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"MORE FUNDAMENTAL" VIEW ON THE ELASTICITIES

OF SUBSTITUTION AND OF DERIVED DEMAND

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1. Introduction

Classical theory of the elasticity of substitution and of derived demand become quite a tangled and perplexing issue when more than two factors contribute to production. This is so because not only substitutability but also complementarity among factors may crawl into the model as Professor J. Hicks [7] correctly pointed out. Marshall's ingenious intuition on the patterns of the change in the elasticity of derived demand with respect to the change in the other economic parameters had been reinvestigated by Pigou [17], and his celebrated "Four Rules" of them have been mathematically well formulated and tested in the analyses of J. Hicks [5, 6] and Bronfenbrenner [3] in their confinement of attention to a two-factor industry. But the validity of those Rules in a general n-factor case is yet to be carefully examined (e.g., in the three-factor case--"Black Labour," "White Labour," and "Capital" à la Hicks, or the combination of not only labour and capital but land for the urban economic concern such as Muth [16]). Although this task was recently tackled by R. Sato & Koizumi [19, 20] and Diewert [4] in a direct aim at the generalization of Hicksian two-factor to a general n-factor model and by Mundlak [15] and Maurice & Ferguson [11] in a different context, their results do not seem to be fully appropriate and complete, especially in their treatment of the elasticity of substitution in the general n-factor case, which is one of the key parameters in determining the elasticity of derived demand.

In his attempt to clear up the ambiguity of an economic meaning of the widely accepted definition of the Allen-Uzawa partial elasticity of substitution, Professor Morishima [13] wove Professor J. Robinson's original idea of elasticity of substitution in [18] into a mathematical formula for the constant returns to scale production function and it was generalized to any satisfactorily smooth production function by Kuga & Murota [8, 9]. It was proved in this new context that Allen-Uzawa's formula makes sense of elasticity of substitution as such if and only if a production function is of constant elasticity of substitution (CES), while the generalized Morishima elasticity of substitution possibly broadens one's scope of economic studies of elasticity to much wider classes of production technology than before.

The purpose of this note is threefold. First we should like to present the author's own account of the unsymmetrical definition(*) of elasticity of substitution in the spirit of J. Robinson and Morishima in contrast to the various symmetrical definitions of it so far widely adopted in the past economic literature. Secondly, we obtain the mathematical representation of the elasticity of derived demand in terms of other related elasticities and factor shares for the three-factor case as a bird's-eye view of a state of general factor market equilibrium under pure competition. We will then confirm that our result encompasses

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(*) The elasticity of substitution of factor j for i may be called <u>unsymmetrical</u> if it is <u>not</u> generally equal to the elasticity of substitution of i for j and symmetrical if it is identically so.

J. Hicks' intuitive formulae of elasticity of substitution and of derived demand in [7] as special cases of our general case and algebraically justifies them. Our three-factor model is squeezed out from the general n-factor model computed in the Mathematical Appendix to this note in a similar fashion to Diewert [4]. Thirdly, we test Marshall's Four Rules on the elasticity of derived demand in our three-factor model and demonstrate some negative accounts of their universal validity by providing a few counterexamples under various classes of CES and non-CES production technology. Our test results show some deviation from the ones given by R. Sato & Koizumi [19] which list affirmative accounts of it. But our intention is not to eclipse Marshall's sparkling achievement in the theory of derived demand at all but solely to strengthen and enrich it on a solid foundation. We also hope that our presentation here brings some positive implication to the recently pronounced Samuelson's suspicion on the substanceless elasticity discussions.(*)

2. Generalized Morishima Elasticity of Substitution

Suppose that an industry output y is produced by using n factors $(x_1, \ldots, x_n)$ through the production function:

(1) $$y = f(x_1, \ldots, x_n) .$$

We assume that the function f is (1) twice differentiable with respect to each input, (2) non-decreasing in input and (3) concave in input. [18; p. 330ff], we have:

DEFINITION I: The elasticity of substitution of factor j for i is defined as the proportionate change in the ratio of the quantities of

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(*)See the postscript to this paper.

the factors i and j, divided by such a proportionate change in the marginal rate of substitution of x_j for x_i as to leave all other marginal rates of substitution and the output level y intact.

