NASA TECHNICAL NOTE

NASA TN D-5239

NASA TN D-5239

THEORY OF THE CORRECTION OF CELESTIAL OBSERVATIONS MADE FOR SPACE NAVIGATION OR TRAINING

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JANUARY 1970

1. Report No. NASA TN D-5239	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle THEORY OF THE CORRECTION OF CELESTIAL OBSERVATIONS MADE FOR SPACE NAVIGATION OR TRAINING		5. Report Date January 1970
		6. Performing Organization Code
7. Author(s) Burnett L. Gadeberg and Kenneth C. White		8. Performing Organization Report No. A-2449
9. Performing Organization Name and Address NASA Ames Research Center Moffett Field, Calif. 94035		10. Work Unit No. 125-17-05-01-00-21
		11. Contract or Grant No.
		13. Type of Report and Period Covered TECHNICAL NOTE
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		
		14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

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17. Key Words (Suggested by Author(s)) Astronomy Space Navigation Libration Refraction	18. Distribution Statement Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	48	$3.00

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SUMMARY

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INTRODUCTION

One of the requirements of a space navigation system is that celestial observations (measurements) of some kind be provided periodically, from which the state vector of the vehicle may be updated or determined directly.

The types of measurements that may be classified as celestial observations are limited by standard usage in astronomy to angular measurements between celestial bodies. The body on which the observer is situated is not included, except when consideration is given to sextant horizon-altitude measurements. In general, however, we will be considering celestial observations as angular measurements between various combinations of stars and bodies of the solar system, all situated some distance from the observer.

Four types of observations are considered in this discussion. They are the measurements between: two stars; a star and the limb of a body displaying an extended disk; a star and some permanent marking on the surface, commonly known as a landmark; or two limbs of the same body or two different bodies. Any of these types of observations may be made with one setting of a sextant or two settings of a theodolite. All four types of observations are of value. To provide position information for space navigation, an observation must include at least one body within the solar system; however, star-star observations are very useful for instrument calibration and for the training of observers. Measurements made between a star and the limb of an extended disk are useful to provide position information. Observations made between a star and a landmark also provide position information, but they may be more accurate when the destination body is close by. Sights between the two limbs of the same body are useful in providing distance information from the body when its apparent diameter is large.

Such measurements have associated errors, which may be classified as either random or systematic. Accurate utilization of the data requires that these errors be compensated as precisely as possible. In the case of random errors, this compensation can be accomplished only by probabilistic means such as a weighted least squares (ref. 1) or a sequential data-processing method in the form of a Kalman filter (ref. 2). Studies describing such optimum compensation for random errors ignore the systematic errors on the premise that they are known and, therefore, can be corrected. However, the proper correction of systematic errors is not trivial and requires detailed attention to the theory.

There has been much previous work, largely by astronomers, on the systematic errors in celestial observations (refs. 3 and 4). This work, however, has been directed to satisfying particular needs, and does not constitute a complete theory for the problems peculiar to space navigation. One purpose of this report is to develop such a theory insofar as it is possible at this time, and to present mathematical techniques suitable for application to specific space-navigation problems. In some cases, astronomers' previous work can be used directly, or with only minor modifications. Other cases require additional analysis, provided herein. Where there is insufficient knowledge for the development of precise correction formulas, the requirements of the correction theory are outlined in general terms.

For most cases in space (position) navigation, the angle desired as an input to the system is the "geometric" angle as seen by the observer, where "geometric" angle means the instantaneous angle between the centers of the masses of the two bodies unaffected by distance, the speed of light, or the velocity of the observer. Since all the bodies of the universe are in a state of dynamical motion, and the angles between them are constantly changing for any observer, a measure of the time of observation is an essential factor.

Even under ideal circumstances, an observer rarely can measure the desired angle. His own velocity causes aberration of the light, the apparent position of the body differs from the true position and ephemeris position due to the finite time of transmission of the light, the light rays may have been refracted by an atmosphere or a window, or the point on the body that really concerns him, the center of mass, is unmarked. There are also systematic errors associated directly with the use of optical instruments in making the measurements, and similar errors introduced by the human operator of the instrument. This report provides definitive descriptions of these errors and outlines some of the techniques whereby corrections may be applied.

The results of this work have been used to develop a series of computer programs for reducing sights made on the surface of the Earth, in a high-flying airplane, or in a near-Earth satellite. These have been developed for the IBM 7094 digital computer, in FORTRAN IV language, and they correct the sights for aberration, refraction, librations, parallax, or changes of epochs. They have been used extensively in connection with studies made at Ames Research Center to evaluate the sextant-observer combination (refs. 5-7).

ANALYSIS OF CELESTIAL OBSERVATION CORRECTIONS

General Procedure for Processing an Observation

When observations are made for navigation purposes, the usual procedure for processing the observation data is to form a "residual," which is the difference between the actually observed quantity and the estimate of this quantity based on the best estimate of the observer's state (i.e., his position and velocity in space) at the time of the observation. The residual is then used, with appropriate weighting, to improve the estimate of the observer's state. When deterministic corrections are to be applied to the observation, the procedure consists of the following steps:

(1) Obtain the estimated state of the observer (assumed given or computable).

(2) Obtain reference data for the sighting from star catalog and ephemeris, or both, etc.

(3) Using (1) and (2), and the appropriate correction formulas, compute the corrected estimate of the line-of-sight vector.

(4) From (3), estimate the observation and form the residual by subtracting this from the actual observation (corrected for instrument error).

Classification of Corrections

The corrections to be applied to celestial observations may be classified as follows:

(1) Corrections due to finite velocity of light (aberration),

(2) Corrections dependent on position of the observer (parallax),

(3) Corrections dependent on time of observation (proper motion, ephemeris time, Earth precession and nutation, and lunar and planetary librations),

(4) Corrections due to medium of transmission (refraction),

(5) Corrections for instrument read-out errors (index, arc, and filter), and

(6) Miscellaneous corrections (irradiance, limb effects, semidiameter, and personal observer effects).

Of the six classes of observations, the first two, aberration and parallax, plus the fifth, instrument error, have been well defined by the astronomers. The only improvement here is to cast the equation for stellar aberration in vector form. The majority of the effects in (3) have been well documented, but this report gives, for the first time, the correct vector matrix procedure for computing the geometrical lunar librations. Refraction through the atmosphere, part of (4), has been well defined. Refraction through an arbitrary window is developed here. The elements of (6) are still only poorly understood, and we will comment on what is known about them.

Table I lists, for all of these types of corrections, the maximum correction magnitude to be expected for the four principal types of celestial observations (star-star, star-limb and occulations, star-landmark, and limb-limb). It must be strictly understood that the values given are applicable only as qualified. Under different circumstances, corrections could be of vastly different magnitude than indicated. For example, local parallax is quoted as 3400", which is an approximate maximum value for an observer on the Earth sighting the Moon. But for other positions of the observer, the parallax may vary from 0° to 180°.

Table I also provides information on the circumstances that affect the requirement for corrections. The applicability of the various corrections depends on: (1) the type of observation taken, (2) the type of mission, or more properly the mission profile variables at the time, and (3) the accuracy requirements of the sight. The first of these conditions may be readily determined from table I. For example, for a star-star sight made from space, the table shows immediately that planetary aberration, local parallax, ephemeris time, Earth orientation, lunar librations, irradiance, limb effects, and semidiameter may all be omitted from consideration. Conditions (2) and (3) then determine the applicability of the remaining corrections, none of which may be safely omitted unless their size is known to be bounded within values that may be considered negligible for the specific case in question.

A detailed analysis of each of the corrections and methods for applying them are given in the following sections.

Corrections Due to the Finite Velocity of Light (Aberration)

Aberration is the error in the apparent direction of the line of sight of an illuminated body due to the vector combination of the velocities of light and the observer relative to the inertial coordinate system. The direction of the observed light is rotated in the direction of the observer's relative velocity vector.

In the field of astronomy, aberration effects are classified in three subdivisions: stellar (also known as annual), diurnal, and planetary. Stellar aberration is that component of aberration due to the velocity of the Earth in its orbit about the Sun (the observer assumed to be on the Earth), and diurnal aberration the component due to the differential velocity imparted to the observer by the rotation of the Earth on its axis. Planetary aberration is the term applied to the aberration phenomenon that exists in the sighting of relatively near bodies whose motion must be taken into consideration along with the motion of the observer.

TABLE I.- MAXIMUM ANGULAR CORRECTIONS DUE TO VARIOUS SOURCES FOR CONDITIONS SPECIFIED

		Star-star	Star-limb and occultation	Star-landmark	Limb-limb
Aberration	Stellar †	20″.4 for ⊕	20″.4 for ⊕	20″.4 for ⊕	N
	Planetary † ǂ	I	0″.67 for ⊕ ☽	0″.67 for ⊕ ☽	N
Parallax	Local ǂ	N	3400″ for ⊕ ☽	3400″ for ⊕ ☽	31″ for ⊕ ☽
	Stellar	0″.78	0″.78	0″.78	I
Proper motion		10″.25/yr	10″.25/yr	10″.25/yr	I
Ephemeris time ǂ		N	18″.1 for ⊕ ☽	18″.1 for ⊕ ☽	N
Earth orientation	Precession	N	50″.27/yr	50″.27/yr	N
	Nutation	N	N ▽	N ▽	N
Lunar librations	Geometrical ǂ	I	See irregularities	112″.5 for ⊕	See irregularities
	Physical ǂ	I	See irregularities	0″.15 for ⊕	See irregularities
Refraction	Atmospheric ◇	1860″ for ⊕	1860″ for ⊕	1860″ for ⊕	4″ for ⊕
	Window	*	*	*	*
Instrument	Index	Δ	Δ	Δ	Δ
	Arc	Δ	Δ	Δ	Δ
	Filter	Δ	Δ	Δ	Δ
Irradiance		I	[16″]	[?]	[32″]
Limb effects	Atmospheric	I	[?]	N	[?]
	Irregularities ǂ	I	[9″ for ☽]	I	[18″ for ☽]
Semidiameter ǂ		I	930″ for ⊕ ☽	I	1860″ for ⊕ ☽
Personal equation ⌂		20″	20″	20″	20″

⊕ Earth
☽ Moon
* Magnitude depends upon window - can be computed from theory provided in this report
▭ Unknown, needs more research
† Depends upon observer's velocity
ǂ Depends upon observer's distance
Δ Depends upon instrument
N Negligible, less than 0″.1
I Inapplicable
◇ Horizon observation from the Earth
⌂ As much as 20 sec of arc has been recorded for the personal equation for a highly trained observer. For an unskilled observer, the personal equation may be much higher.
▽ The maximum coefficient in the expansions for nutation is 9″.21. This has a negligible effect when correcting for atmospheric refraction, but may require consideration if observations are made of a near-Earth satellite.

For our purposes, diurnal aberration will be computed simultaneously with the stellar aberration by simply using the total heliocentric velocity vector of the observer to compute the required correction. Note that for an observer on some other body (e.g., in a spacecraft) we simply use, in like manner, the total inertial heliocentric velocity vector of the observer's platform.

Equations for computing the effect of stellar aberration are developed in appendix A. Either of two equations may be used, depending on the exactness desired. The rigorous equation is

$$\hat{s} = \frac{\left(\frac{V}{c}\right)\hat{V} + \hat{S}}{\left|\left(\frac{V}{c}\right)\hat{V} + \hat{S}\right|}$$

where

$\hat{s}$ observed unit vector of the star

$\hat{S}$ true heliocentric unit vector of the star

$\hat{V}$ heliocentric unit velocity vector of the observer

V velocity of the observer

c velocity of light

If this is not convenient to use, due to the absolute magnitude appearing in the denominator, a very close approximation is given by

$$\hat{s} = \left[1 - \left(\frac{V}{c}\right)(\hat{V} \cdot \hat{S})\right]\hat{S} + \left(\frac{V}{c}\right)\hat{V}$$

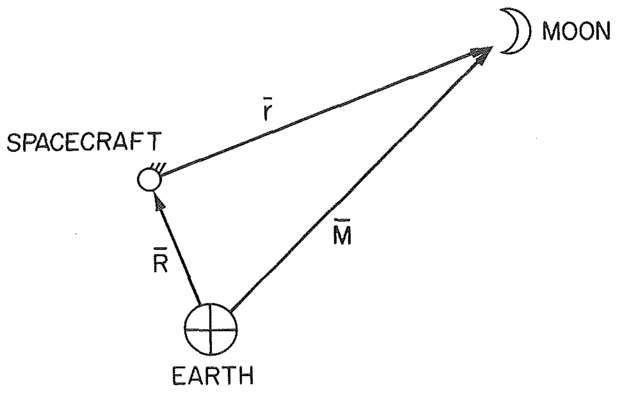
Planetary aberration needs to be treated in a different manner. Whereas the star catalog gives the direction of the unit vector to the star as it would be seen by a heliocentric observer, that is, the "true" line of sight, a planetary ephemeris gives the geometrical direction of the body as it actually exists from a point at the center of the Earth (moving in its orbit about the Sun). A correction must be made because of the time required for light to travel from the observed body to the observer, as well as a correction for the aberration. The correction for planetary aberration may be made by simply reducing the time of observation by the light time and taking the position of the body from the ephemeris at this reduced time. Thus, if an observation had been made of a planet at $09^h30^m20^s$, the ephemeris position of the planet at this time would be used to determine the time required for the light to travel from the planet to the observer. If this light time were 10^s, the time of observation would then be reduced to $09^h30^m10^s$. The ephemeris position of the planet at this reduced time would then give the correct position of the planet as seen by the observer at the time of observation. A more complete discussion of the phenomenon is given in appendix A.

