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The Microeconomics of the Price Mechanism

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Department of Economics
Lectures to Be Given at the Charles University of Prague
Winter 1991/1992

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MICROECONOMICS

PREFACE

Microeconomics comes in two parts, price theory and allocation theory. The present minicourse will develop both parts as well as their late-nineteenth-century integration. The course closes with an elementary restatement of von Neumann's proof of the existence of a general economic equilibrium.

Modern economic theory comes in mathematical form, and no other form will do. The course confines itself to elementary algebra and calculus. A reader needing help will find some in our appendix.

Chapter 1 will be published in the winter 1991 issue of History of Political Economy. Chapters 2 and 3 are newly written but based on material published in chapters 6 and 11, respectively of my Pioneering Economic Theory 1630-1980, A Mathematical Restatement, Baltimore: Johns Hopkins University Press, 1986.

University of Illinois, September 1991

It is not from the benevolence of the butcher, the brewer, or the baker, that we expect our dinner, but from their regard to their own interest

Adam Smith, Wealth of Nations

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MICROECONOMICS

CHAPTER 1

STATICS: RELATIVE PRICE

Abstract

Cantillon tried to build a land theory of value and succeeded. His relative-price solution was self-contained: it had no factor prices in it but only input-output coefficients. Marx tried to build a labor theory of value but failed: his relative-price expression still had the rate of interest in it. Smith and the neoclassicals used the full trinity of capital, labor, and land and made no attempt to reduce it to any single factor. All factor prices appeared in the relative-price expressions—pointing towards a general-equilibrium model.

I. INTRODUCTION

1. Microeconomics

Microeconomic theory considers an economy producing more than one good and comes in two parts, price theory and allocation theory. Price theory determines the relative price of such goods. Allocation theory determines the physical quantities transacted of each good, i.e., how inputs are allocated among outputs and outputs among households. In the late nineteenth century price and allocation theory fell into place and came to be seen as inseparable parts of general equilibrium. We begin with price theory.

2. Variables, Parameters, and Solutions

We try to explain economic variables by building models of them. A model is a system of equations containing variables related to one another *via* parameters. A parameter is a quantity fixed by the investigator using information coming from outside the model. For example, a microeconomic model may use technology, preferences, resources, and legal institutions as its parameters.

Having built our models, we try to solve them. By a solution for a variable we mean an equation having that variable on its left-hand side and nothing but parameters on its right-hand side. Since the distinction between variables and parameters is so important--and may differ among models--we shall open each chapter with a complete list of its variables and its parameters.

Variables and parameters of our first chapter are the following.

3. Variables

h ≡ future cash flow

J ≡ present net worth of capital stock

L ≡ labor

N ≡ land

n ≡ money rent rate

P ≡ price of good

r ≡ rate of interest

S ≡ physical capital stock

w ≡ money wage rate

X ≡ physical output

4. Parameters

a ≡ labor coefficient

α ≡ labor elasticity

$b \equiv$ land coefficient

$\beta \equiv$ land elasticity

$c \equiv$ capital coefficient

$\gamma \equiv$ capital elasticity

$j \equiv$ joint factor productivity

$m \equiv$ labor's manner of living

$u \equiv$ useful life of capital stock

II. CANTILLON

1. Production Technology

Cantillon certainly knew no diminishing returns--indeed nobody knew them before Turgot [1767 (1844: 418-433), (1977: 109-122)].

Did Cantillon know that production takes time? In other parts of his work he was well aware of it, but in the passages [1755 (1931: 41)] developing his famous "Par between Land and Labour" he ignored capital. Let us restate his par mathematically.

Let a Cantillon economy be producing two consumers' goods, i.e., a necessity consumed only by labor and a luxury consumed only by landlords. Both are produced solely from labor and land in processes having fixed input-output coefficients:

$$L_j = a_j X_j \tag{1}$$

$$N_j = b_j X_j \tag{2}$$

where subscripts j = 1, 2 refer to the necessity and the luxury respectively.

There is a third process, a labor-producing one. Like Malthus and von Neumann, Cantillon saw labor as reproducible--produced from necessities at a fixed input-output coefficient equaling labor's subsistence real wage:

$$X_1 = m_1 L \tag{3}$$

Cantillon saw the coefficient m_1, labor's "manner of living," not as a biological minimum but as a social one varying among regions: it was higher in Northern France than in Southern France--as Cantillon [1755 (1931: 71)] described it in such specific detail. However high it was, we treat it as a parameter.

2. Processes Break Even

Now in long-run equilibrium let all processes break even. The two goods-producing processes will break even after freedom of entry and exit has done its work and washed away all profits over and above labor cost at the standard money wage rate w and land cost at the standard money rent rate n. As a result, in each industry revenue equals cost:

$$P_j X_j = wL_j + nN_j$$

Divide by output X_j, use (1) and (2), and write a Cantillon price equation:

$$P_j = a_j w + b_j n \qquad (4)$$

or, in Cantillon's own words [1755 (1931: 41)]: "... the intrinsic value of any thing may be measured by the quantity of Land used in its production and the quantity of Labour which enters into it, ..." But *via* his "Par between Land and Labour" Cantillon reduced his labor to "the quantity of Land of which the produce is allotted to those who have worked upon it." He did it as follows.

The labor-producing process will break even, because [1755 (1931: 83)] "Men multiply like Mice in a barn if they have unlimited Means of Subsistence." Here, too, revenue equals cost or, in more familiar terms, the wage bill equals the value of labor's consumption:

$$wL = P_1 X_1$$

3. Solution for Relative Price

Insert (3), divide L away, and write a Cantillon wage equation:

$$w = m_1 P_1 \qquad (5)$$

Insert (5) into (4) and write the Cantillon price equation:

$$P_j = a_j m_1 P_1 + b_j n$$

which is a system of two equations in three unknowns n, P_1, and P_2. Solve it for P_1 and P_2, divide former by latter, let n cancel, and write Cantillon's relative price

$$\frac{P_1}{P_2} = \frac{b_1}{b_2[1 + (a_2/b_2 - a_1/b_1)b_1m_1]} \qquad (6)$$

which is self-contained: it has no factor prices in it but only the input-output coefficients a_j, b_j, and m_1.

III. SMITH

1. Production Technology

Did Smith assume fixed input-output coefficients, or did he know diminishing returns? Eltis (1984: 107) finds no trace of diminishing returns in Smith. Hollander (1980) finds them only on the basis of a very selective choice of quotes. Samuelson (1977), (1978), on the other hand, assumed Smith to have known diminishing returns. Certainly Smith's "natural price" was phrased generally enough, or vaguely enough, to allow a neoclassical interpretation. For the moment, however, let us assume both consumers' goods to be produced in processes having fixed input-output coefficients.

Smith may or may not have known diminishing returns, but he definitely knew that production takes time. Let his preindustrial capital be all circulating, and let the period of production be one year, i.e., let there be a one-year gap between inputs and outputs:

$$L_j(t) = a_j X_j(t + 1) \tag{7}$$

$$N_j(t) = b_j X_j(t + 1) \tag{8}$$

2. The "Natural Price"

Smith's goods-producing processes will break even after freedom of entry and exit has washed away all profits over and above capital cost at the standard rate of interest r, labor cost at the standard money wage w, and land cost at the standard money rent rate n. As a result in each industry revenue equals cost:

$$P_j X_j(t + 1) = (1 + r)[wL_j(t) + nN_j(t)]$$

Divide by output $X_j(t + 1)$, insert (7) and (8), and find a Smithian price equation:

$$P_j = (1 + r)(a_j w + b_j n) \tag{9}$$

Here is Smith's [1776 (1805: book I, chapter 7)] "natural price," i.e., a price "neither more nor less than what is sufficient to pay the rent of the land, the wages of the labour, and the profits of the stock employed in raising, preparing, and bringing it to market, according to their natural rates."

