

where $q=e^{\pi i\omega}$, from the point of view of the theory of the modular functions. For this purpose it is necessary to find a relation between $\chi(-1/\omega)$ and $\chi(\omega)$. In this way I was led to my paper* on "The value of the definite integral

$$\int_{-\infty}^{\infty}\frac{e^{at^2+bt}}{e^{ct}+d}\,dt",$$

in which I gave without proof the formulæ

$$\int_{-\infty}^{\infty}\frac{te^{\pi i\omega t^2}}{e^{2\pi t}-1}\,dt=-2\sum_1^\infty F(n)\,q^n+\frac{2\sqrt{(-i\omega)}}{\omega^2}\sum_1^\infty F(n)\,q_1^{\,n}+\tfrac{1}{4}\theta_{00}^{\,3}(0,\omega).(1),$$

$$\int_{-\infty}^{\infty}\frac{te^{\pi i\omega t^2}}{e^{2\pi t}+1}\,dt=\sum_1^\infty(-1)^n F(4n-1)\,q^{\frac{1}{4}(4n-1)}$$
$$+\frac{2\sqrt{(-i\omega)}}{\omega^2}\sum_1^\infty(-1)^{n-1}F(n)\,q_1^{\,n}\ldots(2),$$

where $q_1=e^{-\pi i/\omega}$, the radicals are taken with a positive real part, and as usual†

$$\theta_{00}(0,\omega)=1+2q+2q^4+2q^9+\ldots\,.$$

* *Quarterly Journal of Pure and Applied Mathematics*, vol. xlviii., p. 329, referred to hereafter as *Q.J.*

† When there is no doubt as to the argument ω, the abbreviation θ_{00} will be used.

Equation (1) is a relation between $\chi(-1/\omega)$ and $\chi(\omega)$, which apart from the object I had in view, is of a type which seems to be of interest in itself.

These formulæ were deduced from the identity *

$$\omega\theta_{11}(x, \omega)\int_{-\infty}^{\infty}\frac{e^{\pi i\omega t^2-2\pi tx}}{e^{2\pi t}-1}\,dt = f\left(\frac{x}{\omega}, -\frac{1}{\omega}\right)+i\omega f(x, \omega)\ldots(3),$$

where the path of integration may be taken as either the real axis indented by the lower half of a small circle described about the origin as centre, or as a straight line parallel to the real axis and below it at a distance less than unity. Also $f(x)$ is the integral function of x defined by

$$if(x) = \sum_{n\text{ odd}}^{\pm\infty}\frac{(-1)^{\frac{1}{2}(n-1)}q^{\frac{1}{4}n^2}e^{n\pi ix}}{1+q^n},$$

and is of a type which has already been considered in connection with class-relation formulæ.*

If we differentiate equation (3) twice with respect to x and then put $x = 0$, we have

$$-4\pi\omega\theta_{11}'\int_{-\infty}^{\infty}\frac{e^{\pi i\omega t^2}t}{e^{2\pi t}-1}\,dt = \frac{1}{\omega^2}f''\left(0, -\frac{1}{\omega}\right)+i\omega f''(0, \omega)\ldots(3a),$$

where $$if''(0, \omega) = \sum_{n\text{ odd}}^{\pm\infty}\frac{-\pi^2n^2(-1)^{\frac{1}{2}(n-1)}q^{\frac{1}{4}n^2}}{1+q^n},$$

and, as usual, $$\theta_{11}' = \left[\frac{d}{dx}\theta_{11}(x, \omega)\right]_{x=0}.$$

Another form for $f''(0, \omega)$ can be found from a formula in my paper† [putting $g = h = 1$, $\xi = 0$ in the formula on the top of p. 123, and changing the notation by calling now $\phi(x)$ the function there called $f(x)$], namely,

$$\frac{\phi(x)}{\theta_{00}(x)}\theta_{11}(x) = \sum_{n\text{ odd}}^{\pm\infty}\frac{-i(-1)^{\frac{1}{2}(n-1)}q^{\frac{1}{4}n^2}e^{n\pi ix}}{1-q^n}+\frac{i\theta_{00}^2}{2}\frac{\theta_{10}(x)}{\theta_{01}(x)}\ldots(4),$$

where $\phi(x)$ is the integral function of x,

$$\phi(x) = \sum_{n}^{\pm\infty}\frac{q^{n^2}e^{2n\pi ix}}{1+q^{2n}}.$$

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* *Q.J.*, p. 332.

† See my paper "On class-relation formulæ" in the *Messenger of Mathematics* vol. xlvi.

Also

$$\left[\frac{d\phi(x)}{dx}\right]_{x=0}=\phi'(0)=-4\pi i\theta_{00}\sum_{M=1}^{\infty}(-1)^M F(M)q^M\ldots(4a).$$

Differentiating equation (4) twice with respect to x and putting $x=0$, we have

$$\frac{2\phi'(0)\theta_{11}'}{\theta_{00}}=\sum_{n\text{ odd}}^{\pm\infty}\frac{\pi^2 i n^2(-1)^{\frac{1}{2}(n-1)}q^{\frac{1}{4}n^2}}{1-q^n}+\frac{i\theta_{00}^2}{2}\frac{d^2}{dx^2}\left[\frac{\theta_{10}(x)}{\theta_{00}(x)}\right]_{x=0}.$$

But*

$$\frac{\theta_{10}(x)}{\theta_{00}(x)}=\sqrt{(k)}\frac{\operatorname{cn} v}{\operatorname{dn} v}$$

if $v=\pi\theta_{00}^2(x)$, $\sqrt{k}=\theta_{10}/\theta_{00}$, $\sqrt{k'}=\theta_{01}/\theta_{00}$, and since

$$\operatorname{cn} v=1-\frac{v^2}{2}+\ldots,\quad \operatorname{dn} v=1-\frac{k^2v^2}{2}+\ldots,$$

$$\frac{i\theta_{00}^2}{2}\frac{d^2}{dx^2}\left[\frac{\theta_{10}(x)}{\theta_{00}(x)}\right]_{x=0}=\frac{i\theta_{00}^2}{2}\sqrt{(k)}\,\pi^2\theta_{00}^4(k^2-1)$$

$$=\tfrac{1}{2}(-\pi^2 i)\,\theta_{00}\theta_{10}\theta_{01}^4.$$

Hence

$$-8\pi i\theta_{11}'\sum_{M=1}^{\infty}(-1)^M F(M)q^M$$

$$=\sum_{n\text{ odd}}^{\pm\infty}\frac{\pi^2 i n^2(-1)^{\frac{1}{2}(n-1)}q^{\frac{1}{4}n^2}}{1-q^n}-\tfrac{1}{2}\pi^2 i\theta_{00}\theta_{10}\theta_{01}^4.$$

Changing now ω into $\omega+1$ so that q becomes $-q$, this equation becomes

$$-8\pi i\theta_{11}'\sum_{M=1}^{\infty}F(M)q^M=\sum_{n\text{ odd}}^{\pm\infty}\frac{\pi^2 i n^2(-1)^{\frac{1}{2}(n-1)}q^{\frac{1}{4}n^2}}{1-q^n}-\tfrac{1}{2}\pi^2 i\theta_{01}\theta_{10}\theta_{00}^4.$$

Hence the expression for $f''(0,\omega)$ becomes

$$f''(0,\omega)=-8\pi i\theta_{11}'\sum_{M=1}^{\infty}F(M)q^M+\tfrac{1}{2}\pi^2 i\theta_{01}\theta_{10}\theta_{00}^4.$$

Since $\theta_{11}'=\pi\theta_{00}\theta_{01}\theta_{10}$, $\theta_{11}'(-1/\omega)=-i\omega\sqrt{(-i\omega)}\,\theta_{11}'$, etc., equation (3a) becomes

$$-4\pi\omega\theta_{11}'\int_{-\infty}^{\infty}\frac{e^{\pi i\omega t^2}t\,dt}{e^{2\pi t}-1}=A,$$

where

$$A=8\pi\omega\theta_{11}'\sum_{M=1}^{\infty}F(M)q^M-\tfrac{1}{2}\pi^3\omega\theta_{01}\theta_{10}\theta_{00}^4$$

$$-\frac{8\pi}{\omega}\sqrt{(-i\omega)}\,\theta_{11}'\sum_{M=1}^{\infty}F(M)q_1^M-\tfrac{1}{2}\pi^2\omega\theta_{01}\theta_{10}\theta_{00}^4,$$

* *Weber, Algebra*, iii., pp. 138, 139.

so that

$$\int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2} t\,dt}{e^{2\pi t}-1} = -2 \sum_{M=1}^{\infty} F(M)\, q^M$$

$$+ \frac{2\sqrt{(-i\omega)}}{\omega^2} \sum_{M=1}^{\infty} F(M)\, q_1^{\,M} + \tfrac{1}{4}\theta_{00}^{\,3}(0,\ \omega) \ldots (1).$$

This equation can also be proved by starting from a different form of equation (3), namely,*

$$\omega\theta_{00}(x,\ \omega) \int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2 - 2\pi t x}}{e^{2\pi t}-1}\, dt$$

$$= i\omega \sum_{\pm\infty}{}' \frac{q^{n^2} e^{2n\pi i x}}{1-q^{2n}} - i \sum_{\pm\infty}{}' \frac{q_1^{\,n^2} e^{2n\pi i x/\omega}}{1-q_1^{\,2n}} - ix - \frac{i}{2}(1-\omega),$$

where the Σ' denotes that $n=0$ is omitted from the summation; but the right-hand side of equation (1) does not then suggest itself so naturally.

For the proof of equation (2) we start from the identity†

$$\theta_{90}(x,\ \omega) \int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2 - 2\pi t x}}{e^{2\pi t}+1}\, dt$$

$$= \frac{i}{\omega} \sum_{n}^{\pm\infty} \frac{q_1^{\,n^2} e^{2n\pi i x/\omega}}{1+q_1^{\,2n}} - i \sum_{n}^{\pm\infty} \frac{q^{n^2-\frac{1}{4}} e^{(2n-1)\pi i x}}{1-q^{2n-1}}.$$

Differentiating with respect to x and putting $x=0$, we have

$$-2\pi\theta_{00} \int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2} t\,dt}{e^{2\pi t}+1} = \frac{-2\pi}{\omega^2} \sum_{n}^{\pm\infty} \frac{n q_1^{\,n^2}}{1+q_1^{\,2n}} + \pi \sum_{n}^{\pm\infty} \frac{(2n-1)\, q^{n^2-\frac{1}{4}}}{1-q^{2n-1}}.$$

But from equation (4*a*), noting the value of $\theta_{00}(0,\ -1/\omega)$,

$$\sum_{n}^{\pm\infty} \frac{n q_1^{\,n^2}}{1+q_1^{\,2n}} = -2\sqrt{(-i\omega)}\,\theta_{00} \sum_{M=1}^{\infty} (-1)^M F(M)\, q_1^{\,M},$$

and

$$\sum_{n}^{\pm\infty} \frac{(2n-1)\, q^{n^2-\frac{1}{4}}}{1-q^{2n-1}} = \sum_{1}^{\infty} \frac{(2n-1)\, q^{n^2-\frac{1}{4}}}{1-q^{2n-1}} + \sum_{1}^{\infty} \frac{(1-2n)\, q^{(1-n)^2-\frac{1}{4}}}{1-q^{1-2n}}$$

$$= 2\sum_{0}^{\infty} \frac{(2n+1)\, q^{\frac{1}{4}(2n+1)(2n+3)}}{1-q^{2n+1}}.$$

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* This is easily proved by the method used in the *Q.J.*, p. 332.

† *Q.J.*, p. 333.

But by a result due to Hermite*

$$\theta_{00}\sum_0^\infty(-1)^n F(4n+3)q^{\frac{1}{4}(4n+3)}=\sum_0^\infty\frac{(2n+1)q^{\frac{1}{4}(2n+1)(2n+3)}}{1-q^{2n+1}}.$$

Hence

$$\int_{-\infty}^{\infty}\frac{e^{\pi i\omega t^2}t\,dt}{e^{2\pi t}+1}=\frac{2\sqrt{(-i\omega)}}{\omega^2}\sum_{M=1}^{\infty}(-1)^{M-1}F(M)q_1^{M}$$

$$-\sum_{M=0}^{\infty}(-1)^{M}F(4M+3)q^{\frac{1}{4}(4M+3)}\ldots(2).$$

Another result of this kind may be found from equation (3) by differentiating for x and then putting $x=-\frac{1}{2}$. This gives

$$2\pi\omega\theta_{10}\int_{-\infty}^{\infty}\frac{e^{\pi i\omega t^2+\pi t}t\,dt}{e^{2\pi t}-1}=\frac{1}{\omega}f'\left(\frac{-1}{2\omega},\ -\frac{1}{\omega}\right)+i\omega f'(-\tfrac{1}{2},\ \omega).$$

From the series for $f(x,\ \omega)$ we have

$$i\omega f'(-\tfrac{1}{2},\ \omega)=\pi\omega\sum_{n\text{ odd}}^{\pm\infty}\frac{nq^{\frac{1}{4}n^2}}{1+q^n}$$

$$=2\pi\omega\theta_{10}\sum_{M=0}^{\infty}(-1)^M[4F(M)-3G(M)]q^M$$

from a known formula†

Also writing $\omega=-1/\Omega$, $q_1=e^{\pi i\Omega}$

$$\frac{1}{2\pi\omega^2\theta_{10}}f'\left(-\frac{1}{2\omega},\ -\frac{1}{\omega}\right)=\frac{\Omega^2}{2\pi\sqrt{(-i\Omega)}\,\theta_{01}(0,\ \Omega)}f'(\tfrac{1}{2}\Omega,\ \Omega)$$

$$=\frac{\Omega^2}{2\sqrt{(-i\Omega)}\,\theta_{01}(0,\ \Omega)}\sum_{n\text{ odd}}^{\pm\infty}\frac{(-1)^{\frac{1}{2}(n-1)}nq_1^{\frac{1}{4}n^2+\frac{1}{2}n}}{1+q_1^{n}}$$

from the series for $f(x,\ \omega)$. Changing n into $2n+1$ the right-hand side becomes

$$\frac{\Omega^2}{2\sqrt{(-i\Omega)}\,\theta_{01}(0,\ \Omega)}\sum_{n}^{\pm\infty}\frac{(-1)^n(2n+1)q_1^{\frac{1}{4}(2n+1)(2n+3)}}{1+q_1^{2n+1}}$$

$$=\frac{\sqrt{(-i\omega)}}{\omega^2}\sum_0^\infty F(4M+3)q_1^{\frac{1}{4}(4M+3)}$$

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* See, for example, H. J. S. Smith, *Collected Works*, i., p. 340, line 3, and change ω into $\omega+1$.

† *Loc. cit.*, my paper "On class-relation formulæ", pp. 129 and 132 (changing q into $-q$).

on noting the result mentioned as due to Hermite. Hence

$$\int_{-\infty}^{\infty} \frac{e^{\pi i\omega t^2+\pi t}t\,dt}{e^{2\pi t}-1} = \sum_{M=0}^{\infty} (-1)^M [4F(M) - 3G(M)] q^M$$

$$+ \frac{\sqrt{(-i\omega)}}{\omega^2} \sum_{M=0}^{\infty} F(4M+3) q_1^{\frac{1}{4}(4M+3)} \ldots\ldots (5).$$

From the equations of this kind indentities can be deduced in which no integrals occur. Taking equation (2), for example, and putting $\omega = i\xi^2$, where ξ is real and positive, we have

$$\int_{-\infty}^{\infty} \frac{e^{-\pi\xi^2 t^2} t\,dt}{e^{2\pi t}+1} = -\frac{2}{\xi^3} \sum_{M=1}^{\infty} (-1)^{M-1} F(M) e^{-\pi M/\xi^2}$$

$$- \sum_{M=0}^{\infty} (-1)^M F(4M+3) e^{-\frac{1}{4}\pi\xi^2(4M+3)}.$$

Multiplying throughout by $\xi^3 e^{-\pi a^2\xi^2}$ where a is positive, and integrating for ξ between the limits 0 and ∞, and noting that if c and d are positive, the integral

$$\int_0^{\infty} e^{-c\xi^2 - d/\xi^2} d\xi = \tfrac{1}{2}\sqrt{(\pi/c)}\, e^{-2\sqrt{(cd)}},$$

we find

$$a\int_{-\infty}^{\infty} \frac{t}{(t^2+a^2)^2}\,\frac{dt}{e^{2\pi t}+1} = -2\pi^2 \sum_{M=1}^{\infty} (-1)^{M-1} F(M) e^{-2a\pi\sqrt{M}}$$

$$- \sum_{M=0}^{\infty} \frac{a(-1)^M F(4M+3)}{[a^2+\frac{1}{4}(4M+3)]^2}.$$

The inversion in the order of integration and the integration of the series are easily justified.* We note that a series of the form

$$\sum_{M=1}^{\infty} \frac{F(aM+b)}{(M+c)^2},$$

where a, b and c are constants (c not a negative integer), converges absolutely since†

$$\sum_{M=1}^{n} F(M) = \frac{\pi}{6} n^{3/2} + Hn,$$

* Bromwich, *Theory of Infinite Series*, pp. 456, 453, 454.

† Bachmann, *Zahlentheorie*, iii., p. 463, where note that the convention about the form $(a, 0, a)$ being reckoned as $\frac{1}{2}$ instead of 1 is not adopted. This does not affect the result above, since the number of such forms included in the summation is $\leq \sqrt{n}$.

where H is a quantity which is finite for all values of n, and

$$\frac{1}{(M+c)^2} - \frac{1}{(M+1+c)^2} = O\left(\frac{1}{M^3}\right).$$

The series $\sum_{M=1}^{\infty} \frac{G(aM+b)}{(M+c)^2}$ (which is used later on) also converges absolutely, since $F(M)/G(M)$ is finite for all values of M.*

The integral

$$I = \int_{-\infty}^{\infty} \frac{at}{(t^2+a^2)^2} \frac{dt}{e^{2\pi t}+1}$$

is of a type which is well known in the theory of the Gamma Function†, and can be evaluated as follows. We have

$$4iI = \int_{-\infty}^{\infty} \left[\frac{1}{(t-ai)^2} - \frac{1}{(t+ai)^2}\right] \frac{dt}{e^{2\pi t}+1}.$$

Writing $-t$ for t in the first part of the integral

$$4iI = \int_{-\infty}^{\infty} \frac{1}{(t+ai)^2} \frac{e^{2\pi t}dt}{e^{2\pi t}+1} - \int_{-\infty}^{\infty} \frac{1}{(t+ai)^2} \frac{dt}{e^{2\pi t}+1}$$

$$= -2\int_{-\infty}^{\infty} \frac{1}{(t+ai)^2} \frac{dt}{e^{2\pi t}+1}.$$

If the real part of a is positive, we deform the path of integration into the upper half of an infinite circle described about the origin as centre. The poles of the integrand within this contour are at $t=(n+\frac{1}{2})i$ where $n=0, 1, 2, \dots$. Hence

$$2I = -\sum_{n=0}^{\infty} \frac{1}{(a+\frac{1}{2}+n)^2},$$

so that

$$\sum_{n=0}^{\infty} \frac{1}{(a+\frac{1}{2}+n)^2} = 4\pi^2 \sum_{M=1}^{\infty} (-1)^{M-1} F(M) e^{-2a\pi\sqrt{M}}$$
$$+ 2a \sum_{M=0}^{\infty} \frac{(-1)^M F(4M+3)}{[a^2+\frac{1}{4}(4M+3)]^{\frac{3}{2}}} \quad \dots (6),$$

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* This follows from the formulæ (see, for example, H. J. S. Smith, *Collected Works*, i., p. 323)

$$F(4n) = 2F(n),\quad G(4n) = G(n) + F(4n),$$
$$G(n) = F(n) \text{ if } n \equiv 1,\ 2 \pmod 4,$$
$$G(n) = \tfrac{4}{3}F(n) \text{ if } n \equiv 3 \pmod 8,$$
$$G(n) = 2F(n) \text{ if } n \equiv 7 \pmod 8,$$

so that the only case that need be examined is when $n = 4^r N$, where N is not divisible by 4. The result then follows from the first pair of formulæ.

† Cf. Lindelöf, *Le Calcul des Residus*, p. 87.

an expansion which, from the theory of analytical continuation, holds if the real part of a is positive.

From equation (5) we find in exactly the same way

$$\frac{1}{2a^2}+\sum_1^\infty \frac{(-1)^n}{(n+a)^2}=-2\pi^2\sum_{M=0}^\infty F(4M+3)\,e^{-a\pi\sqrt{(4M+3)}}$$
$$+\,2a\sum_{M=0}^\infty (-1)^M\frac{[4F(M)-3G(M)]}{(a^2+M)^2}\quad\ldots(7).$$

To deduce a result of this type from equation (1) we note the well-known expansion*

$$\theta_{00}^{\;3}=\sum_{M=0}^\infty [24F(M)-12G(M)]\,q^M,$$

and we find

$$\frac{1}{2a^2}+\sum_1^\infty \frac{1}{(n+a)^3}=-4\pi^2\sum_{M=1}^\infty F(M)\,e^{-2a\pi\sqrt{M}}$$
$$+\,2a\sum_{M=0}^\infty \frac{[4F(M)-3G(M)]}{(a^2+M)^3}\quad\ldots(8).$$

If, however, we use the theorem that

$$\theta_{00}(0,\,-1/\omega)=\sqrt{(-i\omega)}\,\theta_{00}(0,\,\omega),$$

we find that

$$\frac{1}{2a^2}+\sum_1^\infty \frac{1}{(n+a)^3}=2\pi^3\sum_{M=0}^\infty [4F(M)-3G(M)]\,e^{-2a\pi\sqrt{M}}$$
$$-\,4a\sum_{M=1}^\infty \frac{F(M)}{(a^2+M)^2}\quad\ldots(9).$$

It may be noted that series of the type $\Sigma f(n)\,e^{-a\sqrt{n}}$ are well known in connection with questions† concerning the order of magnitude of $\sum_{n=1}^{M} f(n)$. As is obvious from the way the results (6, 7, 8, 9) were found, similar series for an arithmetical function $f(n)$ can be found by putting

$$\chi(\omega)=\Sigma f(n)\,q^n,$$

and expressing $\chi(-1/\omega)$ as a power series (except for a factor ω^λ) in q.

* See, for example, my paper "Note on class relation formulæ", *Messenger of Mathematics* (1915), p. 78.

† See, for example, Hardy, "On Dirichlet's divisor problem", *Proc. Lond. Math. Soc.*, series 2, vol. xv., 1916; and "On the expression of a number as the sum of two squares", *Quart. Journ.*, vol. xlvi., 1915.

A NOTE ON A THEOREM OF RIEMANN'S.

By *Grace Chisholm Young.*

§ 1. RIEMANN* shewed in the course of his proof of his Third Theorem on Trigonometrical Series that, at any point at which the series

$$\Omega = \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\} = \sum_{n=1}^{\infty} u_n$$

converges, the series

$$V_\alpha = \sum_{n=1}^{\infty} \left\{\frac{\sin n\alpha}{n\alpha}\right\}^2 u_n$$

converges to a value which, when $\alpha \to 0$, tends to the sum of the Ω-series as limit.

As to the points where the Ω-series does not converge, it has been shewn† that, at a point where the upper and lower functions $U(x)$ and $L(x)$ of the Ω-series are finite, the limits of the sum of the V_α-series lie between

$$\tfrac{1}{2}\{U(x) + L(x)\} \pm \tfrac{1}{2}\{U(x) - L(x)\}(1 + c),$$

where c is a certain constant, for which the least value hitherto obtained is $\dfrac{1}{\pi} + \dfrac{1}{\pi^2}$.

In words, *the limits of the sum of the V_α-series lie in the interval $\{L(x), U(x)\}$, elongated at each end by a certain fraction $\frac{1}{2}c$ of the length of that interval.*

I propose to shew that this fraction is less than $\dfrac{1}{\pi^2}$.

§ 2. The method of proof, which is elementary, enables us to state the following general theorem:

THEOREM. *If*

$$\Omega \equiv \sum_{n=0}^{\infty} u_n(x), \quad V_\alpha \equiv \sum_{n=0}^{\infty} \chi(n\alpha)\, u_n(x),$$

* B. Riemann, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe", 1854, *Abh. Ges. W.*, p. 246.

† To the references in Hobson's "Theory of functions of a real variable", p. 732, should be added Hassenfelder, "Zur theorie der trigonometrischen Reihe", 1900, *Jahresber. d. k. Gymn. zu Strassburg*, p. 6.

where $\chi(\phi)$ *is a positive function, having the following properties:*

(i) *It is continuous at the origin, with the value unity there;*

(ii) *it tends to zero as limit, when* $\phi \to \infty$;

(iii) *the points on the right of the origin at which* $\chi(\phi)$ *is a maximum or minimum are such that the distance between any two of these stationary points has a positive* (>0) *lower bound;*

(iv) *denoting these stationary points by* $\varpi_1, \varpi_2, \ldots,$

$$\sum_{\rho=1}^{\infty} \chi(\varpi_\rho)$$

is a convergent series, then the limits, when $\alpha \to 0$, *of the sum of the* V_α*-series lie between*

$$\tfrac{1}{2}\{U(x)+L(x)\} \pm \tfrac{1}{2}\{U(x)-L(x)\}(1+c),$$

where $U(x)$ *and* $L(x)$ *are the upper and lower functions of the* Ω*-series and*

$$\tfrac{1}{2}c \leq \sum_{\rho=1}^{\infty} \chi(\varpi_\rho).$$

§ 3. In the special case of § 1, we have

$$\chi(\phi) = \left(\frac{\sin\phi}{\phi}\right)^2,$$

$$\chi(0) = 1.$$

It is at once evident that this $\chi(\phi)$ satisfies the requirements (i) and (ii) of § 2. That it also satisfies (iii) and (iv) results from the following special cases of these conditions:

(iii a) *The minimum value of* $\chi(\phi)$ *is always zero, and is assumed on the right of the origin at the points* $\varpi_1 = \pi$, $\varpi_3 = 2\pi$, $\varpi_5 = 3\pi$, *and so on; moreover, denoting by* ϖ_{2r} *the point between* $r\pi$ *and* $(r+1)\pi$, *at which* $\chi(\phi)$ *is a maximum, we have, for all values of* ρ,

$$\varpi_\rho - \varpi_{\rho-1} > \tfrac{1}{4}\pi;$$

and (iv a)

$$\sum_{r=1}^{\infty} \chi(\varpi_{2r}) = \sum_{\rho=1}^{\infty} \chi(\varpi_\rho) < \frac{1}{\pi^2}.$$

§ 4. To prove (iii a) and (iv a), we differentiate as usual to determine the stationary values, and obtain for a maximum on the right of the origin

$$\varpi_{2r} = \tan \varpi_{2r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Thus $\tan \varpi_{2r}$ is, like ϖ_{2r}, positive, and therefore

$$r\pi < \varpi_{2r} < (r+\tfrac{1}{2})\pi \quad \text{..................(2)}.$$

Considering the acute angle $(r+\frac{1}{2})\pi - \varpi_{2r}$, we have

$$(r+\tfrac{1}{2})\pi - \varpi_{2r} < \tan\{(r+\tfrac{1}{2})\pi - \varpi_{2r}\} = \cot \varpi_{2r},$$

whence, since by (1) and (2),

$$r\pi < \tan \varpi_{2r} < (r+\tfrac{1}{2})\pi \quad \text{..................(3)},$$

we have
$$(r+\tfrac{1}{2})\pi - \varpi_{2r} < \frac{1}{r\pi} \quad \text{..................(4)},$$

which proves that ϖ_{2r} is very nearly $(r+\frac{1}{2})\pi$ and that (iii a) holds.

Also by (4) and (3)

$$(r+\tfrac{1}{2})\pi - \frac{1}{r\pi} < \varpi_{2r} < (r+\tfrac{1}{2})\pi \quad \text{............(5)},$$

and therefore, squaring and adding unity to each member,

$$r(r+1)\pi^2 + \frac{\pi^2}{4} - 2\left\{1+\frac{1}{2r}\right\}$$
$$+\frac{1}{r^2}\pi^2 + 1 < 1 + \varpi_{2r}^2 < r(r+1)\pi^2 + \frac{\pi^2}{4} + 1,$$

whence, *a fortiori*,

$$r(r+1)\pi^2 < 1 + \varpi_{2r}^2 < r(r+1)\pi^2 + \tfrac{1}{4}\pi^2 + 1 \quad \text{......(6)}.$$

But the maximum values being given by

$$\chi(\varpi_{2r}) = \left(\frac{\sin \varpi_{2r}}{\varpi_{2r}}\right)^2 = (\cos \varpi_{2r})^2 = 1/(1+\varpi_{2r}^2) \text{.....(7)},$$

we see that $\sum\limits_{\rho=1}^{\infty} \chi(\varpi_{2r})$ is convergent, and

$$\sum_{r=1}^{\infty} \chi(\varpi_{2r}) = \sum_{r=1}^{\infty} 1/(1+\varpi_{2r}^2) < \frac{1}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{r(r+1)} < \frac{1}{\pi^2} \text{...(8)}.$$

Thus (iv a) as well as (iii a) has been verified.

We see moreover that the bound assigned to the fraction $\frac{1}{2}c$ in § 1, namely,

$$\tfrac{1}{2}c < \frac{1}{\pi^2}$$

is verified, provided the general theorem is proved.

§ 5. Returning then to the proof of the general theorem, we write in the usual way

$$\lambda(x) = \tfrac{1}{2}\{U(x) + L(x)\} \quad \ldots\ldots\ldots\ldots\ldots (9),$$

$$\mu(x) = \tfrac{1}{2}\{U(x) - L(x)\} \ldots\ldots\ldots\ldots\ldots (10)$$

at a point x where the upper and lower functions $U(x)$ and $L(x)$ of the series $\sum_{r=0}^{\infty} u_r(x)$ are finite. Then

$$s_n(x) - \sum_{r=0}^{n} u_r(x) = \lambda(x) + \beta_n(x) + \mu(x) + \eta_n(x) \ldots (11),$$

where, for all values of n and x,

$$|\beta_n(x)| \leq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

and, for all values of n greater than a certain value $n_{x,\delta}$, depending on x and on δ,

$$|\eta_n(x)| < \delta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

§ 6. Hence, denoting by $V_{n,a}$ the n^{th} partial summation of the series

$$V_a = \sum_{r=1}^{\infty} \chi(r\alpha)\, u_r(x),$$

we may write

$$V_{n,a} = \sum_{r=0}^{n} v_{r,a} = \sum_{r=0}^{n} \chi(r\alpha)\, u_r = s_0 + \sum_{r=1}^{n} (s_r - s_{r-1})\, \chi(r\alpha)$$

$$= \lambda + \beta_0\mu + \eta_0 + \sum_{r=1}^{n} \chi(r\alpha)\, \{(\beta_r - \beta_{r-1})\, \mu + \eta_r - \eta_{r-1}\}$$

$$= \lambda + \sum_{r=0}^{n-1} (\beta_r\mu + \eta_r)\, [\chi(r\alpha) - \chi\{(r+1)\alpha\}]$$

$$+ \chi(n\alpha)\, \{\beta_n\mu + \eta_n\} \quad \ldots\ldots (14).$$

Now by hypothesis (i), $\chi(\phi) \to 1$ when $\phi \to 0$. Therefore each of these term, *for fixed index* r, tends to zero as $\alpha \to 0$. Thus, m denoting any fixed number, and ζ_m being written for the sum of the first m terms,

$$\zeta_m \to 0, \quad (\alpha \to 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Taking $$m > n_{x,\delta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16),$$

we have then by (13) and (14),

$$V_{n,a} = \lambda + \zeta_m + (\mu + \delta) \sum_{r=m}^{n-1} \theta_r [\chi(r\alpha) - \chi\{(r+1)\alpha\}]$$

$$+ (\mu + \delta)\, \theta_n \chi(n\alpha) \ldots (17),$$

where for all values of $n \geq m$,

$$|\theta_n| \leq 1 \quad \text{.......................(18).}$$

§ 7. Now $\chi(r\alpha) - \chi\{(r+1)\alpha\}$ remains as of the same sign as long as the points $r\alpha$ and $(r+1)\alpha$ have between them no stationary points of the function $\chi(\phi)$. Thus, supposing ϖ_ρ and $\varpi_{\rho+1}$ to be consecutive stationary points of $\chi(\phi)$, and

$$(s-1)\alpha < \varpi_\rho \leq s\alpha < t\alpha < \varpi_{\rho+1} \leq (t+1)\alpha,$$

$$\left|\sum_{r=s}^{r=t} \theta_r [\chi(r\alpha) - \chi\{(r+1)\alpha\}]\right|$$

$$\leq \left|\sum_{r=s}^{r=t} [\chi(r\alpha) - \chi\{(r+1)\alpha\}]\right| \leq |\chi(s\alpha) - \chi(t\alpha)|.$$

Thus we may replace our summation of these terms by a single term, say

$$\theta_s\{\chi(s\alpha) - \chi(t\alpha)\},$$

or, as we may say, we may *obliterate* the points intermediate between $s\alpha$ and $t\alpha$.

§ 8. Now, since by hypothesis (iii) the distance between any two stationary points of $\chi(\phi)$ is greater than a certain fixed positive quantity, we may arrange these points in order from left to right, and we may suppose α so small that the interval $\{(s-1)\alpha, s\alpha\}$, which contains the point ϖ_ρ, does not contain any other stationary point. We then write

$$\chi(s\alpha) - \chi\{(s-1)\alpha\} = \{\chi(s\alpha) - \chi(\varpi_\rho)\} + [\chi(\varpi_\rho) - \chi\{(s-1)\alpha\}].$$

By the preceding argument we may then obliterate the point $\{(s-1)\alpha\}$, replacing it, in the summation of terms with indices less than s, by $\varpi_{\rho-1}$. Similarly in the subsequent summation we may obliterate the point $s\alpha$ and replace it by $\varpi_{\rho+1}$. In doing this we have distributed the term

$$\theta_{s-1}[\chi\{(s-1)\alpha\} - \chi(s\alpha)].$$

Similarly we distribute the term

$$\theta_t[\chi(t\alpha) - \chi\{(t+1)\alpha\}].$$

Doing this, we find we have obliterated all the points $r\alpha$ between the stationary points ϖ_ρ and $\varpi_{\rho+1}$, and we have replaced this part of our summation by a single term, which we may write

$$\theta_\rho\{\chi(\varpi_\rho) + \chi(\varpi_{\rho+1})\},$$

since it is numerically

$$\leq \{\chi(\varpi_\rho) - \chi(\varpi_{\rho+1})\} \leq \chi(\varpi_\rho) + \chi(\varpi_{\rho+1}).$$

§ 9. Doing this for all indices ρ, and supposing α so small that

$$m\alpha < \varpi_1,$$

we have by (17)

$$V_{n,\alpha} = \lambda + \zeta_m + (\mu+\delta)\theta_m \chi(m\alpha) + (\mu+\delta)\theta\, 2 \sum_{\rho=1}^{\infty} \chi(\varpi_\rho)$$
$$- (\mu+\delta)\,\theta_{n-1}\chi(n\alpha) + (n+\delta)\,\theta_n \chi(n\alpha) \ldots (19).$$

Now let $n \to \infty$: then since, by hypothesis (ii),

$$\chi(n\alpha) \to 0, \quad (n \to \infty),$$

we get in the limit

$$V_\alpha = \lambda + \zeta_m + (\mu+\delta)\theta_m \chi(m\alpha) + (\mu+\delta)\theta\, 2 \sum_{\rho=1}^{\infty} \chi(\varpi_\rho) \ldots (20).$$

Now letting $\alpha \to 0$, $\chi(m\alpha) \to 1$, by hypothesis (i), thus the part of V_α involving δ may be neglected, since, like δ, it is as small as we please. All the limits of V_α, as $\alpha \to 0$, lie therefore between

$$\lambda \pm \mu \{1 + 2 \sum_{\rho=1}^{\infty} \chi(\varpi_\rho)\},$$

which proves the theorem.

THE TWELVE ELLIPTIC FUNCTIONS RELATED TO SIXTEEN DOUBLY PERIODIC FUNCTIONS OF THE SECOND KIND.

By *E. T. Bell.*

§ 1. GLAISHER* on several occasions has emphasized the advantages of working with the full set of twelve elliptic functions constructed similarly to the special subsets of three functions considered by Abel and Jacobi. Particularly in arithmetical work, where the algebraic calculations must be

* *Messenger of Mathematics*, xi., 81–95, 120–138 (1881–82); xv., 92–148 (1885); xvi., 67–86 (1886); xvii., 1–18 (1887); xviii., 1–84 (1888); *Acta Mathematica*, xxvi., 241–248 (1902).

carried through to the end, these advantages are strikingly apparent in the gain of symmetry.

In this note I propose to show how the twelve elliptic functions arise as degenerate forms, corresponding to zero values of either variable, from what Hermite called the doubly periodic functions of the second kind. The latter functions are not, of course, doubly periodic at all. Hermite's nomenclature, however, is well established, and we shall adhere to it. The ϑ_a notation is that of H. J. S. Smith, which coincides with Jacobi's except that the latter's ϑ is replaced by ϑ_0.

The series written out in § 6 for the twelve elliptic functions are readily seen to be identical, but for notation, with those given by Glaisher in the second of his papers quoted. Nevertheless the form in which they are given here is suggestive in connection with Liouville's numerous writings on the theory of numbers, and they afford a check on the series for the doubly periodic functions of the second kind from which they are here deduced. Regarding the latter series it may be mentioned that equivalent lists for some or all of them in the literature seemed to be marred by misprints, so that for purposes of reference only they are useless. As almost invariably the computations by which the series were derived are suppressed, it frequently is troublesome to run down the errors, which seldom appear until the series are used in numerical work. However, as the calculations by Hermite's* method are all straightforward, I have not thought it necessary to reproduce any of the work, giving only the final forms which have been checked by constant use, and which, therefore, I trust are accurate.

These series for the doubly periodic functions of the second kind are of much use in the theory of numbers. In a paper which will appear elsewhere, it is shown that from them may be inferred the numerous striking theorems which Liouville published without proof in a series of eighteen memoirs, "Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres" (*Journal des Math.*, 1858–65). It is significant in this connection to note that equivalent forms of these series were most probably well known to Liouville from Jacobi's memoir and notes on the theory of rotation, from which they may be derived without much difficulty.

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* Sur quelques applications des fonctions elliptiques, Paris, *Comptes Rendus*, 1877–82; *Oeuvres*, iii., 266–418; especially section viii. The series for the functions of the second kind are reproduced with corrections in *Oeuvres*, iv., but the final form, *ibid.*, pp. 199, 200, still contains misprints.