Corrections Dependent on Position of Observer (Parallax)

Star catalog and ephemeris data give the line of sight to celestial bodies from certain reference points (e.g., center of the Sun for star data), but generally the actual observer is at some other point. This displacement of the observer from the reference point results in a change in direction of the incident light rays from the observed body - the phenomenon called parallax - and requires a correction to be applied to the data.

The parallax corrections may be divided in two classes: one representing the parallax of relatively close objects, such as those within the solar system, and another representing stellar parallax.

Local parallax (within the solar system) may, under certain circumstances, become very large. This type of parallax, however, may be handled very simply by making a linear vector transformation and thus presents no real obstacle to the determination of apparent position of any known object. As an example, from sketch (a), it may be seen that the vector position of the Moon may be corrected for parallax by writing the vector equation



Sketch (a)

$$\bar{r} = \bar{M} - \bar{R} \tag{1}$$

Stellar parallax is in all cases very small. The maximum that has been recorded to date is only 0.78 second of arc, and that is for a relatively inconspicuous star of the eleventh magnitude known as Proxima Centauri. The parallaxes of only a relatively few stars have been measured, due to the very remote positions of the stars in general, and most of these are of the order of 0.1 second of arc or less. Thus, it is a simple matter to avoid the use of stars with sensible parallaxes. If it is necessary to use a star with a high parallax, a linear vector transformation will account for the effect, as it does in the case of local parallax.

Corrections Dependent on Time of Observation

For economy in presentation, star catalog and ephemeris data are given only for certain reference times, and in using these data one must apply corrections that depend on the difference between the reference time and the time of the observation. These corrections are described below.

Proper motion.- Proper motion is defined as the yearly motion of a star normal to the line of sight. In most cases, proper motion is very small. The largest that has been recorded belongs to a tenth magnitude star and is 10.25 seconds of arc per year. Proper motion may be avoided by the judicious selection of stars for navigation purposes, the use of a star catalog of recent epoch; or, since the proper motions are cataloged for both right ascension and declination, simple corrections, linear with time, may be added to the reference stellar position. Possible nonlinear components of the motion need not be considered, for no proper motions have been measured to date with sufficient accuracy to indicate such effects.

Ephemeris time.- Ephemeris time is the smooth flowing measure of time implied by the dynamical physical equations of motion, and is based principally on orbital motions of the planets. All other measures of time commonly used in the field of astronomy, for example, universal time, are associated with the rotation of the Earth, which, for various and unknown reasons, displays some irregularities. Ephemeris time is computed, for our purposes, from

$$ET = UT + \Delta T \qquad (2)$$

where

ET ephemeris time

UT universal time (or Greenwich mean time)

ΔT time interval

The estimated value of ΔT, for a given year may be found in reference 8 (p. vii). The values of ΔT are essentially unpredictable but are only slowly varying from year to year. Final values for the past are determined from a large number of observations of the Sun, Moon, and planets. For the year 1900, the value of ΔT was -4 seconds of time and, for the year 1967, it is estimated to be 37 seconds of time.

Although universal time is the standard used for civil reckoning and most astronomical computations, attention is here directed to ephemeris time because most ephemerides of the bodies of the solar system use it as argument. For example, if the transformation from universal to ephemeris time is neglected, the position of the Moon, as seen from the Earth, will be in error by approximately 18 seconds of arc. If the sight has been made in the lunar orbital plane, the error will show up directly as an error in the measured angle.

Earth precession and nutation.- The orientation of the Earth (i.e., the equatorial coordinate system) changes with time and therefore is not necessarily the same at an observation time as at the reference time of the star catalog or ephemeris. The correct orientation may be important when corrections are being made for parallax and atmospheric refraction for an Earth-based observer. Also, data may be used that are expressed in coordinate systems based on the equatorial system for different epochs. In either case, corrections must be applied that amount to a transformation of coordinate systems.

The gyroscopic forces that produce the Earth orientation changes result in complex motions containing secular terms and periodic terms. The secular terms are easily represented by polynomial expansions in time, These are called precession, which has a maximum motion of approximately 50 seconds of arc per year. The aggregate of the periodic terms, which are of both long and short periods, are called nutation.

A simple matrix equation permits the precession transformation from an initial epoch of 1950.0 to a later epoch:

$$\hat{V} = \begin{bmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix} \hat{V}' \qquad (3)$$

where

$$\left.\begin{aligned} \hat{V}' &= \text{any vector in the original coordinate system} \\ X_x &= 1.00000000 - 0.00029697T^2 - 0.00000013T^3 \\ Y_x = -X_y &= -0.02234988T - 0.00000676T^2 - 0.00000221T^3 \\ Z_x = -X_z &= -0.00971711T + 0.00000207T^2 + 0.00000096T^3 \\ Y_y &= 1.00000000 - 0.00024976T^2 - 0.00000015T^3 \\ Y_z = Z_y &= - 0.00010859T^2 - 0.00000003T^3 \\ Z_z &= 1.00000000 - 0.00004721T^2 + 0.00000002T^3 \end{aligned}\right\} \qquad (4)$$

and T is the time interval between the two epochs expressed in Julian centuries of 36,525 days. Note that this matrix is orthogonal and that the reverse process may be obtained by using the transpose.

The effects of nutation may be incorporated into the computations by premultiplying the above matrix for precession by the matrix

$$\begin{bmatrix} 1 & -\Delta\psi \cos \varepsilon & -\Delta\psi \sin \varepsilon \\ \Delta\psi \cos \varepsilon & 1 & -\Delta\varepsilon \\ \Delta\psi \sin \varepsilon & \Delta\varepsilon & 1 \end{bmatrix} \qquad (5)$$

where

ε true obliquity of the ecliptic

$\Delta\psi$ nutation in longitude

$\Delta\varepsilon$ nutation in obliquity

and

$$\left.\begin{aligned} \varepsilon &= \varepsilon_o + \Delta\varepsilon \\ \varepsilon_o &= 23°27'08''.26 - 46''.845T - 0''.0059T^2 + 0''.00181T^3 \\ &\text{Epoch 1900 Jan 0.313} \end{aligned}\right\} \quad (6)$$

The two quantities $\Delta\psi$ and $\Delta\varepsilon$ must be tabulated values or computed from series expressions in which there are

$$\left.\begin{aligned} &\text{23 long-period terms} \\ &\text{46 short-period terms} \end{aligned}\right\} \text{ for } \Delta\psi$$

$$\left.\begin{aligned} &\text{16 long-period terms} \\ &\text{24 short-period terms} \end{aligned}\right\} \text{ for } \Delta\varepsilon$$

Under the circumstances, one should critically review the accuracy requirements before it is decided to include corrections for nutation, since the coefficient of the largest term in nutation is only 9.21 seconds of arc. Since this represents a distance of only about 900 feet on the surface of the Earth, any effect on the atmospheric refraction corrections would be negligible. Similarly, for observations between natural members of the solar system or the stars, the effect of this 900-foot displacement would be entirely negligible. However, when observing an artificial satellite close to the observer, the effect may be quite large, and then the inclusion of the effects of nutation in the corrections would be appropriate. Most of the coordinate transformations will be made in conjunction with refraction and parallax corrections for more distant bodies, so one may conclude that, in all probability, nutation may be neglected.

For more detailed discussions of precessions and nutation, one should consult references 8, 9, and 10 for the equations and constants presently in use, and reference 4 for a more detailed discussion of the theory and the development of the original constants.

Lunar and planetary librations.- If landmarks are to be used in the navigation process, their locations must be known. Consider the lunar landmarks first. Since the Moon rotates about its own axis in the same period that it revolves in its orbit about the Earth, on the average the same face is presented to an observer on the Earth at all times. For this reason, the practice in recording the locations of lunar landmarks has been to refer them to a coordinate system associated with the mean Earth-Moon line. The presentation of the Moon's face to the observer on the Earth is not exact, however, for three reasons. First, the Moon may be considered to rotate at a constant angular velocity about its axis, but its orbital angular velocity about the Earth is variable because of the eccentricity of the lunar orbit. This, combined with the inclination of the lunar equator to the orbital plane, causes an apparent oscillation in two dimensions about a mean position. These oscillations are known as geometrical librations since they are strictly a function of the Earth-Moon geometry existing at the time of observation.

Second, the figure of the Moon is that of a triaxial ellipsoid, which permits periodic rotational forces to be applied by the Earth and the Sun. These forces result in a complex pendulum-type oscillation known as physical librations. Third, an observer on the Earth is displaced in position, during a 12-hour period, by the diameter of the Earth. The resulting parallactic effect is known as diurnal libration. However, since diurnal libration is strictly a parallactic effect, it can be taken care of by the local parallax corrections.

The method of handling the libration correction is to first compute, separately for both the geometrical and the physical librations, the three selenocentric angles describing the motion, and add the corresponding angles together. The unit vector to the lunar landmark then may be transformed through the three angles and the right ascension and the declination of the lunar position into a unit vector expressed in the equatorial inertial coordinate system.

The equations used to compute the geometrical and physical librations of the Moon are relatively complex. Rather than introduce them at this time, it is expedient to show the complete development of the equations for the geometrical librations and to display them along with the final equations for the physical librations (as taken from ref. 9) in appendix B.

In principle, observations of planetary landmarks could require a reduction similar to that used for lunar librations, except that planetary landmarks locations generally would not be expressed in terms of an Earth-planet mean line-of-sight-coordinate system as are lunar landmarks. Rather, they would usually be given in a planetocentric coordinate system, the orientation of which, in the observer's coordinate system, would have to be known (or computed) at any specific observation time to determine the line-of-sight direction. However, situations could occur in connection with planetary landmarks sightings such that the condition of commensurability between the period of rotation and revolution exists (e.g., a synchronous satellite). Then a librational-type treatment would be reasonable for line-of-sight motions relative to a mean, and corrections similar to those for lunar librations would be required.

Corrections Due to Medium of Transmission (Refraction)

Atmospheric refraction.- Equations developed by astronomers for this purpose, such as those given in reference 11, are obtained by regression analysis techniques whereby empirical functional relationships are established between observed refraction and atmospheric variables. A major difficulty with such equations is that they are cumbersome to use. There is also the theoretical objection that, in reality, refraction depends on the total state of that portion of the atmosphere through which the light rays travel. However, it is not practical to observe the total atmosphere, nor is it necessary, since the corrections obtained from the empirical relationships between refraction and surface conditions fit the observed data well enough for most purposes.

For these reasons, a simplified approximation to the empirical relationships is believed adequate for general use. One simplified equation (ref. 12), developed by Professor Comstock, approximates the refraction to within 1 second of arc for all altitudes of the observed body from the zenith down to 15° above the horizon except for extreme states of the atmosphere. This equation is

$$r = \frac{983b}{460 + T} \text{tgz} \tag{7}$$

where

r refraction, seconds of arc

b barometric pressure, in. Hg

T atmospheric temperature at the observer, °F

z zenith distance of the observed body, degrees of arc

The refraction given by the value of r in the above equation lies in the vertical plane and apparently moves the observed body toward the zenith. This is due to the fact that the observer is situated at the center of the visible portion of the atmosphere, which has the shape of a planoconvex lens. Since any ray entering the eye of the observer must necessarily lie in a plane passing through the axis of rotation of this symmetry, the resulting refracted ray must also lie in the plane of symmetry. Thus, when making corrections for refraction, the ray must be transformed into the observer's topocentric coordinate system to determine the zenith distance of the observed body.

It should be noted that the observer's local altazimuth system is fixed in relation to the equatorial system of the Earth and therefore undergoes continuous precession and nutation along with the Earth. If an inertial system, or epoch, is used representing the equatorial coordinate system as of the beginning of the current year, the effects of precession and nutation on the refraction will be very small. The gyroscopic effects of the Earth can only affect the position of the observer's local coordinate system by approximately 50 seconds of arc in any one year, and since the refraction is a slowly varying function, except when observations are made close to the horizon, the effect will, in general, be negligible.

Window refraction.- The refraction of light rays occurring at the window of a space vehicle is a field that has not received scientific attention heretofore; consequently, although window refraction is amenable to mathematical analysis, it has not been formalized to the same extent as atmospheric refraction. In general, two problems exist with window refraction. First, the design of the spacecraft often dictates that the windows be of irregular outline and that the spacecraft be pressurized. The resulting irregular three-dimensional curve of the window does not lend itself to the normal geometrical optics analysis, which assumes optical surfaces of revolution.

Second, an iterative procedure is involved in the determination of the vector from the sighting instrument to the window and thence to the particular celestial body under observation. A method of handling these calculations is outlined in appendix C; it is necessary to complete some preliminary research on the window before the computations may be carried out. This preliminary work consists basically of an analysis and testing of the deflection of the window under a differential load such that the matrix of the deflection curve polynomial may be found. A knowledge of the differential pressure and the orientation of the spacecraft at the time of observation then permits the calculation of the refraction of the line of sight of any given celestial body which may have been observed.

