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ATTENTION.[1]

AFFECTIVE CONFLICT AND UNITY OF CONSCIOUSNESS.

ALTHOUGH attention may boast of possessing more abundant literature than any other psychical phenomenon, yet it is still far from being fully explained; that is to say, it has not been brought to any extent into relation and association with other psychic phenomena, especially with those to which it is most closely related. And although attention, as Titchener rightly emphasizes, forms the very pivot upon which all psychology hinges, yet to-day the question as to its inmost nature is still very far from solution. What a great loss this branch of science suffers thereby it is easy to conceive.

The cause of this delinquency in the scientific explanation of attention holds true also for all other psychic activities, namely, that the investigation of all these phenomena has been begun at just the point where they are the most complex and intricate instead of beginning with the simplest forms. The question of attention has usually been taken up by means of self-contemplation and at the moment of philosophical reflection, instead of by observing, for instance, the beast of prey, impatient to fall upon the quarry he has espied and for which he has long lain in wait, or the child who would fain put a white pellet in his mouth but is in doubt whether it is a piece of candy as usual, or may turn out to be a bitter pill as was yesterday the case.

[1] Translated from the German which is to appear in the *Archiv für Psychologie.*

The expediency of beginning the investigation with the simplest forms involves the expediency of pursuing the phylogenetic method and following the course of evolution back as far as possible in order to reveal the phenomenon in the very moment of its first appearance. This is the course we pursued when investigating the inmost nature of another psychic phenomenon no less important and fundamental, namely that of affective tendencies, and the phylogenetic research which showed us their mnemonic origin and nature at once threw light upon that class of phenomena previously so obscure.[2]

We believe that this procedure will attain the same success in our study of attention, which however as we shall see is only a secondary phenomenon directly derived from affective tendencies.

In the treatise just mentioned, "On the Mnemonic Origin and Nature of Affective Tendencies," we have seen that these tendencies are originally only expressions of one and the same intrinsic tendency of the organism to preserve or restore the state of its physiological equilibrium, or to reestablish a previous physiological state which had been determined in the past by certain environmental relations. As soon as these relations are even partially repeated they bring about the "discharge" of the mnemonic accumulation which this former physiological system had left behind.

Then from these affective tendencies of direct mnemonic origin which strive to reestablish certain environmental relations as a whole, arise, according to the known law of affective transference of the whole to the

[2] E. Rignano, "Dell' origine e natura mnemonica delle tendenze affettive," *Scientia,* No. XVII, 1, 1911; "Ueber die mnemonische Entstehung und die mnemonische Natur affektiver Neigungen," *Archiv für die gesamte Psychologie,* Vol. XX, No. 1, 1911; "On the Mnemonic Origin and Nature of Affective Tendencies," *Monist,* July, 1911. This treatise later appeared also as Appendix to the English edition of the author's work, *On the Inheritance of Acquired Characters; An Hypothesis of Heredity, Development and Assimilation.* Chicago, The Open Court Publishing Company, 1911.

part, all the other affective tendencies of indirect mnemonic origin which strive to reestablish only very definite parts or details of these environmental relations. Besides the most important environmental relations usually striven for eagerly in their original totality, the higher animals, and especially mankind, always possess a large number of secondary and even quite specific, environmental relations which in this way are capable of becoming in their turn objects of desire.

At this point we must emphasize the fact that when a physiological system has been disturbed by altered environmental conditions and reduced to a potential state in the form of a mnemonic accumulation, it can become fully reactivated and continue active in a stable physiological state only when its internal and external relations are entirely and exactly the same as when they induced this physiological state. Thus the physiological system of an infusorian which has previously lived in a certain temperature or in a salt solution of a certain proportion will generate an affective tendency toward return to its former habitat as soon as it is removed to other environmental relations; and this tendency will be expressed by negative reactions to every other change of its environmental relations which tends to remove it still further from its original habitat, and by positive reactions to every change which brings it nearer to its former habitat (Jennings). But the original physiological state can not be perfectly reestablished and made to persist in regular activity until the little animal by its own movements has succeeded in getting again into an environment identical with the former one.

Likewise the diminution of histogenetic substance in the blood which prevents the continuance of the metabolic state hitherto active and stable, will provoke the affective tendency of hunger and all the acts of seeking and absorbing nourishment proceeding therefrom; but the normal metabolic state can not be completely reestablished until hunger is allayed; that is to say, until the acts carried on for the purpose of seeking and absorbing nourishment and the processes of digestion have endowed the blood with the same intrinsic quality, hence the same proportion of histologic substance, as formerly.

As with all mnemonic evocations in general, a small part of a certain former complex environmental state is sufficient, if not to "satisfy" the associated affective tendency, at least to "discharge" it. That is why the sensations in so far as they represent parts of environmental conditions, become in a very special manner the "dischargers" of affective tendencies. But in this respect there is an essential difference between the "non-distance receptors" and the "distance-receptors" which Sherrington rightly emphasizes, so that a very significant phylogenetic advance was made when the latter gradually developed from the former. For the non-distance receptors (senses with direct contact) usually permit the immediate or almost immediate satisfaction of the affective tendencies which they "discharge." Frequently the sensation discharging a certain affective tendency is identical with its satisfaction. On the other hand the "distance-receptors" usually produce that particular state in which an affective tendency is discharged and held in suspense, and which we are now ready to investigate.

"Between touch and assimilation," says Spencer, "there exists in the lowest creature an intimate connection. In many Rhizopods the tactual surface and the absorbing surface are coextensive. The ameba, a speck of jelly having no constant form, sends out in this or that direction prolongations of its substance. One of these meeting with and attaching itself to some relatively fixed object, becomes a temporary limb by which the body of the creature is drawn forward; but if this prolongation meets with some relatively small portion of organic matter it slowly expands its extremity around this, slowly contracts, and slowly draws the nutritive morsel into the mass of the body, which collapses around it and presently dissolves it. That is to say, the same portion of tissue is at once arm, hand, mouth, and intestine—shows us the tactual and absorbent function united in one."[3]

Sherrington in his turn says: "Animal behavior shows clearly that in regard to these two groups of receptors the one subserves differention of reaction, i. e., swallowing or rejection, of material already found and acquired, e. g., within the mouth. The other, the distance-receptor, smell, initiates and subserves far-reaching complex reactions of the animal anticipatory to swallowing, namely, all that train of reaction which may be comprehensively termed the quest for food. The latter foreruns and leads up to the former. This precurrent relation of the reaction of the distance-receptor to the non-distance receptor" (as well as the 'conative feeling' which the distance-receptor induces) "are typical."[4]

Accordingly non-distance-receptors occasion no "suspended" affective tendencies, no "conative feeling," but instead they bring about the immediate satisfaction of affective tendencies at the moment they are discharged, or the immediate accomplishment of the acts contributing to their satisfaction ("final or consummatory reactions," as Sherrington expresses it). Distance-receptors, on the other hand, discharge the affective tendency involved and keep it active during the entire time of expectation and during the whole series of acts required of the animal before it can carry out the last consummatory act which is to satisfy this affective tendency. Therefore in general

[3] Herbert Spencer, *The Principles of Psychology*, 4th ed., Vol. I, p.307. London, Williams and Norgate.

[4] C. S. Sherrington, *The Integrative Action of the Nervous System*, page 326 f. London, Constable, 1906.

only the distance-receptors but not the non-distance-receptors can bring about a more or less lasting condition of unfulfilled desire: "If all motive impulses could be at once followed up desire would have no place."[5]

Now the question arises how can we explain the fact that the affective tendencies discharged or evoked by the distance-receptors, nevertheless remain "suspended"; in other words, how is it that although they have been evoked and persist in this state, yet for a long time they occasion no actual performance of any of those consummatory acts which to be sure would not now have any result but to which they nevertheless impel, as is shown by the incipient performance of these acts? The beast of prey, for instance, whose appetite is aroused from afar by the scent and sight of his victim coming towards him without presentiment of danger and is whetted constantly more and more, nevertheless does not bound at once toward the longed-for victim, but waits motionless and trembling with all the muscles tense, until the poor victim has come within springing distance. What then prevents the affective tendency so evoked from being at once completely discharged in the consummatory act of springing upon the prey and tearing it to pieces?

This can only be the counteraction of a conflicting tendency by which the first tendency is prevented from accomplishing its consummatory act. And the conflicting tendency in this case can be only the combined result of all consummatory acts which were actually performed in the past at the first awakening of the affective tendency, but every time without result. Accordingly we may make the assertion that it was the "deception" at each premature activation of the affective tendency called forth by the dis-

[5] A. Bain, *The Emotions and the Will*, 4th ed., p. 423. London, Longmans Green, 1899.

tance-receptor, which called into being the opposite tendency by which the other is now held in suspense.

A familiar instance is Möbius's experiment with the pike. By means of a pane of glass he divided a large glass bowl full of water into two parts. In one side he placed the pike and in the other he put tiny whitings which provide the pike's customary food. It now happened that whenever the pike dived after one of the small fishes he fell against the obstructing pane of glass. For a week he continued to make these vain attempts. Then he gave up entirely the pursuit of his unattainable prey and did not change his behavior even when the obstructing pane of glass had been taken away.

Now the constantly repeated deceptions which resulted when the affective tendency released by a distance-receptor produced immediately the performance of a consummatory act which was necessarily unsuccessful, must have a very similar effect on all animals provided with these senses. And so it has come to pass that the very discharge effected by the distance-receptors of any affective tendency and the premature beginning of the movement connected with it, now, thanks to the memory of former unsuccessful attempts, provoke the antagonistic tendency, like that which prevented the pike from falling upon its prey. And this conflict produces that state of an affective tendency "held in suspense" which constitutes the state of attention.

Accordingly we may say that phylogenetically attention originated with the distance-receptors, and that it consists in the conflict of two affective tendencies, the second of which is "discharged" by the first, prevents it for a time from complete activation and hence keeps it "in suspense."

The state of attention therefore does not consist of a single affective state but of the conflict of tendencies arising from the coexistence of two affective states. It is because this fact has been overlooked that it has not been possible heretofore to understand in what the specific nature of this state of attention really consists, and so to understand the real significance of the holding of an affective tendency "in suspense" which is characteristic of attention, nor to understand why all those movements which the first of the two affective tendencies would itself have provoked at once, are arrested "in the nascent state," whereas had this affectivity alone been active they would have proceeded directly to completion.

But aside from the case just considered of a premature performance of the consummatory act involved, the distance-receptors under many other circumstances arouse a second affectivity in conflict with the first which for some time prevents the complete activation of the former, as a consequence of the unexpected, unpleasant results which had some time previously been associated with it. However and whenever such an affective conflict occurs there at once arises also a corresponding state of attention; and *vice versa,* there is no state of attention without such a conflict of tendencies. For we need only consider carefully a few of the most significant cases, selected so as to be as different as possible from one another, in order at once to see in operation this conflict of tendencies in every state of attention.

"A young chick two days old, for example," says Lloyd Morgan, "had learned to pick out pieces of yolk from others of white of egg. I cut little bits of orange-peel of the same sizes as the pieces of yolk and one of these was soon seized but at once relinquished, the chick shaking its head. Seizing another he held it for a moment in the bill but then dropped it and scratched at the base of his beak. That was enough. He could not again be induced to seize a piece of orange-peel. The obnoxious material was now removed and pieces of yolk of egg substituted but they were left untouched, being probably taken for orange-peel. Subsequently he looked at the yolk with hesitation, but presently pecked doubtfully, not seizing but merely touching. Then he pecked again, seized, and swallowed it."[6]

Accordingly we see here how the first act of attention of the newly hatched chicken arose from the conflict between its first tendency to seize the yolk of the egg and the conflicting tendency aroused by the memory of the unpleasant experience produced by picking up the orange-peel. The "effective guidance and control of consciousness," of which Lloyd Morgan speaks as one factor which influenced the instinctive pecking of the chicken, was thus only the arousing of a new affectivity, repugnance, that inhibited the first affectivity, hunger, which of itself impelled toward the completion of the instinctive act.[7]

A little girl is taken out walking by a servant. The child unexpectedly catches a glimpse of her mother on the other side of the street and wishes to run over to her at once. But the maid warns her with a cry, "Look out for the carriage!" and the little one stops. The carriage has hardly passed and she has almost taken a step ahead when another approaching vehicle forces her to give way again. The conflict of the two tendencies of hope and fear, kept alive in the child by the sight of her mother and the repeated passing of vehicles, is shown very clearly by the direction of her steps first forward and then backward. It is faithfully reflected in the expression of the small bright eyes which shine with anticipation and joy as soon as they are turned upon her mother and the child takes a step nearer to her, but at once look anxious and confused when they observe one of the heavy wagons of which there seems to be no end. Finally, however, the street-crossing is unobstructed. The state of fear and also the "state of attention," has entirely disappeared so that the

[6] Lloyd Morgan, *Habit and Instinct,* p. 40 f. New York, Arnold, 1896.
[7] Lloyd Morgan, *op. cit.,* pp. 129-131, 135, 139 f.

little girl may at last satisfy her wish and throw herself into her mother's arms.

The conflict of tendencies is likewise exhibited with great distinctness in certain typical states of attention where it is expressed in the exceedingly subtle choices between almost imperceptible modalities of a certain act.

A billiard player, for instance, who has already directed his cue at the ball, wishes first of all to make a successful stroke. He is ready to make the stroke but the extreme tension of the muscles in his arm causes him to fear that the stroke may turn out to be too strong, as it did shortly before. In consequence of this conflicting affectivity his muscles become somewhat lax. Nevertheless the weaker tension he now feels reawakens in him the memory of an earlier unsuccessful stroke when the movement of the ball had not been swift enough, and now he finds himself perplexed by the opposite fear lest the stroke may be too weak. By the swings of his arm, now longer and now shorter, which precede the stroke and bring the point of the cue nearer to the ball or farther from it, a spectator can discern the rapid alternation of conflicting affectivities which discharge each other and exaggerate or moderate each other in order finally to bring about the result of giving to the ball exactly the necessary force.

The same is true when a person who is writing attempts to remove with his finger a tiny hair from his steel pen. This rarely succeeds at the first attempt because the fear of soiling his finger-tips causes him to press them together before they are near enough to the point of the pen and the hair. The first failure gives rise to care lest the second attempt may also fail, and this opposite fear partly suppresses and moderates the fear of soiling the fingers, so that the wish to remove the hair by this time lends to the arm and fingers exactly the degree of muscular contraction necessary to get hold of the extending end of the hair without touching the inky pen.

From this conflict of tendencies, inevitably occurring as soon as we attempt to perform an act "carefully," arises the well-known fact that attention, when directed to actions which by long practice have become mechanical, makes their execution less rapid and perfect than if they had taken place quite automatically.

"An automatic connection of contents or movements has nothing to gain from the intervention of attention,—nay suffers a very positive loss in accuracy and rapidity of realization, if the attention be directed upon it."[8]

Thus the recitation of a poem which has been learned so well by heart that it can be repeated mechanically becomes uncertain and hesitating when the speaker gives it his whole attention. And a person who writes his name with the greatest facility when he gives no thought to it is pretty sure to do it disconnectedly and without ease as soon as some one asks him for his autograph. For in this case every stroke of the pen needs a short preparation and requires a certain application of the will to begin and complete it, whereas the transference from one stroke to another becomes studied and awkward instead of easy and running as usual.[9]

Nevertheless there are individual cases, even where the attention is greatly aroused, in which the conflict of tendencies appears less distinct. For instance in Sardou's drama, "Tosca," we have the scene where Tosca's lover is tortured. It arouses the keenest sympathy and attention of all the spectators. Where is there any conflict of tendencies in this case? And yet we shall find it if we reflect a little. On the one hand there is the tendency, according

[8] O. Külpe, "The Problem of Attention," *Monist,* XIII, p. 61. Chicago, Oct. 1902.

[9] H. Maudsley, *The Physiology of Mind,* p. 520 f. London, Macmillan, 1876.—*The Pathology of Mind,* p. 143. London, Macmillan, 1895.

to the character of the spectator, either to fall upon the crafty Scarpia and slay him, or to throw oneself at his feet and with Tosca beg his mercy for her lover; or one might hasten to the aid of the unfortunate man and liberate him after driving away or killing the agents of the torturer. On the other hand the cultured man has acquired a tendency by education or custom to do nothing which conventionality does not permit, and not to make himself ridiculous by acts which would be the more ridiculous since every one knows that he is not beholding a reality but a mere invention. And that this is really the case is proved by the village theaters where the actor who plays the part of the tyrant is often hissed by the public, and sometimes even becomes the target of more or less harmless missiles thrown by the more unsophisticated spectators. The author once attended such a spectacle. Some conspirators were in hiding behind a curtain, waiting to kill the king, who by this time had won the favor of the public by his generosity and fearlessness. He had hardly appeared when a voice was heard to call out at the first movement of the curtain, "Look out, they are going to kill you!" The entire audience laughed uproariously, and the simple spectator was overcome with confusion. He will doubtless succeed another time in repressing his magnanimous outburst, thanks to the conflicting tendency not to make himself again the object of derision.

Attention which is aroused by novelty is likewise the result of a conflict of tendencies arising from the fact that just because the object is new, it has not yet been "affectively classified," and therefore arouses both hope and fear at the same time.

If the space at our disposal permitted, we could easily show that any "classification" whatever is based either directly or indirectly upon an affective tendency. The principle upon which it rests consists originally in the fact that no sensation or perception of the distance-receptor has any value for the organism except as a symbol of a possible environmental state, near or remote, to be striven after or avoided. As long as this symbol has not been placed in either category, the conflicting affectivities of hope and fear oppose each other and hold each other in suspense. This opposition is seen distinctly, for instance, in a child who is undecided whether or not he should drink the tea offered him by his mother and which this time has an unusual color, because he is not sure whether it is a sweet or bitter draught; or in a beast of prey that sees a strange looking animal and is in doubt whether it is a dangerous enemy or perhaps a suitable quarry and therefore makes its muscles tense, ready at the same time for either attack or flight.

Curiosity is only one of the least forms of this conflict of tendencies or of this particular state of attention produced by novelty. "The craving for knowledge in its instinctive form is called curiosity. It exists in all degrees, from that of the animal which touches or smells an unknown object, to the all-examining, all-embracing scrutiny of a Goethe." "Curiosity consists of two questions expressed or implied: What is it? What use is it?....The dog brought face to face with an unknown object, looks at it, smells it, approaches, withdraws, ventures to touch it, returns, and begins again; he is pursuing this investigation after his own fashion; he is solving a double problem of nature and utility."[10]

On the other hand the "not new"—and this also may be any specific object when we see it for the first time—comprises everything we know how to classify in one of our various affective categories. It either brings about immediately the evocation and satisfaction of the affectivity con-

[10] Th. Ribot, *Psychologie des sentiments,* pp. 369, 371. Paris: Alcan, 1906. Second English edition, pp. 368, 370. London, Walter Scott, 1911.

cerned, like the little waterfall in the mountain which awakens the desire to drink from it; or it evokes the affective tendency but holds it in suspense for fear lest its immediate complete satisfaction might involve some evil consequences as we have previously seen; or finally it may at that moment be altogether unable to evoke any tendency, like the sight or odor of a familiar dish when we have had enough. In this case the affective activity is reduced to a minimum, the state of attention entirely ceases, and we experience *monotony* or *tedium*. If this state of minimum affective activity is reduced to zero, we have the condition of *sleep*. "Sleep," as Bergson very truly says, "means to disinterest oneself (*se désintéresser*). We sleep in direct proportion to our disinterestedness."[11]

Finally there is only a very slight distinction between "curiosity" and the state of attention of the investigator. The investigator observes a certain object or a certain phenomenon in order to convince himself whether this object or this phenomenon really proves to possess certain properties whose presence has been asserted by others, or which he himself thought he noticed at the first glance, or which in his opinion should exist. The presence or absence of these properties is of exceedingly great value to the observer as is apparent from the fact that he applies himself with such great care to observe them, for they may for instance confirm certain preconceived theories or represent a highly important scientific discovery. Hence on the one hand he cherishes the ardent hope that the supposed properties would really be found to exist. On the other hand he is restrained from prematurely making known a discovery whose accuracy might later be contested by other inquirers to the great injury of his own scientific prestige. Just think for instance with what great

[11] H. Bergson, "Le rêve," *Bulletin de l'Institut Psychologique International*, p. 118. Paris, Alcan, May 1900.

attention—that is to say, with what great care lest he may have been a victim of an optical illusion—Schiaparelli must have carried on his ovservations before he decided to make known his discovery of the canals of Mars. Here too this hope and this care furnish the conflict of two affectivities without which here as elsewhere no actual state of attention would or could be present.

As we have by this time come to recognize the inmost nature of the affective conflict which, as appears from the few examples here adduced, is characteristic of every state of attention, so all other properties which always accompany this state prove at the same time to be so many simple and direct consequences of its nature.

Especially are we able to perceive at once the unconvincing character of Ribot's definition of attention as the state of "relative monoideism." We might if necessary call it a state of "monoaffectivity held in suspense," but as we have seen, it is still better to define it as a state of "double conflicting affectivity."[12]

Ribot's motor or peripheral theory proves to be equally erroneous: "Are the movements of the face, the body and the limbs, and the respiratory modifications that accompany attention, simply effects, outward marks as is usually supposed? Or are they, on the contrary, the necessary conditions, the consistent elements, the indispensable factors of attention? Without hesitation we accept the second thesis."[13]

On the other hand the so-called theories of "central origin" seem to be perfectly correct.[14] Attention is indeed a "central," psychological phenomenon; for the awakening of the primary or active affectivity and the counter-awak-

[12] See Th. Ribot, *Psychologie de l'attention,* pp. 6-8, 6th edition. Paris Alcan, 1902. English edition, p. 10.

[13] Ribot, *op. cit.,* p. 32. English edition, p. 25.

[14] See, e. g., J. Sully, "The Psycho-Physical Process in Attention," *Brain,* July 1890, especially pp. 155-157. London, Macmillan.—Vaschide and Meunier, *La Psychologie de l'attention,* pp. 196 f. Paris, Blond ,1910.

ening of the secondary affectivity which holds the other in suspense, are phenomena of this nature. Attention therefore is first of all an essentially affective phenomenon and only indirectly and in a subordinate manner does it become a motor phenomenon by the fact that the awakening of any affectivity whatever always produces motor and peripheral phenomena which are therefore only accompanying or derived phenomena.

Ribot's error comes from the fact that he has not succeeded in correctly comprehending the nature of affective tendencies, for he sees very well that "attention always depends upon affective states," but he adds soon after: "How are we to represent to ourselves these tendencies? The only positive idea that we can get of them is to consider them as movements (or as inhibitions of movements), be they real or nascent."[15]

Accordingly for this inquirer the motor elements would by themselves constitute the entire essence of affective tendencies. But it is the affective tendencies which are the foundation of the motor elements, and the reverse is false.

As we have seen in our frequently cited treatise "On the Mnemonic Origin and Nature of Affective Tendencies," an affective tendency is only a gravitation, so to speak, toward that environment or those environmental relations which permit the reactivation of the mnemonic accumulation constituting this affective tendency. But of itself, it does not produce any preferential impulse toward one rather than toward another series of movements. For even if these movements were such as could eventually bring the organism back into the desired environmental conditions, yet in themselves they have nothing to do with the ultimate satisfaction of this affective tendency. It is only when one series of movements succeeds in bringing the organism back to the requisite environmental condi-

[15] Ribot, *Psychology of Attention*, pp. 166, 172. English edition, pp. 112, 116.

tions sooner or better than the others and only from this moment, that it becomes preferred to the others. Only from this moment will the awakening of the affective tendency give rise to definite motor elements.

But before this occurs, that is to say before the affective tendency has found preferable any one of the movements capable of leading to the desired end, the affective tendency towards that end will already exist. The very fact of this affective choice proves that in point of time the choosing factor precedes the element chosen, whence it follows that there can be an affective tendency even in the absence of any motor element. For instance a new and unusual indisposition which may attack us arouses the affective tendency to be freed from it, but this does not and cannot initiate any motion whatever.

Hence if affective tendencies and motor elements are two different things, and if the latter are based upon the former but not the reverse, then this is also true with regard to attention for which the motor elements are not an indispensable condition but merely quite secondary phenomena.

Since every conflict of affective tendencies is expressed in a conflict of the motor elements induced by them, so a clear explanation is afforded even with the "central origin" for the fact that "muscular tension," "motor innervation," "tonic contraction," and the "elevation of the entire psychic life," characterize every state of attention, as all have observed.[16]

Affective choice determines not only the particular movements of locomotion, of seizing, etc., which make for the desired object, but also the adjustment of the sense-organs, itself a musculo-motor phenomenon on which depends the more or less successful result of the movements,

[16] Maudsley, *The Physiology of Mind*, p. 313.—Ch. Féré, "Physiologie de l'attention," *Revue philosophique*, Oct. 1800, pp. 401, 404.—K. B. R. Aars, "Notes sur l'attention," *Année psychologique*, VIII, p. 216. Paris, Schleicher, 1902.

of whatever kind they are, and in which therefore both of the two conflicting affectivities cooperate. Now for instance when we are surprised by a sudden noise and direct our glance at once to the distant object from which it seems to come, the state of attention is alert during the whole interval preceding the moment in which the eyes have become adjusted to the new distance, which requires a certain length of time when the object is far away. Thus attention is awakened (here too in conformity with the theory of central origin) before and not after the adjustment of the organ concerned.[17]

Since on the other hand the peripheral sensory relations remain the same, the attention may be directed now to some and now to other sense-perceptions, just as when, confined within our room, we give more heed to certain noises in the street than to others which come from the same direction; for instance, to the hoof-beat of the horses belonging to an equipage that stops before our door, in order to determine by the sound which of our friends has come to call; or to the roll of the wheels in order to find out whether the friend who has come to take us out driving is riding in a closed or open carriage. Attention may even be directed to certain properties of a sense-impression, for instance to the strength or pitch of a note of music, or to certain other characteristics such as its *timbre*. No other examples could demonstrate better than these how entirely attention is independent of the adjustment of the sense, as well as in general of every other "peripheral factor."[18]

From this "central origin" of attention which has been so fully established, and from the inmost nature of the opposition between two mutually conflicting affectivities as above discussed, a conclusion of the utmost importance

[17] See W. B. Pillsbury, *Attention*, p. 13. London, Swan Sonnenschein, 1908.

[18] O. Külpe, *loc. cit.*, p. 50.

may be drawn, namely that the object of attention is observed simultaneously from two quite distinct points of view. Thus a large number of properties and characteristics, of advantages and disadvantages are perceived, observed, recalled and emphasized, which would by no means be the case if only a single affectivity were operative.

Wundt's well-known metaphorical definition of the "apperception" produced by attention as consisting in the transition of the image "from the internal visual field to the internal visual point of consciousness," accordingly, might better be replaced by that of an internal double reflector illuminating the object or the image from several sides at the same time.[19]

That is why attention prevents the mnemonic addition of sensation-evocations, which the affectivity adds to the rough elementary sensation at the moment it is aroused, from distorting the perception produced by this mnemonic contribution into an illusion or hallucination, which on the contrary is always the case when the affectivity thus aroused remains alone.

Sudden and intense fear, for instance, makes any state of attention quite impossible and may give rise—as in the classical case of the wanderer walking at night through a dense forest—to those characteristic hallucinations cited and described in all text-books of psychology and psychopathology. On the other hand that man is "cold-blooded" who does not flee at the sudden rustling of leaves which arouses in him at the first moment the vision of some hidden robber or dangerous beast behind the trees, but who, restrained by his repugnance to so cowardly an action, looks around "with attention" to see whether there really is a living creature there, and what sort of a one it is, or whether indeed it was not the wind that made the noise.

[19] W. Wundt, *Grundzüge der physiologischen Psychologie,* 5th ed., Vol. III, p. 333. Leipsic, Engelmann, 1903.—Ostwald, *Vorlesungen über Naturphilosophie,* 3d ed., pp. 400, 403. Leipsic, Veit, 1905.

Likewise in a state of passion any attention to all that is connected with this passion becomes impossible and the passionate man is therefore exposed to all the auto-suggestions and hallucinations of an Othello because of the very singleness of the control by the hypertrophic affective tendency characteristic of this state. In monomaniacs also as well as in those suffering from a chronic persecution-mania and similar psychical diseases, the thing lacking is the counter-affectivity which would tend to make them fear that they were making a mistake. They are mono-affective in the proper sense of the word, therefore incapable likewise of a real and proper state of attention.

The absence of any counter-affectivity produces in all these cases a total absence of "opposing inhibitors," as Taine would say, which could inhibit the auto-suggestions and hallucinations produced by the one existing affectivity, and permit the latter to reign unhindered and exclusively. On the other hand, great attention always protects from suggestion practised by others just because the opposite affectivity, the fear of being deceived, becomes very strong, as is proved for instance by Binet's experiments on the susceptibility of school children to suggestion.[20]

* * *

Now as we pass to the relations existing between attention and consciousness we must first briefly mention our theory with regard to the conditions which determine the consciousness and those which determine the unconsciousness of the different psychic states.[21]

In the above mentioned treatise we have come to the conclusion that a given psychic state is neither conscious nor unconscious in itself, but that it seems to possess either

[20] H. Taine, *De l'intelligence,* 8th ed., Vol. I, pp. 95 ff. Paris, Hachette, 1897.—A. Binet, *La suggestibilité,* pp. 166, 177 f., 186, 191, 196, 200 etc. Paris, Schleicher, 1900.

[21] E. Rignano, "Qu'est-ce que la conscience?" *Scientia,* 1907, Vol. II, No. IV, 4.

one or the other of these properties only when, having been previously present, it is now referred to another psychic state at present existing. And the necessary and sufficient condition permitting a complex past psychic state to present itself again as "conscious" in relation to a complex present psychic state is that the affective portion of the mnemonic evocation of the former correspond at least in part with the coexisting affective portion of the latter and therefore coalesce with it.

Since, as we have seen in our frequently cited treatise, the possession of a "diffuse seat" is characteristic of affective tendencies—which in this respect are so different from sensations and their images whose seat is localized at a single point or center and which therefore may exist and be active simultaneously in great numbers in one and the same brain—it is difficult even for only two affective tendencies to have their seats in localities which shall not coincide more or less, so that when these tendencies strive to be operative at the same time, they either conflict with each other, or hold each other in suspense, or partially coalesce.

If the discharge of one does not depend on the discharge of the other, and if the respective nervous activities in the part of their seats common to both differ specifically from each other, then the activation of one tendency will of itself imply the exclusion of the other and *vice versa*. If the discharge of the one is caused by the discharge of the other and the two tendencies are antagonistic, we will then have the state in which the primary affective tendency is held in suspense by the secondary; which condition, as we have seen above, is characteristic of the state of attention. If on the other hand the respective nervous activities in that portion of their seats common to both are specifically similar, then their blending together will make the complex psychic state to which one of the tendencies belongs "conscious" with reference to the psychic state to which the other belongs.

Finally a fourth case will occur but much more rarely for reasons given above, in which the two affective tendencies have no part of their seats in common, and accordingly both can be present and operative at the same time without hindering each other or bearing any relation whatever to one another. This case comprises all the phenomena of the so-called double personality. These phenomena nevertheless are by no means always of a pathological character, like the typical ones studied especially by Janet, but they may appear also in normal persons in so-called instances of absent-mindedness. Such was the case, for instance, when we were climbing down into the valley from Ca' di Janzo by a very steep mule path. Leaping from one stone to another constantly demanded our whole attention in order to measure exactly the distance of the leap and lest a foot should slip or dislodge a stone. Yet nevertheless the descent sometimes proceeded "unconsciously" with reference to some other very different affectivity which produced at the same time quite another train of thought.[22]

In the first case the exclusion of all other tendencies with independent discharge as soon as one of them becomes active—an exclusion which persists throughout the whole time during which the first of the two affective tendencies of the state of attention remains "held in suspense"—forms the so-called "unity of consciousness."

In other words, the impossibility for more than one primary affective tendency to be active at any one time results in the impossibility of giving heed to more than one object at one time: "A plurality of stimulations of the nerves may co-exist, but they affect the consciousness only

[22] P. Janet, *L'automatisme psychologique,* pp. 263 ff. Paris, Alcan, 1907.. . Taine, *De l'intelligence,* pp. 16 ff.—Rignano, *Qu'est-ce que la conscience?* pp. 11-13.

by turns, or one at a time. The reason is that the bodily organs are collectively engaged with each distinct conscious state, and they cannot be doing two things at the same instant."[23]

Consequently attention ordinarily is never divided or dispersed. If it is greatly roused it will continue to be directed toward any given objects for a while and hence can not be directed to any others during this entire period. If it is less aroused it passes from one object to another in quick succession and accordingly seems to be divided among many objects at the same time; but in reality even in this case it is directed at each moment to one object only, that is, to the one which corresponds to the momentary affective tendency. Accordingly the speaker who passes judgment upon his own speech, the actor who has command over himself, the chess player who plays several games at one time, Julius Cæsar who dictated several letters at once, do not prove the simultaneous presence of several states of attention, but rather their rapid succession and the alternating predominance of first one and then another.[24]

For this reason the attention directed by self-contemplation upon any affective state brings about the end and disappearance of that state. It is impossible to direct one's attention upon an affectivity. If the attempt is made that particular mood ceases at once, and we are turned aside by a compelling sensation or idea which we have not the slightest desire to observe.[25] For the attention which is directed upon an affectivity within ourselves is a newly originated affectivity, namely the one that impels us to

[23] Bain, *The Emotions and the Will,* p. 5.

[24] E. Meumann, *Intelligenz und Wille,* pp. 22 ff. Leipsic, Quelle & Meyer, 1908.

[25] E. B. Titchener, *The Psychology of Feeling and Attention,* p. 69. New York, Macmillan, 1908.

this observation and investigation, and therefore it displaces the other we wished to observe.

Since the primary affective tendency of the state of attention excludes every other affectivity independently evoked and in this way protects the unity of our consciousness, it makes it possible at the same time for every past state of attention involved to appear conscious to us if we now think back to it and to the object which at that time constituted the end desired. For this memory will now be recalled to the same object by a more or less similar affective tendency which therefore will partially blend with the recollection of the former.

Every state of attention accordingly contains all elements within itself in order later to seem to us to be conscious; but not all past psychic states which now appear conscious were states of attention, as Kohn maintains to whom the state of attention and the conscious state are the same thing. For an affectivity which becomes at once completely active and therefore does not give rise to any state of attention — like a hurried flight caused by sudden terror—is nevertheless able to make the complex psychic state involved appear a conscious one.[26] In other words, the state of attention is a sufficient but not a necessary condition of consciousness. The only condition which is at the same time necessary and sufficient is the presence of some affective tendency, no matter whether it be in the state of suspense or of full activation.

The acts which have become automatic, for instance those which originated through affective choice as conscious movements, and which later by means of attention were perfected under the affective conflict of the tendencies to perform the act but at the same time to avoid one by one its many imperfections, are finally consummated after

[26] See H. E. Kohn, *Zur Theorie der Aufmerksamkeit,* pp. 19, 27. Halle, Niemeyer, 1895.

frequent repetition—according to the mnemonic law that the part gradually becomes independent of the whole—without requiring any "impulsion" or any kind of affective aid whatever, either primarily in the execution or secondarily by way of improvement. For this reason we are accustomed to say that rendering acts automatic liberates the attention so that it may be directed to other objects.[27]

And just because acts which have become automatic do not require attention on our part and take place without the assistance of any affective element, they always seem to us to be unconscious. Consciousness, as Maudsley says, directs the process of adaptation, the efforts to become expert in adjusting the various means to their proper ends and the successive stages of organization; it disappears as soon as the skill has been thoroughly attained.[28]

"Habit," says James, "diminishes the conscious attention with which our acts are performed. One may state this abstractly thus: If an act require for its execution a chain of successive nervous events, then in the first performances of the action the conscious will must choose each of these events from a number of wrong alternatives that tend to present themselves; for consciousness is always and chiefly a selective agency. But habit soon brings it about that each event calls up its own appropriate successor without any alternative offering itself and without any reference to the conscious will, until at last the whole chain rattles itself off as soon as the first event occurs, just as if this and the rest of the chain were fused into a continuous stream."[29]

Just as an act that has become automatic represents a nervous activity which in the absence of any accompanying

[27] Meumann, *Intelligenz und Wille*, p. 23.

[28] Maudsley, *The Pathology of Mind*, p. 9.

[29] Wm. James, *The Principles of Psychology*, Vol. I, pp. 114, 139. London, Macmillan, 1901. The same, briefer course, p. 139. New York, Holt, 1893.

affective tendency remains unconscious, so will every stimulation of our senses remain unconscious when it reaches its sensory seat if it can not arouse any affectivity in us. On the other hand every stimulation of our senses which succeeds in discharging any one of the many affective tendencies potentially present in the brain, will afterwards appear conscious to us; and this may also be expressed by saying that the "stimulation has succeeded in taking possession of the sensorium."[30] Whence it follows that if all objective and sensitive peripheral relations remain the same, it will depend on whether our attention is or is not directed upon something else and on the degree of strength and of opposition of the primary affectivity involved—for thence is derived the power to exclude every other affective tendency which differs from it—whether certain stimuli remain quite unobserved or whether they will appear to us as conscious sensations.[31]

Says James: "A million things in the outside world are present to my senses but do not enter my consciousness. Why? Because they do not interest me. Only that which arouses my attention makes up my experience. Only the objects to which I give heed constitute my understanding. Without selective interest experience is a veritable chaos. Interest first gives color and tone to the image, light and shadow, background and foreground, in a word a distinct perspective."[32]

The primary affectivity of a state of averted attention may be so strong that it can prevent even the most intense irritations, which at other times would seem altogether painful and arouse within us the most strenuous effort to remove them, from reaching our consciousness. Classical,

[30] G. E. Müller, *Zur Theorie der sinnlichen Aufmerksamkeit*, pp. 77. Leipsic, Edelmann.

[31] Müller, *op. cit.*, p. 1.—Külpe, *op. cit.*, p. 40 f.—Ostwald, *Vorlesungen über Naturphilosophie*, pp. 400 ff.

[32] James, *op. cit.*, Vol. I, p. 402.

for instance, is the case of the Christian martyr whose entranced attention was to such a degree absorbed by the beatific visions presented to his eyes, that it prevented him from feeling the pain of the horrible tortures to which his body was subjected. No less significant is the case of Robert Hall, some of whose "most eloquent discourses were poured forth whilst he was suffering under a bodily disorder which caused him to roll in agony on the floor when he descended from the pulpit; yet he was entirely unconscious of the irritation of his nerves by the calculus which shot forth its jagged points through the whole substance of his kidney, so long as his soul continued to be 'possessed' by the great subjects upon which a powerful effort of his will originally fixed it."[33]

However, a large number of facts go to prove that those very irritations which do not discharge any affectivity or are not capable of arousing our attention and therefore remain unconscious, nevertheless likewise succeed in reaching their sensory seats. "The fact that we sometimes become conscious of many sensuous impressions, such as for instance the stroke of a bell, after the stimulus has made itself felt in our sense-organ, tends to show that the excitation reaches its destination rightly enough, but that the sensory center happens at the moment to be in a state not suited for the reception of the afferent stimulus."[34]

The conflict also between the different states of attention which the varied stimuli from the outside world would strive to arouse—owing to the fact that only one single primary affective tendency can ever be operative at any one moment—indicates that, whatever the relation of the stimulations to consciousness may be, they always reach their habitual psychical center; for otherwise they could not all tend to discharge their respective affectivities.

[33] W. P. Carpenter, *Principles of Mental Physiology*, 7th ed., p. 138. London, Kegan Paul, 1896.

[34] Müller, *Zur Theorie der sinnl. Aufm.*, p. 105.

"When one of the various stimuli succeeds in the struggle to obtain possession of consciousness we say that we are attentive to it according to the intensity of the corresponding process of consciousness." "But we can not maintain that excitations which do not enter our consciousness because of averted attention do not enter at all into the organ of consciousness, the cortex of the brain."[35]

It often happens in my own case, for instance, that I am reading a newspaper while the other members of the family are chatting together in the same room or perhaps while one of them reads aloud from a book or a different paper. Sometimes I do not succeed in limiting my attention to what I myself am reading because my interest is aroused by what I hear read aloud. In other cases, however, I succeed very well, and then I no longer hear the words of those in the room. Nevertheless one word pronounced by the reader in exactly the same tone as all the other words—for he is reading right along in the same monotonous voice—suddenly draws me completely away from what I am reading and turns my attention to what he is reading aloud. Thus my attention vibrates constantly back and forth between what I am reading and what I am hearing read. The fact of this conflict between the two states of attention accordingly proves most positively, I repeat, that the irritations produced by the spoken words of another reach their sensory center, their sensory basis, in me even in moments when I am not aware of them; otherwise none of them would be able to rivet my interest or attention.

The same is obviously true for all so-called states of absentmindedness which at bottom, as we have already seen, are only the first physiological indications of that double state of one's own personality which hitherto has

[35] Kohn, *Zur Theorie der Aufm.*, p. 19; and Sigmund Exner, *Entwurf zu einer physiologischen Erklärung der psychischen Erscheinungen.* Part I, p. 72. Vienna and Leipsic, Deuticke, 1894.

been investigated almost exclusively in its pathological forms. As an example of this we mentioned in our essay on consciousness the locking of a drawer while attention was directed elsewhere. This showed that all stimulations of sight proceeding from the key-hole and the key placed in it reached their goal although they remained entirely unconscious. Every one has the experience of walking absentmindedly through the streets and yet without running into people, vehicles, or any other obstructing objects on the way. Our previously mentioned "unconscious" descent from the Ca' di Janzo proves how perfectly in every respect the perception of all the difficulties of the way must have been—the stones, their form, their position, their state of equilibrium—if I were to succeed in leaping from one stone to another without falling or knocking down a stone.

The primary affective tendency which constitutes that state of attention which is directed on a definite object, by no means excludes the intrusion of sensations which at the time have no interest; or, in other words, it does not prevent excitations of a sensory character from reaching their goal, their normal destination, even when we are unconscious of them; but they only oppose the affective tendency which would endeavor to arouse these sensations.

"The entrance of a stimulus into consciousness"—as it is expressed by Kohn and others—does not rest upon the possible intrusion of the stimulus at any particular part of the brain or sensorium whose specific function would be that of consciousness. No more does it depend upon a single "center of perception" as Wundt assumes. But it consists only in the fact that this stimulus evokes some affective tendency relating to the object which it represents. When this evocation takes place the stimulus reaches consciousness; if it does not take place, perhaps because at this moment another affective tendency referring to other sensations is operative, then, although the stimulus may penetrate physiologically to the same point as usual, it cannot reach consciousness and hence remains unobserved and unconscious. The persistence of the mnemonic accumulations of those sensations which remain outside of consciousness and the possibility of evoking them again in the future are at a great disadvantage from the circumstance that they are not able to excite any affective state peculiar to themselves with which they could be connected or associated.

Having thus elucidated the inmost nature of the affective conflict peculiar to attention in its main points, and having seen wherein consists that unity of consciousness which so many inquirers declare to be one of its most essential fundamental properties, space does not now permit us to pass on to the study of the effects arising from this inmost nature and fundamental property of attention upon sensations and ideas, as in general for the whole process of intelligence.

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CHANCE.[1]

"HOW dare we speak of the laws of chance? Is not chance the antithesis of all law?" So says Bertrand at the beginning of his *Calcul des probabilités*. Probability is opposed to certitude; so it is what we do not know and consequently it seems what we could not calculate. Here is at least apparently a contradiction, and about it much has already been written.

And first, what is chance? The ancients distinguished between phenomena seemingly obeying harmonious laws, established once for all, and those which they attributed to chance; these were the ones unpredictable because rebellious to all law. In each domain the precise laws did not decide everything, they only drew limits between which chance might act. In this conception the word chance had a precise and objective meaning: what was chance for one was also chance for another and even for the gods.

But this conception is not ours to-day. We have become absolute determinists, and even those who want to reserve the rights of human free will let determinism reign undividedly in the inorganic world at least. Every phenomenon, however minute, has a cause; and a mind infinitely powerful, infinitely well-informed about the laws of nature, could have foreseen it from the beginning of the centuries. If such a mind existed, we could not play with it at any game of chance, we should always lose.

In fact for it the word chance would not have any mean-

[1] Translated by G. B. Halsted.

ing, or rather there would be no chance. It is because of our weakness and our ignorance that the word has a meaning for us. And, even without going beyond our feeble humanity, what is chance for the ignorant, is not chance for the scientist. Chance is only the measure of our ignorance. Fortuitous phenomena are, by definition, those whose laws we do not know.

But is this definition altogether satisfactory? When the first Chaldean shepherds followed with their eyes the movements of the stars, they knew not as yet the laws of astronomy; would they have dreamed of saying that the stars move at random? If a modern physicist studies a new phenomenon, and if he discovers its law Tuesday, would he have said Monday that this phenomenon was fortuitous? Moreover, do we not often invoke what Bertrand calls the laws of chance, to predict a phenomenon? For example in the kinetic theory of gases we obtain the known laws of Mariotte and of Gay-Lussac by means of the hypothesis that the velocities of the molecules of gas vary irregularly, that is to say at random. All physicists will agree that the observable laws would be much less simple if the velocities were ruled by any simple elementary law whatsoever, if the molecules were, as we say, *organized,* if they were subject to some discipline. It is due to chance, that is to say to our ignorance, that we can draw our conclusions; and then if the word chance is simply synonymous with ignorance what does that mean? Must we therefore translate it as follows?

"You ask me to predict for you the phenomena about to happen. If, unluckily, I knew the laws of these phenomena I could make the prediction only by inextricable calculations and would have to renounce attempting to answer you; but as I may chance not to know, I will answer you at once. And what is most surprising, my answer will be right."

So it must well be that chance is something other than the name we give our ignorance, that among phenomena whose causes are unknown to us we must distinguish fortuitous phenomena about which the calculus of probabilities will provisionally give information, from those which are not fortuitous and of which we can say nothing so long as we shall not have determined the laws governing them. For the fortuitous phenomena themselves, it is clear that the information given us by the calculus of probabilities will not cease to be true upon the day when these phenomena shall be better known.

The director of a life insurance company does not know when each of the insured will die, but he relies upon the calculus of probabilities and on the law of great numbers and he is not deceived since he distributes dividends to his stockholders. These dividends would not vanish if a very penetrating and very indiscrete physician should, after the policies were signed, reveal to the director the life chances of the insured. This doctor would dissipate the ignorance of the director, but he would have no influence on the dividends which evidently are not an outcome of this ignorance.

* * *

To find a better definition of chance we must examine some of the facts which we agree to regard as fortuitous, and to which the calculus of probabilities seems to apply; we then shall investigate what are their common characteristics.

The first example we select is that of unstable equilibrium; if a cone rests upon its apex, we know well that it will fall, but we do not know toward what side; it seems to us chance alone will decide. If the cone were perfectly symmetric, if its axis were perfectly vertical, if it were acted upon by no force other than gravity, it would not fall at all. But the least defect in symmetry will make it lean slightly toward one side or the other, and if it leans, however little, it will fall altogether toward that side. Even if the symmetry were perfect, a very slight tremor, a breath of air could make it incline some seconds of arc; this will be enough to determine its fall and even the sense of its fall which will be that of the initial inclination.

A very slight cause, which escapes us, determines a considerable effect which we cannot help seeing, and then we say this effect is due to chance. If we could know exactly the laws of nature and the situation of the universe at the initial instant, we should be able to predict exactly the situation of this same universe at a subsequent instant. But even then when the natural laws should have no further secret for us, we could know the initial situation only *approximately.* If that permits us to foresee the subsequent situation *with the same degree of approximation,* this is all we require, we say the phenomenon has been predicted, that it is ruled by laws; but it is not always so. It may happen that slight differences in the initial conditions produce very great differences in the final phenomena; a slight error in the former would make an enormous error in the latter. Prediction becomes impossible and we have the fortuitous phenomenon.

Our second example will be very analogous to the first and we shall take it from meteorology. Why have the meteorologists such difficulty in predicting the weather with any certainty? Why do the rains, the tempests themselves seem to us to come by chance, so that many persons find it quite natural to pray for rain or shine, when they would think it ridiculous to pray for an eclipse? We see that great perturbations generally happen in regions where the atmosphere is in unstable equilibrium. The meteorologists are aware that this equilibrium is unstable, that a cyclone is arising somewhere; but where they cannot tell; one-tenth of a degree more or less at any point, and the cyclone bursts here and not there, and spreads its ravages over countries it would have spared. This we could have foreseen if we had known that tenth of a degree, but the observations were neither sufficiently close nor sufficiently precise, and for this reason all seems due to the agency of chance. Here again we find the same contrast between a very slight cause, unappreciable to the observer, and important effects, which are sometimes tremendous disasters.

Let us pass to another example, the distribution of the minor planets on the zodiac. Their initial longitudes can have been any longitudes whatever; but their mean motions were different and they have revolved for so long a time that we may say they are now distributed *at random* along the zodiac. Very slight initial differences between their distances from the sun, or, what comes to the same thing, between their mean motions, have ended by giving enormous differences between their present longitudes. An excess of the thousandth of a second in the daily mean motion will give in fact a second in three years, a degree in ten thousand years, an entire circumference in three or four million years, and what is that to the time which has passed since the minor planets have detached themselves from the nebula of Laplace? Again therefore we see a slight cause and a great effect; or better, slight differences in the cause and great differences in the effect.

The game of roulette does not take us as far as might seem from the preceding example. Assume a needle to be turned on a pivot over a dial divided into a hundred sectors alternately red and black. If it stops on a red sector I win, if not, I lose. Evidently all depends upon the initial impulse I give the needle. The needle will make, suppose, ten or twenty turns, but it will stop sooner or not so soon according as I shall have pushed it more or less strongly. It suffices that the impulse vary only by a thousandth or a two thousandth to make the needle stop over a black sector or over the following red one. These are differences the muscular sense cannot distinguish and which elude even the most delicate instruments. So it is impossible for me to foresee what the needle I have started will do, and this is why my heart throbs and I hope everything from luck. The difference in the cause is imperceptible, and the difference in the effect is for me of the highest importance, since it means my whole stake.

* * *

Permit me, in this connection, a thought somewhat foreign to my subject. Some years ago a philosopher said that the future is determined by the past, but not the past by the future; or, in other words, from knowledge of the present we could deduce the future, but not the past; because, said he, a cause can have only one effect, while the same effect might be produced by several different causes. It is clear no scientist can subscribe to this conclusion. The laws of nature bind the antecedent to the consequent in such a way that the antecedent is as well determined by the consequent as the consequent by the antecedent. But whence came the error of this philosopher? We know that in virtue of Carnot's principle physical phenomena are irreversible and the world tends toward uniformity. When two bodies of different temperature come in contact, the warmer gives up heat to the colder; so we may foresee that the temperature will equalize. But once equal, if asked about the anterior state, what can we answer? We might say that one was warm and the other cold, but not be able to divine which formerly was the warmer.

And yet in reality the temperatures will never reach perfect equality. The differences of temperature only tend asymptotically toward zero. There comes a moment when our thermometers are powerless to make it known. But if we had thermometers a thousand times, a hundred thousand times as sensitive, we should recognize that there still is a slight difference, and that one of the bodies remains a little warmer than the other, and so we could say this it is which formerly was much the warmer.

So then there are, contrary to what we found in the former examples, great differences in cause and slight differences in effect. Flammarion once imagined an observer going away from the earth with a velocity greater than that of light; for him time would have changed sign. History would be turned about, and Waterloo would precede Austerlitz. Well, for this observer, effects and causes would be inverted; unstable equilibrium would no longer be the exception. Because of the universal irreversibility all would seem to him to come out of a sort of chaos in unstable equilibrium. All nature would appear to him delivered over to chance.

* * *

Now for other examples where we shall see somewhat different characteristics. Take first the kinetic theory of gases. How should we picture a receptacle filled with gas? Innumerable molecules, moving at high speeds, flash through this receptacle in every direction. At every instant they strike against its walls or each other, and these collisions happen under the most diverse conditions. What above all impresses us here, is not the littleness of the causes, but their complexity, and yet the former element is still found here and plays an important rôle. If a molecule deviated right or left from its trajectory, by a very small quantity, comparable to the radius of action of the gaseous molecules, it would avoid a collision or sustain it under different conditions, and that would vary the direction of its velocity after the impact, perhaps by ninety degrees or by a hundred and eighty degrees.

And this is not all; we have just seen that it is necessary to deflect the molecule before the clash by only an infinitesimal, to produce its deviation after the collision by a finite quantity. If then the molecule undergoes two successive shocks, it will suffice to deflect it before the first by an infinitesimal of the second order, for it to deviate after the first encounter by an infinitesimal of the first order, and after the second hit, by a finite quantity. And the molecule will not undergo merely two shocks; it will undergo a very great number per second. So that if the first shock has multiplied the deviation by a very large number A, after n shocks it will be multiplied by A^n. It will therefore become very great not merely because A is large, that is to say because little causes produce big effects, but because the exponent n is large, that is to say because the shocks are very numerous and the causes very complex.

Take a second example. Why do the drops of rain in a shower seem to be distributed at random? This is again because of the complexity of the causes which determine their formation. Ions are distributed in the atmosphere. For a long while they have been subjected to air-currents constantly changing, they have been caught in very small whirlwinds, so that their final distribution has no longer any relation to their initial distribution. Suddenly the temperature falls, vapor condenses, and each of these ions becomes the center of a drop of rain. To know what will be the distribution of these drops and how many will fall on each paving-stone, it would not be sufficient to know the initial situation of the ions, it would be necessary to compute the effect of a thousand little capricious air-currents.

And again it is the same if we put grains of powder in suspension in water. The vase is ploughed by the currents whose law we know not, we only know it is very complicated. At the end of a certain time the grains will be distributed at random, that is to say uniformly, in the vase; and this is due precisely to the complexity of these currents. If they obeyed some simple law, if for example the vase revolved and the currents circulated around the axis of the vase, describing circles, it would no longer be the same, since each grain would retain its initial altitude and its initial distance from the axis.

We should reach the same result in considering the mixing of two liquids or of two fine-grained powders. And to take a grosser example, this is also what happens when we shuffle playing-cards. At each stroke, the cards undergo a permutation (analogous to that studied in the theory of substitutions). What will happen? The probability of a particular permutation (for example that bringing to the nth place the card occupying the $\phi(n)$th place before the permutation) depends upon the player's habits. But if this player shuffles the cards long enough, there will be a great number of successive permutations, and the resulting final order will no longer be governed by aught but chance; I mean to say that all possible orders will be equally probable. It is to the great number of successive permutations, that is to say to the complexity of the phenomenon, that this result is due.

A final word about the theory of errors. Here it is that the causes are complex and multiple. To how many snares is not the observer exposed, even with the best instrument! He should apply himself to finding out the largest and avoiding them. These are the ones giving birth to systematic errors. But when he has eliminated those, admitting that he succeeds, there remain many small ones which, their effects accumulating, may become dangerous. Thence come the accidental errors; and we attribute them to chance because their causes are too complicated and too numerous. Here again we have only little causes each of which might produce only a slight effect; it is by their union and their number that their effects became formidable.

* * *

We may take still a third point of view, less important than the first two and upon which I shall lay less stress.

When we seek to foresee an event and examine its antecedents, we strive to search into the anterior situation. This could not be done for all parts of the universe and we are content to know what is passing in the neighborhood of the point where the event should occur, or what would appear to have some relation to it. An examination cannot be complete and we must know how to choose. But it may happen that we have passed by circumstances which at first sight seemed completely foreign to the foreseen happening, to which one would never have dreamed of attributing any influence and which nevertheless, contrary to all anticipation, come to play an important rôle.

A man passes in the street going to his business; some one knowing the business could have told why he started at such a time and went by such a street. On the roof works a tiler. The contractor employing him could in a certain measure foresee what he would do. But the passer-by scarcely thinks of the tiler, nor the tiler of him; they seem to belong to two worlds completely foreign to one another. And yet the tiler drops a tile which kills the man, and we do not hesitate to say this is chance.

Our weakness forbids our considering the entire universe and makes us cut it up into slices. We try to do this as little artificially as possible. And yet it happens from time to time that two of these slices react upon one another. The effects of this mutual action then seem to us to be due to chance.

Is this a third way of conceiving chance? Not always; in fact most often we are carried back to the first or the second. Whenever two worlds usually foreign to one another, come thus to react upon each other, the laws of this reaction must be very complex. On the other hand a very slight change in the initial conditions of these two worlds would have been sufficient for the reaction not to have happened. How little was needed for the man to pass a second later or the tiler to drop his tile a second sooner.

* * *

All we have said still does not explain why chance obeys laws. Does the fact that the causes are slight or complex suffice for our foreseeing, if not their effects *in each case,* at least what their effects will be, *on the average*? To answer this question we had better take up again some of the examples already cited.

I shall begin with that of the roulette. I have said that the point where the needle will stop depends upon the initial push given it. What is the probability of this push having this or that value? I know nothing about it, but it is difficult for me not to suppose that this probability is represented by a continuous analytic function. The probability that the push is comprised between α and $\alpha+\epsilon$ will then be sensibly equal to the probability of its being comprised between $\alpha+\epsilon$ and $\alpha+2\epsilon$, *provided* ϵ *be very small.* This is a property common to all analytic functions. Minute variations of the function are proportional to minute variations of the variable.

But we have assumed that an exceedingly slight variation of the push suffices to change the color of the sector over which the needle finally stops. From α to $\alpha+\epsilon$ it is red, from $\alpha+\epsilon$ to $\alpha+2\epsilon$ it is black; the probability of each red sector is therefore the same as of the following black, and consequently the total probability of red equals the total probability of black.

The datum of the question is the analytic function representing the probability of a particular initial push. But the theorem remains true whatever be this datum, since it depends upon a property common to all analytic functions. From this it follows finally that we no longer need the datum.

What we have just said for the case of the roulette applies also to the example of the minor planets. The zodiac may be regarded as an immense roulette on which have been tossed many little balls with different initial impulses varying according to some law. Their present distribution is uniform and independent of this law, for the same reason as in the preceding case. Thus we see why phenomena obey the laws of chance when slight differences in the causes suffice to bring on great differences in the effects. The probabilities of these slight differences may then be regarded as proportional to these differences themselves, just because these differences are minute, and the infinitesimal increments of a continuous function are proportional to those of the variable.

Take an entirely different example, where intervenes especially the complexity of the causes. Suppose a player shuffles a pack of cards. At each shuffle he changes the order of the cards, and he may change them in many ways. To simplify the exposition, consider only three cards. The cards which before the shuffle occupied respectively the places 123, may after the shuffle occupy the places

$$123, 231, 312, 321, 132, 213.$$

Each of these six hypotheses is possible and they have respectively for probabilities:

$$p_1, p_2, p_3, p_4, p_5, p_6.$$

The sum of these six numbers equals 1; but this is all we know of them; these six probabilities depend naturally upon the habits of the player which we do not know.

At the second shuffle and the following, this will recommence, and under the same conditions; I mean that p_4 for example represents always the probability that the three cards which occupied after the nth shufflle and before the $n+1$th the places 123, occupy the places 321 after the $n+1$th shuffle. And this remains true whatever be the number n, since the habits of the player and his way of shuffling remain the same.

But if the number of shuffles is very great, the cards which before the first shuffle occupied the places 123 may, after the last shuffle, occupy the places

123, 231, 312, 321, 132, 213

and the probability of these six hypotheses will be sensibly the same and equal to 1/6; and this will be true whatever be the numbers $p_1 \ldots . p_6$ which we do not know. The great number of shuffles, that is to say the complexity of the causes, has produced uniformity.

This would apply without change if there were more than three cards, but even with three cards the demonstration would be complicated; let it suffice to give it for only two cards. Then we have only two possibilities 12, 21 with the probabilities p_1 and $p_2 = 1 - p_1$.

Suppose n shuffles and suppose I win one franc if the cards are finally in the initial order and lose one if they are finally inverted. Then, my mathematical expectation will be $(p_1 - p_2)^n$.

The difference $p_1 - p_2$ is certainly less than 1; so that if n is very great my expectation will be zero; we need not learn p_1 and p_2 to be aware that the game is equitable.

There would always be an exception if one of the numbers p_1 and p_2 was equal to 1 and the other naught. *Then it would not apply because our initial hypotheses would be too simple.*

What we have just seen applies not only to the mixing of cards but to all mixings, to those of powders and of liquids; and even to those of the molecules of gases in the kinetic theory of gases.

To return to this theory, suppose for a moment a gas whose molecules cannot mutually clash, but may be deviated by hitting the insides of the vase wherein the gas is confined. If the form of the vase is sufficiently complex the distribution of the molecules and that of the velocities will not be long in becoming uniform. But this will not be so if the vase is spherical or if it has the shape of a cuboid. Why? Because in the first case the distance from the center to any trajectory will remain constant; in the second case this will be the absolute value of the angle of each trajectory with the faces of the cuboid.

So we see what should be understood by conditions *too simple*; they are such as conserve something, which leave an invariant remaining. Are the differential equations of the problem too simple for us to apply the laws of chance? This question would seem at first view to lack precise meaning; now we know what it means. They are too simple if they conserve something, if they admit a uniform integral. If something in the initial conditions remains unchanged, it is clear the final situation can no longer be independent of the initial situation.

We come finally to the theory of errors. We know not to what are due the accidental errors, and precisely because we do not know we are aware they obey the law of Gauss. Such is the paradox. The explanation is nearly the same as in the preceding cases. We need know only one thing: that the errors are very numerous, that they are very slight, that each may be as well negative as positive. What is the curve of probability of each of them? We do not know; we only suppose it is symmetric. We prove then that the resultant error will follow Gauss's law, and this resulting law is independent of the particular laws which we do not know. Here again the simplicity of the result is born of the very complexity of the data.

* * *

But we are not through with paradoxes. I have just recalled the figment of Flammarion, that of the man going quicker than light, for whom time changes sign. I said that for him all phenomena would seem due to chance. That is true from a certain point of view, and yet all these phenomena at a given moment would not be distributed in conformity with the laws of chance since the distribution would be the same as for us, who seeing them unfold harmoniously and without coming out of a primal chaos, do not regard them as ruled by chance.

What does that mean? For Lumen, Flammarion's man, slight causes seem to produce great effects; why do not things go on as for us when we think we see grand effects due to little causes? Would not the same reasoning be applicable in his case?

Let us return to the argument. When slight differences in the causes produce vast differences in the effects, why are these effects distributed according to the laws of chance? Suppose a difference of a millimeter in the cause produces a difference of a kilometer in the effect. If I win in case the effect corresponds to a kilometer bearing an even number, my probability of winning will be 1/2. Why? Because to make that, the cause must correspond to a millimeter with an even number. Now, according to all appearance, the probability of the cause varying between certain limits will be proportional to the distance apart of these limits, provided this distance be very small. If this hypothesis were not admitted there would no longer be any way of representing the probability by a continuous function.

What now will happen when great causes produce small effects? This is the case where we should not attribute the phenomenon to chance and where on the contrary Lumen would attribute it to chance. To a difference of a kilometer in the cause would correspond a difference of a millimeter in the effect. Would the probability of the cause being comprised between two limits n kilometers apart still be proportional to n? We have no reason to suppose so, since this distance, n kilometers, is great. But the probability that the effect lies between two limits n

millimeters apart will be precisely the same, so it will not be proportional to *n,* even though this distance, *n* millimeters, be small. There is no way therefore of representing the law of probability of effects by a continuous curve. This curve, understand, may remain continuous in the *analytic* sense of the word; to *infinitesimal* variations of the abscissa will correspond infinitesimal variations of the ordinate. But *practically* it will not be continuous, since *very small* variations of the ordinate would not correspond to very small variations of the abscissa. It would become impossible to trace the curve with an ordinary pencil; that is what I mean.

So what must we conclude? Lumen has no right to say that the probability of the cause (*his* cause, our effect) should be represented necessarily by a continuous function. But then why have we this right? It is because this state of unstable equilibrium which we have been calling initial is itself only the final outcome of a long previous history. In the course of this history complex causes have worked a great while: they have contributed to produce the mixture of elements and they have tended to make everything uniform at least within a small region; they have rounded off the corners, smoothed down the hills and filled up the valleys. However capricious and irregular may have been the primitive curve given over to them, they have worked so much toward making it regular that finally they deliver over to us a continuous curve. And this is why we may in all confidence assume its continuity.

Lumen would not have the same reasons for such a conclusion. For him complex causes would not seem agents of equalization and regularity, but on the contrary would create only inequality and differentiation. He would see a world more and more varied come forth from a sort of primitive chaos. The changes he could observe would be for him unforeseen and impossible to foresee.

They would seem to him due to some caprice or another; but this caprice would be quite different from our chance, since it would be opposed to all law, while our chance still has its laws. All these points call for lengthy explications which perhaps would aid in the better comprehension of the irreversibility of the universe.

* * *

We have sought to define chance, and now it is proper to put a question. Has chance thus defined, in so far as this is possible, objectivity?

It may be questioned. I have spoken of very slight or very complex causes. But what is very little for one may be very big for another, and what seems very complex to one may seem simple to another. In part I have already answered by saying precisely in what cases differential equations become too simple for the laws of chance to remain applicable. But it is fitting to examine the matter a little more closely, because we may take still other points of view.

What means the phrase "very slight"? To understand it we need only go back to what has already been said. A difference is very slight, an interval is very small, when within the limits of this interval the probability remains sensibly constant. And why may this probability be regarded as constant within a small interval? It is because we assume that the law of probability is represented by a continuous curve, continuous not only in the analytic sense but *practically* continuous, as already explained. This means that it not only presents no absolute hiatus but that it has neither salients nor reentrants too acute or too accentuated.

And what gives us the right to make this hypothesis? We have already said it is because, since the beginning of the ages, there have always been complex causes ceaselessly acting in the same way and making the world tend toward uniformity without ever being able to turn back. These are the causes which little by little have flattened the salients and filled up the reentrants and this is why our probability curves now show only gentle undulations. In milliards of milliards of ages another step will have been made toward uniformity, and these undulations will be ten times as gentle; the radius of mean curvature of our curve will have become ten times as great. And then such a length as seems to us to-day not very small, since on our curve an arc of this length cannot be regarded as rectilineal, should on the contrary at that epoch be called very little, since the curvature will have become ten times less and an arc of this length may be sensibly identified with a sect.

Thus the phrase "very slight" remains relative; but it is not relative to such or such a man, it is relative to the actual state of the world. It will change its meaning when the world shall have become more uniform, when all things shall have blended still more. But then doubtless men can no longer live and must give place to other beings—should I say far smaller or far larger? So that our criterion, remaining true for all men, retains an objective sense.

And on the other hand what means the phrase "very complex"? I have already given one solution, but there are others. Complex causes we have said produce a blend more and more intimate, but after how long a time will this blend satisfy us? When will it have accumulated sufficient complexity? When shall we have sufficiently shuffled the cards? If we mix two powders, one blue the other white, there comes a moment when the tint of the mixture seems to us uniform because of the feebleness of our senses; it will be uniform for the presbyte, forced to gaze from afar, before it will be so for the myope. And when it has become uniform for all eyes, we still could push back the limit by the use of instruments. There is no chance for any man ever to discern the infinite variety which, if the kinetic theory is true, hides under the uniform appearance of a gas. And yet if we accept Gouy's ideas on the Brownian movement, does not the microscope seem on the point of showing us something analogous?

This new criterion is therefore relative like the first; and if it retains an objective character, it is because all men have approximately the same senses, the power of their instruments is limited, and besides they use it only exceptionally.

* * *

It is just the same in the moral sciences and particularly in history. The historian is obliged to make a choice among the events of the epoch he studies; he recounts only those which seem to him the most important. He therefore contents himself with relating the most momentous events of the sixteenth century for example, as likewise the most remarkable facts of the seventeenth century. If the first suffice to explain the second, we say these conform to the laws of history. But if a great event of the seventeenth century should have for cause a small fact of the sixteenth century which no history reports, which all the world has neglected, then we say this event is due to chance. This word has therefore the same sense as in the physical sciences; it means that slight causes have produced great effects.

The greatest bit of chance is the birth of a great man. It is only by chance that meeting of two germinal cells, of different sex, containing precisely, each on its side, the mysterious elements whose mutual reaction must produce the genius. One will agree that these elements must be rare and that their meeting is still more rare. How slight a thing it would have required to deflect from its route the carrying spermatozoon. It would have sufficed to deflect it a tenth of a millimeter and Napoleon would not have been born and the destinies of a continent would have been changed. No example can better make us understand the veritable characteristics of chance.

One more word about the paradoxes brought out by the application of the calculus of probabilities to the moral sciences. It has been proved that no Chamber of Deputies will ever fail to contain a member of the opposition, or at least such an event would be so improbable that we might without fear wager the contrary, and bet a million against a sou.

Condorcet has striven to calculate how many jurors it would require to make a judicial error practically impossible. If we had used the results of this calculation, we should certainly have been exposed to the same disappointments as in betting, on the faith of the calculus, that the opposition would never be without a representative.

The laws of chance do not apply to these questions. If justice be not always meted out to accord with the best reasons, it uses less than we think the method of Bridoye. This is perhaps to be regretted, for then the system of Condorcet would shield us from judicial errors.

What is the meaning of this? We are tempted to attribute facts of this nature to chance because their causes are obscure; but this is not true chance. The causes are unknown to us it is true, and they are even complex; but they are not sufficiently so, since they conserve something. We have seen that this it is which distinguishes causes "too simple." When men are brought together they no longer decide at random and independently one of another; they influence one another. Multiplex causes come into action. They worry men, dragging them to right or left, but one thing there is they cannot destroy, this is their Panurge flock-of-sheep habits. And this is an invariant.

* * *

Difficulties are indeed involved in the application of the calculus of probabilities to the exact sciences. Why are the decimals of a table of logarithms, why are those of the number π distributed in accordance with the laws of chance? Elsewhere I have already studied the question in so far as it concerns logarithms, and there it is easy. It is clear that a slight difference of argument will give a slight difference of logarithm, but a great difference in the sixth decimal of the logarithm. Always we find again the same criterion.

But as for the number π, that presents more difficulties, and I have at the moment nothing worth while to say.

There would be many other questions to resolve, had I wished to attack them before solving that which I more specially set myself. When we reach a simple result, when we find for example a round number, we say that such a result cannot be due to chance, and we seek, for its explanation, a non-fortuitous cause. And in fact there is only a very slight probability that among 10,000 numbers chance will give a round number, for example the number 10,000. This has only one chance in 10,000. But there is only one chance in 10,000 for the occurrence of any other one number; and yet this result will not astonish us, nor will it be hard for us to attribute it to chance; and that simply because it will be less striking.

Is this a simple illusion of ours, or are there cases where this way of thinking is legitimate? We must hope so, else were all science impossible. When we wish to check a hypothesis, what do we do? We cannot verify all its consequences, since they would be infinite in number; we content ourselves with verifying certain ones and if we succeed we declare the hypothesis confirmed, because so much success could not be due to chance. And this is always at bottom the same reasoning.

I cannot completely justify it here, since it would take too much time; but I may at least say that we find ourselves confronted by two hypotheses, either a simple cause or that aggregate of complex causes we call chance. We find it natural to suppose that the first should produce a simple result, and then, if we find that simple result, the round number for example, it seems more likely to us to be attributable to the simple cause which must give it almost certainly, than to chance which could only give it once in 10,000 times. It will not be the same if we find a result which is not simple; chance, it is true, will not give this more than once in 10,000 times; but neither has the simple cause any more chance of producing it.

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THE THEORY OF REVERSIONS.*

SQUARES like those shown in Figs. 1 and 2, in which the numbers occur in their natural order, are known as *natural squares*. In such squares, it will be noticed that the numbers in associated[1] cells are complementary, i. e., their sum is twice the mean number. It follows that any two columns equally distant from the central bar of the lattice are complementary columns, that is, the magic sum

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Fig. 1.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

Fig. 2.

will be the mean of their sums. Further any two numbers in these complementary columns which lie in the same row have a constant difference, and therefore the sums of the two columns differ by n times this difference. If then we raise the lighter column and depress the heavier column by $n/2$ times this difference we shall bring both to the

* This paper was extracted about 18 months ago from three different parts of an unpublished treatise written in 1894. With regard to footnote 6, p. 63, since this was written Sayles and Worthington have independently solved the problem of construction for 6^2.

[1] Two cells are said to be *associated* when the straight line joining their centers intersects the center of the lattice, and they are equally distant from that center.

mean value. Now we can effect this change by interchanging half the numbers in the one column with the numbers in the other column lying in their respective rows. The same is true with regard to rows, so that if we can make $n/2$ horizontal interchanges between every pair of complementary columns and the same number of vertical interchanges between every pair of complementary rows, we shall have the magic sum in all rows and columns. It is easy to see that we can do this by reversing half the rows and half the columns, provided the two operations are so arranged as not to interfere with one another. This last condition can be assured by always turning over columns and rows in associated pairs, for then we shall have made horizontal interchanges only between pairs of numbers previously untouched or between pairs, each of whose constituents has already received an equal vertical displacement; and similarly with the vertical interchanges. By this method, it will be noticed, we always secure magic central diagonals, for however we choose our rows and columns we only alter the central diagonals of the natural square (which are already magic) by interchanging pairs of complementaries with other pairs of complementaries.

Since the $n/2$ columns have to be arranged in pairs on either side of the central vertical bar of the lattice, $n/2$ must be even, and so the method, *in its simplest form,* applies only to orders $\equiv 0 \pmod 4$. We may formulate the rule thus: *For orders of form 4m, reverse m pairs of complementary columns and m pairs of complementary rows, and the crude magic is completed.*

In the following example the curved lines indicate the rows and columns which have been reversed (Fig. 3).

We have said that this method applies only when $n/2$ is even, but we shall now show that by a slight modification it can be applied to all even orders. For suppose n is double-of-odd; we cannot then arrange half the columns in pairs about the center since their number is odd, but we can so arrange $n/2-1$ rows and $n/2-1$ columns, and if we reverse all these rows and columns we shall have made $n/2-1$ interchanges between every pair of complementary rows and columns. We now require only to make the one further interchange between every pair of rows and columns, without interfering with the previous changes or with the central diagonals. To effect this is always

1	58	59	4	5	62	63	8
16	55	54	13	12	51	50	9
17	42	43	20	21	46	47	24
32	39	38	29	28	35	34	25
40	31	30	37	36	27	26	33
41	18	19	44	45	22	23	48
56	15	14	53	52	11	10	49
57	2	3	60	61	6	7	64

Fig. 3.

easy with any orders $\equiv 2 \pmod 4$, (6, 10, 14 etc.), excepting the first. In the case of 6^2 an artifice is necessary. If we reverse the two central diagonals of a square it will be found, on examination, that this is equivalent to reversing two rows and two columns; in fact, this gives us a method of forming the magic 4^2 from the natural square with the least number of displacements, thus:

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Fig. 4.

Applying this idea, we can complete the crude magic

6^2 from the scheme shown in Fig. 5, where horizontal lines indicate horizontal interchanges, and vertical lines vertical interchanges; the lines through the diagonals implying that the diagonals are to be reversed. The resulting magic is shown in Fig. 6.

The general method here described is known as the *method of reversions,* and the artifice used in the double-of-odd orders is called *the broken reversion.* The method of reversions, as applied to all even orders, both in squares and cubes, was first(?) investigated by the late W. Firth, Scholar of Emmanuel, Cambridge.[2]

The broken reversion for 6^2 may, of course, be made in various ways, but the above scheme is one of the most symmetrical, and may be memorialized thus: *For horizontal changes commence at the two middle cells of the bottom row, and progress upwards and divergently along two knight's paths. For vertical changes turn the square on one of its sides and proceed as before.*

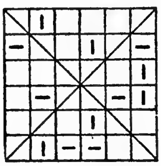

Fig. 5.

36	32	3	4	5	31
12	29	9	28	26	7
13	14	22	21	17	24
19	23	16	15	20	18
25	11	27	10	8	30
6	2	34	33	35	1

Fig. 6.

In dealing with larger double-of-odd orders we may leave the central diagonals "intact" and invert $n/2—1$ rows and $n/2—1$ columns. The broken reversion can then always be effected in a multitude of ways. It must be kept in mind, however, that in making horizontal changes we must not touch numbers which have been already moved horizontally, and if we use a number which has received

[2] Died 1889. For historical notice *vide* section on cubes.

a vertical displacement we can only change it with a number which has received an equal vertical displacement, and similarly with vertical interchanges. Lastly we must not touch the central diagonals.

Fig. 7 is such a scheme for 10^2, with the four central rows and columns reversed, and Fig. 8 shows the completed magic.

It is unnecessary to formulate a rule for making the reversions in these cases, because we are about to consider the method from a broader standpoint which will lead up to a general rule.

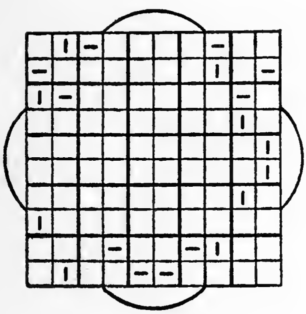

Fig. 7.

1	92	8	94	95	96	97	3	9	10
20	12	13	84	85	86	87	88	19	11
71	29	23	74	75	76	77	28	22	30
40	39	38	67	66	65	64	33	62	31
50	49	48	57	56	55	54	43	42	51
60	59	58	47	46	45	44	53	52	41
70	69	68	37	36	35	34	63	32	61
21	72	73	24	25	26	27	78	79	80
81	82	83	17	15	16	14	18	89	90
91	2	93	4	6	5	7	98	99	100

Fig. 8.

If the reader will consider the method used in forming the magic 6^2 by reversing the central diagonals, he will find that this artifice amounts to taking in every column two numbers equally distant from the central horizontal bar and interchanging each of them with its complementary in the associated cell, the operation being so arranged that two and only two numbers are moved in each row. This, as we have already pointed out, is equivalent to reversing two rows and two columns. Now these skew interchanges need not be made on the central diagonals—they can be made in any part of the lattice, provided the conditions just laid down are attended to. If then we make a second series of skew changes of like kind, we shall have, in effect, reversed 4 rows and 4 columns, and so on, each complete skew reversion representing two rows and columns. Now if $n \equiv 2 \pmod 4$ we have to reverse $n/2-1$ rows and colunms before making the broken reversion, therefore the same result is attained by making $(n-2)/4$ complete sets of skew reversions and one broken reversion.

a *b* *c* *d*

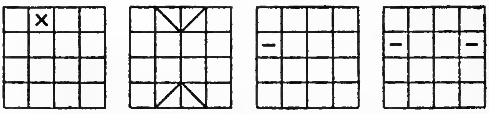

Fig. 9.

In like manner, if $n \equiv 0 \pmod 4$, instead of reversing $n/2$ rows and columns we need only to make $n/4$ sets of skew reversions.

We shall define the symbol [×] as implying that skew interchanges are to be made between opposed pairs of the four numbers symmetrically situated with regard to the central horizontal and vertical bars, one of which numbers

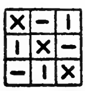

Fig. 10.

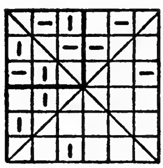

Fig. 11.

36	5	33	4	2	31
25	29	10	9	26	12
18	20	22	21	17	13
19	14	16	15	23	24
7	11	27	28	8	30
6	32	3	34	35	1

Fig. 12.

occupies the cell in which the symbol is placed. In other words we shall assume that Fig. 9*a* indicates what we have hitherto represented as in Fig. 9*b*. Further, it is quite unnecessary to use two symbols for a vertical or horizontal change, for Fig. 9*c* sufficiently indicates the same as Fig. 9*d*. If these abbreviations are granted, a scheme like Fig.

5 may be replaced by a small square like Fig. 10, which is to be applied to the top left-hand corner of the natural 6^2.

Fig. 11 is the extended scheme from Fig. 10, and Fig. 12 is the resulting magic. The small squares of symbols like Fig. 10 may be called *"index squares."*

The law of formation for index squares is sufficiently obvious. To secure magic rows and columns in the resulting square, the symbols — and | must occur once on each row and column of the index, and the symbol × an equal number of times on each row and column; that is, if there are two series ××....× the symbol × must appear twice in every row and twice in every column, and so on. But we already know by the theory of paths that these conditions can be assured by laying the successive symbolic periods along parallel paths of the index, whose coordinates are prime to the order of the index. If we decide always to use parallel diagonal paths and always to apply the index to the top left-hand corner of the natural square, the index square will be completely represented by its top row. In Fig. 10 this is $\boxed{\times}\boxed{-}\boxed{\mid}$, which we may call the index-rod of the square, or we may simply call Fig. 12 the magic $\boxed{\times}\boxed{-}\boxed{\mid}$. Remembering that we require $(n-2)/4$ sets of skew reversions when $n \equiv 2 \pmod 4$ and $n/4$ when $n \equiv 0$, it is obvious that the following rule will give crude magic squares of any even order n:

Take a rod of $n/2$ cells, $n/4$ symbols of the form ×, (using the integral part of $n/4$ only), and if there is a remainder when n is divided by 4, add the symbols | and —. Place one of the symbols × in the left-hand cell of the rod, and the other symbols in any cell, but not more than one in each cell. The result is an index-rod for the magic n^2.

Take a square lattice of order $n/2$, and lay the rod along the top row of the lattice. Fill up every diagonal slanting downward and to the right which has a symbol in its highest cell with repetitions of that symbol. The resulting index-square if applied to the lop left-hand corner of the natural n^2, with the symbols allowed the operative powers already defined, will produce the magic n^2.

The following are index-rods for squares of even orders:

4^2	×						
6^2	×	−	ǀ				
8^2	×		×				
10^2	×		ǀ	×	−		
12^2	×			×	×		
14^2	×	−	×			×	ǀ

When the number of cells in the rod exceeds the number of symbols, as it always does excepting with 6^2, the first cell may be left blank. Also, if there are sufficient blank cells, a × may be replaced by two vertical and two horizontal symbols. Thus 12^2 might be given so [× ǀ ǀ − × −]

[× ǀ ǀ − × −]

×	ǀ	ǀ	−	×	−
−	×	ǀ	ǀ	−	×
×	−	×	ǀ	ǀ	−
−	×	−	×	ǀ	ǀ
ǀ	−	×	−	×	ǀ
ǀ	ǀ	−	×	−	×

Fig. 13.

144	134	135	9	140	7	6	137	4	10	11	133
24	131	123	124	20	127	126	17	21	22	122	13
120	35	118	112	113	31	30	32	33	111	26	109
48	107	46	105	101	102	43	44	100	39	98	37
85	59	94	57	92	90	55	89	52	87	50	60
73	74	70	81	68	79	78	65	76	63	71	72
61	62	75	69	77	67	66	80	64	82	83	84
49	86	58	88	56	54	91	53	93	51	95	96
97	47	99	45	41	42	103	104	40	106	38	108
36	110	34	28	29	114	115	116	117	27	119	25
121	23	15	16	125	19	18	128	129	130	14	132
12	2	3	136	8	138	139	5	141	142	143	1

Fig. 14.

This presentation of 12^2 is shown in Figs. 13, 14, and 14^2 from the index-rod given above, in Figs. 15, 16.

Of course the employment of diagonal paths in the construction of the index is purely a matter of convenience. In the following index for 10^2, (Fig. 17) the skew-symbols are placed along two parallel paths (2, 1) and the symbols — and | are then added so that each shall appear once in each row and once in each column, but neither of them on the diagonal of the index slanting upward and to the left.

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</table>

Fig. 15.

196	13	194	4	5	191	189	8	188	10	11	185	2	183
169	181	26	179	19	20	176	175	23	24	172	17	170	28
168	156	166	39	164	34	35	36	37	159	32	157	41	155
43	153	143	151	52	149	49	50	146	47	144	54	142	56
57	58	138	130	136	65	134	133	62	131	67	129	69	70
126	72	73	123	117	121	78	77	118	80	116	82	83	113
98	111	87	88	108	104	106	105	93	103	95	96	100	85
99	97	101	102	94	90	92	91	107	89	109	110	86	112
84	114	115	81	75	79	119	120	76	122	74	124	125	71
127	128	68	60	66	132	64	63	135	61	137	59	139	140
141	55	45	53	145	51	147	148	48	150	46	152	44	154
42	30	40	158	38	160	161	162	163	33	165	31	167	29
15	27	171	25	173	174	22	21	177	178	18	180	16	182
14	184	12	186	187	9	7	190	6	192	193	3	195	1

Fig. 16.

Crude cubes of even orders we shall treat by the index-rod as in the section on squares. The reader will remember that we constructed squares of orders $\equiv 0 \pmod 4$ by re-

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<tr><td>×·</td><td></td><td>—</td><td>×</td><td>|</td></tr>
<tr><td>|</td><td>×</td><td></td><td>×·</td><td>—</td></tr>
<tr><td>—</td><td>×·</td><td>|</td><td></td><td>×</td></tr>
<tr><td></td><td>—</td><td>×</td><td>|</td><td>×·</td></tr>
</table>

Fig. 17.

versing half the rows and half the columns, and it is easy to obtain an analogous method for the cubes of the same family. Suppose we reverse half the V-planes[3] in asso-

[3] P-plane = Presentation-, or Paper-plane; H-plane = Horizontal plane; V-plane = Vertical plane.

ciated pairs; that is, turn each through an angle of 180° round a horizontal axis parallel to the paper-plane so that the associated columns in each plane are interchanged and reversed. We evidently give to every row of the cube the magic sum, for half the numbers in each row will be ex-

1	62	63	4	17	46	47	20	33	30	31	36	49	14	15	52
5	58	59	8	21	42	43	24	37	26	27	40	53	10	11	56
9	54	55	12	25	38	39	28	41	22	23	44	57	6	7	60
13	50	51	16	29	34	35	32	45	18	19	48	61	2	3	64

Magic in rows only.

Fig. 18. The natural 4^3 with V-planes reversed.

1	62	63	4	17	46	47	20	33	30	31	36	49	14	15	52
56	11	10	53	40	27	26	37	24	43	42	21	8	59	58	5
60	7	6	57	44	23	22	41	28	39	38	25	12	55	54	9
13	50	51	16	29	34	35	32	45	18	19	48	61	2	3	64

Magic in rows and columns.

Fig. 19. Being Fig. 18 with H-planes reversed.

1	62	63	4	32	35	34	29	48	19	18	45	49	14	15	52
56	11	10	53	41	22	23	44	25	38	39	28	8	59	58	5
60	7	6	57	37	26	27	40	21	42	43	24	12	55	54	9
13	50	51	16	20	47	46	17	36	31	30	33	61	2	3	64

Magic in rows, columns and lines.

Fig. 20. Being Fig. 19, with P-planes reversed.

CRUDE MAGIC 4^3.

changed for their complementaries. If we do likewise with H-planes and P-planes the rows and lines[4] will become magic. But as with the square, and for like reasons, these three operations can be performed without mutual interference. Hence the simple general rule for all cubes of the double-of-even orders:

[4] "Line" = a contiguous series of cells measured at right angles to the paper-plane.

Reverse, in associated pairs, half the V-planes, half the H-planes, and half the P-planes.

With this method the central great diagonals, of course, maintain their magic properties, as they must do for the cube to be considered even a crude magic.[5] To make the operation clear to the reader we append views of 4^3 at each

A B C

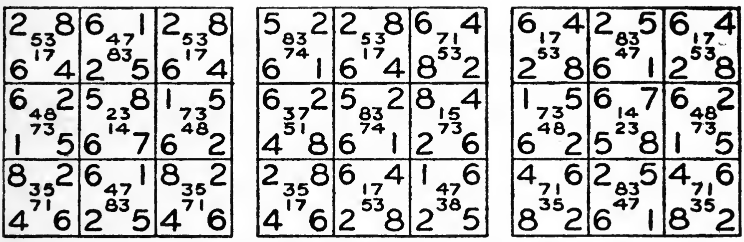


Fig. 21.

separate stage, the central pair of planes being used at each reversion.

By this method the reader can make any crude magic cube of order $4m$. With orders of form $4m+2$ we find the same difficulties as with squares of like orders. So far as we are aware no magic cube of this family had been

1	17	24
23	3	16
18	22	2

15	19	8
7	14	21
20	9	13

26	6	10
12	25	5
4	11	27

Fig. 22.

constructed until Firth succeeded with 6^3 in 1889, and we believe those we shall presently construct are the first which have been published.[6] Firth's original cube was built up by the method of "pseudo-cubes," being an extension to solid magics of Thompson's method. The cube of 216 cells was divided into 27 subsidiary cubes each con-

[5] A cube which is faulty on one of its central great diagonals is no more a magic than is a square which is faulty on one of its central diagonals.

[6] The recent examples published by Willis and Kingery fail in their central great diagonals, a fatal defect.

taining 2 cells in an edge. The 8 cells of each subsidiary were filled with the numbers 1 to 8 in such a way that each row, column, line, *and central great diagonal* of the large cube summed 27. The cube was then completed by using the magic 3^3 in the same way that 6^2 is constructed from 3^2. Firth formulated no rule for arrangement of the numbers in the pseudo-cubes, and great difficulty was encountered in balancing the central great diagonals. His pseudo-

I

2	8	134	129	186	192
6	4	130	133	190	188
182	178	21	24	121	125
177	181	22	23	126	122
144	138	174	169	16	10
140	142	170	173	12	14

II

5	3	132	135	189	187
1	7	136	131	185	191
180	184	18	19	127	123
183	179	17	20	124	128
139	141	172	175	11	13
143	137	176	171	15	9

III

117	114	146	152	62	60
118	113	150	148	64	58
54	50	109	106	168	164
52	56	110	105	162	166
154	160	70	68	97	102
156	158	66	72	98	101

120	115	149	147	63	59
119	116	145	151	61	59
51	55	112	107	161	165
53	49	111	108	167	163
155	157	65	71	100	103
153	159	69	67	99	104

IV

206	204	42	45	78	76
202	208	46	41	74	80
89	93	198	199	38	34
94	90	197	200	33	37
28	30	82	85	212	214
32	26	86	81	216	210

V

201	207	48	43	73	79
205	203	44	47	77	75
95	91	193	196	36	40
92	96	194	195	39	35
31	25	88	83	215	209
27	29	84	87	211	213

VI

Fig. 23.

skeleton is shown in Fig. 21, where each plate represents two P-planes of 6^3, each plate containing 9 pseudo-cubes. The numbers in the subsidiaries are shown in diagrammatic perspective, the four "larger" numbers lying in the anterior layer, and the four "smaller" numbers, grouped in the center, in the posterior layer.

If we use this with the magic of Fig. 22 we obtain the magic 6^3 shown in Fig. 23.

This cube is non-La Hireian, as is frequently the case with magics constructed by this method.

The scheme of pseudo-cubes for 6^3 once found, we can easily extend the method to any double-of-odd order in the following manner. Take the pseudo-scheme of next lower order [e. g., 6^3 to make 10^3, 10^3 to make 14^3 etc.]. To each of three outside plates of cubes, which meet at any corner of the skeleton, apply a replica-plate, and to each of the other three faces a complementary to the plate opposed to it, that is a plate in which each number replaces its complementary number (1 for 8, 2 for 7, etc.). We now have a properly balanced skeleton for the next double-of-odd order, wanting only its 12 edges. Consider any three edges that meet at a corner of the cube; they can be completed (wanting their corner-cubes) by placing in each of them any row of cubes from the original skeleton. Each of these three edges has three other edges parallel to it, two lying in the same square planes with it and the third diagonally opposed to it. In the former we may place edges complementary to the edge to which they are parallel, and in the latter a replica of the same. The skeleton wants now only its 8 corner pseudo-cubes. Take any cube and place it in four corners, no two of which are in the same row, line, column, or great diagonal (e. g. B, C. E, H in Fig. 38), and in the four remaining corners place its complementary cube. The skeleton is now complete, and the cube may be formed from the odd magic of half its order.

This method we shall not follow further, but shall now turn to the consideration of index-cubes, an artifice far preferable.

Before proceeding the reader should carefully study the method of the index-rod as used for magic squares (pp. 57-61).

The reversion of a pair of planes in each of the three aspects, as previously employed for 4^3, is evidently equivalent to interchanging two numbers with their complementaries in every row, line, and column of the natural cube. If therefore we define the symbol $\times$ as implying that such an interchange is to be made not only from the cell in which it is placed, but also from the three other cells with which it is symmetrically situated in regard to the central horizontal and vertical bars of its P-plane, and can make

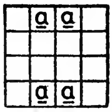


Fig. 24.

one such symbol operate in every row, line and column of an index-cube whose edge is half that of the great cube, we shall have secured the equivalent of the above-mentioned reversion. For example, a $\times$ placed in the second cell of the top row of any P-plane of 4^3, will denote that the four numbers marked *a* in Fig. 24 are each to be interchanged with its complement, which lies in the associated cell in the associated P-plane.

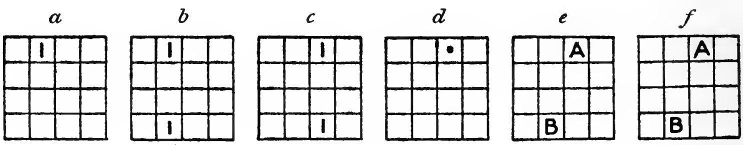


FIG. 25.

From this it follows that we shall have a complete reversion scheme for any order $4m$, by placing in every row, line and column of the index $(2m)^3$, m of the symbols $\times$. In the case of orders $4m+2$, after placing m such symbols in the cube $(2m+1)^3$, we have still to make the equivalent of one reversed plane in each of the three aspects. This amounts to making one symmetrical vertical interchange, one symmetrical horizontal interchange, and one symmetrical interchange at right angles to the paper-plane in every row, line and column. If we use the symbol | to denote such a vertical interchange, not only for the cell in which it stands, but also for the associated cell, and give like meanings to — and ·, for horizontal changes and changes along lines, we shall have made the broken reversion when we allow each of these symbols to operate once in every row, column and line of the index. For example, *a* in Fig. 25 means *b* in its own P-plane, and *c* in the associated P-plane; while *d* indicates that the numbers lying in its own P-plane as in *e* are to be interchanged, A with A and B with B, with the numbers lying in the associated plane *f*. We can always prepare the index, provided the rod does not contain a less number of cells than the number of symbols, by the following rule, *n* being the order.

Take an index-rod of $n/2$ cells, $n/4$ symbols of the form ×, (using the integral part of $n/4$ only), and if there is any remainder when *n* is divided by 4 add the three symbols |, —, ·. Now prepare an index square in the way described on p. 59, but using the diagonals upward and to the right instead of upward to the left,[7] and take this square as the first P-plane of an index-cube. Fill every *great* diagonal of the cube, running to the *right, down* and *away,* which has a symbol in this P-plane cell, with repetitions of that symbol.[8] This index-cube applied to the near, left-hand, top corner of the natural n^3, with the symbols allowed the operative powers already defined, will make the magic n^3.

This method for even orders applies universally with the single exception of 6^3, and in the case of 6^3 we shall presently show that the broken reversion can still be made

[7] Either way will do, but it happens that the former has been used in the examples which follow.

[8] More briefly, in the language of Paths, the symbols are laid, in the square, on (1,1); their repetitions in the cube, on (1, —1, 1).

by scattering the symbols over the whole cube. The following are index-rods for various cubes.

4^3

×	

8^3

×			×

10^3

×	|	–	×	•

12^3

		×	×		×

14^3

	×	–	×	•	×	|

As in the case of index-rods for squares, the first cell may be left blank, otherwise it must contain a ×.

I

64	2	3	61
5	59	58	8
9	55	54	12
52	14	15	49

II

48	18	19	45
21	43	42	24
25	39	38	28
36	30	31	33

III

32	34	35	29
37	27	26	40
41	23	22	44
20	46	47	17

IV

16	50	51	13
53	11	10	56
57	7	6	60
4	62	63	1

Fig. 26.

Fig. 26 is a 4^3, made with the index-rod given above. It has only half the numbers removed from their natural places. Figs. 27 and 28 are the index-rod, index-square and index-cube for 10^3, and Fig. 29 is the extended reversion scheme obtained from these, in which ╲ and ╱ denote single changes between associated cells, and the symbols |, —, and ·, single changes parallel to columns, rows, and lines. Figs. 30 and 31 show the resulting cube.

×	×	•	|	–

Index Rod.

×	×	•	|	–
×	•	|	–	×
•	|	–	×	×
|	–	×	×	•
–	×	×	•	|

Index Square.

Fig. 27.

×	×	•	|	–
×	•	|	–	×
•	|	–	×	×
|	–	×	×	•
–	×	×	•	|

|	–	×	×	•
–	×	×	•	|
×	×	•	|	–
×	•	|	–	×
•	|	–	×	×

×	•	|	–	×
•	|	–	×	×
|	–	×	×	•
–	×	×	•	|
×	×	•	|	–

–	×	×	•	|
×	×	•	|	–
×	•	|	–	×
•	|	–	×	×
|	–	×	×	•

•	|	–	×	×
|	–	×	×	•
–	×	×	•	|
×	×	•	|	–
×	•	|	–	×

Fig. 28. Index Cube.

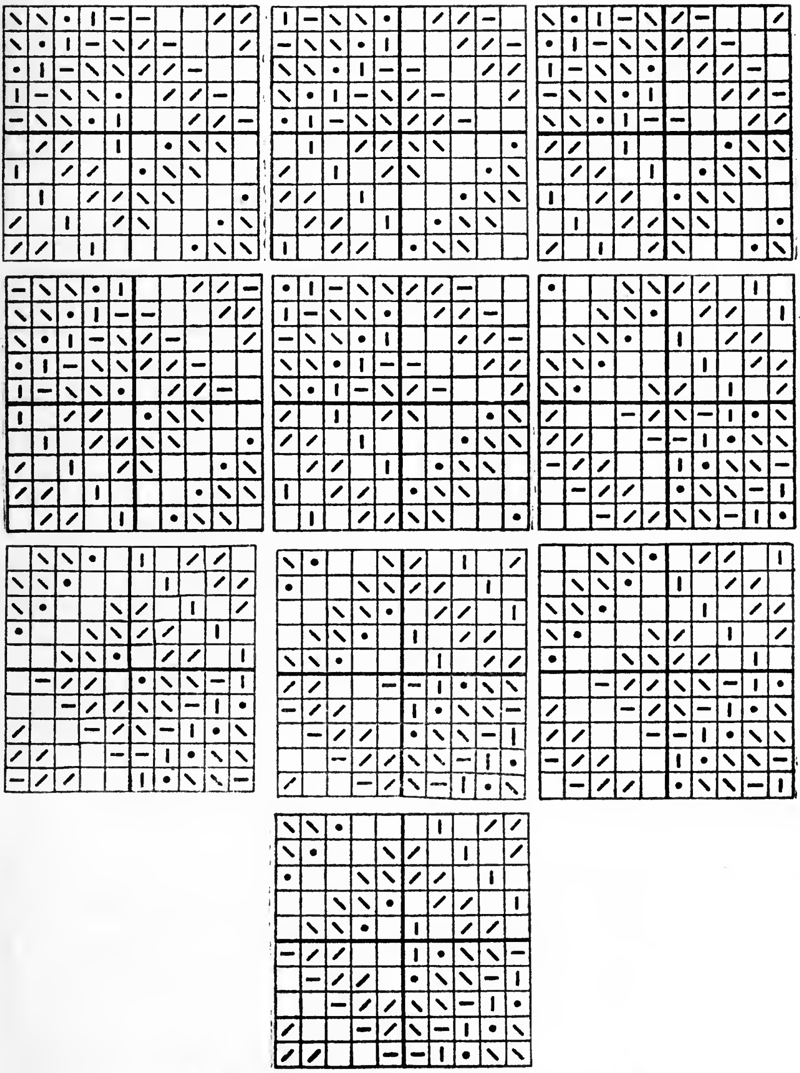


Fig. 29. Extended Reversion Scheme for 10^3.

1000	999	903	94	6	5	7	8	992	991
990	912	83	17	986	985	14	18	19	981
921	72	28	977	976	975	974	23	29	30
61	39	968	967	935	36	964	963	32	40
50	959	958	944	55	46	47	953	952	41
51	949	948	54	45	56	957	943	942	60
31	62	938	937	65	966	934	933	69	70
71	22	73	927	926	925	924	78	79	980
920	82	13	84	916	915	87	88	989	911
910	909	93	4	95	96	97	998	902	901

191	109	898	897	805	106	894	893	102	110
120	889	888	814	185	116	117	883	882	111
880	879	823	174	126	125	127	128	872	871
870	832	163	137	866	865	134	138	139	861
841	152	148	857	856	855	854	143	149	150
151	142	153	847	846	845	844	158	159	860
840	162	133	164	836	835	167	168	869	831
830	829	173	124	175	176	177	878	822	821
181	819	818	184	115	186	887	813	812	190
101	192	808	807	195	896	804	803	199	200

800	702	293	207	796	795	204	208	209	791
711	282	218	787	786	785	784	213	219	220
271	229	778	777	725	226	774	773	222	230
240	769	768	734	265	236	237	763	762	231
760	759	743	254	246	245	247	248	752	751
750	749	253	244	255	256	257	758	742	741
261	739	738	264	235	266	767	733	732	270
221	272	728	727	275	776	724	723	279	280
281	212	283	717	716	715	714	288	289	790
710	292	203	294	706	705	297	298	799	701

310	699	698	604	395	306	307	693	692	301
690	689	613	384	316	315	317	318	682	681
680	622	373	327	676	675	324	328	329	671
631	362	338	667	666	665	664	333	339	340
351	349	658	657	645	346	654	653	342	350
341	352	648	647	355	656	644	643	359	360
361	332	363	637	636	635	634	368	369	670
630	372	323	374	626	625	377	378	679	621
620	619	383	314	385	386	387	688	612	611
391	609	608	394	305	396	697	603	602	400

501	492	408	597	596	595	594	403	409	410
481	419	588	587	515	416	584	583	412	420
430	579	578	524	475	426	427	573	572	421
570	569	533	464	436	435	437	438	562	561
560	542	453	447	556	555	444	448	449	551
550	452	443	454	546	545	457	458	559	541
540	539	463	434	465	466	467	568	532	531
471	529	528	474	425	476	577	523	522	480
411	482	518	517	485	586	514	513	489	490
491	402	493	507	506	505	504	498	499	600

401	502	503	497	496	495	494	508	599	510
511	512	488	487	415	516	484	483	519	590
521	479	478	424	525	576	527	473	472	530
470	469	433	534	535	536	567	538	462	461
460	442	543	544	456	455	547	558	549	451
450	552	553	557	446	445	554	548	459	441
440	439	563	564	566	565	537	468	432	431
580	429	428	574	575	526	477	423	422	571
581	589	418	417	585	486	414	413	582	520
591	592	598	407	406	405	404	593	509	500

Fig. 30. First 6 plates of 10^3, made from Fig. 29. (Sum = 5005.)

601	399	398	304	605	696	607	393	392	610
390	389	313	614	615	616	687	618	382	381
380	322	623	624	376	375	627	678	629	371
331	632	633	367	366	365	364	638	669	640
641	642	358	357	345	646	354	353	649	660
651	659	348	347	655	356	344	343	652	650
661	662	668	337	336	335	334	663	639	370
330	672	673	677	326	325	674	628	379	321
320	319	683	684	686	685	617	388	312	311
700	309	308	694	695	606	395	303	302	691

300	202	703	704	296	295	707	798	709	291
211	712	713	287	286	285	284	718	789	720
721	722	278	277	225	726	274	273	729	780
731	269	268	234	735	766	737	263	262	740
260	259	243	744	745	746	757	748	252	251
250	249	753	754	756	755	747	258	242	241
770	239	238	764	765	736	267	233	232	761
771	779	228	227	775	276	224	223	772	730
781	782	788	217	216	215	214	783	719	290
210	792	793	797	206	205	794	708	299	201

801	802	198	197	105	806	194	193	809	900
811	189	188	114	815	886	817	183	182	820
180	179	123	824	825	826	877	828	172	171
170	132	833	834	166	165	837	868	839	161
141	842	843	157	156	155	154	848	859	850
851	852	858	147	146	145	144	853	849	160
140	862	863	867	136	135	864	838	169	131
130	129	873	874	876	875	827	178	122	121
890	119	118	884	885	816	187	113	112	881
891	899	108	107	895	196	104	103	892	810

100	99	3	904	905	906	997	908	92	91
90	12	913	914	86	85	917	988	919	81
21	922	923	77	76	75	74	928	979	930
931	932	68	67	35	936	64	63	939	970
941	59	58	44	945	956	947	53	52	950
960	49	48	954	955	946	57	43	42	951
961	969	38	37	965	66	34	33	962	940
971	972	978	27	26	25	24	973	929	80
20	982	983	987	16	15	984	918	89	11
10	9	993	994	996	995	907	98	2	1

Fig. 31. Last 4 plates of 10^3, made from Fig. 29. (Sum = 5005.)

If we attack 6^3 by the general rule, we find 4 symbols, ×, —, |, ·, and only 3 cells in the rod; the construction is therefore impossible. Suppose we construct an index-cube from the rod [×|I|—], we shall find it impossible to distribute the remaining symbol [·] in the extended reversion-scheme obtained from this index. The feat, however, is possible if we make (for this case only) a slight change in the meanings of | and —. By the general rule × operates on 4 cells in its own P-plane, where, by the rule of association, the planes are paired thus: $\left|\begin{array}{ccc} 1 & \text{with} & 6 \\ 2 & \text{"} & 5 \\ 3 & \text{"} & 4 \end{array}\right|$. In interpreting the meanings of | and —, in this special case, we must make a cyclic change in the right-hand column of this little table.

Thus for "|"

1	with	5
2	"	4
3	"	6

, and for "—"

1	with	4
2	"	6
3	"	5

. This means that a [|], for example, in the second P-plane has its usual meaning in that plane, and also acts on the two cells which would be the associated cells if the 4th plane were to become the 5th, etc. If we extend this scheme, there will be just room to properly distribute the [·]'s in the two parallelopipeds which form the right-hand upper and left-hand lower quarters of the cube, as shown in Fig. 32.

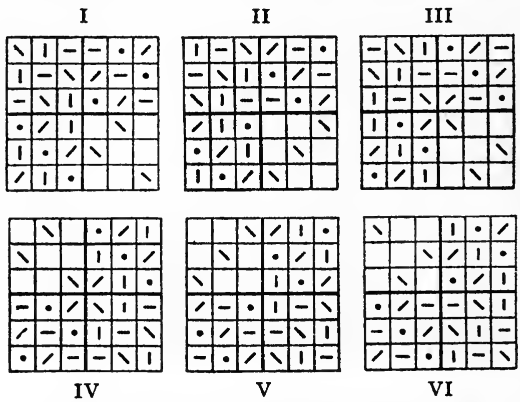


Fig. 32. Extended Reversion-Scheme for 6^3.

This scheme produces the cube shown below, which is magic on its 36 rows, 36 columns, 36 lines, *and on its 4 central great diagonals.*

Fig. 32 is the identical scheme discovered by Firth in 1889, and was obtained a few months later than the pseudo-skeleton shown in Fig. 21. A year or two earlier he had discovered the broken reversion for squares of even order, but he never generalized the method, or conceived the idea of an index-cube. The development of the method as here described was worked out by the present writer in 1894.

About the same time Rouse Ball, of Trinity College, Cambridge, independently arrived at the method of reversions for squares (compare the earlier editions of his *Mathematical Recreations,* Macmillan), and in the last edition, 1905, he adopts the idea of an index-square; but he makes no application to cubes or higher dimensions. There is reason to believe, however, that the idea of reversions by means of an index-square was known to Fermat. In his letter to

I

216	32	4	3	185	211
25	11	208	207	8	192
18	203	21	196	200	13
199	197	15	22	194	24
7	206	190	189	29	30
186	2	213	34	35	181

II

67	41	178	177	38	150
48	173	63	154	170	43
168	56	52	51	161	163
162	50	165	58	59	157
169	155	45	64	152	66
37	176	148	147	71	72

III

78	143	105	112	140	73
138	98	82	81	119	133
91	89	130	129	86	126
85	128	124	123	95	96
120	80	135	100	101	115
139	113	75	106	110	108

109	107	111	76	104	144
102	116	117	136	83	97
121	122	94	93	131	90
132	92	88	87	125	127
84	137	99	118	134	79
103	77	142	141	74	114

IV

145	146	70	69	179	42
151	65	153	46	62	174
60	158	159	166	53	55
54	167	57	160	164	49
61	47	172	171	44	156
180	68	40	39	149	175

V

36	182	183	214	5	31
187	188	28	27	209	12
193	23	195	16	20	204
19	17	202	201	14	198
210	26	10	9	191	205
6	215	33	184	212	1

VI

Fig. 33, made from Fig. 32. Sum = 651.

Mersenne of April 1, 1640, (*Œuvres de Fermat*, Vol. II, p. 193), he gives the square of order 6 shown in Fig. 34. This is obtained by applying the index (Fig. 35) to the *bottom* left-hand corner of the natural square written from below upwards, i. e., with the numbers 1 to 6 in the bottom row, 7 to 12 in the row above this, etc. There is nothing surprising in this method of writing the natural square, in fact it is suggested by the conventions of Cartesian geometry, with which Fermat was familiar. There is a much later similar instance: Cayley, in 1890, dealing with "Latin squares," writes from below upwards, although Euler, in his original Memoire (1782), wrote from above downwards. Another square of order 6, given by Fermat, in the same place, is made from the same index, but is disguised because he uses a "deformed" natural square.

6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
25	29	10	9	26	12
36	5	33	4	2	31

Fig. 34.

—	I	×
I	×	—
×	—	I

Fig. 35.

It is interesting to note that all these reversion magics (unlike those made by Thompson's method), are La Hireian, and also that the La Hireian scheme can be obtained by turning a single outline on itself. To explain this statement we will translate the square in Fig. 12 into the scale

A

55	04	52	03	01	50
40	44	13	12	41	15
25	31	33	32	24	20
30	21	23	22	34	35
10	14	42	43	11	45
05	51	02	53	54	00

B

Fig. 36.

whose radix is 6, first decreasing every number by unity. This last artifice is merely equivalent to using the n^2 consecutive numbers from 0 to n^2-1, instead of from 1 to n^2, and is convenient because it brings the scheme of units and the scheme of 6's digits into uniformity.

If we examine this result as shown in Fig. 36 we find that the scheme for units can be converted into that for the 6's, by turning the skeleton through 180° about the axis AB; that is to say, a single outline turned upon itself will produce the magic.

I

555	051	003	002	504	550
040	014	543	542	011	515
025	534	032	523	531	020
530	524	022	033	521	035
010	541	513	512	044	045
505	001	552	053	054	500

II

150	104	453	452	101	405
115	444	142	413	441	110
435	131	123	122	424	430
425	121	432	133	134	420
440	414	112	143	411	145
100	451	403	402	154	155

III

205	354	252	303	351	200
345	241	213	212	314	340
230	224	333	332	221	325
220	331	323	322	234	235
315	211	342	243	244	310
350	304	202	253	301	255

300	254	302	203	251	355
245	311	312	343	214	240
320	321	233	232	334	225
335	231	223	222	324	330
215	344	242	313	341	210
250	204	353	352	201	305

400	401	153	152	454	105
410	144	412	113	141	445
135	421	422	433	124	130
125	434	132	423	431	120
140	114	443	442	111	415
455	151	103	102	404	450

055	501	502	553	004	050
510	511	043	042	544	015
520	034	522	023	031	535
030	024	533	532	021	525
545	041	013	012	514	540
005	554	052	503	551	000

Fig. 37.

The same is true of the cube; that is, just as we can obtain a La Hireian scheme for a square by turning a single square outline once upon itself, so a similar scheme for a cube can be obtained by turning a cubic outline

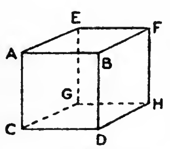


Fig. 38.

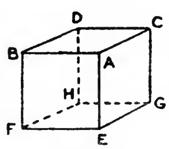


Fig. 39.

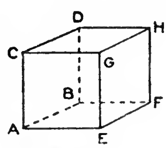


Fig. 40.

twice upon itself. If we reduce all the numbers in Fig. 33 by unity and then "unroll" the cube, we get the La Hireian scheme of Fig. 37 in the scale radix 6.

If now we represent the skeleton of the 6^2's: (left-hand) digits by Fig. 38, and give this cube the "twist" indicated by Fig. 39, we shall get the skeleton of the 6's (middle) digits, and the turn suggested by Fig. 40 gives that of the units (right-hand) digits. Thus a single outline turned twice upon itself gives the scheme.

We can construct any crude magic octahedroid[9] of

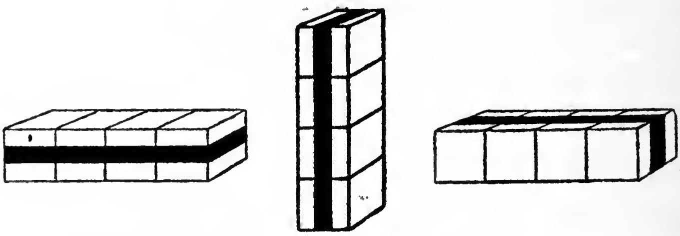


Fig. 41, 1st reversion. Fig. 42, 2d reversion. Fig. 43, 3d reversion.

double-of-even order, by the method of reversions, as shown with 4^4 in Figs. 41 to 44.

The first three reversions will be easily understood from the figures, but the fourth requires some explanation. It actually amounts to an interchange between every pair

A B C D

Fig. 44, 4th reversion.

of numbers in associated cells of the parallelopiped formed by the two central cubical sections. If the reader will use a box or some other "rectangular" solid as a model, and number the 8 corners, he will find that such a change cannot be effected in three-dimensional space by turning the

[9]

Dimensions	Regular Figure	Boundaries
2	*Tetra*gon (or square)	*4* one-dimensional straight lines
3	*Hexa*hedron (cube)	*6* two-dimensional squares
4	*Octa*hedroid	*8* three-dimensional cubes
etc.	etc.	etc.

parallelopiped as a whole, on the same principle that a right hand cannot, by any turn, be converted into a left hand. But such a change can be produced by a single turn in 4-dimensional space; in fact this last reversion is made with regard to an axis in the 4th, or imaginary direction.

1	2	3	4
248	247	246	245
252	251	250	249
13	14	15	16

65	66	67	68
184	183	182	181
188	187	186	185
77	78	79	80

129	130	131	132
120	119	118	117
124	123	122	121
141	142	143	144

193	194	195	196
56	55	54	53
60	59	58	57
205	206	207	208

17	18	19	20
232	231	230	229
236	235	234	233
29	30	31	32

81	82	83	84
168	167	166	165
172	171	170	169
93	94	95	96

145	146	147	148
104	103	102	101
108	107	106	105
157	158	159	160

209	210	211	212
40	39	38	37
44	43	42	41
221	222	223	224

33	34	35	36
216	215	214	213
220	219	218	217
45	46	47	48

97	98	99	100
152	151	150	149
156	155	154	153
109	110	111	112

161	162	163	164
88	87	86	85
92	91	90	89
173	174	175	176

225	226	227	228
24	23	22	21
28	27	26	25
237	238	239	240

49	50	51	52
200	199	198	197
204	203	202	201
61	62	63	64

113	114	115	116
136	135	134	133
140	139	138	137
125	126	127	128

177	178	179	180
72	71	70	69
76	75	74	73
189	190	191	192

241	242	243	244
8	7	6	5
12	11	10	9
253	254	255	256

Fig. 45.

The following four figures (45-48) show each stage of the process, and if the reader will compare them with the results of a like series of reversions made from a different aspect of the natural octahedroid, he will find that the "imaginary" reversion then becomes a real reversion, while one of the reversions which was real becomes imaginary. Fig. 45 is the natural 4^4 after the first reversion, magic in columns only; Fig. 46 is Fig. 45 after the second reversion, magic in rows and columns; Fig. 47 is Fig. 46 after the third reversion, magic in rows, columns and lines; and

1	254	255	4	65	190	191	68	129	126	127	132	193	62	63	196
248	11	10	245	184	75	74	181	120	139	138	117	56	203	202	53
252	7	6	249	188	71	70	185	124	135	134	121	60	199	198	57
13	242	243	16	77	178	179	80	141	114	115	144	205	50	51	208

17	238	239	20	81	174	175	84	145	110	111	148	209	46	47	212
232	27	26	229	168	91	90	165	104	155	154	101	40	219	218	37
236	23	22	233	172	87	86	169	108	151	150	105	44	215	214	41
29	226	227	32	93	162	163	96	157	98	99	160	221	34	35	224

33	222	223	36	97	158	159	100	161	94	95	164	225	30	31	228
216	43	42	213	152	107	106	149	88	171	170	85	24	235	234	21
220	39	38	217	156	103	102	153	92	167	166	89	28	231	230	25
45	210	211	48	109	146	147	112	173	82	83	176	237	18	19	240

49	206	207	52	113	142	143	116	177	78	79	180	241	14	15	244
200	59	58	197	136	123	122	133	72	187	186	69	8	251	250	5
204	55	54	201	140	119	118	137	76	183	182	73	12	247	246	9
61	194	195	64	125	130	131	128	189	66	67	192	253	2	3	256

Fig. 46.

Fig. 48 is Fig. 47 after the fourth reversion, magic in rows, columns, lines and *i*'s, = crude magic 4^4. The symbol *i* denotes series of cells parallel to the imaginary edge.

Fig. 48 is magic on its 64 rows, 64 columns, 64 lines, and 64 *i*'s, and on its 8 central hyperdiagonals. Throughout the above operations the columns of squares have been taken as forming the four cells of the P_1-aspect;[10] the rows of squares taken to form cubes, of course, show the P_2-aspect.

1	254	255	4	65	190	191	68	129	126	127	132	193	62	63	196
248	11	10	245	184	75	74	181	120	139	138	117	56	203	202	53
252	7	6	249	188	71	70	185	124	135	134	121	60	199	198	57
13	242	243	16	77	178	179	80	141	114	115	144	205	50	51	208

224	35	34	221	160	99	98	157	96	163	162	93	32	227	226	29
41	214	215	44	105	150	151	108	169	86	87	172	233	22	23	236
37	218	219	40	101	154	155	104	165	90	91	168	229	26	27	232
212	47	46	209	148	111	110	145	84	175	174	81	20	239	238	17

240	19	18	237	176	83	82	173	112	147	146	109	48	211	210	45
25	230	231	28	89	166	167	92	153	102	103	156	217	38	39	220
21	234	235	24	85	170	171	88	149	106	107	152	213	42	43	216
228	31	30	225	164	95	94	161	100	159	158	97	36	223	222	33

49	206	207	52	113	142	143	116	177	78	79	180	241	14	15	244
200	59	58	197	136	123	122	133	72	187	186	69	8	251	250	5
204	55	54	201	140	119	118	137	76	183	182	73	12	247	246	9
61	194	195	64	125	130	131	128	189	66	67	192	253	2	3	256

Fig. 47.

This construction has been introduced merely to accentuate the analogy between magics of various dimensions; we might have obtained the magic 4^4 much more

[10] Since the 4th dimension is the square of the second, two aspects of the octahedroid are shown in the presentation plane. The 3d and 4th aspects are in H-planes and V-planes. Since there are two P-plane aspects it might appear that each would produce a different H-plane and V-plane aspect; but this is a delusion.

rapidly by a method analogous to that used for 4^3 (Fig. 26). We have simply to interchange each number in the natural octahedroid occupying a cell marked [×] in Fig. 49, with its complementary number lying in the associated cell

1	254	255	4
248	11	10	245
252	7	6	249
13	242	243	16

192	67	66	189
73	182	183	76
69	186	187	72
180	79	78	177

128	131	130	125
137	118	119	140
133	122	123	136
116	143	142	113

193	62	63	196
56	203	202	53
60	199	198	57
205	50	51	208

224	35	34	221
41	214	215	44
37	218	219	40
212	47	46	209

97	158	159	100
152	107	106	149
156	103	102	153
109	146	147	112

161	94	95	164
88	171	170	85
92	167	166	89
173	82	83	176

32	227	226	29
233	22	23	236
229	26	27	232
20	239	238	17

240	19	18	237
25	230	231	28
21	234	235	24
228	31	30	225

81	174	175	84
168	91	90	165
172	87	86	169
93	162	163	96

145	110	111	148
104	155	154	101
108	151	150	105
157	98	99	160

48	211	210	45
217	38	39	220
213	42	43	216
36	223	222	33

49	206	207	52
200	59	58	197
204	55	54	201
61	194	195	64

144	115	114	141
121	134	135	124
117	138	139	120
132	127	126	129

80	179	178	77
185	70	71	188
181	74	75	184
68	191	190	65

241	14	15	244
8	251	250	5
12	247	246	9
253	2	3	256

Fig. 48.

of the associated cube. Fig. 49 is the extended skew-reversion scheme from the index-rod □☒.

All magic octahedroids of double-of-odd order $>10^4$ can be constructed by the index-rod, for just as we construct an index-square from the rod, and an index-cube from the square, so we can construct an index-octrahedroid from the cube. The magics 6^4 and 10^4 have not the capacity for construction by the general rule, but they may be

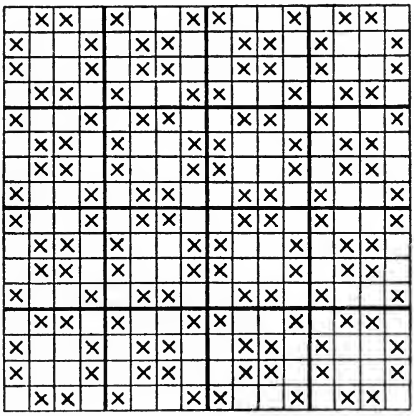


Fig. 49. Skew Reversion for 4^4.

obtained by scattering the symbols over the whole figure as we did with 6^3.

C. Planck.

Hayward's Heath, England.

TWO STUDIES IN SUGGESTION.

THE BOXERS.

ALL the world knows how the North of China was convulsed in the year 1900 by a wave of patriotic feeling stimulated by certain enthusiasts named by foreigners "Boxers." It is not quite so well known that this enthusiasm was propagated by recognized methods of psychical excitement.

This society, known as the *I ho ch'üan* or "Public Harmony Fists," arose in Shantung province, and, by the connivance of certain local officials whose national feelings outran their prudence, expanded and spread throughout that province and into the adjoining one of Chih-Li. In the course of the summer of 1900 all the provinces north of the Yellow River were permeated, the matter coming to a climax in the famous siege of the Peking legations.

All narrators agree that certain rites were performed by the propagators of the movement, which came to receive the vague title of "Boxer drill."

The following quotations will indicate the general nature of this process:

A. "They were not successful in getting the people to take it up at first, so they began with boys ten to twelve years of age.... After a few days it grew very rapidly. The drill, if it may be called so, consists in the boy repeating four short lines of some mystic words, and bowing to the south and falling backwards, when he goes into a trance, remaining lying on his back for an indefinite time, when he rises and is endowed with wonderful strength, boys of twelve being as strong as men. They brandish swords and spears, not seeming to try to be skilful in handling them, but merely to show strength and place themselves under the protection of their symbols. They claim to be invulnerable."—Rev. C. W. Price of Fen-Chou-Fu, Shansi, in *With Fire and Sword in Shan-si* (Diary, June 1, 1900).

B. "Drill consisted in incense before a tablet. . . . and then working themselves by gymnastics, etc., into a state when they were no longer masters of themselves, but became unconscious. After remaining in this state for some time they would rise, declaring themselves possessed by the spirit of one of the heroes of antiquity. In this state they could perform great feats, but the chief mark was that they were invulnerable. Swords did not hurt, and they knocked their heads till great bumps appeared, but never felt it."—Slightly abridged from Mrs. A. H. Mateer, *Siege Days,* New York, Redell.

According to the Rev. G. T. Candlin (author of *Chinese Fiction,* Chicago, Open Court Pub. Co.) who was in Tung Shan during the outbreak, the "four mystic lines" were as follows, and were accompanied by certain postures (bowing in the Chinese ceremonial style of prostrating and beating the head on the ground):

T'ien ta, T'ien chiu k'ai
Ti ta, Ti chiu k'ai
Yao hsüeh I ho ch'üan
Huan tê Shih Fu lai.

"Beat the heaven, the heaven will open;
Beat earth, and earth will open;
Desire to learn the public-harmony-force[1]
Also get the masters to come."

He has also expressed an opinion that Buddhist and Taoist priests were connected in some way with the move-

[1] *Ch'üan* is "fist" but has in this case the sense of the power of the fist.

ment and employed hypnotic methods. In this connection it is interesting to note that Putnam Weale in his famous book *Indiscreet Letters from Peking* speaks of a temple which had been specially frequented by Boxers, and that native Christians had been murdered there, presumably in some sense as sacrifices. He also mentions the large part played by boys in the movement.

If we survey the whole of the information available (of which the above is but a representative selection) it is evident that

1. Ceremonial rites including prostrations and chants formed the initial feature of the process and were prolonged until the cerebral consciousness became dormant;

2. A period of trance supervened;

3. The trance was followed by a period of great excitement in which excessive muscular energy and anesthesia were shown;

4. Boys were more subject to the influence than men, but once started it was very contagious;

5. The dominant idea was to expel the foreigner, and this was readily acceptable to the people at the time on account of public events. This was shown in the motto

Pao ch'ing, mieh yang,

"Guard the Ch'ing Dynasty, destroy the foreigner."

The Chinese are peculiarly subject to the suggestive value of epigrammatic sentences like this, and in this case we have not far to look for the master-thought.

The belief in possession by spirits is of course not peculiar to them, but an example of it in China is given in the Rev. MacGowan's book on *Side Lights on Chinese Life,* quoted in my article on "Chinese Philosophy and Magic" in the *Journal of the Royal Society of Arts,* April 21, 1911. The Confucian philosophy as expounded by Chu Hsi implies that the vital spirit in men is one and the same with that of their ancestors, so that it is not difficult for them to conceive that the peculiar individuality of an heroic ancestor may well up in the soul of his descendant. Such an idea forms a simple (and to them, rational) explanation of the enthusiasm and modification of personality which immediately succeeded the trance.

The words *Shih-Fu,* "master," may be taken as singular, plural or general, just as one thinks fit. Probably the intellects of the I Ho Ch'üan would instruct their followers specially as to the particular incarnation with which they happened to be favored.

The word *ta,* "to beat," is of very great idiomatic power in Mandarin speech, and must not necessarily be taken in its literal sense. It can mean "to appeal to," or "to serve," and undoubtedly is to be so taken.

Heaven and earth are of course the great Chinese polarities, the reservoirs of positive and negative energy.

The general sense of the chants is then that heavenly and earthly powers will respond if called upon, so that one should desire the patriotic vigor and call upon the dead to enthuse one.

The phenomenon of anesthesia (incorrectly regarded as invulnerability) is of course a usual concomitant of hystero-epilepsy. The dauntless frenzy of the Mahdi's followers undoubtedly sprang from the same conviction of personal safety, their master having assured them that neither sword nor bullet could harm them.

The success of the influence with boys indicates the hindrances which the auto-suggestions of reason placed in the way of the submission of adults. Boys have universally been employed as "mediums" in the East.[2]

An interesting point in the whole question is whether it was incepted by intellects who understand more or less well the laws of psychology, or merely arose from the nat-

[2] See Lane's *Modern Egyptians.*

ural aggregation of anti-foreign influences. It will perhaps be useful to consider what are the Chinese notions as to psychology.

Primitive Psychology in China.

The only character in the ancient Chinese hieroglyphics which takes a permanent place in psychological ideas is *hsin,* "the heart." Egyptian and Semitic literature show the same feature. In all three languages other symbols are used for external quasi-psychical phenomena, but the individual's own feelings and thoughts are almost all expressed in terms of the "heart." In other words, the heart was regarded as the seat of the intellect and emotions, presumably because the emotions when of a violent character affect the "sympathetic" or ganglionic nervous system and the heart shows the disturbance most strongly. As example we cite the following compound characters: The term "virtue" consists of a radical meaning "to walk" combined with "straight" and "heart."
The character "like" above "heart" means "reciprocity."
The character "slave" above "heart" means "anger."
The character "receive" above "heart" means "love."
The character "inferior" above "heart" means "hate."
The character "scholar" above "heart" means "will."
The character "mutual" above "heart" means "think."
The character "middle" above "heart" means "sincerity."

Dual Consciousness in Chinese Psychology.

The distinction between the central energies of the cerebro-spinal nervous system and those of the ganglionic (sympathetic) system has only recently been made out (See Hudson's *Law of Psychic Phenomena*) and is clearly adumbrated in the scholastic *animus* and *anima* and the Chinese *hun* and *p'o.* These are the personalized forms of the psychic quantities *shen* and *kuei.* The energies are respectively termed *ch'i* and *ching,* and are regarded as special forms of the positive (*yang*) and negative (*yin*) polarities of energy. The *hun* is supposed to wander at times during life and after death, while the *p'o* controls the animal functions and only persists in a shadowy form after death. Stimulated, the *hun* manifests as *chih* the will, while the *p'o* is the seat of emotion, *ch'ing.*

The ideas outlined above are almost all that can be gleaned from the ordinarily accessible native works. The practice of meditation in Buddhist and Taoist monasteries is undoubtedly based on careful observations of the results of "religious exercises." The Rev. Timothy Richard of Shanghai has translated a book which he calls the "Guide to Buddhahood," *Hsüan Fo p'u* (literally "The Record of the Selection of the Buddha").

This is a graduated statement of the development of the soul on ecstatic lines and reminds one of St. Teresa's *Castillo Interior.* Commencing with introspection of morals, it passes to contemplation of virtue and then through a whole series of meditations on mythological concepts, which will culminate in Nirvana. The analogy with the stages of apotheosis described by Plotinus and the Sufis is obvious.

An acquaintance with such mental conditions (probably accompanied by strange phenomena in various cases due to the nervous idiosyncrasies of the individual) would be quite sufficient to provide a working hypothesis for such a movement as that of the I-ho-ch'üan. By those who care for the terminology it may be legitimately called "black magic," although it amounts to very little more than the control exerted by religious fanatics generally on those of their disciples who have been "worked up" to the point of hysteria. There is this difference, however, that in the East the moving spirits generally know to some extent what they are doing, whereas in the West this knowledge is only possessed by those who have little or no occasion to employ it.

In conclusion the writer would point out that the normal Chinese mind is very acute, but conservative and lacking initiative. When excited however beyond a certain point, it exhibits a wild frenzy which is utterly reckless of consequences. These characteristics of course are not peculiar to the people but seem, at least to the writer, to be more marked than in the European. Speaking broadly, the European in China behaves as if he had little or no self-control in small matters, whereas in important things he generally becomes cool. With the Chinese it is the reverse, perfect nonchalance in ordinary affairs but imperfect balance in large ones. The writer does not of course suggest this is universally true of individuals.

THE MAGICAL USE OF BLOOD.

In the highest and lowest of ceremonial religions, and almost universally in connection with magic, we find references to the potency of blood.

The standard methods of ancestor-worship[3] include a bloody sacrifice to the manes, and an anointing with blood of the eidolon which represents the spirit. Primitively the blood is placed in the mouth of the figure. Almost in all cases it is conceived as providing vitality to the ghost. The invocation of the ghosts in the Odyssey is a typical case. The Pentateuch says "the blood is the life," and to this day the Jews abstain from meat which is not *kosher,* i. e., deprived of blood.

In China there are similar notions. Thus under the character *hsüeh,* "blood," in Giles's Dictionary the following phrase occurs:

Jen hsüeh chih wei yeh huo yeh, "Man's blood causes strange fire."

[3] See Grant Allen's *Evolution of the Idea of God.*

This emanation from blood is also termed *kuei huo,* "ghostly fire."

Again in the medieval books on magic we find that

1. Numerous prescriptions and charms require blood, and even bloody sacrifices are necessary in some cases;

2. Books professing to teach only pure theurgy recommend the practitioner to avoid the use of blood.

The aversion for blood also appears in the practices of bloodless execution employed by the Turks and the Inquisition.

