MUSICAL ACOUSTICS

based on

THE PURE THIRD-SYSTEM

by

THORVALD KORNERUP.

Text-book for the use at
Universities, Polytectical Academies,
Colleges of Music,
and for private Students.

Translated by Phyllis Augusta Petersen.

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Dedicated
The Memory
of

The Englishman
Walter Odington (about 1300 a. C.),

The Spaniard
Bartholomeo Ramis (about 1440—91),

The Italian
Giosepho Zarlino (1517—90),

The Frenchman
Jean Philippe Rameau (1683—1764),

The German
Hermann L. F. v. Helmholtz (1821—94),

who might be described
as the 5 most meritorious pioneers
of the Musical Acoustics.

PREFACE.

When 40 years ago — as 18 year old student — while studying the theory of Harmony I discovered the fact that the interval "c—d" in C, D, F and A major is not $^9/_8$, but $^{10}/_9$, I little realized that by making said discovery I had literally found the key to Modern Acoustics. I contented myself, then, by reporting my discovery to "Tidsskrift for Physik og Chemi" ("Magazine for Physics and Chemistry") Copenhagen 1882, No. 11, page 289—302, and did not follow up my victory. It was not till many years after — in April 1918 — that I once more took up the subject in order to work it out in detail. After having worked at it for three years I am now able to give my results to the public in the shape of the present treatise.

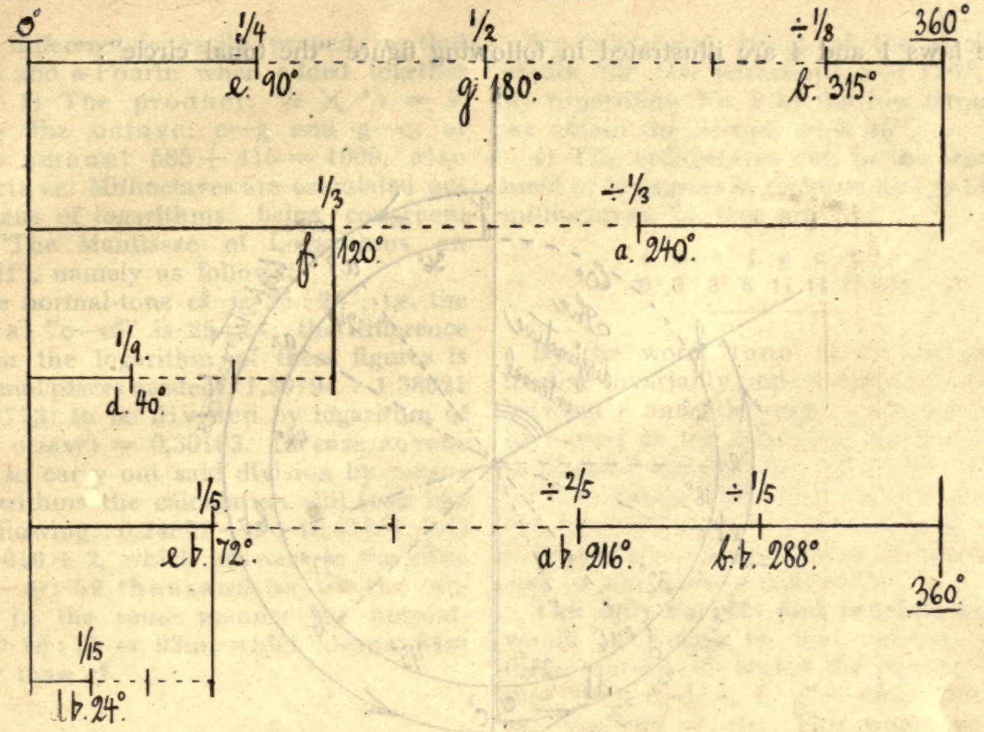
The principal result is indicated by the following "5 fundamental Laws of Acoustics":

Law 1. "The 10 principal intervals are constructed by dividing of the differences between the vibration number of the Tonica (c) and the corresponding numbers of the Octave, the Fourths and the minor Thirds with 2, 3, 4 or 5",
in the following manner (see article VI) in the tonal circle:

By dividing of	by	we obtain from c the normal tones:
c—c′ = 360°	2	g = 180°.
— —	3	f, a = 120 and 240°.
— —	4	e = 90°.
— —	5	e♭, a♭, b♭ = 72, 216 and 288.
c—f = 120° g—c′ = 180°	3 —	d = 40°. a = 180 + 60 = 240°.
c—f = 120° g—c′ = 180°	$^4/_3$ —	e = 90°. b = 180 + 135 = 315°.
c—e♭ = 72° g—b♭ = 108°	3 —	d♭ = 24°. a♭ = 180 + 36 = 216°.

Other divisions of	by	we obtain:
c—e = 90° g—b = 135°	2 —	the Comma-tone d+ = 45°. — — — a+ = 180 + $67^1/_2$ = $247^1/_2$°.
c—f = 120° g—c′ = 180°	$^9/_5$ —	the Comma-t. e♭÷ = $66^2/_3$°. — — b♭÷ = 180 + 100 = 280°.
c—f = 120° g—c′ = 180°	2 —	the Extra-tone $^7/_6$ = 60°. — — "i" = $^7/_4$ = 180 + 90 = 270°.

see following table:



Law 2. "Among these 10 normal tones we recognise c, g, e and b as the overtones No. 1, 3, 5 and 15, — c, f, a♭ and d♭ as the imaginable undertones No. 1, 3, 5 and 15, respectively the C and D♭ major Triads — or the e and f minor Triads".

The C string is divided by 3, 5 and 15, or multiplied by 3, 5 and 15. C is called both over- and under-tone No. 1.

Law 3. "In scales composed by principal tones all the Thirds and Fifths are pure within the octave, — that is to say: we can construct all scales on c by 2 chords of the Seventh from c and c' towards the centre".

The Greek Lydian C major:

c — e — g — b
d — f — a — c'

The Greek Doric c minor

c — e♭ — g — b♭
d♭ — f — a♭ — c'

Law 4. "In the same double scale the intervals in the 2 tetrachords (Fourths) pair off fifth-proportionally ($\frac{3}{2}$)".

d♭ and a♭	$24 + 12 = 36°$
d — a	$40 + 20 = 60°$
e — b	$90 + 45 = 135°$
other examples	
d+ and a+	$45 + 21^1/_2 = 66^1/_2°$
e♭÷ and b♭÷	$66^2/_3 + 33^1/_3 = 100°$
$^7/_6$ and $^7/_4$	$60 + 30 = 90°$

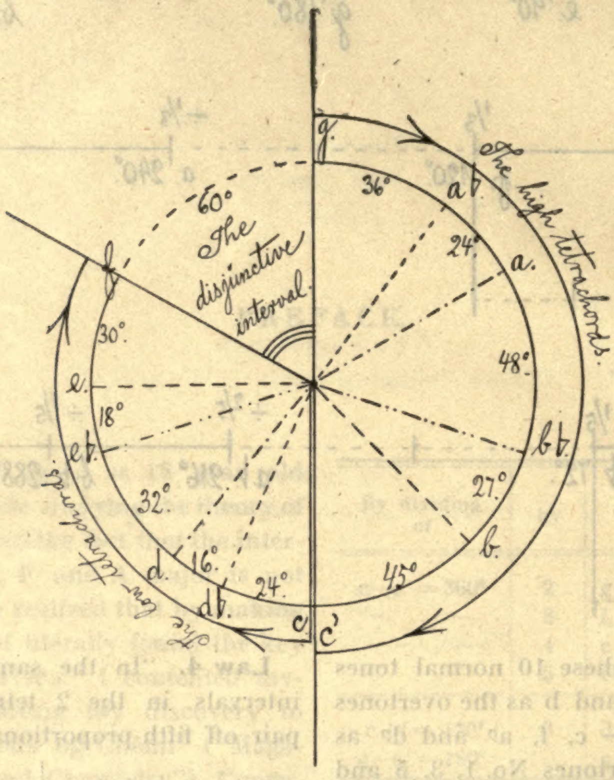
Law 5. "All the normal and Comma-tones pair off as complementary intervals".

d♭ = $^{16}/_{15}$ and b = $^{15}/_8$ = $^{15}/_{16} \cdot 2$
d = $^{10}/_9$ and b♭ = $^9/_5$ = $^9/_{10} \cdot 2$
e = $^5/_4$ and a♭ = $^8/_5$ = $^4/_5 \cdot 2$.

Example:

Low tetrachord in Lydian C major				Ptolemæic tetrachord = high tetrachord in Doric c minor			
c	d	e	f	g	a♭	b♭	c'
1	$^{10}/_9$	$^5/_4$	$^4/_3$	$^3/_2$	$^8/_5$	$^9/_5$	2

The laws 1 and 4 are illustrated in following figure "the tonal circle":



This treatise appeared for the first time in the Danish periodical: "Music", Copenhagen (edited by Godtfred Skjerne), 1920-21. Various additions have been made for the present edition.

Copenhagen, January 1922.

Thorvald Kornerup.

Translations in German and French have been carried summarily out and will be published later on.

INTRODUCTION.

Article 1. Limitations and Terminology of Musical Acoustics.

"Musical Acoustics" are dealing with the problem of the Structure of the System of Tones itself, thus forming the connecting link between Physics and Physiologics etc. on one side[1]) and the Science of Harmony on the other side.

The present treatise does — with some single exception — only avail itself of 4 specifications of intervals, namely

1) as ordinary fraction: the Fifth: $^3/_2$, the Fourth: $^4/_3$, the major Second: $^{10}/_9$, a.s.f., and

2) as millioctaves, or the thousandths of an octave: the Fifth: 585, the Fourth: 415, the major Second: 152, a.s.f.

The difference is easily proved in that a Fifth and a Fourth when added together make: 1) The product: $3/2 \times 4/3 = 2$, namely the octave: c—g and g—c; or 2) the amount $585 + 415 = 1000$, also the octave. Millioctaves are calculated out by means of logarithms, being congruent with "The Mantissae of Logarithms on basis II", namely as follows:

"The normal-tone c♯ is 25 : 24", i.e. the interval "c—c♯" is 25 : 24; the difference between the logarithm of these figures is (5 decimal places needed): 1,39794 ÷ 1,38021 = 0,01773; to be divided by logarithm of 2 (the octave) = 0,30103. In case anyone desire to carry out said division by means of logarithms the calculation will look like the following: 0,24871 ÷ 2 ÷ (0,47861 ÷ 1) = 0,77010 ÷ 2, which answers to the ciffre 0,059 — or: 59 thousandths of the octave. In the same manner the normal-tone d♭ 16 : 15 or 93m, which means: 34m larger than c♯.

"Tone-Logarithms" were made use of in the year 1729 by the Swiss mathematician L. Euler (1707-83), and later M. W. Drobisch (1802-96); they ought to be known by everyone with interest in musical matters as the "minor tabel" of Music. They are indicated in the present treatise by the letter "m".

(The Englishman A. J. Ellis (1814-90) has made use of 1200 parts, "1200 cents", instead of 1000 parts; f.i: c♯ 58,89 m. × 1,2 = 70,67 cents, about 71 cents. — I prefer decidedly 1000 parts, the millioctaves).

3) The ordinary fraction can be replaced by "degrees of arc", untill 360°, as difference between the vibration numbers in about $1^{5}/_{13}$ second, f. i. (see art. VI):

	c	d	e	f	g	a	b	c
	0	**40**	90	**120**	180	240	315	360°
or:	360	400	450	480	540	600	675	720
	c	d♭	e♭	—	—	a♭	b♭	c
	0	**24**	72	—	—	216	288	360°
or:	360	384	432	—	—	576	648	720

By tripartition No. 1 of the circle we obtain the low tetrachord c—f 120°, and by tripartition No. 2 of the low tetrachord we obtain the Second c—d 40°.

4) The millioctaves can in the temperament of 19 degrees be replaced by $t = 52^{63}/_{100}$ millioctaves, f.i. (see art. X):

c	d	e	f	g	a	b	c
0	3	6	8	11	14	17	19t.

By the word "tone" is in the present treatise invariably understood the interval between c and "the note". The notes will be named in the following like customary in England and Holland where "b" is used for the large (major) Seventh from c, the note below c, on the key-board, which in Scandinavia and Germany is called "h", and in the Latin countries "si"[2]).

The only correct and sensible system would of course be that nations, all of them, agreed to accept the letters: "a, b (not "h"), c, d, e, f, g" — and: "cis" for c♯, "ces" for c♭, etc. This would simplify matters very much and prevent many misunderstandings.

By "complementary tones", complement-intervals (inversion-intervals), is meant: two intervals which joined together form an octave; the one of these is constructed by inversion of the other, f.i. "g" and "f" that means: the intervals "c—g" and "c—f"; or "d" and "b♭"; "e" and "a♭"; "b" and "d♭"; they are arranged symmetrically in twos in **annexure I**: "The Aggroupment of the Tones in tonal Zones" with c as centre; here follows another illustration:

f♯♯ 625 : 432 and g♭♭ 864 : 625,
or: f♯♯ 533 + g♭♭ 467 m = 1000m.

The adjective "complementary" is borrowed from physics and mathematics — seeing that the two colours: "red and bluishgreen" together produce white, and therefore are named complementary colours, just like 2 angles together forming 90°, i. e. a right angle, are called complementary angles.

PART I. The INTERVALS.

Article II: The one-sided Pythagorean System: "The Perfect Fifth", based on abstract speculation.

The first to consider the possibility of expressing in figures the mutual relation of tones in the musical system was without doubt the Greek Philosopher Pythagoras (born about 580 b. C. on the island of Samos[3]). His system is called "The Perfect Fifth" because he creates tones by adding the Fifth to itself as many times as desired. By tracing the tones thus constructed back through so many octaves that at last the starting-point (octave) is reached the construction of his tones may be described as follows:

1) 7nth tone after c is c♯ =

$(^3/_2)^7 : 2^4 = \frac{3^7}{2^{7+4}} = \frac{3^7}{2^{11}} = \frac{2187}{2048}$ or 95 m,

or $7 \times 585 \div 4 \times 1000 = 95$ m.

2) 14th tone after c is c♯♯ =

$(^3/_2)^{14} : 2^8 = \frac{3^{14}}{2^{14+8}} = \frac{3^{14}}{2^{22}} = \frac{4782969}{4194304}$ or 189 m

or $14 \times 584{,}96 \div 8 \times 1000 = 189$ m.

3) 19th tone after c is b♯♯ =

$(^3/_2)^{19} : 2^{11} = \frac{3^{19}}{2^{19+11}} = \frac{3^{19}}{2^{30}} = \frac{1162261467}{1073741824}$ or 114 m

or $19 \times 584{,}96 \div 11 \times 1000 = 114$ m.

4) 8th tone counting back from c is f♭ =

$(^2/_3)^8 \times 2^5 = \frac{2^{8+5}}{3^8} = \frac{2^{13}}{3^8} = \frac{8192}{6561}$ or 320 m,

or $5 \times 1000 \div 8 \times 585 = 320$ m.

5) 15th tone counting back from c is f♭♭ =

$(^2/_3)^{15} \times 2^9 = \frac{2^{15+9}}{3^{15}} = \frac{2^{24}}{3^{15}} = \frac{16777216}{14348907}$ or 226 m,

or $9 \times 1000 \div 15 \times 584{,}96 = 226$ m.

All unbiassed readers will be able to grasp, by "spontaneous intuition", that a tonical system which has tones (intervals) indicated by a fraction as "c—b♯♯", — where enumerator and denominator consist of 10ciphered figures, or of 8ciphered figures as in the case of "c—f♭♭", must be wrong; nor is it difficult to point to the exact spot where his mistake comes in. Pythagoras' "Fifth-system" was based on imagination only, — on abstract speculation. — He believes it possible to construct a musical system solely on The Fifths, — just as he believed that "The planets by their rotation are creating tones — same tones being harmoniously connected with each other" . . . . . "The Harmony of the Spheres has tempted many Greek astronomers to speculations which are quite in the clouds" . . . .[4]). But as to that we may say with truth that the whole "Pythagorean Fifth-system" of tones is likewise poised in mid-air — held up solely by the wings of imaginaton! —

The present-day Science, however, is not content by building on imagination — it demands experience as basis; — experience from nature itself, in this present instance from natural tones, differential tones etc. of which subject more will be said in article V. of the present treatise. I shall only permit myself, thus far, to mention three traits all three equally characteristic of the "Pythagorean Fifth-system" as compared with the "Pure Third-system";

1) Pythagoras' c—c♯, 95m, is 36m larger than the normal c♯ = 59m; his "double-sharp", c—c♯♯, 189m i. e. 71 m larger than the normal c♯♯ = 118 m, just as vice versa his "flat", so that his c♭ = 905m, is

36m smaller than the normal c♭ = 941m; his c♭♭ = 811m i.e. 71m smaller than the normal c♭♭ = 882m (see **annexure II**), the result hereof being, amongst other things, that his c♯ = 95 m becomes 20 m larger than his d♭ = 75m, while the normal c♯ = 59m is 34m smaller than the normal d♭ = 93m. Thus his whole system must become lopsided in proportion to the "Third-system" — as the "Leaning Tower of Pisa" when compared to the plumpline!

Consequently the interval "c♭♭—c♯♯" of the "Pythagorean Fifth-system" must be 1189 ÷ 811 = 378 m, thus becoming 2 × 71 = 142m larger than the corresponding interval in the "Third system", which is: 1118 ÷ 882 = 236. Or, to take another instance: As the Pythagorean b is 925m — that means: 18m (called "a comma" = 17,92m = 21,51 cents) larger than the normal one = 907m this must needs result in the Pythagorean b♭♭ = 114 being 18 + 71 = 89m higher than the normal b♯♯ = 25m, in other words: a very discordant tone.

Fig. col. 63 shows the distances in a small whole tone, c—d.

2) Pythagoras being, however, endowed with mathematical talent sufficient to enable him to carry through his errors with the necessary consistency the result is that we find his scales being relatively symmetric f.i. his scale on c, our C major, the primitive major:

			proportion	distance	
c	1 : 1	0 m			
d+	9 : 8	170 —....	9 : 8 ...	= 170 m ...	low tetrachord
e+	81 : 64	340 —....	9 : 8 ...	= 170 —...	
f	4 : 3	415 —....	256 : 243 ...	= 75 —...	
			9 : 8 ...	= 170 —...	disjunctive interval
g	3 : 2	585 —....	9 : 8 ...	= 170 —...	high tetrachord
a+	27 : 16	755 —....	9 : 8 ...	= 170 —...	
b+	243 : 128	925 —....	256 : 243 ...	= 75 —...	
c	2 : 1	1000 —			

3) Also his two tetrachords in the scale, viz: "c d+e+f" and "g a+b+c" are exactly congruent, namely 170+170+75 = 415 m with an interspace called "Diazeuxis" or "disjunctive interval", large 170 m, "f g", same interval being common for both systems in question as "f" and "g" are common for both, see the preface.

But this relative symmetry and congruity is really of no more value than the "established regularity in astronomics" so called by Pythagorean followers; "who endeavoured to trace the connecting link between the relative distances of the various erratic stars — i.e. the planetary conditions of magnitude on one side —, and on the other side the length of the chords producing tones in musical succession" . .[4]).

The Pythagorean Fifths "d+ — a+" and "e+ — b+" pair off "pure", but oblique in proportion to c, placed a comma to right; these 4 tones must be diminished with a comma, indicated in degrees of arc, see the preface:

d+	e+	a+	b+
45	$95^5/_8$	$247^1/_2$	$323^7/_{16}$
d	e	a	b
40	90	240	315°

When notwithstanding all its palpable errors the Pythagorean Fifth-system was able to hold on as long as it did this may be accounted for by the following two facts:

a) In the days of antiquity and during mediæval times the notes of C Major was used almost exclusively, thereby making the errors of the "Pythagorean Fifth-system" less evident.

b) The human ear possesses a truly surprising faculty for adapting itself to discordant tones believing them to be pure and in tune — one proof amongst many of the unending adaptability of mankind[5]). But at the present day the demand for pure tones has grown much more insistent — hence the disinclination to submit to the discordant tones of the old "Pythagorean Fifth-system"!

Article III. Opposition to "The Pythagorean Fifth-system": The Founders of "The Third-System".