§ 2. The four theta functions of Jacobi are

$$\vartheta_0(x) = 1 + 2\Sigma(-1)^n q^{n^2} \cos 2nx,$$
$$\vartheta_1(x) = 2\Sigma(-1|m) q^{m^2/4} \sin mx,$$
$$\vartheta_2(x) = 2\Sigma q^{m^2/4} \cos mx,$$
$$\vartheta_3(x) = 1 + 2\Sigma q^{n^2} \cos 2nx,$$

where the summations are with respect to all positive integers $n = 1, 2, 3, \ldots$, or to all odd positive integers $m = 1, 3, 5, \ldots$, and $(-1|m)$ is Jacobi's extension of Legendre's symbol. Throughout the paper n, m retain this significance. We shall use also the following notation for pairs of conjugate divisors of m or n. Any divisor of either m or n whose conjugate τ is odd is denoted by t. Hence if $m = t\tau$, both t and τ are odd; if $n = 2^a m$, $m = d\delta$, where d, δ are conjugate divisors of m, we write $n = t\tau$; whence $t = 2^a d$, $\tau = \delta$, whether $\alpha = 0$ or $\alpha > 0$. In the notation $n = d\delta$, no restrictions of evenness or oddness are imposed on either d or δ. In any series such as

$$\sec x + 4\Sigma q^n [\Sigma(-1|\tau) \cos(\tau x + 2ty)],$$

the first Σ refers to all values of the exponent of q consistent with the notation above explained; the coefficient of the power of q is in [ ], and the second Σ refers to all divisors, of the type indicated by the notation, of the m or n in the exponent of q. Note particularly that the second Σ refers to the divisors of n, and not of $2n$, when the exponent is $2n$. The constants are

$$\vartheta_a \equiv \vartheta_a(0), \ (\alpha = 0, 2, 3); \quad \vartheta_1' = \vartheta_0 \vartheta_2 \vartheta_3.$$

§ 3. Writing

$$\phi_{ijk}(x, y) = \vartheta_1' \vartheta_i(x+y) / \vartheta_j(x) \vartheta_k(y),$$

we have
$$\phi_{ijk}(x, y) = \phi_{ikj}(y, x);$$

and if x', y' denote $x + \pi/2$, $y + \pi/2$ respectively, the following:

$$\phi_{001}(y, x) = \phi_{010}(x, y), \quad \phi_{331}(y, x) = \phi_{313}(x, y),$$
$$\phi_{212}(y, x) = \phi_{221}(x, y), \quad \phi_{302}(y, x) = \phi_{320}(x, y),$$
$$\phi_{203}(y, x) = \phi_{230}(x, y), \quad \phi_{032}(y, x) = \phi_{023}(x, y);$$
$$\phi_{001}(x', y) = \phi_{331}(x, y);$$
$$\phi_{100}(x, y') = \phi_{203}(x, y), \quad \phi_{001}(x, y') = \phi_{302}(x, y),$$
$$\phi_{111}(x, y') = \phi_{212}(x, y);$$
$$\phi_{111}(x', y') = -\phi_{122}(x, y), \quad \phi_{001}(x', y') = \phi_{032}(x, y),$$
$$\phi_{100}(x', y') = -\phi_{133}(x, y).$$

In this way we generate from the three functions

$$\phi_{100}(x, y), \quad \phi_{001}(x, y), \quad \phi_{111}(x, y),$$

the set of sixteen, disregarding signs,

$$\begin{array}{llll} \phi_{001}(x, y), & \phi_{010}(x, y), & \phi_{023}(x, y), & \phi_{032}(x, y), \\ \phi_{100}(x, y), & \phi_{111}(x, y), & \phi_{122}(x, y), & \phi_{133}(x, y), \\ \phi_{203}(x, y), & \phi_{212}(x, y), & \phi_{221}(x, y), & \phi_{230}(x, y), \\ \phi_{302}(x, y), & \phi_{313}(x, y), & \phi_{320}(x, y), & \phi_{331}(x, y). \end{array}$$

The symmetry of this table, either by rows or columns, is evident. The sixteen functions thus defined are all of the doubly periodic functions of the second kind, which were considered by Hermite. For passing to the elliptic functions, the table is to be read either by rows or columns, the latter being preferable, as shown in a moment. We have not considered changes of q into $-q$; but it may be remarked that it is not difficult, by suitably combining interchanges of x, y into x', y' with changes of q into $-q$, to generate the entire set of sixteen from any one of them. Moreover, except for signs, the set of sixteen is invariant under the group generated by these operations. Thus, if the series for any one of the sixteen is known, those for the rest may be derived from it very simply. This method affords a valuable check on other derivations, notably Hermite's, in which two of the sixteen functions are taken as fundamental. Again, it is of considerable interest otherwise; for, as pointed out in § 7, from the sixteen functions of the second kind may be derived those of the third. Hence any one of the sixteen functions may be taken as the point of departure for developing the entire theory of the doubly periodic functions of the first, second, and third kinds, the first kind, or the elliptic functions, being derived as shown next.

§ 4. From the table in § 3 we deduce the twelve elliptic functions by putting either x or $y = 0$. In either case four of the functions, marked by asterisks in the tables following, become infinite. Putting $x = 0$, and expressing the resulting $\phi_{ijk}(0, y)$ in terms of the thetas, we have, in the same order as in the table of § 3,

$$\begin{array}{llll} \vartheta_2\vartheta_3\vartheta_0(y)/\vartheta_1(y), & *, & \vartheta_3\vartheta_0\vartheta_0(y)/\vartheta_3(y), & \vartheta_0\vartheta_2\vartheta_0(y)/\vartheta_2(y), \\ \vartheta_2\vartheta_3\vartheta_1(y)/\vartheta_0(y), & *, & \vartheta_3\vartheta_0\vartheta_1(y)/\vartheta_2(y), & \vartheta_0\vartheta_2\vartheta_1(y)/\vartheta_3(y), \\ \vartheta_2\vartheta_3\vartheta_2(y)/\vartheta_3(y), & *, & \vartheta_3\vartheta_0\vartheta_2(y)/\vartheta_1(y), & \vartheta_0\vartheta_2\vartheta_2(y)/\vartheta_0(y), \\ \vartheta_2\vartheta_3\vartheta_3(y)/\vartheta_2(y), & *, & \vartheta_3\vartheta_0\vartheta_3(y)/\vartheta_0(y), & \vartheta_0\vartheta_2\vartheta_3(y)/\vartheta_1(y); \end{array}$$

and similarly, for $y=0$,

$$\begin{array}{llll} * & , \vartheta_2\vartheta_3\vartheta_0(x)/\vartheta_1(x), & \vartheta_0\vartheta_2\vartheta_0(x)/\vartheta_2(x), & \vartheta_3\vartheta_0\vartheta_0(x)/\vartheta_3(x), \\ \vartheta_2\vartheta_3\vartheta_1(x)/\vartheta_0(x), & * & , \vartheta_0\vartheta_3\vartheta_1(x)/\vartheta_2(x), & \vartheta_0\vartheta_2\vartheta_1(x)/\vartheta_3(x), \\ \vartheta_0\vartheta_2\vartheta_2(x)/\vartheta_0(x), & \vartheta_3\vartheta_0\vartheta_2(x)/\vartheta_1(x), & * & , \vartheta_2\vartheta_3\vartheta_2(x)/\vartheta_3(x), \\ \vartheta_0\vartheta_3\vartheta_3(x)/\vartheta_0(x), & \vartheta_0\vartheta_2\vartheta_3(x)/\vartheta_1(x), & \vartheta_2\vartheta_3\vartheta_3(x)/\vartheta_2(x), & * \quad . \end{array}$$

In the first of these tables the three functions in any row have the same zeros; in the second, the three functions in any row have the same zeros, and the three in any column the same poles. To complete the identification with the twelve elliptic functions we write

$$u=2Kx/\pi, \quad v=2Ky/\pi, \quad \rho=2K/\pi,$$

and recall that

$$\begin{aligned} \rho &= \vartheta_3^2 \quad , \quad \mathrm{sn}\, z=\vartheta_3\vartheta_1(z)/\vartheta_2\vartheta_0(z), \\ k &= \vartheta_2^2/\vartheta_3^2, \quad \mathrm{cn}\, z=\vartheta_0\vartheta_2(z)/\vartheta_2\vartheta_0(z), \\ k' &= \vartheta_0^2/\vartheta_3^2, \quad \mathrm{dn}\, z=\vartheta_0\vartheta_3(z)/\vartheta_3\vartheta_0(z), \end{aligned}$$

where $z=u$ or v. Whence the twelve theta quotients in the first of the tables above are equivalent, in the same order, to

$$\begin{array}{llll} \rho\,\mathrm{ns}\, v, & *, & k'\rho\,\mathrm{nd}\, v, & k'\rho\,\mathrm{nc}\, v, \\ k\rho\,\mathrm{sn}\, v, & *, & k'\rho\,\mathrm{sc}\, v, & kk'\rho\,\mathrm{sd}\, v, \\ k\rho\,\mathrm{cd}\, v, & *, & \rho\,\mathrm{cs}\, v, & k\rho\,\mathrm{cn}\, v, \\ \rho\,\mathrm{dc}\, v, & *, & \rho\,\mathrm{dn}\, v, & \rho\,\mathrm{ds}\, v; \end{array}$$

and similarly the second table is

$$\begin{array}{llll} * \quad , & \rho\,\mathrm{ns}\, u, & k'\rho\,\mathrm{nc}\, u, & k'\rho\,\mathrm{nd}\, u, \\ k\rho\,\mathrm{sn}\, u, & * \quad , & k'\rho\,\mathrm{sc}\, u, & kk'\rho\,\mathrm{sd}\, u, \\ k\rho\,\mathrm{cn}\, u, & \rho\,\mathrm{cs}\, u, & * \quad , & k\rho\,\mathrm{cd}\, u, \\ \rho\,\mathrm{dn}\, u, & \rho\,\mathrm{ds}\, u, & \rho\,\mathrm{dc}\, u, & * \quad . \end{array}$$

The remarkable symmetry of the last table, which read by columns gives Glaisher's grouping of the twelve functions into particular triads, is apparent. The symmetrical distribution of poles and zeros has been noted in connected with the equivalent theta form of the table.

§ 5. We shall now write down the series for the doubly periodic functions of the second kind, as stated in § 1, using the notation of § 2. The sixteen series are grouped into sets of four each, corresponding to the columns of the table in § 3.

$$n = t\tau; \quad n = d\delta; \quad m = t\tau.$$

$$\phi_{001}(x, y) = \operatorname{cosec} y \qquad + 4\Sigma q^n [\Sigma \sin (2tx + \tau y)]$$
$$\phi_{100}(x, y) = \qquad 4\Sigma q^{m/2} [\Sigma \sin (tx + \tau y)]$$
$$\phi_{203}(x, y) = \qquad 4\Sigma q^{m/2} [\Sigma (-1 \mid \tau) \cos (tx + \tau y)]$$
$$\phi_{302}(x, y) = \sec y \qquad + 4\Sigma q^n [\Sigma (-1 \mid \tau) \cos (2tx + \tau y)]$$

$$\phi_{010}(x, y) = \operatorname{cosec} x \qquad + 4\Sigma q^n [\Sigma \sin (\tau x + 2ty)]$$
$$\phi_{111}(x, y) = \cot x + \cot y + 4\Sigma q^{2n} [\Sigma \sin 2 (dx + \delta y)]$$
$$\phi_{212}(x, y) = \cot x - \tan y + 4\Sigma q^{2n} [\Sigma (-1)^\delta \sin 2 (dx + \delta y)]$$
$$\phi_{313}(x, y) = \operatorname{cosec} x \qquad + 4\Sigma q^n [(-1)^n \Sigma \sin (\tau x + 2ty)]$$

$$\phi_{023}(x, y) = \sec x \qquad + 4\Sigma q^n [(-1)^n \Sigma (-1 \mid \tau) \cos (\tau x + 2ty)]$$
$$\phi_{122}(x, y) = \tan x + \tan y - 4\Sigma q^{2n} [\Sigma (-1)^{d+\delta} \sin 2 (dx + \delta y)]$$
$$\phi_{221}(x, y) = -\tan x + \cot y + 4\Sigma q^{2n} [\Sigma (-1)^d \sin 2 (dx + \delta y)]$$
$$\phi_{320}(x, y) = \sec x \qquad + 4\Sigma q^n [\Sigma (-1 \mid \tau) \cos (\tau x + 2ty)]$$

$$\phi_{032}(x, y) = \sec y \qquad + 4\Sigma q^n [(-1)^n \Sigma (-1 \mid \tau) \cos (2tx + \tau y)]$$
$$\phi_{133}(x, y) = \qquad + 4\Sigma q^{m/2} [(-1 \mid m) \Sigma \sin (tx + \tau y)]$$
$$\phi_{230}(x, y) = \qquad + 4\Sigma q^{m/2} [\Sigma (-1 \mid \tau) \cos (\tau x + ty)]$$
$$\phi_{331}(x, y) = \operatorname{cosec} y \qquad + 4\Sigma q^n [(-1)^n \Sigma \sin (2tx + \tau y)].$$

§ 6. From the preceding list we have the following series for the elliptic functions, the asterisks, as before, marking the places of those functions that become infinite when $y = 0$:

*

$$\phi_{100}(x, 0) = k\rho \operatorname{sn} u \quad = 4\Sigma q^{m/2} [\Sigma \sin tx]$$
$$\phi_{203}(x, 0) = k\rho \operatorname{cn} u \quad = 4\Sigma q^{m/2} [\Sigma (-1 \mid \tau) \cos tx]$$
$$\phi_{302}(x, 0) = \rho \operatorname{dn} u \quad = 1 + 4\Sigma q^n [\Sigma (-1 \mid \tau) \cos 2tx]$$

$$\phi_{010}(x, 0) = \rho \operatorname{ns} u \quad = \operatorname{cosec} x + 4\Sigma q^n [\Sigma \sin \tau x]$$

*

$$\phi_{212}(x, 0) = \rho \operatorname{cs} u \quad = \cot x + 4\Sigma q^{2n} [\Sigma (-1)^\delta \sin 2dx]$$
$$\phi_{313}(x, 0) = \rho \operatorname{ds} u \quad = \operatorname{cosec} x + 4\Sigma q^n [(-1)^n \Sigma \sin \tau x]$$

$$\phi_{023}(x,\,0) = k'\rho\ \text{nc}\ u \quad = \sec x + 4\Sigma q^n\left[(-1)^n\,\Sigma\,(-1\,|\,\tau)\cos\tau x\right]$$
$$\phi_{122}(x,\,0) = k'\rho\ \text{sc}\ u \quad = \tan x - 4\Sigma q^n\left[\Sigma\,(-1)^{d+\delta}\sin 2dx\right]$$

*

$$\phi_{320}(x,\,0) = \quad \rho\ \text{dc}\ u \quad = \sec x + 4\Sigma q^n\left[\Sigma\,(-1\,|\,\tau)\cos\tau x\right]$$

$$\phi_{032}(x,\,0) = \quad k'\rho\ \text{nd}\ u = 1 + 4\Sigma q^n\left[(-1)^n\,\Sigma\,(-1\,|\,\tau)\cos 2tx\right]$$
$$\phi_{133}(x,\,0) = kk'\rho\ \text{sd}\ u = \qquad 4\Sigma q^{m/2}\left[(-1\,|\,m)\,\Sigma\sin tx\right]$$
$$\phi_{230}(x,\,0) = \quad \rho\ \text{dc}\ u = \qquad 4\Sigma q^{m/2}\left[\Sigma\,(-1\,|\,\tau)\cos\tau x\right]$$

*

§ 7. In another paper, which I trust will appear before long, I have stated about 150 expansions for the doubly periodic functions of the third kind useful in the theory of numbers, particularly in the derivation of new class number relations. These expansions can be uniformly deduced from those in § 5 by putting $y = \pm x$, applying then the transformation of the second order and adding or subtracting the results, and by similarly treating the derivatives with respect to x or y.

The series thus obtained include all those for the doubly periodic functions of the third kind given by Hermite, Biehler, Appell, and others, but in a simpler form directly applicable to arithmetic. The distinction lies in the form of the coefficients: reduction of the usual forms leads to what Hermite designated incomplete numerical functions, viz., functions of the divisors of the exponents subject to certain restrictions, such as that no divisor in the function shall exceed the square root of the exponent. Such incomplete functions are well known in arithmetic, especially in Kronecker's and other class number formulæ. In the new forms of the expansions the functions are complete, that is, involve all the divisors of the exponent precisely as in §§ 5, 6. As remarked by H. J. S. Smith, incomplete functions are inherently more complex than the complete, and related arithmetical formulæ more abstruse.

It thus appears that the doubly periodic functions of the second kind hold a central position in the entire theory, leading in one direction to elliptic functions, and in the other to functions of the third kind. On another occasion I hope to indicate briefly the origin of the series for the third kind mentioned in this note.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

LII.

On some definite integrals considered by Mellin.

1. In Note XLIX* I applied the ideas of the modern theory of integration to the pair of reciprocal integral relations

$$f(s)=\int_0^\infty \phi(x)\,x^{s-1}\,dx \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\phi(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s)\,x^{-s}\,ds \quad\ldots\ldots\ldots(2),$$

usually associated with the name of Mellin. Mellin has also considered† integrals of the type

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f_1(s)\,f_2(s)\,f_3(s)\ldots x^{-s}\,ds \quad\ldots\ldots\ldots(3),$$

where $f_1(s)$, $f_2(s)$, ... are functions of the type (1); and it is interesting to extend the results of my earlier note in this direction. I consider here in particular the case of two functions

$$f(s)=\int_0^\infty \phi(x)\,x^{s-1}\,dx,\quad g(s)=\int_0^\infty \psi(x)\,x^{s-1}\,dx \;\ldots(4).$$

2. We shall use the following lemmas.

Lemma 1. *If F, G, F^{1+p}, and $G^{1+1/p}$, where $p>0$, are summable, then*

$$\int_a^b F(x+t)\,G(t)\,dt$$

is a continuous function of t.

This is a known theorem due to W. H. Young‡.

Lemma 2. *If F, G, F^{1+p}, and $G^{1+1/p}$, where $p>0$, are summable in any finite interval, and*

$$\int_{-\infty}^\infty |F|^{1+p}\,dx,\quad \int_{-\infty}^\infty |G|^{1+1/p}\,dx$$

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* *Messenger*, vol. xlvii. (1918), pp. 178—184.

† See, *e.g.*, pp. 336—337 of his memoir in the *Math. Annalen* quoted in Note XLIX.

‡ 'On a class of parametric integrals and their application in the theory of Fourier series', *Proc. Roy. Soc.* (A), vol. lxxxv., 1911, pp. 401—414 (p. 407).

are convergent, then

$$J(T)=\frac{1}{2\pi T}\int_{-\infty}^{\infty}F(u)\,G(v)\left\{\frac{\sin\frac{1}{2}T(u+v)}{\frac{1}{2}(u+v)}\right\}^2 du\,dv$$
$$\rightarrow\int_{-\infty}^{\infty}F(x)\,G(-x)\,dx$$

when $T\rightarrow\infty$.

The lemma rests on the following formal transformations:

$$u+v=2x,\quad u-v=2y,\quad du\,dv=2dx\,dy,$$
$$J(T)=\frac{1}{\pi T}\int_{-\infty}^{\infty}F(x+y)\,G(x-y)\left(\frac{\sin Tx}{x}\right)^2 dx\,dy$$
$$=\frac{1}{\pi T}\int_{-\infty}^{\infty}\left(\frac{\sin Tx}{x}\right)^2 H(x)\,dx\rightarrow H(0),$$

where $$H(x)=\int_{-\infty}^{\infty}F(x+y)\,G(x-y)\,dy.$$

In order to justify these equations, we observe first that

$$\int_{-\infty}^{\infty}|F(x+y)\,G(x-y)|\left(\frac{\sin Tx}{x}\right)^2 dx\,dy$$
$$\leq\int_{-\infty}^{\infty}\mathrm{Min}\left(T^2,\frac{1}{x^2}\right)dx\int_{-\infty}^{\infty}|F(x+y)\,G(x-y)|\,dy$$
$$\leq\int_{-\infty}^{\infty}\mathrm{Min}\left(T^2,\frac{1}{x^2}\right)dx\left\{\int_{-\infty}^{\infty}|F(x+y)|^{1+p}dy\right\}^{1/1+p}$$
$$\times\left\{\int_{-\infty}^{\infty}|G(x-y)|^{1+1/p}dy\right\}^{p/1+p}$$
$$=\int_{-\infty}^{\infty}\mathrm{Min}\left(T^2,\frac{1}{x^2}\right)dx\left\{\int_{-\infty}^{\infty}|F|^{1+p}dx\right\}^{1/1+p}\left\{\int_{-\infty}^{\infty}|G|^{1+1/p}dx\right\}^{p/1+p}.$$

Thus all our integrals are absolutely convergent, and the transformation of $J(T)$ by substitution, and its reduction to a repeated integral, are legitimate. The integral involving $H(x)$ is of Fejér's type, and so all that remains to be proved is that $H(x)$ is continuous for $x=0$. Now

$$H(x)=\int_{-\infty}^{\infty}F(2x-t)\,G(t)\,dt=\int_{-\infty}^{a}+\int_{a}^{b}+\int_{b}^{\infty}=H_1+H_2+H_3.$$

Suppose that $-\delta\leq x\leq\delta$. Then

$$|H_1|<\left\{\int_{-\infty}^{a}|F(2x-t)|^{1+p}dt\right\}^{1/1+p}\left\{\int_{-\infty}^{a}|G(t)|^{1+1/p}dt\right\}^{p/1+p},$$

and may be made less than ϵ by choice of an a independent of x. The same argument applies to H_3; and H_2 is continuous by Lemma 1. Thus H is continuous, and Lemma 2 is proved.

3. THEOREM A. *Suppose that ϕ, ψ, ϕ^{1+p}, and $\psi^{1+1/p}$, where $p>0$, are summable in any interval $0<\xi_1\le x\le\xi_2$. Further, suppose that constants K, α, β, γ, δ exist such that $\alpha<\gamma$, $\beta<\delta$, $\alpha<\delta$, $\beta<\gamma$,*

$$|\phi(x)|<Kx^{-\alpha},\quad |\psi(x)|<Kx^{-\beta},$$

for all sufficiently small positive values of x, and

$$|\phi(x)|<Kx^{-\gamma},\quad |\psi(x)|<Kx^{-\delta}$$

for all sufficiently large positive values of x; so that the functions $f(s)$ and $g(s)$ of $s=\sigma+it$, defined by (4), are regular for

$$\alpha<\sigma<\gamma,\quad \beta<\sigma<\delta$$

respectively. Then the integral

$$J(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s)\,g(s)\,x^{-s}\,ds,$$

where $x>0$, $\alpha<a<\gamma$, $\beta<a<\delta$, is summable $(C, 1)$; and its sum is

$$\int_0^\infty \phi(w)\,\psi\left(\frac{x}{w}\right)\frac{dw}{w}.$$

Suppose that x_1 and x_2 are any two positive numbers whose product is x. Then

$$\frac{1}{2\pi i T}\int_{a-iT}^{a+iT}(T-|t|)\,f(s)\,g(s)\,x^{-s}\,ds$$

$$=\frac{1}{2\pi T}\int_{-T}^{T}(T-|t|)\,dt\int_0^\infty y^{a+it-1}\phi(x_1y)\,dy\int_0^\infty z^{a+it-1}\psi(x_2z)\,dz.$$

The triply repeated integral just written is absolutely convergent, as may be seen at once by comparison with

$$\int_{-T}^{T}dt\int_0^\infty y^{a-1}\,|\phi(x_1y)|\,dy\int_0^\infty z^{a-1}\,|\psi(x_2z)|\,dz.$$

Hence it may be written in the form

$$\frac{1}{2\pi T}\int_0^\infty y^{a-1}\phi(x_1y)\,dy\int_0^\infty z^{a-1}\psi(x_2z)\,dz\int_{-T}^{T}(T-|t|)\,(yz)^{it}\,dt$$

$$=\frac{1}{2\pi T}\int_0^\infty (yz)^{a-1}\phi(x_1y)\,\psi(x_2z)\left\{\frac{\sin(\frac{1}{2}T\log yz)}{\frac{1}{2}\log yz}\right\}^2 dy\,dz$$

$$=\frac{1}{2\pi T}\int_{-\infty}^\infty e^{a(u+v)}\phi(x_1e^u)\,\psi(x_2e^v)\left\{\frac{\sin\frac{1}{2}T(u+v)}{\frac{1}{2}(u+v)}\right\}^2 du\,dv.$$

Let $e^{au}\phi(x_1e^u)=\chi(u)$. Then, since ϕ^{1+p} is summable in any interval $0<\xi_1\le x\le\xi_2$, χ^{1+p} is summable in any finite interval. Also $|\chi(u)| = O\{e^{(a-\alpha)u}\}$ for large negative values of u, and $|\chi(u)| = O\{e^{(a-\beta)u}\}$ for large positive values; and $a-\alpha > 0$, $a - \beta < 0$. Hence

$$\int_{-\infty}^{\infty} |\chi|^{1+p}\, du$$

is convergent. Similarly, if $e^{av}\psi(x_2 e^v) = \omega(v)$,

$$\int_{-\infty}^{\infty} |\omega|^{1+1/p}\, dv$$

is convergent. Thus all the conditions of Lemma 2 are satisfied, and

$$\frac{1}{2\pi i T}\int_{a-iT}^{a+iT} (T - |t|) f(s) g(s) x^{-s}\, ds \to \int_{-\infty}^{\infty} \chi(\rho)\,\omega(-\rho)\, d\rho$$

$$= \int_{-\infty}^{\infty} \phi(x_1 e^{\rho})\,\psi(x_2 e^{-\rho})\, d\rho = \int_0^{\infty} \phi(x_1 w)\,\psi\left(\frac{x_2}{w}\right)\frac{dw}{w}$$

$$= \int_0^{\infty} \phi(w)\,\psi\left(\frac{x}{w}\right)\frac{dw}{w};$$

which proves the theorem.

4. THEOREM B. *The conclusions of Theorem A are also true if the value of the summable integral is defined as the limit of*

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\eta s^2} f(s)\, g(s)\, x^{-s}\, ds \quad (\eta > 0)$$

when η tends to zero.

It is hardly necessary to give the proof in detail. The reader may compare § 3 of Note XLIX., and the corresponding developments in Notes XLVI. and XLVII.

THEOREM C. *If the conditions of Theorem A are satisfied, and the integral is convergent, then its value is that given in Theorem A.*

This is an immediate corollary of Theorem A. To establish the convergence of the integral is (compare § 4 of Note XLIX.) essentially more difficult. In applications it can usually be recognised directly.

Examples.

5. (i) Suppose

$$\phi(x) = \frac{x^{-\alpha}}{(1+x)^{\gamma-\alpha}}, \quad \psi(x) = \frac{x^{-\beta}}{(1+x)^{\delta-\beta}},$$

$$f(s) = \frac{\Gamma(s-\alpha)\,\Gamma(\gamma-s)}{\Gamma(\gamma-\alpha)}, \quad g(s) = \frac{\Gamma(s-\beta)\,\Gamma(\delta-s)}{\Gamma(\delta-\beta)},$$

α, β, γ, δ being real. Then

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(s-\alpha)\Gamma(s-\beta)\Gamma(\gamma-s)\Gamma(\delta-s)x^{-s}ds$$
$$=\Gamma(\gamma-\alpha)\Gamma(\delta-\beta)x^{-\beta}\int_0^\infty\frac{w^{\delta-\alpha-1}\,dw}{(1+w)^{\gamma-\alpha}(w+x)^{\delta-\beta}}$$

if $\alpha<a<\gamma$, $\beta<a<\delta$. Suppose in particular that $x=1$, $a=0$, $\alpha+\gamma=\beta+\delta=0$. Then we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\Gamma(\gamma+it)|^2|\Gamma(\delta+it)|^2dt=\frac{\Gamma(2\gamma)\Gamma(2\delta)\{\Gamma(\gamma+\delta)\}^2}{\Gamma(2\gamma+2\delta)}$$
$$=\frac{1}{2\sqrt{\pi}}\frac{\Gamma(\gamma)\Gamma(\gamma+\frac{1}{2})\Gamma(\delta)\Gamma(\delta+\frac{1}{2})\Gamma(\gamma+\delta)}{\Gamma(\gamma+\delta+\frac{1}{2})},$$

a formula given by Mr. Ramanujan* and valid for all positive values of γ and δ.

(ii) Suppose

$$\phi(x)=x^{-\alpha}(1-x)^{\lambda-1}\ (0<x<1),\quad \phi(x)=0\ (x>1),$$
$$\psi(x)=0\ (0<x<1),\quad \psi(x)=x^{1-\delta-\mu}(x-1)^{\mu-1}\ (x>1).$$

In order that our conditions should be satisfied, it must be possible to choose a positive p such that

$$(1+p)(\lambda-1)>-1,\quad (1+1/p)(\mu-1)>-1;$$

and this will be so if λ and μ are positive and $\lambda+\mu>1$. We have then

$$f(s)=\frac{\Gamma(\lambda)\Gamma(s-\alpha)}{\Gamma(\lambda+s-\alpha)},\quad g(s)=\frac{\Gamma(\mu)\Gamma(\delta-s)}{\Gamma(\mu+\delta-s)}$$

and

$$\frac{1}{2\pi i}\int\frac{\Gamma(s-\alpha)\Gamma(\delta-s)}{\Gamma(\lambda+s-\alpha)\Gamma(\mu+\delta-s)}ds=\frac{1}{\Gamma(\lambda)\Gamma(\mu)}\int_0^1 w^{\delta-\alpha-1}(1-w)^{\lambda+\mu-2}dw$$
$$=\frac{\Gamma(\delta-\alpha)\Gamma(\lambda+\mu-1)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\delta-\alpha+\lambda+\mu-1)}\quad(\alpha<a<\delta).$$

Suppose in particular that

$$\alpha+\delta=0,\quad a=0,\quad \lambda=\mu=\epsilon-\delta.$$

Then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\Gamma(\delta+it)}{\Gamma(\epsilon+it)}\right|^2dt=\frac{\Gamma(2\delta)\Gamma(2\epsilon-2\delta-1)}{\{\Gamma(\epsilon-\delta)\}^2\Gamma(2\epsilon-1)}$$
$$=\frac{1}{2\sqrt{\pi}}\frac{\Gamma(\delta)\Gamma(\delta+\frac{1}{2})\Gamma(\epsilon-\delta-\frac{1}{2})}{\Gamma(\epsilon-\frac{1}{2})\Gamma(\epsilon)\Gamma(\epsilon-\delta)},$$

* S. Ramanujan, 'Some definite integrals', *Messenger*, vol. xliv. (1915), pp. 10–18 (p. 15).

where $\epsilon > \delta + \frac{1}{2} > \frac{1}{2}$. This result also agrees with that given by Mr. Ramanujan*.

(iii) Suppose

$$\phi(x) = x^{-\alpha} e^{-x}, \qquad \psi(x) = x^{-\beta} e^{-x},$$
$$f(s) = \Gamma(1-\alpha+s), \quad g(s) = \Gamma(1-\beta+s).$$

Then we obtain

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(s-\alpha)\,\Gamma(s-\beta)\, x^{-s}\, ds = x^{-\beta}\int_0^\infty e^{-w-x/w}\, w^{\beta-\alpha-1}\, dw,$$

provided $\alpha < a$, $\beta < a$. In particular, if $\alpha = \beta = 0$, we obtain

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \{\Gamma(s)\}^2 x^{-s}\, ds = \int_0^\infty e^{-w-x/w}\,\frac{dw}{w},$$

where a and x are positive. This formula (which can of course be extended at once to complex values of x) is one of importance in the analytic theory of numbers†.

6. The following example is more novel. It is clear that, if we take $\psi(x) = x^{-u}\phi(1/x)$, we have

$$g(s) = \int_0^\infty x^{s-u-1}\phi(1/x)\, dx = \int_0^\infty x^{u-s-1}\phi(x)\, dx = f(u-s).$$

If
$$\phi(x) = \sum_1^\infty e^{-n^2 x},$$
we have
$$f(s) = \Gamma(s)\sum n^{-2s} = \Gamma(s)\,\zeta(2s),$$
and

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(s)\,\Gamma(u-s)\,\zeta(2s)\,\zeta(2u-2s)\, ds = \int_0^\infty w^{u-1}\{\phi(w)\}^2 dw.$$

This formula is valid if $\frac{1}{2} < a < u - \frac{1}{2}$. Now

$$\{\phi(w)\}^2 = \tfrac{1}{4}\{\vartheta(x) - 1\}^2,$$
where
$$\vartheta(x) = 1 + 2e^{-x} + 2e^{-4x} + \ldots;$$
and so is equal to

$$\tfrac{1}{4}\{1 + \sum_1^\infty r(n)\, e^{-nx} - 2 - 4\sum_1^\infty e^{-n^2 x} + 1\} = \tfrac{1}{4}\sum_1^\infty r(n)\, e^{-nx} - \sum_1^\infty e^{-n^2 x},$$

* *Loc. cit.*, p. 12.

† See G. Voronoï, 'Sur une fonction transcendante et ses applications à la sommation de quelques séries', *Annales scientifiques de l'École Normale Supérieure*, ser. 3, vol. xxi., 1904, pp. 207–268, 459–534; G. H. Hardy and J. E. Littlewood, 'Contributions to the theory of the Riemann Zeta-function and the theory of the distribution of primes', *Acta Mathematica*, vol. xli., 1917, pp. 119–196.

where $r(n)$ is the number of representations of n as the sum of two squares. Hence we obtain the formula

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(s)\,\Gamma(u-s)\,\zeta(2s)\,\zeta(2u-2s)\,ds$$

$$=\tfrac{1}{4}\Gamma(u)\left(\sum_1^\infty\frac{r(n)}{n^u}-4\sum_1^\infty\frac{1}{n^{2u}}\right)=\Gamma(u)\,\{\zeta(u)\,\eta(u)-\zeta(2u)\},$$

where $$\eta(u)=1^{-u}-3^{-u}+5^{-u}-\dots\,.$$

7. The corresponding formula for p functions

$$f_1(s),\ f_2(s),\ \dots,\ f_p(s)$$

is

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}f_1(s)\,f_2(s)\dots f_p(s)\,x^{-s}\,ds$$

$$=\int_0^\infty\phi_1(w_1)\,\phi_2(w_2)\dots\phi_{p-1}(w_{p-1})\,\phi_p\left(\frac{x}{w_1w_2\dots w_{p-1}}\right)\frac{dw_1dw_2\dots dw_{p-1}}{w_1w_2\dots w_{p-1}}\,.$$

A NEW INTEGRAL EQUATION SATISFIED BY THE SOLUTIONS OF A CERTAIN LINEAR DIFFERENTIAL EQUATION, WHICH OCCURS IN THE THEORY OF ELECTRICAL OSCILLATIONS AND OF THE TIDES.

By *E. G. C. Poole.*

I.

LET $V=e^{ikct}\pi(\rho,\,r)$ be a solution of the equation of wave-motions $\nabla^2V-\frac{1}{c^2}\frac{\partial^2V}{\partial t^2}=0$, which is symmetrical about the axis of z and simple harmonic with respect to the time. Then π satisfies the equation

$$\frac{\partial^2\pi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\pi}{\partial\rho}+\frac{\partial^2\pi}{\partial r^2}+k^2\pi=0\ \dots\dots\dots\dots(1).$$

If we put $Q=\rho\frac{\partial\pi}{\partial\rho}$ and then pass to elliptic coordinates by the substitution

$$z=a\lambda\mu,\quad \rho=a\sqrt{\{(\lambda^2-1)(1-\mu^2)\}},$$

we find that Q satisfies the equation

$$(\lambda^2-1)\frac{\partial^2 Q}{\partial\lambda^2}+(1-\mu^2)\frac{\partial^2 Q}{\partial\mu^2}+k^2a^2(\lambda^2-\mu^2)\,Q=0\ldots(2),$$

and if Q is of the form $E(\lambda), F(\mu)$, then E, F satisfy equations of identical form

$$(1-\mu^2)\frac{\partial^2 F}{\partial\mu^2}+\beta\,(f^2-\mu^2)\,F=0\ldots\ldots\ldots\ldots(3),$$

where $\beta=k^2a^2$, and f^2 is a constant.

The equation (3) has been studied by Abraham in connection with electrical oscillations on a rod*, and some integral equations satisfied by the solution were obtained by him. If we write $\dfrac{dF}{d\mu}=\zeta$, we obtain the well-known equation, satisfied by the tide-elevation of a free tide on a rotating globe, for an ocean of uniform depth

$$\frac{d}{d\mu}\left\{\left(\frac{1-\mu^2}{f^2-\mu^2}\right)\frac{d\zeta}{d\mu}\right\}+\beta\zeta=0.$$

As the equation is thus of some physical interest it may be worth while to give a new integral equation, satisfied by its solutions, which is derived in a totally different manner from Abraham's.

II.

Bateman has given a general solution of the equation (1) in the form

$$V=\int_{-\pi}^{\pi}f(z+i\rho\cos\theta,\ ct+\rho\sin\theta)\,d\theta,$$

where f is an arbitrary function. And since V is of the form $e^{ikct}\pi(\rho, r)$, we must have

$$\pi=\int_{-\pi}^{\pi}e^{ik\rho\sin\theta}f(z+i\rho\cos\theta)\,d\theta.$$

Therefore

$$Q=\rho\frac{\partial\pi}{\partial\rho}=\int_{-\pi}^{\pi}e^{ik\rho\sin\theta}(ik\rho\sin\theta f+i\rho\cos\theta f')\,d\theta,$$

integrating by parts and introducing a new arbitrary function

$$\phi(z)\equiv\{kf(z)+\frac{1}{k}f''(z)\},$$

* Wiedemann's *Ann.*, vol. lxvi; *Math. Ann.*, vol. lii.

we have $$Q=\int_{-\pi}^{\pi} e^{ik\rho\sin\theta}\phi\,(z+i\rho\cos\theta)\,.\,i\rho\sin\theta\,d\theta\,\ldots\ldots\,(4).$$

And on changing to the variables λ, μ as above and putting $i\rho = a\sqrt{\{(1-\lambda^2)(1-\mu^2)\}}$, we find the symmetrical expression

$$Q=\int_{-\pi}^{\pi} e^{ka\sqrt{\{(1-\lambda^2)(1-\mu^2)\}}\,.\,\sin\theta}\phi\,\{a\,[\lambda\mu+\sqrt{\{(1-\lambda^2)(1-\mu^2)\}}\cos\theta]\}$$
$$\times\, a\sqrt{\{(1-\lambda^2)(1-\mu^2)\}}\sin\theta\,d\theta\,\ldots\ldots\ldots\ldots(5).$$

We now choose a new variable ψ, such that

$$\cos\psi=\lambda\mu+\sqrt{\{(1-\lambda^2)(1-\mu^2)\}}\cos\theta\,\ldots\ldots\ldots(6).$$

If λ, μ, are both real and numerically less than unity, we may write $\lambda=\cos l$, $\mu=\cos m$; then ψ is the base of a spherical triangle, whose vertical angle is θ and whose sides are l, m. We have also

$$\sqrt{\{(1-\lambda^2)(1-\mu^2)\sin\theta\}}=\sin l\sin m\sin\theta$$
$$=\sqrt{\{1-\cos^2 l-\cos^2 m-\cos^2\psi+2\cos l\cos m\cos\psi\}}$$
$$=\sqrt{\{\cos(l-m)-\cos\psi\}\{\cos\psi-\cos(l+m)\}}.$$

Now as θ increases from 0 to π, while l, m remain constant, ψ increases from $\pm(l-m)$ to $(l+m)$, and

$$\sqrt{\{(1-\lambda^2)(1-\mu^2)\}}\sin\theta d\theta=\sin\psi.d\psi.$$

Substituting, we find,

$$Q=\int_0^{\pi} 2\sinh(ka\sin l\sin m\sin\theta)\,.\,\psi(a\cos\psi)\,.\,a\sin l\sin m\sin\theta d\theta$$
$$=\int_{\cos(l+m)}^{\cos(l-m)} 2\sinh ka\sqrt{\{\cos(l-m)-x\}\{x-\cos(l+m)\}}\,.\,\phi(ax)\,.\,adx$$

on writing x instead of $\cos\psi$. Putting $k^2a^2=\beta$, and $a\phi(ax)=\Phi(x)$ we have finally

$$Q=\int_{\cos(l+m)}^{\cos(l-m)} 2\sinh\sqrt{\beta\{\cos(l-m)-x\}\{x-\cos(l+m)\}}\,.\,\Phi(x)\,.\,dn$$
$$\ldots\ldots(7).$$

III.

We will now impose some restrictions on Q and determine the corresponding form of Φ. In the first place, we shall *suppose Q to be of the form $E(\lambda)\times F(\mu)$*; so that E, F are two solutions of the equation (3). If we investigate the latter equation, we find that its exponents at the singularities $\mu=\pm1$

are zero and unity. The solution belonging to the latter exponent is holomorphic, while that belonging to the former has a logarithmic singularity at the point considered. In general, if we continue analytically the solution, which is holomorphic at $\mu=-1$, up to the point $\mu=+1$, it will not be a mere multiple of the solution which is holomorphic at the latter point. But this will be the case if a certain transcendental relation between β and f^2 is satisfied. If β is given, the analysis required to determine f^2 is identical with that for determining the possible frequencies of the free tides in an ocean of uniform depth on a rotating globe. We now suppose, as our second restriction, *that* β, f^2 *satisfy this relation.* There will then be a solution $E(\mu)$ which is holomorphic throughout the μ plane except at infinity. We see by inspection that the points $\mu=\pm 1$ are zeros of such a solution, and that it must further be either an odd or an even function of μ.