The atoning power of blood is referred to in the seventeenth chapter of Leviticus, and developed in Christianity into the eucharistic sacrifice. It is also fairly clearly recognized in all bloody rites performed in the service of spirits.

Other references can be drawn from numerous sources. The marvelous blood-stains which remain on hero's sword and in haunted house; the practice of signing important acts (such as pacts with the devil!) in blood; the impurity of blood when on the person; all illustrate the general conception of its extraordinary properties.

The persistence and generality of such ideas point to some underlying psychical fact. At first sight the common experience of nausea or fainting at the sight of blood might be regarded as the fundamental cause, but a little consideration will show that this is either one of the effects of the cause we seek or a vestigial retro-reminiscence of the beliefs on the subject which dominated our forefathers.

To the writer it appears that the mere continuous juxtaposition of blood with pain and death in common experience, extending through untold generations, is quite sufficient to account for the effects and beliefs which have been referred to, acting in accordance with the laws of psychic change. In minds which have not been trained to oppose the quasi-mechanical suggestions of revived memories by specially developed associations (religious or scientific), the percept of blood will immediately call up memories of pain and death. These again will be followed by memories of incipient insensibility and fear, which will tend to be realized again in the organism by a partial paralysis of the motor centers etc., i. e., the organism will reproduce as far as possible the state remembered.

These changes, proceeding from a cause not immediately apparent to sense, are naturally ascribed to an external source, more particularly in view of the fact that another person (such as a wizard) can by insistent suggestion (with or without hypnosis) set the train of transformation in motion.

Blood has a quite perceptible smell (the extraordinary sensitiveness of carnivorous animals and insects needs only to be mentioned in support of this fact) and a perception of this is sufficient to provide a basis for the belief in peculiar sanguinary emanations. Add to this the obvious connection between blood and vitality, and we have a complete nexus of percepts which will suggest all the magical ideas mentioned, and by the encouragement of such suggestions will tend to realize the psychical counterparts of such magical causes.

Some modifications in this statement may be conceded to those who under the vague name of occultists contend that a whole series of supernormal laws continuously operates on human affairs. Such will say that all the properties attributed to blood in universal spiritualistic belief are real, that spirits (shells) can absorb sanguinary emanations and thereby vitalize themselves, etc. To them it may be said that using the word "spirit" as equivalent to "idea" the difference is merely a matter of terminology.

Herbert Chatley.

T'ang Shan, North China, Oct. 1911.

AUTOMATISM.

IN approaching a subject of such an uncertain nature, of such wide bearing and interest to humanity, and resting on the much disputed border of the unknown, it is only with the greatest regard for fact and approved hypotheses, and the utmost caution in reasoning that I have felt myself at all capable of developing it to any conclusion. The nature of the subject forbids any actual proof by our present facilities and in no place would I wish to assume my own infallibility. While the metaphysics of the question is, at present, of no practical use or bearing, yet a knowledge of the government of our actions and a conception of what this government and its rules should be, I may state to be the thing of highest utility and interest to us. According to Mill, "no belief which is contrary to the truth can be really useful," and so, at least, there is some excuse, aside from complete treatment of the subject, for developing its metaphysical side before proceeding to that of more immediate utility—the educational and moral phases. Many treatises and good have been written upon this subject, and many strong arguments *pro* and *con* adduced, but there is always a last word to be said, and the best inferences and reasons have been put to shame as the truth has slowly come to light.

No man is so presumptuous as to assert that he recognizes all causes which tend toward the production of any phenomenon, but a faith that they exist and are discoverable, is what has led to the present glory and brilliance of science. Man wonders and is curious now even as he was in the dim ages, but he has learned one lesson,—to investigate for natural causes instead of "explaining away" his ignorance by the creation of supernatural powers; and the answers which he gives to the questions of the universe to-day are not mere placebos to console his passion for an answer and to feed his emotions, but passion has been supplanted by a higher and more lasting emotion; namely, the desire for the satisfaction of reason with positive and logically deduced knowledge; and nothing more and nothing less will suffice.

In order to conform to this inner desire and all that is implied with it, it is not necessary to exclude all belief and remain purely agnostic, but to have that belief bounded and governed by the known facts of science and its articles determined by the most plausible inferences adducible therefrom. All men, no matter in what age or circumstances, have with the greatest legitimacy constructed a cosmos and not a chaos as their picture of the nature of things. For do they not see around them at all times direct evidence of law and order in the workings of all material forces? And the least of confirmation is a pillar to belief.

To develop here whatever system of belief might be entertained with the sanction of facts would hardly be within the confines of my subject, but suffice it to say that I agree with Spinoza who says that "an appeal to the interference of a soul (or unknown spiritual force) in order to explain a corporeal state, is an admission that we do not know its cause." I can in no way sympathize with the inert mind of the Orient which, too drunk with sun and plenty, must depend upon the spirit to fill the vacancy in its knowledge,—a spirit about which it has even less of an idea than of the material phenomenon itself. In the absence of knowledge we are only justified by an inference which we believe to be in the direction pointed out by facts.

Now the material and its actions are the only facts with which we are acquainted. Science has classified these facts of experience and induced laws therefrom and in every case the fact has been of a material and causative nature. It therefore becomes our first duty to attempt the reduction of all phenomena to a physical, substantive basis and not, when we have no conception of the cause, to say that its nature is "spiritual," but courageously to assert our ignorance concerning it and work with the faith that it may be reduced to a natural, materially caused phenomenon. Never *ignorabimus*! I shall preserve this rule, and work with this end in view in all that follows.

It may satisfy some to ease their desire for rationality with the following statement of Haeckel, but, however true, it does not make a direct argument against the reason of the indeterminist, which first of all must be shown fallacious before our own can trust the evidence. Professor Haeckel says, "As to the question of free-will which has kept the world busy for two thousand years, and which has produced so many books that encumber our libraries and accumulate dust therein,—this question also is no more than a memory. Of what value are vague suggestions based upon sentiment, in comparison with scientific deductions? The will indeed is not an inert force. It is a power of automatic and conscious reaction which is regulative and actively influential. But the inclinations that are inseparable from life itself explain this attribute, and as to the mode of action inherent in the will we only consider it free because, following the abstract and dualistic method of metaphysicians, we isolate this faculty from the conditions which determine it. We have not, first of all, to consider the will separately, and then examine the circumstances wherein it acts. The will as given is burdened with a thousand determinations which heredity has settled upon it. And each of its resolutions is an adaptation of its pre-existing inclination to actual circumstances. The strongest motive prevails mechanically by virtue of the laws which govern the statics of emotion. If then the merely abstract and verbal will appears free, the concrete will is determined like everything else in the universe."

To say this in the face of the overwhelming number of scientific and unscientific indeterminists is not enough, and it is the object of this essay to adduce such reasons as will lead to the establishment of these statements as facts. In doing so, let me say that I do not consider it inconsistent to accept and reason from the tried theories of science which have stood the test of time and criticism.

We know that during that comparatively simple condition of the earth, before the Laurentian age and the primordial deposits, a simple organic unit was produced. Bernard, who has made the cell his life-study, has reduced the cell, which had been formerly considered the unit of structure, to what he terms the "chromidial unit," a more elementary organic structure, having as definite a morphological significance in its own way as the cell. It can be claimed therefore that some such unit produced all the pre-cellular organisms which built up, among other less successful organisms, the famous cell with which biologists usually start their record of life. Not only was the cell a highly efficient organism in itself, as is shown by the fact that so many unicellular organisms exist to-day, but it had the power of multiplying indefinitely and forming colonies, which colonies have become organisms specialized to numberless more and ever more complicated environments. For the specialization of a large colony of cells as a whole must necessarily be able to reach a level of complexity higher than that to which any single cell could possibly attain. So thus life was raised from one level of complexity to a higher one, and it is by comparatively little reasoning that we reach the age of man.

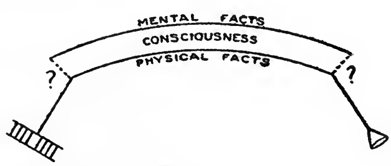
Now in the simple stage of the earth's history and even later in the postcellular age, it is acknowledged that all phenomena obeyed explicitly the omnipotent, omnipresent law of cause-effect. The actions of the ameba are nothing but the simplest of reflexes from external stimuli and this same action is admitted to continue up to the lower vertebrates. All those who have expounded the doctrine of free will have, therefore, consciously or unconsciously, stated that at some unknown instant of time in the slow, gradual evolution of organic life, and also in the growth of the embryo or early life of the infant, the animal has ceased to act according to the natural laws of its previous action and a force has crept into a universe which embraces all space, which is able to produce material phenomena on its own account and aside from the law that all motion possesses a cause of which it is the direct effect. Is it not absurd to hold that the action of a few of the higher animals are not caused and so proceed by "their own virtue?" Is this not exactly how primitive man "explained away" any phenomenon of the cause of which he was ignorant? How unreasonable it is when we realize the complex nature of the subject of our study and the complex environment upon which he must react, to infer that his action is not a more complex one working by the same rules as his simpler action did in past ages at the time of his humble origin. It is a case of realizing that a million phenomena whose cause is known to reside in a certain law, surround one phenomenon,—that of the action of the higher animals, the complexity of which has baffled our investigation, and therefore that we do not assign this one to law, but label it a causeless phenomenon the action of which is based upon the "virtue of the will." It is the insignia and confession of the lack of knowledge and the lack of inductive reasoning power of a great number of our professed scientific thinkers. They should observe their rule, namely, that if the law applies in a thousand cases the probability is a thousand to one that it will apply in the thousand and first case.

There is also another serious difficulty which presents itself to the exponents of free will, and as yet none have replied successfully to Professor Clifford who I believe was the first to discover it. In extract it is this: the will, in being pure and uninfluenced in its choice or production of a material phenomenon, and therefore free, as they say, must, in not being governed according to cause-effect, influence matter through the immaterial; and aside from the fact that the existence of the immaterial is inconceivable, otherwise than that matter should be governed by anything but surrounding matter is also inconceivable, and both are therefore highly improbable. The conclusion therefore is inevitable that the will is a physical manifestation and governed by the laws of physics.

No real boundary exists between the unconscious involuntary actions of instinct born in us or of habits formed, and the subconscious "quasi-voluntary" action of brushing the dust off one's sleeve during a conversation, or between the subconscious and more complex reactions in full consciousness. It is a known fact that when the higher forms of memory appear in animal life, a fuller and more complete consciousness exists. And this is necessarily the case, for in order to obtain the more complex reactions of the higher animals, it is necessary that a greater memory of the results of actions be had and so a fuller consciousness for the revolving of the many memories to obtain the most favorable idea of the would-be consequence and so its enactment. For the most favorable memory or idea of the consequences of actions determines our choice, on account of the self-instinct necessitated by the law of natural selection and whatever social education we have had.

Most indeterminists, believing that *ab extra* the mental and physical processes go along on two parallel platforms—the mental activity opposite the corresponding physical activity—are confronted by this argument: Since we are reasoning beings, there is a chain of mental facts between the incoming and a motor action, and so there is a complete chain of physical facts sufficient to produce the action; for before and along with the mental act of willing there is a parallel brain action which is caused and which causes the motor action. There is then no need for the parallel mental process theory, for by its parallelistic nature it destroys our incapacity for accounting for all phenomena physically, which incapacity caused its creation to "explain away" certain of the higher animal actions. The word mental should signify only in consciousness.

Again, how can pure abstract "will" influence material action? Allowing that not only to us but in abstract that "mental" processes intervene between the sensation and motor action, how is one to get across from the physical to the mental platform and then back onto the physical again? This detour, made by metaphysicians on account of ignorance, leads me to doubt the existence of the "mental," immaterial platform. I fail to see the relation between will and motion by which one can cause the other, unless "will" and "mind" are inherent in it, i. e., a manifestation of molecular or molar motion and therefore governed according to cause-effect and not free. The following diagram will illustrate the point:



If it be asserted that the psychical is inherent in inorganic nature, I have nothing to say, for molecular, atomic, and ionic structure is too little known; but I believe that it is inherent only in the sense that material composition is such that it could produce (by combinations and processes unknown) conscious life. It has been said that consciousness as a form of motion is inconceivable, but admitting its truth I do not consider it a valid argument against materialism, for what kind of an *idea* of consciousness can we have when consciousness can only be the subject and never the object, as we are contained in it?

So in conclusion on the physical facts of the case, the argument may be summed up in these words: In our development from the first transitional organic form to the cell and on through the gastraedic and invertebrate age our condition has resembled that of the monera, amoeba, platode, and up to the lower vertebrates, whose action is so simple that it is readily admitted to be mechanical. But when we come to higher vertebrates and promammals, which we resembled at a more recent period, their constitution and action has become so complex that we must abandon consistency and say: because we see no cause of their actions is there none? No, reason forbids. Upon the fertilization of the ovum and the formation of the stem-cell the life of a human individual begins. This is a mechanical process as is the development of the embryo; the early life of the infant is a combination of instinct and reflex action—purely mechanical. But after the plastic brain substance of the infant has received and held many impressions from the outside world he is equipped for a more complex reaction against it, and since many professed scientists neither realize what his memories are nor see how the most favorable one coupled with the self-preservation instinct sets forth his action, so they assign the action to his pure will to do it and nothing else. Let us use reason in this case. If the action were considered dynamically and an investigation made of its exact molecular cause and its force it would be an operation among those physiological infinitesimals which present calculation must neglect but to which faith must grant an existence. The removal of the cerebral hemisphere reduces all action to a pure and simple automatic nature and no one has had the opportunity, knowledge or facilities to watch and trace the origin of the so-called voluntary actions in the mysterious mazes of the frontal brain. It remains for us but to wait until methods are so perfected and until men, realizing that knowledge is power, educate themselves unhesitatingly to investigate with a view towards their high aim, upon the highest form of living subject obtainable, for a confirmation of those inferences we have deemed reasonable.

"The higher we ascend in the vertebrate series toward man," says Dr. Carpenter, "the more evident does it become that the ordinary course of action is determined rather by the direction given through the cerebrum to the workings of the automatic mechanism than by its (the cerebrum's) own unconscious action." In other words, by reason rather than by instinct. And in man we find that everything is to be learned by experience, save what is imperatively required for the maintenance of life—such as the rhythmical contractions of the heart, the peristaltic movements of the alimentary canal, the acts of swallowing and respiration and the like. It has already been mentioned that memory is the great prerequisite for all "voluntary" action, and it it is also known that the actions of the human embryo are not of that sort until

> "Nature whose heedless might
> Casts like some shipwrecked sailor, the poor babe,
> Naked and bleating on the shores of light."

From that instant the memory is in process of formation, the conscious personality begins, habit adds to the rôle of the involuntary centers, which previously possessed only instinct, and the infant can thus react more perfectly upon complex conditions and exert less effort in the performance of simple and necessarily repeated actions, for their performance has become habitual and subconscious. Thus he is able to direct his higher activities to the more difficult phases of his being—to this end has the law of natural selection, joined with variation, ever worked in the mental field. This is the pregnant fact upon which I shall build my argument from the mental side of the question.

As shown under hypnosis, an impression of every experience, of the sight of every performance of others, of the result of every action, is indelibly recorded in the brain, whether it ever be brought into consciousness or not. We therefore have for our use the knowledge of the result of a thousand actions, whether it be of the tongue or of the hand. Now we also possess from heredity the overwhelming instinct of self-preservation and its brother, the desire for what is productive of the greatest happiness to us. The following mental process is easily discernible by introspection: a condition arises in the environment necessitating a reaction; the memory arises of certain results upon the individual of an action of his own or of some one else; if it be a favorable result his interest in the possible action is aroused and his attention is then directed toward it; the same occurs (from association of ideas in the memory) to four or five (taking an extreme case of indecision) ideas of possible action; the attention is directed then from one to the other and a comparison of them is made according to the individual's belief in the probable nature of their results; one appears more favorable to his happiness and welfare than the others, whereupon it is acted out. Thus truly considering the necessity of memory, Plato has reason to name it a great and mighty goddess. If it were not for this cause-effect mental process, I would fear greatly for the happiness and interests of the individual, if there could be individual life without it. One of the potent factors in causing such a strenuous advocacy of free-will is the pride and vanity of man in himself and his powers. But how often has that pride been humbled and how often must it be in the future when such facts as his low origin or his unlikeness to the image of God are forced upon his realization!

The power of suggestion and association of idea with idea, such as I experience as I sit here writing, must also be thoroughly recognized and considered before any validity, let alone prestige, can be given to the statements of an indeterminist. The previous paragraph has shown the method of the objective or higher faculties in arriving at a conclusion for action, but the memory, or what has been termed the unconscious, subjective mind is always amenable to suggestion and will catch the objective faculties off their guard if possible. A friend related an excellent example of this some days ago: A young man who had determined to stop drinking was invited to step into a saloon and have a glass. He was prepared for this and the suggestion brought up the reply no. A few days later an old school friend met him and said, "Let's go in and sit down and talk over old times." He went in and it is unnecessary to say, succumbed. Taking up the association of memories or ideas, let me ask the free-will exponents if the "chance" were at all probable, of my turning ten minutes ago to the beginning of this paragraph and writing the word *the* with which to start this paragraph, the idea of what I have just written springing spontaneously from my brain? I also ask them to exercise their powers of introspection until they have gained proficiency enough to trace back *why* they did this or thought that the moment before, winding the string (of cause-effect) as they go along and reach, say, their experience an hour ago. My opponents paradoxically admit that they are not reasoning men, for they say they do just as their "free will" pleases and, although moral men, are not governed by duty, responsibility, or fear of consequences. Freedom consists of a recognition of facts and a self-government according to them; bondage, of a struggle against them. I have already determined what pleases us,—namely that of which the consequences are productive of our happiness and well-being. And the proud "free-willers," I believe, have some hedonists in their ranks who will acknowledge what pleases them, so their acts being governed by that, they prove themselves traitors to the cause. Many people may become indeterminists and reach that abnormal state of mind in which they can trust themselves to a universe where law and lawlessness interchange indiscriminately, but I confess myself unable to reach that Nirvana.

Our own immediate mental experience, therefore, has shown that we are no exception to the rule (in that we realize the mental antecedent—the why of our purpose) and they are as worthy of confidence, according to Dr. Carpenter, as are "deductions drawn from phenomena outside ourselves, which we can only rightfully interpret on the basis afforded by those very experiences, the test of the validity of such interpretation being furnished by their conformity to our other immediate experiences." It is well known that the hemisphereless frog or pigeon acts automatically when any thing directly stimulating is administered, but remains perfectly passive until then. The hemispheres, therefore, are the seats of higher consciousness wherein a more complex reaction is aroused from *more distant* and *delicate* stimuli from without—after the formation of the memory within—but not less automatic action. Our consciousness of effort arises from the many and intricate processes of conscious reasoning, judgment, etc., before arriving at a decision or choice, and is accompanied by the feeling of effort arising from muscular movement. It has been often urged that, since neurosis can give rise to psychosis, it is surely quite accordant with the fundamental principle of interaction to affirm that conversely, psychosis can give rise to neurosis, just as the electricity generated in a voltaic battery by chemical change can itself produce chemical change. I quite agree—the psychosis being neurosis consciously felt. The neurosis afferently causes psychosis, i. e., causes will; the psychosis efferently (in regard to the ego) causes neurosis and bodily motion. He simply affirms the chain of cause-effect and the law of the conservation of energy.

In fact, unless combination of memories were used to determine our actions and memory be but a rudiment, or else that memory is used for that purpose now, I can see no object past or present toward which it would be of utility. Darwin and his followers have shown that an animal possesses a function because it was either of use to its ancestors or to itself. Therefore since memory would be useless unless it helped and guided our actions we must concede that it does; and we can also conclude that where animals acted in accordance with a more perfect memory (arising from variation) their actions were more in accordance with the requirements of nature and they more fit to live. Thus natural selection has produced this, as well as all other necessary functions. Who would attempt an explanation of the molecular causes of those imaginary actions in dreams? Memory is involved here but the channels through which we come to those imaginations are so subconscious as to baffle all introspection; yet there is no manifestation of will in them and it is comparatively easy to see the by-cause of conscious volitions.

Under hypnotism the will or judgment is unconscious. The man is under the complete control of the present suggestion. Now we see that it is not a very beneficial reaction when no distinction can be made between the false and true, right and wrong, etc. And thus natural selection gave rise to the will—judgment (comparison of memories) coupled with action. The comparison was and is necessary for existence by a high reaction even as the hemispheres were and are necessary to a reaction from more distant and delicate stimuli. The hemisphereless frog and the hypnotized man are admitted automatons but when there were neither of these conditions and the reaction was complex—from revolving of memories and comparison of them, and from distant stimuli they were thought uncaused as we had no knowledge or perception of them. Now that we see the *why*, we realize that automatic nature in the absence of hypnoses or presence of the hemispheres as well as in the opposite conditions.

On the freedom of choice this is the sole reply which I find from the indeterministic pen. "And yet on the deterministic doctrine, if I am attracted by the temptation of an immediate but immoral pleasure, and am deterred from it either by a sense of duty or by the fear of the remote consequences of the sin, I have no more 'choice' as to the course I shall take than has the piece of iron that is attracted in opposite directions by two unequal equidistant magnets. Now my contention is not merely that I have a choice, but that the very existence of an idea that can be derived from no other source than human experience, confirms that effect." I believe Dr. Carpenter perfectly justified in making this statement. As to the person it is a choice (at the moment he does not figure out all reasons or causes, they being subconscious), but the "choice," not to us but abstractly, is determined and nonexistent. The fact that all experience shows that motives which may exert a preponderating influence at one moment, are comparatively powerless at another, and that, on the other hand, motives whose influence at one moment is scarcely felt, may come to acquire a force that makes them far outweigh those which at first overbalanced them, shows that, although we do not know what is really the best decision, if we can be made to believe that a certain one is (by any means whatever) better, that is the one which the self-instinct, or whatever social education we have had, embodies with the proper action. Indeterminism confesses its inability to trace anything behind the will or existing before it which is in any way connected with it; determinism confesses that it sees and also consciously experiences (and what our consciousness tells us is the surest reality to us) a phenomenon existing before it in time and determinedly related to it. In other words, the will is not a spontaneous and independent thing leaning only against itself.

Santayana says, "Mankind and all its works are undeniably subject to gravity and to the law of projectiles; yet what is true of these phenomena in bulk seems to a superficial observation not to be true of them in detail, and a person may imagine that he subverts all the laws of physics whenever he wags his tongue, only in inorganic matter is the ruling of mechanism open to human inspection; here changes may be seen to be proportionate to the elements and situation in which they occur.... Physics cannot account for that minute motion and pullulation of the earth's crust of which human affairs are a portion. Human affairs have to be surveyed under the categories lying closer to those employed in memory and legend.... That this gulf is apparent only, being due to inadequacy and confusion in human perception rather than to incoherence in things, is a speculative conviction altogether trustworthy.... Now the human senses are not at all fitted to represent an organism on the scale of the human body. They catch its idle gestures but not the inner processes which control its action. The senses are immeasurably too gross. What to them is a *minimum visibile*, a just perceptible atom, is in the body's structure, very likely, a system of worlds, the inner catclysms of which count in producing that so-called atom's behaviour and endowing it with affinities apparently miraculous. What must the seed of animals contain, for instance, to be the ground, as it notoriously is, for every physical and moral property of the offspring?.... Any one who can at all catch the drift of experience—moral no less than spiritual—must feel that mechanism rules the whole world."

According to Spinoza, that masterful combination of reason and intuitional insight, "A thing is said to be free (*libera*) which exists by the mere necessity of its own nature, and is determined in its actions by itself alone." If, then, men can attribute no reason for the willing of anything beyond the immediate cause, then the will is infinite beyond that cause; then the will is equal in power to God, in that He would have no control thereover and all the burden and responsibility of a choice, which may affect the lives of many men, is placed upon this will, infinite in its nature yet limited in its knowledge. It is not just nor right that God should place such responsibility in the ungoverned hands of ignorance. As God is just and righteous it follows "from these premises then, that men think themselves free inasmuch as they are conscious of their volitions and desires, and, as they are ignorant of the causes by which they are led to wish and desire, they do not even dream of their existence." It is then concluded (Prop. 48, Part II) "There is in no mind absolute or free will, but the mind is determined for this or that by a cause which is determined in its turn by another cause, and this one again by another, and so on to infinity. *Proof.*—The mind is a fixed and determined mode of thinking and therefore cannot be the free cause of its actions. It cannot have the absolute faculty of willing or unwilling, but in willing this or that, it must be determined from an infinite line of causation."

Dr. James says in one of his essays, "The sting of the word 'chance' seems to lie in the assumption that it means something positive, and that if anything happens by chance it must needs be something of an intrinsically irrational and preposterous sort." But I confess I can not see that unless *chance* is *governed* (or as he says, "needs be,") by reason or law (a contradiction in itself) how the result of the comparatively few higher animal actions of the future could be anything but "irrational or preposterous." It is a case of to be or not to be. If "chance" is to be governed by reason and by law we may expect the world to continue a part of a universe in the future and if it is not, that it will become participant in a nulliverse. Regret for our past actions and therefore the wish that something might be otherwise takes place in every passing hour and is but a confession that had we been wise enough our act would not have occasioned regret as it would have been governed by that wisdom.

The distinct purposive intervention of the self-conscious ego is what should be designated as will, though the purpose and intervention be caused; it is purely voluntary to us and gives no feeling of oppression although in the true sense not "will." Therefore to say that you cannot perform will as I have re-defined it is untrue, for the memories and instincts—caused causes of the will, are a part and contained in yourself, i. e., to you the act is will. From a point of view outside of the self the ego is not responsible, but you are to yourself since the will is responsible for its conduct to the memories and instincts—the basis of the personality. Yes, and the responsibility is exactly fulfilled.

I can do no better than conclude my argument from the mental point of view with an illustration from the thoughtful pen of Thomas Huxley: "Suppose that an adult man, in the full rigor of his faculties, could be suddenly placed in the world, as Adam is said to have been, and then left to do as he best might. How long would he be left uneducated? Not five minutes. Nature would begin to teach him, through the eye, the ear, the touch, the properties of objects. Pain and pleasure would be at his elbow telling him to do this and avoid that; and by slow degrees the man would receive an education which, if narrow, would be thorough, real, and adequate to his circumstances, though there would be no extras and very few accomplishments. And if to this solitary man entered a second Adam, or, better still, an Eve, a new and greater world, that of social and moral phenomena, would be revealed. Joys and woes, compared with which all others might seem but faint shadows, would spring from the new relations. Happiness and sorrow would take the place of the coarser monitors, pleasure and pain; but conduct would still be shaped by the observation of the natural consequences of actions; or, in other words, by the laws of the nature of man. Nor should I speak of this process of education as past, for any one, be he old as he may. For every man the world is as fresh as it was the first day, and as full of untold novelties for him who has eyes to see them. And nature is still continuing her patient education of us in that great university, the universe, of which we are all members, nature having no Test-Acts. Those who take honors in nature's university, who learn the laws which govern men and things and obey them, are the really great and successful men in this world. The great mass of mankind are the 'Poll,' who pick up just enough to get through without much discredit."

I have quoted this at length because it so admirably conveys the meaning which I have tried to express in other words, and because it contains the foundation of my argument from the moral side of the question which is about to follow. In this domain the exponents of free will have considered themselves least needful of defence, but, as yet, I have not come upon any elucidation of this side of the question which was exactly satisfactory to my demands. Either we are wrong when we blame, or God is immoral, and I greatly suspect that the fault is to be found in our lack of true moral comprehension, rather than in God. It then becomes my duty to whitewash the Devil, although to compel him to keep indoors is the work of the centuries.

Going to the foundations of morality must necessarily give us a truer conception of the import of things, and must also lead, by way of our determinism, to a rational, optimistic, trusting conception of the universe if that doctrine is to be entertained by us for one moment. Before the advent of man, it is easily seen that nothing was moral or immoral, for those terms are merely relative to our mode of thinking and arose with it. Nothing therefore is within itself bad or good and the words signify only the fulfillment of the demands of our nature upon phenomena or the lack of it. That which does not acquiesce to our demands is called bad or evil and if it is shown that our demands are the result of comprehensive reasoning, i. e., that we can see to what end the action or being is directed, and that it is evil (detrimental to happiness and well-being of men) then we have a right to a pessimism toward that universe which would produce good and evil in motley alternation. It therefore devolves upon us here to prove that the tendency of all phenomena is that which would secure the approbation of our moral nature if we could realize their end. Of course we can conceive what would be to us a perfect universe where all pain and evil had a good in its place, but I think we can not censure the scheme or entertain a pessimism if we find that all that is not good is productive of good, regardless of the conscious experience of the individual through whose suffering good is to be realized. Darwin truly states that no species or individual is perfect for its reaction upon its environment (whether of the complex nature or of the continual change of the latter) and Höffding perceives in his "critical" realism the never ending progress of life toward that perfection. Man's becoming a social animal has raised the complexity of his environment a thousandfold, both in relation to the acts and thoughts of other men, and in the increased menace of disease. Now although not so potent a factor as it was thought at first, the law of the natural selection of variant forms, cruel in itself, *has been* the sole means toward that good end, our mind. She selected those who gained and retained from dear experience (a seeming evil) the requisite knowledge. Although the weaklings and the deficients may have been fostered by an unexacting environment and the more fit cut off by accident, yet, in the first case, if the environment remained and nothing stepped in to improve the unfortunates and they generated degenerates which were still fostered by easy surroundings, the time came when the environment changed and their extermination proceeded. If the fit were plucked by accident, and it was exceptional, yet the weeding still went on until ones of just as high a level of fitness were produced. Nature works slowly now through painful education (each age building upon the knowledge gained by the preceding ages from their diligence and to a less extent their lack of it and mistakes) and reaching a higher social scale, after having gone as far as possible with painful extermination, (natural selection has caused reason which has supplanted instinct and selection as factors in our development) toward that consummation, the craving for which has produced the greatest hope of the human breast—knowledge and happiness (perfect adaptation to environment). Thus science confirms with positive knowledge that these beliefs which originated in the heart of primitive man, are not empty and groundless, but even confirms them along with that other—of an omnipresent, omnipotent God, which it explains as a realization of the presence of The Law (of cause-effect).

The morphine fiend could not *help* himself because he was fated from eternity to pain; he had better not been born; but the law of life and of death cares not if a spark of consciousness suffer nor whether the spark know the consequences of its action or not. "Ignorance of the law is no excuse, and the wages of sin is death." Whether the victim is able to *help* his action or not the evil to him exists and although the act is an "evil" which, according to free-will, might not have been, it happened and the universe, be it a monism or dualism, possess we one element (causality) or two elements (causality and free-will), is responsible for its existence and the victim a right to pessimism as long as he regards himself. In either doctrine the only way out of the difficulty that I can see is to take the more comprehensive view, whether you be the victim or not. The invisibility and slow working of the evolutionary law (physical, mental, or social) may make this seem to be closet philosophy, but it is only when we make a retrospection of the ages that the great underlying influences come into broad daylight.

Let us take, by way of illustration, an event told by Dr. James. He says: "At Brockton, the other day, a man, to get rid of the wife whose existence bored him, inveigled her into a desert spot, shot her four times and then as she lay on the ground and said to him, 'You didn't do it on purpose, did you dear?' replied, 'No, I didn't do it on purpose,' as he raised a rock and smashed her skull." The Doctor remarks, "We feel that although a perfect mechanical fit to the rest of the universe, it is a bad moral fit and that something else would really have been better in its place." I do not say that something else would not have been better in its place, but his universe, as well as mine, must account for it and palliate the crime to us with a reason which gains our moral approbation for its existence. I say that our moral view is not the true view, else it would allow the existence of these "evils." No evil is necessary, but as long as we are ignorant or governed by blind passion, we are not perfect in our environment, and the "evils" are bound to exist. The causes of such actions as these are unhealthy bodies, or minds which have not learned from their own or others experience (i. e., educated to a wrong environment), or who do not recognize the stronger demands of society or are guided by passion in lieu of the only legitimate monitor reason. Now, seeing these causes, could we blame the action of this Brockton man? Or could we blame the universe as immoral when it is necessary to evolve slowly into the social state and therefore actions such as this, reversions, come to pass in a state of society where they are immoral—individual strife was not immoral where individualism and natural selection were working as it led to a great good,—the physical and brain development of the race. But reason, experience, and social or moral education are taking place and the future we believe to promise a better condition. There is no more immorality in this mental reversion than in a physical reversion such as the famous Miss Julia Pastrana or a tailed boy. In fact, unless the experience of the past and our possession of reason counted for something in our life, I do not see how any social evolution, an optimistic view of the future, or any reason for our progress thus far can be had, since natural selection has become *nil* to us. The conscious experience of healthy men affirms the potency of reason and experience and as this is the surest of reality to us, I believe no doubt can be had. Even if there were no other palliation to our just desire for a rational and moral universe, the fact of the educational value of this Brockton example as an admonition to posterity would be sufficient.

But this action which I have just explained is rudimentary—the remains of a lower stage of mental evolution. The self-instinct was necessarily the first produced by natural selection and still remains with us although not playing such an important part. The preservation of the young or the family next arose and all actions were sacrificed to it; outside of the family the self-instinct was then guide. So lived our remote ancestors. But the development of the brain meant the birth of memory, comparison, and reason, for those individuals who possessed a little better memory of the consequences of actions were able to determine what would be the most probable result of one not yet performed and so could better serve their self or family instincts. Thus with the birth of reason, instinct became but a secondary factor, and our primitive ancestors, reasoning that a greater surety of food and protection was given by that social institution, the tribe, formed in those more efficient bodies, which had a greater scope of action than was possible for an individual. Natural selection still kept up a certain low standard within the tribe (by rivalry for females and by disease) and also outside of the tribe by selecting those tribes of the greatest population or best organization, thus spreading tribal formation over the continents. But to-day with the decrease of rivalry inside and outside of our social institutions, i. e., decrease of war, disease and personal conflict, natural selection has become almost inert. Our evolution—the evolution of our organization—is proceeding by means of the reasoning powers of man and by the necessity for social action forced upon him by his fellows. In early life he imitates and then sees the reason and expediency of social action. The self-instinct, the love instinct, the family instinct are here to stay, but as social evolution advances all actions are not caused by the first or as later by the first and second or as later when the field of action was divided among the first, second and third, but the field of each of these instincts approaches its limits as the broader fields of service to the nation, and later to society, develop.

Thus we see that natural selection produced a high type of individual, produced the self-instinct, the family-instinct, and had a small part in producing the tribe semi-instinct. Then as reason also developed and partly by it, by instinct, or by imitation, men banded into nations, natural selection slowly subsided and organization and education appeared. The self-instinct of the leaders was limited by the strength of the demands of the others even as it is to-day, the difference being in the strength of the demands. People seldom obtain any more than they demand as self-instinct has the field (*produces* actions) until it is encroached upon by the stronger demands of our fellows. Thus the only moral law, and the only expedient mode of action for ourselves is to comply with the stronger social demands as far as they extend—not so far that we are overcome by the self-action of others. Thus we must fight individually to the extent that individualism is practiced by others and must conform to the growing demand for social action—but not as far as the new twigs which must find nourishment and grow before they will bear our weight. The cause of great suffering has been and will be, (until the limit—utilitarianism—is reached through education) in social evolution, in determining how far, in regard to one's self, social action encroaches our field of expedient self-action. In the most successful lives this dividing line is more approximately determined, and those are unfortunates, who from lack of observation or foresight act either as the criminal, robber, small tyrant, etc., (too much individualism) or such few and unnatural men as Timon of Athens of whom it could be said:

"Poor honest lord, brought low by his own heart,
Undone by goodness! Strange unusual blood,
When man's worst sin is, he does too much good!"

They are to be pitied, but that which caused their action cannot be censured as immoral because it is a necessary accompaniment of the individual-social metamorphosis, and all admit that the end of social evolution is one of the greatest goods attainable by man.

At the present day an excellent example of this is afforded by the action of Germany in European affairs, and is applicable individually as well as nationally. Germany has asserted her self-rights as far as possible. She has exacted Alsace-Lorraine from France and is now endeavoring to shut her out of Morocco. It is a case of get as much as you can without burning your fingers. Now were England, France, and Russia to form a coalition, a strong demand would be created and, being expedient, Germany would have to comply with it. This is of course explaining the extreme expedient selfish case as it exists to-day. But there are others against whom there is not so strong an individual competition and who then can comply also with the lesser demands of society. As social evolution progresses these necessarily become greater in numbers and the evolution gains increasing force as it advances. As the child's first social acts are imitative and educationally induced and as later he sees the expediency of social institutions and demands, so progresses his moral education. And if he has the self-instinct strongly developed, its field of action in him will be limited only by the strongest and most immediate demands of society—demands which require the minimum amount of social action only, and he will not contribute to social progress. But those in whom the instinct is not of such force or who have been educated in highly organized communities, do not stop social action and revert to self-action only at the strongest demands of society, but comply with the lesser demands; themselves create lesser demands and strengthen the pre-existing ones, so that the social evolution of any community or people depends on the number of this type of individual that it contains—if the self-actors predominate evolution would necessarily tend to revert to the remote unsocial period and vice versa. The great factor in producing the less selfish actors in the majority is, that once headed in the direction (usually by education) like habit, the tendency is to let the field of self-instinct be encroached upon gradually more and more (of course retaining as much of the instinct as is required by expediency to combat with the amount of self-action of others at the stage of evolution of the time of the individual). Thus the field of social action widens and limits that of self-action. New demands are created, by a majority; the former weak ones strengthened, and the strong ones are become a matter of course and habit.

It is apparent from this how any set rule for moral action has only been valid for the state of society at its birth, and how in order to lead the most satisfactory life we must comprehend (approximately) the existing state of social evolution—must observe and follow the amount of social action that can be indulged in without neglecting the individual action necessary to maintain one's self. Thus utilitarianism in being the consummation of moral or social evolution—all actions for the good of society and the maximum individual welfare possible for all (the welfare of society's individuals being its own) is not a fit "working hypothesis" to-day as a certain amount of self-action must be mixed with the social. It is, I believe, the goal of social evolution—distant, undiscernible, on the other brow of the earth—and we know the earth is round. Utilitarianism, service substituted for gain, thus seems the far off end of moral action.

I can in no way agree with M. Elie Metchnikoff, who, after showing the insufficiencies of the moral doctrines of Kant and Spencer says, "The ideal will rather be that of men who will be self-sufficient and who will no longer permit others to do them good" — in other words the super-man of Nietzsche. He, a biologist and scientist, fails to scan the field of organic development and does not see that organization is the keyword to all progress in that field. The organization of "chromideals" into cells; the organization of cells into communities or organisms and lastly the organization of organisms into what we call nations and states. The key-word to organization is not self-sufficiency but specialization, cooperation, reciprocal action. The cells perform different functions and loyally work with the welfare of all the other cells (the community) in view, and the organism or community can function where the single cell could not. The analogy is complete. He fails to see that in order for the family to exist, one member must procure food and protection, one must raise the offspring, and the offspring when independent can then become the head of another family (it being necessary for the higher action of the animal that its infantile development be longer). In the tribe some must procure food, others make implements, others protect, etc., in order that individually the tribe may better live and function in accordance with a more complex environment. Would this not be a low social state if each individual had to grow or hunt his own food, manufacture his clothes, his house, his vehicles, etc.? He would be self-sufficient and no one would be doing him good!

The relations of the part to the whole in any highly specialized society are analogous to those of the vital organs to the human body. There is paralysis throughout the system when its functions are interrupted. The lower forms of life are so simple that you cut and subdivide them at will without any impairment of vitality, but as organization develops, with a circulatory system and coordinate functions for the several parts, their independence is lost.

And so in a primitive society the individual is comparatively independent, but as organization takes place and specialization proceeds and the exchanges of civilized life develop, the well-being of the individual becomes more and more dependent upon his cooperation with the other individuals. "Our civilization is based upon the division of labor. Its industrial efficiency, its wealth of production, its comfort and luxuries and variety of opportunity, are the results of cooperative effort. If each member of the community, instead of supplying his own wants, devotes himself to one thing and all exchange the surplus products with each other, the sum-total of their production and possessions is increased." Specialization and not self-sufficiency is the first word in organization, civilization, and social evolution.

Society is automatically regulated, for each man will select as his vocation that mode of action for which society pays most and which he believes himself capable of fulfilling, i. e., to him the strongest demanded (highest paid) mode of action. And according to his ability will he succeed in supplying the demand or descending to a position where he can. To trace the demands of society upon the individual, is to trace the social and moral evolution of the race.

I can see nothing but benefit and increase of happiness from the struggle of the old with the increasing new idea of social duty and in the unhappiness, pain, and sorrow caused by the non-conformity of those unlucky individuals who lacked the wisdom to obey the demands of society as far as these went, or who disregarded the necessary, individual self-action for their happiness in that state of social evolution. The battle has brought and is bringing our more complete organization and individual specialization, and hence greater individual safety from disease, from improper education and from all such mistakes and imperfections as now exist in our governmental and labor organization. The mistakes are a benefit to posterity as it learns from them what should be built upon the present inherited foundation to further the completion of the structure.

So I have shown the reasons, the why, the by-cause of our social actions—which form a great percent of all our actions, and have also shown therefore that they are no less automatic than the others. And not only that but I have palliated to our demands for a completely good, unsullied universe, the number of so-called evils,—the sorrows and pains, which have arisen along with the social evolution as well as those which have arisen from the physical evolution.

It is asked, what is the meaning, the import, the purpose of it all, why the necessity of this development? I can only answer, the universe is infinite. What could be the purpose of the purpose, or the import of the import? Were matter absolutely dense—without motion, we would have no problem; but change is the second most apparent phenomenon. There can be but one kind of change and that is of the position of matter. This may be resolved into molecular and molar motion. If a change in the kind of motion is made it is in the cycle of molecular to molar and by contact of bodies back into molecular motion. There is no purpose, that is too human a mode of thinking. There is but one possible process and that is change. In the universe existence and necessity are the factors; they are not finite as the mind, but free—on account of themselves alone. I sit and watch the development of a crystal—of which we are the molecules, our cells atoms, and our "chromidials" ions. The change of this crystal is molecular into the more substantial molar state accompanying and a part of the earth's change from nebulous to a more solid condition. There is as much import in our development as the development of a grain of salt from solution, and the performance of the experiment is a show continuous. So much for the metaphysics of the question.

I have now, I believe, covered the entire scope of phenomena, have shown the reasons,—the causes of all actions, individual and social, and have shown how each leads life on to "a consummation devoutly to be wished." Thus the dread figure of "evil" has been exposed as a negative quantity while we admit and try to exterminate the to us evil. I have shown that we can blame nothing and that an optimism concerning the universe and its automatism is entertainable. Viscount Amberly has written, "Not in so slovenly a manner has the work of nature been performed. We are no more free to disturb the harmony and beauty of the universe than are the stars in their courses or the planets in their orbits . Our courses and orbits are no less fixed than theirs, and it is but the imperfection of our knowledge, if they have not been and cannot yet be discovered. But it would be a lamentable blot upon a universe, where all things are fixed by a law 'in whom there is no variableness nor shadow of turning' were there permitted to exist a race of creatures who were a law unto themselves." It is already recognized that knowledge repays a hundredfold the sweat that it cost us in this martyrdom of man, for we are thereby enabled to govern our future actions with greater wisdom and with more perfect reasoning, so I need not lay so great stress upon the almost omnipotence of the environment, the education of us all.

Thus, in the belief that "*Alles verstehen ist Alles dulden,*" I widen my moral horizon from that of Dr. James, and find no phenomenon caused by that law-perfect-in-itself: cause-effect, which is not perfect mechanically and morally. I make suffering a good and destroy the word *evil.* Concerning the necessity for "evil"; there is no necessity and its existence is only caused by our imperfection, our ignorance. I no more regret the above incidents than I commend one of the opposite character (except for purpose of encouragement) or blame hydrochloric acid for acting upon zinc. If they say, well then there is no use in our trying, things *will* happen as set from eternity, I say, unless you do act according to that necessary instinct and competent memory you will justly become a victim and you or your life, if nothing of an opposite influence affect you or it, will justly become martyrs and perish in the cause of good. Nature cares nothing for individuals and it is the individual's self-instinct which has brought the free-will and immortality doctrines into being. "The optimism of scientific minds rests in the belief that upon the physical plane—the development of bodily vigor, or upon the intellectual plane—making him capable of reasoning and thinking for himself, or upon the ethical plane—making him a useful, trustworthy human being, all dependent upon beneficial heredity and educational environment, that mankind must be strong, able and free, and that we shall not dwindle into physical weaklings, intellectual nonentities, or spiritual slaves or fanatics." Munro continues, "Life consists in the free exercise of our faculties and happiness in the successful performance of duty and achievement." Indeed I am sure we *can* rely upon that factor which exterminates human inertness, and without which I can see no advancement, no cause for the struggle and no justification of evil to our moral natures.

Some say that the effect of this belief on them would be a feeling of a weight and pressure of the rule of mechanism, that they must feel free in order to remain happy and that there is something uncanny in regarding living creatures as mere complicated machines. These are certain preconceived ideas, arising, not from a change of belief induced by reason or by considering, as I have shown, that the will to us exists, but from a certain fear of the unaccustomed caused by the absence or removal of a belief which had become a habit. Many peoples have lived happy with no feeling of oppression and been fatalists,—such as the old Anglo-Saxons and their *wierd* or fate, the Arabians and Persians who saw in all that took place the inevitable will of Allah, or, in more recent times, the Calvinists and others who betook themselves to this belief as the great and only consolation against the wrongs and injustices of the world. They were taught the belief; it was a part of them the same as the idea of free willing is a part of the majority of people to-day and so the opposite doctrine repulsive. We are thus human. It is a simple matter of attaining the correct attitude of mind and accustoming oneself to the idea, which is facilitated by the fact that will to us exists and that mechanism is more rational, more truthful, and more easily conceived.

A few more remarks will conclude all that I have to say. The belief that events are determinedly related to the condition of things immediately preceding them, is now held by all important thinkers in respect to all kinds of phenomena except higher animal volitions. In each successive department of fact, conflicting modes of thought have receded and faded until at last they have vanished everywhere except from this "mysterious citadel of the will." Then if we have any regard for consistency, and any regard for what facts, so far as we can see, tend to state, it is without the least disturbance of our scientific conscience that we can hold, until otherwise proven, that man is only a more complicated and variously endowed automaton, physical causes solely determining his bodily actions; the molecular activities of his cerebrum producing the succession of his mental states; and brain changes the real origin of those movements he is accustomed to regard as expressing his feelings, or as executing his intentions, those feelings and intentions being the mere "concomitant symbols in consciousness." That the universe ought to be rational is what these conscious feelings tell us, and I think I have ascertained that most rational conception, monism. Reason should be satisfied and I have shown that all things are governed according to that reason which actuates them. Knowing that we cannot *help* doing what our heredity and environment necessitates, I have inferred the direction that may be given to the whole course of a life by a little effort on the part of another to fit the man better to his surroundings and to insure his well-being. And lastly, the most important, I have shown that we may entertain an optimism concerning the universe, a view at once so necessary to our peace of mind and to our obtaining the best out of an existence where life must be thought worth the living and the struggle to repay its cost. In fact I see no reason why we should not welcome with open arms a conception so beneficial to the body, to the understanding and to the craving of the heart.

STEWART P. FOLTZ.

ASHEVILLE, N. C.

GELLERT'S PHILOSOPHICAL POETRY.

ADOPTED BY BEETHOVEN AS THE CONFESSION OF HIS RELIGIOUS FAITH.

BEETHOVEN was born a Roman Catholic and in his early childhood he received impressions exclusively of Catholic traditions, Catholic worship, and Catholic art. It must always have appeared to the boy that the Catholic church was the only religious institution. When he left the city of his childhood and youth whose government was in the hands of a prince-archbishop, one of the electors of the Holy Roman empire, he came to Vienna which is now and was especially in his days a typically Roman Catholic city. It is remarkable that under these circumstances he was not more limited in his religious conviction and art by the ecclesiastical influence which had a strong hold, for instance, on Liszt. Beethoven's religion had broadened under the influence of his acquaintance with other world-conceptions, and it appears that Gellert contributed most to the formation of his views.

Beethoven was a great reader, and we can trace the growth of his conceptions not only by the books he read but also by the very sentences which impressed him, for he had a habit of underlining what struck him forcibly, and thus we can trace his philosophical and religious development. Though he never broke away from the church, he broadened, and his general attitude was not greatly different from that of any other great man of his age. He admired Goethe though the two men were too different in character and disposition to become friends.

Beethoven's religion was strongly tinted by the rationalism of the Kantian school. His God was not the miracle worker, not the God who had revealed himself exclusively to Jews and Christians, and yet Beethoven did not hesitate to lend his art to the composition of a great mass. He was too broad to reject the artistic conception of a religion the dogmas of which he had outgrown.

As a rule when people broaden they become narrow in the very field of their mental growth. They love to parade their breadth of mind by objecting to those forms which characterize the narrower views. Not so Beethoven. He did not frequent the church or attend service, but he did not hesitate, when the opportunity offered, to write a mass for his friend the archduke Rudolf at his installation as archbishop of Olmütz, utilizing the traditional form of service that was customary in the Roman Catholic church. But his composition outgrew the limits of its earlier form. It became a cosmic epic, a doxology of the Creator, a triumphal song of God's glory and a proclamation of his divine dispensation.

The composition of this *Missa Solemnis* is no longer ecclesiastical in style. It has become poetry, and as such the Roman Catholic mode of worship serves as the basis for the presentation of a broader theme. It is like a philosophical drama in music; it is the denouement of the entire world process, an anthem to the infinitude of existence and the victorious advance of evolution, a hymn to the world-order.

In this same sense we have to interpret also Beethoven's compositions of the six religious songs of Gellert. They are Protestant in tone and Protestant in the austerity of their devotion. Beethoven accepts them not in the letter of the word but more as an artistic attitude to express his own sentiments. We cannot doubt that upon the whole he made the thoughts his own, and here in Gellert's songs, if anywhere, is expressed his own religious conviction. From the sentiment of the sixth of these songs, called "Penitential Hymn," the present generation has become estranged, and it will be difficult for us to understand Beethoven's attitude; but it will explain itself if we consider that Beethoven in his constant fear of appearing insincere frequently gave offense to his best friends, and then showed his regret by profuse acknowledgement of his mistake. These outbursts of temper and an ostensible show of discourtesy toward his very best friends, most of whom belonged to the highest circles of the Austrian aristocracy, are mainly due to his democratic pride and to the fear lest he depart from his ideal of independence. It was for the sake of the God within him that he was carried away to brusqueness and rude behavior, and he felt the adjustment had to be made with himself before God alone.

We here insert a translation of the six hymns of Gellert, following mainly the translation of H. Stevens. They read as follows:

PRAYER.

O Lord, thy goodness reaches far,
As far the clouds are guided;
By mercy crown'd, thy creatures are
With needful help provided.
Lord! my defense, my tower and shield,
To me a gracious audience yield,
Approve my supplication.

LOVE THY NEIGHBOR.

If one shall say, "I love the Lord,"
While yet his brother hating,

With mockers he shall reap reward,
God's truth abominating;
For God is love, and wishes me
With all on loving terms to be.

DEATH.

Life is ebbing fast away,
Hourly towards the grave I hasten;
Death may come without delay,
Let this thought my spirit chasten.
Man bethink thee Death is rife,
One thing needful is in life.

NATURE PRAISES GOD.

The Heavens declare the Lord's infinite glory,
The sea and earth sound forth his name,
And tell their origin's wonderful story,
Mark well, O Man, what they proclaim.
Who gave the numberless stars their existence,
Who calls the Sun from his abode,
He cames in brightness and smiles from the distance,
And like a hero keeps his road.

POWER OF GOD.

God is my song!
In strength he reigns victorious,
High is his name,
And all his works are glorious;
Earth, Sea and Heaven to him belong.

PENITENTIAL HYMN.

I.

'Gainst thee alone, God, have I sin committed,
And evil done in thy dread sight,
Thou seest my guilt for which thy wrath is fitted,
See, Lord, my woe and sore affright.

My piteous wail, my sighs are all before thee,
My tears of deep and bitter grief.
O God, my God, shall I in vain implore Thee?
How long wilt thou deny relief?

Lord, do not after my deserts reward me.
Chastise me not! Show me thy face;
I crave for thee! thy pardon, Lord, accord me,
O God of patience and of grace.

II.

O grant me early, God, thy consolation,
Oh Father of mercy, God of love,
For thine own name's sake grant my supplication,
Thou lov'st to bless from Heav'n above.

Let me thy pathway tread; let me be steady
In my obedience to thy word.
To do thy will I shall be always ready,
I am thy servant, thou my Lord.

Lord, hasten thou to shelter and defend me;
Thy light shall lead, point out the goal.
Thy helping hand, O Lord, thy helping hand extend me
And with thy comfort fill my soul.

PAUL CARUS.

CRITICISMS AND DISCUSSIONS.

BUDDHIST LOANS TO CHRISTIANITY.

WITH SPECIAL REFERENCE TO RICHARD GARBE.

In the October *Monist* Professor Garbe, of Tübingen, admits a Buddhist basis for the Christian legends of Saints Christopher and Eustace. In the early part of the same article he also admits Buddhist influence in the Christian Apocryphal Gospels, but denies it in the Canonical ones. I herewith submit two passages from the Gospel of Luke which appear to me to agree as closely with the earliest Buddhist texts as do the saint-legends admitted by Garbe.

The first parallel is taken from my now forgotten pamphlet of 1905, *Can the Pāli Piṭakas aid us in fixing the Text of the Gospels?* The second is from my *Buddhist and Christian Gospels,* as indicated in the first edition (1902) and partially printed in the third and fourth (Tokyo, 1905, and Philadelphia, 1908).

THE ANGELIC HERALDS AND THEIR HYMN.

Luke ii. 8-14.

And there were shepherds in the same country abiding in the field, and keeping watch by night over their flock. And an *angel* of the Lord stood by them, and the glory of the Lord shone round about them: and they were sore afraid. And the angel said unto them, Be not afraid; for behold, I bring you good tidings of *great joy* which shall be to all the people: for *there is born* to you this day *in the city of* David a Saviour, which is *Christ* the Lord. And this is the sign unto you; Ye shall find a babe wrapped in

Sutta Nipāto, Mahāvaggo, Nālaka-sutta (known only in Pāli, but with analogues in later Buddhist books).

The heavenly hosts rejoicing, delighted,
And Sakko the leader and *angels* white-stoled
Seizing their robes, and *praising* exceedingly,
Did Asito the hermit see in noonday rest.

[He asks the angels why they rejoice, and they answer:]

The *Buddha-to-be,* the best and matchless Jewel, swaddling clothes, and lying in a manger. And suddenly there was with the angel a multitude of the *heavenly host praising* God, and saying,

Glory to God in the highest,
And on earth peace, divine favor among men.

Is born for weal and welfare in the world of men,
In the town of the Sākyas, in the region of Lumbini:[1]
Therefore are we joyful and *exceeding glad.*

The parallel is further carried out in the narrative. The hermit, like the shepherds, goes to pay his reverence to the newborn Saviour.

Considering that between the Greek of Luke and the Pāli of the Sutta Nipāto there may lie some lost book, the words in italics are practically identical. The Pāli words *hita-sukhatāya* ("for blessing and happiness") are a convenient phrase, often recurring in the texts. We here translate them "weal and welfare" for the sake of poetic effect, but they mean much the same as the English phrase, "peace and prosperity." Now if Luke, or rather his Oriental intermediary, did actually use the Pāli poem, it is evident that omitting *jāto* ("born"), we find a very good equivalent of the line:

Manussaloke hitasukhatāya jāto,

in the line:

ἐπι της γης εἰρηνη ἐν ἀνθρωποις εὐδοκια.

It is thrown into the form of a Hebrew parallelism, in which peace on earth and divine favor among men are interchangeable terms. It is well known that the oldest manuscripts of the New Testament are at variance here over the word εὐδοκια. Some read εὐδοκιας (genitive) and then we must render: "among men of good will" (or the divine favor, i. e., the elect, as Alford says).

This is the reading of the Vulgate and of the English and American Revised Versions. It is because εὐδοκια in the Septuagint means so often the divine good pleasure that the Revised Version has "men in whom he is well pleased." But the old King James reading (following the *textus receptus* afterwards fixed by the Dutch printers Elzevir) is borne out by the analogy of all Hebrew parallelisms. This is therefore a passage wherein the Pāli Piṭakas can probably aid us in fixing the text of the New Testament.

This parallel is ignored by Garbe, though he mentions that of Asito and Simeon, which is connected with it in the Pāli. But the

[1] A pre-Christian inscription was lately discovered, marking the site of Lumbini.

Lalita Vistara and other late books relied on by Garbe, and by Sanskrit scholars generally, do not contain the Angelic Hymn. I admit the weakness of the Asito-Simeon parallel, when taken by itself; but its strength consists in its organic connection with the Angelic Hymn, both in Luke and the Sutta Nipāto.

In *Buddhist and Christian Gospels* (4th ed. only) I have shown that Luke's alteration of the Buddhist legends is no more than his alteration of the Synoptic tradition (Mark xvi. 7, compared with Luke xxiv. 6).

When all this has been studied as carefully as older points of Gospel criticism, the day will come when school-children will know that "Peace on earth, good will to men" is a Buddhist text.

THE LORD'S THREE TEMPTATIONS.

Luke iv. 1-3.	Classified Collection, Book of Temptations (Pāli and Chinese).

In the Wilderness.

And Jesus, full of the Holy Spirit, returned from the Jordan, and was led by the Spirit *in the wilderness* during forty days, being tempted of the devil. And he did eat nothing in those days; and when they were completed, he hungered.

At one season the Lord was staying in the land of the Kosalā, *among the Himālayas,* in a log-hut. While thus living *in hermitage retired,* the reflection arose within him: "It is really possible to exercise dominion by righteousness, without slaying, or causing slaughter; without oppression or the making thereof; without sorrow or the infliction thereof."

Temptations to Assume Empire and Transmute Matter.
(In different order in Luke and the Pāli.)

And the devil said unto him, If thou art the Son of God, *command this stone that it become bread.* And Jesus answered unto him, It is written, Man shall not live by bread alone. And he led him *up,*[2] and shewed him all the kingdoms of the world in a moment of time. *And the devil said unto him, To thee will I give all this authority,* and the glory of them: for it hath been delivered unto me; and to whomsoever I will I give it. If thou therefore wilt worship before

Then Māro, the Evil One, perceived in his heart the thought which had arisen in the heart of the Lord and he approached the Lord and *spake thus*: *"Lord, may the Lord exercise dominion;* may the Auspicious One exercise dominion by righteousness, without slaying or causing slaughter; without oppression or the making thereof; without sorrow or the infliction thereof."

"What seest thou in me, O Evil One, that thou speakest thus to me?"

[2] Matthew has: *unto an exceeding high mountain* (thus agreeing with the Pāli idea of the Himālayas).

me, it shall all be thine. And Jesus answered and said unto him, It is written, Thou shalt worship the Lord thy God, and him only shalt thou serve.

"Lord, the Lord hath practised the four principles of psychical power, hath developed them, made them active and practical, pursued them, accumulated, and striven to the height thereof. *So, Lord, if the Lord desired, he could turn the Himālaya, the monarch of mountains, into very gold, and gold would the mountain be."*

[Buddha replies:]

"The whole of a mountain of gold, of fine gold,
Twofold, were not enough for one;
Let him who knoweth this govern his life.
He who hath seen Pain and whence its rise,
How could such a one bow to lusts?
He who knoweth that the substratum of existence is what is called in the world 'Attachment,'
Let that man train himself in the subdual thereof."

Then Māro, The Evil One, said, "The Lord knows me; the Auspicious One knows me." And he vanished thence, unhappy and disconsolate.

Temptation to Commit Suicide.

(Continuous in Luke).

And he led him to Jerusalem, and set him on the pinnacle of the temple, and said unto him, If thou art the Son of God, cast thyself down from hence: for it is written,

He shall give his angels charge concerning thee, to guard thee:
and,
On their hands they shall bear thee up,
Lest haply thou dash thy foot against a stone.

Book of the Great Decease: Long Collection, Dialogue 16; Chinese, No. 2. (Three months before Buddha's death).

Now not long after St. Anando had gone, Māro, the Evil One, approached the Lord, and standing beside him, addressed him thus:

"O Master, let the Lord now die the death of an Arahat,[8] let the Auspicious One die the death of an Arahat: now, O Master, is the time for the Lord to die this death; and moreover this word was spoken by the Lord: 'O Evil One, I shall not die the death of an Arahat until my

[8] *Parinibbātu,* literally "become extinct," conveying the double idea of physical and passional death. See note in *Buddhist and Christian Gospels,* fourth ed., Vol. II, p. 99.

And Jesus answering said unto him, It is said, Thou shalt not tempt the Lord thy God.

monks and nuns, my laymen and lay-women become wise and trained disciples, reciters of the Doctrine, walking in the doctrine and the precepts, walking consistently, living out the precepts........

"And now, Master, [is this the case]. O Master, let the Lord now die the death of an Arahat, let the Auspicious One die the death of an Arahat; now, O Master, is the time for the Lord to die this death!"

When he had thus spoken, the Lord said unto Māro, the Evil One: "O Evil One, be content; the Tathāgato's Arahat-death will not be long: at the end of three months is the time for the Lord to die the death of an Arahat."

The Devil Disappears.

And when the devil had completed every temptation, he departed from him for a season.

Classified Collection (in sequence above).

Here we have, in the Pali and the Chinese of the Classified and Long Collections, representing two Buddhist sects of great antiquity, the following root-ideas:

1. Appearance of the Tempter to the Saviour in a wilderness.
2. Temptation to assume empire.
3. To use mystical power to transmute matter.
4. To commit suicide.
5. Disappearance of the Tempter when foiled.

Now Luke has these same root-ideas, though expressed differently in the third case (or, in his text, the first): viz., the transmutation of stones into bread instead of into gold. Matthew also has them, but he interpolates Luke's third temptation (that of suicide) between them. I therefore give the text of Luke, because it agrees with the Buddhist association, as Luke so often does.[4]

It is imperatively necessary to study these parallels by means of their earliest sources; viz., the Pāli and Chinese Hināyāna texts

[4] See the article *Luke and Buddhism,* in the General Index to the fourth edition of *Buddhist and Christian Gospels.* Of course there is the possibility that the Temptation scenes of Luke and Matthew (they are not in Mark, though he mentions the Temptation) belong to a lost book whereto both are indebted. I believe scholars generally consider that these scenes were not in the Logia source. My own belief is that Luke was the first to introduce them, and the editor of Matthew adopted them from his text.

on the one hand and the Greek Gospels on the other. Seydel made the great mistake of dealing with late books like the Lalita Vistara, without distinguishing its lesser value for the comparison. Even so learned a scholar as Garbe still holds to the Seydel tradition, and consequently makes short work of the Temptation parallel by quoting these later legends (*Monist,* October, 1911, pp. 517, 518).

I maintain that there is as much striking agreement between Luke and the Hinayāna texts as there is between the Jātakas and the legends of Saints Christopher and Eustace, except that the latter are much longer and furnish more details for comparison.

In the temptation story there is the same Christian coloring as in the saint-legends, and yet the root-ideas agree. The Christian coloring consists in making the Master quote scripture, whereas the Buddhist idea requires him to state some truth. Again and again in the Jātakas do we find the same magical efficacy ascribed to the calm enunciation of a truth which the Brahmins ascribe to the words of the Veda and the Jews to those of the Torah. In the Zend-Avesta the Tempter uses a similar sacred word, but, as hinted elsewhere (*Buddhist and Christian Gospels,* 4th ed., Vol. I, p. 106), the Mazdean temptation story is only like the Christian one in its theism and its quotation of scripture. The earliest account of the temptation of Zoroaster is in the Vendidād, and it consists of only one, viz., that of empire. Before the temptation the fiend makes a vain attack on the prophet's life, and after it the prophet declares that he will defeat the forces of evil by two things:

1. The eucharistic utensils and sacred drink;
2. A magical word taught him by the Godhead in a past eternity.

While all this is of fascinating interest to the student of religion and of the New Testament in particular, yet it is by no means so close to the Christian stories as are *the earliest* Buddhist ones.

The Classified Collection and the Decease Book represent home-grown primitive Buddhism. And with these does Luke agree rather than with the geographically and theologically nearer Zoroastrian account.

In two other cases does Garbe neglect important parallels from the Pali Nikāyas. On page 521 he gives us interesting evidence, from his Sanskrit reading, of the Hindu character of the idea of walking upon the water, and says (as since amended) that it "belongs not only to the India of Budhism, but to that of Brahminism also." He ought to have added that the power to walk on the water is among the gifts of a pious Buddhist, ascribed to him by Buddha himself, in the sixth sūtra of the Middling Collection in the Pāli (No. 105 in the Chinese version of A. D. 397)—a Hindu book far older than the Brahmin Mahābharata (though not of course than its ancient nucleus).

Again on page 517 Professor Garbe says: "Christ fasts forty days *before* the Temptation, Buddha twenty-eight days *after* the Temptation." But in the thirty-sixth sūtra of the Middling Collection we read that Buddha fasted nearly to death before his illumination, and therefore before his Temptation, which latter occurred after he was Bhagavā (the Lord).[5]

No one who studies the *Periplus of the Erythræan Sea,* a captain's log book of the first century (now newly translated by Wilfred H. Schoff of Philadelphia) will be able to agree with Professor Garbe (p. 524) in his limitation of the probability of Indian influence on Palestine to later times. The Periplus agrees, for the sixties, with Strabo, who saw 120 ships ready to sail from a Red Sea port to India in the twenties of the first century. And, as Wilfred Schoff has shown in his article on another page of this issue, the Roman Empire had a sort of Indian craze at that very time.

In *Buddhist and Christian Gospels,* the Lalita Vistara and other later books are treated in the Appendix as "Uncanonical Parallels," while the body of the book deals with canonical parallels, translated from the Pāli texts by myself and compared with the Chinese version of another ancient recension of the Buddhist scriptures (the Hindu original of which is lost) by Professor Anesaki of Tokyo.

When Rhys Davids's *Buddhist Suttas* (Sacred Books of the East, Vol. XI) were sent me by my bookseller in 1881, I found therein a vigorous protest against any attempt to trace Buddhist loans in the New Testament. This made a great impression upon my youthful mind, and acted as a deterrent in that direction until nearly the end of the century. Then, in 1899, Rendel Harris astonished me by postulating a Buddhist influence in the Acts of Thomas and (save the mark!) in the Gospel of Luke! I was stunned at first, then rallied myself and returned to my old objections. During the next seven years, however, deeper research caused me to change; and when in 1906 I observed the double quotation in John,[6] I admitted that here at least was tangible influence. It was anent the essay which I then wrote that Rhys Davids said

[5] Samyutta Nikāyo, already quoted. Had the Temptation occurred before the Illumination we should have read *Bodhisatto.*

[6] See "Buddhist Texts in the Fourth Gospel," *Open Court,* May, 1911.

to me: "The evidences in favor of intercommunication are growing every day." (I asked his permission to quote this, and he granted it). Paul Carus, in *The Open Court,* October, 1911, has adduced a remarkable picture from a Greek vase, portraying a goddess with water for her lower body, and he thinks that both the Buddhist and Johannine texts may be dependent upon some such ancient idea. So they may, but the strength of my case lies in the fact that the Fourth Gospel's express quotations from sacred literature (*Law* and *Scripture*). Instead of admitting that the quotations are from the Buddhist writings, where I have found them, several of my critics prefer to ascribe them to some lost apocryphal Jewish book. But the time is rapidly passing when scholars will feel compelled to adopt any hypothesis rather than admit the greatness of ancient India and the supremacy of Buddhism which, at the time of Christ, was the most powerful religion on the planet and the dominant spiritual force upon the continent of Asia.

In *Buddhist and Christian Gospels* (4th ed., Vol. II, p. 237) we read:

"A collection of [uncanonical] parallels would probably suggest a Christian influence upon later Buddhism; and indeed we know that, in the eighth century, a Chinese emperor had to forbid the two religions to be mixed. (See Takakusu's note in his *I-Tsing,* Oxford, 1896, p. 224.) This whole field needs very careful working, more than I am able to give."

Two Anglican clergymen, the late Samuel Beal and Arthur Lloyd recently deceased, have maintained this position. The fact is that after Kanishka's Council a new type of Buddhism, predominantly Mahāyāna, gradually supplanted the earlier. This new type was largely foreign, as the primitive type had been native Hindu. Before the Scythian invasions at the end of the first century, the Buddhism of Asoka, with its Pāli texts, had been in the ascendant; and as, in the first century, Christianity was in a formative stage, while Buddhism was settled and aggressive, the loans went from east to west. But afterwards there was a change. In the first place, a different race of sailors appeared in the Red Sea ports,[7] bearing with them the newer Buddhism which they themselves were helping to modify; and, secondly, Christianity itself was becoming a rival to Buddhism, and was beginning to assert itself.

It may be that Buddhism influenced the Roman Empire by

[7] I owe this information to Wilfred H. Schoff, translator of the new edition of the *Periplus.*

means of intermediary books, such as that of Elkesai which had a confessedly Buddhist origin ("Seres of Parthia"); but I maintain that the Nikāyas of primitive Buddhism were strong enough to make themselves felt more directly. In A. D. 149 a Parthian prince headed a long series of scholars who translated them into Chinese; but Buddhism had been established in the Greek empire (*Yonaloko*) since the third century B. C., and was quoted, chapter and verse,[8] by a Greek king, Menander, in the second. Now, the Chinese began to translate Buddhist books immediately upon that religion's introduction into their country in the sixties of the first century; and after a generation or two of translating manuals, lives of Buddha etc., they spent three centuries (circa 150-450)[9] in translating the Nikāyas (or Agamas). Were the Greeks less curious than the Chinese? Had not they also begun to translate the books they admired long before the time of Christ? My thesis is this:[10]

While a religion is in its formative stage, its founders take ideas from their environment, and especially from any system of thought that is paramount, whether in their own country or in those wherewith they have intercourse. But, once knit together, and moving by its own momentum, a religion can no longer add to its primitive documents, though it may give way to new influences in later sectarian developments.

The thesis applied is this:

During the first century Christianity was in its formative stage, and was influenced by the Old Testament, the Greek mysteries, the Philonic philosophy and by Hinayāna Buddhism. After the first century Christianity was strong enough to influence another religion in its formative stage. And such was Mahāyāna Buddhism, which was, in fact, a new religion, with new doctrines and new sacred books. At the same time, Hinayāna Buddhism still existed, and indeed its votaries often cultivated the Mahāyāna too. Consequently there could be and there was a complex interchange between Christianity and Buddhism, both of them giving and taking. But the earliest interchange was when the Hellenizing Evangelists Luke and John borrowed some minor features from the Hinayāna Nikāyas, then in the ascendant.

Before closing, let me add a note on the Wandering Jew legend

[8] So in the Pāli, though Chinese versions do not bear it out.

[9] Anesaki in *Transactions of the Asiatic Society of Japan*, 1908, p. 15.

[10] See my remarks on the Imperfection of the Record (following Darwin) in *Buddhist Texts in John* (2d ed., 1911, p. 27).

among the "Uncanonical Parallels" in my *Buddhist and Christian Gospels.* I lately learned that Sabine Baring-Gould in 1866 pointed out that the germ of the legend is actually found in the canonical Gospels:

Mark ix. 1: "Verily I say unto you, There be some here of them that stand by, who shall in no wise taste of death, till they see the Kingdom of God come with power."

Let me repeat what I said last May in *The Open Court,* and which Professor Garbe does me the honor to quote: *Each religion is independent in the main, but the younger one arose in such a hotbed of eclecticism that it probably borrowed a few legends and ideas from the older, which was quite accessible to it.* The loans are not an integral part of primitive Christian doctrine, as I said in my Tokyo preface (1905), but lie outside of the Synoptical narrative, and occur in the two later Gospels of Luke and John, both open to Gentile influences.

Even now I only put forth these parallels upon the same footing as Gaster, Speyer and Garbe's Christopher and Eustace; and if the scholars of Europe and Asia finally decide that they are wrong, I shall withdraw my venture with a good grace. But if this great admission of Buddhist influence upon the Christian Apocryphal Gospels and the Eustace and Christopher legends receives its "brevet of orthodoxy," the next step will lead a new generation of scholars back to the canonical Gospels and the canonical Nikāyas.

Albert J. Edmunds.

Philadelphia, Pa.

FIRST CENTURY INTERCOURSE BETWEEN INDIA AND ROME.

EDMUNDS VS. GARBE.

In *The Monist* for October, 1911, appears a paper by Prof. Richard Garbe of Tübingen entitled "Contributions of Buddhism to Christianity," the essence of which is that common material is found in the Apocryphal writings of both religions, but that no connection can be proved between the Canonical texts, and that this is due to the fact that active intercommunication between India and the Mediterranean did not exist until the second century, or, as Professor Garbe puts it, "Buddhist influence might have penetrated to Palestine by way of Alexandria, but still more probably by way of Antioch in Syria, but they" (that is, writers pointing out similarities) "are not apt to raise this possibility to a serviceable degree of probability for as early a period as the first post-Christian century."

In thus denying the existence of a rapidly growing and very important stream of commerce between India and Rome, it seems evident that Professor Garbe has overlooked historical facts which, if duly recognized, may compel him to revise his opinion in this matter as he changed his mind in regard to the migration of the fish-symbol from India to Rome.

The incontestable facts of history are that a large Indian influence and an active commerce existed as far as the Mediterranean coast of Syria soon after the conquests of Alexander, and that the conquest of these territories by Roman armies ending in the public triumphs of Pompey the Great, created in the Roman capital a craze for Indian products and luxuries of all kinds which during the actual lifetime of Christ had become a serious problem to the Roman government, leading to numerous efforts at discouragement of the taste for Eastern luxuries which was draining the Empire of its resources. This craze met with a temporary check at the death of Nero. It regained full intensity under Trajan and Hadrian, and was again in a decline during a considerable part of the second Christian century, reviving during the reign of Commodus, and again more seriously declining with the failing powers of the Empire. The existence of this craze for Indian imports and of the substantial remittances of gold coin required to balance the trade, may be surely proved by the hoards of Roman coin unearthed in Southern India and catalogued by the Government Museum at Madras; in which these fluctuating eras of trade prosperity and depression clearly appear. Instead, therefore, of the creation of a new import trade from India in the second century, as Professor Garbe asserts, the most active trade was in the first half of the first century, with two revivals at the beginning and the end respectively of the second; and the drain of specie from Rome to the East had set in even before the birth of Christ.

Space forbids a statement in detail of the almost innumerable facts existing to support the foregoing statement. The following may at least serve as suggestions.

Alexander married a Persian princess, but numbers of his officers took Bactrian and Indian wives.

Greek colonies were established by him along the entire Indian frontier, and colonies of his newly established Indian subjects were similarly established nearer Greece. A Greek dynasty ruled in Bactria after the Parthian revolt disrupted the Seleucid empire, and one of its rulers, Menander, powerfully influenced the spread of Buddhist thought through the Greek-speaking world.

A Greek ambassador at the Maurya court, Megasthenes, wrote a detailed account of its customs, its Brahmin religion, and its capital Pātaliputra; which was widely read and commented upon for centuries.

The conquest of Judea by the Persians and the destruction of the Persian empire by Alexander, reduced the force of Judaism and Mazdaism as world-religions, while the exodus of the Greeks into the East broke down what was left of the distinctive Greek religion. There existed then no faith strongly upheld in the Eastern Mediterranean basin from the third to the first centuries B. C.

Two generations after Alexander's conquests, the Emperor Asoka established Buddhism as the state religion of India, and in his second edict, preserved to us in a rock inscription, he mentions the sending of envoys to all countries with which he entertained relations; particularly mentioning "the dominions of the Greek king Antiochus, and those of the other kings subordinate to that Antiochus." This ruler is identified with Antiochus Theos (B. C. 261-246) in whose capital of Antioch these Indian envoys, physicians and missionaries, for they seem to have held that triple character, were received. In the capital of that ruler who profaned the Jewish Holy of Holies in order to set up the worship of himself, the Buddhist faith was preached by men sent from the head of the Buddhist organization, the ruler of the richest, most powerful and most populous empire in the world at that time.

During the better days of the Seleucidæ, overland communication between India and Syria was unhampered, and there is every indication that it carried an active commerce. The fall of the Seleucid power and the rise of the Parthian monarchy interposed a fiscal obstruction which the Greek rulers in Egypt, the Ptolemies, quickly turned to their advantage. By the establishment of ports on the Red Sea, Egyptian shipping was enabled to trade in the Gulf of Aden and obtain Indian merchandise with less transshipment than had formerly been made, and the opulence of this trade is vividly described by Agatharchides, writing in the closing years of the second century B. C.

For two centuries following Alexander's death we may assume that the Indian trade went no further than the Eastern Mediterranean; but the rise of Rome as a world-power, dating finally from the sack of Carthage and Corinth in 146 B. C., brought the Romans into active trade with the Levantine ports, as evidenced by the growth of piracy in that region, preying on the Roman ships. Pompey's contributions to the Roman state were the suppression of the pirates and the conquest of the Levant; and in his triumphal processions, which are repeatedly mentioned by Pliny in his "Natural History," all the more precious varieties of Indian merchandise were exhibited and brought into popular demand. This point is of importance. Two generations before the birth of Christ the spoils of a conquered land resulted in a fashion for the imports of that land rather than for its own products: for the Indian goods transshipped at the Syrian ports, rather than for the products of Syria itself. The Indian trade had become Syria's richest asset.

The same facts are in evidence upon the conquest of Egypt and the incorporation of the Alexandrian trade into the Roman fiscal system. Primarily grain was the staple export from Egypt to Rome, but the more profitable trade consisted in the incense of Arabia and the gems and spices and textiles of India.

In 22 A. D., in a letter from the Emperor Tiberius to the Roman Senate set forth by Tacitus in his "Annals," the growing drain of specie is pointed out and a remedy demanded. "How," said the Emperor, "are we to deal with the peculiar articles of feminine vanity, and in particular with that rage for jewels and precious trinkets, which drains the Empire of its wealth and sends, in exchange for baubles, the money of the Commonwealth to foreign nations; even the enemies of Rome?"

The geographer Strabo, writing in almost the same year, records having seen a single fleet of 120 ships about to start by the favorable monsoon from an Egyptian Red Sea port to India. Two generations later, according to Pliny, the unfavorable trade-balance had grown more serious still; as he says "in no year does India drain us of less than 550,000,000 sesterces, giving back her own wares, which are sold among us at fully 100 times their first cost."

550,000,000 sesterces in those days was a very considerable sum. In modern valuation it would approach $25,000,000, and this was the state of affairs existing at the end of the reign of Nero. Can one imagine a modern trade requiring so enormous an export of specie without a corresponding influx of merchants, bearing ideas no less than goods, from the producing to the purchasing market? This condition is indeed set forth with sufficient exactness by the writer of the Apocalypse, where he describes, under a veil of fiction, the burning of Rome and the ruin that thereby came upon "every ship-master and all the company in ships, and sailors, and as many as trade by sea," while of the merchandise they handled are specified numerous Indian products, precious stones, pearls, silk, ivory, fragrant wood, iron (Indian steel was known even to the Greeks), cinnamon, odors, ointments. This was in 64 A. D. A year or two before, according to Pliny, at the funeral of Nero's consort Poppæa, there was burned a store of Eastern spices representing a year's imports and valued at millions.

The unknown merchant of this same period who has left us that interesting log of his trading voyages from Roman Egypt to India which we know as the "Periplus of the Erythræan Sea,"[1] enters more specifically into the various articles dealt in and the marked growth in the trade. Briefly following him along his voyage, at the lower western shore of the Red Sea were imported Indian iron and steel, Indian cloth, muslin and lac. On the opposite shore, at the Arabian side of the straits, was a special port established for incoming Indian ships, which were apparently forbidden to trade by the Arabs' port of Muza. On the outer coast, which we know as Somaliland, Indian cinnamon was found and ships of larger size were now required to handle it. Other Indian gums are specified, among them gum *dammar,* and an Indian remedy for tropical disorders, *macir,* which does not again appear in western commercial annals until the days of the Portuguese. At Cape Guardafui was a regular trading rendezvous to which came numerous ships from the Gulf of Cambay bringing cereals, clarified butter, sesame oil, cotton goods, and honey from the reed called "sacchari"; the first known record of sugar as an article of commerce.

On the southern coast of Arabia were two ports at which Indian shipping regularly called. At the one Roman coral, tin, copper and storax were transshipped for the Indian trade, and at the other, more to the east, Indian shipping often wintered. Proceeding with our merchant to the mouth of the Indus, we find these same Roman products recorded among the imports of Northwestern India including, strange to say, Italian wines, preferred to the Syrian, or Arabian; all of which were imported. At the port of Barygaza in the Gulf of Cambay, the newly established Saka government

[1] A new translation, with learned notes, of this document is listed by Longmans for 1912. The translator is the writer of this article.—Ed.

maintained a regular system of pilotage which was necessary to avoid destruction of foreign vessels by the tremendous tides of that estuary. These pilot-boats coasted the shores of the Gulf for 100 miles outside the port, and our merchant records that both Greek and Arabian shipping was guided by them. Here he found among other things, spikenard, highly treasured in the ointments of the time as appears in the Gospel of Mark, chap. xiv. 3-5; and more important still, murrhine, that Indian carnelian, its colors heightened by slow heat and shaped into drinking vessels for which, according to Pliny, fabulous sums were paid in Rome. Petronius broke one of Nero's basins valued at 300,000 sesterces, while Nero himself paid one million sesterces for a single cup. Here at Barygaza were also imported for the Indian markets Italian wine, copper, tin and lead for the coinage of the country, coral and topaz, storax for the Chinese trade, glass, gold and silver coin on which there was a profit when exchanged for the money of the kingdom,—the Roman coinage being superior to the Hindu, which was of base metals only, while the Roman gold coin formed the standard of exchange for all the nations of India. Further down the coast in the back waters of Cochin and Travancore he found especially pepper and malabathrum (cinnamon leaves), on account of the great quantity and bulk of which our merchant tells us, large ships were sent to those ports, Greek and Arabian as well as Hindu. Here were found also great quantities of fine pearls, ivory and precious stones, beryls, diamonds and sapphires, and tortoise-shell, coming from as far distant as the Straits of Malacca in ships specially recorded as "of great size" in comparison with those Roman ships with which our author was familiar. In the adjoining nation, easily recognizable as the Chola Kingdom, whose capital Uraiyūr (Trichinopoly) is recognizable under the author's corruption of Argaru, were found in profusion all the merchandise sent from Egypt; while its ports were a center of shipping not only from Egypt but from the Ganges and Malacca. Here our author digresses to mention Chinese silk brought overland through Bactria to Western India for reshipment to the Roman empire, and among the exports from Rome to balance this trade is again mentioned "a great quantity of coin," fully supporting the testimony of the hoards unearthed in Southern India and recorded at Madras. The coins of Claudius and Nero are among the most numerous of all discovered.

The word which the author of the Periplus uses for the palm oil found by him at Zanzibar, was a word brought from India, the

Prākrit *nargil,* coconut. The most authentic information at the disposal of Lieutenant Speke in preparing for his expedition for the discovery of the sources of the Nile, was a map based on the Hindu Purānas, and setting forth information brought by these same Indian vessels found by the merchant of the Periplus on the African coast. These traders had penetrated the interior and knew of the Nyanza lakes, as the Egyptians did not. The facts already cited are surely sufficient to show a volume of trade not only internationally important, but so great and so one-sided as to be recognized as a serious menace to the prosperity of the newer, poorer, and less populous empire of the West.

Petronius, Nero's crony whom Pliny connects with the mad auction of murrhine cups, has left us *Trimalchio's Dinner,* that inimitable sketch of parvenu society in Rome at the middle of the first Christian century, in which it is mentioned as a matter of course that a rich man sent to India for so slight a thing as mushroom spawn. Pliny tells how Lollia Paulina, wife of the Emperor Caligula, wore at an ordinary betrothal entertainment emeralds and pearls to the value of 40,000,000 sesterces; "indeed, she was prepared to prove the fact by showing the receipts and acquittances." And he goes on to bemoan the prodigality in the use of Indian pearls by Roman women; "now, at the present day" (about 70 A. D.) "the poorer classes are even affecting them.... they put them on their feet, not only on the laces but all over the shoes; it is not enough to wear pearls but they must tread upon them."

The author of the Periplus tells how the Indian trade, as far as western shipping at least was concerned, used to be done in small vessels close to shore; and how Hippalus "by observing the location of the ports and the conditions of the sea, discovered how to lay his course straight across the ocean"—the monsoon being called the "wind of Hippalus"—so that from that time ships steered direct from the Gulf of Aden and Cape Guardafui to the ports of India, "holding their course straight out to sea with a favorable wind, quite away from the land." This discovery of Hippalus occurred in the time of Claudius, and the resulting increase of trade culminated under Nero. Pliny recounts the same story.

The distinction made by Professor Garbe between the parallelisms in the Canonical texts and those in the Apocrypha points to a period of change in the national and religious politics of India which is apparently not realized, and is yet of importance in the study of the interrelations between East and West. At the beginning of the second century came the Council of Kanishka, the Scythian conqueror of the northwest, the second great Buddhist Council. The Scythians were looked upon askance by the native Hindus. It is recorded in the annals of the Andhra dynasty that after a victory over the Scythian or Kushan dominion, a memorial was set up at Kārli telling how the orthodox Andhra king had "destroyed the Sakas, Yavanas and Pahlavas, properly expended the taxes levied in accordance with the sacred law, and prevented the mixing of the four castes." A schism was thus set up in India, racial rather than religious at its root, which later expanded into the great division between the early Buddhist canon and its Mahāyāna corruptions. It was the earlier Buddhism which was carried to the Syrian coast by the messengers of Asoka. It was still a conservative Buddhism, but mingled with various central Asian religions, which was carried to the same region by the subjects of Kanishka; while the great changes of the succeeding centuries brought into Buddhism, no less than into Christianity, a mass of childish apocryphal legends which passed from one faith to the other in much the same way as the earlier ideas, which to some extent at least are found paralleled in the Canonical texts. The distinction is important; but it is a distinction based on changed national politics, rather than newly created trade, as Professor Garbe would infer. This change at the coming of the Scythian shipping into the Indian Ocean is vaguely indicated by Pausanias in a passage not usually understood, where he speaks of the Island of Seria (which was really Masira off the Southern coast of Oman) but which he confuses with the Seres of China. He tells us that "both the Seres and the inhabitants of the neighboring islands of Abasa and Sacæa [the modern Kuria Muria] are of the Ethiopian race. Some say, however, that they are not Ethiopians but a mixture of Scythians and Indians."

At that ancient meeting-point between the Nile trade and that of the Indian Ocean, the Abyssinian highlands, the author of the Periplus gives us the first mention of the Kingdom of Abyssinia, then newly established, and of its capital, "the city of the people called Axumites." The great series of monoliths at Axum dates probably from the first century rather than the second and shows orthodox early Buddhist influence rather than the Buddhism of later ages. James Fergusson's description of the great monolith has not been bettered, "the idea Egyptian but the details Indian, an Indian nine-storied pagoda translated in Egyptian in the first century of the

Christian era." He notes its likeness to such temples as the Bodh Gayā, and says it "represents that curious marriage of Indian with Egyptian art which we should expect to find in the spot where the two peoples came in contact and enlisted architecture to symbolize their commercial union." And so obviously Hindu a ceremony as the Brahman's investiture with the sacred cord is still preserved as the sign of baptism in Abyssinian Christianity.

Now the very existence of the Abyssinian state in the beginning was dependent upon the alliance of the Romans in Egypt, who encouraged its growth in order to counteract the Arabian domination of the Red Sea trade; and this was originally a matter of first-century diplomacy, culminating with the decay of the ancient Sabæan capital Marib, and the conquest of the Nabatæan kingdom under Trajan.

While these relations between India and the West were being developed, a similar connection was formed with the East. The silk-market of the world was in a fertile valley of the Pamirs, whither Chinese merchants brought their goods by the great *Pei-lu* or "Southern way" along the desert of Turkestan. Nomadic marauders hampered the trade, so that the author of the Periplus remarked of China that "few men come from there and seldom"; but the armies of Pan-Chao forged the last link of the great chain, and before the end of the first century communication was unbroken from the English Channel to the Yellow Sea, and the tin of Cornwall exchanged for the silk of Ts'in.

We are therefore forced to the conclusion that the middle of the first century of the Christian era was a time of unexampled commercial activity between East and West, that political turmoil both in Rome and India then caused a lull in this traffic, which did not fully revive until the later years of the second century, and that Professor Garbe's argument, in so far as it affects the general interrelation between Buddhism and Christianity, is to that extent in need of revision.

A Freeman could write "our business is with Europe, and with other parts of the world only so far as they concern Europe." And the Christian Gospels have been read with Western eyes. The Holy Land out of which they came has been conceived as a sort of Ultima Thule, beyond which lay a great void; the country beyond Jordan being remembered as a wilderness, wherein One was tempted of the devil. A barrier is thus set up and maintained, artificial and without foundation, the defence of which some would assert to be a condition of right belief.

For some reason this type of critic would deny that an influx of new commodities carried with it a renascence of ideas, and would draw the old line about Christianity, limiting its environments to the country this side Jordan; inevitably admitting the larger expression which it received from the Gentile peoples of the northern coast of the Mediterranean, but ignoring that which came from the Gentile peoples beyond the Euphrates and the "Erythræan Sea." It is difficult to understand what is gained by so obviously tearing Christianity half out by the roots. The new faith reached out toward the East no less than toward the North and the West, and was so formulated as to be understood by all,—to be part and parcel of the intellectual environment of all. It would therefore be almost a matter of course that Christianity, making its appeal in the centers of trade, at the terminus of the great commercial highways from the East, should express its message in terms likely to be understood by those acknowledging Buddhism, the faith of the countries at the eastern terminus of those highways, and of all the world's faiths at that time, unquestionably the most influential.

Of lasting value, therefore, are all works which help to break down and destroy the ancient but artificial barriers between East and West; and of such works a very notable one is by Mr. Albert J. Edmunds, *Buddhist and Christian Gospels Now First Compared from the Originals* (Philadelphia, 4th edition, 1908-09).

Mr. Edmunds's work goes back to the age in which the Gospels were formulated, and reconstructs the background of world-thought and politics of which they have been so generally deprived.

It is necessary to a clear understanding of the Christian religion that a painstaking study be made of its points of contact with the Buddhist, and of the many thoughts which are their common property. Such a study can detract from neither faith, but must rather serve both, by showing more fully the human ideas and aspirations out of which they arose; by showing them to be living realities in the upward path of mankind, rather than abstractions limited each to its own area. It remains for the individual to make his choice between the two, but he must no longer be hedged in by an artificial barrier, which for centuries has separated peoples closely related at the Christian era, and now by the march of events, once more brought into contact. It is no longer possible for the Teuton to hold aloof from the Tartar, the Anglo-Saxon from the Japanese;

mutual interest requires a closer understanding, a readier sympathy, and a fuller acknowledgment of common aspirations. Present-day commerce has its influence in this direction, and history likewise; but sympathetic comparison of the religions of the two races is among the most important of all such influences.

This work by Mr. Edmunds is therefore especially timely, and the ripe learning which he brings to this great subject assures its permanence.

Previous comparisons, such as those of Hardy and Seydel, had depended on translations and secondary authorities and had necessarily confused primitive writings with commentary and patristics, sometimes of late date; while Mr. Edmunds works with the advantage of an intimate knowledge of both the Pali and Greek originals. He has limited himself to parallels occurring only in the primitive writings of either religion, and his presentment is most convincing. The facts of history would naturally lead the open-minded investigator to look for a certain parallelism growing out of this ancient culture-field, but hardly to expect so formidable a list as 102 parallels of word or thought in the Canonical writings and 13 more in the books relegated to the Apocrypha, but of early date, in both religions. Furthermore, as Mr. Edmunds has shown in another place (*Buddhist Texts in John,* see also *Open Court,* May, 1911) Buddhist writings are actually twice quoted as scripture in the Christian Gospel of John. The proof of intercommunication is abundant.

Mr. Edmunds's comparisons provide a rich field of information for the student of comparative religion, and his conclusion is conservative enough to satisfy scholars of every kind. "No borrowing is alleged on either side—Christian or Buddhist. In these parallels we offer no theory but present them as facts. They at least belong to a world of thought which the whole East had in common."

Were it necessary, many other facts in the history of Syria and Palestine might be cited in support of Mr. Edmunds's argument. The Persianizing tendencies in the later Jewish church, due to the captivity in the Empire of Cyrus, are well known, while recent works by such British investigators as General Sir Thomas Holdich in upper India and Afghanistan, marshal abundant evidence of the eastern extension of the Assyrian Empire and actually of the settlement of Jewish captives in considerable numbers at the very gates of India. Here then was a central administration dominant from the Nile to the Indus seven centuries earlier than the period when

Mr. Edmunds seeks to prove active intercommunication. Six centuries before the same period, one of the last of the Pharaohs opened a canal from the Nile to the Red Sea to bring his country into communication with the Eastern trade in defiance of her Mesopotamian oppressors. Six centuries after the Christian era Buddhist and Christian legends were so mingled in Western Asia, that the Koran absolutely confused the two; while a little later in Eastern Asia a Chinese emperor issued an edict forbidding the same confusion then prevalent in his dominions.

It should hardly be necessary to recall that Palestine was the West-land of the Mesopotamian civilization just as India was the East-land; and that it was at the western rim of that ancient culture-field, and not from the Greek or Roman environment, that the Christian Gospels arose, just as it was at the eastern rim that the Buddhist writings were formulated. Without in any way assuming identity of origin or purpose, it would be strange indeed if there were not identity of expression and parallelism of thought between these two great Canons; and Mr. Edmunds's proof of that identity is a distinct contribution to human knowledge.

WILFRED H. SCHOFF.

PHILADELPHIA, November, 1911.

MR. BERTRAND RUSSELL'S FIRST WORK ON THE PRINCIPLES OF MATHEMATICS.

In *The Monist* for January, 1910,[1] Dr. Carus has criticized an article of Mr. Bertrand Russell's on "Recent Work on the Principles of Mathematics," published in the *International Monthly* for 1901. A copy of the article lately came into my hands, corrected in Mr. Russell's handwriting back again to what he originally wrote.[2] The editor or type-setter occasionally changed Mr. Russell's words to words which he considered more dignified, perhaps. Thus, the *International Monthly* makes Mr. Russell say[3] that in pure mathematics we "take any hypothesis that seems assuring, and deduce its consequences." Mr. Russell had written "amusing," and the substitution of "assuring" rather took away from the force of Mr. Russell's contention that in mathematics we are not in the least con-

[1] Vol. XX, pp. 46-63.

[2] Mr. Russell has since kindly told me that this statement is correct.

[3] Quoted in *The Monist,* Vol. XX, p. 50.

cerned with the truth or otherwise of our hypotheses or consequents, but merely with the truth of the deductions.

The import of another alteration I quite fail to grasp. Mr. Russell wrote that "pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything,* then such and such another proposition is true of that thing." The *International Monthly*[4] put "asseverations" for "assertions"; and so Dr. Carus[5] remarked: "I wish Professor Russell would not describe mathematics as consisting of 'asseverations'; the very idea is jarring on my conception of the nature of mathematics."

When Dr. Carus[6] uses here, as he often has before, the word "anyness" to describe what is the fundamental characteristic of mathematics in his conception, he seems to be in agreement with one of the main tenets of Mr. Russell:[7] the propositions of logic "can be put into a form in which they apply to anything whatever"; "we never know what [which thing] we are talking about" in mathematics; the assertions are that, "if such and such a proposition is true of *anything,* then such and such another proposition is true of that thing."

I am going to try shortly to explain Mr. Russell to my readers. Mr. Russell's work on the principles of mathematics and the relation of mathematics to logic "is by no means," as Couturat said,[8] "like certain philosophical systems in fashion, a brilliant paradox, an individual and ephemeral fantasy, without roots in the past and without fruits in the future, but the necessary culmination and crowning of all the critical researches to which some mathematicians have given themselves up for the last half-century. It is a well-known fact that modern mathematics have constantly tended to deductive rigor of the reasonings and logical purity of the concepts. To these new needs of the scientific spirit a logic more and more exact and refined had to respond; the indispensable instrument of this new logic is the 'symbolic[9] logic' invented by Peano, practised by a whole school of mathematicians, and perfected by Russell.

[4] Quoted in *The Monist,* Vol. XX, p. 50.

[5] *Ibid.,* p. 53.

[6] *Ibid.,* p. 50.

[7] *Ibid.,* pp. 47, 49, 50.

[8] *Les Principes des mathématiques,* Paris, 1905, pp. v-vi. A translation of Couturat's work by the author of this article is in preparation.

[9] As a matter of fact, Peano has always called his system "mathematical logic." The name of Frege ought to be mentioned with Peano's in this connection.

It is owing to this *logistics* (as we will call it) that all mathematical theories have become susceptible of being subjected to a precise and subtle analysis, and of being reconstructed logically with a small number of fundamental data (primitive principles and notions). It is owing to this that Russell has been able, while completing on certain points this work of logical reduction, to systematize all the results acquired in a vast and profound synthesis, which is the quintessence of preceding works, and which manifests the spirit of modern mathematics."

Consider, for a moment, what this logical analysis means. Take the science of arithmetic. All its material and principles have to be reduced to logical terms and expressed unambiguously. This enormously important work is extraordinarily long and often tedious. Processes of thought that most mathematicians perform more or less accurately by "intuition" often take up, in expression, pages of symbols of logical deduction—if such deduction is possible; but then we get complete, and not only "moral," certainty, and an insight into the structure of certain truths. In Dr. Whitehead and Mr. Russell's latest book[10] there are 666 pages, most of them written in symbols, often with abbreviated proofs, and yet the definition of numbers is not yet reached! Things called "1" and "2" are defined, but not till the second volume will it appear that they are numbers!

There is a story current in Cambridge that, after a term's lecturing on the principles of mathematics, Mr. Russell informed his hearers that if they were good they should do simple addition next term.... And so recently as 1888 Dedekind's tract of 58 pages, *Was sind und was sollen die Zahlen?*[11] was derided by some mathematicians because it devoted so much space to the foundations of arithmetic!

Few people can see the immense importance of Mr. Russell's work; fewer know how laborious it has been and by what splendid qualities of mind and character it has been inspired. That is all I can say on this head, as I do not wish to gush and am not writing an obituary notice. Not quite so few people know how brilliant Mr. Russell's work is. Mr. Russell's investigations have revealed some very striking things, and Mr. Russell has said them strikingly—said them, too, in books and articles which are read with delight, and sometimes with profit, by those who are untrained to follow

[10] *Principia Mathematica*, Vol. I, Cambridge, 1910.

[11] English translation by W. W. Beman, in Dedekind's *Essays on the Theory of Numbers*, Chicago, 1901.

Mr. Russell's work. I suppose Mr. Russell has a natural love of paradox, but his paradox is always used to give point to the statement of some truth. In his talk and writings, Mr. Russell is conscientious, truth-loving, keen and witty.

I now propose to analyze the *International Monthly* article and to try to show how the fundamental doctrines of the *Principles of Mathematics* are shortly stated in it. This will continue my article in *The Monist* for January, 1910;[12] and in future I hope to trace Mr. Russell's work beyond 1903.

I.

The first published indication of the effect of Peano's work on Russell appeared in an article by Russell on "Recent Work on the Principles of Mathematics" in the *International Monthly* for 1901.[13] Boole, he said,[14] was "mistaken in supposing that he was dealing with the laws of thought: the question how people actually think was quite irrelevant to him,....His book was in fact concerned with formal logic, and this is the same thing as mathematics." Then came[15] a definition of pure mathematics: "Pure mathematics consists entirely of assertions to the effect that if such and such a proposition is true of *anything,* then such and such a proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true. Both these points would belong to applied mathematics. We start, in pure mathematics, from certain rules of inference, by which we can infer that *if* one proposition is true, then so is some other proposition. These rules of inference constitute the principles of formal logic. We then take any hypothesis that seems amusing, and deduce its consequences. *If* our hypothesis is about *anything,* and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

The reduction of mathematics to logic was spoken of:[16] "Now the fact is that, though there are indefinables and indemonstrables in every branch of applied mathematics, there are none in pure

[12] Vol. XX, pp. 93-118.

[13] Vol. IV, pp. 83-101.

[14] *Ibid.,* p. 83.

[15] *Ibid.,* pp. 83-84. For "assertions" was misprinted "asseverations," and for "amusing" was misprinted "assuring."

[16] *Ibid.,* p. 84.

mathematics except such as belong to general logic. Logic, broadly speaking, is distinguished by the fact that its propositions can be put into a form in which they apply to anything whatever. All pure mathematics—arithmetic, analysis, and geometry—is built up by combinations of the primitive ideas of logic, and its propositions are deduced from the general axioms of logic, such as the syllogism and the other rules of inference."

When dealing with questions of the principles of mathematics, the function of symbolism is exactly the opposite to that of symbolism in the other parts of mathematics. Russell said:[17] "The fact is that symbolism is useful because it makes things difficult. (This is not true of the advanced parts of mathematics, but only of the beginnings.) What we wish to know is, what can be deduced from what. Now, in the beginnings, everything is self-evident; and it is very hard to see whether one self-evident proposition follows from another or not. Obviousness is always the enemy of correctness. Hence we invent some new and difficult symbolism, in which nothing seems obvious. Then we set up certain rules for operating on the symbols, and the whole thing becomes mechanical. In this way we find out what must be taken as premise and what can be demonstrated or defined."

II.

Referring to Peano's three indefinables in arithmetic, Russell remarked:[18] "Even these three can be explained by means of the notions of *relation* and *class*; but this requires the logic of relations which Professor Peano has never taken up."

Russell[19] then indicated his contradiction:

"There is a greatest of all infinite [cardinal] numbers, which is the number of all things altogether, of every sort and kind. It is obvious that there cannot be a greater number than this, because, if everything has been taken, there is nothing left to add. Cantor has a proof that there is no greater number, and if this proof were valid, the contradictions of infinity would re-appear in a sublimated form. But on this one point, the master has been guilty of a very subtle fallacy, which I hope to explain in some future work."

* * *

Russell's statement of Zeno's puzzle about Achilles and the tortoise was:[20]

[17] *Ibid.*, pp. 85-86. [18] *Ibid.*, p. 87. [19] *Ibid.*, p. 95.
[20] *Ibid.*, pp. 95-96.

"The argument is this: Let Achilles and the tortoise start along a road at the same time, the tortoise (as is only fair) being allowed a handicap. Let Achilles go twice as fast as the tortoise, or ten times or a hundred times as fast. Then he will never reach the tortoise. For at every moment the tortoise is somewhere, and Achilles is somewhere; and neither is ever twice in the same place while the race is going on. Thus the tortoise goes to just as many places as Achilles does, because each is in one place at one moment, and in another at any other moment. But if Achilles were to catch up with the tortoise the places where the tortoise would have been would be only part of the places where Achilles would have been. Here, we must suppose, Zeno appealed to the maxim that the whole has more terms than the part. Thus, if Achilles were to overtake the tortoise, he would have been in more places than the tortoise; but we saw that he must, in any period, be in exactly as many places as the tortoise. Hence we infer that he can never catch the tortoise. This argument is strictly correct if we allow the axiom that the whole has more terms than the part. As the conclusion is absurd, the axiom must be rejected, and then all goes well. But there is no good word to be said for the philosophers of the past two thousand years and more, who have all allowed the axiom and denied the conclusion."

* * *

The converse of the Achilles, which Russell called "the paradox of Tristram Shandy," was then described;[21] and the remark was made[22] that the notion of continuity depends upon that of *order,* and that "nowadays, quantity is banished altogether [from mathematics] except from one little corner of geometry, while order more and more reigns supreme." Nowadays, too, a limit is defined ordinally.[23]

Then:[24] "Geometry, like arithmetic, has been subsumed in recent times under the general study of order. It was formerly supposed that geometry was the study of the nature of the space in which we live, and accordingly it was urged by those who held that what exists can only be known empirically, that geometry should really be regarded as belonging to applied mathematics. But it has gradually appeared, by the increase of non-Euclidean systems, that geometry throws no more light upon the nature of space than arithmetic

[21] *Ibid.,* pp. 96-97.

[22] *Ibid.,* p. 97.

[23] *Ibid.,* pp. 97-98.

[24] *Ibid.,* p. 98.

throws upon the population of the United States. Geometry is a whole collection of deductive sciences based on a corresponding collection of sets of axioms. One set of axioms is Euclid's; other equally good sets of axioms lead to other results. Whether Euclid's axioms are true, is a question as to which the pure mathematician is indifferent; and what is more, it is a question which it is theoretically impossible to answer with certainty in the affirmative. It might possibly be shown, by very careful measurements, that Euclid's axioms are false; but no measurements could ever assure us (owing to the errors of observation) that they are exactly true. Thus the geometer leaves to the man of science to decide, as best he may, what axioms are most nearly true in the actual world. The geometer takes any set of axioms that seem interesting, and deduces their consequences. What defines geometry, in this sense, is that the axioms must give rise to a series of more than one dimension. And it is thus that geometry becomes a department in the study of order."

Russell[25] then shortly dealt with the methods used by Peano and Fano in geometry, and finally[26] remarked that "the proof that all pure mathematics, including geometry, is nothing but formal logic, is a fatal blow to the Kantian philosophy."

III.

Let us now point out how this popular article gives indications of his logical work up to 1903.

To begin with, the two great influences on Russell's mathematical and logical work were Georg Cantor and Peano. Cantor had, in 1895 and 1897,[27] brought his researches on transfinite numbers and ordinal types to a close by two articles in which the principles of the subject were stated in an almost perfect logical form. Obviously, the whole question threw a great and welcome light on the principles of arithmetic.[28] Peano invented a symbolic logic which was especially adapted to the analysis and expression of mathematical theories. But Peano's logic was incomplete. It neglected the logic of relations, which was founded and developed by De Morgan, C. S. Peirce, and Schröder; and only contained a symbolical expression of the theory—unused, by the way, in Peano's symbolic

[25] *Ibid.*, pp. 99-100.

[26] *Ibid.*, p. 101.

[27] *Mathematische Annalen*, Vols. XLVI and XLIX. An annotated translation of these articles by the author is in preparation.

[28] Cf. my article on "Transfinite Numbers and the Principles of Mathematics" in *The Monist* for January, 1910.

exposition of arithmetic—of the "representations" of Richard Dedekind.[29] The logic of relations was, as Schröder had observed, necessary for the translation of Cantor's conceptions and proofs into a symbolic (speaking technically) form; and it was necessary in order to complete Peano's theory of arithmetic by defining in logical terms the three indefinables referred to above. Russell completed Peano's logic by a logic of relations in which the Peirce-Schröder ideas were modified so as to fit in with a logic which comprised more subtle distinctions than that of Schröder, in two papers, "Sur la logique des relations, avec des applications à la théorie des séries," and "Théorie des séries bien-ordonnées," which were published in Peano's *Revue de Mathématiques* for 1902,[30] and of the first of which an account was given in Russell's *Principles of Mathematics* of 1903.[31] The logic of relations gave to Russell the means of defining Peano's indefinables of arithmetic, and of proving his primitive propositions of arithmetic.[32]

Peano had emphasized that it was the notion of implication between propositions containing variables—or, as Russell expressed it, of *formal* implications[33] between propositional *functions,*[34] and not implication between (constant) propositions, that is used in mathematics. Further, the development of non-Euclidean geometry had shown in the most striking manner that, in pure mathematics, as in formal logic, we are not concerned with the truth or otherwise of the hypotheses. "Until the nineteenth century," said Russell,[35] "geometry meant Euclidean geometry, *i. e.*, a certain system of propositions deduced from premises supposed to describe the space in which we live....," but now, owing to investigations with premises other than Euclid's, "geometry has become....a subject in which the assertions are that such and such consequences follow from such and such premises, not that entities such as the premises describe actually exist." And all this goes some way

[29] Cf. the English translation of Dedekind's pamphlet in Dedekind's *Essays on the Theory of Numbers,* Chicago, 1901.

[30] An account of Peano's and Russell's logic was given by A. N. Whitehead in his paper "On Cardinal Numbers" in the *Amer. Journal of Math.,* Vol. XXIV, 1902, pp. 367-394.

[31] *The Principles of Mathematics,* Vol. I [the *Principia Mathematica* of Whitehead and Russell, of which the first volume was published in 1910, takes the place of the second volume], pp. 23-26; cf. Couturat, *op. cit.,* pp. 27-34.

[32] *Principles,* pp. 124-128.

[33] *Ibid.,* pp. 5, 11, 14, 36-41; Couturat, *op. cit.,* pp. 4, 21.

[34] *Principles,* pp. 13, 19; Couturat, *op. cit.,* p. 17.

[35] *Principles,* pp. 372-373.

towards explaining the definition of pure mathematics with which Russell's book begins:

"Pure mathematics is the class of all propositions of the form '*p* implies *q*,' where *p* and *q* are propositions containing one or more variables, the same in the two propositions, and neither *p* nor *q* contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that,* the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth."

In this definition culminates the discovery contributed to by Leibniz, Frege, Dedekind, Schröder, and a host of others, that pure mathematics is logic and logic alone. Hence Russell's[36] anti-Kantianism.

* * *

In the question of infinity, we have a discussion of Zeno's puzzles,[37] and meet again the paradox of Tristram Shandy.[38] When discussing continuity, Russell[39] made more explicit Cantor's discovery (1895) that it is a purely ordinal notion; and then, too, Russell succeeded in maintaining his theses that the theory of limits is purely ordinal,[40] that geometry is the study of order,[41] and that the notion of quantity is superfluous in mathematics.[42]

* * *

Finally we come to Russell's[43] contradiction. Starting from a study of Cantor's proof of 1892 that there is no greatest cardinal number, Russell discovered a very simple argument: If w denotes the class of all those entities x such that x is not a member of x; then, obviously, if w is a member of w, w is not a member of w, while if w is not a member of w, w is a member of w. This contra-

[36] *Principles,* pp. 4, 158, 259, 373, 442, 456-461; Couturat, *op. cit.,* pp. 235-308.

[37] *Principles,* pp. 347-353, 358-360.

[38] *Ibid.,* pp. 358-360.

[39] *Ibid.,* pp. 296-303; Couturat, *op. cit.,* pp. 91-97.

[40] *Principles,* pp. 276-277.

[41] *Ibid.,* p. 372; Couturat, *op. cit.,* p. 134.

[42] *Principles,* p. 158; Couturat, *op. cit.,* p. 98.

[43] *Principles,* pp. 364-368, 101-107.

diction, which threw doubt upon the legitimacy of the concept of class, and hence upon that of the science of arithmetic, showed itself as allied in principle to the paradoxes in the theory of aggregates discovered by Burali-Forti, König, Richard, and others, and to the old logical difficulty about the Cretan who said that Cretans were liars, and was only satisfactorily solved by Russell in 1905. Of this more elsewhere.

It only remains at present to refer to the work of Frege. He did his magnificent work on the principles of logic and mathematics alone and almost too independently, and his subtle distinctions and acute analysis have had great influence on modern work. But at first Russell had hardly heard of him, and re-discovered for himself many of his distinctions and views. In his *Principles,*[44] Russell devoted many pages to a careful critical estimate of Frege's work. I hope to give an account of Frege's work later.

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ALFRED BINET.*

OBITUARY.

Readers of *The Monist* are well acquainted with the name of Alfred Binet. That eminent psychologist died at Paris October 18, 1911, at the age of 54, from an attack of cerebral apoplexy. He was born at Nice, July 11, 1857. He first took up the study of law, but later turned his attention to natural sciences, and finally directed all his efforts to psychology. In 1894 in collaboration with Beaunis at the laboratory of physiological psychology of the Sorbonne, he founded the *Année psychologique,* an important publication of permanent value.

His principle works are *Vie psychique des micro-organismes* (English edition, *The Psychic Life of Micro-Organisms,* Open Court Publishing Co., 1894); *Psychologie du raisonnement* (English edition, *The Psychology of Reasoning,* Open Court Publishing Co., 1899); *Le magnetisme animal, Les alterations de la personnalité, Psychologie des grands calculateurs et joueurs d'échechs, Etude experimentale de l'intelligence, L'âme et le corps.* To these we should also add a number of articles on an equal variety of subjects, capil-

[44] Pp. 501-522.

* Translated for *The Monist.*

lary circulation, the pulse, emotions, character, graphology, the mystery of painting, etc.

In his last years he was particularly interested in the "psychological study of the child" and for this purpose founded a society which bore this title. In collaboration with Dr. Simon he published a number of studies on abnormal children. Very recently he suggested a system of "measurement of the development of intelligence in children" which seemed very simple and practical and has been tested by educators in many countries.

Simply to read the list of books and articles published by Binet might give the impression of too great a dispersion of forces. It is further true that the work of Binet does not, like that of other psychologists, present the development of one dominant thought pursued through all the problems of psychology. Nevertheless his work shows a unity of quite another kind, a unity of method. Binet always endeavored to apply the processes of experimentation or direct observation to the most diverse questions, and consequently we may say that inasmuch as his works tended towards the control or invention of facts, they form an important whole and bear constant witness to a truly scientific spirit. Although he did not conceive any broad hypotheses and did not aim at extended or conclusive solutions he was a prudent investigator of broad culture, rich and versatile intelligence and an excellent worker.

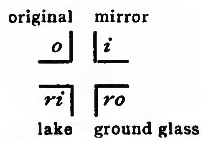
Lucien Arréat.

Paris, France.

MAGIC SQUARES BY REVERSION.

The present number of *The Monist* contains an article on magic squares by Dr. C. Planck entitled "The Method of Reversion." This reminds the Editor of his own contributions to the problem of the construction of magic squares which appeared in Mr. W. S. Andrews's book on *Magic Squares and Cubes* under the title, "Reflections on Magic Squares."

Since these reflections were written I have come to the conclusion that a popular name for the several arrangements of the numbers in their cells would help greatly to make the idea clearer. On page 115 I have called the ordinary order *o,* the reversed ordinary *ro,* the inverse of the ordinary arrangement *i,* and by *ri* is understood the reversed inverted order. Considering the fact that all these arrangements are brought about by a system of inversion which corresponds closely to reading the figures off in mirror writing, we may consider them as originated by placing a mirror on two sides of the original square. If *o* is flanked by a mirror from the top to the bottom it produces the order *i*. If the mirror is placed at the bottom it produces the order *ri* which mirrors the picture as if reflected in the surface of a lake, while the order *ro* lies in the corner between the two mirrors, being the reflection of either mirror in the other and this double inversion which we have called *ro* corresponds directly with the picture which appears on the ground glass of a photographer's camera. Accordingly the several orders on a plane surface might popularly be called the "original," the "mirror" reflection, the "lake" reflection and the "ground glass" picture.



FOUR WAYS OF INVERSION IN A PLANE.

Of course the conditions of such reflections grow more complicated if we venture from the plane into tridimensional space, and it can be extended into 4- and *n*-dimensional spaces. It appears to me that this idea of inversion rests ultimately on the same basis as Dr. Planck's method of reversions. P. C.

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CONTRIBUTIONS OF CHRISTIANITY TO BUDDHISM.[1]

THE Buddhist religion had penetrated to the most extreme northwestern part of India about the middle of the third century B. C. There it developed in the direction which expressed itself most distinctively in the deification of the person of Buddha and in the transformation of the Nirvāna-concept into the idea of a beatified continuous existence; there too arose the most essential points which distinguish northern Buddhism from southern in doctrine and forms of worship. This development found a positive conclusion in the establishment of a new school which assumed the name Mahāyāna, "The Great Vehicle," and which flourished in that region until about the eighth century A. D. After the founding of that school the older original Buddhism in contrast to it was called Hinayāna, "The Small Vehicle."

Expositions of Buddhism usually treat the Mahāyāna disparagingly, first because it places value upon the externalities of worship, and in the second place because in its philosophical speculation it evinces the strongest skepticism in teaching that Nothing is the true essence of things. But more important than these aspects of the varied contents of the Mahāyāna is the new ideal of life with which it has replaced the benevolent but fundamentally egoistic indifference—freedom not only from passions but even from all human emotions. This new ideal, which the early

[1] Authorized translation from the German by Lydia G. Robinson.

Buddhist type of saint no longer satisfied, was that of loving devotion and active compassion. H. Kern[2] says truly: "It is by that feeling of fervent devotion, combined with the preaching of active compassion, that the creed has enlisted the sympathy of numerous millions of people and has become a factor in the history of mankind of much greater importance than orthodox Buddhism." Southern Buddhism, which remained true to the ancient ideal, possessed no such winning power.

Moreover, the Mahāyāna exhibits ideas pleasing to the heart and imagination which run directly counter to the doctrines of the Hināyāna. The old Buddhism acknowledges no soul persisting throughout the course of life and knows no God, for the national gods which it recognizes are transient beings held captive in Samsāra. In the Mahāyāna we find a belief both in a personal soul and in God, at least in a kind of God. In a paradise called Sukhāvati where a reflection of the earthly Buddha, Amitabha, "the one surrounded by immeasurable light," sits enthroned in godlike fashion, the souls of the pious are born again after death in the buds of lotus flowers gradually to grow in the blossoms according to their deserts; and resting upon the lotus leaves they hear the good law preached to them by Amitabha or sung by birds in the leafy trees.[3]

According to the traditional statement, repeated even by Pischel and Edv. Lehmann,[4] the Mahāyāna was founded by Nāgārjuna, whose activity we would place rather in the middle than in the second half of the second century after Christ. But this statement is not correct. Nāgārjuna, who as originator of the Mādhyamika sect introduced into Bud-

[2] *Manual of Indian Buddhism* (Grundriss der Indo-arischen Philologie und Altertumskunde, III, 8, Strassburg, 1896), p. 124.

[3] Teitaro Suzuki, *Outlines of Mahāyāna Buddhism,* London, 1907; H. Hackmann, *Buddhism as a Religion,* London, 1910, pp. 50 ff.; Max Müller, *Last Essays,* II, pp. 304, 305.

[4] Pischel, *Leben und Lehre des Buddha.* Leipsic, 1906, p. 108; 2d ed. by Lüders, p. 104. Lehmann, *Der Buddhismus,* p. 227.

dhism the doctrine of Nothing as the only reality, was indeed one of the most significant and influential exponents of the Mahāyāna[5] and presumably the organizer of that school; but its foundation, that is to say the first literary exposition of its doctrines, must be placed about sixty to seventy years earlier. This was the work of a man who has latterly engaged the attention of the most distinguished Indologues, namely the famous and versatile monk Ashvaghosha, an elder contemporary of King Kanishka, hence in all probability living in the second half of the first century after Christ.[6] Ashvaghosha was an old man at the time of the birth of Nāgārjuna, that is to say, when the last Buddhist council was held at Jālandhara under King Kanishka about 100 A. D., if we may take as a basis of calculation the most probable but not quite assured dating of King Kanishka (last quarter of the first and the beginning of the second century). Cunningham, Pischel, the sinologue O. Franke, Fleet, and Lüders place Kanishka in the first century *before* Christ.

Therefore the appearance and the first propagation of the ideas of the Mahāyāna fall in the last decades before the council at Jālandhara.

It has occurred to many that Christian influences may have had some effect in the transformation of the Buddhist religion into the Mahāyāna form. Thus the sinologue Samuel Beal[7] found "in Ashvaghosha's writings many

[5] H. Kern, *Manual*, 6, pp. 122, 127. Teitaro Suzuki, *Açvaghosha's Discourse on the Awakening of Faith in the Mahāyāna*; translated for the first time from the Chinese version. Chicago, 1900, p. 43.

[6] Besides his best known work, the Buddhacharita which is a poetical biography of Buddha, Ashvaghosha wrote a collection of didactic tales (Sūtrālamkāra) and theological works and was also a successful composer and musician. Lately too by a happy discovery of Heinrich Lüders he has been shown to be a dramatist (*Sitzungsberichte der K. Preussischen Akademie der Wissenschaften*, phil. hist. Klasse, 1911, pp. 388 ff., especially 399; cf. also M. Anesaki, *Encyclopaedia of Religion and Ethics*, II, pp. 159, 160; S. Lévi, "Açvaghosha, le Sūtrālamkāra et ses sources," *Journ. As. S.*, Series X, Vol. XII, pp. 57 ff.

[7] *Abstract of Four Lectures on Buddhist Literature in China.* London, 1882, p. 95.

allusions and illustrations derived apparently from foreign, and perhaps Christian, sources," and arrived at the view "that much in the Buddhist development coming under the name of the Greater Vehicle may be explained on this ground." In another passage[8] he speaks in a more decided tone of the intercommunication in those days between East and West that "shaped the later school of Buddhism into a pseudo-Christian form."

A similar judgment has latterly been the fate of the oldest text-book of the Mahāyāna, Ashvaghosha's *Discourse on the Awakening of Faith in the Mahāyāna,* which is not preserved in the Sanskrit original but only in two Chinese translations.[9] The missionary Dr. Timothy Richard, who has translated this work into English (Shanghai, 1907), finds in it Christian ideas and influences and therefore reproduces the Buddhist terminology very freely in an entirely Christian mode of expression,[10] whereas on the other hand an earlier and more exact translator, the Japanese Teitaro Suzuki, a Buddhist (see above Note 4) has discovered no Christian traces of any kind in the book.

Lately, too, Christian influence in the Mahāyāna has been maintained by the Jesuit Joseph Dahlmann[11] with great determination and with an attempt at detailed scientific proofs. In what follows I shall first have to take his expositions into critical account.

In chapters 25-27 relating to the art of Gandhāra, that is of the Kabul valley and the surrounding country, Dahlmann has undertaken to show that these monuments of Buddhist art which reflect the Mahāyāna thought-cycle betray not only the generally recognized Greco-Roman

[8] *Op. cit.* Introduction, p. xiv.

[9] Bunyiu Nanjio, *Catalogue of the Chinese Translation of the Buddhist Tripitaka.* Oxford, 1883, No. 1249, 1250. There the title of the Sanskrit original is given as *Mahāyāna-shraddhotpāda*[*na*]*-shāstra.*

[10] *The Open Court,* XXV, 1911, pp. 251 ff

[11] *Indische Fahrten* (2 vols., Freiburg, 1908) II, pp. 100 ff.

influence but also a profound Christian influence. From the middle of the first century of the Christian era "that change in worship and art began to be consummated in Gandhāra. The same Buddha whose figure had been painstakingly avoided appears all at once in the monuments of Buddhist art, and not indeed as the simple herald of salvation as in ancient Buddhistic legend, but as in the message of salvation of *the Gospels, as God and as Saviour of the world.* He appears as God and Saviour not in Indian garments but in a garb *such as was worn by the higher classes in Antioch and Alexandria, in Jerusalem and Rome during the first centuries of the Roman empire.*"[12]

True and noteworthy, to be sure, is the circumstance that the likeness of Buddha appears first of all in the art of Gandhāra. Most investigators in Indian archeology have sought the reason for this strange fact and have found it in part (as in the case of Fergusson and Cunningham) in the assumption that the Buddhists had learned idolatry from the Greeks, whereas Grünwedel would fain explain the rise of the Buddha image from the natural development of Buddhism. In early Buddhist art as represented in the monuments of Sanchi, Bharhut and Buddhagayā in Central India, the original home of Buddhism, since the middle of the third century B. C., any likeness of Buddha is entirely absent. Where a likeness of Buddha would naturally be expected in the representations of his life and works we regularly find instead, in strange contrast to the lifelike pictures of all other participants in the scene, a symbol such as the tree of knowledge, a reliquary, or the wheel of the law. In the art of Gandhāra, on the other hand, the likeness of Buddha is the central figure. Here it appears everywhere in a commanding form even in the very same scene in which in ancient art it was replaced by a symbol. This likeness of Buddha

[12] *Op. cit.*, II, p.157.

passed with Buddhism from Gandhāra into all foreign countries which Buddhism conquered—into central Asia, China, Japan and the peninsula of Farther India.

That this surprising change which marks an epoch in Buddhist art can not be explained by external influences alone is obvious, although it must seem very natural that the artists of Gandhāra should rely upon the Greek types already known to them when they felt the need for the production of religious images. But these models would never have been able to accomplish this revolution alone. Such a change presupposes a transformation of Buddhist doctrine. In the original Buddhism Buddha was only a man who by his own power had found salvation from the sorrows of continuous existence and had shown the way by which it might be attained by everyone. Here there could be no worship; here the teaching was more important than the personality of the teacher, just as Buddha himself had said before his death in his last sermon: "The Doctrine and the Order which I have taught and proclaimed unto you — they are your master when I am gone."[13] The art of Gandhāra shows that the personality of Buddha had taken the place of the Doctrine and had become the object of worship. It is the visible witness of a transformation of fundamental views as it had advanced on the road towards the Mahāyāna.

Dahlmann's line of argument, however, places the greatest value on the rôle played in the Mahāyāna by the future Buddha, Maitreya. As we are later to criticise Dahlmann specially it will be better to give his standpoint in his own words. For this reason I have extracted a considerable passage from his work (II, pp. 127, 128):

"Many other Buddhas at long intervals had preceded Gotama Buddha in his calling as teacher of mankind. Gotama himself as the twenty-fifth was claimed to have com-

[13] Oldenberg, *Buddha,* 5th ed., p. 233.

pleted *forever* the series of the teachers of mankind. Therefore all hope of salvation was based on the doctrine he proclaimed. No other Buddha was to be expected in the future as teacher of salvation. To this idea a newly arisen school (the Mahāyāna) took exception, in so far as it supplied a successor as teacher of salvation to the Buddha now worshiped....The Buddha Maitreya constituted the central point of this school. The earlier tradition knew nothing of Maitreya. As simple as it would have been to continue to spin the thread of the Buddhas reappearing at periodical intervals, yet the myth stood still at Gotama as the last Buddha. Buddha *Maitreya* in the form in which he is transmitted to us is a new creation....But in the introduction of the Buddha Maitreya we have not merely to do with a new Buddha. Maitreya became the center of a new cult *in a character fundamentally different from the old Buddha,* and this character was *that of the loving compassionate Saviour who will one day come to liberate the world from the bonds of suffering.* Herewith there entered into this doctrine of salvation *an entirely new element in contradiction to the old tradition.* It directed the cult into the very path which the communities of monks had always resisted hitherto. The *teacher* becomes a *Saviour*; the *human being,* a *divine being* to whom man needs only to turn in trustfulness in order to be saved. In other words it is the *Saviour-idea* as incorporated in the Buddha Maitreya which called the Mahāyāna into existence."

That this conception of Dahlmann is in the main incorrect and easily disproved we shall see later on. At present we shall anticipate only one point. It must be granted that in the Mahāyāna a different character is assigned to the future Buddha Maitreya than formerly to the real Buddha, and that here indeed there exists a new element of which the old tradition knew nothing.

Dahlmann thinks that this new element can be explained only by foreign influence, and to him the only foreign influence worthy of consideration is that of Christianity.

The ardent joy with which Dahlmann proclaims this presumed discovery is easily understood, for in earlier works[14] he had tried to explain the fall of Buddhism in its own country by its intrinsic corruption. How well did this standpoint seem to agree with the knowledge, which Dahlmann thinks he has gained, that Buddhism does not owe its triumphal procession through central and eastern Asia and its dispersion over a third of all mankind to its own power but to Christian ideas by which it was enriched in northwestern India and attained its peculiar world-conquering vitality! Thus it would not be Buddhism which had subjected the peoples of eastern Asia but an offshoot of Christianity in Buddhist garb.

As comprehensible as Dahlmann's joy in his discovery is the enthusiastic applause which his thesis has received from some quarters of the Catholic press. Indeed, the positiveness of the assertion and the brilliant exposition in which Dahlmann has disposed of it seemed once for all "to have made an end of the Buddhism humbug." When we approach Dahlmann's spirited demonstration with a dispassionate critique it vanishes into nothing.

In the first place, what is the *chronological possibility* for the assumption that the appearance of the likeness of Buddha in the art of Gandhāra, the divinity of Buddha as attested by this art, and the conception of Maitreya as a divine Saviour can be explained by Christian influence? It is pretty well established that the art of Gandhāra reached its height at the end of the first and beginning of the second century A. D., but no time can as yet be definitely fixed upon for its beginning. The probability is

[14] *Nirvana, eine Studie zur Vorgeschichte des Buddhismus,* Berlin, 1896; *Buddha, ein Kulturbild des Ostens,* Berlin, 1898.

in favor of the pre-Christian period. The best specialists in this field, Grünwedel and Aurel Stein, have been inclined on account of the new discoveries in Turfan and Khotan to place the beginning of the Gandhāra art in the first or perhaps even in the second century before Christ.[15] And the first contemporary expert of Northern Buddhism, Louis De la Vallée Poussin, has practically settled[16] that the deification of Buddha in mythology and religion had taken place before the Christian era.

But if in spite of this we take Dahlmann's standpoint that the religion and art of Gandhāra originated in the Christian era we must further concede to him that Christianity had penetrated as early as the first century into the valleys of the Kabul and Indus—an assumption whose "possibility is not contested to-day in any quarter(!)"[17] Of course Dahlmann has to base this assumption upon a defense of the historicity of the St. Thomas legend because he needs the apostleship of St. Thomas in the Indo-Iranian territory for his demonstration.

Whereas earlier advocates of the historical character of the legend of St. Thomas, in so far as it relates to the Indo-Iranian territory, based their thesis upon discoveries of coins and one inscription by which the king in the Acts of St. Thomas, Guduphara-Gondaphares, was proved to belong to the first half of the first century after Christ, as well as upon reports of the international commercial relations of that day, Dahlmann brings forward the combination of apostleship and art in the person of St. Thomas as new and in his opinion the strongest evidence that the Christian influence in the art of Gandhāra could be explained through the activity of St. Thomas in India. This idea must be objected to on two grounds: (1) that Christian

[15] Wecker, *Tübinger Theol. Quartalschrift*, 92, note on p. 432.

[16] *Bouddhisme, Opinions sur l'histoire de la dogmatique,* Paris, 1909.

[17] Dahlmann, II, p. 138.

influence can not be proved in the art of Gandhāra; (2) that in the legend of St. Thomas, as O. Wecker justly remarks,[18] "the Christian apostle is not brought into relation with that kind of artistic activity which most clearly betrays the connection between Gandhāra and the west, that is to say with sculpture, but with the work of an architect and carpenter," which may probably be accounted for by the imagery of the construction of church or temple current in Christian modes of speech. Since I have given the reasons in this periodical (October 1911) why there can be no question of an historical nucleus in the Thomas legend, but that on the contrary Christianity did not penetrate into northwestern India at the earliest before the beginning of the third century, Dahlmann's theory becomes for us an historical impossibility.