3. Was Labor Reproducible?

Did Smith, like Cantillon, have a third process producing labor from necessities at a fixed input-output coefficient equaling labor's subsistence real wage? To be sure, Smith [1776 (1805: book I, chapter 8)] did observe that "every species of animals naturally multiplies in proportion to the means of their subsistence..." And, for humans, Smith saw such subsistence not as a biological minimum but as a social one varying among nations. Indeed it was higher in North America than in England.

Yet, if ever tempted to build such a labor-producing process into his price theory, Smith withstood the temptation. Nothing like Cantillon's par between land and labor occurred to Smith. Nowhere did he reduce labor to land.

We, too, shall withstand the temptation, leave Smith's "natural price" the way he left it, and find his relative price.

4. Relative Price

The "natural price" (9) is a system of two equations in five unknowns n, P_1, P_2, r, and w. Write it out for j = 1, 2, divide former by latter, let 1 + r cancel, and write Smith's relative price

$$\frac{P_1}{P_2} = \frac{a_1 w + b_1 n}{a_2 w + b_2 n} \tag{10}$$

The money wage rate w and the money rent rate n are still with us in (10) and will affect relative price P_1/P_2. How? Take the partial derivatives of relative price with respect to factor price

$$\frac{\partial(P_1/P_2)}{\partial w} = \frac{(a_1/b_1 - a_2/b_2)b_1 b_2 n}{(a_2 w + b_2 n)^2} \tag{11}$$

$$\frac{\partial(P_1/P_2)}{\partial n} = \frac{(a_2/b_2 - a_1/b_1)b_1 b_2 w}{(a_2 w + b_2 n)^2} \tag{12}$$

If we think, as we normally do, of necessities (food) as less labor-intensive than luxuries (services), i.e., $a_1/b_1 < a_2/b_2$, then

(11) is negative and (12) positive: a higher money wage rate w will lower but a higher money rent rate n will raise relative price (10).

Only in the special and unlikely case of labor intensities being the same in both goods, i.e., $a_1/b_1 = a_2/b_2$, will (10) collapse into $P_1/P_2 = b_1/b_2$, (11) and (12) be zero, and relative price be insensitive to factor prices.

IV. MARX

1. Fixed Capital

Ricardo had seen that relative price would equal relative man-hours absorbed if all capital was a wage fund, i.e., if all capital was circulating capital. But Ricardo had felt compelled to add his chapter on "machinery" to his third edition. Here he [1821 (1951: 32)] had seen that if fixed capital or its durability varied among industries, relative price would no longer equal relative man-hours. Marx, too, paid much attention to machinery. So--unlike Samuelson (1957: 884) and (1971: 413n)--let us assume Marxian capital to be fixed constituting a third good in our model, "machines," so our $j = 1, 2, 3$.

2. Present Net Worth

Fixed capital involves dynamic planning. Let a firm in the jth industry contemplate the acquisition of a new physical capital stock S_j. Define its future cash flow of revenue minus wage bill as

$$h_j \equiv P_j X_j - wL_j \tag{13}$$

Let the rate of interest used to discount such future cash flows be r. Then at time zero the present worth of a future instantaneous rate of cash flow located at time t is $e^{-rt}h_j dt$, and the present net worth J_j of the contemplated new physical capital stock S_j is the present worth of all future cash flows over its useful life u minus the cost of its acquisition:

$$J_j \equiv \int_0^u e^{-rt} h_j dt - P_3 S_j \tag{14}$$

In a stationary economy the cash flow h_j is not a function of time hence may be moved outside the integral sign. Move it, carry out the integration (14), insert (13), and find the present net worth

$$J_j = \frac{1 - e^{-ru}}{r} (P_j X_j - wL_j) - P_3 S_j \tag{15}$$

3. Production Technology

Ricardo had known diminishing returns but may not have realized that they would make his labor and capital coefficients vary with his margins of cultivation. Marx ignored land and with it diminishing returns. We welcome such simplification allowing us to treat labor and capital coefficients as technological parameters:

$$L_j = a_j X_j \tag{16}$$

$$S_j = c_j X_j \tag{17}$$

Ricardo's durable producers' goods had been made from labor alone. To his credit, to Marx it also took producers' goods to produce producers' goods: $a_j > 0$ and $c_j > 0$ for $j = 1, 2, 3$.

4. Equalization of Rates of Profit

Marx's "values" of volume I [1867 (1908)] resulted from equalization of rates of surplus value among industries. His "prices" of volume III [1894 (1909: 181, 212)] resulted from equalization of rates of profit. As we know [Ott-Winkel (1985: 190)], equalization of rates of profit means nonequalization of rates of surplus value, so we must choose between volume I and volume III. At freedom of entry and exit equalization of rates of profit is more realistic, so we choose volume III and let equalized rates of profit equal the rate of interest common to all borrowers. Then present net worth (15) will be zero. Set (15) equal to zero, divide by physical output X_j, use (16) and (17), rearrange, and find a Marxian price equation

$$P_j = a_j w + c_j P_3 \frac{r}{1 - e^{-ru}} \tag{18}$$

which is a system of three equations in five unknowns P_1, P_2, P_3, r, and w.

5. Was Labor Reproducible?

Did Marx, like Cantillon, have a fourth process producing labor from necessities at a fixed input-output coefficient equaling labor's subsistence real wage? To be sure, in his volume I Marx [1867 (1908: 190)] did apply his labor theory of value to labor itself: labor's value in exchange did equal "the value of the means of subsistence necessary for the maintenance of the labourer."

Yet, if ever tempted to build such a labor-producing process into his price theory, Marx withstood the temptation. For one thing he despised Malthus. For another, in his two-factor model his falling rate of profit implied a rising real wage rate incompatible with a minimum subsistence wage [Ott-Winkel (1985: 214)]. The incompatibility should have bothered Marx, but he never mentioned it. We must agree with Samuelson (1971: 406) that if Marx did have a minimum subsistence wage "it is not well determined by efficacious linkages."

We, too, shall withstand the temptation, leave Marx's price (18) the way he left it, and find his relative price.

6. Relative Price

Write (18) for $j = 3$ and find

$$P_3 = \frac{a_3 w}{1 - c_3 r/(1 - e^{-ru})} \qquad (19)$$

Insert (19) into (18) written for j = 1, 2, divide former by latter, let w cancel, and write Marx's relative price

$$\frac{P_1}{P_2} = \frac{a_1[1 + (c_1/a_1 - c_3/a_3)a_3 r/(1 - e^{-ru})]}{a_2[1 + (c_2/a_2 - c_3/a_3)a_3 r/(1 - e^{-ru})]} \qquad (20)$$

The rate of interest r is still with us in (20) and will affect relative price P_1/P_2. How?

If like Gordon (1961) we think of necessities as more capital-intensive than luxuries and of luxuries as more capital-intensive than machinery, i.e., $c_1/a_1 > c_2/a_2 > c_3/a_3$, then the second terms of the brackets of the numerator and the denominator of (20) will both be positive but the former larger than the latter. In that case a higher rate of interest r, hence a higher $r/(1 - e^{-ru})$, would affect necessities more than luxuries hence raise relative price (20).

Only in the special and unlikely case of capital intensities being the same in both consumers' goods, i.e., $c_1/a_1 = c_2/a_2$, may the brackets of the numerator and the denominator of (20) be divided away and leave us with the pure labor theory of value $P_1/P_2 = a_1/a_2$, having no factor prices in it.

