Faculty Working Papers

THE NASH SOLUTION
AND THE UTILITY OF BARGAINING

Alvin E. Roth

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College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

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Assistant Professor, Business Administration

ABSTRACT

It has recently been shown that the utility of playing a game
with side payments depends on a parameter called strategic risk posture.
The Shapley value is the risk neutral utility function for games with
side payments.

In this paper, utility functions are derived for bargaining games
without side payments, and it is shown that these functions are also
determined by the strategic risk posture. The Nash solution is the risk
neutral utility function for bargaining games without side payments.

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I. Introduction

Recent work has shown that the Shapley value for a game with side payments is a cardinal utility function which reflects the desirability of playing different positions in a game, or in different games (cf. Shapley [14], Roth [9]). A player's utility for playing some position in a game is determined in part by his assessment of the payoff he will receive in a class of games with side payments called bargaining games. Given a player's evaluation of these bargaining games, his utility for playing a position in any game with side payments can be determined (cf. Roth [11]).

It is desirable to extend these results to games without side payments, since the assumption that side payments can be made is not appropriate in many situations. In this paper we will derive a class of utility functions for playing bargaining games without side payments. Games of this sort are studied by Nash [7], who developed a solution to bargaining games which is an extension of the Shapley value for games with side payments. That is, the Nash solution coincides with the Shapley value for bargaining games with side payments.

Somewhat surprisingly, the utility of playing a bargaining game without side payments is determined by the same considerations which determine the utility of playing a game with side payments. Given a player's evaluation of bargaining games with side payments, his utility for bargaining without side payments is determined.

II. Utility Functions for Games with Side Payments

This section summarizes the development of utility functions for games with side payments, as presented in [9] and [11].[1] We begin with some necessary definitions.

A game with side payments consists of a set of positions[2] $N = \{1,\ldots,n\}$ and a superadditive function v from the subsets of N to the real numbers such that $v(\emptyset) = 0$. The function v denotes the amount of wealth which each coalition of players can obtain for itself, and the model assumes that utility is linear in wealth, and that wealth is freely transferable. The set of individually rational outcomes (imputations) is thus the set

$$X = \{(x_1,\ldots,x_n)| \sum x_i = v(N), \ x_i \geq v(i)\}$$

Since we shall be interested in comparing different games, we take N to be the common set of positions for all games. Thus n is the largest number of players who can take part in a game (e.g., n could be the population of the world). In order to distinguish which positions have an active role in a given game, define position i to be a dummy in game v if, for all subsets S of N, $v(S) = v(S \cup i)$. The positions which are not dummies are called the strategic positions of the game v.

If π is a permutation of N, denote the image under π of a subset S of N by πS, and define the game πv by $\pi v(\pi S) = v(S)$. To simplify the exposition, we confine our attention to the class G of non-negative games: i.e., games for which $v(S) \geq 0$.

For any subset R of N, it will be convenient to define a bar-gaining game with side payments, v_R, by $v_R(S) = \begin{cases} 1 & \text{if } R \subset S \\ 0 & \text{otherwise} \end{cases}$. In this game, all players in R are symmetric, and all players not in R are

dummies. Similarly, for each position i in N, denote by v_i the game

$v_i(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$. In this game all positions other than i are

dummies. Denote by v_0 the game $v_0(S) = 0$ for all S, the game in which

all positions are dummies.

In order to make comparisons between positions in a game and in

different games, we shall consider a preference relation defined on the

set N×G of positions in a game. Write $(i,v)P(j,w)$ to mean "it is pre-

fereable to play position i in game v than to play position j in game w."

The letter I will denote indifference, and W will denote weak preference.[3]

We will consider preference relations which are also defined on

the mixture set[4] M generated by N×G. That is, preferences are also

defined over lotteries whose outcomes are positions in a game. Denote

by $[q(i,v);(1-q)(j,w)]$ the lottery which with probability q has a

player take position i in game v, and with probability (1-q) take posi-

tion j in game w. We will henceforth only consider preference relations

which have the standard properties of continuity and substitutability[5]

on M which insure the existence of an expected utility function[6] unique

up to an affine transformation. Denote this function by θ, and write

$\theta_i(v) = \theta((i,v))$, and $\theta(v) = (\theta_1(v),\ldots,\theta_n(v))$.

