THE MODERN GEOMETRY OF THE TRIANGLE

BY

WILLIAM GALLATLY, M.A.

SECOND EDITION

LONDON:
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

PRICE HALF A CROWN NET

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PREFACE.

In this little treatise on the Geometry of the Triangle are presented some of the more important researches on the subject which have been undertaken during the last thirty years. The author ventures to express not merely his hope, but his confident expectation, that these novel and interesting theorems—some British, but the greater part derived from French and German sources—will widen the outlook of our mathematical instructors and lend new vigour to their teaching.

The book includes some articles contributed by the present writer to the *Educational Times* Reprint, to whose editor he would offer his sincere thanks for the great encouragement which he has derived from such recognition. He is also most grateful to Sir George Greenhill, Prof. A. C. Dixon, Mr. V. R. Aiyar, Mr. W. F. Beard, Mr. R. F. Davis, and Mr. E. P. Rouse for permission to use the theorems due to them.

W. G.

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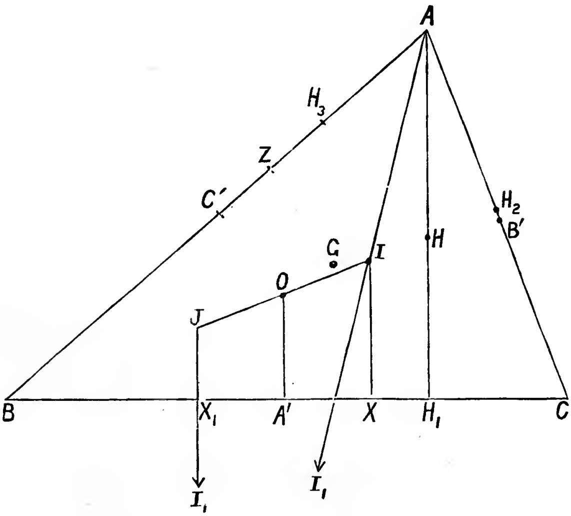
CHAPTER I.

INTRODUCTION: DIRECTION ANGLES.

1. In this work the following conventions are observed:—the Circumcentre of ABC, the triangle of reference, will be denoted by O; the Orthocentre by H; the feet of the perpendiculars from A, B, C on BC, CA, AB respectively by H_1, H_2, H_3 (the triangle $H_1H_2H_3$ being called the Orthocentric Triangle); the lengths of AH_1, BH_2, CH_3 by h_1, h_2, h_3; the Centroid or Centre of Gravity by G; the in- and ex-centres by I, I_1, I_2, I_3; the points of contact of the circle I with the sides of ABC by X, Y, Z; the corresponding points for the circle I_1 being X_1, Y_1, Z_1.



The Circumcentre of $I_1I_2I_3$ is J, which lies on OI; also $OJ = OI$, and the Circumradius of $I_1I_2I_3 = 2R$.

The mid-points of BC, CA, AB are A', B', C'; the triangle $A'B'C'$ being called the *Medial* Triangle. Its Circumcircle is the Nine-Point Circle, with Centre O' and radius $= \frac{1}{2}R$; the Orthocentre of $A'B'C'$ is O, the Centroid is G, while the in-centre is denoted by I'.

The lines drawn through A, B, C parallel to BC, CA, AB form the Anti-Medial Triangle $A_1B_1C_1$.

Its Circumcentre is H, its Centroid is G, its Medial or Nine-Point Circle is ABC; and its Nine-Point Centre is O.

The letters "n.c." stand for *normal* or *trilinear* coordinates denoted by $\alpha\beta\gamma$.

The letters "b.c." stand for *barycentric* or *areal* or *triangular* coordinates, denoted by xyz; also $x \equiv a\alpha$, &c., so that

$$x+y+z = 2\Delta.$$

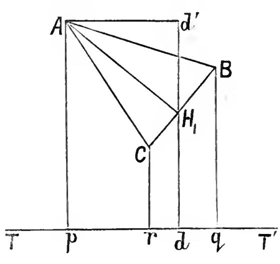
2. Let p, q, r be the lengths of the perpendiculars from A, B, C on a straight line TT'.

Let θ_1, θ_2, θ_3 be the Direction Angles of TT'; that is, the angles which the sides of ABC make with TT', these angles being measured from TT' as axis, and *in the same sense.*

By projecting the sides of ABC along and perpendicular to TT', we obtain $\Sigma . a \cos\theta_1 = 0$, $\Sigma . a \sin\theta_1 = 0$.

The diagram shows that $a \sin\theta_1 = q-r$, &c., so that

$$\Sigma . ap \sin\theta_1 = \Sigma . p(q-r) = 0.$$



Two sets of direction angles should be particularly noted.

For OI, $\cos\theta_1 = A'X/OI = \frac{1}{2}(b-c)/OI \propto (b-c)$.

For OGH,

$$\cos\theta_1 = A'H_1/OH = R\sin(B-C)/OH \propto (b^2-c^2)/a.$$

To express $\cos\theta_1$ in terms of p, q, r.

Since $$CH_1 : H_1B = b\cos C . c\cos B,$$

$$\therefore\quad q.b\cos C + r.c\cos B = (b\cos C + c\cos B)\,H_1d$$
$$= a(p - H_1d') = ap - a.AH_1\cos\theta_1;$$
$$\therefore\quad 2\Delta.\cos\theta_1 = ap - bq\cos C - cr\cos B;$$

When H_1 falls outside dd', the right-hand signs are changed.

3. To determine the condition that $l\alpha + m\beta + n\gamma = 0$, and $l'\alpha + m'\beta + n'\gamma = 0$ may be at right angles.

Let $\theta_1, \theta_2, \theta_3$ and ϕ_1, ϕ_2, ϕ_3 be the direction angles of the two lines, so that $\theta_1 = \phi_1 \pm \frac{1}{2}\pi$.

Let p, q, r, p', q', r' be the perpendiculars on the lines from A, B, C, so that $l \propto ap$, $l' \propto ap'$.

Now $$ap'\sin\phi_1 + \ldots = p'(q' - r') + \ldots = 0.$$

And $$2\Delta.\sin\phi_1 = 2\Delta.\sin(\theta_1 \pm \tfrac{1}{2}\pi) = 2\Delta\cos\theta_1$$
$$= ap - bq\cos C - cr\cos B.$$
$$\therefore\quad ap'(ap - bq\cos C - cr\cos B) + \ldots = 0,$$

or $$ll' + mm' + nn' - (mn' + m'n)\cos A - \ldots = 0,$$

which is the required condition.

4. To determine π, the length of the perpendicular on TT' from a point P, whose b.c. are (x, y, z), in terms of p, q, r.

Since P is the centre for masses at A, B, C proportional to x, y, z. $$\therefore\quad (x + y + z)\,\pi = px + qy + rz;$$
so that π is determined when p, q, r are known.

Note that the *ratios* only of x, y, z are needed.

5. To determine π, when TT' is $l\alpha + m\beta + n\gamma = 0$.

Put $$l^2 + \ldots - 2mn\cos A - \ldots \equiv D^2.$$

Now $$l \propto ap \equiv k.ap.$$

Also $$\Sigma\,(a^2p^2 - qr.2bc\cos A) \equiv 4\Delta^2,$$

and $$x + y + z = a\alpha + b\beta + c\gamma = 2\Delta.$$

Hence $$\pi = (l\alpha + m\beta + n\gamma)/D.$$

A form of little use, as it is almost always difficult to evaluate D.

6. A straight line TT' is determined when any *two* of the three perpendiculars p, q, r are given absolutely. It follows that there must be some independent relation between them.

From elementary Cartesian Geometry we have

$$2\Delta = Ap.qr + Bq.rp + Cr.pq$$
$$= p.a\cos\theta_1 + q.b\cos\theta_2 + r.c\cos\theta_3$$
$$\therefore\quad 4\Delta^2 = ap.2\Delta\cos\theta_1 + \ldots = ap\,(ap - bq\cos C - cr\cos B) + \ldots$$
$$= \Sigma\,(a^2p^2 - 2bc\cos A.qr) = \Sigma\{a^2p^2 - (-a^2+b^2+c^2)qr\}.$$

This is the relation sought.

When TT' passes through A, $p = 0$, so that

$$b^2q^2 + c^2r^2 - 2bc\cos A.qr = 4\Delta^2.$$

7. The points P, P' lying on TT' have *absolute* n.c. (α, β, γ) and $(\alpha', \beta', \gamma')$. It is required to determine d, the length of PP', in terms of these coordinates.

We have $\quad \Sigma\{a^2p^2 - (-a^2+b^2+c^2)qr\} = 4\Delta^2.$

But $\quad a^2 = 2\Delta\,(\cot B + \cot C),$

and $\quad -a^2+b^2+c^2 = 4\Delta\cot A.$

Hence $\quad (q-r)^2\cot A + (r-p)^2\cot B + (p-q)^2\cot C = 2\Delta.$

Now $\quad q - r = a\sin\theta_1 = a.(\alpha-\alpha')/d.$

Hence

$$\Delta/R^2.d^2 = (\alpha-\alpha')^2\sin 2A + (\beta-\beta')^2\sin 2B + (\gamma-\gamma')^2\sin 2C.$$

8. To prove that, when TT' is a circumdiameter,

$$ap/R = b\cos\theta_3 + c\cos\theta_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (G)†$$

From (2) we can express the right side in terms of p, q, r.

Then apply the condition $a\cos A.p + \ldots = 0$, and the result follows.

Hence prove that—

for OI, $\quad p = R/OI.(b-c)(s-a)/a;$

for OH, $\quad p = R/OH.(b^2-c^2)\cos A/a.$

The equation to TOT', which is $p.a\alpha + \ldots = 0$, now takes the form $\quad (b\cos\theta_3 + c\cos\theta_2)\,\alpha + \ldots = 0.$

For OI, $\quad (b-c)(s-a).\alpha + \ldots = 0$

[more useful that $(\cos B - \cos C)\,\alpha + \ldots = 0$].

For OGH, $\quad (b^2-c^2)\cos A.\alpha + \ldots = 0.$

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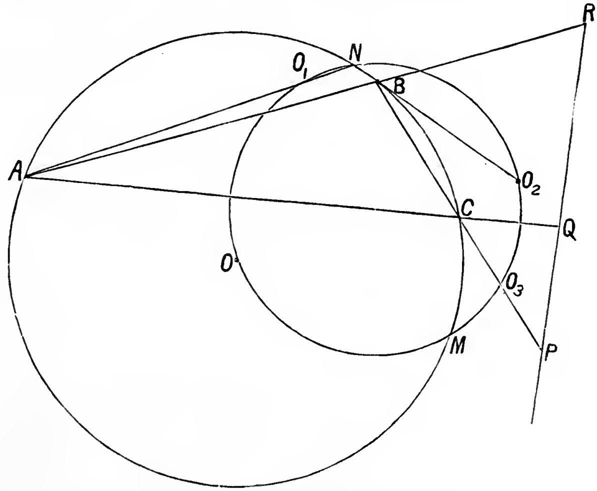
† For theorems and proofs marked (G) the present writer is responsible.

9. The results just obtained are useful in investigating some algebraic relations of a quadrilateral, which has BC, CA, AB for three of its sides: the fourth side being PQR.

The circumcircles of the four triangles AQR, BRP, CPQ, ABC, have a common point (call it M) so that the parabola which touches the four sides of the quadrilateral will have M for focus; while the four orthocentres lie on the directrix, which is also the common radical axis of the circles $(AP)(BQ)(CR)$.

The four circumcentres O, O_1, O_2, O_3 are known to lie on a circle (call it the Centre Circle) which also passes through M.



Let ρ be the radius of this circle and ρ_1, ρ_2, ρ_3 the radii of AQR, BRP, CPQ.

Since the angles at Q, R are θ_2, θ_3,

$$2\rho_1 \sin\theta_3 = AQ \text{ (in circle } AQR) = p/\sin\theta_2;$$

$$\therefore \quad \rho_1 = p\sin\theta_1 . 1/(2\sin\theta_1\sin\theta_2\sin\theta_3) = m.p\sin\theta_1.$$

In circle $AMQR$, $\quad AM = 2\rho_1 \sin ARM$,

and in $BMRP$, $\quad BM = 2\rho_2 \sin(BRM \text{ or } ARM)$;

$$\therefore \quad AM : BM : CM = \rho_1 : \rho_2 : \rho_3 = p\sin\theta_1 : q\sin\theta_2 : r\sin\theta_3;$$

therefore the n.c. of M are as

$$1/p\sin\theta_1, \quad 1/q\sin\theta_2, \quad 1/r\sin\theta_3,$$

or $\qquad a/p(q-r)$, &c. or $a/(1/q-1/r)$. (R. F. Davis)

Note that in the circle $AMQR$,

$$QR = 2\rho_1 \sin A \propto ap\sin\theta_1.$$

10. The join OO_1 of the centres $ABCM$, $ARQM$ is perpendicular to the common chord MA.

Similarly OO_2 is perpendicular to the chord MB.

$$\therefore \quad \angle O_1OO_2 = AMB = ACB \text{ (in } ABCM) = C;$$

$$\therefore \quad O_1O_3O_2 = C, \quad \dots;$$

and the triangle $O_1O_2O_3$ is similar to ABC. But, from

$$MA : MB : MC = \rho_1 : \rho_2 : \rho_3 = MO_1 : MO_2 : MO_3.$$

Hence M is the double point of the similar triangles ABC, $O_1O_2O_3$.

Since O_2O is perpendicular to MB, and O_2O_1 to MR,

$$\therefore \quad \angle OO_2O_1 \text{ or } OMO_1 = BMR = BPR \text{ (circle } BMPR) = \theta_1.$$

Hence the chords OO_1, OO_2, OO_3 in the Centre Circle subtend angles θ_1, θ_2, θ_3, so that

$$OO_1 : OO_2 : OO_3 = \sin\theta_1 : \sin\theta_2 : \sin\theta_3.$$

11. To determine the equation of the 4-orthocentre line. The equation to PQR being $p.a\alpha+q.b\beta+r.c\gamma = 0$, the point P is determined by $\alpha = 0$. $q.b\beta+r.c\gamma = 0$.

The perpendicular from P on AB proves to be

$$\cos A.a\alpha/r+(c/p-a\cos B/r)\beta+\cos A.c\gamma/p = 0,$$

and the perpendicular from R on AC is

$$\cos A.a\alpha/q+\cos A.b\beta/p+(b/p-a\cos C/q)\gamma = 0.$$

By subtraction their point of intersection is found to lie on

$$(1/q-1/r)\cos A.\alpha+(1/r-1/p)\cos B.\beta+(1/p-1/q)\cos C.\gamma,$$

or $$p\sin\theta_1\cos A.a\alpha+\dots = 0,$$

or $$\cos A/\alpha'.a\alpha+\dots = 0,$$

or $$p(q-r)\cos A.\alpha+\dots = 0. \qquad (M \text{ is } \alpha'\beta'\gamma')$$

From the symmetry of this equation the line clearly passes through the other three orthocentres and is therefore the directrix of the parabola.

For example, let PQR be $x/a^2+\dots = 0$, which will prove to be the Lemoine Axis (128).*

Here $p \propto 1/a^2$: so that the focus of the parabola, known as Kiepert's Parabola, has n.c. $a/(b^2-c^2)$ &c., the directrix being $(b^2-c^2)\cos A.\alpha+\dots = 0$, which is OGH.

The mid-points of diagonals lie on

$$(-1/p+1/q+1/r)x+\dots = 0, \quad \text{or} \quad \cot A.x+\dots = 0,$$

the well known Radical Axis of the circles ABC, $A'B'C'$, &c.

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* The bracketed numbers refer to sections.

12. To determine ρ, the radius of the Centre circle. (G)

$$MO_1 = \rho_1 = m.p \sin\theta_1, \text{ so } MO_2 = m.q\sin\theta_2,$$

and $$O_1MO_2 = O_1O_3O_2 = C;$$

$$\therefore\ O_1O_2^2/m^2 = p^2\sin^2\theta_1 + q^2\sin^2\theta_2 - 2pq\sin\theta_1\sin\theta_2\cos C$$

$$= \frac{p^2(q-r)^2}{a^2} + \frac{q^2(r-p)^2}{b^2} - pq\frac{(q-r)(p-r)}{ab}\cdot\frac{a^2+b^2-c^2}{ab};$$

$$\therefore\ O_1O_2^2/m^2c^2 = [\Sigma\, a^2q^2r^2 - \Sigma\, p^2qr\,(-a^2+b^2+c^2)]/a^2b^2c^2 \equiv k^6/a^2b^2c^2.$$

But since $\angle\, O_1O_3O_2 = C,\quad O_1O_2 = 2\rho\sin C,$

$$\therefore\quad \rho = mR.k^3/abc.$$

To determine the length of AP, one of the diagonals of the quadrilateral.

The distance d between $(\alpha\beta\gamma)$ and $(\alpha'\beta'\gamma')$ is given by

$$d^2.\Delta/R^2 = (\alpha-\alpha')^2\sin 2A + \dots. \tag{6}$$

Now, for P, $\quad \alpha = 0;\quad q.b\beta + c.r\gamma = 0;$

$$\therefore\quad \beta = (-r)/(q-r).2\Delta/b,\quad \gamma = q/(q-r).2\Delta/c.$$

Also for A, $\quad \alpha' = 2\Delta/a,\quad \beta' = 0,\quad \gamma' = 0;$

$$\therefore\quad AP^2.\Delta/R^2 = 4\Delta^2/a^2.\sin 2A + r^2/(q-r)^2.4\Delta^2/b^2.\sin 2B$$
$$+ q^2/(q-r)^2.4\Delta^2/c^2.\sin 2C.$$

Hence $\quad AP^2\,(q-r)^2 = q^2b^2 + r^2c^2 - 2qrbc\cos A.$

13. Let $\omega_1, \omega_2, \omega_3$ be the centres of the diameter circles, or mid-points of AP, BQ, CR. To determine the length of $\omega_1\omega_2$.

For ω_1, $\quad \alpha_1 = \Delta/a,\quad \beta_1 = (-r)/(q-r).\Delta/b,\quad \gamma_1 = q/(q-r).\Delta/c.$

For ω_2, $\quad \alpha_2 = r/(r-p).\Delta/a,\quad \beta_2 = \Delta/b,\quad \gamma_2 = (-p)/(r-p).\Delta/c.$

Now, $\omega_1^2\omega_2^2.\Delta/R^2 = (\alpha_1-\alpha_2)^2\sin 2A + \dots$

$$= \frac{p^2}{(p-r)^2}\cdot\frac{\Delta^2}{a^2}.\sin 2A + \frac{q^2}{(q-r)^2}\cdot\frac{\Delta^2}{b^2}.\sin 2B$$
$$+ \frac{r^2(p-q)^2}{(p-r)^2(q-r)^2}\cdot\frac{\Delta^2}{c^2}.\sin 2C;$$

$$\therefore\quad 2.\omega_1^2\omega_2^2/\Delta = \frac{p^2}{(p-r)^2}.\cot A + \frac{q^2}{(q-r)^2}.\cot B$$
$$+ \frac{r^2(p-q)^2}{(p-r)^2(q-r)^2}.\cot C.$$

Hence, since $\quad \cot A = (-a^2+b^2+c^2)/4\Delta,$

$$8.\omega_1\omega_2^2(q-r)^2(r-p)^2 = p^2(q-r)^2(-a^2+b^2+c^2) + \dots;$$

$$\therefore\quad 4.\omega_1\omega_2^2(q-r)^2(r-p)^2 = \Sigma\, a^2q^2r^2 - \Sigma\, p^2qr(-a^2+b^2+c^2) = k^6;$$

$$\therefore\quad \omega_1\omega_2 = \tfrac{1}{2}.k^3/[(q-r)(r-p)] \propto p-q \propto c\sin\theta_3;$$

$$\therefore\quad \omega_2\omega_3 : \omega_3\omega_1 : \omega_1\omega_2 = a\sin\theta_1 : b\sin\theta_2 : c\sin\theta_3.$$

CHAPTER II.

MEDIAL AND TRIPOLAR COORDINATES.

14. *Medial Coordinates.*—If A', B', C' are the mid-points of BC, CA, AB, the triangle $A'B'C'$ is called the Medial Triangle of ABC; its circumcircle $A'B'C'$ is the Nine-Point Circle whose centre O' bisects OH.

For every point P in ABC there is a homologous point P' in $A'B'C'$, such that P' lies on GP, and $GP = 2 . GP'$.

Let $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ be the n.c. of a point P referred to ABC, $A'B'C'$ respectively.

A diagram shows that $\alpha + \alpha' = \frac{1}{2}h_1$.

$$\therefore \quad a\alpha + a\alpha' = \tfrac{1}{2} . ah_1 = \Delta = 4 . \text{area of } A'B'C'$$
$$= 2(a'\alpha' + b'\beta' + c'\gamma') \qquad [a' \equiv \tfrac{1}{2}a]$$
$$= a\alpha' + b\beta' + c\gamma';$$
$$\therefore \quad a\alpha = b\beta' + c\gamma', \quad \&c.,$$

so that
$$2a\alpha' = -a\alpha + b\beta + c\gamma,$$
or, in b.c.,
$$x = y' + z', \quad 2x' = -x + y + z.$$

For example, the $A'B'C'$ b.c. of the Feuerbach Point F being $a/(b-c)$, $b/(c-a)$, $c/(a-b)$, to determine the ABC b.c. of this point.

Here
$$x' \propto a/(b-c), \quad \&c.;$$
$$\therefore \quad x = y' + z' \propto b/(c-a) + c/(a-b) \propto (b-c)(s-a)/(c-a)(a-b)$$
$$\propto (b-c)^2(s-a).$$

If the $A'B'C'$ b.c. are to be deduced from the ABC b.c., then
$$2x' = -x + y + z \propto -(b-c)^2(s-a) + (c-a)^2(s-b) + (a-b)^2(s-c)$$
$$\propto a(a-b)(a-c) \propto a/(b-c).$$

15. If the ABC equation to a straight line is $lx + my + nz = 0$, the $A'B'C'$ equation is
$$l(y' + z') + \ldots = 0, \quad \text{or} \quad (m+n)x' + \ldots = 0.$$

If the $A'B'C'$ equation to a straight line is $l'x' + m'y' + n'z' = 0$, the ABC equation is
$$l'(-x + y + z) + \ldots = 0, \quad \text{or} \quad (-l' + m' + n')x + \ldots = 0.$$

Example.—The well known Radical Axis whose ABC equation is $\cot A.x+\ldots=0$ becomes

$$(\cot B+\cot C)x'+\ldots=0 \quad \text{or} \quad a^2x'+b^2y'+c^2z'=0,$$

when referred to $A'B'C'$.

If H_1, H_2, H_3 are the feet of perpendiculars, the ABC equation to H_2H_3 is

$$-\cot A.x+\cot B.y+\cot C.z=0.$$

Therefore the $A'B'C'$ equation is

$$(\cot B+\cot C)x+(\cot C-\cot A)y+(-\cot A+\cot B)z=0,$$

reducing to $\qquad a^2x+(a^2-c^2)y+(a^2-b^2)z.$

Returning to the quadrilateral discussed in section (13) we see the b.c. of ω_1 given by $a_1=\Delta/a$, &c.

Hence the ABC equation to the mid-point line of the quadrilateral is $(-1/p+1/q+1/r)x+\ldots=0$.

The $A'B'C'$ equation therefore is

$$x'/p+y'/q+z'/r=0, \quad \text{or} \quad aqr.\alpha+\ldots=0.$$

The perpendicular on this from A' is therefore given by

$$\pi_1=aqr.h_1/D,$$

where $\qquad D^2=\Sigma\{a^2q^2r^2-p^2qr(-a^2+b^2+c^2)\}=k^6.$

$$\therefore \quad \pi_1p=2\Delta.pqr/k^3=\pi_2q=\pi_3r.$$

16. To determine the $A'B'C'$ equation to a circumdiameter TOT', whose direction angles are θ_1, θ_2, θ_3.

Let p', q', r' be the perpendiculars from A', B', C' on TOT'.

A diagram shows that

$$p'=OA'\cos\theta_1=R\cos A\cos\theta_1.$$

Hence the required equation is

$$\cos A\cos\theta_1.x'+\ldots=0.$$

Example.—For OI, $\cos\theta_1=\frac{1}{2}(b-c)/OI$.

Hence the $A'B'C'$ equation to OI is

$$(b-c)\cos A.x'+\ldots=0.$$

The ABC equation to the circumcircle ABC is

$$a/\alpha+\ldots=0, \quad \text{or} \quad a^2/x+\ldots=0.$$

Therefore its $A'B'C'$ equation is

$$a^2/(y'+z')+\ldots=0,$$

or $\qquad a^2x^2+b^2y^2+c^2z^2+(a^2+b^2+c^2)(y'z'+z'x'+x'y')=0.$

Referred to $A'B'C'$, the equation of the Nine-Point or Medial Circle is $\qquad a^2/x'+\ldots=0.$

Referred to ABC, this becomes

$$a^2/(-x+y+z)+\ldots=0,$$

reducing to the well known form

$$a\cos A\,.\,\alpha^2+\ldots-a\beta\gamma-\ldots=0.$$

17. *Tripolar Coordinates.*—

The tripolar coordinates of a point are its distances, or ratios of distances, from A, B, C.

To determine the tripolar equation to a circumdiameter whose direction angles are θ_1, θ_2, θ_3.

Let P be any point on the line, x, y, z, the projections of OP on the sides.

Then, if r_1, r_2, r_3 be the tripolar coordinates of P,

$$r_2^2-r_3^2=a\,.\,2x=a\,.\,2\,.\,OP\cos\theta_1,$$

and $$(r_2^2-r_3^2)\,r_1^2+(r_3^2-r_1^2)\,r_2+{}^2(r_1^2-r_2^2)\,r_3^2=0;$$

$$\therefore\quad a\cos\theta_1\,.\,r_1^2+b\cos\theta_2\,.\,r_2^2+c\cos\theta_3\,.\,r_3^2=0,$$

and $$a\cos\theta_1+\ldots=0.$$

So that an equation of the form

$$lr_1^2+mr_2^2+nr_3^2=0,$$

where $$l+m+n=0,$$

represents a circumdiameter.

Note that, for every point P or (r_1, r_2, r_3) on the line, the ratios $r_2^2-r_3^2 : r_3^2-r_1^2 : r_1^2-r_2^2$ are constant; being, in fact, equal to the ratios $a\cos\theta_1 : b\cos\theta_2 : c\cos\theta_3$.

The tripolar equations to OI, OGH should be noted.

(i) The projection of OI on $BC=\frac{1}{2}(b-c)$;

$$\therefore\quad \cos\theta_1 \propto (b-c);$$

and the equation to OI is

$$a(b-c)\,r_1^2+\ldots=0.$$

(ii) The projection of OH on BC

$$=(b^2-c^2)/2a;$$

and the equation to OH is

$$(b^2-c^2)\,r_1^2+\ldots=0.$$

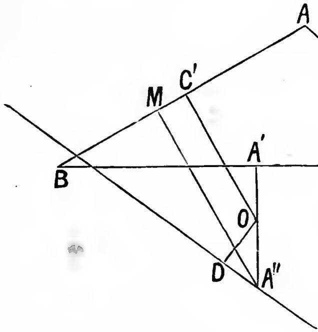
18. To find the equation to a straight line with direction angles θ_1, θ_2, θ_3, and at a distance d from O. (G)

Transferring to Cartesian coordinates, we see that the equation differs only by a constant from that of the parallel circumdiameter.

It must therefore be of the form

$$a\cos\theta_1 . r_1^2 + \dots = k.$$

Let A', B', C' be the mid-points of the sides, and let $A'O$ meet the line in A''.



Then if (ρ_1, ρ_2, ρ_3) be the coordinates of A'',

$$\begin{aligned} k &= a\cos\theta_1 . \rho_1^2 + b\cos\theta_2 . \rho_2^2 + c\cos\theta_3 . \rho_2^2 \quad [\rho_2 = \rho_3] \\ &= a\cos\theta_1 (\rho_1^2 - \rho_2^2) \\ &= a\cos\theta_1 (AM^2 - MB^2) \\ &= a\cos\theta_1 . 2c . C'M \\ &= a(OD/OA'')\, 2c . OA'' \sin B \\ &= d . 2ac \sin B \\ &= 4d\Delta. \end{aligned}$$

Hence the required equation is

$$a\cos\theta_1 . r_1^2 + \dots = 4d\Delta.$$

19. When $(l+m+n)$ is *not zero*.

To prove that if Q be the mean centre of masses l, m, n at A, B, C; or if (l, m, n) are the b.c. of Q, then, for any point P whose tripolar coordinates are (r_1, r_2, r_3),

$$lr_1^2 + mr_2^2 + nr_3^2 = l . AQ^2 + m . BQ^2 + n . CQ^2 + (l+m+n)\, PQ^2.$$

Take any rectangular axes at Q, and let $(a_1 a_2)$, $(b_1 b_2)$, $(c_1 c_2)$, (xy) be the Cartesian coordinates of A, B, C, P.

Then $\quad lr_1^2 = l(a_1 - x)^2 + l(a_2 - y)^2$

$$= l . AQ^2 + l . PQ^2 - 2x . la_1 - 2y . la_2.$$

But, since Q is the mean centre for masses l, m, n,

$$\therefore\ la_1 + mb_1 + nc_1 = 0;\ la_2 + mb_2 + nc_2 = 0;$$

$\therefore\quad lr_1^2+mr_2^2+nr_3^2 = l\,.\,AQ^2+m\,.\,BQ^2+n\,.\,CQ^2+(l+m+n)\,PQ^2.$

This, being true for *any* point P, is true for O;

$$\therefore\quad l\,.\,R^2+m\,.\,R^2+n\,.\,R^2 = \Sigma l\,.\,AQ^2+(l+m+n)\,OQ^2;$$

$$\therefore\quad lr_1^2+mr_2^2+nr_3^2 = (l+m+n)(QP^2-QO^2+R^2).$$

The power Π of the point Q for the circle ABC is R^2-OQ^2 or OQ^2-R^2, according as Q lies within or without the circle. If P describes a circle of radius $\rho\,(=PQ)$ round Q, an internal point, then the tripolar equation to this circle is

$$lr_1^2+mr_2^2+nr_3^2 = (l+m+n)\,(\rho^2+\Pi).$$

If the circle cuts the circle ABC orthogonally, then

$$OQ^2=R^2+\rho^2,$$

so that the circle becomes

$$lr_1^2+mr_2^2+nr_3^2 = 0. \qquad \text{(R. F. Davis).}$$

Examples.—

(A) The circumcircle:

Here $\qquad \rho = R,\quad QO = 0,\quad l \propto \sin 2A;$

$$\therefore\quad \sin 2A\,.\,r_1^2+\sin 2B\,.\,r_2^2+\sin 2C\,.\,r_3^2 = 4\Delta.$$

(B) The inscribed circle:

$$\rho = r,\quad QO^2 = IO^2 = R^2-2Rr,\quad l \propto a;$$

$$\therefore\quad ar_1^2+br_2^2+cr_3^2 = 2\Delta\,(r+2R).$$

(C) The Nine-Point circle:

$$\rho = \tfrac{1}{2}R;\quad QO^2 = \tfrac{1}{4}OH^2 = \tfrac{1}{4}R^2-2R\cos A\cos B\cos C.$$

Since the n.c. of the Nine-Point centre are $\cos(B-C)$...., the b.c. are $\sin A\cos(B-C)$,..., so that

$$l \propto \sin 2B+\sin 2C;$$

$$\therefore\quad \Sigma\,(\sin 2B+\sin 2C)\,r_1^2 = 4\Delta\,(1+2\cos A\cos B\cos C).$$

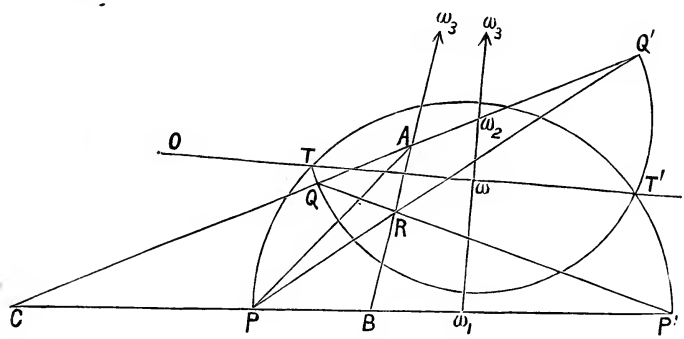
20. To determine the point or points whose tripolar coordinates are in the given ratios $p:q:r$.

Divide BC, CA, AB internally at P, Q, R and externally at P', Q', R', so that

$$BP : CP = q : r = BP' : CP',$$
$$CQ : AQ = r : p = CQ' : AQ',$$
$$AR : BR = p : q = AR' : BR'.$$

Let ω_1, ω_2, ω_3 be the centres of the circles described on PP', QQ', RR' as diameters, and let the circles (PP), (QQ) intersect at T, T'.

Then since $CPBP'$ is harmonic, we have, for every point T or T' on the circle PP'



$$BT : CT = BP : CP = q : r = BP' : CP' = BT' : CT',$$

and TP, TP' bisect the angles at T, while $T'P$, $T'P'$ bisect those at T'.

So for every point T or T' on the circle (QQ')

$$CT : AT = CQ : AQ = r : p, \text{ \&c.}$$

Hence at T, T' the points of intersection of the circles (PP'), (QQ') $\quad AT : BT : CT = p : q : r = AT' : BT' : CT'$.

The symmetry of the result shows that T and T' lie also on the circle (RR').

Hence there are *two* points whose tripolar coordinates are as $p : q : r$, and these points are common to the three circles (PP'), (QQ'), (RR').

Since $(CPBP')$ is harmonic,

$$\therefore \quad \omega_1 P^2 = \omega_1 B . \omega_1 C.$$

Hence the circle (PP'), and similarly the circles (QQ'), (RR'), cut the circle ABC orthogonally, so that the tangents from O to these circles are each equal to R.

It follows that—

(*a*) O lies on TT', the common chord or Radical Axis of the three circles (PP'), (QQ'), (RR').

(*b*) $OT . OT' = R^2$, so that T, T' are *inverse* points in circle ABC.

(*c*) The circle ABC cuts orthogonally every circle through T, T' including the circle on TT' as diameter, so that $\omega T^2 = O\omega^2 - R^2$, where ω is the mid-point of TT'.

(*d*) The centres ω_1, ω_2, ω_3 lie on the line through ω, bisecting TT' at right angles.