V. NEOCLASSICAL RELATIVE PRICE

1. The Smithian Trinity Once Again

Cantillon ignored capital and Marx land. Let us restore the full Smithian trinity of capital, labor, and land. First, extend our future cash flow to include the rent bill:

$$h_j \equiv P_j X_j - wL_j - nN_j \tag{21}$$

Then again define present net worth J_j of a contemplated new physical capital stock S_j as the present worth of all future cash flows over its useful life u minus the cost of its acquisition:

$$J_j \equiv \int_0^u e^{-rt} h_j dt - P_3 S_j \tag{22}$$

Again, in a stationary economy the cash flow h_j is not a function of time hence may be moved outside the integral sign. Move it, carry out the integration (22), insert (21), and find present net worth

$$J_j = \frac{1 - e^{-ru}}{r} (P_j X_j - wL_j - nN_j) - P_3 S_j$$

2. Production Technology

Let us finally come to grips with diminishing returns to the full trinity of capital, labor, and land. Wicksell [1893: V, 121-127 (1954)] and Wicksteed [1894 (1932: 33)] were the first to do so and to show that it doesn't matter who hires whom. With diminishing returns thus generalized we can no longer use input coefficients as technological parameters. But we can use input elasticities as such. Like Wicksell [1901 (1934: 128)] let us do that and choose a Cobb-Douglas form

$$X_j = j_j L_j^{\alpha_j} N_j^{\beta_j} S_j^{\gamma_j} \tag{23}$$

where j_j is joint factor productivity, α_j, β_j, and γ_j are the labor, land, and capital elasticities of output, and where $\alpha_j + \beta_j + \gamma_j = 1$.

3. Optimization

Treating prices of goods and factors as beyond its control, a purely competitive firm will hire another man, rent another acre, or install another machine until such hiring, renting, or installation will add nothing to the present net worth J_j:

$$\frac{\partial J_j}{\partial L_j} = \frac{1 - e^{-ru}}{r} (P_j \frac{\partial X_j}{\partial L_j} - w) = 0 \tag{24}$$

$$\frac{\partial J_j}{\partial N_j} = \frac{1 - e^{-ru}}{r} (P_j \frac{\partial X_j}{\partial N_j} - n) = 0 \tag{25}$$

$$\frac{\partial J_j}{\partial S_j} = \frac{1 - e^{-ru}}{r} P_j \frac{\partial X_j}{\partial S_j} - P_3 = 0 \tag{26}$$

Carry out the partial differentiations of (23), rearrange, and find factor demand to be in inverse proportion to factor price:

$$L_j = \frac{\alpha_j P_j X_j}{w} \tag{27}$$

$$N_j = \frac{\beta_j P_j X_j}{n} \tag{28}$$

$$S_j = \frac{\gamma_j P_j X_j}{P_3 r/(1 - e^{-ru})} \tag{29}$$

Multiply across, add (27), (28), and (29), and notice in passing Wicksteed's [1894 (1932: 37)] product-exhaustion theorem $wL_j + nN_j + P_3 S_j r/(1 - e^{-ru}) = P_j X_j$.

4. Relative Price

Raise (27) to the power α_j, (28) to the power β_j, and (29) to the power γ_j. Multiply the three equations. Use (23) and find an X_j on both the left-hand and the right-hand side of their product. Divide it away, rearrange the rest, and find the neoclassical price equation

$$P_j = \frac{1}{j_j} \left(\frac{w}{\alpha_j}\right)^{\alpha_j} \left(\frac{n}{\beta_j}\right)^{\beta_j} \left(\frac{1}{\gamma_j}\right)^{\gamma_j} \left(\frac{P_3 r}{1 - e^{-ru}}\right)^{\gamma_j} \tag{30}$$

which is a system of three equations in six unknowns n, P_1, P_2, P_3, r, and w. First write it out for j = 3:

$$P_3 = \left[\frac{1}{j_3}\left(\frac{w}{\alpha_3}\right)^{\alpha_3}\left(\frac{n}{\beta_3}\right)^{\beta_3}\left(\frac{1}{\gamma_3}\right)^{\gamma_3}\left(\frac{r}{1-e^{-ru}}\right)^{\gamma_3}\right]^{1/(\alpha_3+\beta_3)} \tag{31}$$

Then write it out for j = 1, 2 and write neoclassical relative price

$$\frac{P_1}{P_2} = \frac{j_2}{j_1}\frac{(w/\alpha_1)^{\alpha_1}}{(w/\alpha_2)^{\alpha_2}}\frac{(n/\beta_1)^{\beta_1}}{(n/\beta_2)^{\beta_2}}\frac{(1/\gamma_1)^{\gamma_1}}{(1/\gamma_2)^{\gamma_2}}\left(\frac{P_3 r}{1-e^{-ru}}\right)^{\gamma_1-\gamma_2} \tag{32}$$

where P_3 stands for (31).

All factor prices, i.e., the money wage rate w, the money rent rate n, and the rate of interest r, are still with us in (32), and we are not surprised. The essence of neoclassical thought is that factors are substitutes and that factor demand depends on factor price--indeed in our (27), (28), (29) was always in inverse proportion to factor price!

In (32) the money wage rate w occurs in the power

$$\alpha_1 - \alpha_2 + \alpha_3 \frac{\gamma_1 - \gamma_2}{\alpha_3 + \beta_3} = \frac{(\alpha_1 - \alpha_2)\beta_3 - \alpha_3(\beta_1 - \beta_2)}{\alpha_3 + \beta_3} \tag{33}$$

If we think of necessities as more land-intensive (food) and more capital-intensive (housing), hence less labor-intensive, than luxuries (services), then $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$. As a result both terms of the numerator of (33) are negative, and a higher money wage rate w will unequivocally lower the relative price of necessities (32).

In (32) the ratio $r/(1 - e^{-ru})$ occurs in the power

$$\frac{\gamma_1 - \gamma_2}{\alpha_3 + \beta_3} \tag{34}$$

If we think of necessities as more capital-intensive (housing) than luxuries (services), then $\gamma_1 > \gamma_2$, and a higher rate of interest r will raise $r/(1 - e^{-ru})$ and unequivocally raise the relative price of necessities (32).

VI. SUMMARY AND CONCLUSION

By ignoring capital and by reducing labor to land, Cantillon tried to build a land theory of value. He succeeded. His relative-price solution (6) was self-contained: it had no factor prices in it but only input-output coefficients. It was indeed a solution.

By ignoring land and by reducing machines to labor, Marx tried to build a labor theory of value. He failed: his relative-price expression (20) still had the rate of interest in it.

Smith and neoclassicals used the full trinity of capital, labor, and land and made no attempt to reduce it to any single factor. Not surprisingly, all their factor prices still appeared in their relative-price expressions (10) and (32).

To Smith natural price was a one-way causal relationship between goods price and factor prices: it was because 2 francs were paid out in rent, 2 francs in wages, and 1 franc in interest that this bottle of wine sells for 5 francs, as Walras [1874-1877 (1954: 211)] put it.

We know better. To us a solution for a variable is an equation having that variable on its left-hand side and nothing but parameters on its right-hand side. But on the right-hand sides of our (10), (20), and (32) we find the rates n, r, or w of rent, interest or wages, and they are not parameters but variables remaining to be explained in a general-equilibrium model.

We have work to do, then. Some of it was first done in Prague, as we shall see in our next chapter.

FOOTNOTE

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CHAPTER 2

STATICS: ALLOCATION AND IMPUTATION

Abstract

The previous chapter pointed towards a general-equilibrium model. The present chapter begins with Wieser's allocation and imputation. Outputs will be substitutes in a utility function. Inputs will be substitutes in a production function. Households will maximize utility and industry will maximize profits. A price mechanism will allocate inputs among outputs and outputs among households. The chapter will build and solve a Wieser model of a stationary economy with two outputs, two inputs, and two households.

I. INTRODUCTION

1. Allocation

We turn to the second part of microeconomics: how are goods allocated among uses? Allocation must reflect preferences, but not until the end of the nineteenth century did preferences enter mainstream microeconomics.

To Menger [1871 (1950)] goods were valued because needed, and their value would depend on the need satisfied by the last unit of goods available. That need would be the least important need: take the last unit away, and the consumer could still satisfy his higher-priority needs and merely go without the satisfaction of his least important one. That was all, but with that Menger had put economic theory on a new foundation.

2. Imputation

But Menger confined himself to households and their demand for outputs. For industry and its demand for inputs we must turn to Friedrich von Wieser (1851-1926) who taught at the Charles University of Prague 1884-1903 where he wrote his Der natürliche Werth

Goods were valued because needed: outputs satisfied needs directly, inputs satisfied them indirectly. How needed inputs were would depend on two things, first, how productive inputs were in producing outputs and, second, how needed such outputs were. Inputs would be valued by the principle of "Zurechnung," i.e., imputation.