We then obtain:

THEOREM: (Morishima [13], Kuga & Murota [8, 9]): Given a production function (1) the generalized Morishima elasticity of substitution of x_j for x_i in accordance to Definition I is formulated as

$$\sigma^M_{ij} = -\partial \log(x_j/x_i) \;/\; \partial \log(f_j/f_i) \tag{2}$$

$$= \frac{f_j}{x_i}\frac{F_{ij}}{F} - \frac{f_j}{x_j}\frac{F_{jj}}{F} \qquad (i,j=1,\ldots,n)$$

where $f_i = \partial f/\partial x_i, \quad f_{ij} = \partial^2 f/\partial x_i \partial x_j,$

$$F = \begin{vmatrix} 0 & f_1 & \cdots\cdots & f_n \\ f_1 & f_{11} & \cdots\cdots & f_{1n} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n1} & & f_{nn} \end{vmatrix}$$

and F_{ij} is the co-factor of f_{ij} in F.(*)

(*) The generalization to non-constant returns to scale production function due to Kuga & Murota can be regarded as an answer to the question raised by Morrissett [14; p. 47] on J. Robinson's "more fundamental" definition of elasticity of substitution in a general case.

As its special case, the above elasticity coincides with:

DEFINITION II (Allen [1], Uzawa [22]): With additional assumption of linear homogeheity of (1), the Allen-Uzawa partial elasticity of substitution of x_j for x_i is given by

$$(3) \qquad {}_{\sigma}A_{ij} = \frac{\sum_{k=1}^{n} f_k x_k}{x_i x_j} \; \frac{F_{ij}}{F} \qquad (i,j=1,\ldots,n)$$

Precisely stating this coincidental relationship, one can prove that ${}_{\sigma}M_{ij} = {}_{\sigma}A_{ij}$ (for all i, j = 1, ..., n; $i \neq j$) if and only if the production function f is of CES (For the proof, see Kuga & Murota [8, 9]).

Comparing the Definition I with II, we believe that the general use of the latter involves the following pitfalls. By means of the well-known theorem of Schwartz or of Young on the interchangeability of the order of differentiation(*) we can claim that $F_{ij} = F_{ji}$ through $f_{ij} = f_{ji}$ for all i, j = 1, ..., n as long as the production function f is twice differentiable with respect to all the factors as we are assuming throughout this note and all the authors of economics of elasticity have been explicitly or implicitly keeping it in mind. Economically this amounts to the symmetry(**) in Allen-Uzawa partial elasticity of substitution:

(*) See, for example, Bartle [2, pp. 241-243].

(**) The symmetry also holds for McFadden's direct partial elasticity of substitution [12] as a reformulation of the idea of Morrisset [14].

$$\sigma^A_{ij} = \sigma^A_{ji} \qquad \text{for all } i, j = 1, \ldots, n.$$

But regardless of how smooth a production function is, why is the elasticity of substitution of "White Labour" for "Black Labour" necessarily identical with the one of "Black Labour" for White Labour" when a third factor, say, "Capital" is substitutable for each of the Labours? Among the available economic literature we can hardly find the reason why. Instead, we are apparently better off if we relax the condition for symmetry and simply admit the unsymmetry, which we encountered in the first Definition;

$$\sigma^M_{ij} \neq \sigma^M_{ji} \quad (i \neq j) \text{ in general.}$$

Another problem involved in Allen-Uzawa's definition is the never explained, purely algebraic creature σ^A_{ii}, i. e., the "Elasticity of Substitution of Factor i for Itself," which R. Sato & Koizumi [19], Diewert [4] and Maurice & Ferguson [11] are using in their expression of the elasticity of derived demand but which Hicks has been deliberately avoiding to write down in his entire series of the analysis of elasticity. If we follow the Robinsonian line of thought stated at the beginning of this section, the "Elasticity of Substitution of a Factor for Itself" should be either equal to zero or undefined. Under Definition I the former holds, i. e.,

$$\sigma^M_{ii} = \frac{f_i}{x_i}\frac{F_{ii}}{F} - \frac{f_i}{x_i}\frac{F_{ii}}{F} = 0,$$

while the mathematical quantity σ_{ii}^{A} cannot be explained in either way.

Taking account of these seeming defects in σ_{ii}^{A}, we use only σ_{ij}^{M} as the measure of elasticity of substitution in the rest of this note and denote it simply by σ_{ij}.