As the equations and procedures for the reduction of the effects of window refraction are lengthy, their development and use is displayed in appendix C, which also compares the present theory with thin lens theory through the computation of a numerical example.

Corrections for Instrument Read-Out Errors

The sextant is used to make angular measurements between two celestial bodies, as needed for space navigation. The sextants of the past, and probably of the future, are light, easy to use, and amazingly accurate, for their size and weight, compared to other astronomical instruments. The best results are obtained when the instrument is carefully adjusted, calibrated, and placed in the hands of a careful, well-trained observer.

After the instrument is fully adjusted, three calibrations, or error corrections, remain to be made: index, arc, and filter. The index error is the reading of the instrument when both lines of sight are brought to bear on the same distant object. If the reading is not zero, an index error exists that must be applied to any sight obtained with the instrument. The arc error is the error made when the markings were scribed on the arc during its manufacture. It is determined by sighting various bodies of known angular displacement and comparing these values with those read from the arc of the instrument. Filter errors are introduced when filters, used to reduce the intensity of bright images, are introduced into one of the lines of sight. Most filters have a slight wedge angle that deviates any ray of light being transmitted through them. The error is determined by measuring the position of an object before and after the filter is placed in the line of sight. One of the fullest discussions on the use, calibration, and adjustment of the sextant is given in reference 13.

For accurate work the sextant must be very carefully calibrated. The index error must be determined sufficiently frequently that the magnitude, drift, and constancy are known within the limits of the precision required. With careful attention to observing technique, determination of the observer's personal equation, and calibration of the instrument, observations may be made to a precision greater than the least count of the instrument.

Miscellaneous Corrections

Irradiance.- Irradiance is a psychophysiological effect wherein the image of a bright body appears to be larger than its true diameter. It is apparently caused by an irregular refraction and scattering of light within the eye, which thereby stimulates the retina outside the immediate area covered by the illuminated image. This well-known effect was once thought to be caused by imperfect optics, as indicated in reference 13 (vol. II, p. 90), which was revised and last published in 1891. Although the phenomenon is now known to be principally psychophysiological in origin, very little is known about it and no satisfactory method has been devised to account for it at the present time.

In reference 14, published recently, the effect was partially studied for very bright extended disks. Here irradiance effects from 38 seconds of arc to over 2° were measured. Reference 9 mentions corrections made for irradiance varying, in different cases, from 1.5 to 2.5 seconds of arc. No reference or explanations are given for these corrections, and the writers suspect that they are empirical, introduced for the purpose of rectifying consistent errors of observation (which is entirely legitimate). One approach to reduce irradiance is to cut down the apparent intensity of the extended body with neutral filters. However, additional research is needed here to indicate the effects. On the other hand, unpublished data of filtered sextant observations of the Moon (reduced and corrected by the authors) display an irradiance effect of 15 seconds of arc. Thus it is apparent that the effects of irradiance, even for filtered observations, are large enough to seriously impair the required accuracy of celestial observations made for space navigation, and that insufficient data are available to correct for this effect.

Limb effects.- Limb effects are similar to irradiance in that they affect the apparent diameter of the observed body. There are two sources of these effects. The first is the effect of an atmosphere on the observed body, which may cause an apparent increase in the visible diameter of the body, such as the Earth, when viewed from space. Very little is actually known about the atmospheric haze effect, except that it seems to be variable, and probably is affected by the terrestrial weather conditions.

Second, the irregularities in the outline of the limb, such as the Moon, may cause either an apparent increase or decrease in the diameter of the body due to local topological features. These irregularities, due to the presence of mountains and valleys situated on the apparent limb, can only be conveniently accounted for, at the present time, when observations are made of the Moon by an observer in the vicinity of the Earth. They have been charted by Dr. C. B. Watts of the U.S. Naval Observatory (see ref. 15) for use with occultation measurements. These charts are also discussed in reference 9. For any other position of the observer, or for any body other than the Moon, this correction may not be considered since sufficient information is presently not available.

Semidiameter.- Since many of the observations made for space navigation or testing will be made to the limb of an extended, illuminated body, a correction for the semidiameter of the body must be applied to represent the observation to the center of the apparent disk. This may be accomplished by one of two methods. First, if the angular measure of the semidiameter is known for the observer's position, this may be directly added to or subtracted from the measured angle depending on whether the observation was made to the near limb or the far limb. Or, if parallax computations must be made, this correction may be incorporated directly into the linear vector transformation, which was discussed under the section on parallax corrections.

In addition to the correction for semidiameter, a small correction of -0.6 second of arc should be applied to the celestial latitude of the Moon to account for the misalinement of the center of mass and the center of figure of the Moon. Further reference may be made to reference 9 (p. 212) for this effect. For many situations, this last correction may be ignored because of its small size.

Personal equation.- One form of systematic error that has had but little thought given to it is the error known to astronomers as the "personal equation." The personal equation of an observer is that correction (error) which must be applied to his observations, after all other corrections have been made, to produce the true quantity that should have been observed. It may be observed as the mean residual error remaining after all other known corrections have been made. Personal equations may be quite sizable and consistent. In some cases, they have amounted to as much as 20 seconds of arc for a highly trained, very careful observer. Although the equation may be consistent, it generally varies over a period of time. It has been found to depend on, among other things, the type of instrument used, its size and position, the position of the observer's body, his degree of fatigue, and the type and velocity of the observed body. Any personal equation applied to a set of observations should have been determined in the recent past. It is obvious that a personal equation cannot be found when observing an unknown quantity. Thus, the personal equation should be determined during the training and retraining periods, and should be further updated by information obtained during the course of a mission. Further discussion on this subject may be found in references 12 and 13.

At this point it should also be remarked that the philosophy of the observer in making high precision observations is also extremely important. High precision observations require not only a highly accurate, well-calibrated instrument, but an observer of knowledge, training, and a philosophy that will not allow him to do anything but his careful best. Only with such an observer will it be possible to have confidence in his personal weighting of individual observations. Discussions on the philosophy of weighting individual observations may generally be found in texts considering the method of least squares (refs. 13 and 16). When a number of sophisticated observations are made, weighting should always be used because a careful observer always has a feeling for the relative merits of his own observations. If he does not, he cannot be considered an accomplished observer and should be trained and practiced with this in mind until he can.

PROCEDURES FOR APPLYING CORRECTIONS

Order of Making Corrections

The order of the application of the corrections is important. A judicious order will eliminate second-order effects, reduce the computations involved, and eliminate the need for iterations. The method of ordering found to give the least confusion may be called the method of "reverse ordering." Consider the path of the light ray as it travels from the illuminated body to the eye of the observer. Along the path the various phenomena under discussion are encountered. Note the order of encounter with these phenomena and then apply their corrections in reverse order. If a case arises where a decision cannot be made between the time priorities for two of the corrections, it is considered good practice to apply the largest correction first; with the above rule of application it is apparent that the instrument corrections, index, arc, and filter should always be applied to the instrument reading first. On the other hand, since they all affect the light ray almost simultaneously, they may be applied in any order.

Consider now the application of corrections to a sight made from the surface of the Earth between a star and the limb of the Moon. Tracing the ray from the star to the eye of the observer, the first phenomenon encountered by the ray is that of aberration as the light approaches the Earth, which is traveling through space with a velocity of the order of 18 miles per second. After this the ray is affected by refraction as it traverses the atmosphere to the observer. It then encounters the effects introduced by the sextant. The corrections are applied in the reverse order. The sextant corrections are first applied to the sextant reading. It is then appropriate to apply the corrections for refraction, which depend on the altitude angle as seen by the observer. To make these computations, one must use the known inertial coordinates of the star, the latitude and longitude of the observer, and the local sidereal time to transform the position of the star into the observer's local altazimuth coordinate system.

After the refraction corrections have been completed, the aberration corrections may be applied as indicated. The above corrections must be applied to both lines of sight when more than one body is observed. However, the corrections applied to the two lines of sight are not necessarily the same, since, as indicated in table I, corrections for planetary aberration, librations, irradiance, limb effects, and semidiameter are not applicable to a line of sight to a star.

A step-by-step outline follows of the recommended procedure for each of the corrections, in the order in which they appear in table I.

<u>Aberration</u>.- (stellar)

1. Compute the velocity vector of the observer in any convenient coordinate system.

2. Transform from that convenient coordinate system to the equatorial system of the desired epoch.

3. Compute the vector of the star as it is seen displaced due to aberration using equation (A1) or equation (A4).

Aberration.- (planetary)

1. Determine the distance of the local body from the fundamental center, using the recorded ephemeris.

2. Using the parallax correction (linear vector transformation) determine the distance between the observer and the local body.

3. Using the velocity of light, compute the time of passage of the light from the local body to the observer (light time).

4. Reduce the time of observation by the light time.

5. Using the reduced time of observation reenter the ephemeris and determine the apparent position of the local body.

Note: Before the ephemeris of the body may be entered, the time of observation and the reduced time of observation must be converted to ephemeris time (see section on ephemeris time).

Parallax.- (earth-based)

1. Compute the radius vector of the observer from the center of the Earth using the equation for an oblate spheroid.

2. For the vector of the observer in the geocentric equatorial system of date.

3. Precess the vector from the equatorial system of date to the equatorial system of the desired epoch (eq. (3)).

4. Make the linear vector transformation as required for parallax (eq. (1)).

Note: The computation of the radius vector of the observer from the center of the Earth must be made using the geocentric latitude of the observer.

The formation of the vector of the observer in geocentric equatorial system of date requires the computation of the local mean sidereal time. The use of mean sidereal time is compatible with the use of precession and the omission of nutation.

Parallax.- (non-earth-based)

1. Compute the position vector from the fundamental center in any convenient coordinae system.

2. Transform from the convenient coordinate system to the equatorial system of the desired epoch.

3. Make the linear vector transformation as required for parallax.

Ephemeris time.-

1. From the American Ephemeris and Nautical Almanac, select the ΔT correction for the year of the observation date.

2. Calculate ephemeris time from the universal time by equation (2): $ET = UT + \Delta T$.

3. Enter the ephemeris as required.

Note: The ΔT correction is given in the ephemeris only for the year of the publication and the past. If the correction is required for a future year, it will have to be estimated as best as possible by extrapolation of the values given.

Past values of the ΔT correction are given to 0.01 second of time, and present values are given only to the nearest second due to the fact that the correction can only be determined by observation. As a consequence, it is really only known after the fact.

Earth orientation.- (precession)

1. Compute and form the transformation matrix from the polynomial expansions in time for the elements (eq. (4)).

2. Transform the vector, using the matrix of equation (3) either from 1950.0 to date, or using the inverse, from date to 1950.0. (Both transformations will have to be used if the fundamental epoch of the computations is some date other than 1950.0.)

Note: The polynomials for the elements and the matrix have been set up for the transformation from the epoch 1950.0 to the time used as the independent variable in the polynomials.

The reverse of this procedure can be used to reverse the direction of the precession (transposed matrix).

The notation 1950.0 represents the beginning of the Besselian year, which is within a fraction of a day of January 1. It actually starts when the mean longitude of the Sun reaches exactly 280°.

The fundamental epoch of the precession matrix (eq. (3)) may be changed to any other epoch by suitably adjusting the coefficients of equation (4).

Libraions.- (geometrical and physical)

1. For the time of observation, compute the three libration angles using the matrix equations developed in appendix B for geometrical librations (eqs. (B5) and (B8)).

2. Compute and add the increments to the libration angles using the equations for physical librations (eq. (B9)).

3. Form the transformation matrix with the three libration angles (eq. (B10)).

4. Transform the selenographic vector of the landmark to the equatorial system of date.

5. Precess the vector from the equatorial system of date to the equatorial system of epoch (eqs. (4) and (3)).

Refraction.- (atmospheric)

1. Transform the star coordinates from the equatorial system of the desired epoch to the equatorial system of date using the precession matrix (eqs. (3) and (4)).

2. Transform from the equatorial system of date to the altazimuth system of the observer.

3. Compute and apply refraction to the altitude of star (eq. (7)).

4. Transform from the altazimuth system of the observer to the equatorial system of date.

5. Transform from the equatorial system of date back to the equatorial system of the desired epoch using the transpose of the precession matrix.

Note: The transformation between the equatorial system of date and the altazimuth system of the observer must be made with the geodetic latitude of the observer.

This transformation also requires the computation of the local mean sidereal time from an expansion in mean solar time, which is compatible with the use of precession and the omission of nutation.

The neglect of nutation produces only very small errors in the computed refraction.

Refraction.- (window)

1. Transform the observed star unit vector from the equatorial inertial coordinates to the window coordinates using equation (C1).

2. Assume any convenient value of the internal unit vector $\hat{V}$. (Values of α, β, and γ in eq. (C10).)

3. Compute the intersection of this vector with the plane of the undeflected window using equation (C11).

4. Set $V_1 = 0$, equation (C12).

5. Using equations (C13) to (C16), iterate to find the intersection of the ray with the deflected surface of the window.

6. Compute the normal to the surface using equation (C17).

7. Compute the refracted ray using equation (C18).

8. Repeat steps 3 to 7 for each refracting surface.

9. Using equation (C19), determine the angle between the free space refracted ray and the star vector determined in step 1.