Wieser's imputation was a one-way causal relationship between goods price and factor prices: it was because this bottle of wine sells for 5 francs that 2 francs could be paid out in rent, 2 francs in wages, and one franc in interest. Wieser reversed the direction of Smith's one-way street. But the street remained one-way!

3. Our Restatement

Functional interdependence was beyond Wieser's ken. But we shall set out his price mechanism as he might have done himself had his form matched his vision. We confine ourselves to the simplest case of two outputs, two inputs, and two households. Let all firms in the same industry have the same production function and let there be constant returns to scale. Competition may then be pure. We need not specify the number of firms but may let a representative firm represent an industry. Our notation will be the following.

4. Variables

$C_k \equiv$ consumption expenditure of kth household

$c_j \equiv$ cost of jth industry

$P_j \equiv$ price of jth output

$p_i \equiv$ price of ith input

$R_j \equiv$ revenue of jth industry

$U_k \equiv$ utility to kth household

$X_j \equiv$ output supplied by jth industry

$X_{jk} \equiv$ jth output demanded by kth household

$x_{ij} \equiv$ ith input demanded by jth industry

$Y_k \equiv$ income of kth household

$Z_j \equiv$ profits of jth industry

5. Parameters

$A \equiv$ elasticity of utility with respect to first output

a_j ≡ joint factor productivity of jth output

α_j ≡ elasticity of jth output with respect to first input

B ≡ elasticity of utility with respect to second output

β_j ≡ elasticity of jth output with respect to second input

q_{ki} ≡ endowment of kth household with ith physical input

II. A NEOCLASSICAL THEORY OF THE HOUSEHOLD

1. Utility Maximization Subject to a Budget Constraint

In our miniature Wieser economy let the two households have the same utility function of Cobb-Douglas form:

$$U_1 = X_{11}{}^A X_{21}{}^B \qquad (1)$$

$$U_2 = X_{12}{}^A X_{22}{}^B \qquad (2)$$

where $0 < A < 1$ and $0 < B < 1$. But A and B do not necessarily sum to one.

Define the consumption expenditures of the two households as

$$C_1 \equiv P_1X_{11} + P_2X_{21} \quad (3)$$

$$C_2 \equiv P_1X_{12} + P_2X_{22} \quad (4)$$

Let our miniature Wieser economy be a stationary one having no saving. Indeed let no household save:

$$C_1 = Y_1 \quad (5)$$

$$C_2 = Y_2 \quad (6)$$

Subject to the budget constraints (3) through (6) let each household maximize its utility (1) and (2). Let the first household change infinitesimally its X_{11} by dX_{11} and its X_{21} by dX_{21}. Let the second household change infinitesimally its X_{12} by dX_{12} and its X_{22} by dX_{22}. In the neighborhood of a utility maximum two things may be said. First, the changes cannot affect utility which is already at its maximum. Second, the changes cannot violate the budget constraint: if the household wants to demand more of one output, it must demand less of another. Consequently

$$dU_1 \equiv \frac{\partial U_1}{\partial X_{11}} dX_{11} + \frac{\partial U_1}{\partial X_{21}} dX_{21} = 0$$

$$dU_2 \equiv \frac{\partial U_2}{\partial X_{12}} dX_{12} + \frac{\partial U_2}{\partial X_{22}} dX_{22} = 0$$

(7)

$$dC_1 \equiv \frac{\partial C_1}{\partial X_{11}} dX_{11} + \frac{\partial C_1}{\partial X_{21}} dX_{21} = 0$$

$$dC_2 \equiv \frac{\partial C_2}{\partial X_{12}} dX_{12} + \frac{\partial C_2}{\partial X_{22}} dX_{22} = 0$$

Use the utility functions (1) and (2) to find the partial derivatives

$$\frac{\partial U_1}{\partial X_{11}} = \frac{AU_1}{X_{11}}$$

$$\frac{\partial U_2}{\partial X_{12}} = \frac{AU_2}{X_{12}}$$

$$\frac{\partial U_1}{\partial X_{21}} = \frac{BU_1}{X_{21}}$$

$$\frac{\partial U_2}{\partial X_{22}} = \frac{BU_2}{X_{22}}$$

Use the budget definitions (3) and (4) to find the partial derivatives

$$\frac{\partial C_1}{\partial X_{11}} = \frac{\partial C_2}{\partial X_{12}} = P_1$$

$$\frac{\partial C_1}{\partial X_{21}} = \frac{\partial C_2}{\partial X_{22}} = P_2$$

Insert these eight partial derivatives into the system (7). Use the latter to express the two marginal rates of substitution dX_{11}/dX_{21} and dX_{12}/dX_{22}, divide U_1 and U_2 away, rearrange, and find

$$AP_2X_{21} = BP_1X_{11} \tag{8}$$

$$AP_2X_{22} = BP_1X_{12} \tag{9}$$

Insert (5) and (6) into (3) and (4), respectively, and write the two budget constraints

$$Y_1 = P_1X_{11} + P_2X_{21} \tag{10}$$

$$Y_2 = P_1X_{12} + P_2X_{22} \tag{11}$$

Insert (8) and (9) into (10) and (11), respectively, and write the four demand functions

$$X_{11} = \frac{A}{A + B} \frac{Y_1}{P_1} \tag{12}$$

$$X_{12} = \frac{A}{A + B} \frac{Y_2}{P_1} \tag{13}$$

$$X_{21} = \frac{B}{A + B} \frac{Y_1}{P_2} \tag{14}$$

$$X_{22} = \frac{B}{A + B} \frac{Y_2}{P_2} \qquad (15)$$

All four demand functions find the demand for output to be in direct proportion to income and in inverse proportion to price. Such simple results follow from the Cobb-Douglas form of the utility functions (1) and (2).

2. Household Income

Let households supply their entire endowment and let the first household be endowed with the first input and the second household with the second input:

$$q_{11} > 0 \qquad (16)$$

$$q_{12} = 0 \qquad (17)$$

$$q_{21} = 0 \qquad (18)$$

$$q_{22} > 0 \qquad (19)$$

Then the incomes of the two households will be

$$Y_1 = p_1 q_{11} \tag{20}$$

$$Y_2 = p_2 q_{22} \tag{21}$$

III. A NEOCLASSICAL THEORY OF A PURELY COMPETITIVE FIRM

1. Profit Maximization

In our miniature Wieser economy let all firms in the same industry have the same production function, and let it be of Cobb-Douglas form:

$$X_1 = a_1 x_{11}^{\alpha_1} x_{21}^{\beta_1} \tag{22}$$

$$X_2 = a_2 x_{12}^{\alpha_2} x_{22}^{\beta_2} \tag{23}$$

where $0 < \alpha_j < 1$, $0 < \beta_j < 1$, and

$$\alpha_1 + \beta_1 = 1 \tag{24}$$

$$\alpha_2 + \beta_2 = 1 \tag{25}$$

The cost of a representative firm is

$$c_1 \equiv p_1 x_{11} + p_2 x_{21} \tag{26}$$

$$c_2 \equiv p_1 x_{12} + p_2 x_{22} \tag{27}$$

The revenue of a representative firm is

$$R_1 \equiv P_1 X_1 \tag{28}$$

$$R_2 \equiv P_2 X_2 \tag{29}$$

The profits of a representative firm is

$$Z_1 \equiv R_1 - c_1 \tag{30}$$

$$Z_2 \equiv R_2 - c_2 \tag{31}$$

and will be maximized with respect to inputs:

$$\frac{\partial Z_1}{\partial x_{11}} = P_1 \frac{\partial X_1}{\partial x_{11}} - p_1 = 0$$

$$\frac{\partial Z_1}{\partial x_{21}} = P_1 \frac{\partial X_1}{\partial x_{21}} - p_2 = 0$$

$$\frac{\partial Z_2}{\partial x_{12}} = P_2 \frac{\partial X_2}{\partial x_{12}} - p_1 = 0$$

$$\frac{\partial Z_2}{\partial x_{22}} = P_2 \frac{\partial X_2}{\partial x_{22}} - p_2 = 0$$

Use the production functions (22) and (23) to find the four partial derivatives, rearrange, and write the four demand functions

$$x_{11} = \alpha_1 P_1 X_1 / p_1 \qquad (32)$$

$$x_{12} = \alpha_2 P_2 X_2 / p_1 \qquad (33)$$

$$x_{21} = \beta_1 P_1 X_1 / p_2 \qquad (34)$$

$$x_{22} = \beta_2 P_2 X_2 / p_2 \tag{35}$$

All four demand functions find the demand for input to be in direct proportion to revenue and in inverse proportion to price. Such simple results follow from the Cobb-Douglas form of (22) and (23).

