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SCHOOL OF EDUCATION

MODERN COURSES IN SECONDARY MATHEMATICS

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Issued by the Committee on High School Relations

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PREFACE

The most widely accepted definition of the courses in secondary mathematics is that formulated by a Committee of the American Mathematical Society in 1903. (See Bulletin of the American Mathematical Society, November, 1903.) The actual practice in progressive schools has departed far from these definitions during the intervening years, so that the definitions of that committee cannot longer be considered the standard. The changes that have taken place have come from within the schools through the efforts of the progressive teachers of mathematics in these schools. These changes are greater than many realize, and entitle the teachers of the subject to greater approval than is often accorded them by some of the critics of the subject.

Among the changes that have taken place should be mentioned the effort to depart altogether from the practice of giving "stratified" instruction in algebra, geometry, etc.,—replacing such instruction by a course in "conglomerate" mathematics, to carry out the figure of speech; this practice, extending over a period of fifteen years, is still in its infancy and must be regarded as experimental. It is so regarded by the mass of experienced teachers of mathematics. In isolated instances, the emphasis has been directed from mathematics to applications of mathematics,—witness the courses in practical mathematics and vocational mathematics as given in trade schools and other special schools; this practice is as yet experimental and unestablished. It must be considered a feature of the special type of schools in which it is found.

Among the majority of progressive teachers, the tendency has been to retain the stratified character of the traditional program of secondary mathematics but to modify the separate courses from within by incorporating as much as seems feasible as well as desirable from the more extreme proposals set forth in the preceding paragraph. For convenience, these modified courses are still referred to as courses in algebra, geometry, etc., but they might properly be termed courses in mathematics, distinguished from each other merely by the characters I, II, etc.

Because these courses do represent great improvements over the courses of fifteen and more years ago, because they are the most widely accepted courses among progressive teachers, and because they can be adopted with a minimum of disturbance in any school, their description is made the special purpose of this bulletin.

The rapidly growing junior high school demands that there shall be a restatement of the program of mathematics for the grades seven to twelve inclusive. Tentative proposals for this program are already available and are well known to workers in junior high schools. A statement of this larger program will appear in the form of a bulletin after current experiments can be viewed with sufficient perspective.

The writer of this bulletin, while accepting full responsibility for it, wishes to acknowledge his indebtedness for counsel received from several members of the Department of Mathematics of the University of Wisconsin, who have read the bulletin. Special acknowledgment is due to a committee consisting of Professors Skinner and Lane, appointed by the department to read the bulletin and to report upon its acceptability to the department.

MODERN COURSES IN SECONDARY MATHEMATICS

1. Definitions of terms. The word *course* will be used to refer to the instruction in any subject during a whole school year (e. g. the first course in algebra), or during a semester (e. g. the course in solid geometry).

The program of courses in mathematics usually consists of a unit course in elementary algebra, an additional one-half unit course in advanced algebra, a one unit course in plane geometry, and one-half unit courses in solid geometry, arithmetic, and trigonometry.

The courses in mathematics may be arranged in various *sequences. A tandem sequence* is one in which a full course, or at least a half course, in one part of mathematics is followed by a full course or half course in some other part of mathematics. Universal custom in this country, to date, has favored a tandem sequence.

Many schools of the state, in the past, have adopted the following tandem sequence, recommended by the State Department of Public Instruction.

SEQUENCE A

First Year:
Algebra

Second Year:
Arithmetic
Bookkeeping

Third Year:
Plane Geometry

Fourth Year:
Advanced Algebra
Solid Geometry

Many cities in this state have used for some time the following tandem arrangement, the one which is in most common use in the country at large:

SEQUENCE B

First Year:
Algebra

Second Year:
Plane Geometry

Third Year:
Advanced Algebra
Solid Geometry

Fourth Year:
Electives

There have been in the past, and are at present, isolated instances of other sequences. One such is the following:

SEQUENCE C

First Year:
Algebra
Plane Geometry

Second Year:
Algebra
Plane Geometry

Third Year:
Advanced Algebra
Solid Geometry

Fourth Year:
Trigonometry
Arithmetic

Another sequence, called a *parallel arrangement*, carries the algebra and geometry side by side for two or three years. Thus, algebra may be given for three hours per week during the first semester of the first year, and plane geometry for two hours per week of that semester; during the second semester of that year algebra may be given for two hours per week and geometry for three hours per week. This type of arrangement has not been very popular in this country.

2. Improvement in secondary mathematical education is most commonly sought through rearrangement of the courses, or of subject matter in the courses. It is not the purpose of this bulletin to discuss the merits or weaknesses of any of the sequences outlined, or the inadequacy of such means of improvement. However, it does seem wise to utter a word of caution against the tendency to change from the old to anything new that may be proposed; to point out that proponents of rearrangements do not find any more valid arguments against the traditional sequences than equally experienced and progressive instructors find against some of the inovations proposed; to suggest, above all, that experimentation with sequences other than A or B should not be undertaken in schools where the principal and the teachers are unlikely to remain long enough to see the experiment carried through.

As a matter of fact, *improvement in the instruction in mathematics* in most schools *is likely to come about* not as a result of rearrangement of the courses, but as a result of *improvements within the several courses*, and of improvements in methods and modes of instruction. Before undertaking any experiments with new sequences, a school should make certain that it has thoroughly modernized the content of each of its mathematical courses and has introduced the modern point of view as to methods and modes of instruction.

This bulletin, therefore, is devoted to the description of the content of the several courses without regard to any sequence in which they may be arranged, although, frankly recognizing conditions as they are, it is assumed that either Sequence A or Sequence B is in use in most of the schools of the state.

3. One fault of the tandem sequences A and B is very common,—*viz.*, the tendency to teach each course in a "water-tight compartment." There has been too much tendency to focus attention in each course upon the one subject alone (algebra or geometry), neglecting opportunities to point out cross relations of that part of mathematics to other parts. It is not implied that this fault is as prevalent now as it was fifteen and more years ago, for great improvements in this respect can be discerned in the practice of many schools. This fault is easily overcome if the teachers of every course conscientiously set for themselves the responsibility of helping pupils maintain the skill they have acquired in previous courses. Among other means for accomplishing this purpose, the teacher will naturally seek occasions in each course to use bits of skill acquired in previous courses. Such efforts are likely to bring about considerable natural unification of algebra and geometry. Quite as important, such efforts will reduce the waste in education which accompanies the attitude that a course once studied and "passed" may be forgotten.