"We see through what wildernesses (Pythagoreas' abstract theories) the human thought has had to work its way out before the real true science of astronomy could see daylight; though it is at the same time a fact that already the Greeks of the classic era proved that they were endowed with power of mind sufficient to emancipate themselves from theories old and defunct, thus working their way towards a true understanding of the phenomenons of the Firmament". Exactly the same words might be made use of in regard to the manner in which the Greeks found their way out of the wilderness known as the "Pythagorean Perfect Fifth". I should like to specially mention the names of the following 5 Greek philosophers and mathematicians, — all belonging to "The School of Harmonists"[6]):

1) Archytas of Taranto, politician and mathematician (about 430—365 b. C.). There is every evidence to make it seem that he would have been the one to introduce the major Third = $^5/_4$ or 322 m (about the year 408 b. C.).

2) Aristoxenos of Taranto, philosopher and mathematician (pupil of Aristoteles). He originated the plan of the equal temperament of 12 degrees (about 350 b. C.); no practical use, however, being made of his system till a few thousand years later on in history,

3) Erastothenes of Cyrene, an Alexandrian (about 275—195 b. C.), introduced the "minor Third" = $^6/_5$ or 263 m (about the year 200 b. C.).

4) Chalcenteros Didymos (pupil of Aristarchos from Samotrace), Alexandrian; grammarian (born about 63 b. C.); was called "The Indefatigable"; introduced the "major Second" = $^{10}/_9$ or 152 m.

5) Claudius Ptolemæus (maior) from Alexandria; invented (year 140 a. C. or thereabout) the tetrachord now used in the melodic minor mode descendant (as "high tetrachord") wherewhith he had **reached the goal** of being able to place the tetrachord with only normal intervals:

Distances of:

	1st degree	2nd degree	3rd degree
e=5 : 4 or 322 m	...93 m	...77 m..	...59 m
f=4 : 3 — 415 —	..170 — ..	..:18 —..	
g=3 : 2 — 585 —	..152 — ..		
a=5 : 3 — 737 —			

The distances are thus in reverse succession to the ones of C major, or to what we in the following shall call the primitive minor, "the Doric minor mode", **the pendent of C major**; the Doric scale of the Greeks; the Phrygian ecclesiastical mode of mediæval days, "c minor with 4♭". From this point and on there came a period of stagnation which lasted for a very long time.

That there is a step forward in evolution from Pythagoras to Ptolemæus is evident when one considers their methods which are so different that they are almost contrasting; — Pythagoras constructed his tones as mentioned above by adding the Fifth to itself, while Ptolemæus starts from the simple fractions: e = $^5/_4$, a = $^5/_3$. But this is forgotten and lying dormant for a period of about 1200 years.

As the discovery made by the astronomer Aristarchos of Samos (3rd century b. C.) that the sun is the centre of the planetary system — was left to be forgotten for many long years until its revival by Kopernicus about year 1500, so the original pure system of "The School of Harmonists" was forgotten for a very long time until it gradually revived. Amongst the men to whom the honour of this revival of system is due we ought not to forget to mention the English Benedictine monk **Walter Odington** (ob. after 1330 a. C.) whose connection with this matter was not known till a much later period. He is attempting to revive the Third-system once more (about the years 1275—1300 — according to papers found in 1864 at Christ College, Cambridge).

The same attempt is made later on, 1480, by the Spanish theorist Bartholomeo Ramis de Pareja (about 1440, ob. 1491), and 1529 the Italian Ludovico Fogliani (ob. about 1539); then 1558 by the musical-theorist of the High-Renaissance **Giosepho Zarlino** of Venice (1517—1590), and 1722 by the French musical theorist **Jean Philippe Rameau** (1683—1764). During these periods, step by step, the Thirds are getting recognised as co-equals with the Fifths as consonances, — and the major and minor scales recognised as being of equal rank.

In the year 1853 the German composer Moritz Hauptmann (1792—1868) brings out the "Aggroupment System", based on both the Fifth and the Third. **The Pythagorean tones** are now arranged in Fifth-succession in a maze of squares, in one horizontal plan-line and places e (the Third) immediately opposite of c; after various oscillations in terminology [7]):

	12 cents according to Ellis:	Milli-octaves m:	Haupt-mann 1853.	Helm-holtz I 1863.	Oettingen 1866.	Helm-holtz II 1870.	Eitz 1891.	Kornerup 1882	Kornerup 1920
Δ 2. Over-Third	773	644	G♯	G♯	g♯	g♯	g♯$^{\div 2}$	g♯	g♯
1. Over-Third	386 182	322 152	e d	e d	e d	e d	e$^{\div 1}$ d$^{\div 1}$	e d	e d
Pythagorean tones	204 0 996	170 0 830	D C B♭	D C B♭	d c b♭	d c b♭	d c b♭	d c b♭	d+ c b♭÷
1. Under-Third	1018 814	848 678	b♭ a♭	b♭ a♭	b♭ a♭	b♭ a♭	b♭$^{+1}$ ab^{+1}	b♭ a♭	b♭ a♭
∇ 2. Under-Third	427	356	F♭	F♭	f♭	f♭	f♭$^{+2}$	f♭	f♭

— the tones are now generally described thus that the Pythagorean tones are looked upon as the normal tones (which is an error) while the tones in the other rows according to Eitz are described by letters to which in each case is added ÷ 1, ÷ 2, a.s.f. — or: + 1, + 2, a.s.f. as power sign, which is here explained:

$f\sharp^{\div 2}$ 474	$c\sharp^{\div 2}$ 59	$g\sharp^{\div 2}$ 644	$d\sharp^{\div 2}$ 229	$a\sharp^{\div 2}$ 814	$b\sharp^{\div 2}$ 399	2. Over-Third.
$d^{\div 1}$ 152	$a^{\div 1}$ 737	$e^{\div 1}$ 322	$b^{\div 1}$ 907	$f\sharp^{\div 1}$ 492	$c\sharp^{\div 1}$ 77	1. Over-Third.
$b\flat$ 830	f 415	c 0	g 585	d 170	a 755	**Pythagorean tones.**
$g\flat^{+1}$ 508	$d\flat^{+1}$ 93	$a\flat^{+1}$ 678	$e\flat^{+1}$ 263	$b\flat^{+1}$ 848	f^{+1} 433	1. Under-Third.
$e\flat\flat^{+2}$ 186	$b\flat\flat^{+2}$ 771	$f\flat^{+2}$ 356	$c\flat^{+2}$ 941	$g\flat^{+2}$ 526	$d\flat^{+2}$ 111	2. Under-Third.
2. Under-Fifth.	1. Under-Fifth.	Vertical central line.	1. Over-Fifth.	2. Over-Fifth.	3. Over-Fifth.	

Enclosed in the italicized lines is the C major scale (see the figures) as seen by Mr. Hauptmann i.e. with d as $^9/_8$; as will be seen: an oblique figure in its proportion to c.

This system was somewhat improved upon by the Japanese Shohé Tánaka (in 1890); he turned the system plan itself 30°, thereby causing c to be placed below the line between the a and e squares — thus:

$a^{\div 1}$ 737	$e^{\div 1}$ 322	$b^{\div 1}$ 907	
f 415	c 0	g 585	d **170**

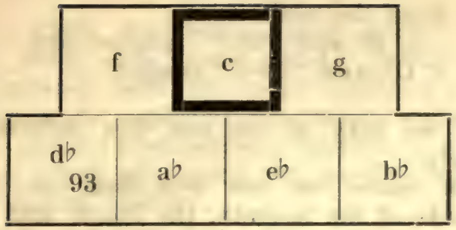
However, — as regards its proportion to C major the figure still remains oblique to C; this is the natural sequence of the fact that Tánaka — and after him the German singing-master Karl Eitz (born 1848) and also the Swiss physiologist Alfred Jonquière (1862—99) all keep to d = 170 in C major[8]).

The alteration effected by author in the Tánaka Aggroupment is a follows:

1) I insert **my perpendicular tonal zones** as described in annexure I. so as to make it clear at once that d = $^{10}/_9$ or 152 m, and not: d = $^9/_8$ or 170 m belongs to C major, seeing that this is the only way of causing all the tones of C major to be symmetrically placed in proportion to c, as will be seen from the figure below:

d **152**	a	e	b
	f	c	g

The same holds good in regard to the primitive minor, the "Doric c minor" (double melodic minor mode descendant) the descriptive figure of which will look as follows:



2) I reject the Pythagorean intervals as starting-point and shall call:

The Middle Zone in annexure I: the normal intervals; this again being flanked by (compare art. VII):

a) The zone to the right: comma-intervals with +, i.e. intervals which are one comma = 18 m higher than the normal intervals, f.i. d+ = 170 m, f+ = 433 m, a♭+ = 696 m, a. s. f.

b) The zone to the left: comma-intervals with ÷, intervals which are one comma lower than the normal intervals, f.i. b♭÷ = 830 m, the complement interval to d+; g÷ = 567; e÷ = 304 m, a.s.f.

Be it noted that annexure I. is cut off c+ and c÷, seeing that we shall rarely have any occasion to use comma-intervals outside these; f. i. rarely for: e+, g+, b+, or: d÷, f÷, a÷, o.s.f.

The signature + and ÷ are to be put immediately after the letters, seeing that in this present instance we have nothing to do with powers in arithmetic, but with: + and ÷ 18 m, f.i. d+ = 152 + 18 = 170 m, b♭÷ = 848 ÷ 18 = 830 m, d+ stands thus for "d + 18 m".

The above represents the Basis of Modern musical Acoustics (the correctness of which statement the following articles are endeavouring to prove). The opposition to the Pythagorean Fifth-system has now brought its struggle to a **final** close; — "per errores ad veritatem", "Truth is reached through errors".

Article IV. The Natural Tones as Fundamental to "The pure Third-system".

The mistake hitherto committed with regard to the use made of the "harmonic over- and under-tones" is that an insufficient number has been made use of in all attempts of building up new systems or improving upon old existing ones. If we just set or mind on a larger number of these over- and undertones we shall at once find it much easier to obtain a general view of their special nature and function — i.e. **the structure** of the whole Musical system, as given to us by the hands of nature itself — not by Pythagoras' abstract imaginations.

The C sounding from the C string of a cello is called "Natural tone No. 1"; the flageolet-tone which is obtained by dividing the C string in two parts of equal longitude, both vibrating, is called "Natural tone No. 2" (or "Over-tone No. 2"); the flageolet tone called forth by dividing the same string into three equal parts is called "Over-tone No. 3", a. s. f. And vice versa: In doubling the length of the C string we shall obtain as a result "Under-tone No. 2", if tripled: "Under-tone No. 3". a. s. f.

C is thus both Over- and Under-tone No. 1. This way of numerating is made use of, for practical reasons.

Now if we were to construct more over-tones of which only some very few are audible we shall at once discover that these tones may be used for composing **various series of triads**, — as well in the major- as in the minor-mode; as, f.i. the following where the figure proceeding each letter indicates the number of over- and under-tones in question:

Series	Over- and under-tones No.	Over-tones giving major mode	Under-tones giving minor mode
1	2, 4, 8, 16	**C**	c
	5, 10, 20	e	a♭
	3, 6, 12, 24	g	**f**
2	5, 10, 20	**E**	a♭
	25, 50, 100	g♯	f♭
	15, 30, 60	b	**d♭**
3	3, 6, 12	**G**	f
	15, 30, 60	b	d♭
	9, 18, 36	d+	**b♭÷**

and vice versa:

Series	Over- and Under-tones No.	Over-tones giving minor mode	Under-tones giving major mode
1	5, 10, 20........	**e**	a♭
	3, 6, 12........	g	f
	15, 30, 60........	b	**D♭**
2	15, 30, 60........	**b**	d♭
	18, 36. 72........	d+	b♭÷
	45, 90, 180.......	f♯+	**G♭÷**

We have thus in the above constructed the following triads

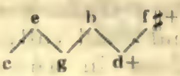
from the over-tones:

C, E and G major as well as
e and b minor,

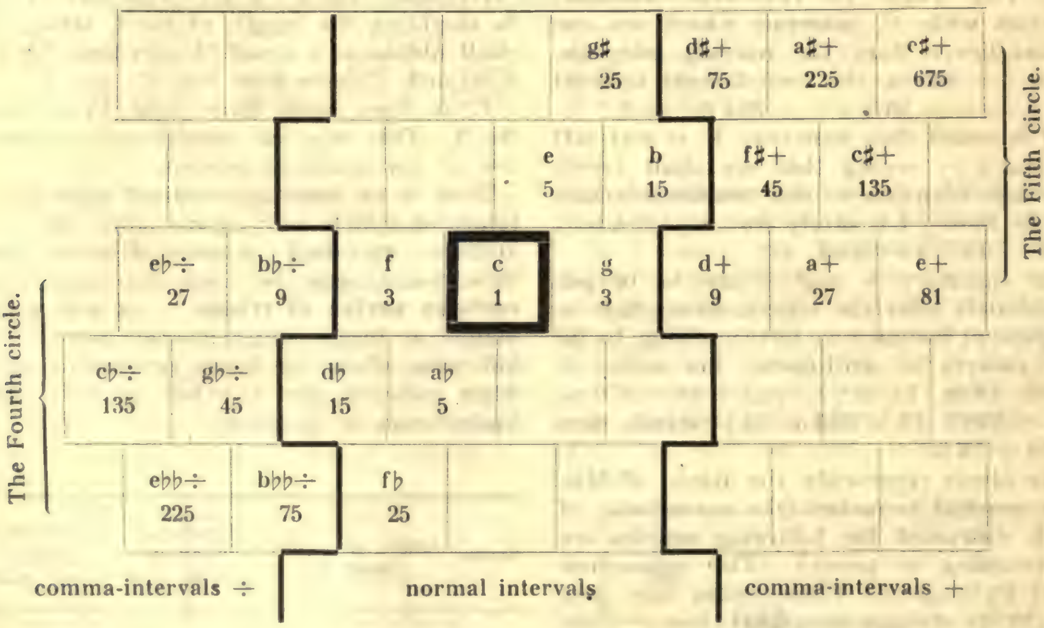
from the under-tones:

f, d♭ and "b♭÷" minor, and
D♭ and "G♭÷" major,

and we may go on like this indeterminately.

Of course we can also thus construct chords of the Seventh, the Ninth and the Eleventh: "c, e, g, d+, f♯+" a. s. f., see annexure 1: 

The best survey over the whole plan is obtained by inserting — in annexure No. 1 — the number of the over- and under-tones respectively (C is looked upon as being both over- and under-tone No. 1), thus:



We discover from the above illustration that by dividing the string by the 3 first prime numbers: 2, 3 or 5 — or by multiplying these figures amongst themselves, as f. i. 4, 6, 8, 9, 10, 12, 15, a. s. f — or vice versa; by multiplying the length of the string by these figures or multiple we obtain over- and under-tones respectively which are able to go on constructing triads or Seventh chords indeterminately by multiplying all the figures by 2, 4, 8, a. s. f.

We further discover that in annexure No. 1 are to be found various axes the assistance of which we shall require in this particular case, namely the following:

1) The horizontal axis, the Fifth-succession, containing the Pythagorean tones,

2) The 60° axis — i. e. major Thirds, f♭—a♭—c—e—g♯, a. s. f.

3) The 30° axis: d♭—c—b a. s. f. being a combination of the two previous axes —: Sevenths.

As there are three primary colours i. e. red, green and "bluish-purple" (indigo) in accordance with the solar spectrum (red, green and "purple" in accordance with the dim spectre from artificial sources of light) from which all colours are constructed —, as "mixed" colours,

— The Trichromatic Colour System —

so there are **Three Primitive Intervals** from which all other normal intervals are constructed as "mixed tones", corresponding to the three figures (the first prime numbers of the numerical series:

1 = 2 = Prime or Octave,
3 = Fifth (major + minor Third),
5 = Major Third (Fifth ÷ minor Third).

If therefore we consider Third, Fifth and Octave to be the three primitive-intervals we have thereby stated and explained the fundamental **structure** of the pure system, — even as it will be seen that the difference between the major and minor triads consists only in the major and minor Thirds following each other in varying succession.

In adding a Third to the triad we obtain a prolongation of the same, viz: the Chord of the Seventh (also called the subsemitone chord), of which there will be 8 different kinds (see article V) which we make use of for the construction of the scales.

Colour-blind people are able to perceive but one or two primary colours; — the "interval deafness" of the Pythagoreans rendered them able to hear only two primitive intervals: the Octave and the Fifth — as tone producing; by division or multiplication of the C string of a cello only by 3 or multiples only of 3, f. i. 3, 9, 27, 81, 243, 729, a. s. f., we have only Pythagorean tones, the exclusive (false) Fifth-succession.

All over- and under-tones produced by multiplication or division by figures other than 2, 3 and 5, or multiple of these, i. e. from all prime numbers other than 2, 3 and 5 (or multiple of all prime numbers other than 2, 3 and 5) f. i. 7, 14, 21, 28, 35 . . . . . 11, 22, 33, 44, 55, a. s. f. we call **extra tones**, because they are outside the frame of the pure Third. But of course even these tones can produce the triads —, f. i. the 7nth over-tone which together with the 21st and 35th over-tone form a genuine triad with the following capacities in milli octaves:

	Prime.	Third.	Fifth.	
	7nth	35th	21st	
	807	1129	1392 m	
pure intervals	322		263	= 585 m.

But this, of course, is only a tautology.

The German musical theorist J. P. Kirnberger (1721—83) has named the 7nth over-tone "i" (the letter after "h" in the alphabet)[9]), which nomenclature is much to be preferred to "the natural Seventh" considering that the extra-tone "i" has nothing to do with the Sevenths of the "Third-system"; millioctaves:

		Temperament of			
"i"	$b\flat$÷	12 degrees:	31 degrees:	19 degrees:	pure:
807	830 also pyth.	$b\flat$ 833	$b\flat$ 839	$b\flat$ 842	$b\flat$ 848 m.

"i" is thus 41 m lower than the normal interval $b\flat$; it is lying outside the "Third-system" between a♯ 796 and a♯+ = 814 m; we might accordingly with some reason be permitted to call "i" "the augmented natural Sixth" a♯; but never on any account the Seventh!

The number of extra-tones is, however, fairly large owing to the great many prime numbers existing in the numerical list:

	Normal-tones and comma-tones	Extra-tones
of the first 10 over-tones are	9	1
- next 10 — -	5	5
- — 10 — -	4	6
- — 10 — -	3	7
- — 10 — -	3	7

a. s. f.

When the French musical theorist Jean Philippe Rameau (1683—1764) constructed the minor triad, "l'accord parfait mineur", of the 3 tones which has c as common over-tone[10]), this was an unnecessary detour seeing that "f—a♭—c" as stated above, are to be found direct as 3rd, 5th and 1st under-tone corresponding to 5th, 3rd and 15th over-tone "e—g—b".

And when somebody holds that minor triads may be constructed only from under-tones then this is an error for — as stated above — the difference between over- and under-tones is not dualism between the major and minor modes but between the tones of the Fifth- and the Fourth-circles — in other words: between ♯ and ♭.

I trust that in the foregoing I have been able to make clear my solution of the problem of "Over- and Under-tones" according to the common law of nature. In the flora and fauna the figures 1, 2, 3 and 5 are ruling according to the law of precedence that 1, 2 and 3 (as well as multiple of these figures with each other) are ruling on a lower stage of development; f. i. in the flowers of monocotyledonous plants and with zoöphytes; while 5 (and multiple as 10, 15, a. s. f.) indicate a higher phase of development, — in the flowers of bicotyledonous plants and with echimoderms (starfish, crinoideans, actiniae), while other prime numbers indicate teratologies. The flower of the horse-chestnut is thus a dicotyledon in which one petal and 3 stamen are missing, hence the numbers 4 and 7; but the seats of the missing parts can be pointed out by means of the symmetrical line in the diagram of the flower. Likewise in the world of tones: The tone "i", — i. e. $^1/_7$ of the string, is outside the Normal, just like the flower of the horse-chestnut tree. Even in chrystals, sonorous figures, acoustic curves, etc. certain proportionate numbers are considered the Normal[11]).