3. Elasticity of Derived Demand in Three-Factor Case

Now that our technical foundation of the elasticity of substitution is constructed, we can proceed to the analysis of the elasticity of derived demand in the world where the market force is fully functioning under the condition of pure competition. When the industry is engaged in the production activity with the production function (1) satisfying the conditions of (i)linear homogeneity in input, (ii)non-decreasingness in input and (iii)concavity in input in the confinement of n = 3, the conditions of industry's cost minimization and market equilibrium for output and input can be shown to yield the elasticity of derived demand for, say, the third factor as:

(4)

$$\begin{aligned}\lambda_3 &= -\frac{p_3}{x_3}\frac{\partial x_3}{\partial p_3} \\ &= \eta k_3 + (k_1\sigma_{13} + k_2\sigma_{23}) \\ &\quad + \begin{bmatrix}\sigma_{13} - k_1\sigma_{13} - k_2\sigma_{23} - \eta k_3 \\ \sigma_{23} - k_1\sigma_{13} - k_2\sigma_{23} - \eta k_3\end{bmatrix}^T\end{aligned}$$

$$\times \begin{bmatrix} -k_2\sigma_{21} - k_3\sigma_{31} - \eta k_1 - e_1 & \sigma_{21} - k_2\sigma_{21} - k_3\sigma_{31} - \eta k_1 \\ \sigma_{21} - k_1\sigma_{12} - k_3\sigma_{32} - \eta k_2 & -k_1\sigma_{12} - k_3\sigma_{32} - \eta k_2 - e_2 \end{bmatrix}$$

$$\times \begin{bmatrix} \sigma_{31} - k_2\sigma_{21} - k_3\sigma_{31} - \eta k_1 \\ \sigma_{32} - k_1\sigma_{12} - k_3\sigma_{32} - \eta k_2 \end{bmatrix},$$

where $\sigma_{ij} = \sigma^M_{ij}$ (Morishima elasticity of substitution of factor j for i)

$\eta = -\frac{\partial}{\partial p} D(p) \frac{p}{y}$ (demand elasticity)

$k_i = p_i x_i / py$ (i-th factor share)

$e_i = \frac{\partial}{\partial p_i} S_i(p_i) \frac{p_i}{x_i}$ (i-th factor's supply elasticity)

and p = the price of output y

p_i = the price of the i-th factor

$D(p)$ = the market demand function for output y

$S_i(p_i)$ = the market supply function of the i-th factor.

In order to have some intuitive justification of this formula (4) obtained from our general formula (A-13) in the Mathematical Appendix, we can compare it with Hicks' equation for a two-factor case:

(5)
$$\lambda = \frac{\sigma\eta + e[k\eta + (1-k)\sigma]}{[k\sigma + (1-k)\eta] + e}$$

in the following manner. In (5) the elasticity λ of derived demand for one factor becomes

(6) $$\lambda = k\eta + (1-k)\sigma$$

when the supply elasticity e of another factor is infinity, and it turns out that

(7) $$1/\lambda = (k/\eta) + (1-k)/\sigma$$

when supply elasticity diminishes to zero.

On the other hand, if we assume that our three-factor production function is of CES, i.e., $\sigma_{12} = \sigma_{21} = \sigma_{23} = \sigma_{32} = \sigma_{31} = \sigma_{13} = \sigma$, the formula (4) yields

(5') $$\lambda_3 = \frac{(\eta-\sigma)[\sigma(k_1e_2+k_2e_1) + k_3e_1e_2] + \sigma(\eta\sigma+e_1e_2)}{(\eta-\sigma)\ (k_1e_2+k_2e_1-k_3\sigma) + \eta\sigma + e_1e_2}$$

Therefore, if the supply elasticities e_1 and e_2 of the first and second factors approach either infinity or zero concurrently, the elasticity λ_3 of derived demand for the third factor takes the following limit values:

(6') $$\lim_{\substack{e_1 \to\infty \\ e_2 \to\infty}} \lambda_3 = k_3\eta + (1-k_3)\sigma$$

(7') $$\lim_{\substack{e_1 \to 0 \\ e_2 \to 0}} (1/\lambda_3) = (k_3/\eta) + (1-k_3)/\sigma,$$

which are perfectly compatible with (6) and (7) of Hicks.