2. Product Exhaustion

Multiply (32) and (33) by p_1 and (34) and (35) by p_2. Then add (32) and (34) and use (24). Then add (33) and (35) and use (25) and write the product-exhaustion theorem

$$p_1 x_{11} + p_2 x_{21} = P_1 X_1 \tag{36}$$

$$p_1 x_{12} + p_2 x_{22} = P_2 X_2 \tag{37}$$

or, in English: under profit maximization, pure competition, and constant returns to scale each input will be paid its marginal value productivity, and the distributive shares thus determined will add up to the pie to be distributed.

We are done with households and firms. All that remains is to let the price mechanism clear all markets.

IV. ALL MARKETS CLEAR

1. Input-Market Clearing

Input-market clearing requires the supply of inputs to equal the demand for them:

$$q_{11} = x_{11} + x_{12} \tag{38}$$

$$q_{22} = x_{21} + x_{22} \tag{39}$$

2. Output-Market Clearing

Clearing in the first output market requires supply to equal demand:

$$X_1 = X_{11} + X_{12} \tag{40}$$

Must the same not be true of the market for the second output? Indeed it must, but not as a new and independent condition. Add the product-exhaustion theorem (36) and (37), use first (38) and (39), then (20) and (21), and find

$$P_1X_1 + P_2X_2 = Y_1 + Y_2$$

Multiply the demand equations (12) through (15) by their respective prices P_j, add all four of them together, use (40), and find

$$P_1X_1 + P_2(X_{21} + X_{22}) = Y_1 + Y_2$$

Thus it follows from equations already written that for the second output as well, supply equals demand:

$$X_2 = X_{21} + X_{22} \qquad (41)$$

So we have encountered Walras's Law: if in an economy with four markets three of them clear, the fourth one will also clear.

Our variables are the following prices, quantities, and money incomes:

P_1, P_2	$X_{11}, X_{12}, X_{21}, X_{22}$
p_1, p_2	$x_{11}, x_{12}, x_{21}, x_{22}$
X_1, X_2	Y_1, Y_2

Our system is homogeneous of degree zero in its absolute prices and money incomes: if satisfied by one set of P_j, p_i, and Y_k the system will also be satisfied by any multiple of that set, hence its absolute prices and money incomes are indeterminable. We must choose a numéraire, say p_1, and divide all equations containing P_j, p_i, or Y_k by it. That will leave us with physical quantities X_j, X_{jk}, and x_{ij} and relative prices and money incomes P_j/p_1, p_i/p_1, and Y_k/p_1 as variables. Can we solve our Wieser system for those variables?

V. SOLUTIONS

1. Allocation of Inputs Among Outputs

Let us begin with the allocation of inputs among outputs x_{ij}. Multiply the demand equations (12) through (15) by their respective prices P_j, add (13) to (12) and (15) to (14), insert the output-market clearing conditions (40) and (41) and the product-exhaustion theorems (36) and (37), and find $A(p_1x_{12} + p_2x_{22}) = B(p_1x_{11} + p_2x_{21})$. Next divide (32) by (34) and find $p_1x_{11} = (\alpha_1/\beta_1)p_2x_{21}$. Divide (33) by (35) and find $p_1x_{12} = (\alpha_2/\beta_2)p_2x_{22}$. Insert all that, apply the result to the input-market clearing conditions (38) and (39), and find the solutions for the allocation of inputs among outputs:

$$x_{11} = \frac{\alpha_1 A}{\alpha_1 A + \alpha_2 B} q_{11} \tag{42}$$

$$x_{12} = \frac{\alpha_2 B}{\alpha_1 A + \alpha_2 B} q_{11} \tag{43}$$

$$x_{21} = \frac{\beta_1 A}{\beta_1 A + \beta_2 B} q_{22} \tag{44}$$

$$x_{22} = \frac{\beta_2 B}{\beta_1 A + \beta_2 B} q_{22} \tag{45}$$

Once we are this far, the rest is easy.

2. Relative Prices

Use the price p_1 of the first input as our numéraire. Divide (34) by (32), insert (42) and (44), and find relative input price

$$\frac{p_2}{p_1} = \frac{\beta_1 A + \beta_2 B}{\alpha_1 A + \alpha_2 B} \frac{q_{11}}{q_{22}} \tag{46}$$

Next find output prices relative to our numéraire p_1. Write (32) as $P_1/p_1 = x_{11}/(\alpha_1 X_1)$. Insert (22), (42), and (44). Write (33) as $P_2/p_1 = x_{12}/(\alpha_2 X_2)$. Insert (23), (43), and (45). The results are

$$\frac{P_1}{p_1} = \frac{1}{a_1 \alpha_1^{\alpha_1} \beta_1^{\beta_1}} \left(\frac{\beta_1 A + \beta_2 B}{\alpha_1 A + \alpha_2 B} \frac{q_{11}}{q_{22}}\right)^{\beta_1} \tag{47}$$

$$\frac{P_2}{p_1} = \frac{1}{a_2 \alpha_2^{\alpha_2} \beta_2^{\beta_2}} \left(\frac{\beta_1 A + \beta_2 B}{\alpha_1 A + \alpha_2 B} \frac{q_{11}}{q_{22}}\right)^{\beta_2} \tag{48}$$

3. Outputs

Simply insert the solutions (42) through (45) for the allocation of inputs among outputs into the production functions (22) and (23) and find the solutions for outputs:

$$X_1 = a_1\left(\frac{\alpha_1 A q_{11}}{\alpha_1 A + \alpha_2 B}\right)^{\alpha_1}\left(\frac{\beta_1 A q_{22}}{\beta_1 A + \beta_2 B}\right)^{\beta_1} \tag{49}$$

$$X_2 = a_2\left(\frac{\alpha_2 B q_{11}}{\alpha_1 A + \alpha_2 B}\right)^{\alpha_2}\left(\frac{\beta_2 B q_{22}}{\beta_1 A + \beta_2 B}\right)^{\beta_2} \tag{50}$$

4. Allocation of Outputs Among Households

Insert (20) and (42) into (12) and find X_{11}. Insert (21), (46), and (47) into (13) and find X_{12}. Insert (20) and (48) into (14) and find X_{21}. Insert (21), (46), and (48) into (15) and find X_{22}. Then the allocation of output among households is

$$X_{11} = \frac{A}{A + B}\, a_1 \alpha_1^{\alpha_1} \beta_1^{\beta_1}\left(\frac{\alpha_1 A + \alpha_2 B}{\beta_1 A + \beta_2 B}\,\frac{q_{22}}{q_{11}}\right)^{\beta_1} q_{11} \tag{51}$$

$$X_{12} = \frac{A}{A + B} a_1 \alpha_1^{\alpha_1} \beta_1^{\beta_1} \left(\frac{\beta_1 A + \beta_2 B}{\alpha_1 A + \alpha_2 B} \frac{q_{11}}{q_{22}}\right)^{\alpha_1} q_{22} \tag{52}$$

$$X_{21} = \frac{B}{A + B} a_2 \alpha_2^{\alpha_2} \beta_2^{\beta_2} \left(\frac{\alpha_1 A + \alpha_2 B}{\beta_1 A + \beta_2 B} \frac{q_{22}}{q_{11}}\right)^{\beta_2} q_{11} \tag{53}$$

$$X_{22} = \frac{B}{A + B} a_2 \alpha_2^{\alpha_2} \beta_2^{\beta_2} \left(\frac{\beta_1 A + \beta_2 B}{\alpha_1 A + \alpha_2 B} \frac{q_{11}}{q_{22}}\right)^{\alpha_2} q_{22} \tag{54}$$

5. Income Distribution

Divide (20) by p_1. Divide (21) by p_1 and insert (46). Then money incomes relative to the *numéraire* p_1 are

$$\frac{Y_1}{p_1} = q_{11} \tag{55}$$

$$\frac{Y_2}{p_1} = \frac{\beta_1 A + \beta_2 B}{\alpha_1 A + \alpha_2 B} q_{11} \tag{56}$$

VI. SUMMARY AND CONCLUSION

We have solved our stylized Wieser model for all its variables. A solution is an equation having a variable on its left-hand side and nothing but parameters on its right-hand side. Our solutions contained four categories of parameters. First, engineering delivered the technology parameters α_j and β_j. Second, tastes delivered the preference parameters A and B. Third, nature delivered the resource parameters q_{ki}, and fourth, legal institutions established private ownership to resources, enabling private persons to earn an income from them.