Finally, we shall suppose that Q is a multiple of $E(\lambda)E(\mu)$, so that it is holomorphic for all finite values of both variables, and vanishes when $(1-\lambda^2)(1-\mu^2)=0$. We thus assume that either

$$E(x) \equiv 1+B_2'x^2+B_4'x^4+\ldots$$
$$\equiv (1-x^2)(1+B_2x^2+B_4x^4\ldots)\quad\ldots\ldots\ldots(8a),$$

or

$$E(x) \equiv x+B_3'x^3+B_5'x^5+\ldots$$
$$\equiv (1-x^2)(x+B_3x^3+\ldots)\quad\ldots\ldots\ldots\ldots(8b).$$

The quantities on the right-hand side are integral functions of x. Let them be such that as $n^2 \to 1$,

$$\underset{x\to 1}{\mathrm{L}} \frac{E(x)}{1-x^2}=k, \qquad \underset{x\to -1}{\mathrm{L}} \frac{E(x)}{(1-x^2)}= \quad k \text{ for } (8a)$$

and

$$\underset{x\to 1}{\mathrm{L}} \frac{E(x)}{1-x^2}=k, \qquad \underset{x\to -1}{\mathrm{L}} \frac{E(n)}{1-n^2} =-k \text{ for } (8b).$$

Now let $\lambda \to 1$ in the equation (1), after division by $(1-\lambda^2)$. We find, for the left-hand side,

$$\underset{\lambda\to 1}{\mathrm{L}} \frac{Q}{1-\lambda^2}=E(\mu)\underset{\lambda\to 1}{\mathrm{L}} \frac{E(\lambda)}{(1-\lambda^2)}=kE(\mu)\quad\ldots\ldots(9),$$

while on the right-hand side we have

$$\underset{l\to 0}{\mathrm{L}} \operatorname{cosec}^2 l \int_{\cos(m-l)}^{\cos(m+l)} 2\sinh\sqrt{[\beta\{\cos(m-l)-x\}\{x-\cos(m+l)\}]}.\Phi(x).dx$$

Now as $l \to 0$ and when x lies within the range of integration

$$|\sqrt{[\beta\{\cos(m-l)-x\}\{x-\cos(m+l)\}]}| \leq |\sin l \sin m \sqrt{\beta}|.$$

Hence, since this quantity is small of the same order as l, we may replace the hyperbolic sine by its argument, with an error of order l^3. Hence

$$\underset{\lambda\to 1}{\mathrm{L}} \frac{Q}{1-\lambda^2} = \underset{l\to 0}{\mathrm{L}}\ 2\operatorname{cosec}^2 l$$

$$\int_{\cos(m+l)}^{\cos(m-l)} \sqrt{[\beta\{\cos(m-l)-x\}\{x-\cos(m+l)\}]}\,.\,\Phi(x)\,.\,dx.$$

If we suppose $\Phi(x)$ to be continuous within the range of integration, we have, by the mean value theorem,

$$\underset{\lambda\to 1}{\mathrm{L}} \frac{Q}{(1-\lambda^2)} = \Phi(\cos m)\ \underset{l\to 0}{\mathrm{L}}\ 2\operatorname{cosec}^2 l \int_{-\sin l\sin m}^{\sin l \sin m} \sqrt{\{\beta(\sin^2 l \sin^2 m - y^2)\}}\,dy$$

$$= \Phi(\mu)\,.\,\pi\sqrt{(\beta)}\,.\,(1-\mu^2) \text{..............(10).}$$

Therefore $\qquad k\,E(\mu) = \pi\sqrt{(\beta)}\,(1-\mu^2)\,\Phi(\mu)$ ....(11).

This gives the required form of Φ, and we have the final result

$$E(\lambda)\,E(\mu)$$

$$= \frac{2k}{\pi\sqrt{(\beta)}} \int_{\lambda\mu-\sqrt{\{(1-\lambda^2)(1-\mu^2)\}}}^{\lambda\mu+\sqrt{\{(1-\lambda^2)(1-\mu^2)\}}} \sinh\sqrt{\{\beta(1-\lambda^2-\mu^2-x^2+2\lambda\mu x)\}}$$

$$\times \frac{E(x)}{(1-x^2)}\,.\,dx \text{....................(12).}$$

IV.

From the general formula (12) several particular results may be derived. In the case corresponding to (8*a*), when $E(\lambda)$ is an even function, we may put $\lambda = 0$, $E(\lambda) = 1$. Then

$$E(\mu) = \frac{2k}{\pi\sqrt{(\beta)}} \int_{-\sqrt{(1-\mu^2)}}^{\sqrt{(1-\mu^2)}} \sinh\sqrt{\{\beta(1-\mu^2-x^2)\}}\,.\,\frac{E(x)}{1-x^2}\,.\,dx$$

or

$$E(\mu) = \frac{4k}{\pi\sqrt{(\beta)}} \int_0^{\sqrt{(1-\mu^2)}} \sinh\sqrt{\{\beta(1-\mu^2-x^2)}\,.\,\frac{E(x)}{(1-x^2)}\,.\,dx \text{...(13}a\text{).}$$

In the case corresponding to (8*b*) we have to differentiate with respect to λ before making $\lambda \to 0$. Since the integrand vanishes at the limits $x = \lambda\mu \pm \sqrt{\{(1-\lambda^2)(1-\mu^2)\}}$, we easily find

$$E(\mu) = \left(\frac{2k}{\pi\sqrt{(\beta)}}\right)$$

$$\times \int_{-\sqrt{(1-\mu^2)}}^{\sqrt{(1-\mu^2)}} \underset{\lambda\to 0}{\mathrm{L}} \frac{\partial}{\partial\lambda} \sinh\sqrt{\{\beta(1-\lambda^2-\mu^2-x^2+2\lambda\mu x)\}} \cdot \frac{E(x)\,dx}{(1-x^2)}$$

$$= \frac{2k}{\pi}\int_{-\sqrt{(1-\mu^2)}}^{\sqrt{(1-\mu^2)}} \frac{\cosh\sqrt{\{\beta(1-\mu^2-x^2)\}} . x\mu . E(x)}{\sqrt{(1-x^2-\mu^2)}\,(1-x^2)}\,dx.$$

Therefore

$$E(\mu) = \frac{4k}{\pi}\int_0^{\sqrt{(1-\mu^2)}} \frac{\cosh\sqrt{\{\beta(1-\mu^2-x^2)\}} . x\mu . E(x)}{\sqrt{(1-x^2-\mu^2)} . (1-x^2)}\,dx \quad \ldots(13b).$$

These are equations of Volterra type, with variable limits. They can easily be converted to Fredholm's type with limits 0 and 1 by introducing a discontinuous factor such as

$$\tfrac{1}{2}\left(1 + \frac{1-x^2-\mu^2}{|1-x^2-\mu^2|}\right),$$

which reduces to unity within the original range of integration and to zero outside it. We thus know the systems of orthogonal functions which satisfy the equations (13*a*) and (13*b*), and their properties can be proved in the ordinary manner. We shall not discuss these here, but we shall conclude by drawing attention to two special cases of the formula (12).

(i) Put $\lambda = \mu$ in (12); we find

$$E^2(\mu) = \frac{2k}{\pi\sqrt{(\beta)}}\int_{2\mu^2-1}^{1} \sinh\sqrt{\{\beta(1-x)(1+x-2\mu^2)\}} \cdot \frac{E(x)}{(1-x^2)} \cdot dx \quad \ldots\ldots(14).$$

(ii) Put $\lambda = \cos\alpha$, $\mu = \sin\alpha$, we find

$$E(\cos\alpha)\,E(\sin\alpha)$$

$$= \frac{2k}{\pi\sqrt{\beta}}\int_0^{\sin 2\alpha} \sinh\sqrt{\{\beta x(\sin 2\alpha - x)\}} \cdot \frac{E(x)}{1-x^2} \cdot dx \quad \ldots(15).$$

If we put $\lambda = \mu = 1/\sqrt{2}$, the formulæ (14), (15) become identical, and give

$$E^2\left(\frac{1}{\sqrt{2}}\right) = \frac{2k}{\pi\sqrt{\beta}}\int_0^1 \sinh\sqrt{\{\beta x(1-x)\}} \cdot \frac{E(x)}{1-x^2} \cdot dx \quad \ldots(16).$$

These integral equations, involving squares and products of the unknown function, appear to be of a somewhat unusual type.

THE TETRAHEDRON AND PENTASPHERICAL CO-ORDINATES.

By *T. C. Lewis, M.A.*

1. For investigating the properties of a tetrahedron in general the most convenient system of pentaspherical co-ordinates is that which has four of the centres of reference, A_1, A_2, A_3, A_4, at the four vertices of the figure, and the fifth, A_5, at the centre of the associated hyperboloid, of which the four perpendiculars from the vertices upon opposite faces are generating lines. It will be seen that the expressions for ρ_1^2, ρ_2^2, ..., ρ_5^2 in terms of the sides are much simplified in this system, and likewise the equations of spheres and other figures related to the tetrahedron.

2. It is known that if O is the circumcentre, and G the centre of gravity of the tetrahedron, the line OA_5 is bisected at G. Hence if a, b, c, d, e, f are the edges of the tetrahedron, and g, h, i, j the distances of A_5 from A_1, A_2, A_3, A_4 respectively, we obtain

$$g^2 = R^2 + \tfrac{1}{4}(b^2 + c^2 + d^2) - \tfrac{1}{4}(a^2 + e^2 + f^2),$$

and similarly for h^2, i^2, j^2; and therefore

$$g^2 + h^2 + i^2 + j^2 = 4R^2,$$

and

$$\left.\begin{aligned} g^2 - \tfrac{1}{2}(b^2 + c^2 + d^2) &= h^2 - \tfrac{1}{2}(a^2 + c^2 + e^2) \\ &= i^2 - \tfrac{1}{2}(a^2 + b^2 + f^2) \\ &= j^2 - \tfrac{1}{2}(d^2 + e^2 + f^2) \\ &= R^2 - \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) \end{aligned}\right\} \ldots\ldots(1).$$

Let this be K.

We also have $3\rho_5^2 = p_5 = 3R^2 - \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2)$.

Moreover, if Δ_1 is the volume of the tetrahedron $A_2A_3A_4A_5$,

and V ,, ,, ,, $A_1A_2A_3A_4$,

we have $\rho_1^2\Delta_1 + \rho_5^2 V = 0,$

and $\Delta_1 = \frac{1}{2}V -$ vol. of tetrahedron $OA_2A_3A_4$

$$= \tfrac{1}{2}V - \tfrac{1}{12}\sqrt{\{R^2(-a^4-e^4-f^4+2e^2f^2+2f^2a^2+2a^2e^2)-a^2e^2f^2\}},$$

therefore

$$2\Delta_1 V = V^2 - \tfrac{1}{6}\sqrt{\{R^2V^2(-a^4-e^4-f^4+2e^2f^2+2f^2a^2+2a^2e^2)-a^2e^2f^2V^2\}}$$
$$= V^2 - \tfrac{1}{144}\{a^2d^2(e^2+f^2-a^2)+b^2e^2(f^2+a^2-e^2)+c^2f^2(a^2+e^2-f^2)-2a^2e^2f^2\}.$$

Therefore

$$\left.\begin{aligned}1+\frac{2\rho_5^2}{\rho_1^2} &= \frac{1}{144V^2}\{a^2d^2(e^2+f^2-a^2)\\ &+b^2e^2(f^2+a^2-e^2)+c^2f^2(a^2+e^2-f^2)-2a^2e^2f^2\}\end{aligned}\right\}\ldots(2).$$

Writing $a^2 = \rho_2^2 + \rho_3^2 - 2\rho_2\rho_3c_{23}$

and similarly for all the edges, including g^2, h^2, i^2, j^2, and substituting in the expressions for g^2, h^2, i^2, j^2, ρ_5^2, we obtain

$$\rho_1\rho_2c_{12}+\rho_1\rho_3c_{13}+\rho_1\rho_4c_{14}+\rho_2\rho_3c_{23}+\rho_2\rho_4c_{24}+\rho_3\rho_4c_{34}$$
$$=3\rho_5(\rho_1c_{15}+\rho_2c_{25}+\rho_3c_{35}+\rho_4c_{45})\ldots(3),$$

which we will call $3l^2$; and the following identities are easily proved, viz.

$$\left.\begin{aligned}2l^2 &= \rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2+4\rho_5^2-4R^2\\ &= \rho_1^2+\rho_2^2+\rho_3^2+\rho_4^2-\tfrac{1}{3}(a^2+b^2+c^2+d^2+e^2+f^2)\\ &= 2\rho_1(\rho_2c_{12}+\rho_3c_{13}+\rho_4c_{14}-2\rho_5c_{15})\\ &= \rho_2\rho_3c_{23}+\rho_2\rho_4c_{24}+\rho_3\rho_4c_{34}+2\rho_1\rho_5c_{15}\end{aligned}\right\}\ldots\ldots(4),$$

and three other expressions similar to each of the last two.

It may also be noted for future use that in general, if $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the areas of the faces opposite to A_1, A_2, A_3, A_4 respectively,

$$4\alpha_1^2 = -\begin{vmatrix}1 & c_{23} & c_{24} & \frac{1}{\rho_2}\\ c_{23} & 1 & c_{34} & \frac{1}{\rho_3}\\ c_{24} & c_{34} & 1 & \frac{1}{\rho_4}\\ \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & 0\end{vmatrix}\rho_2^2\rho_3^2\rho_4^2 \ldots\ldots\ldots(5)$$

and $$36V^2 = -\begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & \frac{1}{\rho_1} \\ c_{12} & 1 & c_{23} & c_{24} & \frac{1}{\rho_2} \\ c_{13} & c_{23} & 1 & c_{34} & \frac{1}{\rho_3} \\ c_{14} & c_{24} & c_{34} & 1 & \frac{1}{\rho_4} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & 0 \end{vmatrix} \rho_1^2\rho_2^2\rho_3^2\rho_4^2$$

$$= -\begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & \frac{1}{\rho_5} \end{vmatrix} \frac{\rho_1^2\rho_2^2\rho_3^2\rho_4^2}{\rho_5}$$

$$= -D_5 \frac{\rho_1^2\rho_2^2\rho_3^2\rho_4^2}{\rho_5} \quad \text{.....................(6)},$$

where D_5 is the last written determinant.

We will also write D_1 for

$$\begin{vmatrix} c_{12} & 1 & c_{23} & c_{24} & c_{25} \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} \\ c_{15} & c_{25} & c_{35} & c_{45} & 1 \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & \frac{1}{\rho_5} \end{vmatrix},$$

and it follows that $\rho_1 D_1 = \rho_5 D_5$.

Moreover, if

$$\Sigma\beta_h x_h \equiv \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & \frac{1}{\rho_5} \end{vmatrix}.$$

Then by (4) $\beta_1\rho_1+\beta_2\rho_2+\beta_3\rho_3+\beta_4\rho_4-2\beta_5\rho_5$

$$\begin{aligned}
&=\beta_1\left(\rho_1 \quad +\rho_2c_{12}+\rho_3c_{13}+\rho_4c_{14}-2\rho_5c_{15}-\frac{l^2}{\rho_1}\right)\\
&+\beta_2\left(\rho_1c_{12}+\rho_2 \quad +\rho_3c_{23}+\rho_4c_{24}-2\rho_5c_{25}-\frac{l^2}{\rho_2}\right)\\
&+\beta_3\left(\rho_1c_{13}+\rho_2c_{23}+\rho_3 \quad +\rho_4c_{34}-2\rho_5c_{35}-\frac{l^2}{\rho_3}\right)\\
&+\beta_4\left(\rho_1c_{14}+\rho_2c_{24}+\rho_3c_{34}+\rho_4 \quad -2\rho_5c_{45}-\frac{l^2}{\rho_4}\right)\\
&+\beta_5\left(\rho_1c_{15}+\rho_2c_{25}+\rho_3c_{35}+\rho_4c_{45}-2\rho_5 \quad -\frac{l^2}{\rho_5}\right)\\
&=\rho_1D_5+2\rho_5D_1\\
&=\frac{D_5}{\rho_1}(\rho_1^2+2\rho_5^2) \quad \text{.......................................(7).}
\end{aligned}$$

We shall also have

$$\beta_1\rho_1-\beta_5\rho_5=-4\alpha_1^2\frac{\rho_1\rho_5}{\rho_2^2\rho_3^2\rho_4^2} \quad \text{............(8).}$$

3. The equation to the circumsphere may be written in the form

$$\begin{vmatrix}
\rho_1^2+\rho_1x_1 & \rho_2^2+\rho_2x_2 & \rho_3^2+\rho_3x_3 & \rho_4^2+\rho_4x_4 & \rho_5^2+\rho_5x_5\\
0 & c^2 & b^2 & d^2 & g^2\\
c^2 & 0 & a^2 & e^2 & h^2\\
b^2 & a^2 & 0 & f^2 & i^2\\
d^2 & e^2 & f^2 & 0 & j^2
\end{vmatrix}=0.$$

If we (i) divide the columns by $\rho_1^2, \rho_2^2, \rho_3^2, \rho_4^2, \rho_5^2$ and add for the last column, which we then multiply throughout by $\frac{2}{3}K$, and (ii) subtract the sum of the first four members of every row from the double of the last member in order to make the last column, it appears that the last column may be replaced (i) by $2K$ for every member, and (ii) by

$$2\rho_5^2+2\rho_5x_5-\rho_1^2-\rho_2^2-\rho_3^2-\rho_4^2-\rho_1x_1-\rho_2x_2-\rho_3x_3-\rho_4x_4$$

in the first row, and $2K$ in every other row; and therefore, by taking the differences of these last columns, it may be replaced by zero in all the rows except the first, where we shall have

$$2\rho_5^2+2\rho_5x_5-\rho_1^2-\rho_2^2-\rho_3^2-\rho_4^2-\Sigma_1^4\rho_kx_k-2K.$$

Therefore the equation to the circumsphere is found by equating this to zero, viz.,

$$\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 + \rho_4 x_4 - 2\rho_5 x_5 + 2l^2 = 0,$$

or

$$(\rho_1^2 - l^2)\frac{x_1}{\rho_1} + (\rho_2^2 - l^2)\frac{x_2}{\rho_2} + \ldots + (\rho_4^2 - l^2)\frac{x_4}{\rho_4} - (2\rho_5^2 + l^2)\frac{x_5}{\rho_5} = 0 \ldots (9),$$

where l^2 has the value given in the preceding paragraph, and therefore vanishes when the tetrahedron is orthocentric.

4. We will now find the equations to the inscribed and escribed spheres. The equation to the plane $A_2A_3A_4$ is

$$\Sigma\beta_h x_h \equiv \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & \frac{1}{\rho_5} \end{vmatrix} = 0.$$

Here $\qquad \Sigma\beta_h^2 + 2\Sigma\beta_h\beta_k c_{hk}$

$$\begin{aligned}
&= \beta_1(\beta_1 \quad + \beta_2 c_{12} + \beta_3 c_{13} + \beta_4 c_{14} + \beta_5 c_{15}) \\
&+ \beta_2(\beta_1 c_{12} + \beta_2 \quad + \beta_3 c_{23} + \beta_4 c_{24} + \beta_5 c_{25}) \\
&+ \beta_3(\beta_1 c_{13} + \beta_2 c_{23} + \beta_3 \quad + \beta_4 c_{34} + \beta_5 c_{35}) \\
&+ \beta_4(\beta_1 c_{14} + \beta_2 c_{24} + \beta_3 c_{34} + \beta_4 \quad + \beta_5 c_{45}) \\
&+ \beta_5(\beta_1 c_{15} + \beta_2 c_{25} + \beta_3 c_{35} + \beta_4 c_{45} + \beta_5 \quad)
\end{aligned}$$

$$= \beta_1 \begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & \frac{1}{\rho_5} \end{vmatrix} - \beta_5 \begin{vmatrix} c_{12} & 1 & c_{23} & c_{24} & c_{25} \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} \\ c_{15} & c_{25} & c_{35} & c_{45} & 1 \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & \frac{1}{\rho_5} \end{vmatrix}$$

$$= \beta_1 D_5 - \beta_5 D_1 = (\beta_1\rho_1 - \beta_5\rho_5)\frac{D_5}{\rho_1}.$$

If we write this $$a_1^2 \frac{D_5^2}{\rho_1^2},$$

$$a_1^2 = (\beta_1\rho_1 - \beta_5\rho_5) \Big/ \frac{D_5}{\rho_1} = \frac{4\rho_1\rho_5\alpha_1^2}{\rho_2^2\rho_3^2\rho_4^2} \cdot \frac{\rho_1^3\rho_2^2\rho_3^2\rho_4^2}{36V^2\rho_5} = \frac{\rho_1^4}{h_1^2},$$

where h_1 is the altitude of A_1 above the base $A_2A_3A_4$. Thus

$$a_1 = \frac{\rho_1^2}{h_1},$$

and $$\Sigma\beta_h^2 + 2\Sigma\beta_h\beta_k c_{hk} = a_1^2 \frac{D_5^2}{\rho_1^2},$$

where $$D_5 = \frac{-36V^2\rho_5}{\rho_1^2\rho_2^2\rho_3^2\rho_4^2}.$$

Let $\Sigma\alpha_h x_h = 0$ be any sphere, and let

$$\Sigma\alpha_h^2 + 2\Sigma\alpha_h\alpha_k c_{hk} = 1 \quad \text{.................(10)}.$$

Then the cosine of the angle of intersection with the plane $A_2A_3A_4$ is $\dfrac{\alpha_1\rho_1 - \alpha_5\rho_5}{a_1}$. If the sphere touches the four faces of the tetrahedron we have

and similarly $$\left.\begin{aligned} \alpha_1\rho_1 - \alpha_5\rho_5 &= a_1 \\ \alpha_2\rho_2 - \alpha_5\rho_5 &= a_2 \\ \alpha_3\rho_3 - \alpha_5\rho_5 &= a_3 \\ \alpha_4\rho_4 - \alpha_5\rho_5 &= a_4 \end{aligned}\right\} \quad \text{......... ....(11)}.$$

Also, by (10),

$$\begin{aligned} 1 = \alpha_1 (\alpha_1 &+ \alpha_2 c_{12} + \alpha_3 c_{13} + \alpha_4 c_{14} + \alpha_5 c_{15}) \\ + \alpha_2 (\alpha_1 c_{12} &+ \alpha_2 \quad + \alpha_3 c_{23} + \alpha_4 c_{24} + \alpha_5 c_{25}) \\ + \&c. \end{aligned}$$

Substitute the values of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in terms of α_5 from (11), therefore

$$1 = \frac{a_1^2}{\rho_1^2} + \frac{a_2^2}{\rho_2^2} + \frac{a_3^2}{\rho_3^2} + \frac{a_4^2}{\rho_4^2} + \frac{2a_1a_2c_{12}}{\rho_1\rho_2} + \ldots$$
$$+ 2\alpha_5\rho_5 \left(\frac{a_1}{\rho_1^2} + \frac{a_2}{\rho_2^2} + \frac{a_3}{\rho_3^2} + \frac{a_4}{\rho_4^2}\right).$$

Therefore $$0 = H + \frac{\alpha_5\rho_5}{r},$$

where r is the radius of the inscribed sphere, and

$$2H = \frac{a_1^2}{\rho_1^2} + \frac{a_2^2}{\rho_2^2} + \frac{a_3^2}{\rho_3^2} + \frac{a_4^2}{\rho_4^2} + \frac{2a_1a_2c_{12}}{\rho_1\rho_2} + \ldots - 1 \ldots\ldots (12).$$

When the system is orthogonal the value of H is unity. Therefore

$$\alpha_5\rho_5 = -Hr,$$

$$\alpha_1\rho_1 = a_1 - Hr,$$

$$\alpha_2\rho_2 = a_2 - Hr,$$

&c. &c.

and the equation to the inscribed sphere is

$$(a_1 - Hr)\frac{x_1}{\rho_1} + (a_2 - Hr)\frac{x_2}{\rho_2} + (a_3 - Hr)\frac{x_3}{\rho_3} + (a_4 - Hr)\frac{x_4}{\rho_4} - Hr\frac{x_5}{\rho_5} = 0$$

or $$\frac{a_1}{\rho_1}x_1 + \frac{a_2}{\rho_2}x_2 + \frac{a_3}{\rho_3}x_3 + \frac{a_4}{\rho_4}x_4 + 2Hr = 0 \ldots\ldots\ldots (13),$$

The four escribed spheres, each touching one face of the tetrahedron on the reverse side to the inscribed sphere, are determined by changing the sign of one of the four quantities a_1, a_2, a_3, a_4, the corresponding change in H being to H_1, H_2, H_3, H_4, and r being changed to r_1, r_2, r_3, or r_4. Therefore, since

$$\frac{1}{r} = \frac{a_1}{\rho_1^2} + \frac{a_2}{\rho_2^2} + \frac{a_3}{\rho_3^2} + \frac{a_4}{\rho_4^2},$$

we have $$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{r}.$$

There are three other escribed spheres, each of which touches two faces on the reverse side, and the radii, r_{12}, &c., are given by

$$\frac{1}{r_{12}} = -\frac{a_1}{\rho_1^2} - \frac{a_2}{\rho_2^2} + \frac{a_3}{\rho_3^2} + \frac{a_4}{\rho_4^2},$$

or, if this is negative, with the signs reversed throughout. Hence $r_{12} = r_{34}$, the spheres being identical. Also

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} = \frac{1}{r^2} + \frac{1}{r_{12}^2} + \frac{1}{r_{13}^2} + \frac{1}{r_{14}^2}.$$

5. We may now prove that the sphere which passes through the points A_2, A_3, A_4 and touches the inscribed sphere will also touch the sphere escribed on $A_2A_3A_4$.

Let the sphere passing through A_2, A_3, A_4 and touching the inscribed sphere be

$$\left(\rho_1 - \frac{l^2}{\rho_1}\right) x_1 + \ldots + \left(\rho_4 - \frac{l^2}{\rho_4}\right) x_4 - \left(2\rho_5 + \frac{l^2}{\rho_5}\right) x_5$$

$$+ k \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & \frac{1}{\rho_5} \end{vmatrix} = 0,$$

or

$$\Sigma \delta_h x_h \equiv \left(k\beta_1 + \rho_1 - \frac{l^2}{\rho_1}\right) x_1 + \ldots$$

$$+ \left(k\beta_4 + \rho_4 - \frac{l^2}{\rho_4}\right) x_4 + \left(k\beta_5 - 2\rho_5 - \frac{l^2}{\rho_5}\right) x_5 = 0 \ldots (14).$$

Here

$$\begin{aligned}
\delta_1 \quad + \delta_2 c_{12} + \delta_3 c_{13} + \delta_4 c_{14} + \delta_5 c_{15} &= kD_5 + \rho_1, \\
\delta_1 c_{12} + \delta_2 \quad + \delta_3 c_{23} + \delta_4 c_{24} + \delta_5 c_{25} &= \rho_2, \\
\delta_1 c_{13} + \delta_2 c_{23} + \delta_3 \quad + \delta_4 c_{34} + \delta_5 c_{35} &= \rho_3, \\
\delta_1 c_{14} + \delta_2 c_{24} + \delta_3 c_{34} + \delta_4 \quad + \delta_5 c_{45} &= \rho_4, \\
\delta_1 c_{15} + \delta_2 c_{25} + \delta_3 c_{35} + \delta_4 c_{45} + \delta_5 \quad &= -kD_1 - 2\rho_5.
\end{aligned}$$

Therefore

$$\Sigma \delta_h^2 + 2\Sigma \delta_h \delta_k c_{hk}$$

$$= \delta_1 (kD_5 + \rho_1) + \delta_2 \rho_2 + \delta_3 \rho_3 + \delta_4 \rho_4 - \delta_5 (kD_1 + 2\rho_5)$$

$$= k^2 (\beta_1 D_5 - \beta_5 D_1) + 2k \frac{D_5}{\rho_1} (\rho_1^2 + 2\rho_5^2) + 4R^2$$

$$= k^2 \frac{a_1^2}{\rho_1^2} D_5^2 + 2kD_5 \frac{\rho_1^2 + 2\rho_5^2}{\rho_1} + 4R^2$$

$$= 4R_1^2,$$

where R_1 is the radius of the sphere (14).

Since this touches the inscribed sphere, $\Sigma \alpha_h x_h = 0$, we have

$$\alpha_1 (kD_5 + \rho_1) + \alpha_2 \rho_2 + \alpha_3 \rho_3 + \alpha_4 \rho_4 - \alpha_5 (kD_1 + 2\rho_5) = 2R_1,$$

therefore $$ka_1 \frac{D_5}{\rho_1} + a_1 + a_2 + a_3 + a_4 - 2Hr = 2R_1 \ldots\ldots\ldots (15).$$

If it also touches the sphere escribed on $A_2A_3A_4$, this equation will remain true for the same values of k and R_1 when the sign of a_1 is changed, and we shall have

$$-ka_1\frac{D_5}{\rho_1}-a_1+a_2+a_3+a_4-2H_1r_1=2R_1.$$

And we should have

$$ka_1\frac{D_5}{\rho_1}+a_1-Hr+H_1r_1=0 \quad \ldots\ldots\ldots\ldots(16)$$

and $$2R_1=a_2+a_3+a_4-(Hr+H_1r_1)\ldots\ldots\ldots\ldots(17).$$

It therefore remains to prove that the above values satisfy (15), or

$$\left\{ka_1\frac{D_5}{\rho_1}+a_1+a_2+a_3+a_4-2Hr\right\}^2=k^2a_1^2\frac{D_5^2}{\rho_1^2}+2k\frac{D_5}{\rho_1}(\rho_1^2+2\rho_5^2)+4R^2.$$

This becomes

$$\left(2ka_1\frac{D_5}{\rho_1}+a_1+a_2+a_3+a_4-2Hr\right)(a_1+a_2+a_3+a_4-2Hr)$$
$$=2k\frac{D_5}{\rho_1}(\rho_1^2+2\rho_5^2)+4R^2.$$

If we substitute the value of k as in (16), we obtain as the identity to be established

$$\left\{a_2+a_3+a_4-rr_1\left(\frac{H}{r_1}+\frac{H_1}{r}\right)\right\}^2-\left\{a_1-rr_1\left(\frac{H}{r_1}-\frac{H_1}{r}\right)\right\}^2$$
$$-2\frac{\rho_1^2+2\rho_5^2}{a_1}\left\{a_1-rr_1\left(\frac{H}{r_1}-\frac{H_1}{r}\right)\right\}=4R^2\ldots(18).$$

Now $$a_1=\frac{\rho_1^2}{h_1}=\frac{\rho_1^2\alpha_1}{3V},$$

where α_1 is the area of the face opposite A_1.

Then, by writing for c_{12}, $\dfrac{\rho_1^2+\rho_2^2-c^2}{2\rho_1\rho_2}$, &c.,

$$2H=\frac{a_1^2}{\rho_1^2}+\ldots+\frac{a_4^2}{\rho_4^2}+\frac{2a_1a_2c_{12}}{\rho_1\rho_2}+\ldots-1$$
$$=\frac{1}{9V^2}\{(\rho_1^2\alpha_1+\rho_2^2\alpha_2+\rho_3^2\alpha_3+\rho_4^2\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$$
$$-a^2\alpha_2\alpha_3-e^2\alpha_2\alpha_4-f^2\alpha_3\alpha_4-b^2\alpha_1\alpha_3-c^2\alpha_1\alpha_2-d^2\alpha_1\alpha_4-9V^2\},$$

whence we have

$$(a_2+a_3+a_4)\frac{1}{rr_1}-\left(\frac{H}{r_1}+\frac{H_1}{r}\right)$$

$$=\frac{1}{27V^3}\{(a^2\alpha_2\alpha_3+e^2\alpha_2\alpha_4+f^2\alpha_3\alpha_4+9V^2)(\alpha_2+\alpha_3+\alpha_4)$$
$$-(b^2\alpha_3+c^2\alpha_2+d^2\alpha_4)\alpha_1^2\},$$

and

$$\frac{a_1}{rr_1}-\left(\frac{H}{r_1}-\frac{H_1}{r}\right)=\frac{\alpha_1}{27V^3}\{(b^2\alpha_3+c^2\alpha_2+d^2\alpha_4)(\alpha_2+\alpha_3+\alpha_4)$$
$$-a^2\alpha_2\alpha_3-e^2\alpha_2\alpha_4-f^2\alpha_3\alpha_4-9V^2\}.$$

Therefore (18) reduces to

$$256(a^2\alpha_2\alpha_3+e^2\alpha_2\alpha_4+f^2\alpha_3\alpha_4+9V^2)^2-256\alpha_1^2(b^2\alpha_3+c^2\alpha_2+d^2\alpha_4)^2$$
$$-2\frac{(\rho_1^2+2\rho_5^2)}{\rho_1^2}.144V^2.16\{a^2\alpha_2\alpha_3+e^2\alpha_2\alpha_4+f^2\alpha_3\alpha_4+9V^2$$
$$-(b^2\alpha_3+c^2\alpha_2+d^2\alpha_4)(\alpha_2+\alpha_3+\alpha_4)\}=576R^2V^2.16\{(\alpha_2+\alpha_3+\alpha_4)^2-\alpha_1^2\}.$$

Here the terms containing the first powers of α_2, α_3, α_4 vanish, and the identity reduces to

$$256(a^4\alpha_2^2\alpha_3^2+e^4\alpha_2^2\alpha_4^2+f^4\alpha_3^2\alpha_4^2+81V^4)-256\alpha_1^2(b^4\alpha_3^2+c^4\alpha_2^2+d^4\alpha_4^2)$$
$$-2\frac{\rho_1^2+2\rho_5^2}{\rho_1^2}.144V^2\{144V^2-16(b^2\alpha_3^2+c^2\alpha_2^2+d^2\alpha_4^2)\}$$
$$-576R^2V^2.16(\alpha_2^2+\alpha_3^2+\alpha_4^2-\alpha_1^2)=0.$$

Since we know the expressions for $16\alpha_1^2$, &c., $144V^2$, $\frac{\rho_1^2+2\rho_5^2}{\rho_1^2}144V^2$ and $576R^2V^2$ in terms of the edges, this identity may now be established by elementary algebra; thus proving the proposition.

It may be noted that every term in the expression which is to be proved to vanish contains either a, e, or f as a factor; and that a, e, f occur symmetrically, associated respectively with d, b, c. Thus if the separate terms in powers of a^2 vanish, so also will those containing e^2 or f^2. It is therefore enough to prove that the terms containing powers of a^2 vanish, and the work of establishing the identity is much shortened.

TWO TRIGONOMETRICAL DETERMINANTS.

By *Prof. E. H. Neville.*

1. WRITING

$$\cos x + i \sin x = t, \quad c = t + t^{-1} = 2\cos x,$$

we have

$$c^n = (t^n + t^{-n}) + \binom{n}{1}(t^{n-2} + t^{-n+2}) + \binom{n}{2}(t^{n-4} + t^{-n+4}) + \dots,$$

$$2ic^{n-1}\sin x = \left\{(t^{n-1} + t^{-n+1}) + \binom{n-1}{1}(t^{n-3} + t^{-n+3}) + \dots\right\}(t - t^{-1})$$

$$= (t^n - t^{-n}) + \left\{\binom{n-1}{1} - 1\right\}(t^{n-2} - t^{-n+2})$$

$$+ \left\{\binom{n-1}{2} - \binom{n-1}{1}\right\}(t^{n-4} - t^{-n+4}) + \dots;$$

that is,

$$c^n = 2\cos nx + \binom{n}{1} 2\cos(n-2)x + \binom{n}{2} 2\cos(n-4)x + \dots,$$

$$c^{n-1}\sin x = \sin nx + \left\{\binom{n-1}{1} - 1\right\}\sin(n-2)x$$

$$+ \left\{\binom{n-1}{2} - \binom{n-1}{1}\right\}\sin(n-4)x + \dots .$$

Elimination of $2\cos(n-2)x$, $2\cos(n-4)x$, $+\dots$ from formulæ of the first kind and of $\sin(n-2)x/\sin x$, $\sin(n-4)x/\sin x$, ... from those of the second kind gives explicitly

$$(1.1) \quad 2\cos nx = \begin{vmatrix} c^n, & \binom{n}{1}, & \binom{n}{2}, & \binom{n}{3}, & \binom{n}{4}, & \dots \\ c^{n-2}, & 1, & \binom{n-2}{1}, & \binom{n-2}{2}, & \binom{n-2}{3}, & \dots \\ c^{n-4}, & 0, & 1, & \binom{n-4}{1}, & \binom{n-4}{2}, & \dots \\ c^{n-6}, & 0, & 0, & 1, & \binom{n-6}{1}, & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

the last row being 1, 0, 0, ..., 0, 1 or c, 0, 0, ..., 0, 1, according as n is even or odd, and

(1.2) $\sin nx / \sin x$

$$= \begin{vmatrix} c^{n-1}, & \binom{n-1}{1}-1, & \binom{n-1}{2}-\binom{n-1}{1}, & \binom{n-1}{3}-\binom{n-1}{2}, & \cdots \\ c^{n-3}, & 1, & \binom{n-3}{1}-1, & \binom{n-3}{2}-\binom{n-3}{1}, & \cdots \\ c^{n-5}, & 0, & 1, & \binom{n-5}{1}-1, & \cdots \\ c^{n-7}, & 0, & 0, & 1, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

the last row in this case being c, 0, 0, ..., 0, 1 if n is even and 1, 0, 0, ..., 0, 1 if n is odd; the last column has no peculiarities in any case, and it must be noticed that the co-factor of the leading element in each determinant is unity.

2. This is a simple way to obtain intelligibly systematic formulæ with small values of x, and numerical determinants are provided for computation:

$$2\cos 7x = \begin{vmatrix} 128\cos^7 x, & 7, & 21, & 35 \\ 32\cos^5 x, & 1, & 5, & 10 \\ 8\cos^3 x, & 0, & 1, & 3 \\ 2\cos x, & 0, & 0, & 1 \end{vmatrix},$$

$$\sin 8x/\sin x = \begin{vmatrix} 128\cos^7 x, & 7-1, & 21-7, & 35-21 \\ 32\cos^5 x, & 1, & 5-1, & 10-5 \\ 8\cos^3 x, & 0, & 1, & 3-1 \\ 2\cos x, & 0, & 0, & 1 \end{vmatrix},$$

$$2\cos 8x = \begin{vmatrix} 256\cos^8 x, & 8, & 28, & 56, & 70 \\ 64\cos^6 x, & 1, & 6, & 15, & 20 \\ 16\cos^4 x, & 0, & 1, & 4, & 6 \\ 4\cos^2 x, & 0, & 0, & 1, & 2 \\ 1\quad, & 0, & 0, & 0, & 1 \end{vmatrix},$$

$$\sin 9x/\sin x = \begin{vmatrix} 256\cos^8 x, & 8-1, & 28-8, & 56-28, & 70-56 \\ 64\cos^6 x, & 1, & 6-1, & 15-\ 6, & 20-15 \\ 16\cos^4 x, & 0, & 1, & 4-\ 1, & 6-\ 4 \\ 4\cos^2 x, & 0, & 0, & 1, & 2-\ 1 \\ 1\quad , & 0, & 0, & 0, & 1 \end{vmatrix}.$$

3. Before developing the determinants we make use of the identity

$$\binom{m}{r} - \binom{m}{r-1} = \frac{m-2r+1}{r}\binom{m}{r-1}$$

to replace (1.2) by

$$(3.1)\quad \sin nx/\sin x = \begin{vmatrix} c^{n-1}, & \frac{n-2}{1}, & \frac{n-4}{2}\binom{n-1}{1}, & \frac{n-6}{3}\binom{n-1}{2}, & \dots \\ c^{n-3}, & 1\ , & \frac{n-4}{1}\ , & \frac{n-6}{2}\binom{n-3}{1}, & \dots \\ c^{n-5}, & 0\ , & 1\ , & \frac{n-6}{1}\ , & \dots \\ c^{n-7}, & 0\ , & 0\ , & 1\ , & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

4. If we write

$$(1+x)^{-n} = 1 - \begin{Bmatrix} n \\ 1 \end{Bmatrix} x + \begin{Bmatrix} n \\ 2 \end{Bmatrix} x^2 - \dots,$$

so that in fact

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = \frac{n\,.\,n+1\,.\,n+2\dots n+r-1}{r\,!} = \binom{n+r-1}{r},$$

then, since in the product $(1+x)^{-(n-r)} \times (1+x)^{n-r}$ the coefficient of x^r is zero if r is not zero,

$$(4.1)\quad \begin{Bmatrix} n-r \\ r \end{Bmatrix}\binom{n-r}{0} - \begin{Bmatrix} n-r \\ r-1 \end{Bmatrix}\binom{n-r}{1} + \begin{Bmatrix} n-r \\ r-2 \end{Bmatrix}\binom{n-r}{2} - \dots = 0,$$

that is,

$$(4.2)\quad \frac{n-1.n-2\dots n-r}{r\,!} - \frac{n-2.n-3\dots n-r}{(r-1)\,!}\cdot\frac{n-r}{1\,!} + \frac{n-3.n-4\dots n-r}{(r-2)\,!}$$

$$\times\frac{n-r.n-r-1}{2\,!} - \frac{n-4.n-5\dots n-r}{(r-3)\,!}\cdot\frac{n-r.n-r-1.n-r-2}{3\,!} + \dots = 0.$$

This, when divided by $n-r$ and multiplied by n, becomes

$$(4.3)\quad \binom{n}{r} - \frac{n}{1\,!}\binom{n-2}{r-1} + \frac{n.n-3}{2\,!}\binom{n-4}{r-2} - \frac{n.n-4.n-5}{3\,!}\binom{n-6}{r-3} + \ldots = 0,$$

and when divided by $n-r$ and multiplied by $n-2r$ can be written

$$(4.4)\quad \frac{n-2r}{r}\binom{n-1}{r-1} - \frac{n-2}{1\,!}\cdot\frac{n-2r}{r-1}\binom{n-3}{r-2} + \frac{n-3.n-4}{2\,!}\cdot\frac{n-2r}{r-2}\binom{n-5}{r-3} - \ldots = 0.$$

It follows from (4.3) that the minors of c^n, c^{n-2}, c^{n-4}, c^{n-6}, ... in the determinant in (1.1) are proportional to

$$(4.5)\qquad 1,\ \frac{n}{1\,!},\ \frac{n.n-3}{2\,!},\ \frac{n.n-4.n-5}{3\,!},\ \ldots,$$

and from (4.4) that the minors of c^{n-1}, c^{n-3}, c^{n-5}, ... in the determinant in (3.1) are proportional to

$$(4.6)\qquad 1,\ \frac{n-2}{1\,!},\ \frac{n-3.n-4}{2\,!},\ \ldots,$$

and, since in each case the value of the first minor in the set is unity, we have in (4.5) and (4.6) the actual values of the minors. Hence come the familiar expansions

$$2\cos nx = c^n - \frac{n}{1\,!}c^{n-2} + \frac{n.n-3}{2\,!}c^{n-4} - \frac{n.n-4.n-5}{3\,!}c^{n-6} + \ldots,$$

$$\frac{\sin nx}{\sin x} = c^{n-1} - \frac{n-2}{1\,!}c^{n-3} + \frac{n-3.n-4}{2\,!}c^{n-5} - \ldots\,.$$

5. To be strictly elementary, a proof should perhaps avoid the use of the expansion for $(1+x)^{-n}$, even though this expansion can be justified, when n is integral, by the actual summation of a finite number of the terms of the series. Since in (4.2) each term is merely a polynomial in n of degree r, the identity is *algebraically* equivalent to that obtained by replacing n by $-m$ throughout, which is

$$(5.1)\quad \binom{m+r}{r} - \binom{m+r}{r-1}\binom{m+r}{1} + \binom{m+r}{r-2}\binom{m+r+1}{2} - \ldots = 0,$$

that is,

$$(5.2)\quad \binom{m+r}{m} - \binom{m+r}{m+1}\binom{m+r}{1} + \binom{m+r}{m+2}\binom{m+r+1}{2} - \ldots = 0.$$

But identically

$$(5.3)\qquad \{x + (1-x)\}^{m+r}(1-x)^{r-1}$$

$$= x^{m+r}(1-x)^{r-1} + \binom{m+r}{1} x^{m+r-1}(1-x)^{r} + \ldots$$

$$+ \binom{m+r}{m} x^{r}(1-x)^{m+r-1} + \binom{m+r}{m+1} x^{r-1}(1-x)^{m+r} + \ldots;$$

the left-hand side of (5.2) is the coefficient of x^r on the right-hand side of (5.3), and this is necessarily zero.