It will be assumed then as an important principle of selection and organization of material that provision will be made for maintaining in each course, as much as possible, the skill acquired in previous courses. In particular, the teacher of algebra and geometry should continue and build upon the work of the teacher of arithmetic in the grades; the teacher of geometry should seek opportunities to recall, to use, and to extend the pupils' knowledge of algebra.

First Course in Algebra—One Unit

4. Fundamental principles to guide in selecting subject matter and methods and modes of instruction.

A. *Make the course maintain and increase the pupils' skill* *in arithmetic.* This aim will be furthered by insisting upon correct and increasingly rapid computation; by using fractional and decimal as well as integral coefficients; by doing considerable evaluation of literal expressions; by checking by substitution; by teaching short cuts in computation that are based on algebraic processes; by acquainting pupils with the use of tables, such as a table of square roots; by utilizing as problem material as much as possible the important informational material taught in the course in arithmetic; and by giving the pupils a deeper understanding of number relations and the fundamental laws of computation.

B. *Develop understanding of and skill in using literal numbers and signed numbers*, (a) in formulae, and (b) in solving problems by means of equations,—the two central topics of algebra proper.

This second principle implies the selection of certain kinds of formulae and certain kinds of equations for study in the first course. Abstract processes should be studied and emphasized only as they tend to produce the skill required to develop these formulae and solve these equations intelligently, accurately, and rapidly. It leads to emphasis upon simple rather than complex examples; it leads to the omission of some material still taught. On the other hand, in applying this principle, the necessity of considerable abstract drill work as a means of developing skill must not be overlooked. Some of this drill work of necessity should be more involved than the manipulations that are likely to arise in the practical computations of the future. The parallels, here, of the finger exercises of one learning to play a piano and the vocal exercises of one receiving vocal instruction are quite obvious.

C. *Introduce the pupils to the concept of functional relationship.* This is accomplished in connection with and gives greater meaning to evaluation of literal expression, and to graphical representation. The essential thought involved is that of a dependence of one number upon one (or more) other numbers, in such manner that a definite value of the first is associated with every specified value of the second. In many cases, this value of the first is not only associated with, but is determined by the value of the second. Such is the case whenever the numerical value of a literal expression is computed for assigned values of the one (or more) letters involved in the expression. It is only a step from this form of determination to that which takes place when a value of the ordinate is determined on a graph as soon as a value of the abscissa is selected. At any rate, these two items,—evaluation and graphical representation,—afford the chief means of giving a clear impression of functional relationship.

D. *Make the course contribute the maximum training in desirable habits of thinking and doing,* (a) by teaching inductively, (b) by emphasizing thoughtful as opposed to mechanical processes, (c) by insisting upon neatness and order in the presentation of all solutions.

Algebra is an invaluable tool in the study of science. Provision is made for this aspect of it in a course organized in accordance with principles A, B, and C. It is also a valuable school discipline with possibilities that are unique. These possibilities can be attained only if attention is given to this principle D.

Mathematics has often been criticised as failing to give training in the inductive type of thinking. Actually, in no other school subject can pupils be given, as effectively, experience in this type of thinking. Arithmetic, elementary algebra, and intuitive geometry are the particular parts of mathematics where this type of thinking is used most appropriately. Teachers unfamiliar with the theory of inductive teaching should study it.[1]

Algebra has been criticized as resulting in meaningless juggling of abstract symbols. That criticism is altogether untrue where, as is often done, pupils are compelled to rationalize the various processes,—where they are not permitted to merely transpose, cancel, clear of fractions, etc., but are expected to explain each such mechanical process in terms of fundamental mathematical processes such as addition, subtraction, etc.

Finally, the joint responsibility of the teacher of mathematics with other teachers in the school for developing desirable habits of workmanship compels attention to the form of presentation as well as to the content of the productions offered by the pupil.

5. Topics for the first unit in elementary algebra. In general, teach the solution of equations of the first degree having one or two unknowns, and of the second degree, having only one unknown; the solution of problems leading to such equations; the study of formulae in which there appear only positive integral exponents and square roots; and all processes required to perform the necessary solutions skillfully.

In particular, teach,—

A. Literal numbers.
B. Positive and negative numbers.
C. Addition, subtraction, multiplication, and division of polynomials.
D. Special products, factoring, and quotients.
E. H. C. F., L. C. M. and fractions.
F. First degree equations and problems in one unknown.
G. Graphical representation of linear functions of one unknown.
H. Simultaneous equations of the first degree in two and possibly three unknowns.
I. Square root and quadratic surds.
J. Quadratic equations and problems in one unknown.

6. Discussion of Topic A. This topic affords the transition from arithmetic to algebra proper; it is more properly called generalized arithmetic. The logical first step is to use letters to represent any numerical value for certain concrete quantities involved in rules of computation, in mensuration and in percentage; thus, the formula $I=P \cdot R \cdot T$, by which interest problems can be solved, affords an easy beginning in literal arithmetic. At this time, emphasize the more important items of notation (e. g., ab means $a \cdot b$; a^3 means $a \cdot a \cdot a$), teach the necessary elementary definitions, develop skill in evaluating simple expressions, thus giving the idea of functional dependence. Equations and problems of certain kinds can and should be solved. These obviously should be solved by the axioms and should not have negative solutions.

7. Discussion of Topic B. Base the instruction on a variety of illustrations of concrete quantities. Follow this with instruction on addition and possibly multiplication of positive and negative numbers. Solve equations having negative results. Instruction on subtraction and division of signed numbers may be postponed until the study of subtraction and division of polynomials. In this way, the number of rules to be learned at the first presentation of the subject is reduced. Do not wait for one hundred per cent efficiency in applying the rules, because the pupils can acquire this skill during the balance of the year's work.