As for price theory we can accept neither Smith's nor Wieser's one-way causal relationships between goods price and factor prices. Goods price is not caused by factor prices as Smith thought. Factor prices are not caused by goods price as Wieser thought. Instead, we must insist, goods prices and factor prices are simultaneously determined within a general-equilibrium model using technology, preferences, resources, and legal institutions for its parameters.

Such functional interdependence was beyond Smith and Wieser but not beyond Walras. But even Walras failed to prove the existence of his general equilibrium. Such proof was first offered by von Neumann to whom we now turn.

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CHAPTER 3

DYNAMICS: EXISTENCE OF GENERAL EQUILIBRIUM

Abstract

Wieser and Walras considered stationary states and believed but never proved general equilibria to exist. Cassel was the first to consider a general equilibrium of a growing economy but still failed to prove its existence. The first to prove it was John von Neumann.

Von Neumann's proof used inequalities. A primal problem was to find the highest rate of growth satisfying the inequality that for every good current input absorbed must be less than or equal to current output supplied. A dual problem was to find the lowest rate of interest satisfying the inequality that in every process cost at time t with interest added must be greater than or equal to revenue at time t + 1. A saddle point would exist at which the maximized rate of growth equaled the minimized rate of interest. The chapter will build and solve a von Neumann model of two goods and two processes.

I. INTRODUCTION

1. Time, Place, and Problem

The late nineteenth century had seen two Vienna breakthroughs, one in surgery by Billroth (1819-1894) and one in economics by Carl Menger (1840-1921). The early twentieth century saw a third breakthrough, logical positivism, by Wittgenstein (1889-1951). Inspired by logical positivism, Kurt Gödel, Karl Menger (son of Carl Menger), John von Neumann, Karl Schlesinger, Abraham Wald, and other mathematicians met in a colloquium that happened to devote some of its time to the very foundation of economic theory: did a general economic equilibrium exist?

Walras [1874-1877 (1954: 43-44)] considered general equilibrium to be determinate "in the sense that the number of equations entailed is equal to the number of unknowns." As pointed out by Karl Menger (1971: 50), for the next sixty years Walras's belief remained unquestioned. Neither uniqueness nor feasibility was ever discussed.

The form of general equilibrium best known to the members of the colloquium was Cassel's [1918 (1932: 32-41 and 152-155)] dynamization of it, "the uniformly progressing state". Like Walras, Cassel failed to prove the existence of a solution.

2. Von Neumann's Breakthrough

Such innocence lasted until the 1930s. In the Viennese colloquium von Neumann [1937 (1945-1946)] formulated a balanced and steady-state growth of a general equilibrium and proved the existence of a solution. The model was slow in reaching print. According to Weintraub (1983: 13n), recollections by Jacob Marschak suggest its genesis to be roughly contemporary with von Neumann's early work on game theory (1928). The model was presented orally to a Princeton mathematics seminar in 1932.

What was new was not the subject matter. The subject matter was allocation and relative price, the heartland of microtheory. There was substitution in both production and consumption. The model could "handle capital goods without fuss and bother," as Dorfman-Samuelson-Solow (1958) put it. There was explicit optimization in the model: its solutions would weed out all but the most profitable process or processes. There were free and economic goods: the solutions would tell us which goods would be free and which economic.

What was new was method rather than subject matter. This time, the matter was in the hands of mathematicians from the very beginning, and the mathematics deployed was very different from the calculus deployed after 1870. The maxima and minima were handled without the use of any calculus at all. What von Neumann taught us was to use inequalities to formulate a primal and a dual problem. What von Neumann offered was a solution of his primal and dual problem displaying a saddle point.

We must convey the flavor of von Neumann's method. But being one of the foremost mathematicians of the twentieth century, von Neumann used nonelementary algebra. Can the von Neumann model be solved by elementary algebra? If collapsed into two goods and two processes, it can, and let its notation be as follows.

3. Variables

$g \equiv$ proportionate rate of growth

$P_i \equiv$ price of ith good

$p \equiv$ relative price

$r \equiv$ rate of interest

$u_i \equiv$ excess supply of ith good

$v_j \equiv$ loss margin of jth process

$X_j \equiv$ level of jth process

$x \equiv$ relative process level

4. Parameters

$a_{ij} \equiv$ input of ith good absorbed per unit of jth process level

$b_{ij} \equiv$ output of ith good supplied per unit of jth process level

II. THE MODEL

1. Goods and Processes

A von Neumann good may be absorbed as an input as well as supplied as an output. A von Neumann process may have several inputs and several outputs, and its unit level is defined as the unit of one of its outputs per unit of time.

Let there be two goods, $i = 1,2$, and two processes, $j = 1,2$. Operated at unit level the jth process converts a_{ij} units of the ith good absorbed as an input into b_{ij} units of that good supplied as an output one year later. The coefficients a_{ij} and b_{ij} are nonnegative technological parameters. But let each process have at least one positive a_{ij}, i.e., be absorbing at least one good as an input. And let each good have at least one positive b_{ij}, i.e., be supplied as an output in at least one process. Let the level of the jth process be the pure number X_j by which unit level should be multiplied in order to get actual output. As in Cassel (1918) let all processes be growing at the stationary rate of growth g:

$$X_j(t + 1) = (1 + g)X_j(t) \tag{1}$$

A von Neumann model can handle joint supply of and joint demand for goods, indeed consists of such supply and demand. Yet the von Neumann model can handle substitution in both production and consumption. First, there is substitution in production, for although each process has parametric input coefficients a_{ij} and output coefficients b_{ij}, the same good may occur as an output in more than one process, hence may be produced in more than one way. Second, there is substitution in consumption, for labor is a good like any other, hence is reproducible:

labor is simply the output of one or more processes whose inputs are consumers' goods. Although each such process has parametric input coefficients a_{ij} and output coefficients b_{ij}, labor may occur as an output in more than one process, hence may be produced in more than one way--by being fed, so to speak, alternative menus.

Does the von Neumann model have capital in it? It does, in fact it incorporates the time element of production in a particularly elegant way. In the von Neumann model all processes have a period of production of one time unit, but this is less restrictive than it sounds: as for circulating capital, if consumable wine has a period of production of two years, simply define two distinct processes and goods as follows. The first process absorbs zero-year-old wine and supplies one-year-old wine; the second absorbs one-year-old wine and supplies two-year-old wine. As for fixed capital, if the useful life of machines is two years, again define two distinct processes and goods. The first process absorbs zero-year-old machines and supplies one-year-old machines; the second absorbs one-year-old machines and supplies two-year old machines!