21. The tripolar coordinates of Limiting Points. (G)

Since $OT.OT' = R^2$, the circle ABC belongs to the coaxal system which has T, T' for Limiting Points, and therefore $\omega_1\omega_2\omega_3$ for Radical Axis; so that, if π_1, π_2, π_3 are the perpendiculars from A, B, C on $\omega_1\omega_2\omega_3$, we have by coaxal theory

$$2.OT.\pi_1 = AT'^2 \quad \text{or} \quad \pi_1 \propto p^2.$$

Hence the equation to the Radical Axis $\omega_1\omega_2\omega_3$ is

$$p^2x+q^2y+r^2z = 0.$$

And conversely, if the Radical Axis be

$$\lambda x+\mu y+\nu z = 0,$$

then the tripolar coordinates of T or T' are $\sqrt{\lambda}$, $\sqrt{\mu}$, $\sqrt{\nu}$.

Examples.—

(1) For the coaxal system to which the circle ABC and the in-circle XYZ belong, the Radical Axis is

$$(s-a)^2 x+\ldots = 0.$$

Hence the tripolar coordinates of the limiting points lying on OI are as $(s-a)$, $(s-b)$, $(s-c)$.

(2) For the circles ABC, $I_1I_2I_3$ the Radical Axis is

$$\alpha+\beta+\gamma = 0 \quad \text{or} \quad x/a+\ldots = 0;$$

$$\therefore \quad p : q : r = 1/\sqrt{a} : 1/\sqrt{b} : 1/\sqrt{c},$$

the limiting points lying on OI.

(3) The circle ABC and the Antimedial circle $A_1B_1C_1$ (1) have Radical Axis $\quad a^2x+b^2y+c^2z = 0$;

$$\therefore \quad p : q : r = a : b : c,$$

the limiting points lying on OGH.

(4) The circles ABC, $A'B'C'$, Polar Circle, &c., have for their common Radical Axis

$$\cot A.x+\cot B.y+\cot C.z = 0;$$

$$\therefore \quad p : q : r = \sqrt{\cot A} : \sqrt{\cot B} : \sqrt{\cot C},$$

the limiting points lying on OGH.

CHAPTER III.

PORISTIC TRIANGLES.

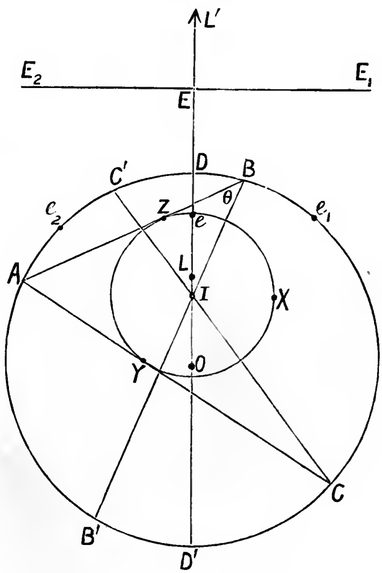
22. Let I be any point on the fixed diameter DD' of the circle $O(R)$. With centre D and radius DI cut the circle at e_1 and e_2; let e_1e_2 cut DD' at e. Let $OI = d$, and $Ie = r$.

Then $De + eI + IO = R$, and $De = De_1^2/2R = DI^2/2R$;

$$\therefore \quad (R-d)^2/2R + r + d = R;$$

$$\therefore \quad r = (R^2-d^2)/2R, \text{ or } OI^2 \equiv d^2 = R^2 - 2Rr.$$

(Greenhill)



An infinite number of triangles can be inscribed in the circle $O(R)$ and described about the circle $I(r)$, provided

$$OI^2 = R^2 - 2Rr.$$

On $O(R)$ take any point A, and draw tangents AB, AC to $I(r)$. Let BI, CI meet $O(R)$ in B', C' respectively.

Then, since $BI.IB' = R^2 - OI^2 = 2Rr$, and $BI = r/\sin\theta$;

$$\therefore \quad B'I = 2R\sin\theta = B'A.$$

So $$C'I = C'A.$$

Hence $B'C'$ bisects AI at right angles, so that

$$\angle B'C'A = B'C'I \text{ or } B'C'C;$$

$$\therefore \quad \angle B'BA = B'BC;$$

$$\therefore \quad BC \text{ touches } I(r).$$

It follows that by taking a series of points I along DD' and calculating r from $r = (R^2 - d^2)/2R$, we have an infinite number of circles $I(r)$; each of which, combined with $O(R)$, gives a poristic system of triangles.

23. The Radical Axis of $O(R)$ and $I(r)$.

Let L, L' be the Limiting Points of the two circles, and let E_1E_2, the Radical Axis, cut OI in E.

Bisect OI in k.

Then $$EO^2 - R^2 = EL^2 = EI^2 - r^2,$$

by ordinary coaxal theory;

$$\therefore \quad 2d.Ek = EO^2 - EI^2 = R^2 - r^2;$$

$$\therefore \quad EO = Ek + \tfrac{1}{2}d = (2R^2 - 2Rr - r^2)/2d,$$

and $$EI = Ek - \tfrac{1}{2}d = (2Rr - r^2)/2d.$$

Also $$EL^2 = EI^2 - r^2 = r^3(4R+r)/4d^2.$$

24. We now proceed to discuss some points which remain unchanged in a system of poristically variable triangles ABC.

(*a*) The inner and outer centres of similitude (S_1 and S_2) of the circles ABC, XYZ.

(*b*) The centre of similitude (σ) of the homothetic triangles XYZ and $I_1I_2I_3$.

(*c*) The orthocentre (H_i) of the triangle XYZ.

(*d*) The Weill Point (G_i), the centroid of XYZ.

25. (*a*) To determine the distances of S_1 and S_2 from the Radical Axis.

Since OI is divided at S_1, so that

$$OS_1 : IS_1 = R : r.$$

$$\therefore \quad ES_1(R+r) = EI\,.\,R + EO\,.\,r;$$

$$\therefore \quad ES_1 = \frac{(4R+r)(R-r)r}{2d(R+r)}.$$

Similarly $\therefore\quad ES_2 = \dfrac{r^2(R+r)}{2d(R-r)}$;

$$\therefore\quad ES_1 . ES_2 = EL^2.$$

So that the circle (S_1S_2) is coaxal with $O(R)$ and $I(r)$, a well known theorem.

26. To show that σ, the centre of similitude of the homothetic triangles XYZ, $I_1I_2I_3$, is poristically fixed, and to determine its distance from the Radical Axis.

Since the circumcentre of XYZ is I, while the circumcentre of $I_1I_2I_3$ is J, lying on OI, and such that $OJ = OI$;

$$\therefore\quad \sigma \text{ also lies on } OI;$$

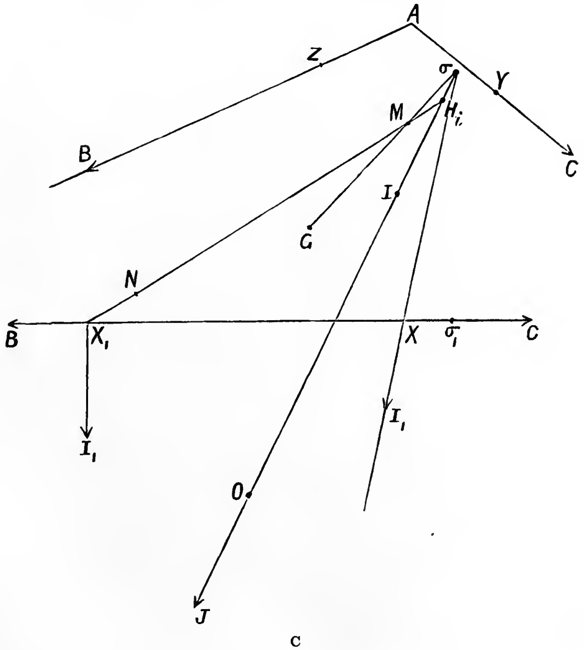
and $\sigma I/\sigma J =$ ratio of circumradii $= r/2R =$ constant;

$$\therefore\quad \sigma \text{ is a } \textit{fixed point.}$$

Again $\sigma I/IJ = r/(2R-r)$, from above;

$$\therefore\quad \sigma I = 2dr/(2R-r);$$

$$\therefore\quad E\sigma = EI - \sigma I = \frac{r^2(4R+r)}{2d(2R-r)}.$$



And $$EI = r(2R-r)/2d.$$
$$\therefore \quad E\sigma . EI = EL^2, \qquad \text{(Greenhill)}$$
so that the circle $I\sigma$ belongs to the coaxal system. Note that the homothetic triangles XYZ, $I_1I_2I_3$ slide on fixed circles, the joins XI_1, YI_2, ZI_3 passing through the fixed point σ.

It will be convenient here to determine the n.c. of the point σ. From figure *p*..., drawing $\sigma\sigma_1$ perpendicular to BC, and noting that X and I_1 are homologous points in XYZ, $I_1I_2I_3$, we have $$\sigma X/\sigma I_1 = \text{ratio of circumradii of the triangles}$$
$$= r/2R;$$
$$\therefore \quad \sigma\sigma_1/I_1X_1 = \sigma X/I_1X = r/(2R-r).$$
$$\sigma\sigma_1 \equiv \alpha = r/(2R-r).r_1;$$
$$\therefore \quad \alpha : \beta : \gamma = r_1 : r_2 : r_3 = 1/(s-a) : 1/(s-b) : 1/(s-c).$$

Note also that, since I and J are the circumcentres of XYZ, $I_1I_2I_3$,
$$\therefore \quad \sigma I/\sigma J = r/2R;$$
$$\therefore \quad \sigma I/IJ = r/(2R-r), \quad \text{and} \quad IJ = 2.OI = 2d;$$
$$\therefore \quad \sigma I = 2dr/(2R-r).$$

27. To prove that H_i, the orthocentre of XYZ, is poristically fixed, and to determine its distance from the Radical Axis.

Since H_i and I are the orthocentres of XYZ, $I_1I_2I_3$,
$$\therefore \quad H_i \text{ lies on } \sigma I, \text{ that is, on } OI;$$
$$\therefore \quad \sigma H_i/\sigma I = r/2R, \text{ a fixed ratio};$$
$$\therefore \quad H_i \text{ is a fixed point.}$$

Again, since $$\sigma I = 2dr/(2R-r), \qquad (26)$$
$$\therefore \quad \sigma H_i = r/2R.\sigma I = \frac{dr^2}{R(2R-r)};$$
also $$E\sigma = \frac{r^2(4R+r)}{2d(2R-r)}; \qquad (26)$$
$$\therefore \quad EH_i = E\sigma + \sigma H_i = \frac{r^2(4R+r)}{2d(2R-r)} + \frac{r^2d}{R(2R-r)},$$
$$= 3r^2/2d.$$

Note also that
$$H_iI = EI - EH_i = \frac{2Rr-r^2}{2d} - \frac{3r^2}{2d},$$
$$= rd/R.$$

To determine the n.c. of H_i.

The orthocentre H_i of XYZ is the centre of masses $\tan X$, $\tan Y$, $\tan Z$ placed at X, Y, Z.

$$\therefore \quad \alpha \propto \tan Y \,.\, XY \sin Z + \tan Z \,.\, ZX \sin Y,$$
$$\propto \cot \tfrac{1}{2}B \cos^2 \tfrac{1}{2}C + \cot \tfrac{1}{2}C \cos^2 \tfrac{1}{2}B,$$
$$\propto (b+c)/(s-a), \text{ \&c.}$$

28. *The Weill Point.*

When an infinite number of n-gons can be inscribed in one fixed circle, and described about another fixed circle, the mean centre of the points of contact X, Y, Z, ... with the inner circle is a *fixed* point, which may be called the Weill Point of the polygon (M'Clelland, p. 96). For a triangle the Weill Point is G_i, the centroid of XYZ.

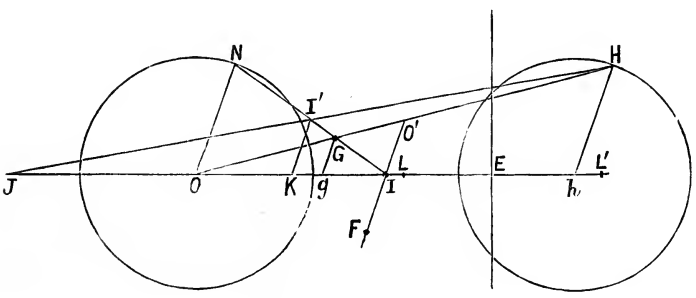
In the triangle XYZ, since I is the circumcentre and H_i is the orthocentre,

$$\therefore \quad G_i \text{ is a fixed point on } OI, \text{ and } G_iH_i = 2\,.\,G_iI.$$

To determine the n.c. of G_i.

Since G_i is the mean centre of masses 1, I, I at X, Y, Z,

$$\therefore \quad \alpha \propto XY \sin Z + ZX \sin Y \propto \cos^2 \tfrac{1}{2}B + \cos^2 \tfrac{1}{2}C.$$



29. We now proceed to discuss the loci of some well known points related to ABC, which are poristically variable:

(*a*) The Feuerbach Point F,

(*b*) The Centroid G,

(*c*) The Orthocentre H,

(*d*) and (*e*) O' the circumcentre, and I' the in-centre of the Medial Triangle $A'B'C'$.

Draw Hh, Gg, $I'k$ parallel to $O'I$:

(*a*) The point F moves along the in-circle,

(*b*) G describes a circle, for $OG = \frac{2}{3}.OO'$;

$$\therefore \quad Og = \tfrac{2}{3}.OI;$$

thus g is fixed, and

$$Gg = \tfrac{2}{3}.O'I = \tfrac{1}{3}(R-2r) = \text{constant},$$

(*c*) H describes a circle, for

$$OH = 2.OO'; \quad \therefore \quad Oh = 2.OI,$$

so that h is fixed, and

$$Hh = 2.O'I = R-2r = \text{constant},$$

(*d*) O' obviously describes a circle, centre I, radius $(\frac{1}{2}R-r)$,

(*e*) I' describes a circle.

For since I, I' are homologous points in the triangles ABC, $A'B'C'$, whose double point is G,

$\therefore$ IGI' is a straight line, and $GI = 2.GI'$,

$\therefore$ $Ik = \frac{3}{2}.Ig$, so that k is a fixed point, and

$$kI' = \tfrac{3}{2}.Gg = \tfrac{1}{2}(R-2r) = IO'.$$

30. (*f*) *The Nagel Point.*—This point also belongs to the series whose circular loci may be found by inspection.

Let XIx be the diameter of the in-circle which is perpendicular to BC, and let the ex-circles I_1, I_2, I_3 touch BC in X_1, CA in Y_2, AB in Z_3 respectively.

Then $$BX_1 = s-c, \quad CX_1 = s-b,$$

so that the equation to AX_1 is

$$y/(s-b) = z/(s-c),$$

and thus AX_1, BY_2, CZ_3 concur at a point N whose b.c.'s are as $(s-a)$, $(s-b)$, $(s-c)$.

This point is called the Nagel Point of ABC.

If the *absolute* n.c. of N are $\alpha\beta\gamma$, then

$$a\alpha/(s-a) = \ldots = 2\Delta/s;$$

$$\therefore \quad \alpha = h_1.(s-a)/s.$$

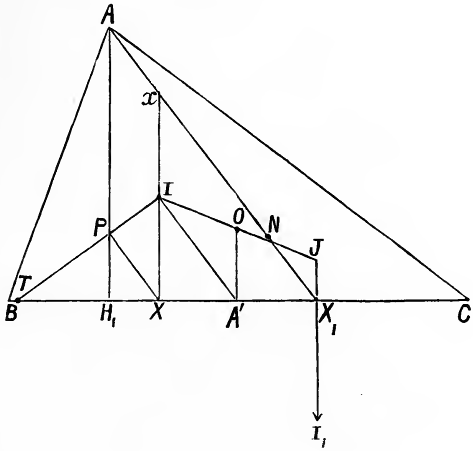
Draw NP, NQ, NR perpendicular to AH_1, BH_2, CH_3, then

$$AP = h_1-\alpha = h_1.a/s = 2r.$$

$\therefore$ the perpendiculars from N on B_1C_1, C_1A_1, A_1B_1, the sides of the anti-medial triangle $A_1B_1C_1$ (...) are each $= 2r$.

Hence N is the in-centre of the triangle $A_1B_1C_1$.

The two triangles ABC, $A_1B_1C_1$ have G for their common centroid and centre of similitude, I and N for their in-centres, O and H for circumcentres, O' and O for Nine-Point centres.



The corresponding joins are parallel, and in the ratio 1:2.

Thus $ON = 2.O'I = R-2r$,

and ON, $O'I$ are parallel.

$\therefore$ N describes a circle, centre O, radius $= R-2r$.

31. An additional note on this interesting point may here be interpolated.

Since AB, AC are common tangents to the circles I and I_1, and Ix, I_1X_1 are parallel radii of the circles, drawn in the same direction,

$\therefore$ AxX_1 or $AxNX_1$ is a straight line.

And since $AP = 2r = Xx$,

$\therefore$ PX is parallel to $AxNX_1$;

$\therefore$ PXX_1N is a parallelogram, and $PN = XX_1 = b-c$.

Again $XI = Ix$ and $XA' = A'X_1$;

$\therefore$ $A'I$ is parallel to AN and PX.

Let IP meet BC in T.

Then $$TH_1 : TX = TP : TI = TX : TA',$$
$$\therefore \quad TX^2 = TH_1 . TA'.$$

$\therefore$ T lies on the Radical Axis (common tangent) of the in-circle and Nine-Point circle.

32. *The Gergonne Point.*

This is another point whose poristic locus is a circle.

Since $BX = s-b$ and $CX = s-c$,

the barycentric equation to AX is $y(s-b) = z(s-c)$, so that AX, BY, CZ concur at a point whose b.c. are $1/(s-a)$, ...

This point, the Centre of Perspective for the triangles ABC and XYZ, is called the Gergonne Point, and will be denoted by M.

To determine the absolute b.c. of M,

$$\frac{x}{1/(s-a)} = \ldots = \frac{2\Delta}{1/(s-a)+\ldots} = \frac{2\Delta^2}{r_1+r_2+r_3} = \frac{2\Delta^2}{4R+r}.$$

The join of the Gergonne Point M and the Nagel Point N passes through H_i. (G.)

Proceeding as usual, the join proves to be

$$a(b-c)(s-a).x+\ldots = 0,$$

which is satisfied by the n.c. of H_i, which are $(b+c)/(s-a)$ ....

The join of M and G passes through σ. (G.)

For this join is $(b-c)(s-a).a\alpha+\ldots = 0$,

which is satisfied by the n.c. of σ, which are $1/(s-a)$ ....

33. In the poristics of a triangle the following formulæ are often required.

(1) $\Delta = rs$. (2) $abc = 4\Delta R = 4Rr.s$.

Put $s-a = s_1$, &c.: so that $a = s_2+s_3$, $s = s_1+s_2+s_3$.

(3) $s_1s_2s_3 = \Delta^2/s = r^2s$.

(4) $1/s_1+1/s_2+1/s_3 = (r_1+r_2+r_3)/\Delta = (4R+r)/rs$.

(5) $s_2s_3+\ldots = s_1s_2s_3(1/s_1+\ldots) = r(4R+r)$.

(6) $$\begin{aligned} a^2s_1+\ldots &= (s_2+s_3)^2 s_1+\ldots \\ &= (s_1+s_2+s_3)(s_2s_3+s_3s_1+s_1s_2)+3s_1s_2s_3\ldots \\ &= s.r(4R+r)+3r^2s = 4r(R+r).s. \end{aligned}$$

34. * The poristic locus of the Gergonne Point M is a circle coaxal with $O(R)$ and $I(r)$. (Greenhill)

Let π_1, π_2, π_3 be the perpendiculars from A, B, C on the Radical Axis E_1E_2.

The power of A for the circle $I(r) = (s-a)^2$.

But by coaxal theory this power is also equal to $2\pi_1 d$;

$$\therefore \quad \pi_1 = (s-a)^2/2d, \quad \&c.$$

But, if π be the perpendicular on the Radical Axis from any point whose b.c. are (x, y, z), then

$$(x+y+z)\,\pi = \pi_1 x + \pi_2 y + \pi_3 z.$$

Hence, for M, whose b.c. are as $1/s_1$, &c.,

$$(1/s_1+1/s_2+1/s_3)\,.\,\pi = s_1^2/2d\,.\,1/s_1 + \dots;$$

and, finally,

$$\pi = \frac{r}{4R+r}\,.\,\frac{s^2}{2d}.$$

Again, the power Π for the circle ABC of a point whose b.c. are (x, y, z) is given by

$$\Pi = \frac{a^2yz+b^2zx+c^2xy}{(x+y+z)^2}. \qquad (60)$$

Hence, for M, whose b.c. are $1/s_1$, ...,

$$\Pi = \frac{(a^2s_1+\dots)\,s_1s_2s_3}{(s_2s_3+\dots)^2} = \frac{4r\,(R+r)}{(4R+r)^2}\,.\,s^2;$$

$$\therefore \quad \Pi = 8\pi d\,.\,\frac{R+r}{4R+r}.$$

Or, the power of M for the circle ABC varies as the distance of M from the Radical Axis. Hence M describes a circle coaxal with $O(R)$ and $I(r)$.

If m is the centre of this circle, then

$$\Pi = 2\,.\,Om\,.\,\pi, \quad \text{by coaxal theory};$$

$$\therefore \quad Om = \frac{4(R+r)}{4R+r}\,.\,d.$$

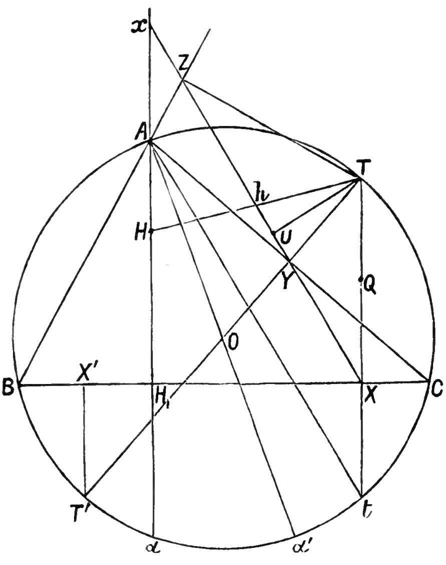
It may be shown that the radius of the M circle $= \dfrac{r(R-2r)}{4R+r}$, but the proof is long.

* The original proof belongs to Elliptic Functions. For the proof here given the present writer is responsible.

CHAPTER IV.

THE SIMSON LINE.

35. From any point T on the circumcircle ABC, draw perpendiculars TX, TY, TZ to the sides BC, CA, AB. Produce TX to meet the circle in t.



Then, since $BZTX$ is cyclic,

$$\angle BZX = BTX \text{ or } BTt = BAt;$$

$$\therefore \quad ZX \text{ is parallel to } At.$$

So $\qquad XY$ is parallel to At.

Thus XYZ is a straight line, parallel to At.

It is called the Simson Line of T, and T is called its Pole.

To draw a Simson Line in a given direction At.

Draw the chord tT perpendicular to BC, meeting BC in X. A line through X parallel to At is the Simson Line required, T being its pole.

Let AH_1, AO meet the circle ABC again in α, α'; it is required to determine the Simson Lines of A, α', α.

(*a*) For A, the point X coincides with H_1, while Y and Z coincide with A; therefore AH_1 is the Simson Line of A.

(*b*) Since $\alpha'BA$, $\alpha'CA$ are right angles, it follows that BC is the Simson Line of α'.

(*c*) Drawing αy, αz perpendicular to AC, AB. it is at once seen that yz, the Simson Line of α, passes through H_1, and that it is parallel to the tangent at A.

36. To prove that XYZ bisects TH (H orthocentre).

If Q be the orthocentre of TBC,

$$TQ = 2R\cos A = AH,$$

and

$$QX = Xt = Ax,$$

since At, XYZ are parallel;

$$\therefore \quad Hx = TX;$$

$$\therefore \quad HxTX \text{ is a parallelogram.}$$

$$\therefore \quad XYZ \text{ bisects } TH, \text{ say at } h.$$

It follows that h lies on the Nine-Point Circle.

TOT' being a circumdiameter, prove that, when the Simson Line of T passes through T', it also passes through G.

(W. F. Beard)

37. Let σ_1, σ_2, σ_3 be the direction angles of XYZ, taking $BXZ = \sigma_1$.

To prove that the base angles of the triangles OAT, OBT, OCT are equal to the *acute* angles which XYZ makes with the sides of ABC; *i.e.*, to σ_1, σ_2, σ_3, or their supplements. as the case may be.

$$\angle OAT \text{ (or } OTA) = \tfrac{1}{2}\pi - \tfrac{1}{2}.AOT$$
$$= \tfrac{1}{2}\pi - AtT = \tfrac{1}{2}\pi - YXT = BXZ = \sigma_1.$$

A relation of *fundamental importance.*

To determine QX or Xt,

$$Xt = Bt.Ct/2R,$$

and

$$Bt = 2R\sin BAt = 2R\sin BZX = 2R\sin\sigma_3;$$

$$\therefore \quad QX = Xt = 2R\sin\sigma_2\sin\sigma_3.$$

38. To determine the n.c. of T, the direction angles of XYZ being σ_1, σ_2, σ_3.

$$TA = 2R.\cos OTA = 2R\cos\sigma_1\,;$$

$$\therefore \quad a.2R = TB.TC = 4R^2\cos\sigma_2\cos\sigma_3\,;$$

$$\therefore \quad a = 2R\cos\sigma_2\cos\sigma_3\,;$$

so that the n.c. are as $\sec\sigma_1 : \sec\sigma_2 : \sec\sigma_3$.

To determine the segments YZ, ZX, XY.

In the circle $AYTZ$, with AT as diameter,

$$YZ = AT\sin A = 2R\cos\sigma_1.\sin A = a\cos\sigma_1.$$

39. To determine p, q, r, the lengths of the perpendiculars from A, B, C on XYZ, the Simson Line of T.

From (37) $\qquad Xt = 2R\sin\sigma_2\sin\sigma_3,$

and $\qquad \angle AtX = ZXT = \frac{1}{2}\pi - \sigma_1\,;$

$$\therefore \quad p = Xt.\sin AtX = 2R\cos\sigma_1\sin\sigma_2\sin\sigma_3 \propto \cot\sigma_1.$$

The equation to XYZ is therefore

$$\cot\sigma_1.x + \cot\sigma_2.y + \cot\sigma_3.z = 0.$$

To determine π, the length of the perpendicular TU on the Simson Line of T.

$$\pi = TU = TX\sin TXZ = 2R.\cos\sigma_2\cos\sigma_3\cos\sigma_1.$$

Or, since $TA = 2R\cos\sigma_1$,

$$p = TA.TB.TC/4R^2.$$

40. Let a parabola be drawn touching the sides of ABC and having T for focus. The Simson Line XYZ is evidently the vertex-tangent and U is the vertex.

Since XYZ bisects TH, the directrix is a line parallel to XYZ and passing through H.

Kiepert's Parabola.—Let T be the pole, found as in (35), of the Simson Line parallel to the Euler Line OGH. Let the direction angles of OGH be θ_1, θ_2, θ_3, then

$$\cos\theta_1 \propto (b^2-c^2)/a, \quad \text{\&c.};$$

so that the n.c. of T are $a/(b^2-c^2)\ldots.$

The directrix will be OGH,

or $\qquad (b^2-c^2)\cos A.a+\ldots.$

41. Denote the vectorial angles OTA, OTB, OTC, OTU by ϕ_1, ϕ_2, ϕ_3, δ respectively, so that ϕ_1, ϕ_2, ϕ_3 are equal to σ_1, σ_2, σ_3 or their supplements.

To prove that $\delta = \phi_1+\phi_2-\phi_3$.

Since $TtT' = \frac{1}{2}\pi$; $\therefore$ arc $Ct = BT'$;

$$\therefore \angle tTC = BTT' = \phi_2;$$

$$\therefore T'Tt = T'TC - tTC = \phi_3-\phi_2;$$

$$\therefore \tfrac{1}{2}\pi - \delta = TYZ = ZXT + T'TX = (\tfrac{1}{2}\pi-\phi_1)+\phi_3-\phi_2;$$

$$\therefore \delta = \phi_1+\phi_2-\phi_3.$$

Making the usual convention that ϕ_3 is to be negative when C falls on the side of TOT' opposite to A and B, we may write

$$\delta = \phi_1+\phi_2+\phi_3.$$

42. Consider the quadrilateral formed by the sides of ABC and a straight line PQR.

The circles ABC, AQR, BRP, CPQ have the common point M, and their centres O, O_1, O_2, O_3 lie on a circle called the Centre Circle, passing through M. (9)

Let θ_1, θ_2, θ_3 be the direction angles of PQR.

If ρ_1 be the circumradius of AQR, the perdendiculars from O_1 on AQ, AR are $\rho_1 \cos R$, $\rho_1 \cos Q$.

So that, if α, β, γ are the n.c. of O_1,

$$\beta : \gamma = \cos R : \cos Q = \sec\theta_2 : \sec\theta_3.$$

Hence AO_1, BO_2, CO_3 meet at a point N whose coordinates are $(\sec\theta_1, \sec\theta_2, \sec\theta_3)$.

And, since $$\frac{a}{\sec\theta_1}+\frac{b}{\sec\theta_2}+\frac{c}{\sec\theta_3} = a\cos\theta_1+\ldots = 0; \quad (2)$$

$$\therefore N \text{ lies on the circle } ABC,$$

and is the pole of the Simson Line parallel to PQR.

Again $$\angle O_1NO_2 = \pi - ANB$$

(BNO_2 being a straight line)

$$= \pi - C;$$

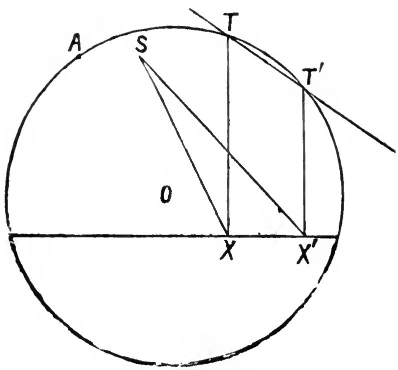
and since OO_2, OO_2 are perpendicular to MA, MB,

$$\therefore O_1OO_2 = AMB = C;$$

so that N lies on the Centre Circle, and is therefore the second point in which the Centre Circle intersects the circle ABC.

43. We will now deal with *pairs* of Simson Lines.

T and T' being any points on the circle ABC, it is required to determine S, the point of intersection of their Simson Lines.



Let $\sigma_1\sigma_2\sigma_3$, $\sigma_1'\sigma_2'\sigma_3'$, $\theta_1\theta_2\theta_3$ be the direction angles of the Simson Lines of T, T', and of the chord TT'.

Then, if α, β, γ are n.c. of S,

$$\alpha = SX \sin\sigma_1 = XX' \sin\sigma_1 \sin\sigma_1'/\sin(\sigma_1 - \sigma_1')$$
$$= TT' \cos\theta_1 . \sin\sigma_1 \sin\sigma_1'/\sin(\sigma_1 - \sigma_1').$$

But $\qquad OAT = \sigma_1, \quad OAT' = \sigma_1'$, (37)

so that $\qquad TAT' = \sigma_1 - \sigma_1'$;

$$\therefore \quad TT' = 2R\sin(\sigma_1 - \sigma_1'),$$

and $\qquad \alpha = 2R\cos\theta_1 \sin\sigma_1 \sin\sigma_1'$. (G.)

The equation to the chord which joins the points $(\alpha_1\beta_1\gamma_1)$ and $(\alpha_2\beta_2\gamma_2)$ on the circle ABC is

$$a\alpha/\alpha_1\alpha_2 + \ldots = 0.$$

Hence the equation to TT' joining the points T, T' whose n.c. are $(\sec\sigma_1, \ldots)(\sec\sigma_1', \ldots)$ is

$$\cos\sigma_1 \cos\sigma_1' . a\alpha + \ldots = 0.$$

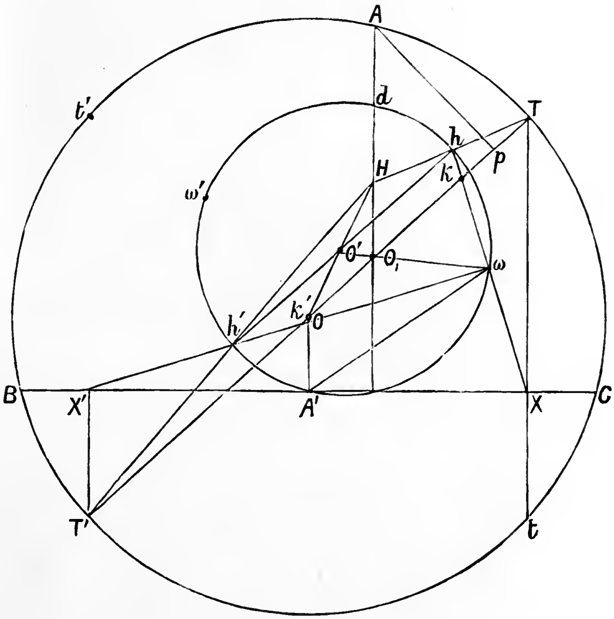
And the tangent at T is

$$\cos^2\sigma_1 . a\alpha + \ldots = 0.$$

44. To prove that the Simson Lines of T and T', the extremities of a circumdiameter, intersect at right angles on the Medial Circle.

Let these Simson Lines Xh, $X'h'$ intersect at ω.

Produce TX, $T'X'$ to meet the circle in t, t', so that $TtT't'$ is a rectangle, and tt' a diameter.



The Simson Lines of T, T' are parallel to At, At' (35), and are therefore at right angles.

Again, since h, h' are mid-points of HT, HT', (36)

$$\therefore \quad hh' = \tfrac{1}{2}TT' = R.$$

But h, h' lie on the Medial Circle; therefore they are the ends of a Medial diameter.

Also $\quad h\omega h' = \tfrac{1}{2}\pi$, as shown above.

Therefore ω lies on the Medial Circle.

Since $\quad A'X = A'X'$, and $X\omega X' = \tfrac{1}{2}\pi$,

$$\therefore \quad A'\omega = A'X \text{ or } A'X' = R\cos\theta_1,$$

where θ_1, θ_2, θ_3 are the direction angles of TOT'.

45. Let TOT' cut the Simson Lines ωh, $\omega h'$ in k, k'; and cut $O'\omega$ in O_1, where O' is the Nine-Point centre.