If the Hindus in the days of antiquity would have made use of 22 tones in the octave, which tones could be constructed by dividing the half C string in 22 equal parts, (according to Hugo Riemann), the result hereof would have been many small and a few large intervals, — "c—f" pure, the other false, — in 4 groups being about equal to 4 temperaments (see art. VIII) with respectively 12, 19, 24 and 31 tones in the octave, as follows:

Millioct.	Name in the Third system		Tones instead of the black digitals	Temperament of
about		about		
0	c			
33		d♭♭		
67		c♯	4	31 degrees.
102		d♭		
138		c♯♯+		
175	d+	....	....	....
212		d♯, e♭♭		
250		e♭÷	3	
290		d♯♯+		
330	e			24 degrees. (Arabian?)
372		e♯, f♭+	1	
415	f			
459		f♯÷, g♭♭		
505		g♭÷	3	
553		f♯♯+		
602	g+	....	....	....
652		g♯	2	
705		g♯♯, a♭+		
759	a+			19 degrees.
816		a♯+	2	
875		a♯♯+		
936	b+	....	....	....
1000	c'	....	0	12 degr.

PART II. The SCALES.

Article V. Construction of Scales with "C" as Tonica (Keynote).

The ordinary way of constructing scales is: to "start" adlibitive with C major and "fill out intervals" adlibitive in same — as "it seems to fit in best"[12]). This is wrong. In opposition to this we shall, in the present treatise, not take C major as our starting point; because so far we know nothing about that particular point. All we know is that triads and the extension of same, i. e. the chords of the Seventh, is given to us straight from the hands of nature with the natural tones, with both minor and major Thirds in varying succession. We will then begin with finding out how many triads and Seventh chords we shall be able to construct with the aid of a dash and a point as indicating the major- and minor Thirds; we discover almost at once that we can construct thus 4 triads and 8 chords of the Seventh, as seen below, with indication of the sum of millioctaves, in order to state the consecutive order, f. i.: 322+644=966 m; and 322+644+966=1932 m:

Triads ⟶				Telegraphic sign	& letter	Sum of m	Sum corresp. to:	transposed to: ⟵		
augmented	c	e	g♯	— —	M	966	b♯	f♭	a♭	c'
major	c	e	g	— -	N	907	b	f	a	c'
minor	c	e♭	g	- —	A	848	b♭	f	a♭	c'
diminished	c	e♭	g♭	- -	I	789	b♭♭	f♯	a	c'
Proof: Difference betw. M and I						177	= 3 × 59			

The major- and minor triads are pendants of each other — - \| - —.

Chords of the Seventh ⟶					Telegraphic sign.	& letter	Sum of m	Sum corresp. to:	transposed to: ⟵				Suggested name:
tempered Octave trisects	c	e	g♯	b♯	— — —	O	1932	a♯♯♯+	d♭♭	f♭	a♭	c'	twice augmented.
harm. minor mode, a III	c	e	g♯	b	— — -	G	1873	a♯♯+	d♭	f	a	c'	augmented.
major mode, C I, G IV.	**c**	**e**	**g**	**b**	— - —	**K**	1814	a♯+	d♭	f	a♭	c'	large major.
major mode, F V	c	e	g	b♭	— - -	D	1755	a+	d	f♯	a	c'	small major.
harm. minor mode, c I	c	e♭	g	b	- — —	W	1755	a+	d♭	f♭	a♭	c'	large minor.
major, B♭ II, A♭ III, E♭ VI	c	e♭	g	b♭	- — -	**R**	1696	a♭+	**d**	**f**	**a**	**c'**	small minor.
maj., D♭ VII, harm. b♭ II	c	e♭	g♭	b♭	- - —	U	1637	a♭♭+	d	f	a♭	c'	diminished.
harm. minor mode, d♭VII	c	e♭	g♭	b♭♭	- - -	S	1578	a♭♭♭+	d♯	f♯	a	c'	twice diminished.
Proof: Difference betw. O and S							354	= 6 × 59					

The chords of the Seventh: O, K, R and S are symmetrical, while G and W are Pendants of each other, likewise D and U.

With this material many kinds of scales may be constructed, "Modes of Scales" or **modi** (from the Latin) namely: by adding triads to the Tonica (keynote) C ascending, and from the octave C′ descending, as also by inserting triads with the Third on the dominant G or on the lower-dominant F; or — what comes to the same:

Law 3: "By adding **chords of the Seventh** (= 3 Thirds) from **C ascending** and from C′ **descending**", — i. e. Constructing the Scales by chords of the Seventh from the terminal points c and c′ **towards the Centre**. What kind of scales we obtain in this way depends upon the chords of the Seventh we are choosing, f. i. will the primitive C major be formed like this (see the preface):

The chord K from C:	C e g b
- — R from C′:	d f a C′
or in millioctaves:	0 322 585 907 152 415 737 1000

If we exchange the chords K and R the primitive c minor will be formed like this:

The chord R from c:	c e♭ g b♭
- — K from c′:	d♭ f a♭ c′
or in millioctaves:	0 263 585 848 93 415 678 1000

Be it noted that both the above scales obtain congruent tetrachords — just because the chords K and R are symmetrical.

The last of these two scales is congruent with the Doric e scale of the ancient Greeks, with the Phrygian ecclessiastical mode of Mediæval times, c minor with 4♭, which by us is indicated with a "small" c with an asterisk: "c* minor". This scale forms the pendant of C major and may be called: "Double reversed major".

These two modes we call **"The 2 primitive modes"**; they border the practical system of scales on both sides — (see art. 3).

But of course many other kinds of major and minor modes may be constructed; we are showing below some of these which have either been made practical use of at some time or other, or which might be used in practical life, at any time; we shall choose 5 scales of the major mode and 6 minor modes and let them have each a number ad libitum but fixing the consecutive order of series by the sum of millioctaves:

Serial No. ad libitum	Chord of Seventh from c′ descend.	Chord of Seventh from c ascending	Greek or Oriental scale	on	Application etc.
1	D	K	Hypolydian	F	"ultra major"[15], obsolete.
2	R	K	**Lydian**	**C**	the primitive major, "**double** major".
3	U	K	Oriental	C	harmonic major, "major and harmonic", Rimski-Korssakow[14]).
4	R	D	Ionic	G	"major and minor", Norwegian "Langeleik", uncertain[15]).
5	K	K	......	..	the symmetric major, "**double** harmonic major".
6	R	W	......	..	melodic minor ascendant, "minor and major", Schoenberg[14]).
7	U	W	......	..	harmonic minor, "minor and harmonic", E. F. E. Richter (1808—79).
8	R	R	**Phrygian**	**d**	the central mode, the symmetric minor, **double** minor".
9	U	R	Æolian	a	melodic minor descendant, "minor and antimajor", Rimski-Korssakow and Schoenberg[14]).
10	K	R	**Doric**	**e**	the primitive minor, "double reversed major", "**double** antimajor".
11	K	U	Mixolydian	b	"ultra minor", obsolete.

In case the tones are given in 2 chords of the Seventh starting from c for the 7 Greek scales these chords will look thus (in succession of major and minor Thirds):

Serial No.	Greek scales	Chord of the Seventh: descending	Chord of the Seventh: ascending	descending ◁—			from c	ascending —▷			Number of ♯ or ♭
1	Hypolydian major	D	K	d	f♯	a	c	e	g	b	1♯
2	**Lydian** —	R	K	d	(f)	a	c	e	g	(b)	0
4	Ionic —	R	D	d	f	a	c	(e)	g	(b♭)	1♭
8	Phrygian minor, the **central** mode	**R**	**R**	d	f	(a)	c	(e♭)	g	b♭	2♭
9	Æolian minor	U	R	(d)	f	(a♭)	c	e♭	g	b♭	3♭
10	**Doric** —	K	R	(d♭)	f	a♭	c	e♭	(g)	b♭	4♭
11	Mixolydian —	K	U	d♭	f	a♭	c	e♭	g♭	b♭	5♭

The missing tones in the 5 pentatonic scales are placed in brackets. All 7 scales are here given in **Third**-succession with c as centre.

No. 2, 5, 8 and 10 are double modes, the 4 natural modes, namely:

No. 2, Lydian, the primitive major,
- 10, Doric, — minor,
- 5, ... - symmetric major,
- 8, Phrygian, - — minor.

No. 2, 8 and 10 are perfect modes (see art. 10).

No. 1 and 11 are "ultra" modes, as "ultra red" and "ultra violet" among the colours.

Indicated with tones from c the 11 modes described on the preceding page will look as follows:

Serial No.	Low tetrachord —▷				Diazeuxis, disjunctive interval	High tetrachord —▷				Number of ♯ or ♭	Sum of m without c′	Sum corresponds to
1	c	d	e	f♯	½	g	a	b	c′	1♯	3177	c♯♯♯
2	c	d	e	f	1	g	a	b	c′	0	3118	c♯♯
3	c	d	e	f	1	g	a♭	b	c′	1♭	3059	c♯
4	c	d	e	f	1	g	a	b♭	c′	1	3059	c♯
5	c	d♭	e	f	1	g	a♭	b	c′	2	3000	c
6	c	d	e♭	f	1	g	a	b	c′	1	3059	c♯
7	c	d	e♭	f	1	g	a♭	b	c′	2	3000	c
8	c	d	e♭	f	1	g	a	b♭	c′	2	3000	c
9	c	d	e♭	f	1	g	a♭	b♭	c′	3	2941	c♭
10	c	d♭	e♭	f	1	g	a♭	b♭	c′	4	2882	c♭♭
11	c	d♭	e♭	f	½	g♭	a♭	b♭	c′	5	2823	c♭♭♭
										Proof: Difference betw 1 and 11:	354	= 6 × 59

As will be seen there are in these 11 modes stated the following 5 kinds of tetrachords, with c as starting point in order to assist comparison, — also statement as to the sum total of millioctaves:

	Tetrachord							Name	Sum of milli octaves	Sum corresponds to:
symmetric	c	—	d	—	e	—	f♯	Tritonus	948	b♯÷
——▷	c	—	d	—	e	f	—	**major**	889	b÷
symmetric {	c	d♭	—	—	e	f	—	"harmonic"	830	b♭÷
	c	—	d	e♭	—	f	—	**minor**	830	b♭÷
◁——	c	d♭	—	e♭	—	f	—	anti-major	771	b♭♭÷
								Proof: Difference betw. Tritonus and anti-major:	177	= 3 × 59

The arrowheads indicate that the major- and the Doric anti-major-tetrachords are "pendants" to each other.

$$\underbrace{\overline{152} + \overline{\overline{170}} + \hat{93}}_{\text{Major with } d = {}^{10}/_{9}} = \underbrace{\hat{93} + \overline{\overline{170}} + \overline{152}}_{\text{Ptolemæic tetrachord.}} = 585 \text{ m}$$

From the above 5 kinds of tetrachords the 11 modes are constructed thus:

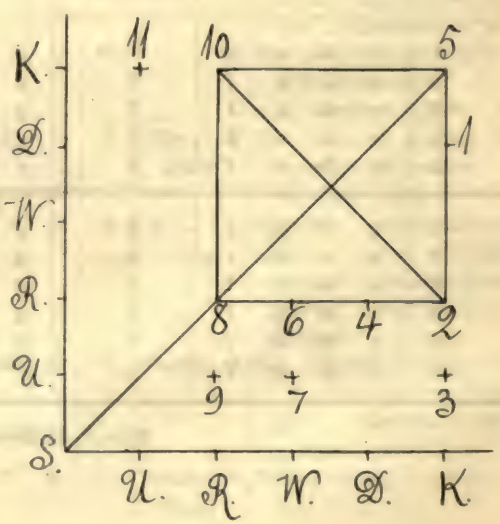
Serial No.	Number of ♯ or ♭	Tetrachord low	high
1	1♯	Tritonus	major
2	0	major	—
3	1♭	—	"harmonic"
4	1	—	minor
5	2	"harmonic"	"harmonic"
6	1	minor	major
7	2	—	"harmonic"
8	**2**	—	minor
9	3	—	anti-minor
10	**4**	anti-major	— —
11	5	— —	Tritonus

Of these 11 modes 4 are "double" i. e. the two tetrachords are congruent (because the K- and R-chords are symmetric); they may accordingly be used for the construction of other scales of the **same** mode on other Tonicae (keynotes) than c, through that

a) ths high tetrachord is made a l o w tetrachord in the following scale, the Fifth-circle c, g, d, a etc.
b) or that the low tetrachord is made h i g h tetrachord in the following scale, the Fourth-circle, c, f, b♭, e♭ etc.

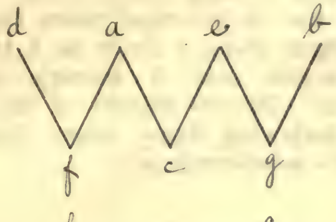
These are the rules which cannot be applied when d+ is included in C major, as it causes the tetrachords to become uneven: ${}^{9}/_{8}$, ${}^{10}/_{9}$, ${}^{16}/_{15}$, and ${}^{10}/_{9}$, ${}^{9}/_{8}$, ${}^{16}/_{15}$.

The 4 "double modes" are lying symmetrically in the following coordinate-system, with the chords of the Seventh downwards (descending) as ordinate axis (perpendicular) and the same chords upwards (ascending) as abscissa axis (horizontal).

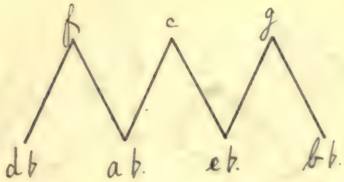


In case these 4 modes are inserted in annexure I. in such a way that the centre of the squares indicate the tone we obtain the following 4 geometrical figures, symmetrical in proportion to c: —

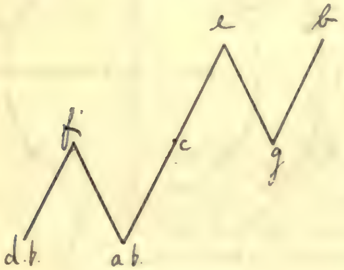
No. 2



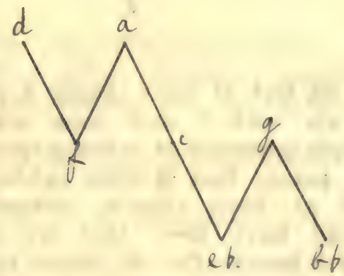
No. 10



No. 5



No. 8



With these figures we are able to construct "reflections" and "pendants" (in two different ways) with No.s 5 & 8, but only reflection (in one way) with No.s 2 & 10; however this is of no immediate interest to us at this point. In the present treatise the description **"pendant"** is used only in the case of "Pendant-Figures of The Telegraphic Signs", f. i. "D" — - - contra "U" - - —, not "K" contra "R".

As will be seen from the above illustration it is not f minor but the Doric c minor which forms the pendant of C major[16]).

I have in the above made use of the ancient Greek names as there seems to me no reason for using the wrong names employed in The Ecclesiastical Modes, — f. i. by Glarean (1488—1563) I thus entirely agree with **Helmholtz** when he says: —

"But I shall not use Glarean's names without expressly mentioning that they refer toan ecclesiastical mode. It would be really better to forget them altogether"[17]).

Article VI. **The 10 Principal Intervals.**

The 11 modes mentioned in article V. have that in common that they altogether make use of only 10 normals intervals which we call **"The 10 Principal Intervals"**, which pair off as complementary intervals — Law 5 —, namely in millioctaves, in "1200 just cents" with 2 decimales, according to Ellis' "Sensations of Tone", 1912, p. 329, and in degrees of arc:

degrees of arc	1200 cents	milli octaves	complement tones		milli octaves	1200 cents	degrees of arc
480°	498,04	415,04	f	g	584,96	701,96	540°
450	386,31	321,93	e	a♭	678,07	813,69	576
432	315,64	263,03	e♭	a	736,97	884,36	600
400	182,40	152,00	d	b♭	848,00	1017,60	648
384	111,73	93,11	d♭	b	906,89	1088,27	675
360	0	0	c	c'	1000	1200	720

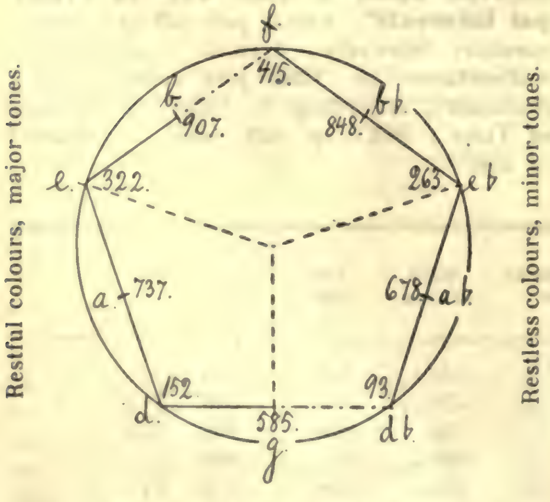
They correspond to the 10 principal colours in the colour circle which is often put up in a regular pentagon by the natural philosophers in such manner that the complementary colours are lying exactly opposite each other[18]). Here we must not forget to take into consideration, however, that while the complementary colours follow each other in the same numerical order in the solar spectrum, the complementary tones (intervals) on the other hand are grouped symmetrically round a tone lying midway between f♯ and g♭, on 500 m, i. e. for instance arranged ad libitum, say according to the influence of tones and colours on the nervous system:—

Restful colours = Major tones.

g bluish-purple	d blue	a bluish-green	e **green**	b yellow-ish-green
yellow f	orange b♭	**red** e♭	purple a♭	violet d♭

Restless colours = Minor tones.

In case we might desire to use f. i. the Dane, Prof. Barmwater's colour-circle for the placing of the 10 principal intervals these are best indicated in Fifth-succession; as f. i. arranged together as below:—



The lines b—f and d♭—g indicate only a diminished Fifth.

From the centre c (corresponding to the white colour in the colour-circle) 3 dotted lines have been made from c to e♭, e and g, the tones of the major- and minor-triad whitout c, corresponding to the 3 primary colours.

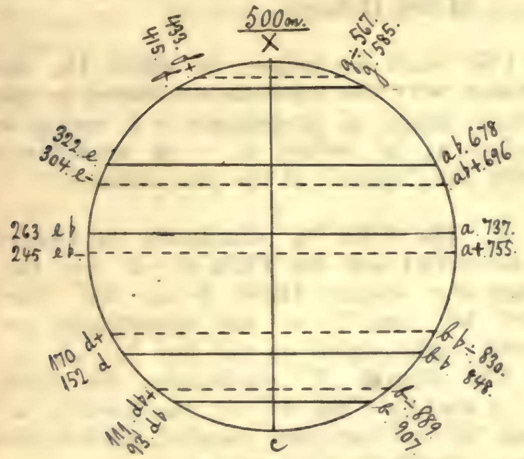
If the 10 principal tones (intervals) are arranged along a circle in 4 tetrachords (2 major and 2 minor), each about 33+37+20=90 degrees of arc, we obtain the following figure where the tones pair off as complementary:

The fact of there being 4 principal intervals with ♭ and none with ♯ is the simple sequence of c being chosen as starting point; had **d** (the centre between a and g), been chosen instead the result would have been that we should have had — beside the 6 other tones on the white digitals of the piano — also c♯, e♭, f♯ and b♭, f. i. in double harmonic D major:

D	e♭÷	f♯	g÷	a	b♭÷	c♯	d
152	245	474	567	737	830	1059	1152

$$\underbrace{\underbrace{}_{93}\ \underbrace{}_{229}\ \underbrace{}_{93}}_{415} \quad 170 \quad \underbrace{\underbrace{}_{93}\ \underbrace{}_{229}\ \underbrace{}_{93}}_{415} = 1000 \text{ m}$$

If the 10 principal tones (intervals) are arranged along a circle in millioctaves (thousandths parts of the circle) we obtain the following figure in which an $\times$ indicates the tone on 500 m:



The drawn lines combine the complementary normal intervals, — and the dotted lines running **parallel** with the drawn ones are combining the complementary comma-intervals: d♭+ with b÷, a.s.f. (compare art. VII).