The utility for a position in a game is given by[7]

$$\theta_i(v) = \theta((i,v)) = \frac{q_{ab}((i,v)) - q_{ab}(r_0)}{q_{ab}(r_1) - q_{ab}(r_0)}$$

where a, b, r_1, and r_0 are elements of M such that aW(i,v)Wb, and

aWr_1Pr_0Wb, and the numbers $q_{ab}(y)$ are probabilities defined for any y

in M such that aWyWb by $yI[q_{ab}(y)a;(1-q_{ab}(y))b]$. The elements r_1 and

r_0 determine the origin and scale, since $\theta(r_1) = 1$, and $\theta(r_0) = 0$.

We assume that the preference relation obeys the following three conditions:

(2.1)   For all $i \in N$, $v \in G$, and for any permutation π, $\pi(i,v)I(\pi i, \pi v)$.

(2.2)   If i is a dummy in the game v, then $(i,v)I(i,v_0)$. Also $(i,v_i)P(i,v_0)$, and for all $(i,v) \in N \times G$, $(i,v)W(i,v_0)$.

(2.3)   For any games v and w, and for any probability q,

$(i,(qw+(1-q)v))I[q(i,w);(1-q)(i,v)]$.

Condition 2.1 says that the names of the positions don't influence their desirability. Condition 2.2 says that a strategic position in a game is always at least as desirable as a dummy position, and that it is equally undesirable to be a dummy in any game. Condition 2.3 expresses indifference between playing position i in the game $(qw + (1-q)v)$, and playing position i in game w with probability q or in game v with probability $(1-q)$.

Condition 2.2 also insures that we may choose the natural normalization for the utility function θ. In what follows, we will consider θ to be normalized so that $\theta_i(v_i) = 1$, and $\theta_i(v_0) = 0$.

It has been shown [9], [11] that any utility function arising from preferences obeying conditions 2.1, 2.2, and 2.3 has the following properties:

Property 1. Symmetry: for any $(i,v) \in N \times G$, and for any permutation
π, $\theta_{\pi i}(\pi v) = \theta_i(v)$.

Property 2. Homogeneity: for any $(i,v) \in N \times G$, and any non-negative
number c, $\theta_i(cv) = c\theta_i(v)$.

Property 3. Additivity: for any $v, w \in G$, $\theta(v + w) = \theta(v) + \theta(w)$.

The utility function θ can now be completely determined by specifying the certain equivalent of playing a bargaining game v_R, as one of r strategic players.[8]

Let $f(r)$ be a number such that

(2.4) $(i,v_R)I(i,f(r)v_i)$ for $i \in R$.

This expresses indifference between receiving $f(r)$ for certain (as the only strategic player in the game $f(r)v_i$) and being one of r strategic players in the game v_R. Note that $f(1) = 1$. Using the terminology of [9], we say that the preference is _neutral_ _to_ _strategic_ _risk_ if $f(r) = 1/r$ for $r = 1,\ldots,n$. The preference is strategic risk _averse_ if $f(r) < 1/r$, and strategic risk _preferring_ if $f(r) > 1/r$

The utility θ for playing an arbitrary game with side payments can now be written in terms of the function $f(r)$.

Theorem 1: $\theta_i(v) = \sum_{T \subseteq N} k(t)[v(T) - v(T-i)],$

where $k(t) = \sum_{r=t}^{n} (-1)^{r-t}\binom{n-t}{r-t}f(r).$

Furthermore, if the preference relation is neutral to strategic risk, then the utility of playing a position in a game is equal to its Shapley value.