We proceed to check what Professor Hicks really means in his attempt to extend his formula (5) from a two-factor to a three-factor case. Rewriting his not necessarily CES, three-factor counterparts [7; p. 291: equations (5) and (6)] to (6) and (7) in our present notation, we get

$$\lambda_3 = k_3\eta + k_1\sigma^H_{13} + k_2\sigma^H_{23} \qquad (e_1 \to \infty,\ e_2 \to \infty) \tag{8}$$

$$(1/\lambda_3) = (k_3/\eta) + (k_1/s_{13}) + (k_2/s_{23}) \quad (e_1 \to 0 \quad e_2 \to 0), \tag{9}$$

where the superfix H is applied to his original notation σ_{ij} by the present author. Hicks considers that σ^H_{ij} and s_{ij} should be some kinds of indicators of elasticity of substitution of factor j for i. Let us tentatively call σ^H_{ij} and s_{ij} Hicksian p-elasticity of substitution and q-elasticity of substitution, respectively.

On the other hand, if the elasticities of supply of first and second factors approach to infinity together in our formula (4), we obtain the limit value of λ_3:

$$\lim_{\substack{e_1 \to \infty \\ e_2 \to \infty}} \lambda_3 = k_3\eta + k_1\sigma_{13} + k_2\sigma_{23}. \tag{8'}$$

Hence the formulae (8) and (8') are formally identical. This may be interpreted to imply that Hicksian p-elasticity of substitution σ^H_{ij} is nothing but Morishima elasticity of substitution:

$$\sigma^H_{ij} = \sigma_{ij} = \sigma^M_{ij},$$

at least in the limit of $e_1 \to \infty$ and $e_2 \to \infty$.

If Professor Hicks feels [7; p. 290 & 296] that his p-definition may be the formulation of elasticity of substitution in the sense of J. Robinson, his intuition is correct!

With respect to the limit value of λ_3 in (4) with e_1 and e_2 approaching to zero, so far we could not find any clear counterpart to (9). Therefore we cannot supply any economic interpretation to Hicksian q-elasticity of substitution s_{ij} at this moment.

4. Some Negative Accounts of Marshall's Four Rules

This section is devoted for the investigation of the workability of Marshall's Four Rules based on our three-factor model(4). To be concrete, we want to know if each of the following:

The First Rule	$\partial\lambda_3/\partial\sigma_{i3} > 0$,	$\partial\lambda_3/\partial\sigma_{3j} > 0$	(i,j=1, 2)
The Second Rule	$\partial\lambda_3/\partial e_i > 0$		(i=1, 2)
The Third Rule	$\partial\lambda_3/\partial k_3 > 0$		
The Fourth Rule	$\partial\lambda_3/\partial\eta > 0$		

universally holds, or if not, under what conditions each may hold. We consider this problem under a few specific types of production technology.

CASE I: CES Production Function

Based on the formula(5') we obtain the following evaluation:

(I-i) $$\partial\lambda_3/\partial\sigma = \frac{1}{g^2}\Big[k_3\eta^2\sigma^2 + \{(\eta-\sigma)\ (k_1e_2 + k_2e_1) + \eta\sigma\}^2$$

$$+ \{ (k_3^2 + k_3 + 1)(\eta-\sigma)^2 + (1-k_3)(2\eta\sigma + e_1e_2)\} \; e_1e_2 \Big]$$

(I-ii) $$\partial\lambda_3/\partial e_1 = \frac{1}{g^2} \; k_3(\eta-\sigma)^2 \{k_1e_2^2 - k_2\sigma^2 + \sigma(k_1e_2 + k_2e_1)\}$$

$$\partial\lambda_3/\partial e_2 = \frac{1}{g^2} \; k_2(\eta-\sigma)^2 \{k_2e_1^2 - k_1\sigma^2 + \sigma(k_1e_2 + k_2e_1)\}$$

(I-iii) $$\partial\lambda_3/\partial k_3 = \frac{1}{g^2} \; (\eta-\sigma)(\sigma^2 + e_1e_2)$$

$$\Big[(\eta-\sigma)\{k_1e_2 + k_2e_1 - k_3(\frac{k_1}{k_3}e_2 + \frac{k_2}{k_3}e_1)\} + \sigma\eta + e_1e_2\Big]$$

