Eugene Schuyler,

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Eng.d by A. H. Ritchie

Very truly yours
Charles Davies

THE

NATURE AND UTILITY

OF

MATHEMATICS,

WITH THE BEST METHODS OF INSTRUCTION EXPLAINED
AND ILLUSTRATED

BY

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PREFACE.

THE following work is not a series of speculations. It is but an analysis of that system of mathematical instruction which has been steadily pursued at the Military Academy nearly half a century, and which has given to that institution its celebrity as a school of mathematical science.

It is of the essence of that system that a principle be taught before it is applied to practice; that general principles and general laws be taught, for their contemplation is far more improving to the mind than the examination of isolated propositions; and that when such principles and such laws are fully comprehended, their applications be then taught, as consequences, or practical results.

This view of education led, at an early day, to the union of the French and English systems of Mathematics. By this union the exact and beautiful methods of generalization, which distinguish the French school, were blended with the practical methods of the English system.

The fruits of this new system of instruction have been abundant. The graduates of the Military Academy have been sought for wherever science of the highest grade has been needed. Russia has sought them to construct her railroads;* the Coast Survey needed their aid; the works of internal improvement of the first class in our country, have mostly been conducted under their direction; and the war with Mexico afforded ample opportunity for showing the thousand ways in which science—the highest class of knowledge—may be made available in practice.

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* Major Whistler, the engineer, to whom was intrusted the great enterprise of constructing a railroad from St. Petersburg to Moscow, and Major Brown, who succeeded him at his death, were both graduates of the Military Academy.

All these results are due to the system of instruction. In that system, Mathematics is the basis—Science precedes Art —Theory goes before Practice—the general formula embraces all the particulars.

Although my official connection with the Military Academy was terminated many years since, yet the general system of Mathematical instruction has not been changed. Younger and able professors have extended and developed it, and it now forms an important element in the education of the country.

The present work is a modification, in many important particulars, of the Logic and Utility of Mathematics, published in the year 1850. The changes in the Text, seemed to require a change in the Title.

It was deemed necessary to the full development of the plan of the work, to give a general view of the subject of Logic. The materials of Book I. have been drawn, mainly, from the works of Archbishop Whately and Mr. Mill. Although the general outline of the subject has but little resemblance to the work of either author, yet very much has been taken from both; and in all cases where it could be done consistently with my own plan, I have adopted their exact language. This remark is particularly applicable to Chapter III., Book I., which is taken, with few alterations, from Whately.

For a full account of the objects and plan of the work, the reader is referred to the Introduction.

FISHKILL LANDING,
    January, 1873.

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INTRODUCTION.

OBJECTS AND PLAN OF THE WORK.

UTILITY and Progress are the two leading ideas of the present age. They were manifested in the formation of our political and social insti- tutions, and have been further developed in the extension of those institutions, with their subduing and civilizing influences, over the fairest portions of a great continent. They are now becoming the controlling elements in our systems of public instruction.

What, then, must be the basis of that system of education which shall embrace within its horizon a Utility as comprehensive and a Progress as permanent as the ordinations of Providence, exhibited in the laws of nature, as made known by science ? It must obviously be laid in the examination and analysis of those laws ; and

Preparatory studies.

primarily, in those preparatory studies which fit and qualify the mind for such Divine Contemplations.

Bacon's Philosophy.

When Bacon had analyzed the philosophy of the ancients, he found it speculative. The great highways of life had been deserted. Nature, spread out to the intelligence of man, in all the minuteness and generality of its laws—in all the harmony and beauty which those laws develop—had scarcely been consulted by the ancient philosophers. They had looked within, and not without. They sought to rear systems on the uncertain foundations of human hypothesis and speculation, instead of resting them on the immutable laws of Providence, as manifested in the material world. Bacon broke the bars of this mental prison-house: bade the mind go free, and investigate nature.

Philosophy of the Ancients.

Foundations of Bacon's Philosophy:

Bacon laid the foundations of his philosophy in organic laws, and explained the several processes of experience, observation, experiment, and induction, by which these laws are made known. He rejected the reasonings of Aristotle because they were not progressive and useful ; because they added little to knowledge, and contributed nothing to ameliorate the sufferings and elevate the condition of humanity.

Why opposed to Aristotle's.

The time seems now to be at hand when the **Practical Philosophy** philosophy of Bacon is to find its full develop-ment. The only fear is, that in passing from a speculative to a practical philosophy, we may, for a time, lose sight of the fact, that Practice without Science is Empiricism; and that all **Its true na-ture.** which is truly great in the practical must be the application and result of an antecedent ideal.

What, then, are the sources of that Utility, **What is the true sys-tem of edu-cation:** and the basis of that Practical, which the pres-ent generation desire, and after which they are so anxiously seeking? What system of training and discipline will best develop and steady the intellect of the young; give vigor and expan-sion to thought, and stability to action? What **Which will develop and steady the intellect?** course of study will most enlarge the sphere of investigation; give the greatest freedom to the mind without licentiousness, and the greatest freedom to action consistent with the laws of nature, and the obligations of the social com-pact? What subject of study is, from its na-ture, most likely to ensure this training, and **What are the subjects of study?** contribute to such results, and at the same time lay the foundations of all that is truly great in the Practical? It has seemed to me that math-**Mathematics** ematical science may lay claim to this pre-emi-nence.

Founda-
tions of math-
ematical
knowledge.

Laws of
Nature.

Number
and
Space.

The first impressions which the child receives of Number and Quantity are the foundations of his mathematical knowledge. They form, as it were, a part of his intellectual being. The laws of Nature are merely truths or generalized facts, in regard to matter, derived by induction from experience, observation, and experiment. The laws of mathematical science are generalized truths derived from the consideration of Number and Space. All the processes of inquiry and investigation are conducted according to fixed laws, and form a science ; and every new thought and higher impression form additional links in the lengthening chain.

Mathemat-
ical knowl-
edge :

What it
does.

The knowledge which mathematical science imparts to the mind is deep—profound—abiding. It gives rise to trains of thought, which are born in the pure ideal, and fed and nurtured by an acquaintance with physical nature in all its minuteness and in all its grandeur : which survey the laws of elementary organization, by the microscope, and weigh the spheres in the balance of universal gravitation.

What
the processes
effect.

The processes of mathematical science serve to give mental unity and wholeness. They impart that knowledge which applies the means of

crystallization to a chaos of scattered particulars, Right knowl-edge applies the means of crystalliza-tion. and discovers at once the general law, if there be one, which forms a connecting link between them. Such results · can only be attained by minds highly disciplined by scientific combinations. In all these processes no fact of science is forgotten or lost. They are all engraved on the great tablet of universal truth, there to be read by succeeding generations so long as the It records and preserves truth. laws of mind remain unchanged. This is strikingly illustrated by the fact, that any diligent student of a college may now read the works of Newton, or the Mécanique Céleste of La Place.

The educator regards mathematical science How the educator re-gards math-ematics. as the great means of accomplishing his work. The definitions present clear and separate ideas, which the mind readily apprehends. The axioms The axioms. are the simplest exercises of the reasoning faculty, and afford the most satisfactory results in the early use and employment of that faculty. The trains of reasoning which follow are combinations, according to logical rules, of what has been previously fully comprehended; and Influence of the study of mathematics on the mind. the mind and the argument grow together, so that the thread of science and the warp of the intellect entwine themselves, and become inseparable. Such a training will lay the foundations

of systematic knowledge, so greatly preferable to conjectural judgments.

How the philosopher regards mathematics: The philosopher regards mathematical science as the mere tools of his higher vocation. Looking with a steady and anxious eye to Nature, and the great laws which regulate and govern all things, he becomes earnestly intent on their examination, and absorbed in the wonderful harmonies which he discovers. Urged forward by these high impulses, he sometimes neglects that thorough preparation, in mathematical science, necessary to success; and is not unfrequently obliged, like Antæus, to touch again his mother earth, in order to renew his strength.

Its necessity to him.

The views of the practical man. The mere practical man regards with favor only the results of science, deeming the reasonings through which these results are arrived at, quite superfluous. Such should remember that the mind requires instruments as well as the hands, and that it should be equally trained in their combinations and uses. Such is, indeed, now the complication of human affairs, that to do one thing well, it is necessary to know the properties and relations of many things. Every thing, whether existing in the abstract or in the material world; whether an element of knowl-

Instruments of the mind

Every thing has a law.

edge or a rule of art, has its connections and its law: to understand these connections and that law, is to know the thing. When the principle is clearly apprehended, the practice is easy.

To know the law is to know the thing.

With these general views, and under a firm conviction that mathematical science must become the great basis of education, I have bestowed much time and labor on its analysis, as a subject of knowledge. I have endeavored to present its elements separately, and in their connections; to point out and note the mental faculties which it calls into exercise; to show why and how it develops those faculties; and in what respect it gives to the whole mental machinery greater power and certainty of action than can be attained by other studies. To accomplish these ends, in the way that seemed to me most desirable, I have divided the work into three parts, arranged under the heads of Book I., II., and III

Mathematics analyzed.

How.

What was deemed necessary.

Book I. treats of Logic, both as a science and an art; that is, it explains the laws which govern the reasoning faculty, in the complicated processes of argumentation, and lays down the rules, deduced from those laws, for conducting such processes. It being one of the leading

Logic.

Explanation.

For what used.

objects to show that mathematical science is the best subject for the development and application of the principles of logic ; and, indeed, that the science itself is but the application of those prin-

The necessity of treating it.

ciples to the abstract quantities Number and Space, it appeared indispensable to give, in a manner best adapted to my purpose, an outline of the nature of that reasoning by means of which all inferred knowledge is acquired.

Book II.

Book II. treats of Mathematical Science. Here I have endeavored to explain the nature of

Of what it treats.

the subjects with which mathematical science is conversant ; the ideas which arise in examining and contemplating those subjects ; the language employed to express those ideas, and the laws of their connection.   This, of course, led to a class-

Manner of treating.

ification of the subjects ; to an analysis of the language used, and an examination of the reasonings employed in the methods of proof.

Book III. Utility of Mathematics.

Book III. explains and illustrates the Utility of Mathematics : First, as a means of mental discipline and training ; Secondly, as a means of acquiring knowledge ; and, Thirdly, as furnishing those rules of art, which make knowledge practically effective.

Having thus given the general outlines of the work, we will refer to the classes of readers for whose use it is designed, and the particular advantages and benefits which each class may receive from its perusal and study.

There are four classes of readers, who may, it is supposed, be profited, more or less, by the perusal of this work :

1st. The general reader ;

2d. Professional men and students ;

3d. Students of mathematics and philosophy ;

4th. Professional Teachers.

First. The general reader, who reads for im- provement, and desires to acquire knowledge, must carefully search out the import of language. He must early establish and carefully cultivate the habit of noting the connection between ideas and their signs, and also the relation of ideas to each other. Such analysis leads to attentive reading, to clear apprehension, deep reflection, and soon to generalization.

Logic considers the forms in which truth must be expressed, and lays down rules for reducing all trains of thought to such known forms. This habit of analyzing arms us with tests by which we separate argument from sophistry—truth from falsehood. The application of these principles,

In the study in the construction of the mathematical science,
of
mathematics. where the relation between the sign (or language)
and the thing signified (or idea expressed), is un-
mistakable, gives precision and accuracy, leads
to right arrangement and classification, and thus
prepares the mind for the reception of general
knowledge.

Advantages   Secondly. The increase of knowledge carries
to profession-
al men.   with it the necessity of classification.   A limited
number of isolated facts may be remembered, or
a few simple principles applied, without tracing
out their connections, or determining the places
which they occupy in the science of general
knowledge.   But when these facts and principles
are greatly multiplied, as they are in the learned
The reason. professions ; when the labors of preceding gen-
erations are to be examined, analyzed, compared ;
when new systems are to be formed, combining
all that is valuable in the past with the stimu-
lating elements of the present, there is occasion
for the constant exercise of our highest facul-
Knowledge ties.   Knowledge reduced to order ; that is,
reduced to
order is knowledge so classified and arranged as to be
science.   easily remembered, readily referred to, and ad-
vantageously applied, will alone suffice to sift
the pure metal from the dust of ages, and fashion
it for present use.   Such knowledge is Science.

Masses of facts, like masses of matter, are capable of very minute subdivisions; and when we know the law of combination, they are readily divided or reunited. To know the law, in any case, is to ascend to the source; and without that knowledge the mind gropes in darkness.

Knowledge may be reduced to its elements.

It has been my aim to present such a view of Logic and Mathematical Science as would clearly indicate, to the professional student, and even to the general reader, the outlines of these subjects. Logic exhibits the general formula applicable to all kinds of argumentation, and mathematics is an application of logic to the abstract quantities Number and Space.

Objects of the work.

Logic and mathematics

When the professional student shall have examined the subject, even to the extent to which it is here treated, he will be impressed with the clearness, simplicity, certainty, and generality of its principles; and will find no difficulty in making them available in classifying the facts, and examining the organic laws which characterize his particular department of knowledge.

Certainty of the results.

Thirdly. Mathematical knowledge differs from every other kind of knowledge in this: it is, as it were, a web of connected principles spun out from a few abstract ideas, until it has become one of the great means of intellectual develop-

Mathematical knowledge.

Its extent.

ment and of practical utility. And if I am per-

mitted to extend the figure, I may add, that the
web of the spider, though perfectly simple, if we
see the end and understand the way in which
it is put together, is yet too complicated to be
unravelled, unless we begin at the right point,
and *observe the law of its formation*. So with
mathematical science. It is evolved from a few
—a very few—elementary and intuitive princi-

ples : the law of its evolution is simple but ex-
acting, and to begin at the right place and pro-
ceed in the right way, is all that is necessary to
make the subject easy, interesting, and useful.

I have endeavored to point out the place of
beginning, and to indicate the way to the math-
ematical student. I am aware that he is start-
ing on a road where the guide-boards resemble
each other, and where, for the want of careful
observation, they are often mistaken ; I have
sought, therefore, to furnish him with the maps
and guide-books of an old traveller.

By explaining with minuteness the subjects
about which mathematical science is conversant,
the whole field to be gone over is at once sur-
veyed : by calling attention to the faculties of

the mind which the science brings into exercise,
we are better prepared to note the intellectual
operations which the processes require ; and by

a knowledge of the laws of reasoning, and an acquaintance with the tests of truth, we are enabled to verify all our results. These means have been furnished in the following work, and to aid the student in classification and arrangement, diagrams have been prepared exhibiting separately and in connection all the principal parts of mathematical science. The student, therefore, who adopts the system here indicated, will find his way clearly marked out, and will recognise, from their general resemblance to the descriptions, all the guide-posts which he meets. He will be at no loss to discover the connection between the parts of his subject. Beginning with first principles and elementary combinations, and guided by simple laws, he will go forward from the exercises of Mental Arithmetic to the higher analysis of Mathematical Science on an ascent so gentle, and with a progress so steady, as scarcely to note the changes. And indeed, why should he? For all mathematical processes are alike in their nature, governed by the same laws, exercising the same faculties, and lifting the mind towards the same eminence.

Fourthly. The leading idea, in the construction of the work, has been, to afford substantial aid to the professional teacher. The nature of

Of a knowledge of the laws of reasoning.

What has been done.

Advantages to the student.

Where he begins.

Order of progress.

Unity of the subject.

Advantages to the professional teacher.

His duties: his duties—their inherent difficulties—the perDiscouragements and difficulties: plexities which meet him at every step—the want of sympathy and support in his hours of discouragement — (and they are many) — are circumstances which awaken a lively interest in the hearts of all who have shared the toils, and been themselves laborers in the same vineyard. He takes his place in the schoolhouse by the roadside, and there, removed from the highways of Remoteness from active life. life, spends his days in raising the feeble mind of childhood to strength—in planting aright the seeds of knowledge—in curbing the turbulence of passion — in eradicating evil and inspiring good. The fruits of his labors are seen but once in a generation. The boy must grow to Fruits of his efforts, when seen manhood and the girl become a matron before he is certain that his labors have not been in vain.

Yet, to the teacher is committed the high trust of forming the intellectual, and, to a certain extent, the moral development of a people. He The importance of his labors. holds in his hands the keys of knowledge. If the first moral impressions do not spring into life at his bidding, he is at the source of the stream, and gives direction to the current. Although himself imprisoned in the schoolhouse, his influence and his teachings affect all conditions of society, and reach over the whole hori-

zon of civilization. He impresses himself on The influence of his labors. the young of the age in which he lives, and lives again in the age which succeeds him.

All good teaching must flow from copious Scurces of good teaching. knowledge. The shallow fountain cannot emit a vigorous stream. In the hope of doing something that may be useful to the professional teacher, I have attempted a careful and full Objects for which the work was undertaken. analysis of mathematical science. I have spread out, in detail, those methods which have been carefully examined and subjected to the test of long experience. If they are the right meth- Principles of all teaching, the same. ods, they will serve as standards of teaching; for, the principles of imparting instruction are the same for all branches of knowledge.

The system which I have indicated is com- System. plete in itself. It lays open to the teacher the entire skeleton of the science—exhibits all its What it presents. parts separately and in their connection. It explains a course of reasoning simple in itself, What it explains. and applicable not only to every process in mathematical science, but to all processes of argumentation in every subject of knowledge.

The teacher who thus combines science with Science combined with art: art, no longer regards Arithmetic as a mere treadmill of mechanical labor, but as a means—

The advan-
tages result-
ing from it.   and the simplest means—of teaching the art and
science of reasoning on quantity—and this is
the logic of mathematics.   If he would accom-
Results of   plish well his work, he must so instruct his
right instruc-
tion.   pupils that they shall apprehend clearly, think
quickly and correctly, reason justly, and above
all, he must inspire them with a love of knowl-
edge.

BOOK I.

LOGIC.

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CHAPTER I.

DEFINITIONS.

§ 1. DEFINITION is a metaphorical word, which literally signifies " laying down a boundary." All definitions are of names, and of names only ; but in some definitions, it is clearly apparent, that nothing is intended except to explain the meaning of the word ; while in others, besides explaining the meaning of the word, it is also implied that there exists, or may exist, a *thing* corresponding to the word.

<div style="float:right">Definition a metaphorical word.

Some definitions explain only words:

others imply *things* corresponding to the words.</div>

§ 2. Definitions which do not imply the existence of things corresponding to the words defined, are those usually found in the Dictionary of one's own language. They explain only the

<div style="float:right">Of definitions which do not imply things corresponding to words.</div>

They explain words by equivaleuts. meaning of the word or term, by giving some equivalent expression which may happen to be better kncwn. Definitions which imply the existence of things corresponding to the words defined, do more than this.

Definition of a triangle; what it implies. For example : "A triangle is a rectilineal figure having three sides." This definition does two things :

1st. It explains the meaning of the word triangle ; and,

2d. It implies that there exists, or may exist, a rectilineal figure having three sides.

Of a definition which implies the existence of a thing. § 3. To define a word when the definition is to imply the existence of a thing, is to select from all the properties of the thing those which are most simple, general, and obvious ; and the

Properties must be known. properties must be very well known to us before we can decide which are the fittest for this purpose. Hence, a thing may have many properties besides those which are named in the definition

A definition supports truth. of the word which stands' for it. This second kind of definition is not only the best form of expressing certain conceptions, but also contributes to the development and support of new truths.

In Mathematics names imply § 4. In Mathematics, and indeed, in all strict sciences, names imply the existence of the things

which they name; and the definitions of those things names express attributes of the things; so that and
express no correct definition whatever, of any mathe- attributes matical term, can be devised, which shall not express certain attributes of the thing correspond- ing to the name. Every definition of this class Definitions is a tacit assumption of some proposition which of this class
are is expressed by means of the definition, and propositions. which gives to such definition its importance.

§ 5. All the reasonings in mathematics, which Reasoning rest ultimately on definitions, do, in fact, rest resting on
definitions; on the intuitive inference, that things corre- sponding to the words defined have a *conceiv-* rests on
intuitive
inferences. *able* existence as subjects of thought, and do or may have *proximately*, an *actual* existence.*

---

* There are four rules which aid us in framing defini- Four rules tions.

1st. The definition must be *adequate :* that is, neither too 1st rule. extended, nor too narrow for the word defined.

2d. The definition must be in itself *plainer* than the word 2d rule. defined, else it would not explain it.

3d. The definition should be expressed in a *convenient* 3d rule. *number of appropriate words.*

4th. When the definition implies the existence of a thing 4th rule. corresponding to the word defined, the certainty of that existence must be intuitive.

OPERATIONS OF THE MIND CONCERNED IN REASONING.

Three opera-
tions of the
mind.

§ 6. There are three operations of the mind which are immediately concerned in reasoning.

1st. Simple apprehension ; 2d. Judgment ; 3d. Reasoning or Discourse.

Simple ap-
prehension.

Incomplex.

Complex.

§ 7. Simple apprehension is the notion (or conception) of an object in the *mind,* analogous to the perception of the senses. It is either Incomplex or Complex. Incomplex Apprehension is of one object, or of several without any *relation* being perceived between them, as of a triangle, a square, or a circle : Complex is of several with such a *relation,* as of a triangle within a circle, or a circle within a square.

Judgment
defined.

Judgment
is
either
affirmative
or
negative.

§ 8. Judgment is the comparing together in the mind two of the notions (or ideas) which are the objects of apprehension, whether complex or incomplex, and pronouncing that they agree or disagree with each other, or that one of them *belongs* or does *not* belong to, the other : for example : that a right-angled triangle and an equilateral triangle belong to the class of figures called triangles ; or that a square is *not* a circle. Judgment, therefore, is either *Affirmative* or *Negative.*

§ 9. Reasoning (or discourse) is the act of proceeding from certain judgments to another *founded* upon them (or the result of them).

Reasoning defined.

§ 10. Language affords the *signs* by which these operations of the mind are recorded, expressed, and communicated. It is also an instrument of thought, and one of the principal helps in all mental operations; and any imperfection in the instrument, or in the mode of using it, will materially affect any result attained through its aid.

Language affords signs of thought:

also, an instrument of thought.

§ 11. Every branch of knowledge has, to a certain extent, its own appropriate language; and for a mind not previously versed in the meaning and right use of the various words and signs which constitute the language, to attempt the study of methods of philosophizing, would be as absurd as to attempt reading before learning the alphabet.

Every branch of knowledge has its own language,

which must be learned.

ABSTRACTION.

§ 12. The faculty of abstraction is that power of the mind which enables us, in contemplating any object (or objects), to attend exclusively to

Abstraction,

some particular circumstance belonging to it, and quite withhold our attention from the rest. Thus,

if a person in contemplating a rose should make the scent a distinct object of attention, and lay aside all thought of the form, color, &c., he would *draw off*, or *abstract* that particular part;

and therefore employ the faculty of *abstraction*. He would also employ the same faculty in considering whiteness, softness, virtue, existence, as entirely separate from particular objects.

§ 13. The term *abstraction*, is also used to denote the *operation* of abstracting from one or

more things the particular part under consideration; and likewise to designate the *state* of the mind when occupied by abstract ideas. Hence, abstraction is used in three senses:

1st. To denote a faculty or power of the mind;

2d. To denote a process of the mind; and,

3d. To denote a state of the mind.

GENERALIZATION.

§ 14. Generalization is the process of con-

templating the agreement of several objects in certain points (that is, abstracting the circumstances of agreement, disregarding the differ-

ences), and giving to all and each of these ob- of several things.
jects a name applicable to them in respect to
this agreement. For example ; we give the
name of triangle, to every rectilineal figure hav-
ing *three* sides : thus we *abstract* this property
from all the others (for, the triangle has three Generaliza-tion
angles, may be equilateral, or scalene, or right-
angled), and name the entire class from the prop-
erty so abstracted. Generalization therefore implies abstraction
necessarily implies abstraction ; though abstrac-
tion does not imply generalization.

TERMS—SINGULAR TERMS—COMMON TERMS.

§ 15. An act of apprehension, expressed in
language, is called a Term. Proper names, or A term.
any other terms which denote each but a single
individual, as " Cæsar," " the Hudson," " the
Conqueror of Pompey," are called Singular Singular terms.
Terms.

On the other hand, those terms which denote
any individual of a whole class (which are form-
ed by the process of abstraction and generaliza-
tion), are called Common or general Terms. For Common terms.
example ; quadrilateral is a common term, appli-
cable to every rectilineal plane figure having
four sides ; River, to all rivers ; and Conqueror,
to all conquerors. The individuals for which a
common term stands, are called its *Significates.* Significates

3

CLASSIFICATION.

Classification.

§ 16. Common terms afford the means of classification; that is, of the arrangement of objects into classes, with reference to some common and distinguishing characteristic. A collection, comprehending a number of objects, so arranged, is called a Genus or Species—genus being the more extensive term, and often embracing many species.

Genus, species.

Examples in classification.

For example: animal is a genus embracing every thing which is endowed with life, the power of voluntary motion, and sensation. It has many species, such as man, beast, bird, &c. If we say of an animal, that it is *rational*, it belongs to the species man, for this is the characteristic of that species. If we say that it has wings, it belongs to the species bird, for this, in like manner, is the characteristic of the species bird.

Subspecies or classes.

A species may likewise be divided into classes, or subspecies; thus the species man, may be divided into the classes, male and female, and these classes may be again divided until we reach the individuals.

Principles of classification.

§ 17. Now, it will appear from the principles which govern this system of classification, that

the characteristic of a genus is of a more exten- Genus more
sive signification, but involves fewer particu- than species,
lars than that ot a species.   In like manner, the
characteristic of a species is more extensive, but
less full and *complete*, than that of a subspecies but less full
or class, and the characteristics of these less full complete.
than that of an individual.

For example ; if we take as a genus the Quadri-
laterals of Geometry, of which the characteristic
is, that they have four sides, then every plane
rectilineal figure, having four sides, will fall under
this class.   If, then, we divide all quadrilaterals
into two species, viz. those whose opposite sides,
taken two and two, are not parallel, and those
whose opposite sides, taken two and two, are
parallel, we shall have in the first class, all irreg-
ular quadrilaterals, including the trapezoid (1 and
2) ; and in the other, the parallelogram, the rhom-
bus, the rectangle, and the square (3, 4, 5, and 6).
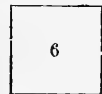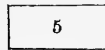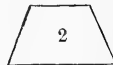
If, then, we divide the first species into two
subspecies or classes, we shall have in the one, the
irregular quadrilaterals (1), and in the other, the
trapezoids (2) ; and each of these classes, being
made up of individuals having the *same char-
acteristics*, are not susceptible of further division.

If we divide the second species into two
classes, arranging those which have oblique an-
gles in the one, and those which have right

angles in the other, we shall have in the first, two varieties, viz. the common parallelogram and the equilateral parallelogram or rhombus (3 and 4) ; and in the second, two varieties also, viz. the rectangle and the square (5 and 6).

Species and classes.

Now, each of these six figures is a *quadri-lateral ;* and hence, possesses the *characteristic* of the genus ; and each variety of both species enjoys all the characteristics of the species to which it belongs, together with some other distinguishing feature ; and similarly, of *all classifications*.

Each individual falling under the genus enjoys all the characteristics.

§ 18. In special classifications, it is often not necessary to begin with the most general characteristics; and then the genus with which we begin, is in fact but a *species of a more extended classification*, and is called a Subaltern Genus.

Subaltern genus.

For example ; if we begin with the genus Parallelogram, we shall at once have two species, viz. those parallelograms whose angles are oblique and those whose angles are right angles ; and in each species there will be two varieties, viz. in the first, the common parallelogram and the rhombus ; and in the second, the rectangle and square.

Parallelogram.

§ 19. A genus which cannot be considered as a species, that is, which cannot be referred

Highest genus.

to a more extended classification, is called the highest genus; and a species which cannot be considered as a genus, because it contains only individuals having the same characteristic, is called the lowest species.

<div style="text-align: right">Highest genus.

Lowest species.</div>

NATURE OF COMMON TERMS.

§ 20. It should be steadily kept in mind, that the "common terms" employed in classification, have not, as the names of individuals have, any *real existing thing in nature* corresponding to them; but that each is merely a name denoting a certain *inadequate notion* which our minds have formed of an *individual*. But as this name does not include any thing wherein that individual differs from others of the same class, it is applicable equally well to all or any of them. Thus, quadrilateral denotes no *real thing*, distinct from each individual, but merely *any* rectilineal figure of four sides, viewed *inadequately ;* that is, after *abstracting* and *omitting* all that is *peculiar* to each individual of the class. By this means, a common term becomes applicable alike to any one of several individuals, or, taken in the plural, to several individuals together.

<div style="text-align: right">A common term has no real thing corresponding:

is an inadequate notion:

does not include any thing in which individuals differ;

but is applicable to many individuals.</div>

Much needless difficulty has been raised respecting the results of this process : many having contended, and perhaps more having taken

<div style="text-align: right">Needless difficulty.</div>

Difficulty in the interpretation of common terms. it for granted, that there must be some really existing *thing* corresponding to each of those common terms, and of which such term is the name, standing for and representing it. For example; since there is a really existing thing corresponding to and signified by the proper and singular name " Ætna," it has been supposed that the *common* term " Mountain" must have some one really existing thing corresponding to it, and of course *distinct* from each individual mountain, yet existing *in* each, since the term, being common, is applicable, separately, to every one of them.

No one real thing corresponding to each.

The fact is, the notion expressed by a common term is merely an inadequate (or incomplete) notion of an individual; and from the very circumstance of its inadequacy, it will apply equally well to any one of several individuals. For example; if I omit the mention and the consideration of every circumstance which distinguishes . Ætna from any other mountain, I then form a notion, that inadequately designates Ætna. This notion is expressed by the common term " mountain," which does not imply any of the peculiarities of the mountain Ætna, and is equally applicable to any one of several individuals.

Merely an inadequate notion partially designating the thing.

" Mountain" is applicable to all mountains.

In regard to classification, we should also bear in mind, that we may fix, arbitrarily, on the

characteristic which we choose to abstract and consider as the basis of our classification, disregarding all the rest : so that the same individual may be referred to any of several different species, and the same species to several genera, as suits our purpose.

May fix on attributes arbitrarily for classification

SCIENCE.

§ 21. Science, in its popular signification, means knowledge.* In a more restricted sense, it means knowledge reduced to order; that is, knowledge so classified and arranged as to be easily remembered, readily referred to, and advantageously applied. In a more strict and technical sense, it has another signification.

Science in its general sense.

Has a technical signification.

" Every thing in nature, as well in the inanimate as in the animated world, happens or is done according to rules, though we do not always know them. Water falls according to the laws of gravitation, and the motion of walking is performed by animals according to rules. The fish in the water, the bird in the air, move according to rules. There is nowhere any want of rule. When we think we find that want, we can only say that, in this case, the rules are unknown to us."†

Views of Kant.

General laws.

Nowhere any want of rule.

Assuming that all the phenomena of nature

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* Section 23.   † Kant.

Science in a technical sense defined: is an analysis of the laws of nature.

are consequences of general and immutable laws, we may define *Science* to be the analysis of those laws,—comprehending not only the connected processes of experiment and reasoning which make them known to man, but also those processes of reasoning which make known their individual and concurrent operation in the development of individual phenomena.

ART.

§ 22. Art is the application of knowledge to practice. Science is conversant about knowledge : Art is the use or application of knowledge, and is conversant about works. Science has knowledge for its object : Art has knowledge for its guide. A principle of science, when applied, becomes a rule of art. The developments of science increase knowledge : the applications of art add to works. Art, necessarily, presupposes knowledge : art, in any but its infant state, presupposes scientific knowledge ; and if every art does not bear the name of the science on which it rests, it is only because several sciences are often necessary to form the groundwork of a single art. Such is the complication of human affairs, that to enable one thing to be *done*, it is often requisite to *know* the nature and properties of many things.

Art, application of science,

and presupposes knowledge.

Many things must be known before one can be done.

CHAPTER II.

SOURCES AND MEANS OF KNOWLEDGE—INDUCTION

KNOWLEGDE.

§ 23. KNOWLEDGE is a clear and certain conception of that which is true, and implies three things :

Knowledge a clear conception of what is true

1st. Firm belief; 2d. Of what is true ; and, 3d. On sufficient grounds.

Implies— 1st. Firm belief; 2d. Of what is true; 3d. On sufficient grounds.

If any one, for example, is in *doubt* respecting one of Legendre's Demonstrations, he cannot be said to *know* the proposition proved by it. If, again, he is fully *convinced* of any thing that is not *true*, he is mistaken in supposing himself to *know* it ; and lastly, if two persons are each *fully confident*, one that the moon is inhabited, and the other that it is not (though one of these opinions must be *true*), neither of them could properly be said to *know* the truth, since he cannot have sufficient *proof* of it.

Examples.

FACTS AND TRUTHS.

Knowledge is of facts and truths.

§ 24. Our knowledge is of two kinds : of facts and truths. A fact is any thing that HAS BEEN or IS. That the sun rose yesterday, is a fact: that he gives light to-day, is a fact. That water is fluid and stone solid, are facts. We derive our knowledge of facts through the medium of the senses.

Truth an accordance with what has been, is, or shall be.

Truth is an exact accordance with what HAS BEEN, IS, or SHALL BE. There are two methods of ascertaining truth :

Two methods of ascertaining it.

1st. By comparing known facts with each other ; and,

2dly. By comparing known truths with each other.

Hence, truths are inferences either from facts or other truths, made by a mental process called Reasoning.

Facts and truths, the elements of our knowledge.

§ 25. Seeing, then, that facts and truths are the elements of all our knowledge, and that knowledge itself is but their clear apprehension, their firm belief, and a distinct conception of their relations to each other, our main inquiry is, How are we to attain unto these facts and truths, which are the foundations of knowledge ?

1st. Our knowledge of facts is derived through

the medium of our senses, by observation, exper-
iment,* and experience. We see the tree, and
perceive that it is shaken by the wind, and note
the fact that it is in motion. We decompose
water and find its elements; and hence, learn
from experiment the *fact*, that it is not a simple
substance. We experience the vicissitudes of
heat and cold; and thus learn from experience
that the temperature is not uniform.

How we arrive at a knowledge of facts.

The ascertainment of facts, in any of the ways
above indicated, does not point out any connec-
tion between them. It merely exhibits them to
the mind as separate or isolated; that is, each
as standing for a determinate thing, whether
simple or compound. The term facts, in the
sense in which we shall use it, will designate
facts of this class only. If the facts so ascer-
tained have such connections with each other,
that additional facts can be inferred from them,
that inference is pointed out by the reasoning
process, which is carried on, in all cases, by com-
parison.

This does not point out a connection between them.

When they have a connection that is pointed out by the reasoning process.

2dly. A result obtained by comparing facts, we
have designated by the term Truth. Truths,
therefore, are inferences from facts; and every

Truth, found by comparing facts;

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* Under this term we include all the methods of inves-
tigation and processes of arriving at facts, except the pro-
cess of reasoning.

and
is inferred
from them.

truth has reference to all the singular facts from which it is inferred. Truths, therefore, are results deduced from facts, or from classes of facts. Such results, when obtained, appertain to all facts of the same class. Facts make a genus: truths, a species; with the characteristic, that they become known to us by inference or reasoning.

How
truths are
inferred from
facts by the
reasoning
process.

§ 26. How, then, are truths to be inferred from facts by the reasoning process? There are two cases.

1st case.

1st. When the instances are so few and simple that the mind can contemplate *all the facts* on which the induction rests, and to which it refers, and can make the induction without the aid of other facts; and,

2d case.

2dly. When the facts, being numerous, complicated, and remote, are brought to mind only by processes of investigation.

INTUITIVE TRUTH.

Intuitive
or
Self-evident
truths.

§ 27. Truths which become known by considering all the facts on which they depend, and which are inferred the moment the facts are apprehended, are the subjects of Intuition, and are called *Intuitive* or *Self-evident Truths*. The

Intuition
defined.

term Intuition is strictly applicable only to that mode of contemplation in which we look at

facts, or classes of facts, and apprehend the relations of those facts at the same time, and by the same act by which we apprehend the facts themselves. Hence, intuitive or self-evi- *How intuitive truths are conceived in the mind.* dent truths are those which are conceived in the mind immediately; that is, which are per- fectly conceived by a single process of induc- tion, the moment the facts on which they depend are apprehended, without the intervention of other ideas. They are necessary consequences of conceptions respecting which they are asserted. *Axioms of Geometry are the simplest kind.* The axioms of Geometry afford the simplest and most unmistakable class of such truths.

" A whole is equal to the sum of all its parts," *A whole equal to the sum of all the parts, an intuitive truth.* is an intuitive or self-evident truth, inferred from facts previously learned. For example; having learned from experience and through the senses what a whole is, and, from experiment, the fact that it may be divided into parts, the mind per- ceives the relation between the whole and the sum of the parts, viz. that they are equal; and then, by the reasoning process, infers that the *How inferred.* same will be true of every other thing; and hence, pronounces the general truth, that " a whole is equal to the sum of all its parts." Here all the facts from which the induction is drawn, *All the facts are presented to the mind.* are presented to the mind, and the induction is made without the aid of other facts; hence,

All the
axioms are
deduced in
the same
way.

it is an intuitive or self-evident truth.   All the
other axioms of Geometry are deduced from
premises and by processes of inference, entirely
similar.   We would not call these experimental
truths, for they are not alone the results of ex-
periment or experience.   Experience and exper-
iment furnish the *requisite information,* but the
reasoning power evolves the general truth.

These
axioms are
general
propositions.

" When we say, the equals of equals are equal,
we mentally make comparisons in equal spaces,
equal times, &c. ; so that these axioms, how-
ever self-evident, are still general propositions :
so far of the inductive kind, that, independently
of experience, they would not present themselves
to the mind.   The only difference between these
and axioms obtained from extensive induction is

Difference
between
them and
other
propositions,
which re-
quire diligent
research.

this : that, in raising the axioms of Geometry,
the instances offer themselves spontaneously, and
without the trouble of search, and are few and
simple : in raising those of nature, they are in-
finitely numerous, complicated, and remote ; so
that the most diligent research and the utmost
acuteness are required to unravel their web, and
place their meaning in evidence."*

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* Sir John Herschel's Discourse on the study of Natural
Philosophy.

TRUTHS, OR LOGICAL TRUTHS.

§ 28. Truths inferred from facts, by the process
of generalization, when the instances do not offer   Truths
themselves spontaneously to the mind, but require   generalized
                                                      from facts,
search and acuteness to discover and point out   and
                                                      truths in-
their connections, and all truths inferred from   ferred from
                                                        truths.
truths, might be called *Logical Truths*.   But as
we have given the name of intuitive or self-
evident truths to all inferences in which *all the
facts* were contemplated, we shall designate all
others by the simple term, TRUTHS.

It might appear of little consequence to dis-   Necessity of
tinguish the processes of reasoning by which   the distinc-
                                                  tion, being
truths are inferred from facts, from those in which the basis of a
                                                  classification.
we deduce truths from other truths ; but this dif-
ference in the premises, though seemingly slight,
is nevertheless very important, and divides the
subject of logic, as we shall presently see, into
two distinct and very different branches.

LOGIC.

§ 29. Logic takes note of and decides upon   Logic
the sufficiency of the evidence by which truths   notes the
                                                   sufficiency of
are established.   Our assent to the conclusion   evidence.
being grounded on the truth of the premises, we
never could arrive at any knowledge by rea-
soning, unless something were known antece-
dently to all reasoning.   It is the province of   Its province.

Furnishes the tests of truth.

Logic to furnish the tests by which all truths that are not intuitive may be inferred from the premises. It has nothing to do with ascertaining facts, nor with any proposition which claims to be believed on its own intrinsic evidence; that is, without evidence, in the proper sense of the word. It has nothing to do with the original data, or ultimate premises of our knowledge; with their number or nature, the mode in which they are obtained, or the tests by which they are distinguished. But, so far as our knowledge is founded on truths made such by evidence, that is, derived from facts or other truths previously known, whether those truths be particular truths, or general propositions, it is the province of Logic to supply the tests for ascertaining the validity of such evidence, and whether or not a belief founded on it would be well grounded. And since by far the greatest portion of our knowledge, whether of particular or general truths, is avowedly matter of inference, nearly the whole, not only of science, but of human conduct, is amenable to the authority of logic.

Has nothing to do with intuitive propositions, nor with original data;

but supplies all tests for general propositions.

The greatest portion of our knowledge comes from inference.