Corollary 1:   If $f(r) = 1/r$, then $\theta_i(v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S-i)].$
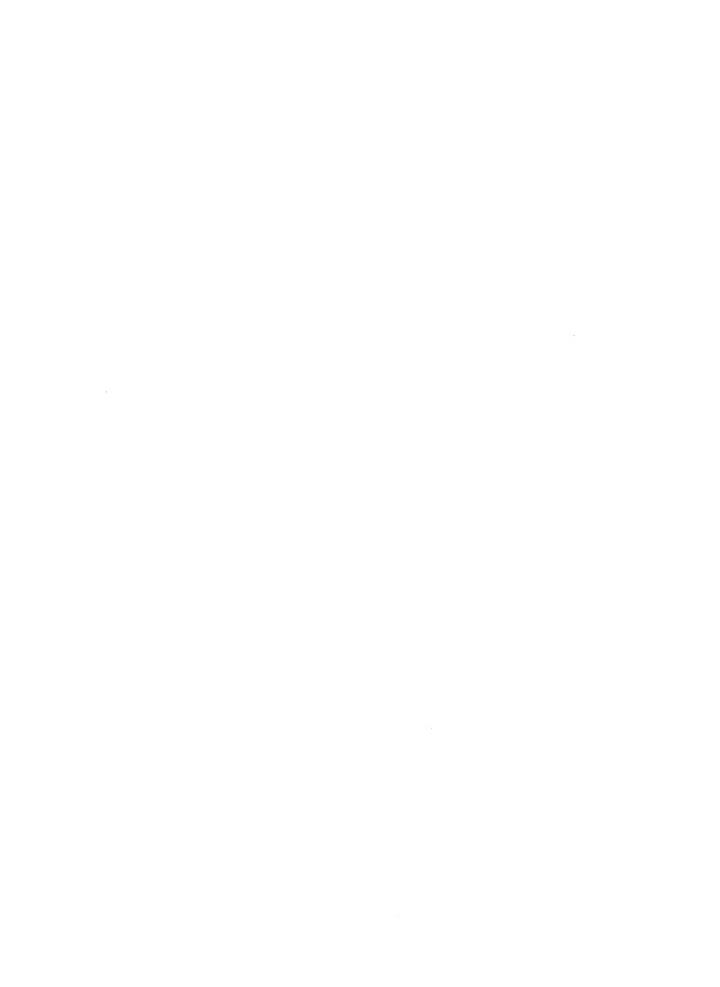
III. Bargaining Games without Side Payments, and Nash's Solution

        An n-person bargaining game without side payments is defined by
a compact convex subset A of n-dimensional Euclidean space, and a point
s contained in A. Any point $x = (x_1,\ldots,x_n)$ contained in A represents
the von Neumann-Morgenstern utility available to each player as the
result of some feasible agreement, and the set A represents the set of
all feasible utility payoffs. The point $s = (s_1,\ldots,s_n)$ represents the
utility of the "status quo"--that is, s gives the utility level achieved
by each player in the absence of any agreement.

        For simplicity, we will assume that the set A contains only
individually rational agreements: i.e., if $x \in A$, then $x \geq s$. We will
also assume that the origin of the utility function for each player
(position) is equal to the status quo payoff; i.e., we assume $s_i = 0$
for all $i \in N$. Denote the class of all such bargaining games by H. An
element of H will be denoted by the feasible set A, with the status quo
being understood to be the origin.

        As in the previous section, take $N = \{1,\ldots,n\}$ to be the common
set of positions for all bargaining games. The set R of strategic
positions in a bargaining game A is the set $R = \{i \in N \mid \exists\, x \in A$ such
that $x_i \neq 0\}$. A position which is not in the set R is a dummy for the
game A.

        A Nash solution to the bargaining problem is a function F,
defined on bargaining games, which associates with each bargaining game
A a single feasible outcome $F(A) \in A$, and which obeys the following
four conditions:

(3.1) Linearity: For any bargaining game A and positive real numbers $a_1,\ldots,a_n$, if $B = \{(a_1 x_1,\ldots,a_n x_n) \mid (x_1,\ldots,x_n) \in A\}$ then $F_i(B) = a_i F_i(A)$ for $i = 1,\ldots,n$.

(3.2) Independence of irrelevant alternatives: If A and B are bargaining games and B contains A, and if $F(B) \in A$, then $F(B) = F(A)$.

(3.3) Symmetry: Let R be the set of strategic positions in a game A, and suppose that for every permutation π of N such that $\pi R = R$, $x \in A$ implies that $\pi x \in A$. Then $F_i(A) = F_{\pi i}(A)$.

(3.4) Pareto Optimality: If x and y are elements of A, and $y_i > x_i$ for all strategic positions $i \in R$, then $F(A) \neq x$.

Nash [7] proved the following theorem.

Theorem 2:   There is a unique function F which satisfies conditions 3.1-3.4. For a bargaining game A, F(A) is the unique element $x \in A$ such that $\prod_{i \in R} x_i > \prod_{i \in R} y_i$ for every $y \neq x$ in A, where R is the set of strategic positions of the game A.

Thus the Nash solution picks the point x in S which maximizes the geometric average of the payoffs x_i for $i \in R$. Note that $F(A) = 0$ if and only if R is empty. It has recently been shown that this condition can replace Pareto optimality in the characterization of the Nash solution (Roth [10]). That is, we have the following theorem.[9]

Theorem 3:   The Nash solution is the unique function F which satisfies conditions 3.1-3.3, and the condition that $F(A) = 0$ only when R is empty.