8. Discussion of Topic C. In the fundamental operations applied to polynomials, avoid very complex examples; avoid expressions involving fractional, negative, and literal exponents; avoid expressions having complicated coefficients and complicated terms. Call attention to the similarity of the ordinary arithmetical processes and the algebraic processes. Solve many simple examples. Give expressions requiring rearrangements of terms. Teach checking by substitution and by other means. Equally, if not more important, teach that the final result is identically equal to the indicated combination of the given polynomials; thus, when $2a+3b$ is added to $5a-4b$, the result $7a-b$ is equal to the sum of the two given binomials for all finite values of a and b. This same thought is to be emphasized throughout the study of the fundamental operations, factoring, and fractions.

Teach and use the proper technical terms, such as addend, sum, subtrahend, quotient, etc., instead of the less definite terms "quantity," "expression," and "answer."

If possible follow each new topic by the application of it to the solving of some new kind of equation of degree one.

Under addition and subtraction, use simple decimal and fractional numerical coefficients.

Under subtraction, teach the definition of subtraction, and teach inductively the usual rule for subtraction.

Under multiplication, do not give any definition of the process itself. Either develop inductively the rules for multiplying signed numbers, or else simply state these rules as laws of multiplication. Teach inductively the law of exponents for positive integral exponents. Give some easy examples in which the "base" as well as the exponent is an arithmetical number.

There is a tendency to reduce the emphasis upon division of polynomials. Undoubtedly too complicated examples often were proposed for solution by pupils. On the other hand, division of polynomials furnishes a valuable review of the three previous processes. Keep the work simple, ordinarily avoiding examples in which the divisor or quotient has more than three terms. Have some examples in which the division is not exact,—examples in which there is a remainder. Develop inductively the law of exponents for position integral exponents, and apply this law also in the case when the base is an arithmetical number.

9. Discussion of Topic D. For many years a considerable group of teachers have eliminated from the first instruction in algebra many of the complicated type forms of special multiplication and factoring. The tendency to do this is now quite widespread. Teach at most the following type forms:

a. $x(a+b+c)$
b. $(x+a)(x-a)$
c. $(x+a)(x+a)$
d. $(x+a)(x+b)$
e. $(ax+b)(cx+d)$
f. $(x+a)(x^2-ax+a^2)$
g. $(x-a)(x^2+ax+a^2)$

It is being proposed at present that types *f* and *g* be omitted from the first year course. Further, two plans are proposed for simplifying the instruction on types *b* to *e* inclusive. One plan is to omit altogether any special reference to types *b*, *c*, and *d*; in this case, Topic D reduces simply to the study of types *a* and *e*. The other plan is to teach type *a*; then teach type *e*; and afterwards call attention to types *b* and *c* as important special cases of type *e*.

When teaching any type, teach the special multiplication, the factoring, and the special quotient examples that accompany that particular type. Omit all generalization of the simple types taught; thus, omit the factoring of $x^2-a^2+2ab-b^2$. Omit all examples having literal, fractional, and negative exponents. Teach the types inductively. Give arithmetical applications when possible. After the types are known, have short periods of individual drill until the pupils can do at least two or three simple examples per minute. Some of the omitted tyes and examples should be done in the course in advanced algebra.

Recall the desirability of teaching that the given expression and the indicated product of its factors are identically equal, and that this fact can be taught best by substituting arithmetical numbers for the letters in both members of the equation. Note that this procedure is more than mere checking by substitution.

Follow this topic with the solution of quadratic equations with one unknown by factoring.

10. Discussion of Topic E. The tendency is to simplify this topic also. First of all, limit the instruction to examples that involve only the type forms studied under topic D; this automatically eliminates the so-called long division method of finding the highest common factor. Teach H. C. F. and L. C. M. as aids in solving examples in fractions.

Teach that a fraction is an indicated division. Base the instruction in each process upon a review of the corresponding process with arithmetical fractions. Emphasize the facts that the numerator and denominator of a fraction may be divided by the same number, and that they may also be multiplied by the same number.

Subordinate reference to cancellation,—possibly going as far as to insist upon division of numerator and denominator by the common factor or factors. Drill extensively upon relatively simple examples,—particularly avoiding complicated combinations in addition-subtraction, and in complex fractions. Apply fractions to the solution of fractional equations of the ordinary sort,—regarding as supplementary material the more complicated ones whose solution is easy only when accomplished by use of special devices.

11. Discussion of Topic F. First degree equations and problems should accompany all the previous topics, starting with very simple kinds and becoming more involved. Each new topic taught should be applied in the solution of some kind of equation. Emphasize the solution of equations by the axioms; in fact, there is no great loss in omitting altogether reference to *transposition* and *clearing of fractions*. To do this last successfully, it is necessary that the pupils use a uniform scheme for indicating the numbers added to or subtracted from both members of the equation, and the ones by which both members are multiplied or divided. Check the solution of equations by substitution. Avoid using only the letter x for the unknown. Use the word "root" to refer to the value of the unknown satisfying the equation.

By this time the pupils are prepared to understand the distinction between equation and identity, and this distinction should be made quite clear. It would be advantageous to use the symbol $\equiv$ to denote "identically equal to," although, at present, few teachers are willing to burden themselves with this undertaking.

Problems are often neglected. They are difficult because of the pupils' inefficiency in reading and understanding English, because the solution of problems calls for concentration of a high order, because problems necessarily refer to a variety of more or less concrete data that is beyond the limited experience of pupils, because skill in solving problems depends in part upon acquaintance with tricks and devices in translating the relations into literal expressions. But, problems should be made an important part of the course.

Problems may be grouped as number problems, age problems, digit problems, distance, rate, and time problems, etc. The groups vary in difficulty. Number problems, information problems, age problems, perimeter problems and area problems are easier for beginners than the other kinds that are commonly given. Make a special effort to teach each new kind when it is first encountered by limiting the instruction on that occasion to problems of that one kind. Emphasize then the meanings of new words and concepts involved and drill at length upon the means of translating the relations into algebraic symbols. If necessary, supplement the textbook by an abundance of oral drill on translating English expressions into algebraic symbols,—such expressions as "B's age 15 years from now," "the rate of a train travelling 2½ times as fast as that of a previously mentioned train," etc. Such expressions are easily extemporized. Do not constantly use the letters x, y and z for the unknowns; instead, use letters suggested by the data of the problem. At first, specialize in class on forming the equations, leaving for the study period review of this "translation" process and the solution of the equations. After two or more types of problems have been taught, have lists of miscellaneous problems from time to time. By all means, however, have problems frequently.