In the physical textbooks we often see the fractions of the intervals being replaced through multiplication by **common denominators** by whole numbers which of course are wrong when d is calculated equal to $^9/_8$ in C major or c minor; in correcting the mistake to $^{10}/_9$ we shall obtain the following 4 common denominators for the 4 double modes: —

	from c	common denominator	Ordre of precedence
No. 2. Lydian major mode	0♭	72	2
- 5. Double harm. major mode	2	120	4
- 8. Phrygian minor mode, the central mode	**2**	**90**	3
- 10. Doric minor mode........	4	30	1

therefore with the following figures of merit for the 7 tones (degrees) together with the octave, when for the sake of getting a comprehensive view we subtract the common denominator from all the 8 figures so that c stands for 0: —

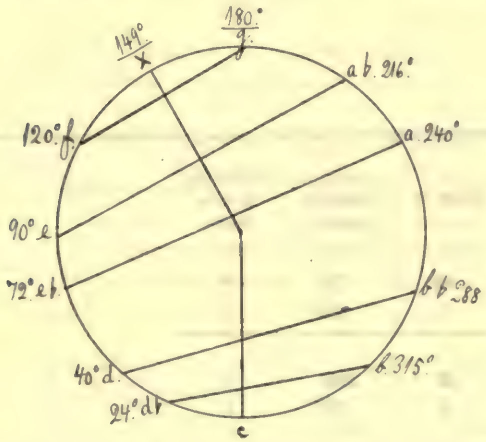
No.	1	2	3	4	5	6	7	8
2	c 0	d 8	e 18	f 24	g 36	a 48	b 63	c' 72
5	c 0	d♭ 8	e 30	f 40	g 60	a♭ 72	b 105	c' 120
8	c 0	d 10	e♭ 18	f 30	g 45	a 60	b♭ 72	c' **90**
10	c 0	d♭ 2	e♭ 6	f 10	g 15	a♭ 18	b♭ 24	c' 30

From a numerical point of view the ideal would be: The principal Scale of the Classic Greeks, the Doric Minor Mode, as it gives us the lowest figures for stating the differences of the vibration numbers, while the harmonic major and minor mode as well as the melodic minor mode ascendant has as common denominator $360 = 12 \times 30$, as far as possible removed from the true ideal.

It will be seen that the division in the extremely cleverly invented phonoscope[19]), constructed in 1885 by the Danish natural philosopher J. C. Forchhammer, born

1861, must be entirely rearranged if we desire to impart to the deaf and dumb the true apprehension of tones.

The mutual proportion between the scales becomes still more evident if we choose the common denominator which is the same for all 4 scales i. e.: $2^3 \times 3^2 \times 5^1 = 8 \cdot 9 \cdot 5 = 360$, the exact number of degrees in a circle, thereby making it possible to state the difference between the vibration numbers ,by **degrees of arc**, with $c = 0$, $c' = 360^0$. We shall then obtain the following figure in which $\times$ stands for the tone midway between f and g about 149^0: —



The drawn lines combine the 10 principal intervals which pair off as complementary (inversion-intervals).

I. Concerning the principal intervals it will easily be seen — by inserting them together with their degrees of arc, in the above circle — that **the order of precedence** of the intervals in the case of the vibration-numbers' difference (degrees of arc) must be as follows, Law 1 in the preface: —

1) Consonances of the 1st degree:
   c forms the Octave
   g dimidiates the Octave
   e quarters the Octave
   } quadrangular sides.
2) Consonances of the 2nd degree: a and f trisects the Octave.
3) Consonances of the 3rd degree: a♭, e♭ and b♭ quinqueparts the Octave.
4) Dissonances i. e. the name of the 3 other principal intervals: d, d♭ and b, which give us still smaller parts of the Octave.

(We shall see below, in article IX, the reason why the intervals are stated in the above order of precedence, — I mean: **e** before **a**; b♭ before d, — in accordance with the "spiral" of the consonances.)

II. The Fifth "g" dimitiates the Octave (the circle) and the Fifth "e♭—b♭" and further the major Third f—a; the lines "e♭—b♭" and "f—a" are accordingly parallel, right-angled on the line "c—g".

The Fourth "f" dimidiates the Sixth c—a and the Fifth d♭—a♭; the lines "c—a" and "d♭—a♭" are also parallel, right-angled on a line from f to centre.

III. Be it noted that by **tri**partition of the circle we obtain the major triad "F—a—c", analogic with the fact that the colours which tripart the colour circle in three even parts are harmonic colours.

Further: the C major triad is constructed by division of the circle by 2 and **4**: $c=0$, $e=90$ and $g=180^0$, **the perfect harmony,** Rameau's "l'accord parfait" from 1722; and:

the c minor triad is constructed by division of the circle by 2 and **5**: $c=0$, $e♭=72$ and $g=180^0$, "l'accord parfait mineur".

IV. The tones "f" and "g" border the 2 **tetrachords** on both sides; within these the principal tones are constructed in the following manner (see the preface):

d♭ and e♭ in dividing by 5 24^0 36^0			
d and a 40^0 60^0	-	—	- 3
e♭ and b♭ 72^0 108^0	-	—	- $^5/_3$
e and b 90^0 135^0	-	—	- $^4/_3$

Law 4 in the preface: these tones pair off "fifth-proportionally": 24+12=36 . . . . . 90+45=135°, likewise the 2 tetrachords themselves: 120+60=180°, see following table:

Name	from	Division of the 2 tetrachords:					
major	c g	d a	1/3	e b	3/4	f c′	low high
harmonic	c g	d♭ a♭	1/5	e b	3/4	f c′	low high
minor	c g	d a	1/3	e♭ b♭	3/5	f c′	low high
anti-major	c g	d♭ a♭	1/5	e♭ b♭	3/5	f c′	low high

From a geometrical point of view **the ideal** would be: The **central** mode, the Phrygian minor mode; degrees of arc 72, 48, 60, 60, 48, 72 are symmetric, and the lines "f—a" and "e♭—b♭" are parallel, as the R-Chord "c—e♭—g—b♭" - — -, forms a symmetrical square with degrees of arc 72, 108, 108 and 72°.

While the major mode contains the octagon face b—c′, beside 2 hexagon-faces, 2 quadrangle-faces and 3 triangle dito, the Doric minor mode contains 3 pentagon-faces; (all the -faces thus mentioned are regular polygons inscribed in the circle. The missing 9 corners in these 5 polygons are partly the comma-interval d+ = 45°, partly 8 extra-intervals, amongst them "i" = 270° nearest to a♯, the one which has caused ever so many theorists to lose their hearts, compare the mathematic definitions: "harmonic points", "harmonic mean-proportionals" and "harmonic series").

The result: By tripartition (No. 1) of the circle we obtain the Fourth c—f, the low tetrachord, and by the tripartion (No. 2) of this Fourth we obtain the small major Second, c—d = 10/9 = 40 degrees of arc, 1/9 of the circle, my discovery in the year 1882[39]).

Article VII. Scales on Other Tonicae (keynotes) than c, c♯ or c♭; Origin of the comma-intervals. The pentatonic scales.

If we were to add triads or chords of the Seventh to the 10 principal intervals ascending (on the right, in annexure 1), or descending (on the left in annexure 1), we will in so doing have crossed the frontier to the middle-zone and would find ourselves on "foreign territory". And it is easy to discern, when studying annexure 1, that all comma-intervals formed "beyond the frontier" are grouped in **2 lateral zones** (compare art. III) as below:

a) The chords of the Seventh on the **right** in annexure 1 from d, f, a and c (the tones in the R-chord in C major) produce normal-intervals only; while chords of the Seventh from e produce the comma-interval d+; from g both d+ and f♯+; and from b both d+, f♯+ and a+. The scales on these 3 tones e, g and b (the tones of the K-chord in C major) will thus get, respectively, 1, 2 or 3 comma-intervals with +, just as the dominant chord of the Seventh in C major (on g) will be g, b, d+ and f+, namely possessing 2 tones which are not congruent with the tones of the scale in C major.

b) Reversed: The chords of the Seventh on the **left** in annexure 1 from b, g, e and c (K-chord) produce normal intervals only; whereas chords of the Seventh from a produce the comma-intervals b♭÷, from f both b♭÷ and g÷; and from d both b♭÷, g÷ and e♭÷; scales in these 3 tones a, f and d (R-chord) will thus get, respectively, 1, 2 or 3 comma-intervals with ÷.

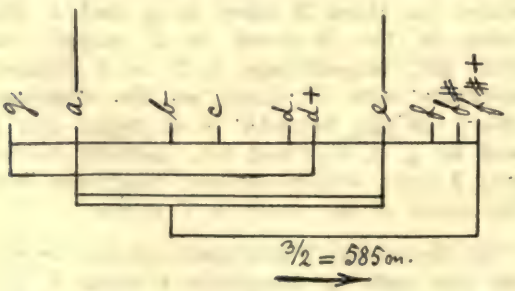
These proportionate conditions have caused an immense amount of unnecessary brain-racking to numerous theorists[20]).

The same result may be obtained by constructing scales — a) in the Fifth-circle by using the high tetrachord in C major as low tetrachord in the G major, the following, a. s. f. — or reversed: b) by constructing scales in the Fourth-circle by using the low tetrachord in C major as high tetrachord in the F major, the following, a. s. f. This state-ment is clearly demonstrated in annexure 1.

By constructing 14 major scales with up to 7 ♯ or 7 ♭ we shall obtain the following scales with up to 3 comma-intervals each:

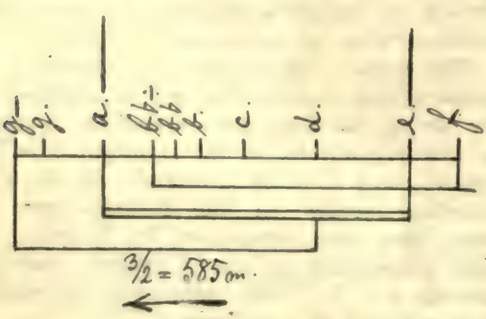
Number of comma-intervals				the Fourth-circle			the Fifth-circle				Number of comma-intervals
2	»	g÷	b♭÷	F major	1♭	1♯	G major	d+	f♯+	»	2
3	d+	f+	a+	B♭ —	2-	2-	D —	e÷	g÷	b÷	3
1	d+	»	»	E♭ —	3-	3-	A —	»	»	b÷	1
1	»	»	b♭÷	A♭ —	4-	4-	E —	d♯+	»	»	1
3	e♭÷	g♭÷	b♭÷	D♭ —	5-	5-	B —	d♯+	f♯+	a♯+	3
2	d♭+	f+	»	G♭ —	6-	6-	F♯ —	»	g♯÷	b÷	2
0				C♭ —	7-	7-	C♯ —				0

The phenomenon: — The origin of the comma-intervals — may also be explained graphically in the following 2 figures:



Of 3 ascendant Fifths on g, a and b only the middle one will come out pure if we are going to use the tones of C major or the deviations of same, seeing that:

g 585 + 585 = 1170 or 170 d+
a 737 + 585 = 1322 or 322 e pure
b 907 + 585 = 1492 or 492 f♯+



And of 3 descendant Fifths from d, e and f only the middle one will come out pure if we are using the tones of C major or the deviations of same, seeing that:

f 1415 ÷ 585 = 830 b♭÷
e 1322 ÷ 585 = 737 a pure
d 1152 ÷ 585 = 567 g÷

It will thus be deemed practical to tune violins to the pitch of A-string seeing that both the Fifth ascendant to e and descendant from a to d are pure, so that only the Fifth descendant from d to g are false to the tones of the e-string, namely one comma too low. (If we had made use of the d, a, e and b strings all 3 intermediate Fifths would have come out pure which may be seen by a glance at annexure 1 which has all these 4 tones arranged beside each other above "f—c—g" stated in milli octaves: d 152 + 3 × 585 = 152 + 1755 = 1907 or 907 = pure b.)

The phenomenon: — The origin of the comma-intervals — may also be explained by the following 2 diagrams:

Diagram No. I. The Greek scales and f. i. the harmonic c minor.

Greek scales	Low tetrachord ⟶				High tetrachord ⟶				Number of comma-intervals	Group
Lydian major	C	d	e	f	g	a	b	C	0	
Phrygian minor	d	e÷	f	g÷	a	b÷	c	d	3÷	I.
Doric minor	e	f	g	a	b	c	d+	e	1+	
Hypo-lydian major	F	g÷	a	b÷	c	d	e	F	2÷	
Ionic major	G	a	b	c	d+	e	f+	G	2+	
Æolian minor	a	b÷	c	d	e	f	g	a	1÷	II.
Mixo-lydian minor	b	c	d+	e	f+	g	a+	b	3+	
Harmonic minor	c	d	e♭	f	g	a♭	b	c	0	III.
Total:	0	3÷	1+	2÷	2+	1÷	3+	0	12	

The comma-intervals group themselves quite symmetrically, with + to the right of c, and ÷ to the left of c.

Diagram No. II. Lydian scales and f. i. harmonic C major.

	Number of ♯	♭	Low tetrachord ⟶				High tetrachord ⟶				Number of comma-intervals
Lydian major:	0	0	C	d	e	f	g	a	b	C	0
	2	—	D	e÷	f♯	g÷	a	b÷	c♯	D	3÷
	4	—	E	f♯	g♯	a	b	c♯	d♯+	E	1+
	—	1	F	g÷	a	b♭÷	c	d	e	F	2÷
	1	—	G	a	b	c	d+	e	f♯+	G	2+
	3	—	A	b÷	c♯	d	e	f♯	g♯	A	1÷
	5	—	B	c♯	d♯+	e	f♯+	g♯	a♯+	B	3+
harm.:	—	1	C	d	e	f	g	a♭	b	C	0
Total			0	3÷	1+	2÷	2+	1÷	3+	0	12

The number of comma-intervals is quite independent of the number of ♯ or ♭; it depends entirely on the Tonica (keynote) in consequence of the fact of the major seconds on the tones of C major having 2 different quantities, f. i. the following in symmetrical succession:

$^{10}/_9$ = 152 m	$^9/_8$ = 170 m	Group.
c — d ——— e — f♯	——— d — e ———	X
———	f — g	centre
g — a ——— b — c♯	——— a — b ———	Y
4	3	7

The groups X and Y are symmetric and congruent.

The consequence 1: The 6 intervals on "d" or "d+" in C major give 3 normal and 3 comma-intervals, namely:

	with d+ = $^9/_8$	but with d = $^{10}/_9$
the Second	d+ — e=152=d	d-e=170=d+
- **Third**	d+ — f=245=e♭÷	d-f=263=e♭ **pure**
- Fourth	d+ — g=415=f	d-g=433=f+
Fifth	d+ — a=567=g÷	d-a=585=g **pure**
- Sixth	d+ — b=737=a	d-b=755=a+
- **Seventh**	d+ =c′=830=b♭÷	d-c′=848=b♭ **pure**
	Errors.	Thruth.

The consequence 2: In all scales 2 Fourths are Comma-Fourths, f. i.:

d b

263 f g 322

170 170 170

Sum: 433 492

= f+ = f♯+

namely: "the pure Third d—f and the false comma-Second f—g" give "the false comma-Fourth f+",

and: "the pure Third g—b and the same false Second f—g" give "the false comma-Fourth f♯+". That is the work of Nature, see annexure 1, in which d and f are placed outmos tto the left; (the pure Fourths from d and f to the left are g÷ and b♭÷).

The 2 comma-Fourths are complementary intervals with 2 comma-Fifths g—d ($^{10}/_9$) and b—f♯ ($^{25}/_{18}$), when g and b in annexure 1 are placed outmost to the right; (the pure Fifths from g and b are d+ = $^9/_8$ and f♯+ = $^{45}/_{32}$).

The consequence 3: The Second in

rightly constructed

D major is 152+152 = 304 = e÷

- - D+ — - 170+152 = 322 = e

wrongly constructed

D+ major - 170+170 = 340 = e+.

But e+ is 2 commas higher as e÷, 2 commas false.

By using chords of the Seventh for the construction of the **5 pentatonic scales,** ascendant as well as descendant, it becomes obvious at once that these scales are but **fragments** of the classic Greek scales on C, G, d, a and e, as the K-, D-, R- and U-chords are imperfect. Helmholtz' serial succession [21]) ought to be arranged accordingly.

wrong No.	Helmholtz' Statement	without 2 intervals	to be played on the black digitals of the piano.	Construction of scales by chords of Seventh of K, D, R and U: descending from Tonica ◁—				Tonica			—▷ ascending from Tonica		My No.
4	Chinese, Scotch major mode	Fourth and Seventh	g♭	R÷ Third	d	(f)	a	C	e	g	(b)	K÷ Sev.	1
1	Chinese	Third and Seventh	d♭	**R complete**	**a**	**c**	**e**	**G**	(b)	d+	(f+)	D÷ Third and Sev.	2
3	Chinese, Gaelic, Irish	Third and Sixth	a♭	R÷ Fifth	e÷	g÷	(b÷)	d	(f)	a	c	R÷ Third	3
2	Scotch minor mode	Second and Seventh	c♭	U÷ Prime and Fifth	(b÷)	d	(f)	**a**	**c**	**e**	**g**	**R complete**	4
5	?	Second and Fifth	b♭	K÷ Prime	(f)	a	c	e	g	(b)	d+	R÷ Fifth.	5

The 5 pentatonic scales are thus fragments of the following scales, indicated in Fifth-succession, seeing that we find between the tones of Fifth of 585 m (g) or 567 m (g÷); but only 526 (g♭) between b÷ and f, or between b and f+, and only 508 (g♭÷) between b and f.

My No.	585 m	567	585	585	585	567		Number of commas	Result: Fragment of	without:	Rest of commas
1	C	g	d	a	e	(b)	(f)	0	Lydian major	maj.4 and maj.7	0
2	G	d+	a	e	(b)	(f+)	c	2+	Ionic —	maj.3 — min.7	1+
3	d	a	e÷	(b÷)	(f)	c	g÷	3÷	Phrygian min.	min.3 — maj.6	2÷
4	a	e	(b÷)	(f)	c	g	d	1÷	Æolian —	maj.2 — min.6	0
5	e	(b)	(f)	c	g	d+	a	1+	Doric —	min.2 — maj.5	1+

The tones b÷, b, f and f+, between each of which are only 526 m or 508 m, will thus be found missing in the pentatonic scales, and the scales on F and b must consequently secede.

1) The differences in milliloctaves between the tones in the 5 pentatonic scales are:

My No.	My Name	I II III IV | V VI VII I	Sum of milliotaves	
1	Lydian	c d e — g a — c 152 170 263 152 263	1796	
2	Ionic	c d — f | g a — c 152 263 170 152 263	1889	93 204 111
3	Phrygian, central mode, symmetric:	c d — f | g — b♭ c 152 263 170 263 152 low tetrachord | high tetrachord	2000 the centre.	—
4	Æolian	c — e♭ f | g — b♭ c 263 152 170 263 152	2111	111 204 93
5	Doric	c — e♭ f — a♭ b♭ c 263 152 263 170 152	2204	

No. 1 and 5 are pendants. No. 2 and 4 have congruent tetrachords. No. 3 is symmetric. All large distances are 263 = a minor Third.

2) The differences in degrees of arc (= differences of vibration numbers) are, indicated in fractions:

No.	Name	I II III IV | V VI VII I	
1	Lydian	c d e — g a — c $^1/_9$ $^5/_{36}$ $^1/_4$ $^1/_6$ $^1/_3$ as Ionic	
2	Ionic	g a — c | d+ e — g $^1/_9$ $^2/_9$ $^1/_6$ $^1/_6$ $^1/_3$	most perfect
3	Phrygian, central mode:	d e÷ — g÷ | a — c d $^1/_9$ $^2/_9$ $^1/_6$ $^3/_{10}$ $^1/_5$ as Ionic | as Æolian.	
4	Æolian	a — c d | e — g a $^1/_5$ $^2/_{15}$ $^1/_6$ $^3/_{10}$ $^1/_5$	
5	Doric	e — g a — c d+ e $^1/_5$ $^3/_{15}$ $^4/_{15}$ $^1/_5$ $^1/_5$ as Æolian	

In Nos. 2 and 4 the tetrachords are fifth-proportional, f. i. in the Ionic pentatonic scale: $^1/_9 \cdot {^3/_2} = {^1/_6}$ and $^2/_9 \cdot {^3/_2} = {^1/_3}$; in the Æolian: $^1/_5 \cdot {^3/_2} = {^3/_{10}}$ and $^2/_{15} \cdot {^3/_2} = {^1/_5}$.

