LOCKHART
ON THE NATURE OF
THE ROOTS
OF
NUMERICAL
EQUATIONS

Florian Cajori

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THE

NATURE OF THE ROOTS

OF

NUMERICAL EQUATIONS.

BY

JAMES LOCKHART, F.R.A.S.,

Author of 'The Resolution of Equations by Inferior and Superior Limits,' inscribed, by permission, to the President and Council of the Royal Astronomical Society.

Longum iter est per præcepta, *breve* et efficax per Exempla.

SENECA.

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PREFACE.

About forty years since, the study of Numerical Equations, in England, was confined to few persons, at the head of whom were Mr. Baron Maseres and the Rev. Mr. Frend. The latter was a gentleman of great mathematical ability, and an accomplished scholar in classical learning and biblical knowledge. The Church-squabbles of the present day would have been odious to him.

Both Maseres and He dismissed imaginary and negative roots from the doctrine of equations, following the example of many learned men, their predecessors in science. They were at no loss to find such real roots as satisfied the conditions of a problem. Mr. Frend's solution of Col. Titus's problem is generally known and admired.

However, as the fashion of the day requires obedience, we have in the following treatise complied, in every respect, with its algebraical dictates. The same was done in the year 1842, when we published the 'Resolution of Equations by means of Inferior and Superior Limits,' in which some ideas, entertained in the present treatise, appeared.

The reader should be acquainted with the theory and solution of equations, particularly with that by Prof. Young. This will enable him to understand each process, and to satisfy any doubt that he may entertain. Besides this, some of the equations are taken from the Professor's admirable work.

It is no part of our plan to discover the figures appertaining to real roots, except in the case of equal roots, nor to find those belonging to imaginary roots, which has been happily effected by our friend Dr. RUTHERFORD, in a work recently published, embracing equations of every degree, and combining truth with simplicity, its chief ornament.

Following the example of Dr. PRIDEAUX, in the Preface to the 'Connections,' we must beg the reader to excuse, on account of our age, now verging towards eighty-seven years, any errors he may discover in this treatise, either of computation or doctrine. No one has assisted in either.

The later homilies of the Archbishop of Grenada have place in our library.

Fasnacloich, Argyllshire;
May, 1850.

NATURE OF THE ROOTS

OF

NUMERICAL EQUATIONS.

THE Equations here proposed for examination have, exclusively, integral indices and coefficients, *unity* being the coefficient of the leading term.

Every such equation can be transformed into another, having its alternate signs beginning at the second term, *minus* and *plus*. This may easily be accomplished by NEWTON'S superior limit, where the signs are all *plus*. Alternating the signs, the nature of the roots is not changed.

None of the preceding equations can have more positive roots between 0 and 1, or between two depressing consecutive integers, or between intervals of such depression, than the integers contained in $\frac{2n}{3}$, n being the highest index of the equation. Thus, an equation of the 41st degree can have only 27 positive roots so situated; those of the 42d and 43d degree only 28 such roots; for since a quadratic can have only one root so situated, it can only impart its two roots to a cubic under the same circumstance, and therefore the cubic can only have two roots so situated; and as all higher equations are compounded of one or both of these, or may be supposed to be so compounded, they partake of the nature of their origin.

Take the triads (2, 3, 4), (5, 6, 7), (8, 9, 10).....(41, 42, 43)..... $n-1$, n, $n+1$; the middle number contains $\frac{n}{3}$ cubics; and since each cubic can only have two positive roots so situated, $\frac{2n}{3}$ positive roots can only be situated in an equation where n is the highest index, and divisible by 3.

The demonstration for the indices $(n-1)$ and $(n+1)$ follows so readily, that we shall not enlarge on it.

If $n-1=41$, it consists of 13 cubics and 1 quadratic, or 27 roots.
$n=42$, of 14 cubics, or 28 roots.
$n=43$, of 13 cubics and two quadratics, or 28 roots,
which are all supposed to be positive.

The foregoing property enables us to decide with certainty that roots are imaginary under circumstances similar to the following:

Let $$x^3 - 2x^2 + 3x - 1 = 0,$$
depress by ·7,
$$1 + \cdot 1 + 1\cdot 67 + \cdot 463,$$
three changes of sign are lost, which would have taken place if the roots had been real, and a superior limit taken; but since three real roots between 0 and 1 cannot exist in a cubic, the loss of the signs is occasioned by two imaginary roots.

This is of frequent occurrence in equations of high degree, pervading many steps of examination. (A.)

Let $\pm r$ be the absolute term of an equation, or of any equation derived therefrom by depression, either of them having one or more roots between 0 and 1.

Let q be the greatest coefficient whose sign is contrary to that of r.

Consider them as positive. Then the least positive root of the equation is greater than $\frac{r}{q + r}$.

See all the treatises on equations. (B.)

Let an equation, whose variations of sign permit the idea of its having two or more real roots between 0 and 1, be depressed by a quantity p less than *unity*, so that the resulting equation may have two or some equal number of signs less than the former.