Since (4.2), unless expressed in the form (4.1), seems wanting in formal elegance, it is worth while to remark that by replacing n, r by $p+2$, $t+1$ we can present the result as

$$\binom{p}{0}\binom{p}{t} - \tfrac{1}{2}\binom{p-1}{1}\binom{p-2}{t-1} + \tfrac{1}{3}\binom{p-2}{2}\binom{p-4}{t-2}$$

$$-\tfrac{1}{4}\binom{p-3}{3}\binom{p-6}{t-3} + \ldots = \frac{1}{p+2}\binom{p+2}{t+1}.$$

THE DISSECTION OF RECTILINEAL FIGURES (*continued*).

By *W. H. Macaulay, M.A.*

In a former paper* I discussed what I called the four-part dissection of a certain pair of hexagons of equal area, which has eleven degrees of freedom. Each hexagon is to have two opposite sides equal and parallel, and arranged so that, when the perimeter is traversed, these sides are traversed in opposite directions. Also the cores of the hexagons are to be identical, the core being defined as the parallelogram formed by joining the middle points of the four inclined sides. I added another restriction, namely that a side of the core of one hexagon which is opposite a parallel side is to be equal to a side of the core of the other hexagon which is opposite a vertex at which inclined sides meet. I shewed that this pair of hexagons has

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* *Messenger of Mathematics*, vol. xlviii., p. 159.

four dissections, in which the dividing lines of each hexagon, either unbroken or broken, are equal and parallel to the four inclined half-sides and one parallel side of the other hexagon. I shall call this the α hexagon dissection.

I propose now to consider another dissection, which I shall distinguish as the β hexagon dissection. This is obtained by the same method as the α dissection, namely interchange of cores, and has the same general properties, but differs from it in being the dissection which is obtained when the hexagons are related so that a side of the core of one hexagon which is opposite a parallel side is equal to a side of the core of the other which is opposite a parallel side. At first I disregarded this case because the α dissection gave what I was seeking, namely the coordination of known four-part dissections of independent figures; but I have since found that an enumeration of these discloses several dissections which I did not know before, which are of the β type.

There are four β dissections of a pair of hexagons suitably related, and the scheme of construction is the same as for α dissections for the same reasons; but in a four-part dissection the parts are two quadrilaterals and two hexagons, instead of being four pentagons. At least two of the dissections must involve broken lines; and the rule for drawing broken lines is the same as before, namely that, when a line which is being drawn meets an inclined side of the hexagon, it is continued in the opposite direction from another point in that side at the same distance from the centre; and that, when it meets a parallel side, it is continued in the same direction from the corresponding point of the other parallel side. The diagram (fig. 1) shews two of the dissections. In one, which is shewn

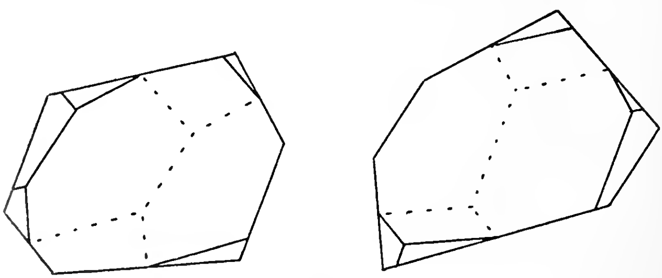

Fig. 1.

by dots, each hexagon is divided by five unbroken lines; in the other, which is drawn with full lines, two of the dividing lines in each hexagon are broken into two parallel portions, in accordance with the rule. The hexagons are placed with their cores parallel, and in each figure four dividing lines are drawn from the centres of the inclined sides, equal and parallel to the inclined half-sides of the other hexagon. In my former paper I counted the number of dissections, of what I called the four-part type, of a pair of independent triangles. I think it certain that this is the only type of dissection of independent triangles which is capable of giving four parts. It is not, however, the only dissection in which each triangle is divided by three lines (broken or unbroken) equal and parallel to half-sides of the other, for I shall now shew that there is also a dissection of the β hexagon type which has this property, though it is incapable of giving four parts.

Let us examine systematically all the ways in which the α and β hexagon dissections give dissections of a pair of independent triangles. The ways in which a hexagon may be drawn so as to form a given triangle, subject to the core retaining one degree of freedom, may be classified according to the way in which the two parallel sides are disposed of. This classification gives five different arrangements of the sides of the hexagon which may be specified as follows: (i) both parallel sides zero; (ii) one parallel side taken for a side of the triangle, and the other cancelled by an inclined side coinciding with it; (iii) one parallel and one adjacent inclined side combined to form a side of the triangle, and the other parallel side cancelled by an inclined side; (iv) one parallel side taken for a side of the triangle, and the other cancelled by the two adjacent inclined sides; (v) the two parallel sides and the two inclined sides which lie betweem them combined to form one side of the triangle. With the arrangement (iii) a half-side of the triangle is a diagonal of the core; with each of the other arrangements a half-side of the triangle is a side of the core.

Let us now take a pair of triangles to dissect, their sides being a, b, c, a', b', c'; and let us suppose, for all purposes of counting, that all coincidences of dimensions which would affect this are excluded. To discover what hexagon dissections they possess, each of the five arrangements of sides must be adopted in turn for each triangle. Their cores must be identical; but the fact that, with each arrangement, the core retains one degree of freedom, makes this possible without the introduction of any relation between the triangles. Accordingly there are fifteen different cases to be considered. In ten of these cases the sides of the common core are respectively half-sides of the two triangles. In four of them the core has a side equal to a half-side of one triangle, and a diagonal equal to a half-side of the other. The remaining case is that in which the arrangement (iii) is adopted for each triangle, in this case the two diagonals of the core are half-sides of the two triangles. The result is that, if we take the nine products aa', ab', ..., we get in the first fourteen cases two possible cores for each product which is greater than twice the area of a triangle. Trial of these cases shews that some are α and some β, and some may be either, but that they all give the four-part type of dissection which has already been investigated. But in the fifteenth case we get a new condition, namely, that there are two possible cores for each product which is greater than four times the area of a triangle. Thus it is clear that we obtain from this case a new dissection of a pair of triangles; this is the β hexagon dissection mentioned above. For this dissection the number of common cores possessed by a pair of triangles may be zero or any even number up to 18; and as each core gives four dissections, the number of dissections of this type is either zero or any multiple of eight up to 72. Combining this with the previous result we see that a given pair of triangles, with no coincidence of dimensions except equal areas, must have at least 48 and may have as many as 144 distinct hexagon dissections. Whether this exhausts the dissections in which each triangle is divided by three lines (unbroken or broken), equal and parallel to half-sides of the other, must perhaps remain unsettled.

An example of the new dissection is shewn in fig. 2. For the triangles in this diagram, only the two longest sides give

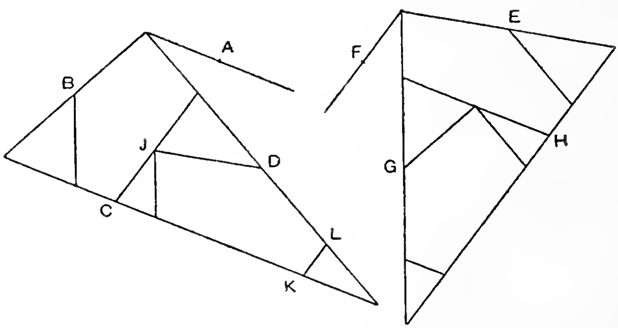

Fig. 2.

a product greater than four times the area, so there are only two cores which give this type of dissection. One of these is $ABCD$ or $EFGH$. The triangles being suitably placed, each of these parallelograms has diagonals equal and parallel respectively to half-sides of the two triangles. Choice of this core determines a pair of hexagons of the β type, each of which has inclined sides bisected by the angular points of the core, and one parallel side cancelled by an inclined side. A dissection of the first triangle can now be made, as shewn in the diagram, by drawing from A, B, C, D lines equal and parallel to inclined half-sides of the other hexagon, and meeting in a point J. A line from A can be drawn in four different ways, each of which gives a dissection. In the diagram it is drawn equal and parallel to half the inclined side which is bisected at H. Starting from A to draw this line, it at once encounters a parallel side of the hexagon, so its continuation is KL, K being the corresponding point of the other parallel side, and the further continuation of it to J is at an equal distance on the other side of D, in accordance with the rule for drawing broken lines. The line from B to J is also broken, the two portions being at equal distances from C, but CJ and DJ happen to be unbroken. The fifth dissecting line of the first hexagon is represented by the broken line JCJ, drawn from J to C and back again, in accordance with the rule. The corresponding dissection of the second triangle can now be drawn in the same way, without any alternatives. For a given core all the alternatives in the construction are those which arise from the choice of the first dissecting line to be drawn.

Let us now count the number of hexagon dissections of a triangle and a parallelogram. It will be found that we get three distinct types of dissection, which I will distinguish by the symbols λ, μ, ν. The first two are known dissections, with four parts in their fundamental cases; the third is not capable of giving less than five parts except in one case noted below. In each of them the triangle is divided by two lines equal and parallel to the sides of the parallelogram, and the parallelogram is divided by three lines equal and parallel to half-sides of the triangle.

We have, as before, five ways in which the sides of a hexagon can be arranged so as to give a triangle. And there are three ways in which the sides of a hexagon can be arranged so as to give a parallelogram, namely: (i) by taking the two parallel sides for sides of the parallelogram, and combining the inclined sides to make the other two sides; (ii) by taking two opposite inclined sides for sides of the parallelogram, and combining each parallel side with one inclined side to make the other sides; (iii) by combining one parallel side with the two adjacent inclined sides to form one side of the parallelogram. With the arrangement (i) a half-side of the parallelogram is a side of the core; with the arrangement (ii) a side of the parallelogram is a diagonal of the core; and with the arrangement (iii) a side of the parallelogram is a side of the core.

Take a triangle with sides a, b, c and a parallelogram with sides p, q; and let us suppose that, for the purpose of counting dissections, any accidental coincidences of dimensions which would affect this are excluded. It must be noticed that the symmetry of a parallelogram reduces the number of dissections obtained from each common core from four to two. Taking in turn each arrangement of sides for each figure, we have fifteen cases to consider. Five of these are cases in which arrangement (i) is adopted for the parallelogram; all these give the same type of dissection, which I will denote by λ. In each of nine other cases we get the type of dissection which I will denote by μ. In one case, namely, that in which arrangement (iii) is adopted for the triangle and arrangement (ii) for the parallelogram, so that the common core has one diagonal equal to a half-side of the triangle and the other diagonal equal to a side of the parallelogram, we get a new type of dissection, obtained from β hexagons, which I denote by ν.

All the dissections of type λ can be obtained from the case in which the first arrangement of sides of a hexagon is adopted for both triangle and parallelogram. Take the six products ap, aq, bp, bq, cp, cq. Each product which is greater than twice the area of a figure gives two parallelograms which can be used as common cores, each of them having sides equal respectively to half-sides of the two figures. Taking one of these parallelograms, it can be placed in the triangle in only one way, but in the parallelogram it can occupy any one of a continuous series of positions. Thus it produces an infinite number of dissections forming a continuous series. In the dissection of the triangle all the members of a given series are obtained from any one of them by continuous shifting of one dissecting line into a succession of parallel positions. As there are two series for each product which is greater than twice the area of a figure, the number of series is either zero or any even number up to 12.

All the dissections of type μ can be obtained from the case in which the first arrangement of the sides of a hexagon is adopted for the triangle, and the second arrangement for the parallelogram. In this case a common core has a side equal to a half-side of the triangle, and a diagonal equal to a side of the parallelogram. Thus, employing the same six products, each product which is greater than the area of a figure yields two common cores, and consequently four distinct dissections. To find the smallest possible number of products greater than the area, Δ, of a figure, suppose $a > b > c$ and $p > q$. Then if $cp < \Delta$, it is clear that a and b must both be greater than p, and this makes ap, bp, aq, bq all greater than Δ. Therefore there cannot be fewer than three products greater than Δ. By taking q small enough the number may be made exactly 3, and by taking c small enough it may be made exactly 4, and obviously it may be 5 or 6. So the number of dissections of type μ must be at least 12, and may be any multiple of 4 from 12 to 24.*

Dissections of the type ν are those given by the case in which arrangement (iii) of the sides of a hexagon is adopted for the triangle, and arrangement (ii) for the parallelogram, so that the core has one diagonal equal to a half-side of the triangle and the other diagonal equal to a side of the parallelogram. For this dissection each product which is greater than twice the area of a figure furnishes two cores, and each core gives two distinct dissections. Starting with a given parallelogram, the construction is made by placing between opposite sides of the parallelogram, produced if necessary, a line equal to a half-side of the triangle and passing through the centre of the parallelogram; then the extremities of this line are two angular points of a core, the other two being the middle points of the other pair of sides of the parallelogram. Except in the cases in which a half-side of the triangle coincides in length with a side of the parallelogram, we get not less than five parts. The number of dissections is either zero or any multiple of 4 up to 24.

The hexagon dissections of a triangle and a parallelogram have now been completely enumerated. There is also a dissection capable of giving four parts, which is derived from the three-part dissection of a pair of parallelograms, but in this case the triangle is not divided by lines equal and parallel to the sides of the parallelogram.

The hexagon dissections of a pair of independent quadrilaterals, each with two parallel sides and two sloping sides, form an interesting series. This pair of figures has seven degrees of freedom. There are three ways in which the sides of a hexagon may be arranged so as to give a quadrilateral with two sides parallel, namely: (i) the arrangement in which

* Diagrams of dissections of the λ and μ types are given in the *Mathematical Gazette*, vol. viii., pp. 73 and 72.

one parallel side of the quadrilateral is a parallel side of the hexagon, and the other is a combination of a parallel and an inclined side of the hexagon, (ii) the arrangement in which each parallel side of the quadrilateral is a combination of a parallel and an inclined side of the hexagon, (iii) the arrangement in which one parallel side of the quadrilateral is a parallel side of the hexagon, and the other is a combination of a parallel side and the two adjacent inclined sides of the hexagon. With the arrangement (i), a sloping half-side of the quadrilateral is a side of core; with the arrangement (ii) the mean of the parallel sides of the quadrilateral is a diagonal of the core; with the arrangement (iii) the mean of the parallel sides of the quadrilateral is a side of the core. Adopting in turn each combination of arrangements for a pair of quadrilaterals, we get six different cases which may be denoted by 11, 22, 33, 12, 23, 31, the first of these cases being that in which arrangement (i) is adopted for each quadrilateral, and so on. Each of these combinations gives a dissection with four parts for a certain range of quadrilaterals. And in every dissection each quadrilateral is divided by three lines (unbroken or broken), such that, if the quadrilaterals are suitably placed, two of them are equal and parallel to the sloping half-sides of the other quadrilateral, and the third is parallel to the parallel sides of the other quadrilateral, and equal to the mean of these two sides. All the six combinations give different constructions when four-part dissections are drawn for the simple cases in which the cores are wholly within the quadrilaterals. But, when this restriction is not made, it will be found that the dissections given by 12 are the same as those given by 31, and those given by 23 are the same as those given by 33, so that all the distinct dissections are given by four combinations, which may be taken to be 11, 22, 33, 31.

Let a, b be the lengths of the sloping half-sides, and c the mean of the parallel sides of one quadrilateral, and a', b', c' the same quantities for the other quadrilateral, and Δ the common area. If these seven quantities are given they specify four alternative pairs of quadrilaterals, which agree in the number of their dissections. With the combination 11, each of the products aa', ab', ba', bb', which is greater than $\frac{1}{2}\Delta$, gives two possible cores, whose sides are the lengths in question. With the combination 22, we get two possible cores with diagonals c and c' if $cc' > \Delta$. With the combination 33, we get two possible cores with sides c and c' if $cc' > \frac{1}{2}\Delta$.

With the combination 31, each of the products ca', cb', $c'a$, $c'b$, which is greater than $\frac{1}{2}\Delta$, gives two possible cores, whose sides are the lengths in question. None of these cores are related to each other in a way which could give rise to repetition, and each of them which exists gives four distinct dissections. It is possible to have a pair of quadrilaterals which possesses 18 of these cores; that is to say, all except those which would be given by the combination 22. Such a pair of quadrilaterals has 72 distinct dissections. If the cores required for the combination 22 exist we cannot have all the others. It is assumed, as before, that none of the dimensions of the figures are accidentally related to each other in a way which would affect the counting.

The diagram, fig. 3, shews a pair of quadrilaterals specified by a, b, c, a', b', c', Δ, and is drawn to shew the way in which

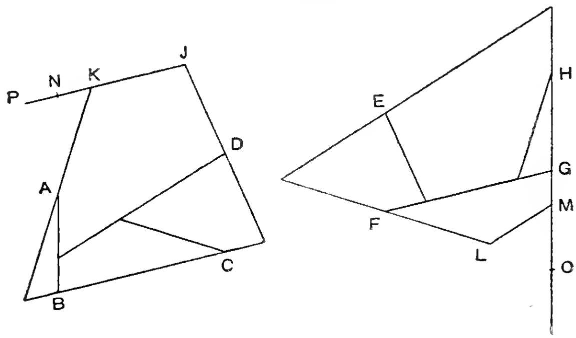

Fig. 3.

the coincidences of dissections obtained from the combination 31 with those obtained from the combination 12 occurs. Taking the arrangement (iii) for the first quadrilateral and (i) for the second, the product cb' is greater than $\frac{1}{2}\Delta$, and therefore gives two common cores, one of which is $ABCD$ or $EFGH$, GH being the half-side which is equal to b'. The interpretation of the quadrilaterals as hexagons is now settled, JK being one of the parallel sides of the first hexagon, and LM one of the parallel sides of the second, and the angular points of the cores being the centres of the inclined sides. In the diagram the quadrilaterals are placed with the identical cores parallel to each other, and one of the dissections is drawn. From each centre a line is drawn equal and parallel to an inclined half-side of the other hexagon, and the lines which join the points in which these meet are equal and parallel to the parallel sides of the hexagons. Corresponding to this scheme of lines there is another pair of identical parallelograms, *NACD* and *EFOG*, which are the cores of another pair of hexagons which represent the pair of quadrilaterals when the combination 21 is adopted. Of these hexagons *N*, *A*, *C*, *D* and *E*, *F*, *O*, *G* are the centres of inclined sides, and *JP* is a parallel side of one, and *LM* a parallel side of the other. This pair of hexagons gives the same dissection. In the first quadrilateral, instead of a line from *B*, we have to draw a line from *N* equal and parallel to half the side whose centre is *G*. This line from *N* at once encounters a parallel side of the hexagon, and is continued from *B*, it is then drawn to *A* and there reversed, and so reaches the same point as the line drawn from *B* in the first dissection. The other lines in the first quadrilateral are the same as before. In the second quadrilateral, instead of a line from *H*, we have to draw the same line from *O*. This line at once encounters, at *O*, the inclined side whose centre is *G*, and is therefore continued from *H*, with the same result as before. The lines from *E*, *F* and *G* are the same as before. Finally the line, in the second quadrilateral, which has to be drawn equal and parallel to a parallel side of the first hexagon, gets the required length, *PJ*, by going to *G* and there being reversed in accordance with the rule for broken lines.

A pentagon with two sides equal and parallel is obtained from a hexagon by combining two inclined sides to form a side of the pentagon, which I will call its base. A pair of independent pentagons can be interpreted as a pair of hexagons of the α type if the product of the bases is not less than twice the area of a figure. There are then two common cores, each giving four dissections with four parts in the fundamental case. One additional cut in each figure gives a dissection of a general pair of quadrilaterals. The interpretation of a pair of pentagons as a pair of β hexagons also demands attention. This requires that the bases of the two pentagons should be equal. There are then an infinite number of common cores. These cores give two continuous series of dissections with not less than four parts; but they also give two single dissections which are the same for all the cores. In these two dissections each pentagon is divided by three lines equal and parallel to the inclined half-sides and one parallel side of the other pentagon, and the number of parts may be three. From this three-part dissection of a pair of pentagons with equal bases various other dissections may be derived, and for the orderly treatment of the subject it is important that it should be recognised as being a hexagon dissection. This shows, for one thing, that the hexagon rule for drawing broken lines applies to it. The simplest case of it is the three-part dissection of a pair of triangles with equal bases, in which each triangle is divided by lines drawn from the middle points of the sides to a point in the base. In a pentagon, lines are drawn from the middle points of the inclined sides and from some point in the base, to meet in a point.

The dissection of a pair of quadrilaterals, mentioned in my former paper as depending on a parallelogram of the same area, may be obtained by two applications of a hexagon dissection. I do not know of any construction to apply to a pair of figures with several degrees of freedom, except the hexagon dissections, and one continuous series for a pair of parallelograms. Of this series one member is of hexagon type.

ON LAPLACE'S INTEGRALS FOR A LEGENDRE POLYNOMIAL.

By *S. Pollard.*

THE various expressions for $P_n(x)$, n being a positive integer, are so well known that it is practically certain nothing more can be done either in the way of obtaining fresh expressions, or in the way of improving the methods by means of which they are obtained. But the question is still open to consideration as to how the various expressions which are known may be linked up, *i.e.* how the remaining expressions may be deduced from any selected one*.

In this paper we propose, not to give a systematic answer to the question—the field of investigation opened up is too wide for that—but to give a particular result which may be of interest. If we define $P_n(x)$ as the coefficient of z^n in the expansion of $(1-2xz+z^2)^{\frac{1}{2}}$, then Taylor's expansion gives at once the formula

$$P_n(x)=\frac{1}{2\pi i}\int_{(\gamma)}\frac{dz}{z^{n+1}(1-2xz+z^2)^{\frac{1}{2}}}\quad\text{.........(1)}$$

when (γ) is any sufficiently small circle with the origin as centre. (γ) is, of course, supposed described in the positive (counter-clockwise) sense. We shall shew how from this formula the two well-known integral expressions

$$(a)\quad P_n(x)=\frac{1}{\pi}\int_0^\pi\{x+(x^2-1)^{\frac{1}{2}}\cos\phi\}^n d\phi,$$

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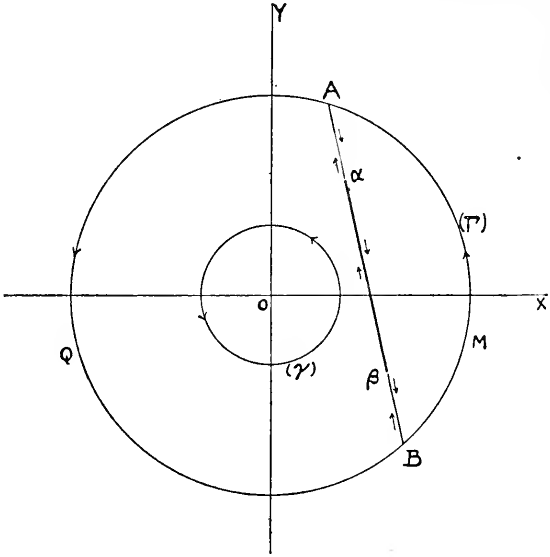
* See, for instance, F. Jackson, "Notes on Laplace's integrals for $P_n(x)$ and $Q_n(x)$", vol. xliv. (1915), p. 117 of this Journal; R. Hargreaves, "Standard relations of Legendre's functions", *Ibid*, vol. xlix. (1919), p. 58.

$$(b)\quad P_n(x) = \frac{1}{\pi}\int_0^\pi \{x + (x^2-1)^{\frac{1}{2}}\cos\phi\}^{-n-1} d\phi$$

can be obtained.

The radical $(1-2xz+z^2)^{\frac{1}{2}}$ is a two-valued function with branch points at $x+(x^2-1)^{\frac{1}{2}}$, $x-(x^2-1)^{\frac{1}{2}}$, which we will



call α, β respectively (see figure). If we make a cut from α to β we obtain two branches which are regular at all finite points of the cut plane and reduce to $+1$, -1 respectively at the origin. It is the one which reduces to $+1$ which is employed in (1).

Make the substitution $z = \frac{1}{\zeta}$. Then as z describes (γ) in the positive sense, ζ describes a concentric circle (Γ) in the negative sense. We have

$$P_n(x) = -\frac{1}{2\pi i}\int_{(\Gamma-)} \frac{\zeta^n d\zeta}{(1-2x\zeta+\zeta^2)^{\frac{1}{2}}}$$

$$= \frac{1}{2\pi i}\int_{(\Gamma+)} \frac{\zeta^n d\zeta}{(1-2x\zeta+\zeta^2)^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots(2)$$

when $(\Gamma-)$, $(\Gamma+)$ denote respectively the negative and positive senses of (Γ).

If ρ is the radius of (γ), that of (Γ) is given by

$$R = \frac{1}{\rho} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Thus, for all sufficiently small values of ρ, (Γ) will enclose the points α and β, and therefore will not cross the cut $\alpha\beta$. It follows that the same branch of $(1-2x\zeta+\zeta^2)^{\frac{1}{2}}$ is used throughout (Γ) in (2). For the sequel the expression $(1-2x\zeta+\zeta^2)^{\frac{1}{2}}$ is to be regarded as referring to this branch.

Let $\alpha\beta$ be produced both ways to cut (Γ) in A and B. Then by Cauchy's Theorem

$$\int_{(ABMA)} \frac{\zeta^n d\zeta}{(1-2x\zeta+\zeta^2)^{\frac{1}{2}}} = 0 \quad \ldots\ldots\ldots\ldots(4),$$

where along $\alpha\beta$ the value of $(1-2x\zeta+\zeta^2)^{\frac{1}{2}}$ is that for the edge near M; and

$$\int_{(BAQB)} \frac{\zeta^n d\zeta}{(1-2x\zeta+\zeta^2)^{\frac{1}{2}}} = 0 \quad \ldots\ldots\ldots\ldots(5),$$

where along $\beta\alpha$ the value of $(1-2x\zeta+\zeta^2)^{\frac{1}{2}}$ is that for the edge away from M.

(4) and (5) give at once

$$\int_{(\Gamma+)} \frac{\zeta^n d\zeta}{(1-2x\zeta+\zeta^2)^{\frac{1}{2}}} = \int_{(\alpha\beta)} \frac{\zeta^n d\zeta}{(1-2x\zeta+\zeta^2)^{\frac{1}{2}}} + \int_{(\beta\alpha)} \frac{\zeta^n d\zeta}{(1-2x\zeta+\zeta^2)^{\frac{1}{2}}},$$

the values of $(1-2x\zeta+\zeta^2)^{\frac{1}{2}}$ being taken as above. The right-hand side reduces at once to

$$2\int_{(\alpha\beta)} \frac{\zeta^n d\zeta}{(1-2x\zeta+\zeta^2)^{\frac{1}{2}}},$$

the value of $(1-2x\zeta+\zeta^2)^{\frac{1}{2}}$ for the edge near M being taken. This is, of course, because the values of $(1-2x\zeta+\zeta^2)^{\frac{1}{2}}$ on the two edges of $\alpha\beta$ differ only by sign. To calculate the last integral put

$$\zeta = x + (x^2-1)^{\frac{1}{2}}\cos\phi.$$

Then
$$1-2x\zeta+\zeta^2 = (1-x^2)\sin^2\phi,$$
$$d\zeta = i(1-x^2)^{\frac{1}{2}}\sin\phi,$$

and the integral reduces to

$$\pm 2i\int_0^\pi \{x+(x^2-1)^{\frac{1}{2}}\cos\phi\}^n d\phi,$$

the ambiguous sign being written because we have not yet ascertained which branch of $(1-2x\zeta+\zeta^2)^{\frac{1}{2}}$ is being used. This gives at once

$$P_n(x) = \pm\frac{1}{\pi}\int_0^\pi \{x+(x^2-1)^{\frac{1}{2}}\cos\phi\}^n d\phi \quad \ldots\ldots(6).$$

To determine which sign is to be taken put $x=1$. Then by (1)

$$P_n(1)=\frac{1}{2\pi i}\int_{(\gamma)}\frac{dz}{z^n(1-z)},$$

as the branch of $(1-2z+z^2)^{\frac{1}{2}}$, which reduces to $+1$ when z is zero, is evidently $1-z$. Thus (1) is the residue of $\frac{1}{z^n(1-z)}$ at 0, *i.e.* 1. Also the right-hand side of (6) is easily found to be 1. Hence the positive sign must be taken.

If x has any value other than 1 we can connect it to 1 by a continuous path in the x-plane, avoiding all the zeros of

$$\frac{1}{\pi}\int_0^\pi \{x+(x^2-1)^{\frac{1}{2}}\cos\phi\}^n d\phi,$$

as these are necessarily finite in number in any finite part of the x-plane, the function concerned being analytic. The zeroes of

$$\frac{1}{2\pi i}\int_{(\gamma)}\frac{dz}{z^{n+1}(1-2xz+z^2)^{\frac{1}{2}}}$$

being finite in number in any finite part of the x-plane for a similar reason, this path can evidently be chosen to avoid them also. It is evident that the same sign must hold in (6) for all the points of such a path, as both sides are continuous, and therefore cannot change sign without going through a zero, and they do not go through a zero. Hence the positive sign must be taken at x as at 1. Thus the formula (a) holds thruoghout.

To obtain (b) we observe that, by arguments based on Cauchy's Theorem similar to those used above, we have

$$\int_{(\gamma+)}\frac{dz}{z^{n+1}(1-2xz+z^2)^{\frac{1}{2}}}+\int_{(\Gamma-)}\frac{dz}{z^{n+1}(1-2xz+z^2)^{\frac{1}{2}}}$$
$$=2\int_{(\alpha\beta)}\frac{dz}{z^{n+1}(1-2xz+z^2)^{\frac{1}{2}}} \quad \text{......(7)}.$$

But $\quad z^{n+1}(1-2xz+z^2)^{\frac{1}{2}}=O(z^{n+2})$ as $|z|\to\infty$.

Thus, since $n\geq 0$,

$$\int_{(\Gamma-)}\frac{dz}{z^{n+1}(1-2xz+z^2)^{\frac{1}{2}}}\to 0 \text{ as } R\to\infty.$$

Hence, by means of (1), (7) gives

$$P_n(x)=\frac{1}{\pi i}\int_{(\alpha\beta)}\frac{dz}{z^{n+1}(1-2xz+z^2)^{\frac{1}{2}}}.$$

Make the substitution, as before,

$$z = x + (x^2 - 1)^{\frac{1}{2}} \cos\phi,$$

and we get at once

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{\{x + (x^2-1)^{\frac{1}{2}}\cos\phi\}^{n+1}},$$

which is (b).

It should be noted that, as we can reduce

$$\int_0^\pi \{x - (x^2-1)^{\frac{1}{2}}\cos\phi\}^n\, d\phi \quad\text{to}\quad \int_0^\pi \{x + (x^2-1)^{\frac{1}{2}}\cos\phi\}^n d\phi,$$

and

$$\int_0^\pi \frac{d\phi}{\{x - (x^2-1)^{\frac{1}{2}}\cos\phi\}^{n+1}} \quad\text{to}\quad \int_0^\pi \frac{d\phi}{\{x + (x^2-1)^{\frac{1}{2}}\cos\phi\}^{n+1}}$$

by changing the variable from ϕ to $\pi - \phi$, it is immaterial which determination of $(x^2 - 1)^{\frac{1}{2}}$ is employed in (a) and (b). This fact frees formulæ (a) and (b) from all possible ambiguity.

A FORM FOR $\frac{d}{dn} P_n(\mu)$, WHERE $P_n(\mu)$ IS THE LEGENDRE POLYNOMIAL OF DEGREE n.

By *A. E. Jolliffe, M.A.*

In a paper by Dr. Bromwich* it is proved that, when n is a positive integer,

$$\frac{d}{dn} P_n(\mu) = P_n \log\tfrac{1}{2}(1+\mu) + A_n P_n + A_{n-1} P_{n-1} + A_0 P_0,$$

where

$$A_{n-1} = -2\left(1 - \frac{1}{2n}\right), \quad A_{n-2} = +2\left(\frac{1}{2} - \frac{1}{2n-1}\right), \ldots$$

$$A_0 = (-1)^n\, 2\left(\frac{1}{n} - \frac{1}{n+1}\right),$$

and

$$A_0 + A_1 + \ldots + A_n = 0.$$

The object of this note is to shew that $\frac{d}{dn} P_n(\mu)$ may be written in the form

$$\frac{2}{2^n\, n!}\left(\frac{d}{d\mu}\right)^n \{(\mu^2-1)^n \log\tfrac{1}{2}(1+\mu)\} - P_n(\mu)\log\tfrac{1}{2}(1+\mu),$$

* "Certain potential functions and a new solution of Laplace's equation", *Proceedings of the London Mathematical Society*, ser. 2, vol. xii., pp. 100–125.

a form analogous to the well-known form of the Q function, viz.

$$Q_n(\mu)=\frac{1}{2^n n!}\left(\frac{d}{d\mu}\right)^n\left\{(\mu^2-1)^n\log\frac{1+\mu}{1-\mu}\right\}-\tfrac{1}{2}P_n\log\frac{1+\mu}{1-\mu}.$$

If we differentiate with respect to n the equation which is satisfied by $P_n(\mu)$, viz.

$$(1-\mu^2)\frac{d^2u}{d\mu^2}-2\mu\frac{du}{d\mu}+n(n+1)u=0,$$

we see that $\frac{d}{dn}P_n(\mu)$ is a solution of the equation

$$(1-\mu^2)\frac{d^2v}{d\mu^2}-2\mu\frac{dv}{d\mu}+n(n+1)v+(2n+1)P_n=0.$$

The form given will then be correct, if we can verify that it satisfies this equation, since it vanishes when $\mu=1$.

If we write

$$z=(\mu^2-1)^n\log\tfrac{1}{2}(1+\mu),$$

$$(1-\mu^2)\frac{dz}{d\mu}+2n\mu z=(1-\mu)(\mu^2-1)^n,$$

therefore

$$(1-\mu^2)\frac{d^{n+2}z}{d\mu^{n+2}}-2\mu z\frac{d^{n+1}z}{d\mu^{n+1}}+n(n+1)\frac{d^nz}{d\mu^n}$$

$$=\left(\frac{d}{d\mu}\right)^{n+1}(1-\mu)(\mu^2-1)^n$$

$$=(1-\mu)\left(\frac{d}{d\mu}\right)^{n+1}(\mu^2-1)^n-(n+1)\frac{d^n}{d\mu^n}(\mu^2-1)^n.$$

If $$w=P_n\log\tfrac{1}{2}(1+\mu),$$

$$(1-\mu^2)\frac{d^2w}{d\mu^2}-2\mu\frac{dw}{d\mu}+n(n+1)w=2(1-\mu)P_n'-P_n.$$

Therefore, if v denotes

$$\frac{2}{2^n n!}\frac{d^nz}{d\mu^n}-P_n\log\tfrac{1}{2}(1+\mu),$$

$$(1-\mu^2)\frac{d^2v}{d\mu^2}-2\mu z\frac{dv}{d\mu}+n(n+1)v+(2n+1)P_n=0.$$

Hence $$v=\frac{d}{dn}P_n(\mu).$$

It is obvious from this form that $\frac{d}{dn} P_n(\mu)$ is of the form $P_n(\mu) \log \frac{1}{2}(1+\mu) +$ a polynomial of degree n in μ.

If we assume the form $A_0 P_0 + A_1 P_1 + A_2 P_2 \ldots A_n P_n$ for this polynomial and substitute in the differential equation satisfied by $\frac{d}{dn} P_n(\mu)$, the coefficients $A_0, A_1, \ldots, A_{n-1}$ are easily found to be those given above. A_n is found from the fact that $\frac{d}{dn} P_n(\mu)$ vanishes when $\mu = 1$, so that $A_0 + A_1 + \ldots + A_n = 0$.

ON A PROPERTY OF ALGEBRAIC NUMBERS.

By *Prof. W. Burnside.*

LIOUVILLE'S proof of the existence of transcendental numbers turns on the expression of the root of an equation in the form of a continued fraction. So far as I know it has not been noticed that the nature of the restrictions on a number, involved in its being the root of an equation, is even more clearly brought out by supposing it expressed as a decimal fraction.

Let $$f(x) \equiv a_0 x^m + a_1 x^{m-1} + \ldots + a_m = 0$$
be an irreducible equation with rational integral coefficients, and x_1 a real positive root of the equation. Suppose x_1 expressed as the sum of the integral part and a decimal fraction, and denote by p_n the approximation to x_1 to n places of decimals.

As in Horner's method of approximation put

$$y = 10^n (x - p_n).$$

In the resulting equation for y, say

$$B_0 + B_1 y + \ldots + B_m y^m = 0,$$

the root y_1, corresponding to x_1, lies between 0 and 1. If y_2 is the root corresponding to another root x_2 of the original equation

$$y_2 = y_1 + 10^n (x_2 - x_1).$$

Since $f(x) = 0$ is irreducible, $x_2 - x_1$ is not zero. Hence, by taking n large enough, it may be insured that the moduli of all the roots of the equation in y, except y_1, are as large as desired. Now

$$\frac{1}{y_1} + \frac{1}{y_2} + \ldots + \frac{1}{y_m} = -\frac{B_1}{B_0} = -\frac{10^{-n} f'(p_n)}{f(p_n)}.$$

Also

$$f(p_n) = 10^{-mn}[a_0(10^n p_n)^m + 10^n a_1(10^n p_n)^{m-1} + \ldots + 10^{mn} a_m].$$

Since $10^n p_n$ is an integer, the quantity in the square bracket is an integer. It cannot be zero, since $f(x) = 0$ is irreducible, and therefore has no rational roots. Hence

$$\left|\frac{1}{y_1} + \frac{1}{y_2} + \ldots + \frac{1}{y_m}\right| \leq |10^{(m-1)n} f'(p_n)|$$

and

$$y_1 \geq \frac{1}{10^{(m-1)n}|f'(p_n)| + \left|\frac{1}{y_2} + \ldots + \frac{1}{y_m}\right|}.$$

As n increases $\left|\frac{1}{y_2} + \ldots + \frac{1}{y_n}\right|$ approaches the limit zero, while $f'(p_n)$ approaches the limit $f'(x_1)$. Hence if r is an integer, such that

$$10^r > |f'(x_1)| \geq 10^{r-1},$$

then when n is large enough

$$y_1 \geq 10^{-(m-1)n-r}$$

or

$$10^{-n} y_1 \geq 10^{-(mn+r)},$$

where $10^{-n} y_1$ is the excess of x_1 over p_n.