10. If the angle determined in step 9 is too large, compute a new internal unit vector using equation (C20), including the gain factor computed from equation (C21).

11. Iterate the entire process from steps 3 through 10, until the angle determined in step 9 is small enough.

12. The final refracted ray is then coincident with the star ray and the proper initial internal ray vector has been determined.

Instrument.- (index, arc, filter)

1. The arc correction should be determined first from the calibration curve of the sextant. Since the arc correction is a function of the location on the arc at which the reading is made, the arc correction should always be taken from the calibration curve for the actual reading of the sextant at the time of observation, that is, the recorded angle.

2. The index correction and the arc correction may then be lumped together and applied to the sextant reading.

3. The filter correction, if applicable, may then be applied.

Note: It has been assumed here that the sextant has been properly adjusted for coplanarity of the principal plane of the sextant, the telescope, and the normal vectors of the horizon and index mirrors.

Irradiance.-

1. At the present time it is impossible to correct for irradiation effects when only limited observations are available.

Note: The best that one can do presently is to use as heavy a filter as possible and hope for the best. The only possible result that can be expected is reduced accuracy.

Limb effects.- (atmospheric, irregularities)

1. Atmospheric limb effects are very poorly known at the present time.

2. Irregularities of the limb are in a condition impossible to use at the present time, except for some of the irregularities near the limb of the Moon, as seen from the Earth.

Semidiameter.-

1. From the relative position vector of the observed body, determined as indicated under parallax corrections, and from the known linear radius of the body, compute the semidiameter of the body.

2. Add or subtract the above computed semidiameter from the measured angle according to whether the measurement has been made to the far limb or the near limb.

3. Or, if direct measurements have been made of the semidiameter, step 1 may be omitted.

CONCLUDING REMARKS

This report has presented new theories concerning window refraction and lunar geometrical librations. Further, we have indicated or collected in one location, and in an easily understood form, essentially all available information on the corrections for celestial observations, so that the knowledge required by the engineer for obtaining high precision observations is readily available. It has been pointed out that a careful analysis should be made of the mission profile to determine which of the many corrections may be safely omitted; some of the corrections require considerable computations and may amount to only a few seconds of arc in themselves, and may have an even smaller effect on the corrected observed angle.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., 94035, Feb. 17, 1969
125-17-05-01-00-21

APPENDIX A

DEVELOPMENT OF VECTOR EQUATIONS FOR THE ABERRATION OF LIGHT

Stellar Aberration

The aberration of a star is the apparent displacement of the unit vector from the observer to the star due to the combination of the velocity of the observer with the velocity of light. This combination is the simple linear vector summation of the two velocities, reduced to the unit vector form.

Since the velocity of light in free space is considered a constant, if the velocity vector of the observer were also constant, all the apparent positions of the stars would be displaced by different but constant angles from their true positions. Under these conditions a slightly erroneous but constant picture of the heavens would be obtained. Now the velocity of the Sun, or the solar system, is essentially constant, at least over very long periods of time, within the accuracy of our astronomical measurements. Consequently, all star catalogs quote star positions as though they were seen by an heliocentric observer moving with the velocity of the Sun. The important point to be remembered here is that the correction for aberration must be based on the heliocentric velocity of the observer.

Two vector equations will be developed for the aberration of light, one will be rigorous, and one will be slightly more convenient to use for machine computations. The geometry of the situation is displayed in figure 1. An observer, located at 0 and moving through space with the velocity vector $\overline{V}$, observes a star whose true direction (heliocentric direction) is given by the unit vector $\hat{S}$. The light from the star approaches the observer with the velocity vector $\overline{c}$ relative to fixed space. The velocity of the star light relative to the moving observer is given by $-\overline{s}$, and the star appears to the observer to be located in direction $\hat{s}$. Now, if we use the convention that any vector is composed of a magnitude and a unit vector, then it is apparent that

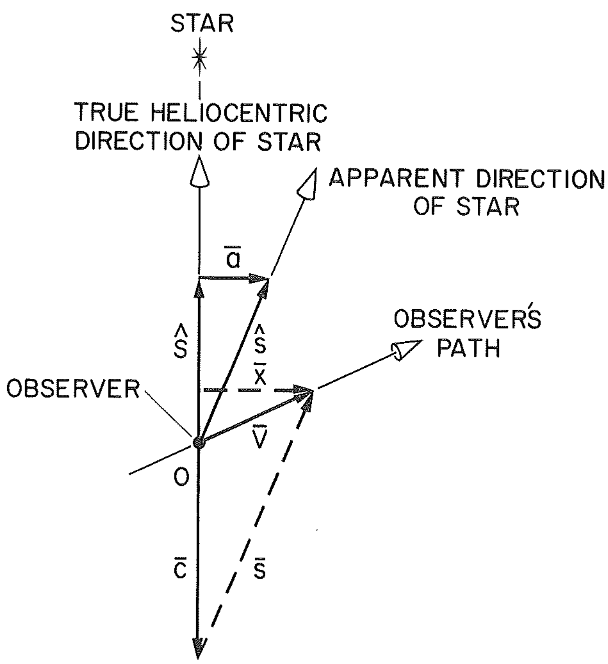


Figure 1.- The aberration of light due to the heliocentric velocity of the observer.

$$\overline{V} = V\hat{V}$$

$$\overline{c} = c\hat{c} = -c\hat{S}$$

Then we may write

$$\overline{s} = \overline{V} - \overline{c}$$

and

$$\hat{s} = \frac{\overline{V} - \overline{c}}{|\overline{V} - \overline{c}|} = \frac{V\hat{V} + c\hat{S}}{|V\hat{V} + c\hat{S}|}$$

or

$$\hat{s} = \frac{(V/c)\hat{V} + \hat{S}}{|(V/c)\hat{V} + \hat{S}|} \tag{A1}$$

This is the rigorous equation for stellar aberration.[1] It has the disadvantage that the magnitude of the vector must be evaluated for the denominator. A slightly more convenient equation may be derived from figure 1 with the help of two additional vectors. These auxiliary vectors are $\overline{a}$ and $\overline{x}$, both of which are normal to the direction to the star as given by $\hat{S}$. Then w may write

$$\hat{s} = \hat{S} + \overline{a} \tag{A2}$$

now

$$a << 1$$

since

$$\overline{a} \cong \frac{\overline{x}}{c} \tag{A3}$$

and

$$\overline{V} \cdot \hat{S} << c$$

Then we may also write

$$\overline{x} = \hat{S} \times (\overline{V} \times \hat{S})$$

which may be reduced to

$$\overline{x} = -V(\hat{V} \cdot \hat{S})\hat{S} + V\hat{V}$$

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[1]These equations have been developed in nonrelativistic mechanics and are thus accurate for velocities of the observer where $V/c << 1$. It has been kindly pointed out by Mr. William M. Adams, Jr., of Langley Research Center, that the correct equation for a relativistic observer would be

$$\hat{s} = \frac{\sqrt{1 - \frac{V^2}{c^2}}}{c\left(1 - \frac{\overline{V} \cdot \overline{c}}{c^2}\right)} \left[\frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\overline{V} - \overline{c} - \left(\frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} - 1\right)\frac{(\overline{V} \cdot \overline{c})\overline{V}}{V^2}\right]$$

This equation reduces to equation (A4) for nonrelativistic observers where $(V/c)^2$ and higher terms may be neglected.

Then, substituting into equation (A3),

$$\bar{a} = -\frac{V}{c}(\hat{V} \cdot \hat{S})\hat{S} + \frac{V}{c}\hat{V}$$

and then into equation (A2)

$$\hat{s} = \left[1 - \frac{V}{c}(\hat{V} \cdot \hat{S})\right]\hat{S} + \frac{V}{c}\hat{V} \qquad \text{(A4)}$$

Although equation (A4) is an approximation, it is a very good approximation, since the values of V are always very small compared to the velocity of light c. The value of $t_g^{-1}(V/c)$ for the Earth, which is known as the constant of aberration, is only 20.4 seconds of arc.

Planetary Aberration

When an observer views a planet, or any other body within the solar system, he sees the body displaced by stellar aberration just as he would when viewing a star. However, a fundamental difference exists in the method of correcting for this error (see ref. 17). This difference arises as a direct result of the fact that the position of the planet, as supplied by the ephemeris, is fundamentally different information than that supplied by a star catalog for the position of the star. A star catalog gives the direction from which the light arrives when it enters the eye of the heliocentric observer. The fact that the observer is considered to be heliocentric is not really germane to this immediate discussion, but the fact that the catalog supplies the direction from which the light is approaching is important. The basic ephemeris of a planet, on the other hand, gives the true geometric position of the planet in space at the time specified, and makes no allowances for the travel time of the light or any other phenomenon. Although this may give the impression that some complication would be involved in the computation of the correction, it will be shown that the apparent position of the planet may be taken directly from the ephemeris by the simple expedient of decrementing the time of observation by the travel time of the light from the planet to the observer.

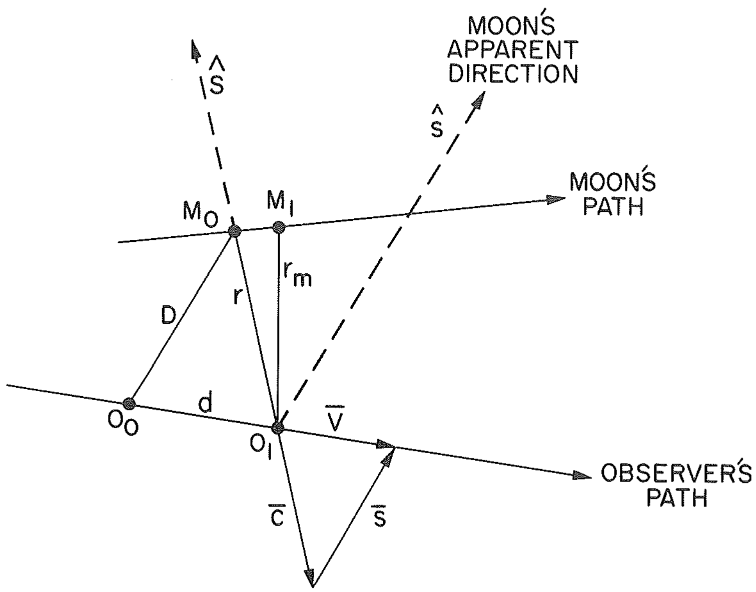
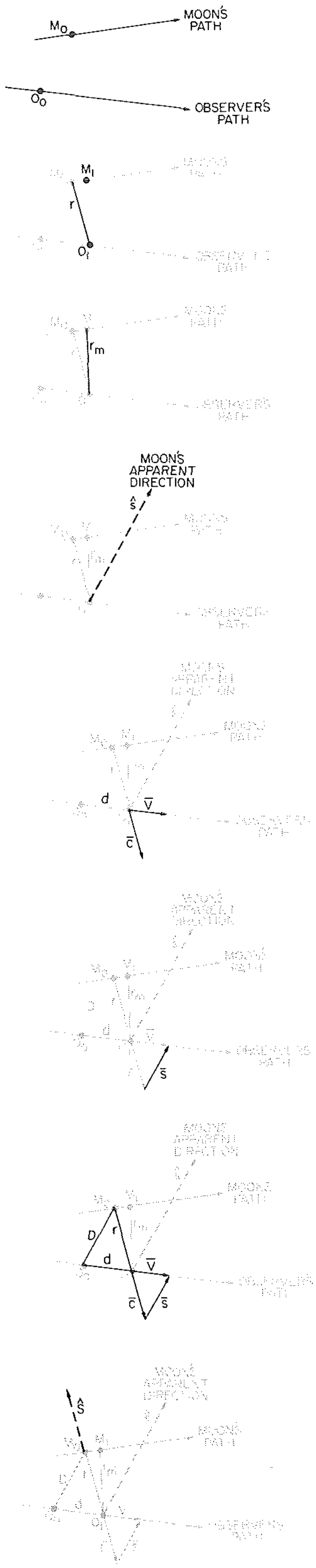


Figure 2.- Planetary and stellar aberration combined.

Figure 2 shows the combined effects of both stellar and planetary aberration. At the time the light leaves the body at M_O (say, the Moon) the observer is located O_O. When the light



reaches the observer at O_1, along the path r, the Moon has moved on to M_1. If at the time of observation we entered the ephemeris of the Moon it would indicate that the Moon was located at M_1 at the distance r_m from the observer. Now, the stellar aberration discussed above gives the observer the impression that the Moon is located in the direction of the unit vector $\hat{s}$. We assume that during the time of light passage from M_o to O_1 that the observer has moved with uniform velocity along d from O_o to O_1 and arrives O_1 with the velocity vector $\bar{V}$, and that the approaching light has the velocity vector $\bar{c}$. Then, as discussed above in conjunction with equation (A1), the body appears to be located in the direction $\hat{s}$, which is parallel to $\bar{s}$. Now it is apparent that $\bar{c}$ and $\bar{r}$ are colinear, as are $\bar{V}$ and $\bar{d}$. Then we may write, using Δt for the light time

$$\frac{r}{c} = \Delta t \ ; \qquad \frac{d}{V} = \Delta t$$

then, on elimination of Δt,

$$\frac{V}{c} = \frac{d}{r}$$

Thus, the two triangles are similar and the side D is parallel to the side $\bar{s}$. Now $\bar{s}$ or $\hat{s}$ is the apparent direction of the Moon, as seen by the observer, and D is the direction of the Moon from the observer at the time the light left the Moon. Consequently, the apparent direction of the Moon, at the time of observation, is the same as the true direction of the Moon at the time the light left the Moon. At the time of observation, the light has traveled the distance r in the time Δt. However, neither of these quantities is known. We do know the distance r_m from the ephemeris, and we may assume that

$$\Delta t = \frac{r_m}{c}$$

since the velocity of the Moon is very small compared to the velocity of light and consequently r and r_m are approximately equal. Thus, the apparent direction of the Moon at the time of observation may be found by entering the ephemeris at the time of observation, determining the distance to the Moon and the light time, and then reducing the time of observation by the light time and then reentering the ephemeris to determine the apparent direction of the Moon at the time of observation.