If the classic lyre or Kithara possessed 5 strings in the following succession: g, a, b, d, e, it follows that it must have given scale No. 1: The Lydian G major without c and f♯, without any Fourth or Seventh, like the East-African "Kissar"[22]).

Helmholtz was not able to solve the pentatonic riddle for the reason that he did not think of constructing scales by means of 2 chords of the S e v e n t h (equal to 3 Thirds) in the directions opposite of the Tonica (keynote).

By looking at annexure 1, another fact we will discover is that "comma-intervals with +" can be **complementary** only with "comma-intervals with ÷" as f. i. "f+ and g÷" or "e÷ and a♭+", a .s. f., which fact is demonstrated graphically in the circle in the cliché, art. V, where the dotted lines connecting the complementary comma-intervals run **parallel** with the drawn lines connecting the complementary normal intervals.

If we were to construct say 15 Doric minor scales in the manner above described the result would be that these scales, together with the corresponding 15 major scales, would get **36 tones**, of which number 21 (or 60 %) are normal-intervals, 8 comma-intervals with + and 7 comma-intervals with ÷; double ♯ or double ♭ do not occur, nor do we find any comma-intervals on c, c+ or c÷, or comma-intervals outside annexure 1.

By playing through these scales once, up to the chord of the Seventh incl., we get $30 \times 7 =$ **210 spaces**, of which 161 (or 77 %) should be filled with normal intervals (which is thus "The Normal"), 49 (or 23 %) with comma-intervals; these 49 spaces are distributed s y m m e t r i c a l l y (with exception of the 49th) over the tones of the white digitals of the piano (and the deviations of these tones) when arranged in Fifth succession, as below:

e	b	f	c	g	d	a	
4÷	12÷	8+	0	8÷	12+	4+	= 48
					and 1		and 1

Also symmetrically in concurrence with the degrees of the scales, thus:

V	VI	VII	I	II	II	IV	
8÷	4+	12÷	0	12+	4÷	8+	= 48
	and 1						and 1

Whatever opinion we may hold in regard to the 2 primitive modes, the C major and the Doric c minor, one fact is undeniable: they are geometrically correct (pendants).

It must be noted that the tones of scales with Tonica (keynote) on different degrees of C major are n e v e r congruent, as f. i. the melodic and harmonic a minor has b÷ = 889 m, while C major has b = 907. This **"sliding"** from normal intervals to comma-intervals has up to the present time erroneously been considered a "dualism" between "harmony" and "melody"; this has been mentioned by Jonquière[28]) as the contrast between:

a) "Repose in Space", by h o m o g e n e o u s polyphony, f. i. the music from a harmonium, unaccompanied choruses, string quartets, etc. as also by slow tempo, and

b) "Change in Time", by h e t e r o g e n o u s polyphony, as f. i. ensemble playing by different instruments, singing accompanied by instrumental music, as also by quick tempo.

This m y s t i c "explanation" has now become quite superfluous, as the sliding from a) normal intervals to b) comma-intervals, or even to extra-intervals, explains the whole thing as

1) The transitory step from music "in scales on the same keynote or its deviations", to "scales on other keynotes and their deviations" (mechanic **modulation**), or

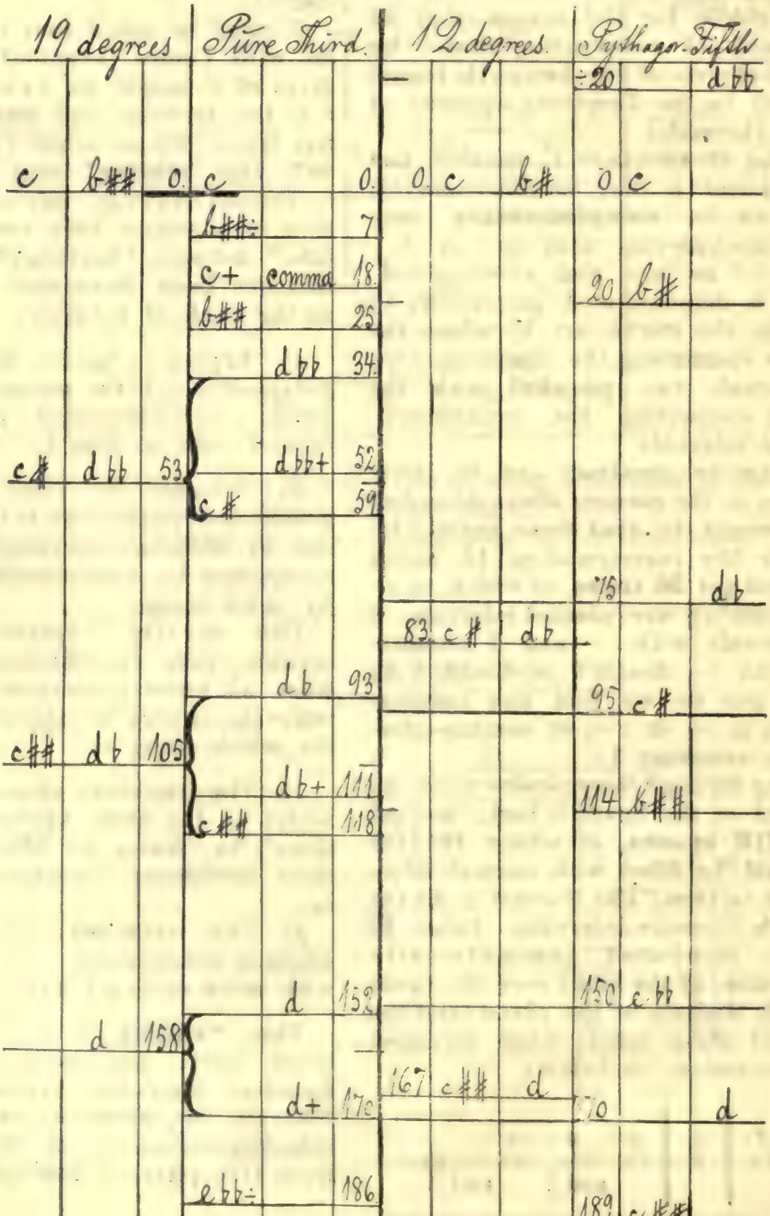
2) The transitory steps from "homogeneous instruments" to "heterogeneous" with more or fewer **extra** tones.

This **"sliding"** is not characteristic of tones only; something corresponding is found in the colour system, the so-called "Wien's Displacement Law", where the intensity-maximum of the c o l o u r spectrum (the physical radiance-maximum) is

displaced (sliding) from red to purple (violet) on the increase in temperature of the luminant (compare herewith the terms: red-hot, white-hot), — only the displacement of the tones is very limited, as it rarely goes beyond the 4 Pythagorean tones congruent with the comma-intervals e♭÷, b♭÷, d+ and a+ (see annexure 1).

A parallel may also be drawn between the different, individual perception of tones by different human beings (and by singing birds)[24]), and between the varying sensitiveness by different human beings (and by different photographic negatives) to the shine, the radiance of colours[25]), and there exists, as we know, an analogy between the microscopic nervefibres of the retina of the eye and the membrana basilaris in the cochlea of the human ear[26]).

The distances within a small whole tone, "c—d", in Pure Third, Pythagorean Fifth and 2 temperaments.



PART III. The MUSICAL PRACTICE.

Article VIII. The Natural Tonical System and the Artificial Temperaments.

In case we are confining ourselves to play only in the 2 primitive modes, in the Lydian major mode and the Doric minor mode, and with up to 7 ♯ or 7 ♭ only, we shall to these 30 scales (as stated in art. VII) need only **36 tones** in the octave[27]), and it will be possible to play these scales with every tone absolutely and mathematically pure and true, nay, heavenly tones they will all be, "silver pure", limpid and bright in their purity, such as the great masters of the violin and cello endeavour to produce on their instruments, or the great singers with their voices. Thes 36 tones are

The tones of C major with ♯ and ♭, 21 normal tones and 15 comma-tones altogether, as follows:

c♭÷	e÷	......:	d♭+	d+	d♯+
g♭÷	g÷	g♯÷	....	f+	f♯+
b♭÷	b÷	....	a♭+	a+	a♯+

If we desire to employ other i. e. more modes as f. i. the melodic minor mode, or if we play with more than 7 ♯ or 7♭, more than 36 tones will be needed. For practical reasons, therefore, a temperature — or, as it is termed in England and by the Latin nations —: a temperament (which term I propose adopted for international use), has been invented.

The most logical of the temperaments have the following number of tones in the octave:

$5 \times 2 + 2 \times 1 = 12$, Aristoxenos, about 350 b. C.

$5 \times 3 + 2 \times 2 = 19$, Elsasz in Vienna, about 1590[28]).

$5 \times 4 + 2 \times 2 = 24$. Arabian system?[29]).

$5 \times 5 + 2 \times 3 = 31$, Vicentino, about 1546[30]).

$\left\{\begin{matrix} 2 \times 6 \\ 3 \times 7 \end{matrix}\right\} + 2 \times 4 = 41$, Paul v. Janko, 1882—1901.

$\left\{\begin{matrix} 2 \times 8 \\ 3 \times 9 \end{matrix}\right\} + 2 \times 5 = 53$, Nicholas Mercator, c. 1675[30]).

I do not propose to take the Arabian system into consideration in this present treatise; the 3 temperaments of 12, 19 and 31 degrees may be characterized as follows below, the tones being indicated by their numbers, with c = 0, c′ = 12, 19 and 31 respectively:

12 degrees:	31 degrees:	19 degrees:
0 c	0 c	0 c
	1 d♭♭	1 c♯
1 c♯, d♭.	2 c♯	
	3 d♭	2 d♭
2 d	4 c♯♯	
	5 d	3 d
11 b	28 b	17 b
	29 c′♭	18 b♯ c′♭
12 c′.....	30 b♯	
	31 c′	19 c′

The temperaments are called "Equal Temperaments" when the intervals between the tones is equal in millioctaves; these may then be calculated out simply by dividing 1000 with resp. 12, 19, 31, 41 and 53, and multiplying with the No. of the tone itself, with c being reckoned equal to 0, c′ = 12, 19, 31, 41 or 53. On the other hand the value of the intervals as fractions must be calculated out by logarithms; and as the tone d is No. 2, 3 and 5 in the 3 systems d will obtain the fractional value of resp. $\sqrt[12]{2^2}$, (12th root of 2 to the second power), $\sqrt[19]{2^3}$, $\sqrt[31]{2^5}$, $\sqrt[41]{2^6}$ and $\sqrt[53]{2^8}$. The logarithms will then be, respectively:

0.30103 × 2 : 12 = 0.050 172......1.1225 or 167 m
0.30103 × 3 : 19 = 0.047 531......1.1157 — 158 —
0.30103 × 5 : 31 = 0.048 553......1.1183 — 161 —
0.30103 × 6 : 41 = 0.044 053......1.1068 — 146 —
0.30103 × 8 : 53 = 0.045 439......1.1103 — 151 —

The acuteness, fineness, of the intonation may be demonstrated in various ways. In annexure II. we will find a statement as to the number of millioctaves by which deviates the 3 first temperaments from the normal intervals, in regard to 35 normal intervals alone:

Pythagor. system		609÷	965+	1574 m
Temp. 12 degrees		416÷	659+	1075 —
— 19	—	190÷	126+	316 —
— 31	—	106÷	161+	267 —
— 41	—	106÷	101+	207 —
— 53	—	63÷	41+	104 —

The temperament of 19 degrees is thus — in regard to the normal intervals alone — 3 and a half times more refined than the temperament of 12 degrees, and the temperament of 31 degrees 4 times more refined than the one of 12 degrees. The difference between the temperaments of 19 and 31 degrees is, however, so small, than it would not be worth while to make use of the 31 degrees' system. Practically speaking our choice must be between the 12 degrees and the 19 degrees systems only. The temperaments of 31, 41 or 53 degrees can not be used on the violin, are consequently only of theoretical interest.

But it is not sufficient for our purpose to give all our attention to the normal intervals alone. The comma-intervals, as we know, have their demand for consideration as well, their place in the musical system, as f. i. d+ in E♭, G and B♭ major, and so on; thus it naturally suggests itself that we should endeavour to find out about **intermediate tones, average tones** between d and d+, and between e♭ and e♭÷, a. s. f., this has been done in the diagram below which is showing clearly **the practical-ideal character** of the system of 19 degrees:

Number of spaces in 30 Lydian and Doric scales	Names	**Intermediate** (avarage) **tones** in the Third system	Temp. 19 degrees	Distances between these in millioctaves ÷	+	Serial No. in regard to 19 degrees "t" =	Inversion No. from c till f♯	Temp. 19 degrees: Ordinary fraction (vibration)	Number of vibrations per second	1200 cents according to Ellis
14	c	0	0	0	0	0		1,000 0	261,02	0
12	c♯	59	53	6		1		1,037 2	270,72	63
8	d♭	102	105		3	2		1,075 7	280,78	126
14	d	161	158	3		3		1,115 7	291,21	189
8	d♯	220	211	9		4		1,157 1	302,03	253
12	e♭	254	263		9	5		1,200 1	313,25	316
14	e	313	316		3	6		1,244 7	324,89	379
6	f♭, e♯	368	368	0	0	7		1,290 9	336,96	442
14	f	424	421	3		8		1,338 9	349,48	505
14	f♯	483	474	9		9		1,388 7	362,47	568
6	g♭	517	526		9	10	9	1,440 2	395.74	632
14	g	576	579		3	11	8	1,493 8	389,90	695

(to be continued.)

(continued.)

10	g♯	635	632	3		12	7	1,549 3	404,39	758
10	a♭	687	684	3		13	6	1,606 8	419,45	821
14	a	746	737	9		14	5	1,666 5	435,00	884
6	a♯	805	789	16		15	4	1,728 4	451,16	947
14	b♭	839	842		3	16	3	1,792 7	467,93	1011
14	b	898	895	. 3		17	2	1,859 3	485,31	1074
6	c♭, b♯	953	947	6		18	1	1,928 4	503,34	1137
—	c'	1000	1000	0	0	19	0	2,000 0	522,04	1200
Total 210		9040	9000	÷70	+30 = ÷40			doubled.		

The false (discordant) character in regard to the comma-intervals alone has also been mentioned in annexure II; it means for the **30 comma**-intervals together:

Pythagor. system.	520÷	876+	1396 m
Temp. 12 degrees	373÷	616+	989 —
— 19 —	223÷	159+	**382** — } NB.
— 31 —	181÷	236+	417 — } NB.
— 41 —	82÷	88+	170 —
— 53 —	59÷	35+	94 —

Consequently the temperament of 19 degrees is — I need only mention its system of comma-intervals — even purer than the temperament of 31 degrees (NB).

The numbers of the 35 + 30 = 65 tones (see annexure II) are:

Pythagor. system.	1129÷	1841+	2970
Temp. 12 degrees	789÷	1275+	2064
— 19 —	413÷	285+	698 } about
— 31 —	287÷	397+	684 } equal.
— 41 —	188÷	189+	377
— 53 —	122÷	76+	198

These systems are thus lying averagely on different planes, the 19- and 53-degrees-systems below, and the 4 others above the system of the Third; they are all more or less imperfect.

The falsity of the 12 degrees piano is, however, varying, according to the scales played. In case we confine ourselves to the Lydian major mode and the Doric minor mode up til 7 ♯ or 7 ♭, — a total of 30 scales —, and add up the temperamental intervals for the first 7 tones in millioctaves the result will be: 231 m in F♯ major, 224 m in Doric d ♯* minor, 219 in C♯ major, 213 in a♯*, 201 in e♯*, 151 in D and b* (complement intervals), 140 in A and e♭*, 133 in G♭ major and f♯* minor, a. s. f., while by using the temp. of 19 degrees we shall not get higher than 97 in B major.

Illustration I. the falsity of the Lydian F♯ major:

	Pythagor. system	Temperaments: 12 deg.	19 deg.	31 deg.	41 deg.	53 deg.
e♯	54	36	13	6	9	4
d♯	54	39	0	15	9	3
c♯	36	24	6	6	10	2
b÷	36	28	6	14	11	2
a♯	54	37	7	10	9	4
g♯÷	54	41	6	19	8	3
F♯	36	26	0	10	11	2
	324	231	38 NB.	80 NB.	67	20

Illustration II. The total amount of falsity in 15 scales with not more than 7 ♯ or 7 ♭, in the Lydian and **Phrygian** modes, total 30 × 7 = 210 spaces (tones), see annexure III:

in	Pyth. syst.	12 deg.	19 deg.	31 deg.	41 deg.	53 deg.
15 Lydian scales	2232	1580	738	661	598	135
15 Phrygian do.	2358	1706	**670**	**757**	595	141
30 scales	4590	3286	**1408** NB.	**1418**	1193	276

With the Lydian and Phrygian scales together the temperament of 19 degrees is 10 m purer than the one of 31 degrees.

Illustr. III. Number of **perfectly pure tones** in 30 Lydian major and Phrygian minor scales:

Temp.	pure tones c	f♯ a e♭ d♯ g♭	number	%	false tones, number	Number total
12, 31, 41 deg.	14	—	14	6,7	196	210
19 —	14	41	55	**26,2**	155	210

Illustr. IV. The Number of false distances in **5 pentatonic scales**:

My No.		Pythag. system	Temperaments of 12 deg.	19 deg.	31 deg.	41 deg.	53 deg.
1	Lydian	54	41	18	19	16	3
2	Ionic	36	29	24	19	11	2
3	Phryg.	90	73	30	47	28	5
4	Æolian	54	41	18	19	16	3
5	Doric	36	29	24	19	11	2
Total:		270	213	**114** NB.	123	82	15

In the case of the 5 pentatonic scales the temperament of 19 degrees is purer than the one of 31 degrees, and can be played on the violin.

Illustr. V. **NB.** In considering **only** Tonica, Third and Fifth in the **triads** we must remember that these often consist of comma-tones; as f. i. the falsity in b♭ minor will be:

	b♭	d♭+	f+	Total	
Pythag. system	18	36	18	72 m	all too small.
Temp. 12 deg.	15	28	16	59 —	
— 19 —	6	6	12	**24** —	
— 31 —	9	14	14	37 —	
— 41 —	6	11	6	23 —	too large.
— 53 —	2	2	1	5 —	

Still greater will be the falsity in the 12 degrees temperament when we are using scales (harmonic, melodic, or with more than 7 ♯ or 7 ♭) which are demanding doubled ♯ or doubled ♭. The distance between f. i.: a♯ = 796 and b♭♭ = 789 is only 7 m, while the distance between a = 737 and b♭♭ = 789 is 52 m.

Of very special interest is the fact that by inserting distances of falsity for the above mentioned 65 tones in annexure I: (compare annexure III.) we shall find that these same distances will group themselves in lines, **parallel with 3 axes**, namely:

Pythag. and
Temp. 12 deg.: the 0° axis, Fifth,
— 19 — : the ÷60° — minor Third
— 31 — : the +60° — major —

Illust. VI. The scales are grouping themselves spontaneously along the same 3 axes, — f. i. in the case of the temperament of 19 degrees:

	Major	Minor	the sum:	
The axis e♯—g♯—b	96	72	168	
a♯—e—b♭	54	42	96	= 168 m.
d♯—c—g♭	36	36	72	
d—a♭—c♭	42	54	96	
d♭—f♭	72	96	168	

The mathematical solution of the above is that e♭ in the temperament of 19 degrees = 5000 : 19 = 263,158 m is only one eight of a millioctave larger than the pure e♭ = $^{6}/_{5}$ = 263,035 m, so that practically speaking all the tones of the axis d♯—c—g♭ (also the comma tones in the extension of the axis) are congruent with the corresponding tones in the temperament of 19 degrees.