(I-iv) $$\partial\lambda_3/\partial\eta = \frac{1}{g^2} \; k_3(\sigma^2 + e_1e_2)^2 > 0,$$

where $$g = (\eta-\sigma)(k_1e_2 + k_2e_1 - k_3\sigma) + \eta\sigma + e_1e_2 \; .$$

Hence only the Fourth Rule universally holds, namely the derived demand for a factor in question is more elastic, the more elastic is the demand for the product. The other three Rules are not always true. A sufficient condition for the validity of the First Rule is that either (a) supply elasticities e_1 and e_2 of factors 1 and 2 are both positive, or (b) both negative or (c) one of them is equal to zero. But this is only a sufficient condition. Depending on the relative magnitude of various

elasticities and factor shares, the First Rule may or may not be true. With respect to the Second and Third Rules it seems almost impossible to work out any simple conditions which guarantee their validity. It is particularly difficult to see the implications of the Third Rule, which required detailed analyses of Hicks [5, 6] and Bronfenbrenner [3] for its test in a two-factor case. For example, even if elasticities of supply e_1 and e_2 are both positive and if the term

$$k_1e_2 + k_2e_1 - k_3(\frac{k_1}{k_3} e_2 + \frac{k_2}{k_3} e_1)$$

is positive, a paradoxical situation may occur, i.e., the reswitching between the case of 'important to be unimportant' and the case of 'important to be important' may occur when the term $(\eta-\sigma)$ changes from a big positive number to a big negative number in the absolute value. These results are different from the conclusion of R. Sato & Koizumi [19; pp. 112-113; Theorems I and II] which relied on the Allen-Uzawa partial elasticity of substitution in their formulation of the elasticity of derived demand.

CASE II: All three factors are perfect substitutes for each other in the sense that all the elasticities of substitution approach infinity.

The formula (5') yields

$$\lambda_3 \rightarrow \frac{\eta - (k_1e_2 + k_2e_1)}{k_3} \tag{11}$$

Therefore we get

(II-ii) $$\partial\lambda_3/\partial e_1 = -\frac{k_2}{k_3} < 0, \qquad \partial\lambda_3/\partial e_2 = -\frac{k_1}{k_3} < 0$$

(II-iii) $$\partial\lambda_3/\partial k_3 = \frac{1}{k_3^{\,2}}\{(k_1e_2 + k_2e_1) - k_3(\frac{k_1}{k_3}e_2 + \frac{k_2}{k_3}e_1) - \eta\}$$

(II-iv) $$\partial\lambda_3/\partial\eta = \frac{1}{k_3} > 0.$$

While the Fourth Rule is always true, the validity of the Third Rule is not clear and the Second Rule is negated in any circumstance. This result (II-ii) forms the opposite to the situation in the two-factor case, i. e., the Hicksian two-factor formula (5) yields

$$\lim_{\sigma\to\infty} = \frac{\eta + e(1-k)}{k}$$

from which we obtain that

$$\partial\lambda/\partial e = (1-k)/k > 0.$$

In summary, when all the factors are perfectly substitutable for each other in our three-factor model, the derived demand for the third factor is more inelastic, the more elastic is the supply of other two factors.

CASE III: The factors 1 and 2 are perfect substitutes in the sense that $\sigma_{12} \to \infty$ and $\sigma_{21} \to \infty$ but other combinations of factors are not.

In this case the formula (4) results in

(13) $$\lambda_3 \to \eta \qquad \text{in the limit.}$$

This amounts to saying that the elasticity of derived demand does not depend on any elasticity of substitution and factor shares but demand elasticity. It is trivial that the Fourth Rule is right and others are irrelevant.

5. Conclusion

This paper was designed, generally to furnish a general theory of the elasticity of derived demand with a solid technical foundation in the Robinsonian spirit of elasticity of factor substitution, and in particular, to answer some of the open problems implicitly posed by Hicks in [7]. At an abstract level we have succeeded to describe a state of competitive factor market equilibrium in terms of various elasticities and factor shares. But the analytical complexity required to test Marshall's Four Rules even in the three-factor CES case leads us to a guess that there may not be any universal rules on the elasticity of derived demand in a general case (possibly except for the Fourth Rule) corresponding to such clear Four Rules as in the Hicksian two-factor case. Entanglement of substitutes and complements among many factors would rather require case-by-case analyses of the magnitude of elasticities and factor shares to execute a comparative statics on the elasticity of derived demand. This observation deviates from the results obtained by R. Sato & Koizume [19], which assert the universality of the Four Rules except for certain reservation on the Third Rule. This gap seems to have arisen from the deviation between the two different concepts of elasticity of factor substitution founded by Allen and Uzawa on the one hand and J. Robinson and Morishima on the other.