The result may be expressed as follows:—

Corresponding to every algebraic number x there are three integers N, m, r, such that when n exceeds N, the excess of x over its approximation to the n^{th} decimal place is greater than $10^{-(mn+r)}$.

A similar result holds supposing the approximation to x carried out in any other scale of notation than the decimal one; and indeed it is easily extended to the case in which x is expressed in the form $\Sigma \alpha_i f^i$, where f is any rational proper fraction, and the α's are integers less than $1/f$, subject to the condition that $\sum_n^\infty \alpha_i f^i$ is less than f^{n-1}, such a representation being unique. In this way it may be shown, as an example, that if a series of increasing integers n_i is formed according to the law

$$n_{i+1} = [n_i^{1+a}] + 1,$$

where $[n_i^{1+a}]$ is the greatest integer in n_i^{1+a}, and a is a given positive quantity, then the sum of the power series $\sum_{i=0}^{i=\infty} x^{n_i}$ for every rational value of x for which it is convergent is a transcendental number.

ON PLANE CURVES OF DEGREE n WITH TANGENTS OF n-POINT CONTACT.

By *Harold Hilton.*

Abstract.

IN this paper are discussed unicursal curves of degree n (i) with a superlinear branch of order $n-1$ and one tangent of n-point contact; (ii) with an $(n-1)$-ple point and two tangents of n-point contact; (iii) with three tangents of n-point contact.

They are capable of projection into one of the curves

$$zy^{n-1} = x^n,$$

$$z(x^{n-1} + x^{n-2}y + \ldots + y^{n-1}) + xy(x^{n-2} + x^{n-3}y + \ldots + y^{n-2}) = 0,$$

$$r \sin(n-1)\theta = a \sin n\theta, \quad x^{1/n} \pm y^{1/n} \pm z^{1/n} = 0, \quad r^{1/n} \cos \theta/n = a^{1/n}.$$

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§ 1. THE properties of the inflexions of a plane cubic curve are well known; and the undulations of a quartic have been discussed by U. Masoni in *Rendiconti Accad. Sci. Fis. Mat. Napoli*, 21 (1882), p. 45. It seems of interest to extend his results to other plane algebraic curves, and we shall study here properties of plane algebraic curves of degree n (n-ics) having one or more tangents of n-point contact.

We shall consider at present unicursal curves (with zero deficiency), leaving other cases for a second paper. To save space we shall content ourselves in the main with the statement of results, leaving the details of the analysis to the reader.

§ 2. First suppose an n-ic has a multiple point $(0, 0, 1)$ of order $n-1$ [an "$(n-1)$-ple point"], and that there is only one distinct tangent $y=0$ at this point, *i.e.* the curve has a superlinear branch of order $n-1$. Suppose, moreover, that the n-ic has a tangent $z=0$ of n-point contact at $(0, 1, 0)$. Then the equation of the curve can be thrown into the form

$$zy^{n-1} = x^n.$$

The curve is its own polar reciprocal with respect to the conic

$$nx^2 - (n-1)y^2 - z^2 = 0.$$

No point of the curve, other than the singular points, is sextactic (*i.e.* no conic has six-point contact with the curve). All conics of closest contact meet $x=0$ and $y=0$ in real points and $z=0$ in unreal points.

§ 3. Now consider the case in which the n-ic has an $(n-1)$-ple point $C(0, 0, 1)$ and two real tangents of n-point contact $y+z=0$ and $x+z=0$ at $A(1, 0, 0)$ and $B(0, 1, 0)$ respectively. The equation of the curve may be put in the form

$$z(x^{n-1}+x^{n-2}y+\ldots+y^{n-1})+xy(x^{n-2}+x^{n-1}y+\ldots+y^{n-2})=0\ldots\text{(i)}.$$

It may be projected into the curve whose Cartesian equation is obtained by replacing z by a in (i). The projection has $x=y$ as an axis of symmetry. It is readily drawn; and we find that, when n is odd, the curve consists of the isolated $(n-1)$-ple point C and a single three-branched circuit meeting every line (other than those through C and the tangents at A, B) in one or three real points. When n is even, the curve consists of a single two-branched circuit through C meeting every line in two real points or none.

The curve is of class $2(n-1)$. Of its $3(n-2)$ inflexions $(n-2)$ coincide at A and $(n-2)$ at B. The rest lie on the line

$$2nz+(n-1)(x+y)=0,$$

which touches the curve at the point $I\{1, 1, (1-n)/n\}$, which is not an inflexion. But none of these inflexions are real, except $J(1, -1, 0)$, when n is odd.

Any point on the curve is

$$\{t(t^n-1),\ t^n-1,\ -t^n+t\},$$

the inflexions being given by the roots, other than $t=1$, of

$$(n-1)(t^{n+1}-1)=(n+1)t(t^{n-1}-1).$$

When n is odd, $n-3$ bitangents pass through J, and there are $(n-3)^2$ others, not counting the tangents at A and B. When n is even, there are $(n-2)$ bitangents through J and $(n-2)(n-4)$ others.

The points of contact of any bitangent lie on a conic through A, B and the intersections of CB, CA with the tangents at A, B. This is obvious for the bitangents through J. If $z=\lambda x+\mu y$ is another bitangent, its points of contact lie on

$$n(x+z)(y+z)=(1+\lambda+\mu)xy.$$

But these points of contact are never real.

The points of contact of the tangents from A (other than the tangents at A) lie on the line IB; and similarly for the tangents from B. But of these points of contact only one is real if n is odd; and none if n is even. The intersections of the tangents from A and B all lie on the curve.

§ 4. Suppose now that the n-ic has an $(n-1)$-ple point and two unreal tangents of n-point contact. We get a form into which such a curve is projectable on replacing x, y, z by $re^{\theta i}$, $re^{-\theta i}$, $-a$ in the curve of § 3, (i). We obtain thus

$$r \sin(n-1)\theta = a \sin n\theta \text{.................(i)},$$

which is symmetrical about the prime vector.

Changing our notation, take O as the pole, and $EOSF$ as prime vector; where $EO = SF = a/(n-1)$, $OS = a$. The curve has tangents of n-point contact at the circular points meeting at the singular focus S. The $n-2$ inflexions, other than the circular points, are real and are the remaining intersections of the curve with the line touching it at F. The $n-2$ real asymptotes pass through E and make angles $k\pi/(n-1)$ with the prime vector, where $k = 1, 2, \ldots, n-2$. If n is odd, one of them is perpendicular to the prime vector and has three-point contact at infinity.

The remaining tangents from E are also real and their points of contact lie on the line bisecting OS at right angles. All the foci (excluding S) lie on the curve. The tangents at O make angles $k\pi/n$ with the prime vector, where $k = 1, 2, \ldots, n-1$. When $n = 2$, the curve is a circle. When $n > 2$, the curve consists of a single $(n-2)$-branched circuit, meeting every line in $n-2$ or n real points.

§ 5. The curve $r \sin(n-1)\theta = a \sin n\theta$ is a particular case of an interesting type of curve. Consider the line $EOSF$, in which

$$OS = a, \quad EO = SF = aq/(p-q),$$

so that E and F divide OS and SO externally in the ratio q/p, where p and q are positive integers prime to one another, p being the greater. In § 4 we have $p = n$, $q = 1$.

Consider now the locus of P when the ratio of the angles POF, PSF is q/p. Taking O as pole, and OSF as prime vector, the equation of the locus is

$$r \sin\frac{p-q}{q}\theta = a \sin\frac{p}{q}\theta \text{.................(i)}.$$

The curve is unicursal, and is symmetrical about the prime vector. Its degree is $p+q-1$, and its class is $2(p-1)$. It has O as a $(p-1)$-ple point, at which the tangents make angles $k\pi/p$ with the prime vector, where $k=1, 2, \ldots, p-1$. Similarly for the $(q-1)$-ple point S. At each circular point there is a singularity whose nature is the same as that of the origin in $y^q=x^p$; and the tangents at the circular points meet at S. All the foci lie on the curve, and the points of contact of tangents from either circular point are collinear.

Of the $3p-q-5$ inflexions each circular point accounts for $p-q-1$. The other $p+q-3$ inflexions are given by $\tan(p\theta/q)=(p/q)\tan\theta$, and are the remaining intersections of the curve with the tangent at F.

There are $p-q-1$ asymptotes making angles $k\pi/(p-q)$ with OF, where $k=1, 2, \ldots, p-q-1$, and all passing through E. The points of contact of the $p+q-1$ other tangents from E are given by $\sin(p+q)\,\theta/q=0$ $(\theta\neq 0)$, and lie on the perpendicular bisector of OS. The lines joining O (or S) to the intersections with the curve of a line through S (or O) are parallel to the sides of a regular polygon.

The curve derived from § 5 (i) in the same way as § 3 (i) from § 4 (i) is the result of taking away the line $x=y$ from

$$y^p(x+z)^q=x^p(y+z)^q,$$

which is the locus of the point

$$\{t^q(t^p-1),\ t^p-1,\ -t^q(t^{p-q}-1)\}.$$

If a curve of degree $p+q-1$ has two q-ple points of the same nature as the origin in $y^q=x^p$, the tangents at which meet at a $(q-1)$-ple point, and has also a $(p-1)$-ple point, it is necessarily of the type discussed in this section.

§ 6. Consider now the unicursal n-ic with three tangents of n-point contact. We may suppose the parameters t of these points to be 0, 1, ∞; and by a suitable choice of homogeneous coordinates the equation of the curve may be thrown into the form

$$x:y:z=t^n:(1-t)^n:-1$$

or

$$x:y:z=t^n:(1-t)^n:1.$$

The former is more convenient when n is odd. The curve is then

$$x^{1/n}+y^{1/n}+z^{1/n}=0.$$

The latter is more suitable when n is even. The curve is then

$$x^{1/n} \pm y^{1/n} \pm z^{1/n} = 0.$$

In either case the curve can be projected to have the symmetry of the equilateral triangle. It has no inflexions other than $t = 0, 1, \infty$, and has no sextactic points other than its intersections $t = -1, 2, \frac{1}{2}$ with the three axes of symmetry. The three points of n-point contact are collinear if n is odd, and such that a conic touches the curve at them if n is even.

The $\frac{1}{2}(n-1)(n-2)$ double points are all acnodes (isolated points) and are given by

$$t = \sin\frac{p\pi}{n} \operatorname{cosec}\frac{(p-q)\pi}{n}\left(\cos\frac{q\pi}{n} \pm i \sin\frac{q\pi}{n}\right) \ldots\ldots \text{(ii)},$$

where p, q are any two of the quantities $1, 2, \ldots, n-1$, such that $p > q$.

The points of contact of the $\frac{1}{2}(n-2)(n-3)$ bitangents are obtained on replacing n by $n-1$ in (i). The bitangents are all "ideal", *i.e.* real with unreal points of contact.

The distribution of the acnodes is as follows. We suppose the curve projected so as to have the symmetry of the equilateral triangle. When n is odd and not a multiple of 3, $\frac{1}{2}(n-1)$ acnodes lie on each axis of symmetry $x = y$, $z = x$, $z = y$, none being at the centre $O(1, 1, 1)$, and the remaining $\frac{1}{2}(n-1)(n-5)$ lie by sixes on circles with centre O. If n is an odd multiple of 3, O is an acnode and each axis of symmetry passes through $\frac{1}{2}(n-3)$ other acnodes, the remaining $\frac{1}{2}(n-3)^2$ lying by sixes on circles with centre O. If n is even and not a multiple of 3, O is not an acnode and each axis of symmetry contains $\frac{1}{2}(n-2)$ acnodes, the remaining $\frac{1}{2}(n-2)(n-4)$ lying by sixes on circles with centre O. If n is an even multiple of 3, O is an acnode, and each axis of symmetry contains $\frac{1}{2}(n-4)$ other acnodes, the remaining $\frac{1}{2}(n^2-6n+12)$ lying by sixes on circles with centre O. Similarly for the bitangents.

§ 7. The curve of § 6 may be projected into the curve with Cartesian equation

$$x = 2^n a t^n, \quad y = 2^n a (1-t)^n,$$

and, if in this we put $re^{\theta i}$ for x and $re^{-\theta i}$ for y, we get the well-known curve

$$r^{1/n} \cos\theta / n = a^{1/n},$$

as the form into which can be projected any unicursal n-ic with one real and two unreal tangents of n-point contact. These tangents are the line at infinity (touching at $\theta = \frac{1}{2}n\pi$) and the circular lines through the pole, which is the only focus.

Of the double points only $\frac{1}{2}(n-1)$ are real if n is odd, and $\frac{1}{2}(n-2)$ if n is even. They are all crunodes (with two real branches) and are collinear (on $\theta = 0$). All the $n-2$ tangents from the real point of n-point contact are real. If n is odd, $n-3$ of them coincide in pairs to form bitangents. Similarly the $n-1$ perpendicular tangents are real, $n-2$ coinciding in pairs when n is even.

AN INTEGRAL EQUATION OCCURRING IN A MATHEMATICAL THEORY OF RETAIL TRADE.

By *Dr. H. Bateman.*

§ 1. A TRADESMAN, who buys and sells various articles, will be supposed to have worked up his business to such an extent that he can be sure of selling his goods at a constant rate so that a new supply of any article will be completely exhausted at the end of an interval of time T after the date of purchase*. Our problem is to find the law according to which goods must be purchased in order that the total value of the stock may remain constant.

To simplify matters the process of buying and selling will be treated as continuous instead of discontinuous. This is approximately true if business is brisk all the time and if the working hours and days are pieced together so that 'business time' can be treated as a continuous variable.

We shall suppose that the initial value of the stock is represented by unity, and that stock of value $\phi(\tau)\,d\tau$, which is purchased in the interval of time from τ to $\tau + d\tau$, is reduced in value by sales in such a way that the value of the remaining portion at time t is $f(t-\tau)\,\phi(\tau)\,d\tau$, where $f(t)$ is a function expressing the rate of sale. According to our simplifying hypothesis we have

$$f(t) = 1 - \frac{t}{T}, \quad (t < T)$$

$$= 0 \qquad (t > T).$$

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* Strictly this should be the date at which the new supply is made available for sale. We shall suppose for simplicity that in the case of any special article the date at which the old supply is exhausted coincides approximately with the date when a new package is opened.

The value at time t of the residual portion of the original stock is clearly $f(t)$, and so the equation for $\phi(\tau)$ is

$$1 = f(t) + \int_0^t f(t-\tau)\,\phi(\tau)\,d\tau \qquad (1).$$

§ 2. To solve an integral equation of type

$$g(t) = \int_0^t f(t-\tau)\,\phi(\tau)\,d\tau \qquad (2)$$

the simplest plan is to form the definite integrals*

$$G(x) = \int_0^\infty e^{-xt} g(t)\,dt,$$

$$F(x) = \int_0^\infty e^{-xt} f(t)\,dt,$$

$$\Phi(x) = \int_0^\infty e^{-xt} \phi(t)\,dt \qquad (3),$$

and to find $\Phi(x)$ by using Borel's relation†

$$G(x) = F(x)\,\Phi(x).$$

The function $\phi(t)$ may then be derived from $\Phi(x)$ by solving the integral equation (3).

In the present case we easily find that

$$F(x) = \frac{1}{x} - \frac{1}{x^2 T}(1 - e^{-xT}), \quad G(x) = \frac{1}{x^2 T}(1 - e^{-xT}),$$

$$\Phi(x) = \frac{1 - e^{-xT}}{xT - 1 + e^{-xT}} = \int_0^\infty e^{-xt}\,\phi(t)\,dt.$$

To determine $\phi(t)$ we assume that $xT > 1$ and expand $\Phi(x)$ in the form

$$\frac{1 - e^{-xT}}{xT - 1} - \frac{e^{-xT} - e^{-2xT}}{(xT-1)^2} + \frac{e^{-2xT} - e^{-3xT}}{(xT-1)^3} - \dots .$$

Writing

$$\frac{e^{-nxT} - e^{-(n+1)xT}}{(xT-1)^{n+1}} = \int_{nT}^{\infty} e^{-xt}\,\psi_n(t - nT)\,dt,$$

we find, on putting $t = nT + \tau$, that

$$\frac{1 - e^{-xT}}{(xT-1)^{n+1}} = \int_0^\infty e^{-x\tau}\,\psi_n(\tau)\,d\tau.$$

* Report on integral equations, *Brit. Assoc. Report* (1910).

† *Leçons sur les séries divergentes* (1901), p. 104; Bromwich's *Infinite Series*, pp. 280–283.

Differentiating with respect to x, we obtain the difference equation

$$\frac{T}{(xT-1)^{n+1}} - \frac{T(1-e^{-xT})}{(xT-1)^{n+1}} - \frac{(n+1)\,T(1-e^{-xT})}{(xT-1)^{n+2}}$$

$$= -\int_0^\infty e^{-x\tau}\,\psi_n(\tau)\,\tau\,d\tau.$$

Therefore

$$\psi_{n+1}(t) = \frac{t^n}{T^{n+1}}\,\frac{1}{(n+1)!}\,e^{t/T} + \frac{1}{n+1}\left(\frac{t}{T}-1\right)\psi_n(t).$$

Now

$$\int_0^\infty e^{-xt+t/T}\,dt = \frac{T}{xT-1},$$

$$\int_T^\infty e^{-xt+(t-T)/T}\,dt = \frac{T}{xT-1}\,e^{-xT};$$

hence

$$\psi_0(t) = \frac{1}{T}e^{t/T} \qquad (t<T)$$

$$= \frac{1}{T}e^{t/T} - \frac{1}{T}e^{(t-T)/T} \qquad (t>T),$$

$$\psi_1(t) = \frac{t}{T^2}\,e^{t/T} \qquad (t<T)$$

$$= \frac{t}{T^2}\,e^{t/T} - \frac{t-T}{T^2}\,e^{(t-T)/T} \qquad (t>T),$$

$$\psi_2(t) = \frac{1}{2!}\,\frac{t^2}{T^3}\,e^{t/T} \qquad (t<T)$$

$$= \frac{1}{2!}\left[\frac{t^2}{T^3}\,e^{t/T} - \frac{(t-T)^2}{T^3}\,e^{(t-T)/T}\right] \qquad (t>T).$$

Summing up we find eventually that

$$\phi(t) = \frac{1}{T}e^{t/T} \qquad (0<t<T)$$

$$= \frac{1}{T}e^{t/T} - \frac{t}{T^2}\,e^{t/T-1} \qquad (T<t<2T)$$

$$= \frac{1}{T}e^{t/T} - \frac{t}{T^2}\,e^{t/T-1} + \frac{t(t-2T)}{T^3 . 2!}\,e^{t/T-2} \qquad (2T<t<3T)$$

....

$$= \frac{1}{T}e^{t/T} - \frac{t}{T^2}\,e^{t/T-1} + \ldots (-1)^n\,\frac{t(t-nT)^{n-1}}{T^{n+1} . n!}\,e^{t/T-n}$$

$$[nT<t<(n+1)T].$$

It may be verified by direct integration that this function satisfies the integral equation (i).

In the case when $f(t) = e^{-at}$ the solution is very much simpler, for then

$$G(x) = \frac{1}{x} - \frac{1}{a+x} = \frac{a}{x(a+x)},$$

$$F(x) = \frac{1}{a+x}, \quad \Phi(x) = \frac{a}{x},$$

$$\phi(t) = a.$$

New goods must thus be purchased at a constant rate.

§ 3. Equations involving definite integrals of type (2) have been used in mathematical physics since the time of Poisson. They occur in the theory of elastic afterworking and dielectric absorption. Equations which also involve derivatives of the unknown function $\phi(t)$ may be solved by the above method. Such equations occur frequently in the theory of recording instruments, especially when there is an irreversible time effect. A good discussion of some typical problems is given in *Aeronautical Instruments Circular*, No. 32, issued by the National Bureau of Standards, Washington, D.C., November, 1918.

An equation of type (2) also occurs in the theory of an aeroplane encountering gusts (British Advisory Committee for Aeronautics, Report No. 121, 1913-14), and may be solved by the above method. An alternative method of solution is given in L. Bairstow's *Applied Aerodynamics* and in the British Advisory Committee's Report for 1916.

ON APOLAR AND CO-APOLAR TRIANGLES FOR A CUBIC, AND ON APOLARLY CONJUGATE TRIANGLES.

By *Prof. E. B. Elliott.*

1. Prof. W. P. Milne has contributed to the *Proceedings of the London Mathematical Society* a series of papers, so briefly expressed as to be difficult reading, on apolar cubic pencils, and triangles apolar with regard to a cubic curve. It may be instructive to prove some of his theorems by elementary analytical methods. Any restrictions on the generality of the theorems will thus present themselves in readily intelligible forms.

The only cases of binary and of ternary apolarity which are relevant are the following:

(1) Two binary cubic pencils

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0,$$
$$a'x^3 + 3b'x^2y + 3c'xy^2 + d'y^3 = 0,$$

are mutually apolar if and only if the invariant relation

$$ad' - 3bc' + 3cb' - da' = 0$$

is satisfied, *i.e.* if

$$(a, b, c, d)\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)^3 \cdot (a', b', c', d')(x, y)^3 = 0.$$

(2) If x, y, z and u, v, w are allied systems of point- and line-coordinates, the class-cubic $(A, \ldots)(u, v, w)^3 = 0$ is apolar with regard to (or for) the order-cubic $(a', \ldots)(x, y, z)^3 = 0$ if and only if the invariant condition

$$(A, \ldots)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^3 \cdot (a, \ldots)(x, y, z)^3 = 0$$

is satisfied. In particular if the class-cubic is a three-point set, the triangle whose vertices are at the three points is apolar for the order-cubic if and only if the condition is satisfied.

We shall deal almost entirely with triangles apolar for a cubic and *inscribed* to it. Cubics for which the triangle of reference is an inscribed apolar triangle constitute the system

$$fy^2z + f'yz^2 + gz^2x + g'zx^2 + hx^2y + h'xy^2 = 0;$$

for the left-hand side may contain no xyz term, being annihilated by $\dfrac{\partial^3}{\partial x \partial y \partial z}$. It can be readily seen that the only sets of three lines included in this system are (1) three concurrent lines, one through each vertex of the triangle of reference ABC, (2) a side, BC say, and two lines, one of them through the opposite vertex A, which divide BC harmonically, and (3) a side counted twice and a line through the opposite vertex.

When Prof. Milne uses the term two apolar triangles or triads he does not mean an ABC and a triad in a relationship (1) or (2) or (3) to it, but two triangles both apolar for some the same cubic.

He treats in particular of what he calls two *co-apolar* triangles for a cubic. As well as being both apolar for the cubic, and in fact inscribed to it, two co-apolar triangles for the cubic form a pair which jointly have an association with it in which the notion of *binary* apolarity is present. The relationship is expressed in what I will call *Milne's first theorem :*

Given two triangles (triads of points) ABC, DEF, the locus of a point P, such that P(ABC) and P(DEF) are mutually apolar pencils, is (in general) a cubic through A, B, C, D, E, F with regard to which ABC and DEF are both apolar triangles. The two triangles are said to be co-apolar for the cubic locus of P, and to have that cubic for their apolar locus.

The employment of a quite elementary method to prove this theorem generally true will incidentally indicate exceptions to it. Given one triangle ABC, it will be seen that we can in an infinity of ways associate with it another DEF, such that ABC and DEF have no restricted apolar locus, but have the property that the pencils $P(ABC)$ and $P(DEF)$ are apolar for *every* position of P. Two such triangles ABC, DEF I will venture to call *apolarly conjugate triangles.*

The exception causes no surprise when we reflect that there are like exceptions to like theorems as to two systems of one point each and two systems of two points each.

To say that $ax+by=0$ and $a'x+b'y=0$ are apolar is to say that $ab'-a'b=0$, *i.e.* that the two lines coincide. Two one-ray systems PA, PD are apolar when and only when P, A, D are collinear. The apolar locus for A, D is then of the first order, the line AD. But if A, D coincide, so do PA, PD wherever P be, and there is no apolar locus. A is apolarly conjugate to itself.

Again the two two-line pencils

$$ax^2+2bxy+cy^2=0, \quad a'x^2+2b'xy+c'y^2=0$$

are apolar when and only when $ac'-2bb'+ca'=0$, *i.e.* when they are pairs of harmonic conjugates. Taking two point-pairs—A, B and D, E—the locus of P, such that $P(AB)$ are harmonic conjugates with regard to $P(DE)$, is of the second order, a conic through A, B, D, E, which degenerates when there are collinearities of three or more of A, B, D, E. This is the apolar locus. But there is no apolar locus when D, E lie on AB and divide it harmonically. Then there is apolarity wherever P lies. The two pairs of harmonic conjugates are apolarly conjugate pairs.

2. Take ABC for triangle of reference, and let D, E, F be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), which may not or may

be collinear—I do not attend to cases when both triads are collinear. If P is (x', y', z') the pencil $P(ABC)$ is

$$\left(\frac{x}{x'}-\frac{z}{z'}\right)\left(\frac{y}{y'}-\frac{z}{z'}\right)\left\{\left(\frac{x}{x'}-\frac{z}{z'}\right)-\left(\frac{y}{y'}-\frac{z}{z'}\right)\right\}=0,$$

or, say, $$\xi\eta(\xi-\eta)=0.$$

The pencil $P(DEF)$ is

$$(\xi\eta_1-\xi_1\eta)(\xi\eta_2-\xi_2\eta)(\xi\eta_3-\xi_3\eta)=0.$$

The two pencils are apolar if and only if

$$\xi_1\eta_2\eta_3+\xi_2\eta_3\eta_1+\xi_3\eta_1\eta_2=\eta_1\xi_2\xi_3+\eta_2\xi_3\xi_1+\eta_3\xi_1\xi_2,$$

i.e.

$$\sum_1^3\left(\frac{x_1}{x'}-\frac{z_1}{z'}\right)\left(\frac{y_2}{y'}-\frac{z_2}{z'}\right)\left(\frac{y_3}{y'}-\frac{z_3}{z'}\right)$$
$$=\sum_1^3\left(\frac{y_1}{y'}-\frac{z_1}{z'}\right)\left(\frac{x_2}{x'}-\frac{z_2}{z'}\right)\left(\frac{x_3}{x'}-\frac{z_3}{z'}\right)\ \ldots\ldots(1),$$

i.e.

$$\frac{1}{y'^2z'}\Sigma z_1y_2y_3+\frac{1}{z'^2x'}\Sigma x_1z_2z_3+\frac{1}{x'^2y'}\Sigma y_1x_2x_3$$
$$=\frac{1}{y'z'^2}\Sigma y_1z_2z_3+\frac{1}{z'x'^2}\Sigma z_1x_2x_3+\frac{1}{x'y'^2}\Sigma x_1y_2y_3;$$

so that, on multiplying through by $x'^2y'^2z'^2$, we find for the apolar locus of ABC, DEF the cubic, which may be degenerate,

$$y^2z\Sigma z_1x_2x_3-yz^2\Sigma y_1x_2x_3+z^2x\Sigma x_1y_2y_3-zx^2\Sigma z_1y_2y_3$$
$$+x^2y\Sigma y_1z_2z_3-xy^2\Sigma x_1z_2z_3=0\ \ldots\ldots(2).$$

It passes through A, B, C, as the form (2) of its equation tells us. It also passes through D, E, F by the form (1). The triangles ABC, DEF are both apolar for it, because

$$\frac{\partial^3}{\partial x\,\partial y\,\partial z}$$

and

$$\left(x_1\frac{\partial}{\partial x}+y_1\frac{\partial}{\partial y}+z_1\frac{\partial}{\partial z}\right)\left(x_2\frac{\partial}{\partial x}+y_2\frac{\partial}{\partial y}+z_2\frac{\partial}{\partial z}\right)\left(x_3\frac{\partial}{\partial x}+y_3\frac{\partial}{\partial y}+z_3\frac{\partial}{\partial z}\right)$$

both annihilate the left of (2).

There is failure here to determine a unique cubic locus when and only when D, E, F have such positions that all the coefficients in (2) vanish. We will see that this can happen.

When it does the condition for apolarity of $P(ABC)$ and $P(DEF)$ is satisfied for every P in the plane. The triangles ABC, DEF are then apolarly conjugate.

3. Let us examine closely the conditions for failure to find a definite apolar locus, *i.e.* for success in determining apolarly conjugate triads to ABC.

First examine possibilities with one of D, E, F at one of A, B, C, say when $y_1 = 0 = z_1$. With these values the coefficients in (2) all vanish if and only if

$$x_1(z_2x_3 + z_3x_2) = 0 \quad \text{..................(3)},$$

$$x_1(x_2y_3 + x_3y_2) = 0 \quad \text{..................(4)},$$

$$x_1y_2y_3 = 0 \quad \text{..........................(5)},$$

$$x_1z_2z_3 = 0 \quad \text{..........................(6)}.$$

We cannot have $x_1 = 0$ with $y_1 = 0 = z_1$. Thus (5) and (6) give either

$$y_2 = 0 = z_2, \text{ or } y_3 = 0 = z_3, \text{ or } y_2 = 0 = z_3, \text{ or } y_3 = 0 = z_2.$$

The first alternative would give from (3) and (4) $z_3 = 0 = y_3$, as $x_2 \neq 0$. This would mean that D, E, F all lie at A. The same would follow from $y_3 = 0 = z_3$. Again $y_2 = 0 = z_3$ would reduce (3) and (4) to $z_2x_3 = 0 = x_2y_3$, *i.e.* as $x_3 = 0 = z_3$ and $x_2 = 0 = y_2$ are both impossible, to the alternative between $z_2 = 0 = y_3$ and $x_2 = 0 = x_3$. Thus either E, F as well as D would lie at A, or E, F would lie one at B and one at C. The same alternative would be given by $y_3 = 0 = z_2$. So far then we have that:

If D, E, F are all at one of A, B, C, or one at each of A, B, C, the pencils $P(ABC)$, $P(DEF)$ are apolar for every P in the plane; and there are no other triads DEF, with one of D, E, F at one of A, B, C, which have not with ABC definite cubic apolar loci.

Next let one of D, E, F be on a side, but not at a vertex, of ABC. Say $z_1 = 0$, but $x_1 \neq 0$, $y_1 \neq 0$. For the vanishing of the coefficients in (2) there have to be satisfied

$$x_2z_3 + x_3z_2 = 0 \quad \text{..........................(7)},$$

$$y_1x_2x_3 + x_1(x_2y_3 + x_3y_2) = 0 \text{..............(8)},$$

$$x_1y_2y_3 + y_1(x_2y_3 + x_3y_2) = 0 \text{..............(9)},$$

$$y_2z_3 + y_3z_2 = 0 \text{..........................(10)},$$

$$z_2z_3 = 0 \quad \text{..............................(11)}.$$

By (11) z_2 or z_3 must vanish. Say $z_2=0$. (7) and (10) then give $z_3=0$, as x_2, y_2 cannot both vanish with z_2. There remain to be satisfied (8) and (9). Also $x_1\neq 0$, $y_1\neq 0$. Thus

$$(x_2y_3+x_3y_2)^2=x_2x_3y_2y_3;$$

and this relation secures consistency for finding x_1/y_1. It gives

$$\frac{x_2}{y_2}=\omega\frac{x_3}{y_3} \text{ or } \omega^2\frac{x_3}{y_3}, \quad \frac{x_1}{y_1}=\omega^2\frac{x_3}{y_3} \text{ or } \omega\frac{x_3}{y_3}.$$

Hence:—*If three lines through a vertex of the triangle of reference ABC have an equation of the form $x^3=ky^3$, and meet the opposite side in points D, E, F, then the pencils P(ABC), P(DEF) are apolar for every P in the plane; and there is no other triad of points D, E, F of which any one is on any side but not at a vertex of ABC which has not with ABC a definite cubic apolar locus.*

Lastly, examine possibilities when none of D, E, F is at a vertex or on a side of ABC. Write

$$x_1', y_1', x_2', \ldots, \text{ for } \frac{x_1}{z_1}, \frac{y_1}{z_1}, \frac{x_2}{z_2}, \ldots,$$

none of which has a vanishing denominator. The vanishing of the coefficients in (2) can be expressed

$$x_2'+x_3'=-x_1' \quad\ldots\ldots(12),$$

$$x_2'x_3'=-x_1'(x_2'+x_3')=x_1'^2 \quad\ldots\ldots(13),$$

$$y_2'+y_3'=-y_1' \quad\ldots\ldots(14),$$

$$y_2'y_3'=-y_1'(y_2'+y_3')=y_1'^2 \quad\ldots\ldots(15),$$

$$x_2'y_3'+x_3'y_2'=-\frac{y_1'}{x_1'}x_2'x_3'=-x_1'y_1' \quad\ldots\ldots(16),$$

$$x_2'y_3'+x_3'y_2'=-\frac{x_1'}{y_1'}y_2'y_3'=-x_1'y_1' \quad\ldots\ldots(17),$$

of which the last two are the same. The first two tell us that x_2', x_3' are the roots of $x^2+x_1'x+x_1'^2=0$, *i.e.* that $\frac{x_2'}{x_1'}, \frac{x_3'}{x_1'}$ are ω, ω^2 in one order or the other. The next two tell us that $\frac{y_2'}{y_1'}, \frac{y_3'}{y_1'}$ are ω, ω^2 in one order or the other. The remaining equation (16) is satisfied if the two orders are different, but not if they are the same. Thus all five equations are satisfied if we take for D, E, F

$$(x_1, y_1, z_1), \;(\omega x_1, \omega^2 y_1, z_1), \;(\omega^2 x_1, \omega y_1, z_1),$$

and in no other way, but for changes of order. Here the first (x_1, y_1, z_1) may be taken at will, not on a side of ABC. Accordingly:

If D is any point (x, y, z) not on a side of ABC, and E, F are the two associated points $(\omega x, \omega^2 y, z)$, $(\omega^2 x, \omega y, z)$, the vertices of the triangles ABC, DEF subtend apolar pencils at any point P in the plane, i.e. are apolarly conjugate; and there is no other triad of points (none on a side of ABC) which has not with ABC a definite cubic apolar locus.

To sum up:—*Point-triads apolarly conjugate to a point-triad $uvw=0$ constitute the system*

$$a^3u^3 + b^3v^3 + c^3w^3 - 3abc\,uvw = 0.$$

The following are facts as to a pair of apolarly conjugate triads ABC, DEF:

(1) Both triads cannot be real, when none of D, E, F is at A or B or C.

(2) If, as we have supposed, A, B, C are not collinear, and D, E, F are not all on a side of ABC, then D, E, F are not collinear. For

$$\begin{vmatrix} 1, & 1, & 1 \\ \omega, & \omega^2, & 1 \\ \omega^2, & \omega, & 1 \end{vmatrix} \neq 0.$$

(3) In the same general case the coordinates of A, B, C referred to DEF must be related just as are those of D, E, F referred to ABC. In fact the connectors of pairs of (a, b, c), $(\omega a, \omega^2 b, c)$, $(\omega^2 a, \omega b, c)$ are

$$\frac{x}{a} + \omega\frac{y}{b} + \omega^2\frac{z}{c} = 0, \quad \frac{x}{a} + \omega^2\frac{y}{b} + \omega\frac{z}{c} = 0, \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0,$$

and, if the left-hand sides here are called $\frac{\alpha}{a'}$, $\frac{\beta}{b'}$, $\frac{\gamma}{c'}$, the α, β, γ coordinates of the x, y, z points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ have respectively the ratios of

$$(a', b', c'), \quad (\omega a', \omega^2 b', c'), \quad (\omega^2 a', \omega b', c').$$

(4) The point-coordinate equations of the sides of ABC, DEF, *i.e.*

$$xyz = 0, \quad \frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{z^3}{c^3} - 3\frac{xyz}{abc} = 0,$$

are related exactly as are the line-coordinate equations of their vertices. Hence two apolarly conjugate triangles have a property correlative to the defining property, viz. *two apolarly conjugate triangles cut any transversal in mutually apolar ranges of points.*

4. Having considered whether two triangles necessarily have a cubic apolar locus, we turn to another question. Is every cubic an apolar locus for two triangles? Again we will mean two triangles of which at least one is not degenerate. The answer is provided by *Milne's Second Theorem:*

Given any cubic, and any inscribed triangle which is apolar for it, it is possible to specify an infinity of triangles DEF, also inscribed, and apolar for the cubic, in such a way that ABC, DEF are co-apolar triangles for the cubic.

Prof. Milne's proof of the existence theorem is not direct. His specification of the triangles DEF is by means of a property which they possess in common. There is a class-cubic which is the envelope of lines cut in mutually apolar ranges by the sides of the triangle ABC and the order-cubic which is given. There are also point-triads D, E, F which have the property that the nine tangents from any one triad to the class-cubic also touch an associated class-cubic whose line-coordinate equation, referred to ABC, is of the form

$$\lambda u^3 + \mu v^3 + \nu w^3 + 6\rho\, uvw = 0\,;$$

and the triangles DEF are those of which any one is co-apolar with ABC for the given cubic. Our object is to specify the individual triangles DEF by our elementary method.

Refer the given cubic to the given triangle ABC as triangle of reference, so that (§ 1) its equation is

$$fy^2z + f'yz^2 + gz^2x + g'zx^2 + hx^2y + h'xy^2 = 0 \;\ldots(18),$$

with known coefficients. We have to see that, by proper choice of (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), this and (2) may be identified.

In the following general investigation it is assumed that the cubic (18) is not so degenerate that all but one of the coefficients vanish. A separate consideration of so special a case, say for instance the case of (18) being $y^2z = 0$, tells us that in that case D, E, F may be either one at A, one at C, and one anywhere on AB, or two at A, and one anywhere on AC.

In general take (x_1, y_1, z_1), with $z_1 \neq 0$, for D. We shall see that it must lie on (17), but that no other condition need be imposed on it for it to be possible to associate with it as E, F points (x_2, y_2, z_2), (x_3, y_3, z_3), which will secure the identity of (2) and (18). Writing

$$x_1', y_1', x_2', \ldots, \text{ for } \frac{x_1}{z_1}, \frac{y_1}{z_1}, \frac{x_2}{z_2}, \ldots,$$

the conditions are

$$\frac{x_2'x_3' + x_3'x_1' + x_1'x_2'}{f} = \frac{y_1'x_2'x_3' + y_2'x_3'x_1' + y_3'x_1'x_2'}{-f'} = \frac{x_1'y_2'y_3' + \ldots}{g}$$

$$= \frac{y_2'y_3' + \ldots}{-g'} = \frac{y_1' + y_2' + y_3'}{h} = \frac{x_1' + x_2' + x_3'}{-h'} = t, \text{ say} \ldots (19).$$

Here t must not be zero. [With $t = 0$ we have in § 3 found solutions, viz. (x_1', y_1'), $(\omega x_1', \omega^2 y_1')$, $(\omega^2 x_1', \omega y_1')$ with any x_1', y_1'. But these points and A, B, C subtend apolar pencils at any point whatever.]

That the point D taken must lie on the cubic (18) is seen at once on multiplying the numerators in (19) by

$$y_1'^2, \quad -y_1', \quad x_1', \quad -x_1'^2, \quad x_1'^2 y_1', \quad -x_1' y_1'^2$$

respectively, and adding. In like manner a possible E, F going with D must also lie on the cubic. A possible DEF will then certainly be inscribed, as it is required to be.

In (19) we have five linear equations, presented as six made consistent by the fact that (x_1', y_1') lies on (18), for the determination of $x_2' + x_3'$, $x_2'x_3'$, $y_2' + y_3'$, $y_2'y_3'$, $x_2'y_3' + x_3'y_2'$ in terms of x_1', y_1' and t. One relation imposed on x_1', y_1', t makes the values of these five functions of four letters consistent. The relation proves to be quadratic in t, determining two values of t, each of which provides values of x_2', y_2', x_3', y_3' in terms of x_1', y_1'. Thus no further condition need be imposed on x_1', y_1'.

The work may proceed as follows. The equations (19) are

$$x_2' + x_3' = -h't - x_1' \quad \ldots\ldots (20),$$

$$x_2'x_3' = ft - x_1'(x_2' + x_3') = (f + h'x_1')t + x_1'^2 \quad \ldots\ldots (21),$$

$$y_2' + y_3' = ht - y_1' \quad \ldots\ldots (22),$$

$$y_2'y_3' = -g't - y_1'(y_2' + y_3') = -(g' + hy_1')t + y_1'^2 \ldots (23),$$

$$\left.\begin{aligned} x_2'y_3' + x_3'y_2' &= (1/x_1')\{-f't - y_1'x_2'x_3'\} \\ &= -(1/x_1')\{(f' + fy_1' + h'x_1'y_1')t + x_1'^2 y_1'\} \end{aligned}\right\} \ldots (24),$$

$$\left.\begin{aligned} x_2'y_3' + x_3'y_2' &= (1/y_1')\{gt - x_1'y_2'y_3'\} \\ &= (1/y_1')\{(g + g'x_1' + hx_1'y_1')t - x_1'y_1'^2\} \end{aligned}\right\} \ldots (25),$$

of which the last two are the same because (x_1', y_1') lies on (18).