But even a person who is not convinced of the unhistorical character of the legend of St. Thomas and who accordingly finds no difficulties in the question of chronology to prevent him from following Dahlmann's lead, can not be convinced by the arguments adduced by Dahlmann for Christian influence on the art and religion of Gandhāra, provided he understands how to pursue with the correct scientific method the beginnings of the development in early Buddhism which led to the later phenomenon of the Mahāyāna in dogma and worship. This has been shown very clearly by O. Wecker,[19] who nevertheless regards the historicity of the fundamental features of the Thomas legend as possible. To him everything that Dahlmann understands only on the assumption of Christian influence is to be accounted for quite spontaneously from the natural development of Buddhism. Some of his statements may follow here in his own words:[20] "In strange contrast to the *theoretical universality of the message of salvation,*

[18] *Tübinger Theol. Quartalschrift,* 92, note on p. 561.
[19] *Loc. cit.,* pp. 441 ff.
[20] Pages 442-444.

there stood from the beginning the difficulty with which the redeeming knowledge is to be gained, a difficulty so great that in fact the salvation of Buddha could never be a salvation for all, especially not for the many small and poor and weak.....As soon as the consequences were drawn from the universality of the salvation which Buddha preached, the exclusiveness of the pure Buddha doctrine must have been shattered; the postulates and ideas must necessarily be leveled and accommodated to the needs of every-day people as soon as the sermon becomes serious with its 'All ye, come unto me.' Is not this what happened? We need only point to the *transformation of the Nirvāna ideal*[21] to illustrate by a classical example the process of conversion which changed the pure teaching of Buddha into a popular religion....*A similar transformation of the person of Buddha was the natural consequence of this evolution.*"

The transformation of the Nirvāna concept, which moreover can not be explained solely by the change of the original doctrine of Buddha into a folk-religion, but

[21] When speaking in this essay of Nirvāna we mean salvation *after death.* Many discussions on the concept of Nirvāna suffer greatly from lack of clearness for the reason that they do not take into consideration the ambiguity of the word Nirvāna, to which attention has been called first by Rhys Davids (*Buddhismus,* 118 ff.) and later by Pischel in an exhaustive argument (*Leben und Lehre des Buddha,* 2d ed., pp. 11 ff.). Even in ancient Buddhism the word Nirvāna was used not only in the sense of salvation proper which took place at the death of the Perfect One, that is in the sense of annihilation of existence, but also to denote *salvation during life,* that is the condition of complete rest and sinlessness which endures until death and is brought about by righteous living and redeeming knowledge. In distinction from this "salvation during life," which has also been a very current idea in the Brahman systems from pre-Buddhistic times until to-day, the real final salvation in death is for the sake of clearness often called Parinirvāna, "perfect Nirvāna"; but usually this distinction is not observed by the language in the texts.

I mention here this ambiguity in the Nirvāna concept because it continues also in the Mahāyāna. What Ashvaghosha says in his *Awakening of Faith* (Teitaro Suzuki, p. 87) about Nirvāna ("As ignorance is thus annihilated, the mind is no more disturbed so as to be subject to individuation. As the mind is no more disturbed the particularization of the surrounding world is annihilated. When in this wise the principle and the condition of defilement, their products, and the mental disturbances are all annihilated, it is said that we attain Nirvāna"), and what the translator (page 119 note) gives as the general conception of the Mahāyānists on the four stages of Nirvāna does not refer to the final Nirvāna but very distinctly to Nirvāna during life.

also by the progress of the doctrine among more active peoples filled with different desires and hopes, would according to Dahlmann's standpoint have to be referred to Christian influence, but strange to say Dahlmann has laid no stress upon the transformation of the Nirvāna ideal in his demonstration.

The deification of the person of Buddha becomes comprehensible from the natural evolution of Buddhistic doctrine not only by means of such general considerations as those we have just discussed. We can also[22] discover quite positive starting points for the path pursued in the alteration of the concept of Buddha. We must remember the charm exercised by the personality of Buddha upon his environment, and the reverence which was shown the master and which of course increased greatly after his death. Even in the formula of admission, "I take my refuge in Buddha, etc.," in the earliest period of Buddhism the person of the founder was placed before the doctrine. Then the worship of sacred places which played a rôle of particular importance in Buddha's life, and the worship of relics which started up in circles of the laity immediately after his death must have contributed to the exaltation of his person, as did also the formation of legends in which not only the life of the historical Buddha but also all the former existences ascribed to him were surrounded by the creations of an unchecked fancy. Even the monuments of early Buddhist art testify that the memory of the founder held the central place in religious thought; for although the likeness of Buddha was avoided (in order to give expression, as a matter of principle, to the thought that the doctrine is more important than the teacher) yet in reality all those old reliefs are "Buddha-centric."[23]

[22] With Wecker, pp. 445 ff.

[23] Wecker, p. 451.

Wecker is right however in laying most emphasis upon the *speculative and dogmatic development* of the old Buddhism. If the form of the one historical Buddha here becomes multiplied, further if beside those Buddhas (called in Sanskrit *pratyeka-*, in Pali *pacceka*-Buddhas) who are capable of attaining saving knowledge only for themselves but have not the ability to bring salvation to others, there appear the *samyak-* (Pali *sammā-*) *sambuddhas,* the holy universal Buddhas who appear at definite times in the various ages of the world in this and in other worlds with quite decided powers and signs in order to preach the saving knowledge, then already "Buddha's form in the belief of the Order had exceeded the limits of earthly human reality."[24] This elevation into the sphere of the supernatural may also have been favored by such stories as that of the conversation with the Brahman Dona[25] in which Buddha expressly states that men who have attained Buddhahood form a special category of beings different from gods, demigods and men.

With the multiplication of the historical Buddha there grew up the faith in future Buddhas for which there is evidence in the canonical Pali literature.[26] The dogma of the Buddha of the future is explained as readily as the deification of the historical Buddha from the evolution of the Buddhist religion. At the same time we do not deny that in the formation of the ideas of the future Buddha analogous foreign elements have cooperated. If the historical possibility and probability of such an influence must be admitted, it even becomes very credible. Dahlmann[27]

[24] Oldenberg, *Buddha,* 5th ed., p. 382; English translation by Wm. Hoey, p. 325.

[25] H. Kern, *Manual of Indian Buddhism* (Grundriss der indoarischen Philologie und Altertumskunde, II, 8) p. 64.

[26] In the Mahāparinibbānasutta, Dighanikāya, XVI, 1, 16, (in the Rhys Davids-Carpenter edition, II, p. 82) according to a kind communication from O. Franke.

[27] II, pp. 131-134.

takes action with great energy but with quite inadequate grounds against the theory that the Iranian ideas of the future Saviour, the Saoshyant (later Sōsiosh) could have influenced the thought-cycle of the Mahāyāna. And yet nothing is more obvious than this, since we are dealing with a time in which Iranian influences upon northwestern India have been plentifully established, as shown for instance on the coins of the Gandhāra period.[28]

Even in the Mahāyāna speculations on the five Dhyānibuddhas, the "Buddhas arisen from meditation," which are reflexes of the earthly Buddhas in transcendent worlds, the influence has been recognized of the Iranian doctrine of the Fravashis, those prototypes of all good creatures existing from eternity to eternity.

The main point against Dahlmann's theory, which brings the whole artificial structure to the ground at one stroke and which, strange to say, has been overlooked by Wecker, I have saved until the last. The foundation upon which Dahlmann's demonstration rests consists of the statement that the older tradition does not know anything at all of Maitreya, but that he is a new creation of the Mahāyāna. This assertion is also found elsewhere. Grünwedel[29] has the following to say about Maitreya: "The northern school is acquainted with him in full detail and puts revelations in his mouth; yes, he is everywhere highly venerated, almost more than Gautama. In the southern canon, as far as I can see, he does not appear, although the Singhalese chronicle Mahāvansa is acquainted with him."[30] Similarly we read in the supplemental volume of

[28] Wecker, *loc. cit.*, pp. 439, 440, 455. Grünwedel, ***Buddhistische Kunst***, 2d. ed., p. 167: "Hence we are perhaps justified in pointing out that here again contact with Iranian ideas has taken place. The similarity of the idea of the future Buddha Maitreya with the Saviour of the Pārsi religion Saoshyant (*Sōsiosh*) is very striking. Although we do not know when the legend of the Saoshyant as it now exists developed among the Iranians yet the dominant position of the Maitreya within the northern church has certainly been influenced by it."

[29] *Buddhistische Kunst*, 2d ed., p. 158.

[30] Except the later continuations, it dates from the end of the fifth cen-

Brockhaus's *Konversationslexikon* (14th edition) in the article "Buddhismus," page 229*a* on Maitreya (Pali, *Metteyya*): "The southern church acknowledges him but the canonical writings do not mention him. The Mahāyāna school which originated in the north betakes itself with peculiar zeal to the Maitreya cult and other Bodhisatvas." Of these two sentences only the second one is correct. A glance into the best known work on Buddhism[31] shows that the idea of the future Buddha Metteyya was not unknown to ancient Buddhism. T. W. Rhys Davids also says expressly that this doctrine already forms part of the system of the Small Vehicle (Hinayāna).[32]

The passage cited by Oldenberg (*loc. cit.*) is taken from the Cakkavattisuttanta, a part of the Dighanikāya and hence belonging to the canonical Pali literature. It reads: "He will be the leader of a band of disciples containing hundreds of thousands as I now am the leader of a band of hundreds."[33]

Further, Metteyya is called the future Buddha in the Buddhavamsa (27. 19),[34] a short poetical biography of the twenty-four former Buddhas which belongs to the appendices of the Suttapitaka. According to the preceding verse Kakusandha, Konāgamana and Kassapa were enumerated as the three Buddhas preceding the historical Buddha in this "blessed eon" (*bhaddaka kappa*). Now to be sure, as the editor observes, the Buddhavamsa orig-

tury after Christ. (See the citations for Metteyya in Childers's *Dictionary of the Pali Language*). Metteyya is moreover mentioned also in the Milindapañha, p. 159, which probably belongs to the second century after Christ.

[31] Oldenberg, *Buddha*, 5th ed., p. 164, note; 384 note 1.

[32] *Der Buddhismus*, translated into German by A. Pfungst, Leipsic, p. 208.

[33] Dighanikāya, Sutta 26. Even a scholar so familiar with canonical Pali literature as Prof. O. Franke considers this passage above suspicion and declares it to be impossible that it could have been interpolated in post-Christian times. Compare further C. A. F. Rhys Davids's review of Carpenter's edition of the Dighanikāya, Vol. III, *Journal of the Royal Asiatic Society*, 1911, p. 557. Professor Oldenberg has kindly called my attention to part of the following passages.

[34] Page 67 of Morris's edition, Pali Text Society.

inally ended with verse 18, and hence the two following verses and the last song (28) would be a later addition; but verse 19 only contains expressly stated what was already implied in the eighteenth verse. For according to the Buddhist doctrine there are not four but five Buddhas in a *bhadda kappa* (Sanskrit, *bhadra kalpa*); hence the mention of such a *kappa* implies the expectation of the fifth Buddha.[35] The eons are divided into "void" (Sanskrit, *shūnya*; Pali, *suñña*) in which no Buddha appears, and "not-void" (Sanskrit, *ashūnya*; Pali, *asuñña*), that is, full periods in which there are one or more Buddhas. The not-void eons bear special names according to the number of the Buddhas which appear in them (from one to five).[36] A *bhadda kappa* with five Buddhas like the present one always comes only after a long interval.

We have no reason to doubt that this entire idea of the different kinds of eons and the "eons blessed" with five Buddhas belonged to Buddhism before its development into the Mahāyāna. And since the name Maitreya-Metteyya, which from what we have said must be old (belonging to about the fourth century before Christ), is derived from the Sanskrit *maitri* (Pali, *mettā*) "love," so we can conclude that even in olden times the idea of loving compassion was bound up with that of the future Buddha.

We see that there is hardly a question in the history of religion which can be decided with greater certainty than that raised by Dahlmann and decided without any doubt, according to his opinion, in the opposite sense. The Mahāyāna has arisen without any influence on the part of Christianity and has overcome the eastern Asiatic world *by its own power* in a mighty triumphal procession, and

[35] Oldenberg, *Buddha,* 5th ed., p. 384, Note 1; Köppen, *Die Religion des Buddha,* I, p. 315.

[36] Spence Hardy, *A Manual of Buddhism,* p. 8; Childers's *Dictionary of the Pali Language,* s. v. "Kappo," p. 186; Pischel, *Leben und Lehre des Buddha,* 2d ed., p. 94.

at the same time to be sure without shedding a drop of blood, solely by the power of conviction and example. How great an influence, lasting even down to the present day, the Mahāyāna has exerted on the higher spiritual development of China, we learn from the great sinologist J. J. M. de Groot who lived in China for years among Buddhist monks and who declared that the Buddhists were the only Chinese who possessed refinement of heart, and the only ones with whom one could discuss spiritual matters.[37]

If we now turn to the question whether *at a later date* the demonstrable contacts with Christianity have left appreciable traces on northern Buddhism, I am inclined to answer in the affirmative, although it is difficult to give a positive proof.

Before I enter upon the subject of the Buddhism of Tibet, which here comes mainly into consideration, I shall add an incidental remark.

To the best known writings of the Mahāyāna literature belong the "Lotus of the Good Law" and the biographies of Buddha called Lalitavistara and Mahāvastu, none of which can be placed before 200 A. D. Most of the parallels with the Gospel stories which have been met with in Buddhist literature are found in these three works[38] (and besides in the Pāli Nidānakathā, the introduction to the Jataka book, dating from the fifth century after Christ).

Nothing more can now be said about these parallels except that it is not impossible that they were borrowed from Christianity. When in the later Mahāyāna writings mention is made of Buddha as a fisherman who catches men like fishes, and this comparison has passed over into Chinese art in which Buddha is represented as a fisherman with rod and hook,[39] we cannot fail to recognize here a

[37] See Edv. Lehmann, *Der Buddhismus,* p. 256.

[38] *The Monist,* XXI, October 1911, p. 520.

[39] Paul Carus, *The Open Court,* June 1911, p. 357.

transference of the Christian symbol into the Buddhist world, because the catching of fish is an entirely un-Buddhistic act. The same is true of the typical representation of the mother with the child Buddha. That this goes back to Christian prototypes one glance at the "Buddhist Madonna" from Chinese Turkestan in the Ethnological Museum at Berlin, is sufficient to prove.[40]

For such transmissions the conditions of those days were particularly favorable. Kennedy mentions,[41] although without stating his source, that in the eighth century a Christian monk and a Bactrian Buddhist together composed a Christian-Buddhist text-book. The fact is that in Singan-fu, the ancient capital of China, the Nestorian missionary Adam, the "presbyter, chorepiscopus and papas of China"—called by the Chinese King Tsing, the "distinguished and pure one"—together with Prajña, a Buddhist from Kapisha in Northern India, translated into Chinese the Buddhist Shatpāramitāsūtra from the Uigurischen.[42] Through the famous Chinese-Syriac inscription of Singan-fu, written in the year 781 by the above mentioned Adam with the aid of other Nestorians, we further learn that at that time in a monastery in that vicinity Buddhist monks and Nestorian Christians were living and working together side by side in a spirit of comradeship.[43] Such friendly intercourse between Buddhists and Christians probably existed in many places in central Asia in those times.

Buddhism did not penetrate into the icy highland of

[40] See frontispiece in A. Foucher's *Beginnings of Buddhist Art and Other Essays on Indian and Central Asian Archæology,* translated by L. A. and W. F. Thomas. Paris, 1912.

[41] *Journal of the Royal Asiatic Society,* 1907, p. 481.

[42] Takakusu, T'oung Tao VII, 1897, pp. 589-591; Berthold Laufer, *The Open Court,* August, 1911, pp. 451-452. According to this the emperor Tai-Tsung (780-804) distinctly issued a warning against the confusion of Christian and Buddhist doctrines.

[43] Max Müller *Last Essays,* I, p. 258; II, pp. 310 ff., according to James Legge, *Christianity in China,* 1888.

Tibet in the form of the Mahāyāna but of the Yogāchāra system,[44] which indeed wishes to be recognized as only a particular school of the Mahāyāna and which according to its own text-books is also called the Tantra school. This school was founded in the sixth century by the monk Aryāsanga of Peshawar, who adopted the Brahman—especially the Shivaitic—gods into Buddhism as defenders of the church against the world of demons, and furnished the religion with a confused theory of witchcraft in which predominated mystical formulas (*dhārani*) for the attainment of supernatural powers and the accomplishment of all possible desires.

In this degenerate form Buddhism reached Tibet in the middle of the seventh century,[45] and about a century later the church known under the name Lamaism, which soon developed into an ecclesiastical state, was founded by the artful "conjurer"" Padmasambhava whom the Indian missionaries of Buddhism called to Tibet from his native land Kafiristan in order to overcome the opposition of the native Shamans.[46] Padmasambhava succeeded in this conquest by incorporating the teachings and usages of these Shamans, who had great influence among the people, into Tibetan Buddhism in which since that time they have formed an important component part.

The possibility of Christian influence upon Buddhism in Tibet and China has existed since 635, for from this year we have evidence of a Nestorian mission which set out for those lands under a leader by the name of Olopan or Alopen.[47] This mission was received in northern India

[44] Literally, "practice of witchcraft," the chief characteristic of this school.

[45] Grünwedel, *"Der Lamaismus,"* p. 141, (In Hinneberg's *Kultur der Gegenwart*, Part I, Section III, 1: "Die Orientalischen Religionen." Berlin and Leipsic, 1906.)

[46] *Ibid.*, p. 143. L. Austine Waddell, *The Buddhism of Tibet or Lamaism*, London, 1895, pp. x, 24 ff.; see index.

[47] Waddell, *op. cit.*, p. 422.

by the famous king Shilāditya at his court in Kanoj in the year 639.[48]

Later there arose in Nepal and Tibet the belief in the Adibuddha, that is, in an omnipotent and omniscient primeval Buddha, who was supposed to have begotten the above-mentioned five Dhyānibuddhas by his meditation—hence a monotheistic transformation of the original atheistic Buddhism. Rhys Davids,[49] following Csoma de Körös, places the rise of this faith in the tenth century, L. de la Vallée Poussin[50] somewhat earlier. At any rate H. Kern and Waddell,[51] who rests upon his authority, are wrong in placing the beginnings of the doctrine of the Adibuddha as early as the first century after Christ.

Poussin regards this entirely theistic (*aishvarika*) Buddhism, which may be divided into several—at least into two—different Adibuddha systems, merely as a final stage in the evolution of the Mahāyāna. He says:[52] "Buddhist in fact only in name and in so far as it employs Buddhist terminology, it nevertheless is, as it were, the consummation of the philosophical, mystical and mythological speculations of the Great Vehicle, and differs from several other systems widespread in the Buddhist world, only by its markedly 'theistic' coloring." He mentions relations with Hinduism but never even alludes to the possibility of Christian influence. We shall have to concede to him that to insert a personal God, inactive in principle but in reality looked upon as creative—and as such we must consider Adibuddha—into the fantastic system of the later Mahā-

[48] Takakusu, I-Tsing XXVIII, note 8; *Athenaeum,* July 3, 1880, p. 8 in the review of Edkins's *Chinese Buddhism*; Grierson, *Encyclopaedia of Religion and Ethics,* II, p. 548 b.

[49] *Buddhismus,* p. 214.

[50] In the scholarly and exhaustive article "Adibuddha," *Enc. of. Rel. and Eth.,* I, pp. 93 ff., at the end of which is appended a comprehensive bibliography.

[51] *Buddhism of Tibet,* pp. 126, 130.

[52] *Loc. cit.,* p. 93 *b.*

yāna is quite comprehensible without foreign influence. As at first the imaginary Dhyānibuddhas and Dhyānibodhisattvas had been placed above the earthly Buddha and his many manifestations in the past and future, which had been accounted for as their earthly reflections, so later a basis might be sought from which those imaginary figures could be deduced, and this basis might be found in a supreme God. It is also conceivable that the desire to obtain adherents for the Buddhist religion among theistically inclined circles has contributed to the production of the Adibuddha. Poussin might have pointed out an analogous phenomenon in the history of Brahman philosophy, namely the introduction of the personal God (*ishvara*) into the atheistic Sāmkhya system, which in a less indirect manner was adopted in the formation of this system into the Yoga doctrine. Nevertheless it must be repeated that the conception of the Adibuddha may possibly be reducible to Christian influence since in Tibetan Buddhism religious discussions with Nestorians had undoubtedly preceded it in point of time.

With greater distinctness we can recognize the often alleged Christian influences on the later development of the Lamaistic form of worship which has been called a caricature of the Catholic service. Yet Catholic missionaries who had penetrated as far as Tibet have reported with horror that the devil had created a caricature of the ritual of the Roman Catholic church there in order to bring it into derision.

From Grünwedel's excellent exposition of Lamaism[53] we learn that the European Christian mission had exerted itself in behalf of Tibet ever since the first half of the fourteenth century. In the year 1330 Odoricus of Pordenone,

[53] In Hinneberg's *Kultur der Gegenwart,* Part I, Sec. III, I: "Die orientalischen Religionen," pp. 136 ff.; X, "Europäische Reisende in Tibet," pp. 156 ff. See also O. Wecker, *Lamaismus und Katholizismus, ein Vortrag.* Rottenburg, 1910; and Hackmann, *Buddhism as a Religion,* pp. 71 ff., 154 ff.

the first European who had succeeded in reaching the place, found Christian missionaries and some converts already in the capital of Tibet,—that is, in Lhasa. At any rate we must understand these missionaries to be Syrian Christians. In 1624 after a long interval the Portuguese Jesuit d'Andrada, coming from Delhi to the city of Chaprang in western Tibet, was received with honor by the ruling king and with his permission laid the corner-stone for a Christian church. We learn then of a series of other missionaries, Dominicans and Jesuits, from the beginning of the eighteenth century, of many hardships with which they had to contend, but also of protection and benevolence on the part of the king. In 1719 begins the missionary activity of the Capuchins, who had been successful at Rome in having the monopoly of the Tibetan mission conferred upon them. It was at once taken in charge by the Capuchins to a much greater extent. In the same year Horatio della Penna came to Tibet with twelve Capuchins, again in 1737 with nine, since most of his first companions had died or had become incapable of work. But towards the middle of the eighteenth century, soon after the death of Horatio, the Capuchins gave up the evangelization of Tibet.

We have no knowledge of any success their exertions may have had. If they had made converts to any considerable extent, surely all accounts of them could not have been so lost as to leave no trace. The missionaries were apparently wise enough to judge the matter correctly and to recognize the hopelessness of any considerable extension of Christianity in Tibet. But from the syncretistic character of Lamaism, which had adopted not only the Brahman gods but also the national divinities of the Tibetans and finally after the conversion of the Mongols even some of their ideas, they must also have been justified in expecting there would be room within it for Christian ideas and Christian forms of worship as well.

With a similar view the Jesuits in China who had come in 1581 under the leadership of Ricci in the garb of Buddhist monks in order to secure a kindly reception, started out towards the end of the sixteenth century, and while publicly participating in Confucian worship diffused Christian ideas so that many Chinese accepted Christianity, but did not for that reason cease being Confucianists, Taoists or Buddhists, until finally a peremptory order from Rome put an end to this adjustment of Christianity to Chinese requirements.[54] So the Christian missionaries in Tibet would naturally have aimed upon the whole at the peaceful infiltration of Christian ideas into Lamaism in the hope of imperceptibly Christianizing it in time. That they succeeded better in this with regard to forms of worship than doctrine may be explained by the fact that Lamaism in contrast to the original Buddhism was directed essentially to externalities. In the high value placed upon sanctimonious observance Lamaism and Catholicism must have met on the same level.

In the year 1760 Tibet closed its doors to European visitors, and since that time only isolated Europeans—usually in the dress of Asiatics—have succeeded in penetrating into that country, but without reaching the capital Lhasa, with the exception of the British expedition under Colonel Younghusband, whose entry into Lhasa in the year 1904 is still fresh in our memories.

At any rate the seclusion of Tibet was complete when the two Lazarist fathers Huc and Gabet, in the garb of Buddhist ecclesiastics, arrived at Lhasa from Mongolia in January 1846 after a toilsome journey of a year and a half, and were compelled to leave again in March upon the demand of the Chinese Resident. The information which

[54] Max Müller, *Last Essays,* II, pp. 315-317.

Huc has given in his famous book[55] on Lamaistic forms of worship is an important source for all who have written on Lamaism.

Rhys Davids's *Buddhism* closes with the following comprehensive description: "Lamaism, indeed, with its shaven priests, its bells, and rosaries, its images, and holy water, and gorgeous dresses; its service with double choirs, and processions and creeds, and mystic rites, and incense, in which the laity are spectators only; its abbots and monks, and nuns of many grades; its worship of the double virgin, and of the saints and angels; its fasts, confessions and purgatory, its images, its idols and its pictures; its huge monasteries and its gorgeous cathedrals, its powerful hierarchy, its cardinals, its pope, bears outwardly at least, a strong resemblance to Romanism, in spite of the essential difference of its teachings and of its mode of thought."

This description could be further supplemented by reference to the crozier and the bishop's mitre, exorcism of demons, the censer with five chains which can be closed or opened at will, the benediction in which the Lama lays his right hand upon the head of the believer, the religious exercises in seclusion, and still other particulars.[56] Furthermore the practice of the higher Lamas to cross themselves before the beginning of a religious service[57] seems to me to deserve special mention, as does also a ceremony which bears a remarkable resemblance to the celebration of the Lord's Supper.[58] In this we have the distribution of consecrated bread and wine to the devout congregation. In place of the bread consecrated pellets of puff-paste are also mentioned, and by wine we must probably understand a

[55] *Souvenirs d'un voyage dans la Tartarie, le Tibet et la Chine,* 2 vols. Paris, 1850 (second edition, 1853); English edition, Chicago, Open Court Publishing Company.

[56] Huc in Wecker, *loc. cit.*, p. 37.

[57] Waddell, *Buddhism of Tibet,* p. 423.

[58] *Ibid.*, pp. 444 ff.

different sort of alcoholic drink.[59] At any rate "bread and wine" are enjoyed by the participants in this ceremony "for the attainment of long life." By long life may be understood a circumlocution for the Christian idea of eternal life.

One other fundamental idea of Lamaism appeals to us as strictly Catholic, namely that the priests "hold the keys of hell and heaven, for they have invented the common saying: 'Without a Lama in front (of the votary) there is no (approach to) God.'"[60]

One might be tempted to account for these correspondences between Catholic and Lamaistic worship as parallel phenomena by the statement that the human mind when moved in the same direction of thought and feeling arrives externally at the same results. But the correspondences are too close and too numerous for us to get along without the assumption of a loan. As at the close of my former essay in this periodical (October, 1911) on "Contributions of Buddhism to Christianity" I could not avoid the conviction that many fundamental features in the worship of the early Christian church have been taken over from Buddhism, so on the other hand at a more recent date many Christian forms of worship of a later stage of development have found acceptance in the most degenerate form of Buddhism, Lamaism.

I have pointed out above (pp. 182-183) how in my opinion this has come about. Huc has called attention to still another possibility.[61] In the thirteenth century in the times of the Mongolian supremacy, ambassadors from the rulers of the world came to Italy, Spain, France and England, and took home, so Huc thinks, a deep impression of the glitter and splendor of the Catholic worship. Per-

[59] "Ambrosia brewed from spirit or beer," Waddell, p. 445; in the middle of page 448 he speaks again of the sacred wine.

[60] *Ibid.*, pp. 422-423.

[61] In Wecker, *Lamaismus und Katholizismus*, pp. 37-39.

haps they did; but the incidental enthusiastic descriptions of these secular ambassadors would have assumed only very general outlines and could hardly have exercised any influence on the later worship of the Mongols. Still less propable is it that the Mongols would have carried traces of this influence to Tibet, since indeed they took Lamaism away from Tibet with them and have remained its true devotees until the present day. Moreover, at the time of their greatest power the Mongols, who were then adherents of Shamanism, were religiously indifferent, and ambassadors of Buddhism, of Islam and even of Christianity waited upon them in vain. When Kubilai Khan was converted to Buddhism in the thirteenth century the Mongolian empire had already fallen to pieces.

For the channels of Christian influence upon Lamaistic worship search must be made within Tibet itself, and at any rate the assumption of Huc[62] can not be ignored that the famous reformer of Lamaism Tsong-Kha-Pa (1356-1418), who introduced clerical vestments and a definitely prescribed ritual, had been under the influence of Christian missionaries, even though we possess no record from this period of a Catholic mission to Tibet. But central Asia was traversed in those days by numerous Christian missionaries, and so the "man from the west with the long nose and eyes gleaming with supernatural fire,"[63] with whom Tsong-Kha-Pa is said to have conversed, may have been a Christian monk who found his way there not from India (for then something more definite would be known of him) but from the north into the interior of Tibet.

At any rate since the Nestorians of the seventh century there have never been wanting channels through which Christian elements of worship might have been introduced into Tibetan Buddhism.

[62] See also Waddell, p. 59; Hackmann, *Buddhism as a Religion*, pp. 74, 75, 180.

[63] Huc, *Souvenirs*, II, 2d ed., p. 106.

In conclusion I should like to deny one possibility which has occasionally been suggested, namely that the Catholic ritual may not have influenced the Lamaistic, but *vice versa* may have been influenced by it.[64] Lamaism has never possessed the requisite strength for this. The side which is much weaker morally and intellectually can not urge its forms of life upon the stronger.

As we have seen, Christian influences upon the development of Buddhism are limited to secondary products of a late day; just as inversely Buddhist influences upon Christianity may be pointed out only in non-essential particulars and from times in which the doctrine of the Christian faith was established as a firm system. All identities and similarities in the teachings of these two great world-religions have, so far as *essential* matters are concerned, originated independently of one another, and therefore are of far greater significance for the science of religion than if they rested upon a loan.

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[64] Waddell, pp. 421-422: "It is still uncertain how much of the Lāmaist symbolism might have been borrowed from Roman Catholicism, or *vice versa*"; Pischel, *Leben und Lehre des Buddha,* p. 124: "Without doubt much has traveled from Lamaism into the Catholic church since even Buddha himself as Josaphat = Bodhisattva has been accepted among its saints in the Roman martyrology"—but not from Lamaism! We have distinct indications that Pischel is also the author of the (anonymous) article on Indian religions in Brockhaus's *Konversations-Lexikon,* 14th ed., XVII, where we read on page 594 b: "....so that the service of Lamaism closely resembles the Catholic service from which many would derive it....But the reverse path of the loan is equally probable."

THE PRINCIPLE OF RELATIVITY.

INTRODUCTORY.

PHYSICAL science seems to have entered into a new phase, the slogan of the new school being THE PRINCIPLE OF RELATIVITY. In some quarters the current modes of thought are declared antiquated, and the promise is made that the old truths will acquire a new meaning. Physicists speak of the relativity of time and space, and we will add that they ought as well speak of the relativity of things, of the whole actual world in all its parts and interrelations.

Many who have watched the origin and rise of the new movement are startled at the paradoxical statements which some prominent physicists have made, and it is remarkable that the most materialistic sciences, mechanics and physics, seem to surround us with a mist of mysticism. The old self-contradictory statements of the Eleatic school revive in a modernized form, and common sense is baffled in its attempt to understand how the same thing may be longer and shorter at the same time, how a clock will strike the hour later or sooner according to the point of view from which it is watched; and the answer of this most recent conception of physics to the question, How is this all possible? is based on the principle of the relativity of time and space.

The man who started this movement and was the first to formulate it in concise language and to base it upon close argument was Professor Einstein,[1] who was followed by Lorentz,[2] and so we hear often of the Einstein-Lorentz theory. The strangest thing about it is that the question is seriously debated whether or not this theory is true, and the answer is expected from experiments; while in our opinion we are here confronted with a method, and the problem is simply how we can best deal with certain difficulties due to the relativity of all things. These difficulties have originated through the need of a greater exactness in measurements, but the underlying truth—the relativity of all things—is not a question of fact, but a recognition of certain complications with which we must learn to deal.

On reading recent expositions of the principle of relativity the man of good education, or the one who has attended universities without being a specialist in either mathematics or physics, feels the *terra firma* give way under his feet, and when he finds that the principle of identity seems to fail in his comprehension of things, a dizziness comes over his intellect and he sinks into the bottomless abyss of the incomprehensibility of existence. A general earthquake seems to quiver through his mind. Everything totters around him and he stands in awe at the significance of the new thought. Nor is there any one who dares to contradict; for the most learned arguments are adduced, the mathematical and logical conclusions of which bristle with formidable formulas,—yea, experiments are made to prove the truth of the relativity of time and space.

For the sake of convenience we will speak of the representatives of this new conception as the "relativity physicists" in contradistinction to the old-fashioned physicists of the old school. It has been said that the former represent more the mathematical aspect of physics while the

[1] *Jahrbuch der Radioaktivität und Elektronik,* 1905-1908.

[2] H. A. Lorentz, *Theory of Electrons* (Teubner) 1910.

latter are the realistic physicists proper, too realistic to understand the significance of the new truth.

In order to facilitate a comprehension of the situation as well as our own conception, we will here at once and dogmatically state that the relativity physicists are perfectly right; what they claim is really and truly a matter of course, and if they only would present their proposition without dressing up their theory in paradoxical statements, nobody would in the least hesitate to accept the new view. But as soon as this is done people will at the same time find out that the new view is not novel. Its importance has been greatly exaggerated, for the principle has been tacitly understood in the correct way by all preceding physicists who, at the time however, ignored, or better did not enter into, the problem, because they had other more pressing work on hand. Nor is it unlikely that they regarded this problem of relativity as a philosophical question which strictly speaking had no place before the forum of physics.

ON THE ABSOLUTE.

Perhaps the easiest way of elucidating the true meaning of the relativity of time and space will be by setting forth our own position as we held it long before the principle of relativity gained prominence or had even been mentioned or alluded to.

The writer's book *Fundamental Problems* contains the following statement under "Definitions and Explanations" (first edition, page 254; seecond edition, page 252):

"Absolute existence (in fact everything absolute) is impossible. Reality is properly called *Wirklichkeit* in German, derived from *wirken,* to take effect. Reality is not immovable and unchangeable absoluteness, but the effectiveness of things in their relations. Reality therefore implies not only existence, but the manifestation of existence also. Existence and its manifestation are not two different things; both are one."

Since the days of Heraclitus it has been a trite truism that all existence is in a flux. There is no rest anywhere, and actuality consists in the effects which these changes exercise upon one another by action and reaction. Upon this lack of stability, resulting from a universal and intrinsic relativity, Mr. Spencer bases one of the strongest, though quite untenable, arguments of his agnosticism. He seems to expect that time, space, motion, and matter are or should be things-in-themselves, and forgets that they represent relations, i. e., certain features of reality. We will here quote his exposition of the unknowableness of motion in space. In his *First Principles* Spencer says:

"Here, for instance, is a ship which, for simplicity's sake, we will suppose to be anchored at the equator with her head to the west. When the captain walks from stem to stern, in what direction does he move? East, is the obvious answer,—an answer which for the moment may pass without criticism. But now the anchor is heaved, and the vessel sails to the west with a velocity equal to that at which the captain walks. In what direction does he now move when he goes from stem to stern? You cannot say east, for the vessel is carrying him as fast towards the west as he walks to the east; and you cannot say west for the converse reason. In respect to surrounding space he is stationary; though to all on board the ship he seems to be moving. But now are we quite sure of this conclusion? Is he really stationary? When we take into account the earth's motion round its axis, we find that instead of being stationary he is traveling at the rate of 1000 miles per hour to the east; so that neither the perception of one who looks at him, nor the inference of one who allows for the ship's motion, is anything like the truth. Nor indeed, on further consideration, shall we find this revised conclusion to be much better. For we have forgotten to allow for the earth's motion in its orbit. This being some 68,000 miles per hour it follows that, assuming the time to be midday, he is moving, not at the rate of 1000 miles per hour to the east, but at the rate of 67,000 miles per hour to the west. Nay, not even now have we discovered the true rate and the true direction of his movement. With the earth's progress in its orbit, we have to join that of the whole solar system towards the constellation of Hercules; and when we do this, we perceive that he is moving neither east nor west, but in a line inclined to the plane of the ecliptic, and at a velocity greater or less (according to the time of the year) than that above named. To which let us add, that were the dynamic arrangements of our sidereal system fully known to us, we should probably discover the direction and rate of his actual movement to differ considerably even from these. How illusive are our ideas of motion, is thus made sufficiently manifest. That which seems moving proves to be stationary; that which seems stationary proves to be moving; while that which we conclude to be going rapidly in one direction, turns out to be going much more rapidly in the opposite direction. And so we are taught that what we are conscious of is not the real motion of any object, either in its rate or direction; but merely its motion as measured from an assigned position—either the position we ourselves occupy or some other."

The same argument of the captain walking the deck of a ship was made before Spencer, though mostly it was a ball rolling on deck; Bradley refers to it as well known in his time, 1727, and the same story has been repeated after Spencer. In fact it is one of the arguments of the relativity of space among modern relativity physicists.

The principle upon which the representatives of the new view take their stand is a consideration of actual life. Things are in a flux, and this is an undeniable fact. We must bear in mind that the way of making knowledge possible at all in the flux of being is to ignore what has nothing to do with the problem under investigation. Our method is based upon a fiction or, if you please, upon an artificial trick, viz., to ignore complications and to consider a certain thing as fixed; but there are cases in which we must remember that we ourselves change and that the very position we assume is moving.

This trick of assuming that our position is stable is easy enough because man does not at once notice that there is any change; but all things are in a flux and he himself changes unconsciously. A primitive unsophisticated man does not know that the earth on which he stands is whirling around itself at the rate of 1037 miles an hour, on the equator, further that it is also revolving with incredible speed around the sun, and that with the sun it is proceeding in a spiral motion towards one of the constellations, probably the constellation Heracles, around an unknown center situated somewhere in the Milky Way. God only knows what else takes place and what kind of whirling dances the Milky Way performs. The savage has not the slightest idea of all this, and so it is easy for him to ignore the motion of which he unconsciously partakes.

If man really were aware of all the events which influence him, his head would swim, and he would be incapable of thinking any sober thought. Fortunately he is concerned solely with his own narrow interests. The more man in the further growth of his mind becomes familiar with these unnoticeable events, the more he discovers that for any particular purpose he must ignore what does not belong to the solution of the special problem under consideration.

This way of ignoring what does not concern us at the time is an artificial process, a process of abstraction and elimination, of cutting off all disturbing incidents, and in doing so the philosophically minded scientist will become aware of the fiction of arbitrarily laying down a point of reference which is treated as if it were stable while in fact, like everything else, it too is caught in the maelstrom of cosmic existence.

There is nothing wrong or harmful in this fiction; on the contrary it is an indispensable part of our method of comprehending things. The universe is too complicated to be understood or viewed at a glance, and knowledge, science, cognition as well as all mental processes become possible merely by concentration, i. e., by selecting a point of view as being a certain fixed location from which we observe a change, an event, a transformation, in order to gain a comprehension of this or that piece of existence in contrast to others of the same or of a different kind. Such is the nature of cognition, and this artificial trick is an essential condition of observation.

Knowledge is relative. It is the relation between subject and object, the thinker and the thing, and this, far from being objectionable, is only the universal condition of all existence; for all existence is relative. All reality is the result of action and reaction; it is a forming and being formed under definite conditions; it is transformation. There is no existence in and by itself. Relativity is the principle of all real and actual being.

TRICKS OF COGNITION.

If the standpoint of an observer changes, the thing observed will naturally change too in its relation to him. Formerly physicists were in the habit of not seriously bearing in mind that the fixedness of their standpoint was an assumption; they did not follow this principle to its ultimate consequences. For their special problems it was not necessary to do so, and there is very little use in bearing it constantly in mind. The difference in time between the moment when the observer looks at an object and that in which the rays of light indispensable for observation strike his eye is too inconsiderable to be taken into account; it is a negligible quantity. But if the object under consideration is at such an enormous distance that it takes the rays of light thousands of years to reach the eye of the astronomer it does make a difference, and so James Bradley was astonished to register the fact that the fixed stars in the sky were not always in the same place but that they pendulated semi-annually above us with the motion of the earth around the sun. The direction in which we see them swings from the aphelion to the perihelion, and a closer consideration of the facts shows that the rays of very distant stars which we catch in the aphelion are not caused at the moment when we see them but started thousands of years prior to the moment in which they strike the lens of the astronomer's telescope, and so the transference of rays of light from the star to the astronomer's eye at this enormous distance represents a relation which most forcibly drives the truth home to us that there is nothing absolute.

The same is true of all things. The object before us seems to stand there in a perfect and quiet completeness, and yet the changes that work unnoticed by our dull senses are constant, continuous and rapid. Heraclitus used to say that he could not come out of the same river into which he had stepped a moment before, because the water was always rushing by. Never is a drop of it the same, and this is true of all things, even of ourselves. The observer has to exclude from his methods of observation the fact that he himself, his senses and his mind, are in a constant flux.

In order to elucidate the significance of the nature of cognition as being a limitation and concentration upon one point and constructing artificial units, the writer has on former occasions used the analogy of the kinematoscope, the machine which produces moving pictures.

In order to make any picture possible we need a lens, and the lens focuses the rays of light so as to throw rays from the same spot upon one and the same place on the plane where the picture appears. The rays of light which proceed from an object scatter in all directions, and unless we use a lens to concentrate the rays, the formation of a picture of the object would remain impossible. Thus the method of producing a picture is by concentration.

The lens produces a picture by focusing rays of light, that is by throwing the same rays upon the same spot; but it would also be possible to produce a picture by cutting off the redundant rays of light and singling out one or very few rays, each one coming from each of the several points of the object. Accordingly we can photograph objects through a pinhole; there is only this difference that the picture is weak and needs long exposure. This proves that the process of concentration is fundamentally a process of abstraction, of leaving out, of omitting the disturbing multiplicity of the innumerable facts of real life as represented in the totality of objective experience.

The kinematoscope involves not only the static form of things, their spatial expression, the juxtaposition of parts, but it also adds the changes that are taking place in time. The film of the kinematoscope consists of a series of pictures, one always a little different from another, and if these are presented in rapid succession the series is fused into one picture in which the succeeding differences appear as motion. This is accomplished by the introduction of a little winged wheel which in rapid succession covers and uncovers the several pictures. If we would take this little wheel with its wings out of the kinematoscope, and if otherwise the pictures on the film would succeed one another in a rapid continuous motion without this artificial separation by the wings of the wheel, we would see no picture at all but simply have a blur on the canvas. In order to have distinct pictures appear on the canvas, we must cut the flux of motion into little separate moments which we may allegorically characterize as atoms of time.

Reality is a continuous flux, but in order to follow it step by step we must do the same thing that the mathematician does with his differential calculus. In the calculus the curve is cut up into infinitesimal lines, which in continuous succession change their directions, and the smaller we conceive these lines to be, the less is the mistake made by this fiction, if they are treated like straight lines.

The method of the calculus, based upon the fiction of substituting for a continuous curve a series of little straight lines constantly changing their direction, is not so very different from the method of cognition in general. Nor is there anything wrong in it, only we must remain conscious of the fiction. In a similar way we must know that existence itself is a continuous system of relations, or in other words, that relativity is the principle of all existence in the world of actual life as well as in the domain of thought. We must cut up the general flux according to the needs of our investigation and lay down artificial limits.

* * *

If we view the new physics under this aspect, it will lose its mystic glamor and at the same time appear intelligible. In fact we shall understand that the principle of relativity is a matter of course, and if we cut up reality into things, as if they were things-in-themselves, into units or atoms, we employ a trick of cognition which makes it possible to focus things and picture them distinctly in our mind.

There are large numbers of scientists possessed of an *odium philosophicum* because philosophy means to them some abstruse metaphysical system of thought which ignores the natural sciences and, spiderlike, spins a world-conception out of pure thought derived from the thinker's subjectivity. The result is that they are soon perplexed in their own science by philosophical problems; for true philosophy—the philosophy of science—is an indispensable factor of cognition, and its influence extends into the fabric of all scientific labors. Thus it happens that problems of a philosophical character arise unexpectedly, and then the information given by nature in reply to experiments is apt to be misunderstood.

If the reference point (R) from which an observer measures is in motion toward R_1, and the object observed (O) also possesses a motion of its own, we are confronted with a complicated phenomenon. If R moves toward O, the object measured will be shorter than if it stands still, and it will be longer if R moves with O in the same direction. We have only to forget, after the fashion of the pragmatist, that there is an ideal of objective cognition, and assume that all there is about size or the objective measure of things consists in the result of our measuring and we have the clue to the paradoxes of the physics of relativity. If the point of reference is not stationary and if we neglect to account for its motion, the result of our measurement is necessarily vitiated thereby as much as the pragmatist's philosophy by his personal equation.

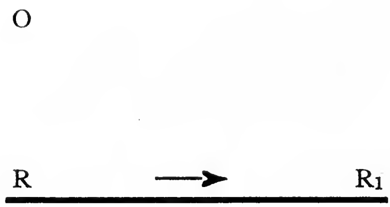


Fig. 1.

There are further complications of measurement. The time needed for the transmission of signals must also be taken into consideration. The rays of light travel at an enormous velocity but the distances in the starry heavens are also enormous and the distance between O and R is less than between O and R_1. The rays which were sent out from O at the moment of measurement have already passed the track of the observer at R, while this same observer has moved on to R_1, and there he catches the rays sent out from O in its position at O; in the meantime however the object O has in its turn also changed its place. From R_1 it appears at O, where it stood while the observer was stationed at R, but in fact it stands no longer at O but has in the meantime proceeded on its own path whithersoever that may have led O, backward or forward, in any other direction than R, possibly in the same direction as R. Such phenomena are necessary results of the relativity of existence, and we must bear them in mind when confronted with complicated conditions which present themselves, for instance in astronomical cases. Here the mistakes rising from the fiction of assuming our reference point to be stable are considerable enough to enforce attention, and in that case we shall have to make allowance for the instability of our reference point, as well as for the time which the rays of light need for their travel through space.

That was exactly Bradley's case as set forth in his essay written in 1727, one hundred and eighty-five years ago, and thus he became the forerunner of the relativity physicists. To state it in other terms, Bradley correctly solved a problem which in our days led to the formulation of the principle of relativity, and he did so without mentioning this theory, yea without feeling the need of formulating it. He simply took it for granted that he had in this case to consider the motion of the earth that served him as a reference point—the place of his observations.

COMSTOCK ON RELATIVITY.

The most popular and at the same time the most exact characterization of the principle of relativity comes from the pen of Prof. D. F. Comstock, of the Massachusetts Institute of Technology. It appeared in *Science* (Vol. XXXI, 1909, p. 767), and we quote from it the passages which contain the statement of the problem:

Professor Comstock starts with the following two postulates:

"The uniform translatory motion of any system can not be detected by an observer traveling with the system and making observations on it alone.

"The velocity of light is independent of the relative velocity of the source of light and observer."

The main passages of his exposition state the problem thus:

"The whole principle of relativity may be based on an answer to the question: When are two events which happen at some distance from each other to be considered simultaneous? The answer, 'When they happen at the same time,' only shifts the problem. The question is, how can we make two events happen at the same time when there is a considerable distance between them.

"Most people will, I think, agree that one of the very best practical and simple ways would be to send a signal to each point from a point half-way between them. The velocity with which signals travel through space is of course the characteristic 'space velocity,' the velocity of light.

"Two clocks, one at A and the other at B, can therefore be set running in unison by means of a light signal sent to each from a place midway between them.

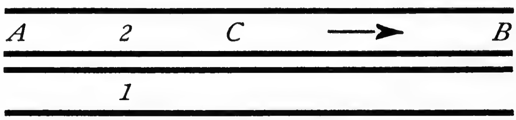


Fig. 2.

"Now suppose both clock A and clock B are on a kind of sidewalk or platform moving uniformly past us with velocity v. In Fig. 2 (2) is the moving platform and (1) is the fixed one, on which we consider ourselves placed. Since the observer on platform (2) is moving uniformly he can have no reason to consider himself moving at all, and he will use just the method we have indicated to set his two clocks A and B in unison. He will send a light flash from C, the point midway between A and B, and when this flash reaches the two clocks he will start them with the same reading.

"To us on the fixed platform, however, it will of course be evident that the clock B is really a little behind clock A, for, since the whole system is moving in the direction of the arrow, light will take longer to go from C to B than from C to A. Thus the clock on the moving platform which leads the other will be behind in time.

"Now it is very important to see that the two clocks are in unison for the observer moving with them (in the only sense in which the word 'unison' has any meaning for him), for if we adopt the first postulate of relativity, there is no way in which he can know that he is moving. In other words, he has just as much fundamental right to consider himself stationary as we have to consider ourselves stationary, and therefore just as much right to apply the midway signal method to set his clocks in unison as we have in the setting of our 'stationary clocks.' 'Stationary,' is, therefore, a relative term and anything which we can say about the moving system dependent on its motion, can with absolutely equal right be said by the moving observer about our system.

"We are, therefore, forced to the conclusion that, unless we discard one of the two relativity postulates, the simultaneity of two distant events means a different thing to two different observers if they are moving with respect to each other."

We quote further:

"It must be emphasized that, because of the first fundamental postulate, there is no universal standard to be applied in settling such a difference of opinion. Neither the standpoint of the 'moving' observer nor our standpoint is wrong. The two merely represent two different sides of reality. Any one could ask: What is the 'true' length of a metal rod? Two observers working at different temperatures come to different conclusions as to the 'true length.' Both are right. It depends on what is meant by 'true.' Again, asking a question which might have been asked centuries ago, is a man walking toward the stern of an eastbound ship really moving west? We must answer 'That depends' and we must have knowledge of the questioner's view-point before we can answer yes or no."

The question of the man walking on a ship not only "might have been asked centuries ago," but it has been asked centuries ago. Our forebears were more conscious of the relativity of existence than the relativity physicists credit them.

Professor Comstock continues:

"It must be remembered that the results of the principle of relativity are as true and no truer than its postulates. If future experience bears out these postulates then the length of the body, even of a geometrical line, in fact the very meaning of 'length,' depends on the point of view, that is, on the relative motion of the observer and the object measured."

Professor Comstock's verdict of the case is summarized in this paragraph:

"The results of the principle for uniform translation are simply as true as its two postulates. If either of these postulates be proved false in the future, then the structure erected can not be true in its present form. The question is, therefore, an experimental one."

Here we demur. We claim that the question is not experimental but belongs to the department of *a priori* reasoning.

Professor Comstock does not enter into questions of mass connected with the principle of relativity but is satisfied with this comment:

"The apparent transverse mass is, I think, best derived by Lewis and Tolman,[3] in their excellent paper on the principle of relativity, and the relation between transverse and longitudinal mass is shown in the most direct and simple way by Bumstead[4] making use of the torsion pendulum. Any one interested in the subject should read these two papers."

THE A PRIORI.

It is characteristic of modern science to denounce the principle of the *a priori* and to extol experiment and experience. Now it is true that experience and experiment are indispensable factors in science, and in all the specialties of science. In experience and experiments we deal with the facts presented to us by nature; but the method of reasoning is not a thing which is derived from sense experience.

The method of reasoning is, as Kant truly said, *a priori* and, let us add, the *a priori* is nothing mystical or mysterious; it is simply the result of pure thought or reflection from which the data of the senses have been excluded. Pure thought (or better, purely formal thought) is a mental construction, or, if you prefer, a fiction. We omit every-

[3] *Phil. Mag.*, 18, 510-523, 1909.

[4] *Am. Jour. of Science*, 26, pp. 493-508, 1909.

thing concrete and thus we retain a field of abstract possibilities. Elsewhere we have called it a field of anyness.[5] Obliterating in our mind all particularity we retain nothing concrete and in this field of nothingness we build up pure relations. From this domain all real things, comprising everything which we subsume under the categories of matter and energy, has been excluded. But these pure relations, i. e., pure forms which are non-material constructions lacking all concrete qualities such as all real things possess, serve us as models for the relations of any possible purely mental or actual existence. Our doings in this field of abstraction consist in the fiction of pure lines, pure numbers, pure motion, pure ideas and their interrelations such as genera and species, and thus we are capable of building up a world of purely formal or relational thought, the totality of which in space is called geometry, and in the domain of numbers which originate by counting a series of single units, arithmetic, etc. In the domain of pure thought, consisting of genera and species, we call the laws that govern their relations logic, and the law of transformation, of which the positive aspect is properly called causality, and its negative counterpart the law of conservation of matter and energy, has been called by Kant pure natural science.

All systems of mental constructions have the advantage of picturing in our mind *any* possible configuration of relativity, and in this sense pure thought (Kant's *a priori*) is a field of anyness. It can be applied to any fact or set of facts of existence, actual or fictitious, and these systems of mental constructions therefore furnish us with the key to determine the relations of real nature. They render possible the systematization of sense impressions and thus

[5] See *Philosophy of Form,* the chapter on "The Foundation of Mathematics and Logic," pp. 7-10. For further details see also the chapter "Form and Formal Thought" in the author's *Fundamental Problems,* pp. 26-60.

these systems of pure thought in the field of anyness are the methods of scientific operation.

Let us not therefore speak contemptuously of the *a priori,* or denounce apriorism as something medieval and elusive, for even here in the attempt at establishing the principle of relativity in time and space, the arguments of the physicists are absolutely aprioristic. There is not one of these so-called experiments, invented to prove the relativity of time and space, which does not ultimately resolve itself into a machine that renders visible aprioristic considerations.

The ultimate arguments in all the experiments made to prove the relativity of time and space move in a domain of purely formal thought, and the force of them is ultimately of the same kind as the Q. E. D. of Euclidean theorems. We think here mainly of such propositions as locate an observer on the sun and another on the earth. Their clocks actually agree, but when compared they are found to differ. About eight minutes have elapsed when the observer on earth registers the time as the rays of the sun reach the earth, and *vice versa* when the clock on earth is observed as the rays from the earth strike the sun. The imitation of the same conditions for the sake of comparing the registration of two moving systems in an actual experiment amounts to nothing more than the pencil drawings of a Euclidean or logical figure in which the *a priori* reasoning is visibly presented as a *demonstratio ad oculos.* The argument remains in either case one of pure thought.

The photograph of such an apparatus built for the purpose of making an experiment in the relativity of time and space to show the difference between a solar clock and a terrestrial clock may be found in the article of Emil Cohn of Strassburg, "Physikalisches über Raum und Zeit" in *Himmel und Erde,* Vol. XXIII. To be sure the instrument does not fulfil the conditions either of distance or of the velocity of the transference of the signal, "but," says Professor Cohn, "that is of secondary importance."

There are two motions both constant and both standing in a definite proportion. The sun with its clocks has been made to stand still. The earth with its two clocks moves, and there is an arrangement by which to represent the transference of signals. The main thing is that "their velocities stand in definite proportions and all that concerns us are these proportions. That we have here replaced the enormous velocity of light by a velocity of a few centimeters per seecond is unessential. It is essential, however, that the velocity of the earth is three-fourths the velocity of light, while the real ratio is 1 : 10,000."

Newton's laws are *a priori,* and Newton proves that these laws hold good in, and are serviceable as, interpretations of the actual world of fact. The empiricist ought to rebel against Newton's laws, because they never have been nor ever can be proved by either experience or experiment. Whoever saw a body moving in a straight line? and has Newton (from the standpoint of the empiricist) any right at all to make such sweeping statements of movements which have never occurred in the experience of anybody?

The most general principle at the bottom of scientific work is perhaps the so-called law of the conservation of matter and energy, and even this law is based on purely *a priori* arguments.

Incidentally we will say that the law does not hold good if we restrict the notion of matter to matter in the sense of the physicist which is mass, i. e., to concrete particles of existence that are extended and possess weight. It holds good only if we understand by matter the substance of being, its objective reality. We had better therefore speak of the conservation not of matter but of substance, for gross matter, consisting of the chemical elements, is constantly being produced before our eyes in the starry heavens where the astronomers can watch the process through their telescopes. In the nebulas we see now the commotion of whirls with which gradually first the lighter and then the heavier chemical elements are being manufactured out of the original world-substance which we assume to be the same as the luminiferous ether.

Therefore we may surrender the law of conservation of gross matter, but we still hold to the conception that there is a conservation of stuff or substance, and the same is true of energy. There may be energy in the shape of a stress incorporated in the same wonderful world stuff, the ether, and this stress may be set free and become actual motion or kinetic energy, by some cause which creates those whirls that start the formation of nebulas.

And what proves the law of this conservation of substance and energy? It is the necessity of *a priori* thought which compels us to assume the principle that nothing originates from nothing and nothing disappears into nothing, which thought rests ultimately on the idea that all processes of existence are transformations. Everything that originates is formed by combination from something that existed before.

It has been maintained that the principle of relativity must be proved experimentally, but this is a mistake. Reality is everywhere a system of interrelations, yea every single concrete thing, every phenomenon, every piece of existence is a bundle of relations. It can be analyzed into its elements, which are actions and reactions; and that is all that reality means. Space as well as time are merely the measures, the former of arrangement or position, the latter of succession. Space denotes the interrelation of parts constituting figures or shapes affording a mode of determining direction and distance. Time measures the duration of events which is done by counting uniform cyclical motions or parts thereof. And so we must grant that the relativity of time and space, as well as of all real things is a universal and inalienable condition of all existence. We can not think of any actuality which would not be dominated by relativity; which means we must regard the principle of relativity as an *a priori* postulate.

The principle of relativity is not established by experience but is ultimately based upon reflection and pure ratiocination. It belongs to the category of purely formal thought as much as all arithmetical and geometrical propositions.

If any proposition of purely formal thought, such as $2 \times 2 = 4$, does not hold good in our experience, we doubt the correctness of our counting or measuring, but we do not doubt our *a priori* proposition. We revise our observation, not our logic, our arithmetic, our mathematics; and suppose our observation proves true, suppose that 2×2 rabbits shut up in a cage are on recounting their number found to be more than four, say six or ten or any higher amount, we do not upset our arithmetic or any of our purely formal propositions, but seek the cause of the irregularity in the objects, in the things or animals counted. In that case we are positive that some transformation of the concrete material has set in which adds to the number to be expected according to arithmetical law.

If the reference point (R) belongs to the same system of motion as the object observed (O), our measurement will be correct and indicate the size of the object adequately. But if R moves in a direction and with a velocity of its own, different from O, the measurement will not be adequate; it will be warped in an exact proportion to the motion of R, and this rule holds good in the same way as all mathematical, logical and generally purely formal theorems.

The reliability of purely formal truths is not merely theoretical, but finds its application in practical life, in the objective world of matter and motion, and can be verified by experience and experiment. And this is true also of the relativity of time and space.

If for instance a photographer takes the picture of a rapid express train in motion with a camera provided with a curtain shutter, the wheels will not be round but oval in the photograph, and the relativity photographer who identifies the picture with the thing, in the same way as the relativity physicist identifies the result of measuring with the objective size of the object measured, will claim that in proportion to the velocity of the train times the inverse proportion of the velocity of the slit in the curtain of the shutter, the wheels will increase their horizontal diameters and become that much more oval. Yea they will insist that the very same wheel will be at the same time in one camera, only a little more, in another one much more oval according to the quickness with which the slit of the curtain passes over the sensitive plate.

The relativity photographer will claim that the wheels in motion *are* oval while common mortals think that they only appear oval in the photograph.

Photographs do not lie; they show the objects photographed without any personal equation on the part of the photographer; their objectivity and impartiality can not be doubted, and here we see the wheels oval. They are oval, and their ovality, viz., their deviation from true circles, depends on the velocity of certain motions. An enthusiast for the principle of relativity can justly claim that every photograph of a rapid train which shows the oval form of the wheels is a successful experiment in the demonstration of the relativity of figure in space.

The truth of the principle of relativity in the domain of photography can be explained by *a priori* considerations. It is a matter of course, and if we argue the subject in our mind in pure reflection, we find out what we must expect, and if finally we make the experiment, the principle proves true.

In the same way all the experiments made by machinery so constructed as to represent terrestrial and solar clocks or yard sticks, and to point out the unavoidable difference of measurements in both time and size resultant from their respective motions of the earth and the sun as well as the time it takes to transmit signals, are not experiments in the physicist's sense but expositions and demonstrations of purely formal truths which belong to the category of mathematics.

If the principle of relativity does not hold good in any domain of actual life, we must seek the cause in the material used and not in the principle of relativity. In other words we would be confronted with a purely physical problem which demands a physical solution, and this seems to be the case of the Fizeau experiment.

Prof. Emil Cohn, of Strassburg,[6] says:

> "It is strange that the relativity principle of mechanics does not hold good in radiation—in radiation and therewith in electrodynamics, for that the spread of radiation is an electrical process we may consider since Heinrich Hertz as an assured matter of experience. The decisive experiment which has been made by Fizeau is this: In a liquid, flowing with a uniform velocity, light is to be propagated in the direction of the current. According to the relativity principle an observer drifting in the current should find the velocity of propagation to be the same as if the liquid were at rest, and an outside observer should find the velocity of the light augmented by the full velocity of the current in the liquid. (Think, e. g., of the ball rolling on the deck of a ship in motion.) But such is not the case. There is added only a certain portion, viz., the index of refraction."

The very result of the experiment proves that one of the determinant factors is the physical property of the fluid.

When the principle of relativity is applied to positive

[6] *Loc. cit.*, p. 7.

facts we reach slippery ground, on which we must be on our guard to avoid mystification, for it would seem as if the law of the conservation of matter and energy were upset and all objectivity of scientific truth were lost. Experiments have been made to prove the principle of relativity with the result that Hupka and Bucherer,[7] the former with cathode rays, the latter with radium rays, demonstrate that mass increases with velocity as the relativity principle demands. Kaufmann, however, comes to the conclusion that there is an increase of mass but not as ought to be expected according to the principle of relativity, while Michelson and Morley demonstrate with great exactness that in spite of the motion of the earth the transmission of light is not changed at all, not within one hundred millionth of its proportion nor even a fraction thereof.

It would lead us too far to discuss the experiments made to apply the principle of relativity to physics and electrodynamics; we will only mention that (as *a priori* might be expected) they tend to corroborate its applicability in these domains.

ON ABSOLUTE MOTION.

Dr. Philipp Frank in his discussion "Does Absolute Motion Exist?"[8] declares that motion in physics always means "motion with reference to some definite body," and he recognizes that "this question is a philosophical one[9] but it is certainly not a physical question." The answer is the first Newtonian law, viz., "A body not affected by an exterior force moves in a straight line with a constant

[7] A. H. Bucherer, "Die experimentelle Bestätigung des Relativitätsprinzips" in *Annalen der Physik,* XXVIII, p. 513; "Messungen an Becquerelstrahlen" in *Physikalische Zeitschrift,* IX, pp. 755-760.

[8] "Gibt es eine absolute Bewegung?" Lecture delivered December 4, 1909, at the University of Vienna before the Philosophical Society. *Wissenschaftliche Beilage,* 1910.

[9] Dr. Frank adds here: "Perhaps the psychologist would call it a psychological one," but this would be a mistake. Psychology has nothing to do with the subject.

velocity which of course may be zero.[10] This is called the law of inertia."

If another force affects the moving body it is subject to the second law, the law of the parallelogram of forces, according to which the body will move along the diagonal of the two forces.

The following extracts translated from Dr. Frank's essay on absolute motion will prove instructive:

"The system of the fixed stars constitutes a fundamental body. Even in shooting a cannon ball towards the south we see no deviation from the law of inertia if we consider it with reference to the fixed stars. The ball remains in the same plane; but this plane does not retain the same relative position to the meridian of the earth, wherefore, of course, with reference to the earth the law of inertia is violated. On the whole it is evident that we really recover all the observed motor phenomena when we refer Newton's laws of motion to the fixed stars. Not until they are referred to the fixed stars do these laws acquire an exact sense which makes it possible to apply them to concrete conditions.

"We shall call those motions which are referred to a fundamental body 'true movements' and those related to any other body of reference 'apparent movements.' For instance the immobility of my chair is only apparent, for when referred to the fixed stars it is in motion.

"We now ask whether there are any other fundamental bodies aside from the system of the fixed stars. Obviously not any body revolving in an opposite direction to the fixed stars can be such a fundamental body, for considered with reference to such a body all rectilinear movements are curved. Therefore the law of inertia could not hold with reference to the body in question if it is valid with reference to the fixed stars. Then too a fundamental body can possess no acceleration with reference to the fixed stars, because otherwise there would be no uniformity of the motion of inertia with reference to it. However, these conditions are not only necessary but they are sufficient to characterize a fundamental body. All bodies moving uniformly and in a straight line with reference to the fixed stars will also be fundamental bodies inasmuch as rectilinearity and

[10] The original reads thus: "Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum nisi quatenus a viribus impressis cogitur statum illum mutare."

uniformity continue to hold for them, as do likewise the supplementary velocities determined by the second law. Accordingly Newton's laws do not indicate one single fundamental body, but an infinite number moving in opposite directions with a uniform and rectilinear motion.

"Hence we may well speak of 'true' in contrast to apparent rotary motion; for all bodies revolving with reference to a fundamental body revolve with reference to all other bodies. The same is true of true acceleration because an acceleration with respect to a fundamental body is also acceleration (i. e., change of velocity) with respect to all the rest. On the other hand, there is no sense in speaking of 'true' uniform rectilinear motion; for if a body possesses a uniform velocity with respect to the fixed stars, it is itself a fundamental body possessing of course with respect to itself a velocity of zero; it is at rest.

"Accordingly there is true acceleration, but not true velocity. From this is easily derived a proposition established by Newton which is called the principle of relativity of mechanics, namely that a uniform rectilinear movement of the system as a whole makes no change in the processes within the system; that is to say, we can not tell from the processes within the system what velocity the uniform rectilinear movement possesses with reference to the fixed stars. On the other hand, the rotary motion of a system has indeed an influence on the processes within the system, as for instance in the phenomena of centrifugal force; thus the earth has become flattened at its poles because of its rotation, or if I revolve a dish full of water the water will rise at the sides."

ABSOLUTE SPACE.

If we make measurements of motions which are limited to terrestrial conditions, the earth is and must be the system which, though not absolute, must for the nonce be so considered, and in that case the earth is called the fundamental or inertial body, of our measurements. But in many purely terrestrial motions we observe in very precise and exact measurements, deviations which compel us to seek for another fundamental body.

This happens in the case of the Foucault pendulum experiments and may also be observed in a cannon ball which if shot south along the meridian will at a great distance show a deviation toward the west. Such experiments point out that the entire system of the fixed stars ought to be regarded as the fundamental body which thus would represent to us absolute space. I say here on purpose "represent to us," not "be," because we are most probably in the same predicament as persons moving in a train to whom the train and its interrelations, so long as the train does not move in a curve, represent the fundamental body or absolute space, viz., the ultimate system of reference.

It stands to reason that bodies in translation (in which the entire system as a whole moves in the same direction with the same velocity and without any internal change even of its smallest particles) behave as if they were at rest, and so the motion of a straight line cannot be observed so long as the observer remains limited to his own system. Every deviation from a straight line, however, implies a retardation on the inner side of the curve, or, what means the same, an acceleration on the outside of the curved path of motion. Accordingly all rotations bear witness to the character of their motion as appears in the Foucault pendulum experiment and in the flattening of the earth at the poles. Since further the idea of a rectilinear motion is a mere *a priori* postulate which can never be realized in actual nature, we see that every motion that takes place anywhere is affected by the totality of the universe. We must assume that its existence (the existence indeed of every particular thing or the recurrence of any event) must be understood to be a part of the whole. It bears traces of all the influences of all masses, and of all forces of the rest of the world according to the way it is interrelated with its surrounding conditions.

The fixed stars have so far proved sufficient for our terrestrial needs to serve us as a fundamental body for calculations of a mechanical nature; but here the problem of absolute space presents itself.

We know positively that though the fixed stars are practically a fundamental body to us for mechanical measurements, they are shifting about among themselves and no more constitute something absolute than does our own earth; and yet there has risen a controversy on this subject in which Ernst Mach applies the principle of relativity throughout the universe while Prof. Alois Höfler stands up for what he calls the absolutist theory. We will hear what Dr. Frank has to say on this point:

"Is it to a certain extent accidental, or is it essential, that the tatality of the fixed stars coincides with that fundamental body in relation to which the laws of Newton hold valid? Or to put it more clearly: If the fixed stars were set violently in motion among each other and hence could no longer constitute a fixed body of reference, would the mechanical processes on earth proceed exactly as they did before? For instance, would the Foucault pendulum move just as at present, even though it now turns with the fixed stars, whereas in that case it would not be quite clear which constellation's revolution it should join?

"Were everything to remain as of old the fundamental system of reference would not be determined by the fixed stars but would only accidentally coincide with them, and would in reality be some merely ideal or yet undiscovered body. In the other case all mechanical occurrences on earth would have to be completely altered to correspond with the promiscuous movements of the fixed stars.

"It is well known that this is the view held by Ernst Mach. It alone holds with consistent firmness to physical relativism, and it alone answers the second main question of physics in the relativistic sense.

"The opposite view is represented by Alois Höfler in his studies on the current philosophy of mechanics, and lately by G. Hamel, professor of mechanics at the technical high school of Brünn, in an essay which appeared in the annual report of the German mathematical society of 1909 on 'Space, Time and Energy as *a priori* Forms of Thought.'

"Before I enter upon the controversy itself I would like further to elucidate Mach's view by carrying out its results somewhat farther. In his well-known essay on the *History and Root of the Principle of the Conservation of Energy*[11] Mach ascribes to the distant masses in space a direct influence on the motor phenomena of the earth which supplements the influence afforded by gravitation. Of course no effect of gravitation from the fixed stars upon the earth can be observed, yet in spite of this they influence, for instance, the plane of oscillation of the Foucault pendulum because in Mach's opinion it remains parallel to them.

"The question now arises according to what general law of nature this influence operates which does not, like gravity, produce accelerations but velocities instead. Obviously this influence must be a property belonging to every mass, for according to our present conception the fixed stars of course are precisely the same sort of masses as earthly bodies.

"However, experience teaches us that terrestrial masses have no more influence on the plane of oscillation of the Foucault pendulum than has the changing position of the moon, sun and planets; but on the other hand it is exactly the most distant masses, the fixed stars, which determine its plane of oscillation. Accordingly we must either assume that the effect is directly proportional to the distance of the masses (which would be very strange indeed) or simply assume that this effect is proportional to the effective masses and independent of the distance, whence the dominant influence of the more remote, as the far greater and more numerous, bodies would naturally follow, and Mach inclines to this latter view.

"Mach's view shows most clearly in his position with regard to Newton's famous bucket experiment. In this Newton intended to show that the centrifugal force produced by a revolving body is due not to its relative but to its absolute velocity of rotation. He suspended a bucket filled with water by a vertical cord, twisted the cord quite tightly and then let it untwist itself, in this way setting the bucket to revolve rapidly. At first the water did not rotate with the bucket and therefore the bucket had a velocity of rotation with reference to the water while in the meantime the surface of the water remained undisturbed. In time, however, friction caused the water to become so affected by the rotary motion that bucket and water revolved like one homogeneous mass whereby the centrifugal

[11] Second edition, Leipsic, 1909; English translation by P. E. B. Jourdain, Chicago, 1911.

force caused the water to rise at the sides of the bucket and the surface became concave.

"Hence it is evident that the centrifugal force reached its greatest strength at the moment when the relative motion of the water with respect to the bucket became zero; hence according to Newton this force can be produced only by the absolute rotary motion of the water.

"To this now Mach justly protests that only the relative rotation of the water with reference to the fixed stars is to be considered, for this system of the fixed stars and not the bucket is the fundamental body. And indeed at first the water was at rest with reference to the fixed stars, but at the close of the experiment it was revolving. The mass of the bucket compared to the mass of the fixed stars is an entirely negligible quantity, so that it does not depend in the least upon the rotation. But we can not know, adds Mach, how the experiment would turn out if the sides of the bucket were miles thick; and by this he apparently means so thick that their mass would be considerable even when compared with the mass of the system of fixed stars. Then indeed might the rotation of the bucket disturb the action of the fixed stars.

"Höfler protests, on the other hand, that a system which is symmetrical round its axis could not according to all our experience in mechanics produce by its rotation that sort of an effect on the water within it.

"This also is quite true. But the effect of the masses assumed by Mach is such that it can not be expressed in our ordinary experiences with mechanics except by means of the facts of the inertia of all motion with reference to the fixed stars. New conditions such as the rotation of an enormously thick bucket might give rise to new phenomena. If we agree with Mach's view that the rotation of the plane of the Foucault pendulum is directly produced by the masses of the fixed stars, we must likewise admit, in order to be consistent, that the relative rotation of the very thick bucket might give rise to similar effects with reference to the water, as the rotation of the system of the fixed stars with reference to the earth to the plane of oscillation.

"Höfler expresses his contention against Mach's thesis in the form of the following question: If in Galileo's time the sky had been clouded over and had never become clear again so that we would never have been able to have taken the stars into our calculation, would it then have been impossible to have established our present mechanics solely by the aid of terrestrial experiments? By this question Höfler means to say that if the connection with the fixed stars were a constituent of the concept of uniform motion, we would never have been able in such an overclouded world to have established the law of inertia, for instance, whereas in reality it is clear that this would nevertheless have been possible.

"I will not dwell on the more psychological question as to whether or how easily this would have been possible, but will only consider now the logical construction of mechanics in such a darkened world on the hypothesis that easily or with difficulty in one way or another we would have attained to our present knowledge of mechanics.

"Let us for a moment imagine ourselves in such a world. Above our heads extends a uniform vault of uninterrupted gray or black. Were we to shoot projectiles toward the south we would see that they describe paths which are curved towards the west; if we started pendulums to vibrating we would see that they would revolve their planes of oscillation in mysterious periods—I say mysterious because we might perhaps be able to perceive the change of day and night as an alternation of light and darkness, but would not be able to refer it to the movements of celestial bodies. Perhaps at first we would surmise that the motion of the pendulum could be ascribed to optical influences. I would like to see placed in such a world one of the philosophers who regard the law of inertia as an *a priori* truth. In the face of these mysterious curvatures and deflections he would probably find no adherents and he would not know himself what to make of his own standpoint.

"Finally, let us assume, there arises a dauntless man, the Copernicus of this starless world, who says that all motions proceed spontaneously in a straight line, but that this straight line is not straight with reference to the earth but with respect to a purely ideal system of reference which turns in a direction opposite to that of the earth. The period of this rotation is supplied by the period of the Foucault pendulum.

"This man would of course deny physical relativism upon the earth, for in his opinion terrestrial processes would not depend only on the relative velocities of terrestrial bodies but on something else besides, viz., their velocities with respect to a purely ideal system of reference. Nevertheless, he would not introduce any nonphysical element because for the purpose of the physicist a purely ideal system of reference whose motion with respect to an empirical system is known serves the same purpose as would the empirical system itself. This bold innovator might finally refer the words 'true rest' and 'true motion' to his ideal fundamental body and so ascribe true motion and only apparent rest to the earth, thus maintaining a mechanics which would coincide literally with that of ours to-day, except that no small luminous points would be seen sparkling in connection with the fundamental body.

"Hence we see that physical relativism is not a necessary tool of the physicist. Apart, perhaps, from the psychological improbability—of which, however, nothing more positive can be said—the possibility of the development here indicated is logically free from objections throughout, and therefore the same is also true of the possibility of a nonrelativistic physics.

"But I would like to strengthen the argument of Höfler even somewhat further. That is to say, I would ask whether the world in which we live is then really so essentially different from that fictitious one. Imagine the dark roof which conceals the sky placed somewhat higher so that there is room beneath it for the fixed stars, perhaps as the dark background which may be seen nightly in the starry sky. The whole difference then consists in the fact that not only the Foucault pendulum and similar appliances move with reference to the earth, but enormously greater masses as well—all the twinkling lights of the sky by which the thought of a fundamental body in motion with respect to the earth is psychologically greatly facilitated, but logically is not much changed. Now imagine the sky of this earlier dark world suddenly illuminated; then we would see that the fictitious system of reference is closely linked to enormous cosmic masses, and it would be easy enough to accept Mach's hypothesis that these masses condition the fundamental system....

"If a distinction must be drawn between the respective values of the conceptions of Mach and Höfler, it is as follows: Mach's view adds decidedly more to the observed facts; for that it retains physical relativism does not involve freedom from hypothesis, because at best this relativism is theory and not fact. Mach sets up, hypothetically of course, a new formal natural law with regard to the action of masses existing side by side with gravitation, affecting the experiment very materially but unable to raise any claim to the simplest description of actual conditions.

"The other view, which simply introduces the system of reference procured by observation of the terrestrial and celestial movements without asking whence all this is derived, represents the present state of our knowledge most adequately without any arbitrary addendum but also without giving the spirit of inquiry any incentive to new experiments.

"It is the old contrast between the most exact and least hypothetical representation possible of the known science, and progressive inquiry after new things in more or less daring and fantastic hypotheses. But Mach in this case stands in the opposite camp as in most other cases where his repugnance to all hypothesis has made him a pioneer in the phenomenological direction....

"I therefore believe I have proved that we can grant the following: Physical phenomena do not depend only on the relative motion of bodies without at the same time admitting the possibility of the concept of an absolute motion in the philosophical sense."*

Strange that Mach, with his reluctance to introduce anything hypothetical except what is absolutely indispensable, should range on the side of the theorists, and after some reflection I believe that there may be a slight hitch in Dr. Frank's interpretation of Mach's view.

First I myself, from my own point of view, would refuse to call the principle of relativity an hypothesis; it is an *a priori* proposition, a theorem, or if you prefer, a postulate of pure thought which either holds good universally, or has no validity whatever. So far as I know, Mach has not discussed this side of the subject but he has instinctively acted upon this view, and I would say that there is a greater hypothetical element in the assumption that the theorem $2 \times 2 = 4$, or any other proposition of the same kind, holds good only for our earth but not for Mars and Venus, than to say that it holds good also for the fixed stars and in the possible worlds outside of our Milky Way. Accordingly, whatever Mach's personal opinion may be, I would regard the universal application of the principle of relativity as less complicated and more free from hypo-

* This last paragraph is printed in spaced letters which indicates the emphasis of the author, and so we print the text of his summary in the original. Dr. Frank says: "Die physikalischen Erscheinungen hängen nicht nur von der Relativbewegung der Körper ab, ohne doch damit die Möglichkeit des Begriffes einer absoluten Bewegung im philosophischen Sinne zuzugeben."

thetical elements than its limitation to a portion of the world.

I can not as yet make up my mind to believe that our system of the Milky Way which furnishes us the grand sight of the fixed stars is an ultimate possessing the characteristics of absolute space.

According to Kant the totality of the fixed stars which are thickest in the Milky Way forms a great system (the system of the Milky Way) and our sun as well as all the visible fixed stars belongs to it. Kant believes that this, our own universe, which in the Milky Way appears to us as an enormous ring but together with the totality of the fixed stars must resemble an oblate spheroid, is not the only cosmic system, but that there are other similar systems outside of it and that they too whirl on through the infinity of space, in company with our Milky Way system, around some center of their own; and this very center of many Milky Ways may partake of a motion the observation of which lies hopelessly beyond our ken. Accordingly the space conditions of the Milky Way may serve *us* as absolute space, but there is a probability that this space is not more absolute than are the space relations in a quick but quietly moving train to the passengers.

Another point where we feel justified in doubting Dr. Frank's exposition is the statement that Mach hypothetically assumes a new law of nature as to the efficacy of masses, besides the law of gravitation. The passage in Mach's writings to which Dr. Frank refers does not (in my opinion) suggest the idea of an additional law of nature according to which the distant fixed stars should exercise a mysterious influence on the Foucault pendulum. We will later on let Mach speak for himself. In our opinion it seems that it would be sufficient to ascribe the rotation of the pendulum to its inertia while the earth revolves round itself, and this takes place in the space in which the earth has its motion, viz., the space of the Milky Way system. The pendulum remains in the plane of oscillation in which it started while the earth turns around underneath. If there are influences at work beyond the expanse of the space of the fixed stars in our Milky Way system, they must affect the totality of our system and would therefore be contained in its space conditions; acting with an unfailing constancy they could not be separated from the properties of our space and would scarcely be discoverable.

There seems to me no need of inventing a new force besides gravitation. The law of inertia seems to explain the Foucault pendulum experiment satisfactorily.

The fixed stars as a totality remain in their places (at least as far as concerns the experiment) and the plane in which the pendulum swings keeps its original direction; thus the apparent motions of both coincide. Their space relations (the space relations of the pendulum and of the fixed stars) are the same, and there is no need to assume the existence of any unknown force exercised by the fixed stars upon the pendulum.

ERNST MACH.

We will let Mach state his views in his own words:

"Obviously it does not matter whether we think of the earth as turning round on its axis, or at rest while the celestial bodies revolve round it. Geometrically these are exactly the same case of a relative rotation of the earth and of the celestial bodies with respect to one another. Only, the first representation is astronomically more convenient and simpler.

"But if we think of the earth at rest and the other celestial bodies revolving round it, there is no flattening of the earth, no Foucault's experiment, and so on—at least according to our usual conception of the law of inertia.

"Now, one can solve the difficulty in two ways: Either all motion is absolute, or our law of inertia is wrongly expressed. Neumann[12] preferred the first supposition, I, the second. The law of

[12] *Ueber die Principien der Galilei-Newton'schen Theorie.* Leipsic, 1870.

inertia must be so conceived that exactly the same thing results from the second supposition as from the first. By this it will be evident that, in its expression, regard must be paid to the masses of the universe.

"In ordinary terrestrial cases, it will answer our purposes quite well to reckon the direction and velocity with respect to the top of a tower or a corner of a room; in ordinary astronomical cases, one or other of the stars will suffice. But because we can also choose other corners of rooms, another pinnacle, or other stars, the view may easily arise that we do not need such a point at all from which to reckon. But this is a mistake; such a system of coordinates has a value only if it can be determined by means of bodies....

"If we wish to apply the law of inertia in an earthquake, the terrestrial points of reference would leave us in the lurch, and, convinced of their uselessness, we would grope after celestial ones. But, with these better ones, the same thing would happen as soon as the stars showed movements which were very noticeable. When the variations of the positions of the fixed stars with respect to one another cannot be disregarded, the laying down of a system of coordinates has reached an end. It ceases to be immaterial whether we take this or that star as point of reference; and we can no longer reduce these systems to one another. We ask for the first time which star we are to choose, and in this case easily see that the stars cannot be treated indifferently, but that because we can give preference to none, the influence of all must be taken into consideration.

"We can, in the application of the law of inertia, disregard any particular body, provided that we have enough other bodies which are fixed with respect to one another. If a tower falls, this does not matter to us; we have others. If Sirius alone, like a shooting star, shot through the heavens, it would not disturb us very much; other stars would be there. But what would become of the law of inertia if the whole of the heavens began to move and the stars swarmed in confusion? How would we apply it then? How would it have to be expressed then? We need not worry about one body as long as we have others enough. Only in the case of a shattering of the universe we learn that all bodies, each with its share, are of importance in the law of inertia....

"Yet another example: A free body, when acted upon by an instantaneous couple, moves so that its central ellipsoid with fixed center rolls without slipping on a tangent-plane parallel to the plane of the couple. This is a motion in consequence of inertia. Here the body makes very strange motions with respect to the celestial bodies. Now, do we think that these bodies, without which one cannot describe the motion imagined, are without influence on this motion? Does not that to which one must appeal explicitly or implicitly when one wishes to describe a phenomenon belong to the most essential conditions, to the causal nexus of the phenomenon? The distant heavenly bodies have, in our example, no influence on the acceleration, but they have on the velocity."

Now follows the passage to which Dr. Frank obviously refers:

"Now, what share has every mass in the determination of direction and velocity in the law of inertia? No definite answer can be given to this by our experiences. We only know that the share of the nearest masses vanishes in comparison with that of the farthest. We would, then, be able completely to make out the facts known to us if, for example, we were to make the simple supposition that all bodies act in the way of determination proportionately to their masses and independently of the distance, or proportionately to the distance, and so on. Another expression would be: In so far as bodies are so distant from one another that they contribute no noticeable acceleration to one another, all distances vary proportionately to one another."

We do not here understand Mach to fall back on the assumption of a new kind of force, and if we must grant that the distant masses exercise a dominant influence while the influence of the nearest ones (of the earth, the moon, and the sun) vanishes, we would say that this is due to the constancy of the distant masses which, as it were, is an inherent and inalienable part of all mass in the entire system and may be said to characterize its space conditions.

In speaking of "space conditions" I am conscious of using a term which Mach would repudiate, for he claims that for a comprehension of the concatenation of events, the notions of time and space are redundant. He says (*loc. cit.* pp. 60-61):

"To say the least, it is superfluous in our consideration of causality to drag in time and space. Since we only recognize what we call time and space by certain phenomena, spatial and temporal determinations are only determinations by means of other phenomena. If, for example, we express the positions of earthly bodies as functions of the time, that is to say, as functions of the earth's angle of rotation, we have simply determined the dependence of the positions of the earthly bodies on one another.

"The earth's angle of rotation is very ready to our hand, and thus we easily substitute it for other phenomena which are connected with it but less accessible to us; it is a kind of money which we spend to avoid the inconvenient trading with phenomena, so that the proverb "Time is money" has also here a meaning. We can eliminate time from every law of nature by putting in its place a phenomenon dependent on the earth's angle of rotation.

"The same holds of space. We know positions in space by the affection of our retina, or our optical or other measuring apparatus. And our x, y, z in the equations of physics are, indeed, nothing else than convenient names for these affections. Spatial determinations are, therefore, again determinations of phenomena by means of other phenomena.

"The present tendency of physics is to represent every phenomenon as a function of other phenomena and of certain spatial and temporal positions. If, now, we imagine the spatial and temporal positions replaced in the above manner, in the equations in question, we obtain simply every phenomenon as function of other phenomena.

"Thus the law of causality is sufficiently characterized by saying that it is the presupposition of the mutual dependence of phenomena. Certain idle questions, for example, whether the cause precedes or is simultaneous with the effect, then vanish by themselves."

We understand that Mach endeavors to eliminate the terms time and space, because he wishes to correct the common notion which regards space as a big box into which the world has been packed. Mach says:

"Space and time are not here conceived as independent entities, but as forms of the dependence of the phenomena on one another. I subscribe, then, to the principle of relativity, which is also firmly upheld in my *Mechanics* and *Wärmelehre*."[13]

We agree with Mach. There is no time in itself; there

[13] Cf. "Zeit und Raum physikalisch betrachtet," in *Erkenntnis und Irrtum*, Leipsic, 1905 (2d ed. 1906, pp. 434-448); See also *Space and Geometry*, pp. 94 ff.

is no space in itself. Nevertheless, Mach has given much attention to physical space and appreciates the important part which it plays not only in the formation of our space-conception, but also in the actual world, for every spot of space possesses physical qualities according to the particles of mass which are there aggregated. Mach says:

"Since the positions in space of the material parts can be recognized only by their states, we can also say that all the states of the material parts depend upon one another.

"The physical space which I have in mind—and which, at the same time, contains time in itself—is thus nothing other than dependence of phenomena on one another. A complete physics, which would know this fundamental dependence, would have no more need of special considerations of space and time, for these latter considerations would already be included in the former knowledge."

The same idea is expressed by Mach in his Essay "Ueber den Zeitsinn des Ohres:[14]

"Physics sets out to represent every phenomenon as a function of time. The motion of a pendulum serves as the measure of time. Thus, physics really expresses every phenomenon as a function of the length of the pendulum. We may remark that this also happens when forces, say, are represented as functions of the distance; for the conception of force (acceleration) already contains that of time. If one were to succeed in expressing every phenomenon—physical and psychical—as a function of the phenomenon of pendulum-motion, this would only prove that all phenomena are so connected that any one of them can be represented as a function of any other. Physically, then, time is the representability of any phenomenon as a function of any other one."

We do not deny the truth of Mach's view. Nevertheless time and space are very convenient terms denoting two categories of certain interrelations (he would call them interdependencies) in the flux of things. Popular terms mostly originate because there is a need of them, and it seems to me it would be wiser to correct the errors connected with them than to drop them. If we pursue the

[14] *Sitzb. der Wien. Akad.*, 1865. Compare *Conservation of Energy*, p. 90.

latter policy we shall find ourselves obliged to reinvent a new collective term for certain classes of relations which belong together and can not be identified with other relations. The space and time relations are radically different from those of a purely physical, chemical or psychological nature.

We need not fear to retain the old terms, space and time, if we only bear in mind that there is neither absolute space nor absolute time but that the words denote relations. It seems to me that when Kant speaks of the ideality of space and time and insists on their non-existence as objective beings (*Wesen* or *Wesenheiten*) he attempts to say the same as Mach who declares that they are not "independent entities."

The conclusion at which we arrive in considering the nature of time and of space, be it from our standpoint of philosophy or from Mach's physical point of view, may be expressed in one word, that their most obvious characteristic is relativity.

CONCLUSION.

Professor Mach says in one of his notes quoted above, "I subscribe then to the principle of relativity," and so do I. Indeed I go one step further. I consider relativity as an inherent quality of existence and so I adopt the principle of it not as a result of experience but on *a priori* grounds. The principle of relativity, however, is frequently stated by relativity physicists as if the old ideal of science in its objective significance had to be abandoned, as if physics had to be remodeled, and as if the proclamation of the principle of relativity indicated a new departure from our traditional methods. This is not so, and I must insist that the principle of relativity has always been subconsciously in the minds of scientists. Only it has lately been forced upon the attention of physicists by the progress in astronomical measurements.

How helpful the emphasis recently laid upon the principle of relativity will prove remains to be seen. Its ardent adherents exhibit great zeal which in many directions seems to be misdirected, and it appears to me that in spite of the correctness of the underlying idea their hopes are greatly exaggerated. After a while when the opponents of the principle of relativity will understand that its truth is as much a matter of course as the truth of the law of conservation of matter and energy, the contentions about it will cease and the evolution of science will no longer show evidence of excitement but will continue in its old quiet way.

There is more philosophy in our science than the school of empiricists are inclined to believe. It is very desirable that in familiarizing themselves with philosophy, these scientists should not fall back on the old systems of a visionary absolute, but they should adopt the philosophy of science, the only philosophy which is not a mere ingenious dream, and possesses objective significance.

The philosophy of science is *the* philosophy. It is the indispensable introduction to the study of any science and furnishes the basis for scientific method as well as a general survey of the assured results of all the several sciences. If the philosophy of science had been better known, the principle of relativity had at once been rightly understood and the vagaries of many mystifying contentions would have been avoided.

* * *

The purpose of this article is to set forth in general outlines the truth and significance of the principle of relativity, not to present an exhaustive treatment of it in all its phases and applications. We must bear in mind that in dealing with the several innumerable problems of existence science introduces a method which possesses certain limitations due to conditions which originate through some fictions of an apparently arbitrary nature assumed for the sake of isolating the object of investigation and concentrating upon it our attention.

We must bear in mind that we behold an object by focusing our eyes upon it and that only thereby can we form a picture of the object. It is a fiction to behold an object as if it were a thing by itself and it is positively impossible to see anything as it is in all its relations and with all its changes, past, present and future. Nor would such a comprehension of the object in all its entirety be desirable, for in the omneity of its relations we would see the whole universe while the special feature which concerns us sinks into insignificance. The same is true of science. Each of the several sciences selects its own field of investigation and thus constitutes a definite domain of abstraction for the sake of concentrating all attention upon it. For mechanics and for the measurements of motion in space, we need a reference point which must be able to be considered stationary, and if that is not the case we must refer both the movable place of observation, viz., the reference point (R) and the object observed (O) to one common system, which could be treated as, or must so far as R and O are concerned, actually be, stable.

We conclude by repeating that there is nothing absolute; all real and actual existences, all concrete things and happenings are relative, and if there is any thing that in a certain sense deserves the name absolute it is the truth as described in our mental fictions, the laws of purely formal thought, the eternal uniformities of purely formal relations such as we know from mathematics and all the other purely formal sciences; but even they are absolute only in the sense of constituting an entire system the truth of which is absolute, viz., it stands aloof and is founded in itself as a world of necessary conclusions built up in the field of anyness to serve as models for any conditions in any world actual or imaginary. And this absolute, this system of mental construction is after all a system of relations.

The more we ponder on the nature of existence, the more we shall understand the sweeping significance of relativity.

P. C.

INVENTORS I HAVE MET.*

ANY one who has been a professor of physics in a large city for several decades, unless he has earned a reputation for the crudest and densest Philistinism, must have made the acquaintance of divers thinkers and inventors who have taken counsel with him in their perplexities—thinkers of all kinds, schooled and unschooled, sanguine and timid, those that solve problems and those that create them; thinkers, suspicious and confiding, ambitious and practical; inventors at any price, and inventors on occasion.

It is obvious that the number of actual or alleged inventors in this company is greater than that of silent studious, self-centered thinkers. Practical discomfort is felt more often and to a greater extent than the rarer purely intellectual discomfort which is the heritage of men on a higher spiritual plane. Many fruitless hours may be spent in such consultations, but many a bit of psychological illumination may be gained and many a glance into the embryology of technique and science. We may add right here that the unlettered, unschooled or wild thinkers and inventors are the most interesting and instructive.

* * *

One day a gentleman was announced who had something of importance to communicate to me. He told me that he had taken a narrow tube full of liquid, closed at the upper end and open below, from which of course

*** Translated from the German by Lydia G. Robinson.**

nothing could flow because of the pressure of air; then he gave it a charge of electricity, whereupon the liquid began at once to flow. From this he drew the rash conclusion that the electric charge removed the air pressure. I gave instructions that an appointment be made with this gentleman for a free hour in the afternoon in order to make the experiment. But since one can easily tell whether or not a man is undertaking something from a purely theoretical interest, I said to the attendant in the laboratory, "The gentleman probably thinks he can drive a railway train with the electrical machine." In the afternoon considerably before the appointed time the stranger put in an appearance. "Are you thinking of driving a railroad train?" the attendant asked him by way of filling the interval with conversation. Immediately and without losing another word the gentleman seized his hat and was gone forever. So I had guessed his purpose correctly, and had deprived him of the pleasure of taking me into his confidence in his alleged lucrative undertaking. Forty years have passed since then, and the man has probably calmed down in the meantime.

* * *

There are people who become greatly excited over every scientific novelty, whose imagination busies itself at once in a new field without any special participation on the part of their intelligence, and whose desire it is to make an invention or a discovery in this field at any cost. So after the discovery of the Foucault rotation of the pendulum's plane of oscillation many experiments were made known by which it was thought this rotation could be perceived in water standing in a cylindrical tub across whose surface coal-dust had been lightly strewn; or again in a horizontal disk suspended by a thread, or in a scale-beam similarly suspended.

But obviously these experiments are not sensible. For instance, if a horizontal disk is actually at rest with reference to the earth it has of course the component of rotation of the earth around the perpendicular corresponding to geographical latitude; therefore the disk can not henceforth alter its position with reference to its terrestrial surroundings. Under other circumstances, however, it has an angular velocity around the perpendicular due to some impulse, to a draft of air, or the thread's momentum of rotation, and hence has no connection whatever to the Foucault rotation. One young man could not accept these reflections at all but persisted in repeating the experiment thus described by which he gained the interest of an old gentleman who observed in them "sometimes" the genuine Foucault rotation.

To be sure, Professor Tumlirz has recently performed an experiment which, while externally similar to this, is *correct*. By this experiment the rotation of the earth can be imitated, if the utmost care is taken, by the direction of the current of water flowing axially out of a cylindrical vessel. Further details are to be found in an article by Tumlirz in the *Sitzungsberichte der Wiener Akademie*, Vol. 117, 1908. I happened to know the origin of the thought that gave rise to this invention. Tumlirz noticed that the water flowing somewhat unsymmetrically in a glass funnel assumed a swift rotation in the neck of the funnel so that it formed a whirl of air in the axis of the flowing jet. This put it in his mind to increase the slight angular velocity of the water at rest with reference to the earth, by contraction in the axis.

The above-mentioned imaginative young man also constructed a telephone by a static electrical charge, and this invention likewise proved a delusion. Experimenting within the space of one room he had heard his own voice both as transmitter and receiver at the same time. Very often an illusory invention bears witness simply to the ardent hopes of its originator.

* * *

Another young man declared that the theories of Galileo with regard to falling bodies and projectiles which he had learned in school were false; that the projected stone forms an entirely different problem from the falling stone; that the stone that is thrown is carried through the air and in the projection gravity is overcome. To this man the Aristotelian distinction between the natural falling motion and the violent motion of throwing is still valid. The fusion of the two primitive ideas into a unified whole had not yet taken place in his understanding.

* * *

Such a reversion to the primitive condition of science is not an isolated one. We may therefore conclude that after a disturbing interruption of the development of civilization science would again pursue almost the same course of evolution it had previously followed, although this of course would not preclude minor accidental discrepancies. Science has also its own natural embryology which is revealed through epistemology. Once I received an inquiry from the United States about the hydrostatic paradox which after Archimedes has been explained by Stevinus and for the third time by Pascal. The American writer declared that he could not understand how the pressure at the bottom of a vessel could depend upon anything else than upon the weight of the liquid resting on the bottom. Of course this was a very natural idea. I now proceeded to expound to the gentleman that the pressure at the bottom can not depend on the weight of the liquid resting on the bottom, but only on that portion of the weight which must be lifted in lifting the bottom, not the whole vessel. This seems to have met with comprehension at once. The ingenious and spontaneous complacency of this American was altogether charming and delightful to me. He answered me in English since he knew no other language. He lived in "Cosmopolis"—street and number were unnecessary, simply the name of the writer sufficed. Hence the place was probably not yet Cosmopolis, but for the time being perhaps an embryo of five or ten houses which had undertaken to become a cosmopolis.

Intercourse with born thinkers of this type is very agreeable to me. Thus I would love to have known that naive Chinaman who, pointing to the street-car in San Francisco, the propelling force of which seemed incomprehensible to him, said (as my colleague B. Brauner tells me), "No pushee, no pullee, but it runs—."

* * *

One day I had a visitor whose external appearance proclaimed him every inch a man conscious of successful achievement. Without any doubt he was also intelligent, a good observer who had used his own eyes and knew how to turn his observations to practical account. He belonged to the class of inventors on occasion who base their constructions on practical and local knowledge and not on the fancy that something must be invented whether or no. He certainly deserved the success of his great business which extended over all Europe. But what surprised me was that he manifested such high theoretical aims at the same time. He felt like the laboratory assistant of Faraday who performed experiments while the great man only delivered the superfluous lecture about them. How could this great lecture, called science, have many difficulties for one who was so successful in his practical life, for that is the proof of the sum? Then too his theory was not in the least without foundation, for it rested on independent observation, that is to say, on what is called the Leidenfrost experiment. But while he ascribed to this one observation an unduly enormous significance, he questioned at the same time the Newton theory of gravitation and all other possible theories, or undertook to base them on different foundations. My word for it, his observation was good, but onesided and incomplete, and therefore inadequate for a foundation of his theories and would not bear much fruit. He had a strong desire to rush at once into print. "If you wish to do that, my dear sir, I advise you at least to publish anonymously or under a pseudonym. In case you are ridiculed you can then join heartily in the laugh without anxiety for your reputation." The sensible man followed this advice and was splendidly successful in his book selling, for there are plenty of imaginative people who take pleasure in crazy theories. "Wisdom and experience in one field," I said also in the course of our conversation, "do not protect us from folly in another. You are efficient in your specialty and we will suppose that I am in mine. Would we not both be astonished and confused if you for instance would come out to-morrow as an obstetrician and I the day after as a dentist? And yet no less schooling and experience are needed for the conquest of a scientific specialty."

* * *

Many people feel that nothing else so cramps and limits their imagination as certain principles in science which are held to be firmly established and which others are used to look upon as providing the most abundant aid. Such a principle for instance is that of the equality of action and reaction, and another is that of the impossibility of perpetual motion.

Once I was urgently invited to visit a man who wished to show me something very remarkable. When I arrived he first told me the following story. He said that he had never doubted the principle of the equality of pressure and counter-pressure. But once he had heard a traveler tell of an animal in South America that sprang with agility from branch to branch without communicating the slightest motion to the branches either as it left one or reached the the other. This aroused his interest so greatly that he went at once to South America in order to observe this squirrel-like animal. Here he convinced himself that the law of the equality of pressure and counter-pressure did not hold good. Upon his return he succeeded in devising an arrangement with which by means of cords fastened to one and the same body a motor tendency was communicated to this body. He showed me a ruler in which a motor impulse would arise by means of threads crossed and stretched in various directions between swivels. As he held it in his hand he said, "Now I feel myself drawn over there towards the door," whereupon he proceeded to step in that direction. "If that is so, sir," answered I, "you will easily be able to convince every one of the fact, if you will let this ruler swim freely on the surface of water so that it can move in a definite direction without your personal intervention." This he promised to do. I now felt myself impelled toward the door and took my leave as I began to feel somewhat uncanny. It was really very disquieting to remain in a place where, because of the inequality of pressure and counter-pressure, a tied-up package or a well-screwed piece of furniture would be able spontaneously and independently to get up and travel and fly at my head. It is now about twenty years since I have heard anything of this wonderful experiment.

* * *

There was an old gentleman of whom I was very fond who took a great interest in the problem of perpetual motion. He held that an instance of it must eventually be found because it was necessary for the progress of humanity. The most diverse hydraulic and mechanical constructions were undertaken. When they were complicated enough so that they could not be seen through he thought he had reached his goal, but each time was of course disillusioned. Since he was an educated man I gave him Huygens's *Horologium oscillatorium* to read in which these conditions are set forth very clearly and simply, but it made no permanent impression. Ever and again his imagination overcame his judgment and ever and again triumphed the

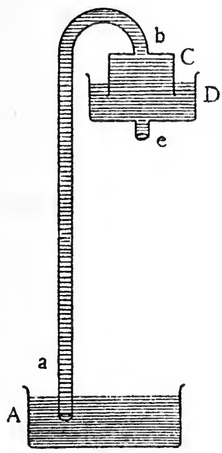


Fig. 1.

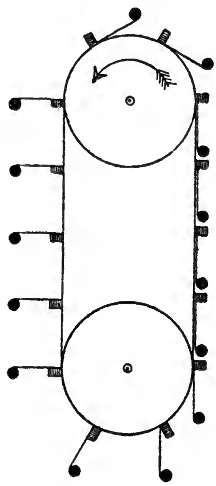


Fig. 2.

unshakable conviction of the necessity of this thing for the good of humanity. Somewhat similarly must Aristotle have thought with regard to the displacing of slave labor by the use of machinery.

One of the constructions of the old gentleman I remember very distinctly. It may be easily understood as presented in Fig. 1. A siphon *ab* dips into the vessel A and at the other end with a bell-shaped expansion C into the vessel D. If the openings *a* and *e* are left unobstructed then, according to the expectation of the inventor, the small mass of water in the tube *ab* would follow the large masses of C and D and flow out at *e*. Instead of this, *Cba* behaved like a normal siphon flowing in the direction indicated by the letters, whereas a part of the water in D, to be sure, descended through *e* so that a break occurred between the water in C and in D, whereat the arrangement had failed to perform its function.

* * *

When I was a boy I had heard so much about perpetual motion that at a time when I had only a very superficial knowledge of the law of the lever I zealously set to work on the construction of a *perpetuum mobile*. The drawing in Fig. 2 will make clear the construction and its error. I was tempted to regard the horizontal bars with weights as somewhat long and efficient levers, although in this case there could be no question of levers and their rotation. Nature does not allow itself to be outwitted like the limited attention of man. To lift a weight P to the height H absolutely requires a weight P′ which reaches the depth H′, so that $P' \times H'$ is at least equal to $P \times H$. I can not say that this effort did me any harm. The mistake taught me to understand machines better than books or instruction could have done.[2] If any theory is of practical value in promoting civilization it is that of the limitation of available mechanical power, and no illusion is more harmful to progress than the idea of its inexhaustibility.[3]

* * *

One of the most remarkable inventors whom I have ever known was an old mechanic. At every detail he noted some advantage in construction and at once applied his idea. He reformed the handles and shape of beer glasses,

[2] The collection of constructions of perpetual motion machines preserved in the Technical Museum at Munich must be very instructive from a psychological point of view, as far as they can be deciphered.

[3] Indeed one of the greatest advances made in natural science rests upon the overthrow of this illusion through a fundamental employment of it.

laundry mangles, theater curtains; he constructed a clock from a barometer-tube closed at both ends in which a short column of mercury was placed at the side of a scale marked off empirically to measure time. He was a funny old fellow who wished to do away with the figures on the tower clock because "anyone would be a fool who would not be able to tell the time by the position of the hands." He was a born physicist. From his simple story I can not doubt that by blowing away the sawdust from a circular saw with perforations in the rim he discovered of his own accord the principle of the disk-shaped siren and the law of tone vibrations.

He was as extremely jealous of Cagniard Latour as if the latter by his much earlier observation had robbed him of the finest discovery. On the principle of the disk-siren he based his invention of a new musical instrument which he called a sirenophone. By means of a weight and a continuous cord a pedal set the system of the siren-disks in uniform rotation and at the same time worked a bellows. Piano keys, sunk more or less deeply with increased pressure, opened one or more tubes which blew with varying degrees of strength into the series of holes of the siren-disks so as to swell individual tones. The difference in pitch was obtained by the proportion of the radii of the pulleys over which the cords of the disks were drawn. This instrument made far more pleasant music than a harmonium and it would be simply impossible for it to get out of tune. It could be manufactured in perfect tune by a simple method of stamping. When a young man proposed to the inventor to sell his invention but keep its name, he received the answer, "The invention is great but unsalable." Hence he apparently preferred that it continue its existence as unique and legendary rather than be a source of profit. When a colleague once tried to play the instrument the inventor fell upon him furiously and declared it was a sacrilege. The inventor surrounded himself with the mystery of a medieval wizard and conjurer. The orders of the minor petty German princes for whom he had arranged various theatrical details he wore with ostentation and listed them carefully upon his visiting cards. This man's vanity greatly diminished the impression of his very considerable talent and disturbed his relations with his hardly less gifted brother.

* * *

In my institute I once had a very gifted young man D. to whom I proposed that he carry on a piece of work in physiological optics in which he made good progress. One day I came to him with the question, "Well, what are you doing?" "Nothing," was the answer, "because I haven't any pasteboard to make a new disk." "Well, if that is all it takes to put a stop to your research you will not get very far," was my reply. This episode would not have remained in my memory if D. had not reminded me of it years later. But it is noteworthy that soon afterwards he completed a series of fine tasks for which he had provided himself with all necessary devices in the simplest way possible; he almost never had need for anything from the materials of the institute. He constructed a Jamin compensator by cutting a slightly curved optical lens. I must add that I have seen many similar accessories in the collection left by Nörrenberg in Tübingen. There stood whole cases full of the cleverest optical apparatus made out of cork and glass. Nörrenberg let the endowment lapse and made his apparatus himself in order not to have to write everything down in the inventory book and keep a strict account of it. Every curator of an institution is familiar with this burden which always intrudes upon his most convenient time for work, or on his vacation.

The young man D., who was the exact opposite of the preceding one in seriousness and simplicity, soon became my assistant and left with me a cheerful memory of his dry humor. When I was demonstrating to beginners the interference bands of the sodium flame by the greater thickness of layers of air of the Newton glass and bade them not to focus their eyes upon the flame but on the glass, they did not all suceeed in this at once. With averted face the assistant scattered a few grains of salt over the glass, with the words, "There now, look at the salt!" When I pointed out the Talbot bands by covering half of the pupil with a piece of mica many looked through the mica and many looked past it. The assistant cut a small hole in a piece of black paste board and covered the half with mica, saying: "There now, look through the hole!" When I called attention to the range of oscillation of a string which vibrated the fundamental tone and the octave at the same time, one of the class was almost misled into considering it two strings. "Put your finger in between quick, then you will have two!" said the assistant.

* * *

In this brief review we have not drawn any sharp distinction between inventors and thinkers, between invention and discovery. Indeed there is no great difference. The liberation from a practical discomfort by a new procedure we call an invention. But if we feel an intellectual discomfort, in that for instance we can not follow in our thought an unaccustomed fact and can not see through it, then we call a serviceable guide of our thoughts which helps us to do so a discovery. When a man fiinds he can not boil water in a pumpkin shell because it catches fire he invents the pot by surrounding the pumpkin with clay. When a man can not understand the light and dark bands in conflicting rays of light from two identical sources because he thinks of light as a uniform stream he discovers interference from the instruction to represent light with periodically changing properties. Discoveries and inventions may be due to an accidental occasional observation, as is shown in the above examples. In other cases they may be the result of prolonged systematic work as has been illuminatingly presented by the Muscovite engineer P. K. v. Engelmeyer in his essay *Der Dreiakt als Lehre von der Technik und der Erfindung* (Berlin, Heymann, 1910).[4] If an invention is to be made there must be the *desire* to remove an inconvenience; there must be the *knowledge* of the means by which this can be done, and the *ability* to make a practical application of them. This is the *Dreiakt* of the *purpose,* the plan for *attaining* it and the material performance which takes place *mutatis mutandis* also whenever a theoretical problem is put to a practical application.

Ernst Mach.

Vienna, Austria.

[4] See a further account of this work in the editorial in this number entitled "A New Theory of Invention."

THE NEW LOGICS.[1]

I. THE RUSSELL LOGIC.

TO justify its pretensions, logic had to change. We have seen new logics arise of which the most interesting is that of Russell. It seems he has nothing new to write about formal logic, as if Aristotle there had touched bottom. But the domain Russell attributes to logic is infinitely more extended than that of the classic logic, and he has put forth on the subject views which are original and at times well warranted.

First, Russell subordinates the logic of classes to that of propositions, while the logic of Aristotle was above all the logic of classes and took as its point of departure the relation of subject to predicate. The classic syllogism, "Socrates is a man," etc., gives place to the hypothetical syllogism: "If A is true, B is true; now if B is true, C is true," etc. And this is, I think, a most happy idea, because the classic syllogism is easy to carry back to the hypothetical syllogism, while the inverse transformation is not without difficulty.

And then this is not all. Russell's logic of propositions is the study of the laws of combination of the conjunctions *if, and, or,* and the negation *not.*

In adding here two other conjunctions *and* and *or,* Russell opens to logic a new field. The symbols *and, or* follow the same laws as the two signs $\times$ and $+$, that is

[1] Translated by George Bruce Halsted.

to say the commutative, associative and distributive laws. Thus *and* represents logical multiplication, while *or* represents logical addition. This also is very interesting.

Russell reaches the conclusion that any false proposition implies all other propositions true or false. M. Couturat says this conclusion will at first seem paradoxical. It is sufficient however to have corrected a bad thesis in mathematics to recognize how right Russell is. The candidate often is at great pains to get the first false equation; but that once obtained, it is only sport then for him to accumulate the most surprising results, some of which even may be true.

II.

We see how much richer the new logic is than the classic logic; the symbols are multiplied and allow of varied combinations *which are no longer limited in number.* Has one the right to give this extension to the meaning of the word *logic*? It would be useless to examine this question and to seek with Russell a mere quarrel about words. Grant him what he demands; but be not astonished if certain verities declared irreducible to logic in the old sense of the word find themselves now reducible to logic in the new sense—something very different.

A great number of new notions have been introduced, and these are not simply combinations of the old. Russell knows this, and not only at the beginning of the first chapter, "The Logic of Propositions," but at the beginning of the second and third, "The Logic of Classes" and "The Logic of Relations," he introduces new words that he declares indefinable.

And this is not all; he likewise introduces principles he declares indemonstrable. But these indemonstrable principles are appeals to intuition, synthetic judgments *a priori.* We regard them as intuitive when we meet them more or less explicitly enunciated in mathematical treatises; have they changed character because the meaning of the word logic has been enlarged and we now find them in a book entitled "Treatise on Logic"? *They have not changed nature; they have only changed place.*

III.

Could these principles be considered as disguised definitions? It would then be necessary to have some way of proving that they imply no contradiction. It would be necessary to establish that, however far one followed the series of deductions, he would never be exposed to contradicting himself.

We might attempt to reason as follows: We can verify that the operations of the new logic applied to premises exempt from contradiction can only give consequences equally exempt from contradiction. If therefore after n operations we have not met contradiction, we shall not encounter it after $n+1$. Thus it is impossible that there should be a moment when contradiction *begins,* which shows we shall never meet it. Have we the right to reason in this way? No, for this would be to make use of complete induction; and *remember, we do not yet know the principle of complete induction.*

We therefore have not the right to regard these assumptions as disguised definitions and only one resource remains for us, to admit a new act of intuition for each of them. Moreover I believe this is indeed the thought of Russell and M. Couturat.

Thus each of the nine indefinable notions and of the twenty indemonstrable propositions (I believe if it were I that did the counting, I should have found some more) which are the foundation of the new logic, logic in the broad sense, presupposes a new and independent act of our intuition and (why not say it?) a veritable synthetic judgment *a priori.* On this point all seem agreed, but what Russell claims, and what seems to me doubtful, is that after these appeals to intuition, that will be the end of it; we need make no others and can build all mathematics without the intervention of any new element.

M. Couturat often repeats that this new logic is altogether independent of the idea of number. I shall not amuse myself by counting how many numeral adjectives his exposition contains, both cardinal and ordinal, or indefinite adjectives such as several. We may cite however some examples:

"The logical product of *two* or *more* propositions is . . . .";

"All propositions are capable only of *two* values, true and false";

"The relative product of *two* relations is a relation";

"A relation exists between *two* terms," etc., etc.

Sometimes this inconvenience would not be unavoidable, but sometimes also it is essential. A relation is incomprehensible without two terms; it is impossible to have the intuition of the relation, without having at the same time that of its two terms, and without noticing they are two, because, if the relation is to be conceivable, it is necessary that there be two and only two.

V.

ARITHMETIC.

I reach what M. Couturat calls the *ordinal theory* which is the foundation of arithmetic properly so called. M. Couturat begins by stating Peano's five assumptions, which are independent, as has been proved by Peano and Padoa.

1. Zero is an integer.
2. Zero is not the successor of any integer.
3. The successor of an integer is an integer.

To this it would be proper to add,

Every integer has a successor.

4. Two integers are equal if their successors are.

The fifth assumption is the principle of complete induction.

M. Couturat considers these assumptions as disguised definitions; they constitute the definition by postulates of zero, of successor, and of integer.

But we have seen that for a definition by postulates to be acceptable we must be able to prove that it implies no contradiction.

Is this the case here? Not at all.

The demonstration cannot be made *by example*. We cannot take a part of the integers, for instance the first three, and prove they satisfy the definition.

If I take the series 0, 1, 2, I see it fulfils the assumptions 1, 2, 4, and 5; but to satisfy assumption 3, it still is necessary that 3 be an integer, and consequently that the series 0, 1, 2, 3, fulfil the assumptions; we might prove that it satisfies assumptions 1, 2, 4, 5, but assumption 3 requires besides that 4 be an integer and that the series 0, 1, 2, 3, 4, fulfil the assumptions, and so on.

It is therefore impossible to demonstrate the assumptions for certain integers without proving them for all; we must give up proof by example.

It is necessary then to take all the consequences of our assumptions and see if they contain no contradiction.

If these consequences were finite in number, this would be easy; but they are infinite in number; they are the whole of mathematics, or at least all arithmetic.

What then is to be done? Perhaps strictly we could repeat the reasoning of number III.

But as we have said, this reasoning is complete induction, and it is precisely the principle of complete induction whose justification would be the point in question.

VI.

THE LOGIC OF HILBERT.

I come now to the capital work of Hilbert which he communicated to the Congress of Mathematicians at Heidelberg, and of which a French translation by M. Pierre Boutroux appeared in *l'Enseignement mathématique,* while an English translation due to Halsted appeared in *The Monist.*[2] In this work, which contains profound thoughts, the author's aim is analogous to that of Russell, but on many points he diverges from his predecessor.

"But," he says (*Monist,* p. 340), "on attentive consideration we become aware that in the usual exposition of the laws of logic certain fundamental concepts of arithmetic are already employed, for example the concept of the aggregate, in part also the concept of number.

"We fall thus into a vicious circle and therefore to avoid paradoxes a partly simultaneous development of the laws of logic and arithmetic is requisite."

We have seen above that what Hilbert says of the principles of logic in the usual exposition, applies likewise to the logic of Russell. So for Russell logic is prior to arithmetic; for Hilbert they are "simultaneous." We shall find further on other differences still greater, but we shall point them out as we come to them. I prefer to follow step by step the development of Hilbert's thought, quoting textually the most important passages.

"Let us take as the basis of our consideration first of all a thought-thing 1 (one)" (p. 341). Notice that in so doing we in no wise imply the notion of number, because it is understood that 1 is here only a symbol and that we do not at all seek to know its meaning. "The taking of this thing together with itself respectively two, three or more times...." Ah! this time it is no longer the same; if we

[2] "The Foundations of Logic and Arithmetic," *Monist* XV, 338-352.

introduce the words "two," "three," and above all "more," "several," we introduce the notion of number; and then the definition of finite whole number which we shall presently find, will come too late. Our author was too circumspect not to perceive this begging of the question. So at the end of his work he tries to proceed to a truly patching up process.

Hilbert then introduces two simple objects 1 and =, and considers all the combinations of these two objects, all the combinations of their combinations, etc. It goes without saying that we must forget the ordinary meaning of these two signs and not attribute any to them.

Afterwards he separates these combinations into two classes, the class of the existent and the class of the non-existent, and till further orders this separation is entirely arbitrary. Every affirmative statement tells us that a certain combination belongs to the class of the existent; every negative statement tells us that a certain combination belongs to the class of the non-existent.

IV.

Note now a difference of the highest importance. For Russell any object whatsoever, which he designates by x, is an object absolutely undetermined and about which he supposes nothing; for Hilbert it is one of the combinations formed with the symbols 1 and =; he could not conceive of the introduction of any thing other than combinations of objects already defined. Moreover Hilbert formulates his thought in the neatest way, and I think I must reproduce *in extenso* his statement (p. 348):

"In the assumptions the arbitraries (as equivalent for the concept 'every' and 'all' in the customary logic) represent only those thought-things and their combinations with one another, which at this stage are laid down as fundamental or are to be newly defined. Therefore in the deduction of inferences from the assumptions, the arbitraries, which occur in the assumptions, can be replaced only by such thought-things and their combinations.

"Also we must duly remember, that through the superaddition and making fundamental of a new thought-thing the preceding assumptions undergo an enlargement of their validity, and where necessary, are to be subjected to a change in conformity with the sense."

The contrast with Russell's view-point is complete. For this philosopher we may substitute for *x* not only objects already known but any thing.

Russell is faithful to his point of view, which is that of comprehension. He starts from the general idea of being, and enriches it more and more while restricting it, by adding new qualities. Hilbert on the contrary recognizes as possible beings only combinations of objects already known; so that (looking at only one side of his thought) we might say he takes the view-point of extension.

VIII.

Let us continue with the exposition of Hilbert's ideas. He introduces two assumptions which he states in his symbolic language but which signify, in the language of the uninitiated, that every quantity is equal to itself and that every operation performed upon two identical quantities gives identical results.

So stated, they are evident, but thus to present them would be to misrepresent Hilbert's thought. For him mathematics have to combine only pure symbols, and a true mathematician should reason upon them without preconceptions as to their meaning. So his assumptions are not for him what they are for the common people.

He considers them as representing the definition by postulates of the symbol (=) heretofore void of all signification. But to justify this definition we must show that these two assumptions lead to no contradiction. For this Hilbert used the reasoning of our number III, without appearing to perceive that he is using complete induction.

IX.

The end of Hilbert's memoir is altogether enigmatic and I shall not lay stress upon it. Contradictions accumulate; we feel that the author is dimly conscious of the *petitio principii* he has committed, and that he seeks vainly to patch up the holes in his argument.

What does this mean? At the point of proving that the definition of the whole number by the assumption of complete induction implies no contradiction, Hilbert withdraws as Russell and Couturat withdrew, because the difficulty is too great.

X.

GEOMETRY.

Geometry, says M. Couturat, is a vast body of doctrine wherein the principle of complete induction does not enter. That is true in a certain measure; we cannot say it is entirely absent, but it enters very slightly. If we refer to the *Rational Geometry* of Dr. Halsted (New York, John Wiley and Sons, 1904) built up in accordance with the principles of Hilbert, we see the principle of induction enter for the first time on page 114 (unless I have made an oversight, which is quite possible).[3]

So geometry which only a few years ago seemed the domain where the reign of intuition was uncontested is to-day the realm where the logicians seem to triumph. Nothing could better measure the importance of the geometric works of Hilbert and the profound impress they have left on our conceptions.

[3] 2d. ed., 1907, p. 86; French ed. 1911, p. 97. G. B. H.

But be not deceived. What is after all the fundamental theorem of geometry? It is that the assumptions of geometry imply no contradiction, and this we can not prove without the principle of induction.

How does Hilbert demonstrate this essential point? By leaning upon analysis and through it upon arithmetic and through it upon the principle of induction.

And if ever one invents another demonstration, it will still be necessary to lean upon this principle, since the possible consequences of the assumptions, of which it is necessary to show that they are not contradictory, are infinite in number.

XI.

CONCLUSION.

Our conclusion straightway is that the principle of induction cannot be regarded as the disguised definition of the entire world.

Here are three truths: (1) The principle of complete induction; (2) Euclid's postulate; (3) The physical law according to which phosphorus melts at 44° (cited by M. Le Roy).

These are said to be three disguised definitions: the first, that of the whole number; the second, that of the straight line; the third, that of phosphorus.

I grant it for the second; I do not admit it for the other two. I must explain the reason for this apparent inconsistency.

First, we have seen that a definition is acceptable only on condition that it implies no contradiction. We have shown likewise that for the first definition this demonstration is impossible; on the other hand we have just recalled that for the second Hilbert has given a complete proof.

As to the third, evidently it implies no contradiction. Does this mean that the definition guarantees, as it should, the existence of the object defined? We are here no longer in the mathematical sciences, but in the physical, and the word existence has no longer the same meaning. It no longer signifies absence of contradiction; it means objective existence.

You already see a first reason for the distinction I made between the three cases; there is a second. In the applications we have to make of these three concepts, do they present themselves to us as defined by these three postulates?

The possible applications of the principle of induction are innumerable; take for example one of those we have expounded above, and where it is sought to prove that an aggregate of assumptions can lead to no contradiction. For this we consider one of the series of syllogisms we may go on with in starting from these assumptions as premises. When we have finished the nth syllogism, we see we can make still another and this is the $n+1$th. Thus the number n serves to count a series of successive operations; it is a number obtainable by successive additions. This therefore is a number from which we may go back to unity by *successive subtractions.* Evidently we could not do this if we had $n=n-1$, since then by subtraction we should always obtain again the same number. So the way we have been led to consider this number n implies a definition of the finite whole number and this definition is the following: A finite whole number is that which can be obtained by successive additions; it is such that n is not equal to $n-1$.

That granted, what do we do? We show that if there has been no contradiction up to the nth syllogism, no more will there be up to the $n+1$th, and we conclude there never will be. You say: I have the right to draw this conclusion, since the whole numbers are by definition those for which a like reasoning is legitimate. But that implies another definition of the whole number, which is as follows: A whole number is that on which we may reason by recurrence. In the particular case it is that of which we may say that, if the absence of contradiction up to the time of a syllogism of which the number is an integer carries with it the absence of contradiction up to the time of the syllogism whose number is the following integer, we need fear no contradiction for any of the syllogisms whose number is an integer.

The two definitions are not identical; they are doubtless equivalent, but only in virtue of a synthetic judgment *a priori*; we cannot pass from one to the other by a purely logical procedure. Consequently we have no right to adopt the second, after having introduced the whole number by a way that presupposes the first.

On the other hand, what happens with regard to the straight line? I have already explained this so often that I hesitate to repeat it again, and shall confine myself to a brief recapitulation of my thought. We have not, as in the preceding case, two equivalent definitions logically irreducible one to the other. We have only one expressible in words. Will it be said there is another which we feel without being able to word it, since we have the intuition of the straight line or since we represent to ourselves the straight line? First of all, we cannot represent it to ourselves in geometric space, but only in representative space, and then we can represent to ourselves just as well the objects which possess the other properties of the straight line, save that of satisfying Euclid's postulate. These objects are "the non-Euclidean straights," which from a certain point of view are not entities void of sense but circles (true circles of true space) orthogonal to a certain sphere. If, among these objects equally capable of representation, it is the first (the Euclidean straights) which we call straights, and not the latter (the non-Euclidean straights), this is properly by definition.

And arriving finally at the third example, the definition of phosphorus, we see the true definition would be: Phosphorus is the bit of matter I see in yonder flask.

And since I am on this subject, still another word. Of the phosphorus example I said: "This proposition is a real verifiable physical law, because it means that all bodies having all the other properties of phosphorus, save its point of fusion, melt like it at 44°." And it was answered: "No, this law is not verifiable, because if it were shown that two bodies resembling phosphorus melt one at 44° and the other at 50°, it might always be said that doubtless, besides the point of fusion, there is some other unknown property by which they differ."

That was not quite what I meant to say. I should have written, "All bodies possessing such and such properties finite in number (to wit, the properties of phosphorus stated in the books on chemistry, the fusion-point excepted) melt at 44°."

And the better to make evident the difference between the case of the straight and that of phosphorus, one more remark. The straight has in nature many images more or less imperfect, of which the chief are the light rays and the rotation axis of the solid. Suppose we find the ray of light does not satisfy Euclid's postulate (for example by showing that a star has a negative parallax), what shall we do? Shall we conclude that the straight being by definition the trajectory of light does not satisfy the postulate, or on the other hand that the straight by definition satisfying the postulate, the ray of light is not straight?

Assuredly we are free to adopt the one or the other definition and consequently the one or the other conclusion; but to adopt the first would be stupid, because the ray of light probably satisfies only imperfectly not merely Euclid's postulate but the other properties of the straight line, so that if it deviates from the Euclidean straight, it deviates no less from the rotation axis of solids which is another imperfect image of the straight line; while finally it is doubtless subject to change, so that such a line which yesterday was straight will cease to be straight to-morrow if some physical circumstance has changed.

Suppose now we find that phosphorus does not melt at 44°, but at 43.9°. Shall we conclude that phosphorus being by definition that which melts at 44°, this body that we did call phosporus is not true phosphorus, or on the other hand that phosphorus melts at 43.9°? Here again we are free to adopt the one or the other definition and consequently the one or the other conclusion; but to adopt the first would be stupid because we cannot be changing the name of a substance every time we determine a new decimal of its fusion-point.

XIII.

To sum up, Russell and Hilbert have each made a vigorous effort; they have each written a work full of original views, profound and often well warranted. These two works give us much to think about and we have much to learn from them. Among their results, some, many even, are solid and destined to live.

But to say that they have finally settled the debate between Kant and Leibnitz and ruined the Kantian theory of mathematics is evidently incorrect. I do not know whether they really believed they had done it, but if they believed so, they deceived themselves.

H. POINCARÉ.

PARIS, FRANCE.

THE WEIRD OF LOVE AND DEATH.

"O inhabitant of Lebanon, that makest thy nest in the cedars, how gracious shalt thou be when pangs come upon thee, the pain as of a woman in travail."—Jeremiah, xxii. 23.

"Then he brought me to the door of the gate of the Lord's house, which was toward the north; and behold, there sat women weeping for Tammuz."—Ezekiel, viii. 14.

"And there followed him a great company of people, and of women, which also bewailed and lamented him.

"But Jesus turning unto them said, Daughters of Jerusalem, weep not for me, but weep for yourselves and for your children.

"For, behold, the days are coming, in which they shall say, Blessed are the barren, and the wombs that never bare, and the paps which never gave suck."—St. Luke, xxiii. 27, 28.

The author of the following verses makes no claim to be a translator, but merely an interpreter of a chapter from the Brick Bible of Babylon. He has relied upon the scholarship of others for his letter, but has sought its spirit not only beneath the text, but in the actual world of love and death. Special students of comparative religions indeed know the truth of Shakespeare's 59th sonnet:

"If there be nothing new, but that which is
Hath been before, how are our brains beguiled,
Which, laboring for invention, bear amiss
The second burthen of a former child!"

But the author has seen no other English version, in poetic form, of this oldest Semitic Gospel of the Resurrection, which, however old it be, is itself a translation, like the Greek Christian Gospels, from earlier originals.[1] Adonis has his Greek Gospels also, but only the apocryphal have come down to us, and these have found ample embodiment in Shakespeare's "Venus and Adonis."

The author has welded to the story of the Descent of Istar, two fragmentary hymns from the same literature invoking the Divine Pair, which common invocation is confirmed by the passage of Jeremiah (xxii. 18) whose Hebrew text should read, according to Prof. T. K. Cheyne: "Alas, my Brother, alas, my Sister! Alas, Adon, [Lord] alas Dodah! [Beloved Lady: a title of Istar]."

The cult of Dumu-zi-abzu (Sumerian or Akkadian "True Son of the Deep Water") whom the Hebrews alternately adored and abhorred as Tammuz, took, in its migration from the shore of the Persian Gulf to the Ægean and Sicilian coasts, only his Semitic title of "Adon." But in the course of transit the divinity first became obscured and then the human reputation. From a benign and mysterious power behind the process of spring, or a general symbol of the life principle of which winter deprives nature and death bereaves love, he became a demi-godlike huntsman and paramour of Aphrodite the goddess of beauty. Finally, in modern parlance, his epithet has dwindled to signify a pretty youth. Though coming from further east, the worship of Tammuz had its most famous seat at Aphaca (now Afka) about fifteen miles from the Phenician coast near the source of a torrent now called Ibrahim. In that ravine a crude but grand cosmic hypothesis was narrowed to a vulgar superstition. The site became a pagan Loretto or Lourdes, and developed a perverse traffic in sacred things which gave to Constantinople in the fourth Christian century the same reason or pretext for suppression that English shrines in the sixteenth century afforded Henry VIII.

The swift stream was miraculously tinged each year with the blood of the dying god whose title it then bore. It is said that the same geologic conditions stll perform the annual miracle. In the Vale of Aphaca the triumph as well as the agony of a divine victim were localized, just as later they were at Jerusalem. From Aphaca to Galilee it is but eighty miles by crow-flight, and to Nazareth less

[1] Dr. Alfred Jeremias has published the original text of the Descent of Istar with a literal German translation. An English version founded on Dr. Jeremias's translation, appeared in *The Open Court.* See Carus, "Babylonian and Hebrew Views of Man's Fate After Death," XV, p. 357.

than one hundred. Indeed, at Bethlehem (the "House of Bread") which lies seventy miles further south, the adoration of Tammuz, as an earlier fruit of the wheat than the Christian eucharist wafer, lingered in the days of St. Jerome. Though an enormous ethical distance separates the personality of Jesus from the mythical boar-chaser of Lebanon, the dogmas of Chaldea show that the traditions of the church rest on more than one foundation. The Egyptian Gospel of Osiris is another corner-stone.

The modest figure of the Virgin Mother Mary has little in common with the proud and passionate image of Istar, Ashtaroth or Astarte. Rather has the concept of her borrowed the attributes of the gracious Egyptian Isis. Istar's exchange of curses with her infernal sister, as told in clay, may nevertheless have stirred the religious feelings of her votaries among the fish-wives of Babylon. But there remains something in the grief of the divine bride for her lost bridegroom which forecasts the plastic pathos of Michelangelo's "Pieta" and is echoed in the rich harmonies of Rossini's "Stabat Mater Dolorosa."

To realm whence no echo is borne,
to region no pioneer showeth;
To the Castle of Darkness Substantial,
to Yesterday's shadowy shore,
Our Lady Astarte, whose beacon
for lovers and mariners gloweth
At morning and even, descended
and smote on the dust-laden door.

"Now open the gate unto me,
grim warden that keepest the marches!
I would enter the Kingdom of Death!"
cried Our Lady, the mystical Bride.
"Unless to my summons thou hearken,
thy gate I will rend from its arches,
Setting free to outnumber the living,
the spirits of men that have died!"

To Lady Astarte, the warden
        that watcheth the entrance of Hades
Made reply: "Till I take to my mistress
        thy word, prithee, Istar, forbear!"
(A feud for eternity lay
        'twixt the lovely and terrible ladies,
So how should Death bid Love be welcome,
        and ope to a rival her lair?)

To pitiless Queen of Irkalla
        the seneschal doubtfully wended:
"Sov'reign Lady of Death, at the precinct
        thy sister Astarte doth stand.
Methinks that in quest of the life-giving
        water the Queen hath descended;
The bars of thy mansion are shaken
        beneath her imperious hand."

To him said Queen Allat: "O warden,
        as grain from the scythe of a reaper
To the Dungeon of Dust falleth Istar
        imploring the water of life!
Like lip of reed that is thirsty
        her need is for Tammuz the Sleeper:
But what are her sorrow and yearning
        to us, or her menace of strife?

"Quoth she: 'For the hero I mourn
        that hath left his wife widow'd and lonely.
I lament for the bride whose embraces
        her husband hath lost and deplored;
For fate of the innocent children
        whose span bore the spring-blossom only;
So lend me the water of life,
        For the healing of Tammuz my Lord!'

"Yet, warden, we grant her caprice!
Suffer Istar to enter our portal
In conformity strict to the letter
of Death's incompassionate law.
Deprive her of every adornment
as if she were humble and mortal;
Extinguish the glory of Istar
that filleth the heavens with awe!"