Change the alternate signs of the equation in p, beginning at the second. Let c be the greatest coefficient under this last circumstance, having its sign contrary to that of the absolute term $\pm d$, where they are both to be taken positive. Then if $\frac{c}{c + d} - (1 - p)$ is less or equal to $\frac{r}{q + r}$ as derived in the last article, the roots passed over are imaginary, agreeing with the equal number of signs lost.

Thus, let $$x^2 - \overset{q}{1}x + \overset{r}{1} = 0,$$

depressed by (·6), $$1 + \overset{c}{\underline{\cdot 2}} + \overset{d}{\cdot 76}$$

$$\frac{r}{q + r} = \frac{1}{2} \text{ and } \frac{c}{c + d} - (1 - \cdot 6) \text{ is negative.}$$

This is explained in the work alluded to in the preface. (C.)

In any part of the process of examination, if a quantity equal to or less than $\frac{r}{q + r}$ (which of course is changed in value at every depression), should, as a depressor, cause two or some equal number of signs to disappear, two or some even number of imaginary roots are discovered. (D.)

The contradiction to article (B) confirms the present.

Two imaginary roots, at least, exist in every equation whose signs are alternately $- +$, beginning at the second term, if the coefficient of the second term, taken positively, is less than the highest index; and even if the second coefficient is equal to the highest, there are imaginary roots, unless all the roots are equal to each other. (E.)

In the equation $x^n - bx + c = 0$, all the roots are imaginary if n is even,

$$\text{and } \frac{n^n}{(n-1)^{n-1}} \text{ greater than } \frac{b^n}{c^{n-1}},$$

if n is odd, there is but one real root, of which $x^3 - 21x + 38 = 0$ is an example. (F.)

The preceding means are employed in respect of the following examples, which the reader may suppose to be derived from equations with greater coefficients, and brought by depression to a proper state for the examination of their roots.

For the most part $\frac{1}{2}$ has been taken for a depressor, when $\frac{r}{q+r}$ was less than $\frac{1}{2}$, for such substitution generally teaches whether we should advance or retrograde.

It often occurs that a single depression is not satisfactory, for no means are yet known, the method of Sturm excepted, to exempt us from trials in almost every department of search after the character of roots.

Example 1.

$$x^8 - x^7 + 4x^6 - \overset{q}{2x^5} + 6x^4 - 2x^3 + 5x^2 - x + \overset{r}{2} = 0,$$

by (B) $x > \frac{1}{2}$, by which depress the roots, the result is ;

$$1 + \cdot5 + 3\cdot75 + + + + + + ;$$

since no change of sign takes place in the depressed equation, all the roots are imaginary. (D.)

Example 2.

$$x^6 - 7x^5 + 19x^4 - \overset{q}{27x^3} + 21x^2 - 9x + \overset{r}{1} = 0,$$

finding that no root exists so great as unity, if a positive root exists it is greater than $\frac{1}{28}$.

Depressing by $\frac{1}{2}$,

$$1 - 4 + \overset{q}{5\cdot25} - 4 + 1\cdot1875 - \cdot75 - \overset{r}{\cdot640625} \ldots\ldots\ldots\ldots (a.)$$

Thus, by the change of sign in the absolute term, there is a positive root between 0 and $\cdot5$; but as five changes of sign still remain, we are yet unacquainted with the nature of four roots.

Depressing the last equation by ·5, making the whole depression *unity*, there results

$1 - 1 - 1 - 1 - 1 - 1 - 1$, showing the loss of four signs.

$\frac{r}{q+r}$ in (a) is a positive quantity;

but by (C), $\frac{c}{c+d} - (1-p) = \frac{1}{1+1} - \frac{1}{2} = 0$:

∴ four roots of the original are imaginary, and two real.

Example 3.

$$x^5 - 7x^4 + 16x^3 - 18x^2 + 9x - 3 = 0$$

in $(x - \cdot 4)$ $\quad 1 - 5 + 6\cdot4 - 4\cdot88 + 0\cdot616 - 1 \ldots\ldots (a)$

in $(x - \cdot 5)$ $\quad 1 - 5\cdot5 + 4\cdot5 - 3\cdot25 - \overset{c}{0\cdot1875} - \overset{d}{1\cdot40625} \ldots\ldots (b)$

in $(x - 1)$ $\quad 1 - 2 - 2 - 2 - 2 - 2 \ldots\ldots (c)$.

Depressing by ·5 two signs are lost; but as one of the terms is very small, it is probable that with ·4 they will be restored, which is found to be the case as above.

The equations (a) and (b) will determine the nature of two roots.

From (a), $\quad x > \frac{1}{7\cdot4} = \cdot13\ldots.$

From (b), $\quad x < \frac{\cdot1875}{1\cdot59375} - (1 - \cdot1) =$ a negative quantity.