Literal equations can be done best after the work in factoring. Keep this work also quite simple in the first course in algebra.

12. Discussion of Topic G. Graphical representation is a valuable tool for expressing numerical relations; it is almost indispensable in the teaching of modern science and mathematics; it is interesting of itself to pupils. These are three important reasons for teaching it to high school pupils. Graphical work should appear not later than just before simultaneous linear equations, for in that topic it becomes indispensable as an aid in instruction.

Make certain that the pupils have a small supply of squared paper. Pieces of paper about four inches square, ruled into squares about 1/5 inch or 1/8 inch on a side, do very well. (Such paper can often be obtained at ten cent stores.) These pieces can be attached with paste to the blank paper usually used in classes in algebra. Or, if desired, get the metrically ruled paper often used in science classes, although this is more expensive. Have a portion of the blackboard about one yard square ruled into squares by lines about 2 inches apart; or get from a school supply house a piece of slated cloth ruled in squares and mounted on an ordinary shade roller. A little yellow, orange, and light green crayon will be helpful but is not indispensable. Begin with graphing statistics, giving the preference to examples having numbers written by only two or at most three significant figures. Presently teach the graphing of the linear function by means of two points, with the third point used as a check upon the accuracy of the solution.

Then teach how to obtain the graph of a linear equation having two unknowns. The difference between the graph of a function, and the locus of a given equation should be pointed out, although much emphasis upon this point during the first course in algebra is probably unwise. Not over two weeks need be spent upon this work. It is important that each pupil do well a few graphs, instead of doing hastily and untidily many exercises.

13. Discussion of Topic H. The first instruction in simultaneous equations should be based upon a graphical solution of a pair of linear equations with two unknowns. The one important new thought is that two such equations have, ordinarily, one and only one common solution, and that this solution corresponds to the one and only one point common to the two lines obtained by graphing the equations. After a few well selected pairs of equations have been solved graphically, one pair should be tried for which the coordinates of the common point are not easily read from the graph. This creates the opportunity to teach the better way of solving simultaneous linear equations, viz., by eliminating one unknown. Then, teach the addition-subtraction method and the substitution method of elimination. Teachers are almost unanimously in favor of omitting the "comparison" method of solution. The tendency is to do very little with systems having three unknowns, and nothing at all with those having more than three unknowns.

14. Discussion of Topic I. Formerly the latter part of the first course in algebra was taken up with an extended study of exponents, radicals, imaginaries, involution and evolution. The last fifteen to twenty years has seen this work become increasingly unpopular with teachers. Now that only one year of algebra, at most, is required in high schools, it is important that the work be so planned that the class will learn how to solve quadratic equations with one unknown. The solution of such equations has been selected as one of the major aims of the course. In order to accomplish this purpose, some of the material formerly given must be omitted. Moreover, since it has been agreed to teach only as much of each abstract process as is necessary to develop the skill needed in solving the equations taught, then radicals, etc., should be limited to the parts needed in solving quadratic equations. The following items only are needed: square root of polynomials (as a means of teaching the next item); square root of numbers, with emphasis upon finding to three decimal places the square root of numbers like 3, 5, 17, 43, etc.; evaluation of radicals of the form $\sqrt{ab^2}$ (e. g., $\sqrt{12} = 2\sqrt{3} = 2x1.732$) and of the form $\sqrt{\frac{a}{b}}$ (e.g. $\sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2} = \frac{1}{2}\times 1.414$) and finally, evaluation to three decimal places of expressions like $\frac{1}{3} \pm \sqrt{\frac{5}{3}}$. The arithmetical evaluation of such radicals affords opportunity for arithmetical calculation, and gives results which have greater meaning to a beginner than those left in radical form. It is granted that such results are only approximately correct; but that in itself is an important point of view. It is granted that the calculation involved takes time; but that time is well spent when the arithmetical needs of the pupils are remembered. After square roots have been computed as suggested above, they should be collected in the form of a table of square roots, for future use; or else a table of square roots should be placed in the hands of the pupils and used by them in future computations.

There is little except logical completeness to justify teaching how to add, subtract, multiply and divide quadratic surd expressions, and nothing at all to justify further work in radicals, exponents and imaginaries in the first course in algebra. The subject matter outlined in this section can be done well in at most two weeks. This saves time for the more important subject that remains for the first course.

15. Discussion of Topic J. The first instruction in quadratic equations is commonly given now after factoring. Then, only equations with rational roots are studied. Now, the subject should be studied more fully. The graph of the simple quadratic function $y=x^2$ should be drawn; also the graphs of a few other quadratic functions of x. Pure quadratics may be studied first because they most easily show the existence of equations with two irrational roots. Take up next the solution of the complete quadratic by completing the square, and finally the solution by the formula. Pupils do not have any trouble with the latter; they learn to use this method intelligently in two days. When irrational roots occur, have them written in their simplest radical form, and then in their approximate decimal form, correct to two or three places. After the real significance of irrational roots is appreciated, it is not longer necessary to compute the decimal values of the roots.

In checking quadratics, no difficulty occurs ordinarily unless the roots are irrational. To check by substituting irrational values for the unknown, leads usually to more inaccuracy than does the original solution. The common sense check in such cases is to go back over the solution or to make use of the well known fact that the sum of the roots of a quadratic equation is the negative of "b/a." It probably is wise to substitute the approximate decimal values of the roots in two or three simple quadratic equations, even though the arithmetical calculation does take some time. After quadratics with real irrational roots can be solved skillfully, teach the solution of quadratics having imaginary or complex roots, as time permits.

16. **The course outlined involves many omissions** of subject matter taught fifteen or more years ago, and unfortunately still taught in some schools. That which remains constitutes certainly the essentials of elementary algebra. The time gained by making the omissions should be utilized for more thorough drill on that which remains, and for more emphasis upon problems.