INDUCTION.

§ 30. That part of logic which infers truths from facts, is called Induction. Inductive reasoning is the application of the reasoning process to a given number of facts, for the purpose of determining if what has been ascertained respecting one or more of the individuals is true of the whole class. Hence, Induction is not the mere sum of the facts, but a conclusion drawn from them.

Induction, to what reasoning applicable.

Induction defined.

The logic of Induction consists in classing the facts and stating the inference in such a manner, that the evidence of the inference shall be most manifest.

Logic of Induction.

§ 31. Induction, as above defined, is a process of inference. It proceeds from the known to the unknown ; and any operation involving no inference, any process in which the conclusion is a mere fact, and not a truth, does not fall within the meaning of the term. The conclusion must be broader than the premises. The premises are facts : the conclusion must be a truth.

Induction proceeds from the known to the unknown.

The conclusion broader than the premises.

Induction, therefore, is a process of generalization. It is that operation of the mind by which we infer that what we know to be true

Induction, a process of generalization ;

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in which we conclude, that what is true under particular circumstances will be true universally.

in a particular case or cases, will be true in all cases which resemble the former in certain assignable respects. In other words, Induction is the process by which we conclude that what is true of certain individuals of a class is true of the whole class; or that what is true at certain times, will be true, under similar circumstances, at all times.

Induction presupposes accurate and necessary observations.

§ 32. Induction always presupposes, not only that the necessary observations are made with the necessary accuracy, but also that the results of these observations are, so far as practicable, connected together by general descriptions : enabling the mind to represent to itself as wholes, whatever phenomena are capable of being so represented.

More is necessary than to connect the observations we must *infer* from them.

To suppose, however, that nothing more is required from the conception than that it should serve to connect the observations, would be to substitute hypothesis for theory, and imagination for proof. The connecting link must be some character which *really exists* in the facts themselves, and which would manifest itself therein, if the condition could be realized which our organs of sense require.

For example ; Blakewell, a celebrated English cattle-breeder, observed, in a great number of

individual beasts, a tendency to fatten readily, Example of Blakewell, the English cattle breeder.
and in a great number of others the absence of
this constitution : in every individual of the for-
mer description, he *observed* a certain peculiar
make, though they differed widely in size, color,
&c.   Those of the latter description differed no
less in various points, but agreed in being of a
different make from the others. These *facts* were How he ascertained the facts : why he inferred.
his data ; from which, combining them with the
general principle, that nature is steady and uni-
form in her proceedings, he *logically* drew the
conclusion that beasts of the specified make have
universally a peculiar tendency to fattening.

The principal difficulty in this case consisted In what the difficulty consisted.
in *making* the observations, and so *collating* and
combining them as to *abstract* from each of a
multitude of cases, differing widely in many re-
spects, the circumstances in which they all
agreed.   But neither the making of the observa-
tions, nor their combination, nor the abstraction,
nor the judgment employed in these processes,
constituted the induction, though they were all
preparatory to it.   The Induction consisted in In what the induction consisted.
the generalization ; that is, in *inferring* from all
the data, that certain circumstances would be
found in the whole class.

The mind of Newton was led to the universal
law, that all bodies attract each other by forces

Newton's
inference of
the law of
universal
gravitation.

varying directly as their masses, and inversely
as the squares of their distances, by Induction.
He saw an apple falling from the tree : a mere
fact ; and asked himself the cause ; that is, if any
inference could be drawn from that fact, which
should point out an invariable antecedent condi-

How he
observed
facts and
their
connections.

tion. This led him to note other facts, to prose-
cute experiments, to observe the heavenly bodies,
until from many facts, and *their connections
with each other*, he arrived at the conclusion,
that the motions of the heavenly bodies were gov-
erned by general laws, applicable to all matter ,
that the stone whirled in the sling and the earth
rolling forward through space, are governed in
their motions by one and the same law. He

The use
which he
made of
exact
science.

then brought the exact sciences to his aid, and
demonstrated that this law accounted for all the
phenomena, and harmonized the results of all ob-
servations. Thus, it was ascertained that the

What was
the result.

laws which regulate the motions of the heav-
enly bodies, as they circle the heavens, also
guide the feather, as it is wafted along on the
passing breeze.

The ways of
ascertaining
facts are
known:

§ 33. We have already indicated the ways in
which the facts are ascertained from which the
inferences are drawn. But when an inference
can be drawn ; how many facts must enter into

the premises; what their exact nature must be; but we do not know certainly, in all cases, when we can draw on inference. and what their relations to each other, and to the inferences which flow from them; are questions which do not admit of definite answers. Although no general law has yet been discovered connecting all facts with truths, yet all the No general law. uniformities which exist in the succession of phenomena, and most of those which prevail in their coexistence, are either themselves laws of causation or consequences resulting and corollaries capable of being deduced from, such laws. It being the main business of Induction to determine the effects of every cause, and the causes Business of Induction. of all effects, if we had for all such processes general and certain laws, we could determine, What is necessary. in all cases, what causes are correctly assigned to what effects, and what effects to what causes, and we should thus be virtually acquainted with the whole course of nature. So far, then, as we How far a science. can trace, *with certainty*, the connection between cause and effect, or between effects and their causes, to that extent Induction is a science. When this cannot be done, the conclusions must be, to some extent, conjectural.

CHAPTER III.

DEDUCTION.

§ 34. We have seen that all processes of

Inductive
processes of
reasoning.

Reasoning, in which the premises are particular facts, and the conclusions general truths, are called Inductions. All processes of Reasoning, in which the premises are general truths and the

Deductive
processes.

conclusions particular truths, are called Deductions. Hence, a deduction is the process of

Deduction
defined.

reasoning by which a particular truth is inferred from other truths which are known or admitted.

Deductive
formula.

The formula for all deductions is found in the Syllogism, the parts, nature, and uses of which we shall now proceed to explain.

PROPOSITIONS.

Proposition,
judgment in
words:

§ 35. A proposition is a *judgment expressed in words*. Hence, a proposition is defined logically, " A sentence indicative :" affirming or

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* Section 30.

denying; therefore, it must not be ambiguous, must not be ambiguous; nor imperfect; nor ungrammatical. for that which has more than one meaning is in reality several propositions; nor *imperfect*, nor *ungrammatical*, for such expressions have no meaning at all.

§ 36. Whatever can be an object of belief, or even of disbelief, must, when put into words, A proposition explained. assume the form of a proposition. All truth and all error lie in propositions. What we call a truth, is simply a true proposition; and errors Its nature,— extent. are false propositions. To know the import of all propositions, would be to know all questions which can be raised, and all matters which are Embraces all truth and all error. susceptible of being either believed or disbelieved. Since, then, the objects of all belief and all inquiry express themselves in propositions, a sufficient scrutiny of propositions and their va- An examination of propositions embraces all questions and all knowledge. rieties will apprize us of what questions mankind have actually asked themselves, and what, in the nature of answers to those questions, they have actually thought they had grounds to believe.

§ 37. The first glance at a proposition shows A proposition is formed by putting two names together. that it is formed by putting together two names. Thus, in the proposition, "Gold is yellow," the property *yellow* is affirmed of the substance *gold*. In the proposition, "Franklin was not born in

England," the fact expressed by the words *born in England* is denied of the man Franklin.

§ 38. Every proposition consists of three parts : the Subject, the Predicate, and the Copula. The subject is the name denoting the person or thing of which something is affirmed or denied : the predicate is that which is affirmed or denied of the subject ; and these two are called the *terms* (or extremes), because, logically, the subject is placed *first*, and the predicate *last*. The copula, in the middle, indicates the act of judgment, and is the sign denoting that there is an affirmation or denial. Thus, in the proposition, " The earth is round ;" the subject is the words " the earth," being that of which something is affirmed : the predicate, is the word *round*, which denotes the quality affirmed, or (as the phrase is) *predicated :* the word is, which serves as a connecting mark between the subject and the predicate, to show that one of them is affirmed of the other, is called the Copula. The copula must be either is, or is NOT, the substantive verb being the only *verb* recognised by Logic. All other verbs are resolvable, by means of the verb " to be," and a participle or adjective. For example :

A proposition has three parts: Subject, Predicate, and Copula.

Subject defined.

Predicate.

Copula must be is or is NOT.

All verbs resolvable into "to be."

" The Romans conquered :"

the word "*conquered*" is both copula and predi-
cate, being equivalent to "*were victorious*."

Hence, we might write,

"The Romans were victorious,"

in which *were* is the copula, and *victorious* the
predicate.

§ 39. A proposition being a portion of dis-
course, in which something is affirmed or denied
of something, all propositions may be divided
into affirmative and negative. An affirmative
proposition is that in which the predicate is af-
firmed of the subject; as, "Cæsar is dead." A
negative proposition is that in which the predicate
is *denied* of the subject; as, "Cæsar is not dead."
The copula, in this last species of proposition, In the last,
the copula is
IS NOT
consists of the words "IS NOT," which is the
sign of negation; "IS" being the sign of affirm-
ation.

SYLLOGISM.

§ 40. A syllogism is a form of stating the con-
nection which may exist, for the purpose of
reasoning, between three propositions. Hence,
to a legitimate syllogism, it is essential that
there should be three, and only three, proposi-

and the third tions.   Of these, two are admitted to be true,
is proved
from them. and are called the *premises :* the third is proved
from these two, and is called the conclusion.
For example :

Example.   " All tyrants are detestable :
Cæsar was a tyrant ;
Therefore, Cæsar was detestable."

Now, if the first two propositions be *admitted,*
the third, or conclusion, necessarily follows from
them. and it is proved that CÆSAR was detestable.

Major Term   Of the two terms of the conclusion, the Predi-
defined.   cate (detestable) is called the *major term,* and
the Subject (Cæsar) the *minor term ;* and these
two terms, together with the term "tyrant,"
make up the three propositions of the syllogism,
Minor Term. —each term being used twice.   Hence, every
syllogism has three, and only three, *different*
terms.

Major   The premise, into which the Predicate of the
Premise
defined. conclusion enters, is called the *major premise ;*
Minor   the other is called the *minor premise*, and con-
Premise. tains the Subject of the conclusion ; and the
other term, common to the two premises, and
with which both the terms of the conclusion were
separately compared, *before* they were compared
Middle Term. with each other, is called the *middle term.*   In
the syllogism above, "detestable" (in the con-

clusion) is the major term, and "Cæsar" the mi-
nor term : hence,

Example,
pointing out
Major
premise,
Minor
premise, and
Middle Term.

"All tyrants are detestable,"

is the major premise, and

"Cæsar was a tyrant,"

the minor premise, and "tyrant" the middle term.

§ 41. The syllogism, therefore, is a mere for-
mula for ascertaining what may, or what may
not, be predicated of a subject. It accomplishes
this end by means of two propositions, viz. by
comparing the given predicate of the first (a
Major Premise), and the given subject of the
second (a Minor Premise), respectively with one
and the same third term (called the middle term),
and thus—under *certain conditions*, or laws of
the syllogism—to be hereafter stated—eliciting
the truth (conclusion) that the given predicate
must be predicated of that subject. It will be
seen that the Major Premise always declares,
in a *general* way, such a relation between the
Major Term and the Middle Term ; and the Mi-
nor Premise declares, in a more *particular* way,
such a relation between the Minor Term and
the Middle Term, as that, in the Conclusion,
the Minor Term must be put under the Major
Term ; or in other words, that the Major Term
must be predicated of the Minor Term.

Syllogism,
a mere
formula.

How applied.

Use of the
Major
premise.

Of the Minor

Of the
Middle Te

ANALYTICAL OUTLINE OF DEDUCTION.

Reasoning
defined.

§ 42. In every instance in which we *reason*, in the strict sense of the word, that is, make use of arguments, whether for the sake of refuting an adversary, or of conveying instruction, or of satisfying our own minds on any point, whatever may be the subject we are engaged on, a certain process takes place in the mind, which is one

The process, in all cases, the same. and the same in all cases (provided it be correctly conducted), whether we use the inductive process or the deductive formulas.

Of course it cannot be supposed that every one is even conscious of this process in his own

Every one not conscious of the process. mind; much less, is competent to explain the principles on which it proceeds. This indeed is,

The same for every other process. and cannot but be, the case with every other process respecting which any system has been formed; the practice not only may exist independently of the theory, but *must* have preceded the theory. There must have been Language

Elements and knowledge of elements, must precede generaliza- tion and classification of principles. before a system of Grammar could be devised; and musical compositions, previous to the science of Music. This, by the way, serves to expose the futility of the popular objection against Logic; viz. that men may reason very well who know nothing of it. The parallel instances adduced show that such an objection may be urged

in many other cases, where its absurdity would Logic
of value. be obvious; and that there is no ground for deciding thence, either that the system has no tendency to improve practice, or that even if it had not, it might not still be a dignified and interesting pursuit.

§ 43. One of the chief impediments to the attainment of a just view of the nature and object of Logic, is the not fully understanding, or not sufficiently keeping in mind the SAMENESS of the reasoning process in all cases. If, as the ordinary mode of speaking would seem to indicate, mathematical reasoning, and theological, and metaphysical, and political, &c., were essentially different from each other, that is, different *kinds of reasoning*, it would follow, that supposing there could be at all any such science as we have described Logic, there must be so many different species or at least different branches of Logic. And such is perhaps the most prevailing notion. Nor is this much to be wondered at; since it is evident to all, that some men converse and write, in an argumentative way, very justly on one subject, and very erroneously on another, in which again others excel, who fail in the former.

Sameness of the reasoning process should be kept in mind.

All kinds of reasoning are alike in principle

Reason of the prevailing errors.

This error may be at once illustrated and re-

moved, by considering the parallel instance of
Arithmetic ; in which every one is aware that
the process of a calculation is not affected by
the nature of the objects whose numbers are
before us; but that, for example, the multipli-
cation of a number is the very same operation,
whether it be a number of men, of miles, or of
pounds ; though, nevertheless, persons may per-
haps be found who are accurate in the results
of their calculations relative to natural philoso-
phy, and incorrect in those of political econo-
my, from their different degrees of skill in the
subjects of these two sciences; not surely be-
cause there are different arts of arithmetic ap
plicable to each of these respectively.

§ 44. Others again, who are aware that the
Some view
Logic as a
peculiar
method of
reasoning: simple system of Logic may be applied to all
subjects whatever, are yet disposed to view it
as a peculiar method of reasoning, and not, as
it is, a method of unfolding and analyzing our
reasoning : whence many have been led to talk
of comparing Syllogistic reasoning with Moral
reasoning ; taking it for granted that it is pos-
sible to reason correctly without reasoning logi-
It is the only cally ; which is, in fact, as great a blunder as if
method of
reasoning any one were to mistake *grammar* for a pecu-
correctly: liar *language,* and to suppose it possible to speak

correctly without speaking grammatically. They have, in short, considered Logic as *an* art of reasoning; whereas (so far as it is an art) it is *the* art of reasoning; the logician's object being, not to lay down principles by which one *may* reason, but by which all *must* reason, even though they are not distinctly aware of them :—to lay down rules, not which *may* be followed with advantage, but which cannot possibly be *departed* from in sound reasoning. These misapprehensions and objections being such as lie on the very threshold of the subject, it would have been hardly possible, without noticing them, to convey any just notion of the nature and design of the logical system.

It lays down rules, not which *may*, but which *must* be followed.

Misapprehensions and objections noticed.

§ 45. Supposing it then to have been perceived that the operation of reasoning is in all cases the same, the analysis of that operation could not fail to strike the mind as an interesting matter of inquiry. And moreover, since (apparent) arguments, which are unsound and inconclusive, are so often employed, either from error or design; and since even those who are not misled by these fallacies, are so often at a loss to detect and expose them in a manner satisfactory to others, or even to themselves; it could not but appear desirable to lay down some gen-

Operation of reasoning should be analyzed:

Because such analysis is necessary to furnish the

rules for the
detection of
error and the
discovery of
truth.

eral rules of reasoning, applicable to all cases; by which a person might be enabled the more readily and clearly to state the grounds of his own conviction, or of his objection to the arguments of an opponent; instead of arguing at random, without any fixed and acknowledged principles to guide his procedure. Such rules

Such rules
are analogous
to the rules of
Arithmetic.

would be analogous to those of Arithmetic, which obviate the tediousness and uncertainty of calculations in the head; wherein, after much labor, different persons might arrive at different results, without any of them being able distinctly to point out the error of the rest. A system of such rules, it is obvious, must, instead of deserv-

They bring
the parties, in
argument, to
an issue.

ing to be called the art of wrangling, be more justly characterized as the "art of cutting short wrangling," by bringing the parties to issue at once, if not to agreement; and thus saving a waste of ingenuity.

Every con-
clusion is
deduced from
two proposi-
tions, called
Premises.

§ 46. In pursuing the supposed investigation, it will be found that in all deductive processes every conclusion is deduced, in reality, from two other propositions (thence called *Premises*); for

If one prem-
ise is sup-
pressed, it is
nevertheless
understood,

though one of these may be, and commonly is, suppressed, it must nevertheless be understood as admitted; as may easily be made evident by supposing the denial of the suppressed premise,

which will at once invalidate the argument.   For
example ; in the following syllogism :

" Whatever exhibits marks of design had an intelligent author:
    The world exhibits marks of design ;
    Therefore, the world had an intelligent author :"

if any one from perceiving that "the world ex-
hibits marks of design," infers that "it must have
had an intelligent author," though he may not be
aware in his own mind of the existence of any
other premise, he will readily understand, if it be
denied that "whatever exhibits marks of design
must have had an intelligent author," that the
affirmative of that proposition is necessary to
the validity of the argument.

§ 47. When one of the premises is suppressed
(which for brevity's sake it usually is), the argu-
ment is called an Enthymeme.   For example :

    " The world exhibits marks of design,
        Therefore the world had an intelligent author,"

is an Enthymeme.   And it may be worth while
to remark, that, when the argument is in this
state, the objections of an opponent are (or rather
appear to be) of two kinds, viz. either objections
to the *assertion* itself, or objections to its *force*
as an argument.   For example : in the above
instance, an atheist may be conceived either de-

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nying that the world *does* exhibit marks of de-
sign, or denying that it *follows* from thence that
it had an intelligent author.   Now it is impor-
tant to keep in mind that the only difference in
the two cases is, that in the one the *expressed*
premise is denied, in the other the *suppressed ;*
for the *force as an argument* of either premise
depends on the other premise : if both be admit-
ted, the conclusion legitimately connected with
them cannot be denied.

§ 48. It is evidently immaterial to the argu-
ment whether the conclusion be placed first or
last ; but it may be proper to remark, that a
premise placed *after* its conclusion is called the
Reason of it, and is introduced by one of those
conjunctions which are called causal, viz. " since,"
" because," &c., which may indeed be employed
to designate a premise, whether it come first or
last.   The illative conjunctions " therefore," &c.,
designate the conclusion.

It is a circumstance which often occasions
error and perplexity, that both these classes of
conjunctions have also another signification, be-
ing employed to denote, respectively, *Cause* and
Effect, as well as *Premise* and *Conclusion*.   For
example : if I say, " this ground is rich, *because*
the trees on it are flourishing ;" or, " the trees are

flourishing, and *therefore* the soil must be rich;" Examples where the conjunctions are used logically.
I employ these conjunctions to denote the con-
nection of *Premise* and *Conclusion ;* for it is
plain that the luxuriance of the trees is not the
cause of the soil's fertility, but only the cause
of *my knowing* it. If again I say, " the trees
flourish, *because* the ground is rich;" or " the
ground is rich, and *therefore* the trees flourish,' Examples where they denote cause and effect.
I am using the very same conjunctions to denote
the connection of *cause* and *effect ;* for in this
case, the luxuriance of the trees being evident
to the eye, would hardly need to be *proved,* but
might need to be accounted for.   There are, Many cases in which the cause and the reason are the same.
however, many cases, in which the cause is em-
ployed to *prove* the existence of its effect ; espe-
cially in arguments relating to *future* events ; as,
for example, when from favorable weather any
one argues that the crops are likely to be abun-
dant, the *cause* and the reason, in that case, co-
incide ; and this contributes to their being so
often confounded together in other cases.

§ 49. In an argument, such as the example In every correct argu- ment, to admit the premise is to admit the conclusion.
above given, it is, as has been said, impossible
for any one, who admits both premises, to avoid
admitting the conclusion.   But there will be fre-
quently an apparent connection of premises with
a conclusion which does not in reality follow

Apparent connection of premises and conclusion must not be relied on.

from them, though to the inattentive or unskilful the argument may appear to be valid ; and there are many other cases in which a doubt may exist whether the argument be valid or not ; that is, whether it be possible or not to admit the premises and yet deny the conclusion.

General rules for argumentation necessary.

§ 50. It is of the highest importance, therefore, to lay down some regular form to which every valid argument may be reduced, and to devise a rule which shall show the validity of every argument in that form, and consequently the unsoundness of any apparent argument which cannot be reduced to it. For example ; if such an argument as this be proposed :

Example of an imperfect argument.

 " Every rational agent is accountable :
    Brutes are not rational agents ;
    Therefore they are not accountable ;"

or again :

2d Example.

" All wise legislators suit their laws to the genius of their nation ;
    Solon did this; therefore he was a wise legislator :"

Difficulty of detecting the error.

there are some, perhaps, who would not perceive any fallacy in such arguments, especially if enveloped in a cloud of words ; and still more, when the conclusion is true, or (which comes to the same point) if they are disposed to believe it ; and others might perceive indeed, but might

be at a loss to explain, the fallacy. Now these
(apparent) arguments exactly correspond, re-
spectively, with the following, the absurdity of
the conclusions from which is manifest:

> " Every horse is an animal :
> Sheep are not horses ;
> Therefore, they are not animals."

And :

> " All vegetables grow ;
> An animal grows ;
> Therefore, it is a vegetable."

These last examples, I have said, correspond
exactly (considered as arguments) with the for-
mer; the question respecting the validity of an
argument being, not whether the conclusion be
true, but whether it *follows* from the premises
adduced. This mode of exposing a fallacy, by
bringing forward a similar one whose conclusion
is obviously absurd, is often, and very advan-
tageously, resorted to in addressing those who
are ignorant of Logical rules ; but to lay down
such rules, and employ *them* as a test, is evi-
dently a safer and more compendious, as well
as a more philosophical mode of proceeding. To
attain these, it would plainly be necessary to
analyze some clear and valid arguments, and to
observe in what their conclusiveness consists.

§ 51. Let us suppose, then, such an examination to be made of the syllogism above mentioned :

" Whatever exhibits marks of design had an intelligent author;
The world exhibits marks of design ;
Therefore, the world had an intelligent author."

In the first of these premises we find it assumed universally of the *class* of " things which exhibit marks of design," that they had an intel-ligent author ; and in the other premise, " the world" is referred to that class as comprehended in it : now it is evident that whatever is said of the whole of a class, may be said of any thing comprehended in that class ; so that we are thus authorized to say of the world, that " it had an intelligent author."

Again, if we examine a syllogism with a negative conclusion, as, for example,

" Nothing which exhibits marks of design could have been produced by chance ;
The world exhibits, &c. ;
Therefore, the world could not have been produced by chance,"

the process of reasoning will be found to be the same; since it is evident that whatever is *denied* universally of any class may be denied of any thing that is comprehended in that class.

§ 52. On further examination, it will be found that all valid arguments whatever, which are based on admitted premises, may be easily reduced to such a form as that of the foregoing syllogisms; and that consequently the principle on which they are constructed is that of the formula of the syllogism. So elliptical, indeed, is the ordinary mode of expression, even of those who are considered as prolix writers, that is, so much is implied and left to be understood in the course of argument, in comparison of what is actually stated (most men being impatient even, to excess, of any appearance of unnecessary and tedious formality of statement), that a single sentence will often be found, though perhaps considered as a single argument, to contain, compressed into a short compass, a chain of several distinct arguments. But if each of these be fully developed, and the whole of what the author intended to imply be stated expressly, it will be found that all the steps, even of the longest and most complex train of reasoning, may be reduced into the above form.

All valid arguments reducible to the syllogistic form.

Ordinary mode of expressing arguments elliptical.

But when fully developed, they may all be reduced into the above form.

§ 53. It is a mistake to imagine that Aristotle and other logicians meant to propose that this prolix form of unfolding arguments should universally supersede, in argumentative discourses,

Aristotle did not mean that every argument should be

thrown into the common forms of expression; and that "to
the form of a
syllogism. reason logically," means, to state all arguments
at full length in the syllogistic form; and Aris-
totle has even been charged with inconsistency
for not doing so. It has been said that he " ar-
gues like a rational creature, and never attempts
That form is to bring his own system into practice." As well
merely a *test*
of truth. might a chemist be charged with inconsistency
for making use of any of the compound sub-
stances that are commonly employed, without
previously analyzing and resolving them into
Analogy to their simple elements; as well might it be im-
the chemist.
agined that, to speak grammatically, means, to
parse every sentence we utter. The chemist
(to pursue the illustration) keeps by him his tests
and his method of analysis, to be employed when
The analogy any substance is offered to his notice, the com-
continued.
position of which has not been ascertained, or
in which adulteration is suspected. Now a fal-
To what a lacy may aptly be compared to some adulterated
fallacy may
be compared. compound; "it consists of an ingenious mixture
of truth and falsehood, so entangled, so intimate-
ly blended, that the falsehood is (in the chemical
phrase) *held in solution :* one drop of sound logic
How detect- is that test which immediately disunites them,
ed.
makes the foreign substance visible, and precipi-
tates it to the bottom."

ARISTOTLE'S DICTUM.

§ 54. But to resume the investigation of the   Form of
principles of reasoning : the maxim resulting from   every real
argument.
the examination of a syllogism in the foregoing
form, and of the application of which, every valid
deduction is in reality an instance, is this :

" That whatever is predicated (that is, affirmed   Aristotle's
dictum.
or denied) *universally*, of any class of things,
may be predicated, in like manner (viz. affirmed
or denied), of any thing comprehended in that
class."

This is the principle commonly called the *dic-*   What the
tum de omni et nullo, for the indication of   principle
is called.
which we are indebted to Aristotle, and which
is the keystone of his whole logical system.   It
is remarkable that some, otherwise judicious   What writers
writers, should have been so carried away by   have said of
this princi-
their zeal against that philosopher, as to speak   ple; and
why.
with scorn and ridicule of this principle, on
account of its obviousness and simplicity ;   Simplicity a
though they would probably perceive at once   test of
science.
in any other case, that it is the greatest tri-
umph of philosophy to refer many, and seem-
ingly very various phenomena to one, or a very
few, simple principles ; and that the more simple
and evident such a principle is, provided it be
truly applicable to all the cases in question, the

greater is its value and scientific beauty. If, indeed, any principle be regarded as *not* thus applicable, *that* is an objection to it of a different kind. Such an objection against Aristotle's dictum, no one has ever attempted to *establish* by any kind of proof; but it has often been *taken for granted;* it being (as has been stated) very commonly supposed, without examination, that the syllogism is a *distinct kind of argument,* and that the rules of it accordingly do not apply, nor were intended to apply, to *all* reasoning whatever, where the premises are granted or known.

§ 55. One objection against the dictum of Aristotle it may be worth while to notice briefly, for the sake of setting in a clearer light the real character and object of that principle. The application of the principle being, as has been seen, to a regular and conclusive syllogism, it has been urged that the dictum was intended to prove and *make evident* the conclusiveness of such a syllogism; and that it is unphilosophical to attempt giving a *demonstration of a demonstration.* And certainly the charge would be just, if we could imagine the logician's object to be, to *increase the certainty* of a conclusion, which we are supposed to have already arrived at by the clearest possible mode

of proof.   But it is very strange that such an
idea should ever have occurred to one who had
even the slightest tincture of natural philosophy ;
for it might as well be imagined that a natural
philosopher's or a chemist's design is to strength-
en the testimony of our senses by *à priori* rea-
soning, and to convince us that a stone when
thrown will fall to the ground, and that gunpow-
der will explode when fired ; because they show
according to their principles those phenomena
must take place as they do.   But it would be
reckoned a mark of the grossest ignorance and
stupidity not to be aware that their object is
not to *prove* the existence of an individual
phenomenon, which our eyes have witnessed,
but (as the phrase is) to *account* for it; that is,
to show according to what *principle* it takes
place ; to refer, in short, the *individual case* to
a *general law* of nature.   The object of Aris-
totle's dictum is precisely analogous : he had,
doubtless, no thought of adding to the force of
any individual syllogism ; his design was to point
out the *general principle* on which that process
is conducted which takes place in each syllo-
gism.   And as the Laws of nature (as they are
called) are in reality merely *generalized facts,* of
which all the phenomena coming under them are
particular instances ; so, the proof drawn from

Aristotle's dictum is not a distinct demonstration brought to confirm another demonstration, but is merely a generalized and abstract statement of *all* demonstration whatever; and is, therefore, in fact, *the very demonstration* which, under proper suppositions, accommodates itself to the various subject-matters, and which is actually employed in each particular case.

§ 56. In order to trace more distinctly the different steps of the abstracting process, by which any particular argument may be brought into the most general form, we may first take a syllogism, that is, an argument stated accurately
and at full length, such as the example formerly given:

" Whatever exhibits marks of design had an intelligent author;
The world exhibits marks of design;
Therefore, the world had an intelligent author:"

and then somewhat generalize the expression, by substituting (as in Algebra) arbitrary unmeaning symbols for the significant terms that were originally used. The syllogism will then stand thus:

" Every B is A; C is B; therefore C is A."

The reasoning, when thus stated, is no less evidently valid, whatever terms A, B, and C respect-

ively may be supposed to stand for ; such terms may indeed be inserted as to make all or some of the assertions *false ;* but it will still be no less impossible for any one who *admits* the truth of the *premises,* in an argument thus constructed, to deny the conclusion ; and this it is that constitutes the conclusiveness of an argument.   .

Viewing, then, the syllogism thus expressed, it appears clearly that "A stands for *any thing whatever* that is affirmed of a certain entire class" (viz. of *every* B), "which class comprehends or contains in it *something else,*" viz. C (of which B is, in the second premiss, affirmed) ; and that, consequently, the first term (A) is, in the conclusion, predicated of the third (C).

§ 57. Now, to assert the validity of this process now before us, is to state the very dictum we are treating of, with hardly even a verbal alteration, viz. :

1. Any thing whatever, predicated of a whole class ;

2. Under which class something else is contained ;

3. May be predicated of that which is so contained.

The three members into which the maxim is here distributed, correspond to the three propo-

sitions of the syllogism to which they are intended respectively to apply.

Advantage of substituting arbitrary symbols for the terms.

The advantage of substituting for the terms, in a regular syllogism, arbitrary, unmeaning symbols, such as letters of the alphabet, is much the same as in geometry : the reasoning itself is then considered, by itself, clearly, and without any risk of our being misled by the truth or falsity of the conclusion ; which is, in fact, accidental and variable ; the essential point being, as far as

Connection, the essential point of the argument.

the *argument* is concerned, the *connection between* the premises and the conclusion.   We are thus enabled to embrace the general principle of deductive reasoning, and to perceive its applicability to an indefinite number of individual cases. That Aristotle, therefore, should have been ac-

Aristotle right in using these symbols.

cused of making use of these symbols for the purpose of *darkening* his demonstrations, and that too by persons not unacquainted with geometry and algebra, is truly astonishing.

Syllogism equally true when abstract terms are used.

§ 58. It belongs, then, exclusively to a syllogism, properly so called (that is, a valid argument, so stated that its conclusiveness is evident from the mere *form* of the expression), that if letters, or any other unmeaning symbols, be substituted for the several terms, the validity of the argument shall still be evident.   Whenever this

is not the case, the supposed argument is either unsound and sophistical, or else may be reduced (without any alteration of its meaning) into the syllogistic form; in which form, the test just mentioned may be applied to it.

When not so, the supposed argument is unsound.

§ 59. What is called an unsound or fallacious argument, that is, an *apparent* argument, which is, in reality, none, cannot, of course, be reduced into this form; but when stated in the form most nearly approaching to this that is possible, its fallaciousness becomes more evident, from its nonconformity to the foregoing rule. For example :

Definition of an unsound argument.

When reduced to the form, the fallacy is more evident.

" Whoever is capable of deliberate crime is responsible ;
    An infant is not capable of deliberate crime ;
    Therefore, an infant is not responsible."

Example.

Here the term " responsible" is affirmed universally of " those capable of deliberate crime ;" it might, therefore, according to Aristotle's dictum, have been affirmed of any thing contained under that class; but, in the instance before us, nothing is mentioned as contained under that class; only, the term "infant" is *excluded* from that class; and though what is affirmed of a whole class may be affirmed of any thing that *is* contained under it, there is no ground for supposing that it may be *denied* of whatever is *not*

Analysis of this syllogism.

Its defective nature pointed out.

so contained; for it is evidently possible that it

may be applicable to a whole class and to some-
thing else besides.   To say, for example, that all
trees are vegetables, does not imply that *nothing
else* is a vegetable.   Nor, when it is said, that

all who are capable of deliberate crime are re-
sponsible, does this imply that no others are
responsible; for though this may be very *true*,

it has not been asserted in the premise before us;
and in the analysis of an argument, we are to
discard all consideration of what *might* be as-
serted; contemplating only what *actually is* laid
down in the premises.   It is evident, therefore,

that such an apparent argument as the above
does not comply with the rule laid down, nor
can be so stated as to comply with it, and is
consequently invalid.

§ 60. Again, in this instance:

" Food is necessary to life;
     Corn is food;
     Therefore corn is necessary to life:"

the term " necessary to life" is affirmed of food,
but *not universally;* for it is not said of *every
kind of food*   ᵗhe meaning of the assertion be-
ing manifestly that *some* food is necessary to
life : here again, therefore, the rule has not been
complied with, since that which has been predi-

cated (that is, affirmed or denied), not of the *whole*, but of a *part* only of a certain class, cannot be, on that ground, predicated of whatever is contained under that class.

Why we cannot predicate of corn what was predicated of food.

DISTRIBUTION AND NON-DISTRIBUTION OF TERMS.

§ 61. The fallacy in this last case is, what is usually described in logical language as consisting in the "non-distribution of the middle term;" that is, its not being employed to denote *all* the objects to which it is applicable. In order to understand this phrase, it is necessary to observe, that a term is said to be "distributed," when it is taken universally, that is, so as to stand for all its significates; and consequently "undistributed," when it stands for only a portion of its significates.* Thus, "*all* food," or *every* kind of food, are expressions which imply the distribution of the term "food;" "*some* food" would imply its non-distribution.

Fallacy in the last example.

Non-distribution of the middle term.

What distribution means.

Non-distribution.

Now, it is plain, that if in each premiss a *part* only of the middle term is employed, that is, if it be not at all distributed, no conclusion can be drawn. Hence, if in the example formerly adduced, it had been merely stated that "*some-*

How the example might have been varied.

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* Section 15.

6

thing" (not *" whatever,"* or *" every thing"*)
" which exhibits marks of design, is the work of
an intelligent author," it would not have fol-

What it
would then
have implied. lowed, from the world's exhibiting marks of de-
sign, that that is the work of an intelligent author

Words mark-
ing distribu-
tion or non-
distribution
not always
expressed. § 62. It is to be observed also, that the words
" all" and " every," which mark the distribution
of a term, and " some," which marks its non-
distribution, are not always expressed : they are
frequently understood, and left to be supplied by
the context; as, for example, "food is neces-
sary ;" viz. "*some* food ;" "man is mortal ;" viz.

Such propo-
sitions are
called
Indefinite. "*every* man." Propositions thus expressed are
called by logicians "*indefinite*," because it is left
undetermined by the form of the expression
whether the subject be distributed or not. Nev-
ertheless it is plain that in every proposition
the subject either is or is not meant to be dis-
tributed, though it be not declared whether

But every
proposition
must be
either
Universal or
Particular. it is or not ; consequently, every proposition,
whether expressed indefinitely or not, must be
understood as either "universal" or "particu-
lar ;" those being called universal, in which the
predicate is said of the whole of the subject
(or, in other words, where all the significates
are included) ; and those particular, in which

Example of
each. only a part of them is included. For example :

"All men are sinful," is universal : "some men are sinful," particular ; and this division of propositions, having reference to the distribution of the *subject*, is, in logical language, said to be according to their "*quantity*." <small>This division relates to quantity.</small>

§ 63. But the distribution or non-distribution of the *predicate* is entirely independent of the *quantity* of the proposition ; nor are the signs "all" and "some" ever affixed to the predicate ; because its distribution depends upon, and is indicated by, the "*quality*" of the proposition ; that is, its being *affirmative* or *negative ;* it being a universal rule, that the predicate of a negative proposition is distributed, and of an affirmative, undistributed. The reason of this may easily be understood, by considering that a term which stands for a whole class may be applied to (that is, *affirmed* of) any thing that is comprehended under that class, though the term of which it is thus affirmed may be of much narrower extent than that other, and may therefore be far from coinciding with the *whole* of it. Thus it may be said with truth, that "the Negroes are uncivilized," though the term "uncivilized" be of much wider extent than "Negroes," comprehending, besides them, Patagonians, Esquimaux, &c. ; so that it would not be allowable to assert, that <small>Distribution of the predicate has no reference to quantity.</small> <small>Has reference to *quality*.</small> <small>When it is distributed:</small> <small>The reason of this.</small> <small>The predicate of affirmative propositions may be applicable to the subject, and yet of much wider extent.</small>

Hence, only a *all* who are uncivilized are Negroes." It is ev-
part of the
term is used. ident, therefore, that it is a *part* only of the
term " uncivilized" that has been affirmed of
" Negroes ;" and the same reasoning applies to
every affirmative proposition.

But it may   It may indeed so happen, that the subject
be of equal
extent with   and predicate coincide, that is, are of equal
the subject: extent; as, for example : " all men are rational
animals ;" " all equilateral triangles are equian-
gular ;" (it being equally true, that " all rational
this not im- animals are men," and that " all equiangular tri-
plied in the
form of the   angles are equilateral ;") yet this is not *implied*
expression.   *by the form of the expression ;* since it would
be no less true that " all men are rational ani-
mals," even if there were other rational animals
besides men.

If any part of   It is plain, therefore, that if *any part* of the
the predicate
is applicable predicate is applicable to the subject, it may be
to the sub-   affirmed, and of course cannot be denied, of that
ject, it may
be affirmed   subject ; and consequently, when the predicate
of the sub-   *is denied* of the subject, this implies that *no*
ject.
part of that predicate is applicable to that sub-
ject ; that is, that the *whole* of the predicate is
If a predicate denied of the subject : for to say, for example,
is *denied* of a
subject, the   that " no beasts of prey ruminate," implies that
whole predi- beasts of prey are excluded from the *whole class*
cate is
denied of   of ruminant animals, and consequently that " no
the subject.
ruminant animals are beasts of prey." And

hence results the above-mentioned rule, that the distribution of the predicate is implied in negative propositions, and its non-distribution in affirmatives.

§ 64. It is to be remembered, therefore, that it is not sufficient for the middle term to *occur* in a universal proposition; since if that proposition be an affirmative, and the middle term be the *predicate* of it, it will not be distributed. For example : if in the example formerly given, it had been merely asserted, that " all the works of an intelligent author show marks of design," and that " the universe shows marks of design," nothing could have been proved ; since, though both these propositions are universal, the middle term is made the predicate in each, and both are affirmative ; and accordingly, the rule of Aristotle is not here complied with, since the term " work of an intelligent author," which is to be proved applicable to " the universe," would not have been affirmed of the middle term (" what shows marks of design") under which " universe " is contained ; but the middle term, on the contrary, would have been affirmed of it.

If, however, one of the premises be negative, the middle term may then be made the predicate

of that, and will thus, according to the above remark, be distributed. For example :

> " No ruminant animals are predacious :
> The lion is predacious ;
> Therefore the lion is not ruminant :"

this is a valid syllogism ; and the middle term (predacious) is distributed by being made the predicate of a negative proposition. The form, The form of this syllogism will not be that prescribed by the dictum, but may be reduced to it. indeed, of the syllogism is not that prescribed by the dictum of Aristotle, but it may easily be reduced to that form, by stating the first proposition thus : " No predacious animals are ruminant;" which is manifestly implied (as was above remarked) in the assertion that "no ruminant animals are predacious." The syllogism will thus appear in the form to which the dictum applies.