The warden returning, threw open
the porch of Irkalla to Istar.
"Thou art welcome to Death, O dread Lady!
Let Ghostland rejoice in its guest!
'T is mine to conduct thee, O Queen,
to the presence of Allat, thy sister!"
But as she stepp'd over the threshold
he plucked from her forehead the crest.

"My crown with the crescent and star
give back to me!" Istar besought him.
"Nay, my Princess, the code of the kingdom
of Death even thou must obey!"
Through Second Gate when they immerged,
as the ruler of Hades had taught him,
The warden of gloom took from Istar
The radiant ear-rings away.

And so at each barrier passed,
the Queen of her robe he divested,
And the necklace, the brooch and the belt
and the bracelets he claim'd as his prey.
Relentless and brutal he was;
When Our Lady Astarte protested,
Repeating: "Nay, Princess, the edict
of Death even thou must obey!"

So into the hall of the hopeless,
the court of Queen Allat the Dreary,
All dishevelled, discrowned and dismantled
Our Lady Astarte he led.
Though her aspect was that of despair,
for her trials were many and weary,
Not dumb was Our Lady at sight
of the sinister Queen of the Dead.

She cursed her with formula dire,
with a torrent of bitter invective,
And she wept more in rage than in sorrow,
recounting the insults of Death.
The face of the monarch of Hades
grew scornfully sweet and reflective,
Nor uttered she one interruption
till Istar expended her breath,

Then spake with a delicate malice
these ominous words unto Istar:
"Since thou quittest the world, not a beast
of the wilderness seeketh a mate,
Nor egg hath been hatched by a fowl,
O gentle and courteous sister
Who threat'nest my realm with invasion,
but leavest thine own desolate!

"The maids of the men are unconscious,
no men to the maids make advances;
And the cradles are empty and rock'd
by the hands of no mothers to-day;
Their music the forests have lost,
the cities are stilled of their dances;
The land of the living is stagnant
since Istar to Death came away.

"My thralls thou hast sought to suborn,
by promising thou wouldst deliver
From the dust of the grave to adore thee
again on thy double-horned throne—
In truth, O Astarte, it seemeth,
now Love hath discarded her quiver,
The task would be light for annexing
the Kingdom of Life to mine own!

"Ho, Namtar! Take Istar and plague her
with sixty-fold measure of illness!
Assail her with chastening agues
and darken the flame of her eyes!
Let agony reign in her bosom,
her ears have the horror of stillness!
Let clouds gather over her spirit!
Let languor her limbs paralyse!"

Through creation there mounted a shudder
to throne of the Father Eternal;
To the One whose dominion is screened
by the awful illusion of space.
All nature cried out at the tyranny
seized by the power infernal;
In conclave aghast at the rumor
the sons of God each took his place.

"'T is June, but the leafage hath fallen;
't is summer, but rime crusteth over
All the meads of the planets with whiteness;
't is season for rain, but a drought
The field of ephemeral life
with a brown desolation doth cover;
The fire of Astarte is dim;
from the tomb cometh Tammuz not out!"

So Pápsukal, angel of light,
unto Marduk the Sun-god repeated,
Who arose and went up to his Father
and bowed in the Presence with tears.
The lord of the hours for grace
of the Infinite Spirit entreated
To call back Our Lady Astarte
from Death to her place in the spheres.

From mind of the Father Eternal
in likeness not man nor yet woman,
Did a messenger come to creation,
with countenance fair and serene.
By myriad titles invoked on the
stammering lips that are human,
Among them "Atsú-su-namír,"
and it meaneth "His Rising is seen."

"To realm whence no echo is borne,
to region no pioneer showeth;
To the Castle of Darkness Substantial;
to Yesterday's shadowy shore
Descend!" quoth the Infinite One,
"for the calm of the tempest that bloweth
From Allat the Queen of Irkalla,
the dame of the seven-fold door!

"Command her, in name of her Father,
to give from the Fount of Revival
Unto Istar her captive a draught
for the raising of Tammuz the Slain.
If pity she will not bestow
on the need of her sister and rival,
Then warn her how fragile Death's fetter
the gods Love and Life to restrain!"

More swift than the flight of a star
        was the radiant herald in falling,
Through the limitless ether convey'd
        on the thought of the Uttermost God.
O'er the Sea of Oblivion borne
        to the Island of Silence appalling
Where hinges of Hell broke asunder
        at touch of a magical rod.

Yet Allat, the Queen of the Dead,
        at the luminous shape hurl'd reviling:
"Though I may not deny nor delay
        my Father's unwelcome behest,
Atsú-su-namír, with the face
        that is evermore hopeful and smiling,
I curse thee, who bringest His will!"
        and she beat her implacable breast.

"Go, Namtar, and knock at the pillars
        that hold up the base of our dwelling;
Bid the gnomes in their cavern assemble
        and sit on their benches of gold;
Let Istar the water receive
        that in Fount of Revival is welling,
And bring back the goddess before us;
        her boon we no more may withhold."

Though grudgingly made the release,
        through the seven-fold gate Lady Istar
In her strength and her beauty renewed,
        from the Castle of Darkness hath gone.
No warden might check or betray,
        no padlock nor bar might resist her;
With mantle and jewels restored
        her figure resplendently shone.

She bore in her hand a bright chalice
for wakening Tammuz the Sleeper;
For Adonis, the First-fruits of Death,
an immortal libation she poured;
While hymns from the farthest confines
of creation grew louder and deeper,
As flowers and fishes and beasts
with mankind her arising adored:

"In Valley of Life there is growing
a tree amaranthine and shady;
From the grail of the crystal abyss
the sap of its verdure is drawn;
In heart of the earth it is rooted,
its leaves form the nest of Our Lady
Whose star in the highway of Heaven
enlight'neth the dusk and the dawn!

"Enshrined in a mystery sweet
is Adonis the Beautiful lying
On the lap of the Mother Divine
who lamented him cruelly slain.
There bloometh the garden of love,
and the flower of life is undying,
Beyond the soft veil of the temple
that hideth the deities twain!

"O Tammuz, our Lord and our Shepherd!
Miraculous Bridegroom of Istar!
Thou hast conquer'd the stronghold of Death
and thou leadest thy people like sheep!
Thou wert as the wheat in the field
that a wind of the desert doth blister,
Like tree of acacia with root
that a treacherous river doth steep!

"Our Lady, whose star in the sky
        bringeth hope to the heart heavy-laden,
And whose justice on earth is a lion,
        whose mercy a lamb at the breast,
O Queen of the House of the Shepherd,
        O Mistress of Love ever-maiden,
May infinite joy be upon thee,
        thy grief be forever at rest!"

EDWARD GILCHRIST.

SWATOW, CHINA.

CRITICISMS AND DISCUSSIONS.

THE REV. JAMES BRADLEY ON THE MOTION OF THE FIXED STARS.

(Reprinted from the *Philosophical Transactions* of 1727.)

[The theory of the relativity of time and space, which is at present uppermost in the minds of physicists, has come into the foreground mainly through the differences of measuring at large distances the time it takes light to reach the observer's eye which is further complicated by the motions of his own standpoint. This happened for the first time in the history of science in the year 1726 when Mr. Bradley discovered that the fixed stars possessed a definite and peculiar motion of their own which was due to the motion of the earth around the sun and depended on the time it takes the light to reach the earth.

This classical exposition of his experiments was published in the form of a letter sent to the *Phil. Trans.* (Vol. XXXIV, p. 637) and has naturally become quite inaccessible. There is probably only one complete file of the *Transactions* west of the Alleghanies, the fortunate possessor of which is the Chicago Public Library. Considering the rarity of this essay we deem it proper to republish it and render it accessible to our readers. We do not doubt the very way in which Mr. Bradley approaches the problem will throw much light on the principle of relativity. In fact this essay will prove sufficient to explain its far-reaching significance, the need of its invention and the limitations of its use. A consideration of the foundaton of this principle and the history of its origin will clear it of the mysticism with which its recent representations have surrounded its statements.—P. C.]

A Letter from the Reverend Mr. James Bradley, Savilian Professor of Astronomy at Oxford, and F. R. S., to Dr. Edmond Halley Astronom. Reg. &c. giving an Account of a new discovered Motion of the Fix'd Stars.

SIR,

You having been pleased to express your Satisfaction with what I had an Opportunity some time ago, of telling you in Conversation, concerning some Observations, that were making by our late worthy and ingenious Friend, the honorable *Samuel Molyneux* Esquire, and which have since been continued and repeated by myself, in order to determine the *Parallax* of the *fixt Stars*; I shall now beg leave to lay before you a more particular Account of them.

Before I proceed to give you the History of the Observations themselves, it may be proper to let you know, that they were at first begun in hopes of verifying and confirming those, that *Dr. Hook* formerly communicated to the publick, which seemed to be attended with Circumstances that promised greater Exactness in them, than could be expected in any other, that had been made and published on the same Account. And as his Attempt was what principally gave Rise to this, so his Method in making the Observations was in some Measure that which *Mr. Molyneux* followed: For he made Choice of the same Star, and his Instrument was constructed upon almost the same Principles. But if it had not greatly exceeded the Doctor's in Exactness, we might yet have remained in great Uncertainty as to the *Parallax* of the *fixt Stars*; as you will perceive upon the Comparison of the two Experiments.

This indeed was chiefly owing to our curious Member, Mr. *George Graham*, to whom the Lovers of Astronomy are also not a little indebted for several other exact and well-contrived Instruments. The Necessity of such will scarce be disputed by those that have had any Experience in making Astronomical Observations; and the Inconsistency, which is to be met with among different Authors in their Attempts to determine small Angles, particularly the annual Parallax of the *fixt Stars*, may be a sufficient Proof of it to others. Their Disagreement indeed in this article is not now so much to be wondered at, since I doubt not, but it will appear very probable, that the Instruments commonly made use of by them, were liable to greater Errors than many times that Parallax will amount to.

The Success then of this Experiment evidently depending very much on the Accurateness of the Instrument that was principally to be taken Care of: In what Manner this was done, is not my present Purpose to tell you; but if from the Result of the Observations which I now send you, it shall be judged necessary to communicate to the Curious the Manner of making them, I may hereafter perhaps give them a particular Description, not only of Mr. *Molyneux's* Instrument but also of my own, which hath since been erected for the same Purpose and upon the like Principles, though it is somewhat different in its Construction, for a Reason you will meet with presently.

Mr. *Molyneux's Apparatus* was compleated and fitted for observing about the End of November 1725, and on the third Day of *December* following, the bright Star at the Head of *Draco* (marked ν by *Bayer*) was for the first Time observed, as it passed near the Zenith, and its Situation carefully taken with the Instrument. The like Observations were made on the 5th, 11th and 12th Days of the same Month, and there appearing no material Difference in the Place of the Star, a farther Repetition of them at this Season seemed needless, it being a Part of the Year, wherein no sensible Alteration of Parallax in this Star could be expected. It was chiefly therefore Curiosity that tempted me (being then at *Kew,* where the Instrument was fixed) to prepare for observing the Star on *December* 17th, when having adjusted the Instrument as usual, I perceived that it passed a little more Southerly this Day than when it was observed before. Not suspecting any other Cause of this Appearance, we first concluded, that it was owing to the Uncertainty of the Observations, and that either this or the foregoing were not so exact as we had before supposed; for which Reason we purposed to repeat the Observation again, in order to determine from whence this Difference proceeded; and upon doing it on *December* 20th, I found that the Star passed still more Southerly than in the former Observations. This sensible Alteration the more surprized us, in that it was the contrary way from what it would have been, had it proceeded from an annual Parallax of the Star: But being now pretty well satisfied, that it could not be entirely owing to the want of Exactness in the Observations; and having no Notion of anything else, that could cause such an apparent Motion as this in the Star; we began to think that some Change in the Materials, &c. of the Instrument itself, might have occasioned it. Under these Apprehensions we remained some time, but being at length fully convinced, by several Trials, of the great Exactness of the Instrument, and finding by the gradual Increase of the Star's Distance from the Pole, that there must be some regular Cause that produced it; we took care to examine nicely, at the Time of each Observation, how much it was: and about the Beginning of *March* 1725, the Star was found to be 20″ more Southerly than at the Time of the first Observation. It now indeed seemed to have arrived at its utmost Limit Southward, because in several Trials made about this Time, no sensible Difference was observed in its Situation. By the Middle of *April,* it appeared to be returning back again towards the North; and about the beginning of *June,* it passed at the same Distance from the Zenith as it had done in *December* when it was first observed.

From the quick Alteration of this Star's Declination about this Time (it increasing a Second in three Days) it was concluded, that it would now proceed Northward, as it before had done Southward of its present Situation; and it happened as was conjectured: for the Star continued to move Northward till *September* following, when it again became stationary, being then near 20″ more Northerly than in *June,* and no less than 39″ more Northerly than it was in *March.* From *September* the Star returned towards the South, till it arrived in *December* to the same Situation it was in at that time twelve Months, allowing for the Difference of Declination on account of the Precession of the Equinox.

This was a sufficient Proof, that the Instrument had not been the Cause of this apparent Motion of the Star, and to find one adequate to such an Effect seemed a Difficulty. A Nutation of the Earth's Axis was one of the first things that offered itself upon this Occasion, but it was soon found to be insufficient; for though it might have accounted for the change of Declination in *ν Draconis* yet it would not at the same time agree with the Phaenomena in other Stars; particularly in a small one almost opposite in right Ascension to *ν Draconis,* at about the same Distance from the North Pole of the Equator: For, though this Star seemed to move the same way, as a Nutation of the Earth's Axis would have made it, yet it changing its Declination but about half as much as *ν Draconis* in the same time (as appeared upon comparing the Observations of both made upon the same Days, at different Seasons of the Year) this plainly proved, that the apparent Motion of the Stars was not occasioned by a real Nutation, since if that had been the Cause, the Alteration in both Stars would have been near equal.

The great Regularity of the Observations left no room to doubt, but that there was some regular Cause that produced this unexpected Motion, which did not depend on the Uncertainty or Variety of the Seasons of the Year. Upon comparing the Observations with each other, it was discovered that in both the fore-mentioned Stars, the apparent Difference of Declination from the *Maxima,* was always nearly proportional to the versed Sine of the Sun's Distance from the Equinoctial Points. This was an Inducement to think, that the Cause, whatever it was, had some Relation to the Sun's Situation with respect to those Points. But not being able to frame any Hypothesis at that Time sufficient to solve all the Phænomena, and being very desirous to search a little farther into this Matter; I began to think of erecting an Instrument for myself at Wansted, that having it always at Hand, I might with the more Ease and Certainty, enquire into the Laws of this new Motion. The Consideration likewise of being able by another Instrument, to confirm the Truth of the Observations hitherto made with *Mr. Molyneux's,* was no small Inducement to me; but the Chief of all was, the Opportunity I should thereby have of trying, in what Manner other Stars were affected by the same Cause, whatever it was. For *Mr. Molyneux's* Instrument being originally designed for observing *ν Draconis* (in order as I said before, to try whether it had any sensible Parallax) was so contrived, as to be capable of but little Alteration in its Direction, not above seven or eight Minutes of a Degree; and there being few stars within half that Distance from the Zenith of *Kew,* bright enough to be well observed, he could not, with his Instrument, thoroughly examine how this Cause affected Stars differently situated with respect to the equinoctial and solstitial Points of the Ecliptick.

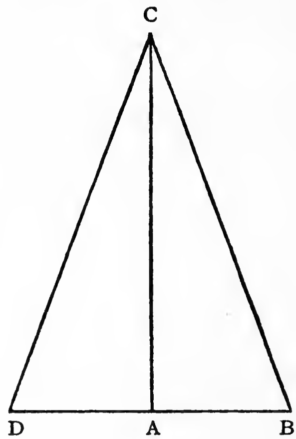
These Considerations determined me; and by the Contrivance and Direction of the same ingenious Person, Mr. Graham, my Instrument was fixed up *August* 19, 1727. As I had no convenient Place where I could made use of so long a Telescope as *Mr. Molyneux's,* I contented myself with one of but little more than half the Length of his (viz. of about 12½ Feet, his being 24¼) judging from the Experience which I had already had, that this Radius would be long enough to adjust the Instrument to a sufficient Degree of Exactness, and I have no reason since to change my Opinion: for from all the Trials I have yet made, I am very well satisfied, that when it is carefully rectified, its Situation may be securely depended upon to half a Second. As the Place where my Instrument was to be hung, in some Measure determined its Radius, so did it also the Length of the Arch, or Limb, on which the Divisions were made to adjust it: For the Arch could not conveniently be extended farther, than to reach to about 6¼° on each Side my Zenith. This indeed was sufficient, since it gave me an Opportunity of making Choice of several Stars, very different both in Magnitude and Situation; there being more than two hundred inserted in the *British* Catalogue, that may be observed with it. I needed not to have extended the Limb so far, but that I was willing to take in *Capella,* the only star of the first Magnitude that comes so near my Zenith.

My instrument being fixed, I immediately began to observe such Stars as I judged most proper to give me light into the Cause of the Motion already mentioned. There was Variety enough of small ones; and not less than twelve, that I could observe through all the Seasons of the Year; they being bright enough to be seen in the Day-time, when nearest the Sun. I had not been long observing, before I perceived, that the Notion we had before entertained of the Stars being farthest North and South, when the Sun was about the Equinoxes, was only true of those that were near the solstitial Colure: And after I had continued my Observations a few Months, I discovered, what I then apprehended to be a general Law, observed by all the Stars, viz. That each of them became stationary, or was farthest North or South, when they passed over my Zenith at six of the Clock, either in the Morning or Evening. I perceived likewise, that whatever Situation the Stars were in with respect to the cardinal Points of the Ecliptick, the apparent motion of every one tended the same Way, when they passed my instrument about the same Hour of the Day or Night; for they all moved Southward, while they passed in the Day, and Northward in the Night; so that each was farthest North, when it came about Six of the Clock in the Evening, and farther South, when it came about Six in the Morning.

Though I have since discovered, that the *Maxima* in most of these Stars do not happen exactly when they come to my Instrument at those Hours, yet not being able at that time to prove the contrary, and supposing that they did, I endeavoured to find out what Proportion the greatest Alterations of Declination in different Stars bore to each other; it being very evident, that they did not all change their Declination equally. I have before taken notice, that it appeared from Mr. *Molyneux's* Observations, that ν Draconis altered its Declination about twice as much as the fore-mentioned small Star almost opposite to it; but examining the matter more particularly, I found that the greatest Alteration of Declination in these Stars, was at the Sine of the Latitude of each respectively. This made me suspect that there might be the like Proportion between the *Maxima* of other Stars; but finding, that the observations of some of them would not perfectly correspond with such an Hypothesis, and not knowing, whether the small Difference I met with, might not be owing to the Uncertainty and Error of the Observations, I deferred the farther examination into the Truth of this Hypothesis, till I should be furnished with a Series of Observations made in all Parts of the Year; which might enable me, not only to determine what Errors the Observations are liable to, or how far they may safely be depended upon; but also to judge, whether there had been any sensible Change in the Parts of the Instrument itself.

Upon these Considerations, I laid aside all Thoughts at that Time about the Cause of the fore-mentioned Phaenomena, hoping that I should the easier discover it, when I was better provided with proper Means to determine more precisely what they were.

When the Year was compleated, I began to examine and compare my Observations, and having pretty well satisfied myself as to the general Laws of the Phaenomena, I then endeavoured to find out the Cause, of them. I was already convinced, that the apparent Motion of the Stars, was not owing to a Nutation of the Earth's Axis. The next Thing that offered itself, was an Alteration in the



Direction of the Plumb-line, with which the Instrument was constantly rectified; but this upon Trial proved insufficient. Then I considered what Refraction might do, but here also nothing satisfactory occurred. At last I conjectured, that all the Phaenomena hitherto mentioned, proceeded from the progressive Motion of Light and the Earth's annual Motion in its Orbit. For I perceived, that, if Light was propagated in Time, the apparent Place of a fixt Object would not be the same when the Eye is at Rest, as when it is moving in any other Direction, than that of the Line passing through the Eye and Object; and that, when the Eye is moving in different Directions, the apparent Place of the Object would be different.

I considered this Matter in the following Manner. I imagined CA to be a Ray of Light, falling perpendicularly upon the Line BD; then if the Eye is at rest at A, the Object must appear in the Direction AC, whether Light be propagated in Time or in an Instant. But if the Eye is moving from B towards A, and Light is propagated in Time, with a Velocity that is to the Velocity of the Eye, as CA to BA; then Light moving from C to A, whilst the Eye moves from B to A, that Particle of it, by which the Object will be discerned, when the Eye in its Motion comes to A, is at C when the Eye is at B. Joining the Points B, C, I supposed the Line CB, to be a Tube (inclined to the Line BD in the Angle DBC) of such a Diameter, as to admit of but one Particle of Light; then it was easy to conceive, that the Particle of Light at C (by which the object must be seen when the Eye, as it moves along, arrives at A) would pass through the Tube BC, if it is inclined to BD in the Angle DBC, and accompanies the Eye in its Motion from B to A; and that it could not come to the Eye, placed behind such a Tube, if it had any other Inclination to the Line BD. If instead of supposing CB so small a Tube, we imagine it to be the Axis of a larger; then for the same Reason, the Particle of Light at C, could not pass through that Axis, unless it is inclined to BD, in the Angle CBD. In like manner, if the Eye moved the contrary way, from D towards A, with the same Velocity; then the Tube must be inclined in the Angle BDC. Although therefore the true or real Place of an Object is perpendicular to the Line in which the Eye is moving, yet the visible Place will not be so, since that, no doubt, must be in the Direction of the Tube; but the Difference between the true and apparent Place will be (*cæteris paribus*) greater or less, according to the different Proportion between the Velocity of Light and that of the Eye. So that if we could suppose that Light was propagated in an instant, then there would be no Difference between the real and visible Place of an Object, although the Eye were in Motion, for in that case, AC being infinite with Respect to AB, the Angle ACB (the Difference between the true and visible Place) vanishes. But if Light be propagated in Time (which I presume will readily be allowed by most of the Philosophers of this Age) then it is evident from the foregoing Considerations, that there will be always a Difference between the real and visible Place of an Object, unless the Eye is moving either directly towards or from the Object. And in all Cases, the Sine of the Difference between the real and visible Place of the Object, will be to the Sine of the visible Inclination of the Object to the Line in which the Eye is moving, as the Velocity of the Eye to the Velocity of Light.

If Light moved but 1000 times faster than the Eye, and an Object (supposed to be at an infinite Distance) was really placed perpendicularly over the Plain in which the Eye is moving, it follows from what hath been already said, that the apparent Place of such an Object will be always inclined to that Plain, in an Angle of 89° 56′½; so that it will constantly appear 3′½ from its true Place, and seem so much less inclined to the Plain, that way towards which the Eye tends. That is, if AC is to AB (or AD) as 1000 to one, the Angle ABC will be 89° 56′½, and ACB = 3′½, and BCD = 2ACB = 7′. So that according to this Supposition, the visible or apparent Place of the Object will be altered 7′, if the Direction of the Eye's Motion is at one time contrary to what it is at another.

If the Earth revolve round the Sun annually, and the Velocity of Light were to the Velocity of the Earth's Motion in its Orbit (which I will at present suppose to be a Circle) as 1000 to one; then tis easy to conceive, that a Star really placed in the very Pole of the Ecliptick, would, to an Eye carried along with the Earth, seem to change its Place continually, and (neglecting the small Difference on the Account of the Earth's diurnal Revolution on its Axis) would seem to describe a Circle round that Pole, every Way distant therefrom 3′½. So that its Longitude would be varied through all the Points of the Ecliptick every Year; but its Latitude would always remain the same. Its right Ascension would also change, and its Declination, according to the different Situation of the Sun in respect to the equinoctial Points; and its apparent Distance from the North Pole of the Equator would be 7′ less at the Autumnal, than at the vernal Equinox.

The greatest Alteration of the Place of a Star in the Pole of the Ecliptick (or which in Effect amounts to the same, the Proportion between the Velocity of Light and the Earth's Motion in its Orbit) being known; it will not be difficult to find what would be the Difference upon this Account, the Difference between the true and apparent Place of any other Star at any time; and on the contrary, the Difference between the true and apparent Place being given; the Proportion between the Velocity of Light and the Earth's Motion in its Orbit may be found.

As I only observed the apparent Difference of Declination of the Stars, I shall not now take any farther Notice in what manner such a Cause as I have here supposed would occasion an Alteration in their apparent Places in other Respects; but, supposing the Earth to move equally in a Circle, it may be gathered from what hath been already said, that a Star which is neither in the Pole nor Plain of the Ecliptick, will seem to describe about its true Place a Figure, insensibly different from an Ellipse, whose Transverse Axis is at Right-angle to the Circle of Longitude passing through the Star's true Place, and equal to the Diameter of the little Circle described by a Star (as was before supposed) in the Pole of the Ecliptick; and whose Conjugate Axis is to its Transverse Axis, as the *Sine* of the Star's latitude to the Radius. And allowing that a Star by its apparent Motion does exactly describe such an Ellipse, it will be found, that if A be the Angle of Position (or the Angle at the Star made by two great Circles drawn from it, thro' the Poles of the Ecliptick and Equator) and B be another Angle, whose Tangent is to the Tangent of A as Radius to the Sine of the Latitude of the Star; then B will be equal to the Difference of Longitude between the Sun and the Star, when the true and apparent Declination of the Star are the same. And if the Sun's Longitude in the Ecliptick be reckoned from that Point, wherein it is when this happens; then the Difference between the true and apparent Declination of the Star (on account of the Cause I am now considering) will be always, as the Sine of the Sun's Longitude from thence. It will likewise be found, that the greatest Difference of Declination that can be between the true and apparent Place of the Star, will be to the Semi-Transverse Axis of the Ellipse (or to the Semi-diameter of the little Circle described by a Star in the Pole of the Ecliptick) as the Sine of A to the Sine of B.

If the Star hath North Latitude, the Time, when its true and apparent Declination are the same, is before the Sun comes in Conjunction with or Opposition to it, if its Longitude be in the first or last Quadrant (viz. in the ascending Semi-circle) of the Ecliptick; and after them, if in the descending Semi-circle; and it will appear nearest to the North Pole of the Equator, at the Time of that *Maximum* (or when the greatest Difference between the true and apparent Declination happens) which precedes the Sun's Conjunction with the Star.

These Particulars being sufficient for my present Purpose, I shall not detain you with the Recital of any more, or with any farther Explication of these. It may be time enough to enlarge more upon this Head, when I give a Description of the Instruments &c. if that be judged necessary to be done; and when I shall find, what I now advance, to be allowed of (as I flatter myself it will) as something more than a bare Hypothesis. I have purposely omitted some matters of no great Moment, and considerd the Earth as moving in a

Circle, and not an Ellipse, to avoid too perplexed a *Calculus,* which after all the Trouble of it would not sensibly differ from that which I make use of, especially in those Consequences which I shall at present draw from the foregoing Hypothesis.

This being premised, I shall not proceed to determine from the observations, what the real Proportion is between the Velocity of Light and the Velocity of the Earth's annual Motion in its Orbit; upon Supposition that the Phaenomena before mentioned do depend upon the Causes I have here assigned. But I must first let you know, that in all the Observations hereafter mentioned, I have made an Allowance for the Change of the Star's Declination on Account of the Precession of the Equinox, upon Supposition that the Alteration from this Cause is proportional to the Time, and regular through all the Parts of the Year. I have deduced the real annual Alteration of Declination of each Star from the Observations themselves; and I the rather choose to depend upon them in this Article, because all which I have yet made, concur to prove, that the Stars near the Equinoctial Colure, change their Declination at this time $1''\frac{1}{2}$ or $2''$ in a Year more than they would do if the Precession was only $50''$, as is now generally supposed. I have likewise met with some small Varieties in the Declination of other Stars in different Years, which do not seem to proceed from the same Cause, particularly in those that are near the solstitial Colure, which on the contrary have altered their Declination less than they ought, if the Precession was $50''$. But whether these small Alterations proceed from a regular Cause, or are occasioned by any Change in the Materials &c. of my Instrument, I am not yet able fully to determine. However, I thought it might not be amiss just to mention to you how I have endeavoured to allow for them, though the Result would have been nearly the same, if I had not considered them at all. What that is, I will shew, first from the Observations of *ν Draconis,* which was found to be $39''$ more Southerly in the Beginning of *March,* than in *September.*

From what hath been premised, it will appear that the greatest Alteration of the apparent Declination of *ν Draconis,* on account of the successive Propagation of Light, would be to the Diameter of the little Circle which a Star (as was before remarked) would seem to describe about the Pole of the Ecliptick as $39''$ to $40''$, 4. The half of this is the Angle ACB (as represented in the Fig.) This therefore being $20''$, 2, AC will be to AB, that is, the Velocity of Light to the Velocity of the Eye (which in this Case may be supposed the same as the Velocity of the Earth's annual Motion in its

Orbit) as 10210 to One, from whence it would follow, that Light moves, or is propagated as far as from the Sun to the Earth in 8′ and 12″.

It is well known, that Mr. *Romer*, who first attempted to account for an apparent Inequality in the Times of the Eclipses of *Jupiter's* Satellites, by the Hypothesis of the progressive Motion of Light, supposed that it spent about 11 Minutes of Time in its Passage from the Sun to us: but it hath since been concluded by others from the like Eclipses, that it is propagated as far in about 7 Minutes. The Velocity of Light therefore deduced from the foregoing Hypothesis, is as it were a Mean betwixt what had at different times been determined from the Eclipses of *Jupiter's* Satellites.

These different Methods of finding the Velocity of Light thus agreeing in the Result, we may reasonably conclude, not only that these *Phaenomena* are owing to the Causes to which they have been ascribed; but also, that Light is propagated (in the same Medium) with the same Velocity after it hath been reflected as before; for this will be the Consequence, if we allow that the Light of the Sun is propagated with the same Velocity, before it is reflected, as the Light of the *fixt Stars*. And I imagine this will scarce be questioned, if it can be made appear that the Velocity of the Light of all the *fixt Stars* is equal, and that their Light moves or is propagated through equal Spaces in equal Times, at all Distances from them: both which points (as I apprehend) are sufficiently proved from the apparent alteration of the Declination of Stars of different Lustre; for that is not sensibly different in such Stars as seem near together, though they appear of very different Magnitudes. And whatever their Situations are (if I proceed according to the foregoing Hypothesis) I find the same Velocity of Light from my Observations of small Stars of the fifth or sixth, as from those of the second and third Magnitude, which in all Probability are placed at very different Distances from us. The small Star, for Example, before spoken of, that is almost opposite to ν *Draconis* (being the 35th *Camelopard. Hevelii* in Mr. *Flamsteed's* Catalogue) was 19″ more Northerly about the Beginning of *March* than in *September*. Whence I conclude, according to my Hypothesis, that the Diameter of the little Circle described by a Star in the Pole of the Ecliptick would be 40″, 2.

The last Star of the great Bear's tail of the 2d Magnitude (marked η by *Bayer*) was 36″ more Southerly about the Middle of *January* than in *July*. Hence the *Maximum*, or greatest Alteration of Declination of a Star in the Pole of the Ecliptick would be 40″, 4, exactly the same as was before found from the Observations of ν *Draconis.*

The Star of the 5th magnitude in the Head of *Perseus* marked τ by *Bayer,* was 25″ more Northerly about the End of *December* than on the 29th of *July* following. Hence the *Maximum* would be 41″. This Star is not bright enough to be seen as it passes over my Zenith about the End of *June,* when it should be according to the Hypothesis farthest South. But because I can more certainly depend upon the greatest Alteration of Declination of those Stars, which I have frequently observed about the Times when they become stationary, with respect to the Motion I am now considering; I will set down a few more Instances of such, from which you may be able to judge how near it may be possible from these Observations, to determine with what Velocity Light is propagated.

α *Persei Bayero* was 23″ more Northerly at the beginning of *January* than in *July.* Hence the *Maximum* would be 40″, 2. α *Cassiopeæ* was 34″ more Northerly about the End of *December* than in *June.* Hence the *Maximum* would be 40″, 8. β *Draconis* was 39″ more Northerly in the beginning of *September* than in *March*; hence the *Maximum* would be 40″, 2. *Capella* was about 16″ more Southerly in *August* than in *Feb.*; hence the *Maximum* would be about 40″. But this Star being farther from my Zenith than those I have before made use of, I cannot so well depend upon my Observations of it, as of the others; because I meet with some small Alterations of its Declination that do not seem to proceed from the Cause I am now considering.

I have compared the Observations of several other Stars, and they all conspire to prove that the *Maximum* is about 40″ or 41″. I will therefore suppose that it is 40″½ or (which amounts to the same) that Light moves, or is propagated as far as from the Sun to us in 8′ 13″. The near Agreement which I met with among my Observations induces me to think, that the *Maximum* (as I have here fixed it) cannot differ so much as a Second from the Truth, and therefore it is probable that the Time which Light spends in passing from the Sun to us, may be determined by these Observations within 5″ or 10″; which is such a degree of exactness as we can never hope to attain from the Eclipses of *Jupiter's* Satellites.

Having thus found the Maximum, or what the greatest Alternation of Declination would be in a Star placed in the Pole of the

Ecliptick, I will now deduce from it (according to the foregoing Hypothesis) the Alteration of Declination in one or two Stars, at such times as they were actually observed, in order to see how the Hypothesis will correspond with the *Phænomena* through all the Parts of the Year.

It would be too tedious to set down the whole Series of my Observations; I will therefore make Choice only of such as are most proper for my present Purpose, and will begin with those of *ν Draconis.*

This Star appeared farthest North about *September* 7th, 1727, as it ought to have done according to my Hypothesis. The following Table shews how much more Southerly the star was found to be by Observation in several Parts of the Year, and how much more Southerly it ought to be according to the Hypothesis.

1727 D.	THE DIFFERENCE OF DECLINATION BY OBSERVATION	THE DIFFERENCE OF DECLINATION BY THE HYPOTHESIS	1728 D.	THE DIFFERENCE OF DECLINATION BY OBSERVATION	THE DIFFERENCE OF DECLINATION BY THE HYPOTHESIS
Oct. 20	4½	4½	Mar. 24	37	38
Nov. 17	11½	12	April 6	36	36½
Dec. 6	17½	18½	May 6	28½	29½
Dec. 28	25	26	June 5	18½	20
1728			June 15	17½	17
Jan. 24	34	34	July 3	11½	11½
Feb. 10	38	37	Aug. 2	4	4
Mar. 7	39	39	Sept. 6	0	0

Hence it appears, that the Hypothesis corresponds with the Observations of this Star through all Parts of the Year; for the small Differences between them seem to arise from the Uncertainty of the Observations, which is occasioned (as I imagine) chiefly by the tremulous or undulating Motion of the Air, and of the Vapours in it; which causes the Stars sometimes to dance to and fro, so much that it is difficult to judge when they are exactly on the Middle of the Wire that is fixed in the common Focus of the Glasses of the Telescope.

I must confess to you, that the Agreement of the Observations with each other, as well as with the Hypothesis, is much greater than I expected to find, before I had compared them; and it may possibly be thought to be too great, by those who have been used to Astronomical Observations, and know how difficult it is to make such as are in all respects exact. But if it would be any Satisfaction to such Persons (till I have an Opportunity of describing my Instrument and the manner of using it) I could assure them, that in above 70 Observations which I made of this Star in a Year, there is but one (and that is noted as very dubious on account of Clouds) which differs from the foregoing Hypothesis more than 2″, and this does not differ 3″.

This therefore being the Fact, I cannot but think it very probable, that the *Phænomena* proceed from the Cause I have assigned, since the foregoing Observations make it sufficiently evident, that the Effect of the real Cause, whatever it is, varies in this Star, in the same Proportion that it ought according to the Hypothesis.

But least ν *Draconis* may be thought not so proper to shew the proportion, in which the apparent alteration of Declination is increased or diminished, as those Stars which lie near the Equinoctial Colure: I will give you also the Comparison between the Hypothesis and the Observations of η *Ursæ Majoris*, that which was farthest South about the 17th Day of *January* 1728, agreable to the Hypothesis. The following Table shews how much more Northerly

1727 d.	The Difference of Declination by Observation	The Difference of Declination by the Hypothesis	1728 d.	The Difference of Declination by Observation	The Difference of Declination by the Hypothesis
Sept. 14	29½	28½	April 16	18½	18
Sept. 24	24½	25½	May 5	24½	23½
Oct. 16	19½	19½	June 5	32	31½
Nov. 11	11½	10½	June 25	35	34½
Dec. 14	4	3	July 17	36	36
1728			Aug. 2	35	35½
Feb. 17	2	3	Sept. 20	26½	26½
Mar. 21	11½	10½			

it was found by Observation in several Parts of the Year, and also what the Difference should have been according to the Hypothesis.

I find upon Examination, that the Hypothesis agrees altogether as exactly with the Observations of this Star, as the former; for in about 50 that were made of it in a Year, I do not meet with a Difference of so much as 2″, except in one, which is mark'd as doubtful on Account of the Undulation of the Air &c. And this does not differ 3″ from the Hypothesis.

The agreement between the Hypothesis and the Observations of this Star is the more to be reguarded, since it proves that the Alteration of Declination, on account of the Precession of the Equinox, is (as I before supposed) regular thro' all Parts of the Years; so far at least, as not to occasion a Difference great enough to be discovered with this Instrument. It likewise proves the other part of my former Supposition, viz. that the annual Alteration of Declination in Stars near the Equinoctial Colure, is at this Time greater than a Precession of 50″ would occasion: for this Star was 20″ more Southerly in *September* 1728, than in *September* 1727, that is, about 2″ more than it would have been, if the Precession was but 50″. But I may hereafter, perhaps, be better able to determine this Point, from my Observations of those Stars that lie near the Equinoctial Colure, at about the same Distance from the North Pole of the Equator, and nearly opposite in right Ascension.

I think it needless to give you the Comparison between the Hypothesis and the Observations of any more Stars; since the Agreement in the foregoing is a kind of Demonstration (whether it be allowed that I have discovered the real Cause of the *Phænomena* or not;) that the Hypothesis gives at least the true Law of the Variation of Declination in different Stars, with Respect to their different Situations and Aspects with the Sun. And if this is the Case, it must be granted, that the Parallax of the fixt Stars is much smaller, than hath been hitherto supposed by those who have pretended to deduce it from their Observations. I believe, that I may venture to say, that in either of the two Stars, last mentioned, it does not amount to 2″. I am of Opinion, that if it were 1″, I should have perceived it, in the great number of Observations that I made especially of ν *Draconis*; which agreeing with the Hypothesis (without allowing anything for Parallax) nearly as well when the Sun was in Conjunction with, as in Opposition to, this Star, it seems very probable that the Parallax of it is not so great as one single

Second; and Consequently that it is above 400000 times farther from us than the Sun.

There appearing therefore after all, no sensible Parallax in the fixt Stars, the *Anti-Copernicans* have still room on that Account, to object against the Motion of the Earth; and they may have (if they please) a much greater objection against the Hypothesis, by which I have endeavoured to solve the fore-mentioned *Phænomena*; by denying the progressive Motion of Light, as well as that of the Earth.

But as I do not apprehend, that either of these Postulates will be denied me by the Generality of the Astronomers and Philosophers of the present Age; so I shall not doubt of obtaining their Assent to the Consequences which I have deduced from them; if they are such as have the Approbation of so great a Judge of them as Yourself. I am

Sir, Your most Obedient
Humble Servant
J. Bradley.

POSTSCRIPT.

As to the Observations of *Dr. Hook,* I must own to you, that before Mr. *Molyneux's* Instrument was erected, I had no small opinion of their Correctness; the Length of his Telescope and the Care he pretends to have taken in making them exact, having been strong Inducements with me to think them so. And Since I have been convinced both from Mr. *Molyneux's* Observations and my own, that the Doctor's are really very far from being either exact or agreeable to the *Phænomena*; I am greatly at a loss how to account for it. I cannot well conceive that an Instrument of the Length of 36 Feet, constructed in the Manner he describes his, could have been liable to an Error of near 30″ (which was doubtless the Case) if rectified with so much Care as he represents.

The Observations of Mr. *Flamsteed* of the different Distances of the Pole Star from the Pole at different Times of the Year, which were through Mistake looked upon by some as a Proof of the annual *Parallax* of it, seem to have been made with much greater Care than those of *Dr. Hook.* For though they do not all exactly correspond with each other, yet from the whole Mr. *Flamsteed* concluded that the Star was 35″ 40″ or 45″ nearer the Pole in *December* than in *May* or *July*: and according to my Hypothesis it ought to appear 40″ nearer in *December* than in *June.* The Agreement therefore of the Observations with the Hypothesis is greater than could reasonably be expected, considering the *Radius* of the Instrument, and the Manner in which it was constructed.

THE PRINCIPLE OF LEAST ACTION.*

REMARKS ON SOME PASSAGES IN MACH'S MECHANICS.

Ernst Mach in his *Mechanics*[1] remarks,[2] with reference to the integral variational principles of Hamilton and of least action, that *other* such principles are possible, which idea has been suggestive to myself, and, as I have obtained some results which throw light on Mach's suggestions, I will try to describe the results here in not too technical language.[3]

I.

We must first of all notice a slight historical inexactitude in Mach's treatment of the principle of least action. "Maupertuis," we are told,[4] "enunciated, in 1747, a principle which he called '*le principe de la moindre quantité d'action.*'" Maupertuis[5] laid before the Paris Academy on April 15, 1744, a memoir in which he explained the reflection and refraction of light by a hypothesis substituted for Fermat's principle of least time.[6]

Maupertuis, like a good follower of Newton, accepted the emission hypothesis of light, and, according to P. Stäckel,[7] the contra-

* Philip E. B. Jourdain, an English scholar who has devoted his life to research in the line of modern logic, mathematics and pure mechanics, submits to us some remarks on Mach's *Science of Mechanics.* He is a devoted and zealous student of Mach's works and is as familiar with them as a theologian with his Bible. Being also well acquainted with the work of Georg Cantor, Peano and Bertrand Russell he is especially fitted to explain the theoretical aspect of pure mechanics. We are confident that his lucubrations serve a good purpose and therefore deem it wise to submit them to specialists by giving them space in our columns.

[1] *Die Mechanik in ihrer Entwickelung historisch-kritisch dargestellt,* 4th ed., Leipsic, 1901, pp. 395-413; Engl. transl. by T. J. McCormack under the title *The Science of Mechanics, a Critical and Historical Account of its Development,* 3d ed., Chicago, 1907, pp. 364-380. (This translation will be referred to as *Mechanics,* and the above German edition as *Mechanik.*)

[2] *Mechanik,* pp. 399, 402, 413; *Mechanics,* pp. 368-369, 371-372, 380.

[3] Cf. note on p. 78 of my paper "On the General Equations of Mechanics," *Quarterly Journal of Mathematics,* 1904, pp. 61-79.

[4] *Mechanik,* p. 395; *Mechanics,* p. 364.

[5] Cf. *Mechanik,* pp. 484-485; *Mechanics,* pp. 454-455.

[6] *Mechanik,* pp. 454-457; *Mechanics,* pp. 422-425.

[7] *Encykl. der math. Wiss.,* IV, I, (1908), p. 49, note 125. Stäckel wrongly refers to the *Berlin Mem.,* 1745, p. 276, for Maupertuis's application of the principle of least action to the motion of light.

diction that Mach found in Maupertuis's application of the principle of least action to the motion of light is due to Mach's mistaken supposition that Maupertuis worked on the basis of the undulatory theory.

On Fermat's principle of least time and Maupertuis's principle of least action, we will quote some passages from E. T. Whittaker's lately published book, *A History of the Theories of Aether and Electricity from the Age of Descartes to the Close of the Nineteenth Century.*[8]

"Descartes's theory of light rapidly displaced the conceptions which had held sway in the Middle Ages. The validity of his explanation of refraction was, however, called in question by his fellow-countryman Pierre de Fermat (b. 1601, d. 1665), and a controversy ensued which was kept up by the Cartesians long after the death of their master. Fermat[9] eventually introduced a new fundamental law, from which he proposed to deduce the paths of rays of light. This was the celebrated *Principle of Least Time,* enunciated[10] in the form, 'Nature always acts by the shortest course.' From it the law of reflection can readily be derived, since the path described by light between a point on the incident ray and a point on the reflected ray is the shortest possible consistent with the condition of meeting the reflecting surfaces.[11] In order to obtain the law of refraction, Fermat assumed that 'the resistance of the media is different,' and applied his 'method of maxima and minima' to find the paths which would be described in the least time from a point of one medium to a point of the other. In 1661 he arrived at the solution.[12] 'The result of my work,' he writes, 'has been the most extraordinary, the most unforeseen and the happiest, that ever was; for, after having performed all the equations, multiplications, antitheses and other operations of my method, and having finally finished the problem, I have found that my principle gives exactly and precisely the same proportion for the refractions which Monsieur

[8] London and Dublin, 1910, pp. 9-11, 102-103.

[9] *Renati Descartes Epistolae, Pars tertia;* Amsterdam, 1683. The Fermat correspondence is comprised in letters xxix to xlvi.

[10] *Epist.* xlii, written at Toulouse in August, 1657, to Monsieur de la Chambre; reprinted in *Œuvres de Fermat* (ed. 1891), Vol. II, p. 354.

[11] That reflected light follows the shortest path was no new result, for it had been affirmed (and attributed to Hero of Alexandria) in the *κεφάλαια τῶν ὀπτικῶν* of Heliodorus of Larissa, a work of which several editions were published in the seventeenth century.

[12] *Epist.* xliii, written at Toulouse on Jan. 1, 1662; reprinted in *Œuvres de Fermat,* Vol. II, p. 457; Vol. I, pp. 170, 173.

Descartes has established.' His surprise was all the greater, as he had supposed light to move more slowly in dense than in rare media, whereas Descartes had (as will be evident from the demonstration given above) been obliged to make the contrary supposition.

"Although Fermat's result was correct, and, indeed, of high permanent interest, the principles from which it was derived were metaphysical rather than physical in character, and consequently were of little use for the purpose of framing a mechanical explanation of light. Descartes's theory therefore held the field until the publication in 1667[13] of the *Micrographia* of Robert Hooke (b. 1635, d. 1703), one of the founders of the Royal Society, and at one time its Secretary."

Further on, we read (p. 102): ". . . . the echoes of the old controversy between Descartes and Fermat about the law of refraction were awakened[14] by Pierre Louis Moreau de Maupertuis (b. 1698, d. 1759).

"It will be remembered that according to Descartes the velocity of light is greatest in dense media, while according to Fermat the propagation is swiftest in free ether. The arguments of the corpuscular theory convinced Maupertuis that on this particular point Descartes was in the right; but nevertheless he wished to retain for science the beautiful method by which Fermat had derived his result. This he now proposed to do by modifying Fermat's principle so as to make it agree with the corpuscular theory; instead of assuming that light follows the *quickest* path, he supposed that 'the path described is that by which the quantity of action is the least'; and this *action* he defined to be proportional to the sum of the spaces described, each multiplied by the velocity with which it is traversed. Thus instead of Fermat's expression

$$\int dt \text{ or } \int \frac{ds}{v}$$

(where t denotes time, v velocity, and ds an element of the path) Maupertuis introduced

$$\int v \, . \, ds$$

as the quantity which is to assume its minimum value when the path of integration is the actual path of light. Since Maupertuis's v, which denotes the velocity according to the corpuscular theory, is

[13] The *imprimatur* of Viscount Brouncker, P.R.S., is dated Nov. 23, 1664.

[14] *Mém. de l'Acad.*, 1744, pp. 417-426 [or *Œuvres de Mr. de Maupertuis*, Vol. IV, Lyons, 1756, pp. 3-18. To Maupertuis's work we will return on another occasion].

proportional to the reciprocal of Fermat's *v,* which denotes the velocity according to the wave-theory, the two expressions are really equivalent, and lead to the same law of refraction. Maupertuis's memoir is, however, of great interest from the point of view of dynamics; for his suggestion was subsequently developed by himself and by Euler and Lagrange into a general principle which covers the whole range of nature, so far as nature is a dynamical system."

* * *

In a memoir of 1746,[15] Maupertuis extended his hypothesis to all motions and called it the universal principle of rest and motion. By way of proving it, he derived the known laws of impact of inelastic and elastic bodies, and of the lever;[16] the motion of light having been dealt with in the memoir of 1744. It is most important to realize that, as A. Mayer[17] pointed out, Euler's discovery, made under the stimulus of the Bernoullis and published in the autumn of 1744 in an appendix to his *Methodus inveniendi,* was independent of Maupertuis, but that later on Euler's own tendency towards metaphysical speculation and the influence of Maupertuis combined to make Euler treat his principle in a less precise and more general way.

II.

Euler observed in 1744 that the differential equations of the motion of a particle are given by the simple requirement that the integral $\int v.ds$, where for the velocity v is substituted its value resulting from the principle of *vis viva,* and the integral is taken between two positions of the particle, should be a minimum. Euler

[15] "Les loix du mouvement et du repos déduites d'un principe métaphysique." *Mém. de l'Acad. de Berlin,* 1746, pp. 267-294. Voss (*Encykl. der math. Wiss.,* IV, 1, p. 95, note 256) has 1745 as the date of this memoir. This memoir was that analyzed by Mach (*Mechanik,* pp. 395-397; *Mechanics,* pp. 364-367). The analogies that exist between the motion of masses and the motion of light, which were noticed by Johann Bernoulli and by Möbius, were dealt with by Mach (*Mechanik,* pp. 402-408, 410-413, 457-459; *Mechanics,* pp. 372-380, 425-427). The principle of least action has been found very useful in optics, by Laplace, for example, in the treatment of astronomical refractions; and the mathematics of the theory of systems of rays built upon this one principle, which was the earliest work of William Rowan Hamilton, were later (in 1834 and 1835) transferred by Hamilton to the general problem of dynamics. Cf. P. Stäckel, *Encykl. der math. Wiss.,* IV, 1, 1908, pp. 489-493.

[16] In a memoir called "Loi du repos des corps" (*Mém. de l'Acad. de Paris,* 1740, pp. 170-176; *Œuvres,* Vol. IV, pp. 45-63) Maupertuis remarked that the work done when a final configuration of equilibrium is reached is generally either a maximum or a minimum (see Mach, *Mechanik,* pp. 69-75; *Mechanics,* pp. 68-73).

[17] *Geschichte des Prinzips der kleinsten Aktion,* Akademische Antrittsvorlesung, Leipsic, 1877; cf. my notes in *Ostwald's Klassiker,* No. 167, pp. 31-37.

expressly emphasized, first, that his theorem only holds if the principle of *vis viva* holds (and therefore cannot hold for motion in a resisting medium), and, secondly, that we must express v in terms of the attracting forces by quantities belonging to the orbit.[18]

Euler's work on this point was influenced adversely by his own tendency toward metaphysical speculation and Maupertuis's discovery—published some months before Euler's—of the obscure and almost theological universal "principle of the least quantity of action."[19]

III.

Lagrange[20] generalized Euler's theorem for the motion of any system of masses in the following way:

Let $m_1, m_2, m_3, \ldots$ be masses which act upon one another in any manner, and also, if we wish, move under the influence of any central forces which are proportional to any functions of the distances; let $s_1, s_2, s_3, \ldots$ be the spaces which are described by these masses in the time t, and let $v_1, v_2, v_3, \ldots$ be their velocities at the end of this time; then[21]

$$\Sigma m . \int v . ds$$

is a maximum or minimum, and thus, by the principles of the calculus of variations,

$$\Sigma m . \int (\delta v . ds + v . \delta ds) = 0. \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Lagrange eliminated the terms involving δv by making use of the equation

[18] Jacobi (see below), by direct generalization of Euler's theorem, reached his theorem.

[19] The early history of the principle of least action is very fully dealt with by me in my notes at the end of *Ostwald's Klassiker der exakten Wissenschaften,* No. 167.

[20] "Application de la méthode exposée dans le mémoire précédent à la solution de différents problèmes de dynamique," *Miscellanea Taurinensia* for 1760 and 1761, Vol. II, pp. 196-298; *Œuvres de Lagrange,* Vol. I, pp. 365-468. This memoir immediately followed Lagrange's first fundamental memoir on the calculus of variations: "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies," *Misc. Taur.,* 1760 and 1761 [published 1762], Vol. II, pp. 173-195; *Œuvres,* Vol. I, pp. 335-362; *Ostwald's Klassiker der exakten Wissenschaften,* No. 47, pp.3-30.

In Lagrange's first publication ("Recherches sur la méthode de maximis et minimis," *Misc. Taur.* for 1759, Vol. I; *Œuvres,* Vol. I, pp. 3-20), he announced (p. 15) his intention of deriving the whole of mechanics, by means of the principle of the least quantity of action, from a method he had of investigating the maxima and minima of indefinite integral formulae.

[21] For convenience of printing, the suffixes to the Σ, m, v, and s are here omitted. Instead of the now more usual Σ Lagrange (see below) used S.

$$\Sigma m . v . \delta v = \delta U \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

got by varying (differentiating with δ) the equation of *vis viva*.

Thus the equation (1), in conjunction with the condition (2), supposing that all the positions at the limits of the integral are given, so that there the variations of the coordinates are zero,[22] gives the fundamental equation[23]

$$\int \mathrm{S} dm \{ (d\frac{dx}{dt} + \Pi dt)\delta x + \ldots \} = 0, \quad \ldots\ldots\ldots\ldots (3)$$

where $\Pi \delta x + \ldots = \delta U,$

and S is a sign of a definite integral which refers to the masses of the system; so that, if there are a finite number of masses m_1, m_2, $m_3, \ldots,$

$$\mathrm{S} dm = \Sigma m.$$

If there is an equation of condition $\phi = 0$ between the coordinates, the equation $\delta\phi = 0$ gives a relation between the δx's, δy's and δz's of (3); and then we can eliminate from (3) all of the variations except a certain number which is the degree of freedom of the system. If, then, we put the coefficient of every independent variation equal to zero, we obtain the necessary number of differential equations for the solution of the problem.

An important point is that, as Hölder[24] remarked, Lagrange[25] drew attention to the fact that, even when the expression for the element of work is not a complete differential, and consequently "δU" can only be regarded as an abbreviation, and not as a notation for the variation of a force-function, that the formula (2), or

$$\delta T = \delta U,$$

can be applied to get an extension of the principle of least action even to non-conservative forces. This wider form was not treated in Lagrange's later work in the *Mécanique analytique* on the principle of least action.

Thus Mach[26] is mistaken in stating that Lagrange "drew express attention to the fact that Euler's principle is applicable only in cases in which the principle of *vis viva* holds." Euler had already made this remark, and subsequently Jacobi strongly emphasized it; but Lagrange, correctly, as we now know, first drew attention to

[22] *Œuvres,* Vol. I, pp. 369-370.

[23] *Ibid.,* pp. 368, 406, 418, 435, 459.

[24] *Gött. Nachr.,* 1896, p. 136. In *Ostwald's Klassiker,* No. 167, last line on p. 39, for "Helmholtz" read "Hölder."

[25] *Œuvres,* Vol. I, pp. 384-385.

[26] *Mechank,* p. 401; *Mechanics,* p. 371.

the fact that the principle of least action, in the very general form which he gave it, does not depend for its validity on that principle of *vis viva,* which only follows from the general equations of mechanics under special conditions.

There was no mention of this extension in Lagrange's later works, and Hamilton, for example, only took from Lagrange the narrower form of the principle of least action which was given in the *Mécanique.*

* * *

Lagrange appears to have noticed that the integrand of (3), put equal to zero, is an expression of d'Alembert's principle; and, in that form, d'Alembert's principle is the fundamental formula of Lagrange's analytical mechanics,[27] and then the principle of least action became, for Lagrange, merely a result of the laws of mechanics, to be got by the integration of the simpler equation.

However, in the early memoir Lagrange had concluded from his generalized principle of least action nearly all the great results which later, in his *Mécanique,* he derived in another way; and so Jacobi[28] remarked that Lagrange's principle became the mother of our whole analytical mechanics.[29]

[27] D'Alembert's principle in combination with the principle of virtual displacements appeared in the above variational form for the first time in a prize essay of 1764 of Lagrange's on the libration of the moon (*Œuvres,* Vol. VI, pp. 5-61); and then, more fully, in a memoir of 1780 (*Œuvres,* Vol. V., pp. 5-122).

The various editions of Lagrange's *Mécanique* are: *Mécanique analitique,* Paris 1788, I vol.; second, greatly enlarged edition, *Mécanique analytique,* Paris, Vol. I, 1811, Vol. II (posthumous), 1815; third edition, with notes by J. Bertrand, 2 vols., Paris, 1853 and 1855; fourth edition, after the third, but with additional notes by G. Darboux, in *Œuvres de Lagrange,* Vols. XI, and XII, Paris, 1888 and 1889.

[28] See *Compt. Rend.,* Vol. V, 1837, pp. 61-67 (*Ges. Werke,* Vol. IV, pp. 129-136); *Vorlesungen über Dynamik, gehalten an der Universität zu Königsberg im Wintersemester* 1842-1843 *und nach einem von C. W. Borchardt ausgearbeiteten Hefte herausgegeben von A. Clebsch,* Berlin, 1866, p. 2 (2d ed., revised by E. Lottner, in Jacobi's *Ges. Werke, Supplementband*). Cf. A. Mayer, *Geschichte des Prinzips der kleinsten Action,* Leipsic, 1877, p. 26 (on Mayer's errors see my notes in *Ostwald's Klassiker,* No. 167).

In this early memoir the problems treated by Lagrange were: the motion of one body attracted by many fixed central forces; general problem of many attracting masses under any other forces; the finding of the orbits of two attracting bodies with respect to a third; a body in a plane under forces and drawing two other bodies by threads; a thread fixed at one end and charged with as many heavy bodies as wished; an inextensible thread, all the points being under any forces; the same problem with an extensible and elastic thread; motion of a body of any figure animated by any forces; laws of the motion of non-elastic and elastic fluids.

[29] However, Lagrange's method of multipliers (Mach, *Mechanik,* pp. 499-500; *Mechanics,* p. 471) appeared first in the *Méchanique analitique* of 1788.

After the publication of the *Mécanique,* the principle of least action fell into the background of interest until Hamilton, in 1834, showed that this principle had also a totally different title to our consideration. The only really important contribution to the exceedingly interesting questions that rise *à propos* of the principle of least action was an almost entirely neglected one made by Olinde Rodrigues in 1816.

IV.

In Lagrange's derivation, the variation of $v(=ds/dt)$ is not carried out, but the terms $m.v.\delta v$ are eliminated by the variational equation obtained from the principle of *vis viva.* Thus it is not necessary to decide whether t must be varied or not, whether we must put

$$\delta v = \frac{d\delta s}{dt} - \frac{ds}{dt}\frac{d\delta t}{dt} \text{ or } \delta v = \frac{d\delta s}{dt}.$$

It is almost beyond doubt that Lagrange would have maintained that the independent variable t was to be varied;[30] but Rodrigues was the first explicitly to say that, in this case, t must be varied.

Lagrange had worked with a *space* integral $\int \Sigma m.v.ds$, and had only remarked, in a short addition to the section on the principle of least action, made in the second edition of the *Mécanique,* that the above space integral transforms into the *time*-integral $\int 2T.dt$, where 2T is the *vis viva* (or, as we now say, double the kinetic energy) of the system.[31] But Lagrange did not actually carry out the calculation of the variation of this time-integral; this was done by Rodrigues.[32] Rodrigues, as E. J. Routh[33] did later and apparently independently, to find the variation of $\int T.dt$ under the condition $T = U + \text{const.}$ for the variation, so that $\delta T - \delta U = 0$, multiplied the left-hand side of this last equation of condition by an undetermined factor, integrated it, added it to the variation of $\int T.dt$, put all equal to zero, and then determined the factor.

[30] Cf. *Œuvres,* Vol. I, pp. 337, 345; and *Ostwald's Klassiker,* No. 167, p. 56.

[31] See *Ostwald's Klassiker,* No. 167, p. 11.

[32] *Correspondance sur l'Ecole polytech.,* Vol. III, 1816, pp. 159-162; German translation, with notes on some errors of Rodrigues, in *Ostwald's Klassiker,* No. 167, pp. 12-15, 41-42, 49-55.

[33] First in *An Elementary Treatise on the Dynamics of a System of Rigid Bodies,* 3d ed., London, 1877, pp. 305-312, 560-562. This passage coincides in essentials with *The Advanced Part* [Part II] *of a Treatise on the Dynamics of a System of Rigid Bodies,* 6th ed., London, 1905, pp. 301-309.

V.

The question as to whether the independent variable should be varied in the calculus of variations is of great importance to our conception of this calculus. According to Mach, [34] the first satisfactory explanation of the meaning of the process of variation used in this calculus was given by J. H. Jellett.[35] The value of the function $y = \phi(x)$ can vary by an (infinitesimal) increment dx of the independent variable, when we obtain the *differential*

$$dy = \phi(x + dx) - \phi(x),$$

or by the varying of the form ϕ of the function without x varying, so that $\phi(x)$ becomes

$$\phi_1(x) = \phi(x) + \epsilon\psi(x),$$

where ψ is an arbitrary function and ϵ is, for the definition of an *infinitesimal* variation, an infinitely small positive number. Then the *variation* of y is defined by

$$\delta y = \phi_1(x) - \phi(x).$$

Thus, if we keep, as is convenient, the term "variation" to denote alterations of value brought about by alteration of the form alone of the function, we see that the independent variable is unaffected by our process of variation. On the other hand, Lagrange, as we have seen, held that the independent variable also was to be affected by the δ of the calculus of variations. Indeed, his claim that his method was more general than that of Euler rested partly on this ground. But other mathematicians appear mostly to have accepted that conception of a variation which Euler gave in a later memoir on Lagrange's method, that a "variation" of a function is brought about by a change in value of the constants occurring in that function. Thus, Jacobi, in his *Vorlesungen über Dynamik*,[36] stated that the variations δq of the generalized coordinates q contain merely the changes in value of the q's which arise from changes in value of the arbitrary constants occurring in the q's. Accordingly, he maintained[37] that the independent variable is not to be "varied," so that $\delta t = 0$.[38]

[34] *Mechanik*, pp. 468-474; *Mechanics*, pp. 437-443.

[35] *An Elementary Treatise on the Calculus of Variations*, Dublin, 1850, pp. 1, 5-6. Cf. A. Kneser, *Lehrbuch der Variationsrechnung*, Brunswick, 1900, pp. 1-2.

[36] *Werke, Supplementband*, p. 145.

[37] *Ibid.*, pp. 50, 59, 146, 149.

[38] Cf. similar views on the nature of a "variation" with Euler, Lagrange, Lacroix, G. W. Strauch, M. Ohm, Cauchy, and Stegmann in I. Todhunter's

So Jacobi, in his *Vorlesungen über Dynamik,*[39] stated that, in the action integral $\int \Sigma m.v.ds$, the time must be eliminated by the principle of *vis viva,* and all be reduced to space-elements. This, as Mayer remarked in his tract of 1877, was required by Euler in the case considered by him. Thus Jacobi's[40] formulation of the principle of least action was: If two positions of the system are given (that is to say, if we know the values which, for $x=a$ and $x=b$, the remaining $3n-1$ coordinates receive), and we extend the integral

$$\int \sqrt{2(U+h)}\ \sqrt{\Sigma m\ .\ ds^2}$$

to the whole path of the system from the first position to the second, then its value is a minimum for the actual path as compared with all possible (consistent with the conditions, if there be any, of the system) paths.[41]

Mayer, in his tract of 1877,[42] accepted Jacobi's view that $\delta t=0$ and consequently that, by means of the principle of *vis viva,* we must reduce all the quantities in the integrand to quantities which refer to the path of the system; and that the theorem of least action without this condition is quite meaningless. Since Lagrange did not eliminate the time, Mayer[43] concluded that Lagrange's theorem was meaningless, and what Lagrange really meant by his theorem was what is known as Hamilton's principle. This view had been previously maintained by M. Ostrogradski.[44]

But, in a memoir of 1886 on the general theorems of the calculus of variations which correspond to the two forms of the principle of least action in dynamics, Mayer[45] remarked, on the variation

work *A History of the Progress of the Calculus of Variations During the Nineteenth Century,* Cambridge and London, 1861, pp. 2, 8, 11, 13, 17-20, 31, 377, 378, 402, 413, 480-481.

[39] *Werke, Supplementband,* p. 44; *Ostwald's Klassiker,* No. 167, p. 17 (on pp. 16-26 is a reprint of Jacobi's sixth and part of his seventh lecture, which relate to the principle of least action).

[40] *Werke, Supplementband,* p. 45; *Ostwald's Klassiker,* No. 167, p. 18 (cf. the note on p. 55).

[41] On the limitations to the minimum-condition, which were pointed out by Jacobi (cf. Mach, *Mechanik,* p. 401; *Mechanics,* p. 371) see *Werke, Suppl.,* pp. 45-49; *Klassiker,* No. 167, pp. 18-22, 58.

[42] See p. 24, and *Klassiker,* No. 167, p. 57.

[43] *Op. cit.,* p. 27.

[44] *Klassiker,* No. 167, pp. 57-58.

[45] "Die beiden allgemeinen Sätze der Variationsrechnung, welche den beiden Formen des Prinzips der kleinsten Aktion in der Dynamik entsprechen," *Berichte der math.-phys. Classe der Kön. Sächs. Ges. der Wiss. zu Leipzig,* Sitzung am 14. November 1886, Vol. XXXVIII, pp. 343-355. The first person correctly to show the importance of Rodrigues's memoir was Th. Sloudsky

of t with Rodrigues: "Now, from the point of view of dynamics, in which we only permit variations from the instantaneous position of the system under consideration, that is so very unusual that I did not think at all of this possibility in my earlier work. But as soon as we neglect a purely dynamical signification (*Deutung*), and vary, not only the coordinates, but also the time, immediately that point which always caused the greatest doubts in Lagrange's derivation becomes clear. It is explained, namely, how the equation of *vis viva,* if it is prescribed as an equation of condition, can yet leave the variations of the coordinates quite unlimited,[46] and we see then that Jacobi's assertion that we must necessarily eliminate the time from the action-integral by means of the theorem of *vis viva* is not so; that, besides Jacobi's principle, there is a second, equally justified form of the principle of least action; and that it is this second form, and not Hamilton's principle inaccurately formulated, which Lagrange proved correctly, though certainly not with his usual clearness.

We may here remark that Routh,[47] from 1877 onwards and apparently independently of Rodrigues, also varied t, "by the fundamental theorem in the calculus of variations," and derived the principle of least action as Rodrigues did.

If t is to be varied, we must regard it, according to the conception of a "variation" derived from Jellett, as a function of another variable, θ, so that $\delta\theta = 0$ but δt is not zero in general. This was done explicitly by Helmholtz[48] in 1887.

Helmholtz also stated the view that Hamilton's principle is a form of Lagrange's principle. The grounds for this view are, as I showed in 1908,[49] more clearly evidenced in an identity established by Réthy under certain restrictions.

VI.

We have dealt with the question as to the relation of the principle of least action to Hamilton's principle, and we have seen how Lagrange, by working with a form which only contained the time through the velocities, and in which the variations of the velocities

(1866); Bertrand, in his notes on Lagrange's *Mécanique,* mentioned Rodrigues, but put $\delta(dq/dt) = d\delta q/dt$.

[46] Cf. *Klassiker,* No. 167, pp. 43-44.

[47] Cf. *ibid.,* pp. 50-51.

[48] "Zur Geschichte des Prinzips der kleinsten Aktion," *Sitzungsber. der Berliner Akad.,* Sitzung vom 10. März 1887, pp. 225-236; *Wiss. Abh.,* Vol. III, pp. 249-263.

[49] *Math. Ann.,* Vol. LXV, pp. 514-516.

could be at once eliminated by means of the varied equation of *vis viva,* allowed it to remain doubtful whether t was to be varied in the principle of least action, or not. We have seen how this question has given rise to discussions and misunderstandings which are connected with the principle of the calculus of variations, in the works of Rodrigues, Jacobi, Ostrogradski, Routh, Mayer, Sloudsky, Bertrand, Helmholtz, and Réthy. We have seen, finally, that Lagrange had attained to a very general formulation of the principle of least action, in which the equation of *vis viva* does not hold, a force-function does not exist, and the equations of condition may depend explicitly on the time. Thus Lagrange's principle is far more general than Jacobi's.

Of late years, the occurrence of differential and non-integrable equations among the equations of condition of a problem has assumed great importance. This happens in certain cases of rolling motion, and systems with such equations of condition were called by Hertz *non-holonomous.* The question arises as to whether the principle of least action and Hamilton's principle can be so formulated as to apply to non-holonomous systems. We shall see that Otto Hölder first succeeded in formulating extended forms of both principles which were completely equivalent to d'Alembert's principle. There were, of course, several points not dealt with by Hölder on which it was essential to be quite clear. Thus, the process of "variation" used by Hölder was not always the one to which we are accustomed in the calculus of variations, and the transformation of the principles from rectangular coordinates—which alone were used by Hölder—to more general coordinates gives rise to interesting questions. However, it seems to me that we have now reached a certain degree of finality in all these subjects, and we will now present the researches whose object was to extend the principles, in their proper order, and, where necessary, comment on them.

VII.

The question as to the extent of the variational principles begins with the publication, in 1894, of Heinrich Hertz's posthumous *Prinzipien der Mechanik.*[50] "The application of Hamilton's prin-

[50] *Gesammelte Werke von Heinrich Hertz,* Vol. III, *Die Prinzipien der Mechanik in neuem Zusammenhange dargestellt* (edited by Ph. Lenard, with a preface by H. von Helmholtz), Leipsic, 1894; English translation by D. E. Jones and J. T. Walley under the title *The Principles of Mechanics,* London, 1899.

ciple," said Hertz,[51] "to a material system does not exclude fixed connections between the coordinates chosen, but it requires that these connections can be exposed mathematically by means of finite equations between the coordinates; it does not permit of such connections as can be expressed only by differential equations. But nature itself appears not simply to exclude connections of the latter kind; for they occur if, for example, three-dimensional bodies roll upon one another without slipping."

Hertz[52] called a material system *holonomous* if between possible positions all thinkable continuous passages are also possible. The name was chosen to indicate that such a system is subject to integral (ὅλος) laws (νόμος), while material systems in general are subject only to differential laws. If the differential equations of condition of a material system can all be integrated, the coordinates of every possible position must satisfy the finite equations. The differences between the coordinates of two neighboring positions therefore satisfy an equal number of homogeneous linear differential equations, and, since these latter cannot contradict the given differential equations (in equal number) of the system, they satisfy the latter too. Thus the displacement between any two possible positions is a possible displacement, and thus the system is holonomous. Inversely, if the system is holonomous, its differential equations of condition allow an equal number of finite or integral equations between the coordinates themselves.

VIII.

Here we may digress to remark that the fact that cases of rolling motion give rise to equations of condition which are not integrable was observed by Routh, Ferrers (1873), and C. Neumann (1888).[53] The usual form of Lagrange's equations then fails. Of the extensions, what I have called, in the paper just quoted, "Routh's form" is the most important form for our present purposes. It involves Lagrange's multipliers, and is the only form of equation valid for non-holonomous systems which can be got directly by development of one of the integral variational principles. In deducing equations of motion from, say, Hamilton's principle,

[51] *Werke,* Vol. III, pp. 22-25; *Principles,* pp. 19-21.

[52] *Werke,* Vol. III, articles 123, 132, and 133 (pp. 91, 95, and 96); *Principles,* pp. 80, 84-85.

[53] Cf. the note on p. 63 of my paper "On the General Equations of Mechanics," *Quart. Journ. of Math.,* 1904, pp. 61-79. Cf. the bibliography in P. Appell's little book on *Les mouvements de roulement en dynamique,* Paris, 1899.

we so to speak divide the material system into a holonomous and a non-holonomous part. Suppose there are $3n$ rectangular coordinates of the system, k finite equations of condition between these coordinates and the time, and l non-integrable equations of condition. We form our integral for a system with $3n-k$ degrees of freedom and then eliminate the l superfluous coordinates by Lagrange's method.

IX.

An important paper on the differential equations of mechanics was written by A. Voss[54] in 1884 and published in 1885. In this paper, the equations of condition were used in their differential form, and were not assumed to be integrable, although the problems of rolling motion which caused such equations to be considered were not mentioned. The part which especially concerns us here is where Voss uses Hamilton's principle for the introduction of more general coordinates. He says[55] that, with non-integrable equations of condition, "the transformation can no longer be reduced to a problem of variations properly so called, but the property of the system of differential equations of condition of being a complete one forms the necessary and sufficient condition for this."

X.

Hertz decided that his own fundamental law[56] holds both for holonomous and non-holonomous systems, and that from this law result the principle of least action[57] and Hamilton's principle[58] only under a limitation to holonomous systems. But this contradicts the general conviction[59] that Hamilton's principle is merely a transformation of d'Alembert's principle, and that the latter holds generally, and is equivalent to Hertz's law.[60] Thus arose the questions as to whether the usual derivation of Hamilton's principle from that of d'Alembert requires any limiting supposition. This question was the origin of the researches of Otho Hölder.[61] The very kernel of

[54] "Ueber die Differentialgleichungen der Mechanik," *Math. Ann.*, Vol. XXV, 1885, pp. 258-286.

[55] *Ibid.*, pp. 263-264.

[56] *Werke*, Vol. III, art. 309, p. 162; *Principles*, p. 144.

[57] *Werke*, Vol. III, arts. 347-356, pp. 174-176; *Principles*, pp. 155-157.

[58] *Werke*, Vol. III, arts. 358-362, p. 177; *Principles*, pp. 158-159.

[59] See, for example, Mach, *Mechanik*, pp. 413-414; *Mechanics*, p. 381.

[60] *Werke*, Vol. III, art. 394, p. 186; *Principles*, p. 166.

[61] "Ueber die Principien von Hamilton und Maupertuis," *Nachr. von der königl. Ges. der Wiss. zu Göttingen, Math. phys. Klasse*, 1896, pp. 122-157.

Hölder's work is his conception of the "variation of the motion of a system;" this it was which allowed him to give such a wide extension to the principles of least action and of Hamilton, so that the reply to the above question is: If d'Alembert's principle holds generally, so also must that of Hamilton, in its completest form; but if we choose Hertz's view that the varied path be a possible one, we get the limitation denoted by him. Hölder's conception of a varied motion is, then, paradoxical in so far that this "motion" need not be a possible one,—need not satisfy the equations of condition. It is, in Hölder's own words, only a mathematical auxiliary conception.

With Hertz, Hölder understood by "the position of a system" the totality of the positions of the material points of the system; the motion consists in a continuous sequence of positions of the system, which are passed through in a definite way with the time. To vary this original motion, we first give every system-position a small displacement, so that a new continuous sequence of positions arises. If the original sequence gives one position twice, we have two positions covering one another which can naturally be displaced in different manners. The starting position A and the final position B are to be fixed, and we refer each position on the varied path to one on the actual path. This correspondence is necessary in order that we may put the variation of an integral taken along the original path equal to the integral of the varied elements. We coordinate the identical initial positions to one another, and similarly with the two final positions.

If we imagine both the actual[62] and the fictitious motion to begin simultaneously at A, then the systems need not arrive at B simultaneously. In this case the corresponding positions on the two paths cannot all be passed simultaneously, and if the passage from an actual position to the corresponding position on the fictitious path be denoted by δ, so that, if the position P is actually reached at the time t, the corresponding position $P+\delta P$ is reached at the time $t+\delta t$ and $\delta(dt)=d(\delta t)$.

Now, in the most general manner of variation of the motion, we can still choose the velocity at each point of the varied path. This must be infinitely little different from the velocity at the corresponding position of the actual path, but is otherwise arbitrary.

Hölder then found the expression for δT in rectangular coordi-

[62] Thus it is assumed that the mechanical problem has one solution and one only.

nates, t and dt being affected by the δ-process, integrated the identity for δT from t_0 to t_1 (the times when the system, in the actual motion, is at A and B respectively), and integrated by parts. Thus, two parts are obtained: one integrated, which vanishes, since the variations of the coordinates at A and B vanish; and the other unintegrated, and we see by d'Alembert's principle, that the integrand of the last integral can be put equal to δU where, as before, "δU" only denotes the variation of a force function U in special cases—provided that the variations of the coordinates represent *virtual* displacements of the system.[63] Thus Hölder obtained the result that, where the δ-process is a process of giving every position P between A and B a virtual displacement to P+δP, and the aggregate of positions P+δP is conceived as a fictitious path, then the equation

$$\int\{2T.d\delta t + (\delta T + \delta U)dt\} = 0, \qquad\ldots\ldots\ldots\ldots(4)$$

where the integral is to be taken between the limits t_0 and t_1, is equivalent to d'Alembert's principle.

We cannot too strongly emphasize the nature of this varied path of the system. It is not necessarily a path that the system, however constrained, could take; that is to say, the connections of the system might have to be distorted from point to point. The displacement δP must be virtual at the instant t, but the position $P + \delta P$ is "reached" by the system, supposed to "move" on a fictitious path in a perhaps impossible way, at the, in general different, time $t + \delta t$. In fact, the fictitious path is only a possible one, of course under new constraints, if the equations of the condition are independent of the time, and the system is holonomous.

This fictitious motion is a useful conception because it enables us to see exactly why Hertz, for example, rather naturally limited the scope of the principle of least action and Hamilton's principle to holonomous systems; and also it allows us to formulate these principles in a perfectly general manner. That the conception of a "variation" is not that of the calculus of variations did not escape Hölder. "At the first glance," he wrote,[64] "the conception is perhaps peculiar, and it has been already said to me that I have no problem of variation properly so called. But that does not concern me. I am only concerned with giving a clear signification to the variations of the coordinates and the time which at the same time is such that

[63] That is to say, displacements consistent with the equations of condition and possible, at the instant considered. Cf., for example, Mach, *Mechanik, p.* 58; *Mechanics,* pp. 49, 56.

[64] In a letter to me of Jan. 15, 1904; cf. *Quart. Journ. of Math.,* 1904, p. 75, last note.

the principles hold as generally as is possible." In conformity with this, Hölder spoke of an "altered" (*abgeänderte*) instead of a "varied" motion.

In the above general principle, we can, without detracting from the equivalence to d'Alembert's principle, specialize the variations. Two ways at once suggest themselves:

(1) We may determine that corresponding positions are to be passed at the same instant, so that $\delta t = 0$, then (4) becomes a generalized Hamilton's principle;

(2) We may determine the velocity at each point of the varied path by fixing that $\delta T = \delta U$, the variation of the time being, of course, not zero; that is to say, using a more restricted phraseology for this wider case, the total energy is constant in a variation; then (4) gives the principle of least action in its most extended form.

XI.

There is one rather important point upon which Hölder only touched very briefly. I mean the introduction of other more general coordinates into the development of equations of motion from the above principles. Voss attempted to do this in 1900, but, as I have shown, [65] he used a method previously used by Routh and Réthy, which preserved the strictly variational character of the δ-process used even when the equations of condition depend explicitly on the time. Thus Voss unintentionally abandoned Hölder's δ-process. The application of Hölder's process to the formulation of the principles in general coordinates was first carried out by myself in the above cited paper of 1904, and more clearly in a paper of 1908.[66] Mathematically speaking, this formulation is not quite so simple as some might suppose; but here we are only concerned with the advantages of Hölder's δ-process over the strictly variational process in the formulation of the principle of least action and Hamilton's principle. The abandonment of the strict conception of a variation may appear to be a disadvantage. But surely this is compensated by greater simplicity; while, in my case, when we come to deal with non-holonomous systems we must abandon this strict conception, as was pointed out—we have seen above—by Voss in 1884 and by others later in somewhat different forms.[67] Further, unless the

[65] *Math. Ann.*, Vol. LXV, 1908, pp. 517-525.

[66] *Math. Ann.*, Vol. LXV, 1908, pp. 525-527.

[67] C. Neumann (1888), Hertz (1894), Hölder (1896), and Appell (1898); see also Boltzmann, *Vorlesungen über die Prinzipe der Mechanik*, Teil II, Leipsic, 1904, pp. 30-34.

equations of condition do not contain the time explicitly, the form of Réthy and Voss requires a condition holding for δt at the limits of integration, whereas in Hölder's generalized principle of least action no such condition is required.

XII.

As we have said at the beginning, Mach has stated, with reference to the principles of least action and Hamilton, that *other* such principles are possible. In this connection there are two investigations to which we must refer. The first was by Voss[68] in 1901, and was inspired by Hölder's work. Voss remarked that if not only the coordinates, but also the time is varied in the most general manner, δt can always be determined subsequently so that if we put the variation of the integral of *any* function of the coordinates and velocities equal to zero, we get the equation of motion. The second was an attempt by myself[69] to solve the problem suggested by Mach, by determining *all the possible integral variational principles.* For this purpose I inquired what was the most general form of the integrand in order that the principle obtained hence should be equivalent to Routh's extension of Lagrange's equations. The result was to find that Hölder's principle (4) was the most general of its kind, and, as Hölder had remarked, his principle may be specialized into Hamilton's principle or the principle of least action. These two principles are, in fact, two special cases out of the manifold of the principles equivalent to d'Alembert's principle and derivable from (4) by determining δt generally in all possible ways.

But there is another aspect of the matter. We have taken Lagrange's equations, or rather Routh's extension of them, as fundamental. But there are other forms of the equations of mechanics involving other quantities than Lagrange's T and U, and which sometimes present advantages over Lagrange's.[70] From these other equations we can derive[71] other variational principles not contained in Hölder's form (4), but since the functions in the integrand now involve differential coefficients with respect to t of the second

[68] "Bemerkungen über die Prinzipien der Mechanik," *Sitzber. der math.-phys. Klasse der k. Bayer. Akad. der Wiss. zu München*, Vol. XXXI, 1901, pp. 167-182, especially pp. 171-175; *Encykl. der math. Wiss.*, IV, 1, 1901, p. 94.

[69] *Quart. Journ. of Math.*, 1904, pp. 76-78.

[70] Cf. my paper on "Alternative Forms of the Equations of Mechanics," in the *Quart. Journ. of Math.*, 1905, pp. 284-296.

[71] Cf. *Ibid.*, pp. 290-295.

order, we must determine the varied path so that not only the variations δq but also the differentials $d\delta q$ of these variations vanish at the limits of integration. Analogous conditions as to the paths arise, if the integrand contains higher differential coefficients than the second.

XIII.

A curious result,[72] by the way, is that if we vary the integral of action $\int 2T.dt$, so that δx means, as with Hölder, a virtual displacement of x, and vary t, we get exactly the same result as if we had not varied t either in T or in dt: the extra terms we get from varying t happen to cancel one another. Hence the faulty derivation, which we sometimes see, of Hamilton's principle from the principle of least action leads to correct results. This derivation is: Since $\delta T = \delta U$, we have

$$\delta \int 2T.dt = \int (\delta T + \delta T)dt = \int (\delta T + \delta U)dt = \delta \int (T + U)dt.$$

It should be noticed that the extra terms above referred to cancel even if the equations of condition contain the time explicitly. Further, we have seen that the identification maintained by Helmholtz and Réthy of Hamilton's principle with the principle of least action depended on the equations of condition not containing the time explicitly; and that the other identifications were based on misunderstandings. Finally, we have seen how in Hölder's other work, the true relation of the principles became clear, and how, at the same time, the principle became generalized.

XIV.

This sketch of the development and gradual generalization of a small part of the theory of mechanics gives us food for meditation. It seems to be necessary, in order that it may be possible to state the principles in question quite generally, to make use of a paradoxical conception—the conception of a generalized, fictitious "motion." It would be easy to say that the principles are, by the laws of logic, valid only under certain conditions; hence the paradox when we attempt to widen those conditions. But the paradox is not logical; it is merely verbal. We speak of a fictitious "path" and "motion" merely for the sake of picturesqueness: a mathematician no more means to imply the existence, in a mystical region of thought, of an impossible and fictitious path or motion, than he means to imply anything more than striking analogies of expression when he speaks,

[72] *Quart. Journ. of Math.*, 1904, pp. 78-79.

in analytical geometry, of "imaginary intersections" or "circular points at infinity." No philosopher wishes to confute a mathematician because, in his technical language, the mathematician may assert that some "real" numbers are not "rational."

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NOTES ON THE CONSTRUCTION OF MAGIC SQUARES
OF ORDERS IN WHICH n IS OF THE GENERAL FORM $4p+2$.

It is well known that magic squares of the above orders, i. e., $6^2, 10^2, 14^2, 18^2$, etc., cannot be made perfectly pandiagonal and ornate with the natural series of numbers.

Dr. C. Planck has however pointed out that this disability is purely arithmetical, seeing that these magics can be readily constructed as perfect and ornate as any others with a properly selected series of numbers.

In all of these squares n is of the general form $4p+2$, but they can be divided into two classes:

Class I. Where n is of the form $8p-2$, as 6^2, 14^2, 22^2 etc.

Class II. Where n is of the form $8p+2$, as 10^2, 18^2, 26^2 etc.

The series for all magics of Class I may be derived by making a square of the natural series 1 to $(n+1)^2$ and discarding the numbers in the middle row and column.

Thus, for a 6^2 magic the series will be:

1	2	3	—	5	6	7
8	9	10	—	12	13	14
15	16	17	—	19	20	21
—	—	—	—	—	—	—
29	30	31	—	33	34	35
36	37	38	—	40	41	42
43	44	45	—	47	48	49

The series for all magics of Class II may be made by writing a square of the natural numbers 1 to $(n+3)^2$ and discarding the numbers in the *three* middle rows and columns. The series for a 10^2 magic, for example, will be:

1	2	3	4	5	. . .	9	10	11	12	13
14	15	16	17	18	. . .	22	23	24	25	26
27	28	29	30	31	. . .	35	36	37	38	39
40	41	42	43	44	. . .	48	49	50	51	52
53	54	55	56	57	. . .	61	62	63	64	65
—	—	—	—	—	. . .	—	—	—	—	—
—	—	—	—	—	. . .	—	—	—	—	—
—	—	—	—	—	. . .	—	—	—	—	—
105	106	107	108	109	. . .	113	114	115	116	117
118	119	120	121	122	. . .	126	127	128	129	130
131	132	133	134	135	. . .	139	140	141	142	143
144	145	146	147	148	. . .	152	153	154	155	156
157	158	159	160	161	. . .	165	166	167	168	169

By using series as above described, pandiagonal magics with double-ply properties, or associated magics may be readily made either by the La Hireian method with magic rectangles, or by the path method as developed by Dr. C. Planck.

Referring now to the La Hireian method and using the 6^2 magic as a first example, the rectangles required for making the two auxiliary squares will necessarily be 2×3, and the numbers used therein will be those commonly employed for squares of the seventh order, i. e., $(6+1)^2$, with the middle numbers omitted thus:

1	2	3	—	5	6	7
0	7	14	—	28	35	42

It may be shown that a magic rectangle having an odd number of cells in one side, and an even number of cells in the other side is impossible with consecutive numbers, but with a series made as above it can be constructed without any difficulty, as shown in Figs. 1 and 2.

Two auxiliary squares may now be made by filling them with their respective rectangles. If this is done without forethought, a plain pandiagonal magic of the sixth order may result, but if attention is given to ornate qualities in the two auxiliaries, these features will naturally be carried into the final square. For example, by the arrangement of rectangles shown in Figs. 3 and 4 both auxiliaries are made magic in their six rows, six columns and twelve

7	2	3
1	6	5

Fig. 1.

42	7	14
0	35	28

Fig. 2.

7	2	3	7	2	3
1	6	5	1	6	5
7	2	3	7	2	3
1	6	5	1	6	5
7	2	3	7	2	3
1	6	5	1	6	5

Fig. 3.

0	42	0	42	0	42
35	7	35	7	35	7
28	14	28	14	28	14
0	42	0	42	0	42
35	7	35	7	35	7
28	14	28	14	28	14

Fig. 4.

7	44	3	49	2	45
36	13	40	8	41	12
35	16	31	21	30	17
1	48	5	43	6	47
42	9	38	14	37	10
29	20	33	15	34	19

Fig. 5.

7	2	3	3	2	7
1	6	5	5	6	1
7	2	3	3	2	7
1	6	5	5	6	1
7	2	3	3	2	7
1	6	5	5	6	1

Fig. 6.

0	42	0	42	0	42
35	7	35	7	35	7
28	14	28	14	28	14
28	14	28	14	28	14
35	7	35	7	35	7
0	42	0	42	0	42

Fig. 7.

diagonals, and they are also 4-ply and 9-ply. Their complementary couplets are also harmoniously connected throughout in steps of

3, 3. These ornate features are therefore transmitted into the finished 6^2 magic shown in Fig. 5. If it is desired to make this square associated, that is with its complementary couplets evenly balanced around its center, it is only necessary to introduce the feature of association into the two auxiliary squares by a rearrangement of their magic rectangles as shown in Figs. 6, 7 and 8, the last figure being a pandiagonal associated magic.

7	44	3	45	2	49
36	13	40	12	41	8
35	16	31	17	30	21
29	20	33	19	34	15
42	9	38	10	37	14
1	48	5	47	6	43

Fig. 8.

The next larger square of Class I is 14^2, and it can be made with the natural series 1 to $(14+1)^2$ arranged in a square, discarding, as before, all the numbers in the central row and column.

The rectangles for this square will necessarily be 2x7 and the numbers written therein will be those ordinarily used for a square of the fifteenth order, $(14+1)^2$, with the middle numbers omitted, thus:

1	2	3	4	5	6	7	—	9	10	11	12	13	14	15
0	15	30	45	60	75	90	—	120	135	150	165	180	195	210

15	2	3	12	11	6	7
1	14	13	4	5	10	9

Fig. 9.

210	15	30	165	150	75	90
0	195	180	45	60	135	120

Fig. 10.

Simple forms of magic rectangles for the auxiliaries are shown in Figs. 9 and 10, but many other arrangements of the couplets will work equally well.

The smallest magic of Class II is 10^2, the series for which is given below. The rectangles used for filling the two auxilliaries of this square are 2x5, and they can be made with the numbers which would be commonly used for a square of the thirteenth order $(10+3)^2$ omitting the three middle numbers in each row thus:

1	2	3	4	5	. . . .	9	10	11	12	13
0	13	26	39	52	. . . .	104	117	130	143	156

Figs. 11 and 12 show these two rectangles with a simple arrangement of the numbers. The two auxiliaries and the finished 10^2

13	2	11	4	5
1	12	3	10	9

Fig. 11.

156	13	130	39	52
0	143	26	117	104

Fig. 12.

magic are given in Figs. 13, 14 and 15. Fig. 15 is magic in its ten rows, ten columns and twenty diagonals. It is also 4-ply and 25-ply. Like the 6^2 magic, this square can also be associated by changing the disposition of the magic rectangles in the auxiliaries.

13	2	11	4	5	13	2	11	4	5
1	12	3	10	9	1	12	3	10	9
13	2	11	4	5	13	2	11	4	5
1	12	3	10	9	1	12	3	10	9
13	2	11	4	5	13	2	11	4	5
1	12	3	10	9	1	12	3	10	9
13	2	11	4	5	13	2	11	4	5
1	12	3	10	9	1	12	3	10	9
13	2	11	4	5	13	2	11	4	5
1	12	3	10	9	1	12	3	10	9

Fig. 13.

The above examples will suffice to explain the general construction of these squares by the La Hireian method with magic rectangles. It may however be stated that although the series previously described for use in building these squares include the lower numerical values, there are other series of higher numbers which will produce equivalent magic results.

0	156	0	156	0	156	0	156	0	156
143	13	143	13	143	13	143	13	143	13
26	130	26	130	26	130	26	130	26	130
117	39	117	39	117	39	117	39	117	39
104	52	104	52	104	52	104	52	104	52
0	156	0	156	0	156	0	156	0	156
143	13	143	13	143	13	143	13	143	13
26	130	26	130	26	130	26	130	26	130
117	39	117	39	117	39	117	39	117	39
104	52	104	52	104	52	104	52	104	52

Fig. 14.

13	158	11	160	5	169	2	167	4	161
144	25	146	23	152	14	155	16	153	22
39	132	37	134	31	143	28	141	30	135
118	51	120	49	126	40	129	42	127	48
117	54	115	56	109	65	106	63	108	57
1	168	3	166	9	157	12	159	10	165
156	15	154	17	148	26	145	24	147	18
27	142	29	140	35	131	38	133	36	139
130	41	128	43	122	52	119	50	121	44
105	64	107	62	113	53	116	55	114	61

Fig. 15.

The following table illustrates another rule covering the selection of numbers for all magic squares of these orders.

ORDER OF SQUARE	NATURAL SERIES	DISCARDING NUMBERS IN
6th	1 to $(6+1)^2$	the middle row and column.
10th	1 to $(10+3)^2$	the 3 middle rows and columns.
14th	1 to $(14+5)^2$	the 5 middle rows and columns.
18th	1 to $(18+7)^2$	the 7 middle rows and columns.
22nd	1 to $(22+9)^2$	the 9 middle rows and columns.
26th	1 to $(26+11)^2$	the 11 middle rows and columns.
		and so forth.

These figures show that this rule is equivalent to taking the numbers of the natural series $\left(\frac{3n-4}{2}\right)^2$ and omitting the central $\frac{n-4}{2}$ rows and columns. In comparing the above with the rules previously given, for which we are indebted to Dr. C. Planck, it will be seen that in cases of magics larger than 10^2 it involves the use of unnecessarily large numbers.

The numerical values of the ply properties of these squares are naturally governed by the dimensions of the magic rectangles used in their construction. Thus the rectangle of the 6^2 magic (Fig. 5) is 2×3, and this square is 2^2-ply and 3^2-ply. The rectangle of the 10^2 magic being 2×5, the square may be made 2^2-ply and 5^2-ply, and so forth.

The formation of these squares by the Path method which has been so ably developed by Dr. C. Planck,[1] may now be considered. The first step is to rearrange the numbers of the given series in such a cyclic order or sequence, that each number being written consecutively into the square by a well defined rule or path, the resulting magic will be identical with that made by the La Hireian method, or equivalent thereto in magic qualities. Starting, as before, with the 6^2 magic, the proper sequence of the first six numbers is found in what may be termed the "continuous diagonal" of its magic rectangle. Referring to Fig. 1, this sequence is seen to be 1, 2, 5, 7, 6, 3, but it is obvious that there may be as many different sequences as there are variations in the magic rectangles.

The complete series given on page 304 must now be rearranged

[1] "The Theory of Path Nasiks," by C. Planck, M.A., M.R.C.S., published by A. T. Lawrence, Rugby, England.

in its *lines and columns* in accordance with the numerical sequence of the first six numbers as above indicated. To make this arrangement quite clear, the series given on p. 304 is reproduced in Fig. 16, the numbers written in circles outside the square showing the numerical order of lines and columns under rearrangement. Fig. 17 shows the complete series in new cyclic order, and to construct a square directly therefrom, it is only necessary to write these numbers consecutively along the proper paths. Since the square will be pandiagonal *it may be commenced anywhere,* so in the present example we will place 1 in the fourth cell from the top in the first column, and will use the paths followed in Fig. 5 so as to reproduce that square. The paths may be written $\begin{vmatrix} 3, & 2 \\ 4, & 3 \end{vmatrix}$ and since we can always write

	(1)	(2)	(6)	(3)	(5)	(4)
(1)	1	2	3	5	6	7
(2)	8	9	10	12	13	14
(6)	15	16	17	19	20	21
(3)	29	30	31	33	34	35
(5)	36	37	38	40	41	42
(4)	43	44	45	47	48	49

Fig. 16.

	(1)	(2)	(3)	(4)	(5)	(6)
(1)	1	2	5	7	6	3
(2)	8	9	12	14	13	10
(3)	29	30	33	35	34	31
(4)	43	44	47	49	48	45
(5)	36	37	40	42	41	38
(6)	15	16	19	21	20	17

Fig. 17.

$-(n-a)$ instead of a, we may write this $\begin{vmatrix} 3, & 2 \\ -2, & 3 \end{vmatrix}$. This only means that the numbers in the first column of Fig. 17 (which may be termed the *leading numbers*) are to be placed in order along the path (3, 2), as in the numbers enclosed in circles in Fig. 5; and then starting from each cell thus occupied, the remaining five numbers in each of the six rows of Fig. 17 are to be written along the path (−2, 3). It will be seen that this is equivalent to writing the successive rows of Fig. 17 intact along the path (−2, 3), or (3, −2) and using a "break-step" (1, −1), as in Fig. 18 where the first break-step is shown with an arrow. The break-step is always given by summing up the coordinates; thus, the paths here being $\begin{vmatrix} 3, & 2 \\ -2, & 3 \end{vmatrix}$, by summing the columns we get (1, 5), that is (1, −1). The resulting square is, of course, identical with Fig. 5.

As previously stated, this square being pandiagonal, it may be commenced in any of its thirty-six cells, and by using the same methods as before, different aspects of Fig. 5 will be produced. Also, since by this method complementary pairs are always sepa-

7		3		2	
			8		
1		5		6	

Fig. 18.

A	B
C	D

Fig. 19.

rated by a step $(n/2, n/2)$, any of the thirty-six squares thus formed may be made associated by the method described in *The Monist,* Vol. XX, No. 3, page 443, under the heading "Magic Squares by

1	2	3	4	9	13	12	11	10	5
14	15	16	17	22	26	25	24	23	18
27	28	29	30	35	39	38	37	36	31
40	41	42	43	48	52	51	50	49	44
105	106	107	108	113	117	116	115	114	109
157	158	159	160	165	169	168	167	166	161
144	145	146	147	152	156	155	154	153	148
131	132	133	134	139	143	142	141	140	135
118	119	120	121	126	130	129	128	127	122
53	54	55	56	61	65	64	63	62	57

Fig. 20.

Complementary Differences,"[2] viz., Divide the square into four quarters as shown in Fig. 19; leave A untouched, reflect B, invert C and

[2] Errata in this article: p. 440, footnote, and p. 443, fourth line from top of page, instead of "for all orders $=4n+2$" read "for orders wherein n is of the general form $4p+2$."—Page 44, last line, for "order $8n$" read "this class."

reflect and invert D. For this concise and elegant method of changing the relative positions of the complementary couplets in a square we are indebted to Dr. Planck.

The next square in order is 10^2. The series of numbers used is given on page 305 and their rearrangement in proper cyclic order

13	160	2	161	11	169	4	158	5	157
27	140	38	139	29	131	36	142	35	133
117	56	106	57	115	65	108	54	109	63
144	23	155	22	146	14	153	25	152	16
130	43	119	44	128	52	121	41	122	50
1	166	12	165	3	157	10	168	9	159
39	134	28	135	37	143	30	132	31	141
105	62	116	61	107	53	114	64	113	55
156	17	145	18	154	26	147	15	148	24
118	49	129	48	120	40	127	51	126	42

Fig. 21.

for direct entry may be found as before in the continuous diagonal of its magic rectangle. The sequence shown in Fig. 11 is 1, 2, 3, 4, 9, 13, 12, 11, 10, 5, and the complete rearrangement of the series in accordance therewith is given in Fig. 20. Various 10^2 magics may be made by using this series with different paths. The paths $\begin{vmatrix} 5, 4 \\ -4, 5 \end{vmatrix}$

21	2	3	4	17	16	15	8	13
1	20	19	18	5	6	7	14	9

Fig. 22.

will produce Fig. 15, and $\begin{vmatrix} 5, 2 \\ 2, 5 \end{vmatrix}$ will make Fig. 21, which is equivalent to Fig. 15 in its ornate features.

These squares and all similarly constructed larger ones of these orders may be changed to the form of association wherein the complementary couplets are evenly balanced around the center of the square, by the method previously explained. It will be unnecessary to prolong the present article by giving any examples of larger squares of this class, but the simple forms of magic rectangles for

23	2	21	4	19	6	17	8	9	10	13
1	22	3	20	5	18	7	16	15	14	11

Fig. 23.

29	2	27	4	25	6	23	8	9	20	11	18	13
1	28	3	26	5	24	7	22	21	10	19	12	17

Fig. 24.

18^2 and 22^2 and 26^2 magics, shown in Figs. 22, 23 and 24, may be of some assistance to those who desire to devote further study to these interesting squares.[3]

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L. S. Frierson.

A NEW THEORY OF INVENTION.

A Russian engineer, P. K. von Engelmeyer of Moscow (Petersburger Chaussée 42), has published a little book on invention and its significance in our industrial life under the title *Der Dreiakt* (Berlin, Carl Heymann's Verlag) in which he claims that man is not only a political being (ζῷον πολιτικόν) as Aristotle claims, but also and mainly a technical being (ζῷον τεχνικόν), and he means it in the same sense in which Franklin called man a "tool-making animal."[1]

Mr. Engelmeyer defines technique as the art of reproducing artificially or intentionally certain desired phenomena (p. 17) and he calls attention to the fact that we are surrounded by the products of invention. Our clothes, the light and heat in our houses, our mode of traveling, in short, all that is called culture and civilization has

[3] More generally, if p, q are relative primes, the square of order pq will be magic on its pq rows, pq columns and $2pq$ diagonals, and at the same time p^2-ply and q-^{2}ply, if it be constructed with the paths $\begin{vmatrix} p, q \\ q, p \end{vmatrix}$, and the period be taken from the continuous diagonal of the magic rectangle $p \times q$. The limitations are dictated by the magic rectangle. Evidently p and q must both be > 1, and consecutive numbers must fail if the order is $\equiv 2 \pmod{4}$; in all other cases consecutive numbers will suffice. C. P.

[1] See the author's *The Philosophy of the Tool*, p. 1.

been invented at various times. Some inventions have been made by conscious endeavor, others by accident.

Our author distinguishes four characteristics of invention: (1) its artificial nature—man interferes with natural conditions and introduces a human element into them: (2) teleology—inventions must be designed, they must serve a purpose; (3) surprise, by which word our author means that they must be something new or original; we do not call invention what is merely an application of former experience; (4) unity—every invention is a kind of a system, an organic whole, and the members must be integral parts of a new entirety. Discovery is somewhat different from invention, but there is a domain which belongs practically to both invention and discovery. Newton's law of gravitation is a discovery, but mathematical formulas are both.

Mr. Engelmeyer quotes Goethe approvingly when he says: "man does not experience or enjoy without at the same time being productive," thus implying that invention is an indispensable element in human existence. There are three fields of human activity. When man devotes his efforts to purposes of utility, the result is called invention; when his efforts are devoted to cognition, the result is called discovery; when this result serves esthetical pleasure it is called a work of art. Just as all three domains are ultimately one, so there must be but one theory of invention which our author calls by the Greek name "Heurology," and in so far as it expresses this union he calls it an act of three, or in German *Dreiakt.*

This theory of the *Dreiakt* is the subject of the main part of the book, and the author has consulted the patent laws of different nations for details and illustrations. From the standpoint of his conception he distinguishes between the product and the method of an invention; the former is the effect accomplished, the latter is the arrangement of parts, the combination of substances in definite proportions, the way in which substances are treated to change their nature. The patent lawyer must consider the principle which comprises the effect together with the way in which it is produced. Examples are furnished by the sewing machine, the bicycle, hydraulic systems, aeronautics, fire arms, chemical inventions, cement, explosives, photography, Bessemer steel, etc.

The concluding chapter of the book is devoted to the application of the *Dreiakt* to patent laws and technical instruction. The universality of the principle of the *Dreiakt* finds appreciation in the proposition that the human will itself is a *Dreiakt.* Our auther gives credit to O. Schanze who has published his views on the same subject under the title *Beiträge zur Lehre von der Patentfähigkeit*, fascicle 2, pages 243-255 (Berlin, Siemens, 1904). He uses the term *Dreiakt* in a slightly different sense and speaks of three fundamental energies: (1) intention or will, (2) reflection or knowledge, (3) practical skill. These characterize every act of creation as a *Dreiakt*, (1) the aim which constitutes the teleology of the work, (2) the plan or design which logically determinates the work and (3) its execution. Schanze applies them to practical problems, especially to these three: *a,* Who among several collaborators is the author of the invention and who merely an assistant; *b,* how far in its application is an invention entitled to protection by patent; and *c,* at what state of completion does an invention acquire the right to be patented.

P. C.

BOOK REVIEWS AND NOTES.

THE PERIPLUS OF THE ERYTHRAEAN SEA. By *Wilfred H. Schoff*. New York: Longmans, Green and Company, 1912. Pp. 323. Price $2.00 net.

"Periplus" means circumnavigation and may be freely translated "log book" or "description of a sea voyage." There are several antique books which bear the same title, and the present work refers to that body of water which in modern times is known as the Indian Ocean, together with the Red Sea and the Persian Gulf. This record describes the voyage from place to place of an ancient merchant vessel and is of great interest in the history of trading. The book itself is not long. It contains only 28 pages of English text, but the translation has been made with great care. Very full notes explain the terms used, the merchandise traded and the historical connections, and these cover pages 50 and 282; tables are appended listing articles of trade and rulers mentioned and dates variously assigned to the original, a map indicates the ports touched at and helps the readers to understand the geography of our travelers. The book is furnished with a very thorough topical index covering thirty two-columned pages. The work is creditable to the spirit of the Commercial Museum of Philadelphia, which has brought it out. W. P. Wilson, the director of the Philadelphia museums, says in his foreword:

"The Periplus of the Erythræan Sea is the first record of organized trading with the nations of the East, in vessels built and commanded by subjects of the Western world. The notes add great interest, giving as they do an exhaustive survey of the international trade between the great empires of Rome, Parthia, India and China, together with a collection of facts touching the early trade of a number of other countries of much interest." *κ*

THE INDIVIDUAL AND REALITY. By *Edward Douglas Fawcett*. New York: Longmans, Green & Co., 1909. Pp. 449.

The author writes as one having authority. He considers himself a free lance, since he is independent of any school of philosophy or religion, and therefore "free to ignore all traditions and conventions and go straight to reality in the search for truth." The present volume is intended to supersede a former one to which he refers as "my *Riddle*." This former work was read with enormous satisfaction by the late Prof. William James, and the fact that this same thinker considers his book "as a great and powerful agency in the spreading of truth" is regarded by the author as sufficient justification for its appearance. Mr. Fawcett credits the source of his thinking to the anti-

Hegelian thought of Schelling and Schopenhauer, being allied to the former's "immemorial being" and Bain's doctrine of relativity. He has not read Bergson with whom his results in part seem to agree. In this later work he has abandoned the monadology of his former production, replacing it with a new form of idealism. Some of the novelties of the former work are here retained. Part I is an introduction to metaphysics; Part II treats of the individual and his universe, appearances, and the individual in his relation to the organism, nature as a whole, and to himself. Part III deals with ultimate questions, such as the ground of appearance, the evolution of nature and individuals, birth, death, destiny and God. *ρ*

Protestant Thought before Kant. By *Arthur Cushman M'Giffert.* New York: Charles Scribner's Sons, 1911. Pp. 261. Price, 75 cents net.

In this volume dedicated to Adolf Harnack, the author's teacher and friend, Professor M'Giffert, of Union Theological Seminary of New York, traces the development of Protestantism from the time of its early workings in medieval Christianity to that of the great Königsberg philosopher. This includes a discussion of the leaders of the Reformation in all countries, Zwingli, Melanchthon, Calvin; also the radical anabaptists and socianians. The Protestant phase of scholasticism is discussed, and pietism in Germany, England and New England. The book closes with a chapter on rationalism in its various phases as found in England, France, Germany and America. Professor M'Giffert's book is the sixth volume in a series entitled "Studies in Theology" which are intended as aids to interpretation in biblical criticism, primarily for the use of ministers and theological students, but still the needs of the general reader are kept in view so that the works shall not become too technical. *ρ*

Ueber Klassiker und Philosophen der Neuzeit. By *Julius Rupp.* Leipsic: Eckardt, 1910. Pp. 796 . Price 6 m.

This is the third volume of Rupp's collected works which are to appear in twelve volumes, and have been edited by P. C. Elsenhans. Each volume has a separate introduction by the editor. The present one on the classicists and modern philosophers contains three different collections of Rupp's essays. The first set, from Lessing to Hegel, are on subjects relating to Lessing, Kant, Herder, Spinoza, Schiller, Fichte and Schleiermacher. The second collection, bearing the general title "Contemporary Philosophy," discusses subjects relating to natural science, controversies about the soul, macrocosm, and microcosm, the relations of soul to body, philosophy, theology, religion of the spirit; and devotes special chapters to Emerson, Gioberti and Alexander Bain. This volume contains also Rupp's "Sketches of a Thinker." *ρ*

The Phenomenology of Mind. By *G. W. F. Hegel.* Tr. by *J. B. Baillie.* 2 vols. London: Sonnenschein, 1910. Pp. 823. Price 21 s. net.

This translation of Hegel's *Phenomenology* is one number of Sonnenschein's "Library of Philosophy" edited by Dr. J. H. Muirhead. The object of the series is to familiarize English readers with results of modern philosophical thought, admitting that in this respect Germany has far excelled England. The editor's purpose, however, besides bringing German philosophy to English thinkers, is to furnish a systematized philosophical library in which English philosophy will receive the consideration due it, as its significance has been largely ignored by the German schools. *p*

DAVID HUME, HANS LIV OG HANS FILOSOFI. Af *Anton Thomsen.* Copenhagen: Nordiske Forfatteres Forlag, 1911. Pp. 458. Price 1.65 kr.

Professor Anton Thomsen of the University of Copenhagen is preparing an extremely comprehensive work on the subject of this great English philosopher. The first volume appeared during last winter, and after a few introductory pages calling attention to the bicentenary of David Hume's birth, takes up in its first book the philosopher's life and works, and in the second book his epistemology and psychology. The philosophical critique is made with due reference to the contemporary thinkers of all lands in connection with the philosophy of past centuries. *p*

L'EVOLUTION DE LA MÉMOIRE. Par *Henri Piéron.* Paris: Flammarion, 1910. Pp. 360. Price 3.50 fr.

This book treats of the extent of the domain of the memory and the relations of the phenomena of inorganic memory to those of psychic memory; of the forms which memory assumes in all the steps of the evolution of animals and the continuity of the series when passing from brute creation to man; of the aspects and limitations of human memory, the cause of its difficulties and its probable future. The discussion of these points is based on the collection of facts actually established by objective psychology, human and comparative.

The conclusion drawn is both pessimistic and optimistic: pessimistic, because it sees no chance for the memory of men regarded individually to increase in capacity, and because the utilization of the traces left by collective memory (i. e., presented by published material) seems likely to become more and more difficult; optimistic, in that the conservation of many recollections will become less and less necessary in the progress of scientific classification which will make possible the substitution of the knowledge of a small number of general laws for that of a large number of particular facts. *p*

DAS KÜNFTIGE JAHRHUNDERT DER PSYCHOLOGIE. Von *G. Heymans.* Aus dem Niederländischen übersetzt von *H. Pol.* Leipsic: Barth, 1911. Pp. 52. Price 1 m. 20.

Prof. G. Heymans, retiring rector of the Groningen University, has published his oration in the translation of Mr. H. Pol, the German teacher of the same university. It bears the title "The Future Century of Psychology," and insists that while progress is rapid in other branches the development of psychology ought not to be neglected because it is more important than our progress in inventions. The main subjects of psychology refer to the nature of our own self, our relation to others and toward the ultimate foundation of the world. He finds that much is to be done and much has been neglected in former ages. In opposition to the common view that competition and war are necessary, that mankind is bad at the core, he quotes Frederick the Great as having said of a prominent educator: "Ah, mon cher Sulzer, vous ne connaissez pas assez cette maudite race, à laquelle nous appartenons!" In opposition to the current view he expects that the future will more and more restrict competition and war, and criticizes the idea that they are necessary for the amelioration of the race; that if the principle were reasonable cattle breeders and hunters might just as well introduce it into the artificial methods of producing higher and better breeds, but what would we think of a hunter who would make his hounds quarrel about a piece of meat in a fierce fight in which half of them would lose their lives, and this simply for the amelioration of the race? He ends his oration by quoting the words of a mystic thinker, "I trust that all will yet be good." κ

In Hamburg, the place of the first monistic congress, a free religious society has been founded which proposes to do a propaganda for a rational world conception. Their aims are through religious devotion to cultivate the true, the good and the beautiful. In politics they favor separation of church from state and of school from church. Their secretary is Bruno Heyer, and their treasurer Adolf Dunkel. κ

The well-known activity of the Leipsic publishing house of Dürr is seen by constant additional contributions to its *Philosophische Bibliothek,* and the value of its productions is attested by the height to which the number of its editions reaches. Among its 1910 publications, besides the seventh edition of Baensch's translation of Spinoza's Ethics, we have an edition by Johannes Schubert of Wilhelm von Humboldt's selected philosophical writings and the Definitions of Christian Wolff collected by Julius Baumann for the purpose of serving as collateral reading in the study of Kant. The centennial of the Berlin University has been celebrated by this enterprising house by a volume introduced by Edouard Spranger and containing the addresses of Fichte, Schleiermacher and Steffins on "The Nature of the University," written or delivered at the time of its opening. A second edition of Dr. Otto Apelt's German translation of Plato's *Theaetetus* bears the date of 1911, and purports to be an entirely new translation of the dialogue. (*Baruch de Spinoza, Ethik,* übers. von Otto Baensch; *Fichte, Schleiermacher, Steffens über das Wesen der Universität,* her. von Edouard Spranger; *Wilhelm von Humboldts ausgewählte philosophische Schriften,* her. von. Johannes Schubert; *Wolffsche Begriffsbestimmungen,* her. von Julius Baumann; *Platons Dialog Theätet,* übers. von Dr. Otto Apelt.) ρ

The scientific publishing house of A. Hermann and Son, at Paris, are publishing a French translation of the sixth German edition of Prof. W. Nernst's large work on theoretical chemistry. It is translated by A. Corvisy, under the title *Traité de chimie générale.* The first part is issued this year, dealing with the general properties of bodies and atoms and molecules. ρ

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THE MONIST

THE PHILOSOPHY OF BERGSON.[1]

I.

THE classification of philosophies is effected, as a rule, either by their methods or by their results: "empirical" and "*a priori*" is a classification by methods, "realist" and "idealist" is a classification by results. An attempt to classify Bergson's philosophy in either of these ways is hardly likely to be successful, since it cuts across all the recognized divisions.

But there is another way of classifying philosophies, less precise, but perhaps more helpful to the non-philosophical; in this way, the principle of division is according to the predominant desire which has led the philosopher to philosophize. Thus we shall have philosophies of feeling, inspired by the love of happiness; theoretical philosophies, inspired by the love of knowledge; and practical philosophies, inspired by the love of action.

Among philosophies of feeling we shall place all those which are primarily optimistic or pessimistic, all those that offer schemes of salvation or try to prove that salvation is impossible; to this class belong most religious philosophies. Among theoretical philosophies we shall place most of the great systems; for though the desire for knowledge is rare, it has been the source of most of what is best in philosophy. Practical philosophies, on the other hand, will be those

[1] The abbreviations of the titles of the works of M. Bergson referred to are: *C. E., Creative Evolution; M. and M., Matter and Memory; Tand F. W., Time and Free Will.* The references are to the English translations of M. Bergson's books.

which regard action as the supreme good, considering happiness an effect and knowledge a mere instrument of successful activity. Philosophies of this type would have been common among Western Europeans if philosophers had been average men; as it is, they have been rare until recent times, in fact their chief representatives are the pragmatists and M. Bergson. In the rise of this type of philosophy we may see, as M. Bergson himself does, the revolt of the modern man of action against the authority of Greece, and more particularly of Plato; or we may connect it, as Dr. Schiller apparently would, with imperialism and the motorcar. The modern world calls for such a philosophy, and the success which it has achieved is therefore not surprising.

M. Bergson's philosophy, unlike most of the systems of the past, is dualistic: the world, for him, is divided into two disparate portions, on the one hand life, on the other matter, or rather that inert something which the intellect views as matter. The whole universe is the clash and conflict of two opposite motions: life, which climbs upward, and matter, which falls downward. Life is one great force, one vast vital impulse, given once for all from the beginning of the world, meeting the resistance of matter, struggling to break a way through matter, learning gradually to use matter by means of organization; divided by the obstacles it encounters into diverging currents, like the wind at the street-corner; partly subdued by matter through the very adaptations which matter forces upon it; yet retaining always its capacity for free activity, struggling always to find new outlets, seeking always for greater liberty of movement amid the opposing walls of matter.

Evolution is not primarily explicable by adaptation to environment; adaptation explains only the turns and twists of evolution, like the windings of a road approaching a town through hilly country. But this simile is not quite adequate; there is no town, no definite goal, at the end of the road along which evolution travels. Mechanism and teleology suffer from the same defect: both suppose that there is no essential novelty in the world. Mechanism regards the future as implicit in the past, since it believes the future to be calculable; teleology also, since it believes that the end to be achieved can be known in advance, denies that any essential novelty is contained in the result.

As against both these views, though with more sympathy for teleology than for mechanism, M. Bergson maintains that evolution is truly *creative*, like the work of an artist. An impulse to action, an undefined want, exists beforehand, but until the want is satisfied it is impossible to know the nature of what will satisfy it. For example, we may suppose some vague desire in sightless animals to be able to be aware of objects before they were in contact with them. This led to efforts which finally resulted in the creation of eyes. Sight satisfied the desire, but could not have been imagined beforehand. For this reason, evolution is unpredictable, and determinism cannot refute the advocates of free will.

This broad outline is filled in by an account of the actual development of life on the earth. The first division of the current was into plants and animals: plants aimed at storing up energy in a reservoir, animals aimed at using energy for sudden and rapid movements. "The same impetus," he says, "that has led the animal to give itself nerves and nerve centers must have ended, in the plant, in the chlorophyllian function" (*C. E.*, p. 120). But among animals, at a later stage, a new bifurcation appeared: *instinct* and *intellect* became more or less separated. They are never wholly without each other, but in the main intellect is the misfortune of man, while instinct is seen at its best in ants, bees, and Bergson. The division between intellect and instinct is fundamental in his philosophy, much of which is a kind of Sandford and Merton, with instinct as the good boy and intellect as the bad boy.

Instinct at its best is called *intuition*. "By *intuition*," he says, "I mean instinct that has become disinterested, selfconscious, capable of reflecting upon its object and of enlarging it indefinitely" (*C. E.*, p. 186). The account of the doings of intellect is not always easy to follow, but if we are to understand Bergson we must do our best.

Intelligence or intellect, "as it leaves the hands of nature, has for its chief object the inorganic solid" (*C. E.*, p. 162); it can only form a clear idea of the discontinuous and the immobile (pp. 163-4); its concepts are outside each other like objects in space, and have the same stability (p. 169). The intellect separates in space and fixes in time; it is not made to think evolution, but represent *becoming* as a series of states (p. 171). "The intellect is characterized by a natural inability to understand life" (p. 174); geometry and logic, which are its typical products, are strictly applicable to solid bodies, but elsewhere reasoning must be checked by common sense, which, as Bergson truly says, is a very different thing (p. 170). Solid bodies, it would seem, are something which mind has created on purpose to apply intellect to them, much as it has created chess-boards in order to play chess on them. The genesis of intellect and the genesis of material bodies, we are told, are correlative: both have been developed by reciprocal adaptation (p. 196). "An identical process must have cut out matter and the intellect, at the same time, from a stuff that contained both" (p. 210).

This conception of the simultaneous growth of matter and intellect is ingenious, and deserves to be understood. Broadly, I think, what is meant is this: Intellect is the power of seeing things as separate one from another, and matter is that which is separated into distinct things. In reality there are no separate solid things, only an endless stream of becoming, in which nothing becomes and there is nothing that this nothing becomes. But becoming may be a movement up or a movement down: when it is a movement up it is called life, when it is a movement down it is what, as misapprehended by the intellect, is called matter. I suppose the universe is shaped like a cone, with the Absolute at the vertex, for the movement up brings things together, while the movement down separates them, or at least seems to do so. In order that the upward motion of mind may be able to thread its way through the downward motion of the falling bodies which hail upon it, it must be able to cut out paths between them; thus as intelligence was formed, outlines and paths appeared (p. 199), and the primitive flux was cut up into separate bodies. The intellect may be compared to a carver, but it has the peculiarity of imagining that the chicken always was the separate pieces into which the carving-knife divides it.

"The intellect," Bergson says, "always behaves as if it were fascinated by the contemplation of inert matter. It is life looking outward, putting itself outside itself, adopting the ways of inorganized nature in principle, in order to direct them in fact" (p. 170). If we may be allowed to add another image to the many by which Bergson's philosophy is illustrated, we may say that the universe is a vast funicular railway, in which life is the train that goes up, and matter is the train that goes down. The intellect consists in watching the descending train as it passes the ascending train in which we are. The obviously nobler faculty which concentrates its attention on our own train, is instinct or intuition. It is possible to leap from one train to the other; this happens when we become the victims of automatic habit, and is the essence of the comic. Or we can divide ourselves into parts, one part going up and one down; then only the part going down is comic. But intellect is not itself a descending motion, it is merely an observation of the descending motion by the ascending motion.

Intellect, which separates things, is, according to Bergson, a kind of dream; it is not *active,* as all our life ought to be, but purely contemplative. When we dream, he says, our self is scattered, our past is broken into fragments (p. 212),[2] things which really interpenetrate each other are seen as separate solid units: the extra-spatial degrades itself into spatiality (p. 218), which is nothing but separateness. Thus all intellect, since it separates, tends to geometry, and logic, which deals with concepts that lie wholly outside each other, is really an outcome of geometry, following the direction of materiality (pages 222-4). Both deduction and induction require spatial intuition behind them (p. 225); "the movement at the end of which is spatiality lays down along its course the faculty of induction, as well as that of deduction, in fact, intellectuality entire." It creates them in mind, and also the order in things which the intellect finds there (p. 228). Thus logic and mathematics do not represent a positive spiritual effort (p. 224), but a mere somnambulism, in which the will is suspended, and the mind is no longer active. Incapacity for mathematics is therefore a sign of grace—fortunately a very common one.

As intellect is connected with space, so instinct or intuition is connected with time. It is one of the noteworthy features of Bergson's philosophy that, unlike most writers, he regards time and space as profoundly dissimilar. Space, the characteristic of matter, arises from a dissection of the flux which is really illusory, useful, up to a certain point, in practice, but utterly misleading in theory. Time, on the contrary, is the essential characteristic of life or mind. "Wherever anything lives," he says, "there is, open some-

[2] It is noteworthy that elsewhere Bergson speaks of dreams as giving us duration more pure than in waking life (*T. and F. W.*, p. 126).

where, a register in which time is being inscribed" (*C. E.*, p. 17). But the time here spoken of is not mathematical time, the homogeneous assemblage of mutually external instants. Mathematical time, according to Bergson, is really a form of space; the time which is of the essence of life is what he calls *duration*. This conception of duration is fundamental in his philosophy; it appears already in his earliest book *Time and Free Will*, and it is necessary to understand it if we are to have any comprehension of his system. It is, however, a very difficult conception. I do not fully understand it myself, and therefore I cannot hope to explain it with all the lucidity which it doubtless deserves.

"Pure duration," we are told, "is the form which our conscious states assume when our ego lets itself *live*, when it refrains from separating its present state from its former states" (*T. and F. W.*, p. 100). It forms the past and the present into one organic whole, where there is mutual penetration, succession without distinction (*ib.*). "Within our ego, there is succession without mutual externality; outside the ego, in pure space, there is mutual externality without succession" (p. 108).

"Questions relating to subject and object, to their distinction and their union, should be put in terms of time rather than of space" (*M. and M.*, p. 77). In the duration in which we *see ourselves acting*, there are dissociated elements; but in the duration in which we *act*, our states melt into each other (*M. and M.*, p. 243). Pure duration is what is most removed from externality and least penetrated with externality, a duration in which the past is big with a present absolutely new. But then our will is strained to the utmost; we have to gather up the past which is slipping away, and thrust it whole and undivided into the present. At such moments we truly possess ourselves, but such moments are rare (*C. E.*, pp. 210-211). Duration is the very stuff of reality, which is perpetual becoming, never something made (*C. E.*, p. 287).

It is above all in *memory* that duration exhibits itself, for in memory the past survives in the present. Thus the theory of memory becomes of great importance in Bergson's philosophy. *Matter and Memory* is concerned to show the relation of mind and matter, of which both are affirmed to be real (p. vii), by an analysis of memory, which is "just the intersection of mind and matter" (p.xii).

There are, to begin with, two radically different things, both of which are commonly called *memory*; the clear distinction between these two is one of the best things in Bergson. "The past survives," he says, "under two distinct forms: first, in motor mechanisms; secondly, in independent recollections" (*M. and M.*, p. 87). For example, a man is said to remember a poem if he can repeat it by heart, that is to say, if he has acquired a certain habit or mechanism enabling him to repeat a former action. But he might, at least theoretically, be able to repeat the poem without any recollection of the previous occasions on which he has read it; thus there is no consciousness of past events involved in this sort of memory. The second sort, which alone really deserves to be called memory, is exhibited in recollections of separate occasions when he has read the poem, each unique and with a date. Here there can be no question of *habit*, since each event only occurred once, and had to make its impression immediately. It is suggested that in some way everything that has happened to us is remembered, but as a rule, only what is useful comes into consciousness. Apparent failures of memory, it is argued, are not really failures of the mental part of memory, but of the motor mechanism for bringing memory into action. This view is supported by a discussion of brain physiology and the facts of amnesia, from which it is held to result that true memory is not a function of the brain (*M. and M.*,

p. 315). The past must be *acted* by matter, *imagined* by mind (*M. and M.,* p. 298). Memory is not an emanation of matter; indeed the contrary would be nearer the truth if we mean matter as grasped in concrete perception, which always occupies a certain duration (*M. and M.*, p. 237).

"Memory must be, in principle, a power absolutely independent of matter. If, then, spirit is a reality, it is here, in the phenomena of memory, that we may come into touch with it experimentally" (*M. and M.*, p. 81).

At the opposite end from pure memory Bergson places pure perception, in regard to which he adopts an ultra-realist position. "In pure perception," he says, "we are actually placed outside ourselves, we touch the reality of the object in an immediate intuition" (p. 84). So completely does he identify perception with its object that he almost refuses to call it mental at all. "Pure perception," he says, "which is the lowest degree of mind—mind without memory—is really part of matter, as we understand matter" (*M. and M.,* p. 297). Pure perception is constituted by dawning action, its actuality lies in its activity (*M. and M.,* p. 74). It is in this way that the *brain* becomes relevant to perception, for the brain is not an instrument of representation, but an instrument of action (*M. anl M.,* p. 83). The function of the brain is to limit our mental life to what is practically useful. But for the brain, one gathers, everything would be perceived, but in fact we only perceive what interests us (cf. *M. and M.*, p. 34). "The body, always turned towards action, has for its essential function to limit, with a view to action, the life of the spirit" (*M. and M.*, p. 233). It is, in fact, an instrument of choice.

We must now return to the subject of instinct or intuition, as opposed to intellect. It was necessary first to give some account of duration and memory, since Bergson's theories of duration and memory are presupposed in his account of intuition. In man, as he now exists, intuition is the fringe or penumbra of intellect: it has been thrust out of the center by being less useful in action than intellect, but it has deeper uses which make it desirable to bring it back into greater prominence. Bergson wishes to make intellect "turn inwards on itself, and awaken the potentialities of intuition which still slumber within it" (*C. E.*, p. 192). The relation between instinct and intellect is compared to that between sight and touch. Intellect, we are told, will not give knowledge of things at a distance; indeed the function of science is said to be to explain all perceptions in terms of touch.

"Instinct alone, he says, "is knowledge at a distance. It has the same relation to intelligence that vision has to touch" (*C. E.*, p. 177). We may observe in passing that, as appears in many passages, Bergson is a strong visualizer, whose thought is always conducted by means of visual images. Many things which he declares to be necessities of all thought are, I believe, characteristic of visualizers, and would not be true of those who think by means of auditory images. He always exalts the sense of sight at the expense of the other senses, and his views on space would seem to be largely determined by this fact. I shall return to this question at a later stage.

The essential characteristic of intuition is that it does not divide the world into separate things, as the intellect does; although Bergson does not use these words, we might describe it as synthetic rather than analytic. It apprehends a multiplicity, but a multiplicity of interpenetrating processes, not of spatially external bodies. There are in truth no *things*: "things and states are only views, taken by our mind, of becoming. There are no things, there are only actions" (*C. E.*, p. 261). This view of the world, which appears difficult and unnatural to intellect, is easy and natural to intuition. Memory affords an instance of what is meant, for in memory the past lives on into the present and interpenetrates it. Apart from mind, the world would be perpetually dying and being born again; the past would have no reality, and therefore there would be no past. It is memory, with its correlative desire, that makes the past and the future real and therefore creates true duration and true time. Intuition alone can understand this mingling of past and future: to the intellect they remain external, spatially external as it were, to one another. Under the guidance of intuition, we perceive that "form is only a snapshot view of a transition" (*C. E.*, p. 319), and the philosopher "will see the material world melt back into a single flux" (*C. E.*, p. 390).

Closely connected with the merits of intuition is Bergson's doctrine of freedom and his praise of action. "In reality," he says, "a living being is a center of action. It represents a certain sum of contingency entering into the world, that is to say, a certain quantity of possible action" (*C. E.*, p. 276). The arguments against free will depend partly upon assuming that the intensity of psychical states is a *quantity,* capable, at least in theory, of numerical measurement; this view Bergson undertakes to refute in the first chapter of *Time and Free Will.* Partly the determinist depends, we are told, upon a confusion between true duration and mathematical time, which Bergson regards as really a form of space. Partly, again, the determinist rests his case upon the unwarranted assumption that, when the state of the brain is given, the state of the mind is theoretically determinate. Bergson is willing to admit that the converse is true, that is to say, that the state of brain is determinate when the state of mind is given, but he regards the mind as more differentiated than the brain, and therefore holds that many different states of mind may correspond to one state of brain. He concludes that real freedom is possible: "We are free when our acts spring from our whole personality, when they express it, when they have that indefinable resemblance to it which one sometimes finds between the artist and his work" (*T. and F. W.*, p. 172).

In the above outline, I have in the main endeavored merely to state Bergson's views, without giving the reasons adduced by him in favor of their truth. This is easier than it would be with most philosophers, since as a rule he does not give reasons for his opinions, but relies on their inherent attractiveness, and on the charm of an excellent style. Like the advertisers of Oxo, he relies upon picturesque and varied statement, and an apparent explanation of many obscure facts. Analogies and similes, especially form a very large part of the whole process by which he recommends his views to the reader. The number of similes for life to be found in his works exceeds the number in any poet known to me. Life, he says, is like a shell bursting into fragments which are again shells (*C. E.*, p. 103). It is like a sheaf (*ib.*, p. 104). Initially, it was "a tendency to accumulate in a reservoir, as do especially the green parts of vegetables" (*ib.*, p. 260). But the reservoir is to be filled with boiling water from which steam is issuing; "jets must be gushing out unceasingly, of which each, falling back, is a world" (*ib.*, p. 261). Again "life appears in its entirety as an immense wave which, starting from a center, spreads outwards, and which on almost the whole of its circumference is stopped and converted into oscillation: at one single point the obstacle has been forced, the impulsion has passed freely" (*ib.*, p. 280). Then there is the great climax in which life is compared to a cavalry charge. "All organized beings, from the humblest to the highest, from the first origins of life to the time in which we are, and in all places as in all times, do but evidence a single impulsion, the inverse of the movement of matter, and in itself indivisible. All the living hold together, and all yield to the same tremendous push. The animal takes its stand on the plant, man bestrides animality, and the whole of humanity, in space and in time, is one immense army galloping beside and before and behind each of us in an overwhelming charge able to beat down every resistance and to clear many obstacles, perhaps even death" (*C. E.*, pp. 285-6).