But the temperamental distance is still smaller when taking into regard the manner in which the normal tones and commatones are distributing themselves over the **210 spaces** in the 30 scales mentioned in art. VII, f. i.: 8 d and 6 d+, namely $8 \times 152 + 6 \times 170$ is 2236, which, when divided by 14 make 159, while the d of the 19 degrees temperament is 158 m, — congruent, in other words. The temperament of 19 degrees is a practical demonstration of **The ideal!** For instance: the falsity of 36 tones used in 15 Phrygian scales may be stated with 20 differences in the following manner:

Pythagor. system	614 m
Temp. 12 degrees	417 -
— 19 —	**80** -
— 31 —	114 -

Thus the temperament of 19 degrees is **the consequence** of the Third system, **the practical ideal**, because it comes nearest to the truth (just as the average solar day-and-night comes nearest to the truth in regard to the astronomically true solar day-and-night), and can be used on the violin and the violoncello. The table page 63 shows the distances in a small whole tone, c—d; the annexures III and IV show the falsity of various scales in the Pythagorean system and 4 temperaments. Relating to the 22 tones of the Hindus, see art. IV.

To the mathematicians it is, of course, a matter of indifference whichever temperament is chosen, as long as it is an equal one; to the musicians it is a matter of deliberating: do we want to give preference to the clearest intonation possible — or do we put the main stress on the difficulty of playing 19 tones to the octave on the violin? Even should we — by taking a plebiscite vote from all lovers and students of music — find that the majority vote be cast in favour of the temperament of 12 degrees, yet none of us can prevent the great masters, the virtuosii, from using a finer intonation. One thing is certain: the schools and colleges of Music ought to **demonstrate** the temperament of 19 degrees seeing that it teaches the pupils to distinguish between the various degrees of perfection as regards intonation.

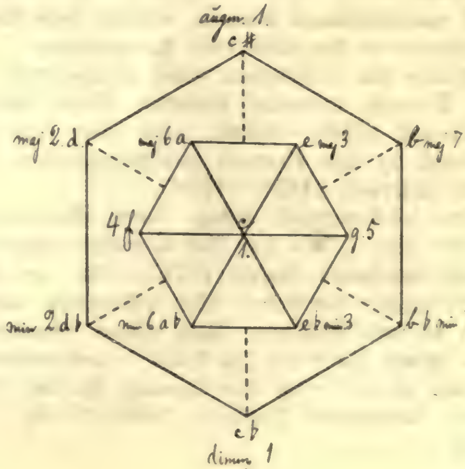
Teachers of the theories of harmony would also do well in employing millioctaves in place of numbers on tones in the various temperaments, among which the temp. of 19 degrees is the **supposition** of theories of harmony, which distinguishes between d♯ and e♭, indicated in 19 numbers, f. i.:

c—d♯—g = 0+4+11 = 15 t
c—e♭—g = 0+5+11 = 16 t
c—e —g = 0+6+11 = 17 t
c—e♯—g = 0+7+11 = 18 t
c—f —g = 0+8+11 = 19 t

"t" = 52,63 m, see art. X.

Article IX. Mutual Relationship of the Intervals: Combinational Tones.

a) A **mechanic** sort of relationship between the intervals has been demonstrated geometrically in the figure below which has been constructed by means of using the central points of the squares in annexure I as indication of the normal intervals.



The 3 pure intervals, Prime, Fourth and Fifth, are to be found on the horizontal line through c; on the line immediately above are to be found the 4 large intervals, Second, Third, Sixth and Seventh; and on the line below: respect. the 4 small ones; and above these the augmented ones: c♯, a. s. f.; lowest down the diminished ones: c♭, a. s. f.

When Helmholtz writes[31]): "Hence while b and d♭ are given with certainty, b♭ and d are uncertain. Either of them may be distant from the tonica (keynote) by the major tone $^9/_8$ (= 204 cents = 170 m) or the minor tone $^{10}/_9$ (= 182 cents = 152 m)", he is absolutely wrong. It goes without saying that the normal tones d and b♭ are just as "certain" as d♭ and b, and the comma tones d+ and b÷ equally so on their special precincts, only we must bear in mind the fact that d and its derivations d♯ and d♭ are to be found in C♭, C, C♯, D♭, D, F, F♯, A♭ and A major; and d+ and derivations from same in E♭, E, G♭, G, B♭ and B major and the corresponding minor.

The axis f—c—g in annexure I contains the Pythagorean tones, the axis f♯—c—g♭, the minor Thirds, congruent with the corresponding tones in the temperament of 19 degrees (compare art. VIII).

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b) An **organic** sort of relationship between the intervals showing their "contents" is brought about by considering the combinational tones (namely differential and summational tones), which owe their existence to the intervals themselves:

While the overtones were discovered by the French musical theorist Pater Marie Mercenne (1588—1648), and later on explained by the French mathematician Joseph Sauveur (1653—1716),

The differential tones were discovered in the year 1745 by the German organist Georg Andreas Sorge (1703—78), as the difference between the vibration numbers of any 2 tones; this same discovery was made later on, in 1754 — independent of the above — by the famous Italian violinist Guiseppe Tartini 1692—1770), — while:

The summational tones as the sum of the vibration numbers of any 2 tones was not discovered till 1854 by the German physiologist H. v. Helmholtz (1821—94).

We speak about combination tones of the 1st, 2nd, 3rd and 4th order, a. s. f., according to a definite plan of succession[32]) stated below, where the interval c—e♭ is being used for illustrative purposes:

	Order	Differential tones: Construction	Serial No.
c÷e♭	I	$^5/_5$ ÷ $^6/_5$ = $^1/_5$ = a♭	1
c ÷No. 1	II	$^5/_5$ ÷ $^1/_5$ = $^4/_5$ = a♭	2
e♭÷No. 1		$^6/_5$ ÷ $^1/_5$ = $^5/_5$ = c	3
c ÷No. 2	III	$^5/_5$ ÷ $^4/_5$ = $^1/_5$ = a♭	4
e♭÷No. 2		$^6/_5$ ÷ $^4/_5$ = $^2/_5$ = a♭	5
c ÷No. 3		$^5/_5$ ÷ $^5/_5$ = 0	6
e♭÷No. 3		$^6/_5$ ÷ $^5/_5$ = $^1/_5$ = a♭	7
No 1÷No 2	IV	$^1/_5$ ÷ $^4/_5$ = $^3/_5$ = e♭	8
No 1÷No 3		$^1/_5$ ÷ $^5/_5$ = $^4/_5$ = a♭	9
	Order	**Summational tones: Construction**	**Serial No.**
c+e♭	I	$^5/_5$ + $^6/_5$ = $^{11}/_5$ = y	1
c +No. 1	II	$^5/_5$ + $^{11}/_5$ = $^{16}/_5$ = a♭	2
e♭+No, 1		$^6/_5$ + $^{11}/_5$ = $^{17}/_5$ = z	3
	III ⋮		

If we take these 9+3 = 12 combinational tones together we shall see that the minor tonality, the interval c – e♭, of 12 combination tones, gets 7 a♭ (variously pitched according to the series $^1/_5$, $^2/_5$, $^4/_5$ and $^{16}/_5$) 1 c, 1 e♭, together creating the A♭ major triad, besides 1 equal to 0 and 2 extra tones: y and z, $^{11}/_5$ and $^{17}/_5$, corresponding to 1?8 and 766 m respectively.

We learn from the above that **"Minor" is the child of "Major"** and is always hankering **back to its parent**, or, as says Helmholtz, almost prophetically[33]): "Every minor Third. . . .becomes at once a major chord". Or, as Jonquière has it[34]): "The minor tonality c—e♭ is tending towards A♭ with a certain amount of perseverance; it evinces a kind of modulated drawing towards a kindred species of major mode", "as part of some next-of-kin (although so far not known) major triad".

The long and short of all this is obviously that the "Dualism" of Hauptmann, v. Oettingen etc. between Major and Minor is exaggerated; we shall have to accustom ourselves to look upon the relations between these two modes as something like the relations existing between father and son (or mother and daughter): further we must accustom ourselves to an entirely different perception of the parrallel scales and the modulation between these, of which subject more will be said in the next article of this treatise.

c) By calculating out the 12 corresponding combinational tones for "the octave and the 10 principal intervals" — 11 altogether —, we obtain "the spiral of the consonances", as explained in the tabel below:

Consonance of	Polygon-faces (art. VI)	The primitifs intervals		Extra-tones	0	Primes	Thirds	Fifths	Sevenths or Ninths	Tending towards major	Serial No.
1st degree	quadrangle	c—c′	maj.	0	3	7c	1e	1g	—	C	1
		c—g	—	1	2	8c	1e	—	—	C	2
		c—e	—	2	1	6c	—	2g	1d+	C	3
2nd deg.	triangle	c—a (—c′)	min.	2	1	8f	—	1c	—	F	4
		c—f (—c′)	maj.	2	1	7f	1a	1c	—	F	5
3th degree	pentagon	c—a♭ (—c′)	maj.	2	1	3a♭	1c	4e♭	1b♭	A♭	6
		c—e♭	min.	2	1	7a♭	1c	1e♭	—	A♭	7
		c—b♭	—	3	1	6a♭	1c	1e♭	—	A♭	8
Dis-sonances	other polygons	c—d (—c′)	min.	4	1	6b♭÷		—	1c	B♭÷	
		c—d♭ (—c′)	maj.	6	1	4d♭	—	—	1c	D♭	
		c—b	—	7	1	3c	—	1g	—	C	
Total 11 × 12 = 132 =				31	14	65	6	12	4	—	

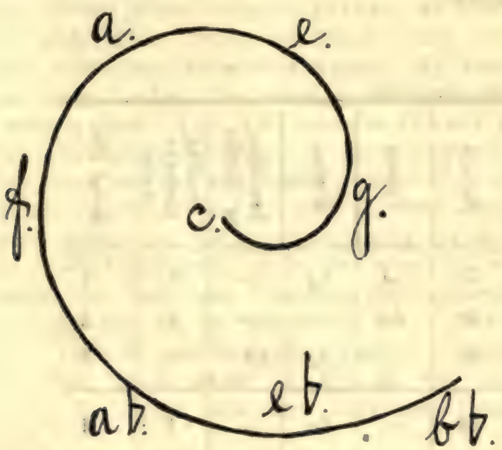
From the octave c—c′ is evolved the C major triad by means only of the first 3 of the summational tones; from the minor Sixth, c—a♭, is evolved the A♭ major triad, analogic with the first 3 differential tones only; the "triads of the over- and undertones", as it says in the present treatise, art. IV, are thus proved to be resting on a still deeper lying basis; the triads of the combinational tones.

d) Finally it will be seen that the order of precedence for consonances and dissonances, as stated in art. VI after the polygon faces, is just exactly in conformity with the number of summation tones equal to 0 (c, g, e, consonances of the 1st degree) — and extra tones of the above tabel (b♭, d, d♭, b) a. s. f., and further it will be seen that of the consonances on c:

3 of the 1st degree are tending towards C major (square)
2 - - 2nd — - — — F — (triangle)
3 - - 3rd — - — — A♭ — (pentagon)

3 Dissonances tending towards others, according to the character of each.

The spiral is easily remembered seeing that it is exactly spiral-shaped, c. shaped, in the annexure I as below:



Article X. **Mutual Relationship of the Scales.** Organic Modulation in 3 orders between Parallel Scales.

When talking about the mutual relationship of scales we distinguish — like in the case of intervals — between mechanic and organic relationship.

a) A **mechanic** relationship may be arranged in different degrees (grades) according to the number of triads in common for two scales; the degree will depend upon the number of ♯ and ♭, for which reason the successional order of the scales is stated in Fifth series in the "family-tree" below:

b) An **organic** relationship we call the relation between modes with an equal number of ♯ and ♭, namely **the Parallel scales**, as f. i. the 5 Greek scales mentioned in article V of the present treatise, played solely on tne white digitals of the piano. By arranging these 5 scales according to the illustration in article V (above the 5 pentatonic scales) we obtain in temperament the following tabel of relationship, which may be continued to both sides at will (I mean: with as many ♯ or ♭ as we please):

			Fourth-circle with ♭								Fifth-circle with ♯						
Sum of ♭ or ♯			7	6	5	4	3	2	1	0	1	2	3	4	5	6	7
No. 2	Lydian	maj.	C♭	G♭	D♭	A♭	E♭	B♭	F	**C**	G	D	A	E	B	F♯	C♯
— 5	Ionic	—	G♭	D♭	A♭	E♭	B♭	F	**C**	G	D	A	E	B	F♯	C♯	G♯
— 8	Phrygian	min.	d♭	a♭	e♭	b♭	f	**c**	g	d	a	e	b	f♯	c♯	g♯	d♯
— 9	Æolian	—	a♭	e♭	b♭	f	**c**	g	d	a	e	b	f♯	c♯	g♯	d♯	a♯
—10	Doric	—	e♭	b♭	f	**c**	g	d	a	e	b	f♯	c♯	g♯	d♯	a♯	e♯
Sum of triads in common with C major			0	0	0	0	0	2	4	7	4	2	0	0	0	0	0
Relationship = degree in regard of C major			8	7	6	5	4	3	2	1	2	3	4	5	6	7	8

The horizontal line C♯ and C♭ and the perpendicular line C—e are both in Fifth succession; the table itself is thus nothing less than ideal.

The transit from a scale to one of its 4 parallel scales we call "**organic** modulation", namely the resolution of a chord of the Seventh into a triad (with doubled Third) of equal value consisting (in a temperament) of the 3 other tones in the scale of the chord of the Seventh. The 2 illustrations below will serve to explain this; a certain amount of importance in this connection is attached to the temperament. The following indication is used below, with V standing for the 5th degree, "the dominant":

Triad	Chord of the Seventh
———	V^4 of "Second"
of "Fourth-Sixth" V_3	V^3 of "Third-Fourth"
of "Sixth" V_2	V^2 of "Fifth-Sixth"
of Tonica (fundamental) V_1	V^1 of "Seventh".

Illustation I:

The chord of the Seventh: "d f **g** b"
of the Lydian C major V
or - Ionic G — I
resolves in the triad "c e **a** c"
of the Phrygian d minor V
or - Doric e — IV
or - Æolian a — I.

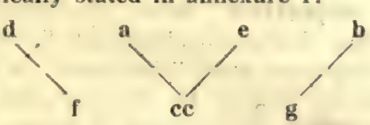
The 2 chords contain in the temperament of 19 degrees 39 "t" (1 "t" = 52,63 m):

d	f	g	b	
3	8	11	17	= 39 t
c	e	a	c	
0	6	14	19	= 39 t

or:

g b d f
a cc e

geometrically stated in annexure I:



The resolution of the Chord of the Seventh is thus carried out through contraction of the wings "d—f" and "g—b" in the central triangle "a—c—e", with intensified c,

Illustration II:

The chords of the Seventh "d♭ f **g** b♭"
of the Phrygian b♭ minor VI
or - Doric c — V
or - Æolian f — II
resolves in the triad "c e♭ **a♭** c"
of the Lydian A♭ major I
or - Ionic E♭ — IV.

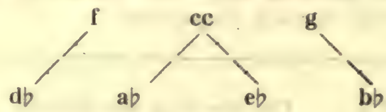
The 2 chords contain in the temperament of 19 degrees 37 "t".

d♭	f	g	b♭	
2	8	11	16	= 37 t
c	e♭	a♭	c	
0	5	13	19	= 37 t

or:

g b♭ d♭ f
a♭ cc e♭

geometrically stated in annexure I:



The wings "d♭—f" and "g—b♭" are contracted in the central triangle "a♭—c—e♭", with intensified c, that is: the child, Doric c minor, is hankering **back to its parent**, Lydian A♭ major (see art. IX) by "a kind of modulated drawing".

All of these triads and chords of the Seventh in the illustration I are in tempenament to be found in all of the 5 above ramed modes without ♯ or ♭ namely **the parallel scales**; and we are thus able to "analyse" (explain) the organic modulation as an expression of a close relationship between these scales mutually.

If next we contemplate the tabels in article V and add up, in pairs, the number of t in temperament of the 2 "halves":

{ the chord of the Seventh d f g b }
{ and the triad C e a C }

as well as the number of the corresponding tonalities in all the other scales we shall get the sums:

Serial No.	Chord of the Seventh	Difference	Triad	Group	Order of the organic modolation	The solution: the chords of
2 8 10	39 t 50 - 61 -	— — —	39 t 50 - 61 -	I	1.	only R or K
1 4 9 11	72 - 82 - 94 - 104 -	÷ 1 t + 1 - ÷ 1 - + 1 -	71 - 83 - 93 - 105 -	II	2.	also D or U
7	39 -	÷ 2 -	37 -	III	3.	only U and W

In other words: we get organic modulation **of 3 orders**. The first 3 scales. in group I, the double scales, are quite perfect; the following 4 scales, in group II, are less perfect, and the harmonic minor mode III is very imperfect.

It is further seen that the chords of the Seventh on the 7 degrees of C major are congruent with the dominant chords of the Seventh of the other **Greek scales**, namely:

Degrees of C major	The chords of the Seventh on the 7 degrees				Number of commas	The chord of the Seventh on the Dominant in		Group
1	c	e	g	b	0	Hypolydian	major	II
2	d	f	a	c	0	Ionic	—	
3	e	g	b	d+	1	Æolian	minor	
4	f	a	c	e	0	Mixolydian	—	
5	g	b	d+	f+	2	Lydian	major	I
6	a	c	e	g	0	Phrygian	minor	
7	b	d+	f+	a+	3	Doric	—	
	0	1	2	3				

We learn from this that the scales ought to be constructed entirely independent of the dominant chords and their resolutions. From the tones g b d+ f+ we are able to construct the Ionic G major but not C major, seeing that neither d+ nor f+ are found in C major.

An objection. Some musicians hold, that "the dominant-chord of Seventh" is not congruent with "the chord of Seventh on the 1st degree in the parallel scale". The question turns upon 3 forms of the chord, f. i. in 3 position:

No. 1	d 152	f 415	g 585	h 907	twice diminished.
No. 2	d+ 170	f 415	g 585	h 907	mixed.
No. 3	d+ 170	f+ 433	g 582	h 907	**right.**

consequently:

in No. 1	in No. 2	in No. 3	
d f = 263 = e♭	d+f=245=e♭÷	d+f+ = 263	"all right"
f g = 170 = d+	f g=170=d+	f+ g = 152	
g h = 322 = e	g h =322=e	g h = 322	
d g = 433 = f+	d+g=415=f	d+ g = 415	
f h = 492 = f♯+	f h =492=f♯+	f+ h = 474	
d h = 755 = a+	d+h=737= a	d+ h = 737	
Total 2 pure, 4 false (+)	3 pure, 3 false (÷ +)	all pure!	

No. 1 presupposes that d in C major is $= {}^{10}/_9$, $f = {}^4/_3$, and that these tones must be placed in the dominant-chord. In milli-octaves: "d f g b" = 152 + 415 + 585 + 907 = 2059 m = "c e a c" = 0+322+737+1000 = 2059 m.

No. 2 presupposes that d in C major is $= {}^9/_8$, $f = {}^4/_3$; 170 + 415 + 385 + 907 = 2077 m. That is wrong, however, because (amongst others) the octave a—a' indicated in vibration numbers (degrees of arc) and triparted gives $d = {}^{10}/_9$:

$$\underbrace{\overset{a}{600}\quad\overset{d}{800}}_{200}\quad\underbrace{\overset{d}{800}\quad\overset{f\sharp}{1000}}_{200}\quad\underbrace{\overset{f\sharp}{1000}\quad\overset{a}{1200^0}}_{200}$$

namely d = 400 × 2 = 800 degrees, — like the Fourth "a—d" triparted gives b÷ = $666^2/_3{}^0 = {}^{50}/_{27}$ pure in a minor.

No. 3 is **quite pure**, wherefore I should like to see No. 3 as the dominant-chord.

Another example. In f minor the dominant chord is: "c e♭ g b♭", although the Phrygian f minor has the tones: "f g÷ a♭ b♭÷ | c d e♭ f". When some musicians prefer b♭÷ for b♭, then it is the temperament of 12 degrees which plays a trick on them. This temperament has a too low minor Seventh, b♭ = 833, close upon b♭÷ = 830, while the pure b in c minor is 848 m. They have accustomed themselves, gradually, to demand a too low minor Seventh, b♭÷ for b♭.