Since $$Hh = hT \quad \text{and} \quad Hh' = h'T',$$

$$\therefore \quad hh' \text{ is parallel to } TOT' \text{ or } kk':$$

And since hh' is parallel to kk', and $O'h = O'h'$,

$$\therefore \quad O_1k = O_1k'; \quad \text{also} \quad k\omega k' = \tfrac{1}{2}\pi:$$

Hence kk' is a diameter of the circle described with O_1 as centre and $O_1\omega$ as radius.

And since $O'O_1\omega$ is a straight line, this circle touches the Nine-Point circle at ω.

46. Let σ_1', σ_2', σ_3' be the direction angles of $X'h'\omega$, the Simson Line of T'.

Then, since the Simson Lines of T and T' are at right angles,

$$\therefore \quad \sigma_1' = \sigma_1 \pm \tfrac{1}{2}\pi,$$

so that the n.c. of T' are $2R\cos(\frac{1}{2}\pi - \sigma_2)\cos(\frac{1}{2}\pi - \sigma_3)$, &c., or $2R\sin\sigma_2\sin\sigma_3$, &c.; which are as $\operatorname{cosec}\sigma_1 : \operatorname{cosec}\sigma_2 : \operatorname{cosec}\sigma_3$.

Or, since $T't$ is parallel to BC,

$$\therefore \quad T'X' = tX = 2R\sin\sigma_2\sin\sigma_3.$$

Therefore the equation to the diameter TOT' is

$$\cos\sigma_1\sin\sigma_1 . x + \ldots = 0.$$

47. Consider the figures $ATOT'$, $\omega XA'X'$.

Since TAT' and $X\omega X'$ are right angles, while O, A' are the mid-points of TOT', $XA'X'$, and

$$\text{the angle } OTA \text{ or } OAT = A'X\omega \text{ or } A'\omega X; \qquad (37)$$

it follows that

the figures $ATOT'$, $\omega XA'X'$ are similar,

a fact to be very carefully noted.

48. Through ω draw $P\omega p'$ perpendicular to BC.

Then from similar triangles AOp, $\omega A'p'$,

$$Op/R = A'p'/A'\omega = A'p'/R\cos\theta_1;$$

$$\therefore \quad A'p' = Op\cos\theta_1 = \text{perpendicular from } p \text{ on } OA';$$

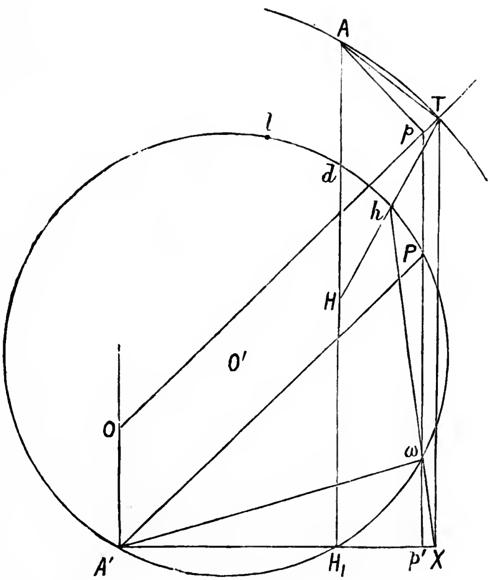
$$\therefore \quad P\omega p' \text{ passes through } p.$$

Hence, if Ap, Bq, Cr be perpendiculars to TOT', and pp', qq', rr' be perpendiculars to BC, CA, AB, then pp', qq', rr' are concurrent at that point ω on the Nine-Point circle, where the Simson Lines of T, T' intersect.

It follows that ω is the *Orthopole* of the diameter TOT' (see Chap. VI).

49. Since $$A'\omega = R\cos\theta_1, \tag{44}$$
and $$A'\omega = R\sin A'P\omega, \text{ in the Nine-Point Circle,}$$
$$\therefore\ A'P\omega = \tfrac{1}{2}\pi - \theta_1 = \text{angle } Opp';$$
$$\therefore\ A'P \text{ is parallel to } TOT'.$$

Hence, to find ω when TOT' is given, draw the chord $A'P$ in the Nine-Point circle, and then draw the chord $P\omega$ perpendicular to BC.



50. The $A'B'C'$ n.c. of ω.

From (35) the Simson Line of ω in the Nine-Point circle is parallel to $A'P$, and therefore to TOT'.

Hence the direction angles of this Simson Line are θ_1, θ_2, θ_3.

It follows from (38) that the $A'B'C'$ n.c. of ω are
$$a' = 2.\tfrac{1}{2}R\cos\theta_2\cos\theta_3, \quad \&\text{c.},$$
which are as $$\sec\theta_1,\quad \sec\theta_2,\quad \sec\theta_3.$$

The ABC n.c. of ω.

From the similar triangles $\omega A'p'$, AOp,
$$\alpha = \omega p' = A'\omega.Ap/AO = p\cos\theta_1.$$

Hence the *absolute* n.c. of ω are $p\cos\theta_1$, $q\cos\theta_2$, $r\cos\theta_3$.

To determine α in terms of p, q, r.

Let p', q', r' be the perpendiculars on TOT' from A', B', C'.
Then since A' is mid-point of BC,

$$\therefore \quad 2p' = q+r\,;$$

also $$p' = A'O.\cos\theta_1 = R\cos A\cos\theta_1\,;$$

$$\therefore \quad a = \frac{p(q+r)}{2R\cos A}\,; \quad \therefore \quad a\alpha = p(q+r)\tan A.$$

The formulæ of (14) and (50) supply us with a very simple proof of (8).

For $$a\alpha = b\beta'+c\gamma'. \qquad (14)$$

$$\therefore \quad a.p\cos\theta_1 = b.R\cos\theta_3\cos\theta_1 + c.R\cos\theta_1\cos\theta_2\,;$$

$$\therefore \quad ap/R = b\cos\theta_3 + c\cos\theta_2.$$

51. To illustrate the use of these formulæ, take the case when TOT' passes through I, the in-centre.

Here $\cos\theta_1 = \frac{1}{2}(b-c)/d$, $p = R/d.(b-c)(s-a)/a$ $[d \equiv OI]$.

So that $$4\alpha' = R(c-a)(a-b)/d^2 \propto 1/(b-c),$$

and $$a\alpha = \tfrac{1}{2}R/d^2.(b-c)^2(s-a) \propto (b-c)^2(s-a).$$

Hence ω is the Feuerbach Point.

52. To determine the $A'B'C'$ n.c. of O_1, the centre of the circle in (45).

Taking $A'B'C'$ as triangle of reference, the equation to TOT' is

$$\cos A\cos\theta_1.a\alpha' + \ldots = 0. \qquad (16)$$

The equation to the diameter of the Nine-Point circle passing through ω, whose n.c. are $(\sec\theta_1, \ldots)$ is

$$\sin\theta_1\cos\theta_1.a\alpha' + \ldots = 0. \qquad (46)$$

Hence at O_1, where these lines intersect,

$$a\alpha'.\cos\theta_1 \propto \cos B\sin\theta_3 - \cos C\sin\theta_2$$
$$\propto \cos B\,(p-q)/\sin C - \cos C\,(r-p)/\sin B\,;$$

$$\therefore \quad \alpha'\cos\theta_1 \propto p(\sin 2B+\sin 2C) - q.\sin 2B - r\sin 2C.$$

But since TOT' is a diameter of ABC,

$$p\sin 2A + q\sin 2B + r\sin 2C = 0.$$

Hence $$\alpha' \propto p\sec\theta_1, \quad \&c.$$

To determine the radius $O_1\omega$ $(\equiv\rho)$ of this circle.

The perpendicular π from ω on the $A'B'C'$ Simson Line of ω

$$= 2.\tfrac{1}{2}R.\cos\theta_1\cos\theta_2\cos\theta_3. \qquad (39)$$

The angle δ' which this makes with the diameter $\omega O_1 O'$

$$= \theta_1+\theta_2+\theta_3. \qquad (41).$$

The perpendicular from ω on TOT'

$$= 2 \text{ perpendicular on Simson Line} = 2\pi\,;$$

$$\therefore\ \rho = O_1\omega = 2\pi \sec \delta' = 2 . R \cos \theta_1 \cos \theta_2 \cos \theta_3 . \sec (\theta_1+\theta_2+\theta_3).$$

The parabola, which touches the sides of $A'B'C'$ and has ω for focus, has the Simson Line of ω for its vertex tangent, and for its directrix the diameter TOT', which is parallel to the Simson Line and passes through the orthocentre O of $A'B'C'$.

53. The envelope of the Simson Line is a Tricusp Hypocycloid.

Since $\angle\, \omega A'H_1 = AOT,$

$$\therefore\quad \omega O'H_1 = 2 . AOT.$$

But Medial Radius $= \frac{1}{2}R\,;$

$$\therefore\quad \text{arc } H_1\omega = \text{arc } AT = 2 \text{ arc } dh,$$

by similar figures Hdh, HAT.

Now take arc $A'L = \frac{1}{3}$ arc $A'H_1$,

and draw Medial diameter $LO'l$.

Then, since $A'd$ is also a Medial diameter,

$$\text{arc } A'L = \text{arc } ld\,;$$

$$\therefore\quad \text{arc } H_1L = 2 \text{ arc } A'L = 2 \text{ arc } ld.$$

Also $\quad$ arc $H_1\omega = 2$ arc hd;

$$\therefore\quad \text{arc } L\omega = 2 \text{ arc } lh.$$

Therefore $X\omega h$ touches a Tricusp Hypocycloid, having O' for centre, the Medial Circle for inscribed circle, and $LO'l$ for one axis.

54. Let DOD' be the circumdiameter through I, the in-centre.

Let ϕ_1, ϕ_2, ϕ_3 denote the vectorial angles $D'DA$, $D'DB$, $D'DC$.

Draw the chord CIP.

Then P is the mid-point of the arc AB; and

$$\therefore\quad \angle\, PDD' = \tfrac{1}{2}(\phi_1+\phi_2).$$

Also $\quad IPD'$ or $CPD' = CDD' = \phi_3$;

$$\therefore\quad IPD = \tfrac{1}{2}\pi - \phi_3.$$

Let $r/R = m$; then

$$\frac{1+m}{1-m} = \frac{ID'}{ID} = \frac{ID'}{IP} \cdot \frac{IP}{ID} = \frac{\sin IPD'}{\sin DD'P} \cdot \frac{\sin PDD'}{\sin IPD}$$

$$= \frac{\sin \phi_3}{\cos \frac{1}{2}(\phi_1+\phi_2)} \cdot \frac{\sin \frac{1}{2}(\phi_1+\phi_2)}{\cos \phi_3}\,;$$

$$\therefore \quad \tan \tfrac{1}{2}(\phi_1+\phi_2) = \frac{1+m}{1-m} \cdot \cot \phi_3.$$

Put $\phi_1+\phi_2-\phi_3 \equiv \delta$; $\sin\delta \equiv S$, $\sin\phi_3 \equiv s$.

Then $S = \sin(\phi_1+\phi_2)\cos\phi_3 - \cos(\phi_1+\phi_2)\sin\phi_3$

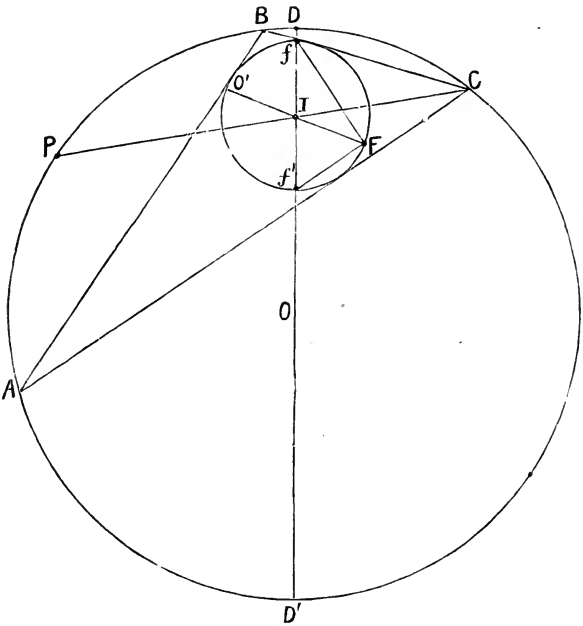
$$= \frac{2\tan\frac{1}{2}(\phi_1+\phi_2)}{1+\tan^2\frac{1}{2}(\phi_1+\phi_2)} \cdot \cos\phi_3 - \frac{1-\tan^2\frac{1}{2}(\phi_1+\phi_2)}{1+\tan^2\frac{1}{2}(\phi_1+\phi_2)} \cdot \sin\phi_3.$$

$$\therefore \quad S/s \,.\, \{(1-m)^2\sin^2\phi_3 + (1+m)^2\cos^2\phi_3\}$$
$$= 2(1-m^2)\cos^2\phi_3 - (1-m)^2\sin^2\phi_3 + (1+m)^2\cos^2\phi_3.$$

And finally,

$$S/s \,.\, \{1+m)^2 - 4ms^2\} = (3-m)(1+m) - 4s^2,$$

or $$s^3 - mS \,.\, s^2 - \tfrac{1}{4}(1+m)(3-m) \,.\, s + \tfrac{1}{4}(1+m)^2 \,.\, S = 0.$$



From (42) the angle δ is the vectorial angle of the perpendicular from D on the Simson Line of D, and this Simson Line passes through the Feuerbach Point F, and through f, one of the fixed points, where the circle $I(r)$ cuts the axis $DIOD'$ (45). The cubic then gives the values of ϕ_1, ϕ_2, ϕ_3, the vectorial angles of DA, DB, DC; and thus the triangle ABC is determined for any given position of F.

Again, if we put $\cos\delta \equiv C$, $\cos\phi_3 \equiv c$, we shall obtain

$$c^3 - mC.c^2 - \tfrac{1}{4}(1-m)(3+m).c - \tfrac{1}{4}(1-m^2).C = 0.$$

(Greenhill).

55. In a poristic system of triangles ABC, the Simson line of a fixed point S on $O(R)$, passes through a fixed point.

Let SS_1, SS_2 be tangents to $I(r)$; then S_1S_2 is also a tangent.

Draw DE touching $I(r)$ and parallel to S_1S_2.

Draw ST perpendicular to DE.

Then shall the Simson Line of S pass through the fixed point T.

Let DE be cut by AB, AC in γ, β.

Let AC, AB cut S_1S_2 in F, G; let AS_2, AS_1 cut DE in H, K.

Now $I\beta$, IE, $I\gamma$, ID are perpendicular respectively to IF, IS_2, IG, IS_1; and hence

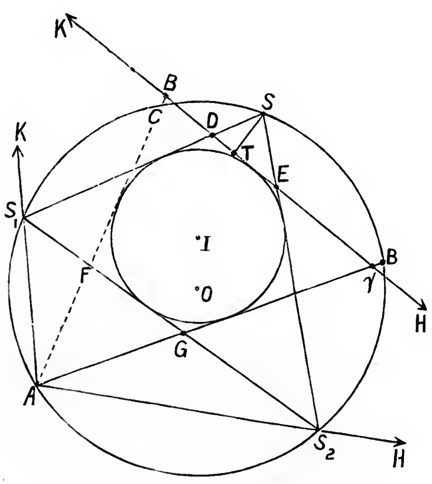
$$(\beta E\gamma D) = (FS_2GS_1) = (\beta H\gamma K).$$

Thus $$\frac{\beta E}{\beta D}\cdot\frac{\gamma D}{\gamma E} = \frac{\beta H}{\beta K}\cdot\frac{\gamma K}{\gamma H},$$

or $$\frac{\beta E.\beta K}{\beta D.\beta H} = \frac{\gamma E.\gamma K}{\gamma D.\gamma H}.$$

Again, $\angle SDH = SS_1S_2 = SAS_2$ or SAH.

$\therefore$ $SDAH$ is cyclic, and similarly $SEAK$.



The relation (i) shows that the powers of β and γ for these circles are in the same ratio; and hence that β, γ lie on a circle coaxal with these circles, and therefore passing through A and S.

That is, $AS\beta\gamma$ is cyclic.

Now the four circles circumscribing the four triangles formed by AB, AC, BC, $\beta\gamma$ have a common point, which must be S, since this point lies on the circles ABC, $A\beta\gamma$.

Hence $\sigma_1\sigma_2\sigma_3$ and T, the feet of the perpendiculars from S on the lines BC, CA, AB, $\beta\gamma$, are collinear.

In other words, the Simson Line $\sigma_1\sigma_2\sigma_3$ of S for the triangle ABC passes through T, a fixed point, being the foot of the perpendicular from S on the fixed line $\beta\gamma$.

(Greenhill and Dixon)

The properties given in (34), (54), (55) present themselves in the Cubic Transformation of the Elliptic Functions.

CHAPTER V.

PEDAL TRIANGLES.

56. FROM a point S within the triangle ABC draw Sd, Se, Sf perpendiculars to BC, CA, AB respectively, so that def is the Pedal Triangle of S with respect to ABC.

Let angle $d = \lambda$, $e = \mu$, $f = \nu$.

Produce AS, BS, CS to meet the circle ABC again in L, M, N.

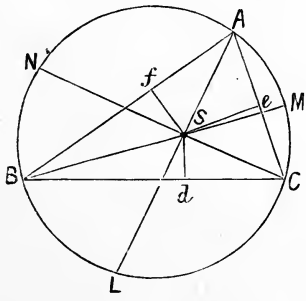
From the cyclic quadrilateral $SdCe$,

$$\angle Sde = SCe \quad \text{or} \quad NCA = NLA,$$

so

$$Sdf = MLA\ ;$$

$$\therefore \quad d \text{ or } \lambda = MLN, \quad \&c.$$



Thus LMN is similar to the pedal triangle of S with regard to ABC.

Similarly ABC is similar to the pedal triangle of S with regard to LMN.

$$\therefore \quad BSC = BMC + SCM$$

$$= BMC + NCM \quad \text{or} \quad NLM = A + \lambda.$$

To determine a point S whose pedal triangle has given angles λ, μ, ν, describe inner arcs on BC, CA, AB respectively, containing angles $A+\lambda$, $B+\mu$, $C+\nu$, any two of these arcs intersect at the point S required.

This construction also gives S as the pole of inversion when ABC is inverted into a triangle LMN with given angles λ, μ, ν.

57. If r_1, r_2, r_3 represent SA, SB, SC, the tripolar coordinates of S, and if p be the circumradius of def,

then $$2p\sin\lambda = ef = SA.\sin eAf$$
(in circle $SeAf$, diameter AS)
$$= r_1 \sin A;$$
$$\therefore\quad MN : NL : LM = ef : fd : de = \sin\lambda : \sin\mu : \sin\nu$$
$$= ar_1 : br_1 : cr_1.$$

Limiting Points. (G.)

In section (21) let TA, TB, TC meet the circle ABC again in LMN.

Then from (57) $\qquad MN \propto \sin\lambda \propto ar_1$.

In (a), $TA \propto s-a$; $\quad\therefore\quad MN \propto a(s-a)$.

In (b), $TA \propto 1/\sqrt{a}$; $\quad\therefore\quad MN \propto a/\sqrt{a} \propto \sqrt{a}$.

In (c), $TA \propto a$; $\quad\therefore\quad MN \propto a^2$.

In (d), $TA \propto \sqrt{\cot A}$; $\quad\therefore\quad MN \propto \sin A\sqrt{\cot A} \propto \sqrt{\sin 2A}$.

Let I_1, I_3 be the in-centres of LMN in (a) and (c); H_2, H_4 the orthocentres in (b) and (d).

It is very easy to prove that
$$OI_1 = OI,\quad OI_3 = OH,\quad OH_2 = OI,\quad OH_4 = OH.$$

58. In (21), since
$$p : q : r \propto \sin\lambda/a : \sin\mu/b : \sin\nu/c,$$
the Radical Axis becomes
$$\sin^2\lambda/a^2.x + \sin^2\mu/b^2.y + \sin^2\nu/c^2.z = 0.$$

To determine J, the pole of this Radical Axis for the circle ABC.

The polar of $\alpha'\beta'\gamma'$ is $(b\gamma'+c\beta')\alpha + \ldots = 0$.
$$\therefore\quad \sin^2\lambda/a \propto b\gamma'+c\beta';$$
$$\therefore\quad \sin^2\lambda \propto ac\beta'+ab\gamma';$$
$$\therefore\quad bc\alpha' \propto -\sin^2\lambda+\sin^2\mu+\sin^2\nu \propto -ef^2+fd^2+de^2$$
$$\propto \cos\lambda.fd.de \propto \cos\lambda.\sin\mu\sin\nu$$
$$\propto \cot\lambda;$$
$$\therefore\quad \alpha' \propto a\cot\lambda;$$
so that the n.c. of J are $a\cot\lambda$, $b\cot\mu$, $c\cot\nu$. (Dr. J. Schick)

59. To show that U, the area of def, is proportional to **Π**, the power of S for the circle ABC.

$$2U = de.df.\sin\lambda = r_2 r_3.\sin B \sin C \sin\lambda,$$

and $$2\Delta = 4R^2 \sin A \sin B \sin C.$$

$$\therefore \quad U/\Delta = \tfrac{1}{4}.\frac{r_2 r_3}{R^2}.\frac{\sin\lambda}{\sin A}.$$

Now $$r_2.SM = \Pi;$$

and in the triangle SMC,

$$r_3 \text{ or } SC = SM.\sin A/\sin\lambda.$$

Hence $$U/\Delta = \tfrac{1}{4}\Pi/R^2 = \tfrac{1}{4}(R^2 - OS^2)/R^2.$$

When U is constant, OS is constant, and S describes a circle, centre O.

Let α, β, γ be the n.c. of S.

Then $$\beta\gamma \sin A + \ldots = 2(\Delta eSf + \ldots)$$
$$= 2U = \tfrac{1}{2}.(R^2 - OS^2).\Delta/R^2.$$

Putting $$4\Delta^2 = (a\alpha + b\beta + c\gamma)^2,$$

we have $\alpha\beta\gamma + \ldots = (R^2 - OS^2)/abc.(a\alpha + b\beta + c\gamma)^2$

as the locus (a concentric circle) of S, when OS is constant.

When $OS = R$, U vanishes, as the pedal triangle becomes a Simson Line, and $\quad a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0.$

60. To determine the power **Π** of a point S in terms of x, y, z, the b.c. of S.

Since $$a\alpha/x = b\beta/y = c\gamma/z = 2\Delta/(x+y+z),$$

$$\therefore \quad \alpha = 2\Delta/(x+y+z).x/a, \quad \&c.;$$

$$\therefore \quad 2U = \beta\gamma \sin A + \ldots = \frac{4\Delta^2}{(x+y+z)^2}.\left\{\frac{y}{b}.\frac{z}{c}.\frac{a}{2R} + \ldots\right\};$$

$$\therefore \quad \Pi \text{ or } (R^2 - OS^2) = 4R^2/\Delta.U = \frac{a^2yz + b^2zx + c^2xy}{(x+y+z)^2}.$$

Note that only the *ratios* $x : y : z$ are required.

Examples.—

(i) For I, $x \propto a$;

$$\therefore \quad \Pi = \frac{abc}{a+b+c} = 2Rr.$$

(ii) For H, $x \propto \tan A$;

$$\therefore \quad \Pi = \frac{a^2 \tan B \tan C + \ldots}{(\tan A + \tan B + \tan C)^2}$$
$$= 8R^2 . \cos A \cos B \cos C.$$

(iii) For G, $x = y = z$;

$$\therefore \quad \Pi = \tfrac{1}{9}(a^2+b^2+c^2).$$

61. When S lies on a known circle for which the powers of A, B, C are simple expressions, the power Π of S for the circle ABC usually takes a simpler form than that given by the above *general* formula.

Let d be the distance between the centres of the two circles Q and ABC; $t_1^2 t_2^2 t_3^2$ the known powers of A, B, C for the circle Q; $\pi\pi_1\pi_2\pi_3$, the perpendiculars from S, A, B, C on the Radical Axis of the two circles.

From (7), we have $\pi = \dfrac{\pi_1 x + \pi_2 y + \pi_3 z}{x+y+z}$.

But, from coaxal theory,

$$t_1^2 = 2d . \pi_1, \text{ \&c.,} \quad \text{while} \quad \Pi = 2d . \pi;$$
$$\therefore \quad \Pi = \frac{t_1^2 x + t_2^2 y + t_3^2 z}{x+y+z}.$$

Equating this to the expression for Π, found in (60) we have $a^2 yz + b^2 zx + c^2 xy = (t_1^2 x + t_2^2 y + t_3^2 z)(x+y+z)$ as the locus of S: *i.e.* the circle Q.

As an example, let us find Π for the point ω (50).

This point lies on the Nine-Point Circle, for which the power of $A = t_1^2 = \frac{1}{2} bc \cos A$, &c.

Also for ω, $\quad x = a\alpha = p(q+r) \tan A$, (50)

and $\quad x+y+z = 2\Delta$.

$$\therefore \quad \Pi = \left\{\tfrac{1}{2} bc \cos A . p(q+r) \tan A + \ldots\right\}/2\Delta$$
$$= qr + rp + pq.$$

62. The Radical Axis of the circles Q and ABC is

$$t_1^2 x + t_2^2 y + t_3^2 z = 0,$$

for $\quad \pi_1 \propto t_1^2$.

Examples.—

(i) When Q is In-circle: $t_1^2 = (s-a)^2$.

(ii) When Q is Circle $I_1 I_2 I_3$: $t_1^2 = bc$.

(iv) When Q is Anti-medial Circle (1): $t_1^2 = a^2$.

(v) When Q is Nine-Point Circle: $t_1^2 = \frac{1}{2}bc\cos A \propto \cot A$.

(vi) When Q is circle $T_1T_2T_3$, circumscribed to the triangle formed by the tangents at A, B, C:

$$t_1^2 = R^2 \tan B \tan C \propto \cot A.$$

(vii) When Q is the circle (GH): $t_1^2 = 2R\cos A.\frac{2}{3}h, \propto \cot A$.

(viii) When Q is Polar Circle:

$$t_1^2 = AH^2 + 4R^2 \cos A \cos B \cos C \propto \cot A.$$

So that circles (v) (vi) (vii) (viii) have the same Radical Axis

$$\cot A.x + \cot B.y + \cot C.z = 0.$$

63. *Feuerbach's Theorem.*

To determine the Radical Axis of the Nine-Point circle and In-circle:

Take $A'B'C'$ as triangle of reference,

The power of A' for the In-circle $= t_1'^2 = A'X^2 = \frac{1}{4}(b-c)^2$;

$\therefore$ the Radical Axis is $(b-c)^2x' + \ldots = 0$.

But this is the tangent to the circle $A'B'C'$ at the point whose n.c. are $1/(b-c)\ldots$.

Hence the two circles touch, and $1/(b-c)\ldots$ are the n.c. of the point of contact.

64. To express Π in terms of λ, μ, ν.

We have $\angle BSC = A + \lambda$; (Fig., p. 37)

$$\begin{aligned}\therefore\quad a\alpha = 2.\Delta BSC &= BS.SC.\sin(A+\lambda)\\ &= BS.SM.\sin(A+\lambda)\sin A/\sin\lambda\\ &= \Pi\left\{\sin^2 A \cot\lambda + \tfrac{1}{2}\sin 2A\right\};\end{aligned}$$

$$\therefore\quad 2\Delta/\Pi \text{ or } (a\alpha + b\beta + c\gamma)/\Pi = \Sigma.\sin^2 A\cot\lambda + \tfrac{1}{2}\Sigma.\sin 2A.$$

Multiply each side by $4R^2$. Then

$$\begin{aligned}8R^2\Delta/\Pi \text{ or } 2R.abc/\Pi &= a^2\cot\lambda + b^2\cot\mu + c^2\cot\nu + 4\Delta\\ &\equiv M.^*\end{aligned}$$

$$\therefore\quad \Pi M = 2R.abc \text{ or } 8R^2\Delta,\quad \Pi = 2R.abc/M;$$

giving Π in terms of λ, μ, ν.

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* The expression "$a^2\cot\lambda + \ldots$" was first used, I believe, by Dr. J. Schick, Professor in the University of Munich. The relations he deals with are different from those treated in this work.

Since $$R^2 - OS^2 = \Pi = 8R^2\Delta/M;$$

$$\therefore \quad OS^2/R^2 = \frac{a^2 \cot\lambda + \ldots - 4\Delta}{a^2 \cot\lambda + \ldots + 4\Delta}.$$

65. Observe also that the area of *def*

$$= U = \tfrac{1}{4}.\Delta/R^2.\Pi = 2\Delta^2/M;$$

so that, if p be the radius of the pedal circle *def*,

$$2p^2.\sin\lambda \sin\mu \sin\nu = U = 2\Delta^2/M;$$

and now all the elements of *def* are found in terms of λ, μ, ν.

From above, $a\alpha = \Pi.\sin(A+\lambda)\sin A/\sin\lambda$.

Hence $$\alpha = \frac{abc}{M}.\frac{\sin(A+\lambda)}{\sin\lambda}.$$

66. To illustrate one of these results, take S, the focus of Artzt's Parabola, which touches AB at B and AC at C (see Figure, p. 88). (G.)

It is known that $\angle\ SAC = SBA,\ SAB = SCA;$

$$\therefore \quad BSK_1 = BAS + ABS = BAS + CAS = A.$$

So $CSK_1 = A$.

$$\therefore \quad BSA = \pi - A = B + C = CSA,$$

and $$BSC = 2A.$$

So that, if λ, μ, ν are the pedal angles of S,

$$A+\lambda = 2A; \quad B+\mu = B+C; \quad C+\nu = B+C;$$

$$\therefore \quad \lambda = A,\ \mu = C,\ \nu = B.$$

$$\therefore \quad M = a^2 \cot A + b^2 \cot C + c^2 \cot B + 4\Delta.$$

Putting $$4\Delta.\cot A = -a^2+b^2+c^2, \quad \&c.,$$

we obtain $$2\Delta.M = a^2(2b^2+2c^2-a^2)$$
$$= 4a^2m_1^2, \quad \text{where } m_1 \equiv AA'.$$

Then $$p^2 \sin A \sin C \sin B = \Delta^2/M;$$

and finally $$p = \tfrac{1}{4}bc/m_1.$$

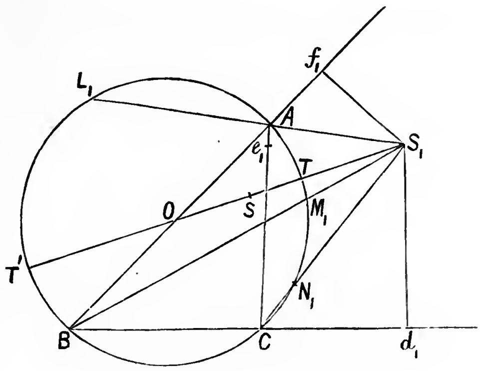
Then $$SA.\sin A = ef = 2p\sin\lambda;$$

$$\therefore \quad SA = \tfrac{1}{2}.bc/m_1; \quad \text{so} \quad SB = \tfrac{1}{2}.c^2/m_1, \quad SC = \tfrac{1}{2}.b^2/m_1.$$

The n.c. of S can be at once obtained from

$$\alpha = abc/M.\sin(A+\lambda)/\sin\lambda, \quad \&c.$$

67. If, with respect to the circle ABC, the point S_1 be the inverse point of S, lying on OS produced; then $OS.OS_1 = R^2$, so that S, S_1 may be taken as the limiting points of a coaxal system, to which the circle ABC belongs.



If OSS_1 cuts this circle in T, T', then AT, AT' bisect the angles between AS and AS_1, &c., so that

$$S_1A : SA = S_1T : ST = S_1B : SB = S_1C : SC.$$

Let $d_1e_1f_1$ be the pedal triangle of S_1; then in the cyclic quadrilateral $S_1e_1Af_1$: $\quad e_1f_1 = S_1A . \sin A$;

$$\therefore \quad e_1f_1 : f_1d_1 : d_1e_1 = S_1A . \sin A : S_1B . \sin B : S_1C . \sin C$$
$$= SA . \sin A : SB . \sin B : SC . \sin C$$
$$= ef : fd : de.$$

Hence the pedal triangles of the inverse points S and S_1 are *inversely similar*: so that

$$d_1 = \lambda, \quad e_1 = \mu, \quad f_1 = \nu.$$

To prove that the triangle $L_1M_1N_1$ is similar to $d_1e_1f_1$, the pedal triangle of S_1.

In the cyclic quadrilateral $S_1d_1ce_1$,

$$\angle S_1d_1e_1 = S_1ce_1 = \pi - N_1CA = N_1L_1A.$$

So $$S_1d_1f_1 = M_1L_1A.$$

$$\therefore \quad e_1d_1f_1 \text{ or } \lambda = M_1L_1N_1, \quad \&c.$$

Similarly ABC is similar to the pedal triangle of S_1 with respect to the triangle $L_1M_1N_1$.

To determine S_1 first find S (56), and then obtain S_1 as the inverse point of S.

We now have a *second* pole (S_1) from which ABC can be inverted into a triangle $L_1M_1N_1$ with given angles λ, μ, ν.

68. To express Π_1, the power of S_1 for the circle ABC, in terms of λ, μ, ν.

$$\angle BS_1C = BM_1C - M_1CS_1 \ (M_1CN_1)$$
$$= BM_1C - M_1L_1N_1 \quad [\text{cyclic quad. } L_1M_1CN_1]$$
$$= A - \lambda.$$

So $$AS_1B = C - \nu.$$

Also $$AS_1C = AS_1B + BS_1C = (A-\lambda)+(C-\nu)$$
$$= \mu - B. \quad [\mu > B]$$

$$\therefore \quad a\alpha_1 = 2.\Delta BS_1C = BS_1 . CS_1 . \sin BS_1C$$
$$= BS_1 . S_1M_1 . \sin(A-\lambda) \sin A/\sin\lambda$$
$$= \Pi_1 . \sin(A-\lambda) \sin A/\sin\lambda.$$

So $$c\gamma_1 = \Pi_1 . \sin(C-\nu) \sin C/\sin\nu,$$

and $$b\beta_1 = \Pi_1 . \sin(B-\mu) \sin B/\sin\mu,$$

giving β_1 a negative value $[\mu > B]$, as the figure indicates.

Proceeding as before, we find

$\Pi_1 M_1 = 2R.abc$ or $8R^2\Delta$, where $M_1 \equiv a^2 \cot\lambda + \ldots - 4\Delta$,

giving Π_1 in terms of M_1, and therefore of λ, μ, ν.