APPENDIX B

DEVELOPMENT OF EQUATIONS FOR LUNAR GEOMETRICAL LIBRATIONS

The rotation of the Moon about its axis may be separated into two components. The first of these is a steady rotation in a counterclockwise direction, as seen from the north, which is in the same direction as the revolution of the Moon about the Earth in its orbit. The period of this rotation is exactly equal to the sidereal period of the lunar orbit. This rotation, combined with the orbital motion, the inclination of the Moon's equator to the ecliptic, the position of the lunar node, and the inclination of the ecliptic to the Earth's equatorial plane gives rise to the geometrical librations. The second part of the lunar rotation is a very low amplitude oscillatory motion, which gives rise to the physical librations. It is the geometrical librations that are of concern here.

Several pertinent facts concerning the interrelationship between the rotation and revolution of the Moon are expressed in the three laws of Cassini. These three laws, which were originally empirical and later proved mathematically by Lagrange and Laplace, state that:

1. The Moon rotates eastward, on an axis fixed in its body, with a constant angular velocity, and the period of rotation is exactly the same as that of the lunar sidereal revolution about the Earth.

2. The inclination of the lunar equator to the ecliptic is a constant.

3. The poles of the lunar equator, the ecliptic, and the lunar orbit all lie in a great circle, in that order.

The rotation of the Moon about its polar axis is constant and the angle through which it has turned after a given period of time is a linear function of the time. Consequently, there is a direct relationship with the mean longitude of the lunar orbital motion. If the mean motion of the Moon and the true motion coincided, then exactly the same face of the Moon would be presented toward the Earth at all times. However, the osculating orbit of the Moon is eccentric, and it is also perturbed by the Sun and the planets. As a consequence, the difference between the true position and the mean position causes the Moon to appear to oscillate about the radius vector of the Moon from the Earth.

The investigation will now proceed along two parallel paths. In one we will develop the transformation matrix for any selenographic vector to the equatorial system via the laws of Cassini and the general geometry of the orbit. In the other we will develop a similar transformation matrix via the three libration angles and the true position of the Moon as given by its right ascension and declination. These will then be equated and a solution obtained for the three libration angles. If we were interested only in determining the vector of a lunar landmark in the equatorial system as disturbed by the geometrical librations, the first transformation would be sufficient. However,

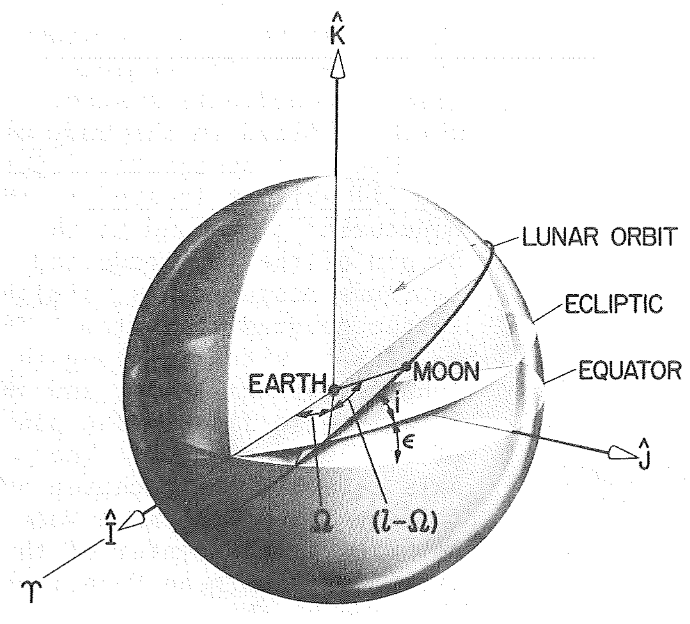


Figure 3.- Orbital geometry of the mean Moon relative to the Earth and the equatorial inertial coordinate system.

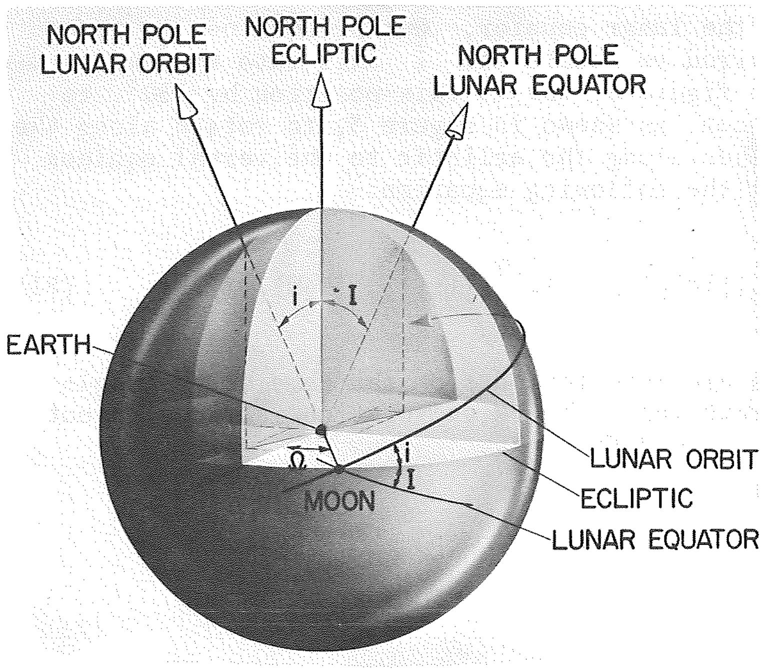


Figure 4.- Illustration of Cassini's Laws with the mean Moon at the orbital node.

the physical librations are given, very conveniently, as a function of time, in terms of the three libration angles. Therefore, it is appropriate to compute the angles for the geometrical librations so that those determined for the physical librations may be added directly to them. The transformation matrix of the landmark is then formed from the total libration angles.

The general geometry of the situation is shown in figure 3. The geocentric equatorial inertial coordinate system is shown as $(\hat{I}\hat{J}\hat{K})$, the obliquity of the ecliptic is ε, the longitude of the lunar orbital node is Ω, inclination of the lunar orbit to the ecliptic is i, the argument of latitude is $\ell - \Omega$. Note that ℓ, the mean longitude of the Moon, is measured from the vernal equinox ♈ along the ecliptic to the node and thence along the orbit to the position of the mean Moon (not the true Moon). The facts expressed by Cassini's third law are shown in figure 4 for the instant the Moon occupies the ascending node. The inclination of the lunar orbit relative to the ecliptic is i and that of the lunar equator is I. The poles of the three planes are coplanar, with that of the ecliptic occupying the central position. The coplanarity of the poles is, of course, not disturbed by the progress of the Moon in its orbit.

Some time after the Moon has passed the node it is situated as shown in figure 3. A close-up of the geometry at

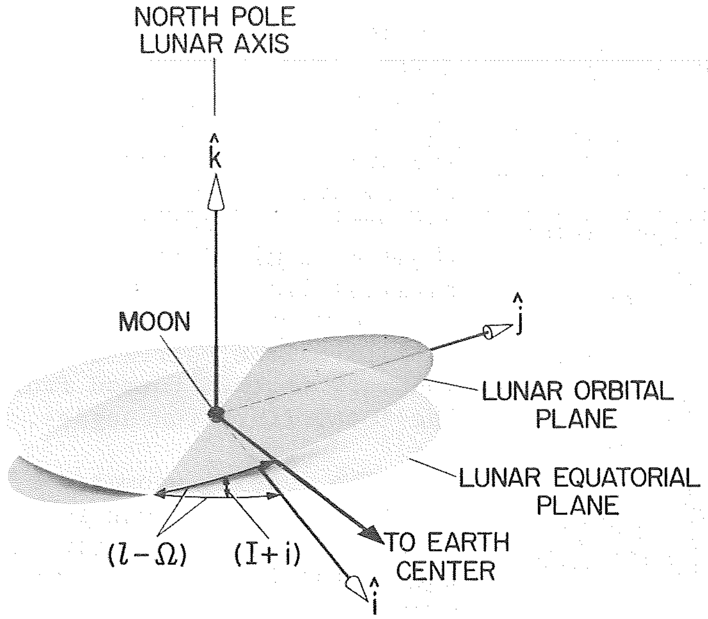


Figure 5.- Geometry between the mean lunar orbital equatorial planes and the terrestrial radius vector.

this lunar position is shown in figure 5. The selenographic coordinate system, which is fixed in the body of the Moon, is designated $(\hat{i}\hat{j}\hat{k})$. The $\hat{i}$ axis is located in the equatorial plane and is the origin of the longitude and latitude measurements, similar to the geographic system. The $\hat{k}$ axis is situated along the lunar rotational pole, and the $\hat{j}$ axis also lies in the plane of the lunar equator. The $\hat{i}$ axis is arbitrarily chosen as that radius of the mean Moon pointing to the center of the Earth when the mean Moon is at the node of the orbit. (Now, since the mean Moon and the true Moon coincide only at perigee and apogee, the $\hat{i}$ axis of the true Moon will point to the center of the Earth only when the true Moon is simultaneously located at the node, either ascending or descending, and at an apse, either perigee or apogee.) The radius vector between the Earth and the Moon (fig. 5) has rotated, in the orbital plane, from the node by the angle $\ell - \Omega$. The $\hat{i}$ axis has also rotated from the nodal line by the angle $\ell - \Omega$ but lies in the plane of the lunar equator. Our process now is to develop the transformations required to rotate the $\hat{i}$ axis into the radius vector to the Earth, as shown in figure 5, rotate this position by 180°, to point away from the Earth, and then, as shown in figure 3, to rotate along the orbital plane to the node, and then along the ecliptic to the vernal equinox. The transformations are given by the following equation:

$$\bar{x} = [\varepsilon_x][\Omega_z][i_x][(\ell-\Omega)_z][180_z][(\ell-\Omega)_z]^T[(I+i)_x][(\ell-\Omega)_z]\bar{x}' \qquad \text{(B1)}$$

The individual rotation matrices are here formed from the name of the angle and subscripted by the axis of rotation. The three rotation matrices, about the three axes, for the angle θ, are defined as:

$$[\theta_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$[\theta_y] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[\theta_z] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The symbol $\overline{x}'$ represents any vector in the selenographic coordinate system, while $\overline{x}$ is that same vector in the equatorial system. The first four matrices of equation (B1), in order from right to left, carry the $\hat{i}$ axis into the line of nodes (fig. 5), the equatorial plane into the orbital plane, the $\hat{i}$ axis into the radius vector, and then rotate it 180° so that it is now pointing directly away from the center of the Earth. The $\hat{i}$ axis is now alined with the radius vector from the Earth to the Moon (fig. 3), and the next four rotation matrices carry the $\hat{i}$ axis along the orbital plane to the node, rotate the orbital plane into the plane of the ecliptic, the $\hat{i}$ axis along the ecliptic to the vernal equinox, and the ecliptic into the plane of the Earth's equator.

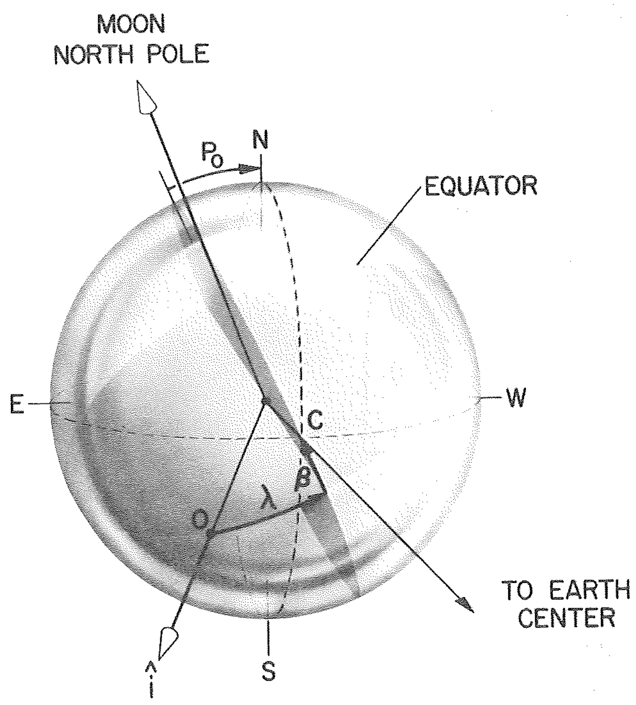
The equation may now be simplified by factoring the rotation matrix $[180_z]$ out to the left.[1] In so doing, all rotations about the z axis are unaffected and any rotations about the x or y axis are transposed. The result and the further steps in the simplifications are given below.