But a cool critic, who feels himself a mere spectator, perhaps an unsympathetic spectator, of the charge in which man is mounted upon animality, may be inclined to think that calm and careful thought is hardly compatible with this form of exercise. When he is told that thought is a mere means of action, the mere impulse to avoid obstacles in the field, he may feel that such a view is becoming in a cavalry officer, but not in a philosopher, whose business, after all, is with thought: he may feel that in the passion and noise of violent motion there is no room for the fainter music of reason, no leisure for the disinterested contemplation in which greatness is sought, not by turbulence, but by the greatness of the universe which is mirrored. In that case, he may be tempted to ask whether there are any reasons for accepting such a restless view of the world. And if he asks this question, he will find, if I am not mistaken, that there is no reason whatever for accepting this view, either in the universe or in the writings of M. Bergson.

II.

The two foundations of Bergson's philosophy, in so far as it is more than an imaginative and poetic view of the world, are his doctrines of space and time. His doctrine of space is required for his condemnation of the intellect, and if he fails in his condemnation of the intellect, the intellect will succeed in its condemnation of him, for between the two it is war to the knife. His doctrine of time is necessary for his vindication of freedom, for his escape from what William James called a "block universe," for his doctrine of a perpetual flux in which there is nothing that flows, and for his whole account of the relations between mind and matter. It will be well, therefore, in criticism, to concentrate on these two doctrines. If they are true, such minor errors and inconsistencies as no philosopher escapes would not greatly matter, while if they are false, nothing remains except an imaginative epic, to be judged on esthetic rather than on intellectual grounds. I shall begin with the theory of space, as being the simpler of the two.

Bergson's theory of space occurs fully and explicitly in his *Time and Free Will,* and therefore belongs to the oldest parts of his philosophy. In his first chapter, he contends that *greater* and *less* imply space, since he regards the greater as essentially that which *contains* the less. He offers no arguments whatever, either good or bad, in favor of this view; he merely exclaims, as though he were giving an obvious *reductio ad absurdum*: "As if one could still speak of magnitude where there is neither multiplicity nor space!" (p. 9). The obvious cases to the contrary, such as pleasure and pain, afford him much difficulty, yet he never doubts or re-examines the dogma with which he starts.

In his next chapter, he maintains the same thesis as regards number. "As soon as we wish to picture *number* to ourselves," he says, "and not merely figures or words, we are compelled to have recourse to an extended image" (p. 78), and "every clear idea of number implies a visual image in space" (p. 79). These two sentences suffice to show, as I shall try to prove, that Bergson does not know what number is, and has himself no clear idea of it. This is shown also by his definition: "Number may be defined in general as a collection of units, or, speaking more exactly, as the synthesis of the one and the many" (p. 75).

In discussing these statements, I must ask the reader's patience for a moment while I call attention to some distinctions which may at first appear pedantic, but are really vital. There are three entirely different things which are confused by Bergson in the above statements, namely: (1) number, the general concept applicable to the various particular numbers; (2) the various particular numbers; (3) the various collections to which the various particular numbers are applicable. It is this last that is defined by Bergson when he says that number is a collection of units. The twelve apostles, the twelve tribes of Israel, the twelve months, the twelve signs of the zodiac, are all collections of units, yet no one of them is the number 12, still less is it number in general, as by the above definition it ought to be. The number 12, obviously, is something which all these collections have in common, but which they do not have in common with other collections, such as cricket elevens. Hence the number 12 is neither a collection of twelve terms, nor is it something which all collections have in common; and number in general is a property of 12 or 11 or any other number, but not of the various collections that have twelve terms or eleven terms.

Hence when, following Bergson's advice, we "have recourse to an extended image" and picture, say, twelve dots such as are obtained by throwing double sixes at dice, we have still not obtained a picture of the number 12. The number 12, in fact, is something more abstract than any picture. Before we can be said to have any understanding of the number 12, we must know what different collections of twelve units have in common, and this is something which cannot be pictured because it is abstract. Bergson only succeeds in making his theory of number plausible by confusing a particular collection with the number of its terms, and this again with number in general.

The confusion is the same as if we confused a particular young man with youth, and youth with the general concept "period of human life," and were then to argue that because a young man has two legs, youth must have two legs, and the general concept "period of human life" must have two legs. The confusion is important because, as soon as it is perceived, the theory that number or particular numbers can be pictured in space is seen to be untenable. This not only disproves Bergson's theory as to number, but also his more general theory that all abstract ideas and all logic are derived from space; for the abstract 12, the common property of all dozens as opposed to any particular dozen, though it is never present to his mind, is obviously conceivable and obviously capable of being pictured in space.

But apart from the question of numbers, shall we admit Bergson's contention that every plurality of separate units involves space? Some of the cases that appear to contradict this view are considered by him, for example successive sounds. When we hear the steps of a passer-by in the street, he says, we visualize his successive positions; when we hear the strokes of a bell, we either picture it swinging backwards and forwards, or we range the successive sounds in an ideal space (*T. and F. W.*, p. 86). But these are mere autobiographical observations of a visualizer, and illustrate the remark we made before, that Bergson's views depend upon the predominance of the sense of sight in him. There is no logical necessity to range the strokes of a clock in an imaginary space: most people, I imagine, count them without any spatial auxiliary. Yet no reason is alleged by Bergson for the view that space is necessary. He assumes this as obvious, and proceeds at once to apply it to the case of times. Where there seem to be different times outside each other, he says, the times are pictured as spread out in space; in real time, such as is given by memory, different times interpenetrate each other, and cannot be counted because they are not separate.

The view that all separateness implies space is now supposed established, and is used deductively to prove that space is involved wherever there is obviously separateness, however little other reason there may be for suspecting such a thing. Thus abstract ideas, for example, obviously exclude each other: whiteness is different from blackness, health is different from sickness, folly is different from wisdom. Hence all abstract ideas involve space; and therefore logic, which uses abstract ideas, is an offshot of geometry, and the whole of the intellect depends upon a supposed habit of picturing things side by side in space. This conclusion, upon which Bergson's whole condemnation of the intellect rests, is based, so far as can be discovered, entirely upon a personal idiosyncrasy mistaken for a necessity of thought, I mean the idiosyncrasy of visualizing successions as spread out on a line. The instance of numbers shows that, if Bergson were in the right, we could never have attained to the abstract ideas which are supposed to be thus impregnated with space; and conversely, the fact that we can understand abstract ideas (as opposed to particular things which exemplify them) seems sufficient to prove that he is wrong in regarding the intellect as impregnated with space.

One of the bad effects of an anti-intellectual philosophy, such as that of Bergson, is that it thrives upon the errors and confusions of the intellect. Hence it is led to prefer bad thinking to good, to declare every momentary difficulty insoluble, and to regard every foolish mistake as revealing the bankruptcy of intellect and the triumph of intuition. There are in Bergson's works many allusions to mathematics and science, and to a careless reader these allusions may seem to strengthen his philosophy greatly. As regards science, especially biology and physiology, I am not competent to criticize his interpretations. But as regards mathematics, he has deliberately preferred traditional errors in interpretation to the more modern views which have prevailed among mathematicians for the last half century. In this matter, he has followed the example of most philosophers. In the eighteenth and early nineteenth centuries, the infinitesimal calculus, though well developed as a method, was supported, as regards its foundations, by many fallacies and much confused thinking. Hegel and his followers seized upon these fallacies and confusions, to support them in their attempt to prove all mathematics self-contradictory. Thence the Hegelian account of these matters passed into the current thought of philosophers, where it has remained long after the mathematicians have removed all the difficulties upon which the philosophers rely. And so long as the main object of philosophers is to show that nothing can be learned by patience and detailed thinking, but that we ought rather to worship the prejudices of the ignorant under the title of "reason" if we are Hegelians, or of "intuition" if we are Bergsonians, so long philosophers will take care to remain ignorant of what mathematicians have done to remove the errors by which Hegel profited.

Apart from the question of number, which we have already considered, the chief point at which Bergson touches mathematics is his rejection of what he calls the "cinematographic" representation of the world. Mathematics conceives change, even continuous change, as constituted by a series of states; Bergson, on the contrary, contends that no series of states can represent what is continuous, and that in change a thing is never in any state at all. This view that change is constituted by a series of changing states he calls cinematographic; this view, he says, is natural to the intellect, but is radically vicious. True change can only be explained by true duration; it involves an interpenetration of past and present, not a mathematical succession of static states. This is what is called a "dynamic" instead of a "static" view of the world. The question is important, and in spite of its difficulty we cannot pass it by.

Bergson's position is illustrated—and what is to be said in criticism may also be aptly illustrated—by Zeno's argument of the arrow. Zeno argues that, since the arrow at each moment simply is where it is, therefore the arrow in its flight is always at rest. At first sight, this argument may not appear a very powerful one. Of course, it will be said, the arrow is where it is at one moment, but at another moment it is somewhere else, and this is just what constitutes motion. Certain difficulties, it is true, arise out of the continuity of motion, if we insist upon assuming that motion is also discontinuous. These difficulties, thus obtained, have long been part of the stock-in-trade of philosophers. But if, with the mathematicians, we avoid the assumption that motion is also discontinuous, we shall not fall into the philosopher's difficulties. A cinematograph in which there are an infinite number of films, and in which there is never a *next* film because an infinite number come between any two, will perfectly represent a continuous motion. Wherein, then, lies the force of Zeno's argument?

Zeno belonged to the Eleatic school, whose object was to prove that there could be no such thing as change. The natural view to take of the world is that there are *things* which *change*; for example, there is an arrow which is now here, now there. By bisection of this view, philosophers have developed two paradoxes. The Eleatics said that there were things but no changes; Heraclitus and Bergson said that there were changes but no things. The Eleatics said there was an arrow, but no flight; Heraclitus and Bergson said there was a flight but no arrow. Each party conducted its argument by refutation of the other party. How ridiculous to say there is no arrow! say the "static" party. How ridiculous to say there is no flight! say the "dynamic" party. The unfortunate man who stands in the middle and maintains that there is both the arrow and its flight is assumed by the disputants to deny both; he is therefore pierced, like St. Sebastian, by the arrow from one side and by its flight from the other. But we have still not discovered wherein lies the force of Zeno's argument.

Zeno assumes, tacitly, the essence of the Bergsonian theory of change. That is to say, he assumes that when a thing is in a process of continuous change, even if it is only change of position, there must be in the thing some internal *state* of change. The thing must, at each instant, be intrinsically different from what it would be if it were not changing. He then points out that at each instant the arrow simply is where it is, just as it would be if it were at rest. Hence he concludes that there can be no such thing as a *state* of motion, and therefore, adhering to the view that a state of motion is essential to motion, he infers that there can be no motion and that the arrow is always at rest.

Zeno's argument, therefore, though it does not touch the mathematical account of change, does, *prima facie*, refute a view of change which is not unlike M. Bergson's. How, then, does M. Bergson meet Zeno's argument? He meets it by denying that the arrow is ever anywhere. After stating Zeno's argument, he replies: "Yes, if we suppose that the arrow can ever *be* in a point of its course. Yes again, if the arrow, which is moving, ever coincides with a position, which is motionless. But the arrow never *is* in any point of its course" (*C. E.*., p. 325). This reply to Zeno, or a closely similar one concerning Achilles and the Tortoise, occurs in all his three books. Bergson's view, plainly, is paradoxical; whether it is *possible*, is a question which demands a discussion of his view of duration. His only argument in its favor is the statement that the mathematical view of change "implies the absurd proposition that movement is made of immobilities" (*C. E.*, p.

325). But the apparent absurdity of this view is merely due to the verbal form in which he has stated it, and vanishes as soon as we realize that motion implies relations. A friendship, for example, is made out of people who are friends, but not out of friendships; a genealogy is made out of men, but not out of genealogies. So a motion is made out of what is moving, but not out of motions. It expresses the fact that a thing may be in different places at different times, and that the places may still be different however near together the times may be. Bergson's argument against the mathematical view of motion, therefore, reduces itself, in the last analysis, to a mere play upon words. And with this conclusion we may pass on to a criticism of his theory of duration.

Bergson's theory of duration is bound up with his theory of memory. According to this theory, things remembered survive in memory, and thus interpenetrate present things: past and present are not mutually external, but are mingled in the unity of consciousness. Action, he says, is what constitutes being; but mathematical time is a mere passive receptacle, which does nothing and therefore is nothing (*C. E.*, p. 41). The past, he says, is that which acts no longer, and the present is that which is acting (*M. and M.*, p. 74). But in this statement, as indeed throughout his account of duration, Bergson is unconsciously assuming the ordinary mathematical time; without this, his statements are unmeaning. What is meant by saying "the past is essentially *that which acts no longer*" (his italics), except that the past is that of which the action is past? The words "no longer" are words expressive of the past; to a person who did not have the ordinary notion of the past as something outside the present, these words would have no meaning. Thus his definition is circular. What he says is, in effect, "the past is that of which the action is in the past." As a definition, this cannot be regarded as a happy effort.

And the same applies to the present. The present, we are told, is *"that which is acting"* (his italics).[3] But the word "is" introduces just that idea of the present which was to be defined. The present is that which *is* acting as opposed to that which *was* acting or *will be* acting. That is to say, the present is that whose action is in the present, not in the past or in the future. Again the definition is circular. An earlier passage on the same page will illustrate the fallacy further. "That which constitutes our pure perception," he says, "is our dawning action....The *actuality* of our perception thus lies in its *activity,* in the movements which prolong it, and not in its greater intensity: the past is only idea, the present is ideo-motor" (*ib.*). This passage makes it quite clear that, when Bergson speaks of the past, he does not mean the past, but our present memory of the past. The past when it existed was just as active as the present is now; if Bergson's account were correct, the present moment ought to be the only one in the whole history of the world containing any activity.

In earlier times there were other perceptions, just as active, just as actual in their day, as our present perception; the past, in its day, was by no means only idea, but was in its intrinsic character just what the present is now. This real past, however, Bergson simply forgets; what he speaks of is the present idea of the past. The real past does not mingle with the present. Our memory of the past does of course mingle with the present, since it is part of it; but that is a very different thing.

The whole of Bergson's theory of duration and time rests throughout on the elementary confusion between the present occurrence of a recollection and the past occurrence which is recollected. But for the fact that time is so famil-

[3] Similarly in *Matter and Memory* (p. 193) he says it is a question whether the past has ceased to exist, or has only *ceased* to be useful. The present, he says, is not that which is, but that which *is* being made. The words I have italicized here really involve the usual view of time.

iar to us, the vicious circle involved in his attempt to deduce the past as what is no longer active would be obvious at once. As it is, what Bergson gives is an account of the difference between perception and recollection—both *present* facts—and what he believes himself to have given is an account of the difference between the present and the past. As soon as this confusion is realized, his theory of time is seen to be simply a theory which omits time altogether.

The confusion between present remembering and the past event remembered, which seems to be at the bottom of Bergson's theory of time, is an instance of a more general confusion which, if I am not mistaken, vitiates a great deal of his thought, and indeed a great deal of the thought of most modern philosophers—I mean the confusion between an act of knowing and that which is known. In memory, the act of knowing is in the present, whereas what is known is in the past; thus by confusing them the distinction between past and present is blurred. In perception, the act of knowing is mental, whereas what is known is (at least in one sense) physical or material; thus by confusing the two, the distinction between mind and matter is blurred. This enables Bergson to say, as we saw, that "pure perception, which is the lowest degree of mind. . . . is really part of matter." The act of perceiving is mind, while that which is perceived is (in one sense) matter; thus when these two are confused, the above statement becomes intelligible.

Throughout *Matter and Memory,* this confusion between the act of knowing and the object known is indispensable. It is enshrined in the use of the word "image," which is explained at the very beginning of the book.[4]

[4] Bergson's use of the word "image" is made clearer by a very penetrating analysis of Berkeley in a recent article, "L'Intuition Philosophique" (*Revue de Métaphysique et de Morale,* Nov. 1911). This article displays very distinctly the profound influence of Berkeley on Bergson's thought. Bergson's "image" is practically Berkeley's "idea."

He there states that, apart from philosophical theories, everything that we know consists of "images," which indeed constitute the whole universe. He says: "I call *matter* the aggregate of images, and *perception of matter* these same images referred to the eventual action of one particular image, my body" (*M. and M.,* p. 8). It will be observed that matter and the perception of matter, according to him, consist of the very same things. The brain, he says, is like the rest of the material universe, and is therefore an image if the universe is an image (p. 9).

Since the brain, which nobody sees, is not, in the ordinary sense, an image, we are not surprised at his saying that an image can *be* without *being perceived* (p. 27); but he explains later on that, as regards images, the difference between *being* and *being consciously perceived* is only one of degree (p. 30). This is perhaps explained by another passage in which he says: "What can be a non-perceived material object, an image not imaged, unless it is a kind of unconscious mental state?" (p. 183). Finally (p. 304) he says: "That every reality has a kinship, an analogy, in short a relation with consciousness—this is what we concede to idealism by the very fact that we term things 'images.'" Nevertheless he attempts to allay our initial doubt by saying that he is beginning at a point before any of the assumptions of philosophers have been introduced. "We will assume," he says, "for the moment that we know nothing of theories of matter and theories of spirit, nothing of the discussions as to the reality or ideality of the external world. Here I am in the presence of images" (p. 1). And in the new Introduction which he wrote for the English edition he says: "By 'image' we mean a certain existence which is more than that which the idealist calls a *representation,* but less than that which the realist calls a *thing,*—an existence placed halfway between the 'thing' and the 'representation'" (p. vii).

The distinction which Bergson has in mind in the above is not, I think, the distinction between the imaging as a mental occurrence and the thing imaged as an object. He is thinking of the distinction between the thing as it is and the thing as it appears, neither of which belongs to the subject. The distinction between subject and object, between the mind which thinks and remembers and has images on the one hand, and the objects thought about, remembered, or imaged—this distinction, so far as I can see, is wholly absent from his philosophy. Its absence is his real debt to idealism; and a very unfortunate debt it is. In the case of "images," as we have just seen, it enables him first to speak of images as neutral between mind and matter, then to assert that the brain is an image in spite of the fact that it has never been imaged, then to suggest that matter and the perception of matter are the same thing, but that a non-perceived image (such as the brain) is an unconscious mental state; while finally, the use of the word "image," though involving no metaphysical theories whatever, nevertheless implies that every reality has "a kinship, an analogy, in short a relation" with consciousness.

All these confusions are due to the initial confusion of subject and object. The subject—a thought or an image or a memory—is a present fact in me; the object may be the law of gravitation or my friend Jones or the old Campanile of Venice. The subject is mental and is here and now. Therefore, if subject and object are one, the object is mental and is here and now; my friend Jones, though he believes himself to be in South America and to exist on his own account, is really in my head and exists in virtue of my thinking about him; St. Mark's Campanile, in spite of its great size and the fact that it ceased to exist ten years ago, still exists, and is to be found complete inside me. These statements are no travesty of Bergson's theories of space and time; they are merely an attempt to show what is the actual concrete meaning of those theories.

The confusion of subject and object is not peculiar to Bergson, but is common to many idealists and many materialists. Many idealists say that the object is really the subject, and many materialists say that the subject is really the object. They agree in thinking these two statements very different, while yet holding that subject and object are not different. In this respect, we may admit, Bergson has merit, for he is as ready impartially to identify subject with object as to identify object with subject. As soon as this identification is rejected, his whole system collapses: first his theories of space and time, then his belief in real contingency, then his condemnation of intellect, then his account of the relations of mind and matter, and last of all his whole view that the universe contains no things, but only actions, movements, changes, from nothing to nothing, in an endless alternation of up and down.

Of course a large part of Bergson's philosophy, probably the part to which most of its popularity is due, does not depend upon argument, and cannot be upset by argument. His imaginative picture of the world, regarded as a poetic effort, is in the main not capable of either proof or disproof. Shakespeare says life's but a walking shadow, Shelley says it is like a dome of many-colored glass, Bergson says it is a shell which bursts into parts that are again shells. If you like Bergson's image better, it is just as legitimate.

The good which Bergson hopes to see realized in the world is action for the sake of action. All pure contemplation he calls "dreaming," and condemns by a whole series of uncomplimentary epithets: static, Platonic, mathematical, logical, intellectual. Those who desire some prevision of the end which action is to achieve are told that an end foreseen would be nothing new, because desire, like memory, is identified with its object. Thus we are condemned, in action, to be the blind slaves of instinct: the life-force pushes us on from behind, restlessly and unceasingly. There is no room in this philosophy for the moment of contemplative insight when, rising above the animal life, we become conscious of the greater ends that redeem man from the life of the brutes. Those to whom activity without purpose seems a sufficient good will find in Bergson's books a pleasing picture of the universe. But those to whom action, if it is to be of any value, must be inspired by some vision, by some imaginative foreshadowing of a world less painful, less unjust, less full of strife than the world of our every-day life, those, in a word, whose action is built on contemplation, will find in this philosophy nothing of what they seek, and will not regret that there is no reason to think it true.

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PSYCHOTHERAPIC CULTS:[1]

CHRISTIAN SCIENCE; MIND CURE; NEW THOUGHT.

THE most noteworthy religious event since the Reformation is perhaps the appearance in the United States of a number of religious movements which may be grouped together under the designation of psychotherapic cults. The foremost of them is "Christian Science," founded by Mrs. Mary Baker Eddy.

I hasten to add that the value of these cults does not, in my mind, belong to their "metaphysics," considered as a philosophical system. It is the product of ignorant and ill-trained minds. Much of it defies logic and offends common sense. But the defects which in the eyes of many wholly damn these movements might conceivably be removed, and there would remain important elements of a new religious faith acceptable to the modern world.

I shall try to show that the psychotherapic movements in their essential teaching are popularized and distorted formulations, on the one hand, of important truths regarding the "power of thought" over body to which psychology has recently given added significance, and, on the other, of a non-theistic philosophy allied to the absolute idealism of modern metaphysics. Although they distort contemporary thought, they do not intend to oppose it. They wish rather to build upon it.

[1] A discussion of other contemporary movements will be found in the author's book, *A Psychological Study of Religion: Its Origin, Function and Future,* Macmillan, 1912.

These new cults are forcible reminders of the fact that belief in a saving power is a condition of the existence of religion, and also that the desire for deliverance from moral and physical miseries and for the realization of ideals continues to be the motive of religious life, just as it was in the days of Gautama the Enlightener, and of Jesus the Healer.

* * *

The mind-cure books announce "the discovery of the might of truth in the treatment of disease as well as of sin," "the vital law of true life, true greatness, power, and happiness." They claim to be "systems of transcendental medicine," or of "psychic therapeutics." They purpose to minister to those who "would exchange impotence for power, weakness and suffering for health and strength, pain and unrest for peace, poverty for fulness and plenty." They proclaim "the birthright of every man born into the world to be physically whole and mentally happy." Their claims have an extravagant sound, but no more so than those made for "faith" by the New Testament writers who declared it would remove mountains and secure eternal blessedness after death. Nothing but vital experiences could have inspired the enthusiasm and the assurance with which these modern zealots proclaim the abounding efficacy of their "truth."

If they call themselves Christians, it is not in the traditional sense. Of traditional Christianity they speak respectfully, but they want a new dogmatics. They say, "The time for thinkers has come. Truth, independent of doctrines and time-honored systems, knocks at the portal of humanity."[2] In another of their aggressive little books one reads: "Unrest is universal. The old landmarks are disappearing. . . . Creed and dogma are things of the past;

[2] Mary G. Baker Eddy, *Science and Health*, 1908, Preface.

religious ceremonial and form no longer interest the masses."[3]

The impression these cults have produced on thoughtful religious people is well expressed in this passage:

"Renan with his usual intuition declared that if it [the religion of the future] were already in our midst, few of us would know it.

"The prediction has proved true. The new religious movement Christian Science has spoken a language so foreign to cultivated ears, its interpretation of the Bible is so false, it is so obviously committed to errors, illusions, and aberrations of every sort, that the intelligent have been disposed to shrug their shoulders in contempt and to ignore it. And yet they have not been able to ignore it altogether. Every once in a while this curious superstition proves its existence with unexpected power. We see a hard-headed business man totally devoid of religious sentiment undergo a new kind of conversion which leaves him as devout and ardent as a Christian of the first century. An ailing wife or daughter whom no physician has been able to help, through some mysterious means is restored to health and happiness. The victim of an enslaving habit, apparently with very little effort and without physical means, sufferings, or relapse, finds himself free. We enter a home where the new belief reigns and we find there a peace to which we are strangers.

"All over the country solid and enduring temples are reared by grateful hands and consecrated to the ideal and name of Mrs. Eddy. And this strange phenomenon has occurred in the full light of day, at the end of the nineteenth and at the beginning of the twentieth century, and these extraordinary doctrines have propagated themselves not in obscure corners of the earth, among an illiterate and fanatical population, but in the chief centers of American

[3] Charles B. Patterson, *A New Heaven and a New Earth,* Preface.

civilization. Such facts may well cause the philosophical student of religion to reflect.[4]

In these movements is restored the alliance between the art of healing the body and the art of healing the soul, which was always a leading characteristic of the higher religions during their period of greatest vitality. To the masses the most impressive aspect of religions has always been their power to heal the body. It was so in the early ministry of Christ and during the first Christian centuries. It is so now with these psychotherapists. And this revival acquires great significance from the fact that it can now be grounded upon the deeper understanding of the interrelation of mind and body, which we owe to modern science.

Speaking of the "four noble truths" of Buddhism, (Satyāni), i. e., the four axioms or certainties: the existence of suffering, the origin of suffering, the emancipation from suffering and the path that leads to the emancipation from suffering, Kern says: "It is not difficult to see that these four Satyas are nothing else but the four cardinal articles of Indian medical science, applied to the spiritual healing of mankind, exactly as in the Yoga doctrine. This connection of the Aryasatyas with medical science was apparently not unknown to the Buddhists themselves." And concerning the twelvefold causal root of the evil of the world, the twelve Nidānas (causes), he declares that they stand to the four Satyas 'in the same relation as pathology to the whole system of medical science.' Now the four truths and the twelve causes are fundamental facts upon which Gautama's scheme of deliverance is built."[5]

* * *

My chief effort will be to get from the writings of the leaders of these therapeutic schools a clear idea of the

[4] Elwood Worcester, Samuel McComb, Isador H. Coriat, *Religion and Medicine,* New York, 1908, pp. 8-10.

[5] Kern, *Manual of Buddhism,* Grundriss der indo-arischen Philologie und Altertumskunde, Vol. III, No. 8, pp. 46-47.

power with which they expect to regenerate humanity, and then to consider its adequacy. Whatever their affiliations, these writers practically agree on the points that most interest us. I do not shrink from putting before my readers, to begin with, brief quotations from two of the most extravagant and crude of these authors; for even they find followers among people who prove themselves intelligent and sensible in the affairs of life.

T. Troward, a leader of Mental Science (not a disciple of Mrs. Eddy), late divisional judge in Punjab and Edinburgh Lecturer on Mental Science, teaches the existence of an unlimited, impersonal, though intelligent power, which man may press into service, or appropriate to himself. His view of man's relation to that power is curious. The individual can call it into action and give it direction, "because it is in itself impersonal though intelligent." "It will receive the impress of his personality, and can therefore make its influence felt far beyond the limits which bound the individual's objective perception of the circumstances with which he has to deal. It is for this reason that I lay so much stress on the combination of two apparent opposites in the Universal Mind, the union of intelligence with impersonality....How do we know what the intention of the Universal Mind may be? Here comes in the element of impersonality. It has *no intention,* because it is *impersonal*.... Combining, then, these two aspects of the Universal Mind,....we find precisely the sort of natural force we are in want of, something which will undertake whatever we put into its hands without asking questions or bargaining for terms, and which, having undertaken our business, will bring to bear on it an intelligence to which the united knowledge of the whole human race is as nothing, and a power equal to this intelligence."[6]

[6] T. Troward, *The Edinburgh Lectures on Mental Science,* The Arcane Book Concern, 1909, Chicago, pp. 66-68.

I find it difficult to conceive an unlimited impersonal intelligence which has no intention and which individual intelligence may direct. But in fairness to the abstruse judge, I must add that this difficulty is no greater than that presented by Hegel's conception of the Absolute Mind.

In the work of W. F. Evans we meet a consistent pantheism. He strives to give to his opinions an impressive background compounded of modern science, antique pantheism, and modern idealism. How vast and accurate is his knowledge will appear in the following passage. I quote it without apology as another instance of a type of conception apparently rational enough to be accepted by many intelligent people. "The soul of man is a part, so to speak, of the *anima mundi,* the soul of the world." The power of the healing thought "issues from the spiritual world of which our minds are a part, for all ideas belong to that boundless realm of life." "It is stored up in exhaustless and overflowing abundance in the bosom of nature....it can be controlled in its lower degrees of manifestation by the intelligent will of man, which is the highest form of its development and expression." "This grand whole....the universal world of spiritual intelligence is called in Sanskrit, *Addi-Budda.* In the writings of Paul it is called the Christ....It is identical with what is called magnetism, and is also that which the philosophers have called the divine *nous.*"[7]

One of the ablest and sanest writers of New Thought, Ralph Waldo Trine, in a book which has passed its seventy-fifth thousand, also announces a pantheistic gospel of an infinite power at the service of man. "The great central fact of the universe is that spirit of Infinite Life and Power that is back of all, that animates all, that manifests itself in and through all; that self-existent principle of life from

[7] W. F. Evans, *The Primitive Mind-Cure: Elementary Lessons in Christian Philosophy and Transcendental Medicine.*

which all has come, and not only from which all has come, but from which all is continually coming."

"This Infinite Power is creating, working, ruling through the agency of great immutable laws and forces that run through all the universe, that surround us on every side. Every act of our every-day lives is governed by these same great laws and forces."

"In a sense there is nothing in all the great universe but law." But the presence of laws indicates a force back of them. "This Spirit of Infinite Life and Power that is back of all is what I call God."

"God, then, is this Infinite Spirit which fills all the universe with Himself alone, so that all is from Him and in Him, and there is nothing that is outside....He is....our very life itself." "In essence the life of God and the life of man are identically the same, and so are one. They differ not in essence, in quality; they differ in degree."

"....if the God-powers are without limit, does it not then follow that the only limitations man has are the limitations he sets to himself, by virtue of not knowing himself?"

"*The great central fact in human life, in your life and in mine, is the coming into a conscious, vital realization of our oneness with this Infinite Life, and the opening of ourselves to this divine overflow.*" This means simply "that we are recognizing our true identity, that we are bringing our lives into harmony with the same great laws and forces, and so opening ourselves to the same great inspirations as have all the prophets, seers, sages, and saviours in the world's history, all men of truly great and mighty power."[8] He does not hesitate to use the term "God-man."

[8] Ralph Waldo Trine, *In Tune with the Infinite or Fullness of Peace, Power, and Plenty,* Thomas Y. Crowell and Co., New York, pp. 11-20.

Christian Science.

It seems almost incredible that one professing to be a Christian should teach the impersonality of the divine nature. And yet this is undoubtedly what Mrs. Eddy does, and in this respect she agrees with those from whom I have just quoted. The term that she prefers as a name for the Divine Power is Principle. As synonyms she uses Life, Truth, Love, God. In the earlier editions of *Science and Health,* it is written that God "is not a person, God is Principle."[9] This is undoubtedly the standpoint of her later writings also. But in them, probably because of the pressure of adverse public opinion, she insists less than at the beginning of her career upon the impersonality of Principle, and the word "person" appears more frequently. "Once in 1898, Mrs. Eddy hints that God may be personal 'if the term personality, as applied to God, means infinite personality,' and Mr. Farlow in 1907 assures the Rev. Edgar P. Hill that Mrs. Eddy does believe that 'God is person in the infinite sense.'"[10] I take the following passages from the same book: "Principle in her theology gathers up into itself all the concepts we habitually associate with God, except the most important—personality. Before her book appeared in 1875, she was telling her pupils, as two of them informed me, that they could make no progress till they had banished from their minds the thought of God as a person. She instructed Richard Kennedy 'to lay special stress' in healing patients on the impersonality of God. This is the commanding thought that rings through the first chapter of the first edition of *Science and Health.*"

"Mrs. Eddy's pantheism is unnecessary, and yet its origin was inevitable in a mind as literal as hers. Quimby often spoke of God as Principle. In the Quimby manu-

[9] Mary G. Baker Eddy, *op. cit.,* 3d ed., 1881, I, 67; II, 27.

[10] Lyman P. Powell, *Christian Science, the Faith and its Founder,* pp. 139-140.

script from which, for several years, Mrs. Eddy taught, no sentence is more startling than the sentence 'God is Principle.'"

"For more than thirty years Mrs. Eddy has been solemnly asserting that in 1866 she received a 'final revelation.' Now this 'final revelation,' which was finally as well as first expressed in 1875, in *Science and Health,* is saturated with thought that God is not a person. In the very first chapter we are informed that 'God is Principle, not person,' [I do not find that expression in the first chapter of the 1908 edition, but it is in *No and Yes,* published in 1909] that Jesus preached the impersonality of God, that the error of believing in the personality of God crucified Jesus, that the trouble with conventional Christianity to-day is that it makes God a person....' (Pages 137-140).

On the other hand, in the seventy-third edition of *No and Yes,* published in 1909, a pamphlet intended "to correct involuntary as well as voluntary error," we read: "Is there a personal Deity? God is Infinite. He is neither a limited mind nor a limited body. God is Love; and Love is Principle, not person. What the person of the Infinite is, we know not; but we are gratefully and lovingly conscious of the fatherliness of this Supreme Being. God is individual, and man is his individualized idea....Limitless personality is inconceivable....Of God as person, human reason, imagination and revelation give us no knowledge.

"When the term divine Principle is used to signify Deity it may seem distant and cold, until better apprehended. This Principle is Mind, Substance, Life, Truth, Love. When understood, Principle is found to be the only term that fully conveys the ideas of God,—one Mind, a perfect Man, and divine Science."[11] This Principle, though not a person, "is intelligence."

Although she wrote, "God is All in all," and "All in all

[11] Eddy, *No and Yes,* 1909, pp. 19, 20.

is God,"[12] she will not be called a pantheist. In the edition of *No and Yes* already quoted, she claims that "Christian Science refutes pantheism, finds Spirit neither in matter nor in the modes of mortal mind. It shows that matter and mortal mind have neither origin nor existence in the eternal Mind. . . . For God to know, is to be; that is, what He knows must truly and eternally exist. If He knows matter, and Matter cannot exist in Mind, then mortality and discord must be eternal."[13]

Her pantheism is in any case not materialistic, since she holds matter to be unreal, a deception of mortal mind. Hers is an idealistic pantheism, such as an ignorant person of a thoroughly optimistic temperament might evolve on the basis of imperfect knowledge of absolute idealism and from observation of the mastery of mind over body.

The writings of Mrs. Eddy's disciples reflect the uncritical, pantheistic idealism of their leader. Their favorite phrases are such as these: "God's presence is the presence of love;" "God is life everywhere present;" "One life fills all, it is the Perfect Life."

The similarity of the essential aspects of New Thought and Christian Science to the mystical element in Christianity is evident. Both give clear expression to the anti-isolation motive, to a dynamic belief in oneness-with-the-whole, and both feel the essence of the cosmic plasma to be love. Man is steeped in all-embracing Love. He need only place himself in unison with the everlasting, all-comprehending life-force and the fulness of life will be his. How love can be an attribute of an impersonal power does not seem to give Mrs. Eddy one moment of uneasiness.

In their curative practices, the psychotherapic cults have the benefit of the recent discoveries concerning the effects of suggestion. Regarding their methods, I may

[12] Eddy, *Science and Health,* 1898, p. 7.

[13] Eddy, *No and Yes,* pp. 15, 16.

say here merely that they tend to place the person, as do the practices of the other ethical religions, in a state of increased suggestibility, a state described in part by the words relaxedness, collectedness, monoideism, meditation, communion. This condition of the subject aids greatly in the realization of the expected benefits. The efficacy of these curative methods is sufficiently demonstrated by the wonderful extension of the movements. In every walk of life people bear witness to the saving grace that is in Christian Science or in New Thought. The forces of a new life have welled up within them; the burdens of existence have lightened, nay, have disappeared; and now they walk through life contented, hopeful, and aggressively benevolent.

The following is an example of what people find in Christian Science apart from the cure of disease:

"I accepted *Science and Health* without expecting it to offer more than a human theory about life,—even the name did not lead me to expect it to be religious; in fact, the chief incentive to my reading it at that time was the great kindness and sincere sympathy evinced by my friend, who placed a copy at my disposal.....I started timidly at first, and prayerfully, lest it should be misleading, but before I had gone very far I experienced that wonderful spiritual quickening which is so often spoken of in our meetings. I wish I could tell exactly what the experience meant to me, the wonderful awakening I had; how old things vanished and all things became new.. It seemed as if the burdens, perplexities, doubts, and fears had all suddenly rolled away; as if the sun had emerged from behind the clouds, and everything was again bright and beautiful.

"And what a feeling of strength, hope, and courage came! Those old troublesome questions, especially the question of death, were explained, and I felt a wonderful release to know that death was not of God. I read and reread the latter part of the chapter on Christian Science Practice, where that glorious truth is explained; it was so beautiful, so natural, and so true. There was such perfect joy to me in that freedom, that I used to declare over and over again, of those who had just passed from us (the members of our home circle), 'They are not dead;' and so free was I made from the old bondage, that never since then has the thought of that change affected me as it did before."[14]

Unnecessary importance is attached by the critical public to the vagaries of Christian Science and of New Thought; for instance, to the denial of the reality of matter, and therefore of disease; to the wild hopes of some of their prophets that "the time will certainly come when the highly developed man will have the power to lay down or take up his life through a conscious knowlege of the laws of eternal being and the direct application of these laws to his own life."

When I say "wild hopes," I speak as the prosaic man that I am. No less a philosopher than Bergson has expressed that same hope of overcoming death.

An apologist of the psychotherapic sects would be justified in making the following claims:

1. The salvation they promise is first of all for *this life.*

2. The soul is not saved independently of the body. The nefarious asceticism of older faiths is impossible on the principles of Christian Science.

3. Their ideal involves efficiency in the conduct of this life.

4. Their conception of salvation is free from anything miraculous. They dispense with the wonders of the Fall,

[14] *Christian Science Sentinel,* Dec. 3, 1901.

[15] Charles B. Patterson, *op. cit.,* Preface.

of the self-sacrifice of a divine personage, and of salvation by his atonement.

5. They divert attention from the sense of guilt and suffering, and direct it to an immediately accessible healing and invigorating power.

6. Although they usually define the aim of life in terms of power, happiness, and love, they cannot fairly be charged either with insensitiveness to moral values, or with indifference to the ethical advancement of mankind.

7. Despite its extravagance, their "metaphysics" may be regarded as a formulation, crude and distorted, of a *Weltanschauung* made unavoidable by modern knowledge, —a *Weltanschauung,* opposed in several important respects to the traditional but no longer acceptable Christian philosophy.

8. These cults have proved their value by their results.

In estimating the chances of continued life of religious movements, one should bear in mind that vitally beneficial beliefs may carry a heavy load of error and even of absurdity. The Christian religion was not destroyed by the expectation of the second coming of the Lord and of the end of the world, by extravagant notions of the power of faith, by absurd or incomprehensible doctrines regarding the means of salvation, the resurrection of the body, and the like. There is enough substantial, practical truth in Christianity to bear the enormous doctrinal dead weight it carries even to this day. It may be possible for the psychotherapic doctrines to be purified in a reformation which would either remove entirely or drive into side-currents most of the offensive tenets.

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THE MYSTERY OF LIFE.

A POETIZATION OF "THE HAKO"— A PAWNEE CEREMONY.

BY HARTLEY BURR ALEXANDER.

PREFATORY NOTE. The 22d Annual Report of the Bureau of American Ethnology contains Alice C. Fletcher's record of "The Hako: a Pawnee Ceremony." This record is the foundation of the present work. Miss Fletcher, in her rhythmic renderings of the Indian songs, has incorporated meanings given in the explanations of the leader of the ceremony as well as the literal sense of the Indian texts; the result being a series of admirable translations, abounding in telling phrases. The version here presented has drawn freely upon Miss Fletcher's fine renderings; but as "The Mystery" was designed to emphasize the universal elements in the Indian thought, it necessarily involved generalization and amplification of the primitive expression, as well as rearrangement of materials.—The piece was conceived as a dramatic pageant, with musical accompaniment, as will appear from its form.

DESCRIPTION OF THE SYMBOLS.

The Persons:

The LEADER, a Priest. He wears leggings and moccasins, and a robe girt about his body, leaving shoulders and arms bare; in his hair is a feather of white eagle's down; he carries the brown-plumed wand.

Five ACOLYTES, dressed like the Leader. They carry the ceremonial articles and act as assistants to the Leader.

The CHORUS, consisting of

(A) The SEMI-CHORUS OF FATHERS, led by the CHIEF OF THE FATHERS. They are dressed in leggings and moccasins and ceremonial shirts, ornamented with blue and white. They wear bonnets of white eagle's plumes. The Chief carries a calumet and his bonnet extends in streamers of plumes down his back. In Part II the bonnets are left off, and all wear blankets, symbolic of night.

(B) The SEMI-CHORUS OF THE CHILDREN, led by the CHIEF OF THE CHILDREN. They are dressed like the Fathers, except that their colors are green and red and their bonnets adorned with brown plumes. In Part II they also leave off the bonnets and wear blankets.

The CHILD.

The PERSONATOR OF THE MORNING-STAR, dressed in red, and wearing a red plume; spread wings are attached to his wrists.

The Powers:

The BLUE SKY, abode of the FATHER OF HEAVEN, the Mighty Power.

The POWERS OF HEAVEN: The MORNING STAR, Herald of Day; the DAWN, Child of Heaven and of Night; the SUN, Father of Day and of Life; the Four WINDS from the Four Quarters of the World, where are the Paths from Earth to Heaven.

MOTHER EARTH, whose Child is the Green Vegetation symbolized by the CORN SPIRIT, and who sustains life with the running Waters which are the WATERS OF LIFE and symbolize the continuing generations of Mankind.

The EAGLE, Chief of the Birds who are the Mediators between the Mighty Power and Man; Conductor of the Visions, dwelling in the lower Heaven, down to Man; Symbol of the care which the Father of Heaven has for his Children,—the brown plumes being emblematic of the Female Eagle in her care for her nestlings, the white plumes of the protecting Male Eagle: the place of the white is always outermost.

Emblems and Ceremonial Articles:

The BROWN-PLUMED WAND, borne by the Leader: a hollow stem, painted blue, emblematic of the Sky, and adorned with a fan of the brown plumes of the Female Eagle; also, with a Duck's head and breast, one end of the stem being thrust through the mandibles; with a tuft of Owl feathers; with red and white streamers, emblematic of Sun and Stars.

The WHITE-PLUMED WAND, borne by an Acolyte: like the preceding except that the stem is green, symbolic of Earth, and the plumes are the white plumes of the Male Eagle.

The SPREAD WINGS OF AN EAGLE, mounted like the wings on the caduceus of Mercury, except that each wing is on a detachable staff so that they can be held separately, simulating flight, or conjoined, forming a banner.

The CORN, a light sheaf of maize with unhusked ears, symbolic of the Corn Spirit and of the Vegetation which is the offspring of life-giving Mother Earth.

The BOWL, hewn from the living wood, a part of Earth's green covering, and painted blue as symbolizing the blue Sky. In it is borne water from a running stream, symbolic of the vigor and strength which Earth gives in the Waters of Life and of the continuance of life in the on-flowing generations of men.

A tray with implements for Fire-making; a tray with ceremonial Tobacco; a tray with four cups, one containing red, one blue, and one green paint, and one with oil and fat. Trays of bread; jugs of water; turfs for building the hearth-altar.

The Place:

The PLACE is a sward of level green on the open prairie. Above is the blue sky with a few fleecy clouds. At the Western side of the sward, forming the background, is a hedge of greenery, with three openings or gates.

The Northern Gate is the Gate of the Fathers,—North is the side of Night and of the need of protection.

The Southern Gate is the Gate of the Children,—the South is the Winter home of the birds, the side of peace and of plenty.

The Middle Gate, facing the Place of Sunrise, is the Gate of the Leader and Acolytes. It opens from the Holy Place.

Color Symbolism:

BLUE symbolizes the abode of the Powers Above and of the Father of Heaven; GREEN symbolizes the Earth and life-giving food; RED is the color of Life, of the life-blood and of the Morning Star who is herald of light and life; WHITE symbolizes Sunlight, the fleecy Clouds and the Winds, and hence the breath of Heaven, the Breath of Life.

PART I. THE COMING OF THE CORN.

THEME I.

Orchestral Prelude. Enter from the Central Gate the Leader and Acolytes; from the North Gate, the Fathers; from the South Gate, the Children. The Acolytes and the Chorus remain at the Rear; the Leader advances to the Forefront.

The Leader intones:

I.

Give heed! Give heed!
Give heed, O ye People!
Unto the Abode of Life give ye heed,
And unto the Powers thereof
Let your hearts be turned in reverence....

II.

Lift up your gaze!
Unto the blue and doming Skies
Lift up your gaze,—
Where dwelleth the Father of Heaven,
Where dwelleth the Father of Life,
Yea, from everlasting to everlasting.
Lift up your gaze
Unto the Father!....

In the Circle of the Heavens He hath set
The manifestations of His glory,—
The bright and shining Sun,
Which giveth forth the Light of Day
And answereth the hymn wherewith His creatures
Waken at Morn,—
In the Circle of the Heavens He hath established the Sun
To be a sign of His presence by Day,
And the quiet Stars hath He set to be His nightly ministers..

The Four Winds
From the Four Pathways of the Skies,—
East, South, North, West,—
Breathe forth His Word and His Life
Throughout the Lodge of Heaven:
Yea, the music of His Word
And the gladness of Life
Breathe they forth
Through the Four Quarters of the World....

Lift up your gaze
Unto the blue and doming Skies!....

III.

Upon the Earth
Let your thoughts descend,—
Our Mother Earth!
From her dark and fruitful womb ye are sprung,
And at her nourishing bosom ye are fed:
She is the Great Mother
Who keepeth us in life
And at death receiveth us:
Think on the Mother!

Her garment is the fair and flowing green,
The verdure of the hills is her habiliment,
Whence they that move obtain their strength
And the Sons of Men their sustenance:
Who is the Giver of Food unto her children.

As milk streameth from the breast,
From her ancient hills
And the cool depths of her yearly snows
The clear and living Waters are poured forth,
To be for her children their drink and their refreshing:
Yea, unto them that thirst She giveth the Waters of Life.
  Think on the Mother!....

IV.

  Upon the Earth
Let your thoughts descend in reverent heed:
Let them be lifted up
  To the blue and doming Skies!
Upon Earth and upon Heaven let your thoughts be placed,
For they are the Abode of Life and of the Powers thereof...

THEME II.

Roll of drums. The Chorus advances a pace, crying in unison:

Look down! We gaze afar on your dwelling!
Ye Mighty Ones, look down!

During the orchestral development that follows, the Semi-Chorus of Fathers advances to the center of the sward where they form a circle, with an opening to the East and one to the West, thus simulating the circular outline of the walls of an earth lodge. With their hands they indicate the building of walls. .During this action they chant:.

Ye of the Winds, behold us!
Ye Thunder gods, behold us!
Wielders of Leven, behold us!
Bringers of Death, behold us!

Ye of the Rains, behold us!
Ye of the Clouds and the Soil!
Givers of Increase, behold us!
Givers of Life, behold us!

We establish here a dwelling,—
A Wall of Defense,
A House of Life,
A Place that is Holy!

Full Chorus:

Look down! We gaze afar on your dwelling!
Ye Mighty Ones, look down!

Semi-Chorus of Children advances, from the Western opening, within the circle—the symbolic lodge—formed by the Fathers. They carry turfs which they build into an hearth-altar at the center of the circle as they file past. They form into two half circles, North and South, within the circle of the Fathers. During the action they chant:

Spirits of Heaven, behold us!
Spirits of Earth, behold us!
Ye Shining Ones, behold us!
Ye Darkling Ones, behold us!
Ye that measure out the ways of men....

Here we build unto you an Altar,
Whereof the flame is the prayer of man
Ascending....

The Leader enters the encircled space from the Eastern opening, three Acolytes bearing fire-making implements enter from the West. At the Altar the Leader lays a fire and sends up a pillar of smoke, like an Indian signal smoke. The music is the music of fire and of prayer. As the smoke ascends—

The Chorus:

See! The Pillar of Smoke ascendeth
Up to the dome of Heaven
Where God abideth....

The Leader:

As riseth the smoke of the Altar,
So the spirit of man upstriveth,
So the cry of his heart upmounteth,
Unto the deeps of the Blue,
Unto the Silence of God....

The Chorus:

Speed aloft!
Bearing our supplication,
Bearing our prayer!

THEME III.

The flutes strike in with the clear piercing music of the Eagle. The Leader gazes into the Eastern Sky. He raises his arm impressively, crying:

Lo, where cometh His answer—
The Eagle of the Chief of Heaven!

The Chorus circles North and South, bringing their faces to the East, and then, during continuous circling motion:

Behold, an Eagle now is circling, widely circling above us!

Semi-Chorus of Children, circling to the South:

As the mother-bird circleth her nestlings, careful for her chicks,
She circleth us, hovering....

Semi-Chorus of Fathers, circling to the North:

She is the Eagle of God!
Of Him who is Father of Heaven,
Who ruleth the quartered Earth,
And sendeth His Will by the Eagle
Over the windy Pathways
That lead from Man up to God....

Semi-Chorus of Children:

She is the Eagle of God!
The sign that He hath sent us
That we are in His eyes
As to the mother-bird are her nestlings....

Semi-Chorus of Fathers:

She is the Eagle of God!
Whose coming is sign of His blessing,—
Of the gift of Food to His children,
Of the gift of Life to His children,
As the mother-bird home circling
Beareth food and life to her nestlings....

Full Chorus:

Helpless are we as are nestlings,
Naked as unfledged eaglets
Lone in their storm-beaten crag....

Semi-Chorus of Fathers:

About them circleth the Eagle,
Strong to protect, ever watchful,
His plumes flashing white in the sunlight,—
The cloud-frothing winds are his coursers!

Semi-Chorus of Children:

Over them hovereth the Eagle,
She of the brown brooding pinions,
Bearing them food in her talons,—
As the Father of Heaven permitteth.

Full Chorus:

We men are as naked and helpless
As the storm-beaten chicks of the Eagle....

He of the wide-encircling Heavens guardeth us,
And the Sun-Father watcheth over us;
Mother Earth bareth Her bosom unto us,
Her bounty is our daily bread....

Amid silence the Leader and the Acolytes retire through the Center Gate, as into an Holy of Holies. The Chorus remains ranged in the North and South Forefronts.

THEME IV.

The oboes and bassoons strike up the droning music of the Corn. The two Chiefs step forward.

Chief of the Fathers:

Father, have pity upon our weakness,
Father, have pity upon our hunger:
We men are as infants before thee,
We men are as helpless children
Weeping for food....

Chief of the Children:

Out of far distant days soft-stepping,
I beheld one coming, a Spirit coming,
Coming to comfort me....

In the tender and caressing night
I beheld my comforter:

Her wings dropped the dews of fragrance;
With the softness of stars was her body beautiful;
In her breast were the singing voices of the fields....

There enter from the North Gate two Acolytes, one bearing a tray of bread, one bearing a jug of water, and from the South Gate two, with bread and water. The Acolytes with the bread offer the bread to the Fathers and to the Children.

Chief of the Fathers:

Lo, they bring ye the Body of Mother Earth:
Take thereof, and eat.

While they partake of the bread, one after the other, the Acolytes with the water advance, offering the water to the Fathers and the Children.

Chief of the Children:

Lo, they bring ye the Waters of Life:
Drink, and be refreshed.

The Fathers and the Children moisten their lips as the water is offered them. The Acolytes pass forth as they entered.

The music becomes tense, vibrant and rapid. The Chorus sways to and fro in a crescendic rhythm.

From the Center Gate enter: Two Acolytes each bearing a staff with a spread eagle-wing attached; the Leader and an Acolyte bearing the plumed wands; an Acolyte bearing aloft the sheaf of maize; an Acolyte with the bowl. They range themselves, the Leader a little in advance and to the South, the Acolytes abreast, the wings at the ends, the Corn in the center, the white-plumed wand at the left hand of the Corn-bearer, the bowl at his right hand.

The Chorus, in animated motion, bursts forth in a lyric Hymn to the Corn:

Daughter of Heaven, Earth's first-born,
Hail to thee! Hail to thee! Spirit of Corn!
Thou at whose bounteous feasts we are fed,
Who givest us life in giving us bread:
Hail to thee! Hail to thee! Spirit of Corn!

Thou who dost welcome the Sun-Father's glance
With tassel and spear flung aloft to His Morn,
With nodding of plume and waving of lance,
Thou who dost make the green gardens to dance
With joy of thee, joy of thee, Spirit of Corn!

Thou who dost gather the sunlight and rain
Till the body of Earth with Heaven is o'erlain,—
Life, life is thy largess, who givest us grain!
Daughter of Heaven, Earth's first-born,
Hail to the thee! Hail to thee! Spirit of Corn!

They cease with nodding plumes.

THEME V.

The grave music of the Way of Life enters as an undertone to the Corn music. The Leader advances to the Altar. He signals to the Acolytes, who uplift the emblems. He addresses the Powers:

Behold us, where we are standing,
Uplifting these emblems,—
Ye Mighty Ones, behold us!....

Out of the Heavens, cometh a flash!
Out of the Heavens, the light of His seeing eye!

At a sign the five Acolytes, abreast, advance to the Altar, before the Leader. They present the emblems to the East, crying:

Ye of the East, behold us!
Ye of the Dawn and the Day!

They advance sixteen paces, wheel, and elevate the emblems to the West, crying:

Ye of the West, behold us!
Ye of the Storm and the Night!

They return sixteen paces, wheel to the left, advance eight paces south, and elevate the emblems to the South:

Ye of the South, behold us!
Ye of the Path of the Sun!

They wheel and advance sixteen paces to the North, elevating the emblems to the North:

Ye of the North, behold us!
Ye of the Mother of Day!

They return eight paces to their original station before the Altar, and once more advance sixteen paces to the East. There they remain, abreast.

The Leader advances eight paces from the Altar, till he stands, as it were, upon the heart of the human figure traced by the evolutions of the Acolytes.

The Leader:

Ye of Heaven and Earth, behold us!
Ye Powers of Life, behold us!....
Who journey the way of man.

Ye have given us for our Strengthener, the Spirit of Corn:
Ye have given us for our Leader, the Spirit of Corn!....
Who journey the toilsome way.

As the Spirit draweth nigh, we bow our heads:
As the Spirit toucheth us, we bow our heads....
Who journey the Way of Life.

While the music grows in depth and gravity, the Chorus moves, forming in a phalanx behind the Leader.

Then the Chorus:

Open our way, Spirit of Corn!
Open our way, Leader in Life!

The Leader:

Open is the Way!
We are led as were our fathers led
Down through the ages:
We follow as they did follow.

The Leader signals; the Chorus moves forward; the Acolytes, abreast, with the emblems upraised, the Corn still at the center, lead the processional, which circles the sward and finally retires, the Leader and Acolytes through the Center Gate, the Fathers through the North, the Children through the South. During this movement, in full choral, is sung the Chant of the Way of Life.

I.

During the advance:

Follow on, O Brothers, follow on!
  The Spirit of the Corn doth lead
  And unto you at your need
    Falleth her benison:
Follow on, O Brothers, follow on.
    Whither your sires are gone....

Your feet one rhythm beating,
Your tongues one song repeating,
Your hearts one boon entreating,
  Follow ye on!
Forth of the ruddy Morn,
  Into the glowing Day,
Where the Spirit of the Corn
  Showeth the way:
Follow on, O Brothers, follow on,
  Whither your sires are gone.

II.

Circling East:

Lo. the Circle of the Earth
  Is the circle of Man's domain,
And he buildeth his puny hearth
  In the midst of her spreading plain,—
    And Morning and Noon and Night
    He kindleth his tiny light.

Circling North:

Heaven hath a myriad stars,
  Heaven hath the burning Sun,
The Day and the Night are their bars,
  And their course is never run:
    In the hour where it began
    Dieth light in the lodge of man.

III.

Circling West:

Man walketh in ways unknown,
  From the darkening East to the West—
As a fledgling that hath flown
  Forth from the Eagle's nest
    To journey the pathless skies
    With the sun of Heaven in his eyes.

Circling South:

Man bareth his head to the rain,
  His breast to the storm layeth bare,
And he stalketh athwart the plain
  Blind in the lightning's glare;

And heavy on his soul
Falls the terrible thunder's roll.

IV.

Circling East:

As an infant that is led
Amid the paths of surprise
By the hand that giveth him bread—
The hand of the foolish or wise—
So is a man in Their care
Who measure the ways he must fare.

Circling North:

The herds of the prairies pass,
At the will of the South and the North,
On the trail of the greening grass,
Where the Spirit of Life floweth forth,—
So man taketh up from the sod
The sacrament of God.

V.

Withdrawing:

Follow on, O Brothers, follow on!
In the ways whereto ye were born,
While leadeth the Spirit of Corn
Granting her benison:
Follow on, O Brothers, follow on,
Whither your sires are gone!....
Your feet one spirit guiding,
Your lives one fate abiding,
In the wisdom of One confiding,
Follow ye on!
Into the sombre Night,
Forth of the flashing Day,
To lands beyond your sight
Lieth the Way....
Follow on, O Brothers, follow on,
Whither your sires are gone.

[*Exeunt omnes.*]

PART II. THE REVELATION.

THEME VI.

The music opens with an eerie prelude, full of whispering notes suggestive of things supernatural. The Chorus, as yet unseen, strike in with their Invocation to the Visions. They enter singing, the Children from the South, the Fathers from the North. They wear no bonnets and they are girt with blankets, symbolic of Night. They circle in opposite directions, passing and repassing, file by file.

The Chorus:

Holy Visions, hither come!
Ye who dwell in rainbow Skies
Hidden from our mortal eyes
By the lights of Paradise,—
Holy Visions, hither come!

Holy Visions, hither come!
To our troubled lives descend,
Draw anigh and o'er us bend,
That our hurts may have an end:
Hither, hither come!

Holy Visions, hither come!
If we wake or if we dream,
Where your flashing pinions gleam
There doth Heaven on us beam:
Holy Visions, come!

Holy Visions, hither come!
Gift of joy your presence brings,
When the music of your wings
To the gladdened spirit sings:
Hither, hither come!

Holy Visions, hither come!
Glorified the spirit blooms
Where the splendor of your plumes
Like a sun its night consumes:
Holy Visions, come!

Holy Visions, hither come!
With the lightnings of your glance
Make the hearts of men to dance
In celestial radiance:
Hither, hither come!

Holy Visions, hither come!
Bearing with you Heaven's Peace,
Bearing every hurt's release
In your healing mysteries:
Holy Visions, hither come!

The Chorus, as the song closes, form a semi-circle facing eastward, the Fathers to the North, Children to the South. They kneel and draw their robes over their heads, as in vigil. The two Chiefs stand, a little in advance.

The music is weird and mysterious, with innumerable fluttering crescendos, as of approaching wings.

Then the Chief of the Fathers:

Hark, the sound of their wings!
Like the wings of mighty eagles:
Like the whistling winds on the prairies:
Like the rushing rain on the corn!....
Hark, the sound of their wings!

A pause. Then the Chief of the Children:

Was it in dreams that our Fathers beheld them?
In wingéd dreams that they came revealing
Unto our sires the Vision of Life?

Chief of the Fathers:

Yea, in their dreams our Fathers beheld them:
In shining dreams they came unveiling
Unto our Sires the Vision of Life....

A pause. Then the Chief of the Children:

Hark, the sound of their wings!
Mighty spirits hither flying:
Mighty spirits here revealing
Visions as in days of yore....
Hark, the sound of their wings!

The mysterious music continues for a time; then dies away into the steady beating of deep-toned drums. An utter silence.

THEME VII.

A burst of drum-beats. The Chorus throw aside their robes from their heads, and rise, crying:

Awake, O Mother, from sleep! The night is far spent.
Awake, O Earth, from your rest! The hills and the valleys stir.
Awake, O World, from your night! Day summoneth Earth and Sky.

The Chorus moves, in a flowing rhythm, while the Chief of the Children sings the Song of the Dawning, the orchestra sustaining with a liquid and lyric mood:

A Wind bloweth forth from the East,
The Wind of the wakening Dawn:
The clutches of Sleep are released
Where the Wind bloweth on, bloweth on....
The liquid Wind of the East,
The living Breath of the Dawn!

Lo, from her crag-built nest
The Eagle glanceth afar!
She preeneth her golden breast,
And with sweep of her pinions doth soar
Over the world's dim crest
Where the lights of the Morning are.

See! In the Eastern sky,
As a herald that runneth swift,
As a chieftain who draweth nigh
With ruddy plume uplift,
One cometh .... and passeth by:
The tidings of Dawn are his gift!

'Tis the Star of the Morn, of the Morn!
A runner whom none shall withstay,
Whose red-shining token doth warn,
As he courseth his luminous way,
That a Child from the Night hath been born:
The Dawn! who foretelleth the Day!

To the growing animation of the music, the Chief of the Fathers:

Behold!
A light in the East!
Behold!
The whitening Dawn!
Unto their morning feast
The creatures of light move on:
In pasture and brake
The world is awake
With browsing herd and with wilding deer:
The Day is here!

The music is rapid and exultant. The Chorus is in swift, swinging motion, with imitative action suiting the words of their choral. All about is an incessant tinkling, as of castanets and little bells.

The Chorus:

Day is here!
Day is here, is here, is here!
Day is here, is here!

Awake, awake! On the hills the light is breaking!
Awake, awake! The heavens are aglow!
The sleepers all, their coverts are foresaking;
The winds of morning freshen as they blow:
Athwart the plain the deer with antlers shaking,
Athwart the sky the singing wildwings go!
Awake, awake! While dewy Earth is making
The springs of life with morning gladness flow!

Day is here!
Day is here, is here, is here!
Day is here, is here!

The song ceases with the Fathers to the North of the Altar, the Children to the South, all facing toward it, their parallel files forming a broad avenue from the Center Gate to the Forefront.

THEME VIII.

The music becomes strong and broad, developing the motive of the creation of light and life and the mystery of revelation.

From the Center there enter, in single file: An Acolyte bearing the spread wings, carried as a banner; an Acolyte bearing the white-plumed wand;

an Acolyte bearing a tray upon which is fire and tobacco; at a distance, the Leader, as one inspired. The three Acolytes advance to the Forefront; the Leader remains before the Altar.

The two Chiefs with their calumets go before the Acolytes. They take the fire and the tobacco and offer a smoke-offering to Heaven. Then the Acolytes, in single file as before, withdraw through the Center; the Chiefs retire to their stations.

The Leader, with uplifted gaze, intones the Psalm of Revelation:

I

With brooding mystery:

As I lay sleeping,
As I lay dreaming,
Out of the distance came one advancing,
Out of the distance came one descending,
As cometh a star from the deep of Heaven,
As cometh a star in a pool of light,
Welling to fullness,
Welling in stillness,
Till resteth its ray
On the brim of the World.

As I lay sleeping,
As I lay dreaming,
Out of the distance one came flying,
Out of the distance, with whirring of wings....
As I lay sleeping,
As I lay dreaming,
Over me drooped her glittering wings,
Over me drooped, while she chanted the mystic
Spell of the riddle that ruleth the World.

As I lay sleeping,
As I lay dreaming,
She sang me the Song of the Eldest of Mornings,
She sang me the deeds of the Father creative,
She sang me the cure of the leaderless life....
As I lay sleeping,
As I lay dreaming,
She read me the riddle that ruleth the World.

II.

With austere solemnity:

How they that were above were in Darkness
And they that were below were in Darkness:
When over all things brooded the Night, heavily....
Silent were all things,
All lay hushed.

Then the Father of Heaven breathed the Breath of Life;
Then the Father of Heaven moved upon the face of Darkness,
Upon the Body of Night,
Upon the body of the Mother of Day,
Moved the Father of Heaven,
Breathing the Breath of Life.

A Child to the Night is born!
Unto the Father of Heaven and unto the Night
Is born the Dawn....
Whose breath is the Breath of Life,
Whose gift is the Gift of Life
Unto all things.

A Child to the Night is born!
Yea, the Dawn,
Whose father is the Father of Heaven
And whose mother is the Night....
And all things above
And all things below
Are quickened into being.

III.

As at first:

As I lay sleeping,
As I lay dreaming,
She sang me the Song of the Eldest of Mornings,
She sang me the deeds of the Father of All.

IV.

Solemnly, but with enthusiasm:

Then the Father of Heaven created the Chieftain Sun:
Who is sire of the shining Day;

Who is leader of the Wardens of Light;
Who holdeth the measures of the years.
His spouse is our Mother Earth,
His warmth is the warmth of all that live,
Gladness is his offspring:
Whom the Father created Chieftain of the Skies.

Yea, the Father of Heaven united Earth and Sun
In Holy Marriage,
Whereof are born her breathing Children—
Bird and beast and mortal men—
And all her living fruits:
The Father of Heaven united Earth and Sun,
Whose Child is mortal Life.

V.

As at first:

As I lay sleeping,
As I lay dreaming,
Lo, in a Vision one came revealing
The Mystery of Life.

VI.

With exaltation:

Give heed! Give heed!
Give heed, O ye People!
Unto the Abode of Life give ye heed,
And unto the Powers thereof
Let your hearts be turned in reverence....

The Leader remains beside the Altar.

THEME IX.

The Chorus moves in stately alternation of the Semi-Choruses, chanting their antiphon to Earth and Sun.

Semi-Chorus of Fathers:

Now behold! Hither cometh the ray of our Father Sun,
Over all the land, us to touch and give us strength!

Semi-Chorus of Children:

We think on Mother Earth who lieth here:
We know she giveth of her fruitfulness.

Semi-Chorus of Fathers:

Now behold! Where mounteth up our Father Sun!
Into the Lodge of Heaven he mounteth up.

Semi-Chorus of Children:

Behold on Mother Earth the growing fields:
Behold the promise of her fruitfulness!

Semi-Chorus of Fathers:

Now behold! Through all the World our Father Sun
Sendeth his rays, the Messengers of Light!

Semi-Chorus of Children:

We think on Mother Earth who lieth here:
We see the promise of her fruitfulness.

Semi-Chorus of Fathers:

Now behold! How all the life of hill and plain
Is quickened by the rays of our Father Sun!

Semi-Chorus of Children:

Give thanks to Mother Earth for trees and streams;
Give thanks to Mother Earth for growing fields;
Give thanks to Mother Earth for ripened corn;
Give thanks to Mother Earth for food and life!

Semi-Chorus of Fathers:

Now behold! Where goeth down our Father Sun,
Who of his strength this day of life hath given!

Semi-Chorus of Children:

We think on Mother Earth who lieth here:
Truly, her power she hath given us!

Semi-Chorus of Fathers:

Now behold! Where sinketh low our Father Sun
Upon the margent of the glowing West!....
So is the life of man led forth
Out of the Night, through Morn and Noon and Eve,
To sink into the silent Night again!

Semi-Chorus of Children:

We think on Mother Earth who lieth here....

THEME X.

The mysterious music of the inaugural is resumed, but with a deeper, more austere meaning. The Chorus forms for the outgoing. Then the Leader, with arms outspread:

There is none persuadeth Death!
The old men have not told how any hath found a way.
The career of a Leader is difficult!

Marching counter, as in their entrance, the Fathers and the Children circle the sward and pass out at their respective gates, chanting:

Holy Visions, ye of yore
To our Fathers came revealing:
Hither come, O come once more,
To our troubled lives with healing!

Holy Visions, ye who bring
From the starlit Sky her glories,
Hither come on shining wing,
Pause ye where the open door is:

Pause ye at the open gate,
Enter at the silent portal,
Bless the hearts of them that wait
With the grace of light immortal:

With the grace of holy sight
To the dream-life of the dreamer
Ye shall come, and guide aright:
He shall know his life's redeemer.

Holy visions! As of yore
To our Sires ye came revealing,
Come, O come to us once more,
With the mystery of healing!

[*As the last of the Chorus is disappearing, the Leader retires, solitary.*]

PART III. THE MYSTERY.

THEME XI.

From the North and South Gates the Fathers and the Children, except their Chiefs, who remain behind, enter in an animated and swift-scattering *movement, giving the effect of individual wheelings and circlings and poisings over the whole plaza. The music is lively and full of bird themes.*

The Chorus:

Hark, hark! The birds!
The birds are a-wing!
Earth and Sky are alive
Where they flit, where they swing!
Where they dip, where they dive,
And down the winds drive,
Till with whir and with whing
Of thunderous wing
The volleying air
Is a-blare, is a-blare!

Rising, circling, dipping, fleeting,
  Now they rest, and now they haste!
Coming, going, parting, meeting,
Bird to bird his cry repeating:
  "Summer nest is Wintry waste!
"Winter stealeth Summer pleasure,
  "Garb of green he turneth gray:
"Where the winds bear Summer's treasure,
  "Thither, thither, haste away!"

Flutt'ring, flocking, flitting, flying,
  Now they rest, and now they haste,
Bird to answering bird a-crying:
  "Summer nest is Wintry waste!"

Individual singers, one by one, sing the songs of the birds, with mimetic action.

The Song of the Nestling:

O'er the prairie, o'er the prairie,
  Round about me as I walk,
How the shadows flit in circles—
  Mischief shadows, making mock!

'Tis the birds above wide circling,
  'Tis their shadows on the ground:
As when parent birds protecting
  Feeble nestlings circle round.

Birds of Heaven, Birds of Heaven,
  We, your nestlings, joyous cry
When His sign of care ye give us,
  Wheeling in the azure sky!

The Song of the Wren:

Whe kee re re wee chee!
Whe kee re re wee chee!
  Joy, joy, joy!
Singeth the tiny Wren:
And shall not men
  Know joy?

The Song of the Duck:

Lo, the Finder-Out of Ways—
The Bird of the Emerald Crest—
The Bird who never strays,
  But doth fare
In arrowy flight and ware
Over water and earth and air,
  North and South,
  East and West....
Oh, the speeding Scout of the Skies
Knoweth their quartering ties:
As the Leader of Men must know
Where the paths of Heaven go!

The Song of the Owl:

He! Hiri Wahoru!
He! Hiri Wahoru!
  Wide-eyed Bird of the Night,
  Who seest invisible things
  And spreadest thy shadowy wings
  In dim and inaudible flight....
He! Hiri Wahoru!
He! Hiri Wahoru!
  Let ours be the gift of thy sight!

Full Chorus:

Oh, the Bird, the Birds!
The Birds are a-wing!

Like sky-blown herds
At the wintry sting
Which the North
Striketh forth....
Where they come,
Where they go,
All the air
Is a-blare,
All the air is a-thrum,
As with beating of drum
And sounding of string
Where drawn is the bow
And the swift arrows sing!....
Oh, the Birds are a-wing!

Summer flown,
Nestlings grown,
Southward blown
Wide a-wing!

The Chorus ends its evolutions with the two divisions forming, as it were, encircling wings, across the Forefront, and facing Northwest and Southwest, so as to view the gates.

THEME XII.

The temper of the music becomes more grave, with the flutes of the Eagle dominant.

Semi-Chorus of Fathers:

Behold, an Eagle now is circling, widely circling above us!

Semi-Chorus of Children:

As the mother-bird circleth her nestlings, careful for her chicks,
She circleth us, hovering....

Full Chorus:

She is the Eagle of God!
Of Him who is Father of Heaven,
Who ruleth the zonéd Earth
And sendeth His will by the Eagle
Over the windy Pathways
That lead from Man up to God!....

The motive of the music is the poignancy of human aspiration.

From the North Gate, the Gate of the Fathers, enter: An Acolyte with the Spread Wings, borne as a banner; an Acolyte with the Corn, one with the Bowl, one with Tray and Cups; the Leader, with wand; an Acolyte with the white-plumed wand. They march in single file, their path a semi-circle from the North to the South Gate, around the Altar. At the Altar they stop.

The Leader:

I know not if the voice of man can reach unto the Skies;
I know not if the Silent One can hear me as I pray;
I know not if my words be foolish words or wise;
I know not if I walk in straight or crooked way.

I only know His power, Who hath made our mortal lot
An hurt and stumbling pace led outward through the dark;
I only know his trust, Who lest He be forgot,
Hath weathered deep the soul of man with an immortal mark.

As they move on toward the South Gate, the Gate of the Children, the Acolytes sing, in choral:

Father, unto thee we cry!
Father of all we hear and see,
Father of all we feel and hope,
Author of life's mystery:
Father, unto thee we cry!

They pass out.

THEME XIII.

The Fathers, pianissimo bass, sing:

With the dawn will I seek my child,
With the tenderly growing dawn;
Where the breath of the morn floweth on
I will go seeking my child,
My little one, my son....

With swelling music, the Children:

Father, come unto me here,
Here where I wait for thee,—
With bread and with morning cheer,
Father, come unto me!

The Fathers:

I come, my child, I come,
Seeking for thee....
Abide me, and nothing fear:
On the wings of the dawn I come
Seeking for thee....

The Children:

See!
The Eagle is flying o'er us!
In the sky above, from the Father's home!
The Eagle descendeth unto us
With the Father's cheer!

In the music is the note of the dawning Light. Then the Chorus:

Behold!
The Star of the East!
The Star of the bursting Morn!

From the Gate of the Children a runner, personator of the Morning Star, clad all in red, the color of life, and in his hair a red plume, symbolic of the breath of life. To his arms are attached spread wings. Sweeping past, he cries:

A Child is born!
Unto Man a Child is born!
Unto Man is born a Son!

He passes forth by the North Gate, the Gate of the Fathers.

The Chorus:

A Child is born! A Child is born!
An holy Child is born!
Stars of the Morning rejoice!
Life is renewed in the World!

The music swells with prophetic exaltation.

Enter from the Holy Gate: The Acolyte with the Spread Wings; the Acolyte with the Corn, he with the Bowl, he with the Tray on which are the four cups; the Acolyte with the white-plumed Wand; the Leader; the Chief of the Fathers, carrying the Child; the Chief of the Children.

When all are entered the Leader takes the Child and holds him aloft, crying:

Behold the Child!

The Chorus:

Behold the Child!
Behold the Promised One!

The Leader returns the Child to the Chief of the Fathers, on either side of whom the Acolytes range themselves, and leads the way to the Altar, while the Acolytes sing:

Here we go singing, singing,....
Looking on the Child—
The little Child who leadeth us,
Borne in his father's arms:
Here we go singing, singing....
Looking on the Child.

THEME XIV.

At the Altar they form: the Leader a few paces in advance, at his left the Chief of the Fathers with the Child and the Chief of the Children; the Acolytes ranged before the Altar, the white-plumed Wand to the North, the Spread Wings to the South.

The Leader spreads his hands, like spread wings, above the Child. He signals to the Acolyte with the tray and cups, who advances. The Leader dips his finger into one of the cups and touches the Child, drawing a semi-circle about his brow.

The Leader:

With the Blue of the Skies I anoint thee....

The Chorus:

That thou may'st long abide beneath the Lodge of Heaven.

The Leader dips his finger into a second cup and draws it across the Child's chin:

With the Green of the Earth I anoint thee....

The Chorus:

That thy feet may be led amid fruitful fields.

Dipping into a third cup and touching the Child's cheeks:

With the Crimson of Life I anoint thee....

The Chorus:

That strength and vigor shall be thine in youth and age.

Dipping into the fourth cup the Leader touches the Child's brow:

With Oil and with Fat I anoint thee....

The Chorus:

That peace and plenty may follow thee all thy days.

The Acolyte retires; the Leader once more spreads his hands above the Child; a second Acolyte advances, bearing the Corn. The Leader taking it, strokes the Child's body:

I stroke thee with the ripened Corn....

The Chorus:

So may thy body's needs be satisfied!

The Acolyte with the Bowl advances. The Leader sprinkles the Child:

I refresh thee with the clear and running stream....

The Chorus:

So may thy generations run onward without ceasing.

The Acolytes retire. The Leader takes from his hair the white eagle-down and fastens it in the Child's hair:

With this sacred token I thee adorn—
Symbol of the fleecy clouds above,
Symbol of the winds of Heaven,
Symbol of the living breath
Into the body of man
Breathed by the Father....

After a pause, his hands resting on the Child's head:

Enter ye into the House of Life, consecrate.

He returns the Child to the Chief of the Fathers. Then triumphantly:

I know now that the voice of man can reach the skies;
I know now that the Mighty One can hear me as I pray;
I know our Father answereth his children's troubled cries,
And pace by pace assigneth us the token of the way.

Give heed! Give heed!
Give heed, O ye People!
Unto the Abode of Life give ye heed,
And unto the Powers thereof
Let your hearts be turned in reverence....

THEME XV.

The music becomes reminiscent of the Chant of the Way of Life. The Chorus moves forward, forming a circle, the Children within, the Fathers without, as in the figure of the lodge. The Leader advances beyond the Altar and paces a small circle, or symbolic lodge. The two Chiefs enter this circle while the Acolytes, with emblems raised as in blessing, form a semi-circle behind.

The Chief of the Children takes the Child from the Chief of the Fathers. Then the Chief of the Fathers moves forward and sings:

Within the House of Life man entereth
  A little Child with slow and faltering feet:
The breathing Heaven is in his fluttering breath,
  The pulse of Earth in his swift blood doth beat.

Within the House of Life man tarrieth,
  As one who for a season taketh rest:
The Blue above, below the grassy Earth,—
  An oriole within a wind-swept nest.

Within the House of Life man offereth
  The simple tokens of his daily need,
His prayer for food and drink, in humble faith
  That some dim distant Power shall give them heed.

Then from the House of Life he hasteneth....
  Aye, as an Eagle in his feathered mail
Battleth adown the blast with windy Death,
  Speedeth the Warrior-Soul with battle-hail!

The Chorus is in motion, moving in a strange dance simulating the flight of eagles. They form in files and circle about the central group. They sing:

Come, ye Fathers!
Come, ye Children!
Come, ye People,—
Mortal men!
Into the House of Life, come enter!
Into the House—the Way is open:
Enter in, O mortal men!....
Like flocking birds,
Like shouting eagles,
Full of joy and lust of life,

Swiftly, swiftly, swiftly come ye,—
Enter in, O mortal men!....
As your Fathers came before you,
As a little child doth come,
Where the Way is open, open,
Enter in, O mortal men!....

As they cease, the two groups, the Fathers and the Children, are formed, on the North and the South, like the spread wings of an Eagle. The two Chiefs, in the center, are the bird's body; the Acolytes, with the emblems, have retreated, forming, as it were, the tail plumes; the Leader, with the Child, has advanced to the head.

There falls an utter stillness. The Leader uplifts the Child, looking upward. In a penetrating voice he cries:

Breathe on him!
Breathe on him!
Life thou alone canst give him:
Long life, we pray, O Father, give unto him!

Mid swelling music, like the march of the tribes and nations of men, exeunt omnes.

THE END.

CRITICISMS AND DISCUSSIONS.

BERGSON AND RELIGION.

Henri Bergson is probably the most potential name in modern philosophy. Prof. William James, who was by common consent our most distinguished thinker, though he was much older, called Professor Bergson "master and teacher." This, certainly, is high praise.

Aside from his speculative capacity, Professor Bergson is a most interesting figure. He is an earnest student of physiology, biology and psychology, and he brings to his philosophical theories a great wealth of scientific illustration and proof. And unlike so many of our great metaphysicians, he has literary power, the gift of musical speech. Whether the *Evolution créatrice* is great art like the Corinthians of Paul, the Divine Comedy, "Lycidas," or "Les Miserables," it may be too soon to decide. But it is certainly a work of art, and of no mean order. Professor Bergson is a personality, and his thought is always suggestive and commands attention.

It is interesting to watch the flight of his speculative arrows, even though we fail to see that they strike any target. Nevertheless, in my judgment he has made one vital suggestion, which I shall indicate in the course of this study. But first I shall attempt to trace his theory of the universe and his theory of truth and show their philosophical and religious meaning and influence.

As every one comes to a study with certain prepossessions, I may say that I am not a materialist, idealist or pragmatist, but conceive there are in man elements not mechanical and that he has, within narrow limits, the power of choice.

Professor Bergson in his theory makes an immeasurable primeval "super-consciousness" the source of all things, of life and matter. This unique creative absolute has will, freedom, and an impulse to create, but strange to say, though it has this consciousness and spontaneity, it has not intelligence. It moves on and on, ever unfolding, ever augmenting, with no design or purpose, seeking no predetermined goal, for M. Bergson frowns upon all forms of teleology.

This theory of a great life-river, if I may so describe it, ever seeking to find new channels of creative opportunity, I found to my surprise was similar to that of my friend, Prof. F. C. Doan, published in the *Journal of Philosophy* about two years before the "Creative Evolution" appeared. I learned, since commencing this paper, that Professor James had made the same discovery. And I may say that, leaving off certain naive features in the book of Genesis, M. Bergson's theory of the origin of the world reminds me of that great sentence: "In the beginning, God."

Whence comes this vast energy with its impulse to create, M. Bergson does not tell. He asserts that from it spring both life and matter and that every living thing, from the lichen on the rock to the golden dandelion nodding in the south wind, from the ameba to the man, possesses consciousness and freedom, and these qualities enmeshed and entangled in matter, reduced or attenuated to the finest threads, are never lost. At times Professor Bergson calls matter "the enemy" of all good. It is ever to be resisted, it must be transmuted into living organisms, it must be saturated with "contingency."

Again he calls the resistance of matter a "stimulus." It is by the reduction of the flesh, by the chastening of the senses, that men become healthy, strong and beautiful.

It will be seen that in this dogma of the life-urge, M. Bergson strenuously opposes the new naturalism so popular at the close of the last century. He affirms that life always has in it the seeds of freedom or contingency, that contingency grows greater as organisms develop. He cannot believe that the high reason that has traced the laws of the earth and measured the stars, that the hope, affection, imagination which blossomed into the melodious words of the Sermon on the Mount are the product of mechanical and unintelligent forces—that blind physical atoms could in time stumble into an orderly living universe!

Has Professor Bergson spoken a deep, living word? Has he made a new synthesis? There are many who believe that he has. He opposes the older idealism of Kant, Hegel and Fichte, and the "absolute" of such teachers as Royce and Bradley in his theory of time. For unlike them, he makes time a reality, and in time creation begins. His theory of this original creative energy makes the universe of life and matter a great Mississippi life-river, ever flowing on. Its course may be traced in the past and in the present, but its course in the future, he says, no one, however wise, can trace.

That the future is impenetrably veiled (an idea I have long contended for), Professor Bergson urges from the fact that the universe is not made but making. There is ever the condition of uncertainty, of spontaneity, of contingency, and thence may come the unexpected. We see now the leaf, the stalk, the bud, but of the glory and beauty of the flower and fruit, none can know.

M. Bergson's opposition to materialism is seen in his radical idea of freedom. He maintains that in all living organisms there is something that cannot be accounted for by the laws of matter. There is in them a power which draws from itself more than it receives, "gives more than has been given to it." There is something free in the violet, the bird, the man, not produced by reflex action. There is a tiny will, a drop of beauty, of will, of love, of intelligence, which is pure creation. There is ever the quality of the contingent, the new, the unforeseen, for this is a "spiritual universe."

Of course the idealist will say that Professor Bergson's theory destroys the timelessness and omniscience of the Absolute, and the naturalist will ask for proof. He will inquire, why it was necessary to invent a "superconsciousness" to start the universe. He will say it is just as easy to think of life evolving from matter, as matter from life; and the idealist will be alarmed at the thought of admitting into the universe the element of imperfection and the unforeseen.

But we must now come to the more original, and more radical part of his theory, his theory of truth. The extreme radicalism of his idea may be seen from the fact that M. Bergson makes "not reason but instinct bring us into the closest touch, the directest relation with what is most real in the universe," to use the words of Mr. Balfour. In this, I may say that Professor Bergson follows the present tendency to distrust the power of the intellect to reach a reasonable explanation of the universe—to prove the existence of God, of freedom, of immortality. All questions of ultimates are beyond intellectual search. The intellect is limited to the sphere of experience.

Professor Bergson agrees with this distrust of the intellect, but affirms that what is impossible to the intellect is possible to instinct. The province of reason is not life, freedom, spirituality, but matter, mechanics and space, "the waste products of the" life-urge. James agrees with his teacher here, for he says that "the reason can know only surfaces." But, one may ask, are not these statements purely dogmatic, speculative?

Professor Bergson, while he admits the immeasurably wider horizons of the human intellect, asserts that instinct, in ants and bees in which it comes to its perfect flower, is in touch with a higher order of truth. Maeterlinck makes a similar assertion in his work on the bees.

But surely there lie innumerable difficulties in the path of this fantastic theory. If the instinct of the Hymenopterae is the infallible organ for the discovery of knowledge, why is it that they do not advance, but keep in the same monotonous round? With this great power, why is their achievement so limited, their vision so narrow? Why should they have so much of this divine power, and man, who is so incomparably greater, have so little? With this great endowment, wherein have they advanced beyond him?

Professor Bergson tells of a certain kind of wasp, the fossorial, which, instead of killing its victim, stings it into unconsciousness by a most delicate surgical act. This mechanical skill, he says, does not come as the result of numberless experiments, and it would be forever impossible to intelligence, but it comes through that instinct which reveals to the wasp the secret of life itself.

Does it not seem fantastic, to say the least, that the instinct of the fossorial wasp can reach a higher truth than the most sustained efforts of a great intellect? Can the work or conquests of the ants and bees compare with the magnificent achievements of the human intellect in mathematical, physical and moral science? Do we come into nearer touch with reality in the cell of an ant than in a painting by Titian?

It is true that the instincts are nearest the primeval forces, and may guide us best in the things of the flesh. Instinct may, by a sort of divine unreason, go straight to the heart of the lower truth, but to solve the supreme problems, the meaning of life, the existence of God, of freedom and of immortality must be an achievement of the highest intellect.

But it would not be fair to M. Bergson, not to explain that these mystical assertions, these speculative dreams, are enmeshed in a profusion of scientific illustration. He shows a minute and wide knowledge of physiology, biology and natural history, and in his boldest speculative flights always makes his final appeal to concrete facts.

But now I come to the question that will arise in many minds:

Is the philosophy of Professor Bergson religious in the highest sense? Does it make its appeal to our spiritual faith and aspirations?

Though this philosophy is radically opposed to the mechanical and atheistical tendencies of naturalism, many will say it cannot be called religious, as M. Bergson certainly means it to be. It is true, the Christian may see theism in the primeval life-urge, which is the source of matter and all living things, and in the exaltation of the instinct a recognition of the validity of the religious intuitions; but it will be difficult for the intelligent man to see a real theism in this primeval creative consciousness, though it has the will to create and freedom, but has no plan or purpose, nor directs the universe to any intelligent goal.

And while in his theory of evolution he escapes the difficulty or dilemma of the old metaphysical systems (that the imperfections, the evil, the sorrows of the universe, had been known to God before He created it, and were of His own selection), it does seem difficult to feel the sense of worship in the thought of a universe ever evolving, yet ever unintelligible and unmoral.

In the pluralism of Professor James, though he calls himself a pupil of Bergson, there is something for the common mind to catch hold of. When he says that God is the deepest power in the universe and is a personality, that "man and God have purposes for which they care and each can hear the other's call," he makes an appeal to the humblest believer. But I fear that the common people will not see the religious element in the philosophy of Professor Bergson. The saints who love and pray will cling to the thought of a transcendent God, leading the world to a wise and happy end, rather than believe in this impersonal life-force that forever unfolds, goes on and on, but knows not whither it is going.

On the other hand, the scientist will have his own thoughts. He sees that M. Bergson, to find an explanation, goes back to that primal sea of life. He will say that he cannot discover wherein that is different from the theologian's going back to God.

Yet, on the whole, I should say that the philosophy of Professor Bergson is theistic rather than atheistic, and spiritual rather than material and mechanical.

I said in the beginning of this study that Professor Bergson had made, in my judgment, a vital suggestion, and that is his recognition of the high function of philosophy. Although in his theory he remands the intellect to a much lower place than instinct, he forgets it in practice when he affirms that the vital, the supreme questions, "What are we; What are we doing here; and whence do we come and whither do we go?" *are the very cause of philosophy's existence*; and that the *future* (italics mine) will give back to philosophy its rightful place—the first.

Professor Bergson does not think that we can arrive at objective certitude or that we can force assent, but he suggests that the collection of many facts and their interpretation may give us a direction, "a direction only." These "lines of facts" will give nothing but a probability; "but all together, by converging on the same point, may give us an accumulation of probabilities which will gradually approximate *scientific certainty*."

It is a pleasure to see the view I have been contending for—that to this present discredit of the intellect, of philosophy, there will come a reaction—confirmed by the high authority of Professor Bergson. How far the reality to be known may exceed the power to know I cannot tell, but this seems reasonable, that the universe has an intellectual answer to those intellectual questions with which it continually confronts us. There is in us the indomitable belief that the terror and mystery of the material world may be transformed by a large knowledge into "transparent formulae." Should we not have the same belief that the terror and mystery of the moral and religious worlds may be also, by a larger intelligence, transformed into "transparent formulae"?

My study must end here, and I am aware how imperfect it has been, but I have tried to represent Professor Bergson kindly and impartially. This task has not been easy for, as Mr. Balfour says, there are parts of his theory, especially his theory of knowledge, difficult to comprehend; but I am sure all will consent that he has broken open new ground, and we can admit even the exaggeration of Professor James: "Open Bergson and new horizons loom on every page you read. It is like the breath of the morning and the song of birds."

James G. Townsend.

Jamestown, N. Y.

THE ANTI-INTELLECTUAL MOVEMENT OF TO-DAY.

Never before in the history of the world has science played such a prominent part and received more recognition as the main factor of civilization. And in truth there is a general agreement as to the hope that we stand at the threshold of the age of science, which means that all problems of life will be solved by scientific inquiry and the old superstitions will be swept away. This principle has been applied to the several domains of life, to transportation, to sanitation, to the preparation of food and medical problems, the building of our homes and public edifices, yea even to the sphere of social and religious life. It is strange, however, that in these very days there have repeatedly appeared philosophical movements which are decidedly anti-intellectual, and treat science with a contempt in favor of the instinctive promptings of sentiment, which is only paralleled among the most old-fashioned dogmatists, in the tendencies of religious faith by such men as Augustine and Luther who treat reason as an enemy to faith, and endorse the old principle *Credo quia absurdum.*

We will here make a few comments in explanation of this movement without taking sides either with the admirers or the critics of the new fashion. The latter, the critical aspect, is most exactly represented by Mr. Bertrand Russell, pages 321 to 347 of the present number of *The Monist*; the other to some extent by the Rev. Dr. James G. Townsend. Mr. Russell points out that "if he (Bergson) fails in his condemnation of the intellect, the intellect will succeed in its condemnation of him."

It seems rather strange that in the days of the dawn of an age of science such movements should be so prominent, but it seems to me that these movements are the natural reaction against the many wrong aspirations of science, for it can not be denied that the prominence which science has gained in our days has also produced a number of narrow-minded scientists, who apply their narrow view of science to the whole of life. To them science is either physics or chemistry or biochemistry, or whatever their specialty may be, and most of them are acquainted with science only in its lower branches, mechanics or physics or some other domain which is void of the higher development of man where it unfolds itself in social and moral ideas. Psychology to such minds is a mere function of the brain, and the truly typical features of the soul are an accidental by-play of its coarsest substratum, or to draw their ultimate conclusion, mind is considered a function of matter. Their view of nature is limited, and while they rob man of his nobility they degrade him into an equality not only with the brute but even with inanimate existence.

The expression of this kind of narrow-minded science which is not true science but the lowest step in the development of science, has caused a distrust in the true nature of science.

Anti-intellectualism has become fashionable in the philosophical world. Prof. William James made a great propaganda for it and succeeded mainly by his amiable personality. He speaks in the name of a certain common sense which stands up for unscientific views and defends a pluralism as well as a subjectivism on the ground that it is based on experience. For the same reason theory is discredited for sheer love of single and unrelated facts. Facts, however, are replaced by interpretations of a very primitive kind, among which even belief in spirits plays an important part. This incoherent system which abhors all moralism and actually represents a reaction to the world-conception of savage life goes under the name of pragmatism. It has made many conquests and gained many brilliant adherents even in the stable circles of European scholarship.

Another center of anti-intellectualism has been established in France of which Henri Bergson has become the leader through his unprecedented brilliancy of style and oratorical talent. He has gained many adherents in his own country, France, and celebrated high triumphs in conservative England. He is expected in the United States, and we have no doubt that he will be welcome in the circles of all who are dissatisfied with the quiet and unpretentious method of patient inquiry and scientific research. Men of this type possess great zeal and they will naturally welcome an ingenious representative of their philosophic tendencies.

In the meantime the spirit of criticism is stirring in England, and we have before us a book which with all soberness reviews the significance of the new star which has risen on the philosophical horizon. It is written by Hugh S. R. Elliot, LL. T., the editor of *The Letters of John Stuart Mill.** Sir Ray Lankester, K. C. B., F. R. S., being invited to write a few words by way of preface to Mr. Elliot's book, says:

"I am glad to do this, not merely because I think that the books in which M. Bergson formulates those illusions are worthless and unprofitable matter, causing waste of time and confusion of thought to many of those who are induced to read them, but also because an unmerited importance has been attached to them by a section of the English public, misled by the ingenious and systematic advertise-

* *Modern Science and the Illusions of Professor Bergson.* By Hugh S. R. Elliot. London, Longmans Green & Co., 1912. Price $1.60 net.

ment of M. Bergson by those who amuse themselves with metaphysical curiosities. He has been introduced to us as a "great French philosopher.' To those who in a thoroughgoing way occupy themselves in collecting and comparing and classifying all the absurdities which have been put forward as 'metaphysics' or 'metaphysical speculation' since the days of Aristotle, this latest effusion has, no doubt, a kind of interest such as a collector may take in a curious species of beetle. To the student of the aberrations and monstrosities of the mind of man, M. Bergson's works will always be documents of value. But it is an injustice as well as an inaccuracy to speak of their author as 'great,' or 'French,' or a 'philosopher.'....

"A main objection to M. Bergson's account of his own performances in the dark chamber [of the metaphysical x] is that he is not content with asserting (and expecting us to accept his bare assertion) that time is a stuff both 'resistant and substantial,' that consciousness is not always dependent on cerebral structure, that intuition is a true guide and the intellect an erroneous guide. Such escapades in the dark room astonish and interest only those who are unacquainted with M. Bergson's numerous predecessors in the maddening hunt of the illusive black cat. It is, however, a speciality of M. Bergson that having by mere assertion attempted to make us believe that he has grasped the black cat, and at any rate has in his hand some hairs from its tail—he proceeds in the same spirit to make absolutely baseless assertions about the domain of scientific fact—a domain 'tabooed' against him and his fraternity. He writes of the facts of physical science with the same careless assurance as that which we tolerate with indifference when he is disporting himself in the extra-territorial region of x. Having made his arbitrary assumptions about x, he proceeds in an inaccurate way to write about some of the well-ascertained facts of the structure of animals and plants. He promulgates novel opinions about them with the air of one who has given serious study to them, which, however, it is abundantly evident he has not. By a light-hearted perversion of the facts as to the structure of the eyes of animals and other such things, he endeavors to make them appear as evidence in support of his arbitrary and preposterous fancies about x! In doing so he ceases to be merely an amusing juggler with the harmless creations of his own and other people's fancy: he becomes a maker of untruth, and for those who listen to him a harmful *Confusionsmeister*.

"M. Bergson is gifted with an admirable facility of diction, and has succeeded in arresting attention. On that account, since he has exceeded the limits of fantastic speculation which it is customary to tolerate on the stage of metaphysics, and has carried his methods into the arena of sober science, it is a matter of urgency that his illusions and perversions should be exposed with uncompromising frankness to the reading public who may be, on their side, under an illusion as to the importance of his teaching. Mr. Elliot's book effects this exposure in a masterly way."

M. Bergson proposes the strange doctrine that perception does not reside in the brain of the perceiver, but in the object perceived, —a proposition which is bewildering, and among his arguments he declares it theoretically not inconceivable that matter should be perceived without sense organs. Such doctrines belong to the cornerstone of his philosophy, and as an instance of Mr. Elliot's critique we will here quote some paragraphs discussing M. Bergson's theory of pain and of memory. M. Bergson defines pain as an "effort to repair damage." Mr. Elliot writes:

"Just as perception is located in the perceived object, so Bergson alleges that pain is located in that part of the body where it appears to be felt. This is of course in opposition to the belief of physiologists, who affirm that the pain is really located in the brain, not at the nerve endings; and who support their contention by pointing, for instance, to the pain which a patient feels and refers to his foot after it has for years been amputated. I am not, however, concerned to defend a well-established fact: I wish only to point out Bergson's mode of refuting it. 'If [the pain] is not at the point where it appears to rise, neither can it be anywhere else: if it is not in the nerve, neither is it in the brain; for to explain its projection from the center to the periphery a certain force is necessary, which must be attributed to a consciousness that is to some extent active. Therefore, he must go further....' Here we get a chain of deductions, every ling of which appears to be false. Why should any force be necessary? Why should that force be attributed to a consciousness? Why should that consciousness be active? It was one of Huxley's chief gifts to biology to have largely banished deduction from that science, by strongly insisting on the danger of traveling outside ascertained facts. A succession of deductions like this, in a physiological inquiry, is *a priori* almost certain to be erroneous. To me *a posteriori* there seems not even *prima facie* evidence in favor of any of them: and they are set against a fact experimentally arrived at!

"The doctrine of two kinds of memory is a complication of natural facts that will not appeal to anybody. But the fundamental objection to it is that so often raised already: that there are no facts to support it. The Professor attacks the physiological view of memory: he adduces a number of facts, such as those of sensory aphasia, in opposition to it; and having destroyed it to his own satisfaction, forthwith we are presented with a new theory which is assumed to be true. This new theory is worked out in extreme detail; it is unaffected by sensory aphasia, but otherwise the only credentials it can produce are those of extreme unintelligibility. We have already had occasion to observe that a doctrine is safest from criticism when it is most difficult to understand. The fog is so thick that the critic is disarmed. I therefore make no specific attack upon it, beyond insisting upon the complete absence of evidence. Moreover, the attack on the physiological theory could scarcely convince ony one but a metaphysician. 'If memories are really deposited in the cortical cells, we should find in sensory aphasia the irreparable loss of certain determined words, the integral conservation of others.' But it is not so. Now, what would a man of science consider himself entitled to deduce from this? Nothing further than that words are not represented in the brain in minute specific areas for each word, but that they are, or may be, represented in some other way, possibly still undiscovered. But what does Bergson infer? That the memories of words are not stored in the brain at all. He refutes a crude physiological hypothesis; he then assumes that the refutation applies to all possible physiological hypotheses, and thence jumps to his own theory. It would have been just as reasonable to found his own theory upon a refutation of Gall's phrenology. For phrenology was a thoroughly materialistic hypothesis; it assumed absolute connection between mind and brain, and definite localization of mental faculties in the brain. Phrenology has long been exploded, but no one (except a metaphysician) infers from that that there is no connection between mind and brain. A belief in that connection is in no wise shaken by the exposure of phrenology; nor is it shaken by the criticism of other crude attempts to localize mental qualities. These criticisms are effective only for the particular theories against which they are levelled. Hence we see that Bergson's theory of mind and matter is founded upon the same fallacy as that of the vital impetus—the fallacy which we stigmatized as the mannikin fallacy at the beginning of the chapter. In bald outline it is like refuting Mahommedanism, and then arguing: (1) Mahommedanism is untrue; (2) therefore all religion is untrue; (3) therefore all morality is a superstition. We have only to point out that (2) does not follow from (1), nor does (3) follow from (2). In Bergson's works the second step (2) is invariably taken silently immediately (1) has been established. The great show of facts in his works are all connected with step (1), the criticism of adverse theories. Step (2) is then slurred over without a word of discussion, and the rest of the philosophy is taken up with step (3), which is just a hypothesis or guess, or intuition, having no connection with foregoing facts, but set out with such a wealth of words and analogies that the unwary reader quickly loses his way and is totally lost. In alliance with the main paralogism is the copious misuse of analogies and of words, the latter especially in the form of materializing abstractions such as time, life, motion, memory. The medieval realists could scarcely have gone farther.

"The tendency to attribute substantial reality to abstractions is conspicuous not only in metaphysics but in the thinking of all primitive races. Thus a Basuto will not walk by a river lest his shadow falling on the water should be seized and devoured by a crocodile. Nearly all children at one time or another attempt to evade their shadows by jumping or running. Names likewise are looked upon as material things: as among the Chinooks, one of whom thought that Kane's desire to know his name proceeded from a wish to steal it. Here, as elsewhere, Bergson does nothing more than systematize and magnify, on an enormous scale, almost universal vices of thought."

As an example of Bergson's method we will quote a few more passages from Mr. Elliot's book. Bergson says:

"'Instinct is knowledge at a distance. It has the same relation to intelligence that vision has to touch.' Why, then, do we owe our knowledge of the stars to intelligence, and not to instinct? Why has astronomy advanced by the gradual triumph of intelligence over bigoted superstition?....

"Bergson's attempt to establish the preeminence of men and hymenoptera takes, in one place, the following form:—'It is unquestionable that success is the most general criterion of superiority, the two terms being, up to a certain point, synonymous. By success must be understood, so far as the living being is concerned, an aptitude to develop in the most diverse environments through the greatest possible variety of obstacles so as to cover the widest possible extent of ground. A species which claims the entire earth for its domain is truly a dominating and, consequently, superior species. Such is the human species, which represents the culminating point of the evolution of the vertebrates. But such also are, in the series of the articulate, the insects, and, in particular, certain Hymenoptera. It has been said of the ants that, as man is lord of the soil, they are lords of the subsoil.'

"Under this definition, birds ought to be a dominating group, for their distribution is wider than that of men. And the most preeminent species of all would not be men, or insects, or even birds, but those simple unicellular creatures like ameba, which are found everywhere all over the earth."

Mr. Elliot sums up the whole book with the following conclusion:

"Professor Bergson's philosophy is contained in three volumes. I here summarize my main objection to the fundamental doctrine of each:

"1. Time is a stuff both 'resistant and substantial.' Where is the specimen on which this allegation is founded?

"2. Consciousness is to some extent independent of cerebral structure. Professor Bergson thinks he proves this by disproving a crude theory of localization of mental qualities. Will he furnish evidence of its existence apart from cerebral structure?

"3. Instinct leads us to a comprehension of life, that intellect could never give. Will Professor Bergson furnish instances of the successes of instinct in biological inquiries, where intellect has failed?

"I venture to think that, until these questions are answered, we are not called upon to consider further the merits of Professor Bergson's philosophy."

EDITOR.

KANT AND BERGSON.[1]

> "It is an incorrect and perverted usage of the word 'symbolic,' but one which is accepted by modern logicians, when it is set in opposition to the 'intuitive' mode of thought; for the symbolic is only a species of the intuitive."—Kant, *Critique of Judgment.*

We have in Kant not only the founder of criticism as a system or a method which would be appreciated for their positive qualities; but on the other hand the purely critical, or if you prefer negative, element is for the most part considered from an historical

[1] Translated from the German by Lydia G. Robinson.

point of view in its application against rationalism and against Locke and Hume. At bottom, however, Kant himself has tirelessly given expression to the propaedeutic character of his critique as among its most essential features. Therefore it seemed to him most important once for all to demonstrate metaphysics, as he found it and understood it, to be futile and impossible, in so far as it laid claim to being a system of cognitions.

Whether or not Kant had come in actual contact with historical metaphysics alone makes no difference. He undoubtedly wished to do away with metaphysics in itself. It may be objected that he opposes his criticism to that kind of metaphysics which he himself has constructed as the object of attack. Nevertheless his critique has a far broader application inasmuch as it makes metaphysics in general the object of investigation. Whoever maintains the mere possibility of a metaphysics must in some way or other decide the question which Kant himself stated and wished to have solved, namely whether metaphysics is at all possible.

In solving this problem it is a matter of indifference whether or not one employs the Kantian method of deducing the possibility of the thing sought from its postulate, from the hypothetical assumptions of the problem. Only in one way or another the critical attitude must be brought to bear upon the question. Hence the nature of metaphysics or its necessity must not be asserted and presented before its possibility is proved. Therefore it is really impossible for a Kantian to admit the methods employed by Bergson in founding a new kind of metaphysics. Nevertheless we shall first accommodate ourselves to his mode of thought so that we can not be subject to the charge of orthodox critique. Yes we will even go one step further. We will hypothetically admit that Bergson's definition of metaphysics is right. He asserts metaphysics to be the science which gets along without symbols; it is intuitive knowledge.

According to Bergson himself intuition is a sort of mental sympathy by means of which one may transfer himself into the midst of an object. Bergson avails himself of still other senses in order to make this kind of cognition comprehensible to us. It is a kind of mental auscultation, an intellectual vision. My present task is to investigate whether such an intuition is possible, whether it is at all thinkable. Bergson is satisfied simply to make the assertion. But I will first show that even granting its possibility it does not accomplish what is claimed for it.

From the very beginning intuition is something more than merely a kind of cognition. It is supposed to transfer us directly into the very being of the object, but in this being is included existence. A comprehension of existence is at the same time a comprehension of the cause of existence. The play is ceaselessly repeated, one direct leap carries us across the abyss of cognition, perception and comprehension. In intuition existence itself is posited. The more intuition is built up upon being, upon existence, the more creative and the more constructive does it itself become. A second process, that of deepening, runs parallel to this development of the concept of intuition. From a comprehension of the object, from a sinking into a strange object, from a constantly greater pouring out of the subject, intuition becomes more and more an internal process; finally, in intuition the subject comes more and more to comprehend itself, its creative nature, its most profound existence.

The stages of this development are well known. Scholasticism saw in intuition the cognition of existence or non-existence. "Intuitive knowledge of a thing is knowledge by means of which it may be known whether a thing is or is not."[2] The logical antithesis of existence and non-existence indicates even beyond that the creative cause of existence. It is only necessary for the factor of necessity to be abstracted from its logical wrappings to make it clear that a decision about existence or non-existence ultimately depends on the foundation, the positing of existence. Existence once posited, the cognition of its necessity is at the same time admitted.

Spinoza goes even one step farther:

"This kind of knowledge, i. e., intuitive knowledge, proceeds from an adequate idea of the absolute essence of certain attributes of God to the adequate knowledge of the essence of things."[3] By reference to God, existence is therefore established more securely so that the *scientia intuitiva*[4] latterly comes to include existence. Intuitive knowledge as knowledge under the form of eternity comprises this, that the essentialities of things follow from the eternal nature of God by eternal necessity.[5] And if we must remove the

[2] "Notitia intuitiva rei est talis notitia, virtute cuius potest sciri, utrum res sit vel non sit."—William of Occam, in *l. sent, prooem.*

[3] "Atque hoc cognoscendi genus (*sc. scientia intuitiva*) procedit ab adaequata idea essentiae, formalis quorundarum Dei attributorum ad adaequatam cognitionem essentiae rerum."—*Ethices*, II, Propos. XL, Schol. 2.

[4] Or *cognitio intuitiva, Eth.* V, Prop. 36 Schol.

[5] *Eth.* V, Prop. 25, 27, and Dem. 32.

factor of the creative, we must nevertheless emphasize with Spinoza himself the power and the force of this third step in cognition from which the *amor intellectualis dei* arises.

The necessity of existence in the *scientia intuitiva* can not be more emphatically expressed than in the words: "Therefore to conceive things under the form of eternity is to conceive things in so far as they are conceived through the essence of God as real entities or insofar as they involve existence through the essence of of God."[6] The climax of this development of the concept of intuition (*Intuitionsbegriff*) is Kant's interpretation of the nature of intellectual intuition (*Anschauung*). According to him it is a non-sensual active "faculty" which produces its intuition directly and at the same time the objects of that intuition by its spontaneous activity. It seems that Kant saw in Plato's Ideas the objective counterpart of this intellectual intuition, for in them as intuitions *a priori* he posits the primitive cause of all things. (*"Von einem neuerdings erhobenen vornehmen Ton in d. Philosophie, Berliner Monatsschrift, Mai,* 1796.")

Kant shares with Spinoza the association of this intellectual intuition with the divine. He differs from him in that he does not admit with Spinoza that it is possible on the part of man.

I pass over entirely the concept of intuition as worked out in mysticism. With this concept the intuition of Bergson has nothing to do.

Granted that intuition is possible, what does it accomplish? It transfers one directly into the midst of objects. What of objectivity it gains it loses in subjectivity. Its climax is its coincidence with the essence of the object, and thus is emphasized as something quite distinct from it. But if it remains distinct then it must always be outside of the center of the object. This transference into a strange object is really only a purposeless example of speculative fancy, for it is absolutely inconceivable how a subject could be so changed into an object that it would take up the object into itself, make itself equivalent to it and yet remain autonomous itself. And even if this procedure were possible we would utterly reject the dualistic theory and be satisfied with the admission that in the center of the essence of an object there is such a comprehension of this center that exactly this comprehension would always be meant and finally would be so

[6] "Res igitur sub specie aeternitatis concipere est res concipere, quatenus per Dei essentiam ut entia realia concipiuntur sive quatenus per Dei essentiam involvunt existentiam."—*Eth.* V, Prop. 30, Dem.

understood again, provided that this procedure could be represented in any way.

The process of intuition can not be presented nor can it be controlled. It withdraws from every attempt at presentation or control. In secret depths there suddenly takes place an escape, a μετάβασις εἰς κρῆμα pursued and extended indeed with effort but in its origin and course unknown and unknowable. Means are entirely lacking to verify its necessity and validity beyond its reality.

Every intuition is isolated, yet we do not see how a methodical and systematic connection can be possible in the sum total of intuitions. Neither an ascent, an increasing deepening, a methodical thought-action, nor a well-constructed systematic connection of cognitions. But we might perhaps waive this: intuitions crowd together in one of the most important, in the intuitive attainment of intuition itself. Thus we would have an undivided apex crowning the structure of cognition. The cognitions themselves might be of another kind. But when and in whom is this intuition to take place? Can any one attain it at any time by making sufficient effort and striving to win it? If so, I should think that exactly these preliminary conditions, the knowledge of the kind of our endeavors, would greatly concern us, and intuition itself would let our endeavor fall from us void of interest like ripe fruit. There is something infinitely wearisome about intuition. At one stroke it tears away the veil from the mystery of mysteries and then all work is performed forever. And yet not for ever. It remains finally, to be sure, the possession of its acquirer who is not in a condition to communicate it to others though he can indeed arrange to put himself in possession of it, but has the possession for himself without being able to compare it or to communicate it. So from this point weighty prospects open before us. We do not exactly see how intuition could remain as a possession with its acquirer. He must ever seek to acquire it anew, for in memory exactly that disappears which makes it intuition, namely, the lack of the symbolic, an everlasting coming and going of intuitive experiences without plan or method, without connection or aim. For each one brings with it as the supreme purpose of cognition, but only as an experience, the truth as it is given, not as it is known, comprehended and perceived.

However, the deeper we descend into the inwardness of the subject which produces the intuition, the stronger is evidenced the characteristic note of the personal life, and the more distinct becomes the absolute in itself. Assuming too that we include in these depths the real, the cosmical center of the spiritual life, then exactly this personal element, this experience, gives it a particularly independent garb. From this point it is quite unimaginable how being and experience are to be associated together. The best we can do is to assert that the Kantian problem of cognition becomes deepened and broadened but it goes no further. Intuition, too, whose legal character and validity must be comprehended or intuitively perceived, is not a datum or a reality; but it is a problem and one that has validity.

What importance for intuition has the character of truth? Since it can not be determined either categorically or by means of ideas and especially not by symbols, it can, to be sure, contain truth in itself—yes, according to its concept it must contain truth; but how and by what intrinsic necessity it contains truth can by no means be expressed without symbols. Only no one needs to know that a cognition or an experience contains truth (for this knowledge would be either accidental or problematic) but only to know by what necessity truth is bound to a cognition or an experience. Then too the mere possession of truth is worthless so long as it is not known that it has its roots in well-grounded associations.

Therefore intuition must be rejected as a postulate because it cannot serve to give any one an accidental experience of finding himself in the center of an object. The primeval dream of humanity to be able to know finality, to be able to possess everything, to penetrate into the innermost kernel of things, is in itself contradictory and untenable. Of what use is it to me to be in the center of an object if I do not have besides an intuition that this is the case, that it actually is the case? Did not Descartes remind us of the possibility that a conception could be produced in us arbitrarily and delusively from an outside cause? He comes to the conclusion that our fancy can actually transport us into a foreign object very vividly and naturally without question, and yet with an easy effort if not simultaneously we can have the consciousness that it was simply an image of our fancy. In the moment when I by means of certain efforts of the imagination live in a vividly portrayed character of romance I have nothing but this imaginative figure within me and it is utterly impossible for me to accompany this process, which likewise is reflected in the very threshold of consciousness, with a particular act of consciousness which includes it as object or even only with the mere idea of the ego. On the other hand it is ~~very~~ possible to emphasize and to comprehend an act of imagination in a particular process of consciousness.

Accordingly since intuition is said to transfer one directly into an object, this is analogous to throwing a piece of sugar into water. The sugar is dissolved; "it" is indeed in the water, but the "piece" of sugar is not in it.

Intuition is a sort of absolute cognition. If intuition is possible, if we could penetrate at even one point into the mystery of the universe, the force of our cognition would be weakened forever at this point. At the most we would still have to assert and communicate the endeavor and the achievement if we could—at best the old traditionalism of the end of the eighteenth century. However, it is clear that intuition has already its results in great men, exceptional personalities, and that definite institutions or school buildings had them in charge. Then the incessant effort to attain possession of these intuitions would always be simply in order to gain the same possession. Of course Bergson himself does not intend this, but it is implied in the consequence of this wearisome intuition.

Bergson has foreseen the dangers that threaten, for instance, his concept of intuition. He constantly asserts the activity of intuition. There is no doubt that such an exceptional event as attaining the absolute is accompanied and introduced by attempts and efforts of an extremely energetic kind. But intuition itself is not for this reason active in any sense, although it is accompanied by activity. Exactly the last point, namely transference, in which the absolute and the comprehending subject almost coincide, must also stand on the lowest step of activity, otherwise the whole process of identity would be incomprehensible.

Bergson will undoubtedly accuse every critic of trying to assail his concept of "pure" intuition with symbols in an inadmissible way. But what if the gift of intuition refuses to come to us in spite of all our efforts? Then in Bergson's estimation we are indubitably lost as metaphysicians. It seems to me that the appeal to intuition would greatly resemble the appeal to the healthy human understanding which Bergson to be sure also invokes (p. 40). But Bergson must show us distinctly and precisely the ways and means that lead to intuition.

The intellectual experience (*Miterleben*) of the real mobility by which thinking is obliged constantly to reverse the work of thought, is claimed to be accomplished methodically. Or, rather, only the reversion is properly claimed to be performed methodically. For does not Bergson see that we are again under the spell of the formulism of symbols which has just been rejected with the greatest energy? Are not mobility, reversion and method symbols just as much as cognition, validity, categories, etc.? Bergson himself sees how difficult it is for "the intuition once attained to find a mode of expression and application corresponding to our habits of thought and offering us in firmly postulated concepts the secure supports of which we are so in need."

But everything finally comes to this, that if one were constantly to imagine that he could transfer himself directly into the midst of an object by exerting a special energy, this procedure would remain epistemologically and metaphysically valueless so long as it does not succeed in establishing the scope and degree of its validity, its internal truthfulness, the origin and structure of its composition, etc. It never depends on the cognition or experience in itself, nor on its kind which may be described as much as one wishes, but always on laying the foundation of cognition on a firm interrelation. Upon what is the certainty of an intuition, and necessarily of its contents, based? In what consists the security that I actually grasp the essence of a thing, that I really am in the center of the object?

Bergson thinks he possesses one means of comparison. He is convinced that the consciousness we have of our own person in its "continuous course leads us into the interior of one reality after whose pattern we must construct the rest."

He also upsets Kant's theory of the unknowability of the ego. ...."Accordingly I have no knowledge of myself, what I am, but merely how I appear to myself."[7]

We nowhere find in Bergson any attempt formally to oppose the well-known deductions of Kant. At the same time he arms himself against Kant's proofs. He accuses Kant of "misconstruing the union of the sciences and metaphysics with intellectual intuition." It would have been more correct to say that Kant has opposed it with all his energy. Kant did not in the least accuse metaphysics of being empty speculation; he was even the first to point out the necessity of the metaphysical impulse. But he has undertaken to show that metaphysics could never stand as a system of cognitions. His problem was formulated: Is metaphysics everywhere possible as a unity of cognition? and his answer was firm

[7] *Cr. of Pure Reason,* II, p. 157, cf. 135, 399 ff., and also the alteration of the first edition. *Proleg.,* 136 ff.

and decided, Not at all. With equal energy he rejected intellectual intuition as cognition. Whoever makes both assertions repeatedly is obliged to shatter and overturn Kant's critique in its fundamentals.

But this is not the case since Natorp's and Cohen's system has been called a "dream" (p. 52). Ultimately we will have to dispose of the idea which ascribes to Kant such a dogmatism as even his own opponents have not consistently perfected, which maintains that after Kant "the main task of criticism is to determine what the intellect is supposed to be and what the object" (p. 52). Equally dogmatic is the postulate that is ascribed to Kant that the intellect is incapable of doing anything but "Platonize, i. e., cast every possible experience into previously existing moulds" (p. 53).

To be sure we are no longer satisfied with the conception of metaphysics as it appeared to Kant. Likewise is it far indeed from us absolutely to deny its possibility as he did. We maintain that greater depths of the soul, which Kant also divined (synthetic oneness of apperception) can become present to us, but not by the help of intuition, of intellectual perception (*Anschauung*), but in an energetic apprehension, in an active realization of its infinite content. Hence we consciously abandon cognition and its ways and means which Bergson desires to broaden and deepen anew. For by means of intellectual perception we fall again and still deeper into the miserable intellectualism in which we long enough have lain imprisoned.

Intuition indeed is to be divested of all intellectuality. Apart from the fact that it thus incurs the loss of all power of cognition, it becomes in addition a kind of assimilation of the object which repeats in some way or other its content, and is everything else, except cognition or comprehension. And yet finally the resultant, the sum total of the intuitive performance must be analogous to "experience." The bare object must be distinguished from the object in the confusion of intuition. And right here lies the problem. For that an object can be concerned with intuition would be possible in itself. But who could undertake to find out by any other means than through intuition what the characteristic feature of the object is, and on the other hand the content of the perfected intuition?

Assuming the possibility of intuition, it does not accomplish what is claimed for it. The leap into the thing buries the one who takes it. Intuition assumes a thing which outside of and independent of itself does not exist. Intuition is not only unfruitful, it is even impossible.

For this statement I hope to bring forward convincing proofs. All the varied results and evolutions of modern epistemology possess the common feature of interpreting cognition as complete and immanent. It deduces all single factors and elements from the problems and laws of cognition itself but does not construct them *a priori* upon metaphysical foundations. For although the constitutive features of the nature of cognition might be based on metaphysical relations yet that which makes cognition cognition can be ascertained only by their surrender. Hence a kind of cognition which assumes the "thing" as given according to its existence and its nature is self-contradictory. Cognition exactly implies that it gains, attains, performs something. A mere transmigration into the center would either signify a mere presence in the thing or a replacement of the objective central point by an assimilating subject. In either case no decision is reached about cognition itself.

The tendency of modern epistemology is to look upon everything as under the law of cognition. Bergson tries to push the thing, the "inwardness" (*Innere*) of cognition, before it and place it outside. Moreover the "being in the center" is the characteristic feature of cognition. But while Bergson stops here the modern epistemologists begin to lay their foundation at just this point. The method by which the center of the object is reached is most important. That cognition reaches this point is implied in its concept and need not be so greatly emphasized. But how it attains it is important, and it makes the matter rather easy if the proper cognitive process in the mysterious leap into the center is allowed to plunge undiscerned. The problem is not how one can be "in" a thing, but how in this center he can be active, and of what kind is the assimilation or establishment of the center.

Then too the idea of a "central point" is an uncertain one because it makes the end disappear and yet holds fast to the goal even though undetermined. Thus the methodical character of cognition is entirely overlooked, and its infinite exertion does not come to its own.

The interrelated cosmos of the objects of cognition is knocked into nothing, and is firmly bound to unchangeable points. Intuition wills everything and is itself nothing.

However greatly much in Bergson's work appeals to us, especially the significance of the real as something moveable (although the last word does not seem to have been spoken even here), yet we must take issue as energetically against the theory of intuition as against his pragmatism (page 54).

I have not formulated the above considerations systematically but have rather adopted the rhetorical style of the French in order to remain as objective as possible. It seems to me the time has not yet come for a far-reaching reflective critique, since Bergson has promised a more conclusive argument for his theory in the future. In any case he must without question come to an understanding with Kant; for to uphold metaphysics according to Kant is difficult, but to introduce intuition again is by far the most difficult.

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MAUPERTUIS AND THE PRINCIPLE OF LEAST ACTION.

The present investigations are concerned with the history of the Principle of Least Action in the hands of Maupertuis, Euler and others. The subject is of great importance in the history of mechanics, both because the principle of least action became, in the hands of Lagrance, "the mother," as Jacobi expressed it, "of our analytical mechanics," and because the animistic tendency displayed in the search for a maximum or a minimum principle in physics undoubtedly had a great influence on such moulders of mechanical theory as Euler, Lagrange (in his early work).[1] Hamilton, Gauss, and, in

[1] Besides Lagrange's early printed works, his correspondence with Euler allows us to form some impression of the stimulating effect which the principle of least action had on Lagrange's mind at the beginning of his career. Lagrange's correspondence with Euler extends from 1754 (probably: the year is not given) to 1775 and is reproduced in the *Œuvres de Lagrange,* vol. xiv, pp. 133-245. Already in 1754 Lagrange announces (*ibid.,* p. 138) that he has made "some observations about the maxima and minima which are in the actions of nature." In a letter of August 12, 1755 (*ibid.,* pp. 138-139) Lagrange informs Euler that he had a new and simpler method of solving isoperimetrical problems and (*ibid.,* pp. 140-144) gives a full statement of it (cf. Euler's reply, *ibid.,* pp. 144-146). This discovery of what was afterwards called "the calculus of variations" certainly gave the principle of least action an additional attractiveness to Lagrange; he speaks, in a letter of May 19, 1756, of his meditations "on the application of the principle of least action to the whole of dynamics" (*ibid.,* p. 155; cf. pp. 156, 158, 161, and the final sentences of Lagrange's first printed paper in the first volume of his *Œuvres*). Lagrange's interest in the principle of least action seems to have evaporated when he observed that, when developed, the integrand is the variational form of d'Alembert's principle, and that it is simpler and equally effective to start with the equations of motion divorced from the integration. This is Lagrange's point of view in 1788. The earliest date at which this change in point of view is shown is, so far as I can find, 1764 (early memoir on the libration of the moon). In a letter of Sept. 15, 1782, to Laplace, Lagrange

our own times, Willard Gibbs. I have avoided, as much as possible, entering into merely biographical details and details of the great controversy between Maupertuis, König, Euler, and Voltaire about this very principle, in so far as they have no value in the history of science. But I have been very careful to give accurate and detailed references to the books and memoirs where everything relevant, so far as I know, may be found. I mention this expressly, because much in this chapter of the evolution of mechanics—one may even say, of thought in general—has been misquoted or misunderstood by even eminent authorities. Unless the contrary is stated, all the books referred to have been consulted either by my assistant, Miss Harwood, or by myself.[2]

I.

Pierre Louis Moreau de Maupertuis[3] was born at Saint-Malo in 1698 and died at Basel in 1759. He was the first French Newtonian;[4] was the author of several papers on the figure of the earth and the leader of that well-known French expedition which measured an arc of the meridian in Lapland, confirming the deduction from the Newtonian theory that the earth is flatter at the poles;[5]

says (*Œuvres*, vol. xiv., p. 116) that he has almost finished a mechanical treatise uniquely founded on "the principle or formula" given in section 1 of his memoir of 1780 on the libration of the moon.

[2] Adolf Mayer (*Geschichte des Princips der kleinsten Action.* Akademische Antrittsvorlesung, Leipsic, 1877, p. 7) reports that among the manuscripts left by Jacobi are fragments of a history of the principle of least action of which he has made use.

[3] There is a biography of Maupertuis by La Beaumelle (*Vie de Maupertuis par L. Angliviel de la Beaumelle; ouvrage posthume, suivi de lettres inédites de Frédéric le Grand et de Maupertuis, avec des notes et un appendice,* Paris, 1856). Cf. also Samuel Formey, *Eloge de M. de Maupertuis* (read in 1760), reprinted, with additions and corrections by de la Condamine and Trublet, in 1766 in the *Histoire de l'Académie de Berlin* for 1759, pp. 464-512; and Emil du Bois-Reymond, *Maupertuis; Rede....*, Leipsic, 1893 (on La Beaumelle's book, see pp. 72-81).

[4] La Beaumelle, *op. cit.*, p. 16; du Bois-Reymond, *op. cit.*, pp. 17-18. See Maupertuis's papers in the Paris *Mémoires* for 1732-1736; and *Discours sur les différentes figures des astres, avec une exposition des systèmes de MM. Descartes et Newton,* published anonymously at Paris in 1732 and again in 1742 (not seen), and the popular part of it is most conveniently consulted in the *Œuvres de Mr. de Maupertuis,* Lyons, 1756, vol. i, pp. 79-170. Cf. La Beaumelle, *op. cit.*, pp. 23-34; I. Todhunter, *A History of the Mathematical Theories of Attraction and the Figure of the Earth from the Time of Newton to that of Laplace,* London, 1873, vol. i, pp. 63-76, 93-102 (this also contains an account of those works which come into the scope of the next note).

[5] La Beaumelle, *op. cit.*, pp. 34-64, 71-75, 457-458, 461-462, 467; Du Bois-Reymond, *op. cit.*, pp. 18-35; and a German translation with notes by myself. of Clairaut's book of 1743 on the figure of the earth, which is soon to appear in *Ostwald's Klassiker.*

and was Frederick the Great's President of the Berlin Academy[6] (from 1746). With Maupertuis's geometrical works we are not concerned here,[7] nor are we with those philological and anatomical speculations of his, which were so ruthlessly and unjustly parodied by Voltaire.

According to Du Bois-Reymond,[8] Maupertuis's teleological tendencies showed themselves early in his career in speculations as to what grounds the Creator could have had for preferring the law of the inverse square to all other possible laws of attraction.

Some words about Maupertuis's personal character are necessary. When Maupertuis returned from Lapland, there was great opposition in some quarters to the reception of his results. This foolish opposition had a bad influence on Maupertuis: his never small feeling of self-importance increased, and he became embittered.[9] On the other hand, he was given, as President of the Berlin Academy, almost unlimited powers, even as regards the payment of the members' pensions,[10] and this may partly explain, as Carlyle suggests in his *Frederick the Great,* the tiring chorus of praise that breaks out in the Berlin *Histoire* whenever any of the members have occasion to mention Maupertuis's name. In the course of our discussions, too, we shall have, in order to correct a strange error about Maupertuis and the principle of least action made by Lord Morley in his well-known book on *Diderot and the Encyclopædists,* to touch upon the question as to whether Maupertuis was a materialist or not.[11]

II.

Maupertuis read to the Paris Academy on the 20th of February, 1740, a memoir entitled: "Loi du Repos des Corps."[12] He began by remarking that demonstrations *a priori* of such principles as that

[6] La Beaumelle, *op. cit.,* pp. 65-68, 76, 91-98, 104; du Bois-Reymond, *op. cit.,* pp. 36, 38, 39-42.

[7] La Beaumelle, *op. cit.,* pp. 15-16. 18-19, 22-23, 460-461; du Bois-Reymond, *op. cit.,* p. 16; M. Cantor, *Vorlesungen über Geschichte der Mathematik,* vol. iii, 2d ed., Leipsic, 1901, pp. 774-775, 786.

[8] *Op. cit.,* p. 18. The place where this speculation is given is in the *Figure des Astres* (*Œuvres,* 1756, vol. i, pp. 166-170).

[9] Du Bois-Reymond, *op. cit.,* p. 33.

[10] *Ibid.,* p. 40; La Beaumelle, *op. cit.,* p. 107.

[11] In the course of this article, we shall refer to Mach's work on mechanics as *Mechanik* and *Mechanics,* as we have done before (*Monist,* April, 1912).

[12] *Histoire de l'Académie royale des sciences. Année 1740. Avec les Mémoires de Math. et de Phys. pour la même Année,* Paris, 1742, pp. 170-176; *Œuvres,* 1756, vol. iv, pp. 45-63.

of the conservation of *vis viva* "cannot apparently be given by physics; they seem to belong to some higher science."

Maupertuis sought for a general law in statics analogous to the known theorem that, in any system of elastic bodies in motion, which act upon one another, $\Sigma m.v^2$ is constant, and found that: In order that a system of bodies of which each is attracted to a center by a force varying as the nth power of the distance from that center, should remain in equilibrium, it is necessary that

$$\Sigma m.f.z^{n+1},$$

where f is the intensity of the force which acts on m, and z is the distance of the mass m from its center of force, is a maximum or a minimum. In the proof, by showing the truth of the principle in two classes of cases, he concludes that as, for equilibrium

$$\Sigma m.f.z^n.dz=0,$$

the above sum must be a maximum or a minimum.[13]

In an "Addition" added to the reprint in the *Œuvres,*[14] Maupertuis remarked that his law holds if the forces are proportional to functions Z of the distances z, and then the law is that

$$\Sigma m.f.\int Z.dz$$

must be a minimum.[15]

III.

Maupertuis's first enunciation of the law of the least quantity of action was in a memoir read to the French Academy on April 15th, 1744, entitled: "Accord de différentes Loix de la Nature qui avoient jusqu'ici paru incompatibles."[16] The laws in question ap-

[13] If there is one constant force on all the masses, and its center is at an infinite distance from the system, the center of gravity of the system must be as far as possible from, or as near as possible to, this center, for equilibrium to subsist.

[14] Vol. iv, pp. 62-63. It should be remarked that Euler, in a paper quoted below in the Berlin *Histoire* for 1751, pp. 171-173, had pointed out: (1) that it is not necessary that the forces are proportional to like powers of the distances, provided that we do not neglect the coefficients $1/(n+1)$ when they are different for the different bodies on which the forces act (p. 171); (2) that the forces need not be supposed to be proportional to functions (*fonctions quelconques*) of the distances, and if the force is V instead of fz^n, $\Sigma f.m.V.dz$ will then be a maximum or a minimum—according to the kind of equilibrium (p. 172); and (3) that the whole distance of each body from the centers of forces need not be considered, but, if convenience of calculation requires it, we need only consider the distances of the bodies from fixed points on the lines of direction of the forces (pp. 172-173).

[15] Maupertuis does not add: "or a maximum." The subject of this memoir of 1740 and its connection with the principle of least action were afterwards greatly developed by Euler. Cf. also Mach, *Mechanik,* pp. 69-75; *Mechanics,* pp. 68-73.

[16] *Histoire de l'Académie*; *Année* 1744 (Paris, 1748), pp. 417-426; *Œuvres,* 1756, vol. iv, pp. 3-18 (with the addition referred to below).

pear[17] to be those of the reflection and of the refraction of light. When a ray of light in a uniform medium travels from one point to another, either without meeting an obstacle or with meeting a reflecting surface, nature leads it by the shortest path and in the shortest time. But when a ray is refracted by passing from a uniform medium to one of different density, the ray neither describes the shortest space nor does it take the shortest time about it. As Fermat showed, the time would be the shortest if light moved more quickly in rarer media, but Newton proved that, as Descartes had believed, light moves more quickly in denser media. Maupertuis's discovery was that light neither takes always the shortest path nor always that path which it describes in the shortest time, but "*that for which the quantity of action is the least.*"

"I must now explain," he went one,[18] "what I mean by *the quantity of action.* A certain action is necessary for the carrying of a body from one point to another: this action depends on the velocity which the body has and the space which it describes; but it is neither the velocity nor the space taken separately. The quantity of action varies directly as the velocity and the length of path described; it is proportional to the sum of the spaces, each being multiplied by the velocity with which the body describes it. It is this quantity of action which is here the true expense (*dépense*) of nature, and which she economizes as much as possible in the motion of light."