All arguments cannot be reduced by so short a process. § 65. It is not every argument, indeed, that can be reduced to this form by so short and simple an alteration as in the case before us. A longer and more complex process will often be required, and rules may be laid down to facilitate this process in certain cases ; but there is no sound argument but what *can* be reduced into But all arguments may this form, without at all departing from the real meaning and drift of it ; and the form will be

found (though more prolix than is needed for ordinary use) the most perspicuous in which an argument can be exhibited. *be reduced to the prescribed form.*

§ 66. All deductive reasoning whatever, then, rests on the one simple principle laid down by Aristotle, that *All deductive reasoning rests on the dictum.*

" What is predicated, either affirmatively or negatively, of a term distributed, may be predicated in like manner (that is, affirmatively or negatively) of any thing contained under that term."

So that, when our object is to prove any proposition, that is, to show that one term may rightly be affirmed or denied of another, the process which really takes place in our minds is, that we *refer* that term (of which the other is to be thus predicated) to some class (that is, middle term) of which that other may be affirmed, or denied, as the case may be. Whatever the subject-matter of an argument may be, the reasoning itself, considered by itself, is in every case the same process; and if the writers against Logic had kept this in mind, they would have been cautious of expressing their contempt of what they call "syllogistic reasoning," which embraces *all* deductive reasoning; and instead of ridiculing Aristotle's principle for its obviousness and simplicity, would have perceived that these are, in fact, its *What are the processes of proof.*

The reasoning always the same.

Mistakes of writers on Logic.

Aristotle's principle

simple and highest praise : the easiest, shortest, and most
general.
evident theory, provided it answer the purpose
of explanation, being ever the best.

RULES FOR EXAMINING SYLLOGISMS.

Tests of the   § 67. The following axioms or canons serve
validity of
syllogisms. as tests of the validity of that class of syllo-
gisms which we have considered.

1st test.   1st. *If two terms agree with one and the same*
third, they agree with each other.

2d test.   2d. *If one term agrees and another disagrees*
with one and the same third, these two disagree
with each other.

The first the   On the former of these canons rests the va-
test of all
affirmative lidity of *affirmative* conclusions ; on the latter,
conclusions.
The second of *negative :* for, no syllogism can be faulty
of negative. which does not violate these canons ; none cor-
rect which does ; hence, on these two canons
are built the following rules or cautions, which
are to be observed with respect to syllogisms,
for the purpose of ascertaining whether those
canons have been strictly observed or not.

Every syllo-   1st. *Every syllogism has three and only three*
gism has
three and *terms ;* viz. the middle term and the two terms
only three of the Conclusion : the terms of the Conclusion
terms.
are sometimes called extremes.

Every syllo-   2d. *Every syllogism h is three and only three*

propositions; viz. the major premise; the minor premise; and the conclusion.

gism has three and only three propositions.

3d. *If the middle term is ambiguous, there are in reality two middle terms, in sense,* though but one in *sound.*

Middle term must not be ambiguous.

There are two cases of ambiguity: 1st. Where the middle term is *equivocal;* that is, when used in different senses in the two premises.   For example :

Two cases

1st case.

> " *Light* is contrary to darkness ;
> Feathers are *light;* therefore,
> Feathers are contrary to darkness."

Example.

2d. Where the middle term is *not* distributed ; for as it is then used to stand for a *part* only of its *significates*, it may happen that one of the extremes is compared with *one part* of the whole term, and the other with another part of it.   For example :

2d case.

> " White is a color ;
> Black is a color ; therefore,
> Black is white."

Examples.

Again :

> " Some animals are beasts ;
> Some animals are birds ; therefore,
> Some birds are beasts."

3d. *The middle term, therefore, must be distributed, once, at least,* in the premises ; that is,

The middle term must be once distributed :

and once is
sufficient.

by being the subject of a universal,* or predi-
cate of a negative ;† and once is sufficient ;
since if one extreme has been compared with a
part of the middle term, and another to the
whole of it, they must have been compared with
the same.

No term must
be distribu-
ted in the
conclusion
which was
not distribu-
ted in a
premise.

4th. *No term must be distributed in the con-
clusion which was not distributed in one of the
premises;* for, that would be to employ the
whole of a term in the conclusion, when *you*
had employed only a *part* of it in the premise ;
thus, in reality, to introduce a fourth term.
This is called an *illicit* process either of the
major or minor term.‡ For example :

Example.

" All quadrupeds are animals,
    A bird is not a quadruped ; therefore,
    It is not an animal."   Illicit process of the major.

Negative
premises
prove noth-
ing.

5th. *From negative premises you can infer
nothing.* For, in them the Middle is pronounced
to disagree with both extremes; therefore they
cannot be compared together : for, the extremes
can only be compared when the middle agrees
with *both ;* or, agrees with *one*, and disagrees
with the *other*.   For example :

Example.

" A fish is not a quadruped ;"
    " A bird is not a quadruped," proves nothing.

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* Section 62.   † Section 63.   ‡ Section 40.

6th. *If one premise be negative, the conclu-* If one prem-
ise is nega-
tive, the
conclusion
will be nega
tive;
sion must be negative; for, in that premise the
middle term is pronounced to disagree with one
of the extremes, and in the other premise (which
of course is affirmative by the preceding rule),
to agree with thé other extreme ; therefore, the
extremes disagreeing with each other, the con-
clusion is negative.   In the same manner it may
be shown, that *to prove a negative conclusion,
one of the premises must be a negative.*

By these six rules all Syllogisms are to be
tried ; and from them it will be evident, 1st,
that *nothing can be proved from two particular
premises ;* (since you will then have *either the
middle term undistributed,* or an *illicit process.*
For example :

> " Some animals are sagacious ;
>   Some beasts are not sagacious ;
>   Some beasts are not animals.")

And, for the same reason, 2dly, that if one of
the premises be particular, the conclusion must
be particular.   For example :

> " All who fight bravely deserve reward ;
> " Some soldiers fight bravely ;" you can only infer that
> " Some soldiers deserve reward :"

for to infer a universal conclusion would be
an illicit process of the minor.   But from two

universal Premises you cannot always infer a universal Conclusion.   For example:

> " All gold is precious ;
> All gold is a mineral ; therefore,
> *Some* mineral is precious.'

And even when we *can* infer a universal, we are always *at liberty* to infer a particular ; since what is predicated of all may *of course* be pre dicated of some.

OF FALLACIES.

Definition of a fallacy.   § 68. By a fallacy is commonly understood " any unsound mode of arguing, which appears to demand our conviction, and to be decisive of the question in hand, when in fairness it is Detection of, not."   In the practical detection of each indi-
depends on
acuteness. vidual fallacy, much must depend on natural and acquired acuteness ; nor can any rules be given, the mere learning of which will enable us to *apply* them with mechanical certainty and Hints and readiness ; but still we may give some hints that
rules useful. will lead to correct general views of the subject, and tend to *engender such a . habit of mind,* as will lead to critical examinations.

Same of Lo-
gic in general.   Indeed, the case is the same with respect to Logic in general ; scarcely any one would, in ordinary practice, state to himself either his

own or another's reasoning, in syllogisms at full length; yet a familiarity with logical principles tends very much (as all feel, who are really well acquainted with them) to beget a habit of clear and sound reasoning. The truth is, in this as in many other things, there are processes going on in the mind (when we are practising any thing quite familiar to us), with such rapidity as to leave no trace in the memory; and we often apply principles which did not, as far as we are conscious, even occur to us at the time.

Logic tends to cultivate habits of clear reasoning.

The habit fixed, we naturally follow the processes.

§ 69. Let it be remembered, that in every process of reasoning, logically stated, the conclusion is inferred from two antecedent propositions, called the Premises. Hence, it is manifest, that in every argument, the fault, if there be any, must be either,

Conclusion follows from two antecedent premises.

Fallacy, if any, either in the premises

1st. In the premises; or,

2d. In the conclusion (when it does not follow from them); or,

or conclusion, or both.

3d. In both.

In every fallacy, the conclusion either *does* or *does not follow from the premises.*

When the fault is in the premises; that is, when they are such as ought not to have been assumed, and the conclusion legitimately follows from them, the fallacy 's called a Material Fal-

When in the premises;

lacy, because it lies in the matter of the argument.

When in the conclusion. Where the conclusion does not follow from the premises, it is manifest that the fault is in the *reasoning*, and in that alone : these, therefore, are called Logical Fallacies, as being properly violations of those rules of reasoning which it is the province of logic to lay down.

When in both. When the fault lies in both the premises and reasoning, the fallacy is both Material and Logical

Rules for examining a train of argument. § 70. In examining a train of argumentation, to ascertain if a fallacy have crept into it, the following points would naturally suggest themselves :

1st Rule. 1st. What is the proposition to be proved ? On what facts or truths, as premises, is the argument to rest ? and, What are the marks of truth by which the conclusion may be known ?

2d Rule. 2d. Are the premises both true ? If facts, are they substantiated by sufficient proofs ? If truths, were they logically inferred, and from correct premises ?

3d Rule. 3d. Is the middle term what it should be, and the conclusion logically inferred from the premises ?

Suggestions serve as guides, These general suggestions may serve as guides in examining arguments for the purpose of de-

tecting fallacies; but however perfect general to detect error. rules may be, it is quite certain that error, in its thousand forms, will not always be separated from truth, even by those who most thoroughly understand and carefully apply such rules

CONCLUDING REMARKS.

§ 71. The imperfect and irregular sketch which Logic corresponds with the reasonings in Geometry. has here been attempted of deductive logic, may suffice to point out the general drift and purpose of the science, and to show its entire correspondence with the reasonings in Geometry. The analytical form, which has here been adopted, Analytical form. is, generally speaking, better suited for *introducing* any science in the plainest and most interesting form; though the synthetical is the more Synthetical form. regular, and the more compendious form for storing it up in the memory.

§ 72. It has been a matter about which wri- Induction: does it form a part of Logic? ters on logic have differed, whether, and in conformity to what principles, Induction forms a part of the science; Archbishop Whately main- Whately's opinion taining that logic is only concerned in inferring truths from known and admitted premises, and that all reasoning, whether Inductive or Deductive, is shown by analysis to have the syllogism

Mill's views. for its type ; while Mr. Mill, a writer of perhaps
greater authority, holds that deductive logic is
but the carrying out of what induction begins ;
that all reasoning is founded on principles of in-
ference ulterior to the syllogism, and that the
syllogism is the test of deduction only.

Without presuming at all to decide defini-
tively a question which has been considered and
Reasons for passed upon by two of the most acute minds of
the course
taken. the age, it may perhaps not be out of place to
state the reasons which induced me to adopt
the opinions of Mr. Mill in view of the par-
tícular use which I wished to make of logic.

Leading ob-   § 73. It was, as stated in the general plan,
jects of the
plan· one of my leading objects to point out the cor-
respondence between the science of logic and
the science of mathematics : to show, in fact,
To show that that mathematical reasoning conforms, in every
mathemati-
cal reasoning respect, to the strictest rules of logic, and is in-
conforms to deed but logic applied to the abstract quantities,
logical rules.
Number and Space.   In treating of space, about
which the science of Geometry is conversant, we
shall see that the reasoning rests mainly on the
Axioms, how axioms, and that these are established by induc-
established. tive processes.   The processes of reasoning which
relate to numbers, whether the numbers are rep-
resented by figures or letters, consist of two parts.

1st. To obtain formulas for, that is, to express in the language of science, the relations between the quantities, facts, truths or principles, what- ever they may be, that form the subject of the reasoning; and,

Two parts of the reasoning process.

2dly. To deduce from these, by processes purely logical, all the truths which are implied in them, as premises.

§ 74. Before dismissing the subject, it may be well to remark, that every induction may be thrown into the form of a syllogism, by sup- plying the major premise. If this be done, we shall see that something equivalent to *the uni- formity of the course of nature* will appear as the ultimate major premise of all inductions ; and will, therefore, stand to all inductions in the relation in which, as has been shown, the major premise of a syllogism always stands to the conclusion ; not contributing at all to prove it, but being a necessary condition of its being proved. This fact sustains the view taken by Mr. Mill, as stated above ; for, this ultimate ma- jor premise, or any substitution for it, is an infer- ence by Induction, but cannot be arrived at by means of a syllogism.

All Induc- tion may be thrown into the form of the Syllogism, by admitting a proper major premise.

How this major prem ise is obtain- ed.

7

OUTLINES OF MATHEMATICAL SCIENCE.

QUANTITY..........

NUMBER..........
{
1. ABSTRACT.
2. CURRENCY.
3. WEIGHT.
4. TIME.
}

SPACE..........
{
1. LINES.
2. SURFACES.
3. VOLUMES.
4. ANGLES.
}

BOOK II.

MATHEMATICAL SCIENCE.

———•◦•———

CHAPTER I.

QUANTITY AND MATHEMATICAL SCIENCE DEFINED—DIFFERENT KINDS OF QUANTITY—LANGUAGE OF MATHEMATICS EXPLAINED—SUBJECTS CLASSIFIED—UNIT OF MEASURE DEFINED—MATHEMATICS A DEDUCTIVE SCIENCE.

QUANTITY.

§ 75. QUANTITY is any thing which can be increased, diminished, and measured.

Quantity defined.

§ 76. MATHEMATICS is the science of quantity; that is, the science which treats of the measures of quantities and of their relations to each other.

Mathematics defined.

§ 77. There are two kinds of quantity; Number and Space.

Kinds of quantity.

NUMBER.

§ 78. A NUMBER is a unit, or a collection of units.

Number defined.

Abstract.   AN ABSTRACT NUMBER is one whose unit is not named; as, one, two, three, &c.

Denominate.   A DENOMINATE NUMBER is one whose unit is named; as three feet, three yards, three pounds. Such numbers are also called CONCRETE NUMBERS.

How we obtain an idea of number.   How do we acquire our first notions of numbers? By first presenting to the mind, through the eye, a single thing, and calling it ONE. Then presenting two things, and naming them TWO; then three things, and naming them THREE; and so on for other numbers. Thus,

It is done by perception, comparison, and reflection.   we acquire primarily, in a *concrete* form, our elementary notions of number, by perception, comparison, and reflection; for, we must first perceive how many things are numbered; then compare what is designated by the word one,

Reasons.   with what is designated by the words two, three, &c., and then reflect on the results of such comparisons until we clearly apprehend the difference in the signification of the words. Having thus acquired, in a concrete form, our conceptions of numbers, we can consider numbers as separated from any particular objects,

Two axioms necessary for the formation of numbers.   and thus form a conception of them in the abstract. We require but two axioms for the formation of all numbers:

1st axiom.   1st. That one may be added to any number, and that the number which results will be

greater by one than the number to which the one was added.

2d. That one may be divided into any num- 2d axiom. ber of equal parts.

§ 79. Under Number, we have four species, or Four kinds subdivisions, each differing from the other three, of number. in the unit of its base: thus,

1. Abstract Number, when the base is the ab- Abstract. stract unit one:

2. Number of Currency, when the base is a Currency. unit of currency, as one dollar:

3. Number of Weight, when the base is a unit Weight. of weight, as one pound:

4. Number of Time, when the base is a unit Time. of time, as one day.

Hence, in number, we have four kinds of Four kinds units: Abstract Units; Units of Currency; Units of units. of Weight; and Units of Time.

SPACE.

§ 80. SPACE is indefinite extension. We ac- Space quire our ideas of it by observing that *parts* defined. of it are occupied by matter or bodies. This enables us to attach a definite idea to the word *place*. We are then able to say, intelligibly, Place: that a point is that which has place, or posi- a point.

tion in space, without occupying any part of it. Having conceived a second point in space, we can understand the important axiom, "A straight line is the shortest distance between two points;" and this line we call *length*, or a *dimension* of space.

Axiom concerning the straight line.

§ 81. If we conceive a second straight line to be drawn, meeting the first, but lying in a direction directly from it, we shall have a second dimension of space, which we call *breadth*. If these lines be prolonged, in both directions, they will include four portions of space, which make up what is called a plane surface, or plane : hence, a plane has two dimensions, length and breadth. If now we draw a line on either side of this plane, we shall have another dimension of space, called *thickness :* hence, space has three dimensions—length, breadth, and thickness.

Breadth defined.

A plane defined.

Space has three dimensions.

§ 82. A portion of space limited by boundaries, is called a *Figure*. If such portion of space have but one dimension, it is called a line, and may be limited by two points, one at each extremity. There are two kinds of lines, straight and curved. A straight line, is one which does not change its direction between any two of its

Figure defined.

Line defined.

Two kinds of lines: straight and curved.

points, and a curved line constantly changes its direction at every point.

§ 83. A portion of space having two dimensions is called a surface. There are two kinds of surfaces—Plane Surfaces and Curved Surfaces. With the former, a straight line, having two points in common, will always coincide, however it may be placed, while with the latter it will not. The boundaries of surfaces are lines, straight or curved. Surface: Plane, curved. Difference. Boundaries of a surface.

§ 84. A limited portion of space, having three dimensions, is called a *Volume*. All volumes are bounded by surfaces, either plane or curved. Volumes. Boundaries.

§ 85. AN ANGLE is the amount of divergence of two lines, of two planes, or of several planes, meeting at a point; and is measured, like other magnitudes, by comparing it with its unit of measure. Hence, in space, we have four units, differing in kind: Angles. Units of measure.

1. Linear Units, for the measurement of lines; Linear.

2. Units of Surface, for the measurement of surfaces; Surface.

3. Units of Volume, for the measurement of volumes; and Volume.

Angle.   4. Units of Angles, for the measurement of angles.

Eight units.   § 86. Besides the eight kinds of units, four of number and four of space, embraced in the above classification, and in which the units of each class are connected by known laws, there are yet isolated denominate numbers, such as five chairs, six horses, seven things, &c., which Units with- do not admit of classification, because they have no law of formation. Neither does this classi- Infinitesimal fication include the Infinitesimal Units, which are specially treated of in Chapter V., Book II., and which are the elements of a very important branch of Mathematical Science.

Language   § 87. The language of Mathematics is mixed. mathema- Although composed mainly of symbols, which are defined with reference to the uses which are made of them, and therefore have a pre- cise signification; it is also composed, in part, of words transferred from our common lan- Symbols guage. The symbols, although arbitrary signs, are, nevertheless, entirely general, as signs and instruments of thought; and when the sense in which they are used is once fixed, by definition, Sense un- they preserve throughout the entire analysis changed. precisely the same signification. The meaning

Units without law.

Infinitesimal units.

Language of mathematics.

Symbols general.

Sense unchanged.

of the words borrowed from our common vocab- Words bor-
ulary is often modified, and sometimes entirely rowed from common
changed, when the words are transferred to the language, are modified
language of science. They are then used in a and used in a technical
particular sense, and are said to have a *technical* sense.
signification.

§ 88. It is of the first importance that the Language
elements of the language be clearly understood, must be understood:
—that the signification of every word or sym-
bol be distinctly apprehended, and that the con-
nection between the thought and the word or
symbol which expresses it, be so well estab-
lished that the one shall immediately suggest Mathemati-
the other. It is not possible to pursue the sub- cal reason-
tle reasonings of Mathematics, and to carry out ings require it.
the trains of thought to which they give rise,
without entire familiarity with those means
which the mind employs to aid its investiga-
tions. The child cannot read till he has learned Cannot use
the alphabet; nor can the scholar feel the deli- any lan-
cate beauties of Shakspeare, or be moved by the till we
sublimity of Milton, before studying and learn- know it.
ing the language in which their immortal
thoughts are clothed.

§ 89. All Quantities, whether abstract or con- Quantities
crete, are, in mathematical science, presented are repre-

5*

sented by
symbols;
and are oper-
ated on by
these sym-
bols.

Signs.

What consti-
tutes the
language.

Its nature.

What it ac-
complishes.

Extent and
power of
Analysis.

to the mind by arbitrary symbols. They are viewed and operated on through these symbols which represent them; and all operations are indicated by another class of symbols called *signs.* These, combined with the symbols which represent the quantities, make up, as we have stated above, the pure mathematical language; and this, in connection with that which is borrowed from our common language, forms the language of mathematical science. This language is at once comprehensive and accurate. It is capable of stating the most general propositions, and of presenting to the mind, in their proper order, all elementary principles connected with their solution. By its generality it reaches over the whole field of the pure and mixed sciences, and gathers into condensed forms all the conditions and relations necessary to the development of particular facts and universal truths. Thus, the skill of the analyst deduces from the same equation the velocity of an apple falling to the ground, and the verification of the law of universal gravitation.

LANGUAGE OF MATHEMATICS.

Language
of
Mathema-
tics.

§ 90. The language of Mathematics embraces, 1st. Parts of our written and spoken language; 2d. The language of Figures,

3d. The language of Lines—straight and curved; Lines.

4th. The language of Letters; and these forms Letters.
of language determine the classification of the
branches of the Science of Mathematics.

LANGUAGE OF NUMBER—ARITHMETIC.

§ 91. The ten characters, called figures, are Language
the alphabet of the language of number. The of
Number.
various ways in which they are combined, form-
ing the exact and copious language of compu-
tation, will be fully explained under the head
of Arithmetic, in a chapter exclusively devoted
to the consideration of numbers, their laws and
their language.

LANGUAGE OF LINES—GEOMETRY.

§ 92. Lines, straight and curved, are the ele- Language
ments of the pictorial language applicable to of
Geometry.
space. The definitions and axioms relating to
space, and all the reasonings founded on them,
make up the science of Geometry, which will
be fully treated under its proper head.

LANGUAGE OF LETTERS—ANALYSIS.

§ 93. ANALYSIS is a general term embracing Analysis.
all the operations which can be performed on

quantities when represented by letters. In this branch of mathematics, all the quantities con-

Quantities represented by letters.

sidered, whether abstract or concrete, are represented by letters of the alphabet, and the operations to be performed on them are indicated by a few arbitrary signs. The letters

Symbols.

and signs are called Symbols, and by their combination we form the Algebraic Notation and Language.

Analysis, Algebra; Analytical Geometry. Calculus.

§ 94. Analysis, in its simplest form, takes the name of Algebra. Analytical Geometry, the Differential and Integral Calculus, extended to include the Theory of Variations, are its higher and most advanced branches.

Term Analysis defined.

§ 95. The term Analysis has also another signification. It denotes the process of separating

Its nature.

any complex whole into the elements of which

Synthesis defined.

it is composed. It is opposed to Synthesis, a term which denotes the processes of first considering the elements separately, then combining them, and ascertaining the results of the combination.

Analytical method.

The Analytical method is best adapted to investigation, and the presentation of subjects in

Synthetical method.

their general outlines; the Synthetical method is best adapted to instruction, because it exhib-

its all the parts of a subject separately, and in their proper order and connection. Analysis deduces all the parts from a whole: Synthesis forms a whole from the separate parts.

Analysis.

Synthesis.

§ 96. Arithmetic, Algebra, and Geometry are the Elementary branches of Mathematical Science. Every truth which is established by mathematical reasoning, is developed by an arithmetical, geometrical, or analytical process, or by a combination of them. The reasoning in each branch is conducted on principles identically the same. Every sign, or symbol, or technical word, is accurately defined, so that to each there is attached a definite and precise idea. Thus, the language is made so exact and certain, as to admit of no ambiguity.

Arithmetic, Algebra, Geometry, elementary branches.

Reasoning the same.

Language exact.

INFINITESIMAL CALCULUS.

§ 97. The language of the Infinitesimal Calculus is very simple. Its chief element is the letter d, which, when written before a quantity, denotes that that quantity increases or decreases according to the law of continuity, and the expression thus arising is one link in that law. Thus, dx denotes that the quantity represented by x, changes according to the law of continuity, and that dx is the unit of that change.

Language of the Infinitesimal Calculus.

What does dx denote.

PURE MATHEMATICS.

Pure Mathe-
matics.

§ 98. The Pure Mathematics embraces all the principles of the science, and all the processes by which those principles are developed from Number and the abstract quantities, Number and Space. All the definitions and axioms, and all the truths deduced from them, are traceable to these two sources.

Number and
Space.

Mathema-
tics, as used
by the an-
cients:

§ 99. Mathematics, in its primary significa-tion, as used by the ancients, embraced every acquired science, and was equally applicable to all branches of knowledge. Subsequently it was restricted to those branches only which were acquired by severe study, or discipline, and its votaries were called Disciples. Those sub-jects, therefore, which required patient investi-gation, exact reasoning, and the aid of the ma-thematical analysis, were called Disciplinal or Mathematical, because of the greater evidence in the arguments, the infallible certainty of the conclusions, and the mental training and de-velopment which such exercises produced.

Embraced
all subjects
which were
disciplinary
in their na-
ture.

Pure Mathe-
matics:
What are its
foundations.

§ 100. The Pure Mathematics is based on definitions and intuitive truths, called axioms, which are inferred from observation and expe-

rience; that is, observation and experience fur- Premises.
nish the information necessary to such intui-
tive inductions.*   From these definitions and
axioms, as premises, all the truths of the science Reasoning.
are established by processes of deductive reason-
ing; and there is not, *in the whole range of* Its tests of
mathematical science any logical test of truth, truths:
but in a conformity of the conclusions to the What they
definitions and axioms, or to such principles or are.
propositions as have been established from them.
Hence, we see, that the science of Pure Mathe- In what the
matics, which consists merely in inferring, by science con-
fixed rules, all the truths which can be deduced sists.
from given premises, is purely a Deductive Is purely
Science.   The precision and accuracy of the Deductive.
definitions; the certainty which is felt in the
truth of the axioms; the obvious and fixed re- Precision of
lation between the *sign* and the *thing signified;* its language.
and the certain formulas to which the reason-
ing processes are reduced, have given to mathe-
matics the name of "Exact Science."   Exact.
                                        Science.

§ 101. We have remarked that all the rea- All reason-
sonings of mathematical science, and all the ing based on
truths which they establish, are based on the definitions
definitions and axioms, which correspond to the and axioms.

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* Section 27.

major premise of the syllogism. If the resemblance which the minor premise asserts to the

Relations not obvious.

middle term were obvious to the senses, as it is in the proposition, "Socrates was a man," or were at once ascertainable by direct observation, or were as evident as the intuitive truth, "A whole is equal to the sum of all its parts;"

Deductive Science, necessary.

there would be no necessity for trains of reasoning, and Deductive Science would not exist.

Trains of reasoning:

Trains of reasoning are necessary only for the sake of extending the definitions and axioms to

What they accomplish.

other cases in which we not only cannot directly observe what is to be proved, but cannot directly observe even the mark which is to prove it.

Syllogism, the final test of deduction.

§ 102. Although the syllogism is the ultimate test in all deductive reasoning (and indeed in all inductive, if we admit the uniformity of the course of nature), still we do not find it convenient or necessary, in mathematics, to throw every proposition into the form of a syllogism.

Axioms and definitions, tests of truth.

The definitions and axioms, and the propositions established from them, are our tests of truth; and whenever any new proposition can be brought to conform to any one of these

A proposition: when proved.

tests, it is regarded as proved, and declared to be true.

§ 103. When general formulas have been framed, determining the limits within which the deductions may be drawn (that is, what shall be the tests of truth), as often as a new case can be at once seen to come within one of the formulas, the principle applies to the new case, and the business is ended. But new cases are continually arising, which do not obviously come within any formula that will settle the questions we want solved in regard to them, and it is necessary to reduce them to such formulas. This gives rise to the existence of the science of mathematics, requiring the highest scientific genius in those who contributed to its creation, and calling for a most continued and vigorous exertion of intellect, in order to appropriate it, when created.

When a principle may be regarded as proved.

Trains of reasoning: why necessary.

They give rise to the science of mathematics.

MIXED MATHEMATICS.

§ 104. The Mixed Mathematics embraces the applications of the principles and laws of the Pure Mathematics to all investigations in which the mathematical language is employed and to the solution of all questions of a practical nature, whether they relate to abstract or concrete quantity.

Mixed mathematics.

8

QUANTITY MEASURED.

Quantity.   § 105. Quantity has been defined, "any thing which can be increased, diminished, and meas-
Increased and diminished, defined.
ured." The terms increased or diminished, are easily understood, implying merely the property of being made larger or smaller. The term measured is not so easily comprehended, because it has only a relative meaning.

Measured.   The term "measured," applied to a quantity, implies the existence of some known quantity of the same kind, which is regarded as a stand-
What it means.
ard. With this standard, the quantity to be measured is compared with respect to its extent or
Standard: magnitude. To such standard, whatever it may
is called unity.
be, we give the name of *unity*, or *unit of measure ;* and the number of times which any quantity contains its unit of measure, is the numerical value of the quantity measured. The
Magnitude: extent or magnitude of a quantity is, therefore,
merely relative.
merely relative, and hence, we can form no idea of it, except by the aid of comparison. Space,
Space: indefinite.
for example, is entirely indefinite, and we measure parts of it by means of certain standards,
Measurement ascertains relation:
called measures ; and after any measurement is completed, we have only ascertained the *relation* or *proportion* which exists between the standard we adopted and the thing measured. Hence,

measurement is, after all, but a mere process *A process of comparison.* of comparison.

§ 106. The quantities, Number and Space, are but *Number and Space:* vague and indefinite conceptions, until we compare *known by* them with their units of measure, and even these *comparison.* units are arrived at only by processes of comparison. Comparison is the great means of discovering the *Comparison* relations of things to each other, as well in general *a general method.* logic, as in the science of mathematics, which develops the processes by which quantities are compared, and the results of such comparisons.

§ 107. Besides the classification of quantity *Quantity.* into Number and Space, we may, if we please, divide it into Abstract and Concrete. An ab- *Abstract.* stract quantity is a mere number, in which the unit is not named, and has no relation to mat- ter or to the kind of things numbered. A con- *Concrete.* crete quantity is a definite object or a collec- tion of such objects.   Concrete quantities are expressed by numbers and letters, and also by *How repre-* *sented.* lines, straight and curved. The number " three" *Example* is entirely abstract, expressing an idea having *of the* *abstract.* no connection with things; while the number " three pounds of tea," or " three apples," pre- *Example* sents to the mind an idea of concrete objects. *of the* *concrete.* So, a portion of space, bounded by a surface, all

Sphere

defined.

the points of which are equally distant from a certain point within called the centre, is but a mental conception of form; but regarded as a portion of space, gives rise to the additional idea of a named and defined thing.

COMPARISON OF QUANTITIES.

Mathematics concerned with Number and Space.

Reasoning involves comparison.

§ 108. We have seen that the pure mathematics are concerned with the two quantities, Number and Space. We have also seen, that reasoning necessarily involves comparison: hence, mathematical reasoning must consist in comparing the quantities which come from Number and Space with each other.

Two quantities can sustain but two relations.

§ 109. Any two quantities, compared with each other, must necessarily sustain one of two relations: they must be equal, or unequal. What axioms or formulas have we for inferring the one or the other?

AXIOMS FOR INFERRING EQUALITY.

Formulas for Equality.

1. Quantities which contain the same unit an equal number of times, are equal.

2. Things which being applied to each other coincide, are equal in all their parts.

3. Things which are equal to the same thing are equal to one another.

4. A whole is equal to the sum of all its parts

5. If equals be added to equals, the sums are equal.

6. If equals be taken from equals, the remainders are equal.

AXIOMS FOR INFERRING INEQUALITY.

1. A whole is greater than any of its parts.

2. If equals be added to unequals, the sums are unequal.

Formulas for Inequality.

3. If equals be taken from unequals, the remainders are unequal.

§ 110. We have thus completed a very brief and general analytical view of Mathematical Science.   We have named and defined the subjects of which it treats—and the forms of the language employed.   We have pointed out the character of the definitions, and the nature of the elementary and intuitive propositions on which the science rests; also, the kind of reasoning employed in its creation, and its divisions resulting from the use of different symbols and differences of language.   We shall now proceed to treat the branches separately.

What features have been sketched.

ONE. UNIT

DIVIDED, according to the SCALE OF TENS
.1 .01 .001 .0001 .00001 &c.

DIVIDED, ACCORDING TO VARYING SCALES. SIMPLE.

$\frac{1}{4}$ $\frac{6}{7}$ $\frac{8}{17}$ $\frac{9}{47}$

$\frac{1}{6}$lb. $\frac{1}{8}$oz. $\frac{1}{7}$cwt. DENOMINATE.

INCREASED, BY THE SCALE OF TENS.
&c. 10000. 1000. 100. 10. 1.

INCREASED BY VARYING SCALES.

£	s.	d.	far.
1	1	1	1

Ton.	cwt.	qr.	lb.	oz.	dr.
1	1	1	1	1	1

&c. for all denominate numbers.

PROPORTION OF UNITY TO ALL NUMBERS,
AND OF NUMBERS TO EACH OTHER.

APPLICATIONS

CHAPTER II.

ARITHMETIC—SCIENCE AND ART OF NUMBERS.

SECTION I.

INTEGRAL UNITS.

FIRST NOTIONS OF NUMBERS.

§ 111. THERE is but a single elementary idea in the science of numbers: it is the idea of the UNIT ONE. There is but one way of impressing this idea on the mind. It is by presenting to the senses a single object; as, one apple, one peach, one pear, &c.

But one elementary idea in numbers.

How impressed on the mind.

§ 112. There are three signs by means of which the idea of one is expressed and communicated. They are,

1st. The word ONE.

2d. The Roman character I.

3d. The figure 1.

Three signs for expressing it.

A word.

Roman character:

Figure.

New ideas which arise by adding one.

§ 113. If one be added to one, the idea thus arising is different from the idea of one, and is complex. This new idea has also three signs; viz. TWO, II., and 2. If one be again added, that is, added to two, the new idea has likewise three signs; viz. THREE, III., and 3.   These

Collections are numbers.

collections, and similar ones, are called numbers. Hence,

A NUMBER *is a unit or a collection of units.*

IDEAS OF NUMBERS GENERALIZED.

Ideas of numbers generalized.

§ 114. If we begin with the idea of the number one, and then add it to one, making two; and then add it to two, making three ; and then to three, making four ; and then to four, making

How formed.

five, and so on ; it is plain that we shall form a series of numbers, each of which will be greater

Unity the basis.

by one than that which precedes it.   Now, one or unity, is the basis of this series of numbers,

Three ways of expressing them.

and each number may be expressed in three ways :

1st way.

1st. By the words ONE, TWO, THREE, &c., of our common language ;

2d way.

2d. By the Roman characters ; and,

3d way.

3d. By figures.

§ 115. Since all numbers, whether integral or All numbers come from one: fractional, must come from, and hence be connected with, the unit one, it follows that there is but one purely elementary idea in the science of numbers. Hence, the idea of every number, Hence but one idea that is purely elementary. regarded as made up of units (and all numbers except one must be so regarded when we analyze them), is necessarily complex. For, since All other notions are complex. the number arises from the addition of ones, the apprehension of it is incomplete until we understand how those additions were made ; and therefore, a full idea of the number is necessarily complex.

§ 116. But if we regard a number as an entirety, that is, as an entire or whole thing, as an entire two, or three, or four, without pausing to When a number may be regarded as incomplex. analyze the units of which it is made up, it may then be regarded as a simple or incomplex idea ; though, as we have seen, such idea may always be traced to that of the unit one, which forms the basis of the number.

UNITY AND A UNIT DEFINED.

§ 117. When we name a number, as twenty What is necessary to the apprehension of a number feet, two things are necessary to its clear apprehension.

First.   1st. A distinct apprehension of the *single thing* which forms the base of the number; and,

Second.   2d. A distinct apprehension of the *number of times* which that thing is taken.

The basis of the number is UNITY.

When it is called UNITY,

and when a UNIT.

The single thing, which forms the base of the number, is called UNITY, or a UNIT. It is called unity, when it is regarded as the *primary base* of the number; that is, when it is the final standard to which all the numbers that come from it are referred. It is called a unit when it is regarded as one of the collection of several equal things which form a number. Thus, in the example, one foot, regarded as a standard and the base of the number, is called UNITY; but, considered as one of the twenty equal feet which make up the number, it is called a UNIT.

OF SIMPLE AND DENOMINATE NUMBERS.

Abstract unit.

§ 118. A simple or abstract unit, is ONE, without regard to the kind of thing to which the term *one* may be applied.

Denominate unit.

A denominate or concrete unit, is *one thing* named or denominated; as, one apple, one peach, one pear, one horse, &c.

Number has no reference

§ 119. Number, as such, has no reference to the particular things numbered. But to dis-

tinguish numbers which are applied to particular *to the things numbered.*
units from those which are purely abstract, we
call the latter Abstract or Simple Numbers, *Simple and Denominate.*
and the former Concrete or Denominate Num-
bers.   Thus, fifteen is an abstract or simple
number, because the unit is *one;* and fifteen *Examples.*
pounds is a concrete or denominate number,
because its unit, one pound, is denominated or
named.

ALPHABET—WORDS—GRAMMAR.

§ 120. The term alphabet, in its most general *Alphabet.*
sense, denotes a set of characters which form
the elements of a written language.

When any one of these characters, or any *Words.*
combination of them, is used as the sign of a
distinct notion or idea, it is called a word ; and
the naming of the characters of which the word
is composed, is called its spelling.

Grammar, as a science, treats of the estab- *Grammar*
lished connection and relation of words, as the
signs of ideas.

ARITHMETICAL ALPHABET.

§ 121. The arithmetical alphabet consists of *Arithmetical Alphabet.*
ten characters, called figures.   They are,

Naught, One,   Two, Three, Four, Five,   Six, Seven, Eight, Nine,
  0   1   2   3   4   5   6   7   8   9

and each may be regarded as a word, since it
stands for a distinct idea.

WORDS—SPELLING AND READING IN ADDITION.

One cannot
be spelled.

§ 122. The idea of one, being elementary, the
character 1 which represents it, is also element-
ary, and hence, cannot be spelled by the other
characters of the Arithmetical Alphabet (§ 121).
But the idea which is expressed by 2 comes from

Spelling by
the
arithmetical
characters.

the addition of 1 and 1 : hence, the word repre-
sented by the character 2, may be spelled by
1 and 1.   Thus, 1 and 1 are 2, is the arithmet-
ical spelling of the word two.

Three is spelled thus: 1 and 2 are 3; and
also, 2 and 1 are 3.

Examples.

Four is spelled, 1 and 3 are 4 ; 3 and 1 are 4 ;
2 and 2 are 4.

Five is spelled, 1 and 4 are 5 ; 4 and 1 are 5;
2 and 3 are 5 ; 3 and 2 are 5.

Six is spelled, 1 and 5 are 6 ; 5 and 1 are 6 ;
2 and 4 are 6 ; 4 and 2 are 6 ; 3 and 3 are 6.

All numbers
may be
spelled in a
similar way.

§ 123. In a similar manner, any number in
arithmetic may be spelled; and hence we see
that the process of spelling in addition consists
simply, in naming any two elements which will
make up the number.   All the numbers in ad-

dition are therefore spelled with two syllables. The *reading* consists in naming only the word which expresses the final idea.   Thus,

0	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
One	two	three	four	five	six	seven	eight	nine	ten.

Examples.

We may now read the words which express the first hundred combinations.

READINGS.

Read.

1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1

Two, three, four, &c.

1	2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2	2

Three, four, &c.

1	2	3	4	5	6	7	8	9	10
3	3	3	3	3	3	3	3	3	3

Four, five, &c.

1	2	3	4	5	6	7	8	9	10
4	4	4	4	4	4	4	4	4	4

Five, six, &c.

1	2	3	4	5	6	7	8	9	10
5	5	5	5	5	5	5	5	5	5

Six, seven, &c.

1	2	3	4	5	6	7	8	9	10
6	6	6	6	6	6	6	6	6	6

Seven, eight, &c.

1	2	3	4	5	6	7	8	9	10
7	7	7	7	7	7	7	7	7	7

Eight, nine, &c.

1	2	3	4	5	6	7	8	9	10
8	8	8	8	8	8	8	8	8	8

Nine, ten, &c.

Ten, eleven, &c.	1	2	3	4	5	6	7	8	9	10
	9	9	9	9	9	9	9	9	9	9

Eleven, twelve, &c.	1	2	3	4	5	6	7	8	9	10
	10	10	10	10	10	10	10	10	10	10

Example for reading in Addition.

§ 124. In this example, beginning at the right hand, we say, 8, 17, 18, 26 : setting down the 6 and carrying the 2, we say, 8, 13, 20, 22, 29 : setting down the 9 and carrying the 2, we say, 9, 12, 18, 22, 30 :

```
 878
 421
 679
 354
 764
————
3096
```

and setting down the 30, we have the entire sum

All examples so solved. 3096. All the examples in addition may be done in a similar manner.