The superior limit being less than the inferior, the two roots are imaginary.

The equations (b) and (c) determine the nature of the two remaining roots:

$$x > \frac{1\cdot40625}{4\cdot90625} = +, \text{ and } x < \frac{2}{4} - \frac{1}{2} = 0.$$

∴ four roots are imaginary.

Operating on the equation (c) for the sake of showing that any equation resulting from a depression may be used, we change the alternate signs:

$$x^5 + 2x^4 - 2x^3 + 2x^2 - \overset{q}{2x} + \overset{r}{2} = 0$$

in $(x - \cdot 5)$ $\quad 1 + 4\cdot5 + 4\cdot5 + 3\cdot25 - \cdot1875 + 1\cdot40625 \ldots\ldots (a)$,

two signs are lost; therefore the two roots related to those signs are imaginary; for (D), the least of them, if real, must be greater than $\frac{1}{2}$; but the signs have disappeared by this depressor.

Two other roots are also unreal, for (a) $x > \frac{1\cdot40625}{1\cdot59375} = \cdot8\ldots\ldots$ but a depression by ·1 leaves no change of sign, the 5th term being ·6158. This last proof requires nothing more than regarding the small term $-\cdot1875$ in contrast with the others.

Example 4.

$$x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0,$$

one root is seen to be positive.

Alternating the signs

$$1 - 3 + 2 + 3 - 2 + 2,$$

if another positive root exists it is $> \frac{2}{5}$ or $\frac{4}{10}$; but a depression by ·3 causes two changes of sign to disappear;

$\therefore$ the two roots are imaginary

in $(x - \cdot 5)$ $\qquad 1 - \cdot 5 - \overset{q}{1\cdot 5} + 2\cdot 75 + 1\cdot 315 + 1\cdot 84375;$

if a real root exist it is $> \dfrac{1\cdot 84375}{3\cdot 34375} = \cdot 55 \ldots\ldots$

but with ·51 as depressor the terms are all positive;

$\therefore$ there are four imaginary roots.

Example 5.

$$x^6 + x^5 - x^4 - x^3 + x^2 - \overset{q}{x} + \overset{r}{1} = 0$$

in $(x - \cdot 5)$ $\qquad 1 + 4 + 5\cdot 25 + 2 + \cdot 1875 - \cdot 75 + \cdot 609375 \ldots\ldots (a),$

there are only two changes of sign in this depression; but as the same number is lost by that of $\frac{1}{2}$ which is equal to $\frac{r}{q+r}$, the contradiction implies two imaginary roots.

Depressing (a) by ·2

$$\left.\begin{array}{l} \text{in } (a) - 2 \\ \text{or } \;\; x - \cdot 7 \end{array}\right\} 1 + 5\cdot 2 + 9\cdot 85 + 7\cdot 96 + 2\cdot 9915 - \overset{q}{\cdot 23308} + \cdot 492619,$$

here $\qquad \dfrac{r}{q+r} = \cdot 6 \ldots\ldots$

but depressing this last by ·3, all the terms are positive, viz.:

$$1 + 7 + 19 + 25 + 17 + 5 + 1;$$

$\therefore$ four imaginary roots exist, as far as this equation has been examined.

The methods of Sturm and Fourier are compared in the 234th page of Professor Young's work by means of the preceding equation; and at the same time he remarks, "that certain steps cannot be made without cautious deliberation, and that sort of tact which experience alone can impart to the analyst."

If it were allowed to frame equations having the absolute term decimal, some of low degree might be formed, requiring long attention; for instance, the following:

$$x^3 - 10x^2 + 10x - 2{\cdot}64163101 = 0;$$

but examples of this kind, obeying the general laws of reduction, take up the time of a student unnecessarily.

Example 6.

In the general equation,

$$x^3 - ax^2 + bx - c = 0$$

if $a^2b^2 + 18abc$ is less than $4ca^3 + 4b^3 + 27c^2$,
two roots are imaginary, as in $x^3 - 10x^2 + 12x - 4 = 0$.

To this we shall add another example:

Let $x^3 + 24x^2 - 30x + 10 = 0$

in $(x - {\cdot}5)$ $\quad 1 + 25{\cdot}5 - 5{\cdot}25 + 1{\cdot}125 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$

if a real root exist in (a), it is greater than $\dfrac{1{\cdot}125}{6{\cdot}375} = {\cdot}17 \ldots\ldots$

a depression by ·17 renders all the signs positive.

$\therefore$ two roots are imaginary.

They are also imaginary by (A) depressing by *unity*.

Budan's correlative is $10 + 0 - 6 + 5$, designating two real roots, which renders necessary a resort to the method contained in the second or third part of his treatise.

When this case of Budan's exception occurs in equations of *high* degree, we know of no means of being aware of it, and have, therefore, lost all confidence in his method on trying occasions, such as when the existence of an even number of real roots is announced. Newton's rule is often more satisfactory.