By (20) and (21) x_2', x_3' are the roots of

$$x^2 + (h't + x_1')\,x + (f + h'x_1')\,t + x_1'^2 \text{.........} (26);$$

and by (22) and (23) y_2', y_3' are the roots of another quadratic. But a definite root x_2' of (26) will have going with it a definite root of the second quadratic as y_2', and not a choice of roots. In fact (22) and (24) determine y_2', y_3' without ambiguity when x_2', x_3' are known. Thus

$$(x_2' - x_3')\,y_2' = x_2'(ht - y_1') + (1/x_1')\{(f' + fy_1' + h'x_1'y_1')\,t + x_1'^2 y_1'\},$$

and

$$(x_2' - x_3')\,y_3' = -x_3'(ht - y_1') - (1/x_1')\{(f' + fy_1' + h'x_1'y_1')\,t + x_1'^2 y_1'\}.$$

Hence, on multiplication of the two sinisters and the two dexters, and use on both sides of (20) and (21),

$$\left.\begin{aligned}&\{(h't + x_1')^2 - 4\,(f + h'x_1')\,t - 4x_1'^2\}\,\{-(g' + hy_1')\,t + y_1'^2\}\\ &= -\{(f + h'x_1')\,t + x_1'^2\}\,(ht - y_1')^2\\ &\quad + (1/x_1')\,(h't + x_1')\,(ht - y_1')\,\{(f' + fy_1' + h'x_1'y_1')\,t + x_1'^2 y_1'\}\\ &\quad - (1/x_1'^2)\,\{(f' + fy_1' + h'x_1'y_1')\,t + x_1'^2 y_1'\}^2\end{aligned}\right\} \text{......} (27).$$

This apparent cubic in t has t for a factor. The relevant values of t are then given, as above stated, by a quadratic. Use of (25) instead of (24) would have given what is in effect the same quadratic. Either root of the quadratic is a value of t, in terms of x_1', y_1', which determines one pair of points (x_2', y_2'), (x_3', y_3'), the second and third or third and second of a triad D, E, F, such that ABC and DEF are co-apolar for the cubic.

Thus (in general) there are *two* inscribed triangles DEF with any chosen point D on the cubic as first vertex, which go with a given inscribed ABC, apolar for the cubic, in constituting a pair having the cubic for apolar locus; and they have been specified.

The class-cubic $\lambda u^3 + \mu v^3 + \nu w^3 + 6\rho\, uvw = 0$ in Mr. Milne's association with DEF is

$$x_1 x_2 x_3 u^3 + y_1 y_2 y_3 v^3 + z_1 z_2 z_3 w^3 + \sum_1^6 x_1 y_2 z_3 \,.\, uvw = 0.$$

5. Another theorem which Prof. Milne proves by somewhat difficult synthetic geometry is the following:

A triangle ABC inscribed in a cubic and apolar for it projects through a point P on the cubic into another inscribed apolar triangle DEF.

Take PAB for triangle of reference, so that the equation of the cubic has the form

$$fy^2z + f'yz^2 + gz^2x + g'zx^2 + hx^2y + h'xy^2 + 2mxyz = 0.$$

The coordinates of A, B, D, E are

$$(1, 0, 0),\ (0, 1, 0),\ (g, 0, -g'),\ (0, f', -f).$$

Let those of C and F be (x, y, z_1) and (x, y, z_2), so that z_1, z_2 are the roots of the quadratic

$$z^2(gx + f'y) + z(g'x^2 + fy^2 + 2mxy) + xy(hx + h'y) = 0 \quad ..(28),$$

and consequently

$$z_1z_2(gx + f'y) = xy(hx + h'y) \quad ....(29).$$

The condition, given to be satisfied, that ABC is apolar for the cubic is

$$hx + h'y + mz_1 = 0 \quad ....(30);$$

and the condition, to be proved consequential, for DEF to be apolar is found to be

$$z_2(mgf' - f'^2g' - g^2f) + f'x(gh - mg') + gy(f'h' - mg) = 0 \quad ..(31).$$

To prove this is, by (29) and (30), to prove that

$$f'g(gx + f'y)(hx + h'y) - mf'g(g'x^2 + fy^2 + mxy) = 0 \quad ...(32).$$

Now if ABC is a non-degenerate triangle, *i.e.* C not on AB, z_1 is not zero, and so, by (30), $hx + h'y$ is not. Also z_1, *i.e.* $-(1/m)(hx + h'y)$, satisfies (28). Expressing this, and dividing by the non-vanishing $hx + h'y$, we obtain that (32) is satisfied.

The proof fails, and the theorem needs modification, if C is on AB, *i.e.* if $z_1 = 0$, and so $hx + h'y = 0$. By (28) z_2 is then given by

$$z_2(gx + f'y) + g'x^2 + fy^2 + 2mxy = 0;$$

and for this z_2, with $hx + h'y = 0$, to satisfy (31), a condition in the coefficients must be satisfied. This proves to be

$$\{2gf'm - f'^2g' - g^2f\}\{hh'm - g'h'^2 - fh^2\} = 0 \quad ...(33).$$

That collinear apolar triads on a cubic differ from inscribed apolar triads in general in not projecting through every vertex on the cubic into apolar triads excites less surprise when we reflect that *every* collinear triad of points on a cubic is apolar for it. That the triad (1, 0, 0), (0, 1, 0), $(h', -h, 0)$ is apolar for the cubic (28) suffices to prove this.

Let us now examine cases when (33) is satisfied, so that collinear triads project through P, *i.e.* (0, 0, 1), into apolar triads.

(i) If $$2gf'm - f'^2g' - g^2f = 0 \quad \text{.................(34)},$$

the polar conic of P is

$$(gx + f'y)\left(2z + \frac{g'}{g}x + \frac{f}{f'}y\right) = 0,$$

i.e. consists of the tangent at P and another line. P is then a point of inflexion. The equation of the cubic becomes

$$z(gx + f'y)\left(z + \frac{g'}{g}x + \frac{f}{f'}y\right) + xy(hx + h'y) = 0,$$

and consequently D, E, F are collinear on $z + \frac{g'}{g}x + \frac{f}{f'}y = 0$. Thus a collinear (and so apolar) triad on a cubic projects through a point of inflexion into another *collinear* (apolar) triad.

(ii) In the other case, when

$$hh'm - g'h'^2 - fh^2 = 0 \quad \text{..............(35)},$$

the tangents at A, B, C are

$$hy + g'z = 0,\quad h'x + fz = 0,\quad hx + h'y + mz = 0,$$

and (35) is the condition that these are concurrent. They are $u_{11} = 0$, $u_{22} = 0$, $u_{12} = 0$, where u is the cubic, so that their intersection is on the Hessian. Accordingly a collinear (apolar) triad on a cubic projects through any point of the cubic into an apolar (not in general collinear) triad on the cubic if the tangents at the triad are concurrent, but not otherwise. The locus of the point of concurrence is the Hessian, and consequently the line of collinearity of the triad taken may be any tangent to the Cayleyan.

6. In his earlier investigations Prof. Milne has much used, without directly proving, the converse fact that the vertices of any apolar inscribed triangle for a cubic may be projected through some point of the cubic into a collinear triad upon it. To verify this it will suffice to see that a conic $\lambda yz + \mu zx + \nu xy = 0$, circumscribed to the triangle of reference ABC, may be so

chosen as to have three-point contact at some point P with the cubic $xy(hx+h'y)+z(g'x^2+fy^2)+z^2(gx+f'y)=0$, for which ABC is apolar. If it can, then the three points where PA, PB, PC meet the cubic again will be residual on the cubic to A, B, C, P, P, P which lie on the conic, and so will be collinear. Now, by elimination of z, we find that the lines from C to the common points other than A, B, C of the conic and cubic are given by

$$(hx+h'y)(\mu x+\lambda y)^2-\nu(\mu x+\lambda y)(g'x^2+fy^2)$$
$$+\nu^2 xy(gx+f'y)=0;$$

and here the left-hand side will be a cube $(y'x-x'y)^3$ if two homogeneous quartic conditions in λ, μ, ν are satisfied. A choice of suitable values for $\lambda:\mu:\nu$ is then provided.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

LIII.

On certain criteria for the convergence of the Fourier series of a continuous function.

1. In this note I propose to examine the logical relations of certain modern criteria for the truth of the equation

$$(1) \qquad \lim_{n\to\infty}\frac{2}{\pi}\int_0^\delta \frac{\sin nx}{x}f(x)\,dx=f(0),$$

where $f(x)$ is a function integrable in the sense of Lebesgue and continuous for $x=0$. It is well known that any test for the convergence of the Fourier series of a continuous* function may be reduced to a set of sufficient conditions for the truth of (1).

I suppose, as we may do without loss of generality, that $f(0)=0$. This being so, there are five tests in question, two 'classical' and three modern. The two classical tests are

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* More generally, of a function all of whose discontinuities are regular, *i.e.* such that $\phi(x-0)+\phi(x+0)=2\phi(x)$.

(D) *Dini's test: the integral*

$$\int_0^\delta \frac{|f(x)|}{x}\, dx$$

is convergent:

(J) *Jordan's test:* $f(x)$ *is of bounded variation.*

The three modern tests are

(V) *de la Vallée-Poussin's test*: the mean value*

$$F(x) = \frac{1}{x}\int_0^x f(t)\, dt$$

is of bounded variation:

(Y) *Young's test*†: $\int_0^x |d\{tf(t)\}| = O(x)$:

(L) *Lebesgue's test*‡: $\int_x^\delta \frac{|f(t+x) - f(t)|}{t}\, dt = o(1)$.

The relations between these tests are of considerable interest, and have not all been stated explicitly. I proceed to consider them more closely.

2. I. *Neither* (D) *nor* (J) *includes the other.*

This is well known. Thus

$$f(x) = \left(\log\frac{1}{x}\right)^{-1}$$

satisfies (J), but not (D); and

(2) $$f(x) = x^\rho \sin\frac{1}{x} \quad (0 < \rho \leq 1)$$

satisfies (D), but not (J).

II. (V) *includes both* (D) *and* (J).

This is proved by de la Vallée-Poussin. For the sake of completeness, and a slight simplification, I repeat the proofs.

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* Ch. J. de la Vallée-Poussin, 'Un nouveau cas de convergence des séries de Fourier', *Rendiconti del Circ. Mat. di Palermo*, vol. xxxi. (1911), pp. 296–299. See also his *Cours d'Analyse*, vol. ii., ed. 2, p. 149.

† W. H. Young, 'Sur la convergence des séries de Fourier', *Comptes Rendus*, 21 Aug. 1916.

‡ H. Lebesgue, 'Recherches sur la convergence des séries de Fourier', *Math. Annalen*, vol. lxi. (1905), pp. 251–280. See also his *Leçons sur les séries trigonométriques*, pp. 59 and 64.

(*a*) If a function is monotonic, its mean value is also monotonic. Hence, if $f(x)$ is of bounded variation, so also is $F(x)$. Thus (V) includes (J).

(*b*) Suppose (D) satisfied. We have

$$(3) \qquad |F'| = \left| \frac{f}{x} - \frac{1}{x^2} \int_0^x f\,dt \right| \leq \frac{|f|}{x} + \frac{1}{x^2} \int_0^x |f|\,dt$$

for $x > 0$, at any rate with the exception of a set of values of x of measure zero. The first term on the right-hand side is integrable in $(0, \delta)$. Also

$$\int_\epsilon^\delta \frac{dx}{x^2} \int_0^x |f|\,dt = \frac{1}{\epsilon} \int_0^\epsilon |f|\,dt - \frac{1}{\delta} \int_0^\delta |f|\,dt + \int_\epsilon^\delta \frac{|f|}{x}\,dx$$

tends to a limit when $\epsilon \to 0$. Hence the second term on the right-hand side of (3) is also integrable in $(0, \delta)$. Therefore $|F'|$ is integrable in $(0, \delta)$; and F is of bounded variation.

III. (Y) *includes* (J).

For
$$\int_0^x |d(tf)| \leq \int_0^x |f|\,dt + \int_0^x t\,|df|$$
$$= o(x) + O\left(x \int_0^x |df|\right) = O(x)$$

if (J) is satisfied.

IV. (Y) *does not include* (D).

This may be shown by means of the function (2). Here we have

$$d(xf) = \left\{(\rho + 1)\,x^\rho \sin\frac{1}{x} - x^{\rho-1} \cos\frac{1}{x}\right\} dx;$$

and (Y) demands that

$$\int_0^x |t|^{\rho-1} \left|\cos\frac{1}{t}\right| dt = O(x).$$

This is only true if $\rho \geq 1$, whereas (D) is satisfied whenever $\rho > 0$.

A fortiori it follows that (Y) *does not include* (V).

3. V. (V) *does not include* (Y).

Write $\log(1/x) = lx$, and consider the function

$$f(x) = \frac{\sin lx}{lx}.$$

Here (Y) demands that

$$\int_0^x \left| \frac{\sin lt}{lt} - \frac{\cos lt}{lt} + \frac{\sin lt}{(lt)^2} \right| dt = O(x),$$

which is plainly true. On the other hand it is easily verified that

$$F = \frac{1}{x} \int_0^x \frac{\sin lt}{lt} dt = \frac{\cos lx + \sin lx}{2lx} + O\left\{\frac{1}{(lx)^2}\right\},$$

$$F' = \frac{f-F}{x} = \frac{\sin lx - \cos lx}{2x\,lx} + O\left\{\frac{1}{x\,(lx)^2}\right\};$$

so that

$$\int_0^\delta |F'|\,dx$$

is divergent. Thus (V) is not satisfied.

4. VI. (L) *includes* (V).

This is stated by de la Vallée-Poussin*, on the ground of a communication of Lebesgue; but no proof seems to have been printed.

It is useful to observe first that (L) may be stated also in the form†

(L′) $$\int_x^\delta |\psi(t+x) - \psi(t)|\,dt = o(1),$$

where $\psi = f/x$. To see this, we observe that the difference of the integrals which occur in (L) and (L′) does not exceed

$$\int_\eta^\delta \frac{|f(t+x) - f(t)|}{t} dt + \int_\eta^\delta |\psi(t+x) - \psi(t)|\,dt + \int_x^\eta \frac{|f(t+x)|}{t(t+x)} dt,$$

where $x < \eta < \delta$. We can choose η so that the last integral is less than

$$\epsilon \int_x^\eta \frac{dt}{t(t+x)} < \epsilon \log 2.$$

When η is fixed, the first two integrals tend to zero, by a fundamental theorem in the theory of Lebesgue.‡ The proposition is thus established.

---

* *Cours d'Analyse*, vol. ii., ed. 2, p. 150.

† This is remarked by Lebesgue, *loc. cit.*

‡ If ϕ is integrable in (a, b), then

$$\int_a^b |\phi(t+x) - \phi(t)|\,dt$$

tends to zero with x. See for example Lebesgue, *Leçons sur les séries trigonométriques*, p. 15.

Suppose now that (V) is satisfied. Then $f = F + xF'$ for almost all values of x, and (L′) will certainly be satisfied if

$$(4) \qquad \int_x^\delta |F'(t+x) - F'(t)|\, dt = o(1)$$

and

$$(5) \qquad \int_x^\delta \left| \frac{F(t+x)}{t+x} - \frac{F(t)}{t} \right| dt = o(1).$$

Of these equations, (4) is satisfied in virtue of the integrability of $F'(x)$, and (5) may (by the same argument that was used above*) be replaced by

$$(6) \qquad \int_x^\delta \frac{|F(t+x) - F(t)|}{t}\, dt = o(1).$$

We write

$$\int_x^\delta \frac{|F(t+x) - F(t)|}{t}\, dt = \int_x^\eta + \int_\eta^\delta = J_1 + J_2.$$

Then

$$J_1 = \int_x^\eta \frac{dt}{t} \left| \int_t^{t+x} F'(u)\, du \right| \leq \int_x^\eta \frac{dt}{t} \int_t^{t+x} |F'(u)|\, du$$

$$= \left[\log t \int_t^{t+x} |F'(u)|\, du \right]_x^\eta - \int_x^\eta \log t \{ |F'(t+x)| - |F'(t)| \}\, dt$$

$$= \log \eta \int_\eta^{\eta+x} |F'|\, du - \log x \int_x^{2x} |F'|\, du - \int_{2x}^{\eta+x} \log(u-x) |F'|\, du$$

$$+ \int_x^\eta \log u\, |F'|\, du$$

$$= \int_x^{2x} \log\left(\frac{u}{x}\right) |F'|\, du + \int_{2x}^\eta \log\left(\frac{u}{u-x}\right) |F'|\, du$$

$$+ \int_\eta^{\eta+x} \log\left(\frac{\eta}{u-x}\right) |F'|\, du.$$

In each of these integrals the logarithmic factor is bounded. Hence J_1 is less than a constant multiple of

$$\int_x^{\eta+x} |F'|\, du,$$

and may therefore be made less than ϵ by choice of η. Finally, when η is fixed, J_2 tends to zero with x. Thus (6), and therefore (L′), is satisfied.

---

* Note that the continuity of f for $x = 0$ involves that of F.

5. VII. (L) *includes* (Y).

Suppose (Y) satisfied, and write $xf=g$; and denote by $V(g)$ the variation of g in $(0, x)$. Then

$$g=o(x), \quad V(g)=O(x);$$

and we can write $$g=g_1-g_2,$$
where g_1 and g_2 are steadily increasing functions of the form $O(x)$. And

$$(7) \quad \int_x^\delta \frac{|f(t+x)-f(t)|}{t}\,dt \leq \int_x^{mx} \frac{|f(t+x)-f(t)|}{t}\,dt$$
$$+\int_{mx}^\delta \left|\frac{g_1(t+x)}{t+x}-\frac{g_1(t)}{t}\right|\frac{dt}{t}+\int_{mx}^\delta \left|\frac{g_2(t+x)}{t+x}-\frac{g_2(t)}{t}\right|\frac{dt}{t}$$
$$=J+J_1+J_2,$$

say. In the first place we have

$$(8) \quad J \leq \int_x^{mx} |f(t+x)|\frac{dt}{t}+\int_x^{mx}|f(t)|\frac{dt}{t} \leq \mu \log m,$$

where μ is the upper bound of $|f|$ in the interval

$$0 \leq t \leq (m+1)x.$$

Next, we have

$$(9) \quad |J_1| \leq \int_{mx}^\delta \frac{|g_1(t+x)-g_1(t)|}{t(t+x)}\,dt + x\int_{mx}^\delta \frac{|g_1(t)|}{t^2(t+x)}\,dt.$$

The second term is

$$(10) \quad O\left\{x\int_{mx}^\delta \frac{dt}{t(t+x)}\right\}=O\left(\log\frac{m+1}{m}\right)=O(\epsilon_m),$$

where ϵ_m is a function of m only which tends to zero when $m\to\infty$. The first term, since g_1 is monotone, is

$$\int_{mx}^\delta \frac{g_1(t+x)-g_1(t)}{t(t+x)}\,dt=\int_{(m+1)x}^{\delta+x}\frac{g_1(u)}{(u-x)u}\,du-\int_m^\delta \frac{g_1(u)}{u(u+x)}\,du.$$

In the first of these integrals we may replace $u-x$ by $u+x$; for the error thus introduced is not greater than

$$2x\int_{(m+1)x}^{\delta+x}\frac{g_1(u)}{(u^2-x^2)u}\,du=O\left\{x\int_{(m+1)x}^{\delta+x}\frac{du}{u^2}\right\}$$
$$=O\left\{\frac{x}{(m+1)x}\right\}=O(\epsilon_m),$$

which may be absorbed into (10). When we make this simplification, we are left with

$$\left(\int_{(m+1)x}^{\delta+x} - \int_{mx}^{\delta}\right) \frac{g_1(u)}{u(u+x)}\,du = \left(\int_{\delta}^{\delta+x} - \int_{mx}^{(m+1)x}\right) \frac{g_1(u)}{u(u+x)}\,du$$

$$= O\left(\int_{\delta}^{\delta+x} \frac{du}{u+x}\right) + O\left(\int_{mx}^{(m+1)x} \frac{du}{u+x}\right)$$

$$= O\left(\log\frac{\delta+2x}{\delta+x}\right) + O\left(\log\frac{m+2}{m+1}\right) = O(\epsilon_x) + O(\epsilon_m),$$

where ϵ_x is a positive function of x alone which tends to zero with x. Combining this result with (9) and (10), we obtain

(11) $$J_1 \leq O(\epsilon_x) + O(\epsilon_m).$$

Plainly J_2 can be treated in the same manner; and so we have, from (7), (8), and (11),

$$\int_x^{\delta} \frac{|f(t+x) - f(t)|}{t}\,dt \leq \mu \log m + O(\epsilon_x) + O(\epsilon_m).$$

The right-hand side may be made as small as we please by choice first of m and then of x; which proves the theorem.

6. Our general conclusion is thus that (L) includes all the remaining tests. The peculiar interest of (V) lies in the fact that so direct a generalisation of (J) should include (D), a test on the face of it of an absolutely different character.

Some of these tests may be extended in such a manner as to apply to functions discontinuous at $x = 0$, the condition

$$f(x) - f(0) = o(1)$$

being replaced by

$$\int_0^x |f(t) - f(0)|\,dt = o(x).$$

But the logical interdependence of the tests, which is all that I am concerned with at the moment, appears most clearly in the simplest case.

FOUR-VECTOR ALGEBRA AND ANALYSIS.

By *C. E. Weatherburn, M.A., D.Sc.*

I.

The object of the following pages is to give a systematic development of the analysis which is founded on the "space-time vectors" introduced by Minkowski,* and at the same

* *Gött. Nach.* (1908), S. 53; also *Math. Ann.*, Bd. 68, S. 472 (1910).

time to extend this analysis in directions that will prove useful for further investigations in the theory of ordinary relativity. After amplifying Minkowski's method of matrices I present the analysis of four-vectors along the lines which I think Gibbs would have followed if he had written upon the subject. In so doing I carry the theory considerably beyond its previous limits; but the results arrived at are the natural extension of the three-dimensional analysis, the differential operator $\Box$ in the present case being analogous to the operator ∇ in the other.

FOUR-VECTOR ALGEBRA.

A.—THE METHOD OF MATRICES.

§ 1. *Orthogonal transformation of four variables.* Consider the set of four variables x_1, x_2, x_3, x_4 which are transformed to x_1', x_2', x_3', x_4' by the homogeneous linear substitution

$$\left.\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ x_2' &= a_{21}x_1 + \dots\dots\dots\dots + a_{24}x_4 \\ x_3' &= a_{31}x_1 + \dots\dots\dots\dots + a_{34}x_4 \\ x_4' &= a_{41}x_1 + \dots\dots\dots\dots + a_{44}x_4 \end{aligned}\right\} \dots\dots\dots\dots (1).$$

The square matrix

$$A = \left\| \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & \dots\dots & \dots\dots & \\ a_{31} & \dots\dots & \dots\dots & \\ a_{41} & \dots\dots & \dots\dots & a_{44} \end{matrix} \right\|$$

is called the matrix of the transformation, and the determinant of this matrix the determinant of the transformation. The transformation itself is completely specified by the relation

$$\|x_1', x_2', x_3', x_4'\| = \|x_1, x_2, x_3, x_4\| \bar{A},$$

where $\bar{A}$ is the matrix conjugate to A. Or, putting x for the matrix $\|x_1, x_2, x_3, x_4\|$ of 1×4 elements, and x' for the corresponding matrix $\|x_1', x_2', x_3', x_4'\|$, we may write the relation briefly

$$x' = x\bar{A} \quad \dots\dots\dots\dots\dots\dots\dots (2).$$

Let us find the properties of the matrix A which ensure that the sum of the squares of the four variables remains unaltered, that is

$$x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = x_1^2 + \dots + x_4^2 \quad \dots\dots\dots (3).$$

Squaring and adding the equations (1) we see immediately that this requires

$$\left.\begin{aligned} a_{1r}a_{1s}+a_{2r}a_{2s}+\ldots+a_{4r}a_{4s}&=0, \quad r\neq s \\ a_{1r}^{2}+a_{2r}^{2}+\ldots+a_{4r}^{2}&=1, \quad r=1,\ldots,4 \end{aligned}\right\} \quad \ldots\ldots(4).$$

In virtue of these relations it follows from the equations (1) that

$$\left.\begin{aligned} x_1&=a_{11}x_1'+a_{21}x_2'+\ldots+a_{41}x_4' \\ x_2&=a_{12}x_1'+\ldots\ldots\ldots\ldots+a_{42}x_4' \\ x_3&=a_{13}x_1'+\ldots\ldots\ldots\ldots+a_{43}x_4' \\ x_4&=a_{14}x_1'+\ldots\ldots\ldots\ldots+a_{44}x_4' \end{aligned}\right\} \ldots\ldots\ldots\ldots(5),$$

which is equivalent to

$$x=x'A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Squaring and adding the equations (5) we see that the property (3) requires

$$\left.\begin{aligned} a_{r1}a_{s1}+a_{r2}a_{s2}+\ldots+a_{r4}a_{s4}&=0, \quad r\neq s \\ a_{r1}^{2}+a_{r2}^{2}+\ldots+a_{r4}^{2}&=1, \quad r=1,\ldots,4 \end{aligned}\right\} \quad \ldots\ldots(7).$$

Hence the coefficients a_{rs} are such that the sum of the squares of those in any row or in any column is equal to unity, while the sum of the products of corresponding terms in two different rows or columns is equal to zero.

On multiplying both members of (6) by the reciprocal of A we have

$$xA^{-1}=x'AA^{-1}=x'I=x',$$

where I is the idemfactor or unit matrix. Comparing this with (2) we see that

$$\bar{A}=A^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

that is, the conjugate of A is the reciprocal of A. Hence

$$A\bar{A}=I \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

A transformation with the property expressed by (3) is called an *orthogonal transformation.* The matrix A, which then satisfies (8) or (9), is called an *orthogonal matrix.* Equate the determinants of both members of (9). Then, since the determinant D of A is also the determinant of $\bar{A}$, and the determinant of I is unity, we have $D^2=1$, or $D=\pm 1$. We take the positive sign so that the identical transformation $A=I$, whose determinant is unity, may be a particular case.

We shall be concerned only with sets of four quantities of which the first three are real and the fourth purely imaginary. In this case the coefficients a_{rs} which contain the suffix 4 once only are purely imaginary, and all the others real. Such an orthogonal transformation is generally called a *Lorentz transformation.* A set of four quantities x_1, x_2, x_3, x_4, of which the first three are real and the fourth purely imaginary, and which is transformed to x_1', x_2', x_3', x_4' by a Lorentz transformation, is called a *four-vector* (Sommerfeld*) or a *space-time vector* of the first kind (Minkowski). The set of coordinates x, y, z, ict is such a four-vector in the ordinary theory of relativity.

§ 2. *Four-vectors.* We shall denote four-vectors by small symbols in black-letter type, using italics for the corresponding 1×4 matrix. Thus the four-vector of the preceding section will be written

$$\mathfrak{x} = (x_1, x_2, x_3, x_4)$$

and its matrix

$$x = \| x_1, x_2, x_3, x_4 \|.$$

The four quantities comprising the vector are its *components.* *Like* components of two vectors $\mathfrak{x}$, $\mathfrak{y}$ are those with equal suffixes. Thus x_3, y_3 are like components.

The *sum* and the *difference* of two vectors are the vectors obtained by adding and subtracting their like components. We borrow the signs + and − from algebra, writing the sum and the difference of $\mathfrak{x}$ and $\mathfrak{y}$

$$\mathfrak{x} \pm \mathfrak{y} = (x_1 \pm y_1, x_2 \pm y_2, \ldots, x_4 \pm y_4).$$

Clearly these are also four-vectors. The vector obtained by multiplying all the components of a vector $\mathfrak{x}$ by the same algebraic number k is denoted by $k\mathfrak{x}$. Thus

$$k\mathfrak{x} = (kx_1, kx_2, kx_3, kx_4).$$

All the laws of ordinary algebra hold for the addition and subtraction of four-vectors and their multiplication by algebraic numbers.

The *scalar product* of two four-vectors $\mathfrak{x}$ and $\mathfrak{y}$ is defined as the sum of the products of their like components. We shall denote† it by $\mathfrak{x} \cdot \mathfrak{y}$. Thus

$$\mathfrak{x} \cdot \mathfrak{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 \quad \ldots\ldots\ldots\ldots (10).$$

* *Ann. der Physik.*, Bd. 32, S. 749 (1910); Bd. 33, S. 649 (1910).

† The bracket notation $(\mathfrak{x}, \mathfrak{y})$ is also used for the scalar product.

This quantity is invariant with respect to an orthogonal transformation; that is to say, $\mathfrak{x}'.\mathfrak{y}' = \mathfrak{x}\cdot\mathfrak{y}$. This may be easily shown by substituting from (1) in the expression $x_1'y_1' + \ldots + x_4'y_4'$ and using the relations (4). Or by means of matrices, if we form the product of x and the transposed of y we find

$$x\bar{y} = \| x_1y_1 + x_2y_2 + \ldots + x_4y_4 \|,$$

a matrix of one element which is the scalar product of $\mathfrak{x}$ and $\mathfrak{y}$. Therefore since

$$x'\bar{y}' = x\bar{A}\,\overline{(y\bar{A})} = x\bar{A}A\bar{y} = x\bar{y},$$

it follows that
$$\mathfrak{x}'.\mathfrak{y}' = \mathfrak{x}\cdot\mathfrak{y},$$

as stated. And it is obvious from the definition that the factors of a scalar product are commutative; that is

$$\mathfrak{x}\cdot\mathfrak{y} = \mathfrak{y}\cdot\mathfrak{x}.$$

If the scalar product of two vectors is zero we say, by analogy with the case of three dimensions, that they are *perpendicular*.

The scalar product of a vector $\mathfrak{x}$ with itself is called the *square* of the vector and is written $\mathfrak{x}^2$. Thus

$$\mathfrak{x}^2 = \mathfrak{x}\cdot\mathfrak{x} = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

from the invariance of which we deduced the properties of the coefficients of the orthogonal transformation. A vector whose square is equal to unity is called a *unit vector;* while one whose square vanishes is said to be *singular*.

The first three components of a four-vector $\mathfrak{x}$ may be regarded as the components of an ordinary vector $\mathbf{x}$. In fact, in all the applications each four-vector is formed from an ordinary real vector and an imaginary scalar. We will therefore sometimes find it convenient to employ the notation

$$\mathfrak{x} = (\mathbf{x}, x_4), \quad \mathfrak{y} = (\mathbf{y}, y_4) \quad \ldots\ldots\ldots\ldots\ldots(11)$$

for the four-vectors $\mathfrak{x}$ and $\mathfrak{y}$, in which $\mathbf{x}$ is the ordinary vector whose components are x_1, x_2, x_3. Then the scalar product defined above is simply

$$\mathfrak{x}\cdot\mathfrak{y} = \mathbf{x}.\mathbf{y} + x_4y_4,$$

$\mathbf{x}\cdot\mathbf{y}$ being the scalar product of the ordinary vectors $\mathbf{x}$, $\mathbf{y}$.

The scalar product follows the *distributive law* of multiplication; that is

$$\mathfrak{x}\cdot(\mathfrak{y} + \mathfrak{z}) = \mathfrak{x}\cdot\mathfrak{y} + \mathfrak{x}\cdot\mathfrak{z}.$$

This is obvious from the definition of such products. Repeated application of this result shows that the scalar product of two sums of vectors may be expanded according to the ordinary distributive law of algebra.

It will be convenient to introduce the *unit* four-vectors

$$\mathfrak{i} = (1, 0, 0, 0), \quad \mathfrak{j} = (0, 1, 0, 0),$$
$$\mathfrak{k} = (0, 0, 1, 0), \quad \mathfrak{h} = (0, 0, 0, 1).$$

The square of any one of these is equal to unity, while the scalar product of any two is zero. They are therefore four mutually perpendicular unit vectors. In terms of these any four-vector $\mathfrak{x}$ may be expressed

$$\mathfrak{x} = x_1\mathfrak{i} + x_2\mathfrak{j} + x_3\mathfrak{k} + x_4\mathfrak{h} \quad \text{..............(12).}$$

If two vectors $\mathfrak{x}$, $\mathfrak{y}$ are such that the components of $\mathfrak{x}$ are proportional to those of $\mathfrak{y}$, the vectors may be said to be *parallel*. In this case either is a multiple of the other.

As a converse to the invariance of the scalar product of two four-vectors we may prove the theorem that if a set of four quantities (s_1, s_2, s_3, s_4) gives an invariant product

$$p = s_1x_1 + s_2x_2 + \ldots + s_4x_4$$

with an arbitrary four-vector $\mathfrak{x}$, it is itself a four-vector. For if s and s' denote the matrix of these four quantities before and after the transformation, we are given that $x'\bar{s}' = x\bar{s}$, and therefore

$$x\bar{A}\bar{s}' = x\bar{s}$$

for an arbitrary four-matrix x. Hence

$$\bar{A}\bar{s}' = \bar{s},$$

from which it follows that

$$s'A = s$$

or

$$s' = s\bar{A},$$

which proves the theorem.

§ 3. *Space-time matrices* or *Lorentz matrices.* Consider the square matrix

$$S = \left\| \begin{matrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & \ldots & \ldots & \ldots \\ s_{31} & \ldots & \ldots & \ldots \\ s_{41} & \ldots & \ldots & s_{44} \end{matrix} \right\| \quad \text{..............(13),}$$

whose 4×4 elements are functions of four variables, which are subject to a Lorentz transformation of matrix A. If the substitution transforms S into the matrix

$$S' = AS\bar{A} \quad \text{......................(14),}$$

S is called a *space-time matrix* (Minkowski) or a *Lorentz matrix*. If then x is any four-vector the matrix xS is the matrix of another four-vector. For the transformation changes this to

$$x'S' = (x\bar{A})(AS\bar{A}) = (xS)\bar{A},$$

which proves the statement.

The product of two Lorentz matrices ST is itself a Lorentz matrix. For

$$S'T' = (AS\bar{A})(AT\bar{A}) = A(ST)\bar{A},$$

as required. The conjugate of S is also a Lorentz matrix. For

$$\bar{S}' = \overline{(AS\bar{A})} = A\bar{S}\bar{A},$$

which proves the statement. In the same way it follows that the reciprocal and therefore also the adjoint of S are Lorentz matrices because $A^{-1} = \bar{A}$. It also follows from (14) that the determinant of S is *invariant*. For the determinant of the product $AS\bar{A}$ is equal to the product of the determinants of the factors, and the determinant of A is unity.

The sum of the elements in the principal diagonal of a Lorentz matrix is another *invariant*; that is to say

$$s_{11} + s_{22} + s_{33} + s_{44}$$

is unaltered by a Lorentz transformation. For

$$S' = AS\bar{A},$$

and if we multiply out the product in the second member, using the value of A given by (1), we find for the sum of the elements in the leading diagonal of S'

$$\sum_m \sum_n a_{1m} s_{mn} a_{1n} + \sum_m \sum_n a_{2m} s_{mn} a_{2n} + \ldots + \sum_m \sum_n a_{4m} s_{mn} a_{4n}$$

$$= \sum_m \sum_n s_{mn} (a_{1m} a_{1n} + a_{2m} a_{2n} + \ldots + a_{4m} a_{4n}).$$

But by the relations (4) the expression in brackets vanishes, except for $m = n$, when it is equal to unity. Thus the value of the sum is

$$s_{11} + s_{22} + s_{33} + s_{44}$$

as required. A simpler proof of this result will be given in § 9 by the method of dyadics.

§ 4. *Six-vectors.* Any anti-selfconjugate Lorentz matrix F has the form

$$F = \begin{Vmatrix} 0 & f_{12} & f_{13} & f_{14} \\ f_{21} & 0 & f_{23} & f_{24} \\ f_{31} & f_{32} & 0 & f_{34} \\ f_{41} & f_{42} & f_{43} & 0 \end{Vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

in which $f_{nm} = -f_{mn}$. This involves six and only six independent elements; and the set of six quantities

$$\mathfrak{F} = (f_{23}, f_{31}, f_{12}, f_{14}, f_{24}, f_{34}) \ldots\ldots\ldots\ldots(16)$$

is called a *six-vector* (Sommerfeld) or a space-time vector of the second kind (Minkowski). We shall employ capital symbols in black-letter type to denote six-vectors. The set of quantities (16) is subject to a linear transformation; but it is best to regard them as the six independent elements of an anti-selfconjugate Lorentz matrix, which is transformed according to the formula $F' = AF\bar{A}$.

Two six-vectors are added or subtracted by adding or subtracting their like components. Thus the sum and the difference of $\mathfrak{F}$ and $\mathfrak{G}$ are

$$\mathfrak{F} \pm \mathfrak{G} = (f_{23} \pm g_{23}, f_{31} \pm g_{31}, \ldots, f_{34} \pm g_{34}).$$

To multiply $\mathfrak{F}$ by an algebraic number k multiply each component of $\mathfrak{F}$ by k. The result is written $k\mathfrak{F}$. These combinations of six-vectors are clearly themselves six-vectors.

The *determinant* $|F|$ of the matrix F is easily calculated to be

$$|F| = (f_{23}f_{14} + f_{31}f_{24} + f_{12}f_{34})^2,$$

and we shall write

$$\sqrt{|F|} = (f_{23}f_{14} + f_{31}f_{24} + f_{12}f_{34}) \ldots\ldots\ldots\ldots(17).$$

The adjoint matrix will similarly be found to be

$$F'' = -\sqrt{|F|} \begin{Vmatrix} 0 & f_{34} & f_{42} & f_{23} \\ f_{43} & 0 & f_{14} & f_{31} \\ f_{24} & f_{41} & 0 & f_{12} \\ f_{32} & f_{13} & f_{21} & 0 \end{Vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(18).$$

We call $-F''/\sqrt{|F|}$ the *dual matrix* to F and denote it by F^*. Thus

$$F^* = \begin{Vmatrix} 0 & f_{34} & f_{42} & f_{23} \\ f_{43} & 0 & f_{14} & f_{31} \\ f_{24} & f_{41} & 0 & f_{12} \\ f_{32} & f_{13} & f_{21} & 0 \end{Vmatrix} \quad \ldots\ldots\ldots\ldots\ldots(19).$$

The six-vector determined by this matrix is

$$\mathfrak{F}^* = (f_{14}, f_{24}, f_{34}, f_{23}, f_{31}, f_{12}) \ldots\ldots\ldots\ldots(20),$$

which is called the *dual six-vector* of $\mathfrak{F}$. It is obtainable from $\mathfrak{F}$ by interchanging the first three with the second three components. And the matrix F^* is obtainable from F in the same way, viz., by interchanging 23 and 14, 31 and 24, 12 and 34.

The *scalar product* of two six-vectors, $\mathfrak{F}$ and $\mathfrak{G}$, is defined as the sum of the products of their like components. It will be denoted† by $\mathfrak{F}\cdot\mathfrak{G}$. Thus

$$\mathfrak{F}\cdot\mathfrak{G} = f_{23}g_{23} + f_{31}g_{31} + \ldots + f_{34}g_{34} \ldots\ldots\ldots(21).$$

From the definition it is clear that the distribution law holds for such products. If the scalar product of two six-vectors vanishes they are said to be *perpendicular* or orthogonal. The scalar product of $\mathfrak{F}$ and its dual is

$$\mathfrak{F}\cdot\mathfrak{F}^* = 2\,(f_{23}f_{14} + f_{31}f_{24} + f_{12}f_{34}) = 2\sqrt{|F|} \ldots(22).$$

The product of the matrices F and F^* is

$$FF^* = -FF'' / \sqrt{|F|} = -FF^{-1}\sqrt{|F|}$$
$$= -\sqrt{|F|}\, I = -\tfrac{1}{2}\mathfrak{F}\cdot\mathfrak{F}^* I.$$

If $|F|$ vanishes the scalar product $\mathfrak{F}\cdot\mathfrak{F}^*$ also vanishes, and the vector $\mathfrak{F}$ is orthogonal to its dual. Such a six-vector is said to be singular.

The scalar product of two six-vectors $\mathfrak{F}$ and $\mathfrak{G}$ is *invariant* with respect to a Lorentz transformation. For the product FG of the corresponding matrices is a space-time matrix, and the sum of the elements in its leading diagonal is easily found to be

$$2\,(f_{12}g_{21} + f_{13}g_{31} + \ldots + f_{34}g_{43}) = -2\mathfrak{F}\cdot\mathfrak{G}.$$

But this sum has already been proved invariant. Hence $\mathfrak{F}\cdot\mathfrak{G}$ is invariant. It follows in particular that

$$\mathfrak{F}^2 = \mathfrak{F}\cdot\mathfrak{F} = f_{23}^{\,2} + f_{31}^{\,2} + \ldots + f_{34}^{\,2}$$

and $$\mathfrak{F}\cdot\mathfrak{F}^* = 2\,(f_{23}f_{14} + f_{31}f_{24} + f_{12}f_{34})$$

are both invariants.