Advantages of reading.

§ 125. The advantages of this method of reading over spelling are very great.

1st. stated.   1st. The mind acquires ideas more readily through the eye than through either of the other senses. Hence, if the mind be taught to apprehend the result of a combination, by merely seeing its elements, the process of arriving at it is much shorter than when those elements are presented through the instrumentality of sound. Thus, to *see* 4 and 4, and *think* 8, is a very different thing from saying, four and four are eight.

2d. stated.   2d. The mind operates with greater rapidity and certainty, the nearer it is brought to the

ideas which it is to apprehend and combine. Therefore, all unnecessary words load it and impede its operations. Hence, to spell when we can read, is to fill the mind with words and sounds, instead of ideas.

3d. All the operations of arithmetic, beyond the elementary combinations, are performed on paper ; and if rapidly and accurately done, must be done through the eye and by reading. Hence the great importance of beginning early with a method which must be acquired before any considerable skill can be attained in the use of figures.

3d. stated.

§ 126. It must not be supposed that the *reading* can be accomplished until the *spelling* has first been learned.

Reading comes after spelling.

In our common language, we first learn the alphabet, then we pronounce each letter in a word, and finally, we pronounce the word. We should do the same in the arithmetical reading.

Same as in our common language.

WORDS—SPELLING AND READING IN SUBTRACTION.

§ 127. The processes of spelling and reading which we have explained in the addition of numbers, may, with slight modifications, be applied in subtraction. Thus, if we are to subtract

Same principle applied in Subtraction.

2 from 5, we say, ordinarily, 2 from 5 leaves **3**;
or 2 from 5 three remains. Now, the word,
three, is suggested by the relation in which 2
and 5 stand to each other, and this word may be
Readings in read at once. Hence, *the reading, in subtrac-*
Subtraction *tion, is simply naming the word, which expresses*
explained. *the difference between the subtrahend and min-*
uend. Thus, we may read each word of the
following one hundred combinations.

<div align="center">

READINGS.

</div>

One from	1	2	3	4	5	6	7	8	9	10
one, &c.	1	1	1	1	1	1	1	1	1	1
Two from	2	3	4	5	6	7	8	9	10	11
two, &c.	2	2	2	2	2	2	2	2	2	2
Three from	3	4	5	6	7	8	9	10	11	12
three, &c.	3	3	3	3	3	3	3	3	3	3
Four from	4	5	6	7	8	9	10	11	12	13
four, &c.	4	4	4	4	4	4	4	4	4	4
Five from	5	6	7	8	9	10	11	12	13	14
five, &c.	5	5	5	5	5	5	5	5	5	5
Six from six,	6	7	8	9	10	11	12	13	14	15
&c.	6	6	6	6	6	6	6	6	6	6
Seven from	7	8	9	10	11	12	13	14	15	16
seven, &c.	7	7	7	7	7	7	7	7	7	7

| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | Eight from |
| 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | eight, &c. |

| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | Nine from |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | nine, &c. |

| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | Ten from ten, |
| 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | &c. |

§ 128. It should be remarked, that in subtraction, as well as in addition, the spelling of the words must necessarily precede their reading. The spelling consists in naming the figures with which the operation is performed, the steps of the operation, and the final result. The reading consists in naming the final result only.

Spelling precedes reading in Subtraction.

Reading.

SPELLING AND READING IN MULTIPLICATION.

§ 129. Spelling in multiplication is similar to the corresponding process in addition or subtraction. It is simply naming the two elements which produce the product ; whilst the reading consists in naming only the word which expresses the final result.

Spelling in Multiplication.

Reading.

In multiplying each number from 1 to 10 by 2, we usually say, two times 1 are 2 ; two times 2 are 4 : two times 3 are 6 ; two times 4 are 8 ; two times 5 are 10 ; two times 6 are 12 ; two

Examples in spelling.

9

times 7 are 14; two times 8 are 16; two times
In reading. 9 are 18; two times 10 are 20. Whereas, we
should merely read, and say, 2, 4, 6, 8, 10, 12,
14, 16, 18, 20.

In a similar manner we read the entire mul-
tiplication table.

READINGS.

Once one is 1, &c.	12	11	10	9	8	7	6	5	4	3	2	1 1
Two times 1 are 2, &c.	12	11	10	9	8	7	6	5	4	3	2	1 2
Three times 1 are 3, &c.	12	11	10	9	8	7	6	5	4	3	2	1 3
Four times 1 are 4, &c.	12	11	10	9	8	7	6	5	4	3	2	1 4
Five times 1 are 5, &c.	12	11	10	9	8	7	6	5	4	3	2	1 5
Six times 1 are six, &c.	12	11	10	9	8	7	6	5	4	3	2	1 6
Seven times 1 are 7, &c.	12	11	10	9	8	7	6	5	4	3	2	1 7
Eight times 1 are 8, &c.	12	11	10	9	8	7	6	5	4	3	2	1 8

12	11	10	9	8	7	6	5	4	3	2	1	
											9	Nine times 1 are 9, &c.

12	11	10	9	8	7	6	5	4	3	2	1	
											10	Ten times 1 are 10, &c.

12	11	10	9	8	7	6	5	4	3	2	1	
											11	Eleven times 1 are 11, &c.

12	11	10	9	8	7	6	5	4	3	2	1	
											12	Twelve times 1 are 12, &c.

SPELLING AND READING IN DIVISION.

§ 130. In all the cases of short division, the quotient may be read immediately without naming the process by which it is obtained. Thus, in dividing the following numbers by 2, we merely read the words below. *In Short Division, we may read:*

2) 4	6	8	10	12	16	18	22
two	three	four	five	six	eight	nine	eleven.

In a similar manner, all the words, expressing the results in short division, may be read. *In all cases.*

READINGS.

2)2	4	6	8	10	12	14	16	18	20	22	24	Two in 2, once, &c.
3)3	6	9	12	15	18	21	24	27	30	33	36	Three in 3, once, &c.
4)4	8	12	16	20	24	28	32	36	40	44	48	Four in 4, once, &c.

Five in 5, once, &c.	5)5	10	15	20	25	30	35	40	45	50	55	60
Six in 6, once, &c.	6)6	12	18	24	30	36	42	48	54	60	66	72
Seven in 7, once, &c.	7)7	14	21	28	35	42	49	56	63	70	77	84
Eight in 8, once, &c.	8)8	16	24	32	40	48	56	64	72	80	88	96
Nine in 9, once, &c.	9) 9	18	27	36	45	54	63	72	81	90	99	108
Ten in 10, once, &c.	10)10	20	30	40	50	60	70	80	90	100	110	120
Eleven in 11, once, &c.	11)11	22	33	44	55	66	77	88	99	110	121	132
Twelve in 12, once, &c.	12)12	24	36	48	60	72	84	96	108	120	122	144

UNITS INCREASING BY THE SCALE OF TENS.

The idea of a particular number is complex.

§ 131. The idea of a particular number is necessarily complex; for, the mind naturally asks:

1st. What is the unit or basis of the number? and,

2d. How many times is the unit or basis taken?

What a figure indicates.

§ 132. A figure indicates how many times a unit is taken. Each of the ten figures, however written, or however placed, always expresses as many units as its name imports, and no more; nor does the *figure itself* at all indicate the kind

of unit. Still, every number expressed by one or ^{Number has} more figures, has for its base either the abstract unit one, or a denominate unit.* If a denominate unit, its value or kind is pointed out either by our common language, or as we shall presently see, by the *place* where the figure is written.

The *number* of units which may be expressed by either of the ten figures, is indicated by the name of the figure. If the figure stands alone, and the unit is not denominated, the basis of the number is the abstract unit 1.

§ 133. If we write 0 on the right of 1, we have - - - - - - - - - $\Big\}$ 10, which is read ONE ten. Here 1 still expresses ONE, but it is ONE ten; that is, a unit ten times as great as the unit 1; and this is called a unit of the *second order*.

Again; if we write two 0's on the right of 1, we have - - - - - - $\Big\}$ 100, which is read ONE hundred. Here again, 1 still expresses ONE, but it is ONE hundred; that is, a unit ten times as great as the unit ONE ten, and a hundred times as great as the unit 1.

§ 134. If three 1's are written by the side of each other, thus - - - - $\Big\}$ 111,

* Section 118.

the ideas, expressed in our common anguage, are these:

1st. *That the* 1 *on the right, will either express a single thing denominated, or the abstract unit one.*

2d. *That the* 1 *next to the left expresses* 1 *ten that is, a unit ten times as great as the first.*

3d. *That the* 1 *still further to the left expresses* 1 *hundred; that is, a unit ten times as great as the second, and one hundred times as great as the first; and similarly if there were other places.*

When figures are thus written by the side of each other, the arithmetical language establishes a relation between the units of their places: that is, the unit of each place, as we pass from the right hand towards the left, increases according to the scale of tens. Therefore, by a law of the arithmetical language, *the place of a figure fixes its unit.*

If, then, we write a row of 1's as a scale, thus:

1 hundred billion,	1 ten billion,	1 billion,	1 hundred million,	1 ten million,	1 million,	1 hundred thousand,	1 ten thousand,	1 thousand,	1 hundred,	1 ten,	1 unit,
1	1	1,	1	1	1,	1	1	1,	1	1	1

the *unit of each place* is determined, as well

as the *law* of change in passing from one place to another. If then, it were required to express a given number of units, of any order, we first select from the arithmetical alphabet the character which designates the number, and then write it in the place corresponding to the order. Thus, to express three millions, we write

<div style="text-align:center">3000000 ;</div>

and similarly for all numbers.

How any number of units may be expressed.

§ 135. It should be observed, that a figure being *a character which represents value*, can have no value in and of itself. The *number* of things, which any figure expresses, is determined by its name, as given in the arithmetical alphabet. The *kind* of thing, or unit of the figure, is fixed either by naming it, as in the case of a denominate number, or by the place which the figure occupies, when written by the side of or over other figures.

A figure has no value in itself.

How the unit is determined.

The phrase "local value of a figure," so long in use, is, therefore, without signification when applied to a figure : the term "local value," being applicable to the *unit of the place*, and not to the figure which occupies the place.

Figure, has no local value.

Term applicable to unit of place.

§ 136. United States Currency affords an ex-

Federal Money :

Its denom- ample of a series of denominate units, increasing
inations. according to the scale of tens: thus,

$$\overset{\text{Eagle,}\ \text{Dollar,}\ \text{Dime,}\ \text{Cent,}\ \text{Mill,}}{1\quad 1\quad 1\quad 1\quad 1}$$

How read. may be read 11 thousand 1 hundred and 11
mills ; or, 1111 *cents* and 1 mill ; or, 111 dimes
1 cent and 1 mill ; or, 11 dollars 1 dime 1 cent
and 1 mill ; or, 1 eagle 1 dollar 1 dime 1 cent
Various kinds and 1 mill.   Thus, we may read the number
of Readings. with either of its units as a basis, or we may
name them all : thus, 1 eagle, 1 dollar, 1 dime,
1 cent, 1 mill.   Generally, in Federal Money,
we read in the denominations of dollars, cents,
and mills ; and should say, 11 dollars 11 cents
and 1 mill.

Examples in   § 137. Examples in reading figures :—
Reading.
1st. Example.   If we have the figures - - - -   89
we may read them by their smallest
unit, and say eighty-nine ; or, we may say 8
tens and 9 units.

2d. Example.   Again, the figures - - - - - -   567
may be read by the smallest unit ;
viz. five hundred and sixty-seven ; or we may
say, 56 tens and 7 units ; or, 5 hundreds 6 tens
and 7 units.

3d. Example.   Again, the number expressed by - 74896

may be read, seventy-four thousand eight hundred and ninety-six. Or, it may be read, 7489 tens and 6 units; or, 748 hundreds 9 tens and 6 units; or, 74 thousands 8 hundreds 9 tens and 6 units; or, 7 ten thousands 4 thousands 8 hundreds 9 tens and 6 units; and we may read in a similar way all other numbers. *Various readings of a number.*

Although we should teach all the correct readings of a number, we should not fail to remark that it is generally most convenient in practice to read by the lowest unit of a number. Thus, in the numeration table, we read each period by the lowest unit of that period. For example, in the number *The best method of reading.* *Each period read by its lowest unit.*

$$874,967,847.047,$$ Example.

we read 874 *billions* 967 *millions* 847 *thousands* and 47.

UNITS INCREASING ACCORDING TO VARYING SCALES.

§ 138. If we write the well-known signs of the English money, and place 1 under each denomination, we shall have *Methods of writing figures having different denominate units.*

$$£. \quad s. \quad d. \quad f.$$
$$1 \quad 1 \quad 1 \quad 1$$

Now, the signs £. *s. d.* and *f.* fix the value of the unit 1 in each denomination; and they also *How the value of each unit is fixed.*

What the language expresses.

determine the relations which subsist between the different units. For example, this simple language expresses these ideas :

The units of the places.

1st. That the unit of the right-hand place is 1 farthing—of the place next to the left, 1 penny —of the next place, 1 shilling—of the next place, 1 pound ; and

How the units increase.

2d. That 4 units of the lowest denomination make one unit of the next higher; 12 of the second, one of the third; and 20 of the third, one of the fourth.

The units in Avoirdupois weight.

If we take the denominate numbers of the Avoirdupois weight, we have

$$\text{Ton. cwt. qr. lb. oz. dr.}$$
$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \;;$$

Changes in the value of the units.

in which the units increase in the following manner : viz. the second unit, counting from the right, is sixteen times as great as the first ; the third, sixteen times as great as the second ; the fourth, twenty-five times as great as the third ; the fifth, four times as great as the fourth ; and the sixth, twenty times as great as the fifth.

How the scale varies.

The scale, therefore, for this class of denominate numbers varies according to the above laws.

A different scale for each system.

If we take any other class of denominate numbers, as the Troy weight, or any of the systems of measures, we shall have different scales for the formation of the different units.

But in all the formations, we shall recognise the application of the same general principles. *The method of forming the scales the same for all numbers.*

There are, therefore, two general methods of forming the different systems of integral numbers from the unit one. The first consists in preserving a constant law of relation between the different unities ; viz. that their values shall change according to the scale of tens. This gives the system of common numbers. *Two systems of forming integer numbers. First system.*

The second method consists in the application of known, though varying laws of change in the unities. These changes in the unities produce the entire system of denominate numbers, each class of which has its appropriate scale, and the changes among the units of the same class are indicated by the different steps of its scale. *Second system. Change in the scales of forming the unities.*

INTEGRAL UNITS OF ARITHMETIC.

§ 139. There are eight classes of units—four of number, and four of space, viz. *Eight classes of units.*

1. Abstract Units;	5. Units of Lines;	Abstract,
2. Units of Currency;	6. Units of Surface;	Currency,
3. Units of Weight;	7. Units of Volumes;	Weight,
4. Units of Time ;	8. Units of Angles.	Time.

First among the Units of Arithmetic stands the simple or abstract unit 1. This is the base of all abstract numbers, and becomes the base, *Abstract unit one, the base.*

The basis of denominate numbers; also, of all denominate numbers, by merely naming, in succession, the particular things to which it is applied.

Also, the basis of all fractions, whether simple or denominate. It is also the basis of all fractions. Merely as the unit 1, it is a whole which may be divided according to any law, forming every variety of fraction ; and if we apply it to a particular thing, the fraction becomes denominate, and we have expressions for all conceivable parts of that thing.

§ 140. It has been remarked* that we can form no distinct apprehension of a number, until we have a clear notion of its unit, and the number of times the unit is taken. *The unit is the great basis.* The utmost care, therefore, should be taken to impress on the minds of learners, a clear and distinct idea of the actual value of the unit of every number with which they have to do. If it be a number expressing currency, one or more of the coins should be exhibited, and the value dwelt upon; after which distinct notions of the other units of currency can be acquired by comparison.

Must apprehend the unit.

Let its nature and kind be fully explained ;

How for a number expressing currency.

Exhibit the unit if it be of weight; If the number be one of weight, some unit should be exhibited, as one pound, or one ounce, and an idea of its weight acquired by actually

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* Section 110.

lifting it. This is the only way in which we can learn the true signification of the terms.

If the number be one of measure, either And also, if it be one of measure. linear, superficial, of volumes or of angles, its unit should also be exhibited, and *the significa-tion of the term expressing it, learned in the only way in which it can be learned, through the senses, and by the aid of a sensible object.*

UNITED STATES CURRENCY.

§ 141. The currency of the United States is Currency of the United States. called United States Currency. Its units are all denominate, being 1 mill, 1 cent, 1 dime, 1 dollar, 1 eagle. The law of change, in passing from one Law of change in the unities. unit to another, is according to the scale of tens. Hence, this system of numbers may be treated, How these numbers may be treated. in all respects, as simple numbers; and indeed they are such, with the single exception that their units have different names.

They are generally read in the units of dollars, How generally read. cents, and mills—a period being placed after the figure denoting dollars. Thus,

<p style="text-align:center">$ 864.849</p> Example.

is read eight hundred and sixty-four dollars, eighty-four cents, and nine mills; and if there were a figure after the 9, it would be read in Of figures after mills. decimals of the mill. The number may, how-

The number read in various ways. ever, be read in any other unit; as, 864849 mills; or, 86484 cents and 9 mills: or, 8648 dimes, 4 cents, and 9 mills; or, 86 eagles, 4 dollars, 84 cents, and 9 mills; and there are yet several other readings.

ENGLISH MONEY.

Sterling Money. § 142. The units of English, or Sterling Money, are 1 farthing, 1 penny, 1 shilling, and 1 pound.

Scale of the unities. The scale of this class of numbers is a varying scale. Its steps, in passing from the unit of the lowest denomination to the highest, are four, How it changes. twelve, and twenty. For, four farthings make one penny, twelve pence one shilling, and twenty shillings one pound.

AVOIRDUPOIS WEIGHT.

Units in Avoirdupois. § 143. The units of the Avoirdupois Weight are 1 dram, 1 ounce, 1 pound, 1 quarter, 1 hundred-weight, and 1 ton.

Scale. The scale of this class of numbers is a varying scale. Its steps, in passing from the unit of the lowest denomination to the highest, are sixteen, sixteen, twenty-five, four, and twenty. Variation in its degrees. For, sixteen drams make one ounce, sixteen ounces one pound, twenty-five pounds one quar-

ter, four quarters one hundred, and twenty hundreds one ton.

TROY WEIGHT.

§ 144. The units of the Troy Weight are, 1 grain, 1 pennyweight, 1 ounce, and 1 pound.   Units in Troy Weight.

The scale is a varying scale, and its steps, in passing from the unit of the lowest denomination to the highest, are twenty-four, twenty, and twelve.   Scale: Its degrees.

APOTHECARIES' WEIGHT.

§ 145. The units of this weight are, 1 grain, 1 scruple, 1 dram, 1 ounce, and 1 pound.   Units in Apothecaries' Weight.

The scale is a varying scale. Its steps, in passing from the unit of the lowest denomination to the highest, are twenty, three, eight, and twelve.   Scale: Its degrees.

UNITS OF MEASURE OF SPACE.

§ 146. There are four units of measure of Space, each differing in *kind* from the other three. They are, Units of Length, Units of Surface, Units of Volume, and Units of Angular Measure.   Four units of measure.

UNITS OF LENGTH.

§ 147. The unit of length is used for measuring lines, either straight or curved. It is a   Units of length.

The stand-
ard.

straight line of a given length, and is often called the standard of the measurement.

What units
are taken.

Idea of
length.

The units of length, generally used as standards, are 1 inch, 1 foot, 1 yard, 1 rod, 1 furlong, and 1 mile. The number of times which the unit, used as a standard, is taken, considered in connection with its value, gives the idea of the length of the line measured.

UNITS OF SURFACE.

Units of
surface.

§ 148. Units of surface are used for the measurement of the area or contents of whatever has the two dimensions of length and breadth. The

What the
unit of
surface is.

unit of surface is a square described on the unit of length as a side. Thus, if the unit of length be 1 foot, the corre-

1 square foot.

Examples.

sponding unit of surface will be 1 square foot; that is, a square constructed on 1 foot of length as a side.

Its connection
with the unit
of length.

If the linear unit be 1 yard, the corresponding unit of surface will be 1 square yard. It will be seen from the figure, that, although the linear yard

1 yard.

Square feet
in a
square yard.

contains the linear foot but three times, the square yard

contains the square foot nine times. The square Square rod and Square mile.
rod or square mile may also be used as the unit
of surface.

The number of times which a surface contains Area or contents of a surface.
its unit of measure, is its area or contents ; and
this number, taken in connection with the value
of the unit, gives the idea of its extent.

Besides the units already considered, there is **a**
special class, called

DUODECIMAL UNITS.

§ 149. The duodecimal units are generally used Duodecimal units.
in board and timber measure, though they may be
used in all measurements of surface and volume.
They are simply the units 1 foot, 1 square foot, What they are.
and 1 cubic foot, divided according to the scale
of 12.

§ 150. It is proved in Geometry, that if the What principle is proved in Geometry.
number of linear units in the base of a rectan-
gle be multiplied by the number of linear units
in the breadth, the numerical value of the pro-
duct will be equal to the number of superficial
units in the figure.

Knowing this fact, we often express it by say- How it is expressed.
ing, that "feet multiplied by feet give square
feet," and "yards multiplied by yards give square

10

This a concise expression. yards." But as feet cannot be taken *feet times*, nor yards, *yard times*, this language, rightly understood, is but a concise form of expression for the principle stated above.

Conclusion. With this understanding of the language, we say, that 1 foot in length multiplied by 1 foot in breadth, gives a square foot; and 4 feet in length multiplied by 3 feet in breadth, gives 12 square feet.

Examples in the multiplication of feet by feet and inches. § 151. If now, 1 foot in length be multiplied by 1 inch $=\frac{1}{12}$ of a foot in breadth, the product will be one-twelfth of a square foot; that is, *one-twelfth of the second unit :* if it
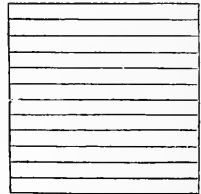
Generalization. be multiplied by 3 inches, the product will be three-twelfths of a square foot; and similarly for a multiplier of any number of inches.

Inches by inches. If, now, we multiply 1 inch by 1 inch, the product may be represented by 1 square inch: that is, *by one-twelfth of one-twelfth of a square foot.* Hence, *the units of this measure decrease according to the scale of* 12. The units are,

How the units change, and what they are.

First. 1st. Square feet—arising from multiplying feet by feet.

Second. 2d. Twelfths of square feet—arising from multiplying feet by inches.

3d. Twelfths of twelfths—arising from multi-   Third.
plying inches by inches.

When we introduce the third dimension, height,
we have, 1 foot being the unit, $1 \times 1 \times 1 = 1$
cubic foot; $1 \times 1 \times \frac{1}{12} = \frac{1}{12}$ cubic foot; $1 \times \frac{1}{12} \times$ Conclusion
general.
$\frac{1}{12} = \frac{1}{144}$ cubic foot; and $\frac{1}{12} \times \frac{1}{12} \times \frac{1}{12} = \frac{1}{1728}$
cubic foot. Hence, the units change by the scale
of 12.

UNITS OF VOLUME.

§ 152. It has already been stated, that if   Units of
volume.
length be multiplied by breadth, the product
may be represented by units of surface. It is   What is
proved in
also proved, in Geometry, that if the length,   Geometry in
breadth, and height of any regular figure, of a   regard to
them.
square form, be multiplied together, the pro-
duct may be represented by units of volume   Units of
volume.
whose number is equal to this product. Each
unit is a cube constructed on the linear unit as
an edge. Thus, if the linear unit be 1 foot, the   Examples.
unit of volume will be 1 cubic foot; that is,
a cube constructed on 1 foot as an edge; and
if it be 1 yard, the unit will be 1 cubic yard.

The three units, viz. the unit of length, the   The three
units essen-
unit of surface, and the unit of volume, are es-   tially differ-
sentially different in kind. The first is a line   ent.
of a known length; the second, a square of a   What they
known side; and the third, a figure, called a   are.

Generally used. cube, of a known base and height.   These are the units used in all kinds of measurement—

Duodecimal system. excepting only angles, and the duodecimal system, which has already been explained.

LIQUID MEASURE.

Units of Liquid Measure. § 153. The units of Liquid Measure are, 1 gill, 1 pint, 1 quart, 1 gallon, 1 barrel, 1 hogshead, 1 pipe, 1 tun.   The scale is a varying

Scale. scale.   Its steps, in passing from the unit of

How it varies. the lowest denomination, are, four, two, four, thirty-one and a half, sixty-three, two, and two.

DRY MEASURE.

Units of Dry Measure. § 154. The units of this measure are, 1 pint, 1 quart, 1 peck, 1 bushel, and 1 chaldron.   The

Degrees of the scale. steps of the scale, in passing from units of the lowest denomination, are two, eight, four, and thirty-six.

TIME.

Units of Time. § 155. The units of Time are, 1 second, 1 minute, 1 hour, 1 day, 1 week, 1 month, 1 year,

Degrees of the scale. and 1 century.   The steps of the scale, in passing from units of the lowest denomination to the highest, are sixty, sixty, twenty-four, seven, four, twelve, and one hundred.

ANGULAR, OR CIRCULAR MEASURE.

§ 156. The units of this measure are, 1 second, 1 minute, 1 degree, 1 sign, 1 circle. The steps of the scale, in passing from units of the lowest denomination to those of the higher, are sixty, sixty, thirty, and twelve.

Units of Circular Measure.

Degrees of the Scale.

ADVANTAGES OF THE SYSTEM OF UNITIES.

§ 157. It may well be asked, if the method here adopted, of presenting the elementary principles of arithmetic, has any advantages over those now in general use. It is supposed to possess the following:

Advantages of the system

1st. The system of unities teaches an exact analysis of all numbers, and unfolds to the mind the different ways in which they are formed from the unit one, as a basis.

1st. Teaches the analysis of numbers:

2d. Such an analysis enables the mind to form a definite and distinct idea of every number, by pointing out the relation between it and the unit from which it was derived.

2d. Points out their relation:

3d. By presenting constantly to the mind the idea of the unit one, as the basis of all numbers, the mind is insensibly led to compare this unit with all the numbers which flow from it, and

3d. Constantly presents the idea of unity.

then it can the more easily compare those num-
bers with each other.

4th. Ex-
plains more
fully the
four ground
rules.

4th. It affords a more satisfactory analysis,
and a better understanding of the four ground
rules, and indeed of all the operations of arith-
metic, than any other method of presenting the
subject.

METRIC, OR FRENCH SYSTEM OF WEIGHTS
AND MEASURES.

Primary
base of
system.

§ 158. The primary base, in this system, for all
denominations of weights and measures, is the
one-ten-millionth part of the distance from the
equator to the pole, measured on the earth's
surface. It is called a METRE, and is equal to
39.37 inches, very nearly.

Scale.

The change from the base, in all the denom-
inations, is according to the decimal scale of
tens: that is, the units increase ten times, at
each step, in the ascending scale, and decrease
ten times, at each step, in the descending scale.

MEASURES OF LENGTH.

Base, 1 metre = 39.37 inches, nearly.

TABLE.

	Ascending Scale.				Descending Scale.		
Myriametre.	Kilometre.	Hectometre.	Decametre.	METRE. UNIT.	Decimetre.	Centimetre.	Millimetre.
1	1	1	1	1	1	1	1

The names, in the ascending scale, are formed by prefixing to the base, Metre, the words, Deca (ten), Hecto (one hundred), Kilo (one thousand), Myria (ten thousand), from the Greek numerals; and in the descending scale, by prefixing Deci (tenth), Centi (hundredth), Mili (thousandth), from the Latin numerals.

SQUARE MEASURE.

Base, 1 Are = the square whose side is 10 metres.
     = 119.6 square yards, nearly.
     = 4 perches, or square rods, nearly.

The unit of surface is a square whose side is 10 metres. It is called an ARE, and is equal to 100 square metres.

MEASURE OF VOLUMES.

Base, 1 Litre = the cube on the decimetre.
     = 61.023378 cubic inches.
     = a little more than a wine quart.

The unit for the measure of volume is the
cube whose edge is one-tenth of the metre—
that is, a cube whose edge is 3.937 inches. This
cube is called a LITRE, and is one-thousandth
part of the cube constructed on the metre, as
an edge.

FOUR GROUND RULES.

System
applied in
addition.

§ 159. Let us take the two following examples
in Addition, the one in simple and the other in
denominate numbers, and then analyze the pro-
cess of finding the sum in each.

	SIMPLE NUMBERS.		DENOMINATE NUMBERS.				
			cwt.	qr.	lb.	oz.	dr.
	874198						
	36984		3	3	24	15	14
Examples.	3641		6	3	23	14	8
	914823		10	3	23	14	6

Process of
performing
addition.

In both examples we begin by adding the units
of the lowest denomination, and then, we *divide
their sum by so many as make one of the denomi-
nation next higher.* We then set down the
remainder, and add the quotient to the units
of that denomination. Having done this, we
apply a similar process to all the other denomina-
tions—*the principle being precisely the same in*
But one
principle.   *both examples.* We see, in these examples, an

illustration of a general principle of addition, Units of the
viz. *that units of the same kind are always added* same kind
unite.
together.

§ 160. Let us take two similar examples in System
Subtraction. applied in
subtraction.

SIMPLE NUMBERS.	DENOMINATE NUMBERS.				
8403	£	s.	d.	far.	
3298	6	9	7	2	Examples.
$\overline{5105}$	3	10	8	4	
	$\overline{2}$	$\overline{18}$	$\overline{10}$	$\overline{2}$	

In both examples we begin with the units of The method
the lowest denomination, and as the number in of perform-
ing the ex-
the subtrahend is greater than in the place di- amples.
rectly above, we suppose so many to be added
in the minuend as make one unit of the next
higher denomination. We then make the sub-
traction, and add 1 to the units of the subtrahend
next higher, and proceed in a similar manner,
through all the denominations. It is plain that
the principle employed is the same in both exam- Principle the
ples. Also, that units of any denomination in same for all
examples.
the subtrahend are taken from those of the same
denomination in the minuend.

§ 161. Let us now take similar examples in Multiplica-
Multiplication. tion.

	SIMPLE NUMBERS.
Examples.	87464
	5
	437320

DENOMINATE NUMBERS.				
℔	℥		℈	gr.
9	7	6	2	15
				5
48	3	2	1	15

Method of performing the examples. In these examples we see, that we multiply, in succession, each order of units in the multiplicand by the multiplier, and that we carry from one product to another, one for every so many as make one unit of the next higher denomination. **The principle the same for all examples.** The *principle* of the process is therefore the same in both examples.

Division. § 162. Finally, let us take two similar examples in Division.

	SIMPLE NUMBERS.
Examples.	3)874911
	291637

DENOMINATE NUMBERS.			
£	s.	d.	far.
3)8	4	2	1
2	14	8	3

Principles involved: We begin, in both examples, by dividing the units of the highest denomination. The unit of the quotient figure is the same as that of the dividend. We write this figure in its place, and then reduce the remainder to units of the next **The same as in the other rules.** lower denomination. We then add in that denomination, and continue the division through all the denominations to the last—the principle being precisely the same in both examples.

SECTION II.

FRACTIONAL UNITS.

FRACTIONAL UNITS.—SCALE OF TENS.

§ 163. If the unit 1 be divided into ten equal parts, each part is called *one tenth.* If one of these tenths be divided into ten equal parts, each part is called *one hundredth.* If one of the hundredths be divided into ten equal parts, each part is called *one thousandth;* and corresponding names are given to similar parts, how far soever the divisions may be carried.

Fraction one tenth defined;

One hundredth;

One thousandth.

Generalization.

Now, although the tenths which arise from dividing the unit 1, are but equal parts of 1, they are, nevertheless, WHOLE tenths, and in this light may be regarded as *units.*

Fractions are whole things.

To avoid confusion, in the use of terms, we shall call every equal part of 1 a *fractional unit.* Hence, tenths, hundredths, thousandths, tenths of thousandths, &c., *are fractional units, each having a fixed relation to the unit* 1, from which it was derived.

Fractional units.

Their nature

§ 164. Adopting a similar language to that used in integral numbers, we call the tenths, fractional units of the *first order;* the hundredths, fractional units of the *second order;* the thousandths, fractional units of the *third order;* and so on for the subsequent divisions.

Is there any arithmetical language by which these fractional units may be expressed? The decimal point, which is merely a dot, or period, indicates the division of the unit 1, according to the scale of tens. By the arithmetical language, the unit of the place next the point, on the right, is 1 tenth; that of the second place, 1 hundredth; that of the third, 1 thousandth; that of the fourth, 1 ten thousandth; and so on for places still to the right.

The scale for decimals, therefore, is

$$.111111111, \&c.;$$

in which the value of the unit of each place is known as soon as we have learned the signification of the language.

If, therefore, we wish to express any of the parts into which the unit 1 may be divided, according to the scale of tens, we have simply to select from the alphabet, the figure that will express the *number* of parts, and then write it in

Fractional units of the first order; second order, &c.

Language for fractional units.

What it fixes.

Names of the places.

Scale.

Any decimal number may be expressed by this scale.

the place corresponding to the *order of the unit.*
Thus, to express four tenths, three thousandths,
eight ten-thousandths, and six millionths, we
write

<div style="text-align:center">.403806 ;</div>

and similarly, for any decimal which can be
named.

§ 165. It should be observed that while the
units of place *decrease,* according to the scale of
tens, from left to right, they *increase* according
to the same scale, from right to left. *This is the*
same law of increase as that which connects the
units of place in simple numbers. Hence, simple
numbers and decimals being formed according to
the same law, may be written by the side of each
other and treated as a single number, by merely
preserving the separating or decimal point.
Thus, 8974 and .67046 may be written

<div style="text-align:center">8974.67046 ;</div>

since ten units, in the place of tenths, make the
unit one in the place next to the left.

FRACTIONAL UNITS IN GENERAL.

§ 166. If the unit 1 be divided into two equal
parts, each part is called a half. If it be divided

A third.   into three equal parts, each part is called a third :
if it be divided into four equal parts, each part is
A fourth.   called a fourth : if into five equal parts, each
A fifth.   part is called a fifth ; and if into any number of
equal parts, a name is given corresponding to the
Generally.
number of parts.

These units   Now, although these halves, thirds, fourths,
are whole   fifths, &c., are each but parts of the unit 1, they
things.   are, nevertheless, in *themselves*, whole things.
Examples.   That is, a half is a whole half; a third, a whole
third ; a fourth, a whole fourth ; and the same
for any other equal part of 1.   In this sense,
therefore, they are *units*, and we call them frac-
Have a rela-   tional units.   Each is an exact part of the unit
tion to unity.   1, and has a fixed relation to it.

§ 167. Is there any arithmetical language by
which these fractional units can be expressed ?

Language for   The bar, written at the right, is the
fractions.   sign which denotes the division of the $\overline{}$.
unit 1 into any number of equal parts.

To express   If we wish to express the number of equal
the number   parts into which it is divided, as 9, for
of equal   example, we simply write the 9 under $\overline{9}$
parts.   the bar, and then the phrase means, that some
thing regarded as a whole, has been divided into
9 equal parts.

If, now, we wish to express any number of these fractional units, as 7, for example, we place the 7 above the line, and read, seven ninths.

$$\frac{7}{9}$$

To show how many are taken.

§ 168. It was observed,* that two things are necessary to the clear apprehension of an integral number.

Two things necessary to apprehend a number.

1st. A distinct apprehension of the *unit* which forms the basis of the number; and,

First.

2dly. A distinct apprehension of the number of times which that unit is taken.

Second

Three things are necessary to the distinct apprehension of the value of any fraction, either decimal or vulgar.

Three things necessary to apprehend a fraction.

1st. We must know the unit, or whole thing, from which the fraction was derived;

First.

2d. We must know into how many equal parts that unit is divided; and,

Second.

3dly. We must know how many such parts are taken in the expression.

Third.

The unit from which the fraction is derived, is called the *unit of the fraction;* and one of the equal parts is called, the *fractional unit.*

Unit of the fraction—of the expression.

For example, to apprehend the value of the

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* Section 117.

What we must know.

fraction $\frac{3}{7}$ of a pound avoirdupois, or $\frac{3}{7}lb.$; we must know,

First.

1st. What is meant by a pound;

Second.

2d. That it has been divided into seven equal parts; and,

Third.

3d. That three of those parts are taken.

In the above fraction, 1 pound is the unit of the fraction; one-seventh of a pound, the fractional unit; and 3 denotes that three fractional units are taken.

Unit when not named.

If the unit of a fraction be not named, it is taken to be the abstract unit 1.

ADVANTAGES OF FRACTIONAL UNITS.

Every equal part of one, a unit.

§ 169. By considering every equal part of unity as a unit in itself, having a certain relation to the unit 1, the mind is led to analyze a fraction, and thus to apprehend its precise signification.

Advantages of the analysis.

Under this searching analysis, the mind at once seizes on the *unit of the fraction* as the principal base. It then looks at the value of each part. It then inquires how many such parts are taken.

Equal units, whether integral or frac-

It having been shown that equal integral units can alone be added, it is readily seen that the

same principle is equally applicable to frac- tional, can alone be added. tional units; and then the inquiry is made: What is necessary in order to make such units equal?

It is seen at once, that two things are neces- Two things necessary for addition. sary:

1st. That they be parts of the *same unit*; and, First.

2d. That they be *like parts*; in other words, Second. they must be of the same denomination, and have a common denominator.

In regard to Decimal Fractions, all that is Decimal Fractions. necessary, is to observe that units of the same value are added to each other, and when the figures expressing them are written down, they should always be placed in the same column.

§ 170. The great difficulty in the management Difficulty in the manage- ment of frac tions. of fractions, consists in comparing them with each other, instead of constantly comparing them with the unit from which they are derived. By considering them as entire things, having a fixed relation to the unit which is their base, How obviated. they can be compared as readily as integral num- bers; for, the mind is never at a loss when it apprehends the unit, the parts into which it is Reasons for greater sim- plicity in integers. divided, and the number of parts which are taken. The only reasons why we apprehend and

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handle integral numbers more readily than fractions, are,

First.   1st. Because the unit forming the base is always kept in view; and,

Second.   2d. Because, in integral numbers, we have been taught to trace constantly the connection between the unit and the numbers which come from it; while in the methods of treating fractions, these important considerations have been neglected.

SECTION III.

PROPORTION AND RATIO.

Proportion defined.   § 171. PROPORTION expresses the relation which one number bears to another, with respect to its being greater or less.

Two ways of comparing.   Two numbers may be compared, the one with the other, in two ways:

1st method.   1st. With respect to their difference, called Arithmetical Proportion; and,

2d method.   2d. With respect to their quotient, called Geometrical Proportion.

Thus, if we compare the numbers 1 and 8, by their difference, we find that the second exceeds the first by 7 : hence, their difference 7, *is the measure of their arithmetical proportion,* and is called, in the old books, their *arithmetical ratio.*

Example of Arithmetical Proportion.

Arithmetical Ratio.

If we compare the same numbers by their quotient, we find that the second contains the first 8 times : hence, 8 is the *measure of their geometrical proportion,* and is called their *geometrical ratio.**

Example of Geometrical Proportion.

Ratio.

§ 172. The two numbers which are thus compared, are called *terms.* The first is called the *antecedent,* and the second the *consequent.*

Terms.

Antecedent.

Consequent.

In comparing numbers with respect to their difference, the question is, *how much* is one greater than the other ? Their difference affords the true answer, and is the measure of their proportion.

Comparison by difference

In comparing numbers with respect to their quotient, the question is, *how many times* is one greater or less than the other? Their quotient or ratio, is the true answer, and is the measure

Comparison by quotient.

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* The term ratio, as now generally used, means the quotient arising from dividing one number by another. We shall use it only in this sense.

Example by
difference.
of their proportion.   Ten, for example, is 9
greater than 1, if we compare the numbers one
and ten by their difference.   But if we compare
them by their quotient, ten is said to be ten
times as great—the language " ten times" having
reference to the quotient, which is always taken
as the measure of the relative value of two
numbers so compared.   Thus, when we say,
that, the units of our common system of numbers
increase in a tenfold ratio, we mean that they so
increase that each succeeding unit shall contain
the preceding one ten times.   This is a conven-
ient language to express a particular relation of
two numbers, and is perfectly correct, when
used in conformity to an accurate definition.