Hicks' attempt to translate his idea of the elasticity of derived demand in a two-factor to a three-factor case was completely reformulated and justified in our new framework, and his vaguely posed p-elasticity of substitution was identified with Morishima's. The meaning of the Hicksian q-elasticity of substitution, as we shall tentatively call so, is left for further research.

Postscript: In his comment on Sato and Koizumi(*), Samuelson(**) pointed out the problem that "most elasticity discussions are empty of substance." From our view in this paper his opinion seems to be perfectly justifiable already at a purely theoretical level before waiting for his words about the problem of empirical measurement of elasticities. The most of the earlier works in this field of economic study have been carried through in terms of partial elasticity of substitution and/or alike. But nobody has ever explained what kind of economic content or relation the partial elasticity of substitution is actually evaluating at least in a theoretical model, if not in dealing with statistical data.

(*)Sato, R., and Koizumi, T., "The Production Function and the Theory of Distributive Shares," American Economic Review, Vol. LXIII, No. 3 (June 1973), pp. 484-9.

(**)Samuelson, P. A., "Relative Shares and Elasticities Simplified: Comment," American Economic Review, Vol. LXIII, No. 4 (September 1973), pp. 770-1.

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Mathematical Appendix

Derivation of the formula of elasticity of derived demand in the general n-factor case

As was stated in the section 3 for the three-factor case, the n-factor production function (1) in section 2 is assumed to be; (i)linearly homogeneous with respect to n factors, (ii)non-decreasing in input and (iii) concave in input. We then denote the industry cost function by $c(p_1, \ldots, p_n)$, where $(p_1, \ldots, p_n)$ stands for the vector of n factor prices. From one of the results of Shepherd [21; p. 47, equation (41)], we can deduce that

$$\text{(A-1)} \qquad x_i(y;\, p_1, \ldots, p_n) = c_i(p_1, \ldots, p_n)\cdot y \qquad (i = 1,\ldots,n)$$

where $c_i = \partial c/\partial p_i$ and x_i is a cost minimizing amount of the i-th factor required for the production of output y given input prices $p_1, \ldots, p_n$. Our formula (2) is then rewritten as

$$\text{(A-2)} \qquad \sigma_{ij} = \frac{p_j}{x_i} c_{ji}\, y - \frac{p_j}{x_j} c_{jj}\, y \, .$$

Following Diewert [3], we introduce the following notation:

$$p = \text{the price of output } y$$

$$D(p) = \text{the market demand function for output } y$$

$$S_i(p_i) = \text{the market supply function of the i-th factor}$$

$$\eta = -\frac{\partial}{\partial p} D(p)\, \frac{p}{y} \qquad \text{(demand elasticity)}$$

$$e_i = \frac{\partial}{\partial p_i} S_i(p_i)\, \frac{p_i}{x_i} \qquad \text{(i-th factor's supply elasticity; } i=1,\ldots,n)$$

$$k_i = p_i x_i / py \qquad \text{(i-th factor share; } i=1,\ldots,n)$$

The market equilibrium conditions for output and input yield

(A-3) $$c_i(p_1, \ldots, p_n)\, D[c(p_1, \ldots, p_n)] - S_i(p_i) = 0 \qquad i=1,\ldots,n$$

(A-4) $$x_n(y;\, p_1, \ldots, p_n) = c_n(p_1, \ldots, p_n)\, D[c(p_1, \ldots, p_n)].$$

Total differenciation of the i-th equation in (A-3) gives us

(A-5) $$\sum_{j=1}^{n} \{c_{ij} D(c) + c_i \frac{\partial}{\partial p} D(c)\, c_j\} dp_j - \frac{\partial}{\partial p_i} S(p_i) dp_i = 0 \, .$$

This is rewritten as

(A-6) $$\sum_{j=1}^{n} \{c_{ij} y - x_i x_j \eta / py\} dp_j - \frac{e_i x_i}{p_i} dp_i = 0 \, .$$