Example 7.

$$x^6 - 11x^5 + 49x^4 - 116x^3 + 164x^2 - 132x + 52 = 0$$

depressed by *unity*,

$$x^6 - 5x^5 + 9x^4 - 10x^3 + 15x^2 - 5x + 7 = 0 \ldots\ldots\ldots\ldots (a),$$

this last having, like the original, six changes of sign, two roots at least are imaginary by (E). But we shall find the nature of all the roots from the equation (a), for the nature of roots is not altered by depression,

in $(x' - {\cdot}5)$ $\quad 1 - 2 + {\cdot}25 - 2 + 8{\cdot}1875 + \overline{5{\cdot}625} + 7{\cdot}421875 \ldots\ldots (b)$

$$x' > \frac{7}{17} = {\cdot}41 \ldots\ldots$$

On referring to (C), the only term changed to *minus* is $5{\cdot}625$,

$$\therefore\ x' < \frac{5{\cdot}625}{13{\cdot}046875} - {\cdot}5, \text{ a negative quantity;}$$

$\therefore$ these two roots are imaginary,

in $(b - {\cdot}8)$ $1 + 2{\cdot}8 + 1{\cdot}85 - 3{\cdot}\overset{q}{7}40 + {\cdot}2835 + 13{\cdot}\underline{27988} + 15{\cdot}84\overset{r}{8}059$,

$$x < \frac{13{\cdot}279}{13{\cdot}279\ldots + 15{\cdot}848\ldots} - {\cdot}5, \text{ negative;}$$

$\therefore$ two roots more are unreal.

The two remaining roots are unreal, because $\cdot 6$, which is less than $\dfrac{r}{q+r}$ in this last, causes all the signs to be *plus*, which being manifest to the eye, needs no operation.

Example 8.

$$x^5 - 5x^2 + 16 = 0.$$

This example is taken from the 'Mem. de Turin,' vol. vi, p. 171.

In $(x - 1)$ $1 + 5 + 10 + 5 - 5 + 12$,

if the two changes of sign denote two positive roots, the least is greater than $\dfrac{12}{17} = {\cdot}7 \ldots$ But if this last equation be depressed by $\cdot 5$, all the signs are positive;

$\therefore$ these two roots are unreal.

Changing the alternate signs of the original equation,

$$1 + 0 + 0 + 5 + 0 - 16 \ldots\ldots\ldots\ldots (a)$$

in $(x' - 1)$ $\quad 1 + \underline{5} + 10 + \underline{15} + 15 - 10 \ldots\ldots\ldots\ldots (b)$.

There are necessarily three changes of signs in (a), and but one in (b).

If the two roots in (a) are real,

the least $\quad > \dfrac{16}{21} = {\cdot}76 \ldots$

that in (b) $\quad < \dfrac{15}{25} = {\cdot}6.$

The contradiction shows that the two roots are unreal.

This example was selected by the learned society in order to clear up some doubts respecting the resolution of the higher orders of equations. It shall now be treated in a different manner,

$4y^5 - 5y^2 - 16 = 0$ is one limiting equation,

$5y^4 - 10y$ is the usual limit.

The root of the former is nearly $1{\cdot}4$,

,, of the latter, nearly $1{\cdot}25$.

But as the first value exceeds the second, the four roots are imaginary.

This method we published in 1837. It is only convenient in binomial and trinomial equations.

Example 9.

$$x^4 + 312x^3 + 23337x^2 - \overset{r}{14874}x + \overset{s}{2360} = 0.$$

An acquaintance with the manner of separating the figures which concur in the real roots of equations is implied in the process of discovering the existence of imaginary roots.

The reader will observe that the following is not given as a rule applicable under all circumstances, nevertheless it is of such frequent occurrence in success, that it is thought advisable to insert it:

$$\frac{2s}{r} = \frac{4720}{14874} = \cdot316\ldots$$

Depress the roots of the equation by ·316, and in the resulting equation take

$$\frac{2s'}{r'} = \frac{\cdot02089}{31\cdot4245} = \cdot000664.$$

The sum of these results is ·316664, which is so far correct for the least root of the original equation; the greater root being ·316665 . . .

This equation is ably managed in Dr. Rutherford's recent work, and in Prof. Young's 'Analysis and Solution of Cubic and Biquadratic Equations.'

Example 10.

$$x^7 \quad 0x^6 - 2x^5 \quad 0x^4 - 3x^3 + 4x^2 - 5x + 6 = 0.$$

The roots of the equation are increased by ·2 in order to avoid the ambiguity of signs preceding the *zeros*,

in $(x + \cdot2)$ $1 - 1\cdot4 - 1\cdot16 + 1\cdot720 - 3\cdot7440 + 5\cdot95328 - 6\cdot975552 + 7\cdot1846272 \ldots$ (*a*)

depressing the roots in x by ·1,

in $(x - \cdot1)$ $\overset{c}{1} + \cdot7 - 1\cdot79 - \cdot965 - 3\cdot1965 + 3\cdot08021 - 4\cdot290993 + \overset{d}{5\cdot5369801} \ldots$ (*b*)

$-\qquad + \qquad - \qquad + \; -$

Here two changes of sign are lost, and it is to be remarked that (*a*) is depressed ·3, so that $p = \cdot3$ . . . . *c*).