Then Maupertuis found, as a consequence of his principle, that the sine of the angle of incidence is to the sine of the angle of refraction in the inverse ratio of the velocity of the light in each medium.[19] After showing that the law of reflection also follows from

[17] Maupertuis afterwards stated (see below, section V) that the agreement was between the laws of the motion of light and mechanical laws. I have given below my grounds for almost suspecting that this was not what Maupertuis originally meant.

[18] *Histoire de l'Académie,* 1744, p. 423; *Œuvres,* vol. iv, p. 17. Notice that *here,* in the *general* definition, *mass* is not mentioned. This is another reason for believing that, at first, Maupertuis only considered the motion of light-corpuscles, and not that of ordinary matter.

[19] Cf. Mach, *Mechanik,* pp. 397-398; *Mechanics,* pp. 367-368. Using Maupertuis's and Mach's figure, CRD is the horizontal refracting plane, AR is the incident and RB the refracted ray (A and B being any points chosen on these respective rays), m the velocity of light along AR and n the velocity along RB. Then Maupertuis says correctly that, according to his principle, $m.\mathrm{AR}+n.\mathrm{RB}$ must be a minimum. That is to say

$$d[m\sqrt{(\mathrm{AC}^2+\mathrm{CR}^2)}+n\sqrt{(\mathrm{BD}^2+\mathrm{DR}^2)}]=0,$$

whence, carrying out the differentiations, observing that AC and BD are constant, and $d(\mathrm{CR})=-d(\mathrm{DR})$,

$$(\mathrm{CR}/\mathrm{AR} : \mathrm{DR}/\mathrm{BR}) :: n : m, \text{ or } (\sin \mathrm{CAR}/\sin \mathrm{RBD}) = (n/m),$$

which is correct on the corpuscular hypothesis; Mach's criticism that the his principle of the least quantity of action, Maupertuis concluded:[20] "We cannot doubt that all things are regulated by a supreme Being, who, while he has imprinted on matter forces which show his power, has destined it to execute effects which mark his wisdom;......" And:[21] "Let us calculate the motion of bodies, but let us also consult the designs of the Intelligence which makes them move."

It is of interest, in connection with the dispute with König which arose afterwards, to read the note which Maupertuis appended to the reprint in his *Œuvres*:[22]

"When I read the preceding memoir in the Paris Academy of Sciences, I only knew of what Leibniz had done on this matter by what M. de Mayran says of it in his memoir on the reflection of bodies in the Paris *Mémoires* for 1723. Like him, I had confused this opinion of Leibniz's with that of Fermat...."

Then he gave,[23] after Euler,[24] the full opinion of Leibniz.[25]

Now we shall see below that Maupertuis in the *Histoire* for 1752 said that he had "adopted" Leibniz's definition of *action*. We have no means of knowing how far, if at all, Maupertuis was indebted to the ideas of Leibniz.

IV.

There is nothing on the subject of the principle of the least quantity of action in the *Histoire de l'Académie de Berlin* (which contains the *Mémoires* of the various classes of the Academy) for 1745; but, in the *Histoire* for 1746, published in 1748, Maupertuis

reciprocal values appear instead of the actual ones is only true, as P. Stäckel observed in the *Encykl. der math. Wiss.*, vol. iv, part i, 1908, p. 491, on the undulatory theory, which Maupertuis, as a good Newtonian, did not adopt.

Further, Maupertuis's principle *does* state that $m.\mathrm{AR}+n.\mathrm{RB}$ (which is what $\int v.ds$ reduces to here) is to be a minimum. This was contested by Mach (but cf. *Mechanik*, p. 406; *Mechanics*, pp. 375-376).

Du Bois-Reymond (*op. cit*)., pp. 48-49) speaks of the example of the motion of light which Maupertuis chose in 1744 to illustrate his principle being "not happily chosen," because experiments have proved that the velocity of light in air is greater than that in water—the opposite state of things to that which the emission theory required.

[20] *Œuvres*, vol. iv, p. 21.

[21] *Ibid.*, p. 22.

[22] *Ibid.*, p. 23.

[23] *Ibid.*, pp. 23-28. In the text of the memoir of 1744, Maupertuis (*ibid.*, p. 15) thus mentioned Leibniz: "Leibniz wished to conciliate the opinion of Descartes [that light moves more quickly in the denser media] with final causes; but he did this only by suppositions which could not be sustained, and which did not square with the other phenomena of nature."

[24] *Hist. de l'Acad. de Berlin*, vol. vii, 1751, pp. 205-209.

[25] *Acta Eruditorum*, 1682 (not seen).

has[26] a memoir: "Les Loix du Mouvement et du Repos, déduites d'un Principe Métaphysique."

This memoir begins with the prefatory remark:[27] "I gave the principle on which the following work is founded on April 15th, 1744, in the public assembly of the Royal Academy of Sciences of Paris, as the *Acta* of this Academy testify." Then Maupertuis refers to Euler's *Methodus inveniendi* of 1744,[28] and the special pleasure that the Appendix gave him, "as," he says, rather patronizingly and in words which led some[29] to suppose that Euler merely applied Maupertuis's principle, "it is a beautiful application of my principle to the motion of the planets, of which this principle is in fact the rule."

The memoir is composed of three parts: (1) Examination of the proofs of the existence of God, which are drawn from the wonders of nature;[30] (2) The thesis that these proofs must be sought in the general laws of motion, and that the laws according to which motion is conserved, distributed, and destroyed are founded on the attributes of a supreme intelligence;[31] and (3) Investigation of the laws of motion and rest.[32] In the third part, Maupertuis[33] states the general principle that "when some change happens in nature, the quantity of action necessary for this change is the smallest possible," and adds: "*The quantity of action* is the product of the mass of the bodies by their velocity and by the space which they describe. When a body is transported from one place to another, the action is greater in proportion as the mass is greater, as the velocity is greater, and as the path by which it is transported is longer." From this principle, Maupertuis deduces the laws of impact of hard (or inelastic) and elastic bodies,[34] and of the lever.[35]

[26] Pp. 267-294. The mathematical (third) part of this memoir is, in part, identical with "Recherche des Loix du Mouvement" in the *Œuvres,* vol. iv, pp. 31-42; the theological part is included in the *Essai de Cosmologie* to which we will soon refer.

[27] *Histoire de l'Acad. de Berlin,* 1746, p. 267. This note was repeated in Maupertuis's *Œuvres,* vol. i (see below).

[28] See below, section IX.

[29] For example La Beaumelle, *op. cit.,* p. 85.

[30] *Histoire de l'Acad. de Berlin,* 1746, pp. 268-277.

[31] *Ibid.,* pp. 277-287.

[32] *Ibid.,* pp. 287-294.

[33] *Ibid.,* p. 290; *Œuvres,* vol. iv, p. 36.

[34] *Histoire,* pp. 290-293; *Œuvres,* vol. iv, pp. 36-42.

[35] *Histoire,* p. 294; not in the *Œuvres.* The explanation of this omission given by Maupertuis (*Œuvres,* vol. i, p. xxvii) is that this problem is too limited (as the directions of the forces of weight are all supposed to be parallel to one another and at right angles to the straight lever); but the "Loi du

When treating of impact of hard (inelastic) bodies of masses A and B, which move with the velocities a and b respectively in a straight line and in the same sense, Maupertuis considers the spaces (a and b) *described in a certain time* (the unit of time), so that $m.v.s$ becomes $m.v^2$, as Mach notices, and so he points out Maupertuis's inconsistency.[36]

Let A move faster than B, so that A catches B up and infringes on it, and let the common velocity of A and B after the impact be x (less than a and greater than b). "The alteration which has happened in the universe consists in that the body A which moved with the velocity a and which in a certain time described a space equal to a only moves with the velocity a and describes a space equal to x, while the body B which only moved with the velocity b and described a space equal to b moves with a velocity x and describes a space equal to x. This change is, then, the same as would have happened if, while A moved with the velocity a and described a space equal to a, it had been carried backwards through a space equal to $a-x$ on an immaterial plane moving with the velocity $a-x$, and while B moved with the velocity b and described a space equal to b, it had been carried forward through a space equal to $x-b$ on an immaterial plane moving with a velocity $x-b$. Now, whether A and B move with their own velocities on movable planes or they are at rest there, as the movement of these planes charged with bodies is the same, the quantities of action produced in nature will be $A(a-x)^2$ and $B(x-b)^2$, and their sum must be as small as possible." This gives

$$2.A.a.dx+2.A.x.dx+2.B.x.dx-2B.b.dx=0,$$

whence

$$x=(Aa+Bb)/(A+B).$$

In this case, where the bodies move in the same direction, the quantity of motion destroyed and the quantity produced are equal, and the total quantity of motion remains, after the impact, the same as it was before. If the bodies move towards one another it is easy to apply the same reasoning; or it is sufficient to consider b as negative with respect to a. Then the common velocity will be

$$x=(Aa-Bb)/(A+B).$$

If A and B are perfectly elastic, and move in the same direction with velocities as before, except that α and β are the respective

repos" of 1740, given in vol. iv of the *Œuvres,* is a *general* principle of statics and "agrees so perfectly with the principle of the least quantity of action that we may say that it is only the same thing."

[36] *Mechanik,* pp. 395-396, 398; *Mechanics,* pp. 365-366, 368.

velocities after impact, "the sum or the difference of these velocities after the impact being the same as it was before," then, by analogous considerations on the change which has happened in nature, Maupertuis arrives at the conclusion that the quantity of action is here

$$A(a-\alpha)^2+B(b-\beta)^2$$

and this, when minimized, since

$$\beta-\alpha=a-b \text{ and thus } d\beta=d\alpha,$$

gives

$$\alpha=(Aa-Ba-2Bb)/(A+B), \qquad \beta=(2Aa+Ab-Bb)/(A+B).$$

Here the sum of the *vires vivae* is conserved on impact, but this is not the case with hard (inelastic) bodies.

To find the law of the lever Maupertuis considers masses A and B attached to the ends of an immaterial lever of length c, and seeks the point, at a distance z from A, around which they are in equilibrium. For this purpose he seeks the point around which, if the lever receives some small movement, the quantity of action is the smallest possible. Then A and B, on this movement being imparted to them, describe small arcs similar to one another and proportional to the distances of these bodies from the point sought. These arcs will be the spaces described by the bodies and at the same time will represent their velocities. Thus the quantity of action will be proportional to

$$Az^2+B(c-z)^2$$

and this, when minimized, gives

$$z=Bc/A+B.$$

V.

In the "Avertissement" to the fourth volume of his *Œuvres,* Maupertuis says of the memoir of 1744: "I show the agreement of the laws which light follows in its reflection and its refraction with those which all other bodies follow in their motion." In point of fact, this is not quite the case: he shows how *both* the law of reflection and that of refraction could, on the corpuscular hypothesis, be deduced from one principle; but, in the whole memoir, other motions than that of light were only referred to shortly. The law that, in a uniform medium, light moves in a straight line is common, he says,[37] to all bodies: they move in a straight line unless some external force deflects them; and the law of reflection is the same as that

[37] *Paris Histoire,* 1744, p. 418; *Œuvres,* vol. iv, p. 7.

followed by an elastic ball impinging on an unbreakable surface. But no like explanation of the law of refraction had been given.

Later on, Maupertuis[38] adds a note to his definition of the quantity of action as $\Sigma\, s.v$: "As here there is only one body, we abstract from its mass."

VI.

Maupertuis's *Essai de Cosmologie* was published in 1751,[39] and consists of three parts: (1) Examination of the proofs of the existence of God, which are drawn from the wonders of nature; (2) Deduction of the laws of motion from the attributes of the supreme intelligence; and (3) Spectacle of the universe. No part of the work is stated mathematically, and the third part is a rhetorical sketch of the solar system, in which the principle of the least quantity of action is not mentioned.[40] The two first parts are practically the two first parts of the memoir of 1746.

[38] *Œuvres,* vol. iv, p. 17. This note is not in the original memoir of 1744 (the paragraph in the text to which the note refers is on p. 423 of this memoir), but was first added, as a marginal note, in the *Essai de Cosmologie* of 1751. These facts suggest that the mechanical applications of Maupertuis's principle were, at least, not clear to Maupertuis in 1744. For my own part, I cannot help almost having the impression from a study of the original memoir of 1744 and its reproduction, with comments, in the *Œuvres* of 1756, that the laws of nature referred to in 1744 are the laws of catoptrics and dioptrics, whereas afterwards Maupertuis, because of the discovery communicated in his memoir of 1746, tried to persuade possibly himself and certainly his readers that the laws were more general laws of nature. Cf. Note 18, Section III, above.

Formey, in the *Eloge* quoted at the beginning of this paper, says (p. 496): "Il y [in the memoir of 1744] étoit principalement question des loix qui suit la lumière, surtout lorsqu'elle passe d'un milieu diaphane dans un autre."

[39] *Essay de Cosmologie.* Par M. de Maupertuis, Leyden, 1751. At the end (pp. 81-104) is a reprint of the 1744 paper with the mathematics (the note referred to in section V, last note, is put in the margin of pp. 97-98); and on pp. 63-80 is a "Recherche mathématique des Loix du Mouvement et du Repos," from, says Maupertuis, the Berlin *Mémoires* for 1747 (a misprint for 1746). The *Essai* was partly reprinted in the first volume of the *Œuvres de Mr. de Maupertuis* (Nouvelle édition, corrigée et augmentée, Lyons, 1756, pp. 3-78, and the mathematical part, which was omitted in the previous editions of Maupertuis's *Œuvres,* is included in vol. iv, pp. 18-19, 36-42. On pp. iii-xxviii, is an "avant-propos" giving, among other things, an account of the Koenig incident of 1751 and its consequences. On pp. xiv-xv is the same notice about his own and Euler's works of 1744 that is at the head of Maupertuis's paper in the Berlin *Mémoires* for 1746. On d'Arcy's objections (see section XV) Maupertuis (*Œuvres,* vol. i, p. xxvi) said that "As the only objection which appears to have some foundation rests on the fact that, in the impact of elastic bodies, he has confused the change which happens to the velocities (which is real) with the change of the quantity of action (which is zero), I will make no other reply than the few words I have said about it in the *Mémoires* of our [Berlin] Academy for the year 1752" (see section XVI).

[40] However, in the second part (*Œuvres,* vol. i, p. 45), we read: "What a satisfaction for the human mind to find in the laws which are the principle

Maupertuis had a low opinion of the proofs of the existence of God from the construction of animals. Thus, somebody[41] found evidence for this existence in the folds of the skin of a rhinoceros—the animal could not move without these folds. Maupertuis[42] rather appositely asked: "What would be said of a man who should deny a Providence because the shell of a tortoise has neither folds nor joints?" And:[43] "It is not in the little details, in those parts of the universe of whose relations are known too little, that we must look for the supreme Being, but in phenomena whose universality suffers no exception and whose simplicity lays them quite open to our sight."

VII.

The reason why Maupertuis laid stress on the deduction from the principle of the least quantity of action of the laws of the impact of inelastic masses was that the law of the conservation of *vis viva* fails in this case.[44] Leibniz[45] recognized Descartes's error in thinking that, in nature, the sum of the products of the masses into their respective velocities is constant, and substituted in it the squares of the velocities for the velocities, so that the sum is what is called the *vis viva* of the system considered. But, in impact, the *vis viva* is only conserved if the bodies are elastic; and, according to Maupertuis:[46] "When we make this objection to the Leibnizians, they prefer to say that there are no hard (*durs*, inelastic) bodies in nature than to abandon their principle. This were to be reduced to the strangest paradox to which love of a system could reduce one: for what can the primitive elementary bodies be but hard bodies?"

In vain, then, said Maupertuis,[47] did Descartes and Leibniz, in

of motion of all the bodies of the universe the proof of the existence of the governor of it!"

[41] *Phil. Trans.*, No. 470. [The paper referred to is entitled: "A Letter from Dr. Parsons to Martin Folkes, Esq., President of the Roy. Soc., containing the Natural History of the Rhinoceros," and is printed in the *Phil. Trans.* for 1743, pp. 523-541].

[42] *Œuvres*, vol. i, p. 12.

[43] *Ibid.*, p. 21.

[44] *Œuvres*, vol. i, pp. xvi-xvii, 44.

[45] On Leibniz's mechanics (the conservation of *vis viva*, and so on), cf. Bertrand Russell, *A Critical Exposition of the Philosophy of Leibniz, with an Appendix of leading Passages*, Cambridge, 1900, pp. 77-99, 226-238; esp. pp. 89-90. The concept of *action* with Leibniz was not mentioned by Russell; on it cf. du Bois-Reymond, *op. cit.*, pp. 48, 51, 89-90; and Helmholtz, "Zur Geschichte des Princips der kleinsten Action," *Sitzungsberichte der Berliner Akad.*, 1887, pp. 225-236, or *Wiss. Abh.*, vol. iii, pp. 249-263. Cf. also L. Couturat, *La logique de Leibniz*, 1901, pp. 229-233, 577-581.

[46] *Op. cit.*, p. xvii.

[47] *Ibid.*, p. xviii.

different ways, imagine a world which could dispense with the hand of a Creator: no quantity which can be regarded as a cause in the distribution of motion subsists unaltered. But "Action" is, so to speak, created at each instant, and always created with the greatest economy possible; and by this the universe announces its dependence on a wise and powerful author.

Maupertuis[48] said that, because he held that the conservation of *vis viva* is not the universal principle of movement, the whole sect of Leibnizians in Germany descended on him (*je vis fondre sur moi toute la secte que M. de Leybnitz a laissée en Allemagne*), and then mentioned[49] König's having attributed some of Maupertuis's and Euler's discoveries to Leibniz. Then follows[50] an account of the incident.

As a justification of the word "action," Maupertuis[51] remarked that he had found this word quite established by Leibniz and Wolff, and did not wish to change the terms.

VIII.

When speaking of Diderot's *Thoughts on the Interpretation of Nature* of 1754, John Morley,[52] now Lord Morley, said:

"Maupertuis had in 1751, under the assumed name of Baumann, an imaginary doctor of Erlangen, published a dissertation on the *Universal System of Nature,* in which he seems to have maintained that the mechanism of the universe is one and the same throughout, modifying itself, or being modified by some vital element within, in an infinity of diverse ways.[53] Leibnitz's famous idea, of making nature invariably work with the minimum of action, was seized by Maupertuis, expressed as the Law of Thrift, and made the starting point of speculations that led directly to Holbach and the *System of Nature*.[54] The *Loi d'Epargne* evidently tended to make unity

[48] *Ibid.,* p. xix.

[49] *Ibid.,* p. xx.

[50] *Ibid.,* pp. xx-xxvi, cf. section XI below.

[51] *Ibid.,* pp. xxvi-xxvii, cf. Maupertuis's paper of 1752, described below in section XVI.

[52] *Diderot and the Encyclopædists,* vol. ii, London, edition of 1905, pp. 262-263.

[53] "As to the precise drift of Maupertuis's theme, see Lange, *Gesch. d. Materialismus,* i, 413, *n.* 37. Also Rosenkranz, *Diderot's Leben,* 1866, vol.i, p. 134."

[54] "In 1765 Grimm describes the principle of Leibnitz and Maupertuis as 'gaining on us on every side'.... *Corr. Lit.,* iv, 186." [Under the date of Feb. 15, 1765, Grimm (*Correspondance littéraire philosophique et critique de Grimm et de Diderot depuis 1753 jusqu'en 1790,* new ed., vol. iv, p. 186) speaks thus

of all the forces of the universe the keynote or the goal of philosophical inquiry. At this time of his life, Diderot resisted Maupertuis's theory of the unity of vital force in the universe, or perhaps we should rather say that he saw how open it was to criticism. His resistance has none of his usual air of vehement conviction. However that may be, the theory excited his interest, and fitted in with the train of meditation which his thoughts about the Encyclopaedia had already set in motion, and of which the *Pensées Philosophiques* of 1746 were the cruder prelude."

Again:[55]

"Diderot was in no sense the originator of the French materialism of the eighteenth century. He was preceded by Maupertuis, by Robinet, and by La Mettrie; and we have already seen that when he composed the *Thoughts on the Interpretation of Nature* (1754) he did not fully accept Maupertuis's materialistic thesis. Lange has shown that at a very early period in the movement the most consistent materialism was ready and developed, while such leaders of the movement as Voltaire and Diderot still leaned either on deism and scepticism."[56]

Lange's[57] work was first published in one volume: *Geschichte des Materialismus und Kritik seiner Bedeutung in der Gegenwart* at Iserlohn in 1866. In the whole book, Maupertuis is only mentioned once. On page 224[58] it is said that people debated whether the Marquis d'Argens (Jean Baptiste de Boyer) or Maupertuis or some personal enemy of Albrecht von Haller, really wrote the *Homme machine* which De la Mettrie ironically dedicated to Von Haller.[59]

The fourth part[60] is devoted to the materialism of the eighteenth century, and consists of three divisions: De la Mettrie's *Homme machine* of 1747;[61] Holbach's *Système de la Nature, ou des lois du monde physique et du monde moral* of 1770, published, according to the title-page, in London, but really at Amsterdam, under the

of the Leibniz-Maupertuis principle of thrift, immediately after speaking of the second volume of Robinet's *De la nature,* published in four volumes 1761-8.

On Holbach's *System of Nature* (1770), see Morley, *op. cit.,* pp. 155-203.

[55] Morley, *op. cit.,* pp. 272-273.

[56] *Gesch. d. Materialismus,* i, 309, 310, etc.

[57] Friedrich Albert Lange.

[58] Cf. the references below the second edition of Lange's work.

[59] Lange, *op. cit.,* p. 72.

[60] *Ibid.,* pp. 163-229.

[61] *Ibid.,* pp. 163-186.

name of Mirabaud who had been dead for ten years;[62] and the German reaction against materialism.[63]

On the other hand, Maupertuis is often spoken of in the second edition of Lange's work, published at Iserlohn in 1873 and 1875 in two volumes under the same title,[64] and it is to this edition that Morley's citations refer. We will continue this reference to Lange's book after having given some information about Maupertuis's work of 1751, which Morley mentions.

In 1751 Maupertuis published at Erlangen, under the pseudonym of "Baumann," a Latin dissertation under the title: *Dissertatio inauguralis metaphysica, de universali naturae systemata,*[65] in which

[62] *Ibid.,* pp. 186-214.

[63] *Ibid.,* pp. 214-229.

[64] There is an English translation of this edition in three volumes, by E. C. Thomas, published at London in 1877, 1880 and 1881 (*History of Materialism and Criticism of its Present Importance*). The passages in this translation parallel to those of Morley's citations are given here.

[65] Another edition, with a French translation and with neither the place nor year of publication has been given; a third, only in French and entitled: *Essai sur la formation des corps organisées* was published by l'Abbé Trublet, with a notice and conjectures about the author, at Berlin (really at Paris) in 1754; and the French version (*Système de la Nature*: *Essai sur la formation des corps organisées*) was published, with a preface, in Maupertuis's *Œuvres,* 1756, vol. ii, pp. 135-168 (between pp. 160 and 161 are pages numbered 145* to *160). Diderot's *Pensées sur l'interprétation de la nature* was published anonymously at Paris in 1754 with "London" as the place of printing (Cf. Karl Rosenkranz, *Diderot's Leben und Werke,* 2 vols., Leipsic, 1866, vol. i, pp. 134-146; *Œuvres complètes de Diderot,* ed. by J. Assézat, vol. ii, Paris, 1875, pp. 1-63; cf. Assézat's "Notice préliminaire," p. 3. Maupertuis's "Réponse aux objections de M. Diderot," was printed in his *Œuvres,* 1756, vol. ii, pp. 169-184 (between pp. 176 and 177 are pages numbered 161* to *176). Cf. on all this, La Beaumelle, *op. cit.,* pp. 178-181, 200-201.

On Maupertuis's theories of generation, see La Beaumelle, *op. cit.,* pp. 86-87, 98-103; du Bois-Reymond, *op. cit.,* pp. 38-39, 44-45. The *Vénus physique* of 1745 (anonymous) was republished in Maupertuis's *Œuvres,* 1756, vol. ii, pp. 1-133. The statement that Maupertuis endeavored to explain the formation of the foetus by gravitation is one of Voltaire's libels on Maupertuis. The truth seems to be that Maupertuis, in his *Vénus* and *Système de la Nature,* as well as in one of his *Letters* ("Lettre xiv, Sur la génération des animaux," *Œuvres,* 1756, vol. ii, pp. 267-282), tried to explain this formation by the different *attractions* or (in the *Système*) psychical tendencies of the different parts. The *Lettres de M. de Maupertuis* (*sur différents sujets*) were published in 1753 and again in the *Œuvres,* 1756, vol. ii, pp. 185-340, after having been grossly caricatured by Voltaire in his *Histoire du docteur Akakia et du natif de Saint Malo* (*Œuvres complètes de Voltaire,* vol. xxiv, Paris, 1892, pp. 358-376). By the way, Letters X and XI ("Sur les loix du mouvement" and "Sur ce qui s'est passé à l'occasion du principe de la moindre quantité de l'action"; *Œuvres,* 1756, vol. ii, pp. 238-242 and 243-251 respectively) refer to the principle of least action; and Letter XII (*ibid.,* pp. 252-257; "Sur l'attraction") contains a short *exposé* of Maupertuis's work in introducing Newtonianism into France.

Maupertuis does not seem, by his published writings, to have been nearly so ridiculous a person as Voltaire, for personal reasons, tried to make him appear to be. And Voltaire's sarcasms have had great influence on the ideas a hypothesis that the parts of matter have something similar to what we call desire, aversion, and memory was advanced to explain certain physiological facts. Maupertuis chose this pseudonymous fashion of giving his thoughts to the public, partly because the work of an unknown author would be less the butt of objections, and partly in order that he should not be obliged to reply to them. But he felt it necessary to reply to Diderot's *Thoughts* because his doctrines were accused of having results contrary to religion. Then he acknowledged the work: he had soon been recognized as its author. What concerns us here is that the law of least action is not mentioned in this work of Maupertuis's. Further, the *Essai de Cosmologie* of 1751 was not published anonymously or pseudonymously. Thus there seem to be no grounds for Morley's strange error.

Lange shows that the Newtonian theory is a combination of materialism in natural science with a religious faith in the spiritual constructor of the material world-machine. "The magnificent phenomena of the seventeenth century were renewed in increased splendor, and to the age of a Pascal and Fermat succeeded with Maupertuis and D'Alembert the long series of French mathematicians of the eighteenth century, until Laplace drew the last consequences of the Newtonian cosmology in discarding even the hypothesis of a creator."[66]

Maupertuis is classed with Robinet and La Mettrie as a materialist[67] on the grounds of his Latin dissertation of 1751. The English translation of the note (37) referred to by Morley is:[68] "*Comp.* Rosenkranz, *Diderot,* i, 134 ff. The pseudonymous dissertation of Dr. Baumann (Maupertuis) I have not seen, and it may be open to some doubt, according to Diderot and Rosenkranz, whether it does really contain the materialism of Robinet—that is, the unconditional dependence of the spiritual upon the purely mechanical series of external events—or whether it inculcates Hylo-

of Maupertuis formed by succeeding generations. Thus Mach (*Mechanik,* pp. 484-485, *Mechanics,* pp. 454-455) gives, I think, Voltaire's version of some of the things dealt with by Maupertuis in a *Letter* published earlier than those just mentioned. Maupertuis's *Lettre sur les progrès des sciences* was published at Berlin in 1752 and again in his *Œuvres,* 1756, vol. ii, pp. 341-399. Here is the project of founding a town where only Latin should be spoken, in order to preserve this most universal of languages (pp. 367-368), and a plea (pp. 394-398) for "metaphysical"—or, as we would say now, psychological—experiments.

[66] Lange, *Geschichte,* 2d ed., vol. i, p. 304; *History,* vol. ii, p. 16.

[67] Lange, *Geschichte,* vol. i, p. 310; *History,* vol. ii, p. 25.

[68] Lange, *Geschichte,* vol. i, pp. 315, 412-413; *History,* vol. ii, p. 31.

zoism—that is, modifications of the natural mechanism by the spiritual content of nature according to other than purely mechanical laws."

Again:[69] "Buffon began the publication of his great work on natural history in the year 1749, with the first three volumes; but it was only in the fourth volume that he unfolded the idea of the unity of principle in the multiplicity of organisms, an idea which occurs again in Maupertuis in an anonymous work in 1751, in Diderot in the *Pensées sur l'Interprétations de la Nature,* 1754, while we find it developed with great clearness and distinctness by La Mettrie as early as the *L'Homme Plante* in 1748."

This, together with the passage referred to above, when we were speaking of the first edition, about Maupertuis being considered by some to be the author of *L'Homme Machine*,[70] completes the list of Lange's references to Maupertuis in the second edition of his book.

We must add that Maupertuis, in his writings and in his life, showed the greatest respect for religion. He was by no means a materialist and atheist,[71] and the only reason, said he, that he had for replying to Diderot's *Thoughts* on his dissertation of 1751 was that Diderot stated that the dissertation, in spite of its carefully religious tone, led to conclusions which were subversive of religion.

IX.

This seems the best place to give some account of the work of a man who will now take a prominent place in the development of Maupertuis's ideas; I mean Leonhard Euler.[72]

The modern period of the discussion of maximal and minimal problems begins with Johann Bernoulli's proposal of the problem of the brachistochrone in 1696 and the consequent rise into importance of the "isoperimetrical" problems.[73] The period 1696 to 1762 of

[69] Lange, *Geschichte,* vol. i, p. 328; *History,* vol. ii, p. 52.

[70] Lange, *Geschichte,* vol. i, p. 398; *History,* vol. ii, p. 137.

[71] Du Bois-Reymond, *op. cit.,* pp. 43-44, 49-50.

[72] On the older period of the history of such problems, see Mach, *Mechanik,* pp. 453-457; *Mechanics,* pp. 421-425. This period is—like all early periods in the history of branches of science—characterized by the fact that the maximal and minimal problems are all *isolated.* This period extends as far as Newton who in 1687 solved "the first problem of the calculus of variations," the determination of the figure of the solid of least resistance (cf. M. Cantor, *op. cit.,* p. 291).

[73] Mach, *Mechanik,* pp. 457-467; *Mechanics,* pp. 425-436. A German annotated translation of some works of Johann Bernoulli, Jakob Bernoulli, and Leonhard Euler, from 1696 to 1744, is given by P. Stäckel in No. 46 of *Ostwalds Klassiker.* Cf. also M. Cantor, *op. cit.,* pp. 237-241, 384, 446-458, 533, 846-848.

the history of such problems is distinguished by the names of Johann Bernoulli, Jakob Bernoulli, and Leonhard Euler, and extends until Lagrange, in 1762, brought all these interrelated methods under the general and abstract analytical form of the calculus of variations. It is to this period that the works of Maupertuis, Euler, and their contemporaries, with which we are concerned here, belong. The leading work published in this period was the famous *Methodus inveniendi lineas curvas maximi minimive proprietate gandentes: sive solutio problematis isoperimetrici latissimo sensu accepti* which was published at Lausanne and Geneva in 1744.[74]

Mathematicians found that various problems of mechanics might be put into isoperimetrical form. Whether their tendency to do this, which was very common at that time, was due to esthetic, theological, or technical reasons, it is hard to say. Daniel Bernoulli —a son of Johann Bernoulli—remarked that certain statical problems can be treated with greater facility by isoperimetrical methods than by the usual mechanical principles; the feeling, too, that the discovery that a problem about natural objects could be put in a maximal or minimal form had a connection with the way the Deity managed things here below in making nature act by the shortest or easiest or readiest paths, and so with what were then called "metaphysical"[75] questions, undoubtedly had an influence on others besides Maupertuis—on Euler for example. But we shall see how piety and humility led Euler, though accurate, judged by the mathematical standards of those days, very cautious, and perhaps a little unimaginative,[76] to accept and admire the bold and not always accurate mechanical generalizations which Maupertuis professed to deduce from "metaphysics." But probably the esthetic satisfaction which

[74] An annotated German translation of a great part of this book was given in No. 46 of *Ostwalds Klassiker.* However, the two appendices (on the elastic curves, and on the motion of a particle round a center of force in a non-resisting medium) with which we shall be especially concerned here were not translated with the main body of the work. But the first appendix was translated, in another connection, in No. 175 of the *Klassiker* (see below, section X). An account of Euler's book of 1744 is given in M. Cantor's *Geschichte,* vol. iii, 2d ed., Leipsic, 1901, pp. 857-867.

[75] In the eighteenth century, "metaphysics" stood for—at least among mathematicians—a branch of learning which included theology, psychology, and logic. Consider the "metaphysical experiments" advocated by Maupertuis, and the "metaphysics of the infinitesimal calculus" (L. N. M. Carnot, Lagrange, and others), which meant what we mean when we say: "the logical principles of the calculus."

[76] D'Alembert, in a letter of March 3, 1766, to Voltaire (quoted by Delambre in his "Notice" in *Œuvres de Lagrange,* vol. i, p. xxi), says of Euler: "c'est un homme peu amusant, mais un très grand géomètre."

arises from stating a problem in a maximal or minimal form influenced mathematicians the most.

However this may be, to this form come many problems of statics, such as the catenary of Johann and Jakob Bernoulli,[77] and Jakob Bernoulli's problem of the elastic curve.[78] From Daniel Bernoulli's letter to Euler and from Euler's first appendix to his book of 1744, we see with what interest Daniel Bernoulli and Euler reduced this problem in the theory of elasticity to isoperimetrical methods.

These problems were all *statical* ones; and it was Daniel Bernoulli who suggested to Euler the putting of a certain *dynamical* problem into isoperimetrical form. It must be remembered that Euler, by his papers published by the St. Petersburg Academy in 1732 and 1736,[79] had placed himself at the head of the mathematical world, in the treatment of isoperimetrical problems. We must now say some words about Daniel Bernoulli and Euler and their relations to one another.

Daniel Bernoulli[80] (1700-1782) was a son of the famous Johann Bernoulli (1667-1748) and was attached to the St. Petersburg Academy from 1725 to 1733. From 1733 to 1782 he was Professor of Anatomy and Botany, and later Experimental Physics and Speculative Philosophy too, at Basel. His mathematical works[81] are on differential equations, the theory of numbers, the theory of probability, series, and mechanics[82]—principally the theorem of *vis viva*,[83] the problem of vibrating cords,[84] and hydrodynamics.[85] Leonhard Euler[86] (1707-1783), whose name as a mathematician is too well known for it to be necessary for us to refer further to his many works, came to St. Petersburg in 1727, owing to the exertions on his behalf of Daniel Bernoulli and Hermann, but left St. Peters-

[77] Cf. Mach, *Mechanik*, pp. 75-77; *Mechanics*, pp. 74-76; *Ostwalds Klassiker*, No. 46, p. 19; M. Cantor, *op. cit.*, pp. 219-220, 228, 235, 289, 384, 455, 853.

[78] Cf. M. Cantor, *op. cit.*, pp. 220-221, and Johann Bernoulli's letter of March 7, 1739, to Euler in Fuss's *Correspondance* referred to below, vol. ii, pp. 23-25.

[79] Cf. M. Cantor, *op. cit.*, pp. 846-856.

[80] M. Cantor, *op. cit.*, pp. 89-90, 550; *Encycl. Brit.*, 9th ed., vol. iii, 1875, pp. 606-607.

[81] *Ibid.*, pp. 477-481, 610, 630-632, 634-635, 640, 642-644, 688, 693, 707, 721, 851, 900, 904-906.

[82] Cf. also Mach, *Mechanik*, pp. 43-49, 326; *Mechanics*, pp. 40-47, 293.

[83] Cf. also Mach, *Mechanik*, pp. 374-379; *Mechanics*, pp. 343, 348.

[84] Cf. also Mach, *Die Principien der Wärmelehre*, 2d ed., Leipsic, 1900, pp. 96-97.

[85] Cf. Mach, *Mechanik*, pp. 440-453; *Mechanics*, pp. 403-420.

[86] M. Cantor, *op. cit.*, pp. 549-551.

burg in 1744 to become Director of the Mathematical Class of Frederick the Great's reformed Academy of Sciences at Berlin. In 1727 Euler met Daniel Bernoulli and was stimulated by him to an investigation on geodesic lines.[87] The letters addressed by Daniel Bernoulli to Euler—those from Euler to Bernoulli are unfortunately lost— from 1726 to 1755 have been published in P. H. Fuss's *Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIième siècle.*[88] From this correspondence we will now make the extracts which concern our present subject.

In a letter to Euler of January 28th, 1741, Daniel Bernoulli asked whether it was not Euler's opinion that orbits about centers of force could be deduced by an isoperimetrical method.[89] As we have said, Euler's replies are lost. In a letter of December 12, 1742, Bernoulli has some further remarks on the same subject;[90] and in a

[87] M. Cantor, *op. cit.*, p. 843.

[88] St. Petersburg, 1843, vol. ii, pp. 407-655. In these letters there is frequently mention of isoperimetrical problems, but the first mention of a *mechanical* problem treated by an isoperimetrical method is on pp. 456-457 (letter of March 7, 1739) where the elastic curve, which requires a certain integral which represents the *"potential vis viva"* to be a maximum, since Bernoulli thinks "that an elastic lamina which takes a certain curvature of itself will bend in such a way that the *vis viva* will be a minimum, since otherwise the lamina would move," is referred to (other references are given on pp. 468-469, 506-507, 512-514, 533-534, 536-537). To this apparently refers what Bernoulli (p. 534) calls an *a priori* method—a speculation which contrasts oddly with the passages quoted below which are rather anti-"metaphysical." The first occurrence of a reference to a *dynamical* problem to be treated by an isoperimetrical method is that given below.

It was Daniel Bernoulli who recommended that Bousquet of Geneva should be chosen as the printer of Euler's "masterly" (*herrlichen*) treatise on the isoperimetrical method—the *Methodus* printed in 1744 (letter of Feb. 9, 1743; *ibid.*, p. 521; cf. pp. 524-525 (see extract below), 528, 529, 533 (see extract below), 541, 550, 553, 578). In a letter of September 4, 1743, Bernoulli (*ibid.*, p. 536) says: "I regret that I could not read through your additions to the treatise on isoperimeters; but I have just (*fugitivo oculo*) glanced at them." This is important in view of Euler's account (section XII below) of the date and circumstances under which these additions were made and printed.

[89] "Von Ew. möchte vernehmen, ob Sie nicht meinen, dass man die orbitas circa centra virium könne methodo isoperimetrica, wie auch die figuram terrae pro theoria Newtoniana herausbringen" (Fuss, *Correspondance,* vol. ii, p. 468).

[90] "Man kann die principia maximorum et minimorum nicht genugsam ausforschen; die trajectoriae circa centrum virium, vel circa plura centra virium, müssen gleichfalls per methodum isoperimetricorum können solviret werden, obschon man das maximum vel minimum, quod natura affectat, nicht einsiehet. Es haben also Ew. einen grossen Nutzen dadurch geschafft, dass Sie die methodum isoperimetricorum so weit perfectionnirt haben. Meiner Meinung nach ist dieses argumentum inter omnia pure analytica utilissimum, und ist dieses ein wahres Exempel, dass vel sola propositio problematis, wenn man auch die Solution nicht hätte, saepe maxima laude digna sey" (*ibid.*, p. 513).

letter of April 23, 1743, speaks[91] with praise of Euler's great treatise on the Isoperimetrical Method, suggests the addition of a treatment of the problem of the elastic curve and others like it, and then comments on Euler's discovery that $\int v.ds$ is a minimum for central orbits, that Euler has obviously communicated to him without proof, as follows:

"The observation about trajectories that $\int v.ds$ must be a maximum or minimum appears to me very beautiful and important; but I cannot see how this principle is demonstrated. Please let me know whether the principle extends to trajectories about many centers of forces. Perhaps it is only an observation *a posteriori,* owing to a discovery you may have made that the trajectories have this property, and you may not have been able to demonstrate it *a priori.*"

In a letter of September 4, 1743, Bernoulli writes:[92]

[91] Wegen Ew. herrlichen Tractat de isoperimetricis werde ich vorläufig mit demselben reden; Sie belieben nur denselben fertig zu halten. Sie könnten das problema de elastica hac methodo invenienda und andere dergleichen noch beyfügen. Ich sehe leicht, dass man die curvaturam catenae et laminae elasticae oscillantis auch darin reduciren kann; auf den modum aber bin ich noch nicht bedacht gewesen. Die meisten curvas mechanicas wird man auch dahin reduciren können. Die Observation von den trajectoriis, dass $\int v.ds$ ein maximum oder minimum seyn müsse, dünkt mich sehr schön und von grosser Wichtigkeit; ich sehe aber die Demonstration dieses principii nicht ein. Ew. belieben mir zu melden, ob sich solches auch ad trajectorias circa plura centra virium erstrecke. Vielleicht ist es nur eine observatio a posteriori, indem Sie angemerkt haben, dass die trajectoriae diese proprietatem haben, ohne solche a priori recht demonstriren zu können" (*ibid.,* pp. 524-525).

[92] "Aus Dero Brief ersehe ich, dass ich in meiner Conjectur mich nicht betrogen, wenn ich gesagt habe, dass Dero Observation circa orbitas planetarum, in quibus $\int v.ds$ vel $\int v.v.dt$ ein minimum ist, vielleicht nur a posteriori sey gemacht worden; denn nach meinen principiis kann ich solches a priori nicht einsehen. Der Herr Clairaut schreibt, dass solches auch schon von einem Engländer sey remarquirt worden. Es scheint, dass dieses nicht sowohl ein principium, als eine proprietas sey, gleich wie es eine proprietas ist elasticae, dass sie das maximum solidum generirt. Doch hab ich nicht untersucht, ob die idea maximi solidi die elasticam in omni extensione begreife. Sie können mich dieser Mühe entheben, denn ich weiss, dass Sie alle dergleichen Untersuchungen allbereits gemacht haben. Von meinem principio a priori, dass die elastica das $\int ds/rr$ ein minimum formire, hab ich mit vieler Erkenntlichkeit ersehen, aber zugleich mit Beschämung, dass Sie in Ihrem supplemento so honorificam mentionem thun. Dieses principium gehet auch an in laminis inaequaliter elasticis, wenn man macht $\int eds/r.r$ ein minimum. Die laminae naturaliter non rectae erfordern zwar einen andern calculum, aber keine andere methodum; wenn aber die laminae proprio pondere zugleich incurvirt werden, so ist es schwer, das maximum oder minimum quod natura affectat zu determiniren. Ich muthmaasse, dass man allhier muss ad maxima maximorum recurriren, wenn zweyerley Considerationen zusammen kommen. Quaeatur brevitatis gratia curva AC, quam lamina naturaliter recta AB et uniformis proprio solo pondere incurvata accipiet: fragt sich, ob nicht curva AC talis seyn könnte, dass inter omnes ejusdem longitudinis, inter eosdemque terminos positas curvas, eandemque $\int ds/rr$ habentes, das centrum gravitatis infimum locum obtineat. Wir haben Beide diese curvam directe determinirt; fragt sich also, ob man ex hoc principio eandem curvam finden

"From your letter I see that I was not mistaken in my conjecture that your observation that $\int v.ds$ or $\int v.v.dt$ is a minimum for the orbits of the planets was perhaps only made *a posteriori*; for I cannot see this *a priori* by the light of my principles. M. Clairaut writes that this property has also been noticed by an Englishman. It appears that this is not so much a principle as a property, just as it is a property of the elastic curve to generate the maximum solid. Still I have not investigated whether the idea of the maximum solid includes that of the elastic curve in all its extension...."

And in a letter of December 25, 1743, Bernoulli writes:[93]

"I doubt whether one can ever show *a priori* that the elastic curve must generate the maximum solid; I consider this as a property which is shown by calculation and that nobody could have foreseen from first principles—as little as the identity of the isochrone and the brachystochrone. Such properties are, as it were, discovered through accident by our reason, and I consider the property observed, that in orbits $\int u.ds$ is a minimum, to be on this level. I was confirmed in this opinion by learning that you only observed this property *a posteriori* and never would have found it if you had not determined the orbit by other means."

Lastly, Bernoulli's anti-"metaphysical" tendency is still more strongly shown in a passage[94] of a letter to Euler of April 29, 1747:

"Herr Ramspeck has written to my father that you are engaged in various public metaphysical controversies. You really ought not to meddle with such matters, for from you we expect only sublime things, and it is not possible to excel in metaphysics."

Euler, we know, had a strong reverence for "metaphysics" and

würde. Der calculus aber wird ohne Zweifel weitläufig seyn, und bin ich von diesem principio nicht convincirt, so dass Ew. sich schwerlich die Mühe werden geben wollen meine Conjectur zu untersuchen. Wenn solche aber richtig wäre, würde es, wie ich glaube, leicht seyn, schièr aller curvarum maxima et minima a priori anzuzeigen" (*ibid.*, pp. 533-534).

[93] "Ich zweifle ob man jemals a priori werde zeigen können, dass die elastica müsse maximum solidum generiren; ich betrachte solches als eine Proprietät, die der calculus ausweiset, und die kein Mensch ex principiis novis jemals würde haben können vorhersehen, eben so wenig als die identitatem isochronae et brachystochronae. Dergleichen proprietates sind ratione nostri gleichsam accidental, und auf diesen Fuss betrachte ich auch die observatam proprietatem orbitarum, in quibus $\int u.ds$ ein minimum macht, worin ich um so viel mehr confirmirt werde, als ich errathen, dass Sie diese proprietatem nur a posteriori observirt haben und niemals würden gefunden haben, wenn Sie nicht die orbitas aliunde determinirt hätten" (*ibid.*, p. 543).

[94] "Herr Ramspeck hat meinem Vater geschrieben, dass Sie in unterschiedenen controversiis metaphysicis publicis stehen. Sie sollten sich nicht über dergleichen Materien einlassen; denn von Ihnen erwartet man nichts als sublime Sachen, und es ist nicht möglich in jenen zu excelliren" (*ibid.*, p. 621).

consequently attached to Maupertuis's *a priori* speculations a value far above his own discovery. We shall see later that, in papers published among the *Mémoires* of the Berlin Academy, he emphasizes, as he apparently did to Daniel Bernoulli, the fact that he had only discovered the minimal condition satisfied by orbital motion in an *a posteriori* manner, as if this was rather a demerit. Nowadays we would say that Euler's great caution in, for example, insisting, in his *Methodus*, that the v in

$$\int v . ds$$

is to be expressed in terms of s by the principle of *vis viva*, so that his minimal principle cannot be extended to the case of motion in a resisting medium, where the principle of *vis viva* does not hold, and, in later publications, the careful enumeration of cases when testing Maupertuis's statical principle, are merits. But the following extract from the first appendix on elastic curves to the *Methodus* of 1744 proves that more general "metaphysical" ideas were by no means foreign to Euler:

"For since the plan of the universe is the most perfect possible and the work of the wisest possible creator, nothing happens which has not some maximal or minimal property, and therefore there is no doubt but that all the effects in nature can be equally well determined from final causes by the aid of the method of maxima and minima as from the efficient causes."[95]

X.

We will now return to the publications of the Berlin Academy. The only paper concerning us here in the *Histoire* for 1747,

[95] "Cum enim Mundi universi fabrica sit perfectissima, atque a Creatore sapientissimo absoluta, nihil omnino in mundo contingit, in quo non maximi minimive ratio quaepiam eluceat; quamobrem dubium prorsus est nullum, quin omnes Mundi effectus ex causis finalibus, ope Methodi maximorum et minimorum, aeque feliciter determinari queant, atque ex ipsis causis efficientibus, *Methodus*, p. 245, and cf. section XII below. (See *Ostwalds Klassiker*, No. 175, p. 18. Cf. Mach, *Mechanik*, p. 485; *Mechanics*, p. 455. Cf. also E. Dühring, *Kritische Geschichte der allgemeinen Principien der Mechanik*, 3d ed., Leipsic, 1887, pp. 293-294, 296-299, 385-400). These reflections of Dühring's are on the effects of philosophy on mechanics and Lagrange's anti-"metaphysical" tendencies. Lagrange's own words are (*Méchanique analitique*, Paris, 1788, p. 187): "....as if vague and arbitrary denominations [such as *the least quantity of action*] made up the essential part of the laws of nature and could by some secret virtue raise simple results of the known laws of mechanics to the position of final causes"; and (p. 189): "....I regard this principle [of least action] not as a metaphysical principle but as a simple and general result of the laws of mechanics."

On the principle of least action with Fermat, Maupertuis, Euler, and Lagrange, and its effect on Gauss, cf. Dühring, *op. cit.*, pp. 100-102, 218-219, 287-302, 425-430.

published in 1749, is one in the class of speculative philosophy by Samuel Formey,[96] entitled: "Examen de la preuve qu'on tire des fins de la nature, pour établir l'existence de Dieu"; in which the author comes, by a rather different way, to the same conclusions as Maupertuis (1746).

In the *Histoire* for 1748, published in 1750, there are two papers relating to our subject by Euler.[97] The first is entitled: "Recherches sur les plus grands et plus petits qui se trouvent dans les actions des forces," and he quoted with approval Maupertuis's memoir of 1746, and remarked[98] that Maupertuis had shown that in the state of equilibrium of bodies, if some small movement were to happen to them, the quantity of action would be the least. He himself, says Euler, had discovered a similar law in the motion of bodies attracted to one or many centers of forces; in this case $\int u.ds$ expresses the quantity of action. In statics[99] this principle has been long recognized. Thus, it is easy to see that a chain suspended by its ends must take such a figure that the center of gravity of the chain is as low as possible; and thus, if x is the distance of the element ds from an arbitrary horizontal plane, $\int x.ds$ will be a minimum for the curve of the chain, and $\int z.ds$ is the quantity of action.[100] Many other analogous cases were, according to Euler, treated by Maupertuis; and Daniel Bernoulli remarked that the curve of an elastic lamina has a minimal property, and this view was developed by Euler in Appendix i of his *Methodus inveniendi* of 1744.[101]

There are, then, two ways of solving mechanical problems: one is the direct method, and the other is, knowing the formula which must be a maximum or a minimum, by the method of maxima and minima; the effect is determined by efficient causes and by final causes respectively. But it is often very difficult to discover the formula which must be a maximum or a minimum, and by which the quantity of action is represented; and this investigation belongs

[96] Pp. 365-384.

[97] Pp. 149-188 and 189-218.

[98] *Ibid.*, p. 150.

[99] *Ibid.*, pp. 150-151.

[100] *Ibid.*, p. 151.

[101] A convenient German translation of this Appendix, with critical and historical notes by H. Linsenbarth, was given in No. 175 of *Ostwalds Klassiker* (*Abhandlungen über das Gleichgewicht und die Schwingungen der ebenen elastischen Kurven von Jakob Bernoulli (1691, 1694, 1695) und Leonh. Euler (1744)*). Very interesting are Euler's (pp. 18-20) theological remarks and references to the frequency with which maximal and minimal problems appeared in the mechanical work of the Bernoullis. (Cf. section IX above.)

rather to metaphysics than to mathematics. "I believe," says Euler,[102] "that we are still very far from that degree of perfection where we are able to assign, for each effect which nature produces the quantity of action which is the smallest, and deduce it from the first principles of our knowledge; and that it will be almost impossible to arrive at it unless we discover, for a great number of different cases, the formulas which become maximal or minimal. Now, knowing the solutions with which the direct method furnishes us, it will not be difficult to find *a posteriori* formulas which express the quantity of action, and then it will not be so difficult to prove their truth by the known principles of metaphysics." With this end in view, Euler investigated several problems as to the curve formed by a flexible string in equilibrium.

Euler[103] arrived at the conclusion that the expression of the quantity of action, which, when supposed to be a minimum, gives the figure of the thread, is in perfect agreement with the *Law of Rest* published by Maupertuis in 1740.

Euler's second memoir on the principle of least action in this volume is entitled: "Réflexions sur quelques Loix générales de la Nature qui s'observent dans les Effets des Forces quelconques." He emphasizes[104] that he was only led *a posteriori* to the discovery of the minimum in the case of the equilibrium of threads, and then[105] remarks: "It is the figure which a fluid mass, all of whose particles are attracted by any forces, which was the principal object of the researches of M. de Maupertuis in order to discover the general law of rest in the Paris *Mémoires* of 1740. Thus I too will consider a fluid mass, all of whose particles are attracted to as many fixed centers as is wished by forces proportional to any functions of the distances to those centers, and I will investigate the figure of equilibrium for this mass. Then I will try to discover what will be a maximum or a minimum in this figure, in order to be in a better state to determine what must be understood by the name of the *quantity of action of the attracting forces*; and afterwards I will show by some reflections the great importance of this quantity in all researches concerning the effects produced by any forces." The expression discovered in this way was again found to agree with Maupertuis's law of 1740.

[102] *Op. cit.*, p. 152.

[103] *Ibid.*, p. 180.

[104] *Ibid.*, p. 190.

[105] *Ibid.*, p. 191; cf. p. 190.

XI.

There is nothing relating to the principle of least action, nor to mechanics (except in astronomy) in the Berlin *Histoire* for 1749 (published in 1751); but in that for 1750 (published in 1752) there is[106] an "Exposé concernant l'examen de la lettre de M. de Leibnitz, alleguée par M. le Prof. Koenig[107] dans les mois de Mars, 1751, des Actes de Leipzig,[108], à l'occasion du principe de la moindre action" by Euler,[109], with the note: "As will easily be seen by reading this memoir, it is one of those whose publication may not be delayed."

König had denied the validity of the principle in the case of equilibrium, and indicated some cases in which what, according to the principle, ought to be a minimum really reduces to nothing. But, says Euler,[110] "this objection is not of great importance, since it is sufficiently recognized in the calculus of maxima and minima that it can often happen what is a minimum vanishes entirely. But although that may be so in certain cases it by no means results that one ought to extend it to all cases of equilibrium, as always necessarily happening in that state; on the contrary, there are numberless cases in which this quantity of action is not zero but is really a minimum; and this puts beyond doubt that the aim of Nature is not the nullity of action, but its minimity." Then Euler quotes the example of the catenary, and says that the quantity of action reduces

[106] Pp. 52-64.

[107] Johann Samuel König (1712-1757); Cf. M. Cantor, *op. cit.*, pp. 599-601. König was a pupil of Johann Bernoulli's at the same time as Maupertuis. (Mayer, *op. cit.*, pp. 17-18).

[108] "De Universali Principio Aequilibrii et motus in Vi viva reperto deque nexu inter Vim vivam et Actionem utriusque Minimo" (*Nova Acta Eruditorum*, 1751, pp. 125-135, 144, 162-176). König affirms that equilibrium is a result of the *nullity* of action and *vis viva* (pp. 126, 164) that in some cases the action is a *maximum*, and this would hardly be reconcilable with Maupertuis's proof of the Creator's wisdom (pp. 126, 165); and that since *action* is *vis viva* into the time, the principle is that *vis viva* is a minimum (p. 127). König, like a thorough Leibnizian, praises the theorem of *vis viva* highly ("Censeo itaque, Theoremate Virium vivarum fundamentum universae Mechanicae contineri," p. 169), and deduces statistics from it. The extract from the letter of Leibniz's is given quite at the end (p. 176) and is: "L'Action n'est point ce que vous pensés, la considération du tems y entre; elle est comme le produit de la masse par le tems, ou du tems par la force vive. J'ai remarqué que dans les modifications des mouvemens elle devient ordinairement un Maximum, ou un Minimum. On en peut deduire plusieurs propositions de grande consequence; elle pourroit servir à determiner les courbes que decrivent les corps attirés à un ou plusieurs centres. Je voulois traiter de ces choses entr'autres dans le seconde partie de ma Dynamique, que j'ai supprimée; le mauvais accueil, que le prejugé a fait à la premiere, m'ayant degouté."

[109] As we learn from a note on p. 63 of the *Histoire* for 1750.

[110] *Ibid.*, p. 53.

to the distance of the center of gravity of the chain from the center of the earth; and[111] Daniel Bernoulli's and his own researches on elastic curves.

As regards dynamics, König quoted from a supposed letter written by Leibniz to Hermann, in which "action" was defined as Maupertuis defined it and the property of being "ordinarily a maximum or a minimum" in dynamical problems remarked. König could not produce the original nor could the original be found by officials. It is not interesting now to follow the controversy much further. König did not charge Maupertuis with plagiarism;[112] but, since the principle was considered by Maupertuis and others to be of the greatest possible importance and to reflect great credit on Maupertuis, its discoverer, the Berlin Academy, of which Maupertuis was president, took up the matter with great zeal, and concluded, like Euler's report, that, on internal and external evidences, the fragment of the letter was forged, either to injure Maupertuis or to exaggerate, by a pious fraud, the merits of Leibniz.[113] The result was an unjust expulsion of König from the Berlin Academy, and the consequent culmination of Voltaire's ill-feeling towards Maupertuis.[114]

XII.

To return to the *Histoire* for 1750. To the literature of the controversy also belongs a "Lettre de M. Euler à M. [Jean Bernard] Merian" of September 3, 1752.[115] Nowadays, the only interesting part of this letter is where Euler[116] gives some details about the publication of his *Methodus inveniendi.* The defenders of König stated that they knew the *Methodus* had been in the publisher's hands at Lau-

[111] *Ibid.,* p. 54.

[112] *Ibid.,* p. 60.

[113] *Ibid.,* p. 62.

[114] On the König incident, see La Beaumelle, *op. cit.,* pp. 139-141, 143-145, 150-167, and, on Voltaire's part in it, pp. 167 *et seq.* Further du Bois-Reymond, *op. cit.,* pp. 35-36, 47, 50-66. It is now known that the fragment of Leibniz's letter was probably genuine, and part of a letter to Varignon; Cf. *ibid.,* pp. 56-57, and the references to Gerhardt's paper in M. Cantor, *op. cit.,* p. 599.

Even in 1877, Mayer (*op. cit.,* p. 19) said that the letter was without doubt forged; but Helmholtz in 1887 (*op. cit.*) showed that its genuineness was probable.

It appears that Euler only made one separately printed contribution to the discussion on König's dissertation; it is entitled: "Dissertatio de principio minimae actionis una cum examinatione objectionum Cl. Prof. König contra hoc principium factorum," Berlin, 1783. We have not seen this work, but only quote it from the Bibliography in Fuss's *Correspondance,* vol. i, p. xciv.

[115] *Ibid.,* pp. 520-532.

[116] *Ibid.,* pp. 525-526.

sanne since 1743, a circumstance which would give Euler priority over Maupertuis. This, says Euler, is correct in so far as it concerns the treatise itself, which he had finished some years before it appeared, but he only made the additions since he had sent the manuscript to Lausanne, and only shortly before the publication of the book towards the end of 1744. Further, he had communicated this supplement to nobody before printing it.

"When," says Euler,[117] "I used the method of maxima and minima to define the trajectories which are described by bodies attracted by any central force, I do not pretend to have been beyond what MM. Bernoulli and others have done when they determined by the help of the same method the curvature of the catenary, that of a piece of linen filled with liquid, and other curves of the same kind. Such investigations only furnish particular principles which can hardly be extended further than the cases to which they are applied. On the other hand, it is a question here of a universal principle, from which all the former principles should result, and which can be regarded as a Law established in all the phenomena of nature; which would render its discussion less the part (*du ressort*) of Mathematics than of Metaphysics, on the principles of which this doctrine should be founded. Also, although for long people have not doubted that, in all natural effects, there is a maximal-minimal principle which determines them, nobody before the Illustrious President of our Academy has even suspected in what elements this principle was contained and how it could be accommodated to all cases.[118] As

[117] *Ibid.*, pp. 526-527.

[118] Cf. *Methodus*, pp. 309, 320. The actual quotations are: (1) "Quoniam omnes naturae effectus sequuntur quandam maximi minimive legem; dubium est nullum, quin in lineis curvis, quas corpora projecta, si a viribus quibuscunque sollicitentur, describunt, quaepiam maximi minimive proprietas locum habeat. Quaenam autem sit ista proprietas, ex principiis metaphysicis a priori definire non tam facile videtur: cum autem has ipsa curvas, ope Methodi directae, determinare liceat; hinc, debita adhibita attentione, id ipsum, quod in istis curvis est maximum vel minimum, concludi poterit. Spectari autem potissimum debet effectus a viribus sollicitantibus oriundus; qui cum in motu corporis genito consistat, veritati consentaneum videtur hunc ipsum motum, seu potius aggregatum omnium motuum qui in corpore projecto insunt, minimum esse debere. Quae conclusio etsi non satis confirmata videatur, tamen, si eam cum veritate jam a priori nota consentire ostendero, tantum consequetur pondus, ut omnia dubia quae circa eam suboriri queant penitus evanescant. Quin-etiam cum ejus veritas fuerit evicta, facilius erit in intimas Naturae leges atque causes finales inquirere; hocque assertum firmissimis rationibus corroborare."...... (2) "Tam late ergo hoc principium patet, ut solus motus a resistentia medii perturbatus excipiendus videatur; cujus quidem exceptionis ratio facile perspicitur, propterea quod hoc casu corpus per varias vias ad eundum locum perveniens non eandem acquirit celeritatem. Quamobrem, sublata omni resistentia in motu corporum projectorum, perpetuo haec constans proprietas locum habebit, ut summa omnium motuum elementarium sit

regards myself, I only knew in a sure manner *a posteriori* the principle I used to determine trajectories; and I have ingenuously confessed that I was not in a position to establish its truth in another manner. All that I have done is to deduce from it the same curves that are commonly found by the direct method, starting from the principles of mechanics. I have not even dared to extend its use unless I could justify by calculation its agreement with known principles. And that is what has led me to separate from this principle motions in a resisting medium and other more complicated ones; for no way presented itself to my mind of discovering the truth with regard to these motions."

Among the *Mémoires* in the Class of Speculative Philosophy in the same volume (1750) of the *Histoire,* are two by Merian[119] entitled: "Dissertation ontologique sur l'Action, la Puissance et la Liberté," and "Seconde Dissertation sur l'Action, la Puissance et la Liberté"; in the first of which[120] Maupertuis's explanation, in the *Essai de Cosmologie,* of the generation of the idea of motive force is quoted.

XIII.

In the Berlin *Histoire* for 1751, published 1753, there are five memoirs we shall have to notice, and all of the Class of Mathematics.[121]

The first is by Euler,[122] and is entitled: "Harmonie entre les Principes généraux de Repos de Mouvement de M. de Maupertuis." Both principles of Maupertuis (of 1740 and 1744) rest, says Euler, on the same foundation, so that if one is proved, the other cannot be

minima. Neque vero haec proprietas in motu unius corporis tantum cernetur, sed etiam in motu plurium corporum conjunctim; quae quomodocunque in se invicem agant, tamen semper summa omnium motuum est minima. Quod, cum hujusmodi motus difficulter ad calculum revocentur, facilius ex primis principiis intelligitur, quam ex consensu calculi secundum utramque Methodum instituti. Quoniam enim corpora, ob inertiam, omni status mutationi reluctantur; viribus sollicitantibus tamparum obtemperabunt, quam fieri potest, siquidem sint libera; ex quo efficitur, ut, in motu genito, effectus a viribus ortus minor esse debeat, quam si ullo alio modo corpus vel corpora fuissent promota. Cujus ratiocinii vis, etiamsi nondum satis perspiciatur; tamen, quia cum veritate congruit, non dubito quin, ope principiorum sanioris Metaphysicae, ad majorem evidentiam evehi queat; quod negotium aliis, qui Metaphysicam prositentur, relinquo."

[119] Pp. 459-485 and 486-516.

[120] *Ibid.,* p. 479.

[121] In this volume, the memoirs in the Classes of Experimental Philosophy and Mathematics are paged (pp. 1-356) separately from those in the Classes of Speculative Philosophy and of Belles Lettres (pp. 1-154).

[122] Pp. 169-198.

doubted. Now, Maupertuis and Euler had established the truth of the law of rest of 1740 by a multitude of different cases. Euler, then, first deduced the principle of motion from that of rest,[123] and then[124] showed that all the elementary theorems of statics follow readily from the law of rest.

The nerve of Euler's investigation is the deduction of the principle of least action from the law of rest. Euler[125] called the integral $\int V.dv$, where V is a central force acting on the body M and v is the distance from M to any fixed point in the direction of V, the *effort* (effort), so that Maupertuis's law is that the sum of all the efforts is a maximum or a minimum.

"What is more natural," exclaims Euler,[126] "than to maintain that this same principle of equilibrium should also subsist in the movement of bodies under like forces? For if the intention of nature is to economize the sum of the efforts as much as possible, this intention must extend also to movements, provided that we take the efforts, not merely as they subsist in an instant, but in all the instants together for which the movement lasts. Thus, if the sum of the efforts is Φ for any instant of the motion, then, putting dt for the element of the time, the integral $\int \Phi.dt$ must be a minimum. If then, for the case of equilibrium the quantity Φ must be a minimum, the same laws of nature seem to exact that, for motion $\int \Phi.dt$ should be the smallest.

"Now it is precisely in this formula that the other principle of M. de Maupertuis, concerning motion, is contained, however different it may appear at the first glance. To show this agreement, I have only to remark that when a body moves under the action of the forces V, V', V'',...., the effort Φ to which the body is subject expresses at the same time the *vis viva* of the body— the product of the mass M of the body and the square of its velocity (u)." Thus the formula which must be a minimum is

$$\int M.u^2.dt = \int M.u.ds.$$

Where $v, v', v'',\ldots.$, are the distances of M from the centers

[123] On pp. 181-182, Euler remarked that, if we wish, inversely, to deduce the principle of rest from that of motion, "we must suppose the motion infinitely small, and this causes great obscurities (*brouïlleries*) in the consideration of infinitely small velocities and of the spaces which are described in an infinitely small time.

[124] *Ibid.*, pp. 183-193.

[125] *Ibid.*, p. 174.

[126] *Ibid.* p. 175.

of forces V, V′, V″,...., which are functions of these distances, Euler[127] gets the equation

$$Mu^2 = \text{const} - \Sigma \int . dv = \text{const} - \Phi;$$

and:[128] "the constant does not disturb this harmony between the effort Φ and the *vis viva* $M.u^2$ of the body; for if $\int \Phi . dt$ is a maximum or a minimum, $\int M.u^2.dt$ or $\int M.u.ds$ will be so also, since the term $\int \text{const}\, dt = \text{const}\ t$ does not enter into the consideration of the maximum or minimum. And, besides that, as the effort Φ is expressed by integral formulae, it already contains in itself any constant, so that I could have neglected this constant entirely and simply put $Mu^2 = -\Phi$, whence the identity would have been more evident. However, if we take the above integrals on a fixed footing (*sur un pied fixe*), so that the effort Φ receives a determined value, the addition of the constant will be necessary; since the velocity of the body at a certain point of its path depends on the initial velocity, and by this initial velocity the constant must be determined in each case proposed. But, of whatever quantity it may be, the determination of the maximum or minimum is not affected." Of course, as Mu^2 is equal to the negative of Φ, if $\int Mu^2.dt$ is a minimum, $\int \Phi . dt$ will be a maximum, and reciprocally.

Euler[129] then proved "the identity between the effort and the *vis viva*" for two or more bodies, connected in any way with one another to make a flexible body: the sum of the *vires vivae* of all the elements of the body always reduces to the sum of the efforts to which all the elements are subject in the same time,—in the case of two bodies of masses M and N, distances to the[130] center of force considered x and y respectively, and the accelerating forces X (a function of x) and Y (a function of y) respectively,

$$\Phi = M \int X.dx + N \int Y.dy.$$

Euler[131] remarked that there are cases of equilibrium in which the sum of the efforts is a *maximum* and[132] classes the cases of equilibrium as of such natures that, if the sum of efforts is a minimum, equilibrium reestablishes itself on an infinitely small dis-

[127] *Ibid.*, p. 177.

[128] *Ibid.*, p. 178.

[129] *Ibid.*, pp. 179-181.

[130] Of course the proof extends to as many centers of force as wished.

[131] *Ibid.*, p. 194.

[132] *Ibid.*, p. 195. There is an example of the sum of efforts being a maximum on pp. 195-196.

placement being given to the system, whereas, if the sum is a maximum, this is not the case.[133]

XIV.

Euler's second paper in the volume for 1751 is entitled: "Sur le Principe de la Moindre Action."[134] This paper is concerned with the opinion that there is a minimum in the actions of nature, with Aristotle and his school, Descartes, Fermat, Leibniz,[135] Wolff, Engelhard, s'Gravesande, and others, and was occasioned by the König affair. It is ridiculous, says Euler,[136] to suppose that König's fragment was written by Leibniz, for it attributes to Leibniz a principle opposed to that which he adopted publicly in the case of the motion of light—that the product of the path described and the resistance is a minimum.

Referring to his own discovery of the minimum of the action—integral for central orbits, Euler[137] remarks: "Besides, I had not discovered this beautiful property *a priori* but (using logical terms) *a posteriori,* deducing after many trials the formula which must become a minimum in these movements; and, not daring to give it more force than in the case which I had treated, I did not believe that I had discovered a wider principle: I was content with having found this beautiful property in the movements which take place around centers of forces."

Euler's third paper in this volume is entitled: "Examen de la Dissertation de M. Le Professeur Koenig, inserée dans les Actes de Leipzig, pour le Mois de Mars, 1751."[138] In this paper Euler examined König's demonstrations with care and pronounced them to be worthless.[139]

The "Essai d'une Démonstration Métaphysique du Principe général de l'Equilibre" of Euler, printed in the same volume,[140] does not mention Maupertuis's name,[141] and is concerned with the deduction from indubitable axioms of the principle that, for equilib-

[133] Cf. Mach, *Mechanik,* pp. 70-75; *Mechanics,* pp. 69-73.

[134] *Loc. cit.,* pp. 199-218.

[135] *Ibid.,* pp. 205-209.

[136] *Ibid.,* p. 209.

[137] *Ibid.,* p. 214.

[138] *Ibid.,* pp. 219-239, "Additions," pp. 240-245.

[139] *Ibid.,* p. 220.

[140] *Ibid.,* pp. 246-254.

[141] It is, however, Maupertuis's "Law of Rest" (Cf. also Mayer, *op. cit.,* p. 23).

rium, where, P, Q,.... are forces and x, y,.... are measured on their respective lines of action,

$$\int P.dx + \int Q.dy + \ldots.$$

is a minimum.

Lastly, there is, in this volume a paper by Nicolas de Beguelin,[142] tutor of Frederick the Great's nephew who was later Frederick William II, entitled: "Recherches sur l'Existence des Corps Durs,"[143] in which Maupertuis is called a great man[144] and the illustrious author of the principle of least action,[145] and the other conclusions are just what Maupertuis would have wished.

XV.

In the Paris *Mémoires* for 1749, the Chevalier d'Arcy[146] published some reflections on the principle of least action, which he had long hesitated to publish, but that he did so in the interests of truth. D'Arcy maintained: (1) That the action of a body is not proportional to $m.v.s$, because this supposition, in a particular case, leads to a result contrary to that which the laws of motion give; (2) That, admitting Maupertuis's definition of action, the quantity of it that nature employs in each change is not a minimum, and that if in some cases this is so, the principle of least action cannot serve to prove it; (3) that Maupertuis's law of equilibrium that Maupertuis deduced from the principle of least action is only established by the introduction of a foreign and gratuitous supposition; (4) that, in general, whatever may be the laws of nature, one could always easily find a function of the masses and velocities which would represent them when it is supposed to be a minimum, but this property would not be enough to give the name of *action* to this function nor to raise the principle thence obtained to the rank of a metaphysical principle;[147] (5) that, if we define the *action of*

[142] Lived from 1714 to 1789. (Cf. Berlin *Histoire*, 1788-9 (not seen); M. Cantor, *op. cit.*, vol. iv, 1908, pp. 174 (article by F. Cajori), 227 (article by E. Netto).

[143] *Ibid.*, pp. 331-355.

[144] *Ibid.*, pp. 344, 346.

[145] *Ibid.*, p. 347.

[146] "Réflexions sur le Principe de la moindre Action de M. de Maupertuis," *Hist. de l'Acad. Roy. des Sci.*, 1749 (Paris, 1753), *Mémoires*, pp. 531-538. There is an account of this memoir in the *Histoire*, pp. 179-181. Patrick d'Arcy was born on Sept. 18 (27), 1725, at Galloway and died on Oct. 18, 1779. He was a count, a field marshal of France, and a "Pensionnaire-Géomètre" of the Paris Academy (*Poggendorff's biog.-lit. Handwörterbuch*, vol. i, p. 57). Cf. M. Cantor, *op. cit.*, vol. iv, 1908, p. 18 (article by S. Günther).

[147] *Ibid.*, pp. 535-536.

a body around a point to be the product $m.v.p$, where p is the perpendicular drawn from this point on the direction of the *body,* then the *total action existing in nature at any instant around a given point, being produced in one given body, the quantity of action of this body will always be the same around this point,*[148] and from this theorem are easily deduced the principle of the conservation of *vis viva,* the case of rest, the centers of oscillation or of percussion, the law of the refraction of light, and so on.

With regard to (1), d'Arcy[149] gave the following considerations. "If two bodies produce equilibrium, that is to say, if rest follows from their direct impact, without our knowing to what the action is proportional, it (the action) must necessarily be equal in the two bodies; for if not, then it would follow that an action was in equilibrium with a lesser action, that is to say that different actions produce the same effect. Now, can we imagine that two equal and similar effects can be produced by unequal quantities of causes? This does not imply that the effect is proportional to its cause, but only that the same effect is always produced by the same quantity of cause and *vice versa.*

"Let there be two hard bodies A and B perfectly equal and proceeding in opposite directions with equal velocities, then clearly rest will follow their impact. If A, proceeding in the same direction with the same velocity, is impinged upon by the body C of different mass and velocity, but such that rest follows impact, I believe that nobody can deny that the action of B is equal to that of C, since both destroy the velocity of A. Can we have another idea of the equality of two quantities than of our being able to substitute one for the other without changing anything?" If B proceeds with double the velocity of, and traverses double the space traversed by, C, the principle of Maupertuis says that the mass of C is four times that of B; and this is contrary to what we find by the laws of motion. "Thus," concludes d'Arcy, "the action is not proportional to the mass multiplied by the velocity and by the space described."

With respect to (2), d'Arcy[150] remarked that if two bodies A and B proceed in the same direction with the velocities a and b,

[148] This theorem was given by d'Arcy in the Paris *Mémoires* for 1747 (published in 1752; pp. 348-356) under the title: "Principe général de Dynamique, qui donne la relation entre les espaces parcourus et les temps, quelque soit le système de corps que l'on considère, et quelles que soient leurs actions les uns sur les autres." This memoir (read in 1746) is part of the paper (of three memoirs) entitled: "Problème de Dynamique" on pp. 344-361.

[149] *Loc. cit.,* pp. 532-533.

[150] *Ibid.,* pp. 533-534.

the action of the bodies A and B will be Aa^2+Bb^2. If after impact they proceed with the velocities x and z, their action after impact will be Ax^2+Bz^2.[151] Now the quantity of action after impact will be either equal to or less than or greater than what it was before impact: if it is equal we have the theorem of *vis viva*, which does not hold for hard bodies; if it is greater it will have increased by the quantity

$$Ax^2+Bz^2-Bb^2-Aa^2;$$

if it is smaller it will be diminished by the quantity

$$Aa^2+Bb^2-Ax^2-Bz^2,$$

and this quantity is the real quantity of action lost, and consequently is that employed by nature to produce the actual change; therefore

$$2Ax.dx+2Bz.dz=0;$$

or, if we suppose $dx=dz$,[152]

$$Ax+Bz=0,$$

which is absurd. It is not, then, the destroyed part of this quantity which is a minimum. Maupertuis's argument is: Suppose that the bodies A and B proceed in the same direction with the velocities a and b and that the plane on which they are moves with the velocity x; evidently A will move on this plane with a velocity $a-x$ and B will move behind with a velocity $x-b$, x being greater than b and less than a. Maupertuis finds that

$$A(a-x)^2+B(x-b)^2$$

will be a minimum when the velocity x is such that

$$A(a-x)=B(x-b),$$

that is to say, when the bodies are in equilibrium on this plane. "I vow," said d'Arcy,[153] "that I do not know what consequence one can deduce from this other than: AP^2+BQ^2 being a minimum and $P^2=\int\Phi.dx$ and $Q^2=\int\Delta.dx$, we will have

$$A.\Phi+B.\Delta=0,$$

and consequently if

$$A.Z=B.X,$$

where Z and X are functions of x, then AZ^2+BX^2 will always be a minimum, and *vice versa*; and this leads me to believe that, when one has found that $A.Z^2+B.X^2$ is a minimum, one knew that $A.Z=B.X$."

[151] "Since a, b, x and z express the spaces as well as the velocities."

[152] For hard bodies $x=z$ and for elastic ones $a-b=z-x$.

[153] *Ibid.*, p. 534.

With regard to (3), when Maupertuis deduced the law of the lever from his principle of least action, he made a gratuitous supposition that the lever moves with a constant angular velocity.[154] To find the point of the lever (of length C) about which two bodies of masses A and B at the ends of the lever produce equilibrium, Maupertuis called Z the distance of A to the sought point, and announced that, to solve the problem, he would suppose the lever to receive some small movement and then express that the quantity of action is the smallest possible. If, remarked d'Arcy, we call V the small velocity of A and suppose that A describes a space a, the velocity of B and the space described by it will be, respectively,

$$V(C-Z)/Z \text{ and } a(C-Z)/Z,$$

and the action of the bodies will be

$$AVa + BVa(C-Z)^2/Z,$$

and the differential equated to zero, supposing that a and V are constant, gives Z=C. Maupertuis gets the correct law by supposing that the lever moves with a constant angular velocity. But this supposition, says d'Arcy, "seems to me absolutely gratuitous, since, to each value of Z, the action or the time necessary for it to describe the constant angle is different."

With regard to (5), d'Arcy[155] remarks that his definition of action is in perfect agreement with d'Alembert's:[156] "The action is the movement that a body produces or tends to produce in another body."

D'Arcy's principle is that the sum of the masses of a system, each mass being multiplied by the sector which it describes around a fixed point in the same time, less the sum of the sectors described in the contrary sense, each being multiplied by the mass of the body which describes it, is proportional to the time. The only difference from the principle that d'Arcy gave in this memoir of 1749 is that instead of (as in 1747) sectors multiplied by masses, were used in 1749 the equivalent expressions $m.v.p$.

Let two bodies A and B move with the velocities a and b before impact and with the velocities x and z after impact. By the above principle the action of A and B round any point O will be the same after as before the impact; thus, where P is the foot of the perpendicular from O on the line joining A and B,

[154] *Ibid.*, p. 535.
[155] *Ibid.*, p. 536.
[156] In the *Encyclopédie* (not seen).

$$A.a.OP+B.b.OP=A.x.OP+B.z.OP,$$

and consequently

$$A(a-x)=B(z-b),$$

and this relation between the velocity lost by A and that gained by B holds whether the bodies are elastic or not. In elastic bodies we easily see that $a-b=z-x$, and hence, from the above equation

$$A(a^2-x^2)=B(z^2-b^2),$$

which is the property of *vires vivae.*[157]

If two bodies A and B strike the ends P and Q of a straight lever with the same velocity *a,* to find the fulcrum-point C of the bar such that A and B remain at rest after the impact, d'Arcy[158] observes that the action of A round C must be equal to the action of B round C, and thus C is the Center of gravity. By the same method we find the centers of oscillation or of percussion, and so on.

When deducing the law of the refraction of light,[159] d'Arcy observes that, in his memoir of 1747, he had proved that it is the same thing whether the bodies are attracted toward the point round which the action is sought or not, as the quantity of this action is not altered thereby. Let FG be the surface of a diaphanous and homogeneous sphere of center C, M a point outside the sphere, and N a point inside. A ray of light—μ being the mass of a corpuscle of light—travels from M to N, its velocity outside the sphere being v and inside the sphere being u, meeting the surface at m. "The action of the surface FG can only be towards the center C; for whatever action this body may have on the corpuscle of light on one side of the perpendicular to the surface, it will have the same action on the other side." Thus we have

$$\mu.v.CR=\mu.u.Cr,$$

and this gives the known law of refraction of light. The case of FG being plane instead of spherical is then treated, and d'Arcy finally remarks that other examples of the application of his principles are given in the memoir of 1747.

XVI.

The Berlin *Histoire* for 1752, published in 1754, contains among the memoirs of the class of Speculative Philosophy a "Réponse à un Mémoire de M. d'Arcy inséré dans le Volume de l'Académie Royale

[157] D'Arcy, *loc. cit.,* p. 537.
[158] *Ibid.*
[159] *Ibid.,* pp. 537-538.

des Sciences de Paris pour l'année 1749" by Maupertuis,[160] which is headed by a notice,[161] in italics, stating that the memoir (1744) in which the principle of the least quantity of action was first communicated was received by the Paris Academy, Maupertuis "dares to say, with some applause (*applaudissment*)." Then Maupertuis refers to his paper of 1746, to his *Essai de Cosmologie,* to the attacks of "un Professeur de la Haye" to whom, as he used libels, he will never reply, and to d'Arcy who "attacks with so much politeness and modesty," that Maupertuis thinks that he ought to reply to him. He appears, says Maupertuis, "to be such a lover of the truth that I will try to introduce him to it."[162]

(1) D'Arcy tried to show that Maupertuis is wrong to call *m.v.s action.* Maupertuis believed that he had good grounds for justifying the name; but, to cut matters short, Maupertuis said that he had adopted Leibniz's definition.[163] D'Arcy's reason against calling the above product *action* reduces to this: In the impact of hard bodies, two different quantities of *action* reduce to rest one and the same body moving with the same velocity. By the same kind of reasoning, says Maupertuis, d'Arcy might object to the name *vis viva*; for two different *vires vivae* can reduce the same hard body to rest." And in fact here the *vis viva* is the same as the *action,* for here "the space is proportional to the velocity."[164] Again, with elastic bodies, if two unequal bodies with the same *vires mortuae* (*m.v*) strike a third body at rest, different *vires mortuae* will come into existence or perish.

(2) D'Arcy, to show that Maupertuis is wrong in holding that the quantity of action necessary to produce any change in nature is a minimum, confuses, when treating of impact, change of the quantity of action with change of velocities.[165] The velocities can change without the quantity of action changing, as is the case in the impact of elastic bodies (when this quantity is the same as the quan-

[160] *Histoire de l'Acad.* de Berlin, 1752, T. VIII, pp. 293-298.

[161] *Ibid.,* pp. 293-294.

[162] "....et paroit si Amateur de la vérité, que je tâcherai de la lui faire connoître" (*ibid.,* p. 294).

[163] "....mais pour trancher court avec M. d'Arcy, je puis dire que ce n'est pas mon affaire. Leibnitz, et ceux qui l'ont suivi, ont appelé ainsi le produit du corps par l'espace et par la vitesse; j'ai adopté une définition établie, contre laquelle on n'avoit point disputé, et que je n'avois aucune raison de changer; voilà ce qu'il me suffiroit de répondre"; *ibid.,* p. 295. Apparently this is upon what E. du Bois-Reymond relies when he says (*op. cit.,* p. 48): "Maupertuis borrowed, as he himself says, the concept and name of *action* from Leibniz..."

[164] *Ibid.,* p. 295.

[165] *Ibid.,* p. 296.

tity of *vis viva*); in the impact of hard bodies, the change of the velocities is neither equal nor proportional to the change in the quantity of action.

If[166] the bodies are elastic, the change is: A which moved before with the velocity a moves afterwards with the velocity α, and the corresponding velocities of B are b and β. If then we wish that afterwards A should move with the velocity a and B with the velocity b, we must transport the A-plane with the velocity $a-\alpha$ and the B-plane with the velocity $\beta-b$; and from this we must get the quantity of action $A(a-\alpha)^2+B(\beta-b)^2$ necessary to produce the change in nature, and which is a minimum. If A and B are hard, and the common velocity after the impact is x, and if we wish each body to move with its original velocity, we proceed as before, and get, for the quantity of action necessary to produce this change, $A(a-x)^2+B(x-b)^2$, the smallest possible.

(3) D'Arcy's criticism on Maupertuis's deduction of the lever is mistaken, for Maupertuis supposed the lever to be in a state of rest and infinitely little displaced from this state.[167]

Finally, Maupertuis[168] mentioned the incompleteness of this theory of the lever, which was not remarked by d'Arcy, but about which we have read in connection with the reprint of the memoir of 1740[169] in Maupertuis's *Œuvres.*[170]

XVII.

In the Paris Mé*moires* for 1752 appeared a reply by d'Arcy[171] to Maupertuis's paper in the Berlin *Mémoires* for 1752. After a few preliminary words in which what looks like sarcasm is veiled in words of compliment—Maupertuis's "modesty," "politeness," and "simplicity" being praised, d'Arcy[172] confesses that if he had need of a proof of an arranging intelligence he would find it just as much in the uniformity of the laws of generation of the vilest insects as in the general laws of mechanics.

[166] *Ibid.*, pp. 296-297.

[167] *Ibid.*, pp. 297-298.

[168] *Ibid.*, p. 298.

[169] Mapertuis here refers to this paper as being in the *Mémoires* for 1743. This is, of course, a misprint.

[170] See section II above.

[171] "Replique à un Mémoire de M. de Maupertuis, sur le principe de la moindre action, inséré dans les Mémoires de l'Académie royale des Sciences de Berlin, de l'année 1752," *Hist. de l'Acad. Roy. des Sci.*, 1752 (Paris, 1756), *Mémoires*, pp. 503-519.

[172] *Ibid.*, p. 503.

With regard to Maupertuis's (correct) classification of d'Arcy's objections under three heads, d'Arcy[173] maintains that the first still holds, for "when someone says that nature economizes action, he clearly means that this quantity expresses this cause or the real force," and d'Arcy[174] even accuses Maupertuis of falling back on the authority of Leibniz. His argument depends for its validity on the principle that a definition is something more than the mere giving of a name.

With regard to d'Arcy's second objection, d'Arcy[175] quoted from the *Encyclopédie*[176] to show that Maupertuis's phrase "change happened in nature" and that his own interpretation of this phrase in the above simple case of impact as

$$Aa^2+Bb^2-A\alpha^2-B\beta^2,$$

which is to be a minimum, so that

$$A\alpha+B\beta=0,$$

is natural and also showed[177] that Maupertuis himself implied this interpretation.

Then d'Arcy[178] showed that the manner in which Maupertuis used his principle in the case of the refraction of light is different from that in which he used it in the case of impact. If we treated the latter case like the former, we should have the result that

$$Aa^2+Bb^2+A\alpha^2+B\beta^2$$

is a minimum, and hence that

$$A\alpha^2+B\beta^2=0.$$

In the case of light, it is the action before the change plus the action after the change which is a minimum; in impact it is the mass by the velocity lost and by the space which will be described in consequence of this velocity.

With respect to Maupertuis's reply to d'Arcy's third objection, Maupertuis, says d'Arcy,[179] has misread the objection: there was not said to be a supposition about an *angular and constant* motion but about a *constant angular* motion. D'Arcy quotes objections nearly the same as his of 1749 from the above cited article on "Cosmo-

[173] *Ibid.*, p. 504.
[174] *Ibid.*, p. 506.
[175] *Ibid.*, pp. 507-508.
[176] Article "Cosmologie," p. 196 [not seen].
[177] D'Arcy, *loc. cit.*, pp. 508-509.
[178] *Loc. cit.*, pp. 509-510.
[179] *Ibid.*, pp. 510-511.

logie": "When Maupertuis applies his principle to the case of equilibrium in the lever, certain suppositions must be made, amongst others, that the velocity is proportional to the distance from the fulcrum,[180] and that the time is constant as in the case of impact...."

In the case of the reflection of light, d'Arcy[181] shows that nature is prodigal or avaricious of action as a mirror is more or less concave respectively, and again quoted the article "Cosmologie" on this point.

Finally, d'Arcy[182] returned to his principle of 1747, which he prepared to substitute for Maupertuis's principle.[183]

XVIII.

In the Berlin *Histoire* for 1753, published in 1755, the only paper[184] relating to the principle of least action is an "Examen des Reflexions de M. le Chevalier d'Arcy sur le Principe de la moindre action" by Louis Bertrand.[185] Bertrand's paper was headed by a note to the effect that, as the Paris Academy of Sciences had, contrary to its custom, hurried to publish in its *Mémoires* of 1749 some reflexions of d'Arcy which he had only given in 1752, the Berlin Academy believed that it might publish this examination one year before it ought to have appeared.

D'Arcy, says Bertrand,[186] undertook to overthrow Maupertuis's principle, but only succeeded in overthrowing the false ideas which he had formed about it. In the first place, d'Arcy objected to Mau-

[180] As d'Arcy expressed it, that the angular velocity is constant.

[181] *Ibid.*, pp. 511-513.

[182] *Ibid.*, pp. 513-519. On p. 513 he emphasized that the memoir containing this principle was read to the French Academy in 1746.

[183] On d'Arcy's memoirs see Mayer, *op. cit.*, pp. 13-15, 21. It seems to me that Mayer's view of these memoirs is too favorable. I will return to this point in my criticisms.

[184] The contrary was stated, owing to a wrong reading of A. Mayer, *op. cit.*, p. 17, by myself in *Ostwalds Klassiker*, No. 167, p. 36; but, of Euler's five papers in this volume, one is on Daniel Bernoulli's papers on vibrating cords (cf. M. Cantor, *op. cit.*, vol. iii, 2d ed., Leipsic, 1901, pp. 904-907), two papers are on spherical and spheroidical trigonometry deduced from the method of maxima and minima (cf. *ibid.*, pp. 867-869), one on the law of refraction of rays of different colors, and one on the paths of projectiles in resisting media;—and in none of these is any reference to the principle of least action except in a passage (p. 306) in the last line but one of these papers, where he refers to the convincing proof of the existence of a Deity given by Maupertuis, and also to the argument from the wonderful structure of the eye.

[185] Pp. 310-320. Louis Bertrand (1731-1812) was then in Berlin and was a friend of Euler's; cf. *Poggendorff*, vol. i, p. 171; M. Cantor, *op. cit.*, vol. iv, Leipsic, 1908, p. 332 (article by V. Bobynin).

[186] *Op. cit.*, p. 311.

pertuis's definition of action. This is a question of words;[187] d'Arcy required that the *action* of different hard bodies should be estimated equal if each of these bodies is capable of reducing to rest the same hard body endowed with a certain velocity, so that the *action* of a body is measured by $m.v$. But, says Bertrand, it is well known[188] that, in the impact of hard bodies, a part of the motion is destroyed —that part which would be reproduced if the bodies were elastic: "hence it follows that, if a hard body (A) of mass 1 and velocity 1 were reduced to rest both by a body (B) of mass 1 and velocity 1 and by a body (C) of mass ½ and velocity 2, we could only affirm positively that the action of B is equal to that of C if we have previously proved that when B impinges on A it loses the same quantity of motion as when C impinges on A. For if it were true that in one case more motion were lost than in the other, the rest in this case ought not to be attributed to the equality of action of the two bodies, but to the greater loss of motion; in fact, if this loss had not been greater, some motion would have been left for the bodies which have impinged, and thus rest would not have followed the impact.

"In order, then, that the reasoning by which M. d'Arcy has wished to prop up his definition of *action* should be conclusive, it would be necessary for him to prove that the same quantity of motion is lost whether B impinges on A or C impinges on A. Now this he will never prove.

"Not being able to do anything in that direction, perhaps he will claim that it is sufficient to attend to the change which happens to the body A after the impact; but, if he only pays regard to the effect produced on the body impinged upon, we can urge against him the impact of elastic bodies, where a body A of mass and velocity both 1 is reduced to rest both by a body B of mass 1 and velocity 0, by a body C whose velocity and mass are both ½, and by a body D whose mass is ⅓ and velocity 1. Now, M. d'Arcy would contradict his own definition of action if he claimed that the actions of B, C, and D were all equal to one another. Thus the foundation on which M. d'Arcy wished to support his manner of estimating action absolutely lacks solidity." In d'Arcy's last paragraph on the definition of action, he wrongly concludes, says Bertrand,[189], that from Maupertuis's definition of *action*, follows that whenever the

[187] *Ibid.*, pp. 311-312, 313.

[188] ". . . . C'est une chose dont tous les Philosophes conviennent. . . ." (*ibid.*, p. 312).

[189] *Ibid.*, pp. 313-314.

velocities and the masses of two hard bodies are such that rest follows the impact of these bodies, the actions of these bodies are equal.

With regard to d'Arcy's attack on Maupertuis's principle, Bertrand[190] remarks that Maupertuis expressly said that not the difference of the actions before and after the impact, but the quantity of action necessary to produce this change is to be a minimum. The quantity of action necessary to produce a change is not the difference of the actions before and after the change; but it is the product of the mass of the bodies whose state is changed, the space that these bodies describe in consequence of (*en suite du*) the change, and the velocity with which they describe it, also in consequence of the change.[191]

With regard to d'Arcy's strictures on Maupertuis's treatment of the lever, Bertrand[192] reproduces d'Arcy's supposition that A moves with a small velocity V and describes a space α, whence the velocity of B is $V(c-z)/z$, the space described by B is $\alpha(c-z)/z$, and the action of the whole system is

$$AV\alpha + BV\alpha(c-z)^2/z^2.$$

Then, before differentiating, d'Arcy supposed V and α constant; and Bertrand inquires why should the velocity of and space described by A be supposed to be constant rather than those of and by B. Maupertuis puts as constant the angle that A and B describe around the fulcrum of the lever; and this supposition does not affect one of the bodies rather than the other, for this angle is the same for both bodies. Still, this supposition appears gratuitous to d'Arcy because for each value of z the action or the time necessary to make A and B describe the angle supposed constant is different. But, says Bertrand, if the action necessary to make A and B describe the angle supposed constant were not different for each value of z, it would be absurd to seek which of these actions is the least.

With regard to d'Arcy's assertion that, whatever the laws of nature might be, it would always be easy to find a function of the velocities and masses such that, when minimized, it would give these laws, Bertrand[193] remarks that "that may be true of many particular cases." Rather earlier in his paper, Bertrand[194] remarks *à propos*

[190] *Ibid.*, p. 314.

[191] *Ibid.*, pp. 314-315.

[192] *Ibid.*, pp. 317-318.

[193] *Ibid.* p. 318.

[194] *Ibid.*, pp. 315-316.

of d'Arcy's suggestion that Maupertuis knew the formula $A(a-x) = B(x-b)$ for impact and concluded that the action must be

$$A(a-x)^2+B(x-b)^2$$

in order that the known formula should result when the action was minimized, and d'Arcy's attempted generalization, that, if Z and X are functions of x, then, if $AZ = BX$,

$$AZ^2+BX^2$$

will always be a minimum and *vice versa,* that this generalization will always be false except when $dZ+dX=0$,—the case which he wished to generalize.

The rest of Bertrand's[195] paper is devoted to d'Arcy's own principle. "This principle," says Bertrand,[196] "can in a certain sense be admitted, but it will never lead to important discoveries; still less will it show us, so to speak, the true ends in view of nature: and these circumstances put it infinitely below that of M. de Maupertuis."

With regard to the way in which Bertrand's paper is written, it seems both magisterial and hasty: attempts at sarcasm against d'Arcy and flattery—or perhaps sincere reverence—for Maupertuis stand out too prominently. Bertrand was young when he wrote it, so there is a greater chance that he was sincere. Still, he was of, or was about to be of, the Berlin Academy.

XIX.

We will now give a brief retrospect of the development of views on the principle of least action, and dispose of all historical questions before trying to elicit what gains have resulted for knowledge by this development.

A. Mayer[197] says of Euler's formulations of the principle of least action: "We shall see that this correct form [in the second appendix to the *Methodus* of 1744] got lost to him in the course of time, and that soon it lost as much in rigor as it appeared to gain in generality." Mayer's[198] grounds for this view were that Jacobi's[199] principle of least action was the "true" principle, owing to

[195] *Ibid.,* pp. 318-320. Just at the end is: "On pourroit faire encore nombre de réfléxions sur l'insuffisance de ce Principe appliqué à la réfraction des rayons de lumiére; mais il semble qu'il y auroit une sorte de mauvaise humeur à examiner si rigoureusement se que M. d'Arcy paroit avoir voulu traiter cavaliérement." I have left the accents unaltered.

[196] *Ibid.,* p. 319.

[197] *Op. cit.,* p. 6.

[198] *Op. cit.,* pp. 6-11.

[199] Cf. *Monist,* vol. xxii, April, 1912.

the difficulty there appeared to be[200] if the time was not eliminated, and this elimination had apparently to be done by the equation expressing the conservation of *vis viva.* Thus the principle of least action is subject to the limitations implied by the subsistence of the theorem of *vis viva.* Thus Euler, in the above mentioned appendix, expressly pointed out that his theorem cannot hold for motion in a resisting medium, and that, in the integrand, the velocity must be expressed "ex viribus sollicitantibus per quantitates ad curvam pertinentes.[201] Consequently Mayer[202] maintained that Lagrange's (1760) generalization of the principle of least action is, in the form in which Lagrange states it, meaningless, and the theorem which he really had in his mind is that known as "Hamilton's principle" given by Hamilton in 1835. We know[203] that later on (in a publication of 1886) Mayer changed this view, owing to acquaintance with a paper of Rodrigues's (1816) in which the time (the t in the integrand) was varied by the δ-process of the calculus of variations, and admitted that there are two forms of the principle of least action: Jacobi's and Lagrange's. This view has been confirmed by the later researches of Hölder.[204]

Now Jacobi's principle may be considered to be a generalized form of Euler's theorem, and Lagrange's principle a more precise and generalized form of Maupertuis's. So it happens that Maupertuis was right in thinking his theorem quite general, and Euler

[200] *Ibid.*

[201] *Methodus,* p. 312. Cf. pp. 318-319 on the necessity for the principle of *vis viva.*

[202] *Op. cit.,* pp. 26-29. Mayer (*ibid.,* p. 24) also remarked that Euler's later (Maupertuisian) form of the principle, in which the condition that all the quantities in the integrand must be reduced, by means of the principle of *vis viva,* to space-elements alone is not stated, is quite meaningless, for the forces acting on the system, on which the path of the system depends, do not occur in the integral of action. Here we will anticipate our criticism by pointing out that in Lagrange's memoir the conditoin

$$\delta T = \delta U,$$

where "T" and "δU" have the meaning already explained in *The Monist,* vol. xxii, April, 1912, p. 290, is *explicitly* given, and what would now be written in the same way was, tacitly or not, presupposed in all Euler's works. Mayer said that the problem of variations only subsisted under the condition

$$T = U + \text{const.},$$

which implies the preceding equation, but, as Lagrange pointed out, is not necessarily implied by it. And it is the preceding equation alone that we require to rescue the principle of least action from meaninglessness. Mayer's remark (*ibid.,* p. 27) that Lagrange completely leaves out the condition is simply an error.

[203] Cf. *Monist,* vol. xxii, April, 1912.

[204] *Ibid.*

was right in doing what Mayer[205] complains of—in dropping the condition about the theorem of *vis viva* holding.[206] Of course, it may have been, and probably was, the case that neither Maupertuis nor Euler had any good grounds for believing that they were right. Indeed, one is forced, against one's will, to the opinion that Euler was in a position in which, as Mayer[207] expresses it, "he could not with propriety retort to the powerful President of his Academy."

The only reason why it is necessary to inquire closely whether Euler really considered Maupertuis's principle to be valid seems to me mainly to be the provision of an example to show the necessity of an additional condition when we wish to deduce properties of motion from the equation of the variation of the integral of action to zero. There is also the possibility of our being given yet another example of the greater power of instinctive beliefs or "metaphysics" over the good man's mind than the love of scientific truth.[208] If we should have to conclude that Euler deliberately hid the truth for the personal favor of Maupertuis, this conclusion will fill us with the same regret and loathing that we feel for the weakness in Galileo's character and the disgraceful exercise of the church's power on him, respectively.

It seems to me true that Euler's love for "metaphysics" alone could not lead him to forsake scrupulous honesty in his search for the truth. It is difficult, but very possible, to acquit Euler of the charge of veiled sarcasm against Maupertuis's principle. In a paper, from which we have quoted above, in the Berlin *Mémoires* for 1748, he expresses his belief that we are still very far from being able to assign, for each effect which nature produces, the quantity of action which is the smallest, and from being able to deduce it from the first principles of our knowledge. Indeed, Euler seems to have no doubt that *something* must be a minimum, but he also thinks that this something may be different— or at least seem to us, without imperfect knowledge, different—in different cases.[209] At any rate Euler goes carefully through single statical cases and determines the equivalent in terms of the forces of "the quantity of

[205] *Op. cit.*, pp. 23-24. Euler did not, however, *explicitly* drop this condition.

[206] Euler had presupposed in 1744 that the principle of *vis viva* held: Maupertuis considered his principle applies to cases—such as the impact of inelastic bodies—where the principle of *vis viva* does not hold.

[207] *Ibid.*, p. 17.

[208] On Euler's "metaphysical" tendencies, cf. Mayer, *ibid.*, pp. 21-23.

[209] Cf. the remark of d'Arcy that, whatever the laws of nature might be one could always find a function of the masses and velocities which, when minimized, would represent them (cf. section XV).

action" in each case. Nowadays, we would say,[210] of course, that this inductive procedure was far more "reasonable" or scientific than Maupertuis's; but we must remember that then the opinion was far more generally held than it is now that knowledge of the truth could be attained by other than scientific methods.

It was, I think we must say, not merely love for "metaphysics" which led Euler to sacrifice important details of his principle. Comparison of Daniel Bernoulli's letter to Euler of September 4, 1743, with Euler's markedly different account in the Berlin *Mémoires* of 1750 of the circumstances about the publication of the *Methodus* of 1744, as well as Euler's obviously unjust attitude towards König, points to a lower influence. If we dismissed—as we would like—thoughts that this sort of influence came in, we would be faced with the insoluble problem that Euler supported a principle which was claimed to embrace cases where the theorem of *vis viva* fails while he had convinced himself that the subsistence of this theorem was a necessary condition for the validity of the principle. And here the suggestion arises of itself that, since Euler, in his papers in the Berlin *Mémoires,* only committed himself to the mathematical support—as distinguished from warmly expressed admiration—of Maupertuis's principle in *statical* cases, he dared not affirm that the action-integral was a minimum in nature even when the principle of *vis viva* did not hold.[211] This stop was reserved for Lagrange, and perhaps it was on this account that Euler in a letter of November 9, 1762, congratulated Lagrange in the words:[212] "What satisfaction would M. de Maupertuis not have, if he were still alive, to see his Principle of least action carried to the highest degree of dignity of which it is susceptible.[213] If this conjecture be true, we must believe that Euler had a childlike faith that "metaphysics" could generalize a theorem so far as to drop a condition which he had satisfied himself, was necessary. We know now that this faith —if indeed it existed—was justified.

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[210] Like Mayer, *op. cit.,* p. 23.

[211] Indeed, where he refers to dynamical cases (in the Berlin *Histoire* of 1751) he explicitly uses the principle of *vis viva.* Euler nowhere refers to the problem of the impact of inelastic bodies, on which Maupertuis and others laid such stress.

[212] *Œuvres de Lagrange,* vol. xiv, p. 201.

[213] "Quelle satisfaction n'aurait pas M. de Maupertuis, s'il était encore en vie, de voir son principe de la moindre action porté au plus haut degré de dignité dont il est susceptible."

THE CAPTURE HYPOTHESIS OF T. J. J. SEE.[1]

In the opinion of Mr. See,[2] the planets were not formed from fragments of the solar nebula, nor did the moon originate from a piece of that of the earth. He believes that the planets had a cosmic origin outside of the solar nebula; that they are foreign bodies captured by the sun while passing near it in their journey; and that in the same way the moon was captured by the earth at a certain remote time.

How was this phenomenon accomplished? Mr. See thinks that the sun was formerly surrounded by a vast atmosphere and that the capture took place as the result of a resistance created by this atmosphere.

Let us therefore study the effect of the resistance of the medium on the motion of a planet.[3] If there were no resistance the motion would be Keplerian, the orbit would be an ellipse of any eccentricity whatever. The density of the resisting medium being by hypothesis very small, this orbit would vary slowly. We shall study the variations of this orbit by the method of the variation of constants.

First let us recall some formulas pertaining to the elliptical motion of planets.

Calling the radius vector r and the true anomaly v, the equation of the orbit is

$$r=\frac{p}{1+e\cos v}, \tag{1}$$

e denoting the eccentricity, and

$$p=a(1-e^2) \tag{2}$$

denoting the parameter of the elliptical orbit whose major axis is $2a$. We have also the equation of the areas

$$r^2\frac{dv}{dt}=\mathrm{C},$$

the constant C of the areas having the value

$$\mathrm{C}=\sqrt{\mathrm{M}p},$$

in which M represents the mass of the sun. (We disregard the

[1] Translated by Lydia G. Robinson from the author's *Leçons sur les hypothèses cosmogoniques,* Chaps. VI and XIII. Paris, Hermann, 1911.

[2] T. J. J. See, *Researches on the Evolution of the Stellar Systems,* Vol. II, "The Capture Theory of Cosmical Evolution." Lynn, Mass., Nichols & Sons; Paris, Hermann, 1910.

[3] See *loc. cit.,* Chap. VII, pp. 134-158.

mass of the planet compared to the sun's mass.) The mean motion n is connected with half the major axis a by Kepler's third law.

(3) $$n^2a^3=M.$$

Finally the equation of the *vis viva* gives

$$T-\frac{M}{r}=-\frac{M}{2a},$$

in which T is half the *vis viva.*

Differentiating equation (1) with reference to time, we have

$$\frac{dr}{dt}=\frac{pe\sin v}{(1+e\cos v)^2}\frac{dv}{dt}$$

$$=\frac{pe\sin v}{(1+e\cos v)^2}\frac{C}{r^2}$$

$$=\frac{pe\sin v}{(1+e\cos v)^2}\frac{C}{p^2}(1+e\cos v)^2$$

$$=\frac{C}{p}e\sin v.$$

Now dr/dt is the component of velocity in the direction of the radius vector. The component perpendicular to this radius vector has for its value

$$r\frac{dv}{dt}=\frac{C}{r}$$

$$=\frac{C}{p}(1+e\cos v).$$

From the two components of the velocity V, we derive the square of this velocity,

$$V^2=\frac{C^2}{p^2}(1+2\,e\cos v+e^2).$$

In short, if we put

$$\rho^2=1+2\,e\cos v+e^2,$$

we shall have

$$V=\frac{C}{p}\rho$$

$$=\rho\sqrt{\frac{M}{p}}.$$

The above formulas belong to Keplerian motion.

Now let us suppose that there is an atmospheric medium with a resistance R directly opposed to the velocity and function of the value V of that velocity. The constant of the *vires vivae* $-M/2a$ during the time dt will undergo a variation

$$\frac{M}{2a^2}da;$$

this variation will equal the work of the resistance R which is

$$-RVdt.$$

Hence we have

$$\frac{M}{2a^2}\frac{da}{dt}=-RV$$

$$=-R\rho\sqrt{\frac{M}{p}},$$

whence we derive

$$\frac{da}{dt}=-\frac{2R\rho a^2}{\sqrt{Mp}};$$

replacing M and p by their values (2) and (3) in this last equation, we obtain

$$(4)\qquad \frac{da}{dt}=-\frac{2R\rho}{n\sqrt{1-e^2}}.$$

This is the equation which gives the variation of the major axis; the second member is necessarily negative. Hence the effect of the resistance of the medium is always to diminish a and consequently according to equation (3) to increase n. The angular velocity of the planet increases[4] at the same time that its mean distance from the sun diminishes.

W shall now study the effect of resistance of the medium on the eccentricity of the orbit.

First of all the derivative dC/dt of the areal constant C would be equal to the momentum of the disturbing force R, with reference to the center of attraction. Now this force R opposed to the velocity has for its components:
in the direction of the vector ray

$$-R\frac{\frac{dr}{dt}}{V},$$

perpendicular to the vector ray

[4] Formula (3) even shows that na increases as a diminishes, whence we have the curious result that resistance of the medium causes an increase in the linear velocity of the planet.

$$-R\frac{r\frac{dv}{dt}}{V};$$

and the momentum of the force R with reference to the sun is

$$-R\frac{r^2\frac{dv}{dt}}{V}=-R\frac{C}{V}.$$

Hence we have

$$\frac{dC}{dt}=-\frac{RC}{V}. \tag{5}$$

Remember that

$$C=\sqrt{Mp}$$
$$=M^{\frac{1}{2}}a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}.$$

Taking the logarithmic derivatives of the two extreme members, we have

$$\frac{dC}{C}=\frac{1}{2}\left(\frac{da}{a}-\frac{2\,ede}{1-e^2}\right).$$

This equation makes it possible for us to obtain de since da and dC have been computed. We find

$$\frac{2e}{1-e^2}\frac{de}{dt}=\frac{1}{a}\frac{da}{dt}-\frac{2}{C}\frac{dC}{dt},$$

an equation which may be written by replacing da/dt and dC/dt by their values (4) and (5),

$$\frac{2e}{1-e^2}\frac{de}{dt}=-\frac{2R\rho}{na\sqrt{1-e^2}}+\frac{2R}{V}. \tag{6}$$

Let us now transform the second member of this equation. We have previously found (page 461)

$$V=\rho\sqrt{\frac{M}{p}}$$
$$=\rho\frac{na}{\sqrt{1-e^2}};$$

hence the second member may assume the form

$$-\frac{2R}{na\sqrt{1-e^2}}\left[\rho-\frac{1-e^2}{\rho}\right],$$

or again, by restoring the value of ρ^2, this other form

$$-\frac{2R}{na\sqrt{1-e^2}}\,\frac{2e\cos v+2e^2}{\rho}.$$

Finally equation (6) then gives

(7) $$\frac{de}{dt}=-\frac{2R\sqrt{1-e^2}}{nap}(e+\cos v).$$

This is the equation which gives the variation of the eccentricity of the orbit.

Formulas (4) and (7) make it possible to compute at any instant the variations of the major axis and of the eccentricity. But here it is only desirable to obtain their *secular* variations, and in order to do this, to compute the value of *da* and *de* during the time of a complete revolution.

Taking as an independent variable the true anomaly v we shall have

(8) $$\begin{cases}\frac{da}{dv}=\frac{da}{dt}\frac{dt}{dv},\\ \frac{de}{dv}=\frac{de}{dt}\frac{dt}{dv}.\end{cases}$$

Now the equation of the areas

(9) $$\frac{dt}{dv}=\frac{r^2}{C}$$

$$=\frac{p^2}{C}(1+e\cos v)^{-2}.$$

Formulas (4), (7) and (9) therefore make it possible to write the values (8) of da/dv and de/dv which, integrated between 0 and 2π will give the variations of half the major axis and the eccentricity during one revolution.

We may here offer certain hypotheses on medial resistance R. This resistance increases as the velocity; we shall suppose it proportional to a certain power of the velocity V. It varies directly as the distance r from the sun, for the density, and consequently the resistance, of the sun's atmosphere increases inversely as the distance; let us suppose R proportional to a certain power (negative) of r. In short let us put

10) $$R=hV^{\alpha}r^{-\beta},$$

h, α and β being positive constants. Since V is proportionate to ρ, and r to $1/(1+e\cos v)$, we can write formula (10) as follows:

$$R=k\rho^{\alpha}(1+e\cos v)^{\beta},$$

k being a new positive constant.

In view of these hypotheses on R, the values (8) of da/dv and

de/dv, computed by means of the formulas (4), (7) and (9), may be written

$$(11)\qquad \begin{cases} \dfrac{da}{dv} = -a\text{H}(1-e^2)^{-\frac{1}{2}}\rho^{\alpha+1}(1+e\cos v)^{\beta-2}, \\ \dfrac{de}{dv} = -\text{H}(1-e^2)^{\frac{1}{2}}\rho^{\alpha-1}(1+e\cos v)^{\beta-2}(e+\cos v); \end{cases}$$

where H denotes the positive constant

$$\text{H} = \frac{2p^2k}{na\text{C}};$$

bear in mind that in these values (11)

$$\rho = (1+2e\cos v+e^2)^{\frac{1}{2}}.$$

In order to study the secular variations of a and e we must develop the second members of the values (11) in trigonometric series according to the cosines of the multiples of v, and integrate between $v=0$ and $v=2\pi$. By integration all the cosines will be 0; therefore we are interested in the constant terms of these trigonometric developments and especially the sign of these constant terms.

We already know that da/dv is necessarily negative, since da/dt is always negative. Therfore we shall work only with de/dv. We must develop in a trigonometric series the expression

$$\rho^{\alpha-1}(1+e\cos v)^{\beta-2}(e+\cos v).$$

Now if we first develop the product of the two first terms we obtain:

$$(12)\qquad \rho^{\alpha-1}(1+e\cos v)^{\beta-2} = \text{A}_0+\text{A}_1\cos v+\text{A}_2\cos 2v+\ldots.$$

We observe that A_0 is necessarily positive because it is the mean value of the first member both of whose terms are always positive. Then multiplying the two members of formula (12) by $(e+\cos v)$ we have

$$\rho^{\alpha-1}(1+e\cos v)^{\beta-2}(e+\cos v) = \left(\text{A}_0e+\frac{\text{A}_1}{2}\right)+\ldots,$$

all the unwritten terms of the second member having their mean value 0.

The second formula (11) therefore gives for the mean value of de/dv during one revolution

$$(13)\qquad \frac{de}{dv} = -\text{H}(1-e^2)^{\frac{1}{2}}\left(\text{A}_0e+\frac{\text{A}_1}{2}\right).$$

Since the second member of equation (13) is generally negative we conclude from it that the medial resistance has the effect of diminishing the eccentricity of the orbit. This would be the case particularly whenever A_1 is positive. Now according to formula (12) we have

$$A_1=\frac{2}{\pi}\int_0^{\pi}(1+2e\cos v+e^2)^{\frac{\alpha-1}{2}}(1+e\cos v)^{\beta-2}\cos v\,dv.$$

If at the same time

$$\alpha \geqslant 1, \qquad \beta \geqslant 2,$$

A_1 will be positive, for of two elements of the integral corresponding to the two values v and $\pi-v$ of the variable of integration, one is positive and the other negative, but the positive element possesses a greater absolute value than the negative.

In an analogous way we know that if the two inequalities

$$\alpha \geqslant 1, \qquad \alpha+2\beta \geqslant 5,$$

are satisfied, we shall likewise have

$$A_1 \geqslant 0.$$

If we suppose the eccentricity e to be so small that we can disregard its square e^2 we shall find more general conditions. The second formula (11) is reduced to

$$\frac{de}{dv}=-\mathrm{H}[1+(\alpha-1)e\cos v+(\beta-2)e\cos v](e+\cos v);$$

whence by retaining only the mean value of the second member we derive

$$\frac{de}{dv}=-\mathrm{H}\left(e+\frac{\alpha+\beta-3}{2}e\right)$$

$$=-\frac{\mathrm{H}e}{2}(\alpha+\beta-1).$$

Then in order to diminish the eccentricity it is sufficient that

$$\alpha+\beta>1.$$

In this case even if $\beta=0$ (that is, if the resistance R does not vary with the distance r from the sun) we need only have

$$\alpha>1,$$

that is to say, R increasing more rapidly than the simple power of the velocity. Now we often grant as an approximation that a medial resistance is proportionate to the square of the velocity.

This diminution of the eccentricity because of a medial resistance might have been foreseen in general and without calculation in the following manner. Suppose the resistance is not felt except in the vicinity of the perihelion P (Fig. 1). In that case the planet undergoes at this point P a sudden diminution of velocity which results in a decrease in the major axis. Since the perihelion remains the same and the aphelion approaches it, it is clear that the eccentricity is lessened. On the other hand, if resistance acts only at the moment of the aphelion, the new orbit would have the same aphelion as the former one, but its perihelion would be nearer that of the sun, and the eccentricity would be increased. In fact the resistance is felt all along the orbit, but two reasons combine to make it felt more strongly at the perihelion: in the first place the velocity is greatest at that point, since the atmosphere which is generally denser nearer the sun offers a greater resistance near the perihelion.

To sum up, the effect of medial resistance on a Keplerian orbit is to diminish both the major axis and the eccentricity.[5] Therefore if we agree with Mr. See that a resisting atmosphere originally extended for vast distances around the sun, we can conceive that a

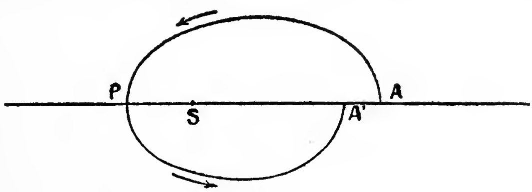

Fig. 1.

body of cosmical origin when passing into the sun's sphere of influence might be able to modify its trajectory. Whether it was parabolic or hyperbolic it now becomes elliptical, because the medial resistance continues to diminish the major axis and the eccentricity of the orbit which approaches the circular form. The resisting atmosphere is gradually absorbed by the sun, and when it finally disappears the smaller body continues to revolve around the sun in its orbit which is almost a circle. Such, according to Mr. See, is the history of all the planets.

Just as the planets have been captured by the sun so also, according to Mr. See, have the satellites been captured by their respective planets.[6]

In order to study this capture we shall take up the comparatively simple case called the restricted problem. The sun S and a planet J

[5] It is easy to recognize that this resistance does not produce any secular effect (at least at the first approximation) on the longitude of the perihelion. To be sure it does not modify the plane of the orbit which retains the same inclination and the same line of nodes with reference to a fixed plane.

[6] *Loc. cit.,* Chap. VIII, pp. 159-182; X, pp 211-236.

(e. g., Jupiter) each revolve around their common center of gravity G in a circular orbit with a constant angular velocity ω (Fig. 2). It is required to study the motion of a small planet P whose mass is negligible with reference to that of the principal planet J and which consequently will not affect the motion of the latter. We will take as origin the center of gravity G, of the system S – J; as plane of the coordinates xy, the plane in which S and J describe their circular orbits; and in this plane rectangular movable axes, the axis of x being the straight line SGJ which connects the sun with Jupiter; the axis of z is the perpendicular to the plane of the orbit at G. The forces acting actually upon the point P (x, y, z) are the attraction of the sun and of Jupiter. These two forces are derived respectively from the two functions of forces[7]

$$U_1=\frac{M}{\rho_1},\quad U_2=\frac{M_2}{\rho_2},$$

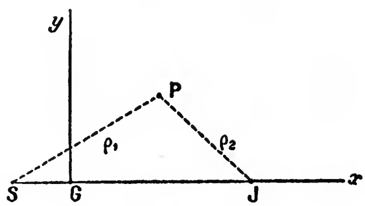

Fig. 2.

M_1, M_2 being the masses of the sun and Jupiter, ρ_1, ρ_2 their distances from P. Since the axes are movable we must add to these forces the centrifugal force and the compound centrifugal force. The components of the centrifugal force are

$$\omega^2 x, \qquad \omega^2 y, \qquad 0.$$

The components of the compound centrifugal force are

$$2\,\omega\frac{dy}{dt}, \qquad -2\,\omega\frac{dx}{dt}, \qquad 0.$$

Hence the equations of the motion of the planet P with relation to the movable axes are

$$\frac{d^2x}{dt^2}=\frac{dU_1}{dx}+\frac{dU_2}{dx}+\omega^2x+2\omega\frac{dy}{dt},$$

$$\frac{d^2y}{dt^2}=\frac{dU_1}{dy}+\frac{dU_2}{dy}+\omega^2y-2\omega\frac{dx}{dt},$$

[7] We assume the mass m of the small planet P to be equal to unity. More exactly, since this mass m is a factor in every case we shall not write it in the formulas.

$$\frac{d^2z}{dt^2}=\frac{dU_1}{dz}+\frac{dU_2}{dz}.$$

If we multiply these three equations

$$dx=\frac{dx}{dt}dt, \qquad dy=\frac{dy}{dt}dt, \qquad dz=\frac{dz}{dt}dt,$$

respectively, and add the results, we obtain a combination immediately integrable which brings us to the following integral

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]=\frac{M_1}{\rho_1}+\frac{M_2}{\rho_2}+\frac{\omega^2}{2}(x^2+y^2)-C,$$

known by the name of the integral of Jacobi.

Since the first member of this last equation is positive, the coordinates x, y, z of the point P will satisfy the inequality

$$\frac{M_1}{\rho_1}+\frac{M_2}{\rho_2}+\frac{\omega^2}{2}(x^2+y^2)-C>0.$$

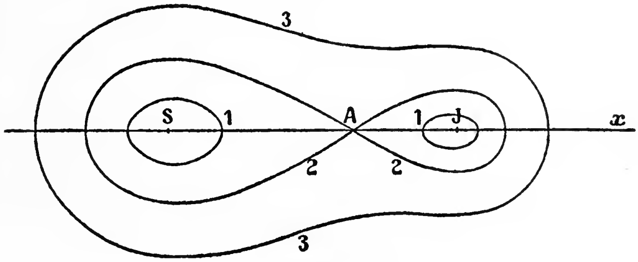

Fig. 3.

Hence the projection (x, y) of the point P on the plane of xy will be within the curve

$$\frac{M_1}{\rho_1}+\frac{M_2}{\rho_2}+\frac{\omega^2}{2}(x^2+y^2)=C;$$

in this equation ρ_1 and ρ_2 denoting the distances of this projection of the point P from the points S and J. For very great values of the constant C this curve comprises two rings (denoted by 1 on Fig. 3) surrounding the points S and J respectively. As C diminishes, these two rings become dilated and finally unite at a double point A (Curve 2). Then when C is further diminished they finally make only one curve (Curve 3) surrounding at the same time both S and J.[8] Hence when the constant C is not too great the small planet is obliged to remain within Curve 3 but still is free to travel in the proximity either of the sun or of Jupiter. On the contrary

[8] We pay no attention here to certain portions of curves which are very far removed from the origin.

if the constant C is very great the small planet will remain within one of the rings 1; it will be a satellite either of the sun or of Jupiter.

Now the effect of a passive resistance like that of a medium is to increase the constant C of the second member of Jacobi's integral. Hence the curve encircling the small planet constantly contracts. If it was originally Curve 3 at a definite moment it will become Curve 2 with the double point. If at this moment the planet is near the sun it will never return to the proximity of Jupiter; it is captured by the sun. If on the contrary it is in the neighborhood of Jupiter it will never return to that of the sun; it will be captured by Jupiter and from that moment will become one of his satellites.

The theory of Mr. See accounts for the smallness of the eccentricities of the orbits of planets and satellites.[9] But why are the movements of almost all the heavenly bodies in a straight line, and why have their orbits such small mutual inclinations? In the hypothesis of Mr. See these two questions remain without any satisfactory answer. To try to explain the smallness of the inclinations we may suppose that the resisting atmosphere of the sun is of a greatly flattened lenticular form; hence a body whose orbit is greatly inclined to the plane of this disk suffers a resistance much smaller than a body moving in the very plane of the disk. The first body has therefore much less tendency to be captured than the second, and is in the plane of the disk in which the captures of the planets are made.

We may also suppose that the resisting medium itself revolves. It will then tend not to counteract the velocity of the planet revolving within it but to impose upon this planet a certain velocity. Since the resistance is no longer directly opposed to the velocity, the plane of the orbit could vary and tend to diminish its inclination to the equatorial plane of the solar atmosphere.

FORMATION OF SPIRAL NEBULAS.

In the work previously referred to,[10] Mr. See is concerned with the formation of nebulas, especially with the origin of spiral nebulas.

Let us imagine two masses of cosmical vapor N and N′, almost equal in size and traveling in opposite directions (Fig. 4*a*). As they

[9] The diminution of the eccentricity because of a resisting medium is of first importance not only in the theory of Mr. See; it is taken into consideration also in the theories of Faye and of Du Ligondès.

[10] *Op. cit.,* Chap. XIX.

approach each other their adjacent extremities will be prolonged each in the direction of the other by mutual attraction (Fig. 4*b*) and may even end in uniting to form a single body (Fig. 4*c*) near whose center attraction combined with friction will tend to produce a condensation, a sort of central nucleus. The two masses of vapor N and N′ will turn in the directions of the arrows around this center like two arms of a windmill.

Such, according to Mr. See, would be the origin of the spiral nebulas. The central nucleus would tend to enlarge more and more

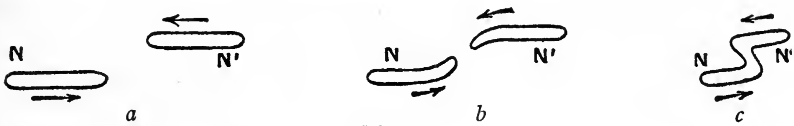

Fig. 4.

at the expense of the matter in the two spiral branches N and N′. Hence we see that in the opinion of Mr. See the motion of the matter in the two arms of the spiral nebula contrary to the usual view would be centripetal and not centrifugal. Moreover whether the motion is convergent or divergent the law of areas accounts equally in both cases for the slowness of the arm's revolution around its pivot, that is to say, the spiral form of both arms.

It may happen that the ends of the two masses of vapor N and N′ do not join as they approach each other, but are merely deviated by attraction. Then the phase following phase 2 of Fig. 4. is not

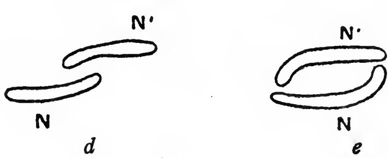

Fig. 5.

phase *c* but phase *d* (Fig. 5) after which it assumes phase *e*. In such a case we have the origin of an annular nebula like that of Lyra. In the two diametrically opposed light portions seen in the ring of Lyra, Mr. See finds an argument for the application of this theory in that adjacent ends of the two masses of vapor N and N′ would not be perfectly united.

Hence Mr. See thinks that an annular nebula is formed by the same mechanical process as spiral nebulas of which it thus proves to be in some sense a particular case. But the annular form is very rare because the conditions for the formation of a perfect ring are not often realized.

One great objection may be offered to this theory. The two arms of a spiral nebula are usually almost symmetrical. In the ordinary hypothesis in which the movement of the arms is assumed to be divergent this symmetry may be explained by the common origin of the two arms. In the hypothesis of Mr. See there is no way to account for it, for the two masses of cosmical vapor N and N′ which give rise to the nebula and which have met accidentally will not usually be equal. They ought then to give birth to an unsymmetrical nebula.

Mr. See thinks that originally the solar system was a spiral nebula of vast extent. The matter at its center first became agglomerated into particles which with the help of the resistance of the medium were condensed into asteroids, according to the process explained above, and then into planets, which are further increased by bombardment.[11]

Mr. See is led by analogy to believe that the spiral nebulas which are less advanced in their evolution than the solar system are composed of a vast number of very small bodies like the planets or even the moon. If we can not analyze these nebulas it will be because of the extremely small size of their component parts and not because these celestial objects are so excessively remote. Mr. Bohlin has tried to measure the parallax of the nebula of Andromeda (which is a spiral nebula of a continuous spectrum) and he has found it equal to 0″, 17, so that this nebula would be comparatively very near us. But considering how little accuracy the points on the nebulas admit of, can we regard this observation as conclusive and certain?

H. Poincaré.

Paris, France.

NOTES ON THE CONSTRUCTION OF MAGIC SQUARES OF ORDERS IN WHICH n IS OF THE FORM $8p+2$.

Referring to the article in the last issue of *The Monist* by Messrs. Andrews and Frierson, under the above heading, it was shown that the minimum series to be used in constructing this class of squares is selected from the series 1, 2, 3, ... $(n+3)^2$, by

[11] Mr. See sees in the lunar craters signs of a bombardment produced at the surface of the moon by the fall of a large number of little satellites. He compares these craters to the marks left by great drops of rain in the mud (*op. cit.*, p. 342, plate XII).

discarding 3 rows and columns from the natural square of the order $n+3$.

It is not necessary, however, to discard the three central rows and columns, as was therein explained, there being numerous variations, the total number of which is always equal to $\left(\frac{n+2}{4}\right)^2$

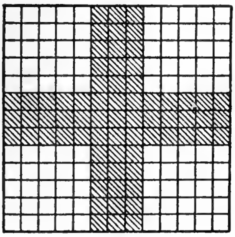

Fig. 1.

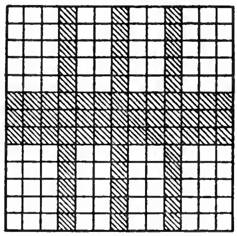

Fig. 2.

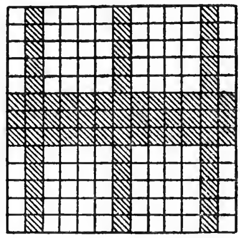

Fig. 3.

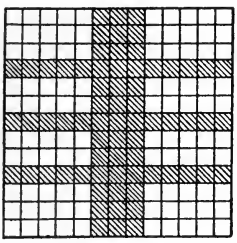

Fig. 4.

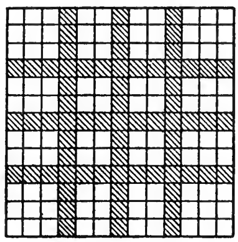

Fig. 5.

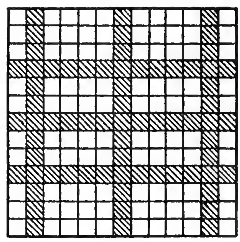

Fig. 6.

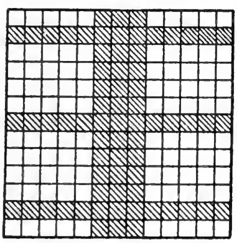

Fig. 7.

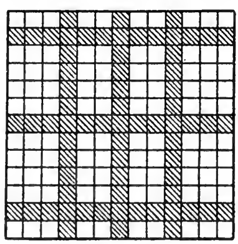

Fig. 8.

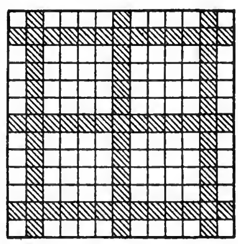

Fig. 9.

therefore the 10^2 can be constructed with 9 different series, the 18^2 with 25 different series, the 26^2 with 49 different series, and so on.

In Figs. 1 to 9 are shown all the possible variations of discarding rows and columns for the 10^2, Fig. I representing the series explained in the foregoing article.

The central row and column must always be discarded, the remaining two rows and columns can be cast out symmetrically in relation to their parallel central row or column and should be an odd number of rows or columns from it. In other words, we cast out the central row, then on each side of it we cast out the 1st, 3d,

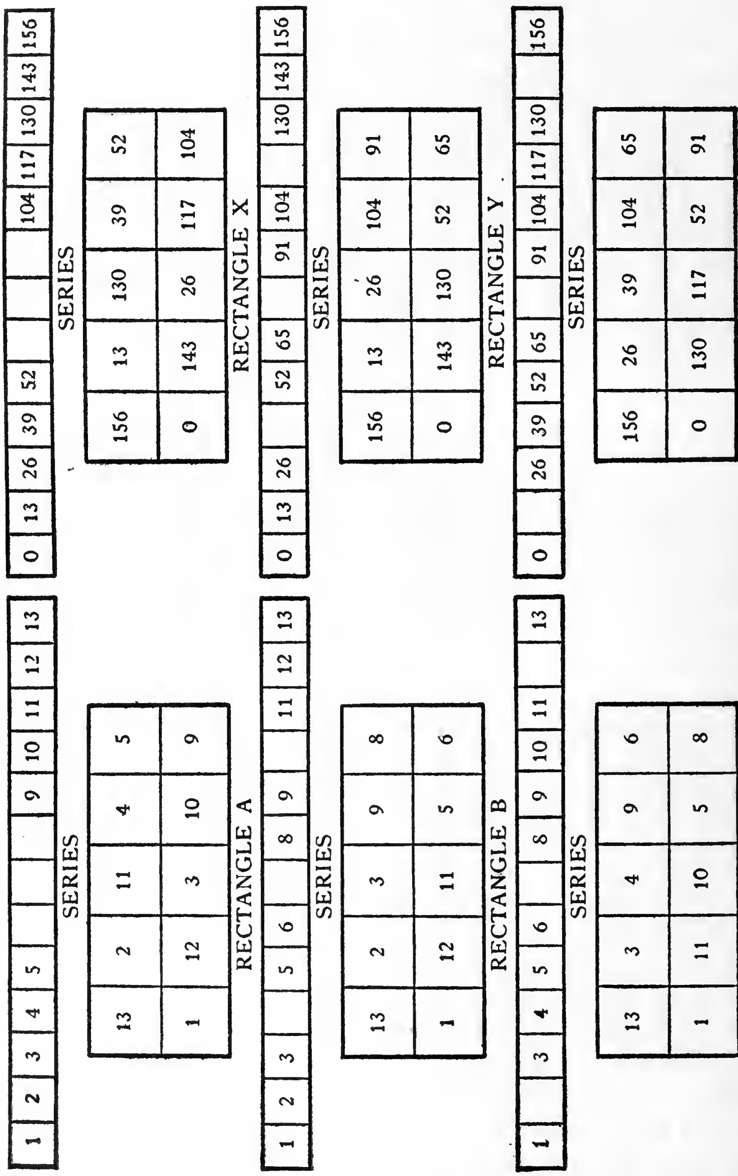

Fig. 10.