The first three components of $\mathfrak{F}$ may be regarded as the components of an ordinary vector $\mathbf{f}_1$, and the remaining three as the components of another $\mathbf{f}_2$. It is convenient to indicate this by the notation

$$\mathfrak{F} = (\mathbf{f}_1, \mathbf{f}_2) \ldots\ldots\ldots\ldots\ldots\ldots(23).$$

† The notation $(\mathfrak{F}, \mathfrak{G})$ is also used for the scalar product.

Then $$\mathfrak{F}^* = (\mathfrak{f}_2, \mathfrak{f}_1).$$
Similarly we should write
$$\mathfrak{G} = (\mathfrak{g}_1, \mathfrak{g}_2), \quad \mathfrak{G}^* = (\mathfrak{g}_2, \mathfrak{g}_1),$$
and with this notation we have
$$\left.\begin{aligned} \mathfrak{F}\cdot\mathfrak{G} &= \mathfrak{f}_1\cdot\mathfrak{g}_1 + \mathfrak{f}_2\cdot\mathfrak{g}_2 \\ \mathfrak{F}^2 &= \mathfrak{f}_1^2 + \mathfrak{f}_2^2 \\ \mathfrak{F}\cdot\mathfrak{F}^* &= 2\mathfrak{f}_1\cdot\mathfrak{f}_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (24).$$

§ 5. *Cross product or vector product of two four-vectors.* Let
$$\mathfrak{x} = (x_1, x_2, x_3, x_4), \quad \mathfrak{y} = (y_1, y_2, y_3, y_4)$$
be two four-vectors. These determine the matrices
$$\overline{X} = \begin{Vmatrix} x_1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 \end{Vmatrix}, \quad Y = \begin{Vmatrix} y_1 & y_2 & y_3 & y_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix},$$
whose product is
$$\overline{X}Y = \begin{Vmatrix} x_1y_1 & x_1y_2 & x_1y_3 & x_1y_4 \\ x_2y_1 & x_2y_2 & \ldots\ldots & \\ x_3y_1 & \ldots\ldots & & \\ x_4y_1 & \ldots\ldots & & x_4y_4 \end{Vmatrix}.$$
This product is a Lorentz matrix. For
$$A\overline{X} = \begin{Vmatrix} x_1' & 0 & 0 & 0 \\ x_2' & 0 & 0 & 0 \\ x_3' & 0 & 0 & 0 \\ x_4' & 0 & 0 & 0 \end{Vmatrix} = \overline{X}',$$
and similarly $$Y\overline{A} = Y'.$$
Therefore $$\overline{X}'Y' = (A\overline{X})(Y\overline{A}) = A(\overline{X}Y)\overline{A},$$
which proves the statement. Similarly $\overline{YX}$ is a Lorentz matrix, and therefore also the difference $\overline{X}Y - \overline{YX}$, which is the anti-selfconjugate matrix

$$\bar{X}Y-\bar{Y}X=\left\|\begin{matrix} 0 & , x_1y_2-x_2y_1, & x_1y_3-x_3y_1, & x_1y_4-x_4y_1 \\ x_2y_1-x_1y_2, & 0 & , \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots \\ x_3y_1-x_1y_3, & \ldots\ldots\ldots\ldots, & 0 & , \ldots\ldots\ldots\ldots \\ x_4y_1-x_1y_4, & \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots, & 0 \end{matrix}\right\|.$$

This skew-symmetric matrix determines the six-vector

$$(x_2y_3-x_3y_2,\ x_3y_1-x_1y_3,\ \ldots,\ x_3y_4-x_4y_3),$$

which is called the *cross product* or *vector product* of $\mathfrak{x}$ and $\mathfrak{y}$ and will be denoted† by $\mathfrak{x}\times\mathfrak{y}$. Thus

$$\mathfrak{x}\times\mathfrak{y}=(x_2y_3-x_3y_2,\ x_3y_1-x_1y_3,\ \ldots,\ x_3y_4-x_4y_3)\ldots(25)$$

is a six-vector. Its dual six-vector is

$$(\mathfrak{x}\times\mathfrak{y})^*=(x_1y_4-x_4y_1,\ x_2y_4-x_4y_2,\ \ldots,\ x_1y_2-x_2y_1)\ldots(26)$$

and it is easily verified that

$$(\mathfrak{x}\times\mathfrak{y})\cdot(\mathfrak{x}\times\mathfrak{y})^*=0\ldots\ldots\ldots\ldots\ldots\ldots(27),$$

so that $\mathfrak{x}\times\mathfrak{y}$ is a singular six-vector. The determinant of the matrix $\bar{X}Y-\bar{Y}X$ is equal to zero.

Since the expansion for $\mathfrak{x}\times\mathfrak{y}$ is linear in the components of $\mathfrak{x}$ and also in those of $\mathfrak{y}$, the *distributive law* holds for the cross product. But the factors of such a product are not commutative, for

$$\mathfrak{x}\times\mathfrak{y}=-\mathfrak{y}\times\mathfrak{x}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28).$$

Reversing the order of the factors changes the sign of the product.

It is worth while noticing that if

$$=(\mathbf{x},\ x_4)\ \text{ and }\ \mathfrak{y}=(\mathbf{y},\ y_4)$$

their cross product is the six-vector

$$\mathfrak{x}\times\mathfrak{y}=(\mathbf{x}\times\mathbf{y},\ y_4\mathbf{x}-x_4\mathbf{y})\ldots\ldots\ldots\ldots\ldots(29),$$

where $\mathbf{x}\times\mathbf{y}$ is the vector product of the ordinary vectors $\mathbf{x}$ and $\mathbf{y}$. Similarly

$$(\mathfrak{x}\times\mathfrak{y})^*=(y_4\mathbf{x}-x_4\mathbf{y},\ \mathbf{x}+\mathbf{y}).$$

§ 6. *Product of a four-vector and a six-vector.* Let

$$\mathfrak{a}=(a_1,\ a_2,\ a_3,\ a_4)$$

† The notation $[\mathfrak{x}, \mathfrak{y}]$ is also used for the vector product.

be a four-vector and $\mathfrak{F}$ a six-vector; and let a, F be the corresponding matrices. The matrix $\overline{F}$ is a Lorentz matrix and therefore, by § 3, the product

$$a\overline{F} = \| a_2 f_{12} + a_3 f_{13} + a_4 f_{14}, \ldots, \ldots, \ldots \|$$

is the matrix of a four-vector. This four-vector

$$(a_2 f_{12} + a_3 f_{13} + a_4 f_{14},\ a_1 f_{21} + a_3 f_{23} + a_4 f_{24},\ \ldots,\ \ldots) \ldots (30)$$

is called the product of the vectors $\mathfrak{a}$ and $\mathfrak{F}$ and will be denoted† by $\mathfrak{a} \times \mathfrak{F}$. If we write

$$\mathfrak{a} = (\mathbf{a}, a_4) \text{ and } \mathfrak{F} = (\mathbf{f}_1, \mathbf{f}_2),$$

the above product may be expressed

$$\mathfrak{a} \times \mathfrak{F} = (\mathbf{a} \times \mathbf{f}_1 + a_4 \mathbf{f}_2,\ -\mathbf{a} \cdot \mathbf{f}_2) \ \ldots\ldots\ldots (31).$$

We might define $\mathfrak{F} \times \mathfrak{a}$ by a product of matrices in the above manner; but for the present it is most conveniently defined by the relation

$$\mathfrak{F} \times \mathfrak{a} = -\mathfrak{a} \times \mathfrak{F} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (32).$$

And, as in previous cases, the distributive law applies to these products.

§ 7. *Infinitesimal vectors and invariants.* We may notice here certain infinitesimal quantities of the nature of four-vectors, six-vectors or invariants. If $l = ict$ we may write $\mathfrak{r} = (x, y, z, l)$ for the position four-vector of a point, and

$$\mathfrak{r} + d\mathfrak{r} = (x + dx,\ y + dy,\ \ldots,\ l + dl)$$

for that of a neighbouring point. Then the difference

$$d\mathfrak{r} = (dx, dy, dz, dl)$$

is also a four-vector; and so also are

$$(dx, 0, 0, 0),\ (0, dy, 0, 0),\ \ldots,\ (0, 0, 0, dl),$$

which are the differences of the position four-vectors of various neighbouring points. The cross product of the first two of these is the six-vector

$$(0, 0, dx\,dy, 0, 0, 0)$$

and five similar ones may be written down by forming the cross products of the others. Hence their sum

† The notation $[\mathfrak{a}, \mathfrak{F}]$ is also used for this product. The product as defined above is the negative of Minkowski's.

$$d\mathfrak{S} = (dy\, dz,\ dz\, dx,\ dx\, dy,\ dx\, dl,\ dy\, dl,\ dz\, dl)$$

is a six-vector, and also its dual

$$d\mathfrak{S}^* = (dx\, dl,\ dy\, dl,\ dz\, dl,\ dy\, dz,\ dz\, dx,\ dx\, dy).$$

The scalar product of these

$$d\mathfrak{S} \cdot d\mathfrak{S}^* = 6\, dx\, dy\, dz\, dl$$

is therefore *invariant*.

Again, since $d\mathfrak{r}$ is a four-vector, each of the quantities

$$(dy\, dz\, dl,\ 0,\ 0,\ 0),\ (0,\ dz\, dl\, dx,\ 0,\ 0),\ \ldots,\ (0,\ 0,\ 0,\ dx\, dy\, dz)$$

is a four-vector, because each gives an invariant scalar product $dx\, dy\, dz\, dl$ with $d\mathfrak{r}$. Hence also their sum

$$d\mathfrak{s} = (dy\, dz\, dl,\ dz\, dx\, dl,\ dx\, dy\, dl,\ dx\, dy\, dz)$$

is a four-vector.

B.—THE METHOD OF DYADICS.

§ 8. *Dyads and dyadics.* Without assuming a knowledge of matrices we might approach the subject from the point of view of dyadics, whose theory in the case of three dimensions was largely developed by Gibbs.† The greater part of his theory is true for any number of dimensions. We shall first consider dyads and dyadics formed with vectors of four components, quite independently of the consideration whether these components are subject to an orthogonal transformation. Then we shall introduce the dyadic which corresponds to such a transformation, and examine the properties of the *four-vectors* which obey it.

To begin with we may speak of the set of four quantities (x_1, x_2, x_3, x_4) simply as a *vector*, reserving the term *four-vector* for a set obeying an orthogonal transformation. Let us denote this vector by $\mathfrak{x}$, and define the sum, the difference, and the scalar product of two vectors just as in § 2. Thus

$$\mathfrak{x} \cdot \mathfrak{y} = x_1 y_1 + x_2 y_2 + \ldots + x_4 y_4,$$
$$\mathfrak{x}^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

and the distributive law holds for such products. In fact, all the results of § 2 apply here, except those connected with the orthogonal transformation.

The *open* product of two vectors $\mathfrak{a}$ and $\mathfrak{b}$ is called a dyad and is written simply $\mathfrak{a}\mathfrak{b}$. The scalar or direct products of this dyad and a vector $\mathfrak{r}$ are defined by

† Cf. E. B. Wilson, *Vector Analysis*, ch. v., New York, 1901.

$$\mathfrak{r}\cdot(\mathfrak{a}\,\mathfrak{b}) = \mathfrak{r}\cdot\mathfrak{a}\,\mathfrak{b},$$
$$(\mathfrak{a}\,\mathfrak{b})\cdot\mathfrak{r} = \mathfrak{a}\,\mathfrak{b}\cdot\mathfrak{r},$$

the vector $\mathfrak{r}$ forming a scalar product with the factor of the dyad next to it. The sum of any number of dyads is called a *dyadic*, and is defined by the property that its product formed in the above manner with a vector $\mathfrak{r}$ is the sum of the products of its separate dyads with this vector. That is to say

$$\mathfrak{r}\cdot(\mathfrak{a}_1\mathfrak{b}_1 + \mathfrak{a}_2\mathfrak{b}_2 + \ldots) = \mathfrak{r}\cdot\mathfrak{a}_1\mathfrak{b}_1 + \mathfrak{r}\cdot\mathfrak{a}_2\mathfrak{b}_2 + \ldots,$$

and so on. The first vector in each dyad is called its *antecedent* and the second its *consequent*. When the dyadic precedes the vector $\mathfrak{r}$ it is called a *prefactor*; when it follows $\mathfrak{r}$ it is called a *postfactor*. Capital Greek letters Φ, Ψ, Ω, ... will be used to denote dyadics. The dyadic obtained by interchanging the antecedent and consequent in each dyad of Φ is called the *conjugate* of Φ and is written Φ_c. It is clear that, for any vector $\mathfrak{r}$,

$$\mathfrak{r}\cdot\Phi = \Phi_c\cdot\mathfrak{r}$$

and

$$\Phi\cdot\mathfrak{r} = \mathfrak{r}\cdot\Phi_c.$$

The properties of dyadics of ordinary vectors have been discussed fairly completely by the author in his *Advanced Vector Analysis* (ch. v.*) Most of the results are true for any number of dimensions, being obvious consequences of the distributive law. To give proofs of these properties when the vectors are of four components would be merely to repeat what I have written there. Attention may be drawn to the following, of which use will be made immediately.

If all the antecedents and all the consequents of a dyadic Φ are expressed in terms of the unit vectors $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$, $\mathfrak{h}$ of § 2, and the separate dyads then expanded according to the distributive law, only 16 different dyads will result; and by collecting similar dyads we may write the expansion

$$\left.\begin{aligned}\Phi = a_{11}\mathfrak{i}\,\mathfrak{i} + a_{12}\mathfrak{i}\,\mathfrak{j} + a_{13}\mathfrak{i}\,\mathfrak{k} + a_{14}\mathfrak{i}\,\mathfrak{h}\\ + a_{21}\mathfrak{j}\,\mathfrak{i} + a_{22}\mathfrak{j}\,\mathfrak{j} + \ldots\qquad\qquad\\ + a_{31}\mathfrak{k}\,\mathfrak{i} + \ldots\qquad\qquad\qquad\quad\\ + a_{41}\mathfrak{h}\,\mathfrak{i} + \ldots\qquad\qquad + a_{44}\mathfrak{h}\,\mathfrak{h}\end{aligned}\right\}\ldots\ldots\ldots\ldots(1),$$

which we shall call the *resolved form* of the dyadic.† Clearly two dyadics are equal if they have the same resolved form.

* Cf. also E. B. Wilson, *loc. cit.*

† Cf. §§ 7–9 of an earlier paper by the author, entitled "Some theorems in four-dimensional analysis", *Quart. Journ.*, vol. xlviii., p. 39 (1917).

The *determinant* of a dyadic is the determinant of the coefficients in its resolved form, arranged in rows and columns as above.

The dyadic I, which is such that, for any vector $\mathfrak{r}$, $I \cdot \mathfrak{r} = \mathfrak{r} = \mathfrak{r} \cdot I$ is called the *idemfactor* or identical dyadic. Its resolved form is

$$I = \mathfrak{i}\mathfrak{i} + \mathfrak{j}\mathfrak{j} + \mathfrak{k}\mathfrak{k} + \mathfrak{h}\mathfrak{h} \quad \text{.................(2).}$$

The *scalar* of a dyadic Φ is the sum of the scalar products of the antecedent and consequent in each of its dyads. It is an invariant, and is written Φ_s. In terms of the coefficients of the resolved form

$$\Phi_s = a_{11} + a_{22} + a_{33} + a_{44} \quad \text{.................(3),}$$

which is the sum of the coefficients in the leading diagonal.

§ 9. *Orthogonal transformation. Four-vectors.* Consider now a dyadic Λ which possesses the property that, for any vector $\mathfrak{r}$, the square of $\mathfrak{r}$ is equal to the square of $\Lambda \cdot \mathfrak{r}$. If this is so,

$$\mathfrak{r}^2 = (\mathfrak{r} \cdot \Lambda_c) \cdot (\Lambda \cdot \mathfrak{r}) = \mathfrak{r} \cdot (\Lambda_c \cdot \Lambda) \cdot \mathfrak{r}$$

for all values of $\mathfrak{r}$. Hence

$$\Lambda_c \cdot \Lambda = I \quad \text{.......................(4),}$$

showing that the conjugate of Λ is equal to the reciprocal Λ^{-1}. If now we express Λ with $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$, $\mathfrak{h}$ as consequents in the form

$$\Lambda = \mathfrak{a}_1\mathfrak{i} + \mathfrak{a}_2\mathfrak{j} + \mathfrak{a}_3\mathfrak{k} + \mathfrak{a}_4\mathfrak{h},$$

then

$$\Lambda_c = \mathfrak{i}\mathfrak{a}_1 + \mathfrak{j}\mathfrak{a}_2 + \mathfrak{k}\mathfrak{a}_3 + \mathfrak{h}\mathfrak{a}_4;$$

and from (4) it follows that

$$\left.\begin{array}{l} \mathfrak{a}_1^2 = \mathfrak{a}_2^2 = \mathfrak{a}_3^2 = \mathfrak{a}_4^2 = 1 \\ \mathfrak{a}_1 \cdot \mathfrak{a}_2 = 0 = \mathfrak{a}_1 \cdot \mathfrak{a}_3 = \mathfrak{a}_2 \cdot \mathfrak{a}_3 = \text{etc.} \end{array}\right\} \text{............(5).}$$

That is to say, $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, $\mathfrak{a}_4$ are mutually perpendicular unit vectors. The relations (5) are equivalent to those found in § 1 among the elements of the matrix A. A transformation

$$\mathfrak{r}' = \Lambda \cdot \mathfrak{r} \quad \text{.........................(6),}$$

which is such that $r'^2 = r^2$ is called an *orthogonal transformation.* The dyadic Λ then satisfies (4) and is called an orthogonal dyadic. Equating the determinants of both members of (4) we find, for the determinant D of Λ,

$$D^2 = 1.$$

We take the positive root $D = 1$, so that the identical transformation $\Lambda = I$ may be a particular case. We shall be concerned only with vectors whose first three components are real, and the fourth purely imaginary. The orthogonal transformation is then called a Lorentz transformation; and the vector $\mathfrak{r}$ which transforms into $\mathfrak{r}'$ according to (6) is called a *four-vector*.

The scalar product of two four-vectors $\mathfrak{r}$ and $\mathfrak{s}$ is *invariant* with respect to an orthogonal transformation. For

$$\mathfrak{r}'\cdot\mathfrak{s}' = \mathfrak{r}\cdot\Lambda_c\cdot\Lambda\cdot\mathfrak{s} = \mathfrak{r}\cdot\mathfrak{s} \quad\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

by (4). This proves the statement. And conversely if the scalar product of an arbitrary four-vector $\mathfrak{r}$ and a vector $\mathfrak{s}$ is invariant, $\mathfrak{s}$ is also a four-vector. For if

$$\mathfrak{r}'\cdot\mathfrak{s}' = \mathfrak{r}\cdot\mathfrak{s}$$

then

$$\mathfrak{r}\cdot\Lambda_c\cdot\mathfrak{s}' = \mathfrak{r}\cdot\mathfrak{s}$$

for an arbitrary four-vector r. Hence

$$\Lambda_c\cdot\mathfrak{s}' = \mathfrak{s}$$

or

$$\mathfrak{s}' = \Lambda\cdot\mathfrak{s},$$

which proves the theorem.

Let Φ be a dyadic composed of four-vectors, say,

$$\Phi = \mathfrak{a}\mathfrak{l} + \mathfrak{b}\mathfrak{m} + \mathfrak{c}\mathfrak{n} + \ldots .$$

Then the transformation (6) changes this to

$$\left.\begin{aligned}\Phi' &= \Lambda\cdot\mathfrak{a}\mathfrak{l}\cdot\Lambda_c + \Lambda\cdot\mathfrak{b}\mathfrak{m}\cdot\Lambda_c + \ldots\\ &= \Lambda\cdot\Phi\cdot\Lambda_c\end{aligned}\right\}\ldots\ldots\ldots\ldots(8).$$

Such a dyadic of four-vectors we shall call briefly a *four-dyadic*. It corresponds to the space-time matrix of § 3. The conjugate Φ_c is obviously also a four-dyadic, for it too consists of four-vectors. And it follows from (8) that the determinant of a four-dyadic is invariant with respect to the orthogonal transformation, because the determinant of Λ is equal to unity.

The *scalar* of a four-dyadic is invariant with respect to the transformation; for the scalar product of the antecedent and consequent of each dyad is invariant. This property was not so simply proved by the matrix method of § 3. The scalar product of a four-vector $\mathfrak{r}$ and a four-dyadic Φ is a four-vector; for the scalar product of $\mathfrak{r}$ and the antecedent of a dyad becomes the (invariant) scalar coefficient of the consequent; or *vice versa*.

The product of two four-dyadics Φ and Ψ is a four-dyadic. For

$$\Phi'\cdot\Psi' = \Lambda\cdot\Phi\cdot\Lambda_c\cdot\Lambda\cdot\Psi\cdot\Lambda_c = \Lambda\cdot(\Phi\cdot\Psi)\cdot\Lambda_c,$$

as required. It is easily shown also that the reciprocal and the adjoint of a four-dyadic are themselves four-dyadics.

§ 10. *Six-vectors. Product of four-vector and six-vector.* An anti-selfconjugate four-dyadic Γ has the resolved form

$$\left.\begin{array}{llll}\Gamma = 0 & + g_{12}\mathfrak{i}\mathfrak{j} & + g_{13}\mathfrak{i}\mathfrak{k} & + g_{14}\mathfrak{i}\mathfrak{h} \\ + g_{21}\mathfrak{j}\mathfrak{i} & + 0 & + g_{23}\mathfrak{j}\mathfrak{k} & + \dots \\ + g_{31}\mathfrak{k}\mathfrak{i} & + \dots & + 0 & + \dots \\ + g_{41}\mathfrak{h}\mathfrak{i} & + \dots & \dots & + 0\end{array}\right\} \quad \dots\dots\dots\dots (9),$$

where $g_{rs} = -g_{sr}$. This involves only six independent coefficients; and the set of six quantities

$$\mathfrak{G} = (g_{23},\ g_{31},\ g_{12},\ g_{14},\ g_{24},\ g_{34})$$

is called a *six-vector*. The adjoint dyadic to Γ is

$$\Gamma'' = -\sqrt{|\Gamma|}\,\Gamma^*,$$

where

$$|\Gamma| = (g_{23}g_{14} + g_{31}g_{24} + g_{12}g_{34})^2 \quad \dots\dots\dots\dots (10)$$

is the determinant of Γ, and Γ^* is the dyadic

$$\left.\begin{array}{llll}\Gamma^* = 0 & + g_{34}\mathfrak{i}\mathfrak{j} & + g_{42}\mathfrak{i}\mathfrak{k} & + g_{23}\mathfrak{i}\mathfrak{h} \\ + g_{43}\mathfrak{j}\mathfrak{i} & + 0 & + g_{14}\mathfrak{j}\mathfrak{k} & + g_{31}\mathfrak{j}\mathfrak{h} \\ + g_{24}\mathfrak{k}\mathfrak{i} & + g_{41}\mathfrak{k}\mathfrak{j} & + 0 & + g_{12}\mathfrak{k}\mathfrak{h} \\ + g_{32}\mathfrak{h}\mathfrak{i} & + \dots & \dots & + 0\end{array}\right\} \quad \dots\dots\dots (11),$$

which is called the *dual dyadic* to Γ, and is also an anti-selfconjugate four-dyadic. The six-vector determined by it is

$$\mathfrak{G}^* = (g_{14},\ g_{24},\ g_{34},\ g_{23},\ g_{31},\ g_{12}) \quad \dots\dots\dots (12).$$

This is the dual six-vector to $\mathfrak{G}$, and is obtainable from it by interchanging the first three with the last three components.

If Φ, Γ are two anti-selfconjugate four-dyadics the scalar of their product $(\Phi\cdot\Gamma)_s$ is invariant with respect to an orthogonal transformation, and is unaltered by expressing the antecedents or consequents of these dyadics in terms of other four-vectors. This invariant has the value

$$2\,(f_{12}g_{21} + f_{23}g_{32} + f_{31}g_{13} + f_{14}g_{41} + \dots).$$

Hence the quantity

$$\mathfrak{F}\cdot\mathfrak{G} \equiv f_{23}g_{23} + f_{31}g_{31} + \dots + f_{34}g_{34} \quad \dots\dots\dots (13)$$

is an *invariant*, and is called the *scalar product* of $\mathfrak{F}$ and $\mathfrak{G}$. And since the distributive law holds for the multiplication of dyadics, it also holds for scalar products of six-vectors.

We may define the *product of a four-vector* $\mathfrak{r}$ *and a six-vector* $\mathfrak{G}$ by the equations

$$\left.\begin{aligned}\mathfrak{r}\times\mathfrak{G}&=-\mathfrak{r}\cdot\Gamma\\ \mathfrak{G}\times\mathfrak{r}&=-\Gamma\cdot\mathfrak{r}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

Each of these products is a four-vector, and because Γ is anti-selfconjugate it follows that

$$\mathfrak{r}\times\mathfrak{G}=-\mathfrak{G}\times\mathfrak{r}.$$

This product may be expressed in terms of the components of $\mathfrak{r}$ and $\mathfrak{G}$, and the value obtained will agree with that given in § 6. Here also the distributive law applies, because it holds for the multiplication of dyadics and vectors.

§ 11. *Cross product (or vector product) of four-vectors.* Let

$$\mathfrak{a}=(a_1,\ a_2,\ a_3,\ a_4),\quad \mathfrak{b}=(b_1,\ b_2,\ b_3,\ b_4)$$

be two four-vectors. With them we may form the anti-self-conjugate dyadic

$$\Pi=(\mathfrak{a}\mathfrak{b}-\mathfrak{b}\mathfrak{a}).$$

The resolved form of this dyadic determines the six-vector

$$\mathfrak{P}=(a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ \ldots,\ a_1b_4-a_4b_1,\ \ldots,\ \ldots).$$

We call this six-vector the cross product or vector product of $\mathfrak{a}$ and $\mathfrak{b}$, and write it

$$\mathfrak{a}\times\mathfrak{b}=\mathfrak{P}.$$

It is clear that by interchanging $\mathfrak{a}$ and $\mathfrak{b}$ we change the sign of the product. Thus

$$\mathfrak{a}\times\mathfrak{b}=-\mathfrak{b}\times\mathfrak{a},$$

while $\mathfrak{a}\times\mathfrak{a}=0$. In fact, the cross product of any two parallel four-vectors is zero. And again the distributive law holds, *i.e.*,

$$\mathfrak{a}\times(\mathfrak{b}+\mathfrak{c})=\mathfrak{a}\times\mathfrak{b}+\mathfrak{a}\times\mathfrak{c},$$

because it holds for the expansion of the open product. The superiority of this method of introducing the vector product over the matrix method of § 5 is apparent.

Corresponding to the scalar of a dyadic Φ we now define the *vector of the dyadic* as the sum of the cross products of the antecedent and consequent in each dyad. Thus, if

$$\Phi=\mathfrak{a}\mathfrak{l}+\mathfrak{b}\mathfrak{m}+\mathfrak{c}\mathfrak{n}+\ldots,$$

its vector, denoted by Φ_v, is

$$\Phi_v=\mathfrak{a}\times\mathfrak{l}+\mathfrak{b}\times\mathfrak{m}+\mathfrak{c}\times\mathfrak{n}+\ldots\quad\ldots\ldots\ldots\ldots(15).$$

And because the distributive law applies alike to open and cross products of four-vectors, it follows that the value of Φ_v

is not altered by expressing the antecedents or consequents of Φ in terms of other four-vectors. And it is clear from the definition that the vector of any dyadic is equal to that of its conjugate with the sign changed. If then the dyadic is selfconjugate its vector is zero. Conversely, if the vector of a dyadic is zero, the dyadic is selfconjugate.

In the case of an anti-selfconjugate dyadic Γ we find from its resolved form (9) that its vector is

$$2\,(g_{23}, g_{31}, g_{12}, g_{14}, g_{24}, g_{34}) = 2\mathfrak{G},$$

where $\mathfrak{G}$ is the six-vector determined by Γ. Thus any six-vector is half the vector of the corresponding anti-selfconjugate dyadic. And if $\mathfrak{r}$ is any four-vector

$$\mathfrak{r}\cdot\Gamma = -\mathfrak{r}\times\mathfrak{G} = -\tfrac{1}{2}\mathfrak{r}\times\Gamma_v \quad\ldots\ldots\ldots\ldots\ldots(16).$$

§ 12. *Products of three-vectors.* Let $\mathfrak{a}$, $\mathfrak{b}$ be two four-vectors, $\mathfrak{F}$ a six-vector, and Φ the corresponding anti-selfconjugate dyadic. Then, since $\mathfrak{a}\times\mathfrak{F} = -\mathfrak{a}\cdot\Phi$, we have

$$\begin{aligned}\mathfrak{a}\times\mathfrak{F}\cdot\mathfrak{b} &= -(\mathfrak{a}\cdot\Phi)\cdot\mathfrak{b} = -\mathfrak{a}\cdot(\Phi\cdot\mathfrak{b})\\ &= \mathfrak{a}\cdot\mathfrak{F}\times\mathfrak{b} = \mathfrak{F}\times\mathfrak{b}\cdot\mathfrak{a}.\end{aligned}$$

Further, by § 10,

$$\begin{aligned}\mathfrak{F}\cdot(\mathfrak{b}\times\mathfrak{a}) &= -\tfrac{1}{2}[\Phi\cdot(\mathfrak{b}\mathfrak{a}-\mathfrak{a}\mathfrak{b})]_s\\ &= -\tfrac{1}{2}(\Phi\cdot\mathfrak{b}\mathfrak{a}-\Phi\cdot\mathfrak{a}\mathfrak{b})_s = -\mathfrak{a}\cdot\Phi\cdot\mathfrak{b},\end{aligned}$$

since Φ is anti-selfconjugate. But this expression agrees with the preceding. Hence in the scalar triple product the dot and cross may be interchanged at pleasure, and the value of the expression depends only on the cyclic order of the factors. We shall denote this value by $[\mathfrak{a}\mathfrak{F}\mathfrak{b}]$. A change in the cyclic order changes the sign of the product. Thus

$$[\mathfrak{a}\mathfrak{F}\mathfrak{b}] = -[\mathfrak{b}\mathfrak{F}\mathfrak{a}].$$

If the two four-vectors are equal (or parallel) to $\mathfrak{a}$ the product has the value $\quad [\mathfrak{a}\mathfrak{F}\mathfrak{a}] = \mathfrak{F}\cdot(\mathfrak{a}\times\mathfrak{a}) = 0.$

We may also have a product of the form $\mathfrak{a}\times(\mathfrak{b}\times\mathfrak{c})$, where $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are four-vectors. By the two preceding §§ this product has the value

$$\begin{aligned}\mathfrak{a}\times(\mathfrak{b}\times\mathfrak{c}) &= -\mathfrak{a}\cdot(\mathfrak{b}\mathfrak{c}-\mathfrak{c}\mathfrak{b})\\ &= \mathfrak{a}\cdot\mathfrak{c}\mathfrak{b} - \mathfrak{a}\cdot\mathfrak{b}\mathfrak{c} \quad\ldots\ldots\ldots\ldots\ldots(17),\end{aligned}$$

a formula which is the same as in the case of ordinary vectors. Similarly

$$\begin{aligned}(\mathfrak{b}\times\mathfrak{c})\times\mathfrak{a} &= -(\mathfrak{b}\mathfrak{c}-\mathfrak{c}\mathfrak{b})\cdot\mathfrak{a}\\ &= \mathfrak{a}\cdot\mathfrak{b}\mathfrak{c} - \mathfrak{a}\cdot\mathfrak{c}\mathfrak{b} \quad\ldots\ldots\ldots\ldots\ldots(17'),\end{aligned}$$

and the reader may easily verify the formulæ

$$(\mathfrak{a} \times \mathfrak{b}) \times (\mathfrak{c} \times \mathfrak{F}) = [\mathfrak{a}\mathfrak{c}\mathfrak{F}]\,\mathfrak{b} - [\mathfrak{b}\mathfrak{c}\mathfrak{F}]\,\mathfrak{a},$$

$$(\mathfrak{a} \times \mathfrak{b})\cdot(\mathfrak{c} \times \mathfrak{d}) = \mathfrak{a}\cdot\mathfrak{c}\;\mathfrak{b}\cdot\mathfrak{d} - \mathfrak{a}\cdot\mathfrak{d}\;\mathfrak{b}\cdot\mathfrak{c}.$$

With the three vectors $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ we may also form the product $\mathfrak{a} \times (\mathfrak{b} \times \mathfrak{c})^*$, involving the dual six-vector of $\mathfrak{b} \times \mathfrak{c}$. By § 6 this is the four-vector

$$\{a_2(b_3c_4 - b_4c_3) + a_3(b_4c_2 - b_2c_4) + a_4(b_2c_3 - b_3c_2)\}\,\mathfrak{i} + \ldots + \ldots$$

$$= \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} \mathfrak{i} + \ldots + \ldots + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \mathfrak{h} \ldots\ldots (18).$$

The value of this expression is clearly unaltered by interchanging the letters a, b, c, provided the same cyclic order is maintained. Therefore

$$\mathfrak{a} \times (\mathfrak{b} \times \mathfrak{c})^* = \mathfrak{b} \times (\mathfrak{c} \times \mathfrak{a})^* = \mathfrak{c} \times (\mathfrak{a} \times \mathfrak{b})^* \ldots\ldots (19).$$

But each of these products is the negative of

$$(\mathfrak{b} \times \mathfrak{c})^* \times \mathfrak{a} = (\mathfrak{c} \times \mathfrak{a})^* \times \mathfrak{b} = (\mathfrak{a} \times \mathfrak{b})^* \times \mathfrak{c} \ldots\ldots (19').$$

Cor.: If two of the vectors are equal (or parallel) the products (19) and (19′) vanish.

§ 13. *Determinant of four four-vectors.* If $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ are any four four-vectors it follows from § 11 that†

$$(\mathfrak{a} \times \mathfrak{b})\cdot(\mathfrak{c} \times \mathfrak{d})^* = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \begin{vmatrix} c_1 & c_4 \\ d_1 & d_4 \end{vmatrix} + \ldots + \ldots$$

$$+ \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} \begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix} + \ldots + \ldots$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (20).$$

From this it is obvious, as it is also from first principles, that

$$(\mathfrak{a} \times \mathfrak{b})\cdot(\mathfrak{c} \times \mathfrak{d})^* = (\mathfrak{a} \times \mathfrak{b})^*\cdot(\mathfrak{c} \times \mathfrak{d}) \ldots\ldots\ldots (21).$$

And by the preceding § each of these is equal to

$$\mathfrak{a}\cdot\mathfrak{b} \times (\mathfrak{c} \times \mathfrak{d})^* = \mathfrak{b} \times (\mathfrak{c} \times \mathfrak{d})^*\cdot\mathfrak{a} = \mathfrak{b}\cdot(\mathfrak{c} \times \mathfrak{d})^* \times \mathfrak{a}$$

or to $\quad (\mathfrak{a} \times \mathfrak{b})^* \times \mathfrak{c}\cdot\mathfrak{d} = \mathfrak{d}\cdot(\mathfrak{a} \times \mathfrak{b})^* \times \mathfrak{c} = \mathfrak{d} \times (\mathfrak{a} \times \mathfrak{b})^*\cdot\mathfrak{c},$

† Cf. § 5 of the author's paper already cited.

but are of opposite sign to

$$(\mathfrak{b} \times \mathfrak{c})^* \cdot (\mathfrak{d} \times \mathfrak{a}) = (\mathfrak{b} \times \mathfrak{c}) \cdot (\mathfrak{d} \times \mathfrak{a})^*.$$

Thus, provided the cyclic order of the factors is maintained, the dot and cross outside the dual six-vector may be interchanged at pleasure, but the sign of the product depends upon the pair of factors grouped to form the dual six-vector. The sign of the grouping $(\mathfrak{a} \times \mathfrak{b})^*$ is the same as that for $(\mathfrak{c} \times \mathfrak{d})^*$, but opposite to that for $(\mathfrak{b} \times \mathfrak{c})^*$ or $(\mathfrak{d} \times \mathfrak{a})^*$. We shall employ the notation

$$[\mathfrak{abcd}] = (\mathfrak{a} \times \mathfrak{b})^* \cdot (\mathfrak{c} \times \mathfrak{d}) = (\mathfrak{a} \times \mathfrak{b}) \cdot (\mathfrak{c} \times \mathfrak{d})^* \quad \text{......(22)},$$

and shall call this expression the *determinant* of the four-vectors $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$.

Cor.: If two of the vectors are equal or parallel their determinant vanishes.

If the determinant of the four-vectors does not vanish, the vectors

$$\left.\begin{aligned} \mathfrak{a}' &= \frac{\mathfrak{b} \times (\mathfrak{c} \times \mathfrak{d})^*}{[\mathfrak{abcd}]}, \quad & \mathfrak{b}' &= \frac{\mathfrak{c} \times (\mathfrak{d} \times \mathfrak{a})^*}{[\mathfrak{bcda}]} \\ \mathfrak{c}' &= \frac{\mathfrak{d} \times (\mathfrak{a} \times \mathfrak{b})^*}{[\mathfrak{cdab}]}, \quad & \mathfrak{d}' &= \frac{\mathfrak{a} \times (\mathfrak{b} \times \mathfrak{c})^*}{[\mathfrak{dabc}]} \end{aligned}\right\} \quad \text{.........(23)}$$

clearly satisfy the relations

$$\mathfrak{a} \cdot \mathfrak{a}' = \mathfrak{b} \cdot \mathfrak{b}' = \mathfrak{c} \cdot \mathfrak{c}' = \mathfrak{d} \cdot \mathfrak{d}' = 1 \quad \text{...........(24)},$$

and, by the above corollary, also

$$0 = \mathfrak{a}' \cdot \mathfrak{b} = \mathfrak{a}' \cdot \mathfrak{c} = \ldots = \mathfrak{b}' \cdot \mathfrak{c} = \mathfrak{b}' \cdot \mathfrak{d} = \text{etc.} \quad \text{......(25)}.$$

In virtue of these relations the vectors $\mathfrak{a}'$, $\mathfrak{b}'$, $\mathfrak{c}'$, $\mathfrak{d}'$ may be called the *reciprocal system* of four-vectors to $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$. The system of unit vectors $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$, $\mathfrak{h}$ is easily shown to be its own reciprocal.

The properties of a system of vectors and its reciprocal have been investigated by the author in an earlier paper†; and it is not necessary here to repeat the results found there.

§ 14. *Other formulæ.* If $\mathfrak{a}$, $\mathfrak{b}$ are two four-vectors and $\mathfrak{F}$ a six-vector, then

$$\mathfrak{b} \times (\mathfrak{a} \times \mathfrak{F}) + \{\mathfrak{a} \times (\mathfrak{b} \times \mathfrak{F}^*)\}^* = -\mathfrak{a} \cdot \mathfrak{b}\,\mathfrak{F} \quad \text{.....(26)}.$$

To prove this let

$$\mathfrak{F} = (\mathbf{f}_1, \mathbf{f}_2), \quad \mathfrak{a} = (\mathrm{a}, a_4), \quad \mathfrak{b} = (\mathrm{b}, b_4).$$

† Ibid. § 6.

Then, by § 6,

$$\mathfrak{a} \times \mathfrak{F} = (\mathbf{a} \times \mathbf{f}_1 + a_4 \mathbf{f}_2, \ -\mathbf{a} \cdot \mathbf{f}_2),$$

and therefore, by (29) of § 5,

$$\mathfrak{b} \times (\mathfrak{a} \times \mathfrak{F}) = \{\mathbf{b} \times (\mathbf{a} \times \mathbf{f}_1 + a_4 \mathbf{f}_2), \ -\mathbf{a} \cdot \mathbf{f}_2 \mathbf{b} - b_4 (\mathbf{a} \times \mathbf{f}_1 + a_4 \mathbf{f}_2)\}.$$

Similarly

$$\mathfrak{a} \times (\mathfrak{b} \times \mathfrak{F}^*) = \{\mathbf{a} \times (\mathbf{b} \times \mathbf{f}_2 + b_4 \mathbf{f}_1), \ -\mathbf{b} \cdot \mathbf{f}_1 \mathbf{a} - a_4 (\mathbf{b} \times \mathbf{f}_2 + b_4 \mathbf{f}_1)\}.$$

Adding then the first and the dual of the second we have

$$\begin{aligned} &\mathfrak{b} \times (\mathfrak{a} \times \mathfrak{F}) + \{\mathfrak{a} \times (\mathfrak{b} \times \mathfrak{F}^*)\}^* \\ &= (-\mathbf{b} . \mathbf{a} \mathbf{f}_1 - a_4 b_4 \mathbf{f}_1, \ -\mathbf{b} . \mathbf{a} \mathbf{f}_2 - a_4 b_4 \mathbf{f}_2) \\ &= -\mathfrak{a} \cdot \mathfrak{b} (\mathbf{f}_1, \mathbf{f}_2) = -\mathfrak{a} . \mathfrak{b} \mathfrak{F}, \end{aligned}$$

as required.