If a positive root exist in (*a*) it is $> \dfrac{7\cdot18\ldots}{7\cdot18\ldots + 6\cdot975\ldots} = \cdot5\ldots$

but in (*b*) the greatest root $> \dfrac{1}{6\cdot536\ldots} - \cdot7 =$ a negative quantity;

$\therefore$ the two roots in this interval are imaginary.

If the equation in (*b*) have a positive root it is $> \cdot5\ldots = \dfrac{r}{q + r}$.

Depressing it by ·6, the result is

$$\overset{c}{1}+4{\cdot}9+8{\cdot}29+5{\cdot}005-4{\cdot}3965-5{\cdot}63053-5{\cdot}387457+\overset{d}{3}{\cdot}1772143 \;.\,.\; (f).$$

Here are only two changes of sign,

by (C) the root in (b) $< \frac{8{\cdot}29}{11{\cdot}467} - {\cdot}4 = {\cdot}7 \ldots - {\cdot}4 \ldots = {\cdot}3 \ldots$ which, being less than ·5, the two roots are imaginary.

The remaining roots of the equation are real, two positive and one negative; for if the equation in x be depressed by *unity*, and again by ·5, the last term of the resulting equation will be $-$ 6·2734375.

The long route we have taken is for the sake of showing how, in higher equations, the signs belonging to the *zeros* may be found.

In the work mentioned in the Preface, the imaginary roots of this equation are found in a compendious way. See the 147th page of Prof. YOUNG's 'Theory of Equations,' where this equation of FOURIER occurs.

EXAMPLE 11.

$$x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x - 5 = 0,$$

taken from the 'Arithmetica Universalis.'

$$\text{In} \qquad (x-1)\; 1 + 12 + \underset{-}{55} + 124 + \underset{-}{150} + 91 + \underset{-}{2} - \underset{+}{45} - 31.$$

Here two changes of sign are lost; if a positive root exist in the interval,

it is $> \frac{5}{10} = \frac{1}{2}$,

but by (C) it is $< \frac{150}{181} - {\cdot}5 = {\cdot}82 - {\cdot}5 = {\cdot}32 \ldots$

∴ the two roots are imaginary.

Alternating the signs of the equation

$$1 - 4 - 1 + 10 + 5 + 5 - 10 + 1 - 5,$$

the least positive root $> \frac{5}{15} = {\cdot}333 \ldots$

But depressing by ·3, in a very early part of the process, two changes of sign are lost, the coefficient related to $-$ 10 becoming +

∴ the two roots in this interval are imaginary.

In the whole process it is readily discovered that the equation contains one positive and three negative roots.

It was with some surprise that we found, on searching for the surd divisors, to show which seems to be the object of the renowned author, that the equation has the following factors:

$$x^6 + 5x^5 + 5x^4 * + 10x^2 + 5x + 5 = 0$$
$$x^2 - 1x - 1 = 0$$

therefore the surd divisors are confined to the equation of the 6th degree.

A method for surd divisors was published in 1658 by Johannes Hudde, Burgomaster of Amsterdam, who gave the following instance:

$$x^4 + 4x^3 - 3x^2 - 8x + 4 = 0,$$

whose factors are

$$x^2 + (2 - \sqrt{3})\,x - 2 = 0$$

$$x^2 + (2 + \sqrt{3})\,x - 2 = 0$$

Example 12.

$$x^6 - 7x^5 + 19x^4 - 22x^3 + 11x^2 - 15x + 25 = 0,$$

the least root $> \dfrac{25}{37} = \cdot 59 \ldots$

depressing by its inferior limit $\cdot 5$.

In $(x - \cdot 5)$ $\quad 1 - 4 + 5\cdot 25 + 1 - 1\cdot 3125 - 13 + 18\cdot 484375 \;\ldots\ldots\; (a)$

$\therefore$ two roots in this interval are imaginary, because two signs are lost.

Now depress (a) by $\cdot 8$, in order to render the second term positive, there results

$$1 + \cdot 8 - 1\cdot 15 + 2\cdot 44 + 6\cdot 9115 - \overset{q}{8\cdot 65392} + \overset{r}{8\cdot 858199} \;\ldots\ldots\; (b)$$

The least root in $(b) > \cdot 5$, by which depress

in $(x' - \cdot 5)$ $\; 1 + 3\cdot 8 + 4\cdot 6 + 4\cdot 64 + 10\cdot 784 - \overset{q}{\cdot 04992} + \overset{r}{6\cdot 532864} \;\ldots\ldots (c).$

Two signs being lost, the roots are imaginary in the interval of (a) and (b).