By quotient.

" Ten times."

Examples of
this use of the
term.

Convenient
language.

In what
all authors
agree :

In what disa-
gree.

Different me-
thods.
Standard the
divisor.

§ 173. All authors agree, that the measure of
the geometrical proportion, between two num-
bers, is their ratio ; but they are by no means
unanimous, nor does each always agree with
himself, in the manner of determining this ratio.
Some determine it, by dividing the first term by
the second ; others, by dividing the second term
by the first.*   All agree, that the *standard*, what-
ever it may be, should be made the divisor.

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* The Encyclopedia Metropolitana, a work distinguished
by the excellence of its scientific articles, adopts the latter
method.

This leads us to inquire, whether the mind What is the
best form. fixes most readily on the first or second number as a standard ; that is, whether its tendency is to regard the second number as arising from the first, or the first as arising from the second.

§ 174. All our ideas of numbers begin at Origin of
numbers. one.* This is the starting-point. We conceive of a number only by measuring it with How we con
ceive of a
number. one, as a standard. One is primarily in the mind before we acquire an idea of any other number. Hence, then, the comparison begins Where the
comparison
begins. at one, which is the standard or unit, and all other numbers are measured by it. When, therefore, we inquire what is the relation of one to any other number, as eight, the idea presented The idea
presented. is, how many times does eight contain the standard ?

We measure by this standard, and the ratio is Standard.
Ratio. the result of the measurement. In this view of the case, the standard should be the first number What they
should be. named, and the ratio, the quotient of the second number divided by the first. Thus, the ratio of 2 to 6 would be expressed by 3, three being the Example. number of times which 6 contains 2.

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* Section 111.

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Other reasons for this method of comparison.

§ 175. The reason for adopting this method of comparison will appear still stronger, if we take fractional numbers. Thus, if we seek the relation between one and one-half, the mind immediately looks to the *part* which one-half is of Comparison of unity with fractions. one, and this is determined by dividing one-half by 1; that is, by dividing the second by the first: whereas, if we adopt the other method, we divide our standard, and find a quotient 2.

Geometrical proportion.

§ 176. It may be proper here to observe, that while the term "geometrical proportion" is used to express the relation of two numbers, compared by their ratio, the term, "A geometrical proportion," is applied to four numbers, in which the ratio of the first to the second is the same as that of the third to the fourth. Thus,

A geometrical proportion defined.

Example. $2 : 4 :: 6 : 12,$

is a geometrical proportion, of which the ratio is 2.

Further advantages.

§ 177. We will now state some further advantages which result from regarding the ratio as the quotient of the second term divided by the first.

Questions in the Rule of Three.

Every question in the Rule of Three is a geometrical proportion, excepting only, that the

last term is wanting. When that term is found, Their nature.
the geometrical proportion becomes complete.
In all such proportions, the first term is used as
the divisor. Further, for every question in the
Rule of Three, we have this clear and simple
solution : viz. that, the unknown term or an- How solved.
swer, is equal to the third term multiplied by
the ratio of the first two. This simple rule, for
finding the fourth term, cannot be given, unless This rule de-
pends on the
we define ratio to be the quotient of the second definition of
Ratio.
term divided by the first. Convenience, there-
fore, as well as general analogy, indicates this as
the proper definition of the term ratio.

§ 178. Again, all authors, so far as I have This defini-
tion of ratio is
used by all
authors, in
one case :
consulted them, are uniform in their definition
of the ratio of a geometrical progression : viz.
that it is the quotient which arises from divid-
ing the second term by the first, or any other
term by the preceding one. For example, in
the progression

$$2 : 4 : 8 : 16 : 32 : 64, \&c.,$$ Example :

all concur that the ratio is 2 ; that is, that it is in which
they all
agree.
the quotient which arises from dividing the sec-
ond term by the first : or any other term by the
preceding term. But a geometrical progression
differs from a geometrical proportion only in

The same should take place in every proportion; for they are all the same.

this : in the former, the ratio of any two terms is the same ; while in the latter, the ratio of the first and second is different from that of the second and third. There is, therefore, no essential difference in the two proportions.

Why, then, should we say that in the proportion

$$2 : 4 :: 6 : 12,$$

Examples.

the ratio is the quotient of the first term divided by the second ; while in the progression

$$2 : 4 : 8 : 16 : 32 : 64, \&c.,$$

the ratio is defined to be the quotient of the second term divided by the first, or of any term divided by the preceding term ?

Wherein authors have departed from their definitions :

As far as I have examined, all the authors who have defined the ratio of two numbers to be the quotient of the first divided by the second, have departed from that definition in the case of a geometrical progression. They have

How used ratio.

there used the word ratio, to express the quotient of the second term divided by the first, and this without any explanation of a change in the definition.

Other instances in which the definition of

Most of them have also departed from their definition, in informing us that "numbers increase from right to left in a tenfold ratio," in

which the term ratio is used to denote the quo- Ratio is not adhered to.
tient of the second number divided by the first.
The definition of ratio is thus departed from,
and the idea of it becomes confused. Such Consequences.
discrepancies cannot but introduce confusion
into the minds of learners. The same term
should always be used in the same sense, and
have but a single signification. Science does What science demands.
not permit the slightest departure from this rule.
I have, therefore, adopted but a single significa-
tion of ratio, and have chosen that one to which The definition adopted.
all authors, so far as I know, have given their
sanction ; although some, it is true, have also
used it in a different sense.

§ 179. One important remark on the subject Important Remark.
of proportion is yet to be made. It is this :

Any two numbers which are compared togeth- Numbers compared must be of the same kind.
er, either by their difference or quotient, must
be of the same kind: that is, they must either
have the same unit, as a base, or be susceptible
of reduction to the same unit.

For example, we can compare 2 pounds with Examples relating to Arithmetical and Geometrical Proportion.
6 pounds : their difference is 4 pounds, and their
ratio is the abstract number 3. We can also
compare 2 feet with 8 yards : for, although the
unit 1 foot is different from the unit 1 yard, still
8 yards are equal to 24 feet. Hence, the differ-

ence of the numbers is 22 feet, and their ratio the abstract number 12.

Numbers with different units cannot be compared. On the other hand, we cannot compare 2 dollars with 2 yards of cloth, for they are quantities of different kinds, not being susceptible of reduction to a common unit.

Abstract numbers may be compared. Abstract numbers may always be compared, since they have a common unit 1.

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SECTION IV.

APPLICATIONS OF THE SCIENCE OF ARITHMETIC.

Arithmetic: In what a science.

Science defined. § 180. ARITHMETIC is both a science and an art. It is a science in all that relates to the properties, laws, and proportions of numbers. The science is a collection of those connected processes which develop and make known the laws that regulate and govern all the operations performed on numbers.

What the science performs. § 181. Arithmetic is an art, in this : the science lays open the properties and laws of numbers, and furnishes certain principles from which

practical and useful rules are formed, applicable in the mechanic arts and in business transactions. The art of Arithmetic consists in the judicious and skilful application of the principles of the science ; and the rules contain the directions for such application.

In what the art consists.

§ 182. In explaining the science of Arithmetic, great care should be taken that the analysis of every question, and the reasoning by which the principles are proved, be made according to the strictest rules of mathematical logic.

In explaining the science: what necessary.

Every principle should be laid down and explained, not only with reference to its subsequent use and application in arithmetic, but also, *with reference to its connection with the entire mathematical science*—of which, arithmetic is the elementary branch.

How each principle should be stated

§ 183. That analysis of questions, therefore, where cost is compared with quantity, or quantity with cost, and which leads the mind of the learner to suppose that a ratio exists between quantities that have not a common unit, is, without explanation, certainly faulty as a process of science.

What questions are faulty.

For example : if two yards of cloth cost 4 dollars, what will 6 yards cost at the same rate ?

Example.

Analysis.—Two yards of cloth will cost twice as much as 1 yard : therefore, if two yards of cloth cost 4 dollars, 1 yard will cost 2 dollars. Again : if 1 yard of cloth cost 2 dollars, 6 yards, being six times as much, will cost six times two dollars, or 12 dollars.

Now, this analysis is perfectly satisfactory to a child. He perceives a certain relation between 2 yards and 4 dollars, and between 6 yards and 12 dollars : indeed, in his mind, he *compares* these numbers together, and is perfectly satisfied with the result of the comparison.

Advancing in his mathematical course, however, he soon comes to the subject of proportions, treated as a science. He there finds, greatly to his surprise, that he cannot compare together numbers which have different units ; and that his *antecedent* and *consequent* must be of the same kind. He thus learns that the whole system of analysis, based on the above method of comparison, is not in accordance with the principles of science.

What, then, is the true analysis ? It is this : 6 yards of cloth being 3 times as great as 2 yards, will cost three times as much : but 2 yards cost 4 dollars ; hence, 6 yards will cost 3 times 4, or 12 dollars. If this last analysis be not as simple as the first, it is certainly more strictly

Side notes:

Analysis:

Satisfactory to a child.

Reason why it is defective.

True analysis:

More scientific.

scientific ; and when once learned, can be ap-
plied through the whole range of mathematical
science.

Its advantages.

§ 184. There is yet another view of this ques-
tion which removes, to a great degree, if not
entirely, the objections to the first analysis. It is
this :

Reasons in favor of the first analysis.

The proportion between 1 yard of cloth and
its cost, two dollars, cannot, it is true, as the
units are now expressed, be measured by a ratio,
according to the mathematical definition of a
ratio. Still, however, between 1 and 2, *regard-
ed as abstract numbers,* there is the same relation
existing as between the numbers 6 and 12, also
regarded as abstract. Now, by leaving out of
view, for a moment, the units of the numbers,
and finding 12 as an abstract number, and then
assigning to it its proper unit, we have a correct
analysis, as well as a correct result.

Numbers must be re-garded as ab-stract:

The analysis then correct.

§ 185. It should be borne in mind, that practi-
cal arithmetic, or arithmetic as an art, selects
from all the principles of the science, the mate-
rials for the construction of its rules and the
proofs of its methods. As a mere branch of
practical knowledge, it cares nothing about the
forms or methods of investigation—it demands

How the rules of arith-metic are formed.

What practical knowledge demands.

the fruits of them all, in the most concentrated
and practical form.   Hence, the best rule of art,
which is the one most easily applied, and which
reaches the result by the shortest process, is not
always constructed after those methods which
science employs in the development of its prin-
ciples.

Best rule of
art.

For example, the definition of multiplication is,
that it is the process of taking one number, called
the multiplicand, as many times as there are
units in another called the multiplier.   This defi-
nition, as one of science, requires two things.

Definition of
multiplica-
tion.

What it de-
mands.

1st. That the multiplier be an abstract num-
ber; and,

First.

2dly. That the product be a quantity of the
same kind as the multiplicand.

Second.

These two principles are certainly correct,
and relating to arithmetic as *a science*, are uni-
versally true.   But are they universally true, in
the sense in which they would be understood by
learners, when applied to arithmetic as a mixed
subject, that is, a science and an art ?   Such an
application would certainly exclude a large class
of practical rules, which are used in the appli-
cations of arithmetic, without reference to par-
ticular units.

May be
differently
considered as
furnishing a
rule of art.

For example, if we have feet in length to be
multiplied by feet in height, we must exclude the

Examples of
such
applications.

question as one to which arithmetic is not applicable ; or else we must multiply, as indeed we do, without reference to the unit, and then assign a proper unit to the product.

If we have a product arising from the three When the three factors are lines. factors of length, breadth, and thickness, the unit of the first product and the unit of the final product, will not only be different from each other, but both of them will be different from the unit of the given numbers.   The unit of the The different units. given numbers will be a unit of length, the unit of the first product will be a square, and that of the final product, a cube.

§ 186. Again, if we wish to find, by the best Other examples. practical rule, the cost of 467 feet of boards at 30 cents per foot, we should multiply 467 by 30, and declare the cost to be 14010 cents, or $140.10.

Now, as a question of science, if you ask, can Considered as a question of science. we multiply feet by cents ? we answer, certainly not.   If you again ask, is the result obtained right ? we answer, yes.   If you ask for the analysys, we give you the following :

1 foot of boards : 467 feet : : 30 cents : Answer.

Now, the ratio of 1 foot to 467 feet, is the ab Ratio. stract number 467 ; and 30 cents being multi-

plied by this number, gives for the product **14010**

cents. But as the product of two numbers is numerically the same, whichever number be used as the multiplier, we know that **467** multiplied by 30, gives the same number of units as 30 multi-

plied by **467** : hence, the first rule for finding the amount is correct.

§ 187. I have given these illustrations to point out the difference between a process of scientific

investigation and a practical rule.

The first should always present the ideas of

the subject in their natural order and connection, while the other should point out the best way of obtaining a desired result. In the latter, the steps of the process may not conform to the order necessary for the investigation of principles ; but the *correctness of the result* must be susceptible of rigorous proof. Much needless and un-

profitable discussion has arisen on many of the processes of arithmetic, from confounding a principle of science with a rule of mere application.

SECTION V.

METHODS OF TEACHING ARITHMETIC CONSIDERED.

ORDER OF THE SUBJECTS.

§ 188. It has been well remarked by Cousin, the great French philosopher, that " As is the method of a philosopher, so will be his system ; and the adoption of a method decides the destiny of a philosophy." *Cousin.* *Method decides Philosophy.*

What is said here of philosophy in general, is eminently true of the philosophy of mathematical science ; and there is no branch of it to which the remark applies, with greater force, than to that of arithmetic. It is here, that the first notions of mathematical science are acquired. It is here, that the mind wakes up, as it were, to the consciousness of its reasoning powers   Here, it acquires the first knowledge of the abstract— separates, for the first time, the pure ideal from the actual, and begins to reflect and reason on pure mental conceptions. It is, therefore, of the highest importance that these first thoughts be impressed on the mind in their natural and proper *True in science.* *Why important in Arithmetic.* *First thoughts should be rightly impressed.*

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Faculties to be cultivated.
order, so as to strengthen and cultivate, at the same time, the faculties of apprehension, discrimination, and comparison, and also improve the yet higher faculty of logical deduction.

First point:
§ 189. The first point, then, in framing a course of arithmetical instruction, is to determine the method of presenting the subject. Is

method of presenting the subject.
there any thing in the *nature* of the subject itself, or the connection of its parts, that points out the order in which these parts should be

Laws of science: what do they require?
studied? Do the laws of science demand a particular order; or are the parts so loosely connected, as to render it a matter of indifference where we begin and where we end? A review of the analysis of the subject will aid us in this inquiry.

Basis of the science of numbers.
§ 190. We have seen* that the science of numbers is based on the unit 1. Indeed, the

In what the science consists.
whole science consists in developing, explaining, and illustrating the laws by which, and through which, we operate on this unit. There

Four classes of operations.
are four classes of operations performed on the unit one.

1st. To increase the unit.
1st. To increase it according to the scale

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* Section 104.

of tens forming the system of common numbers.

2d. To divide it, in any way we please, form-
ing the decimal and vulgar fractions.

2d. To divide.

3d. To increase it according to the vary-
ing scales, forming all the denominate numbers.

3d. To increase.

4th. To compare it with all the numbers which
come from it; and then those numbers with each
other.   This embraces proportions, of which the
Rule of Three is the principal branch.

4th. To compare it.

There is yet a fourth branch of arithmetic;
viz. the application of the principles and of the
rules drawn from them, in the mechanic arts
and in the ordinary transactions of business.
This is called the Art, or practical part, of
Arithmetic.   (See Arithmetical Diagram facing
page 119.)

Fourth branch.

Practical applications;

these the art.

INTEGRAL UNITS.

§ 191. We begin first with the unit 1, and
increase it according to the scale of tens, form-
ing the common system of integral numbers.   We
then perform on these numbers the operations
of the five ground rules; viz. numerate them,
add them, subtract them, multiply and divide
them.

Unit one increased according to the scale of tens.

Operations performed.

FRACTIONAL UNITS.

§ 192. We next pass to the second class of
Divisions of operations on the unit 1; viz. the divisions of it.
the unit.
General Here we pursue the most general method, and
method. first divide it arbitrarily; that is, into any num-
ber of equal parts. We then observe that the
Method ac- division of it, according to the scale of tens, is
cording to
scale of tens. but a particular case of the general law of divi-
sion. We then perform on all the fractional
units which thus arise, every operation of the
five ground rules.

DENOMINATE UNITS.

§ 193. Having operated on the abstract unit 1,
by the processes of augmentation and division,
Next in- we next increase it according to the varying
crease it ac-
cording to scales of the denominate numbers, and thus pro-
varying
scales. duce the system, called Denominate or Concrete
Numbers; after which, we perform on this class of
numbers all the operations of the five ground rules.

Reasons for By placing the subject of fractions directly
placing frac-
tions. after the five ground rules, the two opposite oper-
ations of aggregation and division are brought
into direct contrast with each other. It is thus
seen, that the laws of change, in the two systems
of operation on the unit 1, are the same with
very slight modifications.

This system of classification, has, after expe-
rience, been found to be the best for instruction.

R A T I O , — O R   R U L E   O F   T H R E E .

§ 194. Having considered the two subjects of   Subjects
considered.
integral and fractional units, we come next to
the comparison of numbers with each other.

This branch of arithmetic develops all the   What this
branch de-
relative properties of numbers, resulting from   velops.
their inequality.

The method of arrangement, indicated above.   What the ar
rangement
presents all the operations of arithmetic in con-   does.
nection with the unit 1, which certainly forms
the basis of the arithmetical science.

Besides, this arrangement draws a broad line   What it does
further.
between the science of arithmetic and its ap-
plications ; a distinction which it is very im-
portant to make.   The separation of the prin-   Theory and
practice
ciples of a science from their applications, so   should be
separated.
that the learner shall clearly perceive what is
theory and what practice, is of the highest im-
portance.   Teaching things separately, teaching   Golden rules
for teaching.
them well, and pointing out their connections,
are the golden rules of all successful instruc-
tion.

§ 195. I had supposed, that the place of the

Rule of Three, among the branches of arith-
metic, had been fixed long since. But several
Differences in authors of late, have placed most of the practi-
arrangement; cal subjects *before* this rule—giving precedence,
for example, to the subjects of Percentage, In-
In what they terest, Discount, Insurance, &c. It is not easy
consist. to discover the motive of this change. It is
Ratio part of certain that the proportion and ratio of num-
the science. bers are parts of the *science* of arithmetic ; and
Should pre- the properties of numbers which they unfold,
cede applica- are indispensably necessary to a clear apprehen-
tions. sion of the principles from which the practical
rules are constructed.

We may, it is true, explain each example in
Percentage, Interest, Discount, Insurance, &c.,
Cannot well by a separate analysis. But this is a matter
change the of much labor ; and besides, does not conduct
order. the mind to any general principle, on which
all the operations depend. Whereas, if the Rule
of Three be explained, before entering on the
Advantages practical subjects, it is a great aid and a pow-
of first ex- erful auxiliary in explaining and establishing
plaining the all the practical rules. If the Rule of Three
Rule of
Three. is to be learned at all, should it not rather
precede than follow its applications ? It is a
great point, in instruction, to lay down a gen-
The great eral principle, as early as possible, and then con-
principle of
instruction. nect with it all subordinate operations.

ARITHMETICAL LANGUAGE.

§ 196. We have seen that the arithmetical al- *Arithmetical alphabet.* phabet contains ten characters.* From these elements the entire language is formed ; and we now propose to show in how simple a manner.

The names of the ten characters are the first *Names of the characters.* ten words of the language. If the unit 1 be added to each of the numbers from 0 to 9 in- *First ten combinations.* clusive, we find the first ten combinations in arithmetic.† If 2 be added, in like manner, we have the second ten combinations ; adding *Second ten, and so on for others.* 3, gives us the third ten combinations ; and so on, until we have reached one hundred combinations (page 123).

Now, as we progressed, each set of combina- *Each set giving one additional word.* tions introduced one additional word, and the results of all the combinations are expressed by the words from two to twenty inclusive.

§ 197. These one hundred elementary com- *All that need be committed to memory.* binations, are all that need be committed to memory ; for, every other is deduced from them. They are, in fact, but different spellings of the first nineteen words which follow one. If we extend the words to one hundred, and recollect that

* Section 114.   † Section 116.

at one hundred, we begin to repeat the numbers,
we see that we have but one hundred words to
be remembered for addition; and of these, *all
above ten are derivative.* To this number,
must of course be added the few words which
express the sums of the hundreds, thousands, &c.

§ 198. In Subtraction, we also find one hun-
dred elementary combinations; the results of
which are to be read.* These results, and all
the numbers employed in obtaining them, are
expressed by twenty words.

§ 199. In Multiplication (the table being car-
ried to twelve), we have one hundred and forty-
four elementary combinations,† and fifty-nine
separate words (already known) to express the
results of these combinations.

§ 200. In Division, also, we have one hundred
and forty-four elementary combinations,‡ but
use only twelve words to express their results.

§ 201. Thus, we have four hundred and eigh-
ty-eight elementary combinations. The results
of these combinations are expressed by one hun-
dred words; viz. nineteen in addition, ten in sub-
traction, fifty-nine in multiplication, and twelve

Words to be
remembered
for addition.

Only ten
words primi-
tive.

Subtraction:

Number of
words.

Multiplica-
tion:

Number of
words.

Division:

Number of
words.

Four hun-
dred and
ighty-eight
elementary
combina-
tions.

Words used:
19 in addi-
tion,
10 in subtrac
tion,
59 in multi-
plication,

* Section 127.   † Section 129.   ‡ Section 130.

in division. Of the nineteen words which are 12 in division
employed to express the results of the combina-
tions in addition, eight are again used to express
similar results in subtraction. Of the fifty-nine
which express the results of the combinations
in multiplication, sixteen had been used to ex-
press similar results in addition, and one in
subtraction ; and the entire twelve, which ex-
press the results of the combinations in division,
had been used to express results of previous
combinations. Hence, the results of all the ele-
mentary combinations, in the four ground rules,
are expressed by sixty-three different words ; and Sixty-three
they are the only words employed to translate different
words in all.
these results from the arithmetical into our com-
mon language.

The language for fractional units *is similar* Language
in every particular. By means of a language the same for
fractions.
thus formed we deduce every principle in the
science of numbers.

§ 202. Expressing these ideas and their com-
binations by figures, gives rise to the language Language of
of arithmetic. By the aid of this language we arithmetic:
not only unfold the principles of the science, Its value and
use.
but are enabled to apply these principles to
every question of a practical nature, involving
the use of figures.

But few combinations which change the signification of the figures.
§ 203. There is but one further idea to be presented : it is this,—that there are very few combinations made among the figures, which change, at all, their signification.

Examples.
Selecting any two of the figures, as 3 and 5, for example, we see at once that there are but three ways of writing them, that will at all change their signification.

First:
First, write them by the side of each other - - - - - - - - - - $\left.\right\}$ 3 5, 5 3.

Second:
Second, write them, the one over the other - - - - - - - - - $\left.\right\}$ $\frac{3}{5}$, $\frac{5}{3}$.

Third.
Third, place a decimal point before each - - - - - - - - - - - $\left.\right\}$ .3, .5.

Learn the language by use.
Now, each manner of writing gives a different signification to both the figures. Use, however, has established that signification, and we know it, as soon as we have learned the language.

Its grammar:
Alphabet—words, and their uses.
We have thus explained what we mean by the arithmetical language. Its grammar embraces the names of its elementary signs, or Alphabet,—the formation and number of its words,—and the laws by which figures are connected for the purpose of expressing ideas. We feel that there is simplicity and beauty in this system, and hope it may be useful.

NECESSITY OF EXACT DEFINITIONS AND TERMS.

§ 204. The principles of every science are a collection of mental processes, having established connections with each other. In every branch of mathematics, the Definitions and Terms give form to, and are the signs of, certain elementary ideas, which are the basis of the science. Between any term and the idea which it is employed to express, the connection should be so intimate, that the one will always suggest the other.

Principles of science.

Definitions and terms:

These definitions and terms, when their significations are once fixed, must always be used in the same sense. The necessity of this is most urgent. For, *"in the whole range of arithmetical science there is no logical test of truth, but in a conformity of the reasoning to the definitions and terms, or to such principles as have been established from them."*

When once fixed must always be used in the same sense.

Reason.

§ 205. With these principles, as guides, we propose to examine some of the definitions and terms which have, heretofore, formed the basis of the arithmetical science. We shall not confine our quotations to a single author, and shall make only those which fairly exhibit the general use of the terms.

Definitions and terms examined.

It is said,

Number defined. "*Number* signifies a *unit*, or a *collection of units.*"

How expressed. "The common method of expressing numbers is by the *Arabic Notation*. The Arabic method employs the following *ten characters*, or *figures*," &c.

Names of the characters. "The first nine are called *significant* figures, because each one always has a value, or denotes some number."

And a little further on we have,

Figures have values. "The different values which figures have, are called *simple* and *local* values."

The definition of Number is clear and cor

Number rightly defined: rect. It is a general term, comprehending al. the phrases which are used, to express, either separately or in connection, one or more things

Also figures. of the same kind. So, likewise, the definition of figures, that they are *characters*, is also right.

Definition departed from. But mark how soon these definitions are departed from. The reason given why nine of the figures are called *significant* is, because "each one always has a value, or denotes some number." This brings us directly to the question,

Has a figure value? whether a figure has a *value ;* or, whether it is *a mere representative of value.* Is it a *number* or a *character* to represent number ? Is it a

It is merely a character: *quantity* or *symbol ?* It is defined to be *a char-*

acter which *stands for*, or expresses a number. Has it any other signification? How then can we say that it has a *value*—and how is it possi- ble that it can have a *simple* and a *local value?* The *things* which the figures stand for, may change their value, but not the *figures them- selves*. Indeed, it is very difficult for John to perceive how the figure 2, standing in the sec- ond place, is ten times as great as the same fig- ure 2 standing in the first place on the right! although he will readily understand, when the arithmetical language is explained to him, that the UNIT of one of these places is ten times as great as that of the other.

§ 206. Let us now examine the leading defi- nition or principle which forms the basis of the arithmetical language. It is in these words:

"*Numbers increase from right to left in a* *tenfold ratio; that is, each removal of a figure one place towards the left, increases its value ten times.*"

Now, it must be remembered, that number has been defined as signifying "a unit, or a collection of units." How, then, can it have a *right hand*, or a *left?* and how can it *increase from right to left in a tenfold ratio?*" The explanation given is, that "*each removal of a*

Explanation. *figure one place towards the left, increases its value ten times."*

Number, signifying a collection of units, must Increase of necessarily increase according to the law by numbers has no connection which these units are combined; and that law with figures. of increase, whatever it may be, has not the slightest connection with the *figures* which are used to express the numbers.

Ratio. Besides, is the term *ratio* (yet undefined), one which expresses an elementary idea? And "Tenfold is the term, a *" tenfold ratio,"* one of sufficient ratio:" simplicity for the basis of a system?

Does, then, this definition, which in substance is used by most authors, involve and carry to Four leading the mind of the young learner, the four leading notions of ideas which form the basis of the arithmetical numbers. notation? viz. :

First. 1st. That numbers are expressions for one or more things of the same kind.

Second. 2d. That numbers are expressed by certain characters called figures; and of which there are ten.

Third. 3d. That each figure always expresses as many units as its name imports, and no more.

Fourth. 4th. That the *kind* of thing which a figure expresses depends on the place which the figure occupies, or on the value of the units, indicated in some other way.

PLACE is merely *one* of the forms of language   Place;
by which we designate the unit of a number,   Its office.
expressed by a figure.   The definition attributes
this property of place both to number and fig-
ures, while it belongs to neither.

§ 207. Having considered the definitions and
terms in the first division of Arithmetic, viz. in
notation and numeration, we will now pass to   Definitions in
the second, viz. Addition.   Addition:

The following are the definitions of Addition,
taken from three standard works before me :

" The putting together of two or more num-   First.
bers (as in the foregoing examples), so as to
make one *whole number*, is called *Addition*, and
the whole number is called the *sum*, or *amount*."

" ADDITION is the collecting of numbers to-   Second.
gether to find their sum."

" *The process of uniting two or more num-*   Third.
bers together, so as to form one single number,
is called ADDITION."

" The *answer*, or the number thus found, is
called the *sum*, or *amount*."

Now, is there in either of these definitions   Defects.
any test, or means of determining when the
pupil gets the thing he seeks for, viz. " the sum
of two or more numbers?"   No previous defi-   Reason.
nition has been given, in either work, of the

term SUM.   How is the learner to know what
he is seeking for, unless that thing be defined?

No prin-   Suppose that John be required to find the sum
ciple as a
standard.   of the numbers 3 and 5, and pronounces it to
be 10.   How will you correct him, by showing
that he has not conformed to the definitions and
rules?   You certainly cannot, because *you have
established no test of a correct process.*

But, if you have previously defined SUM to be
a number which *contains as many units* as there
are in all the numbers added: or, if you say,

Correct defi-   "Addition is the process of uniting two or
nition;
more numbers, *in such a way*, that all the units
which they contain may be expressed by a sin-
gle number, called the sum, or sum total;" you
will then have a *test* for the correctness of the
Gives a test. process of Addition; viz. Does the number,
which you call the sum, contain as many units
as there are in all the numbers added?   The
answer to this question will show that John is
wrong.

Definitions of   § 208. I will now quote the definitions of
fractions
Fractions from the same authors, and in the
same order of reference.

First.   "We have seen, that numbers expressing *whole*
things, are called *integers*, or *whole* numbers;
but that, in division, it is often necessary to

divide or *break* a whole thing into *parts*, and that these parts are called *fractions*, or *broken* numbers."

"Fractions are parts of an integer."   Second.

"*When a number or thing is divided into* Third.
equal parts, these parts are called FRACTIONS."

Now, will either of these definitions convey to the mind of a learner, a distinct and exact idea of a fraction?

The term "fraction," as used in Arithmetic, Term fraction
means one or more equal parts of something   defined.
regarded as a whole : *the parts to be expressed in terms of the thing divided* CONSIDERED AS A
UNIT.   There are three prominent ideas which   Ideas
the mind must embrace :   expressed

1st. That the thing divided be regarded as a   First.
standard, or unity ;

2d. That it be divided into equal parts;   Second.

3d. That the parts be expressed in terms of   Third.
the thing divided, regarded as a unit.

These ideas are referred to in the latter part   The defini-
of the first definition.   Indeed, the definition   tions exam
would suggest them to any one acquainted with   ined:
the subject, but not, we think, to a learner.

In the second definition, neither of them is   Is a frac-
hinted at.   Take, for example, the integer num-   tion part of
ber 12, and no one would say that any one part   an integer
of this number, as 2, 4, or 6, is a fraction.

13

Third
definition;

The third definition would be perfectly accu-
rate, by inserting after the word "thing," the
words, "regarded as a whole." It very clearly
expresses the idea of equal parts, but does not

In what de-
fective.

present the idea strongly enough, that the thing
divided must be regarded as unity, and that the
parts must be expressed in terms of this unity.

Necessity of
exact terms.

§ 209. I have thus given a few examples, illus-
trating the necessity of accurate definitions and
terms. Nothing further need be added, except
the remark, that terms should always be used in
the same sense, precisely, in which they are de-
fined.

Objection
to exactness
of thought
and language.

To some, perhaps, these distinctions may ap-
pear over-nice, and matters of little moment.
It may be supposed that a general impression,
imparted by a language reasonably accurate,
will suffice very well; and that it is hardly
worth while to pause and weigh words on a
nicely-adjusted balance.

Any such notions, permit me to say, will lead
to fatal errors in education.

Definitions in
mathematics.

It is in mathematical science alone that words
are the signs of exact and clearly-defined ideas.
It is here only that we can see, as it were, the
very thoughts through the transparent words by
which they are expressed. If the words of the

definitions are not such as convey to the mind of the learner, the fundamental ideas of the science, he cannot reason upon these ideas; for, he does not apprehend them; and the great reasoning faculty, by which all the subsequent principles of mathematics are developed, is entirely unexercised.*

Must be exact to reason correctly.

It is not possible to cultivate the habit of accurate thinking, without the aid and use of exact language. No mental habit is more useful than that of tracing out the connection between ideas and language. In Arithmetic, that connection can be made strikingly apparent. Clear, distinct ideas—diamond thoughts—may be strung through the mind on the thread of science, and each have its word or phrase by which it can be transferred to the minds of others.

Cannot otherwise cultivate habits of thought.

Connection between words and thoughts in arithmetic.

HOW SHOULD THE SUBJECTS BE PRESENTED?

§ 210. Having considered the natural connection of the subjects of arithmetic with each other, as branches of a single science, based on a single unit; and having also explained the necessity of a perspicuous and accurate lan-

What has been considered

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* Section 200.

How ought the subjects to be presented.

guage , we come now to that important inquiry, How ought those subjects to be presented to the mind of a learner ?   Before answering this ques-

Two objects in studying arithmetic:

tion, we should reflect, that two important ob· jects should be sought after in the study of arith-metic :

First.

1st. To train the mind to habits of clear, quick, and accurate thought—to teach it to apprehend distinctly—to discriminate closely— to judge truly—and to reason correctly ; and,

Second.

2d. To give, in abundance, that *practicai* knowledge of the use of figures, in their va-rious applications, which shall illustrate the stri-

Art of arith metic.

king fact, that *the art of arithmetic is the most important art of civilized life—being, in fact, th· foundation of nearly all the others.*

How first im pressions are made.

§ 211. It is certainly true, that most, if not al. the elementary notions, whether abstract or practical—that is, whether they relate to the science or to the art of arithmetic, must be made on the mind by means of sensible objects. Because of this fact, many have supposed that

Is reason-ing to be con-ducted by sensible objects?

the *processes* of *reasoning* are all to be con-ducted by the same sensible objects ; and that every abstract principle of science is to be de-veloped and established ly means of sofas, chairs, apples, and horses.   There seems to be

an impression that because blocks are useful aids in teaching the alphabet, that, *therefore* they can be used advantageously in reading Milton and Shakspeare. This error is akin to that of attempting to teach practically, Geography and Surveying in connection with Geometry, by calling the angles of a rectangle, north, south, east, and west, instead of simply designating them by the letters A, B, C, and D.

This false idea, that every principle of science must be *learned practically*, instead of being *rendered practical by its applications*, has been highly detrimental both to science and art.

A mechanic, for example, knowing the height of his roof and the width of his building, wishes to cut his rafters to the proper length. If he calls to his aid the established, though *abstract principles* of science, he finds the length of his rafter, by the well-known relation between the hypothenuse and the two sides of a right-angled triangle. If, however, he will learn nothing except *practically*, he must raise his rafter to the roof, measure it, and if it be too long cut it off, if too short, splice it. This is the practical way of *learning* things.

The truly practical way, is that in which skill is guided by science.

Do the principles above stated find any appli-

cation in considering the question, How should
arithmetic be taught? Certainly they do. If
arithmetic be both a science and an art, it
should be so taught and so learned.

§ 212. The principles of every science are gen-
eral and abstract truths. They are mere ideas,
primarily acquired through the senses by experi-
ence, and generalized by processes of reflection
and reasoning; and when understood, are certain
guides in every case to which they are applicable.
If we choose to do without them, we may. But
is it wise to turn our heads from the guide-boards
and explore every road that opens before us?

Now, in the study of arithmetic those princi-
ples of science, applicable to classes of cases,
should always be taught at the earliest possible
moment. The mind should never be forced
through a long series of examples, without ex-
planation. One or two examples should always
precede the statement of an abstract principle,
or the laying down of a rule, so as to make the
anguage of the principle or rule intelligible.
But to carry the learner forward through a
series of them, before the principle on which
they depend has been examined and stated, is
forcing the mind to advance mechanically—it
is lifting up the rafter to measure it, when its

exact length could be easily determined by a rule of science.

As most of the instruction in arithmetic must be given with the aid of books, we feel unable to do justice to this branch of the subject without submitting a few observations on the nature of text-books and the objects which they are intended to answer.

Books:

Necessity for treating of them.

TEXT-BOOKS.

§ 213. A text-book should be an aid to the teacher in imparting instruction, and to the learner in acquiring knowledge.

Text-book

It should present the subjects of knowledge in their proper order, with the branches of each subject classified, and the parts rightly arranged. No text-book, on a subject of general knowledge, can contain all that is known of the subject on which it treats; and ordinarily, it can contain but a very small part. Hence, the subjects to be presented, and the extent to which they are to be treated, are matters of nice discrimination and judgment, about which there must always be a diversity of opinion.

What it should be.

Selection of subjects necessary.

Difficulties of selection

§ 214. The subjects selected should be leading ones, and those best calculated to unfold, ex-

Subjects:

plain, and illustrate the principles of the science.
How presented. They should be so presented as to lead the mind to analyze, discriminate, and classify; to see each principle separately, each in its combination with others, and all, as forming an harmonious whole. Too much care cannot be bestowed in forming the *suggestive method of arrangement:* that is, to place the ideas and principles in such a connection, that *each step shall prepare the mind of the learner for the next in order.*

Suggestive method:

Reason for.

Object of a text-book: § 215. A text-book should be constructed for the purpose of furnishing the learner with the keys of knowledge. It should point out, explain, and illustrate by examples, the methods of investigating and examining subjects, but should leave the mind of the learner free from the restraints of minute detail. To fill a book with the analysis of simple questions, which any child can solve in his own way, is to constrain and force the mind at the very point where it is capable of self-action. To do that for a pupil, which he can do for himself, is most unwise.

Nature;

Useless detail;

Should not be historical. § 216. A text-book on a subject of science should not be historical. At first, the minds of children are averse to whatever is abstract, be-

cause what is abstract demands thought, and
thinking is mental labor from which untrained
minds turn away.   If the thread of science be
broken by the presentation of facts, having no
connection with the argument, the mind will
leave the more rugged path of the reasoning,
and employ itself with what requires less effort
and labor.

The optician, in his delicate experiments, ex-
cludes all light except the beam which he uses :
so, the skilful teacher excludes all thoughts
excepting those which he is most anxious to
impress.

As a general rule, subject of course to some
exceptions, but one method for each process
should be given.   The minds of learners should
not be confused.   If several methods are given,
it becomes difficult to distinguish the reasonings
applicable to each, and it requires much knowl-
edge of a subject to compare different methods
with each other.

§ 217. It seems to be a settled opinion, both
among authors and teachers, that the subject of
arithmetic can be best presented by means of
three separate works.   For the sake of distinc-
tion, we will designate them the First, Second,
and Third Arithmetics.

We will now explain what we suppose to be the proper construction of each book, and the object for which each should be designed.

First
Arithmetic:

§ 218. This book should give to the mind its first direction in mathematical science, and its first impulse in intellectual development.

Its
importance.

Hence, it is the most important book of the series. Here, the faculties of apprehension, discrimination, abstraction, classification and comparison, are brought first into activity. Now,

How
the subjects
must be
presented.

to cultivate and develop these faculties rightly, we must, at first, present every new idea by means of a sensible object, and then immediately drop the object and pass to the abstract thought.

Order
of the ideas.

We must also present the ideas consecutively; that is, in their proper order; and by the mere *method of presentation* awaken the comparative and reasoning faculties. Hence, every lesson should contain a given number of ideas. The

Construction
of the lessons.

ideas of each lesson, beginning with the first, should advance in regular gradation, and the lessons themselves should be regular steps in the progress and development of the arithmetical science.

§ 219. The first lesson should merely contain representations of sensible objects, placed opposite names of numbers, to give the impression of the meanings of these names : thus,

One - - - - - - - - -   ✵

Two - - - - - - - - - ✵ ✵

Three - - - - - - - - ✵ ✵ ✵

&c.   &c.

And with young pupils, more striking objects should be substituted for the stars.

In the second lesson, the words should be replaced by the figures : thus,

1 - - - - - - - - - - -   ✵

2 - -   - - - - - - ✵ ✵

3 - - - - - - - - - - ✵ ✵ ✵

&c.   &c.

In the third lesson, I would combine the ideas of the first two, by placing the words and figures opposite each other : thus,

One - - - - 1 | Four - - - - 4

Two - - - - 2 | Five - - - - 5

Three - - - 3 | Six - - - - 6

&c.   &c. | &c.   &c.