2. The Primal Problem: Maximize the Rate of Growth

Feasibility requires overall excess demand for the ith good to be nonpositive: the sum of all inputs of the ith good absorbed in both processes at time t must be less than or equal to the sum of all outputs of it supplied in both processes at time t:

$$a_{i1}X_1(1 + 1) + a_{i2}X_2(t + 1) \leq b_{i1}X_1(t) + b_{i2}X_2(t) \quad (2)$$

Into (2) insert (1), suppress the now redundant time coordinate, and introduce a nonnegative auxiliary variable $u_i \geq 0$. We may then write (2) as the equality

$$(1 + g)(a_{i1}X_1 + a_{i2}X_2) + u_i = b_{i1}X_1 + b_{i2}X_2 \quad (3)$$

or

$$u_i = b_{i1}X_1 + b_{i2}X_2 - (1 + g)(a_{i1}X_1 + a_{i2}X_2) \quad (4)$$

from which the economic meaning of u_i is seen to be current physical output <u>minus</u> current physical input of the ith good, or simply excess supply of the ith good.

We can always make the rate of growth g high enough to generate positive excess demand for at least one good. But how high can we make it <u>without</u> doing that? When the rate of growth reaches its highest possible value, its equilibrium value, excess demand will become zero for at least one good. That good or those goods will then become economic. In other words, the equilibrium rate of growth g will be the rate of growth of the slowest-growing good or goods. Goods growing more rapidly than that will become free.

3. The Dual Problem: Minimize the Rate of Interest

Under pure competition and freedom of entry and exit, profits must be nonpositive, hence for the jth process operated at unit level the sum of all input cost at time t with interest added at a stationary rate r must be greater than or equal to the sum of all revenue at time t + 1:

$$(1 + r)[a_{1j}P_1(t) + a_{2j}P_2(t)] \geq b_{1j}P_1(t + 1) + b_{2j}P_2(t + 1) \qquad (5)$$

As Cassel did, assume all prices to be stationary, suppress the now redundant time coordinates, and introduce a nonnegative auxiliary variable $v_j \geq 0$. We may then write (5) as the equality

$$(1 + r)(a_{1j}P_1 + a_{2j}P_2) = b_{1j}P_1 + b_{2j}P_2 + v_j \qquad (6)$$

or

$$v_j = (1 + r)(a_{1j}P_1 + a_{2j}P_2) - (b_{1j}P_1 + b_{2j}P_2) \qquad (7)$$

from which the economic meaning of v_j is seen to be unit-level cost with interest minus unit-level revenue, or simply loss margin of jth process.

We can always make the rate of interest r low enough to generate positive profits in at least one process. But how low can we make it without doing that? When the rate of interest reaches its lowest possible value, its equilibrium value, profits will become zero in at least one process. That process or those processes will then break even and be operated. In other words, the equilibrium rate of interest will be the profitability of the most profitable process or processes. Processes less profitable than that will remain unused.

4. The Saddle Point

Multiply the excess supply (4) of the ith good by its price P_i and write out the result for both goods i = 1, 2. Multiply the loss margin (7) of the jth process by its level X_j and write out the result for both processes j = 1, 2. The four equations are

$$P_1u_1 = [b_{11} - (1 + g)a_{11}]P_1X_1 + [b_{12} - (1 + g)a_{12}]P_1X_2 \quad (8)$$

$$P_2u_2 = [b_{21} - (1 + g)a_{21}]P_2X_1 + [b_{22} - (1 + g)a_{22}]P_2X_2 \quad (9)$$

$$v_1X_1 = [(1 + r)a_{11} - b_{11}]P_1X_1 + [(1 + r)a_{21} - b_{21}]P_2X_1 \quad (10)$$

$$v_2X_2 = [(1 + r)a_{12} - b_{12}]P_1X_2 + [(1 + r)a_{22} - b_{22}]P_2X_2 \quad (11)$$

Add (8) through (11):

$$P_1u_1 + P_2u_2 + v_1X_1 + v_2X_2 =$$

$$(r - g)(a_{11}P_1X_1 + a_{12}P_1X_2 + a_{21}P_2X_1 + a_{22}P_2X_2) \quad (12)$$

With a zero excess supply u_i the ith good is an economic good having a positive price P_i. With a positive excess supply u_i the ith good is a free good having a zero price P_i. Consequently the product $P_i u_i$ always has one and only one factor equaling zero and must itself be zero.

Likewise with a zero loss margin v_j the jth process will be used and have a positive level X_j. With a positive loss margin v_j the jth process will not be used and will have a zero level X_j. Consequently the product $v_j X_j$, too, always has one and only one factor equaling zero and must itself be zero.

The entire left-hand side of (12), then, is zero. But then at least one of the two factors on the right-hand side of (12) must be zero. Now von Neumann ruled out the uninteresting case that both goods were free and both processes unused. So let there be at least one economic good, i.e., one positive P_i, and let there be at least one process used, i.e., one positive X_j. We have already assumed that each process has at least one positive a_{ij}. Under those three assumptions at least one of the four terms $a_{ij} P_i X_j$ on the right-hand side of (12) will be positive, and that side can be zero only if the maximized rate of growth equals the minimized rate of interest:

$$g = r$$

Such a saddle point was the heart of the von Neumann model. But before finding its coordinates let us be more explicit about its finance than von Neumann was himself.

In a growing economy somebody must be saving. We may think of a von Neumann model as having capitalists in it who are lending money capital to the entrepreneurs to carry them over their one-time unit period of production. At the rate of interest r, capitalists at the beginning of that period lend the entrepreneurs the sum $a_{11}P_1X_1 + a_{12}P_1X_2 + a_{21}P_2X_1 + a_{22}P_2X_2$ financing the purchases of all goods absorbed as inputs.

At the end of the period of production the value of aggregate output will be $b_{11}P_1X_1 + b_{12}P_1X_2 + b_{21}P_2X_1 + b_{22}P_2X_2$, and let us express it in two different ways. First, since the product P_iu_i always has one and only one factor equaling zero, we may set (8) and (9) equal to zero, then add them and find aggregate input to have grown into aggregate output at the rate g. Second, since the product v_jX_j always has one and only one factor equaling zero, we may set (10) and (11) equal to zero, then add them and find aggregate input to have grown into aggregate output at the rate r. But in our saddle point the maximized rate of growth g equaled the minimized rate of interest r. Consequently, out of their sales proceeds the entrepreneurs can pay back with interest the sum they borrowed from the capitalists one time unit earlier.

Now we must find the coordinates of the saddle point.

5. A Quadratic in Relative Process Levels

There was at least one economic good, say the second: $P_2 > 0$, $u_2 = 0$, and one process used, say the second: $X_2 > 0$. We may then safely define relative process levels $x \equiv X_1/X_2$ and write (3) for both goods:

$$1 + g = \frac{b_{11}x + b_{12} - u_1/X_2}{a_{11}x + a_{12}} \qquad (13)$$

$$1 + g = \frac{b_{21}x + b_{22}}{a_{21}x + a_{22}} \qquad (14)$$

Setting the right-hand sides of (13) and (14) equal and multiplying across will give us the quadratic

$$x^2 + Hx + I = 0, \text{ where} \tag{15}$$

$$H \equiv \frac{a_{11}b_{22} + a_{12}b_{21} - a_{21}b_{12} - a_{22}b_{11} + a_{21}u_1/X_2}{a_{11}b_{21} - a_{21}b_{11}}$$

$$I \equiv \frac{a_{12}b_{22} - a_{22}b_{12} + a_{22}u_1/X_2}{a_{11}b_{21} - a_{21}b_{11}}$$

The two roots of the quadratic (15) are

$$x = - H/2 \pm \sqrt{(- H/2)^2 - I} \tag{16}$$

Three cases offer themselves

6. First Case: One Unused Process

The constant term I of (15) will be zero, hence a root x = 0 will exist, if and only if excess supply of first good is:

$$u_1 = \frac{a_{22}b_{12} - a_{12}b_{22}}{a_{22}} X_2 \qquad (17)$$

(17) will be nonnegative if and only if

$$b_{22}/a_{22} \leq b_{12}/a_{12} \qquad (18)$$

which compares growth rates of goods in the only process used, the second one. If the less-than sign holds then in that process the second good is growing less rapidly than the first: the first good will be free. If as an odd piece of luck the equal sign holds, the two goods are growing at the same rate: no free good.