The least root in $(c) > \dfrac{6\cdot 53\ldots}{6\cdot 53\ldots + \cdot 049} = \cdot 9 \ldots$

It is unnecessary to operate further, for it is evident that a very low decimal will occasion the loss of two signs, if (c) is depressed by it.

The six roots are imaginary. The equation has two pairs of equal roots. They form no obstacle to this method of discovering their character. Nor can any other similar equation be proposed that can defeat the means used in respect of the present equation.

The factors of the equation are:

$$x^2 + 1x + 1 = 0$$

$$x^2 - 4x + 5 = 0 \text{ twice.}$$

Example 13.

$$x^{10} - 196830x + 531442 = 0.$$

By (F) all the roots are imaginary.

Example 14.

$$x^6 - 16x^5 + 85x^4 - 144x^3 - 57x^2 + 126x + 54 = 0.$$

This equation was proposed many years since by a learned society in Italy, doubtless with the view that the character of its roots should be ascertained on some general principle. No such object could have been attained. We alluded to this equation in a former work. It is but lately that we have reverted to this equation, and found that it had the following factors :

$$x^2 - 6x + 6 = 0$$

$$x^2 - 5x - 3 = 0 \text{ twice.}$$

Consequently it has six real roots, or four positive and two negative roots, of which two positive roots are equal as well as two negative roots equal, and these are incommensurable roots.

It is easily seen that an equation of the fourth degree, with integral cofficients, the first being *unity*, cannot be constituted with only a single pair of equal incommensurable roots. Two pairs of such roots must simultaneously exist. Thence every equation of a higher degree having one pair of such roots, must have another pair, and such equation must necessarily have a biquadratic factor.

The difficulty of discovering the character of roots that are supposed to be equal by taking the figures of a root to a great extent, and endeavouring, without success, to separate it from another root, is discouraging to the analyst.

If the root on which he is operating belongs to a pair of equal roots, he will find the same obstacle on examining another root belonging to another pair, as well as that connexion between the decimals of the former and the latter, which amounts to all but certainty that two pairs have, separately, equal roots.

We refer to Prof. Young's 'Theory of Equations,' page 116, where the subject of equal roots is amply discussed.

The theorem of Sturm will, undoubtedly, discover the presence of equal roots, but it is impracticable in equations of high degree, notwithstanding the great improvement it has received from Prof. Young.

A thought has occurred to us of finding a substitute for the common measure, in respect of an equation having two pairs of equal incommensurable roots, and no more.

The foundation was a biquadratic, formed from the two equations

$$x^2 + 2ax - 1 = 0$$

$$x^2 + (2a + 1)\,x - 1 = 0$$

taking $a = 998$.

The quadratics are
$$x^2 + 998x - 1 = 0$$
$$x^2 + 999x - 1 = 0$$

whose roots are $-\,499 \pm \sqrt{249002}$

$$-\,499{\cdot}5 \pm \sqrt{249501{\cdot}25}.$$

The decimals in these roots separate almost immediately.

The biquadratic is

$$x^4 + 1997x^3 + 997000x^2 - 1997x + 1 = 0.$$

As the remarks we are about to make are peculiarly connected with the separation of decimals, it may be of use to state that the decimals following very high numbers, whose difference is unity, separate for the most part at the 5th or 6th place; for instance,

$$\sqrt{7896547852144} = 2810079{\cdot}687636$$

$$\sqrt{7896547852145} = 2810079{\cdot}687639$$

Supposing that the biquadratic had been proposed from these numbers, we should have soon discovered, by the early separation of decimals, that the roots were not equal.

The cause of this early separation is the fewness of the figures employed in its construction, connected with their magnitude, and the law of separation is easily traced. It will be remarked that the greatest number of figures in any of the coefficients is *six.*

It is much easier to find eight figures of a root of an equation of high degree than to find what is generally known by the name of the "common measure."

If assent is given, it may be stated that an equation of any degree, having no coefficient, consisting of more than six figures, and which has, or may be supposed to have, two pairs of equal incommensurable roots, and no more, can be freed from the necessity of our finding its common measure.

Let it be supposed that such equation is of the 10th degree, having no more than six figures in any one coefficient. If it have two pairs of equal roots, a biquadratic factor must exist. This cannot be avoided. If it have several concurring figures, a separation must take place in not later than the 6th decimal place, even supposing that the equation has no factors. But we are confident that an equation of the 10th or any higher degree cannot be formed without factors, and with such limit in the figures of one coefficient containing six concurrent decimals in two of its roots, the coefficients being integral, as before mentioned.

This opinion is derived from long study and experience on the subject. We venture, then, to subjoin the following rule in respect of equations of all degrees, above the third of course, whose highest coefficient does not consist of more than six figures:

Find eight correct decimals of a root: increase the last decimal by a unit. Depress the equation resulting from the previous work by the decimal so changed. If two changes of sign are lost, two roots are equal.