17. **Distribution of time during the year.** While it may be unwise to suggest any definite allotment of time to the study of any of the topics outlined, the following schedule approximately will be found feasible in schools whose pupils come without previous acquaintance with algebra. All of topics A, B, C and much of topic F should be completed by Christmas time. By the end of the first semester, complete topic D and more of topic F. Topic E with fractional equations will take about five weeks, topics G and H about five weeks, and topics I and J about five weeks. This leaves time for review for final examination. In many schools, more rapid progress may be

possible in the first half of the year, and thus additional time can be spent on the problems and on the topics G, H and J.

In schools whose pupils have had some algebra in the grades, a peculiar problem arises,—and this problem is going to be increased with the growth of junior high schools. In such schools, avoid re-doing all that has been done in the eighth grade,—granting that the children do come with varied and imperfect preparation. Try to build upon what they already know as rapidly as possible, emphasizing the thinking processes involved in the algebraic manipulations. In general, start in at the beginning of the usual high school text, dwell upon the parts which pupils usually find difficult or fail to appreciate, and get forward as rapidly as possible. The gain from the instruction in the elementary school is not often great, because frequently, too little attention has been given to the Principle D proposed in Section 4 of this monograph. But there is gain. The time so gained should be used for greater emphasis upon Principal A, for doing more complex examples, for doing more problems, and for giving more individual attention to the weak members of the class. If the usual experience with standardized tests means anything at all, it points to the need of more thorough instruction of the kind suggested for the first course in algebra instead of to elaboration of the first years work in high school mathematics by including work in demonstrative geometry, or trigonometry. Stated otherwise, it seems doubtful that sufficient is accomplished in algebra in the eighth grade in most schools to justify attempting to complete the first unit of algebra in any thing less than one full year in high school proper.

Advanced Course in Algebra—One-half Unit

18. Place and Purpose. This course may be regarded now (a) as the college preparatory course, and (b) as the course for those interested in mathematics. It has come to be rather generally an elective course. In large cities, it is usually given during the first semester of the third year. It should follow considerable instruction in demonstrative geometry. It is required of boys who enter the Engineering College at the University.

19. Character and Content in General. First, it should be more than an exercise-solving course, like the first course. Profiting by the mathematical maturity acquired during the course in demonstrative geometry, there should be some attention given to careful definitions of technical terms, to the demonstration of some of the principles applied (e. g., laws of exponents for positive integers) and more emphasis upon the theory involved in certain processes (e. g., the derivation of the formula for solving quadratic equations). None of this implies or necessitates neglect of drill necessary to acquire skill in the processes studied during the course.

As to the content,—during this course the study of more difficult forms of equations encountered in the first course should be continued, and the following new subject matter studied:

A. Special equations of higher degree, whose solution is accomplished by solving easily derived equations of degree one or two.

B. Simultaneous quadratic equations, — including the graphical solution of certain systems.

C. Theory of exponents; radicals; imaginaries.

D. Radical equations that are reducible to linear or quadratic equations.

E. Miscellaneous topics such as the binomial theorem, the progressions, and logarithms.

20. Review Part of the Course. There is necessarily a period of review at the beginning of the course. This should not be a re-doing of the first course; it should be a definite second view of important parts of the first course from a higher point of view, during a limited period of time. The first undertaking is a hasty recalling of some of the algebraic processes taught in the first year. While doing this, some of the generalizations marked for omission in the first course should be included. Thus, expressions with literal exponents should be used in problems of addition, subtraction, etc., of polynomials; under factoring the difference of two squares, examples like $x^2-y^2+2yz-z^2$ should be studied. At this point also extend the subject of factoring by teaching the factoring of certain types which were omitted from the first course. In particular, teach the factoring of the sum and the difference of like powers of two numbers, and possibly teach the factor theorem. Set a definite time limit for this review work. Six or eight weeks probably is all that can be afforded for it. The amount of time needed will depend upon many circumstances such as the thoroughness of the first instruction and the extent to which provision has been made during the year of geometry for keeping up the pupil's skill in algebra. Setting a definite time limit for this work will aid greatly in accomplishing the purpose without frittering away too much of the semester; the use of a text specifically designed for this work will also help materially.

21. Discussion of Topic A. Special equations of higher degree are relatively easy. Usually they appear without notice in connection with quadratics where equations like $x^4-5x^2+6=0$ may be found. The meaning and value of such examples is increased by expressing the fact that an equation of degree n has n roots. This fact is made especially clear and interesting if it is accompanied by a graphical solution of a cubic equation. This work naturally accompanies the new work of factoring.

22. Discussion of Topic B. Simultaneous quadratics is the first new topic in advance of the first course as it has been outlined in this bulletin. Very difficult types of examples under this topic can be devised and have often been taught. For years, the Department of Mathematics of the University of Wisconsin has specifically recommended that instruction in this topic be limited to the combination of one linear and one quadratic equation. Teachers whose classes are skillful will find little difficulty in teaching more complicated combinations, however. The essential consideration is that the pupils shall understand what is being done. This can be accomplished best by free use of graphical representation of the equations. Make certain that the pupils see the analogy between the common solution of the equations and the points of intersection of the graphs obtained from the equations. Have the values of the two variables paired off when writing the solutions, so that a "solution" gives the coordinates of a point of intersection. The solution, when real irrational values, should be expressed to two or three decimal places for some of the equations.

For the graphical work of this course, the metrically ruled squared paper used in science classes is probably better than any other form of squared paper.

23. Discussion of Topic C. Exponents, radicals and imaginaries have always caused pupils trouble. Unusually confusing complications were introduced in the examples in books of fifteen and more years ago. Much of this fault has been removed. Possibly the difficulty was caused by trying to teach these subjects to first year high school children toward the end of their first year when their interest was naturally low anyway. Probably much of the difficulty is due to the fact that many new concepts are involved, and that these, while simple, must be understood and used accurately.

Begin with the proofs of the laws of exponents for positive integers. Follow with an abundance of simple exercises applying these laws. For students who go on in mathematics, whether it be in an engineering school or in the course in commerce mathematics, knowledge of fractional and negative exponents is necessary. Make clear the meaning of these by careful definitions. Impress these meanings by an abundance of easy numerical applications. Then assume that the laws of exponents apply for these new exponents also and give easy exercises in multiplication, division, involution and evolution of radicals. The impression must not be given that fractional and negative exponents do not have great use in practical mathematics. Include some work with decimal exponents, particularly decimal exponents applied to base 10, since these are logarithms and make the natural introduction to logarithms as a separate topic.