IV. The Utility of Bargaining

In this section we will consider the utility of bargaining, by considering a preference relation P defined on the set of positions in bargaining games without side payments. Specifically, take P to be a preference relation defined on N×H, and on the mixture set M' generated by N×H.

It will be convenient to define, for each set R contained in N, the set $A_R = \{x| \sum_{i \in R} x_i \leq 1, x_i \geq 0$ if $i \in R$, and $x_j = 0$ if $j \notin R\}$, and to define for each non-negative vector x, the set A_x to be the line joining x to the origin, i.e., $A_x = \{ax| 0 \leq a \leq 1\}$.

It is easy to see that a bargaining game with side payments is actually a special case of a bargaining game without side payments. In particular, the game v_R with side payments and the game A_R without side payments present the same bargaining opportunities to the set R of strategic players (which is the same for both games). The set of outcomes in which dummies receive a payoff of zero is the same in both games. Similarly, the game v_i can be associated with the game $A_i = A_x$ where $x_i = 1$, and $x_j = 0$ for $j \neq i$, and the game v_0 has the same outcome set as the game A_0, since neither game has any strategic positions.

As in Section II, we confine our attention to preferences P which have the properties of continuity and substitutability necessary to insure the existence of an expected utility function θ. Of course θ is unique only up to affine transformations, so we may set $\theta_i(A_i) = 1$, and $\theta_i(A_0) = 0$, where we denote $\theta_i(A) = \theta(i,A)$.

We also assume that the preference relation obeys the following conditions.

(4.1) For all $i \in N$, $A \in H$, and every permutation π of N, $(i,A)I(\pi i, \pi A)$.

(4.2) For all $i \in N$, $A \in H$, $(i,A)I(i,A_0)$ and $(i,A)I(i,A_0)$ iff i is a

dummy in A. Also, if $x_i > y_i$ then $(i,A_x)P(i,A_y)$, and if R is a

non-empty subset of N such that $R \neq \{i\}$, then $(i,A_i)P(i,A_R)$.

(4.3) If $B = \{(a_1 x_1, \ldots, a_n x_n) \mid x \in A\}$ for $a_j \geq 0$ for $j = 1, \ldots, n$, and

if $a_i > 1$, then $(i,A)I[(1/a_i)(i,B); (1-1/a_i)(i,A_0)]$.

(4.4) If $A \subset B \subset C$, and $(i,A)I(i,C)$ then $(i,A)I(i,B)$.

Condition 4.4 expresses indifference to irrelevant alternatives.
It says that if a player is indifferent between playing in a game A, or
in a game C with a larger set of feasible alternatives, then he is also
indifferent between playing A or any game B which contains A and is con-
tained in C. Condition 4.3 simply says that, if the payoffs available
in a game are multiplied by positive constants, then a player is indif-
ferent between playing one game, or participating in the appropriate
lottery involving the new game. Conditions 4.1 and 4.2 are similar in
form and content to conditions 2.1 and 2.2.

If θ is a utility function reflecting preferences which obey the
above conditions, then it has the following properties.

Lemma 1: If i is a dummy in A, then $\theta_i(A) = 0$, and if $x_i = y_i$, then

$\theta_i(A_x) = \theta_i(A_y)$.

Proof: This follows immediately from condition 4.2.

Lemma 2: If $B = \{(a_1 x_1, \ldots, a_n x_n) \mid x \in A\}$ where all $a_j > 0$, then

$\theta_i(B) = a_i \theta_i(A)$.

Proof: Suppose $a_i \geq 1$. Then by condition 4.3, $\theta_i(A) =$

$\theta_i[(1/a_i)(i,B); (1-1/a_i)(i,A_0)] = (1/a_i)\theta_i(B) + (1-1/a_i)\theta_i(A_0)$

$$= (1/a_i)\theta_i(B). \quad \text{Suppose } a_i \quad 1. \quad \text{Then let } b_j = 1/a_j \text{ for}$$

$$j = 1,\ldots,n. \quad \text{Then } A = \{(b_1 y_1,\ldots,b_n y_n) \mid y \in B\}, \text{ and } b_i > 1.$$

$$\text{So } \theta_i(B) = (1/b_i)\theta_i(A) = a_i \theta_i(A).$$

Lemma 3: For any x, $\theta_i(A_x) = x_i$.