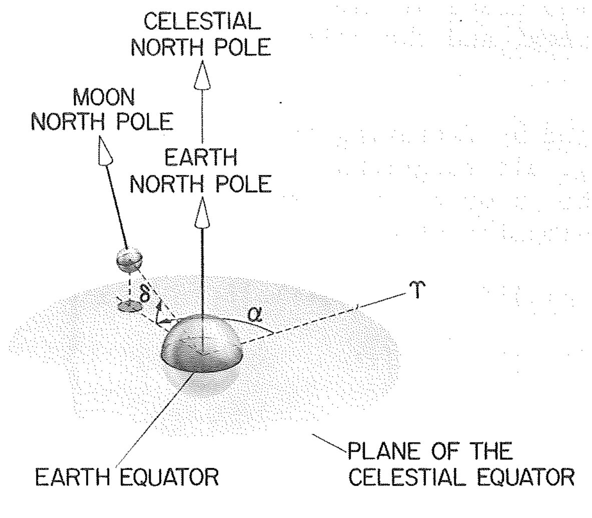
$$\left.\begin{aligned} \overline{x} &= [180_z][\varepsilon_x]^T[\Omega_z][i_x]^T[(\ell-\Omega)_z][(\ell-\Omega)_z]^T[(I+i)_x][(\ell-\Omega)_z]\overline{x}' \\ \overline{x} &= [180_z][\varepsilon_x]^T[\Omega_z][i_x]^T[i_x][I_x][(\ell-\Omega)_z]\overline{x}' \\ \overline{x} &= [180_z][\varepsilon_x]^T[\Omega_z][I_x][(\ell-\Omega)_z]\overline{x}' \end{aligned}\right\} \qquad \text{(B2)}$$

[1]This is a legal operation, because of the special nature of the $[180_z]$ matrix.

Equation (B2) transforms any vector in the selenographic coordinate system into a vector in the equatorial system. As mentioned above, if we were interested only in the effects of the geometrical librations this equation would suffice.

We will now develop the transformation matrix via the libration angles and the true position of the Moon as given by the right ascension and declination. In figure 6, the Moon is shown as it appears in the heavens to an observer on Earth. An hour circle of the equatorial coordinate system is shown passing through the apparent center C of the Moon, and is marked N at its northern part and S at its southern part. A small circle, known as a declination circle is marked E for its eastern part and W for its western part. The central meridian, in the selenographic coordinate system, is shown passing through C and the north pole of the Moon. The position angle of this meridian is designated P_0, and normal to it is the lunar equator. The point O is the origin of selenographic coordinates and extending from it is the principal axis $\hat{i}$. To an observer on the Moon at the point C, the center of the Earth appears to be in the zenith. The selenographic coordinates of the point C are given as longitude λ and latitude β, both positive as shown. The three angles λ, β, and P_0 are the three libration angles in longitude, latitude, and position angle. In addition, the position of the center of the Moon is given as α in right ascension and δ in declination relative to the equator and equinox of our fundamental system. We are





(b) The position of the Moon in the geocentric equatorial system.

Figure 6.- The true Moon as it appears in the heavens to an observer on the Earth.

now in a position to formulate our transformation matrix, which we may write as:

$$\bar{x} = [\alpha_z][\delta_y][180_z][P_{o_x}][\beta_y]^T[\lambda_z]^T\bar{x}' \tag{B3}$$

Here, reading the rotation matrices from right to left, we have rotated the $\hat{i}$ axis through the longitude angle λ and the latitude angle β until it is pointing in the direction of the center of the Earth, and then rotated the plane of the lunar equator through the angle P_o until it is coincident with the declination circle EW and parallel to the equator. Next we have rotated the $\hat{i}$ axis until it is pointing directly away from the Earth, and then brought it into coincidence with the vernal equinox through the successive angles δ and α. As with the previous development we may factor the matrix $[180_z]$ out to the left, producing

$$\bar{x} = [180_z][\alpha_z][\delta_y]^T[P_{o_x}][\beta_y]^T[\lambda_z]^T\bar{x}' \tag{B4}$$

We now have two different expressions (eqs. (B2) and (B4)) containing the same transformation matrix which we may equate and solve for the unknown libration angles.

$$[P_{o_x}][\beta_y]^T[\lambda_z]^T = [\delta_y][\alpha_z]^T[\varepsilon_x][\Omega_z][I_x][(\ell - \Omega)_z] \tag{B5}$$

The quantities on the left are, of course, our unknowns and all the quantities on the right are available for computation. The only problem left is that of unscrambling the three angles from the left side of the expression. This may be readily accomplished by setting

$$[A] = [P_{o_x}][\beta_y]^T[\lambda_z]^T \tag{B6}$$

and then expanding the right-hand side to provide:

$$[A] = \begin{bmatrix} (\cos\beta\cos\lambda) & (\cos\beta\sin\lambda) & (\sin\beta) \\ (\cos P_o\sin\lambda + \sin P_o\sin\beta\cos\lambda) & (\cos P_o\cos\lambda + \sin P_o\sin\beta\sin\lambda) & (-\sin P_o\cos\beta) \\ (-\sin P_o\sin\lambda - \cos P_o\sin\beta\cos\lambda) & (\sin P_o\cos\lambda - \cos P_o\sin\beta\sin\lambda) & (\cos P_o\cos\beta) \end{bmatrix} \tag{B7}$$

The matrix [A] may be computed from the right-hand side of equation (B5) and we may then find our three libration angles from the elements of this matrix.

$$\left.\begin{aligned} tg\lambda &= \frac{A_{12}}{A_{11}} \\ \sin\beta &= A_{13} \\ tgP_o &= -\frac{A_{23}}{A_{33}} \end{aligned}\right\} \tag{B8}$$

The values of the right ascension α and the declination δ of the Moon have been precomputed and are available in tabular form or on magnetic tape suitable for an electronic computer. The other quantities are either constant or have been expressed as functions of time. These quantities I, ε, ℓ, and Ω have been taken from reference 9 and are given below:

$$I = 1^{\circ}32'.1$$

$$\varepsilon = 23^{\circ}27'08''.26 - 46''.845T - 0''.0059T^2 + 0''.00181T^3$$ Epoch = 1900.0 = 1900 Jan $0^{d}.313$

$$\ell = 270^{\circ}.434164 + 13^{\circ}.1763965268d - 0^{\circ}.0000850D^2$$ Epoch = 1900 Jan $0^{d}.5$ ET = JD 2415020.0

$$\Omega = 259^{\circ}.183275 - 0^{\circ}.0529539222d + 0^{\circ}.0001557D^2$$

where

d JD - 2415020.0

D d/10,000

T (d + 0.187)/36525.0

JD Julian Day number

The lunar geometrical librations may be computed from the above simple equations, which obviates the need to store the values on tape and then interpolate them to the desired time.

The physical librations may be computed from relatively simple equations that provide the increments $\Delta\lambda$, $\Delta\beta$, and ΔP_o to be added to the libration angles discussed above. From reference 9,

$$\left.\begin{aligned} \Delta\lambda &= 0^{\circ}.003 \sin(\ell - \Gamma') - 0^{\circ}.005 \sin 2(\Gamma' - \Omega) - 0^{\circ}.016 \sin g_o + \Delta P_o \sin \beta \\ \Delta\beta &= M \cos \lambda + N \sin \lambda \\ \Delta P_o &= (M \sin \lambda - N \cos \lambda) \sec \beta \end{aligned}\right\} \quad \text{(B9)}$$

where

$$M = 0^{\circ}.040 \sin(\Gamma' - \Omega) - 0^{\circ}.003 \sin(\ell - \Omega)$$

$$N = 0^{\circ}.020 \cos(\Gamma' - \Omega) + 0^{\circ}.003 \cos(\ell - \Omega)$$

$$\Gamma' = 334^{\circ}.329556 + 0^{\circ}.1114040803d - 0^{\circ}.0007739D^2$$ Epoch = 1900 Jan $0^{d}.5$ ET = JD 2415020.0

$$g_o = 358^{\circ}.475845 + 0^{\circ}.985600267d - 0^{\circ}.0000112D^2$$

Here ℓ and Ω are as defined above, Γ' is the mean longitude of the lunar perigee, and g_o is the mean anomaly of the Sun; M and N are merely auxiliary quantities.

The total libration angles are then computed by adding the increments found from equations (B9) above to the geometrical libration angles computed from equations (B8).

$$\lambda' = \lambda + \Delta\lambda$$

$$\beta' = \beta + \Delta\beta$$

$$P_o' = P_o + \Delta P_o$$

and the vector of the landmark in equatorial coordinates is then found by substituting into the equation (B4), which becomes

$$\bar{x} = [180_z][\alpha_z][\delta_y]^T[P'_{o_x}][\beta'_y]^T[\lambda'_z]^T\bar{x}' \qquad \text{(B10)}$$

It will be noted that the equations for ε, ℓ, Ω, Γ', and g_o are referenced to a particular epoch and that they are given in terms of the time variables d, D, and T. The quantity d is the number of Julian days that have elapsed since the epoch 1900 January $0^d.5$ (or Julian day number 2415020.0). In Civil time, this epoch is December 31, 1899, at Greenwich noon. The quantity D is the same as d but expressed in terms of 10,000 Julian days. Similarly, T is essentially the same as d expressed in terms of Julian centuries of 36525 days. The quantity 0.187, in the defining equation for T, is an adjustment for the difference in epochs between the one equation containing T (the equation for ε), whose epoch is 1900 January $0^d.313$, and the other equations containing d and D whose epoch is 1900 January $0^d.5$. The difference between ephemeris and universal time at these fundamental epochs, which is $-3^s.9$, has been considered to be sufficiently small to be ignored for the present purposes. It should be carefully noted that all of these equations provide values of their respective variables relative to the mean equator and equinox of date. As a consequence our landmark vector found from equation (B10) is represented in this same coordinate system and must be precessed to the fundamental system before further use is made of it.

APPENDIX C

DEVELOPMENT OF EQUATIONS FOR WINDOW REFRACTION

An observer in a presurrized spacecraft may be required take his navigation observations through a window, or a set of window panes. The windows will undoubtedly be deflected by the differential pressure existing between the interior and the exterior, and any light ray traversing the window will be refracted due to both the difference in pressure and the deflection of the window. The external ray to the object sighted may always be considered to be approximately known, since we assume that a navigation sight to a known object is being made.

The first step to an understanding of the process is to realize that an entire bundle of parallel rays from the object intercepts the window at all points on the surface. And that, even though all these rays are transmitted through the window, only one ray enters the eye of the observer, or the instrument he may be using. The real problem, then, is that of discovering this particular ray. Since the mathematical description of optical phenomenon is reversible we will consider a ray to be emerging from the eye of the observer, extending it to the window, refracting it at the window, and then continuing it on to the body under observation. When a higher order function is used to represent the surface of the refracting window, the problem is mathematically unsolvable in an analytical form and recourse must be had to the process of approximating the solution by some method of iteration. The method that has been set up to be used at Ames in a digital computer program (see ref. 18) is that of first assuming the direction of a ray from the eye of the observer toward the window. This ray is then carried forward by an iterative process until it intersects the surface of discontinuity, at which point the refracted ray beyond the surface is determined. The process is repeated in sequence for each surface of the window. When the last surface has been considered, the direction of the refracted ray beyond the window is then known and can be compared with the direction of the body specified as having been observed. The rotation vector between the refracted ray and the true ray to the object is then determined, modified by the amount of the refraction and applied to the original, assumed, interior ray. The whole process is then repeated as many times as necessary until the refracted ray and the true ray coincide. At this point the last modified ray is the one desired. As many steps in this loop may be added as desired to account for multiple refracting surfaces.

It is necessary then, to formulate equations that will represent: the three-dimensional window surface, the normal to the surface, the refraction of the ray, and the iterations that may be required. All are to be represented in vector-matrix form for ease in computing these three-dimensional entities.

The computation of the refraction of the rays of an object observed through the windows of a pressurized space capsule is complicated by the fact that the window shape must necessarily be dictated by considerations of space available for windows and their placement to accommodate critical modes of flight and maneuver. As a consequence, the windows, as bowed under a differential pressure, are almost certain to have a distinct lack of symmetry, and the rays to be considered are almost certain to be nonradial; previously developed ray trace analyses therefore are not applicable.

The purpose of this appendix is to develop an applicable ray-trace analysis. The development of this analysis will be carried out in window coordinates, since that is the coordinate system in which the window is most easily described and is thus most convenient to use when iterations are required. One vector, however, must be transformed into the window coordinate system before the process may be carried through. This is the unit vector of the observed celestial body, which is provided in the equatorial inertial coordinate system.