Radicals, other than square roots and possibly cube roots, are of theoretical more than practical interest. Instruction in radicals should therefore be kept as simple as possible, with emphasis upon radicals of the second order. As in the first course, express the arithmetical values to three decimal places. Teach the various processes of "simplification" as means of more readily evaluating the original expression; thus, to find the decimal value of $\frac{\sqrt{2}}{2-\sqrt{2}}$ it is advisable to rationalize the denominator, thus producing the result $\frac{2\sqrt{2}+2}{2}$ or $\sqrt{2}+1$, from which the decimal result 2.414 is readily obtained by substituting 1.414 for $\sqrt{2}$.

For radicals of higher order, base the instruction on fractional exponents as is done in most texts. Select simple examples.

Under imaginaries, aside from the limited practical value of the topic to most pupils, the chief educational value lies in appreciating the new extension of the number field. Attention should be called to the different kinds of numbers studied in their past school work,—positive rational integers, fractions, signed numbers, and; more recently, irrational real numbers. For all these there is one common fundamental unit. Certain fundamental processes (addition, subtraction, multiplication, etc.) either may or may not be performed freely, in accordance with certain laws (the commutative, associative, etc.). Now, numbers having the new unit i are to be studied. So approached, interest and attention may be aroused.

Consistent use of this unit i with constant recollection that $i^2 = -1$, helps the pupils avoid the only errors that are likely to occur when studying this topic. As under radicals and exponents, avoid complicated examples.

24. **Discussion of Topic D.** Radical equations of certain simple sorts should now be studied. It should be emphasized that all indicated roots in an equation denote principal roots, —whether the expressions be written with radicals or with fractional exponents. The necessity of substituting in the original equation the value or values of the unknown that are gotten by the solution in order to determine whether "the equation" really is an equation,—i. e., whether there

really is a number satisfying the equation,—that necessity should be made quite clear. This can be done without reference to the subject of "equivalent equations." (See Rietz and Crathorne College Algebra, page 44.) In this case, substituting is more than mere checking; it has the purpose of determining whether the given expression really is an equation. Again, do easy examples. If the radical is of order higher than two, the equation should involve only one radical expression; if the radicals are all of order two, the equation should not contain more than, at most, four terms. In the examples of this latter type, the pupils' most common difficulty is that of failing to properly square a binomial when one or both terms are radicals. This point should be stressed.

25. Discussion of Topic E. When time permits, certain miscellaneous topics may be studied.

Under the binomial theorem, limit the study to the case of positive integral exponents. The use of the binomial theorem in computing the approximate values of such numbers as $(1.01)^{10}$ or $(.98)^{15}$ should not be overlooked, for such computations are frequently necessary in the course in commerce mathematics.

The arithmetical and geometrical progressions are both interesting and valuable topics. This is particularly true of the geometrical progression. The application of it to the derivation of some of the elementary formulae of compound interest may be found possible in some schools. Students who enter the University and take the commerce course, will find this study of the progressions a great help. When studying these topics, the pupils should be expected to derive the formulae encountered as well as use them.

Logarithms are best learned by use so that it becomes a question whether it is profitable to study them in a course in algebra in the high school. If time permits, however, the pupils may be given a first impression of the subject, especially if the teachers of solid geometry and science can be persuaded to use the logarithms in their classes. The course in trigonometry furnishes the first good opportunity for developing skill in the use of logarithms. Only when pupils acquire skill in the use of logarithms do they come to have much respect for the subject.

Plane Geometry—One Unit

20. Purpose and guiding principle. Certain facts taught in geometry are of great use in the work-a-day world,—of varying use to individuals, however, depending upon the particular vocation or interest of the individual. On the other hand, the experience with and the training in deductive reasoning acquired in the study of geometry has always been considered pre-eminent in educational value, and still must be so regarded, because experimental evidence supports belief in "transfer of training." Pupils receive this special training through experience with demonstrative geometry,—not intuitive geometry or constructional geometry. Moreover, they cannot get this training equally well from any other subject in the school curriculum. Therefore, any tendency to substitute for demonstrative geometry some other form of geometry, or to omit the study altogether is distinctly unwise educational policy.

A fundamental principle of instruction in geometry must be not to learn proofs of theorems but to prove theorems. The repetition of proofs found completed in the text must be subordinated; the demonstration of exercises must be emphasized. Such instruction does not necessitate neglect of the utilitarian values of geometry, or other cultural values,—such as attention to historical aspects of the study.

27. The old time description of the course as consisting of five books and exercises was objectionable because there was too much available in the books, for some classes, and because there was nothing to indicate what parts, if any, were of major importance. As a matter of fact, not all the propositions were of equal importance, and not all the exercises were equally suitable as means of accomplishing the major purposes of the course. In particular, the abundance of material provided prevented giving systematic attention to developing power to demonstrate. Such power as was developed in the pupils was chiefly an incidental by-product, and for that very reason was limited.

28. The modern description of the course recognizes that there is:

A. A minimum course consisting of:

(a) The propositions having practical value.

(b) The propositions, definitions, axioms, etc., necessary to weld these practical propositions into a somewhat logical sequence.

(c) Suitably selected exercises designed to aid in teaching these propositions and, further, to give training in independent demonstration.

(d) Historical material and applications to add to the cultural and utilitarian values of the course.

B. Supplementary material designed to add to the educational value of the course for such schools or pupils as have capacity and time to study it.

27. The Minimum Course. The first serious national effort to classify the customary propositions of geometry according to their relative importance was made in the Report of the National Committee of Fifteen on Geometry Syllabus. This Report appeared in School Science and Mathematics in 1911, in the December, 1912, number of The Mathematics Teacher, and in the Report of the Committee published by the United States Bureau of Education. The United States Bureau Report unfortunately is out of print, but was very widely distributed in the country, and can probably be found therefore in the libraries of schools of education and normal schools and in some private libraries. The Manual for Wisconsin High Schools, in 1914, contained the classified list of theorems as recommended by the Committee of Fifteen. Another list of selected theorems will be found in a forthcoming report of the Committee on Mathematical Requirements, which will be published by the United States Bureau of Education in 1921 or 1922. Since the appearance of the first of these reports, a number of texts have been published in which an effort is made either to present a minimum essentials course, or to indicate in some manner the relative importance of the propositions in the text.