Again, if $\mathfrak{F}$, $\mathfrak{G}$ are any two six-vectors and $\mathfrak{r}$ any four-vector,

$$(\mathfrak{r} \times \mathfrak{F}) \times \mathfrak{G} + (\mathfrak{r} \times \mathfrak{G}^*) \times \mathfrak{F}^* = -\mathfrak{F} \cdot \mathfrak{G} \mathfrak{r} \quad \text{.....(27).}$$

For with the notation $\mathfrak{r} = (\mathbf{r}, r_4)$, $\mathfrak{G} = (\mathbf{g}_1, \mathbf{g}_2)$, etc., we have, by § 6,

$$(\mathfrak{r} \times \mathfrak{F}) \times \mathfrak{G} = \{(\mathbf{r} \times \mathbf{f}_1 + r_4 \mathbf{f}_2) \times \mathbf{g}_1 - \mathbf{r} . \mathbf{f}_2 \mathbf{g}_2, \ -(\mathbf{r} \times \mathbf{f}_1 + r_4 \mathbf{f}_2) \cdot \mathbf{g}_2\},$$

and similarly

$$(\mathfrak{r} \times \mathfrak{G}^*) \times \mathfrak{F}^* = \{(\mathbf{r} \times \mathbf{g}_2 + r_4 \mathbf{g}_1) \times \mathbf{f}_2 - \mathbf{r} \cdot \mathbf{g}_1 \mathbf{f}_1, \ -(\mathbf{r} \times \mathbf{g}_2 + r_4 \mathbf{g}_1) \cdot \mathbf{f}_1\}.$$

The sum of these vectors, got by adding their like components, is equal to

$$-(\mathbf{f}_1 \cdot \mathbf{g}_1 + \mathbf{f}_2 . \mathbf{g}_2)(\mathbf{r}, r_4) = -\mathfrak{F} \cdot \mathfrak{G} \mathfrak{r},$$

as required.

If Φ and Γ are the anti-selfconjugate dyadics corresponding to $\mathfrak{F}$ and $\mathfrak{G}$ respectively, the last formula is equivalent to

$$\mathfrak{r} \cdot \Phi \cdot \Gamma + \mathfrak{r} . \Gamma^* . \Phi^* = -\mathfrak{F} \cdot \mathfrak{G} \mathfrak{r} \cdot \mathrm{I}.$$

And since this is true for any four-vector $\mathfrak{r}$ we have the equation of dyadics

$$\Phi \cdot \Gamma + \Gamma^* \cdot \Phi^* = -\mathfrak{F} \cdot \mathfrak{G} \mathrm{I} \quad \text{............(28).}$$

Replacing each six-vector and dyadic by its dual we have also

$$\Phi^* . \Gamma^* + \Gamma \cdot \Phi = -\mathfrak{F}^* \cdot \mathfrak{G}^* \mathrm{I} \quad \text{..........(28').}$$

But obviously $\mathfrak{F}^* \cdot \mathfrak{G}^* = \mathfrak{F} \cdot \mathfrak{G}$. Hence on subtracting the last two equations we have the result

$$\Phi \cdot \Gamma - \Phi^* \cdot \Gamma^* = \Gamma \cdot \Phi - \Gamma^* \cdot \Phi^* \quad \text{.........(29),}$$

which holds for any two anti-selfconjugate dyadics.

ON THE CONGRUENCE $(p-1)! \equiv -1 \pmod{p^2}$.

By *Dr. N. G. W. H. Beeger.*

1. DEMONSTRATION of the congruence

$$(1)\quad \frac{(p-1)!+1}{p} \equiv (-1)^{\frac{1}{2}(p-1)} \cdot \frac{\frac{1}{2}(p-1)!^2+(-1)^{\frac{1}{2}(p-1)}}{p} - \frac{2^p-2}{p} \pmod{p},$$

where p is a prime number;

$$(p-1)! = (p-1)(p-2)\ldots\left(p-\frac{p-1}{2}\right) \cdot \left(\frac{p-1}{2}\right)!\,.$$

Hence we have the congruence

$$(2)\quad (p-1)! \equiv (-1)^{\frac{1}{2}(p-1)}$$

$$\left[\frac{p-1}{2}! - p\frac{p-1}{2}!\left\{1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{\frac{1}{2}(p-1)}\right\}\right] \cdot \frac{p-1}{2}! \pmod{p^2}.$$

By the aid of Wilson's theorem we conclude

$$(3)\qquad \{\tfrac{1}{2}(p-1)\}!^2 \equiv (-1)^{\frac{1}{2}(p-1)} \pmod{p}.$$

Using the binomial theorem it is easy to prove the congruence

$$(4)\qquad \frac{2^p-2}{p} \equiv -\left\{1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{\frac{1}{2}(p-1)}\right\} \pmod{p}.$$

Substituting (3) and (4) in (2), we get (1).

2. With the congruence (1), I have calculated the residue of $\frac{(p-1)!+1}{p}$ for all the prime numbers $p < 300$, making use of the table of the residues of $\frac{2^p-2}{p}$ (mod p) by W. Meissner*. Take for example $p = 101$. I have calculated the products

$$1.2\ldots20;\ 21.22\ldots29.30;\ 31.32\ldots40;\ 41.42\ldots50.$$

Now we calculate the residues of these products (mod 101^2) and take the product (mod 101^2) of these residues. So we get the residue of $50!$ (mod 101^2). By making use of (1) and of the table of Meissner, we find the residue of $101!$ (mod 101^2). This residue ought to be $\equiv -1$ (mod 101), and when this is the case we can, practically, consider the calculation to be exact.

* *Sitz. Ber. d. Berliner Academie*, vol. xxxv., p. 663 (1913).

We see from the following table that *there are no primes* < 300 *that satisfy the congruence* $(p-1)! \equiv -1 \pmod{p^2}$ *except the well-known cases* $p = 5$ *and* 13.

3. Table.

$W_p =$ least pos. residue of $\dfrac{(p-1)!+1}{p} \pmod{p}$,

$w_p =$ „ „ „ „ $\dfrac{\{\frac{1}{2}(p-1)\}! - (-1)^N}{p} \pmod{p}$

for $p = 4n+3$,

$\mathbf{W}_p =$ „ „ „ „ $\dfrac{\{\frac{1}{2}(p-1)\}!^2 + 1}{p} \pmod{p}$

for $p = 4n+1$.

p	W_p	w_p	$(-1)^N$	$\mathbf{W}_p$	p	W_p	w_p	$(-1)^N$	$\mathbf{W}_p$	p	W_p	w_p	$(-1)^N$	$\mathbf{W}_p$
3	1	0	+		79	73	16	−		181	72			6
5	0			1	83	20	25	+		191	159	157	+	
7	5	1	−		89	70			82	193	35			50
11	1	0	−		97	70			87	197	147			103
13	0			6	101	72			54	199	118	194	−	
17	5			14	103	57	36	−		211	173	90	−	
19	2	4	−		107	1	40	+		223	180	56	−	
23	8	2	+		109	30			55	227	113	144	+	
29	18			20	113	95			35	229	131			38
31	19	0	+		127	51	107	−		233	169			183
37	7			9	131	119	11	−		239	147	40		
41	16			21	137	56			25	241	34			77
43	13	10	−		139	67	6	+		251	214	148	+	
47	6	0	−		149	94			137	257	177			113
53	34			0	151	86	52	+		263	73	33	−	
59	27	8	+		157	151			1	269	121			103
61	56			6	163	108	141	−		271	170	194	−	
67	12	16	−		167	115	26	+		277	25			165
71	69	1	+		173	16			112	281	277			144
73	11			50	179	48	74	−		283	164	143	+	
										293	231			94

ON PASCALIAN COLLINEARITIES AND CONCURRENCIES.

By *E. B. Elliott.*

It seems inconceivable that any facts about the configuration of six points on a conic and their connectors should have escaped notice, but also remarkable that, if other properties of the configuration, that are to be found in researches on the subject, can be expressed almost as simply as Pascal's theorem itself, they should not have become almost as widely known. I do not here attend to the complications of 60 cyclical orders of the six points, but only regard one cyclical order *ABCDEF*.

A method which I believe to be exhaustive yields a set of seven theorems associated with that one order. By QR, RP, PQ are meant the sides which lie along AD, BE, CF respectively of the triangle which those 'diagonals' include.

(i) *Pascal's Theorem.* The intersections of pairs of lines (BC, EF), (CD, FA), (DE, AB), which call L, M, N, are collinear.

(ii) The lines LP, MQ, NR cut QR, RP, PQ respectively in collinear points.

(iii) The lines from the intersections of pairs (BF, CE), (CA, DF), (DB, EA) to P, Q, R respectively are concurrent.

(iv) If L', M', N', L_1, M_1, N_1 are the intersections of the pairs (AB, CD), (BC, DE), (CD, EF), (DE, FA), (EF, AB), (FA, BC), then $L'PL_1$, $M'QM_1$, $N'RN_1$ are collinearities of sets of three points on three concurrent lines.

(v) The harmonic conjugates of these three lines severally, with regard to the pairs of sides of PQR which cut on them, meet the remaining sides of PQR in collinear points. [The one of the three through P is the connector of P with the point in which BC is cut by the connector of (AB, CF) and (CD, BE), or again with the point in which EF is cut by the connector of (DE, CF) and (FA, BE); and the others are provided cyclically.]

(vi) If L'', M'', N'', L_2, M_2, N_2 are the intersections of the pairs (AC, BD), (BD, CE), (CE, DF), (DF, EA), (EA, FB), (FB, AC), then $L''PL_2$, $M''QM_2$, $N''RN_2$ are collinearities of sets of three points on concurrent lines.

(vii) The harmonic conjugates of these three lines severally, with regard to the pairs of sides of PQR which cut on them, meet the remaining sides of PQR in collinear points. [To construct the one through P, connect P with the point in which BC is cut by the connector of (AC, BE) and (BD, CF), or again with the point in which EF is cut by the connector of (DF, BE) and (EA, CF); and to construct the others proceed cyclically.]

All seven theorems are easily proved, Pascal's theorem itself being the most refractory, by reference to the triangle PQR.

The remark which has led me to write them down as a set having a certain completeness is that there are just four of the quadratic covariants of six lines,

$$x-ay,\quad x-by,\quad x-cy,\quad x-dy,\quad x-ey,\quad x-fy,$$

which are invariant for permutations of a, b, c, d, e, f, according to the cycle $(abcdef)$ and its reversal $(fedcba)$, the seminvariant leaders of the four being

$$ab - bc + cd - de + ef - fa,$$

$$ab + bc + cd + de + ef + fa - ac - bd - ce - df - ea - fb,$$

$$2(ad + be + cf) - ab - bc - cd - de - ef - fa,$$

$$2(ad + be + cf) - ac - bd - ce - df - ea - fb.$$

The consideration of certain involutions tells us that the connectors of the intersections of the four covariants, in order, with a conic through the origin are lines each through one of the points stated to be collinear in (i), (ii), (v), (vii), respectively; and the cyclical invariancy then tells us that the same connectors also pass through the other points cyclically specified. The other theorems (iii), (iv), (vi) are companion respectively to (ii), (v), (vii), following from them because, when lines through the vertices of a triangle meet the opposite sides in collinear points, the harmonic conjugates of the lines with regard to the sides which meet on them are concurrent. Also (i) and (ii) are polar with regard to the conic.

TRANSITIVE CONSTITUENTS OF THE GROUP OF ISOMORPHISMS OF ANY ABELIAN GROUP OF ORDER p^m.

By *G. A. Miller*.

§ 1. *Introduction.*

To every automorphism of a group G there corresponds some permutation of the operators of G. This permutation may be represented by a substitution on letters corresponding to these operators. The substitutions which correspond to the totality of the possible automorphisms of G constitute a group known as the group of isomorphisms of G, or as a substitution group which is simply isomorphic with this group of isomorphisms, if the latter is regarded as an abstract group. For convenience we shall regard the former as the actual group of isomorphisms of G in the present article, and shall denote it by I.

A necessary and sufficient condition that I is of degree $g - \alpha$, g being the order of G, is that G contains exactly α characteristic operators. As the identity is a characteristic operator of every group, $\alpha \geq 1$. The transitive constituents of I are separately simply isomorphic either with I or with a quotient group of I, and the number of these constituents is equal to the number of complete sets of conjugate operators of G under I, exclusive of the characteristic operators of G. Operators of G which are conjugate under I are known as *I-conjugate* operators, or simply as I conjugates.

It is well known that the group of isomorphisms of any abelian group is the direct product of the groups of isomorphisms of its Sylow subgroups*. Hence the I of any abelian group can be readily found if the I's of its Sylow subgroups are known. On this account we shall restrict ourselves in what follows to the consideration of the case when G is abelian and when g is of the form p^m, p being a prime number, unless the contrary is explicitly stated.

In the special case, when G is of type (1, 1, 1, ...) I contains a single transitive constituent, and when I is transitive G is evidently of this type. As this I is simply isomorphic with the linear homogeneous congruence group, mod p, and has been extensively studied, it will be assumed to be known in what follows. The main object of this paper, is to reduce the determination of the properties of I for a general abelian prime power group to the determination of these properties when G is of type (1, 1, 1, ...). It will be found that this reduction is comparatively simple and involves only a study of groups whose orders are of the form p^k.

An abelian group cannot contain more than two characteristic operators. Hence the I of an abelian group cannot contain more than two transitive constituents which are separately simply isomorphic with I, and a necessary and sufficient condition that it contains two such constituents is that the order of this abelian group is twice an odd number. For every other abelian group the corresponding I contains one and only one transitive constituent which is simply isomorphic with I. This transitive constituent corresponds to the permutations of the operators of highest order.

§ 2. *Constituents of I corresponding to the operators of the same order.*

Suppose that m_1 of the invariants of G are equal to p^{α_1}, m_2 are equal to p^{α_2}, ..., m_λ are equal to p^{α_λ}. Hence $m_1\alpha_1 + m_2\alpha_2 + \ldots + m_\lambda\alpha_\lambda = m$, p_m being the order of G. It may be assumed without loss of generality that $\alpha_1 > \alpha_2 > \ldots > \alpha_\lambda$. All the operators of G, whose orders divide p^r, $0 < r < \alpha_1$, generate a characteristic subgroup of G. We shall first

* G. A. Miller, *Transactions of the American Mathematical Society*, vol. i. (1900), p. 396.

consider the various subgroups of G composed of all the operators of I which leave invariant all the operators of such a characteristic subgroup.

In the case when $r=\alpha_1-1$, $\alpha_1>1$, all such automorphisms of G can be obtained by multiplying successively every operator of highest order in a set of independent generators of G by every operator in the subgroup generated by the operators of order p contained in G. Moreover, all the automorphisms of G which can be obtained in this way are such that every operator whose order divides p^{α_1-1} corresponds to itself. All these automorphisms correspond to operators whose orders divide p in I and any two such automorphisms correspond to two commutative operators of I. Hence it results that the operators of I which correspond to the automorphisms of G in which each of the operators of order p^{α_1-1} corresponds to itself constitute an abelian invariant subgroup of I, which is of type (1, 1, 1, ...) and of order $p^{m_1m'}$, where m_1 is the number of largest invariants of G and $m'=m_1+m_2+...+m_\lambda$ is the total number of the invariants of G.

Let I_1, I_2, ..., I_{α_1} represent the constituents of I which correspond to the permutations of the operators of G of orders p, p^2, ..., p^{α_1} respectively when G is made simply isomorphic with itself in every possible way. The constituent I_{α_1} is known to be transitive. A necessary and sufficient condition that one of the other constituents is transitive is that all of the invariants of G are equal to each other. Hence it results that if one of these constituents is transitive all of them must be transitive. From the preceding paragraph it follows that the constituent I_{α_1} has a $(p^{m_1}, 1^{m'})$ isomorphism with the group formed by the remaining constituents and that its invariant subgroup which corresponds to the identity of the group formed by the other constituents is abelian and of type (1, 1, 1, ...).

To find the invariant subgroup of I_{α_1-1} which corresponds to the identity of the group formed by the constituents I_{α_1-2}, I_{α_1-3}, ..., I_1 when $\alpha_1>2$ it is only necessary to consider all the automorphisms of G in which each operator whose order divides p^{α_1-2} corresponds to itself and to note how the operators of order p^{α_1-1} are permuted under these automorphisms. To do this we may multiply each operator of highest order in a set of independent generators of G by all the operators of G whose orders divide p^2, and, if G contains any independent generator of order p^{α_1-1} we multiply all such independent generators in the given set by all the operators of G whose orders divide p. In each case the operators of order p^{α_1-1} contained in G are multiplied by operators of order p and any two such automorphisms are commutative. In the former case the number of the different multipliers which produce the same automorphism of the operators of order p^{α_1-1} is equal to the order of the subgroup generated by the operators of order p contained in G.

The number of the distinct permutations arising from an operator of highest order in the given set of independent generators in this second case is therefore equal to the order of the quotient group of the subgroup formed by all the operators of G whose orders divide p^2 with respect to the subgroup formed by the operators whose orders divide p. The index of this quotient group is the number of invariants of G which exceed p. The degree of each transitive constituent of the invariant subgroup of I_{α_1} which corresponds to the identity of I_{α_1-1} is the order of the group formed by all the operators of G whose orders divide p. The degree of each of the transitive constituents of the invariant subgroup of I_{α_1} which corresponds to the identity of I_{α_1-2} is the order of the group generated by all the operators of G whose orders divide p^2, etc. The degrees of the corresponding transitive constituents of I_{α_1-1}, I_{α_1-2}, ..., I_2 are not always equal to each other.

When $\alpha_1 > 3$, the subgroup of the constituent of I_{α_1-2} which corresponds to the identity of the constituent I_{α_1-3} may be found by considering all the automorphisms of G in which each operator whose order divides p^{α_1-3} corresponds to itself. These automorphisms result if each operator of highest order in a set of independent generators of G is multiplied by all the operators of G whose orders divide p^3. If this set of independent generators involves operators of order p^{α_1-1} these are separately multiplied by all the operators of G whose orders divide p^2, and if it contains operators of order p^{α_1-2} these are multiplied by all the operators of G whose orders divide p. In each case the operators of order p^{α_1-2} contained in G are multiplied by operators of order p and the former operators are permuted according to an abelian group of type (1, 1, 1, ...). The index of the order of this abelian group is the sum obtained by adding the three products obtained by multiplying the total number of the invariants of G by the number of these invariants which are equal to p^{α_1-2}, the number of the invariants which exceed p by the number of those which are equal to p^{α_1-1}, and the number which exceed p^2 by the number of those which are equal to p^{α_1}.

As this process can clearly be continued until we arrive at I_1 it results that each of the constituents $I_{\alpha_1}, I_{\alpha_1-1}, \ldots, I_2, I_1$, except the last contains an invariant abelian subgroup of order p^l, and of type (1, 1, 1, ...), composed of all its substitutions which correspond to the identity of the following constituent. The index of the order of this invariant subgroup for I_β, $\alpha_1 \geqq \beta > 1$, is the sum of the following products: m_1 times the number of the invariants of G which exceed $p^{\alpha_1-\beta}$, m_2 times the number of these invariants which exceed $p^{\alpha_2-\beta}$, ..., m_s times the number of these invariants which exceed $p^{\alpha_s-\beta}$, provided α_s is the last of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_\lambda$ which is at least equal to β.

The λ transitive constituents of I_1 have the property that one constituent, viz. the largest one, is simply isomorphic with I_1 and involves an abelian invariant subgroup of order p^{l_1} and of type (1, 1, 1, ...) corresponding to those automorphisms of G under which each of the elements of the other transitive constituents of I_1 is left invariant and each of the operators of the quotient group of the subgroup generated by all the operators of G whose orders divide p with respect to the largest characteristic subgroup contained in it is also left invariant. Since each independent generator of this quotient group may be multiplied by each operator in this largest characteristic subgroup $l_1 = m_\lambda\,(m_1 + m_2 + \ldots + m_{\lambda-1})$. In a similar way it may be proved that the second largest transitive constituent of I_1 contains an invariant abelian subgroup of type (1, 1, 1, ...) and of order p^{l_2}, where $l_2 = m_{\lambda-1}(m_1 + m_2 + \ldots + m_{\lambda-2})$, etc. Finally, the next to the smallest transitive constituent of I_1 contains an abelian invariant subgroup of type (1, 1, 1, ...) and of order $p^{l_{\lambda-1}}$ where $l_{\lambda-1} = m_1 m_2$.

We have now accomplished the main object of this paper as stated in the introduction, since the automorphisms of G which remain to be considered are those which correspond to the interchange of the elements of abelian quotient groups of G which are of type (1, 1, 1, ...). These quotient groups are those obtained by considering each of the $\lambda + 1$ characteristic subgroups of G, including the identity, generated by its operators of order p, with respect to the next smaller characteristic subgroup contained in it. We thus obtain λ quotient groups of type (1, 1, 1, ...) and of orders p^{m_λ}, $p^{m_{\lambda-1}}$ ..., p^{m_2}, p^{m_1} respectively. The order of I is the product of the orders of the groups of isomorphisms of these quotient groups and the order of the prime power group corresponding to all the automorphisms considered above.

It should be clearly noted that the preceding method of finding properties of I is based on a consideration of the constituent groups of which I is composed as an intransitive substitution group. In the first place we considered the α_1 constituent groups I_{a_1}, I_{a_1-1}, ..., I_1 which correspond to the permutations of the operators of the same order when G is made simply isomorphic with itself in every possible way. It was noted that each of these constituent groups except the last has a p^l isomorphism with respect to the following constituent, and that this group of order p^l is abelian and of type (1, 1, 1, ...). In this way the study of the factors of composition of I was reduced to the study of these factors for I_1.

In order to study I_1 we made use of smaller constituent groups, viz., of the transitive constituents of I_1. These λ transitive constituents can again be arranged in order such that each of them except the last has a $(k>1)$ isomorphism with the one which follows it. In this case, k is, however, not necessarily a power of a prime number. On the other hand, if only those isomorphisms of G are considered in which each operator of the quotient groups formed by the characteristic subgroups of G generated by operators of order p with respect to the next larger such characteristic subgroups is invariant, then the subgroup of each of these transitive constituents of I_1 which corresponds to the identity of the next smaller such constituent is again an abelian prime power group of type (1, 1, 1, ...) as was noted above.

Hence it results that I contains an invariant subgroup of order p^s, where s may be obtained as follows. For each operator of order p^r, $r>1$, in a set of independent generators of G find the sum of the number of the independent generators of G, the number of these independent generators whose orders exceed p, the number of those whose orders exceed p^2, ..., the number of those whose orders exceed p^{r-2}. The value of s is the sum of these numbers corresponding to all the operators in the given set of independent generators, increased by the sum of the series

$$m_1m_2+(m_1+m_2)\,m_3+(m_1+m_2+m_3)\,m_4+\dots+(m_1+m_2+\dots+m_{\lambda-1})\,m_\lambda\text{*}.$$

The quotient group of I with regard to the invariant subgroup of order p^s noted in the preceding paragraph contains an invariant subgroup which is simply isomorphic with the group of isomorphisms of the abelian group of order p^{m_λ} and

* A. Ranum, *Transactions of the American Mathematical Society*, vol. 8 (1907), p. 87.

of type (1, 1, 1, ...), with respect to this invariant subgroup I has a quotient group involving an invariant subgroup which is simply isomorphic with the group of isomorphisms of the abelian group of order $p^{m_{\lambda-1}}$ and of type (1; 1, 1, ...), ..., with respect to the smallest transitive constituent of I_1, the quotient group of I is simply isomorphic with the group of isomorphisms of the abelian group of order p^{m_1} and of type (1, 1, 1, ...).

To illustrate this result it may be noted that the group of isomorphisms of the abelian group of order p^{25} whose invariants are p^6, p^6, p^4, p^3, p^3, p^3 contains an invariant subgroup of order p^{111} since the index of p to which each of the two independent generators of order p^6 gives rise is $6+6+6+3+2=23$, the index to which the independent generator of order p^4 gives rise is $6+6+6=18$, and the index to which each of the independent generators of order p^3 gives rise is $6+6=12$. Moreover, the sum of the series $m_1m_2+(m_1+m_2)m_3+\ldots+(m_1+m_2+\ldots+m_{\lambda-1})^{m_\lambda}$ in this special case becomes $2+9=11$. With respect to this invariant subgroup of order p^{111} I has a quotient group which contains as an invariant subgroup the group of isomorphisms of the abelian group of order p^3 and of type (1, 1, 1). With respect to this invariant subgroup this quotient group contains an invariant cyclic group of order $p-1$ and the quotient group of this quotient group with respect to this invariant cyclic group is the group of isomorphisms of the abelian group of order p^2 and of type (1, 1). In particular, the order of I is

$$p^{111}(p^3-1)(p^3-p)(p^3-p^2)(p-1)(p^2-1)(p^2-p)$$
$$=p^{115}(p^3-1)(p^2-1)^2(p-1)^3.$$

As a special case of the preceding developments there results the following theorem: *All the factors of composition of the abelian group of order p^m which has m_1 invariants which are equal to p^{α_1}, m_2 which are equal to p^{α_2}, ..., m_λ which are equal to p^{α_λ} are all equal to p except factors of composition of the groups of isomorphisms of the abelian groups of type (1, 1, 1, ...) and of orders p^{m_1}, p^{m_2}, ..., p^{m_λ} respectively.* A necessary and sufficient condition that the group of isomorphisms of an abelian group of order p^m, $p>3$, is solvable is that no two of its invariants are equal to each other. When $p=2$ or 3 a necessary and sufficient condition that the group of isomorphisms of such an abelian group is solvable is that no three of the invariants of this abelian group are equal to each other. A necessary and sufficient condition that the order of this group of isomorphisms is of the form p^k is that $p=2$, and that no two invariants of the abelian group are equal to each other.

IS THERE AN ANALOGUE IN SOLID GEOMETRY TO FEUERBACH'S THEOREM?

By *T. C. Lewis, M.A.*

Is there an analogue to Feuerbach's theorem? Above all what corresponds to the Hart systems? . . . These difficult but important and interesting questions offer ample scope for serious work.—Treatise on the Circle and the Sphere. By J. L. Coolidge, p. 247.

1. Feuerbach's theorem states that there is a circle, viz. the nine-point circle, which touches the inscribed circle of a triangle internally and each of the three escribed circles externally.

Hart's theorem extends this to include triangles formed by circular arcs. Feuerbach's theorem is therefore a particular case of Hart's theorem; and if it were not valid, Hart's theorem would fail with it.

2. It is easy to prove that there is no analogue, in space of three dimensions, to Feuerbach's theorem, and it follows that there is nothing corresponding to the Hart systems.

If there is in general a sphere which touches the four spheres escribed each on one of the four faces of a tetrahedron $A_1A_2A_3A_4$, and also touches the inscribed sphere, as the nine-point circle touches the escribed and inscribed circles of a plane triangle, then there will be such a sphere in the particular cases when the vertex A_4 is removed to an infinite distance at right angles to the base $A_1A_2A_3$. The tetrahedron then becomes a prism having its edges and faces perpendicular to the base, and the opposite edges at right angles to one another.

If r_1, r_2, r_3, r_4 are the radii of the escribed spheres opposite to $A_1, \ldots, A_4$, and r the radius of the inscribed sphere, r_1, r_2, r_3 are equal to the radii of the escribed circles of the base triangle; and $r_4 = r$, which is equal to the radius of the inscribed circle of the same triangle. Let a, b, c be the sides of this triangle.

Denoting spheres by their radii in brackets, the spheres (r_4) and (r) touch the base at the same point, viz. the centre of the inscribed circle. If a sphere touches all the five spheres in the manner suggested, it must touch the base at the same point. Suppose there is such a sphere of radius ρ. Then the condition that it should touch the sphere (r_1) is

$$(\rho - r_1)^2 + a^2 \sec^2 \tfrac{1}{2} A_1 = (\rho + r_1)^2,$$

therefore
$$4\rho r_1 = \frac{a^2bc}{s(s-a)},$$

whence we get
$$\rho = \frac{a}{s} R,$$

where R is the circumradius of the base triangle. Therefore unless $a=b=c$ the same sphere cannot touch the four escribed spheres externally and the inscribed sphere internally, and therefore there is not in general such a sphere; that is, there is no analogue in solid geometry to Feuerbach's theorem.

Since the answer to the first of the proposed questions is in the negative, the second question does not arise.

It may be noted that when $a=b=c$ there is a sphere satisfying the prescribed conditions, and in that case

$$\rho = \tfrac{2}{3} R \qquad \text{..........................(1).}$$

3. In addition to the four escribed spheres already considered there are three other spheres, each of which touches *two* faces of the tetrahedron on the same side as the inscribed sphere, and two faces on the reverse side. If these spheres be indicated by their radii r_{14}, &c., with suffixes corresponding to the faces touched reversely, then there is either a sphere (r_{14}) *or* a sphere (r_{23}), but not both. The analytical expressions for r_{14} and r_{23} differ only in sign, and the one that is positive is that of the actual escribed sphere.

In the particular case when the tetrahedron becomes a prism as in the preceding paragraph, the three additional spheres are (r_{14}), (r_{24}), (r_{34}); and

$$r_{14} = r_1, \quad r_{24} = r_2, \quad r_{34} = r_3.$$

It may be at once proved that in that case there is a sphere which touches each of the eight inscribed and escribed spheres, and that it passes through the three vertices of the base triangle, whether this triangle be equilateral or not.

Let O be the circumcentre of the base, I the centre of the inscribed circle, E_1 the centre of the escribed circle opposite to A_1.

The condition that a sphere with centre at O and radius ρ should touch the inscribed sphere internally is

$$(\rho - r)^2 = OI^2 + r^2 = (R - r)^2$$

or
$$\rho = R.$$

So, if the sphere touches the escribed sphere (r_1) externally, we have

$$(\rho + r_1)^2 = OE_1^2 + r_1^2 = (R + r_1)^2,$$

or again $$\rho = R \qquad\qquad (2).$$

That is, the same sphere of radius R, and having its centre at O, touches the spheres (r) and (r_1); and similarly it also touches the spheres (r_2) and (r_3); and it evidently passes through the vertices of the base triangle. From symmetry we see that it also touches the sphere (r_4) internally, and the spheres (r_{14}), (r_{24}), (r_{34}) externally. It therefore touches all the eight spheres, two internally and six externally.

4. There is not in general a sphere that touches the eight spheres in this manner, for in the case of a regular tetrahedron the sphere which touches (r) and (r_4) internally, having its centre on their common diametral line, touches neither the three escribed circles (r_1), (r_2), (r_3), nor the other three, which are infinite.

But there may be other cases in which a sphere may be found to touch the five spheres as in § 2 or the eight spheres as in § 3; and in each of the two sets of cases a certain very simple formula may be found for the radius of such sphere.

5. It is convenient to adopt the system of pentaspherical co-ordinates of which the centres of reference are the four vertices of the tetrahedron and the centre of the associated hyperboloid*.

The equation to the inscribed sphere is

$$(a_1 - Hr)\frac{x_1}{\rho_1} + (a_2 - Hr)\frac{x_2}{\rho_2} + (a_3 - Hr)\frac{x_3}{\rho_3} + (a_4 - Hr)\frac{x_4}{\rho_4} - Hr \cdot \frac{x_5}{\rho_5} = 0 \qquad (3).$$

The equations to the escribed spheres are derived from this by changing the sign of one, or of two, of the four quantities a_1, a_2, a_3, a_4, H at the same time being changed to H_1, H_2, ..., H_{14}, ..., and r to r_1, r_2, ..., r_{14}, ...

Assume that the three spheres escribed on two faces have the positive radii r_{14}, r_{24}, r_{34}.

Let the equation to a sphere which touches the four singly escribed spheres externally be $\Sigma\alpha_h x_h = 0$, and its radius ρ, where

$$\frac{1}{\rho} = \Sigma \frac{\alpha_h}{\rho_h}.$$

* *Messenger of Mathematics*, vol. xlix, p. 97.

Then the condition of external contact with (r_1) is

$$\alpha_1\left(\frac{-a_1-H_1r_1}{\rho_1}+\frac{a_2-H_1r_1}{\rho_2}c_{12}+\frac{a_3-H_1r_1}{\rho_3}c_{13}+\frac{a_4-H_1r_1}{\rho_4}c_{14}-\frac{H_1r_1}{\rho_5}c_{15}\right)$$

$$+\alpha_2\left(\frac{-a_1-H_1r_1}{\rho_1}c_{12}+\frac{a_2-H_1r_1}{\rho_2}+\frac{a_3-H_1r_1}{\rho_3}c_{23}+\frac{a_4-H_1r_1}{\rho_4}c_{24}-\frac{H_1r_1}{\rho_5}c_{25}\right)$$

$$+\alpha_3\left(\frac{-a_1-H_1r_1}{\rho_1}c_{13}+\frac{a_2-H_1r_1}{\rho_2}c_{23}+\frac{a_3-H_1r_1}{\rho_3}+\frac{a_4-H_1r_1}{\rho_4}c_{34}-\frac{H_1r_1}{\rho_5}c_{35}\right)$$

$$+\alpha_4\left(\frac{-a_1-H_1r_1}{\rho_1}c_{14}+\frac{a_2-H_1r_1}{\rho_2}c_{24}+\frac{a_3-H_1r_1}{\rho_3}c_{34}+\frac{a_4-H_1r_1}{\rho_4}-\frac{H_1r_1}{\rho_5}c_{45}\right)$$

$$+\alpha_5\left(\frac{-a_1-H_1r_1}{\rho_1}c_{15}+\frac{a_2-H_1r_1}{\rho_2}c_{25}+\frac{a_3-H_1r_1}{\rho_3}c_{35}+\frac{a_4-H_1r_1}{\rho_4}c_{45}-\frac{H_1r_1}{\rho_5}\right)=-1.$$

Therefore

$$\alpha_1\left(-\frac{a_1}{\rho_1}+\frac{a_2}{\rho_2}c_{12}+\frac{a_3}{\rho_3}c_{13}+\frac{a_4}{\rho_4}c_{14}\right)$$

$$+\alpha_2\left(-\frac{a_1}{\rho_1}c_{12}+\frac{a_2}{\rho_2}+\frac{a_3}{\rho_3}c_{23}+\frac{a_4}{\rho_4}c_{24}\right)$$

$$+\text{\&c.}\ \ldots\ldots\ldots=\frac{H_1r_1}{\rho}-1\ldots\ldots\ldots(4).$$

Similarly the conditions of contact with the other escribed spheres are

$$\alpha_1\left(\frac{a_1}{\rho_1}-\frac{a_2}{\rho_2}c_{12}+\frac{a_3}{\rho_3}c_{13}+\frac{a_4}{\rho_4}c_{14}\right)$$

$$+\alpha_2\left(\frac{a_1}{\rho_1}c_{12}-\frac{a_2}{\rho_2}+\frac{a_3}{\rho_3}c_{23}+\frac{a_4}{\rho_4}c_{24}\right)+\text{\&c.}=\frac{H_2r_2}{\rho}-1\ldots(5),$$

$$\alpha_1\left(\frac{a_1}{\rho_1}+\frac{a_2}{\rho_2}c_{12}-\frac{a_3}{\rho_3}c_{13}+\frac{a_4}{\rho_4}c_{14}\right)$$
$$+\alpha_2\left(\frac{a_1}{\rho_1}c_{12}+\frac{a_2}{\rho_2}-\frac{a_3}{\rho_3}c_{23}+\frac{a_4}{\rho_4}c_{24}\right)+\&c.=\frac{H_3r_3}{\rho}-1\ldots(6),$$

$$\alpha_1\left(\frac{a_1}{\rho_1}+\frac{a_2}{\rho_2}c_{12}+\frac{a_3}{\rho_3}c_{13}-\frac{a_4}{\rho_4}c_{14}\right)$$
$$+\alpha_2\left(\frac{a_1}{\rho_1}c_{12}+\frac{a_2}{\rho_2}+\frac{a_3}{\rho_3}c_{23}-\frac{a_4}{\rho_4}c_{24}\right)+\&c.=\frac{H_4r_4}{\rho}-1\ldots(7).$$

By addition of these four equations of condition

$$\alpha_1\left(\frac{a_1}{\rho_1}+\frac{a_2}{\rho_2}c_{12}+\frac{a_3}{\rho_3}c_{13}+\frac{a_4}{\rho_4}c_{14}\right)$$
$$+\alpha_2\left(\frac{a_1}{\rho_1}c_{12}+\frac{a_2}{\rho_2}+\frac{a_3}{\rho_3}c_{23}+\frac{a_4}{\rho_4}c_{24}\right)+\&c.$$
$$=\frac{H_1r_1+\ldots+H_4r_4}{2\rho}-2\ldots\ldots\ldots(8).$$

Similarly if the same sphere touches the inscribed sphere internally we have the same expression in α_1, α_2, &c., on the left side of this equation equal to

$$\frac{Hr}{\rho}+1;$$

and therefore

$$6\rho=H_1r_1+H_2r_2+H_3r_3+H_4r_4-2Hr\ldots\ldots\ldots(9).$$

If the opposite edges of the tetrahedron are at right angles to one another the coefficients H, H_1, &c., are all equal to unity, and

$$6\rho=r_1+r_2+r_3+r_4-2r.$$

If the tetrahedron be of the special form considered in § 2, where $r_4=r$, and r_1, r_2, r_3 are equal to the radii of the escribed circles of the base-triangle of the prism, and R is its circum-radius,

$$6\rho=r_1+r_2+r_3-r=4R$$

and $\rho=\frac{2}{3}R$, as already proved in the special case, *vide* (1). Also in a regular tetrahedron, where $r_1=r_2=r_3=r_4=2r$, we shall have $6\rho=6r$, *i.e.*, $\rho=r$, which is satisfied.

6. If the sphere touches (r_1), (r_2), (r_3) externally and (r) and (r_4) internally, the term on the right-hand side of (7) will be

$$\frac{H_4r_4}{\rho}+1,$$

and proceeding then as before we obtain

$$4\rho = H_1 r_1 + H_2 r_2 + H_3 r_3 + H_4 r_4 - 2Hr.$$

For a tetrahedron with opposite edges at right angles to one another this becomes

$$4\rho = r_1 + r_2 + r_3 + r_4 - 2r,$$

and for the special case considered in § 3 we obtain the same result, viz.

$$\rho = R.$$

7. If the same sphere likewise touches (ρ_{14}) externally we have

$$\alpha_1 \left(-\frac{a_1}{\rho_1} + \frac{a_2}{\rho_2} c_{12} + \frac{a_3}{\rho_3} c_{13} - \frac{a_4}{\rho_4} c_{14}\right)$$
$$+ \alpha_2 \left(-\frac{a_1}{\rho_1} c_{12} + \frac{a_2}{\rho_2} + \frac{a_3}{\rho_3} c_{23} - \frac{a_4}{\rho_4} c_{24}\right) + \&\text{c.} = \frac{H_{14} r_{14}}{\rho} - 1,$$

whence it follows that

$$Hr + H_{14} r_{14} = H_1 r_1 + H_4 r_4,$$

with similar conditions for similar contact with (r_{24}) and (r_{34}). Therefore

$$Hr - H_4 r_4 = H_1 r_1 - H_{14} r_{14} = H_2 r_2 - H_{24} r_{24} = H_3 r_3 - H_{34} r_{34} \ldots (10)$$

are conditions that must be satisfied if a sphere can be described to touch (r) and (r_4) internally and the six remaining escribed spheres of the tetrahedron externally. If the opposite edges of the tetrahedron are at right angles to one another these conditions become

$$r - r_4 = r_1 - r_{14} = r_2 - r_{24} = r_3 - r_{34} \ldots\ldots\ldots (11),$$

and these are fulfilled in the case considered in § 3; but in no other case can they be satisfied for such a tetrahedron, as may be proved by writing for r, r_1, &c., their values $\Sigma \frac{a_h}{\rho_h^2}$, with the appropriate signs for a_h.

The conditions (11) are obviously not satisfied in a regular tetrahedron, where $r_1 = r_2 = r_3 = r_4 = 2r$, and r_{14}, &c., are infinite. It has not, however, been proved that the conditions (10) may not be satisfied for special cases of an oblique tetrahedron.

END OF VOL. XLIX.