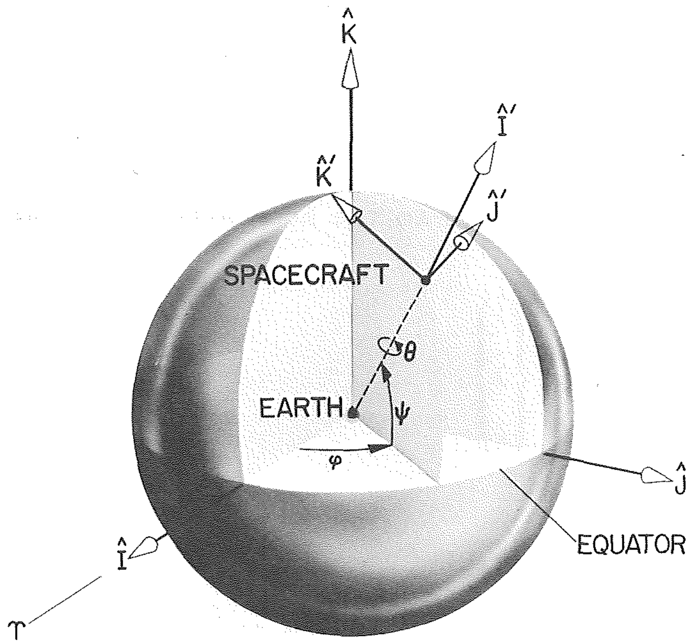


Figure 7.- Orientation of the spacecraft coordinate system $(\hat{I}'\hat{J}'\hat{K}')$ with respect to the equatorial inertial coordinate system $(\hat{I}\hat{J}\hat{K})$.

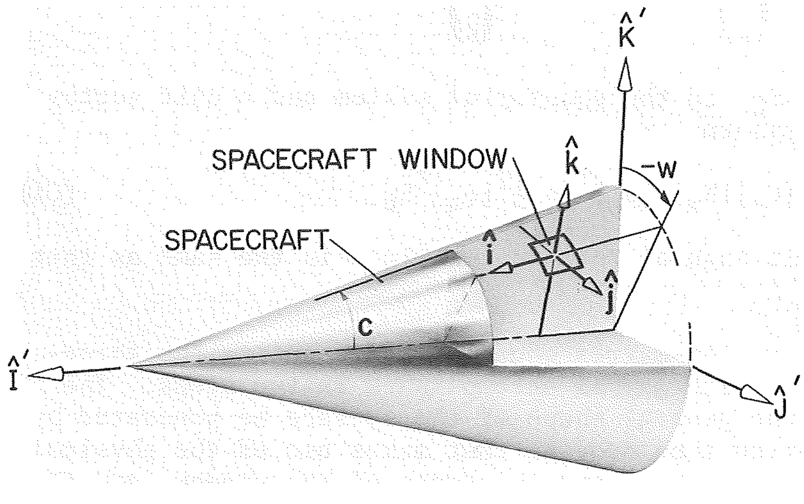


Figure 8.- Orientation of the window coordinate system $(\hat{i}\hat{j}\hat{k})$ with respect to the spacecraft coordinate system $(\hat{I}'\hat{J}'\hat{K}')$.

The geometry of the transformation may be understood by referring to figures 7 and 8. In figure 7, the fundamental inertial coordinate system is designated as $(\hat{I}\hat{J}\hat{K})$, with the principal axis indicated by the direction to the vernal equinox ♈. The body axis of the space capsule is given as $(\hat{I}'\hat{J}'\hat{K}')$, and the orientation angles relative to the fundamental system as ϕ, ψ, and θ. The angles ϕ and ψ are similar to right ascension and declination, and they describe the intersection of the principal body axis with the celestial sphere, while θ is the roll angle of the body. Figure 8 shows the orientation of the window coordinate system $(\hat{i}\hat{j}\hat{k})$ with respect to the body axis system $(\hat{I}'\hat{J}'\hat{K}')$. The space capsule is shown as conical in shape, since that seems to be the type of configuration being considered for space capsules of the near future. The half-cone angle of the body is shown as C, and the roll angle of the window out of the reference plane of the space capsule is given as W. From figure 8, it can be seen that the transformation from the body axis to the window axis system may be written as

$$\begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} = [T_{WB}] \begin{Bmatrix} \hat{I}' \\ \hat{J}' \\ \hat{K}' \end{Bmatrix}$$

where

$$[T_{WB}] = [C_y][W_x]^T$$

And the transformation from the equatorial system to the body axis system may be written as

$$\begin{Bmatrix} \hat{I}' \\ \hat{J}' \\ \hat{K}' \end{Bmatrix} = [T_{BE}] \begin{Bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{Bmatrix}$$

where

$$[T_{BE}] = [\theta_x]^T[\psi_y]^T[\phi_z]^T$$

Then the complete transformation from the equatorial system to the window coordinate system may be given as

$$\begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} = [T_{WB}][T_{BE}] \begin{Bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{Bmatrix}$$

Or in terms of a unit vector $\hat{S}_E$ in the equatorial system and a unit vector $\hat{S}_W$ in the window coordinate system

$$\hat{S}_W = [C_y][W_x]^T[\theta_x]^T[\psi_y]^T[\phi_z]^T\hat{S}_E \tag{C1}$$

The notation used here for unit angular rotation matrices is the same as that given in appendix B.

Next, one must develop a vector-matrix equation representing the three-dimensional refracting surface which may be accomplished in two steps. The first of these requires that the general shape of the surface be generated by means of a structural deformation-type program that makes use of the physical characteristics of the window, the shape and thickness of the window, and the pressure differential across the window. Programs of this type are available for use; one of these is described in reference 19. The output of this program is the deflection of the window at discrete points over its surface for any pressure differential. Although this type of information represents the surface with sufficient accuracy, it is in a nonanalytical form, and if it is used directly a great deal of interpolation is required since the rays will invariably fall between the points for which the discrete data are available Consequently, as a second step, it is considered desirable to convert the data to an analytical form that is smooth and tractable over the portion of the window that will come under consideration, and in addition, may be expressed in the desired vector-matrix form. This may be accomplished by fitting polynomials to cuts through the surface, and fitting the coefficients of these equations with polynomials in the remaining variable. The edges of the windows, as installed in the space capsule, may have clamped or unclamped edges. Our equation should be capable of representing the highest order curve thus generated. This would be the case with clamped edges, which produce curves with two points of inflection. A fourth-degree equation, used here, will represent this type of surface. This technique can be used for higher order equations, for which the necessary modifications will be apparent.

Let us now consider that we have taken cuts through our surface in planes parallel to the yz plane at constant values of the x coordinate. Fitting the resultant deflections, in the z coordinate, by least squares we now have several equations of the form

$$z = A_x y^4 + B_x y^3 + C_x y^2 + D_x y + E_x \tag{C2}$$

where the coefficients are functions of x. Equation (C2) may now be factored into the form

$$z = (y^4 \; y^3 \; y^2 \; y \; 1) \begin{Bmatrix} A_x \\ B_x \\ C_x \\ D_x \\ E_x \end{Bmatrix} = \overline{Y}^T \begin{Bmatrix} A_x \\ B_x \\ C_x \\ D_x \\ E_x \end{Bmatrix} \tag{C3}$$

Since the coefficients are functions of x and are known from the above mentioned least-squares fits to the original data, we may represent them as

$$\left.\begin{aligned} A_x &= m_{11}x^4 + m_{12}x^3 + m_{13}x^2 + m_{14}x + m_{15} \\ B_x &= m_{21}x^4 + m_{22}x^3 + m_{23}x^2 + m_{24}x + m_{25} \\ C_x &= m_{31}x^4 + m_{32}x^3 + m_{33}x^2 + m_{34}x + m_{35} \\ D_x &= m_{41}x^4 + m_{42}x^3 + m_{43}x^2 + m_{44}x + m_{45} \\ E_x &= m_{51}x^4 + m_{52}x^3 + m_{53}x^2 + m_{54}x + m_{55} \end{aligned}\right\} \tag{C4}$$

The values on the left of these equations are known, as is the variable x; hence we may fit these coefficients with a least-square process also. Equations (C4) may now be factored into the form

$$\begin{Bmatrix} A_x \\ B_x \\ C_x \\ D_x \\ E_x \end{Bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} \end{bmatrix} \begin{Bmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{Bmatrix} = [M]\overline{X} \qquad \text{(C5)}$$

Combining equations (C3) and (C5) yields

$$z = \overline{Y}^T[M]\overline{X} \qquad \text{(C6)}$$

where

$$[M] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} \end{bmatrix} \qquad \text{(C7)}$$

$$\overline{X}^T = (x^4 \; x^3 \; x^2 \; x \; 1)$$

$$\overline{Y}^T = (y^4 \; y^3 \; y^2 \; y \; 1)$$

Equation (C7) is known as the surface matrix since it represents the shape of the surface of the bowed window under a differential pressure of unity. When the differential pressure is other than unity, equation (C6) is multiplied by the gage pressure Δp, since we may consider the deflection to be a linear function of the pressure.

$$z = \Delta p\overline{Y}^T[M]\overline{X} \qquad \text{(C8)}$$

Although the quantities $\overline{Y}$ and $\overline{X}$ are not true geometrical vectors, since their elements are not independent, the form of the equation suits our purposes because an expansion of equation (C6) or (C8) produces the mixed polynomial that we require.

As indicated in the above development, the [M] matrix would be developed by first fitting curves as a function of y by least squares at several constant values of x, and then the coefficients from these first fits would themselves be fitted by least squares as a function of x. A more expeditious method may be that of forming matrices from the various given values of x, y, and z, and solving for the [M] matrix in a single matrix least-squares solution. This method may be readily developed as follows from equation (C6).

$$z = \overline{Y}^T[M]\overline{X}$$

By stacking several of these equations under each other for a constant value of x and factoring into vector-matrix form we have

$$\overline{Z} = [Y][M]\overline{X}$$

Now if we stack several of these equations alongside of each other at constant values of y, again factoring into the vector-matrix form, we obtain

$$[Z] = [Y][M][X]$$

The standard method of forming the normal equations in matrix notation may now be applied by multiplying through the equation from the left by $[Y]^T$ and from the right by $[X]^T$. Thus

$$[Y]^T[Y][M][X][X]^T = [Y]^T[Z][X]^T$$

Now if sufficient, redundant, independent observations are available for the least-squares reduction, the quantities $[Y]^T[Y]$ and $[X][X]^T$ are square, nonsingular, and symmetrical, but nonorthogonal. Thus if we solve for the matrix [M] we have

$$[M] = ([Y]^T[Y])^{-1}[Y]^T[Z][X]^T([X][X]^T)^{-1} \qquad \text{(C9)}$$

This is the equation used to determine the surface matrix [M] when the description of the surface is supplied in a rectangular tabular form.

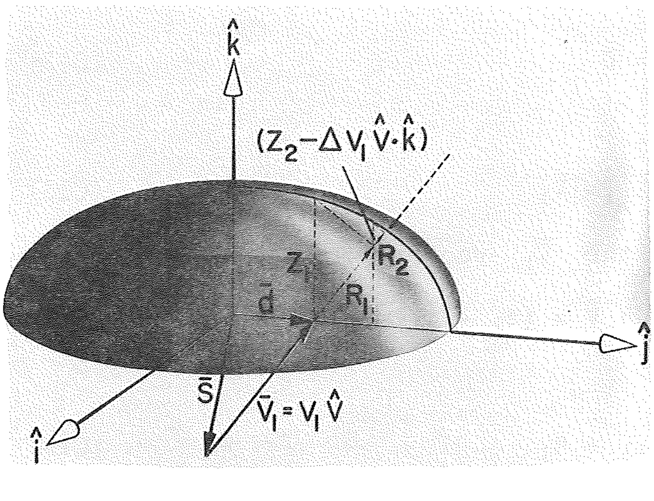


Figure 9.- Geometry of the scheme for iterating from the base plane to the surface of discontinuity.

Now, having the mathematical description of the window surface, we can determine the intersection of the assumed interior ray with this surface. This is an iterative procedure, and is illustrated in figure 9. Here the window coordinate system is given as $(\hat{i}\hat{j}\hat{k})$, the vector from the origin locating either the eye of the observer or some reference of the instrument used, is given as $\overline{S}$, some initial ray vector as $\overline{V}_1$, which terminates in the xy plane, and a vector in the xy plane closing with $\overline{V}_1$ as $\overline{d}$. Let

$$\left.\begin{aligned} \overline{S} &= S_x\hat{i} + S_y\hat{j} + S_z\hat{k} \\ \hat{V}_1 &= \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k} \\ \overline{V}_1 &= V_1\hat{V} \end{aligned}\right\} \quad \text{(C10)}$$

where V_1 is unknown. Now $\overline{d}$ is of the form

$$\overline{d} = x_1\hat{i} + y_1\hat{j} + 0\hat{k}$$

and from figure 9,

$$\overline{d} = \overline{S} + V_1\hat{V}$$

decomposing

$$x_1 = S_x + V_1\alpha$$

$$y_1 = S_y + V_1\beta$$

$$0 = S_z + V_1\gamma$$

then from the third equation

$$V_1 = -\frac{S_z}{\gamma}$$

and from the first two

$$\left.\begin{aligned} x_1 &= S_x - \frac{\alpha}{\gamma} S_z \\ y_1 &= S_y - \frac{\beta}{\gamma} S_z \end{aligned}\right\} \quad \text{(C11)}$$

Now, if we set

$$\Delta V_1 = 0 \quad \text{(C12)}$$

compute

$$z_1 = \Delta p\overline{Y}_1^T[M]\overline{X}_1 \quad \text{(C13)}$$

and project the increment of z_1 above the $\overline{V}_1$ vector onto the $\hat{V}$ direction

$$R_1 = (z_1 - \Delta V_1\hat{V} \cdot \hat{k})\hat{k} \cdot \hat{V} = (z_1 - \Delta V_1\gamma)\gamma \quad \text{(C14)}$$

Now form

$$\Delta V_2 = \Delta V_1 + R_1 \quad \text{(C15)}$$

and compute

$$\left.\begin{aligned} x_2 &= x_1 + R_1\hat{V} \cdot \hat{i} = x_1 + R_1\alpha \\ y_2 &= y_1 + R_1\hat{V} \cdot \hat{j} = y_1 + R_1\beta \end{aligned}\right\} \tag{C16}$$

We may now loop back to the equation for z (eq. (C13)) and continue to compute until $|R| < \varepsilon$. At this time, the last values computed for x, y, and z are the coordinates of the point of intersection of our ray with the refracting surface, to the accuracy which we have specified for ε. The basic scheme of this iteration is that by successively projecting the value of $(z - R\hat{V} \cdot \hat{k})$ onto the vector $\hat{V}$ we approach the point of intersection with the window surface. This system appears to be stable for all continuous smooth surface functions.