Since $$\Pi_1 = OS_1^2 - R^2 = 8R^2\Delta/M_1,$$

$$\therefore \quad \frac{OS_1^2}{R^2} = \frac{a^2 \cot\lambda + \ldots + 4\Delta}{a^2 \cot\lambda + \ldots - 4\Delta}.$$

[or from $OS . OS_1 = R^2$]

Hence $$OS^2/R^2 = M_1/M, \text{ and } OS_1^2/R^2 = M/M_1;$$

$$\frac{ST}{ST'} = \frac{R-OS}{R+OS} = \frac{\sqrt{M}-\sqrt{M_1}}{\sqrt{M}+\sqrt{M_1}}.$$

The area of $\triangle d_1e_1f_1 = U_1 = 2\Delta^2/M_1$.

Hence $$1/U - 1/U_1 = 4/\Delta.$$

If p_1 be the radius of the pedal circle $d_1e_1f_1$,

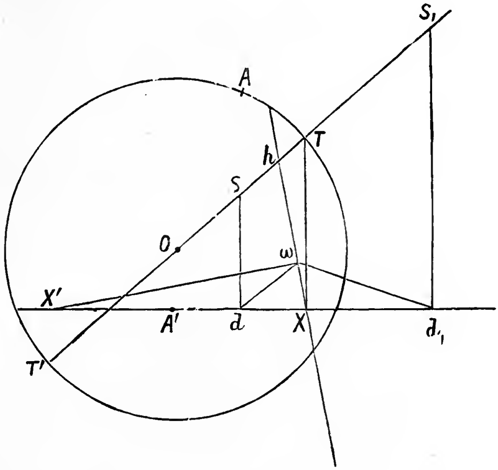
$$2p_1^2 . \sin\lambda \sin\mu \sin\nu = U_1 = 2\Delta^2/M_1.$$

And now all the elements of $d_1e_1f_1$ are found in terms of λ, μ, ν.

From above, $a\alpha_1 = \Pi_1 . \sin(A-\lambda) \sin A/\sin\lambda$.

Hence $$\alpha_1 = \frac{abc}{M_1} . \frac{\sin(A-\lambda)}{\sin\lambda}.$$

69. In section (47) it was proved that the figures $ATSOT'$ and $\omega XdA'X'$ are similar, and just as S is homologous to d, so is S_1 homologous to d_1.



Now AT bisects the angle SAS_1;

$\therefore$ ωX bisects the angle $d\omega d_1$.

It follows that

$$\omega d : \omega d_1 = Xd : Xd_1 = TS : TS_1, \text{ \&c.},$$

so that $$\omega d : \omega e : \omega f = \omega d_1 : \omega e_1 : \omega f_1.$$

Hence ω is the double point of the similar figures; and ωX, $\omega X'$, the Simson Lines of T', T', are the axes of similitude for the *inversely* similar triangles def and $d_1e_1f_1$. (Neuberg)

CHAPTER VI.

THE ORTHOPOLE.*

70. LET p, q, r be the lengths of the perpendiculars Ap, Bq, Cr from A, B, C on any straight line TT', whose direction angles are θ_1, θ_2, θ_3.

Draw pS_1 perpendicular to BC, qS_2 perpendicular to CA, rS_3 perpendicular to AB.

These lines shall be concurrent.

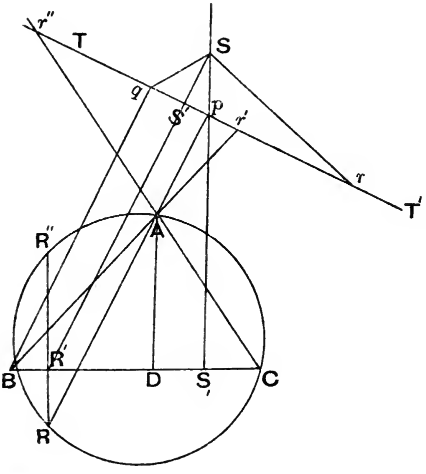
For $\quad BS_1 = BD + p\sin\theta_1, \quad CS_1 = CD - p\sin\theta_1,$

and $\quad \Sigma . ap\sin\theta_1 = 0; \qquad (2)$

$\therefore \quad \Sigma(BS_1^2 - CS_1^2) = \Sigma(BD^2 - CD^2) + 2 . \Sigma\, ap\sin\theta_1 = 0.$

Hence S_1p, S_2q, S_3r are concurrent.

The point of concurrence, denoted by S, is called (by Professor J. Neuberg) the Orthopole of TT'.



* The Orthopole theorems are nearly all due to Professor J. Neuberg.

Let ∂ be the length of the perpendicular SS' on TT'.

Now, for this one particular case, take θ_1, θ_2, θ_3 to represent the *acute* angles which the sides of ABC make with TT'.

We have $$pq = c\cos\theta_3.$$

Also $$pSS' = \theta_1;$$

since Sp, SS' are perpendicular to BC, TT';

also $$qSS' = \theta_2 \text{ and } pSq = C;$$

since Sp, Sq are perpendicular to BC, CA.

$$\therefore\quad Sp = c\cos\theta_3 . \sin(90^\circ - \theta_2)/\sin C = 2R\cos\theta_2\cos\theta_3$$

and $$\partial = Sp\cos\theta_1 = 2R.\cos\theta_1\cos\theta_2\cos\theta_3.$$

As TT' moves parallel to itself, the figure $SqS'pr$ remains unchanged in shape and size.

71. To determine the Orthopole geometrically.

Let Ap meet the circle ABC again in R.

Draw the chord $RR'R''$ perpendicular to BC.

Let BA, CA meet TT' in r', r''.

Then $$BR = 2R.\sin(BAR \text{ or } r'Ap) = 2R\cos\theta_3,$$

so $$CR = 2R\cos\theta_2.$$

$$\therefore\quad RR = BR.CR/2R = 2R\cos\theta_2\cos\theta_3 = Sp.$$

Hence S is found by drawing $R'S$, pS parallels to Ap, RR'.

As TT' moves parallel to itself, S slides along $R'S$ perpendicular to TT'.

Also $R'S'S$, being parallel to AR, is the *Simson Line of* R''.

72. To determine the ABC n.c. of S, (G.)

$$\begin{aligned} a = SS_1 &= \text{projection of } BqS \text{ on } SS_1 \\ &= q\cos\theta_1 + Sq\cos pSq \\ &= q\cos\theta_1 + 2R\cos\theta_1\cos\theta_3 . \cos C. \end{aligned}$$

$$\therefore\quad a\sec\theta_1 = q + 2R\cos C\,(cr - ap\cos B - bq\cos A)/2\Delta,$$
from (2).

Multiply first term on right side by $\sin A\sin B\sin C$ and the other terms by $2\Delta/4R^2$.

Then

$$\begin{aligned} a\sec\theta_1 . \Delta/2R^2 &= \sin A\sin C . q\sin B - \cos A\cos C . q\sin B \\ &\quad + \cos C . r\sin C - \cos B\cos C . p\sin A \\ &= \cos B . q\sin B + \cos C . r\sin C - \cos B\cos C . p\sin A. \end{aligned}$$

Also $2\Delta . \cos\theta_1 = ap - \cos C . bq - \cos B . cr.$ (2)

*$\therefore \alpha = R/2\Delta^2 . (ap - bq\cos C - cr\cos B)$

$\times (\cos B \cos C . ap - \cos B . bq - \cos C . cr).$

Multiplying the two factors, and using the form

$$\Sigma . a^2p^2 - \Sigma . qr . 2bc\cos A = 4\Delta^2, \quad (5)$$

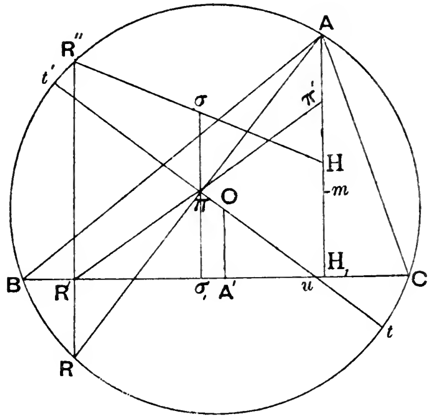
we obtain

$$2\Delta\alpha = abc\cos B\cos C + aqr - rpb\cos C - pq . c\cos B.$$

When TT', whose equation is $ap . \alpha + \ldots = 0$, passes through the circumcentre O, then $ap . \cos A + \ldots = 0$.

In which case

$$\alpha = R/2\Delta^2 . 2\Delta\cos\theta_1 . (\cos B\cos C . ap + ap . \cos A)$$
$$= p\cos\theta_1.$$



73. Let tOt' be the circumdiameter parallel to TT'.

Then σ, the orthopole of tt', lies on the Nine-Point circle.

Let H be the orthocentre of ABC.

Since $A\pi = \pi R$, $\quad \therefore \quad R'\sigma_1 = \sigma_1 H_1$.

Again $R''R' = RR' + 2 . OA' = \sigma\pi + AH$;

$\therefore \quad R''R' + HH_1 = \sigma\pi + AH_1$.

---

* It was Mr. T. Bhimasena Rao who first pointed out that the expression for each coordinate involves two linear factors in p, q, r. His method is different from that given above.

But $A\pi' = RR'$ (equal triangles $A\pi\pi'$, $R\pi R'$) $= \sigma\pi$.

And $\pi' H_1 = 2 . \pi\sigma_1$;

$\therefore \quad R''R' + HH_1 = 2(\sigma\pi + \pi\sigma_1) = 2 . \sigma\sigma_1$.

Hence σ is the mid-point of HR'', and therefore lies on the Nine-Point circle.

74. To find $\alpha_0\beta_0\gamma_0$, the ABC n.c. of σ. (G.)

Bisect $\pi' H_1$ in m.

Then $\sigma\pi = A\pi'$ and $\pi\sigma_1 = \pi' m$.

$\therefore \quad \alpha_0 = \sigma\sigma_1 = Am = A\pi \cos mA\pi = p \cos\theta_1$, &c.,

as found in preceding article.

To find $\alpha_0'\beta_0'\gamma_0'$, the $A'B'C'$ n.c. of σ. (G.)

Since $\sigma\sigma_1$ is $=$ and parallel to Am,

$\therefore \quad A\sigma$ is $=$ and parallel to $\sigma_1 m$.

Also $\pi\sigma_1 H_1 m$ is a rectangle;

$\therefore \quad AH_1\pi\sigma$ is a *symmetrical trapezium*.

Hence the circle (Au), passing through H_1 and π, passes also through σ.

$\therefore \quad 90° - \sigma H_1 A' = \sigma H_1 A = \sigma\pi A = \theta_1$;

$\therefore \quad \sigma A' = R \sin \sigma H_1 A' = R \cos\theta_1$;

$\therefore \quad \alpha_0' = \sigma B' . \sigma C' / R = R \cos\theta_2 \cos\theta_3$, &c.,

so that σ is $(\sec\theta_1, \sec\theta_2, \sec\theta_3)$ referred to $A'B'C'$.

It follows that the point ω of sections (44–50) coincides with the Orthopole σ of a circumdiameter TOT'.

75. The Simson Lines of the extremities of any chord TT' of the circle ABC pass through S, the orthopole of TT'.

For, since $\angle TpR = 90° = TR_1R$,

$\therefore \quad TR_1pR$ is cyclic;

$\therefore \quad pR_1X$ or $pR_1T_1 = TRp$ or $TRA = TT_1A$;

$\therefore \quad R_1p$ is parallel to AT_1.

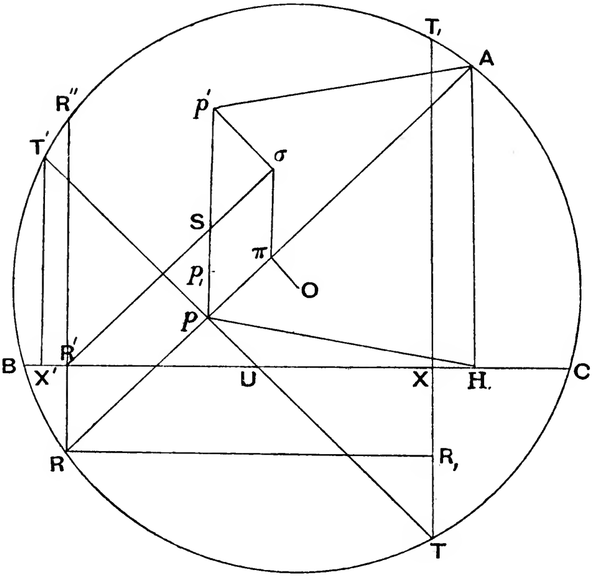
But $XR_1 = R'R = Sp$;

$\therefore \quad XS$ is parallel to R_1p, and therefore to T_1A.

Hence XS is the Simson Line of T.

So for T'.

Thus we have three Simson Lines SX, SX', SR', all passing through S, their poles being T, T', R'' respectively.



They are the three tangents drawn from S to the tricusp-hypocycloidal envelope of the Simson Lines, so that each of the points T, T', R'' has similar relations to the other two; for example:

(a) Just as $R'S$ is perpendicular to TT', so XS is perpendicular to $T'R''$, and $X'S$ to TR''; or each Simson Line is perpendicular to the join of the other two poles.

(b) As p, the foot of the perpendicular from A on TT', lies on the perpendicular to BC from S: so also do p_1 and p_2, the feet of perpendiculars from A on TR'', $T'R''$.

Hence pp_1p_2 is the Simson Line of A in the triangle $TT'R''$.

Thus (S. Narayanan) the Simson Lines of A, B, C for th triangle $TT'R''$ as well as the Simson Lines of T, T', R'' for ABC all pass through S.

76. Let TT' cut BC in U, and let the circle (AU), passing through H_1 and p, cut Sp in p'.

Then $Sp.Sp'$ = twice product of perpendiculars from O and S on TT'. (G.)

In the circle (AU) the chords pp', H_1A are parallel;

$\therefore$ the trapezium $Ap'pH_1$ is symmetrical.

But the trapezium $A\sigma\pi H_1$ is also symmetrical;

$$\therefore \quad p'\sigma = p\pi = d \text{ and } p\pi = S\sigma;$$

$$\therefore \quad Sp' = 2S\sigma \cos \sigma Sp' = 2d \cos \theta_1.$$

But $$Sp = 2R \cos \theta_2 \cos \theta_3 = \partial \sec \theta_1;$$

$$\therefore \quad Sp.Sp' = 2d\delta.$$

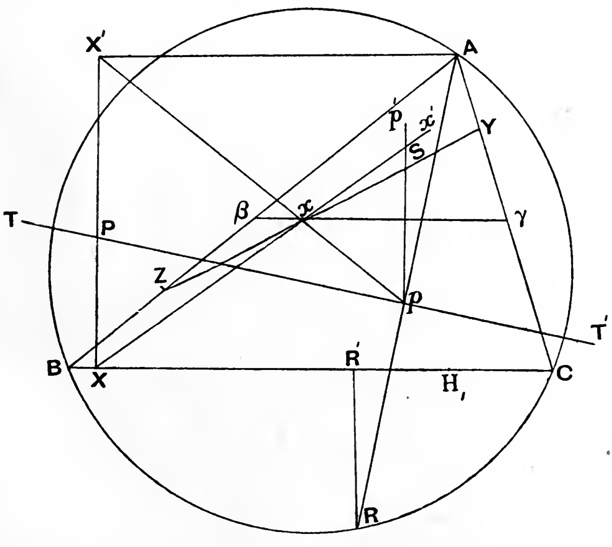
Note that $2d\delta$ is therefore the power of S for the circle (AU).

Similarly, if TT' cut CA in V, AB in W, the power of $S = 2d\delta$ for each of the circles (BV), (CW).

Therefore in the quadrilateral formed by BC, CA, AB, TT', the orthopole (S) of TT' lies on the 4-orthocentre line, or common Radical Axis of the three diameter circles.

77. *Lemoyne's Theorem.*

If P be any point on the straight line TT', whose orthopole



is S, then the power of S with regard to XYZ, the pedal circle of P, is constant.

Draw $p\beta$, $p\gamma$ parallel to RB, RC, so that the figures $A\beta p\gamma$, $ABRC$ are homothetic.

Draw AX' parallel to BC, XPX' perpendicular to BC.

Let YZ cut $\beta\gamma$ in x.

Since R lies on the circle ABC, therefore p lies on the circle $A\beta\gamma$.

$$\therefore \quad \angle p\gamma x \text{ or } p\gamma\beta = pAB \text{ or } pAZ$$
$$= pYZ \text{ or } pYx$$
$$\text{(circle } AYpZPX'\text{, diameter } AP\text{)};$$
$$\therefore \quad Yxp\gamma \text{ is cyclic};$$
$$\therefore \quad px\gamma = pY\gamma = 180^\circ - pYA = pX'A;$$
$$\therefore \quad pxX' \text{ is a straight line.}$$

Let Xx cut pS in δ.

Then $\delta p/XX' = px/xX'$

$= $ ratio of perpendiculars from p and A on $\beta\gamma$

$= \quad$,, $\quad$,, $\quad$,, $\quad$ R and A on BC

(from similar figures $A\beta p\gamma$, $ABRC$)

$= RR'/XX'$;

$\therefore \quad \delta p = RR' = pS$;

$\therefore \quad \delta$ coincides with S.

Observe that, since $H_1pp'A$ is symmetrical,

$\therefore \quad Xpp'X'$ is symmetrical, and therefore cyclic.

Denote the circles XYZ, $XX'pp'$, $AYpZPX'$ by L, M, N respectively.

Let L and M intersect again in x'.

Now the common chord of L and M is Xx'.

,, $\quad$,, $\quad$,, $\quad$ M and N is $X'p$.

,, $\quad$,, $\quad$,, $\quad$ L and N is YZ.

These common chords are concurrent.

But $X'p$, YZ pass through x;

$\therefore \quad Xx'$ passes through x, and therefore through S.

Finally, since $Xpx'p'X'$ is cyclic,

$$\therefore \quad SX.Sx' = Sp.Sp' = 2d\delta.$$

But X, x' are two points on the circle XYZ.

Hence the power of S for the circle $XYZ = 2d\delta$.

Note that the circles (AU), (BV), (CW) are the *pedal circles* of U, V, W.

78. When TT' is a circumdiameter tOt', then d vanishes, and then the pedal triangles *all pass through the orthopole* σ of tOt'.

Since in this case $A\pi = \pi R$, the dimensions of the homothetic figures $A\beta\pi\gamma$, $ABRC$ are as $1:2$, so that $\beta\gamma$ becomes $B'C'$.
Hence $X\sigma$ and YZ intersect on $B'C'$.

79. Returning to the general case, the diagram shows that if x, y, z are the b.c.'s of S with regard to the triangle XYZ, then

$$\frac{x}{x+y+z} = \frac{\Delta SYZ}{\Delta XYZ} = \frac{Sx}{xX} = \frac{SP}{XX'}$$
$$= \frac{2R\cos\theta_2\cos\theta_3}{2R\sin B\sin C};$$

$\therefore \quad x : y : z = \sec\theta_1 \sin A : \sec\theta_2 \sin B : \sec\theta_3 \sin C.$

Thus the Orthopole has constant b.c. for every one of the pedal triangles XYZ. (*Appendix I.*)

80. In section (20) we determined the inverse points T, T' whose tripolar coordinates are p, q, r by dividing BC at P, P'; CA at Q, Q'; AB at R, R'; so that $BP : PC = q : r$, &c., and describing circles on PP', QQ', RR'.

Conversely we may begin by taking two inverse points T and T', whose tripolar coordinates are p, q, r. Then the internal and external bisectors of BTC, $(BT'C)$ meet the sides in the points P, P', &c.; for

$$BP : PC = BT : TC = q : r, \quad \&c.$$

It is known that the triads of points $P'Q'R'$, $P'QR$, $Q'RP$, $R'PQ$ lie on four straight lines, the equation of $P'Q'R'$ being

$$px + qy + rz = 0,$$

while that of $P'QR$ is $-px+qy+rz = 0$, &c.

To prove that the point common to the four circumcircles of the four triangles formed by these four straight lines is ω, the Orthopole of OTT'.

Describe a parabola touching $P'Q'R'$ and the sides of the Medial Triangle $A'B'C'$, and take $A'B'C'$ as triangle of reference.

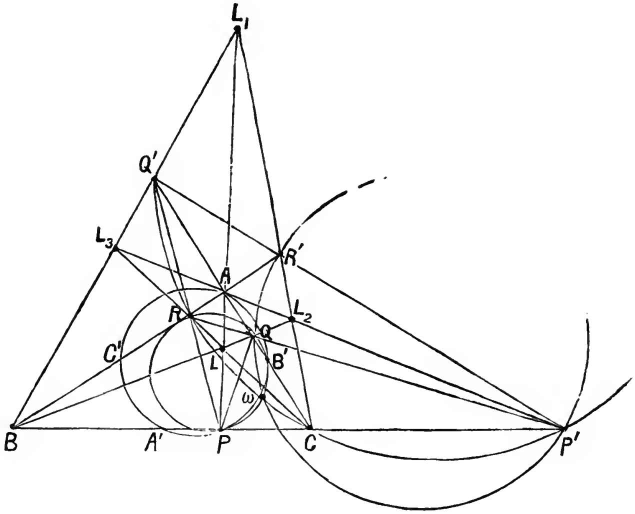
From (15) the $A'B'C'$ equation of $P'Q'R'$ is

$$(q+r)x' + (r+p)y' + (p+q)z' = 0.$$

Then from (9) the n.c. of the focus are as $\dfrac{a}{p'(q'-r')}$, &c., where $[q+r = p']$, ..., or as $\dfrac{a}{q^2-r^2}$, ...

Changing $+p$ into $-p$ makes no change in the focus, therefore the parabola also touches $P'QR$, and similarly $Q'RP$ and $R'PQ$

Now let θ_1, θ_2, θ_3 be the direction angles of OTT', and draw Tm perpendicular to BC.



Then $q^2-r^2 \propto BT^2-CT^2 \propto 2a.A'm \propto 2a.OT\cos\theta_1$;

$$\therefore \quad \frac{a}{q^2-r^2} \propto \sec\theta_1.$$

Hence the orthopole ω, whose n.c. are $(\sec\theta_1, \ldots)$ is the focus of the parabola: the Simson Line of ω (in $A'B'C'$) being the vertex tangent, and OTT' the Directrix. And, the circumcircles of the triangles formed by any three of the four tangents $P'Q'R'$, ..., pass through the focus ω.

On the fixed Directrix TOT' may now be taken an *infinite number* of inverse pairs (TT'). For each pair we have a set of four harmonic lines touching the parabola, the circumcircles of the four triangles passing through ω, and the four orthocentres lying on TOT'.

In addition to each set of four harmonic tangents, there are also the three tangents $B'C'$, $C'A'$, $A'B'$. The student may develop this hint. (G.)

Remember also that the *pedal circles* of all points T or T' also pass through ω. (*Appendix II*).

CHAPTER VII.

ANTIPEDAL TRIANGLES.

81. If S be any point within the triangle ABC, the angles BSC, CSA, ASB are called the Angular Coordinates of S; they are denoted by X, Y, Z.

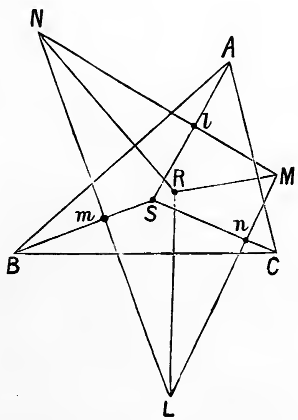
If λ, μ, ν are the angles of the pedal triangle of S, then

$$X = BSC = A + \lambda.$$

So that, if α, β, γ are the n.c. of S,

$$a\alpha = \Pi . \sin(A+\lambda) \sin A / \sin \lambda \qquad (65)$$
$$= \Pi . \sin X \sin A / \sin (X - A).$$

82. *Orthologic Triangles.*



Draw any straight lines MN, NL, LM perpendicular to SA, SB, SC, so that the perpendiculars from A, B, C on the sides of LMN are concurrent (at S). Then shall the perpendiculars from LMN on the sides of ABC be concurrent.

Let AS, MN intersect at l, &c.

Then $$SM^2-SN^2 = Ml^2-Nl^2 = AM^2-AN^2;$$
$$SN^2-SL^2 = BN^2-BL^2;$$
$$SL^2-SM^2 = CL^2-CM^2;$$
$$\therefore \quad (BL^2-CL^2)+(CM^2-AM^2)+(AN^2-BN^2) = 0.$$

Hence the perpendiculars from L, M, N on BC, CA, AB are concurrent, say at R.

Triangles ABC, LMN which are thus related, are said to be mutually *Orthologic.*

To determine the trigonometrical relation between S and R.

Since SB, SC are perpendicular to LN, LM,
$$\therefore \quad \angle BSC = X = \pi - L;$$
so that $$a\alpha = \Pi . \sin(\pi - L) \sin A/\sin(\pi - L - A), \quad \text{from (65)}$$
or $$x = \Pi . \sin L \sin A/\sin(L+A).$$

Now let x', y', z' be the b.c. of R referred to LMN, and let Π' be the power of R for the circle LMN.

Reasoning as before, we have
$$x' = \Pi' . \sin(\pi - A) \sin L/\sin(\pi - A - L)$$
$$= \Pi' . \sin A \sin L/\sin(A+L);$$
$$\therefore \quad x/x' = y/y' = z/z' = \text{area } ABC/LMN$$
$$= \Pi/\Pi'.$$

Hence the b.c. of S with reference to ABC are as the b.c. of R with reference to LMN; and the proportion is that of the areas of the triangles, or of the powers of S and R for their respective circumcircles.

83. *Antipedal Triangles.*

Let def be the pedal triangle of S.

Then, since dS is perpendicular to BC, eS perpendicular to CA, fS perpendicular to AB are concurrent at S, the triangles ABC, def are orthologic, so that the perpendiculars from A on ef, from B on fd, from C on de, are concurrent—say at S'.

Through A, B, C draw perpendiculars to $S'A$, $S'B$, $S'C$, forming the triangle $D'E'F'$—the *Antipedal* Triangle of S'.

The sides $E'F'$, ef (being each perpendicular to $S'A$) are parallel; and therefore def, the *pedal* triangle of S, and $D'E'F'$, the *antipedal* triangle of S', are homothetic.

It follows that
$$D' = d = \lambda, \quad E' = e = \mu, \quad F' = f = \nu.$$

Also

$$BS'C = \pi - D' = \pi - \lambda, \quad CS'A = \pi - \mu, \quad AS'B = \pi - \nu.$$

Again, since ABC and def are orthologic, the b.c. of S' for ABC are as the b.c. of S for def.

Therefore, taking $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ as the n.c. of S, S', respectively,

$$\frac{\Delta . BS'C}{\Delta . eSf} = \ldots = \frac{\text{area of } ABC}{\text{area of } def} = \frac{\Delta}{U};$$

$$\therefore \quad \alpha\alpha' = \beta\gamma \sin A . \Delta/U.$$

But, from (65),

$$\beta = \frac{abc}{M} . \frac{\sin(B+\mu)}{\sin \mu}; \quad \gamma = \frac{abc}{M} . \frac{\sin(C+\nu)}{\sin \nu};$$

and $$\Delta/U = M/2\Delta; \tag{65}$$

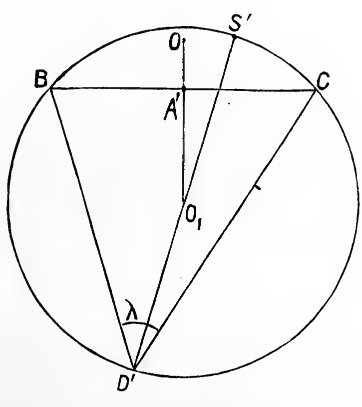
so that, finally, $\alpha' = \dfrac{abc}{M} . \dfrac{\sin(B+\mu)\sin(C+\nu)}{\sin \mu \sin \nu}$.

But $$\alpha = \frac{abc}{M} . \frac{\sin(A+\lambda)}{\sin \lambda};$$

$$\therefore \quad \alpha\alpha' = \frac{a^2b^2c^2}{M^2} . \frac{\sin(A+\lambda)\sin(B+\mu)\sin(C+\nu)}{\sin\lambda \sin\mu \sin\nu}$$

$$= \beta\beta' = \gamma\gamma', \quad \text{from symmetry.}$$

Thus S, S' are the foci of a conic inscribed in ABC.



In Germany these points are called "*Gegenpunkte.*" In this work the name "Counter Points" will be used.

84. To determine the area (V') of $D'E'F'$, the antipedal triangle of S', in terms of λ, μ, ν. (G.)

Henceforth the areas of the pedal triangles of S, S', &c., will be denoted by U, U', &c., antipedal triangles by V, V', &c.

Let O_1, O_2, O_3 be the circumcentres of the circles $S'BD'C$, $S'CE'A$, $S'AF'B$; $S'D'$, $S'E'$, $S'F'$ being diameters.

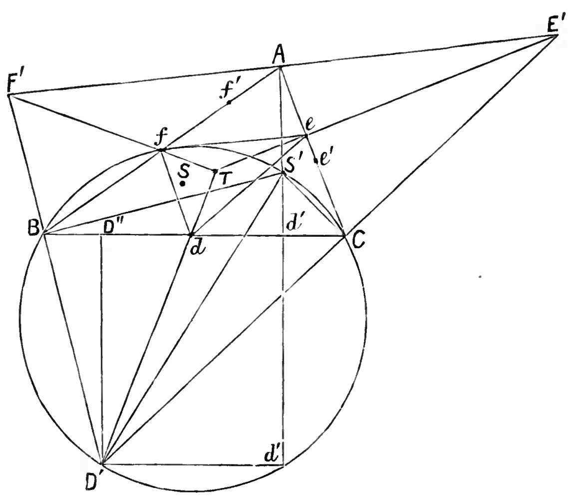
Then $$O_1A' = \tfrac{1}{2}.a \cot D' = \tfrac{1}{2}.a \cot \lambda;$$
$$\therefore \triangle O_1BC = \tfrac{1}{4}.a^2 \cot \lambda.$$

Now O_1 is the mid-point of $S'D'$;

$$\therefore \quad \text{area } S'BD'C = 2.BO_1CS' = 2(\tfrac{1}{4}.a^2 \cot\lambda + S'BC);$$
$$\therefore \quad V' = S'BD'C + S'CE'A + S'AF'B$$
$$= \tfrac{1}{2}(a^2 \cot\lambda + b^2 \cot\mu + c^2 \cot\nu + 4\Delta)$$
$$= \tfrac{1}{2}M.$$

But $$U = 2\Delta^2/M. \qquad (65)$$
$$\therefore \quad UV' = \Delta^2;$$

or, the area of ABC is a geometric mean between the area of the *pedal* triangle of any point, and the homothetic *antipedal* triangle.



This is a particular case of the more general theorem given in (158).

85. To determine the n.c. (u, v, w) of T, the centre of similitude of the homothetic triangles def and $D'E'F'$. (G.)

The ratio of corresponding lengths Td, TD' in def and $D'E'F'$

$$= \sqrt{U} : \sqrt{V'} = \sqrt{UV'} : V' = \Delta : V' = U : \Delta.$$

Therefore, since d, D' are homologous points,

$$u/D'D'' = Td/D'd = U/(\Delta - U).$$

Now the perpendicular from d' on $AC = \beta' + \alpha' \cos C$,

$$\text{,,} \quad AB = \gamma' + \alpha' \cos B;$$

$$\therefore \quad d'C = (\beta' + \alpha' \cos C)/\sin C, \quad d'B = (\gamma' + \alpha' \cos B)/\sin B;$$

$$\therefore \quad D'D'' = d'd'' = d'B.d'C/d'S';$$

$$\therefore \quad u = m.\frac{(\beta' + \alpha' \cos C)(\gamma' + \alpha' \cos B)}{\alpha'.\sin B \sin C},$$

where $m = U/(\Delta - U)$.

Example.—The orthocentric triangle, or *pedal* triangle of H, is homothetic to $T_1T_2T_3$ (Fig., p. 89), which is formed by the tangents at A, B, C, and is therefore the *antipedal* triangle of O.

Here $\alpha' = R \cos A$, &c., and the formula gives

$$u = 2R.\frac{\sin^2 A \cos B \cos C}{1 - 2 \cos A \cos B \cos C} \propto \sin A \tan A.$$

Since H and O are homologous points, being the in-centres of the two triangles, the point T lies on OH.

86. Let S_1 be the inverse of S for the circle ABC, as in (67), and let $d_1e_1f_1$ be the pedal triangle of S_1.

Then, since d_1S_1, e_1S_1, f_1S_1, perpendiculars to BC, CA, AB, are concurrent at S_1, therefore the triangles ABC and $d_1e_1f_1$ are orthologic.

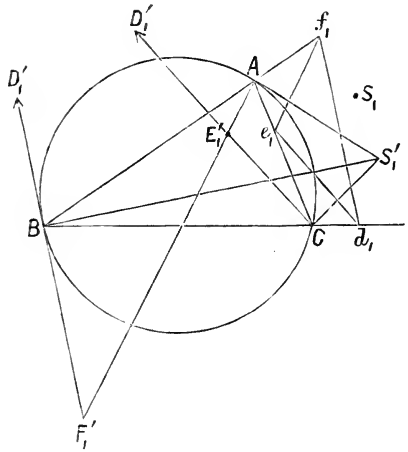
It follows that the perpendiculars from A, B, C on e_1f_1, f_1d_1, d_1e_1 respectively meet at a point—call it S_1'.

Through A, B, C draw perpendiculars to $S_1'A$, $S_1'B$, $S_1'C$ forming the triangle $D_1'E_1'F_1'$, the antipedal triangle of S_1'.

The sides $E_1'F_1'$ and e_1f_1, being each perpendicular to $S_1'A$, are parallel, so that $d_1e_1f_1$, the pedal triangle of S_1, and $D_1'E_1'F_1'$, the antipedal triangle of S_1', are homothetic.

Hence $D_1' = d_1 = \lambda$, $E_1' = e_1 = \mu$, $F_1' = f_1 = \nu$.

We have now *four* triangles, viz.: the *pedal* triangles of S, S_1, and the *antipedal* triangles of S', S_1', each of which has angles λ, μ, ν.



Let $\alpha_1\beta_1\gamma_1$ and $\alpha_1'\beta_1'\gamma_1'$ be the n.c. of S_1 and S_1'.