For example: let the derived decimals be ·31479305; depress by ·00000006.

This may cause three changes of signs to be lost, provided another root, such as ·314793058 . . . be passed over. This, however, would be no obstacle; it would only show the existence of another positive root. If, then, we have succeeded in the attempt to dispense with the "common measure" when the highest coefficient contains six figures only, we can equally do so when the highest coefficient contains any number of figures, because a biquadratic could be easily found accommodated to the requirements of the case.

Finally, we would remark, that all idea of perplexity, arising from taking the real part of an imaginary root and proceeding with it as if it were one of the real roots, is totally excluded. See Prof. YOUNG's 'Theory of Equations,' page 309, where a *constant* is immediately preceptible. See also the 12th Example, which has been expressly constructed for the purpose of showing that we are enabled to decide that the equal roots are imaginary, if such be the case, in any proposed interval; and our previous reasoning is subject to the preceding remark if a case should occur requiring the interference; but this, we are of opinion, can never happen, under the limited magnitude of the highest coefficient.

Any number of equal roots, *mutatis mutandis,* can be discovered to exist in an equation of any degree.

EXAMPLE 15.

$$x^4 - 72x^3 + 1344x^2 - \overset{p}{1728}x + \overset{q}{576} = 0.$$

Two roots of some kind appear to take place between ·6 and ·7, and a separation of the decimals not being yet attainable, we take $\frac{2q}{p}$, the result is ·66.

In $(x - \cdot 66)\ 1 - 69\cdot 36 + 1204\cdot 0536 - \overset{p'}{46\cdot 859616} + \overset{q'}{\cdot 45643536}$, taking $\frac{2q'}{p'} = \cdot 0194$, and adding it to ·66, there is derived ·6794.

Depressing the last equation by ·0194, found as before, and taking $\frac{2q''}{p''}$, and adding, as before, $x = \cdot 679491924 \ldots$

This is so far the value of two equal roots; for, depressing again by ·000000005, two changes of sign are lost. (See the last Example).

Observation on the Limiting Equations.

Let $$x^4 - 10x^3 + 31x^2 - 30x + 9 = 0,$$

then $$\frac{3x^4 - 20x^3 + 31x^2 - 9}{4x^3 - 30x^2 + 62x - 30} = \frac{0}{0}.$$

There are two pairs of equal roots, which in this case are

$$\left.\begin{array}{l} 2\cdot 5 + \sqrt{3\cdot 25} \\ 2\cdot 5 - \sqrt{3\cdot 25} \end{array}\right\} \text{twice.}$$

(See the 8th Example.)

EXAMPLE 16.

$$x^6 + 370x^5 + 33736x^4 - 90160x^3 + 116333x^2 - 50858x + 6962 = 0.$$

No means are yet known of obtaining, without trial, the first figure of the root of an equation.

This example is given to exemplify the fact, for it obtains in the first and second figures; therefore recourse has been had to trial for the first three figures.

There are also some peculiarities in the roots, the development of which is left to the student. The three following steps are given, unabridged, as well for his assistance, as to give him confidence in the accuracy of the calculations in this treatise.

Depressed by ·3,

$$1 + 371{\cdot}8 + 34292{\cdot}35 - 49343{\cdot}260 + 53506{\cdot}4615 - 1742{\cdot}91242 + 14{\cdot}411429.$$

This by ·01,

$$1 + 371{\cdot}86 + 34310{\cdot}9415 - 47971{\cdot}194180 + 52046{\cdot}74282875 - 687{\cdot}4489800094 + 2{\cdot}283950650681.$$

This by ·006,

$$1 + 371{\cdot}896 + 34322{\cdot}097840 - 47147{\cdot}597709080 + 51190{\cdot}673299529040 - 68{\cdot}039307979776544 + {\cdot}0226222043671500l6.$$

ADDENDA.

ON THE PECULIAR EQUATION $x^3 - bx = 1$.

1. Let $p^3 - rp = s$ be a cubic equation having three real roots, and let it be represented by

$$p^3 - \left(\frac{b^4 - 6b}{3}\right)p = \frac{2b^6}{27} - \frac{2b^3}{3} + 1 \ldots\ldots\ldots\ldots (A)$$

then will

$$\frac{r^3}{s^2} = 6{\cdot}75 + \frac{b^3 - 6{\cdot}75}{s^2} \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

where b is consequently greater than $(6{\cdot}75)^{\frac{1}{3}}$. If $b = 9$, the quotient is $6{\cdot}750000\ldots$, and the greater b is assumed, the number of *zeros* following $6{\cdot}75$ increases in a known ratio, for such number is obvious when the values of b and s are given. The number of *zeros* would be unlimited if the magnitude of b were unlimited, because s is greater than b.