The Roman method of representing numbers should next be taught, making the fourth lesson ؛ viz.,

One - - - - I.	Four - - - IV.
Two - - - - II.	Five - - - V.
Three - - III.	Six - - - VI.
&c. &c.	&c. &c.

§ 220. We come now to the first ten combinations of numbers, which should be given in a separate lesson. In teaching them, we must, of course, have the aid of sensible objects. We teach them thus:

One	and	one	are how many ?
*		*	
One	and	two	are how many ?
*		* *	
One	and	three	are how many ?
*		* * *	
&c.		&c.	&c.,

through all the combinations: after which, we

pass to the abstract combinations, and ask, one and one are how many? one and two, how many? one and three, &c.; after which we express the results in figures.

We would then teach in the same manner, in

a separate lesson, the second ten combinations; then the third, fourth, fifth, sixth, seventh, eighth, ninth, and tenth. In the teaching of these com-

binations, only the words from one to twenty will have been used. We must then teach the

combinations of which the results are expressed by the words from twenty to one hundred.

§ 221. Having done this, in the way indi- cated, the learner sees at a glance, the basis on which the system of common numbers is con- structed. He distinguishes readily, the unit one from the unit ten, apprehends clearly how the second is derived from the first, and by com- paring them together, comprehends their mutual relation.

Having sufficiently impressed on the mind of the learner, the important fact, that numbers are but expressions for one or more things of the same kind, the unit mark may be omitted in the combinations which follow.

§ 222. With the single difference of the omis- sion of the unit mark, the very same method should be used in teaching the one hundred combinations in subtraction, the one hundred and forty-four in multiplication, and the one hundred and forty-four in division.

When the elementary combinations of the four ground rules are thus taught, the learner looks back through a series of regular progression, in which every lesson forms an advancing step, and where all the ideas of each lesson have a

mutual and intimate connection with each other.

Will not such a system of teaching train the mind to the habit of regarding each idea sepa-

rately—of tracing the connection between each new idea and those previously acquired—and of comparing thoughts with each other?—and are not these among the great ends to be attained, by instruction ?

§ 223. It has seemed to me of great import-

ance to use figures in the very first exercises of arithmetic. Unless this be done, the operations must all be conducted by means of sounds, and

the pupil is thus taught to regard sounds as the proper symbols of the arithmetical language.

This habit of mind, once firmly fixed, cannot be easily eradicated ; and when the figures are learned afterwards, they will not be regarded as the representatives of as many things as their names respectively import, but as the rep-resentatives merely of familiar sounds which have been before learned.

This would seem to account for the fact, about which, I believe, there is no difference of

opinion ; that a course of oral arithmetic, ex-tending over the whole subject, without the aid and use of figures, is but a poor preparation for operations on the slate. It may, it is true,

sharpen and strengthen the mind, and give it What
it may do. development : but does it give it that language and those habits of thought, which turn it into What it
does not do. the pathways of science ? The language of a science affords the tools by which the mind Language
of arithmetic: pries into its mysteries and digs up its hidden treasures. The language of arithmetic is formed from the ten figures. By the aid of this lan- Its uses. guage we measure the diameter of a spider's web, or the distance to the remotest planet What
it performs. which circles the heavens ; by its aid, we cal- culate the size of a grain of sand and the mag- nitude of the sun himself : should we then aban- don a language so potent, and attempt to teach Its value. arithmetic in one which is unknown in the higher departments of the science ?

§ 224. We next come to the question, how Fractions the subject of fractions should be presented in an elementary work.

The simplest idea of a fraction comes from Simplest
idea. dividing the unit one into two equal parts. To ascertain if this idea is clearly apprehended, put How
impressed. the question, How many halves are there in one ? The next question, and it is an import- Next
question. ant one, is this : How many halves are there in one and one-half ? The next, How many halves in two ? How many in two and a half ? In

three ?   Three and a half ? and so on to twelve.

Results.   You will thus evolve all the halves from the units of the numbers from one to twelve, inclusive.   We stop here, because the multiplication table goes no further.   These combina-

First lesson.   tions should be embraced in the first lesson on fractions.   That lesson, therefore, will teach the

Its extent.   relation between the unit 1 and the halves, and point out how the latter are obtained from the former.

Second lesson.   § 225. The second lesson should be the first, reversed.   The first question is, how many

Grades of questions.   whole things are there in two halves ?   Second, How many whole things in four halves ? How many in eight ? and so on to twenty-four halves, when we reach the extent of the division

Extent of the lesson.   table.   In this lesson you will have taught the pupil to pass back from the fractions to the unit from which they are derived.

Fundamental principles: § 226. You have thus taught the two fundamental principles of all the operations in fractions : viz.

First.   1st. To deduce the fractional units from integral units ; and,

Second.   2dly. To deduce integral units from fractional units

§ 227. The next lesson should explain the law
by which the thirds are derived from the units
from 1 to 12 inclusive; and the following lesson
the manner of changing the thirds into integral
units.

Lessons
explaining
thirds.

The next two lessons should exhibit the same
operations performed on the fourth, the next
two on the fifth, and so on to include the
twelfth.

Fourths
and other
fractions.

§ 228. This method of treating the subject of
fractions has many advantages :

Advantages
of the
method.

1st. It points out, most distinctly, the relations
between the unit 1 and the fractions which are
derived from it.

First.

2d. It points out clearly the methods of pass-
ing from the fractional to the integral units.

Second.

3d. It teaches the pupil to handle and com-
bine the fractional units, as entire things.

Third.

4th. It reviews the pupil, thoroughly, through
the multiplication and division tables.

Fourth.

5th. It awakens and stimulates the faculties
of apprehension, comparison, and classification.

Fifth.

§ 229. Besides the subjects already named,
the First Arithmetic should also contain the
tables of denominate numbers, and collections
of simple examples, to be worked on the slate,

What
else the First
Arithmetic
should con-
tain.

14

Examples,
how taught.
under the direction of the teacher. It is not supposed that the mind of the pupil is sufficiently matured at this stage of his progress to understand and work by rules.

What
should be
taught in
the First
Arithmetic.
§ 230. In the First Arithmetic, therefore, the pupil should be taught,

1st. The language of figures;

Second.
2d. The four hundred and eighty-eight elementary combinations, and the words by which they are expressed;

Third.
3d. The main principles of Fractions;

Fourth.
4th. The tables of Denominate Numbers; and,

Fifth.
5th. To perform, upon the slate, the elementary operations in the four ground rules.

SECOND ARITHMETIC.

Second
Arithmetic.
§ 231. This arithmetic occupies a large space in the school education of the country. Many study it, who study no other. It should, therefore,

What it
should be.
fore, be complete in itself. It should also be eminently practical; but it cannot be made so either by giving it the name, or by multiplying the examples.

Practical
application of
principle.
§ 232. The truly practical cannot be the antecedent, but must be the consequent of science.

Hence, that general arrangement of subjects demanded by science, and already explained, must be rigorously followed.

Arrangement of subjects

But in the treatment of the subjects themselves, we are obliged, on account of the limited information of the learners, to adopt methods of teaching less general than we could desire.

Reasons for departures.

§ 233. We must here, again, begin with the unit one, and explain the general formation of the arithmetical language, and must also adhere rigidly to the method of introducing new principles or rules by means of sensible objects. This is most easily and successfully done either by an example or question, so constructed as to show the application of the principle or rule. Such questions or examples being used merely for the purpose of illustration, one or two will answer the purpose much better than twenty : for, if a large number be employed, they are regarded as examples for practice, and are lost sight of as illustrations. Besides, it confuses the mind to drag it through a long series of examples, before explaining the principles by which they are solved. One example, wrought under a principle or rule clearly apprehended, conveys to the mind more practical information, than a dozen wrought out as independent

Basis.

Method.

How carried out.

Few examples.

Reasons.

One example under a rule.

Principle.
Practice.

exercises. Let the principle precede the practice, in all cases, as soon as the information acquired will permit. This is the golden rule both of art and morals.

Subjects embraced.

§ 234. The Second Arithmetic should embrace all the subjects necessary to a full view of the science of numbers; and should contain an abundance of examples to illustrate their Reading: practical applications. The reading of numbers, so much (though not too much) dwelt upon, is an invaluable aid in all practical operations.

Its value in Addition:

By its aid, in addition, the eye runs up the columns and collects, in a moment, the sum of Subtraction: all the numbers. In subtraction, it glances at the figures, and the result is immediately suggested. Multiplication: In multiplication, also, the sight of the figures brings to mind the result, and it is reached and expressed by one word instead of Division. five. In short division, likewise, there is a corresponding saving of time by reading the results of the operations instead of spelling them. The method of reading should, therefore, be constantly practised, and none other allowed.

THIRD ARITHMETIC.

§ 235. We have now reached the place where arithmetic may be taught as a science. The pupil, before entering on the subject as treated here, should be able to perform, at least mechanically, the operations of the five ground rules.

Third Arithmetic

Preparation for.

Arithmetic is now to be looked at from an entirely different point of view. The great principles of generalization are now to be explained and applied.

View of it.

Primarily, the general language of figures must be taught, and the striking fact must then be explained, that the construction of all integer numbers involves but a single principle, viz. *the law of change in passing from one unit to another.* The basis of all subsequent operations will thus have been laid.

What is taught primarily.

General law

§ 236. Taking advantage of this general law which controls the formation of numbers, we bring all the operations of reduction under one single principle, viz. this law of change in the unities.

Controls formation of numbers

Passing to addition, we are equally surprised and delighted to find the same principle controlling all its operations, and that integer numbers of all kinds, whether simple or denominate, may be added under a single rule.

Its value in Addition.

Advantages of knowing a general law.   This view opens to the mind of the pupil a wide field of thought. It is the first illustration of the great advantage which arises from looking into the laws by which numbers are Subtraction. constructed. In subtraction, also, the same principle finds a similar application, and a simple rule containing but a few words is found applicable to all the classes of integral numbers.

In multiplication and division, the same striking results flow from the same cause; and General law of numbers: thus this simple principle, viz. *the law of change in passing from one unit of value to another, is the key to all the operations in the four ground rules*, whether performed on simple or denominate numbers. Thus, all the elementary opera- Controls every operation. tions of arithmetic are linked to a single principle, and that one a *mere principle of arithmetical language*. Who can calculate the labor, intellectual and mechanical, which may be saved by a right application of this luminous principle?

Design of the higher arithmetic: § 237. It should be the design of a higher arithmetic to expand the mind of the learner over the whole science of numbers; to illustrate the most important applications, and to make manifest the connection between the science and the art.

It will not answer these objects if the methods of treating the subject are the same as in the elementary works, where science has to compromise with a want of intelligence. An elementary is not made a higher arithmetic, by merely transferring its definitions, its principles, and its rules into a larger book, in the same order and connection, and arranging under them an apparently new set of examples, though in fact constructed on precisely the same principles.

Its requisites.

Must have a distinctive character.

§ 238. In the four ground rules, particularly (where, in the elementary works, simple examples must necessarily be given, because here they are used both for illustration and practice), the examples should take a wide range, and be so selected and combined as to show their common dependence on the same principle.

Construction of examples in the four ground rules.

§ 239. It being the leading design of a series of arithmetics to explain and illustrate the science and art of numbers, great care should be taken to treat all the subjects, as far as their different natures will permit, according to the same general methods. In passing from one book to another, every subject which has been fully and satisfactorily treated in the one, should be transferred to the other with the fewest pos-

Design of a series.

Subjects to be transferred when fully treated

...

How common subjects may be studied.

sible alterations ; so that a pupil shall not have to learn under a new dress that which he has already fully acquired. They who have studied the elementary work should, in the higher one, either omit the common subjects or pass them over rapidly in review.

Reasons.

The more enlarged and comprehensive views which should be given in the higher work will thus be acquired with the least possible labor, and the connection of the series clearly pointed out.

Additional reason stated.

This use of those subjects, which have been fully treated in the elementary work, is greatly preferable to the method of attempting to teach every thing anew : for there must necessarily be much that is common ; and that which teaches no new principle, or indi ates no new method of application, should be precisely the same in the higher work as in that which precedes it.

A contrary method leads to confusion:

§ 240. To vary the examples, in form, without changing in the least the principles on which they are worked, and to arrange a thousand such collections under the same set of rules and subject to the same laws of solution, may give a little more mechanical facility in the use of figures, but will add nothing to the stores of arithmetical knowledge. Besides, it deludes the learner with the hope of advancement, and when

he reaches the end of his higher arithmetic, he It misleads
the pupil: finds, to his amazement, that he has been conducted by the same guides over the same ground through a winding and devious way, made It com-
plicates the
subject. strange by fantastic drapery : whereas, if what was new had been classed by itself, and what was known clothed in its familiar dress, the subject would have been presented in an entirely different and brighter light.

<div align="center">CONCLUDING REMARKS.</div>

We have thus completed a full analysis of the Conclusion language of figures, and of the construction of numbers.

We have traced from the unit one, all the What
has been
done. numbers of arithmetic, whether integer or fractional, whether simple or denominate. We have developed the laws by which they are derived Laws. from this common source, and perceived the connections of each class with all the others.

We have examined that concise and beautiful Analysis
of the lan-
guage. language, by means of which numbers are made available in rendering the results of science practically useful ; and we have also considered Methods
of teaching
indicated. the best methods of teaching this great subject —the foundation of all mathematical science— Import-
ance of the
subject. and the first among the useful arts.

GEOMETRY.

LINES.

STRAIGHT AND CURVED.
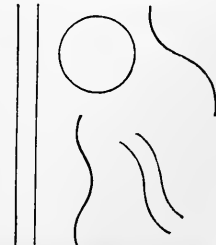
SURFACES.

PLANE.

CURVED.

VOLUMES.

BOUNDED BY PLANES.

BY CURVED SURFACES.

ANGLES.

BOUNDED BY LINES.

BOUNDED BY PLANES.
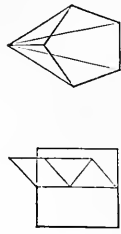
CHAPTER III.

GEOMETRY DEFINED—THINGS OF WHICH IT TREATS—COMPARISON AND PROP-
ERTIES OF FIGURES — DEMONSTRATION — PROPORTION — SUGGESTIONS FOR
TEACHING.

GEOMETRY.

§ 241. GEOMETRY treats of space, and com- Geometry.
pares portions of space with each other, for the
purpose of pointing out their properties and mu-
tual relations.   The science consists in the de- Its science.
velopment of all the laws relating to space, and
is made up of the processes and rules, by means
of which portions of space can be best compared
with each other.   The truths of Geometry are a Its truths.
series of dependent propositions, and may be di- Of three
                                                  kinds.
vided into three classes :

1st. Truths implied in the definitions, viz. that 1st. Those
things do exist, or may exist, corresponding to implied in
                                                  the defini-
the words defined.   For example : when we say, tions.
" A quadrilateral is a rectilinear figure having four
sides," we imply the existence of such a figure.

2d. Self-evident, or intuitive truths, embodied 2d. Axioms.
in the axioms ; and,

3d. Truths inferred from the definitions and 3d. Demon-

strative truths.

When demonstrated.

axioms, called Demonstrative Truths. We say that a truth or proposition is proved or demonstrated, when, by a course of reasoning, it is shown to be included under some other truth or proposition, previously known, and from which is said to *follow ;* hence,

Demonstration.

A DEMONSTRATION is a series of logical arguments, brought to a conclusion, in which the major premises are definitions, axioms, or propositions already established.

Subjects of Geometry.

§ 242. Before we can understand the proofs or demonstrations of Geometry, we must understand what that is to which demonstration is applicable : hence, the first thing necessary is to form a clear conception of space, the subject of all geometrical reasoning.*

Names of forms.

The next step is to give names to particular forms or limited portions of space, and to define these names accurately. The definitions of these names are the definitions of Geometry, and the portions of space corresponding to them are

Figures.

called Figures, or Geometrical Magnitudes ; of

Four kinds.

which there are four general classes :

First.   1st. Lines ;

Second.   2d. Surfaces ;

Third.   3d. Volumes ;

Fourth.   4th. Angles.

* Sections 80 to 84.

§ 243. Lines embrace only one dimension of space, viz. length, without breadth or thickness. The extremities, or limits of a line, are called points. Lines.

There are two general classes of lines—straight lines and curved lines. A straight line is one which lies in the same direction between any two of its points ; and a curved line is one which constantly changes its direction at every point. There is but one kind of straight line, and that is fully characterized by the definition. From the definition we may infer the following axiom : " A straight line is the shortest distance between two points." There are many kinds of curves, of which the circumference of the circle is the simplest and the most easily described. Two classes: Straight and Curved.

One kind of straight line

many of curves.

§ 244. Surfaces embrace two dimensions of space, viz. length and breadth, but not thickness. Surfaces, like lines, are also divided into two general classes, viz. plane surfaces and curved surfaces. Surfaces:

Plane and Curved.

A plane surface is that with which a straight line, any how placed, and having two points common with the surface, will coincide throughout its entire extent. Such a surface is perfectly even, and is commonly designated by the term " A plane." A large class of the figures A plane surface :

Perfectly even.

Plane Figures.   of Geometry are but portions of a plane, and all such are called plane figures.

§ 245. A portion of a plane, bounded by three straight lines, is called a triangle, and this is the simplest of the plane figures. There are several kinds of triangles, differing from each other, however, only in the relative values of their sides and angles. For example: when the sides are all equal to each other, the triangle is called equilateral; when two of the sides are equal, it is called isosceles; and scalene, when the three sides are all unequal. If one of the angles is a right angle, the triangle is called a right-angled triangle.

A triangle, the most simple figure.

Kinds of triangles.

§ 246. The next simplest class of plane figures comprises all those which are bounded by four straight lines, and are called quadrilaterals. There are several varieties of this class:

Quadrilaterals.

1st species.   1st. The mere quadrilateral, which has no special mark, called a trapezium.

2d species.   2d. The trapezoid, which has two sides parallel and two not parallel;

3d species.   3d. The parallelogram, which has its opposite sides parallel and its angles oblique;

4th species.   4th. The rectangle, which has all its angles right angles and its opposite sides parallel; and,

5th. The square, which has its four sides equal
to each other, each to each, and its four angles
right angles.

§ 247. Plane figures, bounded by straight lines,
having a number of sides greater than four, take
names corresponding to the number of sides, viz.
Pentagons, Hexagons, Heptagons, &c.

§ 248. A portion of a plane bounded by a
curved line, all the points of which are equally
distant from a certain point within called the
centre, is called a circle, and the bounding line
is called the circumference. This is the only
curve usually treated of in Elementary Geometry.

§ 249. A curved surface, like a plane, em-
braces the two dimensions of length and breadth.
It is not even, like the plane, throughout its whole
extent, and therefore a straight line may have
two points in common, and yet not coincide with
it. The surface of the cone, of the sphere, and
cylinder, are the curved surfaces treated of in
Elementary Geometry.

§ 250. A volume is a limited portion of space,
combining the three dimensions of length, breadth,
and thickness. Volumes are divided into three
classes:

1st class.   1st. Those bounded by planes ;

2d class.   2d. Those bounded by plane and curved sur·
faces ; and,

3d class.   3d. Those bounded only by curved surfaces.

What figures   The first class embraces the pyramid and
fall in each
class.   prism with their several varieties ; the second
class embraces the cylinder and cone ; and the
third class the sphere, together with others not
generally treated of in Elementary Geometry.

Magnitudes   § 251. We have now named all the geomet-
named.   rical magnitudes treated of in elementary Ge-
What they   ometry.   They are merely limited portions of
are.   space, and do not, *necessarily*, involve the idea
A sphere.   of matter.   A sphere, for example, fulfils all the
conditions imposed by its definitions, without any
reference to what may be within the space en-
Need not be   closed by its surface.   That space may be oc-
material.   cupied by lead, iron, or air, or may be a vacuum,
without at all changing the nature of the sphere,
as a geometrical magnitude.

It should be observed that the boundary or
Boundaries   limit of a geometrical magnitude, is another geo-
of solids.   metrical magnitude, having one dimension less.
For example: the boundary or limit of a volume,
Examples.   which has three dimensions, is always a surface
which has but two; the limits or boundaries of

all surfaces are lines, straight or curved; and the extremities or limits of lines are points.

§ 252. We have now named and shown the   Subjects named.
nature of the things which are the subjects of
Elementary Geometry. The science of Ge-   Science of Geometry.
ometry is a collection of those connected pro-
cesses by which we determine the measures,
properties, and relations of these magnitudes.

COMPARISON OF FIGURES WITH UNITS OF MEASURE.

§ 253. We have seen that the term measure   Measure.
implies a comparison of the thing measured with
some known thing of the same kind, regarded
as a standard; and that such standard is called
the unit of measure.* The unit of measure for   Unit of measure
lines must, therefore, be a line of a known length: For Lines,
a foot, a yard, a rod, a mile, or any other known   A Line.
unit. For surfaces, it is a square constructed   For Surfaces,
on the linear unit as a side: that is, a square   A Square.
foot, a square yard, a square rod, a square mile;
that is, *a square described on any known unit
of length.*

The unit of measure, for volumes, is a volume,   For Solids.
and therefore has three dimensions. It is a cube   A Cube.

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* Section 94.

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constructed on a linear unit as an edge, or on
the superficial unit as a base. It is, therefore,
a cubic foot, a cubic yard, a cubic rod, &c.
Three units Hence, there are three units of measure, each
of measure : differing in kind from the other two, viz. a known
A Line, length for the measurement of lines ; a known
A Square, square for the measurement of surfaces ; and a
A Cube. known cube for the measurement of volumes.
Contents : The measure or contents of any magnitude, be-
how ascer- longing to either class, is ascertained by finding
tained. how many times that magnitude contains its
unit of measure.

§ 254. In the fourth class of the Geometrical
magnitudes, there are several varieties. First,
the inclination of lines to each other ; 2d, of
planes ; and 3d, the space included by three or
more planes meeting at a point. In Geometry,
Angles: the right angle is the simplest unit,—in Trig-
their unit. onometry, the degree, with its subdivisions.

§ 255. We have dwelt with much detail on
the unit of measure, because it furnishes the
Importance only basis of estimating quantity. The con-
of the unit of
measure: ception of number and space merely opens to
the intellectual vision an unmeasured field of
investigation and thought, as the ascent to the
summit of a mountain presents to the eye a

wide and unsurveyed horizon.   To ascertain the height of the point of view, the diameter of the surrounding circular area and the distance to any point which may be seen, some standard or unit must be known, and its value distinctly apprehended.   So, also, number and space, which at first fill the mind with vague and indefinite conceptions, are to be finally measured by units of ascertained value.

§ 256. It is found, on careful analysis, that every number may be referred to the unit one, as a standard, and when the signification of the term ONE is clearly apprehended, that any number, whether large or small, whether integral or fractional, may be deduced from the standard by an easy and known process.

In space, also, which is indefinite in extent, and exactly similar in all its parts, the faculties of the mind have established ideal boundaries. These boundaries give rise to the geometrical magnitudes, each of which has its own unit of measure; and by these simple contrivances, we measure space, even to the stars, as with a yard-stick.

§ 257. We have, thus far, not alluded to the difficulty of *determining* the exact length of that

Conception of the unit of measure: which we regard as a standard. We are presented with a given length, and told that it is called a foot or a yard, and this being usually done at a period of life when the mind is satisfied with mere facts, we adopt the conception At first, a mere impression. of a distance corresponding to a name, and then by multiplying and dividing that distance we are enabled to apprehend other distances. But this by no means answers the inquiry, What is the standard for measurement?

Standard of measurement. The common standards of measurement, 1 yard, 1 foot, with their multiples and subdivisions, are derived from the English Exchequer and the laws of Great Britain. The one common standard from which they are all deduced is the English A brass rod. yard. The positive standard yard, is a brass rod of the year 1601, deposited in the British Ex-All weights and measures come from it. chequer. All the weights and measures in the United States, in general use, are derived from this standard. Besides this standard, there is yet Metre also a standard. another, in very general use, and consequently another system of Weights and Measures, known as the Metric system of France.

Primary base. The primary base of this system, for all denominations of Weights and Measures, is the one-ten-millionth part of the distance from the equator to the pole, measured on the meridian.

It is called a METRE, and is equal to 39.37 inches, very nearly.

The imperial yard has also been referred to an invariable standard, viz. the distance between the axis of suspension and the centre of oscillation of a pendulum which shall vibrate seconds in vacuo, in London, at the level of the sea. This distance is found, and declared to be, 39.1393 *imperial inches;* that is, 3 imperial feet and 3.139 inches.

§ 258. The standard unit of length is not only important, as affording a basis for all measurements of surface and capacity, but is also the element from which we deduce the unit of weight. The weight of a cubic foot of pure rain-water, is divided into one thousand equal parts, and each part is called one ounce. Sixteen of these ounces make the pound avoirdupois, which is our common unit of weight. Hence, the existing weights and measures of the United States, are derived from the English Exchequer and the laws of Great Britain.

§ 259. Two geometrical figures are said to be *equal,* when they contain the same unit of measure an equal number of times. Two figures are said to be *equal in all their parts, each to each,*

equal in all their parts: when they can be so applied to each other as to coincide throughout their whole extent. The term equal, is thus used in Geometry in the same sense in which it is used in Arithmetic and in Analysis: viz. to denote the relation between two quantities each of which contains the same unit an equal number of times. If two geometrical magnitudes can be applied, the one to the other, so as to coincide, they are not only equal in measure, but each part of the one is equal to a corresponding part of the other: hence, they are said to be equal in all their parts.

Difference between equal

and

equal in all parts.

PROPERTIES OF FIGURES.

Property of figures.

§ 260. A property of a figure is a mark common to all figures of the same class. For example: if the class be "Quadrilateral," there are two very obvious properties, common to all quadrilaterals, besides the one which characterizes the figure, and by which its name is defined, viz. that it has four angles, and that it may be divided into two triangles. If the class be "Parallelogram," there are several properties common to all parallelograms, and which are subjects of proof; such as, that the opposite sides and angles are equal; the diagonals divide each other into equal parts, &c. If the class be "Triangle," there are many properties common to all triangles, besides the characteristic that

Quadrilaterals.

Parallelogram.

Triangle:

they have three sides. If the class be a par- Equilateral,
ticular kind of triangle, such as the equilateral, Isosceles.
isosceles, or right-angled triangle, then each class Right-angled.
has particular properties, common to every indi-
vidual of the class, but not common to the other
classes. It is important, however, to remark, Every prop-
that every property which belongs to " triangle," erty which
belongs to a
regarded as a genus, will appertain to every genus will be
common to
species or class of triangle; and universally, every spe-
cies :
every property which belongs to a genus will
belong to every species under it; and every
property which belongs to a species will be-
long to every class or subspecies under it; and also to every
every property which belongs to one of a sub- subspecies,
and to every
species or class will be common to every indi- individual.
vidual of the class. For example : " the square Examples.
on the hypothenuse of a right-angled triangle is
equal to the sum of the squares described on
the other two sides," is a proposition equally
true of every right-angled triangle : and " every
straight line perpendicular to a chord, at the Circle.
middle point, will pass through the centre," is
equally true of all circles.

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MARKS OF WHAT MAY BE PROVED.

§ 261. The characteristic properties of every Characteris-
geometrical figure (that is, those properties with- tic proper-
ties.

out which the figures could not exist), are given
in the definitions.   How are we to arrive at all
the other properties of these figures?   The
propositions implied in the definitions, viz. that
Marks: things corresponding to the words defined do or
may exist with the properties named ; and the
Of what may self-evident propositions or axioms, contain the
be proved.
only marks of what can be proved ; and by a
How ex-   skilful combination of these marks we are able
tended.
to discover and prove all that is discovered and
proved in Geometry.

Definitions and axioms, and propositions de-
Major   duced from them, are major premises in each
Premiss.
The science: new demonstration ; and the science is made up
in what it   of the processes employed for bringing unfore-
consists.
seen cases under these known truths ; or, in syl-
logistic language, for proving the minors neces-
sary to complete the syllogisms.   The marks
being so few, and the inductions which furnish
them so obvious and familiar, there would seem
to be very little difficulty in the deductive pro-
cesses which follow.   The connecting together
of several of these marks constitutes Deductions,
Geometry,   or Trains of Reasoning ; and hence, Geometry
a Deductive   is a Deductive Science.
Science.

DEMONSTRATION.

§ 262. As a first example, let us take the first proposition in Legendre's Geometry :

"*If a straight line meet another straight line,* *the sum of the two adjacent angles will be equal to two right angles.*"

Let the straight line DC meet the straight line AB at the point C, then will the angle ACD plus the angle DCB be equal to two right angles.
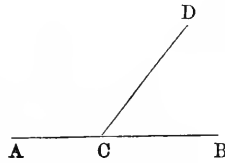
To prove this proposition, we need the defini- tion of a right angle, viz. :

When a straight line AB meets another straight line CD, so as to make the adjacent angles BAC and BAD equal to each other, *each of those angles is called a* RIGHT ANGLE, *and the line AB is said to be* PERPENDICULAR *to CD.*
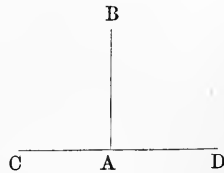
We shall also need the 2d, 3d, and 4th axioms, ior inferring equality,* viz. :

2. Things which are equal to the same thing are equal to each other.

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* Section 109.

3. A whole is equal to the sum of all its parts.

4. If equals be added to equals, the sums will be equal.

Now before these formulas or tests can be ap-

plied, it is necessary to sup-
pose a straight line CE to be
drawn perpendicular to AB
at the point C: then by the
definition of a right angle,
the angle ACE will be equal to the angle ECB.
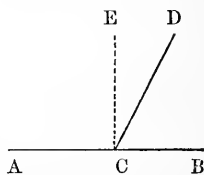
By axiom 3rd, we have,

ACD equal to ACE plus ECD: to each of these equals add DCB; and by the 4th axiom we shall have,

ACD plus DCB equal to ACE plus ECD plus DCB; but by axiom 3rd,

ECD plus DCB equals ECB: therefore by axiom 2d,

ACD plus DCB equals ACE plus ECB.

But by the definition of a right angle,

ACE plus ECB equals two right angles: there-fore, by the 2d axiom,

ACD plus DCB equals two right angles.

It will be seen that the conclusiveness of the proof results,

1st. From the definition, that ACE and ECB are equal to each other, and each is called a

right-angle : consequently, their sum is equal to
two right angles ; and,

2dly. In showing, by means of the axioms, that
ACD plus DCB equals ACE plus ECB; and
then *inferring* from axiom 2d, that, ACD plus
DCB equals two right angles.

§ 263. The difficulty in the geometrical rea-
soning consists mainly in showing that the prop-
osition to be proved contains the marks which
prove it.   To accomplish this, it is frequently
necessary to draw many auxiliary lines, forming
new figures and angles, which can be shown to
possess marks of these marks, and which thus
become connecting links between the known
and the unknown truths.   Indeed, most of the
skill and ingenuity exhibited in the geometrical
processes are employed in the use of these auxil-
iary means.   The example above affords an illus-
tration.   We were unable to show that the sum
of the two angles possessed the mark of being
equal to two right angles, until we had drawn a
perpendicular, or supposed one drawn, at the
point where the given lines intersect.   That be-
ing done, the two right angles ACE and ECB
were formed, which enabled us to *compare* the
sum of the angle ACD and DCB with two right
angles, and thus we *proved* the proposition.
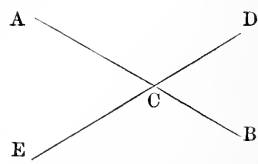
Proposition.   § 264. As a second example, let us take the
following proposition :

Enunciation.   *If two straight lines intersect each other, the
opposite or vertical angles will be equal.*

Let the straight line AB
intersect the straight line
Diagram.   ED at the point C : then
will the angle ACD be
equal to the opposite an-
gle ECB ; and the angle ACE equal to the an-
gle DCB.

Principles   To prove this proposition, we need the last
necessary.   proposition, and also the 2d and 5th axioms, viz. :
" If a straight line meet another straight line,
the sum of the two adjacent angles will be equal
to two right angles."

Axioms.   " Things which are equal to the same thing
are equal to each other."
" If equals be taken from equals, the remain-
ders will be equal."

Now, since the straight line AC meets the
straight line ED at the point C, we have,

Proof.   ACD plus ACE equal to two right angles.

And since the straight line DC meets the
straight line AB, we have,

ACD plus DCB equal to two right angles :
hence, by the second axiom,

ACD plus ACE equals ACD plus DCB : ta-

king from each the common angle ACD, we **Conclusion.**
know from the fifth axiom that the remain-
ders will be equal ; that is, the angle ACE
equal to the opposite or vertical angle DCB.

§ 265. The two demonstrations given above
combine all the processes of proof employed in **Demonstra-**
every demonstration of the same class. When **tions general.**
any new truth is to be proved, the known tests
of truth are gradually extended to auxiliary **Use of auxil-**
quantities having a more intimate connection **iary quanti-**
ties.
with such new truth than existed between it and
the known tests, until finally, the known tests,
through a series of links, become applicable to
the final truth to be established : the interme-
diate processes, as it were, bridging over the
space between the known tests and the final
truth to be proved.

§ 266. There are two classes of demonstra- **Direct dem**
tions, quite different from each other, in some **onstration.**
 respects, although the same processes of argu-
mentation are employed in both, and although
the conclusions in both are subjected to the
same logical tests.   They are called Direct, or **Negative,**
Positive Demonstration, and Negative Demon- **or**
Reductio ad
stration, or the Reductio ad Absurdum. **Absurdum.**

Difference.   § 267.   The main differences in the two
methods are these : The method of direct demon-
Direct Dem-   stration rests its arguments on known and ad-
onstration.
mitted truths, and shows by logical processes
that the proposition can be brought under some
previous definition, axiom, or proposition : while
Negative   the negative demonstration rests its arguments
Demonstra-
tion.   on an hypothesis, combines this with known pro-
positions, and deduces a conclusion by processes
Conclusion : strictly logical.   Now if the conclusion so de-
duced agrees with any known truth, we infer
With what   that the hypothesis, (which was the only link in
compared.
the chain not previously known), was true ; but
if the conclusion be excluded from the truths
previously established ; that is, if it be opposed
to any one of them, then it follows that the hy-
pothesis, being contradictory to such truth, must
Determines   be false.   In the negative demonstration, there-
whether the
hypothesis is fore, the *conclusion* is compared with the truths
true or false.
known antecedently to the proposition in ques-
tion : if it agrees with any one of them, the hy-
pothesis is correct ; if it disagrees with any one
of them, the hypothesis is false.

Proof by   § 268.   We will give for an illustration of this
Negative
Demonstra-   method, Proposition XVII. of the First Book of
tion.   Legendre : " When two right-angled triangles
have the hypothenuse and a side of the one equal

to the hypothenuse and a side of the other, each
to each, the remaining parts will be equal, each to
each, and the triangles themselves will be equal."

In the two right-angled triangles BAC and
EDF (see next figure), let the hypothenuse AC
be equal to DF, the side BA to the side ED:
then will the side BC be equal to EF, the angle
A to the angle D, and the angle C to the angle F.
To prove this proposition, we need the follow-
ing, which have been before proved ; viz. :

Prop. X. (of Legendre). "When two triangles
have the three sides of the one equal to the three
sides of the other, each to each, the three an-
gles will also be equal, each to each, and the
triangles themselves will be equal."

Prop. V. "When two triangles have two
sides and the included angle of the one, equal
to two sides and the included angle of the other,
each to each, the two triangles will be equal."

Axiom I. "Things which are equal to the
same thing, are equal to each other."

Axiom X. (of Legendre). "All right angles
are equal to each other."

Prop. XV. "If from a point without a straight
line, a perpendicular be let fall on the line, and
oblique lines be drawn to different points,

1st. "The perpendicular will be shorter than
any oblique line ;

2d. "Of two oblique lines, drawn at pleasure, that which is farther from the perpendicular will be the longer."

Now the two sides BC and

Beginning of the demonstration. EF are either equal or unequal. If they are equal, then by Prop. X. the remaining parts of the two triangles are also equal, and the triangles themselves are equal. If the two sides are unequal, one of them must be greater than the other: suppose BC to be the greater.
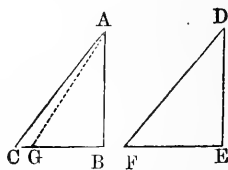
Construction of the figure. On the greater side BC take a part BG, equal to EF, and draw AG. Then, in the two triangles BAG and DEF the angle B is equal to the angle E, by axiom X (Legendre), both being right angles. The side AB is equal to the side DE, and by hypothesis the side BG is equal to the side EF: then it follows from Prop. V. that the side AG is equal to the side DF. But the side

Demonstration. DF is equal to the side AC: hence, by axiom I, the side AG is equal to AC. But the line AG cannot be equal to the line AC, having been shown to be less than it by Prop. XV.: hence,

Conclusion. the conclusion contradicts a known truth, and is therefore false; consequently, the supposition (on which the conclusion rests), that BC and EF are unequal, is also false; therefore, they are equal

§ 269. It is often supposed, though erroneous- Negative
Demonstra-
tion:
ly, that the Negative Demonstration, or the dem-
onstration involving the "reductio ad absurdum,"
is less conclusive and satisfactory than direct or Conclusive.
positive demonstration. This impression is sim-
ply the result of a want of proper analysis. For
example : in the demonstration just given, it was Reasons.
proved that the two sides BC and EF cannot
be *unequal;* for, such a supposition, in a logi-
cal argumentation, resulted in a conclusion di- Conclusion
corresponds
to, or is op-
posed to
known truth.
rectly opposed to a known truth ; and as equality
and inequality are the only general conditions
of relation between two quantities, it follows
that if they do not fulfil the one, they *must* the
other. In both kinds of demonstration, the
premises and conclusion agree ; that is, they are Agreement.
both true, or both false ; and the reasoning or
argument in both is supposed to be strictly logi-
cal.

In the direct demonstration, the premises are
known, being antecedent truths ; and hence,
the conclusion is true. In the negative demon- Differences in
the two
kinds.
stration, *one element* is assumed, and the con-
clusion is then compared with truths previously
established. If the conclusion is found to agree
with any one of these, we infer that the hy- When the
hypothesis is
true.
pothesis or assumed element is true ; if it con-
tradicts any one of these truths, we infer that

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When false. the assumed element is false, and hence that its
opposite is true.

Measured: § 270. Having explained the meaning of the
its significa-
tion.   term measured, as applied to a geometrical mag-
nitude, viz. that it implies the comparison of a
magnitude with its unit of measure ; and having
also explained the signification of the word Prop-
General   erty, and the processes of reasoning by which,
Remarks.
in all figures, properties not before noticed are
inferred from those that are known ; we shall
now add a few remarks on the relations of the
geometrical figures, and the methods of compar-
ing them with each other.

PROPORTION OF FIGURES.

Proportion.   § 271. Proportion is the relation which one
geometrical magnitude bears to another of the
same kind, with respect to its being greater or
less. The two magnitudes so compared are called
Its measure. terms, and the measure of the proportion is the
quotient which arises from dividing the second
term by the first, and is called their Ratio.   Only
Ratio.
Quantities of quantities of the same kind can be compared
the same   together, and it follows from the nature of the
kind com
pared.   relation that the quotient or ratio of any two
terms will be an abstract number, whether the
terms themselves be abstract or concrete

§ 272. The term Proportion is defined by most *Proportion: how defined.* authors, "An equality of ratios between four numbers or quantities, compared together two and two." A proportion certainly arises from such a comparison : thus, if

$$\frac{B}{A} = \frac{D}{C} \; ; \text{ then,}$$ *Example.*

$$A : B : : C : D$$

is a proportion.

But if we have two quantities A and B, which *True definition.* may change their values, and are, at the same time, so connected together that one of them shall increase or decrease just as many times as the other, their ratio will not be altered by such changes; and the two quantities are then said *Two proportional quantities.* to be in proportion, or proportional.

Thus, if A be increased three times and B three times, then,

$$\frac{3\,B}{3\,A} = \frac{A}{B} \; ;$$

that is, 3 A and 3 B bear to each other the same proportion as A and B. Science needed a gen- *Term needed.* eral term to express this relation between *two* quantities which change their values, without altering their quotient, and the term "proportional," or "in proportion," is employed for that *How used.* purpose.

Reasons for modification.