5th, or 7th, etc. rows from it, and irrespective of the rows, we do likewise with the columns.

In a manner already explained, numbers are selected according to the series desired and arranged in rectangles with which the magic square is constructed.

A set of rectangles with their respective series is shown in Fig. 10, and the following table will give directions for their use.

Series	Rectangles (See Fig. 10)
Fig. 1	A and X
Fig. 2	B and X
Fig. 3	C and X
Fig. 4	A and Y
Fig. 5	B and Y
Fig. 6	C and Y
Fig. 7	A and Z
Fig. 8	B and Z
Fig. 9	C and Z

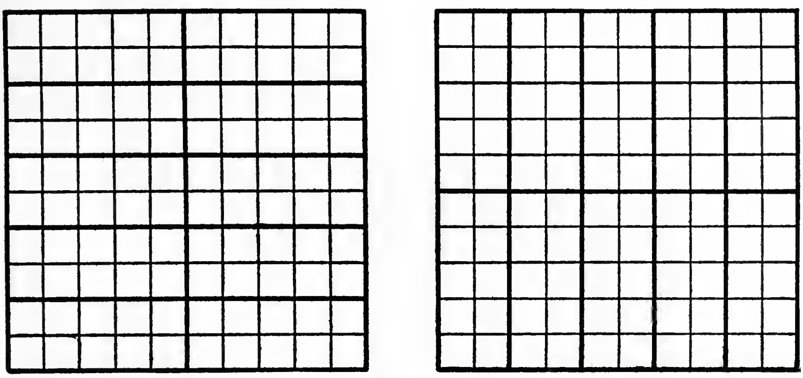

Fig. 11.

For example, suppose we were to construct a square, using the series denoted in Fig. 3. By referring to the table it is seen that we must employ rectangles C and X. By using the La Hireian method these rectangles are placed as shown in Fig. 11, care being taken to arrange them in respect to the final square, whether it is to be associated or non-associated.[1]

A non-associated square resulting from rectangles C and X is shown in Fig. 12. Another example by Mr. Andrews, using the path method is shown in Figs. 13, 14 and 15. Here a series corres-

[1] See preceding article.

ponding to Fig. 8 has been selected and the natural square is shown in Fig. 13, the heavy lines indicating the discarded rows and columns. The rows and columns are re-arranged according to the nu-

65	107	56	113	58	117	55	108	61	110
40	128	49	122	47	118	50	127	44	125
143	29	134	35	136	39	133	30	139	32
14	154	23	148	21	144	24	153	18	151
169	3	160	9	162	13	159	4	165	6
53	115	62	109	60	105	63	114	57	112
52	120	43	126	45	130	42	121	48	123
131	37	140	31	138	27	141	36	135	34
26	146	17	152	19	156	16	147	22	149
157	11	166	5	164	1	167	10	161	8

Fig. 12.

1	2	3	5	6	8	9	11	12	13
27	28	29	31	32	34	35	37	38	39
40	41	42	44	45	47	48	50	51	52
53	54	55	57	58	60	61	63	64	65
66	67	68	70	71	73	74	76	77	78
92	93	94	96	97	99	100	102	103	104
105	106	107	109	110	112	113	115	116	117
118	119	120	122	123	125	126	128	129	130
131	132	133	135	136	138	139	141	142	143
157	158	159	161	162	164	165	167	168	169

Fig. 13.

merical sequence of the continuous diagonals[1] of rectangles B and Z of Fig. 10, this re-arrangement being shown in Fig. 14.

1	2	11	9	6	13	12	3	5	8
27	28	37	35	32	39	38	29	31	34
118	119	128	126	123	130	129	120	122	125
105	106	115	113	110	117	116	107	109	112
92	93	102	100	97	104	103	94	96	99
157	158	167	165	162	169	168	159	161	164
131	132	141	139	136	143	142	133	135	138
40	41	50	48	45	52	51	42	44	47
53	54	63	61	58	65	64	55	57	60
66	67	76	74	71	78	77	68	70	73

Fig. 14.

5	162	1	168	11	161	6	157	12	167
100	73	104	67	94	74	99	78	93	68
57	110	53	116	63	109	58	105	64	115
126	47	130	41	120	48	125	52	119	42
135	32	131	38	141	31	136	27	142	37
9	164	13	158	3	165	8	169	2	159
96	71	92	77	102	70	97	66	103	76
61	112	65	106	55	113	60	117	54	107
122	45	118	51	128	44	123	40	129	50
139	34	143	28	133	35	138	39	132	29

Fig. 15.

[1] See article in *Monist* of April, 1912.

In constructing the final square, Fig. 15, an advance move −4, −5 and a break move 1, 1 was used.

It wiil be unnecessary to show examples of higher orders of these squares, as their methods of construction are only extensions of what has been already described. It may be mentioned that these squares when non-associated can be transformed into associated squares by the method given in Messrs. Andrews and Frierson's article.

HARRY A. SAYLES.

SCHENECTADY, N. Y.

POSTSCRIPT ON BUDDHISM AND CHRISTIANITY.

My article on the "Contributions of Buddhism to Christianity," which appeared in *The Monist* of October, 1911, called forth two criticisms in the following number (January 1912). One was by Albert J. Edmunds, "Buddhist Loans to Christianity," pp. 129 ff., and the other by Wilfred H. Schoff, "First Century Intercourse Between India and Rome," pp. 138 ff.

Even before these criticisms reached me, I began to doubt whether my standpoint that Buddhist influences were "not yet to be found in the canonical Gospels, but first in the Apocryphal Gospels," could be maintained in this categorical form.[1] The historical *possibility* for the infiltration of Buddhist material into the canonical Gospels I have never denied, but only its *probability*. I take pleasure in using this opportunity to grant that by the lucid critique of Edmunds the probability of the hypothesis of Buddhist loans in the New Testament has increased in my opinion.

The connection of the Asita-Simeon parallel with the praise of the heavenly hosts in both the Suttanipāta and in the Gospel of Luke has strongly impressed me even though I can not concede to Edmunds that this connection is an "organic" one on both sides. The connection is organic only in the Pali source and not in Luke, where in the second chapter the Simeon story does not stand in an intrinsic connection with the angelic hymn but only *near* it. But even this correspondence is certainly remarkable enough.

The exposition which Edmunds has given of the temptation parallels (Samyuttanikāya and Luke iv. 1-2) also decidedly increases the probability of the loan hypothesis. Because of this the Buddhist origin of some other New Testament stories, to which I have heretofore only with hesitancy granted a remote possibility that they

[1] See my article, "Buddhistisches im Neuen Testament," in *Das Freie Wort*, Frankfort, December 1911, pp. 674 ff.

might have been borrowed from India, also becomes of course more probable.

Edmunds is entirely mistaken in his explanation of the Wandering Jew (pp. 137-138).[2] Mark ix. 1: "Verily I say unto you, There be some here of them that stand by, who shall in no wise taste of death, till they see the kingdom of God come with power," does not in the least contain the germ of this legend but simply expresses like the other passages on the Parousia (Matt. x. 23; xvi. 28; Luke ix. 27) the conviction of Jesus that the end of the world was at hand.

The article of W. H. Schoff elucidates in a clear exposition well-known facts about the commercial intercourse between India and the Occident in the first century after Christ, but he brings no positive proof that an exchange of ideas necessarily went hand in hand with the extensive commercial intercourse. Especially, he, as the translator of the Periplus, ought to have inferred from this text that the mariners and traders of those days had but little thought for anything but their merchandise. The author of the Periplus, who describes his journey to India between 70 and 75 A. D., treated only of what would be interesting to the merchant and mariner, but otherwise shows that he was uninformed about the most commonplace things and says not one word about religion. Likewise the Indian merchants who had settled in Alexandria were according to the testimony of Dio Chrysostom (*Orat.* 35) ignorant people and probably of Dravidian race. They would have taken no more interest in religious questions than the Greek or Roman merchants of their time.

When Schoff (page 141) describes the merchants as "bearing ideas no less than goods," this is simply begging the question.

More important for our purpose than all reports of ancient commercial relations seems to me the observation of Max Müller expressed in the following words:[3] "Though we have no tangible evidence of anything like translations, whether Oriental or Occidental, at that time, we seem perfectly within our right when we look upon the numerous coincidences between the fables of Æsop and the fables occurring in Sanskrit and Pāli literature as proving the fact that there was a real literary exchange between India, Persia, Asia Minor and Greece beginning with the 6th century B. C."

TÜBINGEN, GERMANY. R. GARBE.

[2] Compare also *Buddhist and Christian Gospels,* 4th ed., II, pp. 264 ff.

[3] In the article "Coincidences" in *Last Essays,* I, 269-270.

POINCARE'S COSMOGONIC HYPOTHESES.

Prof. H. Poincaré has just published an important book[1] which treats the interesting problem of the origin of the world according to the scientific views of modern philosophers and naturalists. Professor Poincaré in the first chapter discusses Kant's hypothesis and subjects it to a critical analysis. The second chapter is devoted to La Place; the third analyzes La Place's hypothesis and discusses the work of La Roche, especially the theory of the stability of rings and the formation of satellites. Subdivisions of this third chapter treat the hypothesis of a uniform notation, the rings of Saturn, the rupture of rings according to La Place and the formation of planets and satellites, and the author sums up the objections to the theory of La Place.

The fourth chapter is devoted to the hypothesis of H. Faye, according to which the earth is much older than the sun. Chapter five discusses the hypothesis of du Ligondès who claims that Kant's hypothesis stand in contradiction to the principle of the gases. The sixth chapter treats the hypothesis of Prof. T. J. J. See, which will be of special interest to American readers because he is a native American and is the astronomer of the Naval Observatory, Mare Island, California. This chapter together with the thirteenth is reproduced in an English translation on another page of this issue. The seventh chapter discusses the theory of Sir George Howard Darwin, his theory of tides, especially the internal tides of the earth, the accelerative influence of cooling down, and his hypothesis of the formation of the moon. The eighth chapter treats the theory of solar and terrestrial heat, as well as the adiabatic equilibrium of a perfect gas.

Chapter nine treats of the theory of Sir Norman Lockyer, Chapter ten of Schuster and Chapter eleven of Arrhenius's theories; Chapter twelve compares the mass of the Milky Way with a gaseous mass. Its substance is comparable to the radiant matter of Krookes, rather than to a true gas. He then treats possible causes of the flattening of the Milky Way and concludes with a consideration of the star clusters of Kapteyn and Schiaparelli.

In the thirteenth chapter our author returns to Professor See and discusses his view of the formation of the nebular spirals; and the last chapter is devoted to the hypothesis of Emile Belot. P. C.

[1] *Leçons sur les hypothèses cosmogoniques.* Paris: Hermann, 1911. Price 12 francs.

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FOR LOGISTICS.[1]

INTRODUCTORY NOTE.—M. POINCARÉ AND M. COUTURAT.

Soon after Mr. Bertrand Russell's *Principles of Mathematics* of 1903 was published, M. Louis Couturat gave an exceedingly interesting popular account of this and other works in the *Revue de Métaphysique et de Morale* for 1904 and 1905, which was afterwards published in book form in 1905[2] with an appendix on Kant's philosophy of mathematics read at the celebrations in 1904 of the centenary of the death of Kant. Then M. Henri Poincaré thought fit to publish, in the above *Revue,* a series of articles of which this is a list:

"Les mathématiques et la logique," *Revue,* Vol. XIII, 1905, pp. 815-835; Vol. XIV, 1906, pp. 17-34; and *ibid.,* pp. 294-314; "La logique de l'infini," *ibid.,* Vol. XVII, 1909, pp. 461-482.

In connection with some of the subjects so lightly and gracefully touched upon by M. Poincaré appeared the following:

Mario Pieri, "Sur la compatibilité des axiomes de l'arithmétique," *Rev. de Métaphys.,* Vol. XIV, 1906, pp. 196-207.

Louis Couturat, "Pour la Logistique (réponse à M. Poincaré)," *ibid.,* pp. 208-250.

B. Russell, "Les paradoxes de la logique," *ibid.,* pp. 627-650.

The writings of M. Poincaré are well known to the readers of *The Monist.* His criticisms are refreshingly light and gay and he never allows profundity to obscure his wit. It is, however, unfortunate that his airy remarks on modern logic—which, by the way, he confesses rather needlessly that he has not studied—have been taken so seriously by many. It is, as newspaper editors know, a tendency of the public to read with interest and even to accept uncritically the opinions of an eminent person on matters about which he is not an expert. The views of a well-known football player on

[1] Translated by Philip E. B. Jourdain.

[2] *Les principes des mathématiques avec un appendice sur la philosophie des mathématiques de Kant.*

the science of anatomy would no doubt be widely read, and the views of M. Poincaré on the philosophical questions at the root of mathematics are not, in essentials, of a very different nature. It is part of the business of anatomy to study deeply these faculties which the athlete uses unconsciously. The analogy is quite evident. M. Poincaré is one of our greatest mathematicians, and centuries have proved that a man who is a great mathematician need be neither a great philosopher nor a great logician. We do not expect such a combination of qualities,....nor, as a rule, do we find them.

M. Louis Couturat gave a very full and adequate reply to the first two of M. Poincaré's articles. In spite of this M. Poincaré reproduced, in the same words, his refuted arguments in his lately published book *Science et Méthode.* The chapter entitled "Les Mathématiques et la Logique" on pp. 152-171 of the book is almost identical with pp. 815-824 of the first article; the chapter entitled "Les Logiques nouvelles" on pp. 172-196, which is that translated on pp. 243-256 of *The Monist* for April, 1912, is an abridged version of pp. 826-835 of the first paper and the second article. The chapter entitled "Les derniers efforts des Logisticiens" on pp. 192-214 reproduces much of the less technical parts of his third article, and this article, which is translated in the present number, was replied to by Mr. Russell in his above cited paper.

The fourth article of M. Poincaré is concerned principally with a memoir on the theory of "logical types" published by Russell in 1908 and with one on the foundations of the theory of aggregates published by Zermelo in the same year.

It is quite obvious that nobody should allow himself to speak or write in terms of approval or disparagement of a branch of study with which he has only a superficial acquaintance. Each of us is free to dislike or like a particular subject and to leave it alone or to cultivate it accordingly, and if he finds good reasons for so doing he ought to publish them. But not even the most eminent can really think that a brightly written condemnation of a subject, based on a very superficial acquaintance with it, is of any real value. Indeed, the more eminent a person is, the more able he generally is to prevent us from seeing the truth. And then, besides the thought of the efforts of others to perceive the truth, there is the very noble sentiment with which M. Poincaré begins his book *La valeur de la science*: "The search for truth ought to be the aim of our activity; it is the only end which is worthy of it." Very nice, but with regard to what the French call "logistics" or "mathematical logic," and everybody used to call "symbolic logic," M. Poincaré has not been as true to his lofty sentiment as his admirers have learned to expect and demand.

Under these circumstances it seems only fair—I do not mean to logistics but to the public—to give people the opportunity to read M. Couturat's answer as well as M. Poincaré's attack.

In the following translation, any bibliographical or other notes which I have added are enclosed in square brackets. Where possible I have abbreviated the translation and avoided the use of symbols.

There are a few passages in M. Couturat's article which may possibly give rise to a wrong impression. Thus, he speaks of logical demonstrations making true the chance finds of the intellect. Of course the process of demonstration does not do this: It gives the finder and other people certainty as to whether the find is true or not. But we must not accuse M. Couturat of being a pragmatist on the slender grounds of a loosely expressed sentence; especially as in other parts of this article he has protested in the clearest possible way against the confusion between creation and discovery.

Near the end of the second section there is a reference to a number of mathematicians who failed adequately to deal with the paradox discovered by Burali-Forti, among whom are mentioned Russell and myself. The article of Russell referred to contains, implicitly, a criticism of certain views widely held by mathematicians at that time and also—again implicitly—the solution of the paradox and others like it. This of course was familiar to M. Couturat, but the citation of Russell in that connection might mislead some people. With regard to myself, at the time (1903-1904) that I wrote the papers referred to I was hardly, as M. Couturat says, "totally a stranger to logistics," but I freely grant that I was not as familiar with it as is necessary even to grasp the full bearings of the question. My attempt at the solution, though I believe it has one small merit in distinguishing between what may be called *entity* and *existence,* I have since then abandoned.

The discussion, in the third of M. Couturat's sections, of the question of *existence* does not appear to me to be satisfactory, and I have added a note referring to some former remarks of mine on this subject in *The Monist* for January, 1910. P. E. B. J.

REPLY TO M. POINCARÉ.

I thank M. Poincaré for the honor which he has done me by taking me in particular as the subject of his articles on "Mathematics and Logic,"[3] but I must say that I do not deserve this honor, for the ideas which I have presented are not my own and I fear that M. Poincaré has done them a great wrong by discussing them from a work in which they are given at second-hand. In fact, as I have been careful to warn my readers, my articles[4] were mainly only an account of Mr. Russell's book; and wherever I have been led to add an analysis of the works of other logisticians I have not omitted to refer to them. Now it is not customary to criticize works of this class from a simple analysis of them, above all when the value of these works consists in the rigor of demonstrations, and these demonstrations are necessarily absent from my summary exposition. For example, I have analyzed long memoirs of Peano, Pieri and Whitehead by limiting myself to the enunciation of their chief theorems, without the quotation of a single demonstration. It is constantly assumed that the reader who wishes to see the demonstration of such and such a theorem has only to seek out the original memoirs, and it would obviously be pointless to reproach me for not having given it. Similarly I have thought I ought to describe in my book, to complete one of my articles, Peano's space-filling curve in an elementary and intuitive form which is accessible to the first comer and without speaking of the rigid analytical demonstration. What would one think of a mathematician who, only knowing this curve by my account of it, allowed himself to criticize its construction, to doubt the rigor of the demonstration, or to declare that this demonstration does not exist and that the proposition in question rests on intuition?

Also I had warned my readers that in my work I would

[3] [See Dr. G. B. Halsted's translation of "The New Logics" in *The Monist* of April 1912, and of "The Latest Efforts of the Logisticians" in the present number.]

[4] Published, with some corrections and additions, in a volume bearing the title, *Les principes des mathématiques,* Paris, Alcan, 1905.

sacrifice rigor to clearness, not to that *logical* clearness which is inseparable from rigor and which can only be obtained by logistical symbolism, but to that clearness in the common accèptation of the term which is called *intuition* and which M. Poincaré esteems so highly. It must be granted that I am very badly rewarded for the concessions which I have made to intuition, since M. Poincaré profits by them to reproach me with a lack of rigor. In any case, I wished to do the work of a commentator and a popularizer and to compose for the use of the laity a kind of introduction to the works of which I gave a short account. That is to say, it was not for M. Poincaré that I wrote, and I did not pretend to teach him anything about these works. In all cases, a work of the kind I wrote may serve—I hope so at least—to *teach* the elements of the doctrines in question, but it cannot serve as a sufficient basis to *criticize* these doctrines; to be just and effective, the criticism ought to be on the original works from which I drew my inspiration. What would M. Poincaré say if some one took it upon himself to discuss Hilbert's principles of geometry[5] from the analysis—however exact and complete it may be—which he has given of it to the French public?

I might stop with these remarks, and perhaps I ought to do so; for if I have already compromised the doctrines in question by my attempt at popularization, I run the risk of compromising them still more by undertaking to defend them against an adversary like M. Poincaré. If I have resolved so to defend them, it is, on the one hand, because it has pleased M. Poincaré to substitute me for the masters of logistics, and, on the other hand, because these masters have believed that I would suffice for the task and have left to me the care of justifying them. I thank them for

[5] [*The Foundations of Geometry,* trans. by Townsend. Chicago, The Open Court Publishing Co.]

their confidence; but the reader ought to know that if there is any thing good and enduring in my work it is to those masters that I owe it, and that all that is feeble and defective comes from myself. If then I succeed in justifying logistics against the criticisms of M. Poincaré, so much the better; if not, it will be my fault and will prove nothing against logistics.

I.

In the first place, we must not confuse logistics with what M. Poincaré calls "the logic of M. Hilbert." M. Poincaré has not made this confusion, but many of his readers may do so when they see him associate these two doctrines in the same discussion and in a common condemnation. Now it must be clearly realized that Hilbert is a complete stranger to logistics and has never used any logical calculus in his researches. If then the criticisms that M. Poincaré makes against him are just, they have no bearing against logistics, but rather tend to prove the superiority of logistics over verbal reasoning and simple common sense.

It is important also to correct a historical error to which the following phrase of M. Poincaré may give rise: "What Hilbert has done for geometry others wish to do for arithmetic and analysis." We might believe from this passage that the logisticians attack the subject of arithmetic and analysis *after* the works of Hilbert on geometry, and in imitation of them. The *Grundlagen der Geometrie*[6] of Hilbert were published in 1899. Now, ten years before this (in 1889) Peano had published not only his *Arithmetices principia nova methodo exposita* but also *I Principii di Geometria logicamente esposti,* both of which were written in the symbolism which he had invented in the year before. In 1891 he published in the first volume of the *Rivista di Matematica* two articles on the concept of num-

[6] [English translation as noted above.]

ber which already contained the five fundamental axioms of arithmetic. In 1894 he published in the fourth volume of the same *Rivista* the memoir on the foundations of geometry which I have analyzed in *Les Principes des Mathématiques*.[7] Lastly, in 1899 Pieri published his logical reconstruction of projective geometry and of metrical geometry in the Memoirs of the Turin Academy. These dates are enough to prove that, if Hilbert has not *wished* to profit by the works of the logisticians, the logisticians *could* not have profited by his work and had no need of his example not only in arithmetic and analysis but even in geometry. Consequently M. Poincaré commits a historical error in attributing to the "works of M. Hilbert" the "triumph" of logistics in geometry. I content myself, on this point, with stating a fact: In 1900 Hilbert elaborated for arithmetic a complicated system of *eighteen* axioms,[8] when *eleven years before this* arithmetic had been built up on *five* axioms only, which Padoa in 1902 reduced to four. Finally, to render to each person the "chronological" justice which is due to him, I should record that Frege stated, in his *Grundlagen der Arithmetik* of 1884, the theory of the integer number which Russell has adopted in principle, and undertook to prove that the principles of arithmetic are purely logical—analytical in Kant's sense.

M. Poincaré writes: "This invention of M. Peano was called *pasigraphy*," and adds: "This name exactly defines its bearing." The first phrase contains an error of fact. *Never* did Peano call his logical symbolism by the name of "pasigraphy"; he always called it "mathematical logic".[9] If I call it "logistics," it is first, because of the equivocal-

[7] Chap. VI, pp. 159-180.

[8] "Ueber den Zahlbegriff," *Jahresber. der deutsch. Math.-Ver.*, Vol. VIII, 1900. [This essay was reprinted in an appendix to the 3d German edition of the *Grundlagen der Geometrie*, Leipsic and Berlin, 1909, pp. 256-262.]

[9] See all the editions of Peano's *Formulaire de Mathématiques*, and the *Notations de Logique mathématique* (Turin, 1894) which forms the introduction to the first edition.

ness of the expression "mathematical logic," and, secondly, not because "this new name implies the purpose of revolutionizing logic," but because this good old word, which Vieta gave to algebra, indicates, by its very etymology, the general art of reasoning and calculating. In this sense it was employed in the eighteenth century by Lambert to denote his own logical calculus.[10] It was Schröder who first called it "pasigraphy" in a communication made to the first congress of mathematicians at Zürich in 1898, and that probably with a depreciative intention.[11] Now this word is quite inexact, whatever M. Poincaré may say. People call any written universal language a "pasigraphy"; thus the international code of maritime signals[12] is a "pasigraphy." I myself formerly used this word when speaking of Peano's symbolism but I corrected it at once: "A system of *pasigraphy* or, better, of *ideography*";[13] this means that the symbols translate not words or phrases but ideas. I concluded the same article by the words: "We would restrict incorrectly the value of Peano's symbolism if we only regarded it as a kind of stenography. It is also and chiefly an instrument of logical analysis, of deduction and of verification"; and I recalled, *à propos* of this, the "universal characteristic" of Leibniz. It is, then, entirely to misinterpret the nature and bearing of logistics to regard it as a mere pasigraphy.

For the rest, M. Poincaré speaks of logistics in the way in which a *bel esprit* might speak of algebra or mathematics in general. For example, he says: "The essential elements of this language are certain algebraic signs which represent the different conjunctions *if, and, or,* and *then.* That these signs may be convenient is possible, but that

[10] "Versuch einer Zeichenkunst in der Vernunftlehre, *Logische und philosophische Abhandlungen,* edited by John Bernoulli, Berlin, 1782.

[11] Translated into English in *The Monist* for October, 1898.

[12] Cf. Couturat and Leau, *Histoire de la langue universelle* (Paris, 1903), preliminary chapter on "Les pasigraphies."

[13] *Bulletin des Sciences mathématiques,* Vol. XXV, 1901.

they are destined to revolutionize the whole philosophy is another question. It is difficult to admit that the word *if* acquires, when it is written Ↄ, a virtue which it did not have when it was written *if*." In the first place we must not believe that logistical symbols are limited to the literal translation of some words.[14] The sign Ↄ translates *if* no more than *then*, it expresses the idea of implication; the same sign may translate *and* in certain cases and *or* in other cases. Inversely, the word *and* has not the same meaning in the three following cases: "Peter is rich and happy," "Peter and Paul are rich," "Peter and Paul are brothers"; and consequently it is not translated by the same logistical symbol. It is, then, quite unjust to consider "the new language" as a mere tracing of ordinary language and consequently as having no value and no utility of its own.[15]

M. Poincaré believes that I attach "an exaggerated importance which would astonish M. Peano himself" to the use of symbols. I can reassure him on this point. M. Peano writes to me on this subject: "I have always affirmed the importance of symbolic notation in all mathematical propositions, its great utility in difficult and delicate questions, and its indispensability in the study of principles. That is written down in all the volumes of the *Formulaire* . . . ."Everywhere and always he insists upon the necessity of expressing every mathematical proposition and every definition *entirely in symbols*.[16]

[14] Like the childish notations of Hérigone, who wrote, for example, "5<" for *pentagon*; or like any system of analogous abbreviations that a mathematical student may invent for taking notes.

[15] In 1895 Peano wrote: "Mathematical logic....does not reduce merely to an abbreviated symbolical writing, to a kind of tachygraphy; it allows us to study the laws of these signs and the transformations of propositions.... The two objects of mathematical logic, the formation of a symbolical script and the study of the forms of transformations (or of reasoning) are closely connected" ("Sur la définition de la limite d'une fonction," *American Journal of Mathematics,* Vol. XVII). This memoir was meant (as its subtitle "Exercice de logique mathématique" shows) to make the new logic known to mathematicians. Mathematicians then cannot be excused for still ignoring it, and it is doubly inexcusable for them to criticize it without knowing it.

[16] Cf. his memoir printed at Paris in 1900 among those read at the first international congress of philosophy.

However that may be, there was some one who had an opinion which is as "exaggerated" as that of Peano and myself of the importance of symbolism, and that was Leibniz. He went as far as to say that the discoveries in mathematics that he had made arose solely from the fact that he had perfected the use of symbols, and his discovery of the infinitesimal calculus was, for him, only a specimen of his *characteristica universalis.*[17] In fact, we know that he did not invent infinitesimal *ideas*; he only invented a symbolism to represent them and an algorithm to manipulate them. We might say of him: "He only introduced two new signs, *d* and $\int$. That these signs may be convenient is possible; that they could revolutionize the whole of mathematics is incredible." We might also say of algebra: "It consists simply in representing by signs the words *plus, minus, multiplied by,* and *divided by.* But it is not to be seen how it constitutes a progress beyond arithmetic; it is difficult to admit that the word *plus* when it is written + acquires a virtue that it did not possess when written *plus.*" And yet, could the theory of equations and the theory of algebraic forms have been elaborated with *words*?

M. Poincaré asserts that "pasigraphy does not preserve us from error." Without doubt it does not, any more than the rules of algebra or arithmetic do. Does it follow that these rules are false or that we ought to defy them? Because we make mistakes in addition, must we condemn the four rules of arithmetic and even the arithmetical signs, and only count on our fingers or with little balls? The mistakes which a logistician may commit do not weaken the value of logistics any more than mistakes in calculation shatter the certainty of arithmetic. It is enough that logistics allows us to reason more easily and more surely and to discover faults of reasoning more easily; and that

[17] See Couturat, *La Logique de Leibniz,* pp. 84-85, the texts cited in the note and the third appendix.

is what it does. In this sense it is, as Leibniz said, an art of infallibility—not that logisticians are infallible, but they are less exposed to error than those who trust to simple common sense, that is to say to intuition.

Besides, M. Poincaré forms quite a false idea of logistics by considering it as a mechanism from which intelligence is nearly excluded; and his comparison of it with the "logical piano" of Stanley Jevons is not exact. We must first of all know that this logical piano merely concerns logical classes and that it only effects the least important —and the most mechanical—part of reasoning. Its office consists in suppressing the elementary classes which are annulled in virtue of the given premises. But it leaves almost all the rest to be done; thus, on the one hand, we have to put the logical problem into equations, and, on the other hand, we have to combine the subsisting classes in such a way as to obtain the consequences in the desired form. Thus the algebra of logic does not reduce to a blind mechanism. This is still more true of logistics which surpasses the algebra of logic and is much less "mechanical."

Another comparison is no happier: "Are the rules of perfect logic the whole of mathematics? We might just as well say that the whole art of the player of chess reduces to the rules for moving the pieces." But nobody ever asserted that all mathematics reduces materially to logic, that is to say that there is *nothing more* in a treatise on mathematics than in a treatise on logic. We maintain only that all mathematical reasonings are effected in virtue of the rules of logic *alone,* in the same way that all the games of chess that have been and can be played are effected according to the rules of the game...., otherwise the rules would be worthless. The comparison returns then against the adversaries of logistics, for it shows how a small number of elements, combined according to some few fixed laws, can generate an unlimited variety of consequences. People have asserted that logistics put leading-strings on invention, and have urged against logistics the rights of genius. How could mathematics constantly evolve and progress if it is always condemned to rest on a small number of principles and "logical constants"? M. Poincaré does not use this argument and leaves on one side the question of invention; but it is clearly visible that the theory of "logical constants" inspires in him an instinctive repugnance, and that every attempt to catalogue the primitive notions and principles of mathematics appears to him to be an insupportable pretension and a restriction on the "liberty" of the scientific man. It is for that reason that he opposes to logical and demonstrative reason the "sure instinct" of the inventor and the "more profound geometry" which guides him; and these kinds of considerations are very much in fashion. It is, at the present time, fashionable to put the "logic of nature and of life" in opposition to formal logic that is disdainfully called "dialectical," "abstract," and "verbal."

There is here a confusion which it is important to dissipate. To oppose to logic the psychological fact of invention is to commit the most gross *ignoratio elenchi.* Logic has neither to inspire invention nor to explain it; it contents itself with controlling it and *verifying* it in the proper sense of the word (making it true). Do we reproach metrical science for not giving poetical genius or the science of harmony for not conferring musical genius? And do we therefore conclude that the rules of both have no value and no utility? As for the theory of "logical constants," the liberty of the mathematical discoverer is no more restricted by formulating the primitive principles and notions on which his science rests than the libetry of the musician, of the painter and of the poet is restricted by saying to them in turn: "As for you, you will never be able to do anything but combine the seven fundamental notes with their accidentals; as for you, the seven colors of the spectrum, and as for you, the twenty-six letters of the alphabet." That is exactly in what measure logistics clogs invention and clips the wings of genius. People should really stop throwing invention at the head of logicians, as if invention could be contrary to logic. Besides, this "sure instinct" and this "more profound geometry" which guide the discoverer are only unconscious forms of the logical reason and proceed according to the same laws. The reason which invents is conformable, and at bottom identical, with the reason which demonstrates, and without it the latter could not *verify* what the former has by chance found; and these chance finds are only *true* on this condition. It is, then, conformity with the laws of logic "which alone gives value to the edifice which has been built."

M. Poincaré speaks of "the logic of Russell" and opposes it to the logic of Aristotle, as if Mr. Russell was the first to go beyond the confines of the Aristotelian logic. He appears besides to have an inexact notion of the Aristotelian logic when he says: "The logic of propositions of Russell is the study of the laws according to which the conjunctions *if, and, or* and the negation *not* are combined. It is a considerable extension of the ancient logic." I can assure M. Poincaré that Aristotle was already acquainted with the conjunctions *if, and, or* and negation, and that he took account of them in his logic. All the classical logicians knew and studied hypothetical judgments (where *if* figures), copulative judgments (where *and* figures), and disjunctive judgments (where *or* figures); and classical logic has always admitted *negative* judgments. If M. Poincaré means that Mr. Russell is the first who has translated these judgments into symbols and submitted them to an algorithm he is at least half a century out of his reckoning: for it is to Boole (without speaking of his fore-runners) that this honor is due. It is, then, not Mr. Russell who has "adjoined" to syllogistics "the conjunctions *and* and *or*" and who has thus "opened up a new domain to logic."

M. Poincaré believes that he can establish a fundamental difference between the two logics by remarking that "the symbols are multiplied and permit of varied combinations *which are no longer finite in number*," and he adds: "Have we any right to give this extension to the meaning of the word *logic*?" It would, then, seem that for him logic is characterized by the *limited number* of the combinations which it admits. But I do not see that there is a radical difference. Besides, in what sense did the ancient logic only admit a limited number of combinations? Is it a question of the number of valid moods of the syllogism? But modern logic, too, only admits a limited number of simple types of reasoning. Is it a question, on the other hand, of the infinite diversity of complex reasonings that one can obtain by combining these types? But classical logic too could form an infinity of reasonings by combining syllogisms. In all cases the two logics have the same character and only differ in respect of the more or less. Besides, how is the number relevant in this matter? If a logical principle is true, whether it be the principle of the syllogism or any other, is it not capable of justifying an infinite number of reasonings just as well as a finite number? Does its demonstrative virtue by some chance become exhausted after *n* applications? Lastly, what means this reproach addressed to logistics of admitting an infinite number of combinations, when, on the other hand, it is reproached with only having a very limited number of principles? Is it not, rather, for it, just as it is for geometry (according to a well-known phrase), a glory to deduce from so small a number of principles so great a number of consequences? How can this fact scandalize a mathematician who is familiar with the incredible fruitfulness of the theory of combinations?

When M. Poincaré opposes the old and the new logic to one another and considers the latter as an enormous and perhaps illegitimate "extension" of the former, he appears to forget the fact that the *domain* of a science may receive an extension—even a considerable one—without the *notion* and the definition of this science changing. Otherwise we could never speak of the progress of the sciences: M. Poincaré seems to suppose by that that a science remains in essentials identical with itself in the course of its historical development. The reasoning of M. Poincaré would serve to prove that the infinitesimal calculus is not a part of mathematics; that electricity is not relevant to physics, and that the theory of organic compounds is not relevant to chemistry. Now it is for this reason that the extension of the "field" of classical logic becomes an extension of the "meaning of the word *logic*." M. Poincaré says again: "It seems that there is nothing new to write about formal logic and that Aristotle saw to the bottom of it." If he means by that (as Kant did) that logic has made no progress since Aristotle, it is nowadays a simple error of fact; but if he means that logic ought to remain (or ought to have remained) confined in the domain assigned to it by Aristotle, he maintains implicitly that logic was perfect and complete at its birth, and this is contrary to the analogy of all the other sciences and to probability. We would only smile at a man who claimed to reduce mathematics to what it was in the time of Euclid, and physics to Aristotle's physics. How then dare any one maintain or insinuate that Aristotle has said the last word about logic and that it is forbidden to develop this science beyond the narrow limits assigned to it by its founder?

Besides, if "the new logic is richer than the classical logic," it is not so much by the extension of its domain as by the deep study of the principles that have *always* directed those reasonings which have been recognized as just by that rational instinct to which M. Poincaré attaches so much value. He seems to reproach the logisticians with "introducing" into logic indefinable notions and indemonstrable principles. It would be more just to say that they have discovered or recognized them; just as Aristotle did not invent but discovered and recognized the principle of the syllogism. M. Poincaré is in too great a hurry to assert that these indemonstrable principles "are appeals to intuition, are synthetic *a priori* judgments." Perhaps he would have been of another opinion if he had taken the trouble to run through the enumeration of these principles. Why should the principle of composition: "If a is b, and a is c, then a is bc" constitute an appeal to intuition rather than the principle of the syllogism: "If a is b, and b is c, then a is c"? In what is the principle of simplification: "ab is a" more synthetic than the principle of identity with which it has been so often confused? In any case, it has been considered by Kant as the type of analytic judgments. Is it of these principles that M. Poincaré said: "We regarded them as intuitive when we met them, more or less explicitly enunciated, in treatises on mathematics. Have they changed character because the meaning of the word logic is enlarged and we now find them in a book called *Treatise on Logic*"? In what treatise of mathematics has M. Poincaré seen them formulated? And his argument returns on himself, for even if they were put in a treatise on mathematics, would that change their character as logical principles? *"They have not changed their nature, they have only changed place,"* writes M. Poincaré in italics; but it is he who has changed place. It is not enough that they should be used in mathematical reasonings to call them mathematical, and it is not enough that they are not found in treatises on classical logic to refuse to them the title of logical principles. Otherwise it would be necessary to say that *logical* principles are, by definition, those which Aristotle and the schoolmen have discovered and formulated; and that all the logical principles discovered by modern logicians are *intuitive*. The distinction of the logical and the intuitive would then reduce to a question of chronology.

Besides, the vague conception of *intuition* is out of place as a weapon against the logicians, especially when the intuition spoken of is not specified. Is intellectual intuition meant, which bears upon the relations of ideas, or sensible intuition, which necessarily clothes the spatial form? These two intuitions are wholly different. All logicians are ready to recognize that their principles proceed from intellectual intuition, that is to say they are objects of immediate knowledge by the reason; but very few will agree that they proceed from sensible intuition, and rest, for example, as Lange has maintained, on spatial schemata. For the rest, whatever the solution of this "metalogical" problem may be, all the logical principles ought to have the same fate; and the traditional principles of identity, contradiction and so on will be "appeals to intuition" in the same sense and in the same measure as the others. The logisticians then, must not be accused of altering logic by introducing intuition into it; for if this accusation has any value, it is Aristotle who began this introduction.

In any case it is inexact to say that "living" reasonings, the only ones "in which our mind remains active," are "those in which intuition still plays a part." Purely logical reasonings need more mental effort and ingenuity than M. Poincaré believes, and, even with the mediocre aid of Jevon's logical piano, a certain cleverness is necessary to combine the brute results of mechanism and to draw the conclusion wished. Besides, why reproach logistics with making reasonings easier and more sure?[18] If, like algebra, it condenses into short formulae the result of long and complicated reasonings, it is to relieve the powers of the mind and to allow it to embrace a greater number of data and to draw vaster and more distant conclusions. Consequently, far from paralyzing the faculty of invention or rendering it useless logistics lends it stilts or wings. The discovering mind will always find something to exercise itself upon, but it will do so on data which are more and more complex. That is what happens in analysis, where each new theory combines formulae which sum up the results of simpler and more elementary theories. M. Poincaré may then be reassured: logistics does not exclude genius.

M. Poincaré makes a curious reproach to logistics: "The part of intelligence is restricted to choosing among a limited arsenal rules posited beforehand, and has not the right to invent new ones." If we remark that the "rules" are none other than the principles of logistics, this phrase appears to me to mean that intelligence "has the right" to invent new logical principles. It is a strange conception of logic to consider it as always evolving and as never finished.[19] It evidently proceeds from the psychological confusion between the science and what we know of it at a given moment. No one will ever "invent" new rules of logic; some of these rules which had not been noticed but were quite as "ancient" as the others and equally "posited" beforehand, that is to say *a priori,* will perhaps be "discovered." And the logisticians do not do anything else. But then, why does M. Poincaré reproach

[18] "More sure," for M. Poincaré confesses that in living reasonings "it is difficult not to introduce an axiom or postulate which is unperceived." Must we conclude from that that "life" is incompatible with logic?

[19] To use the favorite comparison of M. Poincaré, what would we say of a chess-player who wished to invent a new rule in the middle of a game,—for example, to make his king move several squares when in check? Such an "invention" would be called trickery and nothing else.

them with innovating? With respect to the nine indefinable notions and the twenty indemonstrable propositions of Russell, he says: "I believe that....I would have found some more." He is quite at liberty to do so: the logisticians do not ask for anything better, and will register his discoveries —or, if he prefers to say, his inventions—with gratitude. But what do these contradictory reproaches mean if not that M. Poincaré claims for himself "the right" to "invent" logical principles at the very moment when he refuses this right to the logisticians?

For the rest, what good is it to discuss *in abstracto* the qualities of logistics? M. Poincaré grants that "pasigraphy can furnish us with a criterion to decide the question which occupies us. If every treatise on mathematics can be translated into the Peanian language, the logisticians are right." Now the logisticians replied in advance, long ago, to this ironical invitation. Ten years ago Peano published the first edition of his *Formulaire de Mathématiques*, which is precisely a treatise or manual entirely written in logistics; the fourth edition (1903-1904) comprises Logic, Arithmetic, Theory of Numbers, Algebra, the Theory of Real Numbers, the Theory of Definite Functions, the Infinitesimal Calculus, the Theory of Complex Numbers, the Theory of Circular Functions, the Geometrical Calculus (comprising the theory of vectors and the theory of Quaternions), and Differential Geometry; the "Additions" even contain the elements of kinematics. The fifth edition of the *Formulaire* is in course of publication.[20] The principal theorems are accompanied by their logistical demonstrations. I will add that this mathematical manual is a collective work which M. Peano and his collaborators are incessantly revising and perfecting. Consequently the proof

[20] Professor Peano has published, besides, a classical manual entitled: *Aritmetica generale e algebra elementare,* drawn up in logistics (Turin, 1902).

which M. Poincaré requires of logisticians was given long ago and is being completed from day to day.

It is true that M. Poincaré soon seems to repent of his rash concession and adds: "Again we must examine the translation closely. It is not sufficient that we should be presented with a single page where there are only formulae and not a single word of ordinary language, in order that we must bow down.... It will be necessary, when we are in the presence of a pasigraphical reasoning, even when this reasoning is correct, to examine if an appeal to intuition is not hidden away in some corner." These reserves are evidently very just in so far as they are counsels of critical method. But why does M. Poincaré not conform to them? It is not enough to express these general reserves which are applicable to any demonstrative work, to weaken the value of logistics and throw disfavor and suspicion on the work of logisticians. The logisticians have given to the public not "one page" but more than three hundred pages of logistical formulae and demonstrations. Let those who have doubts on the value of these demonstrations "examine" them as closely as they wish and let them point out lacunae and errors,—for that is their right and even their duty. But the burden of proof falls on them, and it is not enough, in order to get rid of this burden, to shake their heads with a smile of incredulity.

II.

I pass on to the objections aimed at logistics in so far as it is applied to mathematics. Here again I must say that M. Poincaré wrongs it by judging it merely from the "popular" exposition which I have given of it. In effect, the logistical formulae which constitute, as M. Poincaré says, a "new language" are sufficient by themselves and are intelligible wholly by themselves; if it were necessary to add to them a single word of ordinary language, it would prove their incompleteness or defectiveness. Besides, this "new language" was invented expressly to avoid the equivocation or the beggings of the question implied more or less confusedly in ordinary language. Consequently the logistical formulae are the *only* ones which can be exact, rigorous and exempt from the above logical vices. Thus, when an author thinks that he ought to translate them into ordinary language, it is merely to make them more accessible to the "laity"; but it must be understood that this verbal translation is always imperfect, approximate and by no means allows the proper appreciation of the logical value of the formulae. Just because language cannot equal the precision and the rigor of the formulae, I have made no scruples about introducing into my verbal translations apparent beggings of the question in order to make them more clear and more "French." What does an inexactitude more or less matter when the logistical formula alone counts from the logical point of view? I could not expect that any one should judge and condemn these formulae from the mere inspection of the verbal translation which I gave of them for the use of novices. All translation is a betrayal; but that is still more true when the translation makes exactly those qualities of the original on which study and discussion bear vanish. It is exactly as if some one wished to study the meter of Virgil in a French translation of the *Aeneid.*

Now, it is of these verbal translations, *and only of these verbal translations,* that M. Poincaré has taken account in his criticism; he does not appear to have noticed the logistical formulae; "It is Greek, so it is unread." He may then "amuse himself by counting how many numerical adjectives my exposition contains": that will prove absolutely nothing against "pasigraphy." Nevertheless we will examine his arguments one by one in order to show better that they all miss the point. On the subject of the logical definition of zero, he says: "to define *zero* by something *null* and something *null* by *none* is indeed to misuse the richness of the French language." Then he recognizes that I have "introduced an improvement in *my* definition" (a double inexactitude, for this definition is not my own and the "improvement" in question is due to Mr. Russell) by writing "what," according to M. Poincaré "means, in French, zero is the number of objects which satisfies a condition which is never satisfied. But as '*never*' signifies '*in no case*' I do not see that the progress is very great." I will confine myself to recalling the verbal translation that I have given of this formula: "if φx is always false, Λ is the class of x's which verify φx." The verbal translation of that is: Λ is the class of objects which satisfy a condition which is *always false,* that is to say, false for *all* the values attributed to x. Where is to be seen in this formula the idea of the number zero or even of any number? And are we to be reproached for introducing into logic mathematical notions, when classical logic was acquainted with universal judgments and used the word *all*? To be able to attribute to us a begging of the question—even one that is simply verbal—M. Poincaré has had to transform our translation by replacing "always false" by "never true." If, then, somebody here abuses the French language it is not I.

But this reproach is even more undeserved if it is addressed to the logistician, who writes neither in French nor in Italian nor in English but in a symbolism made expressly to liberate ideas from the tacit implications that language introduces into them by custom. M. Poincaré himself says: "It is impossible to give a definition without enunciating a phrase and difficult to enunciate a phrase without putting in it a name of a number or at least the word *many* or a word in the plural.[21] And then the roof

[21] On the subject of the plural, it may be remarked that Peano has, follow-

is slippery and at every moment there is a risk of falling into a begging of the question." These very just reflections bear only on the logical defects of language and on the faults that language can make us commit. It is precisely to avoid these faults and to cure these defects that the logisticians have invented their rigorously defined signs which have no meaning but that which they are given by definition.[22] Put shortly, M. Poincaré's argument comes to this: "All those who reason with the words of ordinary language are *liable* to commit beggings of the question; now the logisticians use, not words, but symbols rigorously defined; consequently they too *must* commit beggings of the question." The syllogism is not conclusive for it has four terms. And even if it had only three, that is to say when one could legitimately conclude from words to symbols, the two words which I have italicized would still render it invalid; the major says that we *may* commit errors; the conclusion asserts that certain authors have *necessarily* committed them.

The criticism of the definition of the number 1 is no firmer. "*One* is the number of elements of a class of which any two elements are identical"; such is the verbal translation that M. Poincaré gives of this definition. "It is more satisfactory....in the sense that, in order to define 1, we do not use the word *one*;—but still the word *two* is used"; and M. Poincaré rightly suspects that *two* can only be defined by means of *one*.[23] But he makes an unjust use of the fact that I have used the word *two* to make a phrase in ordinary language. The more exact translation of the logistical formula is: "1 is the class of classes *u* which are

ing Leibniz's recommendation, excluded it from the "uninflected Latin" which he has given out as a form of international language, and which has been adopted by many.

[22] Cf. the beginning of the preface to Peano's *Arithmetices principia* of 1889.

[23] By means of the general formula by which we define $n+1$ by means of n; cf. *Les Principes des Mathématiques,* Chap. II, § B, p. 59.

not null and such that if x is a u and y is a u then x is identical with y whatever x and y may be." Where is there, I do not say the word, but the idea of two in this formula? M. Poincaré will say perhaps that *two* (problematic) elements x and y of the class u are made to appear in it; but the fact that they are *two* does not come into the question in any way; and the proof of it is that in reality they are only *one*: x and y are merely two names (excuse me, *names*) for a single individual. This criticism obviously has no bearing on another equivalent formula which I have given,[24] and which may be translated: "*One* is the class of classes u which are not null and such that if x is a u the class of the elements of u which are not identical with x is null." That presupposes of course the definition of the null class; but, as we see, there is no more even a problematical *two* elements of u, but only *one*, and we only express that there is no other.[25]

Will anybody say that, by the mere fact that *an* element is spoken of, the number *one* is implied?[26] But that is an objection which M. Poincaré does not formulate and to which I have replied in advance in the following passage: "We must not believe that the definition of the number *one* constitutes a vicious circle, for the definition of the singular class rests solely on the relation of identity. If it is true that it implies in a sense the *unity* or rather the *individuality* of the element considered, this unity cannot be identical with the *number one* which is to be defined: for this unity is a property of each element while the *number one* is the property of a class. . . . consequently, in all cases

[24] *Ibid.* I have logistically deduced this from the preceding one on p. 60.

[25] Here is a more fundamental definition that Mr. Russell has communicated to me: "*One* is the class of classes u such that the proposition: 'x is a u' is equivalent, for all values of x, to 'x is identical with c' is not false for all values of c.'" Notice that this definition does not presuppose the notion of the null class. As for the formula "x is a u", cf. its definition farther on.

[26] [In French, the same word *un* stands for both *an* and *one*].

the *units* which constitute a cardinal number are different from the number *one*."[27]

The confusion which exists in many minds between these two ideas arises, I believe, from the double meaning of the word for one, which is used both as the name of a number and as an indefinite article.[28] In the latter case it would be better to use the word *some* as the logicians do.[29] This equivocalness exists in French and German, but not in English. If then, somebody is inclined to invoke it, he should take care to abuse not "the richness" but the poverty of the French language. To sum up, it is not enough to conceive *any one* object to conceive the *number one,* nor to think of two objects together to have by that alone the idea of the *number two.* From the fact that a logical formula contains two or many symbols we must not conclude that it implies by that alone the idea of two or of some other number. When we say: "Peter and Paul are wise," we mean to say: "Peter is wise and Paul is wise"; we do not think the *number two* and we have no need to think it nor to notice that that makes "two wise men." In the same way when we say: "x and y are elements of the class u,"[30] we do not think the number two and no idea of number is implied in this proposition.[31]

[27] *Les Principes des Mathématiques,* Chap. II, § A, pp. 47-48. M. Poincaré seems to propose or to accept such a justification when, after having quoted the phrase of Hilbert: "Let us consider the object 1," he adds: "Remark that by doing this we by no means imply the notion of number, for it is understood that 1 here is only a symbol...." Doubtless, but it is *a* symbol, that is to say *one* object. Will M. Poincaré say that that implies the *number one?* Or will he grant to the logisticians the same liberty as to Hilbert?

[28] [Cf. note 26].

[29] And also M. Méray, thus giving example in logic to other mathematicians.

[30] Notice that it is only grammar which makes us use the sign of plural in *are elements.*

[31] Here is the rigorous definition of the proposition "x is a u," that Mr. Russell has communicated to me: "x is a u" means: "The proposition: 'ϕx is true, and u has the relation of a class to the property which defines it' is not false for all values of x." There is not here the shadow of the idea of the number one, but, as in my enunciation, the purely logical notions of *false, negation* and *all.* This definition was already given by G. Frege, *Grundgesetze der Arithmetik,* Vol. I, 1893, p. 53.

These considerations reply to this objection of M. Poincaré's: "A relation is incomprehensible without two terms; it is impossible to have the intuition of the relation without having at the same time the intuition of its two terms." That proves nothing, and M. Poincaré adds: "And without remarking that they are two, for in order that the relation may be conceivable, it is necessary that they should be two and two only." It is not the question to know—and it is a psychological question—if we "remark" or not that they are two, but if the notion of the relation implies that of the number *two*. Now for that it would be necessary that it implied the notion of the class formed by its "two" terms; and that is obviously not the case. The proposition: "x is the father of y" by no means implies the idea of the class formed by x and y. Besides, it often happens that a relation (which is then called reflexive) exists between a term and itself. Would it be maintained then that it has still two terms? That would be to say that x is at the same time *one* and *two*.

The only logistical formula that M. Poincaré has criticized in itself and not in its verbal translation is one given by Burali-Forti. M. Poincaré says on this point: "I understand the Peanian language too little to dare to risk a criticism." This confession would disarm us if he did not "risk" this criticism immediately afterwards: "I fear that this definition begs the question, for I see the figure 1 in the first member and '*Un*' in the second member." M. Poincaré has trusted too much to his "intuition," and it has deceived him. Instead of "risking" this criticism on the mere witness of his eyes, he ought, conformably to the fundamental rule of mathematical method, to have substituted for what is defined the phrase which defines it; and to ascertain if this definition really begs the question, he had only to refer to the definition of the symbol "*Un*."

Now M. Burali-Forti defines "*Un*" as the class of singular classes, which in Russell's definition of the *cardinal number 1*. This definition is equivalent to the one which I have given above and neither of them implies the idea of that which is defined. As to the formula which M. Poincaré has criticised, it means: "1 is the ordinal type of the ordered classes of which the cardinal number is unity." Thus it consists in defining the *ordinal* number 1 by means of the *cardinal* number, and this explanation is enough to do away with any appearance of a vicious circle. So we see how "risky" the criticism of M. Poincaré is.

He seems to consider as insignificant the formula

$$1 \varepsilon \mathrm{No}$$

which M. Burali-Forti deduces from his definition. M. Poincaré translates it inaccurately as: "One is a number"; and then makes merry at the expense of pasigraphy, which "is sufficient to demonstrate that one is a number." If he had read the memoir of M. Burali-Forti—even in the "interlinear Italian translation"—more attentively he would have known that "No" means ordinal number, and perhaps he would have found the formula which teaches us that 1 is an ordinal number less ridiculous. Even if this formula "taught" nothing to M. Poincaré, he had no grounds for judging it to be insignificant, and that for two reasons. On the one hand, this formula is sufficient to prove that the class "No" exists, and this result is not to be despised, since M. Poincaré attaches so much importance to existence-theorems and wrongly reproaches the logisticians with neglecting them. On the other hand to prove that all the finite whole numbers are ordinal numbers, we are obliged to use the principle of induction, and for that purpose to set out from the fact that 1 is such a number. However evident or trivial this fact may appear to M. Poincaré, it was important to demonstrate it, and the formula at which he mocks proves the conscientiousness and the rigor of the logisticians. The pleasantries of M. Poincaré are then quite pointless.

As for the paradox discovered by M. Burali-Forti in the theory of transfinite ordinal numbers, and from which M. Poincaré deduces an argument against logistics, I will only say that this contradiction can by no means be imputed to the use of logical symbols; and the proof of this is that mathematicians who are total strangers to logistics recognize it, discuss it, and have for years past spent vain efforts to solve it.[32] It is a purely logical difficulty which resides in the principles of the logic of classes, that is to say in the old and traditional part of logic. M. Burali-Forti, in a communication made to me,[33] believes that the contradiction arises from the different meanings that are given to the word "ordinal number," and that it depends, at bottom, on the extension and the properties attributed to the concept of *class*. Mr. Russell believes that it can only be solved by restricting or even sacrificing the notion of *class*; broadly speaking, we must give up the principle—apparently so evident and clear to intuition—that each concept determines a class which is its extension.[34] If logistics has enabled us to discover this contradiction, it can only be considered as a merit and not as a reproach for it proves that it is an instrument of precision for thought. But M. Poincaré is more exacting. He summons logistics to re-

[32] Bernstein, *Math. Ann.*, Vol. LX; Jourdain, *Phil. Mag.*, 1904-1905; Russell, *Mind*, 1905; Hadamard, Borel, Lebesgue, Baire, *Bulletin de la Société Math. de France*, 1905; Zermelo, Borel, König, Schönflies, *Math. Ann.*, Vol. LIX, LX.

[33] [Cf. for fuller details pp. 228-229 of Couturat's original paper].

[34] See Russell, "On some difficulties in the theory of transfinite numbers and order types," *Proc. Lond. Math. Soc.* (2), Vol. IV, 1905, pp. 29-53. M. Poincaré concludes hastily: "Burali-Forti and Cantor have arrived at contradictory conclusions; thus one or the other is mistaken." It cannot be said that one of them is mistaken if it is a question, as Russell shows, of a contradiction of principle, of a kind of antinomy. Thus we can see how much the conclusion is worth: "consequently pasigraphy does not preserve us from error." For the rest, logistics is only a "method of infallibility" (as Leibniz said) if certain premises are granted; it cannot be made responsible for a contradiction inherent in the premises.

solve here and now the contradiction which has become the crux of mathematicians. He says of Mr. Russell and Dr. Whitehead: "If they could have....purged it [the theory of infinite numbers] of every contradiction, they would have rendered us a signal service." The logisticians are not obliged to solve difficulties which stop all mathematicians, M. Poincaré included, and—as if they were modern Oedipuses — to reply to the riddles of all the sphinxes which are encountered in science; but if they succeed where others have failed, M. Poincaré will be good enough to remember this phrase, and do honor to logistics for the solution.

III.

I now come to the special criticisms that M. Poincaré addresses to the logisticians on the subject of their philosophy of mathematics and, in particular, of their theory of whole number. In the first place, there are certain arguments which it is astonishing to find him using, but which fortunately are not likely to impress philosophers. For example: "The definitions of number are very numerous and very varied....If one of them were satisfactory, no new ones would be given." The same objection might be urged not only to every philosophical speculation—and that is the usual argument of sceptics and positivists—but to every scientific *theory*; M. Poincaré knows this quite well. If this argument had any value, it would be the negation of all progress, even scientific progress. In mathematics in particular there exist numerous definitions of the irrational number, of the limit, of the definite integral, and so on. Has ever any one concluded from this that all these definitions are bad? Certainly not, but simply that certain ones are better than the others, without these others being properly speaking defective or wrong. For the rest, if this argument were to be taken literally, it would prove at the outside that all the definitions proposed are bad *except one,* the last. Consequently, the argument has no bearing against Mr Russell's definition as long as this definition is the last proposed.

M. Poincaré is surprised that the logisticians define arithmetical addition by means of logical addition which appears to him to rest on an act of intuition which is analogous but "more complex." But in the first place if an act of intuition is really necessary for one or the other of these operations, is it not advantageous and meritorious to define the one by the other, so as to reduce to a minimum the number of acts of intuition? Logical addition is not an invention of the logisticians; it has existed at all times and in all minds. It is the combination which the conjunction *and* expresses in the phrases, " the French and the English," "philosophers and mathematicians." Logic, even classical logic, cannot dispense with it. Thus it is not arbitrarily, as M. Poincaré seems to believe, that this notion is introduced "into the chapter headed 'Logic.'" Given that it is indispensable to logic, the whole question is to know if it can be used to define arithmetical addition. This idea is too natural for Peano and Russell to have been the first to do it; it is already clearly expressed in the work of Lambert. To refute it, M. Poincaré ought to have shown how and why arithmetical addition cannot be defined by means of logical addition, and consequently ought to have criticized Whitehead's[35] formal definition of it. Or, if he believes that the notion of logical addition is "more complex" than that of arithmetical addition, he should try to define the first by means of the second. That is the best means of proving that mathematics is independent of logic. Meanwhile, he ought to allow the logisticians to observe the classical precept that principles must not be multiplied without necessity.

[35] *Amer. Journ. of Math.,* Vol. XXIV, 1902.

M. Poincaré solemnly accuses the logisticians of having violated two rules of method. The first consists in this: Every mathematical definition supposes the existence of the object defined and is only valid on this condition. But this condition is by no means a necessary rule. It is useless to invoke the opinion of John Stuart Mill, whose authority is rather mediocre in the logic of mathematics. The condition that M. Poincaré wishes to impose on logisticians is absolutely gratuitous and is not acknowledged by the most rigorous mathematicians. A definition is no more than the giving of a name; it by no means supposes the existence of its object. We can very well define a problematical object, and then prove that it does not exist. Thus Euclid denotes by a certain sign "the greatest prime number," and then demonstrates that it does not exist. We define the derivative or the integral of a function in general, without supposing that every function has a derivative or an integral. What M. Poincaré wished to say or ought to have said is, on the other hand, that a definition does not *imply* the existence of the object defined, and this existence must be proved or postulated if we wish to be able to use it in further reasonings. This[36] is a well-known rule of mathematical method, and it is enough to run through Peano's *Formulaire* to see that each definition is accompanied, when there is occasion, by an existence-theorem which usually determines the conditions under which the object defined exists.

M. Poincaré says that "in mathematics the word *exists* can only have one meaning, it means '*is exempt from contradiction.*' " I am sorry to contradict him on so elementary and essential a point: logical—or mathematical, it is all one—existence is quite another thing from the absence of contradiction.[37] It consists in the fact that a class is not

[36] Cf. *Les Principes*, 39.

[37] It is a curious thing that this conception of logical existence only appears admissible in a *panlogism* analogous to that of Leibniz, and where the exten-

empty; that is to say that at least one member of it exists, and this means by definition that the class in question is not null. It is exactly for that reason that it is the custom of mathematicians to prove the existence of a class by giving an *example,* that is to say by indicating an individual which belongs to this class; and they have no other means for proving an existence-theorem—unless they reduce it to a preceding theorem or postulate of existence.[38]

But, it will be said, how is the existence of the individual which is used as an example proved? Must not this existence be established in order that the existence of the class of which it is a part may be deduced? Although this assertion may seem paradoxical, the existence of an individual as such is not demonstrated. The individuals, by the mere fact that they are individuals, are always considered as existing; or rather the question does not arise for them since logical existence is a property of classes and not of individuals.[39] We never have to express that an individual exists, absolutely speaking, but only that it exists in a class, that is to say, is an element of it.[40] When an individual is defined by means of general terms, its existence is demonstrated in two stages: This individual being defined as *the* u (u being a certain class), we demonstrate that the class u is not null, and then that it is a singular class. The defi-

sion of concepts would be absolutely determined by their comprehension. For example, Leibniz and his disciples believed that if "No man is a stone," that is to say, if no "men-stones" exist, it is because the concepts *man* and *stone* respectively contain contradictory elements (such as *living* and *not-living*).

[On the following discussion of the "existence" of classes and individuals, cf. my remarks in *The Monist,* Jan., 1910, Vol. XX., pp. 113-116.—Tr.]

[38] It is enough to have a proposition of the form: "x is a member of u," to be able to conclude that the class u exists.

[39] Of course, classes themselves can be considered as individuals with respect to classes of classes, but then they "exist" even when they are null.

[40] M. Poincaré thinks it necessary to add to Peano's postulate the following: "Every integer has one which follows it." He does not see that this postulate, which he believes new, is contained in the third postulate: "The consecutive of an integer is an integer." In fact, this implies that the consecutive referred to exists as an individual of a class, and even that it is unique, for otherwise we would say that the consecutives are *contained in* the class of integers.

nition of the individual is then justified.[41] But what we really demonstrate is not the existence of the individual as such but the existence of the class to which it belongs.[42]

In all of this there is no question of contradiction. What then is the relation between the existence of a class and the absence of contradiction in its definition? It consists in this: If a definition is contradictory, *no* individual fulfils its conditions, and consequently the corresponding class does not exist. Contradiction is then a purely negative criterion of existence; it is the criterion of non-existence. And reciprocally, if a class exists, that is to say contains an element, we can conclude from that, as M. Poincaré says, that its definition is not contradictory. Thus existence appears as the criterion of non-contradiction. But it is to be noticed that the relation between existence and contradiction is exactly the inverse of that which M. Poincaré affirms; it is not non-contradiction that proves existence, but it is existence that proves non-contradiction.[43]

It is then arbitrary and misleading to maintain that a definition is only valid if we first prove that it is not contradictory. Besides, it would be interesting to know how we could prove *directly* that a definition or a system of postulates is not contradictory. The presence of a contradiction can certainly be proved, but the absence of every contradiction is, like the innocence of an accused person, a negative fact which cannot be proved directly. Hilbert

[41] It must be noticed that, though the definition of a class has no need of justification (as this class may be null) the definition of an individual must be justified by the double demonstration of the *existence* and *uniqueness*. There is, then, no contradiction here.

[42] Cf. Russell, "The Existential Import of Propositions," *Mind,* July, 1905.

[43] After having written these lines, I found the same doctrine stated by G. Frege in his *Grundlagen der Arithmetik,* §§ 94, 95 (1884). "A concept is admissible, even when its marks contain a contradiction; only we must not suppose that anything falls in its extension. But from the mere fact that a concept does not contain a contradiction, we cannot conclude that something falls in its extension. . . ." (§ 94). "The non-contradiction of a concept can only be established rigorously if we prove that something falls in its extension. The inverse would be an error" (§ 95).

stated in 1900 that we can find a direct demonstration of the compatibility of the axioms of arithmetic;[44] and in 1904 he believed that he had found such a demonstration.[45] But this demonstration is not satisfactory, in the opinion of M. Poincaré himself. If "M. Hilbert hides himself," it is not "because the difficulty is too great," but because the problems which he has proposed to himself appear insoluble. M. Padoa[46] has already replied to Hilbert by recalling that, in his own theory of algebraic numbers,[47] he has demonstrated, by the exemplary method which is the only one possible, the irreducibility of his postulates and their reciprocal independence. And he concluded with this phrase: "The contradictions or the dependencies of propositions can only be demonstrated by deductive reasoning while non-contradiction or independencies of propositions can only be demonstrated by verifications (we verify that properly chosen interpretation of the symbols satisfy or do not satisfy the propositions in question)." In fact, a contradiction or a dependence is translated by a proposition of non-existence or by an implication; while a non-contradiction or an independence is translated by a proposition of existence or by a non-implication. And this difference is equivalent to that of universal and particular propositions in classical logic. We know that we can only establish really universal propositions by demonstration, but that to establish a particular proposition it is enough to cite a single case in which it is true. In general we have no other means, for we cannot deduce it from universal premises without the adjunction of some particular proposition.

[44] Communication to the second international congress of mathematicians at Paris in 1900; cf. *Bulletin of the Amer. Math. Soc.*, 1902.

[45] Communication to the third congress at Heidelberg in 1904; cf. *Monist*, July, 1905. [Hilbert's paper is reprinted on pp. 263-279 of the third edition of his *Grundlagen der Geometrie*, published at Leipsic and Berlin in 1909].

[46] "Le problème no. 2 de M. D. Hilbert," *L'Enseignement mathématique*, Vol. V, 1903, pp. 85-91.

[47] *Bibliothèque du (1er) Congrès int. de Philosophie*, Vol. III, pp. 309-365; *Revue de mathématiques*, Vol. VII, 1901, pp. 73-84.

Now, just as it is impossible to deduce a particular from universal premises—that is to say a negation from many affirmations—it is impossible to prove deductively an existence or a non-implication if we set out from non-existences or from implication. Thus the direct method that Hilbert and Poincaré recommend is impracticable. M. Poincaré has no right, then, to require of the logistician a demonstration which Hilbert could not furnish. He might just as well convict them of impotence by summoning them to take a bite out of the moon.

In default of a direct demonstration M. Poincaré suggests a very curious method of verification. To prove that a system of postulates is not contradictory, it would be necessary, according to him, to compare two by two all their consequences to prove that "there are not two which are contradictory to each other." But, as he himself immediately recognizes, this method is impracticable if the consequences to be examined are infinite in number, as is the case in arithmetic. I add that, in fact, it has never been applied. Nobody has ever seen a mathematician spend his time in comparing among themselves all the propositions of a theory to assure himself that the definitions from which he started do not contain some contradiction, and that *consequently* the entities defined really exist. Where would we be if we had to make such a verification for each new definition? But, it would be replied, it is the whole of mathematics which constitutes this verification; it is a fact that no contradiction between any two propositions has ever been met. Very well, but this verification *a posteriori* is as valid for logistics as for mathematics, since logistics merely claims to formulate the primitive principles and definitions of mathematics. For example, we may ask if the postulates by which whole number is defined are not contradictory. Logistics has only to reply: I deduce from them all the theorems of classical arithmetic; you have never found the least contradiction in these theorems when you made them rest on vague and confused intuitions; why do you wish that there should be any more contradiction in them at the present time? They are the same propositions, merely reduced deductively to some principles. In any case, the burden of proof falls on those who believe that these principles are contradictory; for contradiction may be proved, but non-contradiction may not.

The method in question is not only practicably inapplicable and unapplied in fact: it is logically illegitimate. In fact, it is not enough to bring two propositions together, to discover that they are contradictory, unless the contradiction is formal and explicit. For example, there is no formal contradiction between the two propositions: "ABCD is a non-rectangular parallelogram" and "ABCD is a quadrilateral which can be inscribed in a circle"; the contradiction only appears when we know the properties of the inscribable quadrilateral, that is to say, when the consequences of the second proposition are deduced. To bring to light the implicit and latent contradiction of two postulates, it would be necessary, accordingly, to deduce all the possible consequences (in number infinite) from these postulates. That presupposes the following definition: "Two propositions are contradictory to one another when they have consequences which are contradictory to one another." But such a definition is illogical because it contains a circle. Thus M. Poincaré's criterion of non-contradiction implies, not merely an infinite regress, but also a vicious circle.

M. Poincaré well knows that the method which he proposes is impracticable. He tries to correct it by means of the principle of induction: "Perhaps there may be a means of showing that a new reasoning cannot introduce contradiction, provided that we suppose that, in the series of preceding reasonings, we have hitherto met with none." Notice the very doubtful form in which this hypothesis is stated. Indeed, it is a hypothesis in the air, which rests on no example and on no precedent, and which seems to be invented merely to charge the logisticians with a vicious circle. Now, not only is it not true, that is to say, nobody has ever used so strange a kind of reasoning, but it is improbable. To show this, let us see what further indications M. Poincaré gives. He supposes that "a series of syllogisms" can be formed from the starting-point of the axioms as premises; then, "when we have finished the nth syllogism, we see that we can make a $(n+1)$th"; lastly, we can show that, "if there has been no contradiction at the nth syllogism, there will not be at the $(n+1)$th." All these hypotheses are absolutely gratuitous and contrary to all probability. In the first place, mathematical reasonings do not, in general, consist in a *linear* series of syllogisms; otherwise the type of mathematical reasoning would be the *sorites.* Must we repeat that the syllogism is by no means the only type of deduction, and that there are many other logical principles or rules which enter into reasoning? Then, the simple deductions which compose a reasoning to not arrange themselves in a linear series, as M. Poincaré imagines; the image of mathematical reasoning is not a chain but rather a genealogical tree.[48] What, then, does the *number* of reasonings made at a given moment signify if their linear order is always more or less arbitrary, and arises solely from the practical necessity of enunciating them in speech, because time has only one dimension? "The number n serves to count a series of successive operations," says M. Poincaré; what becomes of his argument if these operations are not successive or are only so by accident? Can we affirm that this number n exists at each instant and that it is well determined? We can count simple deductions if they all reduce to the type of the syllogism; but

[48] Cf. *Les Principes,* p. 286 [and the long note on p. 238 of M. Couturat's article, which we have not here translated. It contains the detailed writing out of a simple theorem in the mathematical logic of Peano].

how are we to count heterogeneous deductions which proceed from various rules? Will it be said that each application of a logical principle constitutes a unity? But, besides the fact that all the principles have not the same deductive importance, it can happen that many principles occur at once in an elementary deduction. That is what happens, notably, when one intervenes as premise and the other as a rule of deduction. All this proves that the *number* of deductions, whether syllogistic or not, has no objective reality, and that any numbering of them is arbitrary. Consequently, how are we to admit that a proposition depending on this number can be established and concluded from n to $n+1$? And then M. Poincaré relies on his hypothetical case to attribute to the logisticians a vicious circle which they have not committed. "With an *if*," says common sense, "we could put Paris in a bottle." It is with an *if* that M. Poincaré arrives at attributing a paralogism to the logisticians.

Unfortunately, M. Poincaré seems to forget elsewhere all his *if*'s when he asserts categorically that the principle of induction is necessarily used in every demonstration of the compatibility of the axioms of arithmetic or of any system of axioms. For example, he says: "We must have recourse to processes of demonstration where, in general, we have to use the same principle of complete induction which is the one to be verified." Would it not be believed that the fantastic method which he proposes is in current use? Elsewhere he cites it as one of the "possible applications of the principle of induction," as if this application had been actually made. Finally he says, on the subject of the theorem of Bernstein: "If ever another demonstration is invented, it must still rest on this principle, since the possible consequences of the axioms which are to be shown to be non-contradictory are infinite in number." Thus, it is enough that we have to do with an infinity of propositions (or of any objects) in order that, according to M. Poincaré, the principle of induction *necessarily* intervenes. He has quite forgotten that the application of this principle proposed by him is subject to extremely restrictive hypotheses.

At the bottom, he seems to confuse mathematical induction with induction pure and simple. For how are we to conceive that from the absence of contradiction in a series of reasonings we can infer the absence of contradiction in the following reasonings? Without doubt, if this inference was certain and could be expressed by the precise formula: "If no contradiction has been found in the first n reasonings it will not be found in the $n+1$ first ones," there would be an occasion to apply the principle of induction, and the conclusion would be equally certain. But the inference in question can at most only be probable, and consequently it only constitutes a common induction and not a mathematical induction. To borrow an example from M. Poincaré, the geometry of Lobachevski, since it only comprises a finite number of theorems, did not absolutely prove that the postulate of Euclid is independent of the other geometrical axioms (that is to say that its negation is compatible with them); it only gave this proposition a probability which was greater as the number of theorems of the new geometry became greater. But there is always an abyss between a probability, however great it may be, and an apodictic certainty. Now, the results of common induction are characterized by probability, while mathematical induction is a rigorous process which engenders certainty. If then the inference that is drawn from reasonings already made to future reasonings has only a probable value (as common sense—that "sure instinct" to which M. Poincaré refers—thinks), it rests on an induction pure and simple and not on the principle of mathematical induction.[49]

[49][The fourth section of M. Couturat's paper occupies pp. 241-247, and contains a detailed refutation of M. Poincaré's remark that Mr. Russell had not demonstrated the existence of the integers. M. Poincaré's opinion rested

The second principal fault with which M. Poincaré charges the logisticians consists in that they surreptitiously change a definition: "You give a subtle definition of number and then you think nothing more about it. . . . and when the word 'number' is found farther on, you attach the same meaning to it as the first comer would. . . . Here is a word of which we have given an explicit definition A. We then make use of it, in discourse, in such a way that it implicitly supposes another definition B." That is a very serious reproach that must not be urged without proof against logicians so rigorous and so practised as Peano and his collaborators. Now M. Poincaré gives no proof and confines himself to general reflections on method which affect logisticians less than anybody else, for there is continually in these reflections a question of "words" and of "phrases." Mathematicians who reason with words and phrases are doubtless liable to attribute to a term, instead of the meaning assigned to it by its definition, the meaning which current use gives it. But it is exactly to avoid these illogical associations and implications that the logisticians use symbols whose meaning is solely determined by their formal relations, and which are manipulated in virtue of formal rules of calculation. Has M. Poincaré already forgotten that he reproached logisticians with reducing reasoning to a blind mechanism, that is to say, with neglecting the meaning of their symbols? "To demonstrate a theorem it is not necessary nor even useful to know what it means"; "the mathematician has no need of understanding what he

partly on a misreading and partly on the fact that, in M. Couturat's popular book, the question of existence was rather neglected in comparison with Mr. Russell's work. However, in Mr. Russell's early work, while *existence* was treated at length, the far more important question of *entity* was not considered. Thus the justification of Mr. Russell's early existence-theorems does not now appear to be quite satisfactory, and accordingly is here left untranslated. The second part of the fourth section is also untranslated here. It contains a refutation of M. Poincaré's hasty judgment that the principle of induction is not the definition of finite number, and is slightly more technical than the rest of M. Couturat's paper. What follows is, in essentials, M. Couturat's fifth section.]

does." The two reproaches are contradictory; let M. Poincaré leave to the logisticians at least the advantage of "the logical correction of reasonings" which compensates for its "formal" and almost "unintelligent" character. In any case, all the general and vague reasons which he alleges to support his criticism return against it, for they tend to prove that the logisticians are exempt from the causes of error which he points out.

I have long sought in the articles of M. Poincaré for the precise proofs of his accusation. I believe that I have found one, and yet I am not quite sure. M. Poincaré reproaches Mr. Russell with using two different formulae of the principle of induction, and with confusing them illegitimately: "A number may be defined by recurrence; on this number we may reason by recurrence: these are two distinct propositions. The principle of induction does not teach us that the first is true, it teaches us that the first implies the second." He says again: "The principle of induction does not mean that every whole number can be obtained by successive additions; it means that, for all the numbers that can be obtained by successive additions, we can demonstrate any property by recurrence." In the first place, the expression "successive additions" is not precise. The question necessarily arises, "How many additions?", and the reply is, "a finite number"; but the finite numbers are characterized by the principle of induction. Consequently, M. Poincaré's proposed enunciation means: "For all the numbers which can be defined by recurrence (or by complete induction), we can demonstrate a property by recurrence." Now that is a wholly analytical proposition, and almost a tautology: "All the numbers which verify the principle of induction verify the principle of induction." If this were the formula of the principle of induction, it would be an analytic judgment, and not a synthetic judgment as M. Poincaré maintains.

But that is not the true formula of the principle of induction, and it is incomprehensible how a mathematician like M. Poincaré could have made such a mistake. It is not with him just an airy remark, for he returns to this important question at the end of his second article and gives precise expression to his thoughts in the following terms: "A whole number is that which can be obtained by successive additions, it is that which can be defined by recurrence.... A whole number is that on which we can reason by recurrence....The two definitions are not identical; without doubt they are equivalent, but they are so in virtue of a synthetic *a priori* judgment; we cannot pass from one to the other by purely logical processes."[50]

Will it be said that the logisticians have invented a new enunciation of the principle of induction, which they set up against the classical enunciation? By no means, they have only translated the traditional enunciation into symbols. And what is still stronger, M. Poincaré himself quoted this traditional enunciation at the beginning of his first article: "We know the enunciation of this principle. If a property is true of a number 1,[51] and if we establish that it is true of $n+1$ provided that it is true of n, it will be true of all the whole numbers." Now that is one of the verbal translations of the formula of the principle of induction.[52] M. Poincaré cannot then dispute the exactness of the symbolic formula. Thus he accuses the logisticians of surreptitiously changing a definition; and it is he himself who, in one and the same article, changes the definition, or rather the enunciation, of the principle of induction!

[50] [M. Couturat, on p. 249 of his article, formulates these two definitions in symbols, and shows that the passage from one to the other is effected by a process as analytic as the passage from the proposition, "Pompey is one of the x's such that Cæsar conquered x," to the proposition, "Pompey was conquered by Cæsar," or, "Cæsar conquered Pompey."]

[51] Or of the number 0; that comes to the same thing here.

[52] It is one of the verbal translations that I have given in *Les Principes*, p. 55.

To prove his accusation he himself commits the paralogism which he wrongly attributes to them, and all his reproaches of illogicality fall on himself alone. If I had the wit of M. Poincaré, I would say that his "adventure" is quite as instructive as that of M. Burali-Forti, and that it ought to "warn" the adversaries of logistics of the necessity of being circumspect.

I will not bring up the conclusion of the articles of M. Poincaré because I do not see the utility of carrying the discussion into history where it is complicated by questions of interpretation. The controversy is not "between Kant and Leibniz,"[53] but between M. Poincaré and the logisticians. Besides, the question, as M. Poincaré has put it, is not a question of general philosophy or of epistemology, but of pure logic. Admitting the principles and the primitive ideas of the logisticians, M. Poincaré has maintained that, setting out from these data, they cannot build up mathematics without another postulate—an appeal to intuition or a synthetic *a priori* judgment; and he has thought that he has discovered in their *logical* construction certain paralogisms (beggings of the question or vicious circles). I believe that I can conclude from the above discussion that not one of these theses is proved, and that, in particular, the logisticians have not committed any of the logical errors that are so lightly imputed to them. I have too high an idea of the wit and the character of M. Poincaré not to believe that he will form a more just and more favorable opinion of logistics...when he has studied it.

LOUIS COUTURAT.

PARIS, FRANCE.

[53] [*Monist,* April, 1912, Vol. XXII, p. 256.]

THE LATEST EFFORTS OF THE LOGISTICIANS.[1]

THE logicians have attempted to answer the preceding[2] considerations. For that, a transformation of logistic was necessary, and Russell in particular has modified on certain points his original views. Without entering into the details of the debate, I should like to return to the two questions to my mind most important: Have the rules of logistic demonstrated their fruitfulness and infallibility? Is it true they afford means of proving the principle of complete induction without any appeal to intuition?

THE INFALLIBILITY OF LOGISTIC.

On the question of fertility, it seems M. Couturat has naive illusions. Logistic, according to him, lends invention "stilts and wings," and on the next page: "*Ten years ago,* Peano published the first edition of his *Formulaire.*" How is that, ten years of wings and not to have flown!

I have the highest esteem for Peano, who has done very pretty things (for instance his "space-filling curve," a phrase now discarded); but after all he has not gone further nor higher nor quicker than the majority of wingless mathematicians, and would have done just as well with his legs.

On the contrary I see in logistic only shackles for the inventor. It is no aid to conciseness—far from it, and if

[1] Translated by George Bruce Halsted.

[2] "The New Logics," in *The Monist,* April, 1912.

twenty-seven equations were necessary to establish that 1 is a number, how many would be needed to prove a real theorem? If we distinguish, with Whitehead, the individual *x*, the class of which the only member is *x* and which shall be called l *x*, then the class of which the only member is the class of which the only member is *x* and which shall be called ll *x*, do you think these distinctions, useful as they may be, go far to quicken our pace?

Logistic forces us to say all that is ordinarily left to be understood; it makes us advance step by step; this is perhaps surer but not quicker.

It is not wings you logisticians give us, but leading-strings. And then we have the right to require that these leading-strings prevent our falling. This will be their only excuse. When a bond does not bear much interest, it should at least be an investment for a father of a family.

Should your rules be followed blindly? Yes, else only intuition could enable us to distinguish among them; but then they must be infallible; for only in an infallible authority can one have a blind confidence. This, therefore, is for you a necessity. Infallible you shall be, or not at all.

You have no right to say to us: "It is true we make mistakes, but so do you." For us to blunder is a misfortune, a very great misfortune; for you it is death.

Nor may you ask: Does the infallibility of arithmetic prevent errors in addition? The rules of calculation are infallible, and yet we see those blunder *who do not apply these rules*; but in checking their calculation it is at once seen where they went wrong. Here it is not at all the case; the logicians *have applied* their rules, and they have fallen into contradiction; and so true is this, that they are preparing to change these rules and to "sacrifice the notion of class." Why change them if they were infallible?

"We are not obliged," you say, "to solve *hic et nunc* all possible problems." Oh, we do not ask so much of you.

If, in face of a problem, you would give *no* solution, we should have nothing to say; but on the contrary you give us *two* of them and those contradictory, and consequently at least one false; this it is which is failure.

Russell seeks to reconcile these contradictions, which can only be done, according to him, "by restricting or even sacrificing the notion of class." And M. Couturat, discovering the success of his attempt, adds: "If the logicians succeed where others have failed, M. Poincaré will remember this phrase, and give the honor of the solution to logistic."

But no! Logistic exists, it has its code which has already had four editions; or rather this code is logistic itself. Is Mr. Russell preparing to show that one at least of the two contradictory reasonings has transgressed the code? Not at all; he is preparing to change these laws and to abrogate a certain number of them. If he succeeds, I shall give the honor of it to Russell's intuition and not to the Peanian logistic which he will have destroyed.

THE LIBERTY OF CONTRADICTION.

I made two principal objections to the definition of whole number adopted in logistic. What says M. Couturat to the first of these objections?

What does the word *exist* mean in mathematics? It means, I said, to be free from contradiction. This M. Couturat contests. "Logical existence," says he, "is quite another thing from the absence of contradiction. It consists in the fact that a class is not empty." To say: a's exist, is, by definition, to affirm that the class a is not null.

And doubtless to affirm that the class a is not null, is, by definition, to affirm that a's exist. But one of the two affirmations is as denuded of meaning as the other, if they do not both signify, either that one may see or touch *a*'s which is the meaning physicists or naturalists give them, or that one may conceive an *a* without being drawn into contradictions, which is the meaning given them by logicians and mathematicians.

For M. Couturat, "it is not non-contradiction that proves existence, but it is existence that proves non-contradiction." To establish the existence of a class, it is necessary therefore to establish, by an *example,* that there is an individual belonging to this class: "But, it will be said, how is the existence of this individual proved? Must not this existence be established, in order that the existence of the class of which it is a part may be deduced? Well, no; however paradoxical may appear the assertion, we never demonstrate the existence of an individual. Individuals, just because they are individuals, are always considered as existent....We never have to express that an individual exists, absolutely speaking, but only that it exists in a class." M. Couturat finds his own assertion paradoxical, and he will certainly not be the only one. Yet it must have a meaning. It doubtless means that the existence of an individual, alone in the world, and of which nothing is affirmed, cannot involve contradiction; in so far as it is all alone it evidently will not embarrass any one. Well, so let it be; we shall admit the existence of the individual, "absolutely speaking," but nothing more. It remains to prove the existence of the individual "in a class" and for that it will always be necessary to prove that the affirmation, "Such an individual belongs to such a class," is neither contradictory in itself, nor to the other postulates adopted.

"It is then," continues M. Couturat, "arbitrary and misleading to maintain that a definition is valid only if we first prove it is not contradictory." One could not claim in prouder and more energetic terms the liberty of contradiction. "In any case, the *onus probandi* rests upon those who believe that these principles are contradictory." Postulates are presumed to be compatible until the contrary is proved, just as the accused person is presumed innocent. Needless to add that I do not assent to this claim. But, you say, the demonstration you require of us is impossible, and you cannot ask us to jump over the moon. Pardon me; that is impossible for you but not for us, who admit the principle of induction as a synthetic judgment *a priori*. And that would be necessary for you, as for us.

To demonstrate that a system of postulates implies no contradiction, it is necessary to apply the principle of complete induction; this mode of reasoning not only has nothing "bizarre" about it, but it is the only correct one. It is not "unlikely" that it has ever been employed; and it is not hard to find "examples and precedents" of it. I have cited two such instances borrowed from Hilbert's article. He is not the only one to have used it and those who have not done so have been wrong. What I have blamed Hilbert for is not his having recourse to it (a born mathematician such as he could not fail to see a demonstration was necessary and this the only one possible), but his having recourse without recognizing the reasoning by recurrence.

THE SECOND OBJECTION.

I pointed out a second error of logistic in Hilbert's article. To-day Hilbert is excommunicated and M. Couturat no longer regards him as of the logistic cult; so he asks if I have found the same fault among the orthodox. No, I have not seen it in the pages I have read; I know not whether I should find it in the three hundred pages they have written which I have no desire to read.

Only, they must commit it the day they wish to make any application of mathematics. This science has not as sole object the eternal contemplation of its own navel; it has to do with nature and some day it will touch it. Then it will be necessary to shake off purely verbal definitions and to stop paying oneself with words.

To go back to the example of Hilbert: always the point at issue is reasoning by recurrence and the question of knowing whether a system of postulates is not contradictory. M. Couturat will doubtless say that then this does not touch him, but it perhaps will interest those who do not claim, as he does, the liberty of contradiction.

We wish to establish, as above, that we shall never encounter contradiction after any number of deductions whatever, provided this number be finite. For that, it is necessary to apply the principle of induction. Should we here understand by finite number every number to which by definition the principle of induction applies? Evidently not, else we should be led to most embarrassing consequences. To have the right to lay down a system of postulates, we must be sure they are not contradictory. This is a truth admitted by *most* scientists; I should have written *by all* before reading M. Couturat's last article. But what does this signify? Does it mean that we must be sure of not meeting contradiction after a *finite* number of propositions, the *finite* number being by definition that which has all properties of recurrent nature, so that if one of these properties fails—if, for instance, we come upon a contradiction—we shall agree to say that the number in question is not finite? In other words, do we mean that we must be sure not to meet contradictions, on condition of agreeing to stop just when we are about to encounter one? To state such a proposition is enough to condemn it.

So, Hilbert's reasoning not only assumes the principle of induction, but it supposes that this principle is given us not as a simple definition, but as a synthetic judgment *a priori.*

To sum up:

A demonstration is necessary.

The only demonstration possible is the proof by recurrence.

This is legitimate only if we admit the principle of induction and if we regard it not as a definition but as a synthetic judgment.

THE CANTOR ANTINOMIES.

Now to examine Russell's new memoir. This memoir was written with the view to conquer the difficulties raised by those Cantor antinomies to which frequent allusion has already been made. Cantor thought he could construct a science of the infinite; others went on in the way he opened, but they soon ran foul of strange contradictions. These antinomies are already numerous, but the most celebrated are:

1. The Burali-Forti antinomy;
2. The Zermelo-König antinomy;
3. The Richard antinomy.

Cantor proved that the ordinal numbers (the question is of transfinite ordinal numbers, a new notion introduced by him) can be ranged in a linear series, that is to say that of two unequal ordinals one is always less than the other. Burali-Forti proves the contrary; and in fact he says in substance that if one could range *all* the ordinals in a linear series, this series would define an ordinal greater than *all* the others; we could afterwards adjoin 1 and would obtain again an ordinal which would be *still greater,* and this is contradictory.

We shall return later to the Zermelo-König antinomy which is of a slightly different nature. The Richard antinomy (*Revue générale des sciences*, June 30, 1905) is as follows: Consider all the decimal numbers definable by a finite number of words; these decimal numbers form an aggregate E, and it is easy to see that this aggregate is countable, that is to say we can *number* the different decimal numbers of this assemblage from 1 to infinity. Suppose the numbering effected, and define a number N as follows: If the *n*th decimal of the *n*th number of the assemblage E is

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

the *n*th decimal of N shall be:

1, 2, 3, 4, 5, 6, 7, 8, 1, 1

As we see, N is not equal to the *n*th number of E, and as *n* is arbitrary, N does not appertain to E and yet N should belong to this assemblage since we have defined it with a finite number of words.

We shall later see that M. Richard has himself given with much sagacity the explanation of his paradox and that this extends, *mutatis mutandis,* to the other like paradoxes. Again, Russell cites another quite amusing paradox: *What is the least whole number which cannot be defined by a phrase composed of less than a hundred English words?*

This number exists; and in fact the numbers capable of being defined by a like phrase are evidently finite in number since the words of the English language are not infinite in number. Therefore among them will be one less than all the others. And, on the other hand, this number does not exist, because its definition implies contradiction. This number in fact is defined by the phrase in italics which is composed of less than a hundred English words; and by definition this number should not be capable of definition by a like phrase.

ZIGZAG THEORY AND NO-CLASS THEORY.

What is Mr. Russell's attitude in presence of these contradictions? After having analyzed those of which we have just spoken, and cited still others, after having given them a form recalling Epimenides, he does not hesitate to conclude: "A propositional function of one variable does not always determine a class." A propositional function (that is to say a definition) does not always determine a class. A "propositional function" or "norm" may be "non-predicative." And this does not mean that these non-predicative propositions determine an empty class, a null class; this does not mean that there is no value of x satisfying the definition and capable of being one of the elements of the class. The elements exist, but they have no right to unite in a syndicate to form a class.

But this is only the beginning and it is needful to know how to recognize whether a definition is or is not predicative. To solve this problem Russell hesitates between three theories which he calls

A. The zigzag theory;

B. The theory of limitation of size;

C. The no-class theory.

According to the zigzag theory "definitions (propositional functions) determine a class when they are simple and cease to do so when they are complicated and obscure." Who, now, is to decide whether a definition may be regarded as simple enough to be acceptable? To this question there is no answer, if it be not the loyal avowal of a complete inability: "The rules which enable us to recognize whether these definitions are predicative would be extremely complicated and cannot commend themselves by any plausible reason. This is a fault which might be remedied by greater ingenuity or by using distinctions not yet pointed out. But hitherto in seeking these rules, I have not been able to find any other directing principle than the absence of contradiction."

This theory therefore remains very obscure; in this night a single light—the word zigzag. What Russell calls the "zigzaginess" is doubtless the particular characteristic which distinguishes the argument of Epimenides.

According to the theory of limitation of size, a class would cease to have the right to exist if it were too extended. Perhaps it might be infinite, but it should not be too much so. But we always meet again the same difficulty; at what precise moment does it begin to be too much so? Of course this difficulty is not solved and Russell passes on to the third theory.

In the no-classes theory it is forbidden to speak the word "class" and this word must be replaced by various periphrases. What a change for logistic which talks only of classes and classes of classes! It becomes necessary to remake the whole of logistic. Imagine how a page of logistic would look upon suppressing all the propositions where it is a question of class. There would only be some scattered survivors in the midst of a blank page. *Apparent rari nantes in gurgite vasto.*

Be that as it may, we see how Russell hesitates and the modifications to which he submits the fundamental principles he has hitherto adopted. Criteria are needed to decide whether a definition is too complex or too extended, and these criteria can only be justified by an appeal to intuition.

It is toward the no-classes theory that Russell finally inclines. Be that as it may, logistic is to be remade and it is not clear how much of it can be saved. Needless to add that Cantorism and logistic are alone under consideration; real mathematics, that which is good for something, may continue to develop in accordance with its own principles without bothering about the storms which rage outside it, and go on step by step with its usual conquests which are final and which it never has to abandon.

THE TRUE SOLUTION.

What choice ought we to make among these different theories? It seems to me that the solution is contained in a letter of M. Richard of which I have spoken above, to be found in the *Revue générale des sciences* of June 30, 1905. After having set forth the antinomy we have called Richard's antinomy, he gives its explanation. Recall what has already been said of this antinomy. E is the aggregate of *all* the numbers definable by a finite number of words, *without introducing the notion of the aggregate E itself.* Else the definition of E would contain a vicious circle; we must not define E by the aggregate E itself.

Now we have defined N with a finite number of words, it is true, but with the aid of the notion of the aggregate E. And this is why N is not part of E. In the example selected by M. Richard, the conclusion presents itself with complete evidence and the evidence will appear still stronger on consulting the text of the letter itself. But the same explanation holds good for the other antinomies, as is easily verified. Thus *the definitions which should be regarded as not predicative are those which contain a vicious circle.* And the preceding examples sufficiently show what I mean by that. Is it this which Russell calls the "zigzaginess"? I put the question without answering it.

THE DEMONSTRATIONS OF THE PRINCIPLE OF INDUCTION.

Let us now examine the pretended demonstrations of the principle of induction and in particular those of Whitehead and of Burali-Forti.

We shall speak of Whitehead's first, and take advantage of certain new terms happily introduced by Russell in his recent memoir. Call *recurrent class* every class containing zero, and containing $n+1$ if it contains n. Call *inductive number* every number which is a part of *all* the recurrent classes. Upon what condition will this latter definition, which plays an essential rôle in Whitehead's proof, be "predicative" and consequently acceptable?

In accordance with what has been said, it is necessary to understand by *all* the recurrent classes, all those in whose definition the notion of inductive number does not enter. Else we fall again upon the vicious circle which has engendered the antinomies.

Now *Whitehead has not taken this precaution.* Whitehead's reasoning is therefore fallacious; it is the same which led to the antinomies. It was illegitimate when it gave false results; it remains illegitimate when by chance it leads to a true result.

A definition containing a vicious circle defines nothing. It is of no use to say, we are sure, whatever meaning we may give to our definition, zero at least belongs to the class of inductive numbers; it is not a question of knowing whether this class is void, but whether it can be rigorously deliminated. A "non-predicative" class is not an empty class, it is a class whose boundary is undetermined. Needless to add that this particular objection leaves in force the general objections applicable to all the demonstrations.

* * *

Burali-Forti has given another demonstration.[3] But he is obliged to assume two postulates: First, there always exists at least one infinite class. The second is thus expressed:

$$u \epsilon \mathrm{K}(\mathrm{K} - \iota\Lambda) \,.\, \supset \,.\, u < v'u.$$

The first postulate is not more evident than the principle to be proved. The second not only is not evident, but it is false, as Whitehead has shown; as moreover any recruit would see at the first glance, if the axiom had been

[3] In his article "Le classi finite," *Atti di Torino,* Vol. XXXII.

stated in intelligible language, since it means that the number of combinations which can be formed with several objects is less than the number of these objects.

ZERMELO'S ASSUMPTION.

A famous demonstration by Zermelo rests upon the following assumption: In any aggregate (or the same in each aggregate of an assemblage of aggregates) we can always choose *at random* an element (even if this assemblage of aggregates should contain an infinity of aggregates). This assumption had been applied a thousand times without being stated, but, once stated, it aroused doubts. Some mathematicians, for instance M. Borel, resolutely reject it; others admire it. Let us see what, according to his last article, Russell thinks of it. He does not speak out, but his reflections are very suggestive.

And first a picturesque example: Suppose we have as many pairs of shoes as there are whole numbers, and so that we can number *the pairs* from one to infinity, how many shoes shall we have? Will the number of shoes be equal to the number of pairs? Yes, if in each pair the right shoe is distinguishable from the left; it will in fact suffice to give the number $2n-1$ to the right shoe of the nth pair, and the number $2n$ to the left shoe of the nth pair. No, if the right shoe is just like the left, because a similar operation would become impossible—unless we admit Zermelo's assumption, since then we could choose *at random* in each pair the shoe to be regarded as the right.

CONCLUSIONS.

A demonstration truly founded upon the principles of analytic logic will be composed of a series of propositions. Some, serving as premises, will be identities or definitions; the others will be deduced from the premises step by step.

But though the bond between each proposition and the following is immediately evident, it will not at first sight appear how we get from the first to the last, which we may be tempted to regard as a new truth. But if we replace successively the different expressions therein by their definition and if this operation be carried as far as possible, there will finally remain only identities, so that all will reduce to an immense tautology. Logic therefore remains sterile unless made fruitful by intuition.

This I wrote long ago; logistic professes the contrary and thinks it has proved it by actually proving new truths. By what mechanism? Why in applying to their reasonings the procedure just described—namely, replacing the terms defined by their definitions—do we not see them dissolve into identities like ordinary reasonings? It is because this procedure is not applicable to them. And why? Because their definitions are not predicative and present this sort of hidden vicious circle which I have pointed out above; non-predicative definitions cannot be substituted for the terms defined. Under these conditions *logistic is not sterile, it engenders antinomies.*

It is the belief in the existence of the actual infinite which has given birth to these non-predicative definitions. Let me explain. In these definitions the word "all" figures, as is seen in the examples cited above. The word "all" has a very precise meaning when it is a question of an infinite number of objects; to have another one, when the objects are infinite in number, would require there being an actual (given complete) infinity. Otherwise *all* these objects could not be conceived as postulated anteriorly to their definition and then if the definition of a notion N depends upon *all* the objects A, it may be infected with a vicious circle, if among the objects A are some indefinable without the intervention of the notion N itself.

The rules of formal logic express simply the properties of all possible classifications. But for them to be applicable it is necessary that these classifications be immutable and that we have no need to modify them in the course of the reasoning. If we have to classify only a finite number of objects, it is easy to keep our classifications without change. If the objects are *indefinite* in number, that is to say if one is constantly exposed to seeing new and unforeseen objects arise, it may happen that the appearance of a new object may require the classification to be modified, and thus it is we are exposed to antinomies. *There is no actual* (*given complete*) *infinity.* The Cantorians have forgotten this, and they have fallen into contradiction. It is true that Cantorism has been of service, but this was when applied to a real problem whose terms were precisely defined, and then we could advance without fear.

Logistic also forgot it, like the Cantorians, and encountered the same difficulties. But the question is to know whether they went this way by accident or whether it was a necessity for them. For me, the question is not doubtful; belief in an actual infinity is essential in the Russell logic. It is just this which distinguishes it from the Hilbert logic. Hilbert takes the view-point of extension, precisely in order to avoid the Cantorian antinomies. Russell takes the view-point of comprehension. Consequently for him the genus is anterior to the species, and the *summum genus* is anterior to all. That would not be inconvenient if the *summum genus* was finite; but if it is infinite, it is necessary to postulate the infinite, that is to say to regard the infinite as actual (given complete). And we have not only infinite classes; when we pass from the genus to the species in restricting the concept by new conditions, these conditions are still infinite in number. Because they express generally that the envisaged object presents such or such a relation with all the objects of an infinite class.

But that is ancient history. Russell has perceived the peril and takes counsel. He is about to change everything, and, what is easily understood, he is preparing not only to introduce new principles which shall allow of operations formerly forbidden, but he is preparing to forbid operations he formerly thought legitimate. Not content to adore what he burned, he is about to burn what he adored, which is more serious. He does not add a new wing to the building, he saps its foundation.

The old logistic is dead, so much so that already the zigzag theory and the no-classes theory are disputing over the succession. To judge of the new, we shall await its coming.

HENRI POINCARÉ.

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THE PHILOSOPHY OF RELATIVITY

IN THE LIGHT OF THE PHILOSOPHY OF SCIENCE.

Objectivity.

SINCE the dawn of civilization man has groped after truth. He has investigated it; he has pondered on it; he has made guesses and proposed hypotheses; he has approximated truth by allegories, foreshadowing it in verse and fable; and since he began to count and to measure he has reduced the results of his inquiry to exact statements.

All observations are necessarily subjective, but man is not satisfied with subjective truth, he wants objective truth and objectivity of statement is the ideal of science.

Is objectivity impossible? Must we abandon our ideal of science? It seems to us that science has more and more in its various fields approached its ideal of objective truth. Standard measures have been invented and perfected. Time is measured by a pendulum of definite size, even apparently trivial factors have been considered such as latitude and altitude; and our precision machines testify to the ingenuity of man's genius in his attempt to eliminate personal equations as much as possible. The reliability of scientific computation has reached a marvelous degree, but it is almost more astonishing that we are still dissatisfied and that our measurements of minute fractions of the wave lengths of light are no longer exact enough for our needs.

In the face of the enormous accomplishments of science in approximating the ideal of objectivity, a new school has risen which goes so far as to deny all objectivity, and insisting upon the truth of relativity, it would make us believe that objectivity is a phantom.

The relativity principle was first pronounced by Einstein in the *Jahrbuch der Radioaktivität* (Vol. IV, pp. 411 ff., 1907). It was invented to account for certain difficulties in the explanation of optical and electrical phenomena by considering the relativity of the movements in a system that is not at rest, called a disturbed system in contrast to quiet systems. In all quiet systems the common laws of dynamics hold good and the proposition of the relativity principle has been made for the sake of accounting for the laws of disturbed systems.

The principle of relativity is an *a priori* postulate from which certain theorems are derived whose truth is to be verified or refuted by experiment. Mr. Norman Campbell says:[1]

"The principle is what is more often termed a 'theory' —that is to say, it is a set of propositions from which experimental laws may be logically deduced. It can be proved to be true or false in a manner convincing to everybody only by comparing the laws so deduced with those found experimentally; but a theory which never conflicted with experiment might yet (as I hold) be judged objectionable on other grounds, and, conversely, a theory which was not in complete accord with experiment might yet be judged satisfactory."

Among the postulates of the principle of relativity there is one, counted the second, which presents great difficulties. It proclaims that "The velocity of light determined by all observers who are not accelerated relatively to each other is the same whatever may be the relative velocities of the observer."

[1] See "The Common Sense of Relativity" in *The Philosophical Magazine* for April 1911, pp. 502 ff.

An unsophisticated thinker would naturally assume that the velocity of light must be expected to increase or decrease according to the velocity of the observer. But the relativist assures us that light is an exception; on his assumption light is like a shadow whose motion depends upon the motion of its body representing the observer. The relation of the shadow to its body remains the same, however its body's (the observer's) velocity may change.

The question as to the velocity of light is a question of physics, not of philosophy, and we will touch upon it later. Here we will state only that the main objection to the relativity principle is the inference which implicates our objective ideal of science.

Not all the relativists agree on all points of their doctrine, and contradictory statements are not uncommon. We can here only characterize the general tendency and will not enter into the individual interpretations too closely.

Relativists try to avoid a difficulty which we grant exists, but is not insurmountable. Idealists of former days have used more subtle methods to dispose of the belief in objectivity of things, of time, and of space. They have produced only quibbles and the relativists have succeeded no better; only it is strange that the movement has originated among the physicists.

In a former article[2] we have demonstrated the paramount importance of relativity, but for all that we see no necessity for abandoning the old ideal of science. On the contrary we feel inclined to insist on it more strongly than ever. We do not deny the relativity of all existence throughout and without exception, but we still cling to the old scientific ideal of objectivity and we can not see that the relativity principle, in the one-sided sense in which the relativity physicists uphold it, is well established.

Having discussed in the article mentioned the part

[2] "The Principle of Relativity," *Monist,* April, 1912.

which relativity plays in scientific method, we feel inclined to add a few suggestions concerning the significance of the recent movement among physicists who emphasize the principle of relativity and prophesy that through it a new era in the scientific interpretation of the world will have to begin.

We have seen that many of the paradoxes which are proclaimed by the relativity physicists disappear on close inspection, for the contradictions resolve themselves into purely verbal contrasts. The same object is not in itself longer or shorter, but the result of measurement will be different according to the conditions under which the measurements take place. And further, although time can be eliminated, although it may be treated as a function of space, or even be treated as a kind of fourth dimension, the conception of time will nevertheless still remain of great convenience. The truth is that we must subsume time and space under one common category which, with Kant and other thinkers of well-established classical tradition since the days of Aristotle, has been called "form." We must always bear in mind the interrelation between time and space and view the two as the forms of one and the same reality. Time is the form of doing, of progressive action, of change, of events, and space is the form of being, of existence in its juxtaposition of parts. The former is the order of procedure in which the latter is transformed. Neither can be thought without the other, and the two are one. The principle of simplicity requires us to consider both in their interrelation. But for all that the traditional notion of time still proves the best method for rendering measurements of changes intuitively clear while an elimination of time as proposed by the Relativity Physicists is apt to obscure the issue; and we come to the conclusion that experience has not without good reasons found in the proper terms "space" and "time" a very convenient, yea, as it seems to me, the most appropriate, mode of representation.

It is strange that the relativity principle has been proposed for the very purpose of approximating objective truth with greater exactness, but instead of accounting for inexactness or inaccuracies in results and for apparent contradictions by taking into consideration the mistakes in calculation on account of the shifting conditions of this world which is a constant flux, a *panta rhei,* the leaders of the new movement cancel the old ideal of science which has guided us thus far and propose a new standard strongly tinged with subjectivism, built upon the basis of the relativity of all existence.

All experience is a mixture of objectivity and subjectivity: it is due to the interrelation between a sentient subject and the sensed objects. So far science has tried to eliminate the subjective side, the personal equation, while the relativity physicists deny the legitimacy of the ideal of objectivity, or as they call it, the concept of the real. It is true that in clinging to the facts of observation without trying to eliminate the subjective elements and thereby to unify our results in an objective statement, we simplify our calculations, but it is very doubtful whether this procedure can be generally applied to other than optical and electrical phenomena. Relativists deem the theory justified if they simplify their own line of labors. Mr. Campbell exclaims in his enthusiasm:

"Anything more beautifully straightforward it would be hard to conceive. Not only is the result magnificently simple, but it furnishes us with a mathematical instrument of extraordinary power. In place of the elaborate calculations which have hitherto been necessary in dealing with moving systems, all that we have to do now is to solve the problem under consideration for the limiting case of infinitesimal velocity, and then effect a mere algebraical transformation. The only objection that seems likely to be raised is that the principle proves too much, that it appears impossible that such far-reaching conclusions can be drawn from such simple assumptions: the only difficulty, in fact, is that the thing is too easy."

"The crudest arguments based on the oldest theory of light lead to the conclusion that the rate of a clock as observed by a certain observer must change with the relative motion of clock and observer. For, it will be argued, the observer does not see the clock 'as it really is at the moment,' but 'as it was a time T earlier, where T is the time taken for light to reach the observer.' And on these lines it is easy to show that the apparent rate of a clock moving away from the observer with a velocity v is $(1 - v/c)$ times[3] the rate of the same clocks observed at rest. It is only the magnitude of the change concerning which the two theories differ.

" 'Yes,' says our objector, 'that is all very well: of course the apparent rate of the clock changes with motion, but does the real rate change?' We immediately inquire what the 'real rate' means. He is at first inclined to assert that it is the rate observed by an observer traveling with the clock, but when we inquire relative to what clock that observer is to measure the rate he becomes uneasy. He cannot compare another clock traveling with him, for if the 'real rate' of one clock has changed, so has the 'real rate' of the other; and he cannot use a clock which is not traveling with him, because he admits that he does not see such a clock 'as it really is.'

"Pressing our inquiries, I think we shall get an answer of this nature. 'If I take a pendulum clock to some place where gravity is different, the rate of the clock will change. It is a change of this nature which I call a change in the

[3] c denotes the universal velocity whatever it may turn out to be. See *ibid.* p. 508.

"real rate," and I want to know whether there is any change of that kind, on the theory of relativity, when the clock is set in motion.' Now why does our objector call a change of the first kind a change in the 'real rate'? The reply is to be found in the history of the word 'real.' The word is intimately associated with the philosophic doctrine of realism, which holds that the most important thing that we can know about any body is not what we observe about it, but its 'real nature,' which is something that is independent of observation.

Now, of course, a quantity which is wholly independent of observation cannot play any part in an experimental science, but there are quantities which are independent of observation in the more limited sense that they are observed to be the same by whatever observer the observation is made. The term 'real' has come to be transferred from the philosophical conception to such quantities. The 'real rate' of the clock is said to change when it is transferred to a place where gravitation is different, because all observers agree that the rate of the clock which has been moved has undergone an alteration relatively to that which has not been moved.

"Now in the conditions which we are considering the observers do not agree. If A and B, each carrying a clock with him, are moving relatively to each other, they will not agree as to the rate of either of their clocks relative to A's standard or to B's standard or to any other standard. The conditions which, in the case of the alteration of gravitation, gave rise to the conception of a 'real rate' are not present: in this case there is no 'real rate,' and it is as absurd to ask whether it has changed as it would be to ask a question about the properties of a round square. However, some people, who in their eagerness to escape the reproach of being metaphysicians have adopted without inquiry the oldest and least satisfactory metaphysical doctrines, are so enamoured of the conception of 'reality' that they refuse to give it up. Finding that the observations of different observers do not agree, they define a new function of those observations, such that it is the same for all observers, and proceed to call this the 'real rate.' This function, according to the principle of relativity, is $\beta n'$ where n' is the rate of the clock as seen by an observer relative to whom it is traveling with the velocity v: according to that principle, if we substitute in that function the appropriate values for any one observer, the resulting number will always be the same. So far no overwhelming objection can be raised."

What the relativists call "real" we would call objective, and we deem the ideal of objectivity to be the goal of science. Mr. Campbell has much to say on the concept of reality:

"It is the great merit of the principle of relativity that it forces on our attention the true nature of the concepts of 'real time' and 'real space' which have caused such endless confusion. If we mean by them quantities which are directly observed to be the same by all observers, there simply is no real space and real time. If we mean by them, as apparently we do mean nowadays, functions of the directly observed quantities which are the same for all observers, then they are derivative conceptions which depend for their meaning on the acceptance of some theory as to how the directly observed quantities will vary with the motion, position, etc. of the observers. 'Real' quantities can never be the starting point of a scientific argument; by their very nature they are not quantities which can be determined by a single observation: the term 'real' has always kept its original meaning of some property of a body which is not observed simply.

"All the difficulties and apparent paradoxes of the principle of relativity will vanish if the attention is kept rigidly fixed upon the quantities which are actually observed. If any one thinks he discovers that that principle predicts some experimental result which is incomprehensible, let him dismiss utterly from his mind the conception of reality. Let him imagine himself in the laboratory actually performing the experiment: let him consider the numbers which he will record in his note-book and the subsequent calculation which he will make. He may then find that the result is somewhat unexpected—to meet with unexpected results is the usual end of performing experiments,—but he will not find any contradiction or any conclusion which is not quite as simple as that which he expected.

"There is one further point sometimes raised in connection with the principle on which a few words may be said.

"It is sometimes objected that the principle 'has no physical meaning,' that it destroys utterly the old theory of light based on an elastic ether and puts nothing in its place, that, in fact, it sacrifices the needs of the physical to the needs of the mathematical instinct. That the statement is true there can be no doubt, but the absence of any substitute for the elastic ether theory of light may simply be due to the fact that the principle has been developed so far chiefly by people who are primarily mathematicians. It is well to ask, can any physical theory of light be produced which is consistent with the principle?

"The answer depends on what is meant by a 'physical theory.' Hitherto the term has always meant a 'mechanical theory,' a theory of which the fundamental propositions are statements about particles moving according to the Newtonian dynamical formulæ. In this sense a physical theory is impossible if the principle of relativity be accepted, for the same reason that a corpuscular theory of light is impossible, if the undulatory theory of light be accepted. Newtonian dynamics and the principle of relativity are two theories which deal in part with the same range of facts;

they both pretend to be able to predict how the properties of observed systems will be altered by movement. If they are not logically equivalent they must be contradictory: in either case an 'explanation' of one in terms of the other is impossible. It can be easily shown that they are contradictory: if the principle of relativity is true, Newtonian dynamics must be abandoned."[4]

We start with "the facts of observation," and try to establish the objective state of things, called also "the real"; but relativists ignore the latter, and since every observer has his own particular observation, they declare that there is neither real time nor real space. The real is ruled out from observation.

Suppose, however, that the clocks which the relativist observes were the heartbeats of the relativist himself and the observer were the diagnosing physician, would the relativist insist that the physician had better drop out of sight the notion of reality, that there is as little sense in asking for "the real rate" of his heartbeat as it is absurd "to inquire whether, if all triangles had four sides, all circles would be square"?[5] If we can not attain an absolutely correct objective statement, we keep at least the ideal in view and this ideal is not an empty dream.

The relativity principle is a mathematical view of certain problems worked out for the sake of most minute measurements; and the attitude of the relativists is stern. If the facts can not be clearly represented by it, the worse for the facts, and if the physicists declare that their physical theories are incompatible with it, a new brand of physicists has to be manufactured who will inaugurate a relativist reform in physics.

[4] This conclusion is reached by Sommerfeld in a paper, *Ann. d. Phys.*, XXXIII, p. 684, etc. (1910).

[5] See Campbell, *loc, cit.*, p. 509. The comparison is not appropriate.

Primary Concepts.

The relativity problem would never have originated had the philosophy of science been clearly and distinctly understood by physicists, but they have familiarized themselves very little with even the problems, let alone reached proper solutions which explain the elementary concepts of our scientific terms, the difference between substance and form, between energy and matter, and the significance of the purely formal sciences.

As mathematicians are in the habit of starting with axioms, so the relativists begin with postulates and these postulates come in collision with the primary concepts such as have been formulated among the orthodox physicists and mathematicians of the present day.

A truly scientific view will brook neither axioms in mathematics, nor postulates in philosophy, nor primary concepts in physics.

There has been much talk about primary concepts, and arguments have been offered why time is not a primary notion or why we should let it pass as such. The truth is that time as well as space are two methods of describing definite relations. Time is not so much a fourth dimension of space, though we might look upon it as if it were such, time is the measure of motion and space is the scope of motion. Both time and space are presupposed in the idea of motion. There is no time in itself, there is no space in itself. What Newton and others with him call absolute space is "space conception" and what they call absolute time is "time conception." Such are the ideas which by pure deduction on *a priori* arguments, physicists form of time and of space, just as mathematicians formulate the general conception of numbers, of distances and of other relations, angles, areas, etc.

The idea of primary concepts is a very unfortunate de-vice to lay a foundation for science. The faults of this method will not show so long as specialists are concerned about specialist problems, but the carelessness of taking anything for granted shows itself as soon as any problem broadens out into a general inquiry when its connection with universal problems is questioned. Such primary concepts are assumed to be undefinable and self-evident. That opens the door to an arbitrary interpretation as to the nature of space and time and energy, and gives a wide berth to mysticism.

Science brooks neither axioms nor primary concepts. Science starts with experience; it quarries out of experience the stones of the purely formal sciences which furnish all the methods of both common sense knowledge and scientific inquiry. The most general characteristic of experience is activity. Activity manifests itself in change. Change implies motion; it means either change of place, i. e., moving from here to there, or change of combination, viz., a moving of particles among themselves. Change interferes with existing relations, it modifies the old interrelations and establishes new interrelations.

The nature of relations in one terse term is called form. The word "form" comprises both outer shape and inner structure, and all interrelations of things as well as thoughts can be determined by the laws of pure form, arithmetic, geometry, logic, etc. Under all circumstances change modifies relations and means "transformation." There is a transformation in the juxtaposition of things or their parts, and there is a succession of events. The scope of the former we call "space," of the latter "time"; or better from the former we deduce our notion of space, from the latter our notion of time.

Physical inquiry is not helped by calling certain features of experience "primary concepts" and least of all (as has been done) should space, time and force,—these highly complicated constructions of *a priori* thought— be beclouded by this mystifying name. Both time and space are features of the form of existence, and force is a general term for that feature of existence which marks its activity as motion, viz., as change of place, or rather as that which causes changes and is measured by the resistance overcome.

If we adopt the relativist principle to ignore the scientific ideal of objectivity, i. e., if we define size as the result of measurement and moments of time as determinations of measurement by units of duration, without regard to the ideal of coincidental happenings, and a common standard of time, we may produce incredible statements against which common sense rebels, and Professor Magie in his Presidential Address,[6] delivered before the Physical Society and Section B of the American Association for the Advancement of Science, at Washington, D. C. (December 28, 1911), says in comment thereof:

"A description of phenomena in terms of four dimensions in space would be unsatisfactory to me as an explanation, because by no stretch of my imagination can I make myself believe in the reality of a fourth dimension. The description of phenomena in terms of a time which is a function of the velocity of the body on which I reside will be, I fear, equally unsatisfactory to me, because, try I ever so hard, I can not make myself realize that such a time is conceivable......I do not believe that there is any man now living who can assert with truth that he can conceive a time which is a function of velocity or is willing to go to the stake for the conviction that his 'now' is another man's 'future' or still another man's 'past.'

"One of the members of this society, recognizing our present inability to conceive of relative time, and conceiving our intuitions of space and time to be the result of

[6] Published in *Science,* February 23, 1912, pp. 281 ff:

heredity operating through many generations of men who lacked the light of relativity, once proposed to me that every one who could get even a glimmer of the notion of relative time should persistently exercise his mind therein and teach it to his students, in the hope that in a few generations the notion would emerge with the force of an intuition. It would not be fair to leave the impression that he was solemnly serious when he made this suggestion."

Form (i. e., relativity) is, as much as matter and energy, an ultimate generalization and may be called a fundamental concept (not a primary concept), and all the work of science is a tracing of transformations.

It is essential for the measurement of space and time to employ as measures uniform units, for space of distance and for time of duration. In the same way we need uniform units to measure force.

Besides a quantitative analysis of experience, there is a qualitative analysis which traces such transformations as build up parts into a higher unit, whereby through the interrelation or the interaction of the parts a new thing originates possessed of properties which are absent in the parts before their combination.[7]

The law of change is called causality. Cause is the motion which starts the process of transformation; effect is the result of the change; and reason is the general rule (formulated as a so-called law of nature) from which we understand why the cause must have this effect.[8]

The so-called law of the conservation of matter and energy is a deduction from the law of causality, which can be made as soon as we understand that all happenings are transformations, for if all changes are transformation, the

[7] See for instance the author's exposition of the nature of quality in *The Monist,* Vol. XV, p. 375. See also *Philosophy of Form,* p. 12.

[8] This has been repeatedly discussed, e. g., in the author's *Fundamental Problems,* pp. 79 ff.

amount of existence, its *that*, remains the same, only its form changes.

While investigating the several problems of our experience, scientists assume that they deal with real occurrences and thus they implicitly grant the *that* of existence, popularly denoted "matter" and "energy," viz., thingishness (or with a Latin term "reality") and actuality. The existence of ether is but an extension of the concept matter and so physicists have so far believed in the existence of ether; but the relativity physicists, in their anxiety to propound original ideas, deny the existence of ether. Says Prof. William Francis Magie in his above mentioned Presidential Address:

"The principle of relativity in this metaphysical form professes to be able to abandon the hypothesis of an ether. All the necessary descriptions of the crucial experiments in optics and electricity by which the theories of the universe are now being tested can be given without the use of that hypothesis. Indeed the principle asserts our inability even to determine any one frame or reference that can be distinguished from another, or, what means the same thing, to detect any relative motion of the earth and the ether, and so to ascribe to the ether any sort of motion; from which it is concluded that the philosophical course is to abandon the concept of the ether altogether. I may venture to say that in my opinion the abandonment of the hypothesis of an ether at the present time is a great and serious retrograde step in the development of speculative physics. The principle of relativity accounts for the negative result of the experiment of Michelson and Morley, but without an ether how do we account for the interference phenomena which made that experiment possible? There are only two ways yet thought of to account for the passage of light through space. Are the supporters of the theory of relativity going to return to the corpuscles of Newton? ......There is choice only between corpuscles and a medium, and I submit that it is incumbent upon the advocates of the new views to propose and develop an explanation of the transmission of light and of the phenomena which have been interpreted for so long as demonstrating its periodicity. Otherwise they are asking us to abandon what has furnished a sound basis for the interpretation of phenomena and for constructive work in order to preserve the universality of a metaphysical postulate."

The concepts substance, i. e., matter or mass, and energy are ultimate generalizations as much as form, but they are very different from form. We could do without the words "matter" or "ether" by the use of some other indication to be introduced in our formulas which denotes reality; but that would not disprove the truth of the popular view, which describes every concrete bodily existence as material, nor is it likely that the old method of nomenclature will be rendered antiquated or erroneous.

We must not forget what matter means. Matter is a word which denotes that quality of objects which all of them have in common, viz., objectivity. An object is a thing that is objected to us, that offers us resistance, that impresses itself upon our existence and thereby affects our senses, and by objectivity we understand the general property of concrete existence, the *that* of experience, or its reality, viz., its thingishness. To deny the reality of the real, the thingishness of things, is as ridiculous as the opposite mistake, i. e., to think of reality, or objectivity, or of matter as a mysterious entity in itself. There is no reality *in abstracto*, for every *that* of existence is of a definite form which acts somehow, and the activity of things we call their actuality, or, as we call it in physics, energy.

The same problem presents itself in the domain of the phenomena of ether, i. e., of light and electricity. There are some good reasons to assume that concrete matter has originated by a contraction or condensation of a more primitive substance which for all we know may prove to be the luminiferous ether, that thin substance which has been assumed to be the medium of light and electricity. If it is claimed by modern physicists that the principle of relativity disposes of the ether, that we no longer need it and can discard a belief in it as a superstition, that all physical phenomena can be accounted for without the assumption of an ether, we confront the same situation as in the theory of energetics, where the claim is made "There is no matter, all is energy."

The truth of this position, so far as we freely grant it, is this, that all scientific explanation describes the transformation of things; it traces the changes that take place according to the laws of form (mathematics and mechanics). In experience we are confronted with the fact that it is so, but the scientist inquires into the factors how it has become so, *how* it acts, and *how* it changes. By describing the *how* in formulas (so-called laws of nature) we denote the several factors with algebraic letters, such as $g =$ gravity, $t =$ seconds of time, $d =$ the distance traversed by a falling body and $v =$ the velocity of the fall, etc., and express their interrelation in equations, as

$$v = gt \text{ and } d = \tfrac{1}{2}gt^2.$$

By this method the essential features of natural phenomena are expressed in symbols, and he who has been initiated into the secret meaning of the symbols and the method of using them, will be able to predict the course of events if he is in possession of the necessary data.

What we here call with one word "essential" Kirchhoff characterizes in two words "most complete and most terse," or to use the common version "the most exhaustive and simplest." We deem our term preferable, and we understand by "essential" all that which is efficient to produce the result, not more, not less.

We speak of the three laws of Kepler and of the condensed statements of the law of gravitation as "formulas," and this term truly expresses the nature of these generalized descriptions of certain types of uniformities. They are reductions of events to their purely formal (i. e., purely relational) conditions, and these purely formal conditions are the determinant (i. e., the causative) factors in all possible phenomena of a special type.

This is not a new truth. How old it is may be inferred from the Greek term "formal"[9] which in its etymology means "the causal" or "the causative" because the Greek philosophers describe the formal factors as efficient in causation.

When we have traced the essential factors of a certain type of changes, the scientist's work is finished. Whether mankind will ever be able to complete a scientific comprehension of the world in all its details, must be regarded as doubtful, but wherever science has succeeded in discovering the essential factors and has reduced them to formulas, we have been enabled to offer for every such phenomenon a satisfactory explanation.

This procedure affords us an insight into the reason why the course of a certain phenomenon must be so, why it can not be otherwise, and in this procedure the *that* is the basis, the *how* is the method of cognition. There is no explanation possible for the *that,* for the reality of the real, for the actuality of the fact; all explanations refer to the *how.* The *that* is a brutal fact, and the ultimate goal of science is the how, the answer being the formulation of laws of nature which explain to us by a use of the law of pure form that under given circumstances definite transformations will take place. Knowledge of the laws of na-

[9] τὸ αἰτιῶδες, derived from αἰτία = cause.

ture helps man to adapt himself to nature and also to adjust his surrounding natural conditions to himself.

In our explanation we can omit the *that* as a matter of course, for it is understood that reality is real. We can describe the purely formal relations only, which are the essential part of explanations. There is no sense in explaining the *that*. We have simply to state whether or not a formula covers actual facts, but to deny the *that* and say that there is only a how the world wags, but there is no world, seems to us a proposition that misconceives the situation.

We must not forget that such a word as substance, denoting here both "matter" and "ether" or existence in general, is a term that stands for objective reality. Ether is the *that* of the phenomena of electricity and light, as matter is the *that* of bodily objects, declaring that they are real, that they are concrete, and the term "substance" covers any kind of existence, it embraces both matter and ether or whatever the ultimate world-stuff may be called. There is no sense in denying their actuality, and all that may be meant by such a denial can only be either the redundancy of an express declaration that the formulas of physics refer to real processes, or a denial of ether or of matter as existences in themselves apart from their manifestations in definite configurations or modes of motion — a proposition which nowadays no one will seriously dispute.

A denial of the existence of substance (of matter and ether) is a purely verbal quibble. We might as well deny the existence of energy and declare that there is no energy, that there are only changes of place. The truth is that the faculty of existence which manifests itself in changes of place is called energy. We must not conceive of energy as something in itself.

* * *

I am told that my own view is the gist of the principle of relativity, and if that be true, I would gladly hail a philosophy of relativity as another name for the philosophy of science. I have myself characterized the philosophy of science as a philosophy of form, and form denotes the relations in their totality. However, I would add that the system in which I have formulated this philosophy of science is simpler than the world-conception of the relativity physicists, besides it rests on a more solid foundation and is absolutely free from paradoxes.

While I deny that we can dispense with the idea of objectivity (be it called matter, or ether, or substance) I claim that we need make no mention of it in our formulas. In this sense we can dispense with the mention of ether. While I would not take the several paradoxes of time and space as serious and deny their objective truth, I grant that by a little confusion of thought in calling time or space relations the results of our different measurements, we can legitimately produce these paradoxes by exhibiting the inevitable discrepancies which originate through measurements from different standpoints as objective contradictions. Finally I consider it the ideal of a scientific philosophy to reduce all possible occurrences to relations, to resolve them into questions of form, to look upon them as transformations, and therefore I say that the ultimate aim of science is to describe everything in formulas. I see no objection to the relativist claim that this is a postulate of science. In fact, I deduce this postulate directly from my conception of reality which presents itself everywhere in our experience as transformation. Thus we would justify the principle of relativity on the basis of the old traditional basis of exact science.

The main claim of the relativists is based upon their simplification of the electromagnetic equations, and this is granted even by the adversaries of the principle of relativity. Professor Magie says:

"It is surely true that if it were not for this demand of simplicity, immediately attainable and at present expressed in the electromagnetic equations, the chief incentive to the development of the theory of relativity would be wanting."

The one simplification of formulas is attempted by certain relativists by a generalization of time and space into a higher four-dimensional system, and they call it a four-dimensional space. We may note incidentally that Wagner's Parsival has anticipated the doctrine of relativity, for in his search he utters the mysterious words: *"Zum Raum wird hier die Zeit!"* (Into space here changeth time!) The relativists might as well have called their four-dimensional space a four-dimensional time. We abstain from giving it a name, but subsume time and space under one and the same category as "form" which enables us to view time and space as two inseparable factors of the cosmic system of interrelations, and we deem it wise to remember that they are different. If the relativity physicists have this in mind and do not mean ulterior mystifications, I would not hesitate to join their ranks on this point.

* * *

We may add one more comment about simplification. Logical possibilities are wider than actualized reality. Reality is one instance among many others which are not actualized. The fictions of fairy tales, of Gulliver's Travels, and of religious myth are instances of it. But in the domain of pure logic even actually absurd conditions parade as legitimate potentialities. Actual space has three dimensions, but metageometricians have invented more-dimensional spaces. Why not? We have in the construction of purely logical systems the undeniable right to generalize into the not actualized logical possibilities and mathematicians can not be restrained from building up a pangeometry. While Euclidean space is homaloidal, they may create all kinds of curved spaces, which are all legitimate before the tribunal of pure logic, if they are but consistent in themselves. The main gain derived from such constructions which will naturally appear to the average man of average common sense as gratuitous, if not positively nonsensical, consists in rising to a higher level and understanding from this higher point of view the actualized reality better than if he remains on the *terra firma* of a limited sense-experience.

It might help our comprehension of causality as a transformation according to the laws of form to conceive the chain of causation as reversible, that the condition of causes are turned into effects and that the final factors that bring about the effect become the causes. This view has been humorously worked out by Fechner who for this purpose assumes that the pendulum of events will go on for a while in the direction it takes now, but the time will come when it will swing back. And then it will appear to us as quite natural and necessary that the decayed and waste material from fields and polluted rivers pass into our bodies and are changed in our bowels into juice to go forth from our mouths on the dinner table as lovely fruit or cheese, with bread and butter, and as roast venison or fish to go back and constitute useful parts in the revived animal. It would please us to see all this come about and the thought of the resurrection of the lamb under the butcher's knife would demonstrate that there is a purpose in the law of causation. We would be accustomed to the outcome and deem it natural. In fact some notions of an inverse world order in the golden age when the lamb will feed on the wolf, when the deer will hunt the hunter, when the rich shall be poor and the poor rich, when the miserable will be comforted while the fortunate will be tortured has now and then received serious support in the religious hopes of the disinherited classes of mankind, and we may find in the New

Testament an echo of this belief in those traditions which come down to us from Ebionite sources, the parables of the foolishness of the rich and the benediction of the poor. Dives goes to Hell while Lazarus is carried by angels to Abraham's bosom. Abraham says in Luke xvi: 25: "Son, remember that thou in thy lifetime receivedst thy good things, and likewise Lazarus evil things: but now he is comforted, and thou art tormented." No mention is made that Dives was wicked and that Lazarus was good; the only argument is that the other world must be reverted in its order.