Without losing generality we can focus our interest on the investigation of own elasticity of derived demand for the n-th factor:

$$\lambda_n = - \frac{p_n}{x_n} \frac{\partial x_n}{\partial p_n} \, .$$

Our definitional relationship (A-2) contains

(A-7) $$\sigma_{in} = \frac{p_n y}{x_i} c_{ni} - \frac{p_n y}{x_n} c_{nn} \qquad (i=1,\ldots,n-1).$$

We also have

(A-8) $$0 = \sum_{k=1}^{n} p_k c_{nk}$$

because of linear homogeneity of the unit cost function c with respect to factor prices $p_1, \ldots, p_n$. Regarding n equations (A-7) and (A-8) as the simultaneous equation system for n unknowns $c_{n1}, \ldots, c_{nn}$, we solve for them to obtain

$$c_{nj} = -\frac{x_j}{p_n y}(\Omega_n - \sigma_{jn}), \qquad i=1,\ldots,n$$

or in general,

(A-9)
$$c_{ij} = -\frac{x_j}{p_i y}(\Omega_i - \sigma_{ji}), \qquad i=1,\ldots,n$$

where

(A-10)
$$\Omega_i = \sum_{j=1}^{n} k_j \sigma_{ji}.$$

Using this result, we can reduce the equations (A-7) into

(A-11)
$$\sum_{j=1}^{n} x_j(\sigma_{ji} - \Omega_i - \eta k_i)dp_j - e_i x_i dp_i = 0 \qquad i=1,\ldots,n.$$

Hence we obtain

(A-12)
$$\nabla g = -\hat{p}\hat{k}^{-1}(\Sigma^T - \phi 1^T - \eta k 1^T - \hat{e})^{-1}(\sigma_{n.} - \phi - \eta k)\cdot k_n/p_n ,$$

where

$$\nabla g = \left[\partial p_1/\partial p_n, \partial p_2/\partial p_n, \ldots, \partial p_{n-1}/\partial p_n\right]^T$$

$\hat{p}$ = (n-1) x (n-1) diagonal matrix with i-th diagonal element equal to p_i

$\hat{k}$ = (n-1) x (n-1) diagonal matrix with i-th diagonal element equal to k_i

Σ = (n-1) x (n-1) matrix $[\sigma_{ij}]$

ϕ = the transpose of $[\Omega_1, \Omega_2, \ldots, \Omega_{n-1}]$ with Ω_i being defined by (13)

1^T = $1 \times (n-1)$ vector $[1, 1, \ldots, 1]$

k = the transpose of $[k_1, k_2, \ldots, k_{n-1}]$

$\hat{e}$ = $(n-1) \times (n-1)$ diagonal matrix with i-th diagonal element equal to e_i

$\sigma_{n\cdot}$ = the transpose of $[\sigma_{n1}, \ldots, \sigma_{n\,n-1}]$

We then calculate the elasticity λ_n of derived demand for the n-th factor based on the generalized Morishima elasticity of substitution as:

$$\lambda_n = -\frac{p_n}{x_n}\frac{\partial x_n}{\partial p_n} = -\frac{p_n}{x_n}\frac{\partial}{\partial p_n}\{c_n(p_1, \ldots, p_n)\, D[c(p_1, \ldots, p_n)]\}$$

$$= -\frac{p_n}{x_n}\{c_{nn}y + c_n \frac{\partial D}{\partial p} c_n + y \sum_{j=1}^{n-1} c_{nj}\frac{p_j}{p_n} + c_n \frac{\partial D}{\partial p}\sum_{j=1}^{n-1} c_j \frac{p_j}{p_n}\} .$$

Using the equations and notations above, this is summarized as:

THEOREM:

$$\text{(A-13)} \qquad \lambda_n = \eta k_n + \Omega_n$$

$$+ (\sigma_{\cdot n} - \Omega_n 1 - \eta k_n 1)^T (\Sigma^T - \phi 1^T - \eta k 1^T - \hat{e})^{-1} (\sigma_{n\cdot} - \phi - \eta k)$$

where $\sigma_{\cdot n}$ = the transpose of $[\sigma_{1n}, \ldots, \sigma_{n-1\,n}]$.