2. The equation $x^3 - bx = 1$ is brought into relation with $p^3 - rp = s$, by assuming

$$x = \frac{p + \frac{b^2}{3}}{1 - bp - \frac{b^3}{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(C)}.$$

Let $x^3 - 30x = 1$; then $b = 30$, and by (A) we get the equation

$$p^3 - 269940\,p = 53982001;$$

and by (B) the quotient

$$\frac{r^3}{s^2} = 6{\cdot}75000000000\ldots$$

When the roots of the equation $x^3 - bx = c$ are equal, then $\frac{b^3}{c^2} = 6{\cdot}75$ exactly, and the positive root $= \frac{3c}{b}$.

The equation in p is so closely allied to this case that $\frac{3s}{r}$ can be safely taken as the positive root in p to a great extent of figures. The result is 599·933329, which is so far correct, and had the division been further extended, it would probably have afforded a still closer approximation.

Applying this value of p to the formula (C), it gives $x = -{\cdot}033334568038$, which is one of the roots of the equation $x^3 - 30x = 1$, and correct to the last decimal figure.

When one of the roots is known, the others are derived by applying such root to the formula $-\frac{x}{2} \pm \left\{b - \frac{3x^2}{4}\right\}^{\frac{1}{2}}$. The remaining roots in p are $-$ 299·9664599 and $-$ 299·966869, which, if somewhat more extended, would give the other roots in x.

3. The equation $x^3 + bx = 1$ may be treated in the same manner as the foregoing equation, for the three formulas remain unaltered, but b and its odd powers are here negative.

Let, for example, $x^3 + 12x = 1$ be the proposed equation; then

$$p^3 - 6936\,p = 222337;$$

$$\therefore\ p = 96{\cdot}1665224, \text{ and thence } x = {\cdot}083285191\ldots$$

There are some very remarkable features connected with the resolution of all cubic equations arising from the present case.

A few small values of b are, in the present state of this problem, not suited to the resolution. The effectiveness of $b = 6$ may be shown in the following manner, and as this number answers the purpose, all superior integers and mixed fractions will be equally effective.

Let $$x^3 - 6x = 1,$$

$$x'^3 + 6x' = 1,$$

and as the equations in p were introduced only to render the subject more intelligible, they will now be dispensed with.

Now $$p \text{ and } p' = \frac{3s}{r} = \frac{2b^2}{3} - \frac{2}{b} - \frac{3}{b^4 - 6b};$$

$$\therefore\ p = 23{\cdot}664, \quad p' = 24{\cdot}331.$$

and by $$\text{(C) } x = -\frac{35{\cdot}664}{212{\cdot}984} = -{\cdot}16744919\ldots\ldots$$

$$x' = +\frac{36{\cdot}331}{218{\cdot}986} = +{\cdot}16590558\ldots\ldots$$

It is apparent that these approximating roots may be derived from each other, an unusual circumstance in ordinary operations.

By means of the resolution of the proposed equation, the following are resolved:

1. The equation $y^3 + by^2 = 1$, whose roots are $-(a^2 + ab)$, $-(b^2 + ab)$, $+ ab$ when the roots of the proposed are $a + b$, $-a$, $-b$.
2. The equation $y^3 - b^3y = b^3$, since $bx = y$.
3. The equation $y^3 - ry = s$, when $\frac{r^3}{s^2} = b^3$, for $\frac{rx}{s} = y$, reference being had to the equation $x^3 - b^3x = b^3$.

Another class comprehends all cubic equations having roots real or imaginary. The same formulas are used as in the preceding resolutions, with the exception that a variable quantity is employed, which in the former was a constant, and that constant was $-b$.

Keeping to the equation $x^3 - bx = 1$, having three real roots, and where the value of b is not less than 6, let it be connected with

$$p^3 - \left(\frac{b^2}{3}n^2 + 3n + b\right)p = \left(1 - \frac{2b^3}{27}\right)n^3 + bn^2 + \frac{2b^2}{3}n + 1,$$

where n may be any number or fraction whatever, positive or negative. Having found the approximate values of the roots of $x^3 - bx = 1$, equations are obtained in p, which are brought into relation with those in x by assuming

$$\frac{p + \frac{2bn}{3} + n^2}{np - \frac{b}{3}n^2 + 1} = x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(D.)}$$

Thus let the equation be $x^3 - 6x = 1$, one of whose roots has been found above, and if

$n = +1$, then we have $p^3 - 21p = 16$,
$n = -1$ „ „ $p^3 - 15p = -2$,
$n = +2$ „ „ $p^3 - 60p = -47$,
$n = -2$ „ „ $p^3 - 48p = 97$, &c., $p = 7{\cdot}776$.

And the roots of these equations can readily be obtained from those of the equation $x^3 - 6x = 1$. Selecting the equation $p^3 - 15p = -2$, we have, by (D),

$$\frac{p - 3}{-p - 1} = -{\cdot}16744919 = x;$$

hence $$p = +3{\cdot}8045115.$$

C. and J. Adlard, Printers, Bartholomew Close.