We learn from this:

a) that the C **major triad** in the 3 positions has these distances of vibration numbers:

Position:												Sum:
1:	c	90	e	90	g	180	c'		1/4	1/4	1/2	360°
2:	e	90	g	180	c'	180	e'		1/5	2/5	2/5	450°
3:	g	180	c'	180	e'	180	g'		1/3	1/3	1/3	540°

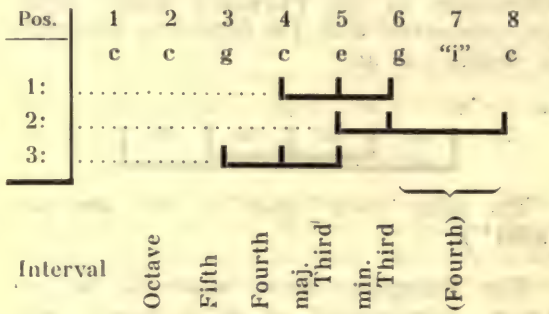
The division

by 2 and 4 gives the 1st position

- 5 — - 2nd —

- 3 — - 3rd —

with which discovery I supplement the axiom of Rameau of 1722; the 3 positions are reminiscent of the over-tones No. (with same distances):



b) that the c **minor triad** has these distances:

Position:												Sum:
1:	c	72	e♭	108	g	180	c'		1/5	3/10	1/2	360°
2:	e♭	108	g	180	c'	144	e♭'		1/4	5/12	1/3	432°
3:	g	180	c'	144	e♭'	216	g'		1/3	4/15	2/5	540°

The science of Harmony must undertake the explanation of these conditions in detail. In this present connection (acoustics) I shall just mention that Richter's "harmonic" minor mode and Rimski-Korssakow's "harmonic" major mode are both lying outside the natural system of scales and for this reason ought not to be made use of as basis to the Theory of Harmony — they ought to be replaced by the Phrygian minor mode, for instance. This last named mode has many advantages in its favour, namely:

The Phrygian minor mode — **the central mode** — is lying midway between the above mentioned 5 Greek scales, fragments of which are our pentatonic scales (see the tabels in articles V and VI).

The Phrygian minor mode is not only a double scale i. e. the two tetrachords are congruent — but these tetrachords are symmetric with the following distances in m:

low tetr.	c	d	e♭	f
high —	g	a	b♭	c′
difference:	152	111	152	= 415 m.

Relating to the difference between the vibration numbers (as degrees of arc) the figure in art. VI indicates, that these differences in the Phrygian minor mode are symmetric with the following distances in degrees of arc:

c		e♭		f		g		a		b♭		c
	72		48		60		60		48		72	

The lines "f—a" and "e♭—b♭" are parallel[35]).

The Phrygian minor mode is, as we know, constructed of 2 R-chords viz: c, e♭, g, b♭, each of which is forming a symmetrical square, inserted in a circle, with arcs: 72, 108, 108 and 72°; comp. art. V and the fifth-proportional tetrachords in Doric c minor:

c 0° —— d♭ 24° —— e♭ 72° —— f 120°
Difference...24..........48..........48

g 180° —— a♭ 216° —— b♭ 288° —— c′ 360°
Difference...36..........72..........72

with both Third and Seventh pure, namely fifth-proportional ($24 + 12 = 36$ and $48 + 24 = 72°$).

Further: the Phrygian minor mode is melodious, ascending and descending, — as a lovely melody; I feel tempted to name this mode: "The fairest rose of the garden!" — while the harmonic modes are only to be likened to artificial flowers, scentless, or, as Helmholtz expresses it: "A mixed mode, as a compromise between different kinds of claims"[36]).

All this makes one think that time must now be ripe for reforming the Theory of Harmony by discarding the harmonic minor mode as basis and replace it with, let us say, the Phrygian minor mode:

c	d	e♭	f	g	a	b♭	c′
—	152					848	—

"the central mode",

which comes to us straight from the hands of nature.

At all events, I propose the following diagram:
"chords of the Seventh as basis of the scales",
indicating the proportionate relation between vibration numbers differences.

Telegraph. letter, col. 33—34	Chords of the Seventh	Precedence					Positions 1st.	2nd.	3th.	4th.
R	small minor	1	c	e♭	g	b♭	. 2, 3, 3. .	. . 3, 3, 2 .	. . . 3, 2, 4	1, 2, 3 . . *)
K	large major	2	c	e	g	b	2, 2, 3 . . .	. 2, 3, 1 . .	. . 3, 1, 4 .	. . . 1, 4, 4
D	small —	3	c	e	g	b♭	. . 5, 5, 6.	. . . 5, 6, 4	3, 2, 5 . . **)	. 2, 5, 5 . .
G	augmented	4	c	e	g♯	b	4, 5, 5 . . .	. 5, 5, 2 . .	. . 5, 2, 8 .	. . . 1, 4, 5
U	diminished	5	c	e♭	g♭	b♭	5, 6, 9 . . .	. 6, 9, 5 . .	. . 9, 5, 10 .	. . . 5, 10, 12
W	large minor	6	c	e♭	g	b	8, 12, 15 . .	. 12, 15, 5. .	. . 15, 5, 16 .	. . . 5, 16, 24

*) Explanation:	b♭	c′	e♭′	g′	
	648	720	864	1080°	sum:
Differance		72	144	216	= 432°
abbreviated		1	2	3	6 × 72°

**) Explanation:	d+	f+	g	b	
	405	486	540	675°	sum:
Difference		81	54	135	= 270°
abbreviated		3	2	5	= 10 × 27°

RESUMÉ.

Preface.

Explanation of "the result": the 5 fundamental Laws of the Acoustics, corresponding to Keppler's 3 astronomical Laws.

Introduction.

Article I. As international terminology is proposed the English letters "a **b** c d e f g" for the white digitals of the piano, instead of the German and Scandinavian **h** for b, and the Latin "la, si, do (ut), re, mi, fa, sol", — and at the same time cis for c♯, ces for c♭, bis for French si♯, bes for si♭, as in Holland.

Part I. The Intervals.

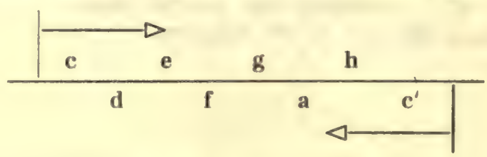
Articles II and III. In the Pythagorean C major we find 4 false (discordant) tones of which only three were corrected during the Renaissance period while $d = {}^{10}/_{9}$ and $d+ = {}^{9}/_{8}$ were used at random in the place of d. The Danish school-master Hans Mikkelsen Ravn, called Corvinus (1610—63), is mentioning both these tonic values as co-ordinate, in his book: "Logistica Harmonica"; later on preference is given to $^{9}/_{8}$; Rameau in the year 1722 is seen to have used "$d = {}^{10}/_{9}$" and in 1726 "$d = {}^{9}/_{8}$"[37]).

Helmholtz is beginning to vacillate: "b♭ and d are uncertain"[38]). It was not till 1882 that it was pointed out by the author of this present treatise[39]) that the normal tone $^{10}/_{9}$ belongs to C, D, F and A major, the comma tone $^{9}/_{8}$ to E♭, G and B♭ major a. s. f. I insert (1918) **perpendicular tonal zones** in the "Tone-Aggroupment" of the Japanese **Tanaka**, in order to demonstrate my system.

Article IV. The difference between over- and under-tones is congruent with the difference between the Fourth- and Fifth-circles, i. e. between ♯ and ♭. Both these kinds of natural tones produce both major and minor chords of the Seventh (Law 2).

Part II The Scales.

Article V. By playing through these triads and chords of the Seventh from Tonica (keynote) and from the octave **towards the centre** all sorts of scales are constructed (Law 3).



Article VI. By tripartition No. 1 of the tonal circle in C major we obtain the major triad F—a—c, analogic with the colours triparting the colour-circle being harmonic colours. Likewise in A major (Law 1):

$$\underset{600}{a} \text{———} \underset{800}{\mathbf{D}} \text{———} \underset{1000}{f\sharp} \text{———} \underset{1200^\circ}{a}$$
$$\underbrace{\quad}_{200}\ \underbrace{\quad}_{200}\ \underbrace{\quad}_{200}$$

By tripartition No. 2 of the Fourth, the low tetrachord, we obtain the major Second; — f. i.:

C major:	c 360 —	d 400 —	f 480°
Difference		40	80
G major:	g 540 —	a 600 —	c 720°
Difference		60	120
A major:	a 300 —	b÷ 333⅓ —	d 400°
Difference		33⅓	66⅔

that is c—d = $^{10}/_9$ in C major, my discovery in the year 1882[39]).

Article VII. If C as Tonica (keynote) is exchanged for any other tone belonging to C major (or their deviations) we get the **comma-tone**

g and g = d+ or $^3/_2 \times {}^3/_2 = {}^9/_8 \cdot 2$
b and g = f♯+ or $^{15}/_8 \times {}^3/_2 = {}^{45}/_{32} \cdot 2$

The 5 pentatonic scales are fragments of 5 ancient Greek scales.

Part III. The Musical Practise.

Article VIII. The temperament of 19 degrees I suggest introduced at Music Schools, Colleges, Academies etc. for the demonstration of a **more exquisite intonation** than the piano of 12 tones is able to give; to mention an example: there is difference of only 7 m between the normal tones d♯♯ 270 m and e♭ 263 m, or between f♯ 474 and g♭♭ 467, or between a♯♯ 855 and b♭ 848 m.

The axis of the minor Thirds "f♯, a, c, e♭, g♭", contains the purest tones in the temperament of 19 degrees.

Article IX. The Bohemian Dr. Otokar Hostinsky (1847—1910) admits in 1879 that the minor mode tonality is tending towards the major mode; "is showing a kind of a modulated drawing towards a kindred species of major mode"[40]). Consequently: "Minor" is the child of "Major" and is hankering back to its parent. This is **Monism** — the very soul of modern natural science:

c—e♭ = minor
A♭—c—e♭ = major.

Article X. When the theorists of the Renaissance gave up the ecclesiastical modes of the Church of Rome they were feeling their way, with faltering steps, when looking for new scales to replace the discarded ones. At the present time voices have been raised in favour of constructing new scales, to increase the number of scales. My suggestion in this present treatise is, by all means to discard the harmonic minor mode as basis to the Theory of Harmony. I suggest its being replaced by f. i. **the Phrygian minor** mode — the central mode — which is lying mid-way between the border-scales: Lydian major and Doric minor mode.

Complementary diagram
to the axiom of Rameau of 1722
about the invertion of chords.

Telegraph. letter, col.33—34	Triads	Precedence		Positions 1st.	Positions 2nd.	Positions 3th.
N	major***)	1	c e g c′	. 1, 1, 2. . .	. . 1, 2, 2	1, 1, 1 . . *)
A	minor	2	c e♭ g c′	2, 3, 5. . **)	. 3, 5, 4 .	. . 5, 4, 6
J	diminished	3	c e♭ g♭ c′	. . 5, 6, 14 .	3, 7, 5 . .	. 7, 5, 6 .
M	augmented	4	c e g♯ c′	4, 5, 7 . . .	. 5, 7, 8 .	. . 7, 8, 10

					equal difference:
*) Explanation:	c 360	f 480	a 600	c′ 720°	120°
or:	f 480	b♭♭÷ 640	d′ 800	f′ 960°	160°
or:	a 600	d′ 800	f♯′ 1000	a′ 1200°	200°
Difference abbr.		1	1	1	1

**) Explanation:	d 400	f 480	a 600	d′ 800°	sum:
Difference		80	120	200	= 400°
abbreviated		2	3	5	=10×40°

***) Position	the first sector	the rest bisected (halved)
1	$^1/_2$: g—c′	c—e—g
2	$^1/_6$: c—e♭	e♭—a♭—c′
3	$^1/_8$; c—f	f—a—c
		1. 1.

NOTES.

Column

Preface:

Article I.

11 1) Helmholtz: "Tonempfindungen" 1863; Tyndall: "Sound" 1867; Ellis: "Sensations of tone" 1875; Stumpf: "Tonpsychologie" 1883; the Dane, Prof. Alfred Lehmann (1858—1921): "Grundzüge der Psychophysiologie", Leipzig, 1912.

14 2) Compare Grove: "Dictionary of Music and Musicians", Volume I., page 20 ... "for the sake of uniformity".

Part I:

Article II.

15 3) Compare A. Hammerich: "Mediæval Musical Relics of Denmark", Copenhagen 1912; and G. Skjerne: "Plutarc's Dialogue about Music", Copenhagen 1909.

16-17 4) P. Heegaard: "Popular Astronomy", Copenhagen 1911, page 27.

19 5) A. Hammerich: "Studies in Icelandic Music", Copenhagen 1900; and: Hjalmar Thuren (1874—1912): "On The Eskimo Music", in informations about Greenland, XL, Copenhagen 1911, page 17, when mentioning the predilection of the Swiss peasants for false Thirds, Fourths and Sevenths.

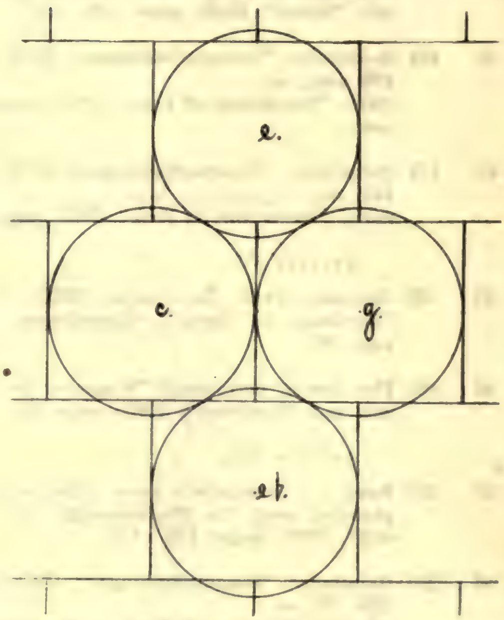
Article III.

19 6) Compare Jonquière's "Grundriss der musikalischen Akustik". Leipzig 1898, pages 93—100 and 154.

22 7) Hermann L. F. Helmholtz: "Tonempfindungen", Augsburg, edition 1913, pages 532—36 or:
Ellis: "Sensations of Tone", 1912, pages 329—331;
Hugo Riemann's "Musiklexikon", Leipzig, ed. 1916, page 889;
Riemann: "Akustik", page 13; and
Jonquière: "Akustik", page 36, note. —
C. E. Naumann (1832—1910) has contributed towards the building up of the Hauptmann-system.

Column

23 8) Jonquière's "Akustik", pages 38 and 155. The sqares must be shorter, than the are long by about $^1/_7$ for constructing regular hexagons, as in art. IX, as the shortest line from the centre of a cirele to the hexagon side is about $^1/_7$ shorter than this (= Radius).



Dr. Möhring has in 1855 suggested d = $^{10}/_9$; but he evidently did not carry through his suggestion.

Article IV.

30 9) Riemann's "Musiklexikon", ed. 1916, page 492.

31 10) Panum and Behrend: "Illus. History of Music", Copenhagen 1905, vol. I, pages 449—51;
Helmholtz: "Tonemp.", 1913, page 380.

31 11) Margaret Watts-Hughes: "The Eidophone voice figures", London 1904.

Column **Part II.**

Article V.

33 12) Compare f. i. A. Paulsen (1833—1907): "Natural Forces", Copenhagen, vol. I, 1874, page 348; also edition 1891, page 381.

36 13) Used, nevertheless, by Beethoven and Chopin, see Panum and Behrend: "History of Music", 1905, I, page 13.

36 14) Rimski-Korssakow: "Praktisches Lehrbuch der Harmonie", by Withol, Steinberg & Hans Schmidt, Leipzig, 1913, pages 6 and 56, — and
Arnold Schönberg: "Harmonielehre", Leipzig, Vienna, 1913, pages 34, 115 and 125.

36 15) A. Hammerich: "Illus. catalogue of the Museum for Musical History", Copenhagen 1909 II, page 47, and
Erik Eggen's essay in the Danish periodical: "Music", 1919, pages 149—152.

42 16) Helmholtz: "Tonempfindungen", 1913, page 498 note, or:
Ellis: "Sensations of Tone", 1912, page 308 note.

42 17) Helmholtz: "Tonempfindungen", 1913, page 441; or;
Ellis: "Sensations of Tone", 1912, page 269.

Article VI.

43 18) Compare Ferd. Barmwater (1857—1918): "Textbook in Optic", Copenhagen 1916, page 76.

46 19) The Danish periodical: "Magazine for Physics and Chemistry", 1887, page 100.

Article VII.

50 20) Read f. i. Jonquière's vain attemps of explaining away the Phenomenon, in "Akustik", 1898, pages 132—137.

56 21) Helmholtz: "Tonempfindungen", 1913, pages 428—31, or:
Ellis: "Sensations of Tone", pages 258—61, further
Hjalmar Thuren: "Old Folksongs of the Faroe Islands": Copenhagen 1908, pages 193—203.

61 22) Hammerich: "Catalogue" 1919, No. 545.

62 23) Jonquière: "Akustik", pages 146—154.

64 24) Compare The Perception of Tones by the Arabs.

64 25) The Preparing of Negatives with various Chemicals.

64 26) Helmholtz: "Sensation of Sound" versus Ewald; "Sonorous Figures".

Column **Part III.**

Article VIII.

65 27) The Temperament of 36 degrees, used by the Organist G. D. Berlin, was an Equal Temperament, see "The Publications of the Trondhiem Society", III, Copenhagen 1765, pages 542 and 562

66 28) Elsasz's "Universal Clavicymbal" with 19 tones to the octave has been seen before the year 1600 at Carl Luyton's, Organist To The Court, Prague, by M. Prætorius; compare Prætorius: "Syntagma Musicum", IV, 1619, pages 63—64.

A Harmonium with 19 tones to the octave, constructed about the year 1845 by P. S. Munck of Rosensköld, Professor at Lund, Sweden, is to be seen at the Stockholm Museum.

F. W. Opelt (1794—1863) has also suggested, in; "Allgem. Theorie der Musik", 1852, the introduction of the Temperament of 19 degrees (equal temp.), comp. Jonquière "Akustik", page 116.

Further the Norwegian Erik Eggen has been agitating for the reinstatement of the 19 degr. temperament, in: "Seen and Heard" (Syn og Sagn), Christiania 1911, and in: "Music", 1920, page 112; Dr. P. S. Wedell has done the same in 1914, and has constructed a Harmonium with 19 and 31 tones to the octave.

66 29) Compare Idelsohn: "Die Maqamen (Scales) der arabischen Musik", in: "Sammelbände der internationalen Musikgesellschaft", 1913 —14, pages 1—63.

66 30) Vicentino's "Clavicymbal" (Archicembalo); a specimen from 1606, a "Clavicymbalum omnitonans" of Otto de Transuntini from Venice, with 31 tones is to be seen at the Bologne Museum, advised by Augul Hammerich; and the French priest, father M. Mersenne, speaks in 1636 of a piano of 31 tones, constructed by himself, see Riemann: "Akustik", 1914, pages 47—52. Wedell has also advocated the temp. of 31 degrees in "Music", 1917, page 61 and 98; 1918. page 165, and 1920, pag. 110 and 137.
A Harmonium wit 53 tones to the octave is constructed by Bosanquet, London, 1875.

Article IX.

75 31) Helmholtz: "Tonempfindungen", 1913, page 452 or:
Ellis: "Sensations of Tone", page 276.

76 32) Helmholtz: "Tonempfindungen", 1913, pages 353—55, or:
Ellis: "Sensations of Tone", pages 215—17: also: Jonquière: "Akustik", page 321.

Column

76 33) Helmholtz, page 355, or:
Ellis', pages 215—17.

76 34) Jonquière: "Akustik", pages 144—45 and 316—17.

Article X.

87 35) The sentence in Jonquière's "Akustik", page 60, line 2—4, is explained by the word: "gewöhnt" (accustomed) page 68, line 13.

88 36) Helmholtz: "Tonempfindungen, 1913, page 498, notes, 586, notes, and 587; or:
Ellis; "Sensations of Tone", 1912, page 308, (notes), 365 and 365 (notes).

Resumé:

89 37) In "Heptacordum Danicum", Copenhagen 1646, last part, pages 12 and 14, compare to M. Mersenne: "Harmonicorum Instrumentorum, 4 libri, Paris, 1636, page 5, and: Descartes: "Musicæ Compendium", Amsterdam 1656, page 23; also
Rameau: "Code de Musique Pratique", Paris 1760, page 218, notes.