From (82) the b.c. of S_1' in ABC are as the b.c. of S_1 in $d_1e_1f_1$.

$$\therefore \quad \frac{a\alpha_1'}{2 . \Delta e_1 S_1 f_1} = \ldots = \frac{\Delta}{U_1}; \quad (U_1 = \text{area of } d_1e_1f_1)$$

$$\therefore \quad a\alpha_1' = \beta_1\gamma_1 . \sin A . \Delta / U_1.$$

But, from (68), $$\beta_1 = \frac{abc}{M_1} . \frac{\sin(B-\mu)}{\sin\mu};$$

so for γ_1; and $$U_1 = 2\Delta^2/M_1.$$

Hence, finally,

$$\alpha_1' = \frac{abc}{M_1} . \frac{\sin(B-\mu)\sin(C-\nu)}{\sin\mu\sin\nu};$$

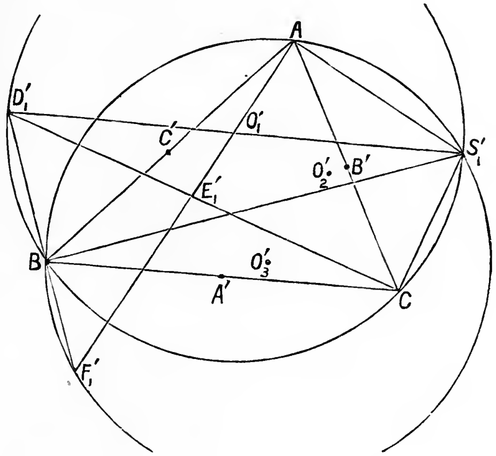
$$\therefore \quad \alpha_1\alpha_1' = \frac{a^2b^2c^2}{M_1^2} . \frac{\sin(A-\lambda)\sin(B-\mu)\sin(C-\nu)}{\sin\lambda\sin\mu\sin\nu}$$

$$= \beta_1\beta_1' = \gamma_1\gamma_1', \quad \text{from symmetry.}$$

Hence S_1 and S_1' are a second pair of Counter Points, being the foci of a conic touching the sides of ABC.

87. To determine the area (V_1') of the antipedal triangle $D_1'E_1'F_1'$ in terms of λ, μ, ν. (G.)

Let O_1', O_2', O_3' be the circumcentres of the circles $S_1'BD_1'C$, $S_1'CE_1'A$, $S_1'AF_1'B$; $S_1'D_1'$, $S_1'E_1'$, $S_1'F_1'$ being diameters, and O_1', O_2', O_3' lying on $A'O$, $B'O$, $C'O$ respectively.



Then $$O_1'A' = \tfrac{1}{2}a \cot D_1' = \tfrac{1}{2}a \cot \lambda;$$
$$\therefore \quad \Delta O_1'BC = \tfrac{1}{4}a^2 \cot \lambda.$$

Now O_1' is the mid-point of $S_1'D_1'$.

Therefore, if $(\alpha_1'\beta_1'\gamma_1')$ be the n.c. of S_1', and k_1 be the perpendicular from D_1' on BC,
$$k_1 + \alpha_1' = 2 . A'O_1';$$
$$\therefore \quad k_1 - \alpha_1' = 2(A'O_1' - \alpha_1');$$
$$\therefore \quad \Delta . D_1'BC - S_1'BC = 2(O_1'BC - S_1'BC)$$
$$= 2(\tfrac{1}{4}a^2 \cot \lambda - S_1'BC).$$

So $$E_1'CA + S_1'CA = 2(\tfrac{1}{4}b^2 \cot \mu + S_1'CA);$$
$$F_1'AB - S_1'AB = 2(\tfrac{1}{4}c^2 \cot \nu - S_1'AB).$$

Adding, we have on the left side
$$D_1'BC + F_1'AB - (ABC - E_1'CA); \; [S'BC + S'AB - S'CA = ABC]$$
$$= D_1'BC + F_1'AB - (E_1'BC + E_1'AB)$$
$$= (D_1'BC + E_1'BC) + (F_1'AB - E_1'AB)$$
$$= E_1'BD_1' + E_1'BF_1' = D_1'E_1'F_1' = V_1'.$$

Hence $V_1' = \frac{1}{2}(a^2 \cot\lambda + b^2 \cot\mu + c^2 \cot\nu - 4\Delta)$
$= \frac{1}{2}M_1.$

Now area (U_1) of $d_1e_1f_1$ is $2\Delta^2/M_1$; (68)

$$\therefore \quad U_1V_1' = \Delta^2.$$

Note also that difference of antipedal triangles of S', S_1'

$$= V' - V_1' = \tfrac{1}{2}M - \tfrac{1}{2}M_1 = 4\Delta.$$

The points S' and S_1', having similar antipedal triangles, are called "Twin Points."

Of the four points S, S', S_1, S_1':

(*a*) S and S_1 are Inverse Points, with similar *Pedal* triangles.

(*b*) S' and S_1' are Twin Points, with similar *Antipedal* triangles.

(*c*) (SS') and (S_1S_1') are pairs of Counter Points, the *Pedal* triangle of either point of a pair being homothetic to the *Antipedal* triangle of its companion point.

CHAPTER VIII.

THE ORTHOGONAL PROJECTION OF A TRIANGLE.

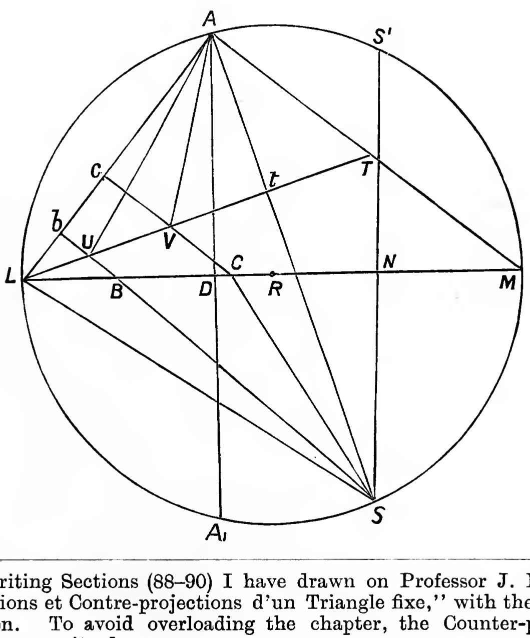
88. * AL being a given axis in the plane of the triangle ABC, it is required to determine the shape and size of the orthogonal projection of ABC on a plane X, passing through AL and inclined at an angle θ to the plane ABC.

Draw AM perpendicular to AL.

In AM take a point T, such that $AT/AM = \cos\theta$. Draw BUb, CVc perpendicular to AL, cutting LT in U and V.

Then, since $Ub/Bb = Vc/Cc = AT/AM = \cos\theta$,

AUV represents, in *shape* and *size*, the orthogonal projection of ABC on the plane X.



* In writing Sections (88–90) I have drawn on Professor J. Neuberg's "Projections et Contre-projections d'un Triangle fixe," with the author's permission. To avoid overloading the chapter, the Counter-projection theorems are omitted.

On either side of the common base BC a series of triangles is described similar to the orthogonal projections of ABC on a series of planes X passing through the common axis AL.

It is required to determine the locus of the vertices of these triangles.

Draw AtS perpendicular to LT, cutting the circle ALA_1M in S. The triangle SBC will be similar to AUV, and therefore to the projection of ABC on the plane X.

Draw SNS' perpendicular to BC.

Then $$\angle ALT = tAT \text{ or } SAM$$
$$= SLM.$$

And A, t, S, N are right angles.

Therefore the figures $ALtT$, $SLNM$ are similar.

And $$LU : LB = LV : LC = LT : LM.$$

Therefore the figures $ALUVtT$ and $SLBCNM$ are similar.

Therefore the triangle SBC is similar to AUV.

Hence the vertices of all triangles SBC, described on BC, and having $SBC = AUV$, $SCB = AVU$, $BSC = UAV$, lie on the circle ALA_1.

The angles of projection range from
$$\theta = 0 \quad \text{to} \quad \theta = \tfrac{1}{2}\pi.$$

When $\theta = 0$, $\cos\theta = 1$, so that T coincides with M, S with A_1, and S' with A.

As θ increases from 0 to $\frac{1}{2}\pi$, T travels from M to A, S from A_1 to M, and S' from A to M.

Hence the locus of S is the arc AMA_1 on the side of AA' *remote from* the axis AL.

89. When a series of *variable* axes AL_1, AL_2, ... are taken, the point A_1, being the image of A in BC, remains unchanged.

Hence each position of the axis AL gives rise to a circle ALA_1M passing through the two fixed points A and A_1, so that this family of circles is *coaxal*.

Let DA or $DA_1 = k$, and let $DR = h$, where R is the centre of the circle ALA_1.

Then, with D as origin and DA as y-axis, the equation of the circle is $$x^2+y^2-k^2 = 2hx,$$
h being the variable of the coaxal system.

We may now deal with the problem: To determine the plane X on which the orthogonal projection of ABC has given angles λ, μ, ν.

Construct the triangle SBC, so that $SBC = \mu$, $SCB = \nu$, and hence $BSC = \lambda$.

Let the circle ASA_1 cut BC in L and M.

Then, if AL and S are on opposite sides of AA_1, AL is the required axis.

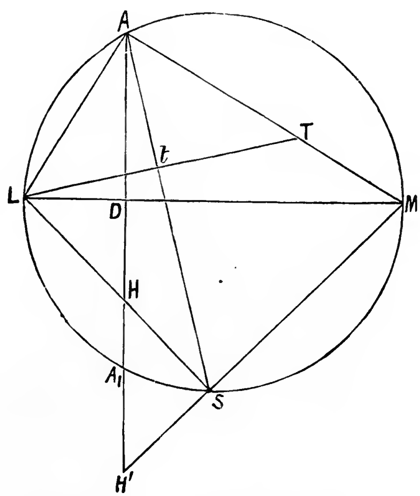
Draw LtT perpendicular to SA.

Then the required inclination θ of the plane X to the plane ABC is given by

$$\cos\theta = AT/AM.$$

90. The triangle ABC being projected on a series of planes making a *constant* angle α with the plane of ABC, and the triangles SBC being drawn, as before, similar to the successive projections, it is required to determine the locus of S.

Draw AL parallel to the line of intersection of the planes; then the original projection is equal and similar to the projection on a parallel plane through AL.



Determine S as before, taking T such that

$$AT/AM = \cos\alpha.$$

Join SL, SM, cutting AA_1 in H and H'.

Then H is the orthocentre of $LH'M$, so that

$$DH.DH' = DL.DM = k^2.$$

Again,
$$\cos\alpha = \frac{AT}{AM} = \frac{\tan ALT}{\tan ALM}$$
$$= \frac{\tan SLM}{\tan A_1LM} = \frac{HD}{A_1D};$$
$$\therefore \quad HD = k\cos\alpha = \text{constant},$$
$$H'D = k\sec\alpha = \text{constant},$$
so that H and H' are fixed points.

Hence, since HSH' is a right angle, the point S describes a circle on HH' as diameter.

The equation to the circle HSH' is
$$x^2 + (y - k\cos\alpha)(y - k\sec\alpha) = 0.$$

For another series of planes inclined at a constant angle α', the points H and H' would be changed to H_1 and H_1', where
$$DH_1 = k\cos\alpha', \quad DH_1' = k\sec\alpha',$$
and
$$DH_1 . DH_1' = k^2.$$

The series of circles are therefore coaxal, having A and A_1 for limiting points, and therefore cutting orthogonally the former series of circles, which pass through A and A_1.

91. A triangle ABC, with sides a, b, c and area Δ, is projected orthogonally into a triangle $A'B'C'$, with sides a', b', c', angles λ, μ, ν, and area Δ'; the angle of projection being θ, so that
$$\Delta' = \Delta\cos\theta.$$

To prove
$$\Sigma . a'^2 \cot A = 2\Delta(1+\cos^2\theta);$$
$$\Sigma . a^2 \cot\lambda = 2\Delta(\sec\theta + \cos\theta).$$

Let h_1, h_2, h_3 be the heights of the points A, B, C above the plane $A'B'C'$.

Then
$$h_2 - h_3 = \sqrt{a^2 - a'^2};$$
$$\therefore \quad \sqrt{a^2-a'^2} \pm \sqrt{b^2-b'^2} \pm \sqrt{c^2-c'^2} = 0,$$
the signs of the surds depending on the relative heights of A, B, C.

This leads to
$$4b'^2c'^2 - (b'^2+c'^2-a'^2)^2 + 4b^2c^2 - (b^2+c^2-a^2)^2 \text{ or } 16\Delta^2 + 16\Delta'^2$$
$$= 2\Sigma . a'^2(b^2+c^2-a^2) = 2\Sigma . a^2(b'^2+c'^2-a'^2).$$

Now
$$b^2+c^2-a^2 = 4\Delta . \cot A;$$
$$b'^2+c'^2-a'^2 = 4\Delta' . \cot\lambda = 4\Delta . \cos\theta\cot\lambda.$$

Hence
$$\Sigma . a'^2\cot A = 2\Delta(1+\cos^2\theta);$$
$$\Sigma . a^2\cot\lambda = 2\Delta(\sec\theta+\cos\theta).$$

The Equilateral Triangle and the Brocard Angle.

When ABC is equilateral, we have

$$(a'^2+b'^2+c'^2).1/\sqrt{3} = 2\Delta(1+\cos^2\theta) = 2\Delta'(\sec\theta+\cos\theta).$$

But, if ω' be the Brocard Angle of $A'B'C'$,

$$\cot\omega' = (a'^2+b'^2+c'^2)/4\Delta'; \qquad (131)$$

$$\therefore\quad \cot\omega' = \sqrt{3}/2.(\cos\theta+\sec\theta).$$

The Brocard Angle therefore depends solely on the angle (θ) of projection. It follows that all equilateral coplanar triangles project into triangles having the same Brocard Angle.

92. Antipedal Triangles and Projection. (G.)

Let $l'm'n'$, $l_1'm_1'n_1'$ be the sides; V', V_1' the areas; and λ, μ, ν the angles of $D'E'F'$, $D_1'E_1'F_1'$, the antipedal triangles of S', S_1'.

From (84),

$$2V' = \Sigma.a^2\cot\lambda+4\Delta = 2\Delta(\sec\theta+\cos\theta)+4\Delta;$$

$$\therefore\quad V'/\Delta = (\sqrt{\sec\theta}+\sqrt{\cos\theta})^2.$$

So $$V_1'/\Delta = (\sqrt{\sec\theta}-\sqrt{\cos\theta})^2;$$

$$\therefore\quad \sqrt{V'}-\sqrt{V_1'} = 2.\sqrt{\Delta\cos\theta} = 2.\sqrt{\Delta'};$$

and $$\sqrt{V'}+\sqrt{V_1'} = 2.\sqrt{\Delta\sec\theta} = 2.\sqrt{\Delta_1};$$

where Δ_1 is the area of the counter-projection of ABC: that is, the triangle whose projection (for θ) is ABC.

Now the triangles V', V_1', with the projection and counter-projection, are all similar, having angles λ, μ, ν; so that their corresponding sides are as the square roots of their areas;

$$\therefore\quad l'-l_1' = 2.a'; \quad l'+l_1' = 2.a_1.$$

Hence, if two antipedal triangles be drawn having the same angles as the projection, the sides of the projection are half the difference of the corresponding sides of the antipedal triangles.

This theorem, of which the above is a new proof, is due to Lhuillier and Neuberg.

93. The triangle ABC is projected orthogonally on to a plane inclined at an angle θ to the plane of ABC.

If U, the line of intersection of the planes, make direction angles u_1, u_2, u_3 with the sides of ABC, it is required to determine the lengths a', b', c' of the sides of the projection $A'B'C'$ in terms of u_1, u_2, u_3, and θ.

Let $\cos\theta = k$.

Draw perpendiculars Bb, Cc to the line U, and take

$$bB' = k.bB, \quad cC' = k.cC,$$

so that $B'C'$ $(= a')$ is equal to the projection of BC.

Now the projection of $B'C'$ on Bb

$$= B'b - C'c = k(Bb - Cc) = k.a \sin u_1;$$

and the projection of $B'C'$ on $U = a \cos u_1$;

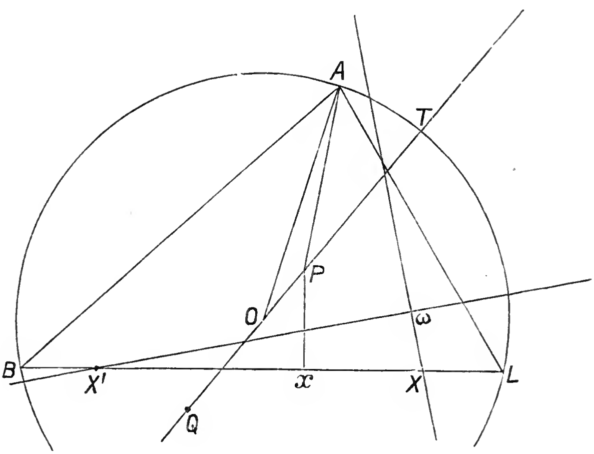
$$\therefore \quad a'^2 = B'C'^2 = a^2(\cos^2 u_1 + k^2 \sin^2 u_1)$$
$$= a^2\{1-(1-k^2)\sin^2 u_1\}$$
$$= a^2(1-\sin^2 u_1 \sin^2\theta).$$

The dimensions of the projection depending, of course, only on the *direction*, not on the position, of U.

94. Pedal Triangles and Projection. (G.)

To prove that, as the plane of projection revolves round U, the projections of ABC are similar to the pedal triangles of points lying on the circumdiameter TOT', where T is the pole of the Simson Line parallel to U.

Let XYZ, $X'Y'Z'$ be the Simson Lines of T, T'.



Then XYZ, being parallel to U, makes angles u_1, u_2, u_3 with the sides of ABC, and therefore

$$\angle\, OTA \text{ or } OAT = u_1. \qquad (37)$$

In TOT' take any point P, and let $OP = k'.R$.
Let u, v, w be the sides of the pedal triangle of P.
Then, in the triangle OAP,

$$AP^2 = R^2 + k'^2R^2 - 2k'R^2 \cos(\pi - 2u_1)$$
$$= R^2(1+k')^2 \left\{1 - \frac{4k'}{(1+k')^2} \sin^2 u_1\right\}.$$

So that

$$4u^2 = 4.AP^2 \sin^2 A = a^2(1+k')^2 \left\{1 - \frac{4k'}{(1+k')^2} \sin^2 u_1\right\}.$$

But $$a'^2 = a^2 \{1-(1-k^2)\sin^2 u_1\}. \tag{93}$$

Hence $$\frac{a'^2}{4u^2} = \frac{1}{(1+k')^2},$$

and $$a' : b' : c' = u : v : w$$

(so that the projection on the plane θ is similar to the pedal triangle of P), provided that

$$\frac{4k'}{(1+k')^2} = 1-k^2,$$

or $$k = \frac{1-k'}{1+k'}, \quad k' = \frac{1-k}{1+k};$$

so that P, and its inverse point P', are given by

$$TP/T'P = TP'/T'P' = \cos\theta.$$

From (36) the point T is found by drawing chord At parallel to U, and the chord tXT perpendicular to BC.

95. Next, let the plane of projection be inclined at a *constant* angle α to the plane of ABC. (G.)

In this case, $$k' = \frac{1-k}{1+k} = \frac{1-\cos\alpha}{1+\cos\alpha} = \tan^2 \tfrac{1}{2}\alpha;$$

$$\therefore \quad OP = R \tan^2 \tfrac{1}{2}\alpha, \quad OP' = R \cot^2 \tfrac{1}{2}\alpha.$$

If, therefore, circles be drawn with centre O and radii $\tan^2 \frac{1}{2}\alpha . R$ and $\cot^2 \frac{1}{2}\alpha . R$, any point P on one circle, and its inverse point P' on the other circle, will have their pedal triangles similar to the projections of ABC.

Another Proof.—Let ABC be projected on the plane α into $A'B'C'$, whose angles are λ, μ, ν. Let P (or its inverse P') be a point whose pedal triangle has angles λ, μ, ν.

$$\text{Then}\quad OP^2/R^2 = \frac{a^2\cot\lambda + b^2\cot\mu + c^2\cot\nu - 4\Delta}{a^2\cot\lambda + b^2\cot\mu + c^2\cot\nu + 4\Delta} \qquad (64)$$

$$= \frac{2\Delta(\sec\alpha + \cos\alpha) - 4\Delta}{2\Delta(\sec\alpha + \cos\alpha) + 4\Delta} = \frac{(1-\cos\alpha)^2}{(1+\cos\alpha)^2}.$$

$$\therefore\quad OP/R = \tan^2\tfrac{1}{2}\alpha;\quad \text{so}\quad OP'/R = \cot^2\tfrac{1}{2}\alpha,$$

as before.

96. A triangle XYZ is projected orthogonally on a series of planes inclined to its own plane at a given angle θ. A point is taken such that its pedal triangle with respect to ABC is similar to one of these projections.

To determine the locus of the point.

Draw inner arcs BQC, CQA, AQB containing angles $A+X$, $B+Y$, $C+Z$: a point Q being thus determined whose pedal triangle has angles X, Y, Z.

Let AQ, BQ, CQ (or AQ', BQ', CQ') meet the circle ABC again in xyz (or $x'y'z'$): then it is known that

$$\text{the angle } x \text{ or } x' = X,\quad y \text{ or } y' = Y,\quad z \text{ or } z' = Z. \qquad (56)$$

Take Q as inversion centre, and let

$$(\text{inversion-radius})^2 \text{ or } k^2 = \text{power of } Q \text{ for } ABC$$
$$= R^2 - OQ^2 = OQ.OQ' - OQ^2$$
$$= QO.QQ';$$

so that in this system Q' is inverse to O.

Also $\qquad k^2 = AQ.Qx = BQ.Qy = CQ.Qz;$

so that xyz inverts into ABC,

97. *Lemma.*

If with any centre K and any radius p, any four points D, E, F, G are inverted into D', E', F', G', then the pedal triangle of G with respect to DEF is similar to the pedal triangle of G' with respect to $D'E'F'$. For

$$D'G'/DG = KD'/KG, \text{ and } E'F'/EF = KE'/KF;$$

$$\therefore\quad (D'G'.E'F')/(DG.EF) = (KD'.KE')/(KG.KF)$$
$$= (KD'.KE'.KF')/(p^2.KG)$$
$$= (E'G'.F'D')/(EG.FD)$$
$$= (F'G'.D'E')/(FG.DE),$$

by symmetry, so that the theorem is proved.

In Section (95) substitute the triangle xyz for ABC. Then the circles with common centre O and radii Rt^2, R/t^2, are the loci of points whose pedal triangles with respect to xyz are similar to the projections of XYZ on planes inclined to the plane of XYZ at an angle θ.

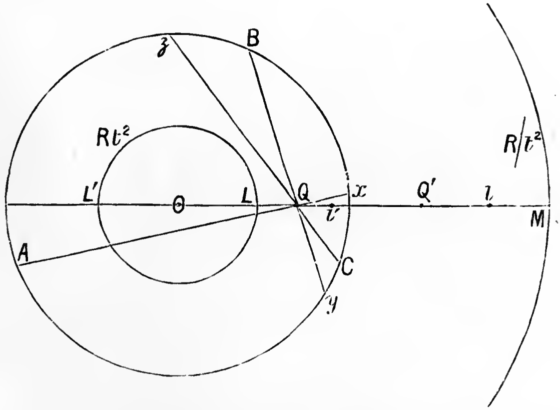
Now invert, with centre Q, radius k. Then xyz inverts into ABC, O into Q', and the two concentric circles (O, Rt^2) and $(O, R/t^2)$ into circles of the coaxal system which has Q and Q' for its limiting points.

Let R be a point on either concentric circle, inverting into S, a point on one or other of the coaxal circles. Then, by the Lemma, the pedal triangle of S with respect to ABC is similar to the pedal triangle of R with respect to xyz; that is, to one of the projections of XYZ.

Hence the required locus consists of these two coaxal circles.

98. Let the circumdiameter OQQ' be cut by the concentric circles in LL', MM': and by the coaxal circles in ll', mm',: so that L and l, ..., are inverse points in the (Q, k) system.

The centres ω, ω' and the radii ρ, ρ' of the circles ll', mm' will now be determined in terms of X, Y, Z, t,



$$Ql = k^2/QL = k^2/(OQ-Rt^2), \quad Ql' = k^2/(OQ+Rt^2);$$

$$\therefore \quad Ol = OQ+Ql = [R(R-OQ.t^2)]/(OQ-Rt^2),$$

$$Ol' = [R(R+OQ.t^2)]/(OQ+Rt^2), \qquad [k^2 = R^2-OQ^2];$$

$$\therefore \quad \rho/R = \tfrac{1}{2}(Ol-Ol')/R = k^2t^2/(OQ^2-R^2t^4),$$

$$O\omega/OQ = \tfrac{1}{2}(Ol+Ol')/OQ = [R^2(1-t^4)]/(OQ^2-R^2t^4).$$

And, writing $1/t^2$ for t^2,

$$Om = [R(OQ-Rt^2)]/(R-OQ.t^2),$$
$$Om' = [R(Rt^2+OQ)]/(R+OQ.t^2),$$
$$\rho'/R = k^2t^2/(OQ^2.t^4-R^2),$$
$$O\omega'/OQ = [R^2(1-t^4)]/(R^2-OQ^2.t^4).$$

Hence $$Ol.Om = R^2 = Ol'.Om',$$
so that the circles ll', mm', which are inverse to LL', MM' in the (Q, K) system, are mutually inverse (as are LL', MM') in the (O, R) system. Since the pedal triangle of Q has angles X, Y, Z, it follows, from Section (64), that

$$OQ^2/R^2 = \frac{a^2\cot X + b^2\cot Y + c^2\cot Z - 4\Delta}{a^2\cot X + b^2\cot Y + c^2\cot Z + 4\Delta},$$

so that now ρ, ρ', $O\omega$, $O\omega'$ are expressed in terms of X, Y, Z, t.

When $OQ = Rt^2$, or when Q lies on the inner concentric circle LL', then $O\omega$ and ρ are infinite.

Hence the (Q, k) inverse of the circle (O, OQ) is DED', the Radical Axis of the coaxal system, bisecting QQ' at right angles.

And if N be the pole of DED' for the circle ABC, it is easily proved that the (Q, k) inverse of the circle (O, OQ') is the circle ON.

99. A case of great beauty and interest presents itself when the triangle XYZ is *equilateral.* For then Q, Q', having *equilateral* pedal triangles, are the Isodynamic points δ, δ_1, lying on OK, the Radical Axis of the coaxal system is now the Lemoine axis; and N, the pole of the Lemoine axis for the circle ABC, is the Lemoine point K.

It follows that in the (δ, k) system, the Lemoine axis is the inverse of the circle $(O, O\delta)$; while the Brocard circle is the inverse of $(O, O\delta_1)$.

Suppose the equilateral triangle XYZ to be projected into a triangle with angles $\lambda\mu\nu$, sides lmn, area Δ', and Brocard angle ϕ.

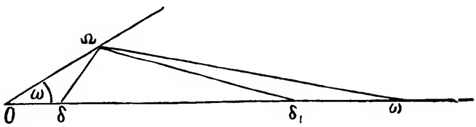
Then, from Section (91),

$$\cot\phi = (l^2+m^2+n^2)/4\Delta' = \tfrac{1}{2}\sqrt{3}.(\sec\theta+\cos\theta)$$
$$= \sqrt{3}.(1+t^4)/(1-t^4); \qquad [t \equiv \tan\tfrac{1}{2}\theta]$$
$$\therefore\quad t^4 = (\cot\phi-\sqrt{3})/\cot\phi+\sqrt{3}).$$

Therefore ϕ is constant, as θ is constant.

Hence the remarkable property of this coaxal system of Schoute circles, viz., the pedal triangles of all points on the circumference of any one circle have the same Brocard angle.

100. To prove that the Brocard angle ϕ is equal to the acute angle (less than 30°) between $\omega\Omega$ and $O\Omega$. (G.)



From Section (98),

$$O\delta^2/R^2 = (a^2 \cot 60° + \ldots - 4\Delta)/(a^2 \cot 60° + \ldots + 4\Delta)$$
$$= (\cot\omega - \sqrt{3})/(\cot\omega + \sqrt{3}).$$

And $t^4 = (\cot\phi - \sqrt{3})/(\cot\phi + \sqrt{3})$ (Section 99).

Therefore, from Section (98),

$$O\omega/O\delta = [R^2(1-t^4)]/(O\delta^2 - R^2t^4)$$
$$= (\cot\omega + \sqrt{3})/(\cot\omega - \cot\phi).$$

In our calculations, Section (98), we took $Rt^2 < OQ$, so that ω lies to the right of Q, and therefore of Q' (here δ_1).

Let $O\Omega\omega = 180 - \chi$.

It is known that

$$O\Omega\delta = 30°, \quad O\Omega\delta_1 = 150°; \qquad (160f.)$$
$$\therefore \quad O\omega/O\delta = O\omega/O\Omega \,.\, O\Omega/O\delta$$
$$= \sin\chi/\sin(\chi-\omega) \,.\, \sin(\omega+30°)/\sin 30°$$
$$= (\cot\omega + \sqrt{3})/(\cot\omega - \cot\chi);$$
$$\therefore \quad \phi = \chi.$$

Many years ago Prof. Dr. P. H. Schoute proved that the locus of a point whose pedal triangle had a constant Brocard angle is a circle of the coaxal system whose limiting points were δ, δ_1.

His proof, I believe, was analytical.

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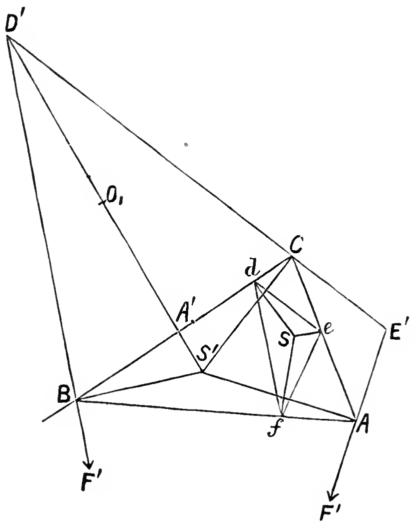
* Read sections 99, 100 after Chapter XI.

CHAPTER IX.

COUNTER POINTS.

101. We now proceed to a further examination of the two pairs of Counter Points (S, S') and (S_1, S_1'), where the pedal triangle of S is homothetic to the antipedal triangle of S', while the pedal triangle of S' is homothetic to the antipedal triangle of S, and so for S_1 and S_1'.



Let $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ be the n.c. of S, S'.

It has been shown that

$$\alpha\alpha' = \beta\beta' = \gamma\gamma' = \frac{a^2b^2c^2}{M^2} \cdot \frac{\sin(A+\lambda)\sin(B+\mu)\sin(C+\nu)}{\sin\lambda\sin\mu\sin\nu}. \quad (83)$$

It follows that S and S' are the foci of a conic inscribed in ABC; that the two pedal triangles def, $d'e'f'$ have the same circumcircle, the Auxiliary Circle of the conic; that the major axis $= 2p$, the diameter of this circle, the centre lying midway between S and S', while the semi-minor axis q is given by $q^2 = \alpha\alpha'$, &c.

The points S, S' are often called "Isogonal Conjugates," because, by a well known property of conics,

$$\angle SCB = S'CA, \quad SBA = S'BC, \quad SAC = S'AB,$$

the pairs $(SA, S'A)$, &c., being *equally inclined* to the corresponding sides.

The line $S'A$ is then said to be isogonal to SA, $S'B$ to SB, $S'C$ to SC; so that S', the counter point of S, lies on any line isogonal to SA or SB or SC.

Let $\alpha\beta\gamma$ be any point L on BC, and $\alpha'\beta'\gamma'$ its counter point.

Then $$\beta\beta' = \gamma\gamma' = \alpha\alpha' = 0. \quad (\alpha = 0)$$

But β, γ are not zero;

$$\therefore \quad \beta' = \gamma' = 0;$$

i.e. the counter point of L is A.

If S be on the circle ABC, draw chord ST perpendicular to BC, and draw diameters SS_1 and TT_1.

Obviously AT_1 is isogonal to AS; also AT_1 is parallel to the Simson Line of S_1, and therefore perpendicular to the Simson Line of S.

Hence the counter point of S is at infinity, being common to the three parallels AT_1, BT_2, CT_3 which are perpendicular to the Simson Line of S.

102. To determine the relations between λ, μ, ν, the angles of def, and λ', μ', ν', the angles of $d'e'f'$.

It is known that $\angle BSC = A+\lambda$; so $BS'C = A+\lambda'$. (56)

But $BS'C = \pi - BD'C = \pi - \lambda$; so $BSC = \pi - \lambda'$.

$$\therefore \quad BSC + BS'C = (A+\lambda) + (\pi - \lambda) = \pi + A.$$

And $$A+\lambda = BSC = \pi - \lambda';$$

$$\therefore \quad \lambda + \lambda' = \pi - A.$$

$$\therefore \quad \sin\lambda' = \sin(A+\lambda), \quad \sin\lambda = \sin(A+\lambda');$$

also $$a = \frac{abc}{M} \cdot \frac{\sin \lambda'}{\sin \lambda}; \quad a' = \frac{abc}{M} \cdot \frac{\sin \mu' \sin \nu'}{\sin \mu \sin \nu}; \tag{83}$$

and $q^2 = aa' = m^2 . \dfrac{a^2b^2c^2}{M^2}$, where $m^2 \equiv \dfrac{\sin \lambda' \sin \mu' \sin \nu'}{\sin \lambda \sin \mu \sin \nu}$.

Let Π' be the power of S'; U' the area of $d'e'f'$;

$$M' \equiv a^2 \cot \lambda' + \ldots + 4\Delta.$$

Then $$\Pi M = 2R . abc = \Pi' M'. \tag{64}$$

And $$2p^2 . \sin \lambda \sin \mu \sin \nu = U = 2\Delta^2/M; \tag{65}$$

$$2p^2 . \sin \lambda' \sin \mu' \sin \nu' = U' = 2\Delta^2/M'.$$

$$\therefore \quad m^2 = U'/U = M/M' = \Pi'/\Pi;$$

$$\therefore \quad U' = m^2 U = m^2 . 2\Delta^2/M,$$

and $$M' = M/m^2, \quad \Pi' = m^2 \Pi.$$

Since $$aa = \Pi . \frac{\sin (A+\lambda) \sin A}{\sin \lambda} = \Pi . \frac{\sin \lambda' \sin A}{\sin \lambda};$$

and, similarly,

$$aa' = \Pi' . \frac{\sin (A+\lambda') \sin A}{\sin \lambda'} = \Pi' . \frac{\sin \lambda \sin A}{\sin \lambda'}.$$

$$\therefore \quad 4R^2 q^2 = 4R^2 . aa' = \Pi \Pi';$$

$$\therefore \quad (R^2 - OS^2)(R^2 - OS'^2) = 4R^2 q^2;$$

a result due to Professor Genese.