Guided either by the reports named, or by one of the modern texts embodying the spirit of such a report, every teacher of geometry should mark in the text used by the class the propositions that do not fall within the minimum course as described. In order to present a logical sequence, the list of theorems will depend upon the sequence selected. Propositions marked as supplementary material should be omitted, if necessary, to gain time in which to do well the minimum course.

30. **The first part of the course is a critical period.** The danger then is that pupils will be overburdened with a multiplicity of definitions, concepts, notation and terminology to be learned, together with a certain logical attitude to be acquired. To avoid this, there should be a period of from two to three weeks of informal, intuitional study of geometry, devoid of insistence upon logical niceties, during which emphasis should be centered on acquisition of a limited number of necessary concepts, terminology and notation. Many teachers are unfamiliar with this kind of geometry because their own instruction in geometry has been almost exclusively of the deductive form.

The effectiveness of this part of the course depends in part upon the selection of material but chiefly upon the method of instruction. First of all, the method should be inductive—not deductive. This necessitates considerable drawing, measuring, paper folding, tracing figures on transparent paper, etc. Above all else, **mere memorizing** of technical terms, definitions, axioms, and postulates must be avoided. Sometimes suitable material for this part of the course can be found in the introduction to a text, but not always. If the introduction contains a collection of definitions, axioms and postulates needed in Book One, postpone as many as possible until they are really needed in the course.

Following this introductory period should come a transition period during which theorems are learned first inductively and then demonstrated. A common fault in instruction during this period is the tendency to go rapidly. Slow progress at this time, due to giving much individual instruction to pupils who do not find starting the subject easy, is compensated for by the more certain grasp the class as a whole gets of the subject. At this time, the supplementary material should be used for the brighter pupils.

This is the time also to start desirable habits of drawing figures, arranging proofs, and giving proofs. The figures should be drawn carefully, and completely lettered with capital printed letters; every statement made in a demonstration should be supported by the citation of an authority,—preferably quoted in full as found in the text; correct use and spelling of new technical terms should be expected; attention should be given to the general English expression of the pupils. All this takes time and vigilant attention on the part of the teacher.

31. **The balance of the course,** in fact, the bulk of the course consists of the study of the selected theorems and exercises. The success or failure of the course depends most emphatically upon the mode of instruction, i. e., upon the management of the class.

If the theorems proved in the text have very complete demonstrations, the class should be made responsible for reading, understanding, and rapidly reproducing the proof. There is, indeed, great educational value in these exercises, if the class is not allowed to get the impression that such an assignment is a difficult one, and if they are not allowed to slow up the recitation on such a demonstration because of faulty or inadequate study. To be sure, perfection cannot be expected. In general, the teacher should be exacting where experience has shown that no difficulty really exists in the proof, and be encouraging and helpful where experience has shown that even conscientious pupils do have difficulty. As a matter of fact, pupils can do much better work here than they usually do.

If the demonstrations in the text are incomplete, and this is indeed preferable, then greater consideration for the pupils is due from the teacher. In such cases, an effort should be made to get the pupils to do their best, to state frankly where they have trouble, and to volunteer in class whenever they think they have a demonstration for some difficult part of the proof. Frank questioning and good spirited cooperation in the class should be encouraged.

Fundamental principles of demonstration and the various types of proofs should be emphasized.

32. **The exercises should clear up and impress the meaning** of the propositions; they should prepare for new propositions; they should afford drill in independent demonstration; they should exhibit some of the uses of geometry. A first requisite is that carefully selected and well graded exercises should accompany every proposition, if possible. By no means should they be studied only periodically. At first they may be taken up in class, but later on they should be assigned for study outside of class. By classifying the exercises so as to teach definitely some principle of demonstration and by teaching the various methods of analysis, pupils of even mediocre ability can be trained to solve some originals.

33. **The distinction between undefined concepts,** defined concepts, axioms, and theorems should be made quite clear. The impossibility of defining some elements should be taught. Exactness in those definitions which are given should be expected, avoiding on the one hand incompleteness of definition, and on the other, redundancy. Axioms should be treated in their modern form as unproved theorems.

34. **Limits and incommensurables** may properly be relegated to the supplementary part of the course, or omitted altogether. In this latter case, the treatment of certain of the mensuration theorems will reduce to the study of only the commensurable cases.

35. **The supplementary part of the course** will include propositions which may have great mathematical interest but little practical value; more difficult exercises, including, among others, some which show applications of geometry in drawing, in design, and in indirect measurement; more extended study of methods of analysis, and methods of proof; the study of geometrical fallacies; etc.

Such material may properly be used for the further training of the brighter classes, or bright pupils in any class. In this latter case, much of the material must be assigned for study to individuals or to groups in the class, while the teacher is giving attention to weaker members of the class who are doing routine work.

The Course in Solid Geometry—One-half Unit

36. This course should be organized in a manner similar to that followed in the organization of the plane geometry course; that is, a minimum essentials course should be selected and other material should be relegated to the supplementary part of the course to be done if time and circumstances permit. As to **the chief purposes of the course,** the pupils have already gotten training in logical demonstration through their course in plane geometry. This course, therefore, cannot be expected to do more than strengthen any impressions and habits of such nature created in the course in plane geometry. This course, therefore, becomes more particularly a study of the subject itself. Both its utilitarian value and the possibility of creating and satisfying mathematical interest during the course are great. It may be expected to develop the power to visualize space relations.

37. The first part of the course is again a critical period. For pupils well trained in plane geometry, there is then only the new problem of visualizing the space figures. This difficulty is a real one. In fact pupils who have been quite successful in plane geometry often find the problem a greater one than do other pupils of lesser ability. As an aid to overcoming this difficulty, models of the first figures studied should be made from cardboard, string, wire, etc. It is undesirable, however to carry this form of assistance very far, for pupils may easily fail then to acquire the power of visualizing the three dimensional figures. Moreover, for most pupils who take this course, since it is elective, such help is unnecessary.