Proof: Let y be the vector such that $y_i = x_i$ and $y_j = 0$ for

$j \neq i$. Then lemma 2 implies that $\theta_i(A_y) = \theta_i(x_i A_i) = x_i \theta_i(A_i)$

$= x_i$, and lemma 1 implies that $\theta_i(A_x) = \theta_i(A_y)$.

As is the case for games with side payments, the function θ will
be completely determined by the posture towards strategic risk. For
bargaining games with side payments, condition 2.4 stated $(i,v_R)I(i,f(r)v_i)$
for $i \in R$. The equivalent condition for bargaining games without side
payments is

$$(4.5) \qquad\qquad (i,A_R)I(i,f(r)A_i) \quad \text{for } i \in R.$$

This expresses indifference between playing a strategic position in the
game A_R (as one of r strategic players) or receiving the utility $f(r)$
for certain (as the only strategic player in the game A_i). By condition
4.2 we know that $f(1) = 1$, and $0 < f(r) < 1$ for $r > 1$. As in the case
of games with side payments, we say the preference relation is neutral
to strategic risk if $f(r) = 1/r$, averse to strategic risk if $f(r) < 1/r$,
and strategic risk preferring if $f(r) > 1/r$. We will show that the Nash
solution is the utility function reflecting risk neutrality.

An immediate consequence of condition 4.5 is that $\theta_i(A_R) = f(r)$.
More generally, we have the following result.

Lemma 4: If $B_R = \{y \geq 0 \mid \sum_{i \in R} b_j y_i \leq 1, \ y_j = 0 \text{ for } j \notin R\}$ where $b_i > 0$

for each $i \in R$ then $\theta_i(B_R) = f(r)/b_i$ for $i \in R$.

Proof: $B_R = \{(a_1 x_1, \ldots, a_n x_n) \mid x \in A_R\}$ where $a_i = 1/b_i$ for $i \in R$ and $a_j = 1$ for $j \notin R$. So lemma 2 implies that $\theta_i(B_R) = a_i \theta_i(A_R) = f(r)/b_i$.
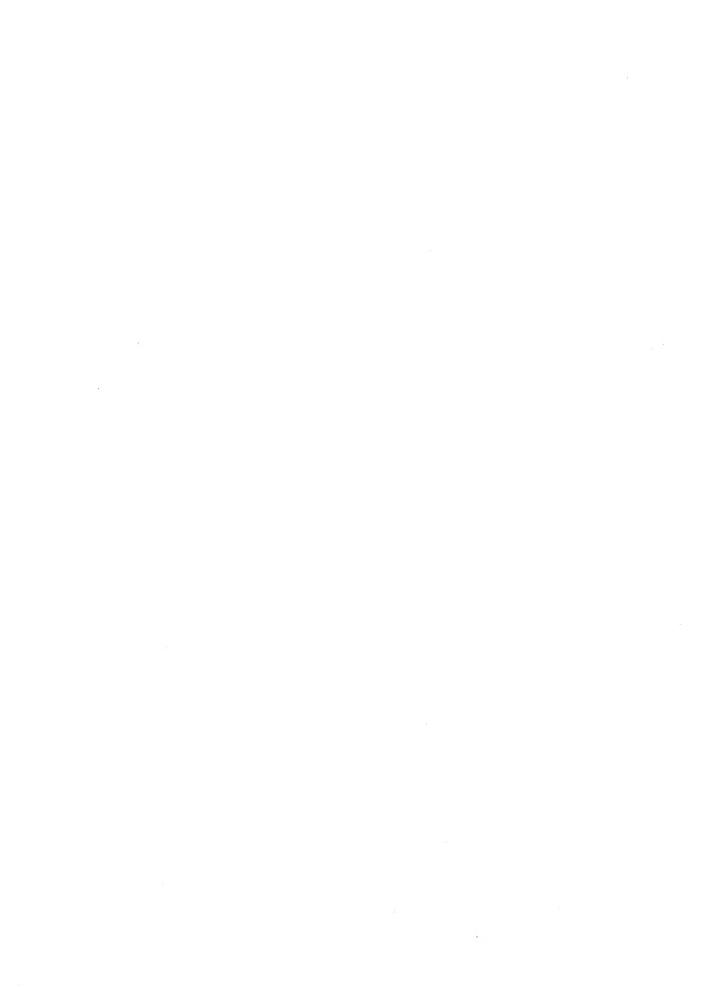
We can now specify the function θ for an arbitrary bargaining game A.