A view of this kind which generalizes the mechanical constitution of the world and sees the possibility of an inverted causation, just as an engine may be reversed, may widen our comprehension and simplify our formulas of moral action, but we need not for that reason believe in its actualization. It is simply an instructive *lusus imaginationis,* an ingenious and helpful fiction—like our conception of four-dimensional space.

The mathematician who limits his studies to the Euclidean plane will understand his problems better if he becomes familiar with the theorems of stereometry, or if he views the figures of plane geometry as projections; or again if he regards a certain set of curves as conic sections. And further many problems of stereometry find a simpler formulation if viewed from the more comprehensive, though purely imaginary, view-point of a four-dimensional geometry. All this indicates that the simplifications of which the relativity physicists boast, may be (and I am inclined to believe that they are) very harmless. For all I can say, judging merely from the acceptance they have found, they must be true, but I can not see why they should be subversive of the scientific world-conception of the past.

A peculiar view of time which has been proposed in all seriousness, although common sense might consider it as absurd, is the concept of time and space as consisting of discrete ultimate units. Do not our years, and days, and our hours too begin at definite moments? We become fifty or sixty years old suddenly with the beginning of a definite minute, According to this, time would run in jerks like the jumping second hands, and it would ultimately consist of infinitesimally small units of duration. Space also would be stippled and not continuous. Every motion would have to proceed in hopping from spot to spot, and the surface of a plane would be not unlike a half-tone picture which produces the impression of a continuous level but consists in reality of different dots more or less deeply tinged with ink. Such conceptions of time and space are quite conceivable although our classical and well-established views of both present them as continua. If space and time were actual entities endowed with positive qualities, if they were not merely potentialities of motion, a scope in which we move about, we could discover the nature of space by experiment. However, as they are constructions made in the abstract domain of anyness we should not refuse to consider seriously all kinds of propositions as to the nature of time and space.[10]

In comment on theories of this kind we would say that duration is continuous, but time consists of discrete units of duration; and again the scope of motion shows us an uninterrupted expanse while geometry exhibits definite lines

[10] The present number of *The Monist* contains an article on "Atomic Theories of Energy" by Mr. Arthur E. Bostwick, which will prove of great interest even to those who do not accept this theory. In comment we would say that Mr. Bostwick's defense of an atomic theory of energy is certainly true of definite amounts of energy, and his theory holds good also in his comparison of energy to amounts of money values deposited in a bank account. If deposits were made in specie, we could trace every dollar of a deposit. It is true we can not do so, but this we can not do only because no one cares to receive definite and individual coins, but is satisfied with money in any form. Therefore the bank is like a reservoir of water which receives and gives out water as it happens to come. The bank gives credit for amounts received and pays out amounts according to request. Thus the individual coin is lost sight of as the many drops of water are definite and concrete masses, and every dollar in a bank represents some concrete value somewhere.

of definite direction and of definite length. Geometrical space in its classical Euclidean form is not stippled, nevertheless every construction is particular. Geometrical points have no extension, but they possess a definite location, being determined, e. g., by two crossing lines. Thus space is not the totality of all points, but the totality of our scope of motion and anywhere in space points may be laid down. In a word: Time and geometrical space are constructions invented for the purpose of making measurements possible in a scope of potentialities.

Actual existence is always definite, pure forms however as well as purely formal thoughts, are always potential.

It seems as if the beginning of actuality must consist in establishing something that is limited and concrete. In this way it appears plausible that a potential world would be continuous as an ocean of pure ether might be, but an actual world ought to consist of a group of units, of atoms, of definite particular specks of existence endowed with definite amounts of energy, and we ought to be able to trace every definite amount of existence through all the changes which in the process of evolution it will undergo; and this ought to be true as regards every amount of both matter and energy.

SOME PHYSICAL PROBLEMS OF RELATIVITY.

The physical problems presenting themselves in the experiments which have become connected with the movement of relativity do not seem to have any direct bearing on the principle of relativity itself. Relations are of a purely formal nature and relativity therefore belongs to the same kind of knowledge as arithmetic, geometry and logic. Relativity can and must be applied to physics just as much as there is an applied mathematics, but as the Pythagorean theorem is independent from its applications in experience, so applied relativity can neither establish nor refute the principle of relativity. This is true above all of the well-known and most important Michelson-Morley experiment.

The instrument made in Berlin by Schmidt & Haensch was so delicate that it was of no use in Berlin, and even when placed upon the foundation for the pier of the equatorial in the Astrophysical Observatory at Potsdam the fringe of interference rings disappeared by stamping upon the pavement at a distance of about 100 meters. Every detail of consequence was taken into consideration, not only the motion of the earth through the ether but also the motion of the whole solar system towards the constellation of Heracles. The expansion of the brass arms of the instrument through a change in temperature, and also the bending of the arms through rotation were duly considered and the difficulties arising therefrom met. A scale ruled on glass was employed in order to dispense with the micrometer screw which here proved useless. Yellow light was used, because its wavelength is least difficult to measure.

If the ether is at rest while the earth moves through it, the time required for light to pass from one point to another on the earth's surface would depend on the direction in which it travels. Two pencils of light that travel over paths at right angles to each other will interfere; the one traveling in the direction of the earth's motion will travel 0.04 of a wave length farther than it would have done were the earth at rest, while the other pencil at right angles to the motion of the earth would not be affected.

The results of Professor Michelson's experiment are negative. He found very small displacements in the fringes of his ray of light, so small that they must be accounted as mere errors of the experiment. While we ought to expect a displacement of 0.05 we have only such as lie between 0.004 and 0.015. Professor Michelson says:[11]

"The interpretation of these results is that there is no displacement of the interference bands. The result of the hypothesis of a stationary ether is thus shown to be incorrect, and the necessary conclusion follows that the hypothesis is erroneous.

"This conclusion directly contradicts the explanation of the phenomenon of aberration which has been hitherto generally accepted, and which presupposes that the earth moves through the ether, the latter remaining at rest."

In another article Professor Michelson states his result thus:[12]

"The luminiferous ether is entirely unaffected by the motion of the matter which it permeates."

Professor Michelson has varied the conditions of his experiment by trying whether deviations could be detected through a change of level, by throwing pencils of light upward and by repeating it at different hours of the day, but the displacements remained insignificant. One of Professor Michelson's articles ends thus:[13]

"In any case we are driven to extraordinary conclusions, and the choice lies between these three:

"1. The earth passes through the ether (or rather allows the ether to pass through its entire mass) without appreciable influence.

"2. The length of all bodies is altered (equally?) by their motion through the ether.[14]

[11] "The Relative Motion of the Earth and the Luminiferous Ether" in *The American Journal of Science,* Vol. CXXII, page 128.

[12] "Influence of Motion of the Medium on the Velocity of Light," in *The American Journal of Science,* Vol. CXXXI, page 386.

[13] "The Relative Motion of the Earth and the Ether," *The American Journal of Science,* Vol. CLIII, p. 478.

[14] This would be the case according to the theory of H. A. Lorentz, whose views are mainly presented in the *Encyclopädie der math. Wissenschaften.*

"3. The earth in its motion drags with it the ether even at distances of many thousand kilometers from its surface."

Another article by Professor Michelson on the same subject is published in *The American Journal of Science*, Vol. CXXXIV, p. 333.

What this famous experiment has to do with the principle of relativity except in a most general way, is not yet clear to those who have not joined the ranks of the relativity physicists; but the relativity physicists insist very vigorously and dogmatically that it proves, or at least favors, their theory. Professor Michelson himself has not joined their ranks, though he recognizes the difficulties of the situation.

It is strange that Michelson's experiment seems to stand in contradiction to another and older experiment made first by Bradley, which is known as the aberration of light. If the earth passes through the ether with its own velocity (e) while the rays of the sun come down upon the earth with the velocity of light (l) there ought to be a deflection of light amounting to e/l, viz., the velocity of the earth divided by the velocity of the light in its path from the sun towards the earth, and though this relation is very small, it has actually been observed and determined to amount to a trifle over twenty seconds.

This conclusion which could be anticipated according to the logic of mechanics seems to be contradicted by Michelson-Morley's experiment in which the attempt is made to measure with a ray of light the motion of the earth while passing through the ether.

The discrepancy between the two experiments will perhaps find a proper explanation in the proposition that if the source of light lies outside the earth as in the case of the rays of the sun, they will show the deflection. As is to be expected they would come down in straight lines like raindrops falling in an absolutely quiet air which would be caught by a moving body as if they came down at an angle; but if the source of light moves along with the earth there would be no difference whichever way they turn, first towards the east or first towards the west, or at right angles, and the sources of the light would partake of the acceleration of the earth so as to show no difference, as raindrops dripping down within the car would fall down in straight lines from its top to the floor, assuming that the doors and windows of the car are hermetically closed and there be no draft which would deflect their perpendicular dripping.

It almost seems as if some ether were carried along by the earth to a considerable distance beyond its surface while the other ether in outer space would remain at rest, but it would be bold for any one but a specialist to venture the proposition of any theory on so new a subject of which few facts only have been ascertained. Yet most assuredly the topic under investigation has nothing to do with the principle of relativity, unless relativity is a misnomer for the phenomena attributed to the luminiferous ether.

The question of relativity is a philosophical problem, but the Michelson-Morley experiment is of a purely physical nature, and so we must expect that the last word as to its explanation should be given by physicists.

The other experiment which is assumed to verify the principle of relativity is the one first made by Kauffmann, and afterwards repeated in a modified form by Bucherer. This experiment too has little or nothing to do with relativity. On the contrary it seems to prove the existence of something absolute for it reaches a limit of velocity.

There is at present a tendency in the world of thought, noticeable in pragmatism and other anti-intellectual movements, which seems to annihilate the very existence of objectivity, and with it science, man's endeavor after a purely objective cognition. Everything is relative, and the general belief has spread that an absolutely objective description is impossible. To speak of the size of objects seems to have lost its sense, for size has become to the present generation merely the result of measurement, and thus an objective determination is in some quarters looked upon as a superstition of prescientific tradition, an inheritance from the dark ages. But it is not true that there is no objectivity, for one of the greatest accomplishments of Michelson was the establishment of a definite measure by calculating the size of a meter in wave-lengths or red cadmium light in a vacuum. The waves of light are absolutely definite, and thus we have here a result of measurement in truly objective terms. If the Kauffmann-Bucherer experiments prove, as is claimed, that an increase of velocity means an increase of mass and that the limit which is reached is the velocity of light, we only learn that relativity is not without bounds, and that on the contrary a climax is reached which can not be surpassed. The highest velocity is the velocity of light.

The conclusion that the highest velocity is the velocity of light seems to be contradicted by the facts of gravitation for according to the Newtonian theory gravitation is possessed of a practically infinite velocity in that the gravity of the sun exercises its influence upon the planets without any perceptible difference of time. But this is no objecttion, for consider: The action of gravity formulated in the well-known law of falling bodies and of their acceleration which describes true motions is very slow in comparison to the velocity of light. The influence which is exercised in the strain between two gravitating bodies, say between the moon and the earth, is not a motion at all, but a condition, and this condition is the same between the two centers of the thus interrelated bodies. It is a state of tension and there is no transference of a wave motion either from the moon to the earth or from the earth to the moon. The tension is simultaneous. The misconception seems to rise from the error that there are two bodies and there is a third item which manifests itself as a passing from the one to the other under the name of gravitation. We must view the whole system as one field of action in which several bodies in motion are balanced among themselves according to their mass. Their mutual attraction is not transferred motion but a simultaneous interaction. Newton retarded the general acceptance of the law of gravitation, first definitely proposed by Hooke, for eighteen long years because he could not make up his mind to believe in an *actio in distans,* and when he was finally convinced, he still expressed his misgivings how to overcome this objection, but is there any *actio in distans* at all? Is not the whole system of the universe an interrelated whole and does not a center of gravity (howsoever it may have originated) extend so far as its stress reaches? Where its strain produces a tension, there it affects its surroundings. If we look upon the phenomena of gravitation in this light we need not make the fantastical assumption that gravity is possessed of an infinite velocity.

The relation between the increase of velocity and the increase of mass promises to throw light on the ultimate constitution of matter, but the result of the experiment is only the first step to a solution of this tremendous problem, concerning which at the present stage of science we can have only vague suggestions. When the man appears who can read the facts aright, he may be able to point out how by a mere stress the aboriginal world-stuff which, for all we know, may be, or even must be, the ether, produces a tension within this mysterious infinitely elastic and incredibly thin substance, and the tension between two centers of such contraction would, like the strain between nodes within thin tridimensional rubber, act in all directions according to the Newtonian formula of gravitation, as being directly proportional to the product of their amounts of contraction, and inversely proportional to the square of their distance between two centers. Thus the origin of matter would be due to an unknown force which with a velocity only inferior to the velocity of light would drive infinitely small corpuscles around in a whirling dance with such a regulated speed that conglomerated multitudes of such whirls would appear to us as solid masses.

Here again we would be confronted by an ultimate limit. We would discover that objective reality, our world of matter in motion, is built up of ultimate particles; or perhaps better, of ultimate activities, that below the atom there are smaller units, the hypothetical electrons, which may be characterized as centers of force, and that they are due to condensation which produces the phenomena of gravitation. All further phenomena of physics and chemistry would have to be explained as the result of these elementary actions.

Formerly thinkers were inclined to see infinity all around. They thought of the atomic structure not only as infinitesimally small, but also as truly infinite; the molecules being analyzable into atoms and the atoms again into still smaller units, say into electrons or monads, and that the monads were again compounds of monadules and so forth—all this being argued on the poetic notion that

> "Great fleas have little fleas
> Upon their backs to bite 'em,
> And little fleas have lesser fleas,
> And so *ad infinitum.*"

The molecule is a kind of planetary system, with atoms as satellites, so is the atom with its circling electrons; why should not the electron be of the same construction and why should not the component parts of the electron be assumed to be made after the same pattern world without end? On the other hand our solar system is one among uncountably many others of the Milky Way; and the Milky Way in its turn is one universe of an enormously larger system of many Milky Ways. This is the conclusion which astronomy has deduced from actual facts. Why then should not this in our opinion enormous system of the many Milky Ways be only a tiny item in a still larger system, and why should we not be justified in the assumption that we are confronted with an infinite vista into both directions toward the infinitely small and the infinitely great?

This notion has been brought out in the second quatrain which reads:

"And the great fleas themselves in turn
Have greater fleas to go on,
While these again have greater still,
And greater still and so on."

A vista into infinitudes, going out into the infinitely small and the infinitely great, now seems to become untenable, and definite limits loom up, which condition, so it seems to us, would reveal, not a bottomless and undefinable relativity but a definite world of an objective reality with definite interrelations and limits. If there are definite limits in either direction we may fairly well assume that they are in both directions. Further, if the universe is definite in its space relation, it should also be definitely limited in time. The world may have originated in an immeasurable ocean of uniformities as a definite commotion and may terminate again in a general dissolution by dissipation. If such be the case the relativity principle would not apply to the whole. Relativity would mean the interrelationship of all things, but the whole as a whole would be of a definite particularity with definite boundaries while the constitution of the world would exhibit a structure of extremely tiny ultimate units of a determinably definite size, endowed with a definite velocity and at every given point of a definite form of motion.

While the totality of existence, the sum total of our Milky Ways, appears to have had a beginning and may after the lapse of immeasurable ages come again to an end, we do not deem it excluded that the same process of world-formation may start again, as it probably was repeated long before the origin of this our present universe. While thus everything existent, even the ether itself in its totality, would have to be regarded as particular and concrete with definite boundaries and as being limited to a definite time both in its beginning and in its end, there would after all loom up in the background of this world an infinitude of space, an eternity of time and an unfathomable wealth of potentialities as to new formations which in spite of all the light which the most advanced science will ever shed on the world problem will keep this great All of existence with its inexhaustible resources and its mysterious order an object of constant wonder and awe.

The relativity problem as such is a philosophical problem, but the relativity physicists have made a physical problem of it, and the philosophical problem of relativity is not a new problem, it is as old as science; it is only the lack of philosophical training which has led to the enunciation of some baffling paradoxes which if they were true would make objective science impossible, for they would abolish definiteness of any kind and do away with objectivity. And strange to say, claims of this kind are upheld on the ground of experiments which tend to establish the existence of an absolute, or as we would prefer to say, of some ultimate, which would prove that our experience does not float as a local tangle in an endless infinitude, but that there is a beginning and end, and also a boundary of all concrete reality at every definitely given moment. No mysticism is needed. Infinitude and eternality are potentialities, not actualities. They are vistas of what may be, not what is. They constitute the inexhaustible wealth of nature and of life without robbing science of its validity.

There is a tendency in mankind to think of the present moment as the climax of the past, which ushers in a new era by being an unprecedented and unique start. Every new generation passes through such a period of self-sufficiency and of an intoxication with their own incomparable selfhood. The old problems seem new to them, and trying to formulate them in an original way, they applaud their own mistakes as something extraordinary and wonderful. Goethe characterizes this tendency in the young graduate who has just taken his degree of Bachelor (See *Faust,* Second Part, Act II) where this young man vents his ambitious conceit in these words:

"This is Youth's noblest calling and most fit!
The world was not, ere I created it;
The sun I drew from out the orient sea;
The moon began her changeful course with me;
The Day put on her shining robes, to greet me;
The Earth grew green, and burst in flower to meet me,
And when I beckoned, from the primal night
The stars unveiled their splendors to my sight.
Who, save myself, to you deliverance brought
From commonplaces of restricted thought?
I, proud and free, even as dictates my mind,
Follow with joy the inward light I find,
And speed along, in mine own ecstasy,
Darkness behind, and Glory leading me!"

It is apparent that the relativity physicists confront an important problem, but they have not succeeded in solving it; they have not even as yet properly formulated the question and their propositions are still in a state of fermentation. It is difficult to say what will come of it. It is to be hoped, however, that the movement will follow the usual course of mental growth. The relativists will drop their extravagant claims, they will mature the truth which they grope after and will at last formulate it into clear statements so as to justify the prophecy of Mephistopheles, who comments upon the proud words of the young Bachelor thus:

> "Go hence, magnificent Original!—
> What grief on thee would insight cast!
> Who can think wise or stupid things at all,
> That were not thought already in the Past?
> Yet even from him we're not in special peril;
> He will, ere long, to other thoughts incline:
> The must may foam absurdly in the barrel,
> Nathless it turns at last to wine."

At the present state of our knowledge it would be fantastical to suggest a solution of the physical problems connected with the relativity movement, and we must leave the discussion of them to the future, for ere we can approach a solution we must know much more about the ultimate constituents of matter.

Who will furnish the key to the lock of the closed door at which the relativity physicists are knocking?

CONCLUSION.

The details of the physical problems and their solution have only a slight interest for philosophy. The philosopher, however, expects that the physicist's solutions shall be consistent and that our scientific world-conception shall tolerate no contradictions.

If we consider the all-importance of form and the enormous significance which the formal sciences possess, we are inclined to regard the philosophy of relativity as a synonym and parallel development of the philosophy of science — the philosophy of form. But before we can definitely say so, we would expect the relativists to work out their philosophical substructure in a conservative way, to rid themselves of their paradoxical propositions, give up false pretensions to originality, recognize the past traditions of science, and rather than abandon the past, join their cause to the legitimate progress that follows from the tendencies, the ideals and aspirations of the established sciences.

We do not deny the relativity of all existence throughout and without exception, but we still cling to the old scientific ideal of objectivity and we can not see that the relativity principle is well established.

The great question before the world of thinkers is this: Is it possible to construct a philosophy of science? The author of this essay has answered this question in the affirmative, and has worked in this field for fully a quarter of a century. He has worked out the details of a philosophy of science, and has submitted to the world in both *The Open Court* and *The Monist* his answers to the several philosophical questions. These questions are: the nature of the soul; the origin of sentiency and of thought; the nature of reason, especially in its origin and in its relation to language, the mechanism with which reason manifests itself; the nature of ethics and the foundation of morality as it is found in the laws of the objective world; the significance of the God-conception as the authority of conduct, as the ideal of right and wrong, as the standard of truth and error, as the object of devotion, of gratitude, of reverence mainly as the factor which determines good and evil. All these questions are not beyond the scope of scientific inquiry and in the philosophy of science definite solutions are propounded which, though based on radical principles of unbiased thought, lead to a justification of the historical growth of religion and science.

The whole scope of existence as it presents itself in human experience can become an object of scientific inquiry, and all scientific problems admit ultimately of a definite solution without equivocation or prevarication, yet at the same time science is only one attitude among several others from which the world can be confronted. The noetic conception is the ideal of understanding the world in its pure objectivity represented in mental terms to the exclusion of sentimental subjectivity. But man is not a child of reason only. He is also endowed with sentiments, with will and with artistic tendencies. While the scientific world-conception is absolutely indispensable for the man of thought who works for a constant elevation of mankind upon a higher level, we must at the same time recognize the rights of the large masses who naturally are non-scientific and are swayed by sentiment, by devotion, by art, by ethical aspirations, by a religious comprehension of life; and thus we see in artistic and religious conceptions ways of treating the world problem which are by no means unjustified and ought not to be repudiated on the ground that they are non-scientific, sometimes unscientific, or even anti-scientific and purely sentimental. Religious cosmogonies, ecclesiastical ceremonies, religio-poetical fictions possess values of their own which can not and should not be measured by the standards of scientific method. The mystic also has his right to confront the world with his emotions and visions. Nevertheless, even here the philosophy of science will be capable of investigating various products of these tendencies and has a right to evaluate their truth or untruth by tracing the meaning of allegorical poetry as well as the wholesomeness of ethical attitudes which they encourage. In this way the philosophy of science as worked out by the present writer has by no means been narrow but has granted a free scope to all legitimate tendencies of the human mind, and if the philosophy of science has been properly understood, leaders of thought in the movements of pragmatism, relativism, Bergsonianism and other modern tendencies, would have been able to avoid at least some of their aberrations, and could have devoted their energies to efforts in the right direction. At any rate they would have been better understood; instead of being classified with philosophy, they would more properly have been regarded as a new species of poetry, or as literary ebullitions. Such they are; as such they possess value. They are not philosophy, certainly not philosophy in the strict sense of the word; they are not scientific world-conceptions.

It may appear strange to class the movement which proclaims the principle of relativity in the same category with pragmatism and other antiscientific tendencies. We do so because the relativists have much in common with pragmatists, because both cancel the ideal of objectivity, both identify truth with the subjective conception of the real or with the observer's statement of facts. They identify size with result of measurement and think that the traditional view of truth is an error.

We do not overlook the fact that the relativists are of a highly intellectual type and employ scientific methods, but their aim is after all a denial of the old ideal of science, of the objectivity of truth, and of clearness of thought. All this is surrendered for the sake of a purely subjective simplification of statement which recommends itself in their own specialty. Certainly there is a great difference between relativists and pragmatists, but we recognize in both a subjectivist tendency and a subjectivist aim. Neither of them feel the need of approximating objectivity and both indulge in ideal constructions, both build air castles, the former of mathematical fiction, the latter of philosophical poetry.

All these modern anti-scientific isms may have originated through the one-sided tendencies of a misapplied scientism or even through the lack of comprehension of the principles and the significance of science among naturalists. These isms emphasize therefore certain contentions which have a nucleus of truth, by insisting on the rights of sentiment though they go too far when attacking science itself and claiming a superiority for unscientific sentiment over clear and methodical thought.

There is no question that all these modern movements try each in its own way to satisfy legitimate tendencies, but in doing so they have mostly gone astray; partly they misunderstand their own aspirations, partly they lack sufficient depth of comprehension and width of horizon in encompassing the whole realm of human endeavor.

We do not expect that in this partisan scramble of various prejudices, the whole world of thinkers can be induced to recognize the common ideal of philosophical thought, but we hope that there will be enough minds to understand the several movements, to appreciate them so far as their aspirations are legitimate, and to discover their weak points in which they stray away from the straight path that leads forward to a truer, deeper and a broader conception of the world.

EDITOR.

ATOMIC THEORIES OF ENERGY.

A THEORY involving some sort of a discrete or discontinuous structure of energy has been put forward by Prof. Max Planck of the University of Berlin. The various aspects of this theory are discussed and elaborated by the late M. Henri Poincaré in a paper entitled "L'Hypothèse des Quanta," published in the *Revue Scientifique* (Paris, Feb. 21, 1912).

A paper in which a discontinuous or "atomic" structure of energy was suggested was prepared by the present writer fifteen years ago but remains unpublished for reasons that will appear later. Although he has no desire to put in a claim of priority and is well aware that failure to publish would put any such claim out of court, it seems to him that in connection with present radical developments in physical theory the paper, together with some correspondence relating thereto, has historical interest. Planck's theory was suggested by thermodynamical considerations. In the paper now to be quoted the matter was approached from the standpoint of a criterion for determining the identity of two portions of matter or of energy. The paper is as follows:

SOME CONSIDERATIONS ON THE IDENTITY OF DEFINITE PORTIONS OF ENERGY.

It has been remarked recently that physicists are now divided into two opposing schools according to the way in which they view the subject of energy, some regarding it as a mere mathematical abstraction and others looking upon it as a physical entity, filling space and continuously migrating by definite paths from one place to another. It may be added that there are numerous factions within these two parties; for instance, not all of those who consider energy to be something more than a mere mathematical expression would maintain that a given quantity of it retains its identity just as a given quantity of matter does. In fact a close analysis would possibly show that opinions are graded very closely and continuously from a view hardly differing from that of Lagrange, who clearly saw and freely used the mathematical considerations involving energy before the word had been invented or its physical meaning developed, up to that stated recently in its extreme form by Professor Ostwald, who would replace what he terms a mechanical theory of the universe by an "energetical" theory, and would dwell exclusively on energy as opposed to its vehicles.

Differences of opinion of this sort very frequently reduce to differences of definition, and in this case the meaning of the word "identity" or some similar word or phrase has undoubtedly much to do with the view that is taken of the matter. It may be interesting, for instance, to look for a moment at our ideas of the identity of matter and the extent to which they are influenced by the accepted theory of its constitution.

Very few persons would hesitate to admit that the matter that now constitutes the universe is identical in amount with that which constituted it one million years ago, and that any given portion of that matter is identical with an equal amount of matter that then existed, although the situations of the parts of that portion might be and probably were widely different in the two cases. To assert this is of course a very different thing from asserting that the identity of the two portions or any parts thereof could have been practically shown by following them during all their changes of location or state. That cannot be done even in the case of some simple changes that are effected in a fraction of a second. For instance, if water from the pail A be mixed with water from the pail B there is no possible way of telling which pail any given portion of the mixture came from or in what proportions, yet it is certain that such portion is identical with a portion of equal mass that recently occupied part of one or both pails.

How far our certainty as to this is influenced by our ideas regarding the ultimate constitution of the water is worthy of investigation. All who accept the molecular theory, for instance, will regard our inability to trace the elements of a mixture as due to purely physical limitations. A set of Maxwell's "demons" if bidden to watch the molecules of the water in pail A, one demon being assigned to each molecule, would be able to tell us at any time the precise proportions of any given part of the mixture. But if we should not accept the molecular theory and believe for instance, that water is a continuum, absolutely homogeneous, no matter how small portions of it be selected, then our demons would be as powerless as we ourselves now are to trace the constituents in the mixture.

We are now in a position to ask the question: Is the matter in a mixture of two continua identical with that of its constituents? The identity certainly seems of a different kind or degree from that which obtains in the first case, for there is no part, however small, that was derived from one pail alone. The mixture is something more than a mere juxtaposition of elements each of which has retained its identity; it is now of such nature that no part of it is identical with any part of A alone or of B alone, nor of A+B, where the sign + denotes simple juxtaposition. It is identical, to be sure, with a perfect mixture of certain parts of A and B, but this is simply saying that it is identical with what it is now, that is, with itself, not with something that went before.

Probably no one now believes that water or any other kind of matter is a continuum, but the bearing of what has been said may be seen when we remember that this is precisely the present stage of our belief regarding energy.

No one, so far as I know, has ventured to suggest what may be termed a molecular theory of energy, a somewhat remarkable fact when we consider the control now exercised over all thought in physics by molecular theories of matter. While we now believe, for instance, that a material body, say a crystal, can by no possibility increase continuously in mass, but must do so step by step, the minimum mass of matter that can be added being the molecule, we believe on the contrary that the energy possessed by the same body can and may increase with absolutely perfect continuity, being hampered by no such restriction.

It is not the purpose of this paper to discuss whether we have grounds for belief that there is such a thing as a minimum quantity, or atom, of energy, that does not separate into smaller parts, no matter what changes it undergoes. Suffice it to say that there appears to be no *a priori* absurdity in such an idea. At first sight both matter and energy appear non-molecular in structure. But we have been forced to look upon the gradual growth of a crystal as a step-by-step process, and we may some day, by equally cogent considerations, be forced to regard the gradual increase of energy of an accelerating body as also a step-by-step process, although the discontinuity is as invisible to the eye in the latter case as in the former.

Without following this out any farther, however, the point may be here emphasized that it is hardly possible for one who, like the majority of physicists, regards matter as molecular and energy as a continuum, to hold the same ideas regarding the identity of the two. Efforts to show that definite portions of energy, like definite portions of matter, retain their identity have hitherto been made chiefly on the lines of a demonstration that energy travels by definite and continuous paths in space just as matter does. This is very well, but it would appear to be necessary to supplement it with evidence to show that the lines representing these paths do not form at their intersections continuous blurs that not only forbid any practical attempt at identification on emergence, but make it doubtful whether we can in any true sense call the issuing path identical with the entering one. Otherwise the identity of energy can be admitted to be only that kind of identity that could be preserved by matter if its molecular structure did not exist. One who can admit that this sort of identity is the same sort that can be preserved by molecular matter may be able to hold the identity of energy in the present state of the evidence, but the present attitude of physicists would seem to show that, whether they realize the connection of the two subjects or not, they cannot take this view. In other words, modern views of the identity of matter seem closely connected with modern views of its structure, and the same connection will doubtless hold good for energy.

Regarding the probable success of an attempt to prove that energy has a "structure" analogous to the molecular structure of matter, any prediction would doubtless be rash just now. The writer has been unable, up to the present time, to disprove the proposition, but the subject is one of corresponding importance to that of the whole molecular theory of matter and should not be entered upon lightly.

* * *

The writer freely acknowledges at present that the illustrations in the foregoing are badly chosen and some of the statements are too strong, but it still represents essentially his ideas on the subject. No reputable scientific journal would undertake to publish it. The paper was then sent to Prof. J. Willard Gibbs of Yale, and elicited the following letter from him:

"NEW HAVEN, June 2/97.

"MY DEAR MR. BOSTWICK:

"I regret that I have allowed your letter to lie so long unanswered. It was in fact not very easy to answer, and when one lays a letter aside to answer, the weeks slip away very fast.

"I do not think that you state the matter quite right in regard to the mixture of fluids if they were continuous. The mixing of water as I regard it would be like this, if it were continuous and not molecular. Suppose you should take strips of white and red glass and heat them until soft and twist them together. Keep on drawing them out and doubling them up and twisting them together. It would soon require a microscope to distinguish the red and white glass, which would be drawn out into thinner and thinner filaments if the matter were continuous. But it would be always only a matter of optical power to distinguish perfectly the portions of red and white glass. The stirring up of water from two pails would not really mix them but only entangle filaments from the pails.

"To come to the case of energy. All our ideas concerning energy seem to require that it is capable of gradual increase. Thus the energy due to velocity can increase continuously if velocity can. Since the energy is as the square of the velocity, if the velocity can only increase discontinuously by equal increments, the energy of a body will increase by unequal increments in such a way as to make the exchange of energy between bodies a very awkward matter to adjust.

"But apart from the question of the increase of energy by discontinuous increments, the question of relative and absolute motion makes it very hard to give a particular position to energy. Since the 'energy' we speak of in any case is not one quantity but may be interpreted in a great many ways. Take the important case of two equal elastic balls. One, moving, strikes the other at rest, we say, and gives it nearly all its energy. But we have no right to call one ball at rest and we can not say (as anything absolute) which of the balls has lost and which has gained energy. If there is such a thing as absolute energy of motion it is something entirely unknowable by us. Take the solar system, supposed isolated. We may take as our origin of coordinates the center of gravity of the system. Or we may take an origin with respect to which the center of gravity of the solar system has any (constant) velocity. The kinetic energy of the earth, for example, may have any value whatever, and the principle of the conservation of energy will hold in any case for the whole solar system. But the shifting of energy from one planet to another will take place entirely differently when we estimate the energies with reference to different origins.

"It does not seem to me that your ideas fit in with what we know about nature. If you ask my advice, I should not advise you to try to publish them.

"At best you would be entering into a discussion (perhaps not in bad company) in which words would play a greater part than precise ideas.

"This is the way I feel about it.

"I remain

"Yours faithfully,

J. W. Gibbs."

Professor Gibbs's criticism of the illustration of water-mixture is evidently just. Another might well have been used where the things mixed are not material—for instance the value of money deposited in a bank. If A and B each deposits \$100 to C's credit and C then draws \$10, there is evidently no way of determining what part of it came from A and what from B. The structure of "value," in other words, is perfectly continuous. Professor Gibbs's objections to an "atomic" theory of the structure of energy are most interesting. The difficulties that it involves are not overstated. In 1897 they made it unnecessary, but since that time considerations have been brought forward, and generally recognized, which may make it necessary to brave those difficulties.

Planck's theory was suggested by the apparent necessity of modifying the generally accepted theory of statistical equilibrium involving the so-called "law of equipartition," enunciated first for gases and extended to liquids and solids.

In the first place the kinetic theory fixes the number of degrees of freedom of each gaseous molecule, which would be three for argon, for instance, and five for oxygen. But what prevents either from having the six degrees to which ordinary mechanical theory entitles it? Furthermore, the oxygen spectrum has more than five lines, and the molecule must therefore vibrate in more than five modes. "Why," asks Poincaré, "do certain degrees of freedom appear to play no part here; why are they, so to speak, 'ankylosed'?" Again, suppose a system in statistical equilibrium, each part gaining on an average, in a short time, exactly as much as it loses. If the system consists of molecules and ether, as the former have a finite number of degrees of freedom and the latter an infinite number, the unmodified law of equipartition would require that the ether should finally appropriate all energy, leaving none of it to the matter. To escape this conclusion we have Rayleigh's law that the radiated energy, for a given wavelength, is proportional to the absolute temperature, and for a given temperature is in inverse ratio to the fourth power of the wave-length. This is found by Planck to be experimentally unverifiable, the radiation being less for small wave-lengths and low temperatures, than the law requires.

Still again, the specific heats of solids, instead of being sensibly constant at all temperatures, are found to diminish rapidly in the low temperatures now available in liquid air or hydrogen and apparently tend to disappear at absolute zero. "All takes place," says Poincaré, "as if these molecules lost some of their degrees of freedom in cooling —as if some of their articulations froze at the limit."

Plank attempts to explain these facts by introducing the idea of what he calls "quanta" of energy. To quote from Poincaré's paper:

"How should we picture a radiating body? We know that a Hertz resonator sends into the ether Hertzian waves that are identical with luminous waves: an incandescent body must then be regarded as containing a very great number of tiny resonators. When the body is heated, these resonators acquire energy, start vibrating and consequently radiate.

"Planck's hypothesis consists in the supposition that each of these resonators can acquire or lose energy only by abrupt jumps, in such a way that the store of energy that it possesses must always be a multiple of a constant quantity, which he calls a 'quantum'—must be composed of a whole number of quanta. This indivisible unit, this quantum, is not the same for all resonators; it is in inverse ratio to the wave-length, so that resonators of short period can take in energy only in large pieces, while those of long period can absorb or give it out by small bits. What is the result? Great effort is necessary to agitate a short-period resonator, since this requires at least a quantity of energy equal to its quantum, which is great. The chances are, then, that these resonators will keep quiet, especially if the temperature is low, and it is for this reason that there is relatively little short-wave radiation in 'black radiation.'... The diminution of specific-heats is explained similarly: When the temperature falls, a large number of vibrators fall below their quantum and cease to vibrate, so that the total energy diminishes faster than the old theories require."

Here we have the germs of an atomic theory of energy. As Poincaré now points out, the trouble is that the quanta are not constant. In his study of the matter he notes that the work of Prof. Wilhelm Wien, of Würzburg, leads by theory to precisely the conclusion announced by Planck that if we are to hold to the accepted ideas of statistical equilibrium the energy can vary only by quanta inversely proportional to wave-length. The mechanical property of the resonators imagined by Planck is therefore precisely that which Wien's theory requires. If we are to suppose atoms of energy, therefore, they must be variable atoms. There are other objections which need not be touched upon here, the whole theory being in a very early stage. To quote Poincaré again:

"The new conception is seductive from a certain standpoint: for some time the tendency has been toward atomism. Matter appears to us as formed of indivisible atoms; electricity is no longer continuous, not infinitely divisible, it resolves itself into equally-charged electrons; we have also now the magneton, or atom of magnetism. From this point of view the quanta appear as *atoms* of *energy*. Unfortunately the comparison may not be pushed to the limit; a hydrogen atom is really invariable. . . . The electrons preserve their individuality amid the most divers vicissitudes, is it the same with the atoms of energy? We have, for instance, three quanta of energy in a resonator whose wave-length is 3; this passes to a second resonator whose wave-length is 5; it now represents not 3 but 5 quanta, since the quantum of the new resonator is smaller and in the transformation the number of atoms and the size of each has changed."

If, however, we replace the atom of energy by an "atom of action," these atoms may be considered equal and invariable. The whole study of thermodynamic equilibrium has been reduced by the French mathematical school to a question of probability. "The probability of a continuous variable is obtained by considering elementary independent domains of equal probability.... In the classic dynamics we use, to find these elementary domains, the theorem that two physical states of which one is the necessary effect of the other are equally probable. In a physical system if we represent by q one of the generalized coordinates and by p the corresponding momentum, according to Liouville's theorem the domain $\int\int dpdq$, considered at a given instant, is invariable with respect to the time if p and q vary according to Hamilton's equations. On the other hand p and q may, at a given instant take all possible values, independent of each other. Whence it follows that the elementary domain is infinitely small, of the magnitude $dpdq$.... The new hypothesis has for its object to restrict the variability of p and q so that these variables will only change by jumps....Thus the number of elementary domains of probability is reduced and the extent of each is augmented. The hypothesis of quanta of action consists in supposing that these domains are all equal and no longer infinitely small but finite and that for each $\int\int dp\, dq = h$, h being a constant."

Put a little less mathematically, this simply means that as energy equals action multiplied by frequency, the fact that the quantum of energy is proportional to the frequency (or inversely to the wave-length as stated above) is due simply to the fact that the quantum of action is constant—a real atom. The general effect on our physical conceptions, however, is the same: we have a purely discontinuous universe—discontinuous not only in matter but in energy and the flow of time. M. Poincaré thus puts it:

"A physical system is susceptible only of a finite number of distinct states; it leaps from one of these to the next without passing through any continuous series of intermediate states."

He notes later:

"The universe, then, leaps suddenly from one state to another; but in the interval it must remain immovable, and the divers instants during which it keeps in the same state can no longer be discriminated from one another; we thus reach a conception of the discontinuous variation of time —the *atom of time*."

I quote in conclusion, Poincaré's final remarks:

"The present state of the question is thus as follows: the old theories, which hitherto seemed to account for all the known phenomena, have met with an unexpected obstacle. Seemingly a modification becomes necessary. A hypothesis has presented itself to M. Planck's mind, but so strange a one that one is tempted to seek every means of escaping it; these means, however, have been sought vainly. The new theory, however, raises a host of difficulties, many of which are real and not simply illusions due to the indolence of our minds, unwilling to change their modes of thought.....

"Is discontinuity to reign throughout the physical universe, and is its triumph definitive? Or rather shall we find that it is but apparent and hides a series of continuous processes?....To try to give an opinion just now on these questions would only be to waste ink."

It only remains to call attention again to the fact that this conception of the discontinuity of energy, the acceptance of which Poincaré says would be "the most profound revolution that natural philosophy has undergone since

Newton" was suggested by the present writer fifteen years ago. Its reception and serious consideration by one of the first mathematical physicists of the world seems a sufficient justification of its suggestion then as a legitimate scientific hypothesis.

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CRITICISMS AND DISCUSSIONS.

HENRI BERGSON, PRAGMATISM AND SCHOPENHAUER.

The history of philosophy is like one of the ancestral galleries in ancient European castles. At first glance you find yourself before a bewildering variety of individuals, but if you look closer you discover that certain family traits reappear again and again. In the history of philosophy a similar variety of individual systems at first confuses the student. But upon closer scrutiny he will find that here too the bewilderment ceases, that certain outlines are typical for the structure of a great number of systems and that almost every individual system belongs to such a type of structure. Considered in this light the history of philosophy presents a few types of thought which undergo slight changes and show a slow development according to the intellectual conditions of the century in which the philosopher moulded his system.

The systematic structure to be considered in this paper is a very modern one, namely Henri Bergson's philosophy. From a purely philosophical standpoint it was severely criticized in the last number of this magazine.[1] In this number it may be scrutinized from a purely historical standpoint.

Among laymen Bergson's name carries with it a certain feeling of mystic refinement. However little they may know about him, they instinctively expect such an appeal to their artistic natures as they would from a sculpture by Auguste Rodin or a drama by Maeterlinck. They instinctively feel the kind of a man who is about to confront them and their feeling is probably correct.

Feeling, however, is quite a different thing from knowing. Strange to say, philosophers are greatly at variance as to the place of Bergson's philosophy in the gallery of philosophical systems.

[1] See "The Philosophy of Bergson" by Mr. Bertrand Russell, *Monist,* July, 1912. Cf. also in the same number, "Bergson and Religion" by Dr. James G. Townsend, and "Kant and Bergson" by Dr. Bruno Jordan.

Here is the struggle of a man's mind, some philosophers say, which can be compared in importance only with the philosophy of Kant, while still very different. Others are reminded of Hegel and others of Berkeley; while American pragmatists say that Bergson is a pragmatist. Others again are of the opinion that it is quite out of the question to compare Bergson at all with philosophers of any other school. They say he is unique; he is no type, but has an individuality of his own.

We shall see. It is certain however that it would be rather strange if Bergson were not a type, notwithstanding his marked individuality; if his philosophy really bore no relation to philosophies of former days. This would mean that in the development of philosophical traditions a structure of philosophy had arisen no likeness to which had ever been seen before, that a child without ancestors had been brought to light; but this is not very probable. To inquire therefore after the historical antecedents of Bergson's philosophy would mean to ask whether its structure is entirely new or only a transformation of what already existed, and in the latter case the questions arise, who used this structure before Bergson? Who among the philosophers of former days may be called nearest akin to him? Which philosophical tradition is continued by Bergson's thought? These are the questions which I shall try to answer in this paper.

* * *

Which philosophical tradition is continued by Bergson? "The pragmatistic one," say the American pragmatists; "Bergson is a pragmatist." I do not hesitate to agree that Bergson really is a pragmatist, and here are the proofs.

Pragmatism holds that what we call reality, world, object of knowledge, is not something independent of us, but rather a man-made picture, a raw material transformed into a complicated instrument for action. "The world is only an opportunity to do our duty," Fichte was wont to say. "The world is only an expedient for our action," the pragmatists of to-day tell us. Seeing, hearing, smelling, touching the world, as well as considering it as a multitude of atoms, electrons, ions, means nothing else than a preparation to grapple with the world.

This doctrine becomes very evident in Bergson's philosophy. For Henri Bergson the essence of man is life. What we call man is not so much the human body nor a soul-substance within that body, but rather a dramatic performance, a continual course of events, of visions, sounds, feelings, images, thoughts and especially of actions. Man means an *élan vital,* a life-current running along seventy, eighty or ninety years.

If this is so, what is the business of that life-current? To what purpose is that action acting? In agreement with the biologists Bergson would answer that the business of life is to find its way among other life-currents. But if we look closer we shall see that these other life-currents differ widely from one another, that man treats them at least in very different ways, some of them as matter only and others as matter but also as souls.

For the present we shall disregard life-currents treated as souls. As far as their treatment as pure matter is concerned, Bergson is among the pragmatists. Man is different from most animals in that he has hands and uses them not only for climbing trees as the monkeys do but for providing instruments. It would not be too bold to say that man has made his hands longer and stronger by utilizing matter for instruments. The axe, the arrow, the target are such artificial hands, not to mention the enormous development of inventions during the last centuries, all tending to the same end.

This leads to the question of how this mechanical point of view influences our knowledge of reality. The answer is very simple. Since man intends to use reality for instrumental purposes it appears to him entirely as an object of action—either to act upon or to act with.

We seldom realize how much difference such a point of view makes in our knowledge. A desk in my study may serve as a simple instance. Everybody agrees that it is a desk. But "desk" means an *instrument* either to put a manuscript on, to read a book on, to fit a lamp on, etc. In other words, from the very beginning we all acknowledge a certain piece of wood as a *practical* object destined for *practical* purposes, and it would be surprisingly hard for us to free ourselves from this impression, to look at that object and not identify it with a desk. But desk means instrument and instrument means possibility of action. I shall return to this point very soon.

Meanwhile another reflection may serve to clear up this situation still further. I say that to all of us the desk appears as a desk. We cannot help it. Still the impression of the desk is by no means the same to all of us, for in addition to its being acknowledged as a desk it may be recognized in many other ways. Let me suppose that a dealer in woods happens to examine my desk. He will see many features in it which other people do not observe. He will not only infer but actually see the quality of the wood and probably tell its cash-value off-hand; while all that other people see is that it is "apparently of oak" and possibly that it looks "rather nice." The merchant being a connoisseur actually sees more than other people. Now perhaps a carpenter looks at the desk, then an artist, then a botanist, then a chemist, then a physicist. The layman would simply laugh when the botanist says that the desk is a conglomeration of cells, or the physicist that it is a heap of molecules, or the chemist that it is a multitude of atoms. Still these are the ways in which the desk would be regarded by these men. It is only the *point of view* which is different.

In other respects however we all are taking the same point of view, for to all of us, whether layman, merchant or physicist, the desk appears as a desk, and if it does not appear as a desk, it would at least appear as a table. And if it does not appear as a table it would at least appear as a "thing," and "thing" always means, if we believe Bergson, *something to act upon or to act with.*

It is only the *point of view* which is different. In other respects however all are taking the same point of view, for to all of us, whether layman, merchant or physicist, the desk appears as a desk, or if not as a desk, at least as a table. And if it does not appear as a table it would at least appear as a "thing," and "thing" always means, if we believe Bergson, *something to act upon or to act with.*

Hence, in considering the world as an accumulation of things we have already taken a certain limited point of view; we are on the way to treat the universe as an object and eventually as a means of *action.* We have taken the decisive step, and cannot now go back. We have made the start in our calculation and it will proceed accordingly.

* * *

It is interesting that Bergson considers three-dimensional space as one of the most important elements in the method by which the world is conceived as a means for action. To see things in space means to consider them as objects to act upon. Space is a kind of uniform which we put upon the world in order to control it, for to control things we cannot care what they are in themselves; we must care what they may be to us. Hence we deprive them of their very essence; we treat them by a scheme, which enables us to divide things up quite at will—while according to Bergson things in themselves are indivisible, although, as he expresses it, they have a certain "ballast" of that scheme.

Our three-dimensional space is a scheme for division and nothing else; its very homogeneity is the means by which we divide things. This is most apparent in theoretical physics where all plurality of our pluralistic world is eliminated and only shadowy colorless sections of space remain where before we saw blue and red and green and yellow, where we heard noise and music, smelled odors and tasted sweet and bitter. Theoretical physics is space triumphant, which means it is a triumph of practical handling of the world. Plurality once eliminated from the world there is no limit for divisions. We may divide indefinitely, and dividing will doubtless contribute much toward our practical control over the world.

* * *

It is a very significant fact that we treat time in a similar way again and again; we spatialize it. If somebody knocks at the door three times in succession and we try to recall that succession of knocks, we discover in ourselves a certain inability to do so. Instead of representing the three knocks as a real sequence in time we find ourselves fancying them one beside or behind the other, quite as though they were three pearls on a cord or three blots on a line. We are almost compelled to do so; for to recall something means to see it all at once, and to see all at once is precisely not to see one part after the other in succession, but to perceive them simultaneously. Hence we spatialize time, and in so doing we are tending again towards its homogeneity. Of course we are not treating time as three-dimensional space—that would not do; but modern scientists treat it as a "fourth dimension" of space, and common sense regards it as a one-dimensional line. The next step in this procedure will be to eliminate all plurality of concrete experience from that line, to divide up time quite as arbitrarily as space and to deal with time too, regardless of its contents, as suits our purpose. The dial on a watch is nothing but a graphic demonstration of such a one-dimensional time-line, its straightness being turned into a circle.

Bergson's idea of the business of understanding is now sufficiently clear. Understanding is an instrument by which human life works its way through its surroundings. Moreover it is an instrument by which human life continually makes use of homogeneous schemes, treating simultaneous impressions by space and successive impressions by spatialized time. This threefoldness of understanding

—practical use, homogeneous space and spatialized time—may still become important for our later discussion. For the present it suffices to state that understanding is to Bergson an instrument for practical use. This is the reason why in America he is called a pragmatist.

* * *

And now in seeming contradiction to my own words, and in open contradiction to the theory of most American philosophers, I say that *Bergson is not a pragmatist.* He is the very *foe* of the pragmatists; for to tell the truth, what pragmatism advocates, Bergsonian philosophy *opposes.* It is one of the main features of this philosophy to *disregard* the whole realm of understanding as a realm for practical use. To Bergson's mind it is precisely the pragmatic attitude that *hinders* understanding from entering philosophy—and Bergson is after all a philosopher. He leaves to pragmatism the realm of science and common sense, but *in philosophy he protests against it.*

To Bergson's mind *philosophy begins where pragmatism ceases.* To be a Bergsonian philosopher I must rid myself of pragmatistic habits. It is pragmatistic to look at everything from the point of view of action, hence it is philosophic to discard that point of view. It is pragmatistic to consider the world as wrapped in space, hence it is philosophic to free the world from space. It is pragmatistic to spatialize time, hence it is philosophic to remove that spatialization.

A Bergsonian philosopher is a thinker freed from all pragmatism. He no longer looks for the practical use of things, but looks to things for their own sake. His mind no longer works to make headway for life, but it turns itself round and looks at life itself as it goes on within him. Bergsonian philosophy is consciousness of life itself freed from its practical service.

* * *

And now our thoughts do not leave Bergson in passing over to Schopenhauer and attempting to throw light on Bergson's ideas by the philosophy of Schopenhauer and on Schopenhauer's ideas by the philosophy of Bergson.

Like Bergson, Schopenhauer is a pragmatist; like the pragmatism of Bergson, that of Schopenhauer holds good for understanding only; and as for Bergson, so for Schopenhauer, philosophy begins just where understanding and pragmatism cease.[2]

[2] Cf. Schopenhauer, *Werke* (ed. Grisebach), I, p. 242.

Thus *understanding* plays quite a similar rôle with Schopenhauer as with Bergson. It is very significant that Bergson and Schopenhauer use the same simile. Both call understanding "a lantern" which life has kindled in order to find its way through the world. For both this "lantern," originally used for a limited service, has extended its light more and more until now it shines over the whole universe, or at least what it calls a "universe."

Now the "universe" over which this lantern sheds its light is matter. Our earth and all that lives on earth is matter. The moon, the sun, the planets, all solar systems are matter. But for Bergson as for Schopenhauer "matter" is nothing but reality treated by the methods of understanding, covered with three-dimensional space, put into spatialized time and considered pragmatically for the single purpose of action.

For Bergson matter is simply correlative to understanding; it is the only means by which understanding knows reality. Schopenhauer takes exactly the same position. "Matter," he writes, "and hence the appearance of the whole universe, is there for understanding only. Understanding is its support, the condition of its very existence; it is its necessary correlatum."[3]

It is important to realize that this coincidence between Bergson and Schopenhauer is not insignificant, but indicates a very interesting and far-reaching identity in the main structure of their systems. That understanding is limited to matter and matter limited to understanding is of decisive consequence for the whole development of both philosophies.

* * *

But the coincidence between Bergson's and Schopenhauer's idea of understanding goes much farther than this; for precisely the same *three functions of understanding* with regard to matter pointed out by Bergson are likewise pointed out by Schopenhauer. If Bergson declares that the only aim of understanding is to have the possibility of *acting* upon its environment, and that this is the reason why it *materializes* everything, Schopenhauer would say that *causality* is the only category of understanding, for understanding is there solely to *act* upon its environment. That is the service forced upon understanding by the all-powerful will to live, and that too is the main reason why understanding transforms everything into *matter*. In other words, Bergson's pragmatism of understanding and Schopenhauer's doctrine of causality as the only

[3] *Werke,* II, p. 160.

function of understanding tend precisely towards the same point, a point of greatest consequence, in fact a cornerstone in the structure of both systems.

For Schopenhauer as for Bergson understanding stands for activity and for activity only. Its functions to provide for *space* and *time,* are subordinate to that main function. They are the means by which understanding succeeds in materializing the world for the purpose of action.

When Bergson says that in order to handle the world for action understanding covers it with three-dimensional homogeneous space, Schopenhauer would say that three-dimensional homogeneous *space* is one of the two *indispensable intuitions a priori* of understanding. That space is *a priori* for Schopenhauer and possibly *a posteriori* for Bergson is of no consequence for this part of our comparison. But it certainly is of very great consequence that in the philosophy of both understanding uses space as its most important instrument in materializing the world for the purpose of action.

The other instrument by which understanding materializes the world is *time.* Bergson says that for this purpose time is spatialized. Schopenhauer declares that time is the other of the two *"intuitions a priori"* of understanding. Now there is apparently no similarity at all between the ideas of Schopenhauer and Bergson; and still, on looking closer, a careful observer will discover that there is a similarity, and indeed a far greater one than would be suspected. I venture to call special attention to this point.

What distinctive feature of space makes it appear to Bergson particularly adapted for the practical purpose of understanding? Certainly not that space is three-dimensional, but that three-dimensional space is *homogeneous.* Space is fit for action solely because of its homogeneity. With homogeneity arbitrary divisibility is possible, and with arbitrary divisibility understanding finds its way to handle reality instrumentally—that is all.

Let us pass to time. What distinctive feature of time makes it appear to Bergson as being particularly adapted for the practical purposes of understanding? Bergson finds this distinctive feature in the fact that we "spatialize" time. But what does he mean by spatializing time? Again nothing but an attempt to treat time like space, namely as a *homogeneous* something. It is not the one dimension nor the straightness of the line, that "spatializes" time; but its uniformity, its homogeneity, its divisibility at any point, and hence its possibility to be handled instrumentally. Homogeneity, divisibility at any point, is the only quality for which understanding cares. It is wholly indifferent to everything else in space and in spatialized time.

Passing to Schopenhauer we discover that the reason why he, like his master Kant, considers space and time as "*intuitions a priori*" is again their all-embracing *homogeneity.* The reader remembers Kant's demonstration of this in his Critique of Pure Reason. Space and time have the peculiarity of not being "concepts," but "intuitions," because they are spread out with absolute uniformity, with absolute homogeneity, which does not allow the distinction between different (three-dimensional) "spaces." Furthermore these "intuitions" are *a priori* because they are necessary; they are necessary because they are supposed to cover everything; and they are supposed to cover everything once more because they are absolutely homogeneous—a uniform scheme which nothing can escape. Space and time are for Schopenhauer and Kant *intuitions a priori* in so far only as they are *homogeneous schemes,* to be placed over a world of immediate images.

Schopenhauer's idea of time, as an "*intuition a priori*" is the idea of a homogeneous scheme to be placed over a world of immediate images. But this is exactly what Bergson calls *spatialized time.* Therefore when Bergson contends that understanding as the correlative of matter and an instrument for action makes use not only of space but also of "spatialized time," and Schopenhauer contends that understanding as the correlative of matter and an instrument for action makes use not only of space but also of time as an "intuition *a priori*" they mean the very same thing. They only use a different vocabulary. It is essential for both that space and time are homogeneous schemes and belong to understanding as means of turning immediate images into matter and of handling matter practically. I shall very soon return to this point.

In the meantime I venture to sum up the result, thus far reached in our investigation, in the statement that the theories of Bergson and Schopenhauer with regard to understanding, matter, space and time are essentially the same. And since half of the entire structure of the philosophies of these two men is constituted by those theories, their sameness means a corresponding sameness of half that structure.

* * *

We now proceed to the second half of that structure. Opposed to understanding and matter there stands in Schopenhauer's philosophy the "*will to live.*" Opposed to understanding and matter there stands in Bergson's philosophy a "*life-current,*" the "*élan vital.*"

For Bergson as well as for Schopenhauer the world of understanding is only a world of "appearance." It is not true reality, but reality prepared for action. Hence, by understanding we are not in a position to penetrate into the essence of reality. We are getting only to the surfaces. We go around things, but cannot enter them.

We previously noted that Bergson used a simile originally used by Schopenhauer. Both compare understanding to a "lantern" which life has kindled in order to find its way through the world. Now we discover for a second time that Bergson and Schopenhauer use one and the same simile in precisely the same way. Schopenhauer writes with regard to the inability of understanding to enter life: "We see that it is *impossible to reach the inner essence of things by external means*; however much we thus investigate we find only images and empty names. We are like the man who walks around a castle, looks in vain for an entrance into it and in the meantime *sketches its walls.*"[4] Compare with this Bergson's account of understanding trying in vain to enter life: "It is like the work of an artist," he writes, "who traveling through Paris *sketches a tower* of Notre Dame....This designer *replaces the true inner essence of the object by an external schematic reproduction.*"[5]

It is the way of understanding to approach its objects by external means only, and in doing so it tries to comprehend the life inherent in them by dividing the object up into small parts. This, however, is in vain. Understanding can never enter life in this way, for the only true way of coming into contact with life is by feeling it instinctively.

Schopenhauer illustrates this by an impressive simile: "Abstract knowledge," he writes, "compares with instinctive feeling *as a mosaic compares with a painting* by Van der Werft or Denner; for *however nicely the mosaic may be put together,* the outlines between the stones remain and *no continual transition* from one shade of color to another is possible."[6] Bergson holds the same view, and it is surprising that in another context, which however carries the same meaning, he again uses the very same comparison originally used by Schopenhauer. "A gifted artist paints a picture," he writes.

[4] *Werke,* I, p. 150.

[5] Bergson, "Introduction à la philosophie," *Revue de métaphysique et de morale,* 1903, p. 10.

[6] *Werke,* I, p. 98 f.

"We could *imitate his picture by many colored stones. The more nicely our stones are put together* the more manifold and different they are in color, the better we shall be able *to reproduce their curves and shades. But we should need an infinite number of infinitely small and infinitely differentiated stones* in order to reproduce the picture exactly."[7]

The idea in common, expressed by Schopenhauer and Bergson by exactly the same simile, is that life cannot be perceived by external dissection and recomposition but only by instinctive feeling. This idea is of decisive importance for Bergson's own thought as well as for Schopenhauer's, and our discovery that Bergson and Schopenhauer use the same similes in this context several times indicates that their coincidence is more than a mere curiosity.

* * *

For Schopenhauer as for Bergson life is an object not of understanding but of instinct. In fact, Schopenhauer's philosophy might well be headed "A Study in Instinct" as well as Bergson's. It is therefore quite characteristic for both of these thinkers that they are so greatly concerned with instinct as a biological fact; that both of them pay special attention to insects, because in them instinct is most developed. For Bergson as for Schopenhauer life is not only "the object" of instinct. It would be more just to say that to them instinct is the *living* of life; it is life itself; it is really as it is seen directly, while the knowledge of understanding is neither life nor reality but only an indirect way of preparing reality for practical use.

For Schopenhauer as for Bergson the world of instinct as a world of reality and life stands opposed to the world of understanding as a world of appearance and death. This distinction is of greatest importance for Schopenhauer's philosophy as well as for Bergson's.

So Bergson writes: "If one compares the definitions of metaphysics and of the Absolute he observes that all philosophers in spite of their controversies are at one in distinguishing two absolutely different ways of knowing. The one way stops at the relative; the other penetrates to the Absolute, where it is approachable.... The Absolute is perceived by intuition, everything else by analysis. I call intuition the instinctive sympathy by which we put ourselves into the heart of an object in order to unite with its particular in-

[7] Bergson, *loc. cit.*, p. 2.

expressible essence."[8] With this introduction Bergson plunges the reader into what he considers one of the most central points of his philosophy.

Precisely the same way of "knowing the Absolute" is called by Schopenhauer "the philosophical truth κατ' ἐξοχήν" for his own system,[9] and Bergson and Schopenhauer furthermore agree that this central truth of philosophy, instinctive knowledge as a key for life, is limited at first to the knower's own being. "There is at least one reality," Bergson tells us, "which we all perceive from the inside by intuition....that is we, ourselves."[10]

It is the instinct of our own life of which we are thus aware; but unfortunately this instinct is the only reality of which we dispose directly. We have to transfer our own living instinct to the external world in order to conceive the entire life of the universe. This indirect way of knowledge set forth by Professor Bergson is one of the chief principles of Schopenhauer's philosophy. What Schopenhauer calls "will" is nothing else but what Bergson describes as man's own life immediately experienced by instinctive feeling as the "one reality which we all perceive from the inside," and then transferred to the external world. "Will" is a *determinatio a potiori* for "what we immediately experience as the innermost essence" of our own life, Schopenhauer declares,* and furthermore contends that "to him who knows the most immediate datum of consciousness is will....this conviction will become the key for a knowledge of the inmost essence of all nature. For he will now transfer his immediate experience of life to all those objects which are not given to him in immediate experience."[11]

From this it will be more evident why Schopenhauer's "will" and Bergson's *élan vital* are precisely the same thing. The reason is that both are brought forth by the same "philosophical truth κατ' ἐξοχήν." Both "will" and *élan vital* are philosophical expressions for instinctive feeling conceived as "the living of life" and thence transferred to the external world.

* * *

In this world of "will" or *élan vital* the problem of time again appears. We have already dealt with this problem as far as under-

[8] Bergson, *loc. cit.*, p. 1.

[9] *Werke*, I, p. 154.

[10] *Ibid.*, p. 4.

* *Werke*, I, p. 164.

[11] *Werke*, I, p. 162.

standing is involved in it; we have now to deal with it as far as it extends to the instinctive feeling of life.

In this latter realm the problem takes up the form of an antithesis. Life as instinctively felt is according to Bergson, *durée réelle,* true duration, true time, and to Schopenhauer it is "timeless." Here Bergson and Schopenhauer evidently seem to contradict each other, but as in our former discussion the contradiction is rather in words than in thought. "True duration" means for Bergson time *freed from its spatialization.* "Timelessness" means for Schopenhauer time *freed from past and future.* It means "eternal presence."

It certainly is interesting to compare this conception of Schopenhauer's with Bergson's *durée réelle.* If you ask Bergson why we spatialize time, he would answer that we do so in order to bring past and future to the same level; in order even to conceive of a future and of a past. Hence *durée réelle,* as time freed from its spatialization, is time *deprived of the conception of past and future.* But that is precisely what Schopenhauer calls "*timelessness.*" If consistently carried out, Bergson's *durée réelle* viewed from the point of view of the being himself who exists in *durée réelle* is adequately represented only by Schopenhauer's idea of "timelessness" or "eternal presence." Bergson and Schopenhauer, contradicting each other in their expressions, are logically bound to agree in the fact. Schopenhauer's "timelessness" is *durée réelle,* and Bergson's *durée réelle* ought necessarily be conceived as timelessness.

It is very probable that Professor Bergson himself would contradict this statement. He would point to the obvious fact that again and again he has characterized *durée réelle* as involving both past and future, and hence as something entirely different from Schopenhauer's timelessness. This is very true. But it is no less true that if meant as an objection to Schopenhauer's timelessness Bergson's statement would be very inconsistent.

There are two points of view which should not be confounded. Either *durée réelle* is viewed by an observing outsider or by the being himself who exists in *durée réelle.* An outsider can realize that what this being experiences involves past and future, the being himself however can not. The simple reason for this is that according to Bergson's own doctrine spatialization of time is necessary in order to conceive, however vaguely, of any past or any future. But spatialization of time is not *durée réelle.* Hence a being existing in *durée réelle* does not realize that what he experiences involves past and future. He does not know either "past" or "future." He lives in timelessness, because he lives in *durée réelle.*

Durée réelle or "timelessness," viewed from the point of view of an outsider, leads towards the conception of evolution. Learning that his own doctrine of evolution was very much like that of Schopenhauer,[12] Professor Bergson thought this coincidence "a happy inconsistency" on the part of Schopenhauer because of the latter's doctrine of "timelessness"; while in truth the apparently missing coincidence in the doctrine of "timelessness" and *durée réelle* was an unhappy inconsistency on the part of Professor Bergson. There can be no thought of evolution from the point of view of the being who himself lives in *durée réelle,* for no evolution can be thought of except in spatialized time. But, in full compliance with the philosophy of Schopenhauer as well as of Bergson himself, an observing outsider may see a very obvious evolution where the being existing in *durée réelle* sees "timelessness" only.

* * *

I return from the problem of timelessness, *durée réelle* and evolution to the main idea of Schopenhauer's and Bergson's thought, to their "philosophical truth *κατ' ἐξοχην.*" True reality, life, is the instinctive feeling of one's own life-current transferred to the outer world and the world of philosophy as a world of instinct and life is opposed to the dead world of understanding.

This philosophical truth *κατ' ἐξοχην* is founded on the instinctive feeling of everybody's own life. Hence everybody is a philosopher in so far as he is aware of his own life. He only does not try to express his feelings in conceptual language. Technical philosophy therefore is essentially constituted of everybody's instinctive and intuitive feelings enlarged, systematized and changed into knowledge.

Here again Schopenhauer and Bergson express themselves in a very similar way. So Schopenhauer writes: "By intuition or *in concreto* everybody is conscious of all philosophical truths, but to demonstrate them in abstract concepts and reflective thought is the business of philosophy."[13] Compare with this Bergson's words: "Every lasting system of philosophy is enlivened by intuition at

[12] Compare the very interesting article of Prof. Arthur O. Lovejoy of Johns Hopkins University in *The Monist,* XXI, pp. 216 ff.

[13] *Werke,* I, p. 491.

least in some parts. Dialectic is necessary to demonstrate intuition, to mirror it in concepts and to communicate it to others."[14]

The similarity between Bergson's and Schopenhauer's idea is striking even in its expression. Indeed both philosophies belong to the same well-known type of thought that calls itself "intuitive." Every intuitive philosophy somewhat despises abstract concepts.[15] Still when intuition becomes a philosophy abstract concepts are needed, and they are generously used in the writings of our two thinkers in spite of their contention that nobody enters the Absolute in this way. The gulf between understanding and instinct, theoretically established, is practically bridged by philosophic discourse. There the work of understanding is enlivened by instinct, and the life of instinct is expressed in terms of understanding. None the less instinct and understanding remain theoretically entirely different for both Bergson and Schopenhauer, and it is not their difference but their combination which by the latter is considered as the "riddle of the universe."

This riddle of the universe, and corresponding to it the "philosophical truth *κατ' ἐξοχην*" of Bergson and Schopenhauer, fairly sums up what I wanted to state in this part of my paper, namely, that these two philosophies are no less similar in the second half of their structure than they were in its first half. A theory of understanding common to both constitutes the first half, a theory of instinctive life as opposed to understanding no less common to both constitutes the second half of their systems. Thus the entire structure of the thought of both is identical in its main outlines.

* * *

It would be possible to show how in some corollaries further instructive similarities between Bergson's and Schopenhauer's philosophy spring up from that main identity.

I shall allude only to Bergson's theory of evolution, which, as Prof. Arthur O. Lovejoy has pointed out in his very interesting paper, exhibits a striking similarity to Schopenhauer's evolutionism.[16] Bergson's doctrine of the unity of life is strangely similar to Schopenhauer's doctrine of the unity of will. For both thinkers this unity forms the basis of their evolutionism. On the other hand it has been pointed out in this paper that Bergson's theory of dis-

[14] Bergson, *Evolution créatrice,* p. 259; "Introduction à la philosophie," *Revue de métaphysique et de morale,* 1903, p. 4.

[15] The only true language of intuition is silence.

[16] See note 12.

section as opposed to the unity of life goes back also to Schopenhauer.

Another point of similarity is to be found in the theory by which Bergson and Schopenhauer explain laughter. Both find its main cause in the sudden perception of a discord between torpid understanding and flexible life.

In their theory of art both lay stress on the artist's power to do away with the pragmatistic narrowness of understanding and to bring man into immediate contact with life itself.

And Schopenhauer's theory of freedom although apparently very different from that of Bergson is likewise founded upon the idea that causality exists only in the realm of understanding and appearance, while life in itself, will, is free and may manifest its freedom to the living being—although the realization of this freedom is conceived very differently by Schopenhauer and Bergson, the former being far more consistent in this respect as in others.

* * *

Professor Bergson is by no means a consistent thinker. On the other hand it is well known that although more consistent than Bergson, Schopenhauer too has justly been accused of great inconsistency. This is one of the chief reasons why in spite of their striking similarity the philosophies of Schopenhauer and Bergson differ widely in more than one respect. Schopenhauer's Kantianism and Neo-Platonism, which are incongruous with the rest of his system, are happily avoided by Bergson. On the other hand Bergson's inconsistencies with regard to his activism and the confusion of his terminology with regard to time and some minor points are avoided by Schopenhauer. That Schopenhauer was more pessimistic than Bergson is of much consequence for the external appearance but of comparatively little consequence for the inner structure of both systems.

Important, however, is the fact that Schopenhauer is far more systematic than Bergson. His philosophy is an attempt to furnish an all-embracing *Weltanschauung* from the point of view of the will to live and its servant, understanding. Bergson's philosophy consists rather of several specific investigations, more or less loosely connected. That is the reason why on the whole Schopenhauer's philosophy is much more imposing and in almost all details much richer than Bergson's thought, while Bergson has the advantage of being less dogmatic and possibly still more stimulating than Schopenhauer.

As to elegance of style, figurative language and happy choice of comparisons no modern philosopher can equal either Schopenhauer or Bergson. They rival each other and both are unsurpassed. However, it seems to me that Professor Bergson's style, especially his figurative language and his predilection for comparisons, is not quite independent of the stilistic habits of Schopenhauer. Our discovery that Bergson uses three similes originally used by Schopenhauer is a strong indication in that direction. Bergson carries a stilistic habit of Schopenhauer to an extreme; but this is only a symptom of a general feature of Bergson's style which is far more onesided and far less varied, but at the same time still more surprising and stimulating, than Schopenhauer's. Thus the differences of thought in the two men are peculiarly mirrored in the differences of their style.

* * *

This leads to the question as to the psychological background for the similarity between Bergson's and Schopenhauer's thought. When Bergson worked out his thought did he plagiarize Schopenhauer? The only answer to this very natural question of a layman in the history of philosophy would be an unmistakable "Certainly not!"

Personally I have good reasons to contend that Professor Bergson is not even aware of most of these similarities, and very likely never was. He himself is probably inclined to think that most of what he actually took from Schopenhauer is his own original thought —original in the popular sense of "creation."

In fact, however, Bergson's thought appears to be "original" not in the popular, but only in the Bergsonian sense of creation. Popular creation is creation out of nothing; Bergsonian creation is the past prolonged into the future. For Bergson all life is creative because it is saturated with the life that has preceded it. Bergson's own philosophy certainly is Bergsonian life *par excellence*. Hence it is creative, because it is saturated with preceding life; and I contend that the preceding life with which it is saturated is Schopenhauer's philosophy.

This is not very difficult to prove, for Bergson himself tells us that he formerly studied Schopenhauer's philosophy closely. Moreover, when Bergson was at the most impressionable and decisive stage of his mental development a Schopenhauer craze was prevalent in France, and *Schopenhaueristes* were seen even in literature and society. Finally, Bergson's teacher Ravaisson was a follower of Schelling. Indeed some residua of Schelling's philosophy—especially with regard to his theory of matter—are easily discoverable in Bergson's thought. But what is still more important for our investigation, there was no better way to prepare Bergson for Schopenhauer's philosophy than by a knowledge of Schelling, out of whose thought Schopenhauer's own ideas emerged, just as to-day there is perhaps no better way to prepare a mind for Bergson's philosophy than by a knowledge of Schopenhauer, out of whose thought Bergson's ideas have emerged.

* * *

All this leads us back to where we started. The history of philosophy is like one of the ancestral galleries in ancient European castles. Certain family traits reappear again and again in the structure of philosophical systems and Henri Bergson's philosophy bears the family traits of Schopenhauer's "World as Will and Idea."

But Schopenhauer's own philosophy shows family traits as well. It emerged out of the philosophy of Schelling and the general trend of German romantic thought in the early 19th century, and German romanticism again owed a great deal to Herder and Goethe. Indeed I know a passage by Goethe which contains Bergson's entire thought in a nutshell. Using "reason" for "intuition" according to the terminology of his time Goethe says to Eckermann: "The godhead is active in the living, but not in the dead; it works in the growing, the developing, but not in the finished, the torpid. Therefore reason with its tendency for the Divine has to do with the growing, the living; understanding with the finished, the torpid that it may use it for practical purposes."[17] Change Goethe's terminology into the language of Bergson, and the thought expressed by Goethe is almost as Bergsonian as Bergson's own.

This Bergson-Schopenhauer-Schelling-Goethe-Herder type of philosophy could easily be traced back to the old German mysticism and still further back to the ancient philosophy of the Vedanta, with both of which Schopenhauer and most of his German predecessors knew that their own thought was more or less closely connected.

Viewed from its first beginning to its present stage the development of this type of thought goes on exactly in the way which Bergson himself terms *évolution créatrice*. As a living process it enters the thought of a philosopher in the shape last given to it by the preceding generation. As a living process it is itself "creative," i. e., it assumes a new shape different from what it had before; and

[17] *Conversations with Eckermann,* Febr. 13, 1829.

in the brain of the following generation it certainly will change again as a creative power.

Bergson's philosophy proceeds from an *élan vital* of thought. This explains why it is saturated with the past and, as we may hope, pregnant with the future. The past with which it is saturated, however, is neither pragmatism nor any American nor English philosophy, for all these mean typical work of "understanding," while for Bergson philosophy begins where understanding ceases. The *élan vital* in Bergson's own philosophy is German and characteristic of the close affinity between German and French philosophy—an affinity which may be traced back all through the history of human thought. In former ages the influence of French thought on Germany preponderated over the influence of German thought on France. Since the beginning of the nineteenth century it is the influence of German thought on France which has preponderated because there was a feeling that the *élan vital* of German thought is creative and pregnant with a future. Never was its creative power developed with more splendor and force than in Henri Bergson's philosophy.

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HENRI POINCARE: OBITUARY.

On July 17, 1912, the world lost the great French mathematician whom Karl Weierstrass—one of the greatest mathematicians of the nineteenth century—when writing to Sophie Kowalevsky, specially singled out as one of the most eminent of the school of younger French mathematicians.[1]

Jules Henri Poincaré was born at Nancy on April 29, 1854. He came of a family of which various members have risen to eminence. His father was professor in the Faculty of Medicine at Nancy, and wrote, among other works, on neurology at a time at which such researches were only pursued by a small number of scientific men. An uncle, Antoni Poincaré, wrote on meterology; and, of his two sons, one is M. Raymond Poincaré, the present President of the Ministerial Council, and the other is M. Lucien Poincaré, who is Director of Secondary Education and Minister of Public Instruction. Henri Poincaré's only sister married M. Emile

[1] Cf. G. Mittag-Leffler, *Compte rendu du deuxième congrès international des mathématiciens tenu à Paris....1900*, Paris, 1902, pp. 145-148.

Boutroux, the distinguished philosopher, and their son, M. Pierre Boutroux, is a well-known mathematician.

Henri Poincaré was precocious, intellectually, and entered the Ecole Polytechnique in 1873, and in 1875 the School of Mines as engineering pupil; in 1879 he gained the degree of Doctor of Mathematical Sciences at the University of Paris; in the same year he joined the Service of Mines as engineer; in 1881 he became professor at the Faculty of Sciences in Paris; in 1887 he was elected a member of the Academy of Sciences; and in 1908 he was elected one of the forty "immortals" of the French Academy.

A biography of Poincaré and a bibliography of his works has been published by Ernest Lebon.[2] Poincaré's first original researches were in pure mathematics. In 1880 the Academy of Sciences proposed the theory of differential equations as the subject of the great prize. Poincaré sent in a sketchy memoir with the title "Non inultus premor"—that of the town of Nancy—which did not gain the prize but which Charles Hermite mentioned encouragingly in his report. From the beginning of 1881 the subject—the integration of certain linear differential equations—was developed with surprising genius and rapidity in a series of papers presented weekly to the Academy of Sciences. Weierstrass, who admired these papers so warmly, thought it a pity that Frenchmen published their discoveries in a succession of little papers. But surely the psychological interest is heightened by this mode of publication. We know that Poincaré worked almost subconsciously, and often had no idea of what he was going to discover. Gauss's motto was, *Pauca sed matura,* and even now almost every publication of his is an almost perfect and complete classic; and yet how greatly do we feel the need of some indication as to how these discoveries grew. Weierstrass reminds us in many respects of Gauss. His works, too, were never quickly published, and very many important things he found or views he held were either not published at all, or only long after he announced them, and then by his pupils. The case is different with Poincaré. One of the many reasons for which he will live is because he has made it possible for us to understand him as well as to admire him.

Poincaré's name is associated, for the pure mathematician, with the "Fuchsian," "Thetafuchsian," and "Zetafuchsian" functions. We now call them, after Felix Klein, "automorphic" functions. But we

[2] *Henri Poincaré: biographie, bibliographie analytique des écrits*; 2d ed., Paris, Gauthier-Villars, 1912 (collection "*Savants du Jour*").

can only refer to his other researches on the theory of functions and his allied work on the theory of numbers, and will now turn to his works on astronomy and physics.

Poincaré's investigations on the form taken by a gravitating mass of fluid in rotation (1885-1901) led him to interesting theories on the parting of the earth and moon and the formation of variable stars. His researches on the stability of the solar system, which consisted in the revision of Laplace's calculations and the carrying of them to a higher order of approximation, showed that Laplace's theory of 1784 was quite just. These and other results are contained in Poincaré's three volumes on the new methods of celestial mechanics.[3] Here we must also refer to his works on the tides and on the problem of three bodies. On mathematical physics, Poincaré published many volumes of lectures given at the University of Paris and elsewhere on light, electricity—including the theory of Maxwell—capillarity, vortices, potential, thermodynamics, the theory of the conduction of heat, elasticity, and the theory of wireless telegraphy. Besides these, his lectures on the calculus of probabilities and on various subjects in celestial mechanics have been published.

Some very interesting psychological and physical details about Poincaré were published in 1900 by Dr. Toulouse as the second volume—the first was chiefly occupied by a study of Emile Zola—of his *Enquête medico-psychologique sur la supériorité intellectuelle.*[4] The help given to the scientific answering of the question: "Le génie est-il une névrose?" by such studies is, of course, immense; but most of my readers are more concerned with the qualities associated with the great mathematical capacities of a man who took such a keen interest in questions on the border-line between mathematics and philosophy.[5] It is impossible to read Dr. Toulouse's book without gaining a very vivid picture of the personality of Henri Poincaré. It is always deeply interesting to read authentic accounts of the methods of work of mathematicians, and for some years past, MM. H. Fehr, Th. Flournoy and E. Claparède have conducted an inquiry on this subject in the columns of *L'Enseignement mathématique.* Poincaré himself, in a well-known article published in 1908, has made some striking observations on his own process of mathematical discovery. And we must, I think, bear in mind, when

[3] *Les Méthodes nouvelles de la Mécanique céleste,* Paris, 1892-1899.

[4] *Henri Poincaré*; Paris, Ernest Flammarion.

[5] Poincaré's work in this direction is well known to readers of *The Monist* by the translations of George Bruce Halsted.

reading Poincaré's articles on the logic of mathematics, that they are the work of a man who was primarily—perhaps almost exclusively—interested in the faculties of invention. When mathematical logicians asserted that the whole of mathematics follows by logical principles alone from concepts which can be logically defined and from the primitive propositions of logic alone, Poincaré and many other mathematicians objected that "intuition" was left out of account. There is a great likelihood that this is not really Kantianism in mathematics; only phrases make it seem so. Kant clearly recognized the distinction between the question as to whether a truth B is logically implied by a truth A and that as to whether B is discovered by a certain person who starts from the premise A alone and uses only purely logical considerations. The mathematical logicians do not deny to the seeker of truth either genius or the creative power—if such exist—of the artist; they are concerned with an epistemological question, and psychological objections are irrelevant there. The case is analogous to this: If someone were to point out that the properties of logarithms are simple consequences of the conception of one number as a power of another, he would not be confuted by the remark that Napier did not invent logarithms in that way; or again, it is not relevant to the student of Keats's poetry, as such, to know what porridge John Keats ate.

If this interpretation of the attitude of the "creative" mathematicians is correct, their position with respect to mathematical logic is easily explained. That the interpretation is correct seems supported by Poincaré's last controversial work on mathematical logic which he gave this year as a lecture to London University, and which has just been printed in *Scientia*.[6] In the previous discussions on the use of the infinite in mathematics, in which Poincaré joined, each side kept on repeating the same arguments. There seems, in fact, a fundamental difference in mentality among mathematicians. Some, whom Poincaré called "pragmatists," believe that the infinite is derived from the finite, and all verification and all definition is performed with a finite number of words; others, the "Cantorians," believe that there are objects and truths which cannot be defined or demonstrated in a finite number of words. The Cantorians are realists and believe that the truth of a proposition does not depend on its verification by us. It is not difficult to place Poincaré, on the

[6] "La Logique de l'infini," *Scientia* (*Rivista di Scienza*), July, 1912, pp. 1-11.

grounds of some of his writings, among those whom he not inappropriately calls "pragmatists."

When Poincaré was five years old, he had a severe attack of diphtheria, and partial paralysis. All this made him rather weak for a long time, and perhaps was the origin of his lifelong clumsiness. Of his absence of mind, many stories are told. Once during a walk, he was suddenly surprised to find a wicker bird-cage in his hand. He had unconsciously removed it from a wayside stall.

As regards religion, at the moment of his first communion he was a believer; then belief left him gradually, and, from the age of eighteen he was a sceptic. In politics he was a republican; he held to the principle of personal property; he believed in political equality and the political rights of women,—but here he feared clerical influence.

In mathematics, he cannot be said to belong to any school. In a short life not without physical drawbacks, he has, by regular work, produced about 500 writings—some of them of the very first order.

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HENRI POINCARE: AN APPRECIATION.

The foremost of Frenchmen is no more. When Laplace was asked to name the greatest German mathematician he answered, "Pfaff." "But how about Gauss?" said the inquirer. "Ah," replied Laplace, "he is the greatest of all mathematicians." Similarly we might modify our first statement and declare that the foremost of all men is no more. For on July 17, having apparently recovered almost completely from a surgical operation undergone only a few days before, Henri Poincaré, while dressing himself in the morning, was suddenly smitten with an embolism and fell dying in the arms of his wife. While the delegates of learning from all quarters of the globe were assembling in London to celebrate the 250th anniversary of the Royal Society, instantly the brightest star in the galaxy of the sciences was eclipsed forever. The sad intelligence was at once flashed around the world, but the details as set forth in the Paris journals of the 18th are but lately at hand.

Commanding the homage and admiration of all, so generous, so pure-hearted, so noble-minded was Poincaré that he aroused the envy and jealousy of none. If "Freedom shrieked when Kosciusko fell," with far more propriety may universal Science, may Philosophy herself, weep at the passing of her illustrious son. For Poincaré was not a mere specialist, an isolated summit of technical learning, but rather a mountain range of knowledge. His Andean intellect traversed the whole continent of science. In physics, in mathematics, in astronomy, in logic, in philosophy even, he strode from peak to peak in the heights of thought, and wherever his feet touched, there was a blaze enkindled. His compatriots say that France has borne no equal in a hundred years, not since d'Alembert and Laplace. To a foreigner it may be questionable whether the limit may not be pushed much further back, even to the days of Descartes. For while the mind of Poincaré did indeed cast off no single orb of thought to match at once in largeness and in luster the *Mécanique céleste,* or still more the *Théorie analytique des probabilités,* yet it has studded the firmament of exact science with a host of splendors. Scarcely if at all inferior to Laplace or even to Lagrange as analyst, as geometer, as physicist, as astronomer, Poincaré was what they were not —he was a logician of the first order and a philosopher, profound, penetrating, and spiritual. Nor was this all; for his genius in exposition allied him with Clifford and brought him into livelier sympathy with the lay intellect than almost any of his peers in the realm of pure science, while his fine artistic nature and literary sense expressed themselves in a style at once clear and concise, nervous, vivid, picturesque and animated. As subtle as Hume, as comprehensive as Helmholtz, he was least of all a dry-as-dust savant; the keenest of logicians, he did not disdain the graces of rhetoric, but poured out for his fellows the divine draughts of his thought in golden goblets of speech.

It was not, however, as mathematical physicist, as analyst of Fuchsian functions, not as student of the stability of the solar system, not as discoverer of unsuspected figures of equilibrium, not as master of metageometry, not as preeminent logician of science, not as any nor as all of these, that Poincaré rendered his highest service to humanity. It is his supreme merit to have recognized explicitly the inalienable rights of the human spirit, to have opposed firm as Gibraltar the rising tide of naturalism and the pride of knowledge which, intoxicated with the triumphs of physical science and its applications, refuses to see any mystery beyond sense remaining in the world, and boldly aspires by means of mind to pull down mind from its throne and to reduce the universe to a molecular maelstrom, to a wisp of granulated ether. When Napoleon asked Laplace about having written so great a book without once naming the name of God therein the savant replied, "Sire, I have no need of that hypothesis." Lagrange, a finer spirit, on hearing of this commented: "But it is a beautiful hypothesis that explains many things." In his famous *mot* Laplace has declared his kingship and at the same time defined his kingdom. True, in the realm of matter he had no need of the beautiful hypothesis; in the kingdom of causality, of mass and motion, there is no purpose, no reason, and hence no need of God, the Reason of all. But Poincaré saw through all the phantasies of "scientists," as the astronomer sees through the nebula in Orion, and beheld far behind the phenomena of time and space the eternal realities of self and of soul; while peering into the processes of physical nature deeper even than Laplace himself, he never forgot that they are after all an unsubstantial pageant, that

> "On earth is nothing great but man,
> In man is nothing great but mind."

So he became in a sense the moderator of the assembly of the sciences. As no other living man he could say, "Thus far and no further;" for he spake as one having authority. Even the Germans, who are seldom over-quick to acknowledge the hegemony of others in the ranks of thought, forgot all national and racial prejudices in the presence of Poincaré and freely declared him to be "the first authority of this age" (*die erste Autorität von dieser Zeit*).

The savant closed his eyes at the age of fifty-eight in the full flush of his powers, at the fever-heat of his intellectual activity. What more he might have done, who knows? But assuredly his mantle will fall upon worthy if not upon equal shoulders; in the paths he has broken there will follow increasing throngs. It is our human form of speech to deplore an irreparable loss. But in some larger, deeper and higher, though indefinable, sense there is perhaps only gain forevermore.

> "One accent of the Holy Ghost
> The heedless world hath never lost."

Le roi est mort: vive le roi. Poincaré is dead: but deathless is Poincaré.

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THE CAPTURE THEORY OF COSMICAL EVOLUTION CONFIRMED BY THE LATEST RESEARCHES ON THE ORIGIN OF STAR CLUSTERS.

Introductory Remarks.

It is a very remarkable fact that since the epoch of Sir William Herschel little serious consideration has been given to the origin of star clusters, until within the last few years. Accordingly the thoughtful suggestions thrown out by that unrivaled man long proved largely if not entirely barren of fruitful results for the development of a science of cosmogony, because his early ideas were lost sight of or forgotten. We have labored under a strange delusion, of preferring the theories of Laplace to those of Herschel, but have at last found the way from darkness to light, from error to truth. And it happens that of all the investigations yet made in cosmogony those relating to the neglected subject of star clusters are the most convincing and least open to objection.

In a paper entitled "Dynamical Theory of the Globular Clusters and of the Clustering Power Inferred by Herschel from the Observed Figures of Sidereal Systems of High Order," recently communicated to the American Philosophical Society held at Philadelphia, and just published in the *Proceedings* of that illustrious society, I have examined the whole problem of the origin of clusters in a somewhat exhaustive manner. By the use of mathematical methods of rigorous character, I was able to develop the most convincing proofs that these aggregations of stars have arisen by the process of capture in the course of millions of ages. It will be the main purpose of this article to discuss the results arrived at in this general investigation of the origin of star clusters; but before taking up this subject in detail it will be allowable to treat briefly of the conclusions reached by the illustrious Poincaré in his *Leçons sur les Hypothèses Cosmogoniques,* 1911, and to notice also the unimportant objections advanced by Prof. Charles André in *Scientia,* No. 2, 1912.

The Views of Poincaré.

In his *Leçons sur les Hypothèses Cosmogoniques,* 1911, M. Poincaré gives a summary of the various theories of cosmogony, and in two chapters discusses results arrived at in my *Researches on the Evolution of the Stellar Systems,* Vol. II, 1910.* He exam-

* These chapters appeared in translation in *The Monist* of July, 1912.

ines and adopts the proof therein given that the roundness of the orbits of the planets and satellites is due to the secular effects of the action of a resisting medium. He concurs in the capture of satellites, as well as in the new theory of spiral and ring nebulae. M. Poincaré especially remarks how well the resisting medium explains the roundness of the orbits of the planets and satellites; and altogether is favorable to the recent development of cosmogony into a new science of the starry heavens.

One point in my work, however, has been slightly misunderstood by Poincaré, and I will therefore dwell upon it here. After outlining the leading principles of the Capture Theory he adds that while it fully explains the roundness of the orbits, it does not give a satisfactory explanation of the inclinations of the planetary orbits. But here he has evidently lost sight of the nature of the spiral nebula from which our solar system is supposed to have arisen.

The Capture Theory means primarily that all revolving bodies are added on from without,—the planets being added to the sun and the satellites added to the several planets; but it is not held that the entrance into our system was from all directions over the celestial sphere and thus entirely at random. On the contrary it is carefully explained in my work that the system from the beginning had a fundamental plane of maximum areas, due to the fact that a spiral nebula is formed by two principal streams coiling about one another; and the plane of maximum areas is the plane determined by these predominant streams.

In the condensation of a nebula an infinite number of minor streams probably are involved; but the whirling motion is made possible only by two predominant streams, as shown in photographs of spiral nebulae. That is to say, there is more matter in the two large streams than in the smaller ones; and this gives a fundamental plane to the system when it becomes mature, just like that in which the planets of the solar system are found to move. The planetary orbits ought not to lie exactly in the same plane, but near an invariable plane, such as Laplace in 1784 proved to exist in every system of bodies subjected to the mutual gravitation of its parts.

The comets or smaller masses of nebulosity naturally should be inclined at all angles to the invariable plane; but as they intersect that plane twice in their orbital motion about the sun, they will sooner or later pass near a planet revolving in an orbit lying near the fundamental plane of the system, and their orbits are thus subjected to profound changes of position as well as of form and extent. Count-less comets are destroyed in building up the planets; so that much matter not originally lying in the plane of the planets is finally captured and drawn into that plane, as the masses of the planets gradually augment, and they are drawn nearer the sun in orbits becoming ever smaller and smaller, and rounder and rounder.

Thus on the one hand the resistance of cometary matter reduces the size of the planetary orbits and makes them rounder, while on the other hand the growth of these masses increases their mutual attraction; and if they were originally near the plane of the predominant streams, in time they come to move almost exactly in one plane, as now observed in our actual planetary system.

For as our system has shrunk, and the original orbits were hundreds of times larger than at present, the nuclei at the outset were not necessarily very near the plane in which they now move, but may have departed from it considerably. Mutual inclinations of a few degrees now found in our system are magnified at hundredfold primordial distances into very great absolute distances; so that the original streams need not have been at all compressed, but may have been exceedingly diffuse, just as actual nebulae appear to be.

Accordingly it is a remarkable fact that the theory which accounts for the roundness of the orbits of the planets also explains the small mutual inclinations of their orbits, and the rotation of the sun about an axis nearly perpendicular to the plane in which the planets revolve. The explanation of the origin of our system from a spiral nebula thus appears to be entirely satisfactory.

The Views and Objections of André.

The objections to the Capture Theory advanced by André are easily shown to be without the slightest foundation. It is quite unnecessary to consider most of them, and I will therefore content myself with the three chief ones, which will sufficiently show the weakness of the rest.

1. André claims that the spherical expansion in Babinet's criterion as I have used it is not strictly in accordance with Laplace's theory, because Laplace did not imagine the sun's atmosphere to be expanded in a spherical form, but rather in the form of a flat disc. This objection is quite devoid of foundation, as will appear from the following simple considerations.

a. If the expansion be spheroidal, as a flat disc, more of the matter is at greater distance from the center, for given volume, than in a spherical expansion; so that the moment of inertia is increased, and with constant moment of momentum the angular velocity is therefore *decreased.* Hence a discoidal expansion of the sun is more unfavorable to Laplace's hypothesis than the spherical expansion used by me. For in case of a sphere the moment of inertia is shown in works on the calculus to be $2/5(Mr^2)$, where r is the radius and M the mass; in an ellipsoid with equatorial axes a and b it is $M/5(a^2+b^2)$, and when $a=b$, as in an ellipsoid of revolution, this becomes $2/5(Ma^2)$, a being the equatorial axis.

b. To reduce this to numbers I took ellipsoids with meridian sections of eccentricity 0.10, 0.25, 0.5, and 0.8, giving oblatenesses of 0.00501, 0.03176, 0.13397, and 0.40000 respectively; and found $a^2=1.00336r^2$; $a^2=1.0217r^2$; $a^2=1.1525r^2$; $a^2=1.4057r^2$. *This shows how the moment of inertia increases as the oblateness increases, and thus proves a corresponding decrease of the angular velocity of rotation below that published in my tables of Babinet's criterion.* The objection of André therefore has not the slightest foundation, because my calculations are more favorable to Laplace's theory than those based on the theory of an oblate spheroid.

2. André dwells on the fact that Laplace imagined only *the atmosphere* of the sun expanded to the orbits of the planets. But as the sun itself when so expanded becomes much rarer than most atmospheres we are familiar with, it is readily seen that this point is not well taken. When the sun is expanded to Neptune's orbit, the average density of the nebula is 260 million times less than that of air at sea level. Nothing more need be said on this point. Such a medium could exert little or no hydrostatic pressure from the center, and Laplace's theory of the detachment of zones of vapor under conditions of hydrostatic pressure implies that he overlooked the rarity of this medium, which makes such a thing as hydrostatic pressure quite impossible. No alteration of central arrangement of density would materially change this result, and we may thus dismiss it without further comment.

3. As the centrifugal force, by Babinet's criterion, is only a ten millionth part of that required to detach the earth, and a three hundred millionth of that required to detach Neptune, while the hydrostatic pressure likewise is insensible, it is clear that no such detachment as Laplace imagined ever took place. André, Ligondes and other French writers are simply injuring the memory of Laplace by presenting to the Paris Academy of Sciences conclusions which would be immediately rejected by Laplace himself if he were living to-day.

After having studied the works of this great master of celestial mechanics from the days of my youth, I believe I have followed his spirit in rejecting what is now known to be false. Professor André is in the unfortunate position of having written books favorable to the abandoned theory of Laplace; but he should aim at truth rather than perpetual consistency, and modify his views to meet the latest discoveries in science. For a true philosopher does not aim at supporting his earlier writings, but at gradually attaining the truth, even if his first work has to be modified or entirely abandoned. The successors of Laplace obviously should act upon this laudable principle.

4. Even if the retrograde satellites and a multitude of other phenomena did not tell us unmistakably that all the satellites have been captured, and we still tried to explain these bodies by the detachment theory of Laplace, we should remain quite in the dark as to the origin of the observed rotations. They would be simply assumed, and not explained; and so we should have no rational theory of the formation of the solar system; whereas the Capture Theory gives a simple and natural explanation of the rotations and obliquities as well as the orbital motion of the satellites, and the variations of their brightness, the lunar craters and maria and kindred phenomena; and all the phenomena are so woven together that it is impossible to doubt the truth of the new theory.

In the same way, even if the solar nebula could have rotated rapidly enough to detach zones of vapor as Laplace imagined, it would still be impossible to account for so rapid a rotation. Fortunately Babinet's criterion shows that no such rapid rotation for the detachment of zones of vapor ever took place; and that Laplace was deceived by the roundness of the planetary orbits, which we now recognize to be due to the secular action of the nebular resisting medium formerly pervading our solar system.

Necessity for Wider View of all Sidereal Systems.

It requires no elaborate argument to convince any philosophic investigator that the laws of cosmical evolution can best be deduced from the study of nature in the widest sense. The narrowness of the cosmogony of Laplace arose from the fact that it was based wholly on our solar system, and that too before the system was fully understood. The roundness of the orbits of the planets and satellites and the survival of a ring about Saturn led to the idea that all these bodies had originated by the detachment of rings. Yet as soon as the orbits of the double stars were determined, they were found to have eccentricities of every degree, between the round orbits characteristic of the planets and satellites and the very elongated orbits characteristic of the comets. The development of double stars obviously could not have been by the formation of rings as imagined by Laplace.

Accordingly without such a comprehensive view of the different types of systems it would be vain to hope for the deduction of a general law of nature. The folly of adhering to the old methods of Laplace based on an imperfect knowledge of the solar system alone is thus apparent; and after what is now shown, from Babinet's criterion, as to the impossibility of detaching masses or rings, there is no course open to us but to reject Laplace's hypothesis once for all. It does not give us a general law of nature, and is not true even for the special case of the solar system.

Our hope for finding the law of nature must be based on the study of double and multiple stars, and sidereal systems of higher order. Now it happens that of the various sidereal systems known to the astronomer, the globular clusters are the most complex, and at the same time the most symmetrical and regular in their constitution. If therefore any light can be obtained on the formation of sidereal systems of such high order, it might be possible to derive principles which could be applied to less symmetrical systems of lower order. This is what I have done in my recent investigation of the origin of clusters. Having deduced the law of nature from the highest and most complex systems, with wonderful regularity of figure, I have proceeded to apply it also to systems of the lowest type, as the solar system and the double and multiple stars. This new method of procedure is so important, that it becomes advisable to explain it in some detail.

Nature of Clusters, Average Distance of the Stars Apart, Increase of Density Towards Center.

Sir William Herschel always considered the globular clusters to be the most wonderful of all sidereal systems. He never ceased to marvel at the existence of these swarms of stars, which were known to be aggregations of suns; and he inferred that at length they had been moulded into the spherical form by the action of central powers.

Even in the time of Herschel it was recognized that the clusters are very far from the earth, and thus that the component stars are not really close together, but separated by intervals which are very great compared to those which separate the planets from the sun.

More modern discovery has confirmed the sagacious conjectures of the great Herschel. The latest investigation of the profundity of the Milky Way, which I finished in November, 1911, and have just published in the *Proceedings* of the American Philosophical Society at Philadelphia, shows that the remotest clusters are removed from us by at least a million light-years. Indeed this determination of the depth of the Milky Way shows that the remotest stars may be removed from us by distances of five or ten million light-years; but even with most of the clusters at distances of hundreds of thousands of light-years, it is possible to say with certainty that the average space between the stars in globular clusters is of the order of a light-year, which is 63275 times the distance of the earth from the sun. We thus have the spectacle of systems of stars separated by great intervals, but so remote as to be drawn together by perspective into a small angular space on the surface of the sky.

The density in these masses of stars was found by Herschel to be always greatest towards the center; and in fact to be in excess of that corresponding to the supposition of equal scattering. Herschel therefore inferred that the accumulation in the centers of the clusters must be due to the secular action of a clustering power, which he believed to be nothing else than universal gravitation working over millions of ages. He remarked that the Milky Way presented the aspect of a clustering stream traversing the heavens as an irregular band of milky light; and as he had found the sidereal universe to be greatly extended in the direction of the plane of the Milky Way, he correctly inferred that the clustering stream thus presented to the eye was the effect of distance and of local aggregations of the stars into star-clouds and clusters. The stars are spread out into a comparatively thin stratum, and at great distance the effect is to give the appearance of the Milky Way, which thus appears as a clustering stream several degrees in width.

How the Stars are Captured in Clusters.

In the memoir above referred to I have established the capture of stars by a cluster, and the secular shrinkage of the cluster, by the use of Green's theorem for the transformation of a triple integral appropriate for space into a double integral over the surface of the cluster. By showing that the surface shrinks as the result of close appulses among the stars, and also as the outcome of mutual gravitation, even when no close approach occurs, it is found that the cluster becomes more and more compressed, with density accumulating towards the center.

The attraction of members of a cluster is analogous to surface tension in working to decrease the volume of a bubble, or in rounding up a drop of dew, to give minimal surface for a given volume. In the same way gravity tends to make a planet perfectly round, except as modified by rotation into an oblate figure. Herschel used such analogies in his argument for a clustering power, which he inferred to be moulding the figures of clusters. And recently I have tested his suggestion mathematically, and found a conclusive proof that the argument is correct.

To give a simple analogy for the capture of stars in clusters, with known processes in the solar system, we may remark that Jupiter captures the comets crossing over his orbit, and transforms their paths till they lie wholly within that of the planet. In this way he has captured quite a family of comets and thrown their orbits within his own orbit. Now in the memoir above referred to I have shown that a *shell of stars* in a cluster acts very much as Jupiter does on the comets—and thus tends to reduce the path of an oscillating star till it comes within the confines of the shell.

Accordingly if a star from without once enters a cluster, and thus begins to traverse the series of shells of which the cluster is made up, it will never quit the swarm but be gradually drawn in, and captured, during one or more complete oscillations. The extent of its outward journey from the cluster, if any occurs, will be decreased, until finally it is dragged down to the level of the shell, and becomes a member of the cluster. This is one of the most remarkable results of our dynamical theory of clusters. The Capture Theory being thus verified for these globular masses of stars, it naturally may be expected to operate in systems of lower order.

No Possible Origin of Clusters Except that Outlined by the Capture Theory.

The globular clusters are so perfectly symmetrical that they become of high interest in elucidating the problems of cosmogony. For it is not conceivable that systems of such large mass, great extent and perfect symmetry, can have arisen, except by the gathering together of stars from a wider extent of space.

No process of collision, for example, would account for the globular clusters; for by impact the matter of two hypothetically disrupted masses would neither be symmetrically distributed nor dispersed over such a vast space as that now occupied by the thousands of suns composing a cluster. Then, again, to be effective such hypothetical collision would have to be between approximately equal giant suns; and there are too few stars of such enormous mass for pairs of them ever to come into bodily collision.

Accordingly, a little consideration shows us, on the one hand, that such giant collisions would not occur; and, on the other, that even if they could take place such widely diffused and symmetrical swarms of stars could not arise by this process. The globular clusters therefore are due to the aggregation of stars once symmetrically and widely distributed in space. This gives us a good illustration of the Capture Theory on the most stupendous scale. Similar views were reached by Herschel, without mathematical investigation of the subject, such as I have recently developed; and it may be remarked that he found the evidence of a clustering power most convincing.

The *New General Catalogue of Nebulae and Clusters,* published by the Royal Astronomical Society of London in 1888, contains a list of more than one hundred globular clusters, mostly distributed along the course of the Milky Way. The clustering of the stars into great systems about so many centers shows how general and widespread this tendency is in nature.

If now we recall that only the oldest sidereal systems can have attained a state of perfect symmetry, it is obvious that a larger number of sidereal systems might be expected to have an irregular and unsymmetrical aspect. The globular clusters are therefore only a part of the aggregations of stars exhibiting the effect of the clustering power; but the perfection of this type of system renders it eminently adapted to disclosing the process by which all clusters are formed. For if the law of nature can be deduced from the perfect type of sidereal development, it may with equal certainty be inferred to operate in those sidereal systems which have not yet attained to full maturity. By investigating the different types of sidereal systems our studies may thus disclose the general law of cosmical evolution and embrace phenomena extending over millions of ages!

The Law of Nature Embraces also Systems of Lower Order, and Therefore the Planetary System and the Systems of Double and Multiple Stars.

Those who believe in the uniformity and continuity of the laws of nature, as laid down by Newton in the *Principia,* 1687, will quickly realize that the law of cosmical evolution established for the globular clusters should necessarily hold also for systems of lower order. Rule I: "We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances." Rule II: "Therefore to the same natural effects we must, as far as possible, assign the same causes."

Accordingly, in line with these rules of Newton, I have shown that the Capture Theory will explain the formation of the solar system, as well as the double and multiple stars: and having found the principle to be the same throughout the sidereal universe, I have inferred that nature's law everywhere is one of adding on from without. The component stars are added to the clusters, and drawn nearer and nearer the center; the planets added to the sun and made to revolve in smaller and smaller and rounder and rounder orbits. Likewise the satellites were added on to their several planets, and the moon captured by the earth. The double and multiple stars were formed on the same principle—the nuclei having originated in the distance, and subsequently approached the centers about which they now revolve. This gives us a general law of nature of the utmost simplicity.

And not only is the generality of the law proved by force of analogy, but also by direct mathematical demonstrations in the solar system, deduced from Babinet's Criterion; while in the clusters the proof is so obvious that it need scarcely be emphasized. The demonstration of this law in the double and multiple stars is similar to that available in the solar system; and moreover is supported by the analogy of the clusters, into which the multiple stars merge by insensible degrees, when the number of bodies in a group is increased indefinitely.

Accordingly nothing is more certain than that the law of cosmical evolution now recognized is the true law of nature. It does not even resemble the abandoned theory of Laplace, but has considerable resemblance to the general outline of the nebular hypothesis as traced by Sir William Herschel. In particular Herschel's theory of clusters, as originating by the aggregation of isolated stars is deserving of attention; for this is the earliest outline of a process of capture similar to that now worked out in detail and shown to be applicable to all types of systems observed in the sidereal universe.

In conclusion it seems advisable to close this discussion by the following summary quoted from my latest memoir on *The Dynamical Theory of Clusters.*

Summary and Conclusions.

Without attempting, in this closing section, to recapitulate the contents of this memoir in detail, it may yet be well to draw attention to some of the most significant conclusions at which we have arrived.

1. As intimated in the first section of this paper the problem of n-bodies, under ideal dynamical conditions, remains forever beyond the power of the most general methods of analysis; but the dynamical theory of clusters gives us the one secular solution of this problem found under actual conditions in nature. For when n *is* of the order of 1000, so as to give rise to a cluster, the clustering power observed by Herschel operates to exhaust the mutual potential energy of the system, and bring about increasing accumulation in the center, so that the cluster finally unites into a single mass of enormous magnitude. Probably the giant stars of the type of Canopus and Arcturus have arisen in this way.

2. And since attendant bodies of every class—as satellites, planets, comets, double and multiple stars—tend everywhere to approach the centers about which they revolve, as an inevitable effect of the growth of the central masses and of the action of the resisting medium over long ages, it follows that the secular solution of the problem of clusters is more or less valid for all cosmical systems. They finally end by the absorption of the attendant bodies in the central masses which now govern their motions.

3. The dynamical theory of globular clusters shows that the clustering power inferred by Herschel is nothing else than the action of universal gravitation; and that it operates on all sidereal systems, but does not produce the cumulative effect which Herschel ascribed to the ravages of time inside of millions of ages.

4. The globular clusters are formed by the gathering together of stars and elements of nebulosity from all directions in space; and this points to the expulsion of dust from the stars of the Milky Way,

and its collection about the region of the formation in such manner as to give essential symmetry in the final arrangement of the cluster, which doubtless has some motion of rotation, and originally a tendency to spiral movement.

5. The stars and smaller masses are captured by the mutual action of the other members of the cluster, and worked down towards the center of the mass. This gives a central density in excess of that appropriate to a sphere of monatomic gas in convective equilibrium (A. N. 4053 and A. N. 4104).

6. The density of the clusters is greater on the outer border than in a globe of monatomic gases, which shows that stars are still collecting from the surrounding regions of space. The starless aspect of the remoter regions about clusters is an effect of the ravages of time, as correctly inferred by Herschel in the course of his penetrating sweeps of the starry heavens.

7. And just as clusters under the mutual gravitation of the component stars contract their dimensions, with time, chiefly owing to the growth of the central masses, so also do other systems, whether the mass-distribution be *single,* giving a system made up of a sun and planets, or *double, triple* and *multiple,* giving binary, triple or multiple stars, or sidereal systems of still higher order. The tendency everywhere is from a wider to a narrower distribution of the large bodies; while the only throwing off that ever occurs is of particles driven away from the stars by the action of repulsive forces.

8. The orbits of the stellar and planetary systems are decreased by the growth of the central masses and rounded up by the action of the nebular resisting medium. And in like manner all clusters tend to assume spherical or globular figures, so as to justify the expression of Plato, that the Deity always geometrizes; or Newton's remark that the agency operating in the construction of the solar system was "very well skilled in mechanics and geometry."

9. Newton required the intervention of the Deity to give the planets revolving motion in their orbits, because in the absence of repulsive forces he could not account for the dispersion of the matter so as to produce the tangential motions actually observed. By means of the theory of repulsive forces, however, it is now possible to explain these projectile motions, which Herschel likewise pointed to as the chief agency for the preservation of sidereal systems. The only assumption necessary is an unsymmetrical figure of the primordial nebula, giving a whirling motion about the center as the system develops; and since the dust gathers from all directions it is certain

that this lack of perfect symmetry will always develop, as we see also by the spiral nebulae.

10. It is this unsymmetrical form of the spiral nebulae produced by the gathering of the dust from the stars, or the slight relative tangential motion of stars formed separately but finally made to revolve together as a binary system, that gives the projectile forces with which they are set revolving in their orbits. In no case have they resulted from the rupture of a rotating mass of fluid under conditions of hydrostatic pressure as formerly believed by Darwin, Poincaré and See.

11. Even if the rotation could become rapid enough to produce a separation, under conditions of hydrostatic pressure, by rupture of a figure of equilibrium, there would still be the equal or greater difficulty of explaining the origin of the primitive rapid rotation. This last difficulty escaped notice till we came to assign the cause of rotations, and found that mechanical throwing off was impossible under actual conditions in nature. It is therefore recognized, from the definite proof furnished by Babinet's criterion in the solar system, that such a thing as a throwing off never takes place; but that all planetary and stellar bodies are formed in the distance, and afterwards near the centers about which they subsequently revolve.

12. This gives us a fundamental law of the firmament—the planets being added on to the sun, the satellites added on to their planets, the moon added on to the earth, and the companions added on to the double and multiple stars—which now is found to be beautifully confirmed by the dynamical theory of the globular clusters. *It is not often that such a great law of nature can be brought to light, and it is worthy of the more consideration from the circumstance that it explains all classes of stellar systems by a single general principle.*

13. As sidereal systems of lower order are conserved by projectile forces, it is probable that the clusters likewise have a spiral motion of rotation, with similar projectile forces tending to counteract simple progressive collapse. The period of the orbital revolution of the stars of a cluster is found to be common to all, without regard to the dimensions of the elliptical orbits described; and thus the whole system may have a common period of oscillation, after which the initial condition is perfectly restored. This possibility in the dynamics of a cluster is exceedingly wonderful and results from the central attraction depending directly on the distance.

14. The equality of brightness in star clusters shows that some process of compensation between the attractive and repulsive forces

has produced stars of wonderful uniformity of luster. Thus the present investigation confirms the previous researches on the evolution of the stellar systems, which have laid the foundations for a New Science of the Starry Heavens.

15. Accordingly the Capture Theory of cosmical evolution being now firmly established for the clusters, where the nature of the process is entirely clear, it becomes at once a guide to us in dealing with systems of lower order; and we see that the law of nature is uniform and everywhere the same, the large bodies working in towards the centers of attraction, while the only throwing off that ever takes place is of small particles driven out of the stars by the action of repulsive forces. All planetary bodies are formed in the distance, and have their orbits reduced in size by increase of the central masses, and rounded up by moving in a resisting medium. *This is a perfectly general law of the sidereal universe. It verifies the early conjectures of Plato and Newton concerning the stability of the order of the world and shows that these illustrious philosophers were quite justified in concluding that the Deity always geometrizes.* The spiral nebulae tend to develop systems with rounder and rounder orbits, and the clusters made up of thousands of stars assume globular figures with minimal surfaces and internal density so arranged as to give maximum exhaustion of the potential energy.

16. This is geometry of the most marvelous kind, as we find it impressed on the systems of the sidereal universe; and the perfection of this most beautiful science of celestial geometry may be considered the ultimate object of the labors of the astronomer. The philosophic observer is not and never can be content with mere observations of details which do not disclose the living, all-pervading spirit of nature.

17. If, then, the mystery of the gathering of stars into clusters is now penetrated and traced to the clustering power of universal gravitation, so also is the mystery of the *converse problem of starless space,* which was a subject of such profound meditation by the great Herschel.

18. This incomparable astronomer likewise correctly concluded that the breaking up of the Milky Way into a clustering stream is an inevitable effect of the ravages of time; but we are now enabled to foresee the restorative process, under the repulsive forces of nature, by which new nebulae, clusters and sidereal systems of high order will eventually develop in the present depopulated regions of starless space.

19. If there be an incessant expulsion of dust from the stars to form the nebulae, with the condensation of the nebulae into stars and stellar systems, while the gathering of stars drawn together by a clustering power operating over millions of ages gives at length a globular mass of thousands of stars accumulating to a perfect blaze of starlight in the center, but surrounded externally by a desert of starless space resulting from the ravages of time, certainly the building of these magnificent sidereal systems may well engage the attention of the natural philosopher.

20. The foremost geometers of the 18th century, including Lagrange, Laplace and Poisson, were greatly occupied with the problem of the stability of the solar system; and in his historical eulogy on Laplace the penetrating Fourier justly remarks that the researches of geometers prove that the law of gravitation itself operates as a preservative power, and renders all disorder impossible, so that no object is more worthy of the meditation of philosophers than the problem of the stability of these great celestial phenomena.

But if the question of the stability of our single planetary system may so largely absorb the talents of the most illustrious geometers of the age of Herschel, how much more justly may the problem of the stability of clusters, involving many thousands of such systems, claim the attention of the modern geometer, who has witnessed the perfect unfolding of the grand phenomena first discovered by that unrivaled explorer of the heavens?

The grandeur of the study of the origin of the greatest of sidereal systems is worthy of the philosophic penetration of a Herschel! The solution of the dynamical problem presented surpasses the powers of the most titanic geometers, and would demand the inventive genius of a Newton or an Archimedes!

Yet notwithstanding the transcendant character of the problem, and the hopelessness of a rigorous solution in our time, even an imperfect outline of nature's laws may aid the thoughtful astronomer, in penetrating the underlying workings of the sidereal universe, and thus enable him to perceive the great end subserved by the development of the cosmos. If so, he may well rejoice, and exclaim with Ptolemy:

"Though but the being of a day,
When I the planet-paths survey,
My feet the dust despise;
Up to the throne of God I mount
And quaff from an immortal fount
The nectar of the skies."—Transl. by W. B. Smith.

T. J. J. See.

THE PROGRESS OF BUDDHIST RESEARCH;

WITH SOMETHING ABOUT PENTECOST.

Since the writing of my note on the "Buddhist-Christian Missing Link" in the fall of 1911 (*Open Court,* Chicago, and *Mahā Bodhi Journal,* Colombo, both for January, 1912), great events have happened in the field of Buddhist learning. We are now hot on the trail of the Missing Link, if we have not yet found it. For, besides the selected documents, to be presently described, there are thousands more reposing in the libraries of Pekin, London, Paris and Berlin, which we know to contain many more canonical Sūtras translated into Sogdian,[1] and there are doubtless more forthcoming in Bactrian also.[2]

In a book published in 1908, I said this:

"Menander, in the second century before Christ, showed an interest in, and a knowledge of, the Buddhist scriptures which may have been founded upon a knowledge of Pāli; but even then one would expect such a patron to have some specimens of the lore he admired translated into Greek, or into some vernacular. Strabo says that nearly the same language pervaded Media and parts of Persia, Bactria and Sogdiana. Strabo also says that the Corybantes had come from Bactria, and Euripides pictures them as passing the Bactrian Gates. When Buddhist ideas were carried westward, they would as surely be translated as the Bacchic had been."

These words were written not later than 1907, and since then my prediction has been abundantly verified. We have actually found fragments in Chinese Turkestan of Buddhist scriptures both in Bactrian and Sogdian, the latter coming from a Chinese library that was closed up in 1035, while documents from a near-by tower were dated A.D. 1 and A. D.20![3] Bactrian or Tokharish was the language of ancient Tukhāra, i. e., northern Afghanistan and parts of Chinese Turkestan. Sogdian was spoken in Russian Turkestan, where the city of Samarkand had been the center of a Greek civilization since the time of Alexander.

In Tokharish we have found Pācittiya 92 of the Vinaya, in the recension of the Sarvâstivādins, thus confirming the words of Yuan

[1] M. Aurel Stein, *Ruins of Desert Cathay.* London, 1912, Vol. II, p. 213.

[2] Sylvain Lévi, in *Le Temps,* Paris, May 19, 1911. Reprinted in the *Revue Archéologique.*

[3] Stein, *Ruins of Desert Cathay.* Among these documents, though undated, are some Sogdian epistles in Aramaic letters, now being read by Gauthiot.

Chwang, who said that all Tukhāra was Sarvâstivādin. In Sogdian we have found the Vessantara Jātaka, that great favorite about the Bodhisat prince who gave all he had away.[4] It was this very Jātaka that was graven upon the Great Tope at Anurādhapura, when visitors from Alexandria came to see the opening ceremonies, in the second century before Christ.

Other portions of scripture—the Nidāna and Dasabala Sūtras, the Dharmapada—and a patristic hymn, have been found in Sanskrit;[5] while fragments of patristics have also turned up in Eastern Turkish, written in characters of Syrian origin, side by side with a Christian legend about the Wise Men from the East in the Gospel of Matthew![6]

All this means that in the early centuries of the Christian era the religion of the Buddha was actively at work in languages spoken by the Medes and Parthians who were present at Jerusalem in the thirties of the first century (Acts ii. 9): "Parthians and Medes and Elamites, and the dwellers in Mesopotamia, in Judea and Cappadocia, in Pontus and Asia." It is to be noted that *Judea,* in this verse, is tautological, for the scene narrated is laid in Judea. As foreign countries are being represented, we must probably read *India,* though Dean Alford defends our present text on geographical grounds. Now the New Testament writer who tells us this is Luke, the Antioch physician, the author of a Gospel whose aim was to take Christianity outside the narrow pale of Judaism and put it into line with the Gentile religions. It is Luke alone who has the story of the Penitent Thief, corresponding to the Angulimālo of the Sūtras.[7] And in order to introduce this story into the Gospel, Luke is compelled to violate the text of his master Mark, who says that both the malefactors reviled the Lord. A scholar of the English church, in a recent number of the *Hibbert Journal,* has shown that Luke was utterly unscrupulous in literary matters, and again and again did violence to his sources to carry out his aims. I have suggested, both in my Tokyo edition (1905) and in my Philadelphia edition (1908) that Luke did violence to the text of Mark on purpose to introduce these Buddhist legends wherewith he was familiar.

It is true that our present Bactrian and Sogdian manuscripts

[4] Gauthiot, in the Paris *Journal Asiatique,* January-February, 1912.

[5] *Journal Asiatique,* Nov.,-Dec., 1910.

[6] *Abhandlungen* of the Royal Academy of Berlin, 1908 and 1911: article "Uigurica," by F. W. K. Müller.

[7] Middling Collection, No. 86, in the Pāli; but in the Numerical Collection in Chinese.

are probably to be dated between the third century and the eighth. But this is in Chinese Turkestan, whither their archetypes had been brought from regions to the westward. We know, from coins and from Buddhist history, that the religion was flourishing in Bactria both at and before the time of Christ; and the inference is irresistible that, when the missionaries began their Chinese translations in the sixties of the first century, they had already acquired experience as translators in the tongues of the Parthian empire. The only difference is, that the well-established civilization of China, and the continuance of Buddhism therein, have preserved and dated the Chinese versions, whereas the extinction of Buddhism by Islam in Afghanistan and elsewhere has destroyed those older ones.

What we have actually found of them is due to Chinese care, in Chinese dominions; but we are entitled to infer a whole lost literature in Bactrian, Sogdian AND GREEK,[8] which was the vehicle of Buddhist propaganda in the days of the Christian Evangelists.

We do not need to wait until a Greek Sūtra is dug up in Afghanistan, as I have hitherto anticipated. We now have actually in our hands a series of Buddhist documents translated by missionaries into languages that were understood by the very people whom Luke records as present at a feast which his authorities had witnessed. Could we but find, in these languages, the Buddhist Angelic Heralds and their Hymn, as recorded in the Sutta-Nipāto; the Lord's Three Temptations, viz., to transmute matter, to assume temporal power and to commit suicide, as recorded in the Classified and Long Collections; the Penitent Brigand aforesaid; and the Charge to the Sixty-one Missionaries, so like Luke's Charge to the Seventy, we should have in our hands the key to the riddle which Max Müller said he had spent his life in trying to solve; viz., the indebtedness of our proud religion of humility and peace, which has been spread over the planet by the swords of Europeans, to the meek and lowly cult of our brown brethren across the sea—that cult which, alone among the faiths of mankind, has never dipped its hands in the blood of animals or men.

Albert J. Edmunds.

Philadelphia, Pa.

[8] "There is little doubt that in Bactria, Buddhist literature was actually translated into Greek." (*Buddhist and Christian Gospels,* 4th ed., Philadelphia, 1908-1909. Vol. I (1908), p. 154). See also my remarks on "that lost version of the Sūtras which traveled westward." (*Buddhist Texts in John,* 1906, pp. 26-28.

BUDDHIST LOANS TO CHRISTIANITY.
WITH SPECIAL REFERENCE TO RICHARD GARBE.
SECOND ARTICLE.

To my remarks in *The Monist* and *The Open Court* for January, 1912, I should like to add a few words to congratulate Professor Garbe upon the conclusion of his learned monograph. His final summary I heartily endorse, except that I would modify one statement. The following is the paragraph referred to (*Monist,* April, 1912, p. 187):

"As we have seen, Christian influences upon the development of Buddhism are limited to secondary products of a late day; just as inversely Buddhist influences upon Christianity may be pointed out only in non-essential particulars and from times in which the doctrine of the Christian faith was established as a firm system.[ ] All identities and similarities in the teachings of these two great world-religions have, so far as *essential* matters are concerned, originated independently of one another, and therefore are of far greater significance for the science of religion than if they rested upon a loan."

These are essentially my own conclusions, stated many times since February, 1900; but I would add, at the brackets, the words: [*except a few passages of minor import which found their way from organized and aggressive Buddhism into formative Christianity.*]

The passages especially in my mind are the Angelic Heralds and their Hymn in Luke ii; the Lord's Three Temptations in Luke and Matthew; two texts in John expressly quoted as Law and Scripture, but not found in the Old Testament or any other Jewish book (John vii. 38; xii. 34); and the phrase *æon-lasting* (or "eternal") *sin* at Mark iii. 29—a phrase so foreign to Christian ideas that the copyists altered it to "eternal damnation," as Dean Alford admitted. Moreover, as said in *Buddhist and Christian Gospels* (Ed. 4, vol. 1, p. 157), Luke was probably influenced by such stories as the Charge to the Sixty-one Missionaries (his "Seventy") and the Penitent Brigand. As shown in my Tokyo edition (p. 48: the only important passage not repeated in the Philadelphia one) each of these stories of Luke is demonstrably fiction, and he moreover can be proved to have altered the Marcan or Synoptic tradition to suit his own ideas (as in Mark xvi. 7 = Luke xxiv. 6). To my mind the case is precisely analogous to that of the moons of Uranus being perturbed by the presence of Neptune.

When in Philadelphia last fall, Franz Cumont told us that there is a set of technical phrases in ancient Greek books on astrology which have now been shown to be literal translations from the Babylonian. In precisely the same way, such Buddhist phrases as *æon-lasting sin* and others gained similar currency among the ancients, *who persistently sought out the distinctive teachings of the great nations, just as we do now.*

With these reservations, I wish, as a student of Buddhism, to give my most cordial adhesion to the conclusions of the learned Brahmin scholar, who has dealt with a knotty problem in a masterly manner and summarized the researches of many specialists.

PHILADELPHIA, PA. ALBERT J. EDMUNDS.

A POSTSCRIPT TO INDO-ROMAN RELATIONS IN THE FIRST CENTURY.

In *The Monist* for December, 1911, Professor Garbe denied the existence of Buddhist loans to canonical Christianity, and gave as one of his reasons the following question and answer:

> "Do the evidences of intercommunication at all permit the *assumption*[1] that as early as the first century after Christ, or earlier, Buddhist legends and ideas had found their way into Palestine?....
>
> "They are not apt to raise this possibility to a serviceable degree of probability for as early a period as the first post-Christian century."

To this assertion I replied in the following number, avoiding reference to the canonical literature, which was simultaneously considered by my friend Mr. Albert J. Edmunds, but assembling various evidences of a large, important and rapidly increasing intercommunication between Rome and India during the first century of the Christian era, as indicating the possibility of the assumption which Professor Garbe had outlined.

In *The Monist* for July, 1912, in a postscript to his most instructive discussion, Professor Garbe acknowledges the probability of closer intercommunication than he has heretofore admitted, and accepts one of the canonical parallels offered by Mr. Edmunds; but he thinks that I "beg the question" by *assuming* the possibility of an interchange of ideas as well as goods.

To this objection I would reply that I was but addressing my-

[1] Italics mine.

self to the *assumption* which he had declared to be unwarranted; so that if there has been any begging of the question it would seem to have been in the same degree on each side of the discussion.

Professor Garbe objects to any citation of the Periplus because it does not mention religion; but the citation was merely to show the existence of an active commerce, and it is well known that the missionary and the trader have gone close together in many ages of the world. They have not always respected one another, but they have usually followed the same paths. Surely Professor Garbe would not expect a future historian of our own times to deny the *assumption* of Christian missions in China because some surviving consular report on the Shanghai trade might omit a reference to the Nicene Creed!

Professor Garbe objects, also, that the Hindu traders to the Roman Empire were Dravidians and stupid, and therefore not likely to talk of their religion. But in the first century of our era they were increasingly Indo-Ṣcythian, from a portion of India that professed a liberal and proselytizing Buddhism, and I repeat that for that date and race, a spreading of ideas together with an interchange of goods, was not only a possible *assumption,* but a probable fact.

The extent of such intercommunication is made much more evident by Mr. J. Kennedy's paper "The Secret of Kanishka," begun in the *Journal of the Royal Asiatic Society* for July 1912. The long-drawn discussion as to the so-called Vikrama era of 58 B. C. seems brought to a reasonable conclusion through Mr. Kennedy's brilliant assembling of Chinese and numismatic evidence. It was the era of the second Buddhist Council and of the Kushan king Kanishka. His power over northwestern India, built up by his control of the transcontinental silk-trade, was fortified by his becoming the protector of the Buddhist faith; and under him and his immediate successors, just before the Christian era, it is highly probable that his faith was expounded to the east as far as Turfan, and to the west as far as Charax Spasini, Antioch and Alexandria.

The truth is, that during the period between 50 B. C. and 100 A. D., approximately, India was a leading factor in the world's thought, industry, commerce, and wealth; and, this being the case, to repeat Professor Garbe's own words, "the evidences of intercommunication permit the assumption of the migration of Buddhist legends and ideas into Palestine as early as the first century after Christ."

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BOOK REVIEWS AND NOTES.

THE RELIGION OF THE IRANIAN PEOPLES. By *C. P. Tiele.* Part I (From the German). With Darmesteter's Sketch of "Persia," and Goldziher's "Influence of Parsism on Islam" (From the French). Translated by *G. K. Nariman.* Bombay: Parsi Publishing Co., 1912.

G. K. Nariman, who is known to readers of the *Revue de l'Histoire des Religions* and other magazines, is a Parsi scholar who is also acquainted at first-hand with Buddhism, both from its Pāli and its Sanskrit sides. Indeed, one of the features of this excellent translation is an appendix by the translator entitled, "Some Buddhistic Parallels." The subjects dealt with are: (1) the triple formula: body, speech and mind; (2) Buddhist allusions to consanguineous marriages; (3) Hindu exposure of the dead, as described in the Pitakas; (4) Māra and Ahriman; (5) questions addressed to Mazda and Buddha; (6) the idea of a counterfeit creation (Parsi) or a counterfeit religion (Buddhist); (7) the use of the name Ormazd by Turanian Buddhists; (8) the killing of noxious creatures by the Kambojas; and some other points.

The following note on the Turkish Buddhist literature recently found in Central Asia deserves reprinting:

"*Ein in turkischer Sprache bearbeitetes buddhistisches Sūtra,* by Radloff and Stael-Holstein (St. Petersburg, 1910). This important work is typical of the avoidable Babel which Western philologists seem unfortunately determined to create in their otherwise fascinating field of marvelous investigations. The original text is in the, up to now, almost unknown Uigurian, which the Russian scholars have made accessible to us through a German translation; but the transliteration is in Russian character, and the interesting notes on the Brahmi gloss are made unserviceable to the average student of Buddhism by the introduction of two sets of unknown alphabets, besides Chinese, Arabic and Nāgarī scripts. Eastern students, however, must be grateful to the Imperial Academy of Sciences, St. Petersburg, for the employment of the Nāgarī character in the publication of its admirable series of *Bibliotheca Buddhica.*"

It is studies like this which help to break down the former provincialism of religious treatises, wherein (to borrow a phrase of Cumont's) each religion was regarded as an experiment in a closed vessel.

We could wish that the learned translator had had a little more regard to the naturalization of Oriental adjectives, and so given us the familiar English forms *Achæmenian, Sassanian,* instead of *Achæmenide, Sassanide.* The lack of diacritical marks is also confusing. But such small flaws by no means detract from the great value of Tiele's able essay in an English dress, accompanied by the other essays indicated, which make the little book an interesting companion for the student of Comparative Religion.

Albert J. Edmunds.

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altutonish (pangerman). bei elias molee. Tacoma, 1911. Pages 32.

The advocate of a new language called "altutonish" or "pangerman" together with an abolition of all capital letters is vigorously continued by "elias molee, ph. b., 1554 'd' street, tacoma, wash., u, s. a." by sending out a pamphlet containing exercises in his new language which he characterizes as " ein (a) union spiek (language), makn up ov deuch, english, skandinavish and hollandi, for to agenfererein (re-unite) al tutonish folka (people) into ein spiek mitin (within) feivti (50) jiera (years)."

He believes that the world is mainly Teutonic and that a combination of all Teutonic languages should be the best international language. As a motto he selects a word of Victor Hugo, who has said somewhere: "The German character hovers over the nations," or as it reads in German: *"Die deutsche Natur schwebt über den Völkern."*

Such a combination might have been possible at the time the English language originated after the Norman conquest through the breakdown of the literal Anglo Saxon; but times are changed through the wide-spread use of written and printed language which has added power to the resistance of the established language such as was impossible in the times when language was still purely speech, when it was limited to the spoken word which is more flexible and would admit easily of radical changes. *κ*

Readers of *The Monist* will be interested to hear that Prof. Hugo de Vries is making another visit to the United States this autumn. He reached New York about September 12, where he was to give a lecture at the Botanical Garden on September 14. From there he goes to the University of Pennsylvania to see the work which Professor Davis is doing with *Oenotheras,* and then spends a short time in Washington. His next objective point is Dixie Landing, Ala., where he goes with Professor Tracy to visit the type locality of *Oenothera grandiflora* to study its possible mutants in its original habitat. He then goes to Biloxi, Miss., where he will make his headquarters while visiting the "mud lumps" near the mouth of the Mississippi River, and a number of islands near Biloxi. After that he and Professor Tracy will go to San Antonio, Brownsville, and other points in southern Texas, where Professor de Vries goes to study the relation of the flora to the geological and geographical conditions. On October 14, Professor de Vries is to deliver the dedicatory address of the Rice Institute, at Houston, Texas.