As some apology for the modification of the definition of proportion, which has been so long accepted, it may be proper to state that the term has been used by the best authors in the exact

Use of the term.

sense here attributed to it. In the definition of the second law of motion, we have, "Motion, or change of motion, is proportional to the force impressed;"[*] and again, "The inertia of a body is proportioned to its weight."[†] Similar examples may be multiplied to any extent. Indeed,

Symbol used to represent proportion.

there is a symbol or character to express the relation between two quantities, when they undergo changes of value, without altering their ratio. That character is $\propto$, and is read "proportional to." Thus, if we have two quantities denoted by A and B, written,

Example.

$$A \propto B,$$

the expression is read, "A proportional to B."

Another kind of proportion.

§ 273. There is yet another kind of relation which may exist between two quantities A and B, which it is very important to consider and understand. Suppose the quantities to be so connected with each other, that when the first is increased according to any law of change, the second shall decrease according to the same law; and the reverse.

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* Olmsted's Mechanics, p. 23.   † Ibid. p. 23.

For example : the area of a rect-
angle is equal to the product of its
base and altitude. Then, in the
rectangle ABCD, we have
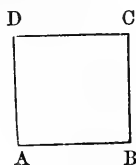
$$\text{Area} = AB \times BC.$$

Take a second rectangle EFGH, having a
longer base EF, and a less altitude FG, but such
that it shall have an equal
area with the first : then we
shall have
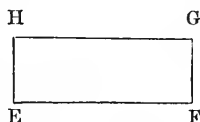
$$\text{Area} = EF \times FG.$$

Now since the areas are equal, we shall have

$$AB \times BC = EF \times FG ;$$

and by resolving the terms of this equation into
a proportion, we shall have

$$AB : EF :: FG : BC.$$

It is plain that the sides of the rectangle ABCD
may be so changed in value as to become the
sides of the rectangle EFGH, and that while
they are undergoing this change, AB will in-
crease and BC diminish. The change in the
values of these quantities will therefore take place
according to a fixed law : that is, one will be di-
minished as many times as the other is increased,

since their product is constantly equal to the area of the rectangle EFGH.

Denote the side AB by x and BC by y, and the area of the rectangle EFGH, which is known, by a; then

$$xy = a ;$$

and when the product of two varying quantities is constantly equal to a known quantity, the two

quantities are said to be *Reciprocally* or *Inversely* proportional ; thus x and y are said to be inversely proportional to each other. If we divide 1 by each member of the above equation, we shall have

$$\frac{1}{xy} = \frac{1}{a} ;$$

and by multiplying both members by x, we shall have

$$\frac{1}{y} = \frac{x}{a} ;$$

and then by dividing both numbers by x, we have

$$\frac{\frac{1}{y}}{x} = \frac{1}{a} ;$$

that is, the ratio of x to $\frac{1}{y}$ is constantly equal to $\frac{1}{a}$; that is, equal to the same quantity, however x or

y may vary, for, a and consequently $\frac{1}{a}$ does not change. Hence,

Two quantities, which may change their values, are reciprocally or inversely proportional, when one is proportional to unity divided by the other, and then their product remains constant.

We express this reciprocal or inverse relation thus:

$$A \propto \frac{1}{B}.$$

A is said to be inversely proportional to B: the symbols also express that A is directly propor-

tional to $\frac{1}{B}$. If we have

$$A \propto \frac{B}{C},$$

we say, that A is directly proportional to B, and inversely proportional to C.

The terms Direct, Inverse or Reciprocal, apply to the *nature* of the *proportion*, and not to the Ratio, which is always a mere quotient and the measure of proportion. The term Direct applies to all proportions in which the terms increase or decrease together; and the term Inverse or Reciprocal to those in which one term increases as the other decreases. They cannot, therefore, properly be applied to ratio without changing entirely its signification and definition.

COMPARISON OF FIGURES.

Geometrical magnitudes compared.   § 274. In comparing geometrical magnitudes, by means of their quotient, it is not the quotient alone which we consider. The comparison implies a general relation of the magnitudes, which is measured by the Ratio. For example : we

Example.   say that " Similar triangles are to each other as the squares of their homologous sides." What do we mean by that ? Just this :

Formula of Comparison.   That the area of a triangle

Is to the area of a similar triangle

As the area of a square described on a side of the first,

To the area of a square described on an homologous side of the second.

Thus, we see that every term of such a pro-

Changes of value : how affected   portion is in fact a surface, and that the area of a triangle increases or decreases much faster than its sides ; that is, if we double each side of a triangle, the area will be four times as great : if we multiply each side by three, the area will

Results.   be nine times as great ; or if we divide each side by two, we diminish the area four times, and so on. Again,

Circles compared.   The area of one circle

Is to the area of another circle,

As a square described on the diameter of the first

To a squar• described on the diameter of the
second.

Hence, if we double the diameter of a circle, How their areas change
the area of the circle whose diameter is so in-
creased will be four times as great : if we mul-
tiply the diameter by three, the area will be nine Principle general.
times as great; and similarly if we divide the
diameter.

§ 275. In comparing volumes together, the Comparison of volumes.
same general principles obtain. Similar volumes
are to each other as the cubes described on their
homologous or corresponding sides. That is,

A prism
Formula.
Is to a similar prism,

As a cube described on a side of the first,

Is to a cube described on an homologous side
of the second.

Hence, if the sides of a prism be doubled, the con- How the volumes change.
tents of volume will be increased eight-fold. Again,

A sphere

Is to a sphere, Sphere:

As a cube described on the diameter of the first,

Is to a cube described on a diameter of the
second.

Hence, if the diameter of a sphere be doubled, How its volume changes.
its contents of volume will be increased eight-
fold; if the diameter be multiplied by three, the

contents of volume will be increased twenty-seven fold: if the diameter be multiplied by four, the contents of volume will be increased sixty-four fold; the contents of volume increasing as the cubes of the numbers 1, 2, 3, 4, &c.

Ratio: § 276. The relation or ratio of two magnitudes an abstract to each other, may be, and indeed is, expressed number. by an abstract number.   This number has a When hav- fixed value so long as we do not introduce a ing a fixed value. change in the contents of the figures; but if we wish to express their ratio under the supposition that their contents may change according to fixed laws (that is, so that the volumes How varying shall continue similar), we then compare them volumes are compared. with similar figures described on their homologous or corresponding sides; or, what is the same thing, take into account the corresponding changes which take place in the abstract numbers that express their contents.

RECAPITULATION.

General   § 277. We have now completed a general outoutline. line of the science of Geometry, and what has been said may be recapitulated under the following heads. It has been shown,

Geometry: 1st. That Geometry is conversant about space,

or those limited portions of space which are to what it relates. called Geometrical Magnitudes.

2d. That the geometrical magnitudes embrace four species:

    1st. Lines—straight and curved;   Lines.

    2d. Surfaces—plane and curved;   Surfaces.

    3d. Volumes—bounded either by plane sur-   Volumes. faces or curved, or both; and,

    4th. Angles, arising from the positions of   Angles. lines with each other; or of planes with each other—the lines and planes being boundaries.

3d. That the science of Geometry is made up   Science: of those processes by means of which all the   how made up. properties of these magnitudes are examined and developed, and that the results arrived at constitute the truths of Geometry.

4th. That the truths of Geometry may be di-   Truths vided into three classes: three classes.

    1st. Those implied in the definitions, viz.   First class. that things exist corresponding to certain words defined;

    2d. Intuitive or self-evident truths em-   Second. bodied in the axioms;

    3d. Truths deduced (that is, inferred) from   Third. the definitions and axioms, called Demonstrative Truths.

5th. That the examination of the properties of   Geometrical the geometrical magnitudes has reference,   magnitudes,

Comparison.   1st. To their comparison with a standard
or unit of measure ;

Properties.   2d. To the discovery of properties belong-
ing to an individual figure, and yet common to
the entire class to which such figure belongs ;

Proportion.   3d. To the comparison, with respect to mag-
nitude, of figures of the same species with each
other; viz. lines with lines, surfaces with sur-
faces, volumes with volumes, and angles with
angles.

SUGGESTIONS FOR THOSE WHO TEACH GEOMETRY.

Suggestions.   1. Be sure that your pupils have a clear ap-
First.   prehension of space, and of the notion that Ge-
ometry is conversant about space only.

2. Be sure that they understand the significa-
Second.   tion of the terms, lines, surfaces, and volumes,
and that these names indicate certain portions
of space corresponding to them.

3. See that they understand the distinction
Third.   between a straight line and a curve; between a
plane surface and a curved surface; between a
volume bounded by planes and a volume bounded
by curved surfaces.

4. Be careful to have them note the charac-
· Fourth.   teristics of the different species of plane figures,
such as triangles, quadrilaterals, pentagons, hexa-
gons, &c.; and then the characteristic of each

class or subspecies, so that the name shall recall, at once, the characteristic properties of each figure.

5. Be careful, also, to have them note the characteristic differences of the volumes. Let them often name and distinguish those which are bounded by planes, those bounded by plane and curved surfaces, and those bounded by curved surfaces only. Regarding Volume as a genus, let them give the species and subspecies into which it may be divided. *Fifth.*

6. Having thus made them familiar with the things which are the subjects of the reasoning, explain carefully the nature of the definitions ; then of the axioms, the grounds of our belief in them, and the information from which those self-evident truths are inferred. *Sixth.*

7. Then explain to them, that the definitions and axioms are the basis of all geometrical reasoning : that every proposition must be deduced from them, and that they afford the tests of all the truths which the reasonings establish. *Seventh.*

8. Let every figure, used in a demonstration, be accurately drawn, by the pupil himself, on a blackboard. This will establish a connection between the eye and the hand, and give, at the same time, a clear perception of the figure and a distinct apprehension of the relations of its parts. *Eighth.*

9. Let the pupil, in every demonstration, first
Ninth.   enunciate, in general terms, that is, without the
aid of a diagram, or any reference to one, the
proposition to be proved; and then state the
principles previously established, which are to
be employed in making out the proof.

10. When in the course of a demonstration,
Tenth.   any truth is inferred from its connection with
one before known, let the truth so referred to be
fully and accurately stated, even though the
number of the proposition in which it is proved,
be also required.   This is deemed important.

11. Let the pupil be made to understand that
Eleventh.   a demonstration is but a series of logical argu-
ments arising from comparison, and that the
result of every comparison, in respect to quan-
tity, contains the mark either of equality or
inequality.

12. Let the distinction between a positive
Twelfth.   and negative demonstration be fully explained
and clearly apprehended.

13. In the comparison of quantities with each
Thirteenth.   other, great care should be taken to impress the
fact that proportion exists only between quan-
tities of the same kind, and that ratio is the
measure of proportion.

14. Do not fail to give much importance to
Fourteenth.   the *kind* of quantity under consideration.   Let

the question be often put, What kind of quantity
are you considering? Is it a line, a surface, or
a volume? And what kind of a line, surface, or
volume?

15. In all cases of measurement, the unit of
measure should receive special attention. If
lines are measured, or compared by means of a
common unit, see that the pupil perceives that
unit clearly, and apprehends distinctly its rela-
tions to the lines which it measures. In sur-
faces, take much pains to mark out on the
blackboard the particular square which forms
the unit of measure, and write unit, or unit of
measure, over it. So in the measurement of
volumes, let the unit or measuring cube be ex-
hibited, and the conception of its size clearly
formed in the mind ; and then impress the im-
portant fact, that, *all measurement consists in
merely comparing a unit of measure with the
quantity measured ; and that the number which
expresses the ratio is the numerical expression
for that measure.*

16. Be careful to explain the difference of the
terms *Equal* and *Equal in all their parts,* and
never permit the pupil to use the terms as sy-
nonymous. An accurate use of words leads to
nice discriminations of thought.

ANALYSIS.

DIFFERENTIAL AND INTEGRAL CALCULUS.

ANALYTICAL GEOMETRY.

ALGEBRA.

CHAPTER IV.

ANALYSIS—ALGEBRA—ANALYTICAL GEOMETRY.

ANALYSIS.

§ 278. ANALYSIS is a general term, embracing that entire portion of mathematical science in which the quantities considered are represented by letters of the alphabet, and the operations to be performed on them are indicated by signs.

Analysis defined.

§ 279. We have seen that all numbers must be numbers of something;* for, there is no such thing as a number without a basis : that is, every number must be based on the abstract unit one, or on some unit denominated.   But although numbers must be numbers of *something*, yet they may be numbers of *any thing*, for the unit may be whatever we choose to make it.

Numbers must be of things;

but may be of many kind of things.

* Section 112.

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§ 280. All quantity, made up of definite parts, can be numbered exactly or approximatively, and, in this respect, possesses all the properties of number. Propositions, therefore, concerning numbers, have the remarkable peculiarity, that they are propositions concerning all quantities whatever. That half of six is three, is equally true, whatever the word six may represent, whether six abstract units, six men, or six triangles. Analysis extends the generalization still further. A number represents, or stands for, that particular number of things of the same kind, without reference to the *nature* of the thing; but an analytical symbol does more, for it may stand for *all numbers*, or for all quantities which numbers represent, or even for quantities which cannot be exactly expressed numerically.

As soon as we conceive of a thing we may conceive it divided into equal parts, and may represent either or all of those parts by a or x, or may, if we please, denote the thing itself by a or x, without any reference to its being divided into parts.

§ 281. In Geometry, each geometrical figure stands for a class; and when we have demonstrated a property of a figure, that property is considered as proved for every figure of the class.

(margin notes:)
All quantity consists of parts.

Propositions in regard to number apply also to quantity.

Algebraic symbols more general.

Any thing conceived may be divided.

Each figure stands for a class.

For example: when we prove that the square   Example.
described on the hypothenuse of a right-angled
triangle is equal to the sum of the squares de-
scribed on the other two sides, we demonstrate the
fact for all right-angled triangles. But in analy-   In analysis
sis, all numbers, all lines, all surfaces, all vol-   the symbols
stand for
umes, may be denoted by a single symbol, a or x.   things of all
classes.
Hence, all truths inferred by means of these
symbols are true of all things whatever, and not
like those of number and geometry, true only
of particular classes of things. It is, therefore,
not surprising, that the symbols of analysis do
not excite in our minds the ideas of particular   Hence, the
things. The mere written characters, a, b, c, d,   truths infer-
red are gen-
x, y, z, serve as the representatives of things in   eral.
general, whether abstract or concrete, whether
known or unknown, whether finite or infinite.

§ 282. In the uses which we make of these   Symbols
come to be
symbols, and the processes of reasoning carried   regarded as
things.
on by means of them, the mind insensibly comes
to regard them as *things*, and not as mere signs;
and we constantly predicate of them the prop-
erties of things in general, without pausing to
inquire what kind of thing is implied. Thus,   Example.
we define an equation to be a proposition in   The equa-
tion.
which equality is predicated of one thing as
compared with another. For example:

$$a + c = x,$$

What axioms necessary to its solution. is an equation, because x is declared to be equal to the sum of a and c. In the solution of equations, we employ the axioms, " If 'equals be added to equals, the sums will be equal ;" and, " If equals be taken from equals, the remainders will be equal." Now, these axioms do not ex-

They express qualities of things. press qualities of language, but properties of quantity. Hence, all inferences in mathemat-

Hence, inferences relate to things. ical science, deduced through the instrumentality of symbols, whether Arithmetical, Geometrical, or Analytical, must be regarded as concerning quantity, and not symbols.

Quantity need not always be present to the mind. As analytical symbols are the representatives of quantity in general, there is no necessity of keeping the idea of *quantity* continually alive in the mind ; and the processes of thought may, without danger, be allowed to rest on the symbols themselves, and therefore, become to that extent, merely mechanical. But, when we look

The reasoning is all based on the supposition of quantity. back and see on what the reasoning is based, and how the processes have been conducted, we shall find that every step was taken on the supposition that we were actually dealing with things, and not symbols ; and that, without this understanding of the language, the whole system is without signification, and fails.

§ 283. There are three principal branches of Analysis : Three branches

　1st. Algebra. Algebra,

　2d. Analytical Geometry. Analytical Geometry,

　3d. Differential and Integral Calculus. Calculus.

ALGEBRA.

§ 284. Algebra is, in fact, a species of universal Arithmetic, in which letters and signs are employed to abridge and generalize all processes involving numbers.   It is divided into two parts, corresponding to the science and art of Arithmetic : Algebra:
Universal
Arithmetic.
Two parts

　1st. That which has for its object the investigation of the properties of numbers, embracing all the processes of reasoning by which new properties are inferred from known ones ; and, First part.

　2d. The solution of all problems or questions involving the determination of certain numbers which are unknown, from their connection with certain others which are known or given. Second part.

ANALYTICAL GEOMETRY.

§ 285. Analytical Geometry examines the properties, measures, and relations of the geometrical magnitudes by means of the analytical Analytical Geometry.
Its nature.

symbols. This branch of mathematical science
originated with the illustrious Descartes, a cele-
brated French mathematician of the 17th cen-
tury. He observed that the positions of points,
the direction of lines, and the forms of surfaces,
could be expressed by means of the algebraic
symbols; and consequently, that every change,
either in the position or extent of a geometrical
magnitude, produced a corresponding change in
certain symbols, by which such magnitude could
be represented. As soon as it was found that,
to every variety of position in points, direction
in lines, or form of curves or surfaces, there cor-
responded certain analytical expressions (called
their Equations), it followed, that if the processes
were known by which these equations could be
examined, the relation of their parts determined,
and the laws according to which those parts
vary, relative to one another, ascertained, then
the corresponding changes in the geometrical
magnitudes, thus represented, could be imme-
diately inferred.

Hence, it follows that every geometrical ques-
tion can be solved, if we can resolve the corre-
sponding algebraic equation; and the power over
the geometrical magnitudes was extended just in
proportion as the properties of quantity were
brought to light by means of the Calculus. The

Marginal notes:

Descartes, the original founder of this science.

What he observed.

All position expressed by symbols.

The equation develops the properties of the mag- nitude.

Power over the magni- tude extend- ed by the equation.

applications of this Calculus were soon made to the subjects of mechanics, astronomy, and indeed, in a greater or less degree, to all branches of natural philosophy; so that, at the present time, all the varieties of physical phenomena, with which the higher branches of the science are conversant, are found to answer to varieties determinable by the algebraic analysis.

§ 286. Two classes of quantities, and conse- quently two sets of symbols, quite distinct from each other, enter into this Calculus; the one called *Constants*, which preserve a fixed or given value throughout the same discussion or investigation; and the other called Variables, which undergo certain changes of value, the laws of which are indicated by the algebraic expressions or equations into which they enter. Hence,

Analytical Geometry may be defined as that branch of mathematical science, which examines, discusses, and develops the properties of geometrical magnitudes by noting the changes that take place in the algebraic symbols which represent them, the laws of change being determined by an algebraic equation or formula.

DIFFERENTIAL AND INTEGRAL CALCULUS.

<div style="float:left">Quantities considered.</div>

§ 287 In this branch of mathematical science,
as in Analytical Geometry, two kinds of quan-

<div style="float:left">Variables, Constants.</div>

tity are considered, viz. Variables and Constants ;
and consequently, two distinct sets of symbols

<div style="float:left">The Science.</div>

are employed.   The science consists of a series
of processes which note the changes that take
place in the value of the Variables.   Those
changes of value take place according to fixed
laws established by algebraic formulas, and are

<div style="float:left">Marks.</div>

indicated by certain *marks* drawn from the va-

<div style="float:left">Differentials.</div>

riable symbols, called *Differentials*.   By these
marks we are enabled to trace out with the
accuracy of exact science the most hidden prop-
erties of quantity, as well as the most general
and minute laws which regulate its changes of
value.

<div style="float:left">Analytical Geometry, and Calculus :</div>

§ 288. It will be observed, that Analytical
Geometry and the Differential and Integral Cal-
culus treat of quantity regarded under the same
general aspect, viz. as subject to changes or va-

<div style="float:left">How they regard quantity :</div>

riations in magnitude according to laws indica-
ted by algebraical formulas ; and the quantities,
whether variable or constant, are, in both cases,

<div style="float:left">by what represented.</div>

represented by the same algebraic symbols, viz.
the constants by the first, and the variables by

the final letters of the alphabet. There is, how- *Difference;*
ever, this important difference : in Analytical
Geometry all the results are inferred from the *In what it
consists.*
relations which exist between the quantities
themselves, while in the Differential and Integral
Calculus they are deduced by considering what
may be indicated by *marks* drawn from variable
quantities, under certain suppositions, and by
marks of such marks.

§ 289. Algebra, Analytical Geometry, the Dif- *Analytical
Science.*
ferential and Integral Calculus, extended into the
Theory of Variations, make up the subject of
analytical science, of which Algebra is the ele-
mentary branch. We shall, in this chapter,
limit our remarks to the subject of Algebra ; *Algebra.*
reserving a separate chapter for the Differential *Differential
Calculus.*
and Integral Calculus. This subject embraces a
very remarkable class of quantities.

ALGEBRA.

§ 290. On an analysis of the subject of Alge- *Algebra.*
bra, we think it will appear that the subject itself
presents no serious difficulties, and that most of *Difficulties.*
the embarrassment which is experienced by the
pupil in gaining a knowledge of its principles, as *How over-
come.*
well as in their applications, arises from not at

Language. tending sufficiently to the *language* or *signs* of the thoughts which are combined in the reasonings. At the hazard, therefore, of being a little diffuse, I shall begin with the very elements of the algebraic language, and explain, with much minuteness, the exact signification of the characters that stand for the quantities which are the subjects of the analysis; and also of those signs which indicate the several operations to be performed on the quantities.

Characters which represent quantity.

Signs.

Quantities. § 291. The quantities which are the subjects
How divided. of the algebraic analysis may be divided into two classes: those which are known or given, and those which are unknown or sought. The known are uniformly represented by the first letters of the alphabet, a, b, c, d, &c.; and the unknown by the final letters, x, y, z, v, w, &c.

How represented.

May be increased or diminished. § 292. Quantity is susceptible of being increased or diminished; and there are six operations which can be performed upon a quantity that will give results differing from the quantity itself, viz.:

Five operations.

First. 1st. To add it to itself or to some other quantity;

Second. 2d. To subtract some other quantity from it;

3d. To multiply it by a number ;   Third.

4th. To divide it ;   Fourth.

5th. To raise it to any power ; and   Fifth.

6th. To extract a root of it.   Sixth.

. The cases in which the multiplier or divisor is 1, are of course excepted ; as also the case   Exception. where a root is to be extracted of 1.

§ 293. The six signs which denote these oper-   Signs. ations are too well known to be repeated here. These, with the signs of equality and inequality,   Elements the letters of the alphabet and the figures which   of the Algebraic are employed, make up the elements of the alge-   language braic language.   The words and phrases of the   Its words algebraic, like those of every other language, are   and phrases to be taken in connection with each other, and are not to be interpreted as separate and isolated   How inter- preted. symbols.

§ 294. The symbols of quantity are designed   Symbols of to represent quantity in general, whether abstract   quantity . or concrete, whether known or unknown ; and the signs which indicate the operations to be   General. performed on the quantities are to be interpreted in a sense equally general.   When the sign plus is written, it indicates that the quantity before   Examples. which it is placed is to be added to some other   Signs plus quantity ; and the sign minus implies the exist-   and minus

ence of a minuend, from which the subtrahend
is to be taken.   One thing should be observed in
Signs have
no effect on
the nature of
a quantity.
regard to the signs which indicate the operations
that are to be performed on quantities, viz. *they
do not at all affect or change the nature of the
quantity before or after which they are written
but merely indicate what is to be done with the
Examples:
In Algebra.
quantity.*   In Algebra, for example, the minus
sign merely indicates that the quantity before
which it is written is to be subtracted from
In Analytical
Geometry.
some other quantity ; and in Analytical Geom-
etry, that the line before which it falls is esti-
mated in a contrary direction from that in which
it would have been reckoned, had it had the sign
plus ; but in neither case is the *nature* of the
quantity itself different from what it would have
been had it had the sign plus.

Interpreta-
tion of the
language:
The interpretation of the language of Algebra
is the first thing to which the attention of a pupil
should be directed ; and he should be drilled on
the meaning and import of the symbols, until
their significations and uses are as familiar as
Its necessity.
the sounds and combinations of the letters of the
alphabet.

Elements
explained.
§ 295. Beginning with the elements of the
language, let any number or quantity be desig-
nated by the letter a, and let it be required to

add this letter to itself, and find the result or sum.
The addition will be expressed by

$$a + a = \text{the sum.}$$

But how is the sum to be expressed ? By simply
regarding a as *one a*, or $1a$, and then observing
that one a and *one a* make *two a*'s or $2\,a$: hence,

$$a + a = 2\,a ;$$

and thus we place a figure before a letter to in-
dicate how many times it is taken.   Such figure
is called a *Coefficient.*

§ 296. The product of several numbers is in-
dicated by the sign of multiplication, or by sim-
ply writing the letters which represent the num-
bers by the side of each other.   Thus,

$$a \times b \times c \times d \times f, \text{ or } abcdf,$$

indicates the continued product of a, b, c, d, and
f, and each letter is called a factor of the prod-
uct : hence, a *factor* of a product is one of the
multipliers which produce it.   Any figure, as **5**,
written before a product, as

$$5\,abcdf,$$

is the coefficient of the product, and shows that
the product is taken **5** times.

Equal factors: what the product becomes.

§ 297. If in the product *abcdf*, the numbers represented by a, b, c, d, and f were equal to each other, they would each be represented by a single letter a, and the product would then become

$$a \times a \times a \times a \times a = a^{5};$$

How expressed.

that is, we indicate the product of several equal factors by simply writing the letter once and placing a figure above and a little at the right of it, to indicate how many times it is taken as a factor. The figure so written is called an *exponent*. Hence, an exponent denotes how many equal factors are employed. The result of the multiplications, is called the 5th *Power* of a.

Exponent: where written.

Division: how expressed.

§ 298. The division of one quantity by another is indicated by simply writing the divisor below the dividend, after the manner of a fraction ; by placing it on the right of the dividend with a horizontal line and two dots between them ; or by placing it on the right with a vertical line between them : thus either form of expression,

Three forms.

$$\frac{b}{a}, \qquad b \div a, \qquad \text{or} \qquad b \mid a,$$

indicates the division of b by a.

Roots: how indicated.

§ 299. The extraction of a root is indicated by the sign $\sqrt{}$. This sign, when used by itself indicates the lowest root, viz. the square root.

If any other root is to be extracted, as the third, fourth, fifth, &c., the figure marking the degree of the root is written above and at the left of the sign ; as,

$$\sqrt[3]{} \text{ cube root, } \sqrt[4]{} \text{ fourth root, \&c.}$$

The figure so written, is called the *Index* of the root.

We have thus given the very simple and general language by which we indicate every one of the six operations that may be performed on an algebraic quantity, and *every process in Algebra* involves one or other of these operations.

MINUS SIGN.

§ 300. The algebraic symbols are divided into two classes entirely distinct from each other, viz. the letters that are used to designate the quantities which are the subjects of the science, and the signs which are employed to indicate certain operations to be performed on those quantities. We have seen that all the algebraic processes are comprised under addition, subtraction, multiplication, division, and the extraction of roots ; and it is plain, that the *nature* of a quantity is not at all changed by prefixing to it the sign which indicates either of these opera-

tions. The quantity denoted by the letter *a*, for example, is the same, in *every respect*, whatever sign may be prefixed to it; that is, whether it be added to another quantity, subtracted from it, whether multiplied or divided by any number, or whether we extract the square or cube or any other root of it. The algebraic signs, therefore, must be regarded merely as indicating *opera-tions* to be performed on quantity, and not as affecting the *nature* of the quantities to which they may be prefixed. We say, indeed, that quantities are plus and minus, but this is an abbreviated language to express that they are to be added or subtracted.

Algebraic signs: how regarded.

Plus and Minus.

Principles of the science:

From what deduced.

Example.

What we wish to discover.

§ 301. In Algebra, as in Arithmetic and Geometry, all the principles of the science are deduced from the definitions and axioms ; and the rules for performing the operations are but directions framed in conformity to such principles. Having, for example, fixed by definition, the power of the minus sign, viz. that any quantity before which it is written, shall be regarded as to be subtracted from another quantity, we wish to discover the process of performing that subtraction, so as to deduce therefrom a general *prin-ciple*, from which we can frame a rule applicable to all similar cases.

SUBTRACTION.

§ 302. Let it be required, for example, to Subtraction.
subtract from b the difference be-
tween a and c. Now, having writ-
ten the letters, with their proper

$$b$$
$$a - c$$

Process.

signs, the language of Algebra expresses that it
is the *difference* only between a and c, which is
to be taken from b; and if this difference were Difference.
known, we could make the subtraction at once.
But the nature and generality of the algebraic
symbols, enable us to *indicate operations*, merely, Operations indicated.
and we cannot in general make reductions until
we come to the final result. In what general
way, therefore, can we indicate the true differ-
ence?

If we indicate the subtraction of
a from b, we have $b - a$; but then
we have taken away too much from

$$b - a$$
$$b - a + c$$

Final formula.

b by the number of units in c, for it was not a,
but the *difference* between a and c that was to
be subtracted from b. Having taken away *too
much*, the remainder is *too small* by c: hence,
if c be added, the true remainder will be express-
ed by $b - a + c$.

Now, by analyzing this result, we see that the Analysis of the result.
sign of every term of the subtrahend has been
changed; and what has been shown with re-

<div align="center">18</div>

spect to these quantities is equally true of all others standing in the same relation : hence, we have the following general rule for the subtraction of algebraic quantities :

Change the sign of every term of the subtrahend, or conceive it to be changed, and then unite the quantities as in addition.

MULTIPLICATION.

§ 303. Let us now consider the case of multiplication, and let it be required to multiply $a - b$ by c. The algebraic language expresses

that the difference between a and b is to be taken as many times as there are units in c. If we knew this difference, we could at once

$$\begin{array}{r} a - b \\ c \\ \hline ac - bc \end{array}$$

perform the multiplication. But by what gen-

eral process is it to be performed without finding that difference ? If we take a, c times, the product will be ac; but as it was only the *difference* between a and b, that was to be multiplied by c,

this product ac will be too great by b taken c times; that is, the true product will be expressed by $ac - bc$: hence, we see, that,

If a quantity having a plus sign be multiplied by another quantity having also a plus sign, the sign of the product will be plus ; and

if a quantity having a minus sign be multi-
plied by a quantity having a plus sign, the sign
of the product will be minus.

§ 304. Let us now take the most general
case, viz. that in which it is required to multi-
ply $a - b$ by $c - d$.

Let us again observe that the algebraic lan-
guage denotes that $a - b$ is
to be taken as many times
as there are units in $c - d$;
and if these two differences
were known, their product

$$
\begin{array}{l}
a - b \\
c - d \\
\hline
ac - bc \\
\qquad - ad + bd \\
\hline
ac - bc - ad + bd
\end{array}
$$

would at once form the product required.

First : let us take $a - b$ as many times as there
are units in c; this product, from what has al-
ready been shown, is equal to $ac - bc$. But
since the multiplier is not c, but $c - d$, it follows
that this product is too large by $a - b$ taken d
times; that is, by $ad - bd$: hence, the first prod-
uct diminished by this last, will give the true
product. But, by the rule for subtraction, this
difference is found by changing the signs of the
subtrahend, and then uniting all the terms as in
addition : hence, the true product is expressed
by $ac - bc - ad + bd$.

By analyzing this result, and employing an
abbreviated language, we have the following gen-

eral principle to which the signs conform in mul-
tiplication, viz.:

General
Principle.

*Plus multiplied by plus gives plus in the prod-
uct ; plus multiplied by minus gives minus ; mi-
nus multiplied by plus gives minus ; and minus
multiplied by minus gives plus in the product.*

Remark.   § 305. The remark is often made by pupils
that the above reasoning appears very satisfac-
tory so long as the quantities are presented un-
der the above form ; but why will $-b$ multiplied
Particular   by $-d$ give plus bd ? How can the product of
case.
two negative quantities *standing alone* be plus ?

Minus sign: In the first place, the minus sign being pre-
fixed to b and d, shows that in an *algebraic sense*
they do not stand by themselves, but are con-
Its interpre- nected with other quantities; and if they are
tation.
not so connected, the minus sign makes no dif-
ference ; for, it in no case affects the quantity,
but merely points out a connection with other
quantities.   Besides, the product determined
above, being independent of any particular value
attributed to the letters a, b, c, and d, must be
Form of the of such a form as to be true for all values; and
product :
must be true hence for the case in which a and c are both
for quantities equal to zero.   Making this supposition, the
of any value.
product reduces to the form of $+ bd$.   The rules
for the signs in division are readily deduced from

the definition of division, and the principles al-
ready laid down.

<center>ZERO AND INFINITY.</center>

§ 306. The terms zero and infinity have given
rise to much discussion, and been regarded as
presenting difficulties not easily removed. It may
not be easy to frame a form of language that shall
convey to a mind, but little versed in mathe-
matical science, the precise ideas which these
terms are designed to express; but we are un-
willing to suppose that the ideas themselves arc
beyond the grasp of an ordinary intellect. The
terms are used to designate the *two limits of
Space and Number.*

§ 307. Assuming any two points in space, and
joining them by a straight line, the distance be-
tween the points will be truly indicated by the
length of this line, and this length may be ex-
pressed numerically by the number of times
which the line contains a known unit. If now,
the points are made to approach each other, the
length of the line will diminish as the points
come nearer and nearer together, until at length,
when the two points become one, the length of
the line will disappear, having attained its *limit,*

which is called *zero*. If, on the contrary, the points recede from each other, the length of the line joining them will continually increase ; but so long as the length of the line can be expressed in terms of a known unit of measure, it is not infinite.   But, if we suppose the points removed, so that any known unit of measure would occupy no *appreciable portion* of the line, then the length of the line is said to be *Infinite*.

Illustration, showing the meaning of the term Infinity.

§ 308. Assuming one as the unit of number, and admitting the self-evident truth that it may be increased or diminished, we shall have no difficulty in understanding the import of the terms zero and infinity, as applied to number. For, if we suppose the unit one to be continually diminished, by division or otherwise, the fractional units thus arising will be less and less, and in proportion as we continue the divisions, they will continue to diminish.   Now, the limit or boundary to which these very small fractions approach, is called Zero, or nothing.   So long as the fractional number forms an appreciable part of one, it is not zero, but a finite fraction ; and the term zero is only applicable to that which forms no appreciable part of the standard. If, on the other hand, we suppose a number to be continually increased, the relation of this

The terms Zero and Infinity applied to numbers.

Illustration.

Zero :

Illustration.

number to the unit will be constantly changing.
So long as the number can be expressed in
terms of the unit one, it is finite, and not infi- Infinity;
nite; but when the unit one forms no appre-
ciable part of the number, the term infinite is
used to express that state of value, or rather,
that limit of value.

§ 309. The terms zero and infinity are there- The terms,
fore employed to designate the limits to which how
employed.
decreasing and increasing quantities may be
made to approach nearer than any assignable
quantity; but these limits cannot be compared, Are limits.
in respect to magnitude, with any known stand-
ard, so as to give a finite ratio.

§ 310. It may, perhaps, appear somewhat par- Why limits?
adoxical, that zero and infinity should be defined
as "the limits of number and space" when they
are in themselves not measurable.   But a limit
is that "which sets bounds to, or circumscribes;" Definition of
and as all finite space and finite number (and a limit.
such only are implied by the terms Space and Of Space and
Number), are contained between zero and in- Number
finity, we employ these terms to designate the
limits of Number and Space.

OF THE EQUATION.

Deductive reasoning.   § 311. We have seen that all deductive rea-
soning involves certain processes of comparison,
and that the syllogism is the formula to which
those processes may be reduced.*   It has also
Comparison of quantities.   been stated that if two quantities be compared
together, there will necessarily result the condi-
Condition.   tion of equality or inequality.   The equation is
an analytical formula for expressing equality.

Subject of equations: how divided.   § 312. The subject of equations is divided
into two parts.   The first, consists in finding
First part: the equation ; that is, in the process of express-
ing the relations existing between the quantities
considered, by means of the algebraic symbols
Statement.   and formula.   This is called the Statement of
Second part: the proposition. ·The second is purely deduc-
tive, and consists, in Algebra, in what is called
Solution.   the solution of the equation, or finding the value
of the unknown quantity ; and in the other
branches of analysis, it consists in the discus-
Discussion of an equation.   sion of the equation ; that is, in the drawing out
from the equation every thing which it is ca-
pable of expressing.

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* Section 100.

§ 313. Making the statement, or finding the Statement: equation, is merely analyzing the problem, and what it is. expressing its elements and their relations in the language of analysis. It is, in truth, collating the facts, noting their bearing and connection, and inferring some general law or principle which leads to the formation of an equation.

The condition of equality between two quantities is expressed by the sign of equality, which is placed between them. The quantity on the left of the sign of equality is called the first member, and that on the right, the second member of the equation. The first member corresponds to the subject of a proposition ; the sign of equality is copula and part of the predicate, signifying, IS EQUAL TO. Hence, an equation is merely a proposition expressed algebraically, in which equality is predicated of one quantity as compared with another. It is the great formula of analysis.

Equality of two quantities:
How expressed.
1st member.
2d member.

Subject.
Predicate.

Proposition.

§ 314. We have seen that every quantity is either abstract or concrete :* hence, an equation, which is a general formula for expressing equality, must be either abstract or concrete.

Abstract.
Concrete.

An abstract equation expresses merely the

* Section 78.

relation of equality between two abstract quan
tities : thus,

Abstract
equation.

$$a + b = x,$$

is an abstract equation, if no unit of value be
assigned to either member ; for, until that be
done the abstract unit one is understood, and the
formula merely expresses that the sum of a and b
is equal to x, and is true, equally, of all quantities.

Concrete
equation.

But if we assign a concrete unit of value, that
is, say that a and b shall each denote so many
pounds weight, or so many feet or yards of
length, x will be of the same denomination, and
the equation will become concrete or denominate.

Five opera-
tions may be
performed.

§ 315. We have seen that there are six oper-
ations which may be performed on an algebraic
quantity.*   We assume, as an axiom, that if
the same operation, under either of these pro-
cesses, be performed on both members of an
equation, the equality of the members will not be
changed.   Hence, we have the five following

Axioms.   A X I O M S.

First.   1. If equal quantities be added to both mem-
bers of an equation, the equality of the members
will not be destroyed.

---

* Section 292.

2. If equal quantities be subtracted from both Second members of an equation, the equality will not be destroyed.

3. If both members of an equation be multi- Third. plied by the same number, the equality will not be destroyed.

4. If both members of an equation be divided Fourth. by the same number, the equality will not be destroyed.

5. If both members of an equation be raised to Fifth. the same power, the equality of the members will not be changed.

6. If the same root of both members of an Sixth. equation be extracted, the equality of the members will not be destroyed.

Every operation performed on an equation Use of
axioms. will fall under one or other of these axioms, and they afford the means of solving all equations which admit of solution.

§ 316. The term Equal, in Algebra, implies that Equal. each of the two quantities of which it is predi- Its meaning
in Algebra. cated, contains the same unit an equal number of times. So in Geometry, two figures are equal when Its meaning
in Geome-
try. they contain the same unit of measure an equal number of times. If in addition to this equality of measure, they are capable of superposition, they are then said to be *equal in all their parts.* Equal in all
parts.

§ 317. We have thus pointed out some of the
marked characteristics of analysis.   In Algebra,
the elementary branch, the quantities, about
which the science is conversant, are divided,
as has been already remarked, into known and
unknown, and the connections between them,
expressed by the equation, afford the means of
tracing out further relations, and of finding the
values of the unknown quantities in terms of the
known.

Classes of
quantities in
Algebra.

In the other branches of analysis, the quanti-
ties considered are divided into two general
classes, Constant and Variable ; the former pre-
serving fixed values throughout the same pro-
cess of investigation, while the latter undergo
changes of value according to fixed laws ; and
from such changes we deduce, by means of the
equation, common principles, and general prop-
erties applicable to all quantities.

How divided
in the other
branches of
Analysis.

§ 318. The correspondence between the pro-
cesses of reasoning, as exhibited in the subject of
general logic, and those which are employed in
mathematical science, is readily accounted for,
when we reflect, that the reasoning process is
essentially the same in all cases ; and that any
change in the language employed, or in the sub-
ject to which the reasoning is applied, does not

Correspond-
ence in
methods of
reasoning
accounted
for.

at all change the nature of the process, or mate-
rially vary its form.