When the coordinates of the point of intersection of the ray vector with the surface has been established, the unit vector normal to the surface $(\hat{N})$ may be determined from the gradient of the scalar function F

$$\hat{N} = \frac{\overline{\nabla}F}{|\overline{\nabla}F|} \tag{C17}$$

where

$$F(x,y,z) = z - \Delta p \overline{Y}^T [M] \overline{X} \equiv 0$$

$$\overline{\nabla}F = \left(\frac{\partial F}{\partial x}\right) \hat{i} + \left(\frac{\partial F}{\partial y}\right) \hat{j} + \left(\frac{\partial F}{\partial z}\right) \hat{k}$$

$$\left(\frac{\partial F}{\partial x}\right) = -\Delta p \overline{Y}^T [M] \left(\frac{\partial \overline{X}}{\partial x}\right)$$

$$\left(\frac{\partial F}{\partial y}\right) = -\Delta p \left(\frac{\partial \overline{Y}^T}{\partial y}\right) [M]\overline{X}$$

$$\left(\frac{\partial F}{\partial z}\right) = +1$$

$$\left(\frac{\partial \overline{X}^T}{\partial x}\right) = (4x^3 \;\; 3x^2 \;\; 2x \;\; 1 \;\; 0)$$

$$\left(\frac{\partial \overline{Y}^T}{\partial y}\right) = (4y^3 \;\; 3y^2 \;\; 2y \;\; 1 \;\; 0)$$

and

$$|\overline{\nabla}F| = \left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right]^{1/2}$$

After the unit vector normal to the surface has been established, the refracted ray may be computed by an equation which will be developed with the help of figure 10. The unit vector $\hat{N}$ is the normal to the surface of discontinuity, on either side of which exist a medium of transmission whose indices of refraction, relative to free space, are n_2 and n_1. The incident ray has unit vector $\hat{I}$ and angle of incidence i, while the refracted ray has unit vector $\hat{R}$ and refraction angle r. The two principal precepts of geometrical optics are that the vectors $\hat{I}$, $\hat{R}$, and $\hat{N}$ are coplanar, and that Snell's Law holds:

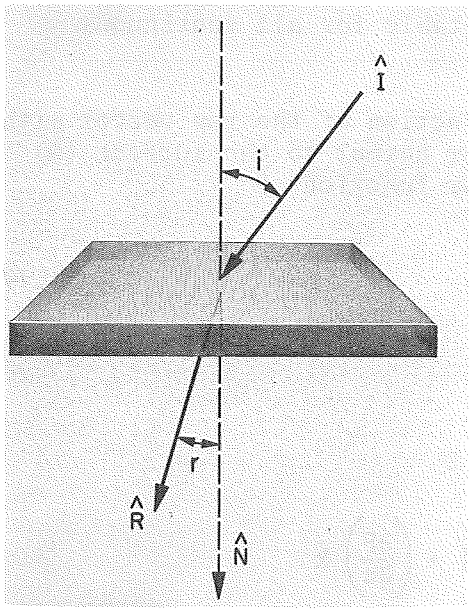


Figure 10.- Refraction at a surface of discontinuity.

$$\frac{\sin i}{\sin r} = \frac{n_2}{n_1}$$

From the requirement of coplanarity, we may write

$$\hat{I} \times \hat{N} = \sin i \; \hat{c}$$

and

$$\hat{R} \times \hat{N} = \sin r \; \hat{c}$$

where $\hat{c}$ is a unit vector normal to $\hat{I}$, $\hat{R}$, and $\hat{N}$. The elimination of $\hat{c}$ from the two equations leaves

$$(\hat{R} \times \hat{N}) = \frac{\sin r}{\sin i}(\hat{I} \times \hat{N}) = \frac{n_1}{n_2}(\hat{I} \times \hat{N})$$

If now, we cross through this equation with $\hat{N}$ from the left, we may solve for the vector $\hat{R}$, which gives us

$$\hat{R} = \left(\frac{n_1}{n_2}\right)\hat{I} + \left[\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 [1 - (\hat{I} \cdot \hat{N})^2]} - \left(\frac{n_1}{n_2}\right)(\hat{I} \cdot \hat{N})\right]\hat{N} \qquad \text{(C18)}$$

Although this equation has been developed on the basis of refraction, and we will use it for that purpose, it applies equally well to reflection by merely setting n_1 and n_2 equal to each other and taking the negative sign on the radical.

The star unit vector, already transformed into the window coordinate system, is now compared with the refracted ray $\hat{R}$. (For simplicity, we will call the star vector in the window coordinate system $\hat{S}$.) This may be accomplished by comparing the angle with an arbitrary assigned limit, such as

$$|\hat{R} \times \hat{S}| \leq \varepsilon \tag{C19}$$

Now if the refracted ray $\hat{R}$ is not alined with the star vector (fig. 11), it is necessary to modify the internal vector $\hat{V}$ and continue with another iteration. The modification of the old internal vector may be accomplished in the following manner. Two assumptions are made. The first is that the internal vectors and the external vectors are coplanar; that is

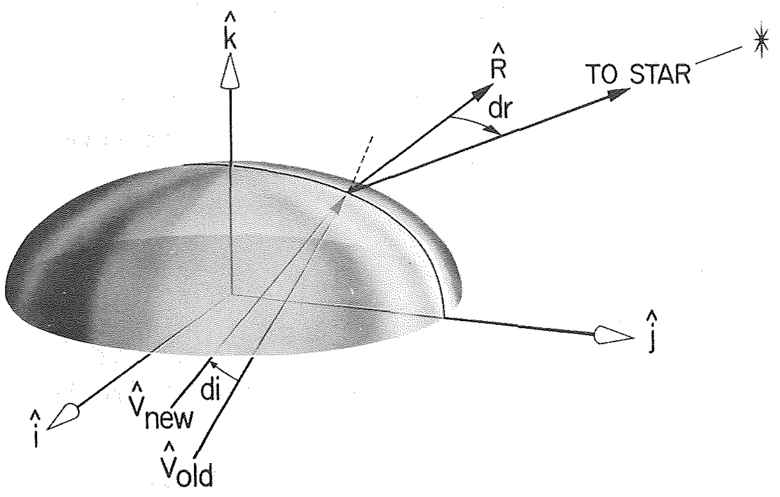


Figure 11.- Rotation of $\hat{R}$ into the star vector and generation of $\overline{V}_{new}$ from $\overline{V}_{old}$.

$$\frac{\hat{R} \times \hat{S}}{|\hat{R} \times \hat{S}|} = \frac{\hat{V}_{old} \times \hat{V}_{new}}{|\hat{V}_{old} \times \hat{V}_{new}|}$$

The second is that the magnitude of the angle change is modified by the ratio of the change in the angle of incidence to the change in the angle of refraction. If G represents this ratio, then

$$G|\hat{R} \times \hat{S}| = |\hat{V}_{old} \times \hat{V}_{new}|$$

When the second of these equations is substituted into the first the result is

$$\hat{V}_{old} \times \hat{V}_{new} = G(\hat{R} \times \hat{S})$$

This may be solved for $\hat{V}_{new}$ by crossing through the equation from the right with $\hat{V}_{old}$. The result is

$$\hat{V}_{new} = G[(\hat{R} \times \hat{S}) \times \hat{V}_{old}] + \hat{V}_{old}$$

with the assumption that

$$\hat{V}_{new} \cdot \hat{V}_{old} = 1$$

Since this assumption will destroy the unit magnitude of the vector, the unit vector may be found from

$$\hat{V}_{new} = \frac{G[(\hat{R} \times \hat{S}) \times \hat{V}_{old}] + \hat{V}_{old}}{|G[(\hat{R} \times \hat{S}) \times \hat{V}_{old}] + \hat{V}_{old}|} \tag{C20}$$

The gain factor G may be derived simply by starting with the equation

$$\frac{\sin i}{\sin r} = \frac{n_2}{n_1}$$

and differentiating, we have

$$\cos i \, di = \frac{n_2}{n_1} \cos r \, dr$$

and this we may reduce to

$$\left(\frac{di}{dr}\right) = \left(\frac{n_2}{n_1}\right) \frac{(\hat{R} \cdot \hat{N})}{(\hat{I} \cdot \hat{N})}$$

which is the gain factor through a single refracting surface. Now if several refracting surfaces are in the path of the ray, then we may write, with sufficient approximation

$$G = \left(\frac{di_1}{dr_n}\right) = \left(\frac{di_1}{dr_1}\right)\left(\frac{di_2}{dr_2}\right)\left(\frac{di_3}{dr_3}\right) \cdots \left(\frac{di_n}{dr_n}\right) \tag{C21}$$

since the change in the incident ray at a surface is equal to the change in the refracted ray at the preceding surface.

The entire process in its proper order may be conveniently listed in step form as follows:

1. Transform the observed star unit vector from equatorial inertial coordinates to window coordinates using equation (C1).

2. Assume any convenient value of the internal unit vector $\hat{V}$ (values of α, β, and γ in eq. (C10)). A convenient value is $\alpha = \beta = 0$, and $\gamma = 1$.

3. Compute the intersection of this vector with the plane of the undeflected window using equation (C11).

4. Set $V_1 = 0$, equation (C12).

5. Using equations (C13) to (C16), iterate to find the intersection of the ray with the deflected surface of the window.

6. Compute the normal to the surface using equation (C17).

7. Compute the refracted ray using equation (C18).

8. Repeat steps 3 to 7 for each of the refracting surfaces.

9. Using equation (C19), determine the angle between the free-space refracted ray and the star vector determined in step 1.

10. If the angle determined in step 9 is too large, compute a new internal unit vector using equation (C20), including the gain factor computed from equation (C21).

11. Iterate the entire process from steps 3 through 10 until the angle determined in step 8 is small enough.

12. The final refracted ray is then coincident with the star ray and the proper initial internal ray vector has been determined.

This theory has been programmed in FORTRAN IV for operation on an IBM 7094 digital computer. The operation of the program is completely discussed in reference 14. An indication of the accuracy of the theory, as programmed, relative to thin lens theory may be gained by comparing the figures in table II. These were obtained by computing the deflection of

TABLE II.- COMPARISON BETWEEN DEFLECTION ANGLE COMPUTED BY THIN LENS THEORY AND THE PRESENT THEORY AND COMPUTER PROGRAM

Angle of incident of ray	Vertical location of ray	Deflection angle from thin lens theory (sec of arc)	Deflection angle from present theory and program (sec of arc)
	2.5	12.176	12.182
	1.25	6.088	6.094
0	0	0	.006
	-1.25	-6.088	-6.082
	-2.5	-12.176	-12.170

para-axial rays passing through a thin spherical lens 6 inches in diameter, 0.38 inch thick at the edges with two spherical surfaces of 11.60×10^{-5} inch sagittal height. The dimensions of this lens corresponds to a portion of a simulated spacecraft window for which other test data were computed. The equation, in thin lens theory, used as a check was

$$\text{Defl} = -\left[2 + \frac{n_1}{n_2}\left(\frac{1 - \frac{n_2}{n_1}}{R}\right) D\right]\left(\frac{1 - \frac{n_2}{n_1}}{R}\right) y$$

where

$$n_2/n_1 = 1.458$$

$$D = 0.380 \text{ in.}$$

$$R = 38{,}793.10 \text{ in. (radius of curvature)}$$

The matrix [M] used in equation (C7) for the computation in the present theory was

$$[M] = \begin{bmatrix} -\frac{15}{64}R^{-7} & 0 & -\frac{3}{16}R^{-5} & 0 & -\frac{1}{8}R^{-3} \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{16}R^{-5} & 0 & -\frac{1}{4}R^{-3} & 0 & -\frac{1}{2}R^{-1} \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{8}R^{-3} & 0 & -\frac{1}{2}R^{-1} & 0 & h \end{bmatrix}$$

This was obtained from the equation for a spherical surface of sagittal height h

$$z = h - R + [R^2 - x^2 - y^2]$$

by expanding in the binomial theorem. It is apparent from the two right-hand columns of table II that the two methods of computation differ by a maximum of only 0.″006. This difference could well be due to round off errors in the computation.

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