7. A Quadratic in Relative Prices

There was at least one process used, say the second: $v_2 = 0$, $X_2 > 0$, and one economic good, say the second: $P_2 > 0$. We may then safely define relative prices $p \equiv P_1/P_2$ and write (6) for both processes:

$$1 + r = \frac{b_{11}p + b_{21} + v_1/P_2}{a_{11}p + a_{21}} \qquad (19)$$

$$1 + r = \frac{b_{12}p + b_{22}}{a_{12}p + a_{22}} \qquad (20)$$

Setting the right-hand sides of (19) and (20) equal and multiplying across will give us the quadratic

$$p^2 + Jp + K = 0, \text{ where} \qquad (21)$$

$$J \equiv \frac{a_{11}b_{22} - a_{12}b_{21} + a_{21}b_{12} - a_{22}b_{11} - a_{12}v_1/P_2}{a_{11}b_{12} - a_{12}b_{11}}$$

$$K \equiv \frac{a_{21}b_{22} - a_{22}b_{21} - a_{22}v_1/P_2}{a_{11}b_{12} - a_{12}b_{11}}$$

The two roots of the quadratic (21) are

$$p = - J/2 \pm \sqrt{(- J/2)^2 - K}, \tag{22}$$

offering our second case.

8. Second Case: One Free Good

The constant term K of (21) will be zero, hence a root $p = 0$ will exist, if and only if the loss margin of the first process is:

$$v_1 = \frac{a_{21}b_{22} - a_{22}b_{21}}{a_{22}} P_2 \tag{23}$$

(23) will be nonnegative if and only if

$$b_{21}/a_{21} \leq b_{22}/a_{22} \tag{24}$$

which compares profitabilities of processes for the sole economic good, the second one. If the less-than sign holds then the first process is less profitable than the second in producing that good: first process will be unused. If as an odd piece of luck the equal sign holds, the two processes are equally profitable: no unused process.

In our first case one unused process would occur with either one free good or none. In our second case one free good would occur with either one unused process or none. The only alternative left is the case of no free good occurring with no unused process. We turn to that case.

9. Third Case: No Free Good, No Unused Process

If the first good is to be economic, its excess supply $u_1 = 0$, and H and I are purged of their last terms $a_{21}u_1/X_2$ and $a_{22}u_1/X_2$, respectively. If the first process is to be used, its loss margin $v_1 = 0$, and J and K are purged of their last terms $a_{12}v_1/P_2$ and $a_{22}v_1/P_2$, respectively.

May (16) have a positive real root leaving no process unused? It may as follows. Compare growth rates of goods in the same process. Suppose, first, that in the first process the second good is growing less rapidly than the first good:

$$b_{21}/a_{21} < b_{11}/a_{11} \tag{25}$$

but, second, that in the second process the first good is growing less rapidly than the second good:

$$b_{12}/a_{12} < b_{22}/a_{22} \tag{26}$$

Then the numerator of the purged I will be positive, the denominator negative, and I itself negative. As a result no root (16) can be complex. A positive and a negative real root of x will exist regardless of the sign of H. In short, there will be no unused process.

Similarly may (22) have a positive real root leaving no good free? It may as follows. Compare profitabilities of processes producing the same good. Suppose, first, that first process is less profitable than second process in producing the first good:

$$b_{11}/a_{11} < b_{12}/a_{12} \tag{27}$$

but, second, that second process is less profitable than first process in producing the second good:

$$b_{22}/a_{22} < b_{21}/a_{21} \tag{28}$$

Then the numerator of the purged K will be negative, the denominator positive, and K itself negative. As a result no root (22) can be complex. A positive and a negative real root of p will exist regardless of the sign of J. In short, there will be no free good.

10. Summary

Our (3) and (6) would remain satisfied if process levels X_j, prices P_i, excess supplies u_i, and loss margins v_j were multiplied by an arbitrary constant. Reduced to (3) and (6), then, the von Neumann system was homogenous of degree zero in its absolute process levels, prices, excess supplies, and loss margins and could only be solved for its relative ones. Since the numbering of goods and processes is arbitrary, we assumed that at least the second good would be economic and at least the second process be used. In that case dividing by P_2 or X_2 would always be meaningful, and the system could be solved for its relative process level x, its relative price p, its relative excess supply u_1/X_2, and its relative loss margin v_1/P_2. Solving for those four coordinates of the von Neumann saddle point we found three cases. May the cases coexist?

We notice at once that technologies satisfying (18) cannot satisfy (26) and vice versa: first and third case cannot coexist. Similarly, technologies satisfying (24) cannot satisfy (28) and vice versa: second and third case cannot coexist.

But could first and second case coexist with one another?

The answer is an easy yes if we ignore our odd pieces of luck, the equal signs of (18) and (24). Technologies may then exist satisfying the inequality parts of (18) and (24) at the same time. As a result a homogenous (15) and a homogenous (21) may coexist and have the roots $x = 0$ and $p = 0$, respectively: one unused process may coexist with one free good.

The answer is no if we cannot ignore our odd pieces of luck. The equal sign of (18) would mean one unused process coexisting with no free goods--ruling out the second case. The equal sign of (24) would mean one free good coexisting with no unused process--ruling out the first case.

In short, we may have, first, one unused process coexisting with one free good, second, one unused process coexisting with no free good, third, one free good coexisting with no unused process, and fourth, no unused process coexisting with no free good. We have discussed all four possibilities and specified the technologies that would generate them.

11. Preferences?

How did von Neumann treat consumption? Who consumed? Assuming all his goods to be reproducible, von Neumann excluded natural resources their owners, and the consumption by such owners. Like Walrasian ones von Neumann's entrepreneurs didn't consume anything, because their income qua entrepreneurs was zero--pure competition and freedom of entry and exit saw to that. Capitalists did have an interest income but saved all of it. That left labor as the only consumer in a von Neumann model. Labor was a good like any other, hence was reproducible: labor was simply the output of one or more processes whose inputs were consumers' goods. Labor might occur as an output in more than one process and thus be produced in more than one way--by being fed, so to speak, alternative menus. The alternative menus did represent substitution in consumption, but how was the choice among them made? Labor-producing processes displaying zero loss margins would be operated at positive levels representing the consumption choice of the economy. But that choice did not express anybody's preference; it merely minimized the cost, including interest, of breeding labor. Labor was bred as cattle!

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APPENDIX: A MATHEMATICAL REMINDER

A MATHEMATICAL REMINDER

Let a and C be constants, u, v, x, and y variables, f and ϕ functional forms, t time, and e Euler's number, the base of natural logarithms.

1. Rules of Differentiation

Chain Rule: $$\frac{df(u)}{dx} = \frac{df(u)}{du}\frac{du}{dx}$$

Constant Rule: $$\frac{da}{dx} = 0$$

Euler's Rule: $$\frac{de^{ax}}{dx} = ae^{ax}$$

Inverse Rule: $$\frac{du}{dx} = \frac{1}{dx/du}$$

Power Rule: $$\frac{dx^{a}}{dx} = ax^{a-1}$$

Product Rule: $$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Proportion Rule: $$\frac{d(ax)}{dx} = a$$

Quotient Rule: $$\frac{d(u/v)}{dx} = \frac{v(du/dx) - u(dv/dx)}{v^2}$$

Sum or Difference Rule: $$\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

2. Rule of Integration

The indefinite integral $\int f(x)dx$ of the integrand $f(x)$ will equal $\phi(x) + C$, where C is the constant of integration, if

$$\frac{d\phi(x)}{dx} = f(x)$$

From Euler's Rule of differentiation it then follows that

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

3. Partial Derivatives

Consider a function of more than one variable, say, $u = f(x, y)$. The partial derivatives of that function are

$$\frac{\partial u}{\partial x} \equiv \frac{du}{dx} \text{ treating y as a constant}$$

$$\frac{\partial u}{\partial y} \equiv \frac{du}{dy} \text{ treating x as a constant}$$

4. The Total Differential

For increments dx and dy the total differential of $u = f(x, y)$ is

$$du \equiv \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

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