The mensuration theorems should be accompanied by an abundance of numerical exercises, and these can be of a very practical sort. If logarithms have been taught, the numerical exercises furnish a good opportunity to use the knowledge so gained.

The Course in Arithmetic—One-half Unit

38. Place and character of the course. A semester course in arithmetic has been given in many Wisconsin High Schools in the second year. In other schools in this state, and elsewhere, such a course often is given in the third or even in the fourth year.

Frequently no well defined purpose for this course is apparent. It is often merely a review of arithmetic which has been studied in the lower grades. In other cases, the course is definitely designed to meet the needs of pupils who expect to enter a normal school or to go into some form of commercial work. Besides providing for these two classes of pupils, the course may most properly be made the time to teach other pupils as well some of the more intricate parts of business arithmetic, parts which, most wisely, are now usually omitted from the eighth grade course in arithmetic. While the development of skill in computation should be a by-product of this course, it must not be regarded the major purpose. The informational content of the course should be emphasized. In no other course that is likely to be taken by pupils can clearer ideas about certain business and governmental practices be conveyed.

These purposes of the course justify postponing it until the third or fourth year of high school, preferably the fourth, if the program of the school permits that arrangement. The policy of giving the course in the first or second year seems to be dictated by a belief that pupils are deficient in arithmetic, —meaning in this case, lacking in skill in computaton. That they are often deficient in that respect is indeed true—but overcoming that deficiency is scarcely defensible as the major purpose of the course. As a matter of fact, that purpose often defeats itself for it is not one in which the pupils can be greatly interested. Even more so is it questionable as a matter of policy to give such a course to pupils in the first year of the high school. There is a justifiable tendency in the country at large to reduce the time given to instruction in mere arithmetic,—partly because the pupils become wearied with the repetitions involved even in the present elementary and grammar school courses.

39. **What to teach** in the course in arithmetic depends on the pupils in the class. A brief time should be spent on drill designed to revive skill in computation. This period should, by no means, be more than one-fifth or, at most, one-fourth of the semester devoted to arithmetic. At this time attention should be given to "short cuts" in computation, to the use of tables, and to rational approximations in computing.

The balance of the course should be devoted to a study of the following two large groups of material.

A. Mensuration and its applications.

B. Percentage and its applications.

Parts of each group should be given to every class. If the class consists chiefly of pupils likely to go into commercial work of some sort, then group B should be emphasized. If the pupils are chiefly boys, who may go into trades, or shops, or on the farm, then group A may well be emphasized. For the latter, special attention should be given to problem material having significance in an agricultural community. If no such pronounced classification of the pupils can be discerned, that is, if there are many in the class of both kinds, then about equal attention should be given to the two groups of material.

The Course in Plane Trigonometry—One-half Unit

40. **Plane trigonometry**, with emphasis upon the solution of problems of indirect measurement, is always interesting to properly qualified pupils because of its obviously practical nature. In schools where conditions permit, the course should be offered as an elective to pupils who have had the course in advanced algebra. Solid geometry is not a necessary prerequisite for the course.

It is rather futile to give a syllabus of this course. The solution of problems by trigonometric means should receive the major attention. This will necessitate the definition of the trigonometric functions, the derivation of certain well known formulae, the solution of selected abstract exercises designed to impress the meaning of these formulae, the solution of problems involving right triangles and oblique triangles, both by natural functions and by logarithms of the values of the functions. Complicated abstract exercises to impress the meaning of the formulae should be avoided. This part of trigonometry is called goniometry. Like the older courses in algebra, goniometry was unduly elaborated. The solution of problems by logarithms takes time and patience. If possible, do some outdoor work, either with a transit or, if the school cannot afford a transit, with home made instruments for measuring both vertical and horizontal angles.

Junior High School Courses

41. **At the present time** of experimentation with the junior high school and its various courses, any degree of agreement about the content and arrangement of the courses cannot be expected. Enthusiastic proponents of a particular scheme may for awhile make that scheme appear desirable. It may be urged with safety that there should not be any radical departures from existing practices; that, instead, there should be a **gradual evolution toward** schemes that, theoretically, appear ideal. Just as efforts to improve the high school instruction in mathematics as a whole should begin with modernizing the separate courses, so efforts to improve the course in grades seven to nine inclusive should start with an overhauling of the courses in the separate grades. As a guide to such revision, attention may be directed to Bulletin No. 11, 1913, of the United States Bureau of Education.

Some enrichment of the work in grades seven and eight has been demanded since the time of the Report of the Committee of Ten (1896). That committee proposed some constructional geometry and some algebra for this purpose. It should not be overlooked that capable school people tried to carry out these proposals,—and later on dropped them to such extent that the very same proposals today seem novel to persons unacquainted with the previous attempts. The failure was due possibly if not probably to merely transferring certain high school material into the grades,—and that failure should have been expected. A similar result will probably follow a similar attempt today.

Some constructional, intuitional geometry for these grades is certainly desirable both for its direct value and for its indirect training value. It should be associated chiefly with the mensuration work of these grades. It may suffice even if the mensuration material is taught more concretely, for such instruction demands much drawing and measuring, and that is the chief characteristic of this intuitional geometry.

In the case of the algebra,—the use of literal numbers in formulae, the concepts of the signed number and the equation have become so widely used among intelligent people that instruction concerning them may properly be regarded part of "elementary" or junior high school education. Moreover, the pupils of these grades, especially the eighth, need the stimulus of contact with a new type of mathematical material. This justifies then some instruction in these topics. To carry such instruction far into the course in abstract algebra has not yet been approved by superintendents of elementary schools of wide experience.

REFERENCES TO COURSES IN MATHEMATICS

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2. Reports of the American Commission on the Teaching of Mathematics:
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   (b) *The Teaching of Mathematics in Secondary Schools.*
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3. Reports of the Central Association of Science and Mathematics Teachers:
   (a) Of the Committee on Algebra.
   School Science and Mathematics, VII; 694.
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6. Report of the National Committee of Fifteen on Geometry.
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   (a) *On the Content of Geometry.* E. B. Skinner, Professor of Mathematics, Chairman.
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