Theorem 4: If A is a bargaining game with R the set of strategic positions, then for $k \in R$, $\theta_k(A) = x_k$, where x is the unique element of A such that $\prod_{i \in R} x_i^{q_i} > \prod_{i \in R} y_i^{q_i}$ for all $y \in A$ such that $y \neq x$, where $q = (q_1, \ldots, q_n)$ is any non-negative vector such that $q_k = f(r)$ and $\sum_{i \in R} q_i = 1$.

The element x named in the theorem maximizes the geometric average with weights $q_1, \ldots, q_n$ over the set A. (The weighted geometric average[10] is concave, so it has a unique maximum of A.) The statement of the theorem implies that $\theta_k(A) = x_k$ depends only on q_k. Explicitly, the following technical proposition follows as a corollary of the theorem.

Proposition: If x maximizes $\prod_{i \in R} x_i^{q_i}$ and y maximizes $\prod_{i \in R} y_i^{p_i}$ over the set A, where q and p are non-negative vectors such that $\sum_{i \in R} q_i = \sum_{i \in R} p_i = 1$, then $x_k = y_k$ if $q_k = p_k$.

Proof of theorem: Let A be a bargaining game without side payments, and let $R \subset N$ be the set of strategic positions of A, and let $k \in R$. Let $q = (q_1, \ldots, q_n)$ be a non-negative

vector such that $\sum_{i \in R} q_i = \sum_{i \in N} q_i = 1$, and $q_k = f(r) > 0$.

Let x be the element of A which maximizes $\prod_{i \in R} x_i^{q_i}$. That

is, $\prod_{i \in R} x_i^{q_i} = c > \prod_{i \in R} y_i^{q_i}$ for all $y \in A$ such that $y \neq x$.

Let $H = \{y \mid \prod_{i \in R} y_i^{q_i} \geq c\} = \{y \mid \sum_{i \in R} q_i \log y_i \geq \log c\}$.

Then H and A are convex sets whose intersection is

the point x, and so there is a plane which separates H and

A. This plane is the tangent to H at x, i.e., the set

$T = \{z \mid z \cdot n = x \cdot n\}$ where $n = (q_1/x_1, \ldots, q_n/x_n)$. So $T =$

$\{z \mid (q_1/x_1)z_1 + \ldots + (q_n/x_n)z_n = \sum_{i \in N} q_i = 1\}$.

Let $B = \{z \mid (q_1/x_1)z_1 + \ldots + (q_n/x_n)z_n \leq 1\}$. Then

$A \subset B$, since T separates A from H. Lemma 4 implies that

$\theta_k(B) = (x_k/q_k)f(r) = (x_k/f(r))f(r) = x_k$. So $(k,B)I(k,A_x)$,

since $\theta_k(A_x) = \theta_k(B) = x_k$. Thus we have $A_x \subset A \subset B$, and

$(k,A_x)I(k,B)$. By condition 4.4, this implies that

$(k,A_x)I(k,A)$, and so $\theta_k(A) = x_k$. This completes the proof.

Corollary 2: When $f(r) = 1/r$, θ is equal to Nash's solution.

Proof: If A is a bargaining game with strategic positions

R, then for $k \in R$, $\theta_k(A) = x_k$ where x maximizes $\prod_{i \in R} x_i^{q_i}$

on A, for $q_k = f(r) = 1/r$ and $\sum_{i \in R} q_i = 1$. In particular,

x maximizes $\prod_{i \in R} x_i^{1/r}$, and, since $r > 0$, x maximizes $\prod_{i \in R} x_i$.

Thus we have shown that the utility of playing a bargaining game

without side payments is determined by the posture towards strategic

risk. Since the Shapley value and the Nash solution agree on bargaining games with side payments, it is natural to observe that they result from the same risk posture.

The treatment presented here permits us to observe not only the similarities between utility functions for games with and without side payments, but the differences as well. The most significant difference seems to be that, for bargaining games without side payments, there is no parallel to condition 2.3 for games with side payments. That is, if A, B, and C are bargaining games without side payments, such that $C = \frac{1}{2} A + \frac{1}{2} B = \{(\frac{1}{2} x + \frac{1}{2} y) \mid x \in A, y \in B\}$, then the utility of a lottery between A and B is not in general equal to the utility of C. That is $\theta_i[\frac{1}{2} A; \frac{1}{2} B] = \frac{1}{2}\theta_i(A) + \frac{1}{2}\theta_i(B) \neq \theta_i(C)$. A discussion of this phenomenon in the context of the Nash solution[11] is given by Harsanyi [2, pp. 330-332].