103. The equation to the minor axis of the conic which has S, S' for foci. (H. M. Taylor)

Let π_1, π_2, π_3 be the perpendiculars on the minor axis from A, B, C, so that the equation is

$$\pi_1 a . \alpha + \pi_2 b . \beta + \pi_3 c . \gamma = 0.$$

A diagram shows that

$$SA^2 - S'A^2 = 2 . SS' . \pi_1;$$

also $$SA \sin A = ef = 2p \sin \lambda; \tag{57}$$

$$\therefore \quad \pi_1 . \sin^2 A \propto (\sin^2 \lambda - \sin^2 \lambda') \propto \sin (\lambda - \lambda') \sin A.$$

$$[\lambda + \lambda' = \pi - A]$$

Hence the equation to the minor axis is

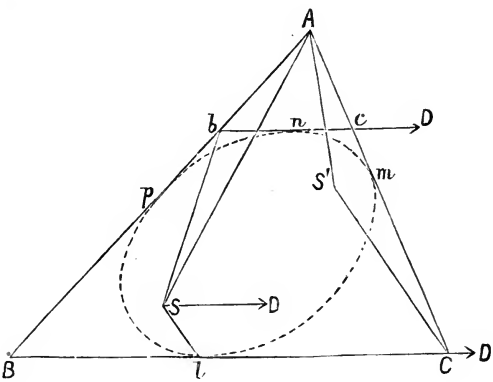
$$\alpha . \sin (\lambda - \lambda') + \beta . \sin (\mu - \mu') + \gamma . \sin (\nu - \nu') = 0.$$

The proof here given is by the present writer.

104. *Lemma.*—The tangent bc to the conic being drawn parallel to BC, to prove

$$SA.S'A = AC.Ab = AB.Ac.$$

Let α, β, γ, δ be the angles subtended at S by the tangents from B, C, c, b.



Then $\quad ASb = ASp - bSp = \frac{1}{2}(2\gamma + 2\delta) - \delta = \gamma;$

and, regarding the parallel tangents as drawn from a point D at infinity, so that SD is parallel to BC,

$$\angle DSl = DSn = \tfrac{1}{2}(2\beta + 2\gamma) = \beta + \gamma;$$

$$\therefore \quad DSC = DSl - CSl = (\beta+\gamma) - \beta = \gamma;$$

$$\therefore \quad S'CA = SCl = DSC = \gamma = ASb.$$

Also $\quad S'AC = SAb.$

Hence the triangles ASb, ACS' are similar;

$$\therefore \quad AS : Ab = AC : AS';$$

$$\therefore \quad SA.S'A = AC.Ab = AB.Ac.$$

The theorem is due to Mr. E. P. Rouse, the demonstration to Mr. R. F. Davis.

105. The coordinates of the centre σ_0 of the conic. (G.)

Let $\alpha_0\beta_0\gamma_0$, $\alpha_0'\beta_0'\gamma_0'$ be the n.c. of σ_0, the centre of the conic, referred to ABC and to the mid-point triangle $A'B'C'$ respectively.

Let h_1 be the perpendicular from A on BC.

Then $$\alpha_0+\alpha_0' = \tfrac{1}{2}h_1;$$

$$\therefore\quad 4R.\alpha_0' = 2R(h_1-2\alpha_0) = 2R\times\text{perp. from } A \text{ on } bc$$
$$= 2R\times Ab\sin B \text{ or } 2R\times Ac\sin C$$
$$= AC.Ab \text{ or } AB.Ac$$
$$= SA.S'A \quad \text{(from Rouse's Theorem).}$$

But $$SA.\sin A = ef = 2p.\sin\lambda; \tag{57}$$

$$\therefore\quad 4R.\alpha_0' = 4p^2.\sin\lambda\sin\lambda'/\sin^2 A;$$

$$\therefore\quad a\alpha_0' = 2p^2.\sin\lambda\sin\lambda'/\sin A,$$

giving the *absolute* $A'B'C'$ b.c. of σ_0; and since

$$a\alpha_0'+b\beta_0'+c\gamma_0' = \Delta,$$

$$\therefore\quad p^2 = \tfrac{1}{2}\Delta/N, \quad \text{where} \quad N = \Sigma.\sin\lambda\sin\lambda'/\sin A.$$

106. The conic touching BC at l, to prove

$$Bl : Cl = \sin\mu\sin\mu'/\sin B : \sin\nu\sin\nu'/\sin C.$$

Project the conic (ellipse) into a circle of radius q, the angle of projection being $\theta = \cos^{-1} q/p$.

Let the centre σ_0 be projected into σ_1, and ABC into $A_1B_1C_1$.

Then $\Delta.B_1\sigma_1C_1 = \Delta.B\sigma_0C\times q/p$; $\therefore$ $q.a_1 = a\alpha_0\times q/p$;

$$\therefore\quad a_1 \propto a\alpha_0;$$

$$\therefore\quad Bl : Cl = B_1l_1 : C_1l_1 = s_1-b_1 : s_1-c_1$$
$$= a\alpha_0-b\beta_0+c\gamma_0 : a\alpha_0+b\beta_0-c\gamma_0$$
$$= b\beta_0' : c\gamma_0' \tag{14}$$
$$= \sin\mu\sin\mu'/\sin B : \sin\nu\sin\nu'/\sin C.$$

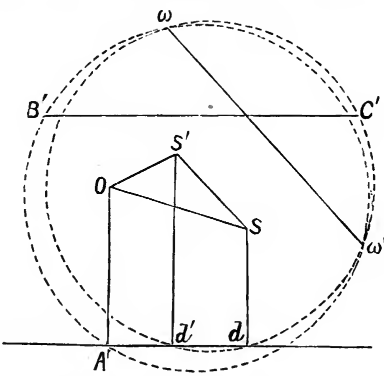
The theorem is by Mr. H. M. Taylor; the proof is by the present writer.

107. Let ω, ω' be the orthopoles of the circumdiameters passing through S and S'; *i.e.* the points on the Nine-Point circle whose Nine-Point circle Simson Lines are parallel to the diameters OS and OS' (49).

The pedal circle of every point on the diameter through S passes through ω; therefore def, the pedal circle of S, passes through ω; similarly the circle $d'e'f'$ passes through ω'. (78)

Therefore $defd'e'f'$, the common pedal circle of S and S', passes through ω and ω'.

And therefore $\omega\omega'$ is the Radical Axis of the two circles. (G.)



When OS' falls on OS, ω' coincides with ω, and the pedal circle touches the Nine-Point circle at ω.

108. *Aiyar's Theorem.*

If O' be the Nine-Point centre

Then $$OS.OS' = 2R.O'\sigma_0.$$

Let $\theta_1\theta_2\theta_3$ and $\theta_1'\theta_2'\theta_3'$ be the direction angles of OS, OS'.

The power of A' for the pedal circle $= A'd.A'd'$

$$= OS\cos\theta_1 . OS'\cos\theta_1'.$$

But since $\omega\omega'$ is the Radical Axis of the two circles, the power of A' for pedal circle

$$= \text{perpendicular from } A' \text{ on } \omega\omega' \times 2.O'\sigma_0.$$

And this perpendicular from A'

$$= A'\omega.A'\omega'/R = R\cos\theta_1.R\cos\theta_1'; \quad (44)$$

$$\therefore \quad OS.OS' = 2R.O'\sigma_0.$$

(V. Ramaswami Aiyar.)

Note that the $A'B'C'$ equation to $\omega\omega'$ is

$$\cos\theta_1 \cos\theta_1' . \alpha\alpha' + \ldots = 0:$$

for, from (50), ω and ω' have n.c. ($\sec\theta_1 \ldots \sec\theta_1' \ldots$).

109. M'Cay's Cubic. For this occasion take (lmn) as the n.c. ABC coordinates of S; so that ($1/l$, $1/m$, $1/n$) are the n.c. of S'.

The equation to SS' then is

$$l(m^2-n^2)\alpha + \ldots = 0.$$

When SS' passes through O, we have

$$l(m^2-n^2)\cos A + \ldots = 0.$$

So that S and S' lie on M'Cay's Cubic,

$$\alpha(\beta^2-\gamma^2)\cos A + \ldots.$$

And conversely, if any diameter TOT' cut this cubic at S, S' (the third point is O), then S, S' are counter points, and their pedal circle *touches* the Nine-Point circle at ω, the orthopole of TOT'.

We have already met with this circle in Section (45): its centre is O_1, $O'O_1\omega$ is a straight line, so that now Aiyar's Theorem may be written $OS . OS' = 2R(R-p)$, and if the circle cut $TSS'OO_1T'$ in k, k', then ωk, $\omega k'$ are the Simson Lines of T, T'.

110. *Counterpoint Conics.* (G.)

The point P moving along a given line TT', it is required to determine the locus of its counter point Q.

Let TT' cut the sides of ABC in L, M, N.

Then the Q locus passes through A, B, C the counter point of L, M, N. (101)

Draw the chord $A\alpha$ parallel to TT', and αt parallel to BC.

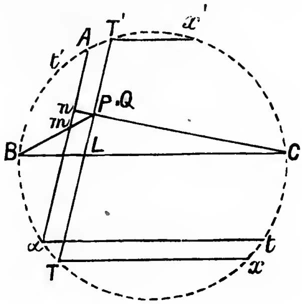
The counter point of t (101) is the point at ∞ on $A\alpha$, and therefore on TT'. But this point being on TT', its counter point must be on the locus of Q.

Hence t is a point where the Q locus cuts the circle ABC; also (101) t is the pole of the Simson Line perpendicular to TT'.

Denote perpendiculars from Q, the counter point of P, on AC, AB, Bt, Ct by β, γ, β', γ'.

Then $$\frac{\beta}{\gamma}=\frac{\sin QAC}{\sin QAB}=\frac{\sin PAB}{\sin PAC};$$

$$\frac{\beta'}{\gamma'}=\frac{QB.\sin QBt}{QC.\sin QCt}=\frac{\sin QCB}{\sin QBC}\cdot\frac{\sin QBt}{\sin QCt}=\frac{\sin PCA}{\sin PBA}\cdot\frac{\sin QBt}{\sin QCt}.$$



Let BP, CP cut $A\alpha$ in m and n.

Then $QBt = QBC+CBt = PBA+BC\alpha = PBA+BAm$

$= Bm\alpha = BPT.$

So $QCt = QCB+BCt = PCA+CB\alpha = PCA+nAC$

$= Pn\alpha = CPT.$

$$\therefore\quad \frac{\beta\beta'}{\gamma\gamma'}=\frac{PA}{PC}\cdot\frac{PB}{PA}\cdot\frac{\sin BPT}{\sin CPT}=\frac{PB.\sin BPT}{PC.\sin CPT}=\frac{q}{r},$$

where p, q, r are perpendiculars from A, B, C on TT'.

Hence the locus is a *conic* passing through A, B, C, t.

If the sides of the quadrilateral were BA, BC, tA, tC, we should have $\gamma\gamma'/\alpha\alpha' = r/p$, ...

111. To determine the directions of the asymptotes and axes.

Draw Tx, $T'x'$ parallel to BC.

The points at infinity of the Q conic are the counter points of T, T'; and therefore lie on Ax, Ax', which are isogonal to AT, AT', so that Ax, Ax' are parallel to the asymptotes.

But the arc $tx = \alpha T = AT'$.

Hence tT, tT' are parallel to Ax', Ax, and therefore to the asymptotes.

112. To determine the Counter Point Conic of a *circum-diameter* TOT'.

From (111), the Asymptotes are at right angles, since they are parallel to tT, tT'; therefore the conic is a Rectangular Hyperbola.

If TOT' be $l\alpha + m\beta + n\gamma = 0$ or $p.a\alpha + \ldots = 0$,

the conic is $$l/\alpha + \ldots = 0.$$

To find the centre, we have

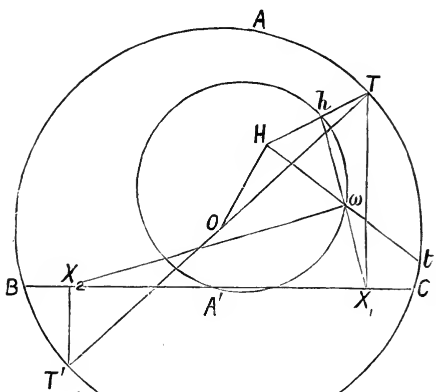
$$m\gamma + n\beta \propto a, \quad \&c.;$$

$$\therefore \quad \alpha \propto l\,(-al + bm + cn).$$

But $l \propto ap$, and, from (8), $ap/R = b\cos\theta_3 + c\cos\theta_2$;

whence $$\alpha \propto p\cos\theta_1, \quad \&c.$$

Hence, from (50), the centre is ω, the Orthopole of TOT'.



113. To determine the Asymptotes. (G.)

Let ωX, $\omega X'$ be the Simson Lines of T, T'.

From Section (41), if u_1, v_1, w_1, h_1 are the perpendiculars from A, B, C, H on ωX_1; and u_2, v_2, w_2, h_2 those on ωX_2; then

$$u_1 = 2R.\cos\sigma_1 \sin\sigma_2 \sin\sigma_3,$$

$$u_2 = 2R.\cos\left(\tfrac{1}{2}\pi - \sigma_1\right)\ldots$$

$$= 2R.\sin\sigma_1 \cos\sigma_2 \cos\sigma_3;$$

$$\therefore \quad 2.u_1u_2 = R^2.\sin 2\sigma_1 \sin 2\sigma_2 \sin 2\sigma_3$$
$$= 2.v_1v_2 = 2.w_1w_2.$$

Also $\quad h_1 =$ perp. from H on ωX_1

$=$ perp. from T on ωX_1

$= 2R.\cos\sigma_1 \cos\sigma_2 \cos\sigma_3$.

So $\quad h_2 = 2R.\cos(\frac{1}{2}\pi-\sigma_1)\ldots$

$= 2R.\sin\sigma_1 \sin\sigma_2 \sin\sigma_3$;

$$\therefore \quad 2.u_1u_2 = \ldots 2h_1h_2.$$

Hence ωX_1, ωX_2 are the Asymptotes of the Rectangular Hyperbola and the square of the semi-axis

$$= R^2.\sin 2\sigma_1 \sin 2\sigma_2 \sin 2\sigma_3.$$

Since $\quad OTA$ or $OAT = \sigma_1$; (37)

$$\therefore \quad p = R.\sin AOT = R.\sin 2\sigma.$$

Hence the square of semi-axis $= pqr/R$.

114. Produce $H\omega$ to cut the circle ABC in t'.

Then, since ω lies on the Nine-Point Circle,

$$H\omega = \omega t'.$$

But H is on the Rectangular Hyperbola, and ω is the centre.

Therefore t' lies on the Rectangular Hyperbola.

Therefore t' coincides with t, the fourth point where the Rectangular Hyperbola cuts the circle ABC.

Let α_1, β_1, γ_1 be the n.c. of S_1, inverse to S; and α_1', β_1', γ_1' the n.c. of S_1', the Twin Point of S'.

It has been shown that

$$\alpha_1 = \frac{abc}{M}\cdot\frac{\sin(A-\lambda)}{\sin\lambda}; \quad \alpha_1' = \frac{abc}{M}\cdot\frac{\sin(B-\mu)\sin(C-\nu)}{\sin\mu\sin\nu}.$$

(68) and (86)

$$\therefore \quad \alpha_1\alpha_1' = \beta_1\beta_1' = \gamma_1\gamma_1'.$$

So that S_1' is the counter point of S_1, and therefore lies on the Rectangular Hyperbola, which is the counter point conic of TOT'.

115. To prove that ω is the mid-point of $S'S_1'$.

We have shown that the pedal circles of all points on TOT' pass through the orthopole ω.

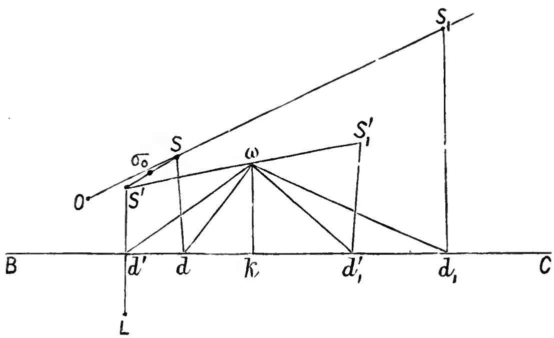
Therefore the pedal circles of (SS') and (S_1S_1') pass through ω.

Also, from Section (69), ω is the Centre of Similitude of def, the pedal circle of S, and of $d_1e_1f_1$, the pedal circle of S_1.

Hence, if p, p_1 are the circumradii of these circles,

$$\frac{\omega d}{p} = \frac{\omega d_1}{p_1},$$

d, d' being homologous points in the similar triangles def, $d_1e_1f_1$.



Draw ωk perpendicular to BC.

Then, since ω is on the circle $defd'e'f'$,

$$\omega k = \frac{\omega d \cdot \omega d'}{2p}, \quad \text{and similarly,} \quad \omega k = \omega d_1 \cdot \omega d_1'/2p_1;$$

$$\therefore \quad \omega d' = \omega d_1';$$

so that the projection of ω on BC is midway between the projections of S' and S_1' on BC.

So for CA, AB.

Therefore ω is the mid-point of S' and S_1'.

Hence, as the inverse points S and S_1 travel along TOT' in contrary directions from T, their counter points S' and S_1' travel along the Rectangular Hyperbola which passes through A, B, C and has ω for its centre. S', S_1' are always at the extremities of a diameter of the Rectangular Hyperbola, and the difference between the areas of their antipedal triangles is always 4Δ.

116. Let l', m', n' be the images of S' in BC, CA, AB. (G.)
Then, because σ_0 is the centre of the pedal circle of (SS'),

$$\therefore \quad \sigma_0 S = \sigma_0 S'; \qquad \therefore \quad Sl' = 2.\sigma_0 d' = 2p.$$

Therefore a circle, centre S, radius $2p$, passes through $l'm'n'$.
Again, since $S'\sigma_0 = \sigma_0 S$ and $S'\omega = \omega S_1'$,

$$\therefore \quad SS_1' = 2.\sigma_0 \omega = 2p,$$

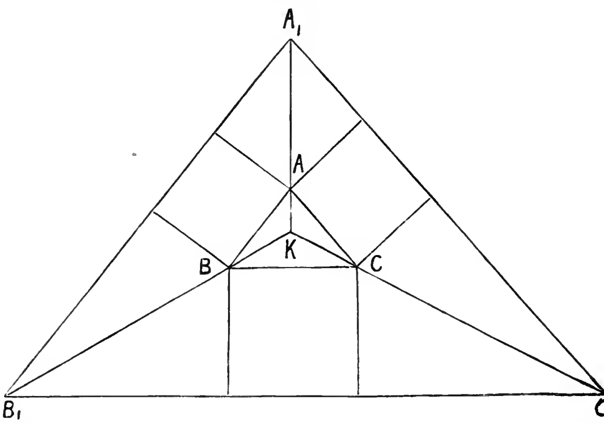
since the circle def passes through ω.

Hence a circle described with centre S and radius $2p$ passes through $l'm'n'$ and S_1'.

CHAPTER X.

LEMOINE GEOMETRY.

117. *The Lemoine Point.*—On the sides of the triangle ABC construct squares externally, and complete the diagram as given, the three outer sides of the squares meeting in A_1, B_1, C_1.



The perpendiculars from A_1 on AC, AB being b and equal to c, the equation to AA_1 is $\beta/b = \gamma/c$.

Hence AA_1, BB_1, CC_1 meet at the point whose n.c. are (a, b, c), and whose b.c. are therefore (a^2, b^2, c^2).

This point is called the *Lemoine* or Grebe or Symmedian Point and will be denoted by K.

AK, BK, CK are called the Symmedians of A, B, C.

The absolute values of the n.c. are given by $\alpha = ka$, &c.,

where $$k = \frac{2\Delta}{a^2+b^2+c^2}.$$

Produce AK to meet BC in K_1; then,

$$BK_1 : CK_1 = \triangle AKB : \triangle AKC$$
$$= \text{ratio of b.c. } z \text{ and } y$$
$$= c^2 : b^2;$$

so that the segments BK_1, CK_1 are as the squares of adjacent sides.

118. K is the centroid of its pedal triangle def.

For $$\triangle eKf = \tfrac{1}{2}.Ke.Kf \sin A$$
$$\propto bc \sin A;$$
$$\therefore \quad \triangle eKf = \triangle fKd = \triangle dKe.$$

So that K is the centroid of def.

Since the n.c. of O are as $(\cos A, \ \cos B, \ \cos C)$, while those of K are as $(\sin A, \sin B, \sin C)$, therefore the equation to OK is $\quad \sin(B-C).\alpha+\sin(C-A).\beta+\sin(A-B).\gamma,$

or $$(b^2-c^2)/a^2.x+\ldots = 0.$$

The Power of K for the circle ABC.

Using the form of Section (60), we have
$$R^2-OK^2 = \Pi = \frac{a^2yz+\ldots}{(x+y+z)^2} = \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2}.$$

119. If α, β, γ are the n.c. of any point, then $\alpha^2+\beta^2+\gamma^2$ is a minimum at K.

Since $(a^2+b^2+c^2)(\alpha^2+\beta^2+\gamma^2)$
$$= (a\alpha+b\beta+c\gamma)^2+(b\gamma-c\beta)^2+(c\alpha-a\gamma)^2+(a\beta-b\alpha)^2$$
$$= 4\Delta^2+\ldots;$$

therefore $(a^2+b^2+c^2)(\alpha^2+\beta^2+\gamma^2)$ is always greater than $4\Delta^2$ except when $$\alpha/a = \beta/b = \gamma/c$$
or when the point coincides with K.

In this case, therefore, $\alpha^2+\beta^2+\gamma^2$ has its minimum value, which is $$\frac{4\Delta^2}{a^2+b^2+c^2}.$$

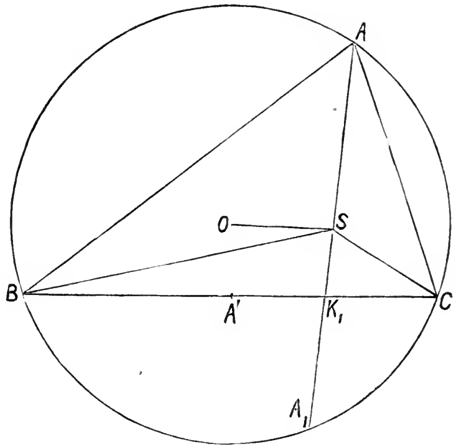
The sides of the pedal triangle of a point S are u, v, w. To show that $u^2+v^2+w^2$ is a minimum when S coincides with the Lemoine Point K.

From (57), $$u = r_1.\sin A; \qquad (SA \equiv r_1)$$
$$\therefore \quad 4R^2(u^2+v^2+w^2) = a^2r_1^{\,2}+b^2r_2^{\,2}+c^2r_3^{\,2}.$$

But, since a^2, b^2, c^2 are the b.c. of K, we have, from (19),
$$a^2.r_1^{\,2}+b^2.r_2^{\,2}+c^2.r_3^{\,2} = (a^2+b^2+c^2)(KS^2-KO^2+R^2);$$
and the right-hand expression is a minimum when $KS = 0$, or when S coincides with K.

The minimum value of $u^2+v^2+w^2$
$$= \frac{a^2r_1^{\,2}+\ldots}{4R^2} = \frac{a^2+b^2+c^2}{4R^2}.\Pi = 3\Delta.\tan\omega.$$
(118) and (131)

120. Let figures Y and Z, directly similar, be described externally on AC, AB, so that A in Y is homologous to B in Z, and C in Y to A in Z.



Then, if S be the double point of Y and Z, the triangles SAC, SBA are similar, having

$$\angle SBA = SAC; \quad SAB = SCA; \quad ASB = ASC,$$

so that S is the focus of a parabola known as Artzt's First Parabola, touching AB, AC at B, C.

Let p_2, p_3 be the perpendiculars from S on AC, AB.

Then, from similar triangles SBA, SAC,

$$p_2 : p_3 = AC : AB,$$

so that S lies on the A-Symmedian.

Again, $\quad \angle BSK_1 = SAB + SBA$
$\qquad = SAB + SAC$
$\qquad = A;$

$$\therefore \quad BSC = 2 . BSK_1 = 2A = BOC;$$

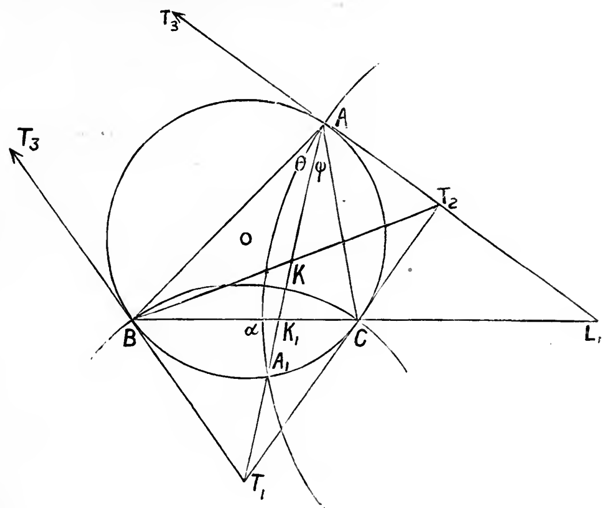
so that S lies on the circle BOC.

Also, $\quad OSK_1 = OSB + A = OCB + A$ (in circle $BOSC$)
$\qquad = 90°;$

$$\therefore \quad SA = SA_1.$$

Hence the double point S of the two directly similar figures on AB, AC may be found by drawing the Symmedian chord AA_1 through K and bisecting it at S.

121. Let $T_1T_2T_3$ be the Tangent Triangle, formed by drawing tangents to the circumcircle at A, B, C.



Then AT_1, BT_2, CT_3, are concurrent at K.

For, if q, r be the perpendiculars from T_1 on AC, AB,

$$\frac{q}{r}=\frac{T_1C\sin ACT_2}{T_1B\sin ABT_3}=\frac{\sin B}{\sin C}=\frac{b}{c}.$$

Hence AT_1 passes through K; so also do BT_2, CT_3.

Note that T_1 is the point of intersection of the tangents at B and C, whose equations are $\gamma/c+\alpha/a=0$, $\alpha/a+b/\beta=0$, so that the n.c. of T_1 are $(-a, b, c)$.

122. The Lemoine Point Λ of $I_1I_2I_3$.

In the figure of the preceding section, it will be seen that the Lemoine Point K of the inscribed triangle ABC, is the Gergonne Point (32) of $T_1T_2T_3$.

Therefore the point M, which is the Gergonne Point of ABC, is the Lemoine Point of XYZ.

But XYZ and $I_1I_2I_3$ are homothetic, the centre of similitude being σ. (26)

Therefore the Lemoine Point—call it Λ—of $I_1I_2I_3$ lies on σM, and $\sigma M:\sigma\Lambda=$ linear ratio of XYZ, $I_1I_2I_3=r:2R$.

The point Λ has n.c. $(s-a)$, ... ; for the perpendiculars from the point $\{(s-a), (s-b), (s-c)\}$ on the sides of $I_1I_2I_3$ are found to be proportional to the sides of this triangle.

Note the following list of "$(s-a)$" points :—

(1) Nagel Point: b.c. are $(s-a)$, $(s-b)$, $(s-c)$. (30)
(2) Gergonne Point: b.c. are $1/(s-a)$, &c. (32)
(3) Lemoine Point Λ of $I_1I_2I_3$: n.c. are $(s-a)$, ...
(4) Centre of Sim. σ of XYZ, $I_1I_2I_3$: n.c. are $1/(s-a)$,.... (26)

123. To prove that AK bisects all chords of the triangle ABC, which are parallel to the tangent at A, or perpendicular to OA, or parallel to the side H_2H_3 of the orthocentric triangle $H_1H_2H_3$.

Let $BAK = \theta, \quad CAK = \phi.$

Then $\sin\theta/\sin\phi = \gamma/\beta = c/b$

$= \sin C/\sin B = \sin BAT_2/\sin CAT_2.$

Therefore AK and T_2AT_3 are harmonic conjugates with respect to AB and AC.

It follows that AK *bisects all chords which are parallel to* T_2T_3 *or* H_2H_3, *or are perpendicular to* OA.

124. *The Harmonic Quadrilateral.*—The angles of the harmonic pencil at A are seen to be B, ϕ, θ or C, θ, ϕ (Fig., p. 89).

The same angles are found, in the same order, at B and C and A_1.

Hence the pencils at B, C, A and A_1 are harmonic; ABA_1C being called, on this account, a Harmonic Quadrilateral.

In the triangle ABA_1 the tangent at B is harmonically conjugate to BK_1, so that BK_1 is the B-symmedian for this triangle.

Similarly, A_1K_1 is the A_1-symmedian in BA_1C_1; and CK_1 the C-symmedian in ACA_1.

To prove that rectangle $AB.A_1C$ = rectangle $AC.A_1B$.

$$\sin\theta/\sin\phi = \sin BCA/\sin CBA_1 = A_1B/A_1C.$$

$$\therefore \quad AB.A_1C = AC.A_1B.$$

If x, y, z, u are the perpendiculars from K_1 on AB, AC, A_1B, A_1C, then $\qquad x/AB = y/AC = z/A_1B.$

For, since AK_1 is the A-symmedian of ABC,

$$\therefore \quad x/AB = y/AC.$$

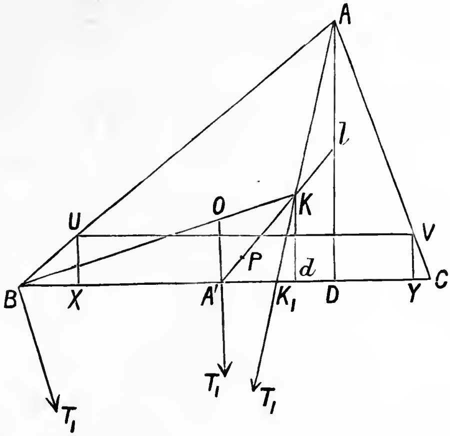
And, since BK_1 is the B-symmedian of $\triangle ABA_1$,

$$x/AB = z/A_1B.$$

And, since A_1K_1 is the A_1-symmedian of A_1BC,

$$\therefore \quad z/A_1B = u/A_1C.$$

125. The Lemoine Point is the point of intersection of lines joining the mid-points of the sides of ABC to the mid-points of corresponding perpendiculars.



Let A', B', C' be the mid-points of BC, CA, AB, and let $A'K$ meet AD in l.

Then, since BK and the tangent BT_1 at B form a harmonic pencil with BA, BC, therefore the range (AKK_1T_1) is harmonic; therefore the pencil $A'(AlD\infty)$ is harmonic, so that AD is bisected at l.

A rectangle $XYVU$ being inscribed in ABC with the side XY on BC, to find the locus of its centre P.

The diagram shows that the mid-point of UV lies on $A'A$, and that P lies on $A'l$.

The Lemoine Point K, common to $A'l$, $B'm$, $C'n$ is the common centre of three inscribed rectangles, standing on BC, CA, AB respectively.

126. To determine the direction angles θ_1, θ_2, θ_3, which OK makes with the sides of ABC.

From the diagram,

$$OK \cos\theta_1 = A'd = A'D\,\frac{Kd}{lD}.$$

Now $$A'D = \frac{b^2-c^2}{2a}, \quad Kd = \frac{2\Delta}{a^2+b^2+c^2}\,a, \qquad (117)$$

and $$lD = \tfrac{1}{2}h_1 = \frac{\Delta}{a};$$

$\therefore \quad \cos\theta_1 = m.a(b^2-c^2)$, where $1/m = OK.(a^2+b^2+c^2)$;

$\therefore \quad \cos\theta_1 : \cos\theta_2 : \cos\theta_3 = a(b^2-c^2) : b(c^2-a^2) : c(a^2-b^2)$.

Hence the tripolar equation to OK is

$$a^2(b^2-c^2)r_1^2+b^2(c^2-a^2)r_2^2+c^2(a^2-b^2)r_3^2 = 0. \qquad (17)$$

127. *The Apollonian Circles.*—Let the several pairs of bisectors of the angles A, B, C meet BC in α, α'; CA in β, β'; AB in γ, γ'.

The circles described on $\alpha\alpha'$, $\beta\beta$, $\gamma\gamma'$ as diameters are called the Apollonian Circles.

Let the tangent at A to the circle ABC meet BC in L_1.

Then angle $L_1\alpha A = \alpha BA + BA\alpha = B+\frac{1}{2}A$,

$L_1A\alpha = L_1AC + CA\alpha = B+\frac{1}{2}A$;

so that $L_1\alpha = L_1A$.

And, since $\alpha A\alpha' = 90°$,

$\therefore \quad L_1A = L_1\alpha'$;

so that L_1 is the centre of the Apollonian Circle $(\alpha\alpha')$, passing through A and orthogonal to the circle ABC.

Since AL_1 is a tangent, the polar of L_1 passes through A; and, since (BK_1CL_1) is harmonic, the polar of L_1 passes through K_1.

It follows that $AKK_1A_1T_1$ is the polar of L_1; so that L_1A_1 is the other tangent from L_1.

Hence the common chords AA_1, BB_1, CC_1 of the circle ABC and the Apollonian Circles intersect at K, which is therefore equipotential for the four circles.

Note that OL_1 bisects AKK_1A_1 at right angles, and therefore passes through the point S of Section (120).

128. *The Lemoine Axis.*

Since the polars of L_1, L_2, L_3 pass through K, therefore $L_1L_2L_3$ lie on the polar of K.

The equation to the tangent at A is $\beta/b+\gamma/c = 0$, &c.