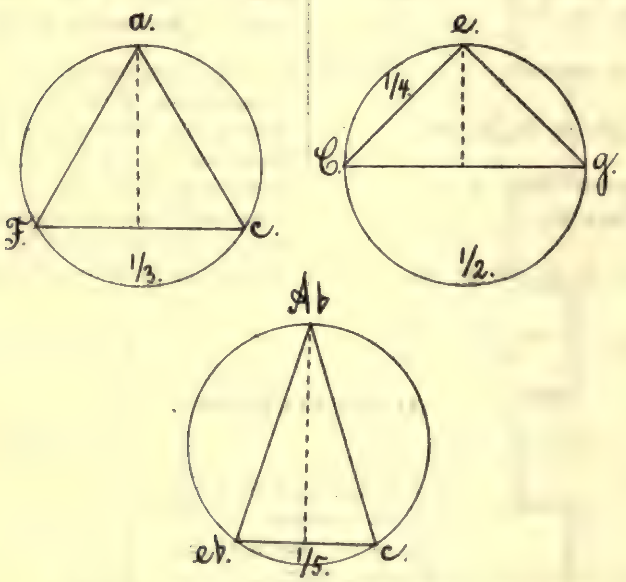
Column

89 38) Helmholtz: "Tonempfindungen", 1913, page 452, or:
Ellis: "Sensations of Tone", 1912, page 276.

49,89 39) Th. Kornerup in the Danish periodical "Magazine for Physics and Chemistry", 1882, pages 289—302.

92 40) Riemann: "Akustik", page 91;
Jonquière; "Akustik", pages 144—45 and 316—17.

The axiom of Rameau of 1722
completed 1922 (see colm. 86 and 92),
F, C and A♭ major triads in 3th, 1st and 2nd position,
(3 isosceles triangles):



Explaination of complementary differences (law 5):

	or:
d♭ 24° + b 315° = 339°	d 40° + b♭ 288° = 328°
Sliding of b 1/15 · 315 = 21°	Sliding of b♭ 1/9 · 288 = 32°
The Octave 360°	The Octave 360°

SOME PROFESSIONAL TERMS.

ABBREVIATIONS.

a. s. f., and so forth, etc.
f. i., for instance.
i. e., id est, that is to say.
ob., obiit, dead.
a. C., after Christ.
b. C., before —
col., column.

Annexure I.

Aggroupment of tones
in 3 tonal zones.

The zone of the left: Comma-intervals with ÷		The middle zone: Normal intervals.				The zone to the right; Comma-intervals with +	
2. c♯♯÷ 100.	13. g♯♯÷ 685.	5. d♯♯ 270.	16. a♯♯ 855.	8. e♯♯ 440.	19. b♯♯ 25.	11. f♯♯♯+ 610.	3. c♯♯♯+ 195.
7. e♯÷ 363.	18. b♯÷ 948.	10. f♯♯ 533.	2. c♯♯ 118.	13. g♯♯ 703.	5. d♯♯+ 288.	16. a♯♯+ 873.	
1. c♯÷ 41.	12. g♯÷ 626.	4. d♯ 211.	15. a♯ 796.	7. e♯ 381.	18. b♯ 966.	10. f♯♯+ 551.	2. c♯♯+ 136.
6. e÷ 304.	17. b÷ 889.	9. f♯ 474.	1. c♯ 59.	12. g♯ 644.	4. d♯+ 229.	15. a♯+ 814.	
0. c÷ 982.	11. g÷ 567.	3. d 152.	14. a 737.	6. e 322.	17. b 907.	9. f♯+ 492.	1. c♯+ 77.
5. e♭÷ 245.	16. b♭÷ 830.	8. f 415.	0. c 0.	11. g 585.	3. d+ 170.	14. a+ 755.	
18. c♭÷ 923.	10. g♭÷ 508.	2. d♭ 93.	13. a♭ 678.	5. e♭ 263.	16. b♭ 848.	8. f+ 433.	0. c+ 18.
4. e♭♭÷ 186.	15. b♭♭÷ 771,	7. f♭ 356.	18. c♭ 941.	10. g♭ 526.	2. d♭+ 111.	13. a♭+ 696.	
17. c♭♭÷ 864.	9. g♭♭÷ 449.	1. d♭♭ 34.	12. a♭♭ 619.	4. e♭♭ 204.	15. b♭♭ 789.	7. f♭+ 374.	18. c♭+ 959.
3. e♭♭♭÷ 127.	14. b♭♭♭÷ 712.	6. f♭♭ 297.	17. c♭♭ 882.	9. g♭♭ 467.	1. d♭♭+ 52.	12. a♭♭+ 637.	
÷		Normal intervals.					+

The figure before the letter indicates the number in the temperament of 19 degrees, about equal to number of "t" = 52,$_{63}$ millioctaves.

The figure underneath the letter indicates millioctaves.
F. i. "g" equal 11 × 52,$_{63}$ = 579, about 585 m.

Annexure II.

65 tones in millioctaves, apportioned in 19 groups.

Pythagorean system	Temperament of 12 degrees	The **pure** tones of the Third system	Temperament of 19 degrees	Temperament of 31 degrees	Inversion number	The Third system: Name	Nume-rator	Denomi-nator	Falsity, too small: Pythag. system	Temperament 12 degr.	Temperament 19 degr.	Temperament 31 degr.	Falsity, too large: Pythag. system	Temperament 12 degr.	Temperament 19 degr.	Temperament 31 degr.	Difference between temperaments 12 and 19 degrees ÷	+
0	0	0	0	0	35	c	1	1										
÷20	—	34	53	32	34	d♭♭	128	125	54	34		2			19		53	
—	—	52	—	—	33	d♭♭+	648	625	72	52		20			1		53	
95	83	59	—	65	32	c♯	25	24			6		36	24		6		30
75	—	93	105	97	31	d♭	16	15	18	10					12	4	22	
—	—	111	—	—	30	d♭+	27	25	36	28	6	14					22	
189	167	118	—	129	29	c♯♯	625	576			13		71	49		11		62
170	—	**152**	158	161	26	**d**	10	9					18	15	6	9		9
—	—	170	—	—	25	d+	9	8	0	3	12	9	0					9
150	—	186	211	194	24	e♭♭÷	256	225	36	19					25	8	44	
—	—	204	—	—	23	e♭♭	144	125	54	37		10			7		44	
265	250	211	—	226	22	d♯	125	108			0		54	39	0	15		39
—	—	229	—	—	21	d♯+	75	64			18	3	36	21				39
245	—	245	263	258	20	e♭÷	32	27	0				0	5	18	13	13	
—	—	263	—	—	19	e♭	6	5	18	13	0	5			0		13	
359	333	270	—	290	18	d♯♯	3125	2592			7		89	63		20		70
—	—	288	—	—	17	d♯♯+	625	512			25		71	45		2		70
226	250	297	316	—	16	f♭♭	768	625	71	47		7			19		66	
340	333	304	—	323	15	e÷	100	81					36	29	12	19		17
226	250	315	—	290	14	f♭♭+	3888	3125	89	65		25			1		66	
340	333	**322**	—	323	13	**e**	5	4			6		18	11		1		17
320	—	356	368	355	12	f♭	32	25	36	23		1			12		35	
435	417	363	—	387	11	e♯÷	625	486					72	54	5	24		49
320	333	374	—	355	10	f♭+	162	125	54	41	6	19					35	
435	417	381	—	387	9	e♯	125	96			13		54	36		6		49
415	—	**415**	421	419	8	**f**	4	3	0				0	2	6	4	4	
529	500	422	—	452	7	e♯♯÷	15625	11664			1		107	78		30		79
415	417	433	—	419	6	f+	27	20	18	16	12	14					4	
529	500	440	—	452	5	e♯♯	3125	2304			19		89	60		12		79
396	417	449	474	—	4	g♭♭÷	512	375	53	32					25	3	57	
—	—	467	—	—	3	g♭♭	864	625	71	50		15			7		57	
510	500	474	—	484	2	f♯	25	18			0		36	26	0	10		26
—	—	492	—	—	1	f♯+	45	32			18	8	18	8				26

490	500	508	526	516	1	g♭÷	64	45	18	8					18	8	26	
—	—	526	—	—	2	g♭	36	25	36	26	0	10			0		26	
604	583	533	—	548	3	f♯♯	625	432			7		71	50		15		57
—	—	551	—	—	4	f♯♯+	375	256			25	3	53	32				57
585	—	567	579	581	6	g÷	40	27					18	16	12	14		4
—	—	**585**	—	—	8	**g**	3	2	0	2	6	4	0					4
565	—	619	632	613	9	a♭♭	192	125	54	36		6			13		49	
680	667	626	—	645	10	g♯÷	125	81					54	41	6	19		35
565	583	637	—	613	11	a♭♭+	972	625	72	54	5	24					49	
680	667	644	—	645	12	g♯	25	16			12		36	23		1		35
660	—	678	684	677	13	a♭	8	5	18	11		1			6		17	
774	750	685	—	710	14	g♯♯÷	3125	1944			1		89	65		25		66
660	667	696	—	677	15	a♭+	81	50	36	29	12	19					17	
774	750	703	—	710	16	g♯♯	625	384			19		71	47		7		66
755	—	**737**	737	742	19	**a**	5	3			0		18	13	0	5		13
—	—	755	—	—	20	a+	27	16	0	5	18	13	0					13
735	—	771	789	774	21	b♭♭÷	128	75	36	21					18	3	39	
—	—	789	—	—	22	b♭♭	216	125	54	39	0	15			0		39	
850	833	796	—	806	23	a♯	125	72			7		54	37		10		44
—	—	807	—	—	—	"i"	7	4										
—	—	814	—	—	24	a♯+	225	128			25	8	36	19				44
830	—	830	842	839	25	b♭÷	16	9	0				0	3	12	9	9	
—	—	848	—	—	26	b♭	9	5	18	15	6	9					9	
944	917	855	—	871	27	a♯♯	3125	1728			13		89	62		16		75
—	—	873	—	—	28	a♯♯+	1875	1024			31	2	71	44				75
811	833	882	895	—	29	c♭♭	1125	625	71	49		11			13		62	
925	917	889	—	903	30	b÷	50	27					36	28	6	14		22
—	—	**907**	—	—	31	**b**	15	8			12	4	18	10				22
905	—	941	947	935	32	c♭	48	25	36	24		6			6		30	
1020	1000	948	—	968	33	b♯÷	625	324			1		72	52		20		53
—	—	966	—	—	34	b♯	125	64			19		54	34		2		53
1000	—	**1000**	1000	1000	35	**c**	1	1										
1114	1083	1007	—	1032	36	b♯♯÷	15625	7776			7		107	76		25		83
—	—	1025	—	—	37	b♯♯	3125	1536			25		89	58		7		83
34.892	34.666	34.180	34.052	34.290	+								1841	1275	285	397		1574
		Pure tones			÷	The Third system.			1129	789	413	287					960	
									too small.				too large					

Proof: 34.180 + 285 ÷ 413 = 34.052 m (temp. 19 degrees),
or: 34.666 ÷ 34.052 = 1574 ÷ 960 = 614 m.

Annexure III.

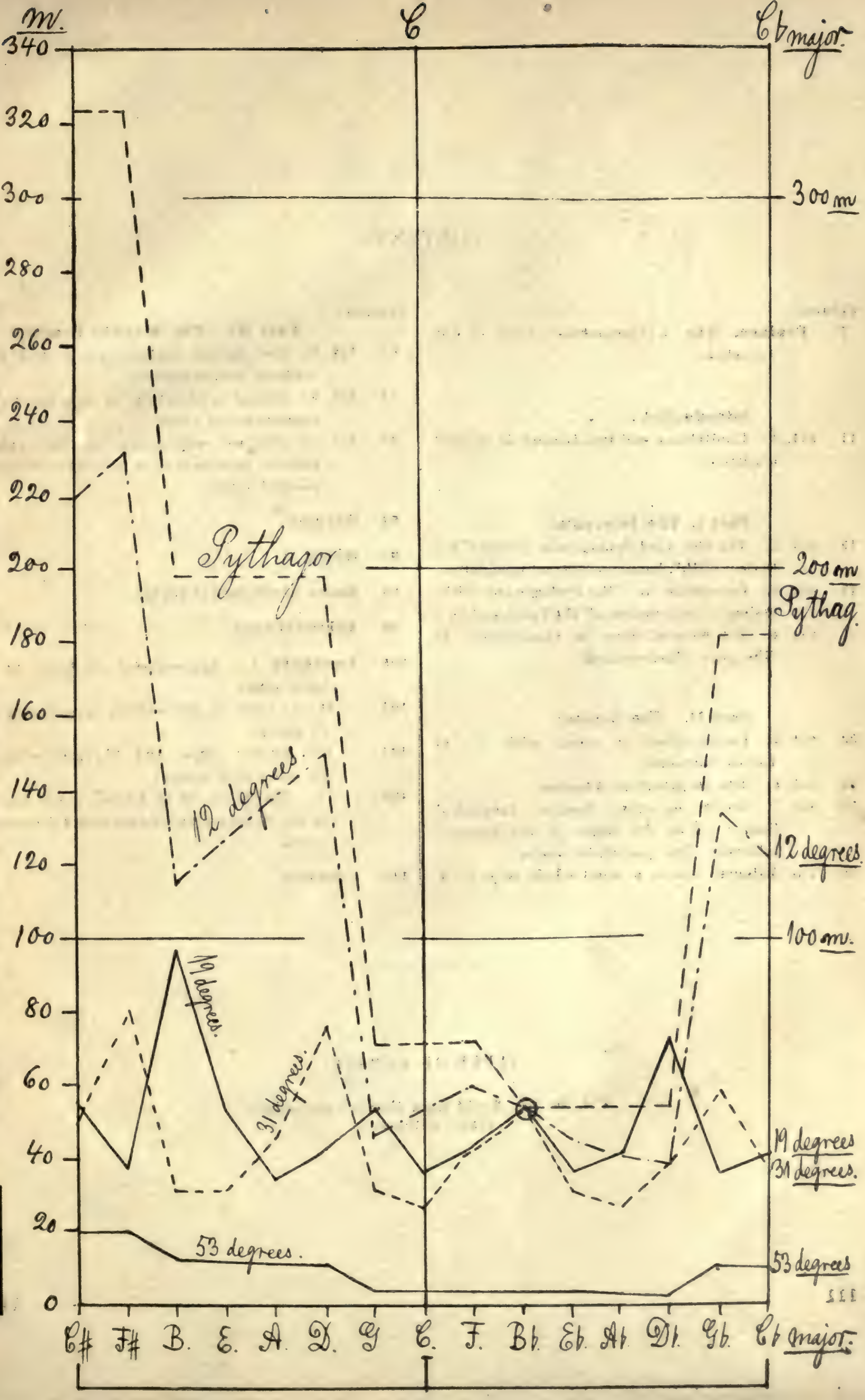
Lydian major and Phrygian minor.

30 scales (210 spaces):

The modes:	Number of ♯ or ♭	I	II	III	IV	V	VI	VII	Number of comma-tones:	Pythag. system	12 degr.	**19 degr.**	31 degr.	53 degr.
		The degrees of the scale:								Distance of falsity by the Temperament of				
Major	7♯	C♯	d♯	e♯	f♯	g♯	a♯	b♯	—	**324**	219	57	50	21
Circle of Fifth	6	F♯	g♯÷	a♯	b÷	c♯	d♯	e♯	2÷	324	**231**	38	**80**	20
	5	B	c♯	d♯+	e	f♯+	g♯	a♯+	3+	198	1.6	**97**	31	13
	4	E	f♯	g♯	a	b	c♯	d♯+	1+	198	128	54	30	12
	3	A	b÷	c♯	**d**	e	f♯	g♯	1÷	198	140	36	46	11
	2	**D**	e÷	f♯	g÷	a	b÷	c♯	3÷	198	151	42	77	11
	1	G	a	b	c	d+	e	f♯+	2+	72	47	54	31	4
	—	C	**d**	e	f	g	a	b	—	72	53	36	27	4
Circle of Fourth	1♭	F	g÷	a	b♭÷	c	**d**	e	2÷	72	60	42	42	4
	2	B♭	c	d+	e♭	f+	g	a+	3+	**54**	**54**	**54**	**54**	4
	3	E♭	f	g	a♭	b♭	c	d+	1+	54	46	36	32	4
	4	A♭	b♭÷	c	d♭	e♭	f	g	1÷	54	41	42	27	**3**
	5	D♭	e♭÷	f	g♭÷	a♭	b♭÷	c	3÷	54	39	72	39	**3**
	6	G♭	a♭	b♭	c♭	d♭+	e♭	f+	2+	180	133	36	59	11
	7	C♭	d♭	e♭	f♭	g♭	a♭	b♭	—	180	122	42	36	10
Minor	7♯	d♯	e♯÷	f♯	g♯÷	a♯	b♯÷	c♯	3÷	**378**	**273**	25	**104**	24
Circle of Fifth	6	g♯	a♯	b	c♯	d♯+	e♯	f♯+	2+	252	159	**86**	38	17
	5	c♯	d♯	e	f♯	g♯	a♯	b	—	252	170	43	47	15
	4	f♯	g♯÷	a	b÷	c♯	d♯	e	2÷	252	182	24	70	14
	3	b	c♯	d+	e	f♯+	g♯	a+	3+	126	84	84	42	7
	2	e	f♯	g	a	b	c♯	d+	1+	126	89	42	39	7
	1	a	b÷	c	**d**	e	f♯	g	1÷	126	95	24	43	7
	—	**d**	e÷	f	g÷	a	b÷	c	3÷	126	103	42	65	7
Circle of Fourth	1♭	g	a	b♭	c	d+	e	f+	2+	72	60	42	42	5
	2	c	**d**	e♭	f	g	a	b♭	—	72	**60**	**24**	**36**	5
	3	f	g÷	a♭	b♭÷	c	**d**	e♭	2÷	72	60	42	42	**4**
	4	b♭	c	d♭+	e♭	f+	g	a♭+	3+	126	103	42	65	8
	5	e♭	f	g♭	a♭	b♭	c	d♭+	1+	126	95	24	43	8
	6	a♭	b♭÷	c♭	d♭	e♭	f	g♭	1÷	126	89	42	39	7
	7	d♭	e♭÷	f♭	g♭÷	a♭	b♭÷	c♭	3÷	126	84	84	42	6
Number of comma-tones		—	13÷	4+	9÷	8+	5÷	12+	24+ 27÷ 51	4590	3286	**1408 NB.**	1418	276

Example, d♯ minor:

Pyth.		54	72	36	54	54	72	36	378	
Temp.	12	39	54	26	41	37	52	24	273	
	19	—	5	—	6	7	1	6	25	NB.
	31	15	24	10	19	10	20	6	104	
	41	9	3	11	8	9	3	10	53	
	53	3	5	2	3	4	5	2	24	



Annexure IV. The falsity of 15 Lydian major scales in Pythagorean system and various temperaments.

CONTENTS.

TABLE OF ERRATA:

Col. 39, line 9—10 from bottom: anti-minor
read: anti-major.

The Reform of the Calender.

The proposal of Thorvald Kornerup, published in "Scandinavian Astronomical Review", Copenhagen, October 1918, and in "Popular Astronomy", Northfield, Minnesota, U.S.A., Novbr. 1920, is the following:

1. March, June, September and December each 31 days, the other 8 months each 30 days, the 4 quarters of the year are equal: $4 \times 91 = 364$ days.

2. The 365th day is kept seperated from the date and the days of the week and placed between December 31 and January 1.

3. The intercalary day, which necessarily will appear each 4th year, is likewise kept without daily and weekly indication and placed between December 32 and January 1. December will consequently in each 4th year contain 33 days.

4. Easter Sunday shall always be appointed on April 7, as I take it for granted that the normal calendar is so begun that January 1 is a Sunday, that is celebrated as a holy day all over the world. Whitsunday will thus occur on May 27.

According to Kornerup's proposal the Sundays will always be coincident with the following dates:

1- 8-15-22-29—Jan., April, July, Oct.
6-13-20-27 —Feb., May, Aug., Nov.
4-11-18-25 —March, June, Sept., Dec.

The 1st in each month will thus always be:

A Sunday in Jan., April, July, Oct.
A Tuesday in Feb., May, Aug., Nov.
A Thursday in March, June, Sept., Dec.