Nash originally interpreted his solution as applying to players of equal bargaining ability, but subsequently modified this interpretation [7,8]. Our results support Nash's original interpretation. The attitude of neutrality to strategic risk, which gives rise to the Nash solution as a utility function, simply expresses a player's belief that he will receive the average reward in a bargaining situation. As we have seen, any other risk posture gives rise to a utility function different from the Nash solution.

1. For related results, see Roth [12,13].

2. We speak of "positions" rather than the more customary "players" since we are interested here only in the structural properties of the game. We shall be concerned with the problem of evaluating the different positions from the point of view of a player who must choose among different positions.

3. So aIb means neither aPb or bPa, and aWb means aPb or aIb.

4. A mixture set has the properties that for all $a,b \in M$ $[1a;0b] = a$, $[qa;(1-q)b] = [(1-q)b;qa]$, and $[q[pa;(1-p)b];(1-q)b]$ $= [pqa;(1-pq)b]$. (Cf Herstein and Milnor [4].)

5. Cf Herstein and Milnor.

6. A utility function has the property that $u(a) > u(b)$ if and only if aPb. An expected utility function on a mixture space has the property that $u([qa;(1-q)b]) = qu(a) + (1-q)u(b)$. That is, the utility of a lottery is its expected utility.

7. Cf Herstein and Milnor.

8. The cardinality of sets R, S, T is denoted r, s, t.

9. This statement of the theorem makes use of the fact that we have already assumed individual rationality.

0. Harsanyi and Selten [3, lemma 10.1] and Kalai [5] both show that weighted geometric averages of this sort obey all of Nash's conditions except symmetry.

11. For different approaches to the bargaining problem see Brito, et. al. [1] or Kalai and Smorodinsky [6].
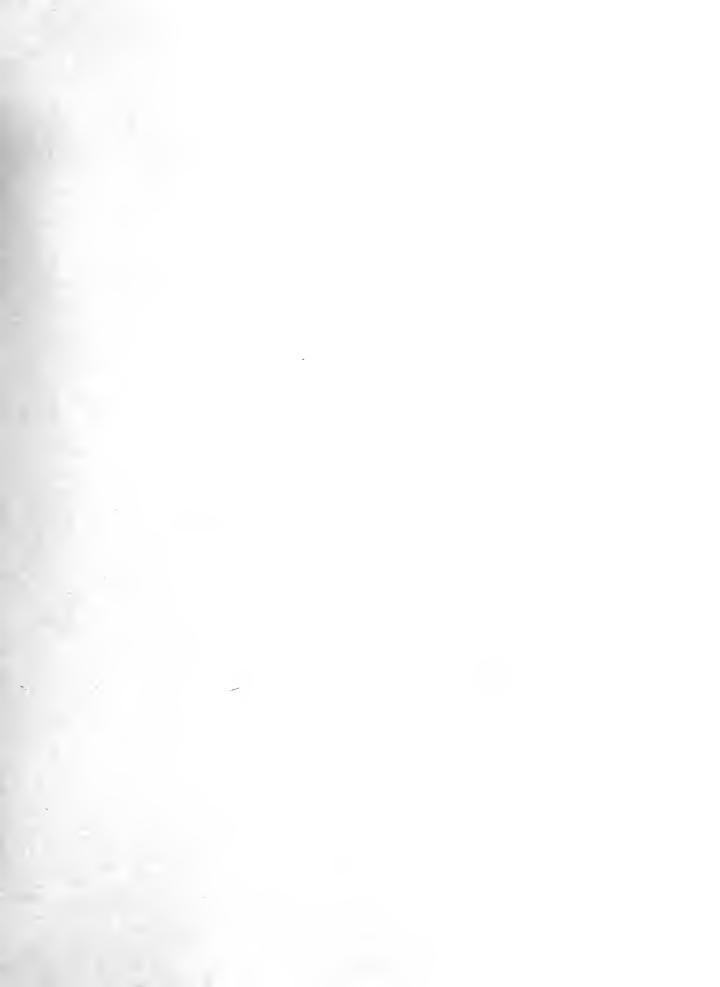
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