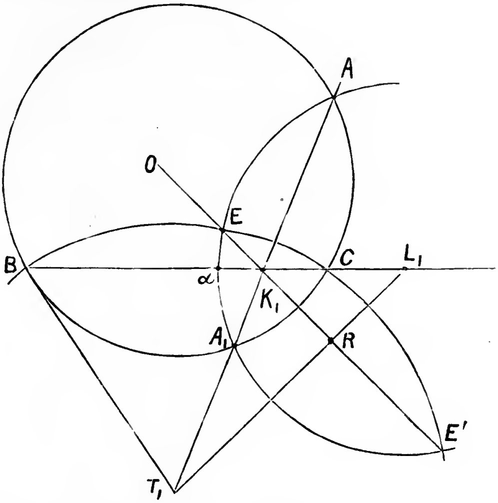
Hence $L_1L_2L_3$ is

$$\alpha/a+\beta/b+\gamma/c = 0, \quad \text{or} \quad x/a^2+y/b^2+z/c^2 = 0.$$

This is called the Lemoine Axis.

129. A Harmonic Quadrilateral such as ABA_1C can be inverted into a square.

Let the circle described with centre T_1 and radius T_1B or T_1C cut the Apollonian Circle L_1 in E and E'.



Then, since the tangents OA, OB, OC, OA_1 to the two circles are equal, O lies on their common chord EE', and

$$OE.OE' = R^2;$$

so that E, E' are inverse points for the circle ABC.

Let AE, BE, CE, A_1E meet the circle again in $LMNL'$.

Then, taking E as pole and $\sqrt{\Pi}$ as radius of inversion (where Π = power of $E = R^2 - OE^2$), we have

$$LM = AB.\frac{\Pi}{EA.EB}, \quad LN = AC.\frac{\Pi}{EA.EC}.$$

But $$BE : EC = BA : AC,$$

since E is on the Apollonian Circle aAa';

$$\therefore \quad LM = LN, \text{ \&c.}$$

Hence $LMNL'$ is a square.

Another square may be obtained by taking E' as pole.

In the above figure, E is on the Apollonian Circle aa';

$$\therefore \quad r_2 : r_3 = BE : CE = Ba : aC = c : b;$$

$$\therefore \quad br_2 = cr_3; \qquad \therefore \quad de = df \text{ or } \mu = \nu.$$

Also $$A + \lambda = BEC = \pi - \tfrac{1}{2}.BT_1C = \tfrac{1}{2}\pi + A;$$

$$\therefore \quad \lambda = \tfrac{1}{2}\pi, \text{ so that } \mu = \nu = \tfrac{1}{4}\pi.$$

CHAPTER XI.

LEMOINE-BROCARD GEOMETRY.

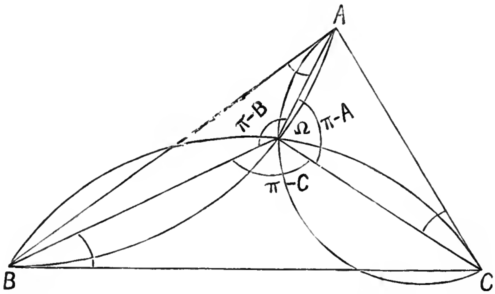
130. *The Brocard Points.*—On the sides BC, CA, AB let triads of circles be described whose *external segments* contain the angles

(a) A, B, C;
(b) C, A, B;
(c) B, C, A;

the cyclic order (ABC) being preserved.

The first triad of circles intersect at the orthocentre H.

Let the second triad intersect at Ω, and the third at Ω'. Ω and Ω' are the *Brocard Points* of ABC.



Since the external segment of $B\Omega C$ contains the angle C, this circle touches CA at C.

Similarly, $C\Omega A$ touches AB at A, and $A\Omega B$ touches BC at B. (Memorize the order of angles for Ω by the word "CAB.")

In like manner, $B\Omega' C$ touches AB at B, $C\Omega' A$ touches BC at C, $A\Omega' B$ touches CA at A.

Again, since $A\Omega B$ touches BC at B, the angle

$$\Omega BC = \Omega AB.$$

So $$\Omega CA = \Omega BC.$$

Similarly, $$\Omega' AC = \Omega' CB = \Omega' BA.$$

Denote each of the equal angles ΩBC, ΩCA, ΩAB by ω, and each of the equal angles $\Omega' CB$, &c., by ω'.

131. To determine ω and ω'.

In the triangle $A\Omega B$,

$$\Omega B = c \sin\omega / \sin B = 2R \sin\omega . c/b.$$

So $\qquad \Omega A = 2R \sin\omega . b/a, \quad \Omega C = 2R \sin\omega . a/c;$

$$\therefore \quad \frac{\sin(A-\omega)}{\sin\omega} = \frac{\sin \Omega AC}{\sin \Omega CA} = \frac{\Omega C}{\Omega A} = \frac{a^2}{bc};$$

$$\therefore \quad \cot\omega = \frac{a^2+b^2+c^2}{4\Delta} = \cot A + \cot B + \cot C$$

$$= \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2 \sin A \sin B \sin C} = \frac{1+\cos A \cos B \cos C}{\sin A \sin B \sin C}.$$

The same expressions are found for ω';

$$\therefore \quad \omega' = \omega.$$

The angle ω, which is equal to each of the six angles ΩAB, ΩBC, ΩCA, $\Omega' BA$, $\Omega' CB$, $\Omega' AC$, is called the Brocard Angle of ABC.

Since $\omega' = \omega$, it follows that

$$\Omega' A = 2R \sin\omega . c/a, \quad \Omega' C = 2R \sin\omega . b/c, \quad \Omega' B = 2R \sin\omega . a/b.$$

Observe that the n.c. of K may now be written

$$\alpha = \frac{2\Delta}{a^2+b^2+c^2} . a = \tfrac{1}{2} a \tan\omega = R \sin A \tan\omega, \ \&c. \quad (117)$$

132. To determine the n.c. and b.c. of Ω and Ω'.

From the diagram,

$$\alpha = \Omega B \sin\omega = 2R \sin^2\omega . c/b.$$

So $\qquad \beta = 2R \sin^2\omega . a/c, \quad \gamma = 2R \sin^2\omega . b/a.$

And for Ω',

$$\alpha' = 2R \sin^2\omega . b/c, \quad \beta' = 2R \sin^2\omega . c/a, \quad \gamma' = 2R \sin^2\omega . a/b,$$

$\therefore \quad \alpha\alpha' = \beta\beta' = \gamma\gamma'$: so that Ω, Ω' are Counter Points.

The b.c. of Ω are given by

$$x : y : z = 1/b^2 : 1/c^2 : 1/a^2;$$

and for Ω', $\qquad x' : y' : z' = 1/c^2 : 1/a^2 : 1/b^2.$

The line $\Omega\Omega'$ is then found to be

$$(a^4 - b^2c^2) . x/a^2 + (b^4 - c^2a^2) . y/b^2 + (c^4 - a^2b^2) . z/c^2 = 0.$$

The power Π of $\Omega = \dfrac{a^2yz + \ldots}{(x+y+z)^2} = \dfrac{a^2b^2c^2}{b^2c^2+c^2a^2+a^2b^2}$

$$= \frac{1}{1/a^2 + 1/b^2 + 1/c^2} = \Pi', \text{ from symmetry.}$$

$$\therefore \quad O\Omega = O\Omega'.$$

133. The Brocard Angle is never greater than 30°.

For $$\cot\omega = \cot A + \cot B + \cot C,$$

and $$\cot B \cot C + \ldots = 1;$$

$$\therefore \quad \cot^2\omega = \cot^2 A + \ldots + 2;$$

$$\therefore \quad (\cot B - \cot C)^2 + \ldots = 2(\cot^2 A + \ldots) - 2$$
$$= 2(\cot^2\omega - 3).$$

Hence $\cot\omega$ is never less than $\sqrt{3}$, and therefore ω is never greater than 30°.

134. Some useful formulæ.

(*a*) $$\operatorname{cosec}^2\omega = 1 + \cot^2\omega = 1 + (a^2+b^2+c^2)^2/16\Delta^2$$
$$= (b^2c^2 + c^2a^2 + a^2b^2)/4\Delta^2;$$

and $$1 - 4\sin^2\omega = \frac{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2}{b^2c^2+c^2a^2+a^2b^2}.$$

This expression will be denoted by e^2.

(*b*) $$\cos\omega = \frac{a^2+b^2+c^2}{2(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}}.$$

(*c*) $$\sin 2\omega = \frac{2\Delta(a^2+b^2+c^2)}{b^2c^2+c^2a^2+a^2b^2}.$$

(*d*) $$\cos 2\omega = \frac{a^4+b^4+c^4}{2(b^2c^2+c^2a^2+a^2b^2)}.$$

(*e*) $$\cot 2\omega = \tfrac{1}{4}\cdot\frac{a^4+b^4+c^4}{\Delta(a^2+b^2+c^2)}.$$

(*f*) Since $$\sin(A-\omega)/\sin\omega = a^2/bc;$$

$$\therefore \quad \sin(A-\omega) : \sin(B-\omega) : \sin(C-\omega) = a^3 : b^3 : c^3,$$

and $$\sin(A-\omega)\sin(B-\omega)\sin(C-\omega) = \sin^3\omega.$$

(*g*) $$\sin(A+\omega)/\sin\omega = \sin A(\cot\omega + \cot A)$$
$$= (b^2+c^2)/bc.$$

Note that, when $b = c$, $\sin(A+\omega) = 2\sin\omega$.

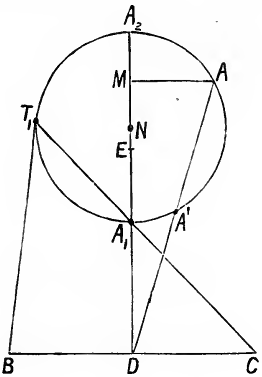
(*h*) $$\cos(A+\omega)/\sin A\sin\omega = \cot A\cot\omega - 1$$
$$= (-a^2+b^2+c^2)(a^2+b^2+c^2)/16\Delta^2 - 1;$$

$$\therefore \quad \cos(A+\omega) = \sin A\sin\omega/8\Delta^2.(b^4+c^4-a^2b^2-a^2c^2)$$
$$\propto \sin(A-B)\sin B + \sin(A-C)\sin C.$$

135. Neuberg Circles.

The base BC of a triangle ABC being fixed, to determine the locus of the vertex A, when the Brocard Angle of the triangle ABC is constant.

Bisect BC in D: draw DA_1A_2 perpendicular, and AM parallel to BC.



Then $$AB^2+AC^2 = 2.AD^2 + \tfrac{1}{2}.BC^2.$$

$$\therefore \quad \cot\omega = \frac{BC^2+CA^2+AB^2}{4.\text{area of } ABC} = \frac{3a^2+4.AD^2}{4a.DM}.$$

$$\therefore \quad AD^2 - a.DM.\cot\omega + 3/4.a^2 = 0.$$

Take $DN = \frac{1}{2}a.\cot\omega$, so that $\angle BND = CND = \omega$.

Then
$$\begin{aligned} NA^2 &= AD^2+ND^2-2.DN.DM \\ &= AD^2-2.\tfrac{1}{2}a\cot\omega.DM+DN^2 \\ &= DN^2-3/4.a^2 = 1/4.a^2(\cot^2\omega-3) \\ &= \text{constant.}\end{aligned}$$

Hence the locus of A is a circle, called a *Neuberg* Circle, centre N, and radius $\rho = \frac{1}{2}a\sqrt{\cot^2\omega-3}$.

136. Let BEC, $BE'C$ be equilateral triangles on opposite sides of the common base BC, so that $DE = \frac{1}{2}a.\sqrt{3}$.

Let the Neuberg Circle cut DE in A_1, A_2.

Then $$DA_1.DA_2 = DN^2-\rho^2 = 3/4.a^2 = DE^2.$$

And thus, for different values of ω, the Neuberg Circles form a coaxal family, with E and E' for Limiting Points, and BC for Radical Axis.

Let CA_1 cut the circle in T_1.

Then, since E is a limiting point,

$$\therefore \quad CA_1 . CT_1 = CE^2 = CB^2,$$

so that the triangles CBA_1, CT_1B are similar;

$$\therefore \quad BT_1 : A_1B = BC : A_1C.$$

But $\quad A_1B = A_1C, \quad \therefore \quad BT_1 = BC = BE;$

$$\therefore \quad BT_1 \text{ is a tangent at } T_1.$$

Similarly, if BA_2 cuts the circle at T_2;

then CT_2 is a tangent at T_2.

137. The Steiner Angles.

From the similar triangles BA_1C, T_1BC,

$$\angle T_1BC = BA_1C.$$

Also, from the cyclic quadrilateral BT_1ND_1, with right angles at T_1 and D, $\quad \angle BT_1D = BND$ or ω;

$$\therefore \quad \angle T_1DC = BT_1D + T_1BC = \omega + BA_1C = A_1 + \omega;$$

so that $\quad \dfrac{\sin(A_1+\omega)}{\sin\omega} = \dfrac{\sin T_1DC}{\sin BT_1D} = \dfrac{BT_1}{BD} = \dfrac{BC}{BD} = 2.$

Similarly, $\quad \dfrac{\sin(A_2+\omega)}{\sin\omega} = 2.$

Thus A_1, A_2 are the values of x obtained from

$$\sin(x+\omega) = 2\sin\omega.$$

This gives $\quad \cot^2 \tfrac{1}{2}x - 2\cot\tfrac{1}{2}x . \cot\omega + 3 = 0;$

whence $\quad \cot\tfrac{1}{2}A_1 = \cot\omega - \sqrt{\cot^2\omega - 3},$

$$\cot\tfrac{1}{2}A_2 = \cot\omega + \sqrt{\cot^2\omega - 3};$$

as is obvious from the diagram.

For $\quad A_1D/DB = DN/DB - NA_1/DB;$

$$\therefore \quad \cot\tfrac{1}{2}A_1 = \cot\omega - \rho/\tfrac{1}{2}a = \cot\omega - \sqrt{\cot^2\omega - 3}.$$

So $\quad \cot\tfrac{1}{2}A_2 = \cot\omega + \sqrt{\cot^2\omega - 3}.$

The angles $\tfrac{1}{2}A_1$, $\tfrac{1}{2}A_2$ will be called the Steiner Angles, and denoted by S_1, S_2.

138. Either Brocard Point Ω or Ω' supplies some interesting illustrations of the properties of pedal triangles.

For Ω, $A+\lambda = B\Omega C = 180 - C = A+B$;

$\therefore \quad \lambda = B, \quad$ so $\quad \mu = C, \quad \nu = A.$

And for Ω', $\lambda' = C, \quad \mu' = A, \quad \nu' = B.$

So that the pedal triangles of the Brocard Points *are similar to ABC.*

To determine ρ, the circumradius of the pedal triangle of Ω.

$$ef = 2\rho \sin\lambda = 2\rho \sin B.$$

But $\quad ef = \Omega A \sin A = 2R \sin\omega . b/a . \sin A \qquad (131)$

$$\therefore \quad \rho = R \sin\omega.$$

Hence, the triangles def, ABC have their linear ratio equal to $\sin\omega : 1$.

$$\therefore \quad U = \Delta \sin^2\omega.$$

Also $\quad \Pi = 4R^2/\Delta . U$;

$$\therefore \quad R^2 - O\Omega^2 = \Pi = 4R^2 \sin^2\omega,$$

$$\therefore \quad O\Omega^2 = R^2(1 - 4\sin^2\omega) = e^2R^2. \qquad (134\,a)$$

139. Lemma I.

Let XBC, YCA, ZAB be isosceles triangles, described all inwards or all outwards, and having a common base angle θ.

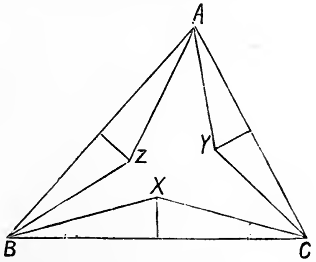
To prove that AX, BY, CZ are concurrent.

Let $(\alpha_1\beta_1\gamma_1)$, $(\alpha_2\beta_2\gamma_2)$, $(\alpha_3\beta_3\gamma_3)$ be the n.c. of X, Y, Z.

Then $\quad \alpha_1 = \frac{1}{2} a \tan\theta$;

$\beta_1 = XC \sin(C-\theta) = \frac{1}{2} . a \sec\theta . \sin(C-\theta)$;

$\gamma_1 = \frac{1}{2} a \sec\theta . \sin(B-\theta)$.



So that $\quad \alpha_1 : \beta_1 : \gamma_1 = \sin\theta : \sin(C-\theta) : \sin(B-\theta)$;

$\alpha_2 : \beta_2 : \gamma_2 = \sin(C-\theta) : \sin\theta : \sin(A-\theta)$,

$\alpha_3 : \beta_3 : \gamma_3 = \sin(B-\theta) : \sin(A-\theta) : \sin\theta$.

The equation to AX is $\beta/\beta_1 = \gamma/\gamma_1$,

or $$\beta.\sin(B-\theta) = \gamma\sin(C-\theta), \text{ \&c.}$$

Hence AX, BY, CZ concur at a point δ, the centre of Perspective for triangles ABC, XYZ, whose n.c. are as $1/\sin(A-\theta)$, $1/\sin(B-\theta)$, $1/\sin(C-\theta)$.

The point δ obviously lies on Kiepert's Hyperbola, the Counter Point conic of OK, for the K.H. equation is

$$\sin(B-C).1/a + \ldots = 0. \quad (\textit{Appendix III. a})$$

140. Lemma II.

The centroid (G') of XYZ coincides with G.

For if $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ be the n.c of G'.

$$3.a\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$$
$$= \tfrac{1}{2}\sec\theta\,\{a\sin\theta + b\sin(C-\theta) + c.\sin(B-\theta)\}$$
$$\propto 2b\sin C \propto 1/a; \quad \text{\&c.}$$

$\therefore$ G' coincides with G. (*See Appendix III.b*)

141. *Illustrations.*—(A) In the diagram of Neuberg's Circle (p. 97), change N into N_1, and on BC, CA, AB describe the isosceles triangles N_1BC, N_2CA, N_3AB (all inwards), with the common base angle $(\frac{1}{2}\pi - \omega)$, so that $N_1N_2N_3$ are the centres of the three Neuberg Circles, corresponding to BC, CA, AB.

Then since $\sin(A-\theta)$ becomes $\cos(A+\omega)$, the lines AN_1, BN_2, CN_3 concur at a point whose n.c. are as $\sec(A+\omega)\ldots$, that is, at the Tarry Point. (143)

For a second illustration take the triangles PBC, QCA, RAB having the common base angle ω measured inwards.

The triangle PQR, called the First Brocard Triangle, has G for centroid from (140), while AP, BQ, CR meet at a point D, the Centre of Perspective for PQR, ABC, its n.c. being as $1/\sin(A-\omega)$ &c., or $1/a^3$, $1/b^3$, $1/c^3$, the b.c. being as $1/a^2$, $1/b^2$, $1/c^2$.

For the n.c. of P,

$$\alpha_1 : \beta_1 : \gamma_1 = \sin\omega : \sin(C-\omega) : \sin(B-\omega);$$
$$= 1 : c^2/ab : b^2/ac,$$

so that the b.c. of P are as a^2, c^2, b^2.

Similarly those of Q are c^2, b^2, a^2, and those of R are b^2, a^2, c^2.

The equation of QR is found to be

$$(a^4-b^2c^2)\,x+(c^4-a^2b^2)\,y+(b^4-c^2a^2)\,z=0.$$

This meets BC at a point p, for which

$$(c^4-a^2b^2)\,y+(b^4-c^2a^2)\,z=0,$$

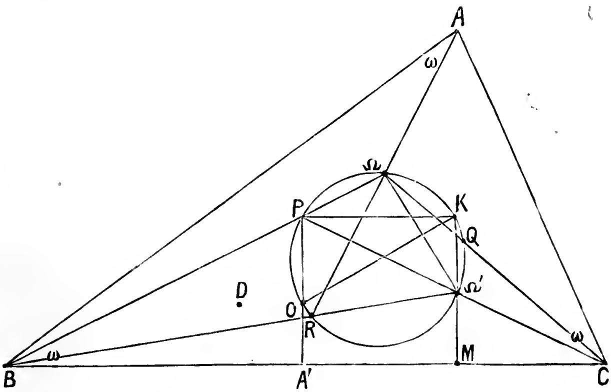
or, $$y/(b^4-c^2a^2)+z/(c^4-a^2b^2)=0.$$

Hence the Axis of Perspective pqr of the triangles PQR, ABC is

$$x/(a^4-b^2c^2)+\ldots=0. \qquad (\textit{Appendix III.c})$$

142. Ω, Ω', K.

Some of the relations between Ω, Ω', and K, will now be investigated.



(*a*) Since $PA'=BA'.\tan PBA'=\frac{1}{2}a.\tan\omega=KM$,

$\therefore$ KP is parallel to BC,

so KQ, KR are parallel to CA, AB respectively.

Since $OPK=90^\circ=OQK=ORK$,

it follows that P, Q, R lie on the Brocard Circle (OK).

(*b*) Since the angles PBC, ΩBC are each $=\omega$,

$\therefore$ BP passes through Ω.

Similarly CQ, AR pass through Ω, while CP, AQ, BR pass through Ω'.

(*c*) Since KQ, KR, are parallel to AC, AB,

$$\therefore \quad QPR = QKR = A.$$

So $$RQP = B, \quad PRQ = C;$$

and thus PQR is inversely similar to ABC, the triangles having the common centroid G as their double point.

(*d*) Since

$$P\Omega R = \Omega AB + \Omega BA = \omega + (B - \omega) = B = PQR,$$

$\therefore$ Ω (and similarly Ω') lies on the Brocard Circle.

(*e*) $$\angle \Omega OK = \Omega PK = \Omega BC = \omega; \quad \text{so } \Omega' OK = \omega.$$

$\therefore$ OK bisects $\Omega\Omega'$ at right angles (at Z).

From Section (138).

$$O\Omega = R(1 - 4\sin^2\omega)^{\frac{1}{2}} \equiv eR;$$

$\therefore$ OK (diameter of Brocard Circle) $= eR \sec\omega$.

$$\Omega\Omega' = 2 \, . \, O\Omega \sin\omega = 2eR\sin\omega.$$

143. The Steiner and Tarry Points.

The ABC Steiner Point denoted by Σ is the pole of the ABC Simson Line which is parallel to OK.

To determine Σ geometrically, draw $A\sigma$ parallel to OK, and $\sigma\Sigma$ perpendicular to BC. (35)

If θ_1, θ_2, θ_3 are the direction angles of OK, it has been proved that $$\cos\theta_1 \propto a(b^2 - c^2). \qquad (126)$$

The n.c. of Σ are $2R\cos\theta_2\cos\theta_3$, &c., which are as $\sec\theta_1$, &c., or as $\dfrac{1}{a(b^2-c^2)}$; and the b.c. are as $1/(b^2-c^2)$.

The point diametrically opposite to Σ on the circle ABC is called the Tarry Point, and is denoted by T; therefore the Simson Lines of Σ and T are at right angles. Hence the n.c. of T are $2R\sin\theta_2\sin\theta_3$, &c. (46)

Now $$OK\sin\theta_1 = KM - OA' \qquad (KM \text{ perp. to } BC)$$
$$= R\sin A\tan\omega - R\cos A$$
$$\propto \cos(A + \omega).$$

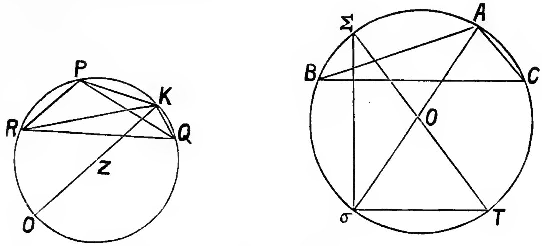
Hence the n.c. of T are as $\sec(A+\omega)$, $\sec(B+\omega)$, $\sec(C+\omega)$.

In (42) let PQR be the Lemoine Axis; then N is the pole of the Simson Line parallel to this axis, and therefore perpendicular to OK.

Hence N is the Tarry Point.

144. PQR being the First Brocard Triangle, inversely similar to ABC, to prove that the figure $KPROQ$ is inversely similar to $\Sigma ACTB$.

Since ΣOT is a diameter, $\Sigma\sigma T$ is a right angle, so that σT is parallel to BC, and arc $B\sigma = CT$.



Now, since KR is parallel to AB (142 *a.*), and KO to $A\sigma$ (as above), $\therefore \angle OKR = \sigma AB = T\Sigma C$, from the equal arcs.

Similarly, $OKQ = T\Sigma B$;

also ABC, PQR are inversely similar.

Hence O and T are homologous points in these two triangles.

Therefore K and Σ are homologous.

Hence the figures $KPROQ$, $\Sigma ACTB$ are inversely similar.

Since $\angle \Sigma BA = KQP$ (similar figures)

$= KRP$,

and AB is parallel to KR,

$\therefore$ ΣB is parallel to RP, &c.

Hence TA is perpendicular to QR, TB to RP, TC to PQ.

145. Lemma.

The points L, M, N have b.c. proportional to yzx, zxy, xyz, arranged in cyclic order; to prove that G', the mean centre of LMN, coincides with G.

Let $(\alpha_1\beta_1\gamma_1)$, $(\alpha_2\beta_2\gamma_2)$, $(\alpha_3\beta_3\gamma_3)$, $(\alpha'\beta'\gamma')$ be the *absolute* n.c. of L, M, N and G' respectively.

Then, for L,

$$a\alpha_1/y = b\beta_1/z = c\gamma_1/x = 2\Delta/(x+y+z).$$

$$\therefore \quad \alpha_1 = 2\Delta/(x+y+z).y/a.$$

So $$\alpha_2 = 2\Delta/(x+y+z).\,z/a,$$

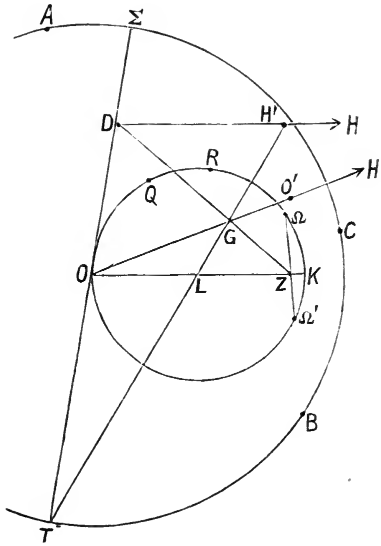
and $$\alpha_3 = 2\Delta/(x+y+z).x/a;$$

$$\therefore \quad 3\alpha' = \alpha_1+\alpha_2+\alpha_3 = 2\Delta/a, \quad \&c.$$

Hence G' coincides with G.

146. The b.c. of P, Q, R being $a^2c^2b^2$, $c^2b^2a^2$, $b^2a^2c^2$, in cyclic order, it follows that G is the centroid of PQR, as already proved, and is therefore the double point of the inversely similar triangles ABC, PQR.

Let L be the circumcentre of PQR, and therefore of the Brocard Circle (OK); then L in PQR is homologous to O in ABC. Therefore the axes of similitude of the two inversely similar triangles bisect the angles between GO and GL.



Again, the b.c. of D, Ω, Ω' are $1/a^2$, $1/b^2$, $1/c^2$; $1/b^2$, $1/c^2$, $1/a^2$; $1/c^2$, $1/a^2$, $1/b^2$, in cyclic order.

Hence G is the centroid of $D\Omega\Omega'$.

Bisect $\Omega\Omega'$ in Z, then G lies on DZ, and $GZ : GD = 1 : 2$.

But $$OG : GH = 1 : 2. \quad (H \text{ orthocentre of } ABC)$$

$\therefore$ DH is parallel to OK, and $DH = 2 . OZ = 2 . eR \cos \omega$.

Let LG meet DH in H'.

Since L is the circumcentre, and G the centroid of PQR,

$\therefore$ LGH' is the Euler Line of PQR.

But $$LG : GH' = ZG : GD = 1 : 2;$$

$\therefore$ H' is the orthocentre of PQR.

Again, $HH' = 2.OL = OK,$

$\therefore$ $H'HKO$ is a parallelogram

and $H'K$ bisects OH at the Nine-Point centre.

147. Since G is the double point of the inversely similar figures $\Sigma ACTB$, $KPROQ$, therefore the points G, O, T in the former figure are homologous to G, L, O in the latter.

Hence angle $OGT = LGO$, so that G, L, T are collinear.

Again, if $\rho\,(=\frac{1}{2}eR\sec\omega)$ be the radius of the Brocard Circle,

then $GO : GL = R : \rho$ (by similar figures).

So $GT : GO = R : \rho$;

$\therefore$ $GO^2 = GL.GT$;

and $GT^2 : GO^2 = R^2 : \rho^2 = GT : GL.$

148. To prove that D lies on the circumdiameter ΣOT.

From (146) $H'D : ZL = H'G : GL = 2 : 1$;

and $ZL = OZ - \rho = eR\cos\omega - \frac{1}{2}eR\sec\omega = \rho\cos 2\omega$;

$\therefore$ $H'D = 2.ZL = 2\rho\cos 2\omega.$

Again, $H'T : LT = GT + 2.GL : GT - GL$;

and, from above, $GT : GL = R^2 : \rho^2$;

$$\therefore\quad H'T : LT = R^2 + 2\rho^2 : R^2 - \rho^2$$
$$= 2\cos 2\omega : 1 = H'D : LO \text{ (or } \rho\text{)},$$

$\therefore$ D lies on ΣOT.

To determine OD.

$$DT : OT = H'T : LT = 2\cos 2\omega : 1,$$

$$\therefore\quad OD : R = 2\cos 2\omega - 1 : 1;$$

$$\therefore\quad OD = e^2R.$$

Note also that

$$OD.O\Sigma = e^2R^2 = O\Omega^2 \quad\text{or}\quad O\Omega'^2.$$

So that $O\Omega$, $O\Omega'$ are tangents to the circles $\Omega D\Sigma$, $\Omega' D\Sigma$ respectively.

149. The Isodynamic Points.

These are the pair of inverse points δ and δ_1, whose pedal triangles are *equilateral;* so that

$$\lambda = \mu = \nu = 60^\circ.$$

In this case, $M = a^2 \cot\lambda + \ldots + 4\Delta$ (64)

$$= 4\Delta\,(\cot\omega \cot 60^\circ + 1).$$

So for δ_1, $M_1 = 4\Delta\,(\cot\omega \cot 60^\circ - 1).$ (68)

The Powers Π (Π_1) are given by

$$\Pi\ (\Pi_1) = 8R^2\Delta/M\,(M_1) = 2R^2/(\cot\omega \cot 60^\circ \pm 1).$$

$$\text{Areas of pedal triangles} = 2\Delta^2/M\,(M_1)$$

$$= \tfrac{1}{2}\Delta/(\cot\omega \cot 60^\circ \pm 1).$$

$$\text{Absolute n.c.} = a\,(a_1) = \frac{abc}{M\,(M_1)} \cdot \frac{\sin(A \pm 60^\circ)}{\sin 60^\circ}$$

$$\propto \sin(A \pm 60^\circ).$$

The circumradii of the pedal triangles are given by

$$2p^2\,(2p_1^2) . \sin 60^\circ \sin 60^\circ \sin 60^\circ = \text{area of pedal triangles}$$

$$= \tfrac{1}{2}\Delta/(\cot\omega \cot 60^\circ \pm 1).$$

Let $(\rho_1\rho_2\rho_3)$, $(\rho_1'\rho_2'\rho_3')$ be the tripolar coordinates of δ, δ_1.

Then $\rho_1 \sin A = ef = p \sin 60^\circ = \text{constant}.$

$$\therefore\ \ \rho_1 : \rho_2 : \rho_3 = 1/a : 1/b : 1/c = \rho_1' : \rho_2' : \rho_3'.$$

The tripolar equation to OK is

$$a^2(b^2-c^2)\,r_1^2 + \ldots = 0, \quad \text{for} \quad \cos\theta_1 \propto a\,(b^2-c^2)\,; \qquad (126)$$

and this is satisfied by $r_1 \propto 1/a$.

Hence δ, δ_1 lie on OK.

150. Consider the coaxal system which has δ and δ_1 for its limiting points.

From (149), since $p : q : r = 1/a : 1/b : 1/c$, therefore the Radical Axis of the system becomes

$$x/a^2 + y/b^2 + z/c^2 = 0,$$

which is the Lemoine Axis $L_1L_2L_3$.

Let this Radical Axis cut OK in λ;

then, since K is the pole of $L_1L_2L_3$ for the circle ABC,

$$\therefore\ \ \lambda O . \lambda K = \text{square of tangent from } \lambda \text{ to } ABC$$

$$= \lambda\delta^2 \text{ or } \lambda\delta_1^2,$$

since ABC belongs to the coaxal system.

Therefore the Brocard Circle (OK) belongs to this system, and therefore is coaxal with ABC.

151. The Isogonic Points.

These are the Counter Points of δ and δ_1; they are therefore denoted by δ' and δ_1'.

Their *antipedal* triangles are equilateral, having

$$\text{areas } \tfrac{1}{2}M\,(M_1) = 2\Delta\,(\cot\omega\cot 60°\pm 1).$$

Their n.c. are

$$a'\,(a_1') = \frac{abc}{M\,(M_1)}\cdot\frac{\sin(B\pm 60°)\sin(C\pm 60°)}{\sin 60°\sin 60°}$$
$$\propto 1/\sin(A\pm 60°).$$

Hence, from (139), if equilateral triangles XBC, YCA, ZAB be described inwards on BC, CA, AB, then XA, YB, ZC concur at δ'; for the outward system, the point of concurrence is δ_1'.

These points lie on Kiepert's Hyperbola, whose equation is

$$\sin(B-C)/a+\ldots = 0.$$

152. The Circum-ellipse.

Let $l/\alpha+m/\beta+n/\gamma = 0$, be the ellipse, with axes $2p$, $2q$.

Let $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ be the n.c. of the centre Ω, referred to ABC and $A'B'C'$, so that $2a\alpha' = -a\alpha+b\beta+c\gamma$.

Project the ellipse into a circle, centre ω, the radius of the circle being therefore q, while the angle of projection θ is $\cos^{-1} q/p$.

Let LMN, with angles $\lambda\mu\nu$, be the triangle into which ABC is projected.

$$\therefore\quad \Delta.M\omega N = \Delta.B\Omega C\times\cos\theta;$$
$$\therefore\quad q^2\sin 2\lambda = a\alpha\times q/p;$$
$$\therefore\quad a\alpha = pq\sin 2\lambda;$$

so that the ABC b.c. of Ω are as $\sin 2\lambda$, $\sin 2\mu$, $\sin 2\nu$.

Also $$\tfrac{1}{2}(a\alpha+b\beta+c\gamma) = \Delta;$$
$$\therefore\quad 2pq\sin\lambda\sin\mu\sin\nu = \Delta.$$

Again $$2a\alpha' = -a\alpha+b\beta+c\gamma;$$
$$\therefore\quad a\alpha' = 2pq\sin\lambda\cos\mu\cos\nu;$$

so that the $A'B'C'$ b.c. of Ω are as $\tan\lambda$, $\tan\mu$, $\tan\nu$.

Since $$\Delta = 2pq\sin\lambda\sin\mu\sin\nu;$$
and $$-a\alpha+b\beta+c\gamma = 4pq\sin\lambda\cos\mu\cos\nu;$$
and $$a\alpha = pq\sin 2\lambda;$$
it follows that

$$\Delta.(-a\alpha+b\beta+c\gamma)(a\alpha-b\beta+c\gamma)(a\alpha+b\beta-c\gamma)$$
$$= 2p^4q^4.\sin^2 2\lambda\sin^2 2\mu\sin^2 2\nu;$$

$$\therefore \quad \Delta . p^2q^2 (-a\alpha+b\beta+c\gamma)(a\alpha-b\beta+c\gamma)(a\alpha+b\beta-c\gamma) = 2.a^2b^2c^2.\alpha^2\beta^2\gamma^2;$$

giving the locus of the centre, when the area of the ellipse (πpq) is constant.

Let PQR be the triangle formed by tangents to the Ellipse at A, B, C, then AP, BQ, CR have a common point—call it T—whose b.c. are as al, bm, cn.

Let PQR be projected into pqr, whose sides touch the circle LMN.

* Then the projection of T is evidently the *Lemoine Point* of LMN (fig., p. 89), and therefore its b.c. are as

$$MN^2, NL^2, LM^2 \quad \text{or as} \quad \sin^2\lambda, \sin^2\mu, \sin^2\nu;$$

$$\therefore \quad al : bm : cn = \sin^2\lambda : \sin^2\mu : \sin^2\nu.$$

Hence the Ellipse is

$$\sin^2\lambda/a.\beta\gamma+\ldots = 0;$$

and its Counter Point Locus is

$$\sin^2\lambda/a.\alpha+\ldots = 0.$$

This is the Radical Axis of the coaxal system, whose Limiting Points have $\lambda\mu\nu$ for the angles of their pedal triangles.

To calculate the axes of the ellipse in terms of λ, μ, ν.

From (91),

$$a^2\cot\lambda+b^2\cot\mu+c^2\cot\nu = 2\Delta(\sec\theta+\cos\theta)$$
$$= 2\Delta(q/p+p/q) = 2\Delta(p^2+q^2)/pq.$$

And from (152), $\Delta = 2pq\sin\lambda\sin\mu\sin\nu$.

$$\therefore \quad (p+q)^2 = \tfrac{1}{4}.\frac{a^2\cot\lambda+\ldots+4\Delta}{\sin\lambda\sin\mu\sin\nu} = \tfrac{1}{4}.\frac{M}{\sin\lambda\sin\mu\sin\nu}. \qquad (64)$$

So
$$(p-q)^2 = \tfrac{1}{4}.\frac{M_1}{\sin\lambda\sin\mu\sin\nu} \qquad (68).$$

A well known example is the Steiner Ellipse, whose centre is G, so that LMN is equilateral, and

$$\lambda = \mu = \nu = \tfrac{1}{3}\pi.$$

It will be found that

$$p^2+q^2 = \tfrac{2}{9}(a^2+b^2+c^2) = \tfrac{8}{9}.\Delta\cot\omega; \quad pq = 4\Delta/3\sqrt{3};$$
$$p/q = (\cot\omega+\sqrt{\cot^2\omega-3})/\sqrt{3} = \cot S_1\sqrt{3};$$
$$q/p = (\cot\omega-\sqrt{\cot^2\omega-3})/\sqrt{3} = \cot S_2\sqrt{3};$$

where S_1, S_2 are the Steiner Angles.

The Counter Point Locus of the Steiner Ellipse is

$$\alpha/a+\beta/b+\gamma/c = 0,$$

which is the Lemoine Axis.

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* The point T may be called the *Sub-Lemoine* Point of the conic.

CHAPTER XII.

PIVOT POINTS. TUCKER CIRCLES.

153. Let DEF be any triangle inscribed in ABC, and let its angles be λ, μ, ν.

The circles AEF, BFD, CDE meet in a point—call it S.

Let def be the pedal triangle of S, U its area, and p its circumradius.

In the circle $SDBF$, angle $SDF = SBF$ or $SBf = Sdf$.

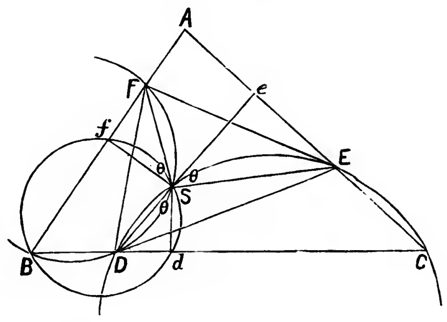
So $$\angle SDE = Sde;$$

$$\therefore \quad d = \lambda, \quad \text{so} \quad e = \mu, \quad f = \nu;$$

and thus the triangle DEF is similar to the pedal triangle of S.

The point S can be found, as in (56) by drawing inner arcs

$$(A+\lambda)\ldots \quad \text{on} \quad BC,\ CA,\ AB.$$



Again, $\angle dSf = \pi - B = DSF$ (circle $SDBF$).

Hence $$\angle dSD = fSF = eSE.$$

Denote each of these angles by θ.

Then $SD = \alpha \sec\theta$, $SE = \beta \sec\theta$, $SF = \gamma \sec\theta$,

where $(\alpha\beta\gamma)$ are the n.c. of S.

Hence S is the double point for any pair of the family of similar triangles DEF, including def, so that it may be fitly named the "Pivot Point" (*Drehpunkt*) of these triangles, which rotate about it, with their vertices on the sides of ABC, changing their size but not their shape.

The linear dimensions of DEF, def are as $\sec\theta : 1$; so that, if M, m are homologous points in these triangles; then

$$MSm = \theta, \quad MS = mS \sec\theta;$$

and the locus of M for different triangles DEF is a line through m perpendicular to Sm.

An important case is that of the centres of the triangles DEF. These lie on a line through σ_0, the centre of the circle def, perpendicular to $S\sigma_0$; and, if σ is the centre of DEF, then $\sigma S \sigma_0 = \theta$.

154. All the elements of DEF may now be determined *absolutely* in terms of $\lambda\mu\nu$ and θ.

For $$a = \frac{abc}{M} \cdot \frac{\sin(A+\lambda)}{\sin\lambda} \text{ \&c.} \tag{65}$$

$$U = 2\Delta^2/M;$$

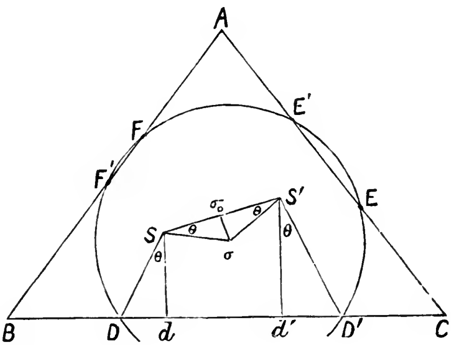
where $$M = a^2 \cot\lambda + b^2 \cot\mu + c^2 \cot\nu + 4\Delta;$$

and $$2p^2 . \sin\lambda \sin\mu \sin\nu = U;$$

so that p is known.

Hence $SD = a \sec\theta$, circumradius of $DEF \equiv \rho = p \sec\theta$,
$EF = 2\rho \sin\lambda$; area of $DEF = U \sec^2\theta = 2\Delta^2/M . \sec^2\theta$.

155. The circle DEF cutting the sides of ABC again in $D'E'F'$, let λ', μ', ν' be the angles of the family of triangles $D'E'F'$, and S' their Pivot Point.



In the triangle $AF'E$,

$$180° - A = FF'E + F'EE' = FDF + F'D'E' = \lambda + \lambda',$$

So $$180° - B = \mu + \mu', \quad 180° - C = \nu + \nu'.$$

It follows that S' is the Counter Point of S, and that the angles of the family $D'E'F'$ are $180°-A-\lambda$, ... (102)

The triangles def, $d'e'f'$ therefore have the same circumcentre σ_0 and the same circumradius ρ_0.

To find σ', the centre of $D'E'F'$, we draw a perpendicular to $S\sigma_0S'$ through σ_0, and take

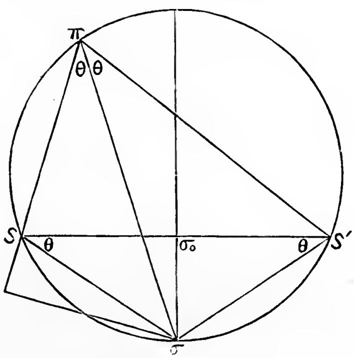
$$\angle \sigma' S' \sigma_0 = \theta' ;$$

where $\theta' = d'S'D'$.

But σ' coincides with σ, either being the centre of the circle $DD'EE'FF'$.

Hence $$\theta' = \theta.$$

156. To prove that the circle $DD'EE'FF'$ touches the conic which is inscribed in ABC, and has S, S' for foci.



Let π be a point where the circle $S\sigma S'$ meets the conic.

Then since $$\text{arc } S\sigma = S'\sigma ;$$

$$\therefore \quad \angle S\pi S' \text{ is bisected by } \pi\sigma.$$

Therefore $\sigma\pi$ is normal to the conic at π.

Now, in the cyclic quadrilateral $\sigma S\pi S'$,

$$\sigma\pi . SS' = S'\sigma . S\pi + S\sigma . S'\pi = S\sigma\,(S\pi + S'\pi) ;$$

$$= S\sigma . 2p : \quad \text{for } 2p = \text{major axis of conic.}$$

But $$SS'/S\sigma = 2 . S\sigma_0/S\sigma = 2\cos\theta ;$$

$$\therefore \quad \sigma\pi = p\sec\theta = \rho.$$

Also σ is the centre of the circle DD'....

Hence this circle touches the conic at π.

157. Triangles circumscribed about ABC.

Through A, B, C draw perpendiculars to $S'A$, $S'B$, $S'C$, forming pqr, the *Antipedal* Triangle of S'.

This triangle (S, S' being Counter Points) is known to be homothetic to def, the pedal triangle of S, and therefore to have angles λ, μ, ν.

Obviously Sp, Sq, Sr are diameters of the circles $BS'Cp$, &c.

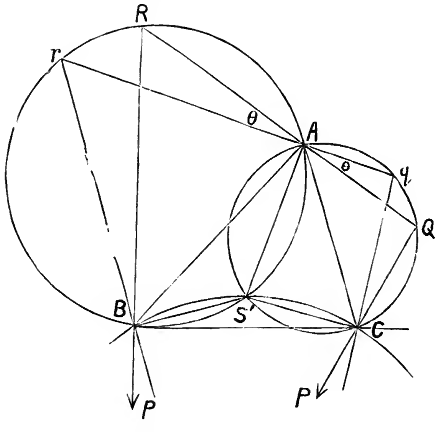
Through A draw QR parallel to EF, and therefore making an angle θ with qr.

Let QC, RB meet at P.

Since $\quad AQC = AqC = \mu, \quad$ and $\quad ARB = \nu$;

$$\therefore \quad BPC = \lambda;$$

so that P lies on the circle $BS'Cp$.



Hence, as the vertices of DEF slide along the sides of ABC, the sides of PQR, homothetic to DEF, rotate about A, B, C, and its vertices slide on fixed circles.

Since $S'q$ is a diameter of $S'qQC$,

$$S'Q = S'q \cos\theta, \ldots.$$

Therefore S' is the double point of the family of triangles PQR, including pqr.

The linear dimensions of the similar triangles PQR, pqr are as $\cos\theta : 1$; so that, if N, n be homologous points in the two triangles, N describes a circle on $S'n$ as diameter.

158. To determine the elements of PQR.

From (84) $\qquad V' = \text{area of } pqr = \frac{1}{2}M;$

so that, if p' be the circumradius of pqr,

$$2p'^2 . \sin\lambda \sin\mu \sin\nu = \tfrac{1}{2}M.$$

Then, for PQR, circumradius $\rho' = p' \cos\theta$.

$$\text{Area of } PQR = V' \cos^2\theta = \tfrac{1}{2}M\cos^2\theta.$$

But $\qquad \text{area of } DEF = 2\Delta^2/M . \sec^2\theta.$

Hence, the area of ABC is a geometric mean between the areas of any triangle DEF inscribed in ABC, and the area of the triangle PQR which is homothetic to DEF, and whose sides pass through A, B, C.

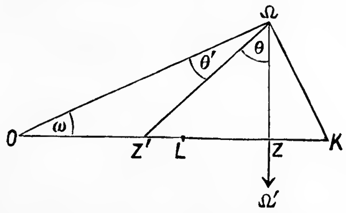
159. Tucker Circles.

An interesting series of circles present themselves, when for "Pivot Points" we take the Brocard Points Ω and Ω'. The circle $DD'EE'FF'$, is then called a *Tucker Circle*, from R. Tucker, who was the first thoroughly to investigate its properties.

Since def, $d'e'f'$ are now the pedal triangles of Ω, Ω';

$$\therefore \quad D = d = B, \quad E = e = C, \quad F = f = A;$$
$$D' = d' = C, \quad E' = e' = A, \quad F' = f' = B.$$

Denote by Z (corresponding to σ_0) the centre of the common pedal circle of Ω, Ω', and by Z' (corresponding to σ) the common circumcentre of DEF, $D'E'F'$.



The line of centres ZZ', bisecting $\Omega\Omega'$ at right angles, falls on OK; also $\qquad Z'\Omega Z = \theta = D\Omega d = E\Omega e = F\Omega f.$

And since $\quad \Omega OZ = \omega = \Omega AF = \Omega BD = \Omega CE,$

it follows that the figures $\Omega OZ'Z$, ΩAFf, ΩBDd, ΩCEe are similar.

Let $O\Omega Z' = \theta' = A\Omega F = B\Omega D = C\Omega E.$

Then $\theta + \theta' + \omega = \frac{1}{2}\pi;$

$$\therefore \quad \theta = \tfrac{1}{2}\pi - \omega - \theta'.$$

Now the radius of the pedal circle of $\Omega\Omega' = p = R \sin \omega$, (138)

$\therefore$ circumradius of $DD' \ldots = \rho = p \sec \theta = R . \sin \omega / \sin (\omega + \theta')$.

The quadrilateral $BD\Omega F$ being cyclic, $BFD = B\Omega D = \theta'$; therefore the arcs DF', FE', ED' subtend each an angle θ' at the circumference, and are therefore equal.

Hence the chord $E'D$ is parallel to AB, $F'E$ to BC, $D'F$ to AC: and a circle with centre Z' and radius $\rho \cos \theta'$ will touch the three equal chords.

In the cyclic quadrilateral $E'EDF$, $AE'F = EDF = B$; so that the equal chords are anti-parallel to the corresponding sides of ABC.

$$\angle DE'D' = F'E'D' - F'E'D = A - \theta'.$$

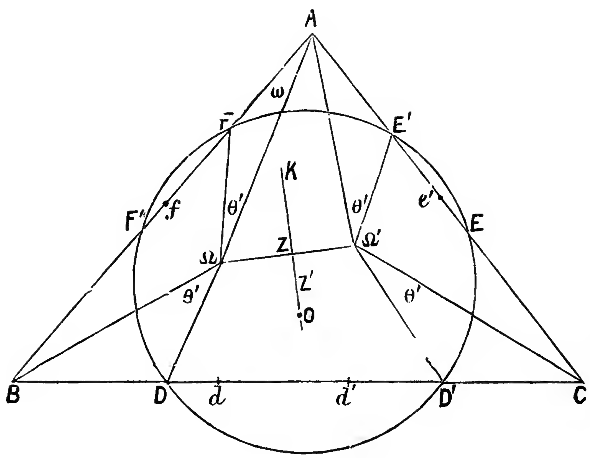
$$F'E'E = F'E'D' + D'E'E = A + \theta'.$$

Chord EF' parallel to $BC = 2\rho \sin F'E'E = 2\rho \sin (A + \theta')$;

chord DD' cut from $BC = 2\rho \sin DE'D' = 2\rho \sin (A - \theta')$.

And if $\alpha\beta\gamma$ be the n.c. of the centre Z',

$$\alpha = \rho \cos \tfrac{1}{2} DZ'D' = \rho \cos DE'D' = \rho \cos (A - \theta').$$



160. The following list of formulæ will be found useful.

(*a*) Radius of circle $DD'\ldots = \rho = R \sin \omega/\sin(\omega+\theta')$.

(*b*) Radius of circle touching equal chords $= \rho' = \rho \cos \theta'$.

(*c*) N.c. of centre Z'; $\alpha = \rho \cos(A-\theta')$.

(*d*) Length of equal anti-parallel chords $= 2\rho \sin \theta'$.

(*e*) Chord DD' cut from $BC = 2\rho \sin(A-\theta')$.

(*f*) Chord EF' parallel to $BC = 2\rho \sin(A+\theta')$.

(*g*) If d and d_1 are points on OK such that

$$O\Omega d = 30^\circ,\quad O\Omega d_1 = 150^\circ;$$

then $$\alpha \propto \cos(A-30) \propto \sin(A+60^\circ),$$

and $$\alpha_1 \propto \cos(A-150) \propto \sin(A-60^\circ).$$

Hence d and d_1 coincide with the Isodynamic Points δ and δ_1.

161. The Radical Axis of the Tucker circle (parameter θ'), and the circle ABC.

If t_1^2, t_2^2, t_3^2 are the powers of A, B, C for the Tucker circle, then the required Radical Axis is $t_1{}^2x + \ldots = 0$. (62)

Now,

$$AF = A\Omega.\frac{\sin\theta'}{\sin(\omega+\theta')} = 2R\sin\omega.\frac{\sin B}{\sin A}.\frac{\sin\theta'}{\sin(\omega+\theta')}.$$

$$= 2\rho.\frac{\sin B}{\sin A}.\sin\theta'.$$

And since EF' is parallel to BC,

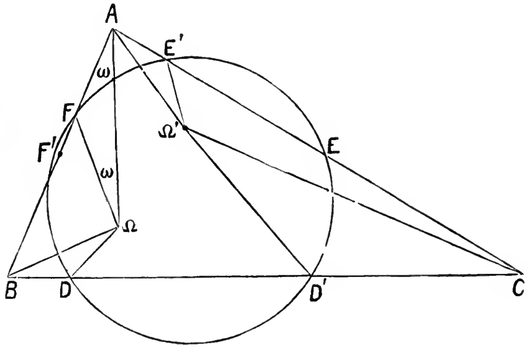
$$\therefore\quad AF' = \frac{\sin C}{\sin A}.EF' = \frac{\sin C}{\sin A}.2\rho\sin(A+\theta');$$

$$\therefore\ t_1^2 = AF.AF' = 4\rho^2.\sin A\sin B\sin C\sin\theta'.\sin(A+\theta')/\sin^3 A,$$

so that the Radical Axis is

$$\sin(A+\theta')/a^3.x + \ldots = 0.$$

162. The properties of four Tucker Circles, whose centres are certain standard points on OK, will now be discussed.



(A) The First Lemoine Circle, or Triplicate Ratio Circle.

This has its centre at L, the mid-point of OK, or the centre of the Brocard Circle.

so that $$\theta' = \omega.$$

Then, (*a*) $\rho = R \sin \omega / \sin 2\omega = \frac{1}{2} R \sec \omega.$

(*b*) $\rho' = \rho \cos \omega = \frac{1}{2} R.$

(*c*) $a = \frac{1}{2} R \sec \omega . \cos (A - \omega).$

(*d*) Anti-parallel chord $= R \sec \omega . \sin \omega = R \tan \omega.$

(*e*) Chord DD' cut from $BC = R \sec \omega \sin (A - \omega).$

So that

$$DD' : EE' : FF' = \sin (A - \omega) : \sin (B - \omega) : \sin (C - \omega)$$
$$= a^3 : b^3 : c^3.$$

Hence the name "Triplicate Ratio Circle."

(*f*) Chord EF' parallel to $BC = R \sec \omega . \sin (A + \omega).$

These chords pass through K. For since chord ED' is anti-parallel to AB, $ED'C = A$,

$\therefore$ perp. from E on $BC = ED' \sin A = R \tan \omega . \sin A$, from (*d*)

$= $ perp. from K on BC. (131)

163. (B) The Pedal Circle of $\Omega\Omega'$.

The centre being Z, $\theta' = \frac{1}{2}\pi - \omega$.

(a) $\rho = R \sin \omega$.

(b) $\rho' = R \sin^2 \omega$.

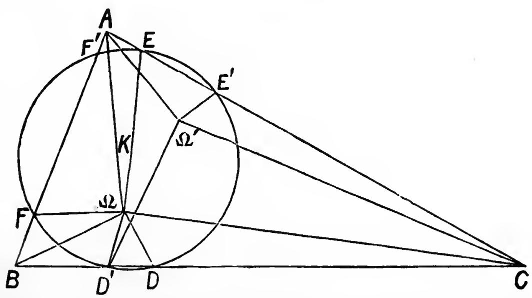
(c) $a = R \sin \omega . \sin (A+\omega) = R \sin^2 \omega . \{b/c + c/b\}$.

(d) Anti-parallel chord $= R \sin 2\omega$.

(e) Chord cut from $BC = 2R \sin \omega \cos (A+\omega)$.

(f) Chord parallel to $BC = 2R \sin \omega \cos (A-\omega)$.

164. (C) The Second Lemoine Circle, or Cosine Circle.



The centre of this circle is K, so that $\theta' = O\Omega K = \frac{1}{2}\pi$.

(a) $\rho = R \tan \omega$.

(b) $\rho' = 0$.

(c) $a = R \tan \omega \sin A$. (131)

(d) Anti-parallel chords $E'F$, $F'D$, $D'E$ each equal the diameter $2R \tan \omega$; so that they each pass through the centre K, as is also obvious from (b).

(e) $DD' = 2R \tan \omega \cos A$;

$\therefore$ $DD' : EE' : FF' = \cos A : \cos B : \cos C$.

Hence the name "Cosine Circle."

(f) Chord parallel to $BC = 2R \tan \omega \cos A$
$= $ chord cut from BC.

165. The Taylor Circle.

Let H_1, H_2, H_3 be the feet of the perpendiculars from A, B, C on the opposite sides.

Draw H_1F perpendicular to AB, H_2D to BC, H_3E to CA.

Let $$A\Omega F = \phi.$$

Then $$AF = AH_1 \sin B = 2R \sin^2 B \sin C$$

and $$A\Omega = 2R \sin\omega \sin B/\sin A; \qquad (131)$$

$$\therefore \quad \frac{\sin(\omega+\phi)}{\sin\phi} = \frac{A\Omega}{AF} = \frac{\sin\omega}{\sin A \sin B \sin C}.$$

Also $$\cot\omega = \frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}. \qquad (131)$$

Hence $$\tan\phi = -\tan A\tan B\tan C.$$

Similarly it may be shown that

$$\tan B\Omega D \text{ or } \tan C\Omega E = -\tan A\tan B\tan C;$$

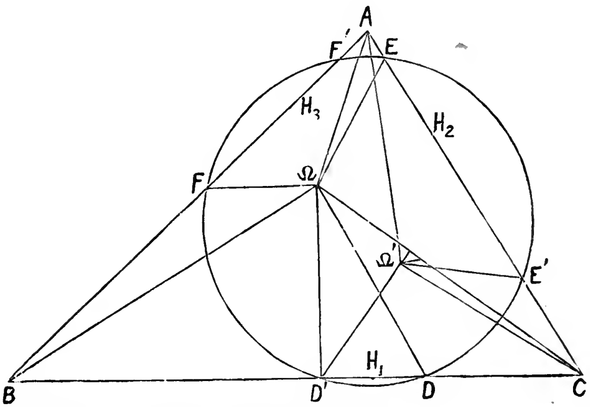
so that $$A\Omega F = B\Omega D = C\Omega E.$$

Next, draw H_1E' perpendicular to CA, H_2F' perpendicular to AB, H_3D' perpendicular to BC.

Then it may be shown that

$$A\Omega'E' = B\Omega'F' = C\Omega'D' = \phi.$$

The six triangles $A\Omega F$, ... being all similar, it follows that $DD'EE'FF'$ lie on a Tucker circle, called the Taylor circle, after Mr. H. M. Taylor.



The angle ϕ is called the Taylor Angle.

Since ϕ is less than π, we have

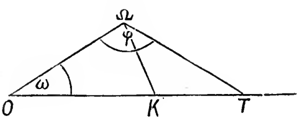
$$D\sin\phi = +\sin A \sin B \sin C, \quad D\cos\phi = -\cos A \cos B \cos C,$$

where $\quad D^2 = \cos^2 A \cos^2 B \cos^2 C + \sin^2 A \sin^2 B \sin^2 C.$

A diagram shows that, T being the centre of this circle,

$$OT : TK = -\tan\phi : \tan\omega = \tan A \tan B \tan C : \tan\omega.$$

Note the equal anti-parallel chords DF', FE', ED'; also the chord $E'D$, parallel to AB, $F'E$ to BC, $D'F$ to AC.



166. The list of formulæ is now—

(a) Radius of T-circle $= R\dfrac{\sin\omega}{\sin(\omega+\phi)} = RD.$

(b) $\rho' = RD\cos\phi = R\cos A \cos B \cos C.$

(c) $a = RD\cos(A-\phi)$
$\quad = R(\cos^2 A \cos B \cos C - \sin^2 A \sin B \sin C).$

(d) Anti-parallel chord $E'F$ or $F'D$ or $D'E$
$\quad = 2RD\sin\phi = 2R\sin A \sin B \sin C.$

(e) Chord cut from $BC = 2RD\sin(A-\phi)$
$\quad = R\sin 2A\cos(B-C).$

(f) Chord $F'E$ parallel to $BC = R\sin 2A\cos A$;
the other chords being $D'F$ and $E'D$.

To determine the Radical Axis of the circle ABC and the Taylor Circle.

$$\sin(A+\phi) = \sin A.\cos\phi + \cos A\sin\phi$$
$$= D(-\sin A.\cos A\cos B\cos C + \cos A.\sin A\sin B\sin C)$$
$$\propto \sin A\cos^2 A.$$

Hence, from (61), the Radical Axis is

$$\cot^2 A.x + \ldots = 0.$$

So that the tripolar coordinates of the Limiting Points of these two circles are as $\cot A : \cot B : \cot C$. (21)

APPENDIX I.

Let $LMN\ldots$, $L'M'N'\ldots$ be two systems of n points.

Place equal masses p, p at L, L'; q, q at M, M', &c.

To determine the condition that the two systems shall have the same mass-centre.

Project $LMN\ldots$ and the mass-centre on any axis, and let $lmn\ldots\bar{x}$ be the distances of these projections from a given point O on the axis.

Then $$(p+q+r+\ldots)\bar{x} = pl+qm+rn+\ldots$$
$$(p+q+r+\ldots)\bar{x} = pl'+qm'+rn'+\ldots.$$
$$\therefore\quad p(l-l')+q(m-m')+r(n-n')+\ldots = 0.$$

and so for any number of axes.

But $l-l'$, &c., are the projections of LL', MM', $NN'\ldots$.

Therefore the required condition is that a closed polygon may be formed, whose sides are parallel and proportional to $p.LL'$, &c.

In the case of a triangle

$$p.LL' \propto \sin(MM', NN').$$

Now, in the case under discussion, take a second point P' on TT', and let its pedal triangle be $d'e'f'$.

Here $$LL' = dd' = PP'.\cos\theta_1,$$

and angle (MM', NN') is (ee', ff') or A;

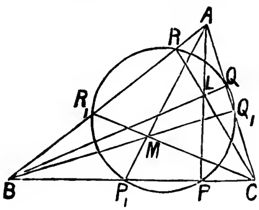
$$\therefore\quad p\cos\theta_1 \propto \sin A, \quad \text{or} \quad p \propto \sin A \sec\theta_1.$$

Hence all pedal triangles def of points P on TT' have the same mass-centre for the constant masses $\sin A\sec\theta_1$, &c., placed at the angular points d, e, f.

APPENDIX II.

To determine the *second* points in which the four circles cut the Nine-point Circle.

Let the circle PQR cut the sides of ABC again in P_1, Q_1, R_1.



Then $$AQ.AQ_1 = AR.AR_1, \text{ \&c.};$$

$$\therefore\ AQ.AQ_1.BR.BR_1.CP.CP_1 = AR.AR_1.BP.BP_1.CQ.CQ_1.$$

But, by Ceva's Theorem, since AP. BQ, CR are concurrent,

$$BP.CQ.AR = CP.AQ.BR;$$

$$\therefore\quad CP_1.AQ_1.BR_1 = BP_1.CQ_1.AR_1.$$

Therefore AP_1, BQ_1, CR_1 are concurrent.

Again, since $$CQ : QA = r : p,$$

and $$AR : RB = p : q,$$

$$\therefore\quad AQ = p/(r+p).b; \quad AR = p/(p+q).c.$$

So $$AQ_1 = p_1/(r_1+p_1).b; \quad AR_1 = p_1/(p_1+q_1).c.$$

But $$AQ.AQ_1 = AR.AR_1, \text{ \&c.};$$

$$\therefore\quad \frac{pp_1.b^2}{(r+p)(r_1+p_1)} = \frac{pp_1.c^2}{(p+q)(p_1+q_1)};$$

$$\therefore\quad \frac{a^2}{(q+r)(q_1+r_1)} = \frac{b^2}{(r+p)(r_1+p_1)} = \frac{c^2}{(p+q)(p_1+q_1)};$$

$$\therefore \quad p_1 \propto -\frac{a^2}{q+r}+\frac{b^2}{r+p}+\frac{c^2}{p+q}.$$

So $$q_1 \propto +-+\,; \quad r_1 \propto ++-\,;$$

$$\therefore \quad q_1+r_1 \propto 2.a^2/(q+r) : q_1-r_1 \propto 2b^2/(r+p)-2c^2/(p+q).$$

In (80) it was shown that the circle PQR cuts the Nine-Point Circle at a point ω, whose b.c. are as $a^2/(q^2-r^2)$, ...

Similarly the circle $P_1Q_1R_1$ (the *same* circle) cuts the Nine-Point Circle at a point ω', where the b.c. of ω' are given by

$$x \propto \frac{a^2}{(q_1+r_1)(q_1-r_1)} \propto \frac{1}{b^2(p+q)-c^2(p+r)}, \text{ \&c.}$$

So, if the circle $PQ'R'$ cuts the Nine-Point Circle again at ω_1, the b.c. of ω_1 are given by

$$x_1 \propto \frac{1}{b^2(-p+q)-c^2(-p+r)}, \text{ \&c.,}$$

writing $-p$ for $+p$.

APPENDIX III.

(*a*) To determine the area of XYZ.

Since

$$A'O = R\cos A,\quad A'X = \tfrac{1}{2}a\tan\theta = R\sin A\tan\theta.$$

$$\therefore\quad OX = R/\cos\theta.\cos(A+\theta).$$

$$\therefore\quad 2.\text{area } YOZ = OY.OZ.\sin A.$$

$$= R^2/\cos^2\theta.\cos(B+\theta)\cos(C+\theta)\sin A.$$

$$\therefore\quad 2.\Delta XYZ = 2(YOZ+ZOX+XOY) = \dots;$$

and by some easy reduction we obtain,

$$\Delta XYZ = \Delta/4\cos^2\theta.\{2\sin\omega - \sin(2\theta+\omega)\}/\sin\omega.$$

When θ is equal to either Steiner Angle, (137)

then $$\sin(2\theta+\omega) = 2\sin\omega,$$

and the triangle XYZ vanishes, so that XYZ is a straight line.

But this triangle always has G for its centroid.

Hence, in this particular case, XYZ passes through G.

(*b*) Instead of the base angles being equal, suppose that

$$\angle XBC = YCA = ZAB = \theta,$$

$$BCX = CAY = ABZ = \phi,$$

and $$BXC = CYA = AZB = \chi.$$

Then $$a_1 = a.\sin\theta\sin\phi/\sin\chi,$$

$$a_2 = b\sin\phi.\sin(C-\theta)/\sin\chi,$$

$$a_3 = c\sin\theta.\sin(B-\phi)/\sin\chi;$$

$$\therefore\quad 3\bar{a} = a_1+a_2+a_3 = h_1,\quad \&c.$$

Hence G is the centroid of XYZ.

(c) Let YZ, ZX, XY meet BC, CA, AB in x, y, z respectively.

Then, since AX, BY, CZ are concurrent, xyz is a straight line, being the axis of perspective of the triangles ABC, XYZ.

To show that the envelope of xyz is Kiepert's Parabola.

The equation of yz is

$$(\beta_2\gamma_3-\beta_3\gamma_2)\,\alpha+(\gamma_2\alpha_3-\gamma_3\alpha_2)\,\beta+(\alpha_2\beta_3-\alpha_3\beta_2)\,\gamma = 0.$$

Now $\gamma_2\alpha_3-\gamma_3\alpha_2 \propto \sin(A-\theta)\sin(B-\theta)-\sin\theta\sin C$;

$$\propto \sin A\sin B-\sin C\sin 2\theta.$$

So $\alpha_2\beta_3-\alpha_3\beta_2 \propto \sin C\sin A-\sin B\sin 2\theta.$

Therefore at x we have

$$\beta/(\sin C\sin A-\sin B\sin 2\theta)+\gamma/(\sin A\sin B-\sin C\sin 2\theta) = 0,$$

so that xyz is

$$a\alpha/(\sin A\sin B\sin C-\sin^2 A\sin 2\theta)+\ldots = 0.$$

Writing this as $px+qy+rz = 0$, we know, from (9), that this line touches the parabola, the n.c. of whose focus are $a/(1/q-1/r)$, &c.

Here $1/q-1/r \propto (\sin^2 B-\sin^2 C)\sin 2\theta.$

Hence the focus has n.c. $a/(b^2-c^2)$ &c., and the directrix is

$$(b^2-c^2)\cos A-\alpha+\ldots = 0.$$

Hence the envelope of xyz is Kiepert's Parabola, having for focus the point whose Simson Line is parallel to OGH, and OGH for directrix.

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London: Printed by C. F. Hodgson & Son, 2 Newton Street, Kingsway, W.C.