§ 319. We shall not pursue the subject of
algebra any further; for, it would be foreign
to the purposes of the present work to attempt   Objects of
more than to point out the general features and   the present
                                                    work:
characteristics of the different branches of math-
ematical science, to present the subjects about
which the science is conversant, to explain the
peculiarities of the language, the nature of the
reasoning processes employed, and of the con-
necting links of that golden chain which binds   How far
                                                   extended.
together all the parts, forming an harmonious
whole.

SUGGESTIONS FOR THOSE WHO TEACH ALGEBRA.

1. Be careful to explain that the letters em-   Letters are
ployed, are the mere symbols of quantity.   That but mere
                                                    symbols.
of, and in themselves, they have no meaning or
signification whatever, but are used merely as
the signs or representatives of such quantities
as they may be employed to denote.

2. Be careful to explain that the signs which   Signs indi-
                                                   cate opera-
are used are employed merely for the purpose   tions.
of indicating the six operations which may be
performed on quantity; and that they indicate

operations merely, without at all affecting the
nature of the quantities before which they are
placed.

Letters and
signs
elements of
language.

3. Explain that the letters and signs are the
elements of the algebraic language, and that the
language itself arises from the combination of
these elements.

Algebraic
formula:

4. Explain that the finding of an algebraic
formula is but the translation of certain ideas,
first expressed in our common language, into
the language of Algebra ; and that the interpre-

Its interpret-
ation.

tation of an algebraic formula is merely transt
lating its various significations into common
language.

Language.

5. Let the language of Algebra be carefully
studied, so that its construction and significa-
tions may be clearly apprehended.

Coefficient,
Exponent.

6. Let the difference between a coefficient
and an exponent be carefully noted, and the
office of each often explained ; and illustrate fre
quently the signification of the language by at-
tributing numerical values to letters in various
algebraic expressions.

Similar
quantities.

7. Point out often the characteristics of sim-
ilar and dissimilar quantities, and explain which
may be incorporated and which cannot.

Minus sign.

8. Explain the power of the minus sign, as
shown in the four ground rules, but very par-

ticularly as it is illustrated in subtraction, multi-
plication, and division.

9. Point out and illustrate the correspondence
between the four ground rules of Arithmetic   Arithmetic
and Algebra; and impress the fact, that their   and Algebra
                                                 compared.
differences, wherever they appear, arise merely
from differences in notation and language : the
principles which govern the operations being
the same in both.

10. Explain with great minuteness and par-   Equation.
ticularity all the characteristic properties of the   Its proper
                                                       ties.
equation ; the manner of forming it; the differ-
ent kinds of quantity which enter into its com-
position ; its examination or discussion ; and
the different methods of elimination.

11. In the equation of the second degree, be   Equation of
careful to dwell on the four forms which em-   the second
                                                   degree.
brace all the cases, and illustrate by many ex-
amples that every equation of the second de-
gree may be reduced to one or other of them.   Its forms.
Explain very particularly the meaning of the
term root; and then show, why every equation   Its roots.
of the first degree has one, and every equation
of the second degree, two.   Dwell on the prop-
erties of these roots in the equation of the sec-
ond degree.   Show why their sum, in all the   Their sum
forms, is equal to the coefficient of the second
term, taken with a contrary sign ; and why their

Their prod- product is equal to the absolute term with a
uct.   contrary sign.   Explain when and why the roots
are imaginary.

General   12. In fine, remember that every operation
Principles : and rule is based on a principle of science, and
that an intelligible reason may be given for it.
Find that reason, and impress it on the mind
Should be of your pupil in plain and simple language, and
explained. by familiar and appropriate illustrations.   You
will thus impress right habits of investigation
and study, and he will grow in knowledge.   The
broad field of analytical investigation will be
opened to his intellectual vision, and he will
have made the first steps in that sublime science
They lead to which discovers the laws of nature in their most
general laws. secret hiding-places, and follows them, as they
reach out, in omnipotent power, to control the
motions of matter through the entire regions of
occupied space.

CHAPTER V.

DIFFERENTIAL CALCULUS.

§ 320. THE entire science of mathematics is Science of Mathematics. conversant about the properties, relations, and measurement of quantity. Quantity has already been defined. It embraces everything which can be increased, diminished, and measured.

In the elementary branches of mathematics, Discontinuous quantity. quantity is regarded as made up of parts. If the parts are equal, each is called a unit, and the measure of a quantity is the number of times which it contains its unit. Such quantities are called *discontinuous;* because, in passing from one state of value to another, we go by the steps of the unit, and hence, pass over all values lying between adjacent units.

Thus, if we increase a line from one foot to Example in discontinuous quantity. forty feet, by the continued addition of one foot, we touch the line, in our computation, only at its two extremities, and at thirty-nine intermediate points, of which any two adjacent points are one foot apart. In the scale of ascending numbers, 1, 2, 3, 4, 5, 6, etc., we pass over all quantities less

19

than that which is denoted by the unit, one. Discontinuous quantities are generally expressed by numbers, or by letters, which stand for numbers.

Continuous   § 321. In the higher branches of mathematics, the laws which regulate and determine the changes of quantity, from one state of value to another, are quite different. Suppose, for example, that instead of considering a right line to be made up of forty feet, or of 480 inches, or of 960 half inches, or of 1920 quarter inches, or of any number of equal parts of the inch, we regard it as a quantity having its origin at 0, and increasing according to such a law, as to pass through or assume, in succession, all values between 0 and forty feet. This supposition gives us the same distance as before, but a very different law of formation. A quantity so formed or generated, is called a *continuous quantity.* Hence,

and

Discontinu-
ous.

A DISCONTINUOUS QUANTITY is one which is
Dincontinu- made up of parts, and in which the changes, in
ous.
passing from one state of value to another, can be expressed in numbers, either exactly, or approximately ; and

A CONTINUOUS QUANTITY is one which in
Continuous. changing from one state of value to another, according to a fixed law, passes through or

assumes, in succession, all the intermediate values.

Thus, the time which elapses between 12 and 1 o'clock, or between any two given periods, is continuous. All space is continuous, and *every quantity* may be regarded as continuous, which can be subjected to the required law of change.

Space continuous.

LIMITS.

§ 322. The LIMIT of a variable quantity is a quantity towards which it may be made to approach nearer than any given quantity, and which it reaches, under a particular supposition.

Limits.

LIMITS OF DISCONTINUOUS QUANTITY.

§ 323. The limits of a discontinuous quantity are merely numerical boundaries, beyond which the quantity cannot pass.

General

For positive quantities, the minimum limit is 0, and the maximum limit, infinity. For negative quantities, they are 0, and minus infinity; and generally, using the algebraic language, the limits of all quantities are,

Limits.

Minimum limit, — infinity; maximum limit, + infinity.

We can illustrate these limits, and also what

Examples.

we mean by the terms, 0 and infinity, plus or
minus, by reference to the trigonometrical func-
tions. Thus, when the arc is 0, the sine is 0.
When the arc increases to 90°, the sine attains
its maximum value, the radius, R. Passing into
the second quadrant, the sine diminishes as the
arc increases, and when the arc reaches 180°, the
sine becomes 0. From that point, to 270°, the
sine increases numerically, but *decreases algebraic-
ally,* and at 270°, its minimum value is — R.
From 270° to 360°, the sine decreases numeric-
ally, but increases algebraically. Hence, the nu-
merical limits of the sine, are 0 and R; and its
algebraic limits, — R and + R.

Illustra-
tions.

Limits.

Let us now consider the tangent. For the arc
0, the tangent is 0. If the arc be increased from
0 towards 90°, the length of the tangent will
increase and as the arc approaches 90°, the pro-
longed radius or secant becomes more nearly par-
allel with the tangent; and finally, at 90° it
becomes absolutely parallel to it, and the length
of the tangent becomes greater than any *assign-
able* line. Then we say, that the tangent of 90°
is *infinite ;* and we designate that quantity by ∞.
After 90°, the tangent becomes minus, and con-
tinues so to the end of the second quadrant,
where it becomes — 0; and at 270° it becomes
equal to, + ∞. The secant of 90° is also equal to

Tangent.

$+ \infty$; and of 270°, to $- \infty$. These illustrations When it
indicate the significations of the terms, *zero* and
infinity.   They denote the *limits* towards which reaches its
variable quantities may be made to approach
nearer than any given quantity, and which limits   Limits.
are reached under particular suppositions.

§ 324. The term, *given*, or *assignable quantity*,   Quantity.
denotes any quantity of a limited and fixed value.

The term, *infinitely great*, or *infinity*, denotes   Infinite.
a quantity greater than any assignable quantity
of the same kind.

The term, *infinitely small*, or *infinitesimal*, de-   Infinitesi-
notes a quantity less than any assignable quan-   mal.
tity of the same kind.

CONTINUOUS QUANTITIES.

§ 325. A continuous quantity has already been Continuous.
defined (Art. 325). By its definition it has two
attributes:

1st. That it shall change its value according to Quantity.
a fixed law ; and

2d. That in changing its value, between any Attributes.
two limits, it shall pass through all the interme-
diate values.

§ 326. CONSECUTIVE VALUES.—Two values of Consecutive.

a continuous quantity are consecutive when, if the greater be diminished, or the less increased, *according to the law of change,* the two values will become equal.

Let A be the origin of a system of rectangular co-ordinate axes, and C a given point on the axis of X.

If we suppose a point to move from A, in the plane of the axes, and with the further condition, that it shall continue at the same distance from the point C, it will generate the circumference of a circle, $APBDEA$, beginning and terminating at the point A. The moving point is called the *generatrix*.

Quantities generated.
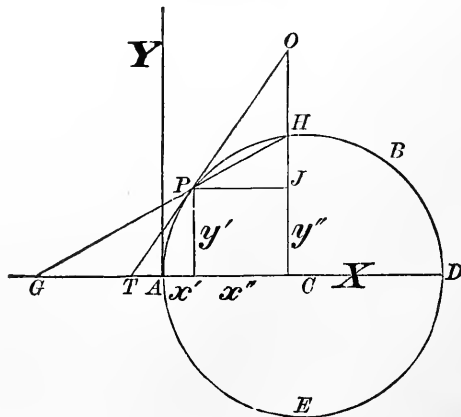
The circumference of this circle may also be generated in another way, thus:

Circle.   Denote the straight line AD by $2R$, and sup-

pose a point to move uniformly from A to D. Denote the distance from A to any point of the line AD, by x: then, the other segment *Tangent* will be denoted by $2R - x$. Now, *at every point* of AD, suppose a perpendicular to be drawn to AD. Denote each perpendicular by y, and *line and* suppose y always to have such a value as to satisfy the equation *Equation.*

$$y^2 = 2Rx - x^2.$$

Under these hypotheses, it is plain that the ex- *Supposition* tremities of the ordinate y will be found in the circumference of the circle, which will be a continuous quantity. The ordinate y will be contained, in the first quadrant, between the *on the* numerical limits of $y = 0$ and $y = +R$; in the second, between the numerical limits of $y = + R$, and $y = 0$; in the third, between $y = -0$ and $y = -R$; and in the fourth, between $y =$ *Equation.* $- R$ and $y = -0$.

The circumference $ABDEA$, may be regarded under two points of view:

First. As a discontinuous quantity, expressed *When con-* *tinuous.* in numbers : viz. by $AD \times 3.1416$; or it may be expressed in degrees, minutes, or seconds, viz. $360°$, or $21600'$, or $1296000''$. In the first case, *1st Case.* the step, or change, in passing from one value to the next, will be the unit of the diameter AD.

2d Case.   In the second, it will be one degree, one minute, or one second. In neither case, will the parts of the circumference less than the unit be reached by the computation. Or,

Secondly.   Secondly: We may regard the circumference as a continuous quantity, beginning and terminating at A. Under this supposition, the gene-

When dis- ratrix will occupy, in succession, every point of
continuous. the circumference, and will, in every position, satisfy the equation

$$y^2 = 2Rx - x^2.$$

Hence, if we measure a quantity by a *finite*
Idfinitesi- unit, that quantity is discontinuous; but if we
mal. measure it by an *infinitesimal* unit, the quantity becomes continuous.

TANGENT LINE AND LIMIT.

§ 327. Take any point of the circumference of
Tangent this circle, as P, whose co-ordinates are x' and y',
limit. and a second point H, whose co-ordinates are x''
Secant line. and y'', and through these points draw the secant line, HPG.

Then, $HJ = y'' - y'$, and $PJ = x'' - x'$; and $\frac{HJ}{PJ} = \frac{y'' - y'}{x'' - x'} = \text{tang.}$ of the angle HPJ, or HGC.

Let us now suppose a tangent line TP to be
drawn to the circle, touching it at P. If we
suppose the point H to approach the point P, it
is plain that the value of y'' will approach to
the value of y', and the value of x'' to that of x';
and when the point H becomes consecutive with
the point P, y'' and y' will become consecutive,
and so also will x'' and x'. When the point H
becomes consecutive with the point P, the secant
line, HG, becomes the tangent line PT. For,
since the arc is a continuous quantity, no point
of it can lie between two of its consecutive
values; and hence, at P, no point of the curve
can lie above the line TP; therefore, by the de-
finitions of Geometry, TP is a tangent line to
the circle at the point P.
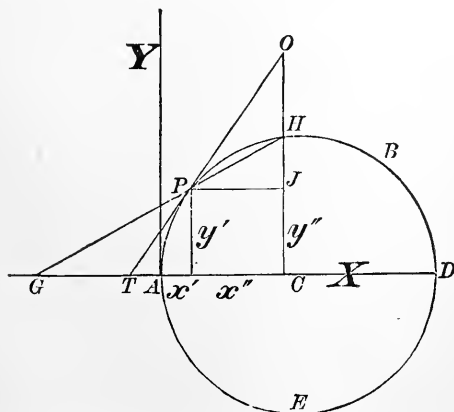
But the definition of a tangent line to a circle,

in elementary Geometry, viz. that it touches the
circumference in one point, is incomplete.   It
is provisional only.   For, as we now see, the
tangent line touches the circle in *two consecu-*
tive points, which, in discontinuous quantity,
are regarded as one, because the distance between
them, expressed numerically, is zero.

If we prolong JH till it meets the tangent line
at O, we see that,

$$\frac{JO}{x'' - x'} - = \text{tangent of } OPJ = \text{tangent of } OTC;$$

and that,

$$\frac{JH}{x'' - x'} = \text{tangent of } HPJ = \text{tangent of } HGC.$$

When the point H approaches the point P
nearer than any given distance, the angle HGC
will approach the angle PTC nearer than any
given angle, and when H becomes consecutive
with P, the angle HGC will become equal to
the angle PTC, which is therefore its *limit.*
Under this hypothesis, the point H falls on the
tangent line, and JH becomes equal to JO.
Under the same hypothesis, y'' and y' become
consecutive, and also x'' and x'; hence, $y'' - y'$
becomes less than any given quantity; and so,
also, does $x'' - x'$. This difference between con-
secutive values is expressed by simply writing
the letter d before the variable. Thus, the dif-

ference of the consecutive values of y is de- Value
denoted by dy; and is read *differential* of y;
and the difference between the consecutive
values of x is denoted by dx, and is read *differ-* of the
ential of x. Hence, we have

$$\frac{dy}{dx} = \text{tangent } PTC; \text{ viz.}$$

the tangent of the angle which the tangent at Angle.
the point P makes the axis of X.

By the definition of a limit, dy becomes the Limit.
limit of $y'' - y'$, and dx the limit of $x'' - x'$, under
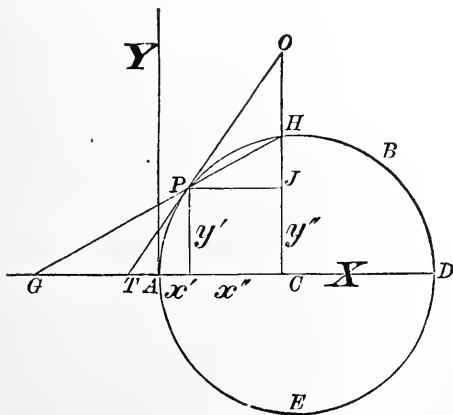
the supposition that y'' and y', and x'' and x' be- When it is
reached.
come, respectively, consecutive. The term *limit*,
therefore, used to designate the ultimate differ-
ence between two values of a variable, denotes
the actual difference between its two consecu- Infinitely
small.
tive values; this difference is infinitely small,

and consecutive with zero. For, if after y'' has become consecutive with y', it be again diminished, according to the law of change expressed by the equation

Equation. $$y^2 = 2Rx - x^2,$$

When the it will, from the definition of consecutive values, become equal to y', and then x'' will become equal x', and we shall have

tangent $$y'' - y' = 0 \qquad \text{and} \qquad x'' - x' = 0.$$

Under this hypothesis the line PT has, at P, becomes a but a *single point*, common with the circumference of the circle ; it then ceases to be a tangent, and becomes any secant line passing through Secant. this point and intersecting the circumference in a second point.

Generally § 328. What we have here shown in regard to true. the circumference of the circle, and its tangent, is equally true of any other curve and its tangent, as may be shown by a very slight modification of the process.

A straight The fact, that a straight line tangent to a line; when curve, has two consecutive points common with a tangent. it, appears in all the elementary problems of tangents. The conditions are, an equality between the co-ordinates of the point of contact

and the first differential co-efficients, at the same point, of the straight line and curve. These conditions fix the consecutive points common to the straight line and curve.

Analysis, therefore, by its searching and microscopic powers—by looking into the changes which take place in quantity, as it passes from one state of value to another, develops properties and laws which lie beyond the reach of the numerical language. Thus, the distance between two consecutive points, on the circumference of a circle, cannot be expressed by numbers; for, however small the number might be, chosen to express such a distance, it could be diminished, and hence, there would be intermediate points.

The introduction, therefore, of continuous quantity, into the science of mathematics, brought with it new ideas and the necessity of a new language. Quantity, made up of parts, and expressed by numbers, is a very different thing from the continuous quantity treated of in the Differential and Integral Calculus. Here, the law of continuity, in the change from one state of value to another, is the governing principle, and carries with it many consequences.

Time and space are the continuous quantities with which we are most conversant. If we take a moment in time, and look back to the past, or

forward to the future, there is no interruption.
The law of continuity is unbroken, and the in-
Examples finite opens to our contemplation. If we take a
point in space, and through it conceive a straight
line to be drawn, the law of continuity is also
there, and the imagination runs along it, to the
of the infinite, in either direction. The attraction of
gravitation is a continuous force; and all the
motions to which it gives rise, follow the law of
law of continuity. All growth and development, in the
vegetable and animal kingdoms, so far as we
know, conform to this law. This, therefore, is
Continuity. the great and important law of quantity, and the
Higher Calculus is conversant mainly about its
development and consequences.

CONSEQUENCES OF THE LAW OF CONTINUITY.

1. The most striking consequence of the law
Conse- of continuity, is the fact, that whatever be the
quences. quantity subjected to this law, or whatever be
the law of change, the difference between any two
First. of the consecutive values is an infinitesimal,
and hence cannot be expressed by numbers.

2. Since a continuous quantity may be of any
value, and be subjected to any law of change, the
Second. infinitesimal which expresses the difference be-
tween any two of its consecutive values, is a *vari-*

able quantity; and hence, may have any value between zero and its maximum limit.

3. The law of continuity in quantity, therefore, introduces into the science of mathematics a class of variables called *infinitesimals*, or *differentials*. Every variable quantity has, at every state of its value, an infinitesimal corresponding to it. This infinitesimal is the connecting link, in the law of continuity, and will vary with the value of the quantity and the law of change.

<div style="text-align: right">Third.</div>

4. In the Infinitesimal Calculus, the properties, relations, and measurement of quantities are developed by considering the laws of change to which they are subjected. The elements of the language employed, are symbols of those infinitesimals.

<div style="text-align: right">Fourth.</div>

NEWTON'S METHOD OF TREATING CONTINUOUS QUANTITY.*

LEMMA I.

§ 329. *Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer, the one to the other, than by any given difference, become ultimately equal.*

<div style="text-align: right">Rations

of

limits.</div>

---

* *Principia,* Book I., Section I.

If you deny it, suppose them to be ultimately unequal, and let D be their ultimate difference. Therefore, they cannot approach nearer to equality than by that given difference D; which is against the supposition.

LEMMA II.

§ 330. *If in any figure, AacE, terminated by*
right lines Aa, AE, and the curve acE, there be inscribed any number of parallelograms Ab, Bc, Cd, etc., comprehended under the equal bases, AB, BC, CD, etc., and the sides Bb, Cc, Dd, etc., parallel

to one side *Aa of the fig-*
ure ; and the parallelograms aKbl, bLcm, cMdn, dDEo are completed ; then, if the breadth of these par-
allelograms be supposed to be diminished, and their number to be augmented

in finitum ; *I say, that the ultimate ratios*
which the inscribed figure AKbLcMdD, the circumscribed figure AalbmcndoE and the curvilinear figure AabcdE will have to one another, are ratios of equality.
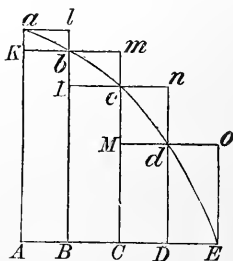
For, the difference of the inscribed and cir-
cumscribed figures is the sum of the parallelo-

grams, Kl, Lm, Mn, Do, that is, (from the equal- of the
ity of their bases), the rectangle under one of
their bases Kb and the sum of their altitudes
Aa; that is, the rectangle $ABla$. But this rect-
angle, because its breadth AB is supposed dim- general
inished *in finitum*, becomes less than any given
space. And therefore, (by Lemma I.) the figures
inscribed and circumscribed, become ultimately Proposition.
equal one to the other; and much more will the
intermediate curvilinear figure be ultimately
equal to either.

Lemma III.

§ 331. *The same ultimate ratios are also ra-* Statement.
tios of equality, when the breadths AB, BC, DC,
etc., of the parallelograms are unequal and are
all diminished in finitum.

For, suppose AF to be the greatest breadth, Demonstra-
and complete the paral-
lelogram $FAaf$. This
parallelogram will be
greater than the differ-
ence of the inscribed and
circumscribed figures;   tion of the
but because its breadth
AF is diminished *in fini-*
tum, it will become less than any given rect-
angle.

20
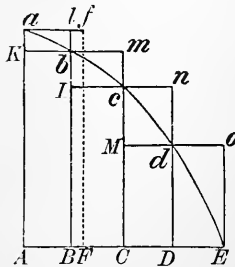
general   COR. 1. Hence, the ultimate sum of these evan-
escent parallelograms will, in all parts, coincide
Proposition. with the curvilinear figure.

                       COR. 2. Much more will the rectilinear figure
Ultimately   comprehended under the chords of the evanes-
equal.   cent arcs, *ab*, *bc*, *cd*, etc., ultimately coincide with
the curvilinear figure.

Also,   COR. 3. And also, the circumscribed rectilinear
figures.   figure comprehended under the tangents of the
same arcs.

Limit of   COR. 4. And therefore, these ultimate figures
areas.   (as to their perimeters, *acE*) are not rectlinear,
but curvilinear limits of rectilinear figures.

LEMMA IV.

Statement   § 332. *If in two figures, AacE, PprT, you in-*
of
Proposition. *scribe (as before) two ranks of parallelograms, an*
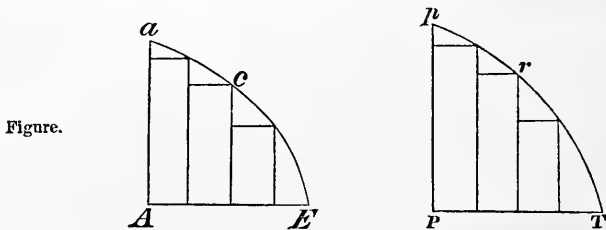
Figure.

*equal   number in each rank, and, where their
breadths are diminished,* in finitum, *the ultimate
ratios of the parallelograms in one figure to those
in the other, each to each respectively, are the*

*same ; I say, that those two figures, AacE, PprT,
are to one another in that same ratio.*

For, as the parallelograms in the one figure are
severally to the parallelograms in the other, so
(by composition) is the sum of all in the one to
the sum of all in the other ; and so is the one
figure to the other ; because (by Lemma III.),
the former figure to the former sum, and the
latter figure to the latter sum, are both in the
ratio of equality.

Cor. Hence, if two quantities of any kind are anyhow divided into an equal number of parts,
and those parts, when their number is aug-
mented, and their magnitude diminished *in fini-
tum*, have a given ratio one to the other, the first to the first, the second to the second, and so on
in order, the whole quantities will be one to the
other in that same given ratio.   For if in the
figures of this lemma, the parallelograms are taken one to the other in the ratio of the parts,
the sum of the parts will always be as the sum of
the parallelograms ; and therefore, supposing the number of the parallelograms and parts to be
augmented, and their magnitudes diminished *in
finitum*, those sums will be in the ultimate ratio
of the parallelogram in the one figure to the cor- responding parallelogram in the other ; that is
(by the supposition), in the ultimate ratio of

any one part of the one quantity to the corre-
sponding part of the other.

LEMMA V.

§ 333. *In similar figures all sorts of homolo-
gous sides, whether curvilinear or rectilinear, are
proportional ; and their areas are in the duplicate
ratio of their homologous sides.*

LEMMA VI.

§ 334. *If the arc ACB, given in position, is
subtended by the chord AB, and in any point A
in the middle of the
continued curvature,
is touched by a right
line AD, produced
both ways; then, if
the points A and B
approach one another
and meet,* [become consecutive] *I say, the angle
BAD contained between the chord and the tan-
gent will be diminished* in finitum, *and ultimately
will vanish.*

Statement

of

general
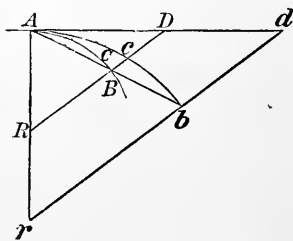
Proposition.

For, if it does not vanish, the arc *ACB*, will
Demonstra- contain, with the tangent *AD,* an angle equal to
tion.   a rectilinear angle ; and therefore, the curvature

at the point A will not be continued, which is against the supposition.

LEMMA VII.

§ 335. *The same thing being supposed, I say* Statement. *that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality.*

For, while the point B approaches towards the Demonstra-
point A, consider always AB and AD as pro- tion.
duced to the remote points b and d, and paral-
lel to the secant BD draw bd: and let the arc
Acb be always similar to the arc ACB. Then,

supposing the points
A and B to coin-
cide, [become con-
secutive], the angle
dAb will vanish, by
the preceding lem-
ma; and therefore,
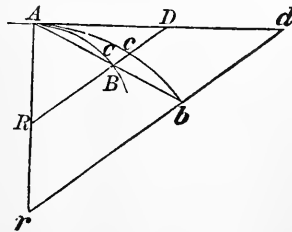
Figure.

the right lines Ab, Ad (which were always
finite), and the intermediate arc Acb, will co-
incide, and become equal among themselves.
Wherefore, the right lines AB, AD, and the Conclusion.
intermediate arc ACB (which are always pro-
portional to the former), will vanish, and ulti-
mately acquire the ratio of equality.

Corollary.   COR. 1. Whence, if through B we draw BF parallel to the tangent, always cutting any right line AF passing through A and F, this line BF will be, ultimately, in the ratio of equality with the evanescent arc ACB; because, completing the paral **Ratio.** lelogram $AFBD$, it is always in the ratio of equality with AD.
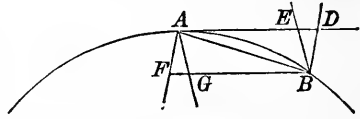
COR. 2. And if through B and A more right **Ultimate** lines be drawn BE, BD, AF, AG, cutting the tangent AD and its parallel BF, **Ratio of** the ultimate ratio of the abscissas AD, AE, BF, BG, and of the arc AB, **Equality.** any one to any other, will be the ratio of equality.

COR. 3. And therefore, in any reasoning about **Ultimate** ultimate ratios, we may freely use any one of **Ratios.** those lines for any other.

*   *   *   *   *   *

§ 336. *Scholium.*—Those things which have **Scholium.** been demonstrated of curve lines, and the superficies which they comprehend, may be easily applied to the curve superficies, and contents of **Use of** **Lemmas** solids. These lemmas are premised to avoid the

tediousness of deducing perplexed demonstrations to avoid the
ad absurdum, according to the method of the
ancient geometers.   For demonstrations are more
contracted by the method of indivisibles: but be-
cause the indivisibles seem somewhat harsh, and   use of
therefore, that method is reckoned less geometri-
cal, I chose rather to reduce the demonstrations
of the following propositions to the first and last
sums and ratios, of nascent and evanescent quan-   Reductio ad
tities; that is, to the limits of those sums and
ratios; and so, to premise, as short as I could,
the demonstration of those limits.   For, hereby   absurdum.
the same thing is performed as by the method
of indivisibles; and now those principles being
demonstrated, we may use them with more safety.
Therefore, if hereafter I should happen to con-   Ultimate
sider quantities as made up of particles, or should
use little curve lines for right ones, I would not
be understood to mean indivisibles, but evanes-   not
cent divisible quantities; not the sums and ra-
tios of determinate parts, but always the limits of
sums and ratios; and that the force of such dem-   Indivisible.
onstrations always depends on the method laid
down in the foregoing lemmas.

Perhaps it may be objected, that there is no   Objections
ultimate proportion of evanescent quantities; be-
cause the proportion, before the quantities have   to
vanished, is not the ultimate, and when they are

Ultimate   vanished, is none. But by the same argument it
may be alleged, that a body arriving at a cer-
tain place, and there stopping, has no ultimate
Quantities   velocity; because, the velocity, before the body
comes to the place, is not its ultimate velocity;
when it has arrived, is none. But the answer is
answered.   easy; for, by the ultimate velocity is meant, that
with which the body is moved, neither *before* it
arrives at its last place and the motion ceases,
nor *after;* but, at the *very instant* it arrives;
that is, that velocity with which the body arrives
at its last place, and with which the motion
Ultimate   ceases. And in like manner, by the ultimate
ratio;   ratio of evanescent quantities is to be under-
stood the ratio of the quantities not before they
vanish, not afterwards, but with which they
vanish. In like manner, the first ratio of nas-
cent quantities is that with which they begin
How it is   to be. And the first or last sum, is that with
which they begin to be (or to be augmented or
diminished). There is a limit which the velocity
at the end of the motion may attain, but not
to be   exceed. This is the ultimate velocity. And there
is the like limit in all quantities and proportions
that begin and cease to be. And since such lim-
its are certain and definite, to determine the same
understood.   is a problem strictly geometrical. But whatever
is geometrical we may be allowed to use in de-

termining and demonstrating any other thing Geometri-cal.
that is likewise geometrical.

* * * * * *

FRUITS OF NEWTON'S THEORY.

§ 337. The main difficulties in the higher ma- Newton's
thematics, have arisen from inadequate or erro- Theory.
neous notions of ultimate or evanescent quanti-
ties, and of the ratios of such quantities. After
two hundred years of discussion, of experiment
and of trial, opinions yet differ widely in regard
to them, and especially in regard to the forms
of language by which they are expressed.

One cannot approach this subject, which has Difficulty
so long engaged the earnest attention of the
greatest minds known to science, without a feel-
ing of awe and distrust. But tapers sometimes of the
light corners which the rays of the sun do not
reach ; and as we must adopt a theory in a sys-
tem of scientific instruction, it is perhaps due subject.
to cthers, that we should assign our reasons
therefor.

§ 338. An ultimate, or evanescent quantity, Ultimate
which is the basis of the Newtonian theory, is Quantity.
not the quantity " *before* it vanishes, nor *after-
wards ;* but, *with which it vanishes.*"

I have sought, in what precedes and follows, to define this quantity—to separate it from all other quantities—to present it to the mind in a crystallized form, and in a language free from all ambiguity; and then to explain how it becomes the key of a sublime science.

As a first step in this process, I have defined continuous quantity (Art. 322), and this is the only class of quantity to which the Differential Calculus is applicable. The next step was to define consecutive values, and then, the difference between any two of them (Art. 327). These differences are the ultimate or evanescent quantities of Newton. They are not quantities of determinate magnitudes, but such as come from variables that have been diminished indefinitely. They form a class of quantities by themselves, which have their own language and their own laws of change; and they are called, *Infinitesimals*, or *Differentials*.

Since the difference between any two values of a variable quantity, which are near together, but not consecutive, will depend on the relative VALUES of the quantities and the *law* of change, it is plain, that when we pass to the limit of this difference, such limit will also depend for its value on the variable quantity and the law of change: and hence, the infinitesimals are un-

Very Important.

First step.

Next step.

Ultimate of Newton.

Infinitesimals.

On what Difference
-
of values
depends.

equal among themselves, and any two of them may have, the one to the other, any ratio whatever.

These infinitesimals will always be quantities of the same *kind* as those from which they were derived ; for the *kind* of quantity which expresses a difference, is the same, whether the difference be great or small.

Quantities

of

same kind.

LIMITS.

§ 339. Marked differences of opinion exist among men of science in regard to the true notion of a *limit ;* and hence, definitions have been given of it, differing widely from each other. We have adopted the views of Newton, so clearly set forth in the lemmas and scholium which we have quoted from the *Principia.* He uses, as stated in the latter part of the scholium, the term limit, to designate the ultimate or evanescent value of a variable quantity ; and this value is reached under a particular hypothesis. Hence, our definition (Art. 323).

Limits.

Difference

of

Definitions.

How defined

by Newton.

Let us now refer again to the case of tangency.

Let APB be any curve whatever, and TPF a tangent touching it at the point P. Draw any chord of the curve, as PB, and through P and B

Case of

tangency

draw the ordinates PD and BH.   Also draw PC
parallel to TH.

Then, $\dfrac{CF}{PC} = $ tang. $FPC = $ tang. the angle

PTH, which the tangent line TPF makes with
the axis TH.

But, $\dfrac{BC}{PC} = $ tangent of the angle BPC.

If now we suppose BH
to move towards PD, the
angle BPC will approach
the angle FPC, which is
its limit.   When BH be-
comes   consecutive   with
PD, BC will reach its ul-
timate value: and since by

Lemma VII., the ultimate ratio of the arc, chord,
and tangent, any one to any other, is the ratio of
equality, it follows that they must then all be
equal, each to each.   Under this hypothesis the
point B must fall on the tangent line TPF; that
is, the chord and tangent, in their ultimate state,
have two points in common; hence they coin-
cide; and as the two points of the arc are con-
secutive, it must also coincide with the chord and
tangent.
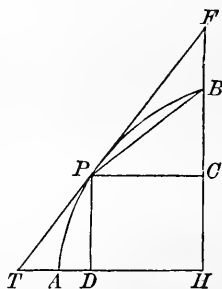
This, at first sight, seems impossible.   But if
it be granted that two points of a curve can be

consecutive and that a straight line can be drawn
through *any* two points, we have the solution.
If we deny that two points of the curve can be
consecutive, we deny the law of continuity.

The method of Leibnitz adopted the simple   Method
hypothesis that when the point *B* approached the
point *P*, infinitely near, the lines *CF* and *CB*
become infinitely small, and that then, either may   of
be taken for the other; under which hypothesis
the ratio of *PC* to *CB*, becomes the ratio of *PC*   Leibnitz.
to *CF*.

WHAT THE LEMMAS OF NEWTON PROVE.

§ 340. The first lemma, which is "the corner-   Lemmas
stone and support of the entire system," predi-
cates *ultimate* equality between any two quanti-
ties which continually approach each other in
value, and under such a law of change, that, in   of
any finite time they shall approach nearer to
each other than by any given difference. The
common quantity towards which the quantities
separately converge, is the limit of each and both
of them, and this limit is always reached under   Newton.
a particular supposition.

Lemmas II., III., and IV. indicate the steps by
which we pass from discontinuous to continuous   What
quantity. They introduce us, fully, to the law

they
prove.

Links in

the law of

Continuity.

Common

limit.

of continuity. They demonstrate the great truth, that the curvilinear space is the common limit of the inscribed and circumscribed parallelograms, and that this limit is reached under the hypothesis that the breadth of each parallelogram is infinitely small, and the number of them, infinitely great. Thus we reach the law of continuity; and each parallelogram becomes a connecting link, in passing from one consecutive value to another, when we regard the curvilinear area as a variable. That there might be no misapprehension in the matter, corollary 1, of Lemma III., affirms, that, "the ultimate sum of these evanescent parallelograms, will, in all parts, coincide with the curvilinear figure." Corollary 4, also, affirms that, "therefore, these ultimate figures (as to their perimeters, acE), are not rectilinear, but curvilinear limits of rectilinear figures:" that is, the curvilinear area AEa is the common limit of the inscribed and circumscribed parallelograms, and the curve $Edcba$, the common limit of their perimeters. This can only take place when the ordinates, like Dd, Cc, Bk, become consecutive; and then, the points o, n, m and l fall on the curve.

What the
law of

The law of continuity carries with it, necessarily, the ideas of the infinitely small and the infinitely great. These are correlative ideas, and

in regard to quantity, one is the reciprocal of the other. The inch of space, as well as the curved line, or the curvilinear surface of geometry, has within it the seminal principles of this law.

continuity implies.

If we regard it as a continuous quantity, having increased from one extremity to the other, without missing any point of space, we have, the law of change, the infinitely small (the difference between two consecutive values, or the link in the law of continuity), and the infinitely great, in the number of those values which make up the entire line.

Continuity.

It has been urged against the demonstrations of the lemmas, that a mere *inspection* of the figures proves the demonstrations to be wrong. For, say the objectors, there will be, always, *obviously*, "a portion of the exterior parallelograms lying without the curvilinear space." This is certainly true for any *finite* number of parallelograms.

Objections.

But the demonstrations are made under the express hypothesis, that, "the breadth of these parallelograms be supposed to be diminished, and their number to be augmented, *in finitum.*" Under this supposition, as we have seen, the points, *o, n, m,* and *l,* fall in the curve, and then the areas named are certainly equal.

Objections

fully

answered.

NEWTON'S METHOD IN HARMONY WITH
THAT OF LEIBNITZ.

§ 341. The method of treating the Infinitesi-

Harmony   mal Calculus, by Leibnitz, subsequently ampli-

of   fied and developed by the Marquis L'Hopital, is

Methods.   based on two fundamental propositions, or de-
mands, which were assumed as axioms.

I. That if an infinitesimal be added to, or sub-

First   tracted from, a finite quantity, the sum or differ-
ence will be the same as the quantity itself. This

demand.   demand assumes that the infinitesimal is so small
that it cannot be expressed by numbers.

II. That a curved line may be considered as

Second.   made up of an infinite number of straight lines,
each one of which is infinitely small.

It is proved in Lemma II. that the sum of the

What the   ultimate rectangles Ab, Bc, Cd, Do, etc., will be
equal to the curvilinear area AaE. This can only
be the case when each is "less than any given

Lemma   space," and their number infinite. What is
meant by the phrase, "becomes less than any
given space"? Certainly, a space too small to be

proves.   expressed by numbers; for, if we have such a
space, so expressed, we can diminish it by dimin-
ishing the number, which would be contrary to

Ultimate   the hypothesis. This ultimate value, then, of
either of the rectangles, is numerically zero: and

hence, its addition to, or subtraction from, any finite quantity, would not change the value. The ultimates of Newton, therefore, conform to the first demand of Leibnitz, as indeed they should do; for, they are not numerical quantities, but connecting links in the law of continuity.

value of

Rectangles.

It is proved in Lemma VII., that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality: hence, their ultimate values are equal. When this takes place, the two extremities of the chord become consecutive, and the remote extremity of the tangent falls on the curve, and coincides with the remote extremity of the chord: that is, F falls on the curve, and PB and PF, coincide with each other, and with the curve. The length of this arc, chord, or tangent, in their ultimate state, is

Ratio

of chord,

tangent,

and

arc, equal.

$$\sqrt{dx^2 + dy^2},$$

Value of
chord.

a value familiar to the most superficial student of the Calculus.

Behold, then, one side of the inscribed polygon, when such side is infinitely small, and the number of them infinitely great.

Coincidence.

That such quantities as we have considered, have a conceivable existence as subjects of thought, and do or may have, *proximatively*, an *actual* existence, is clearly stated in the latter

Quantities

have a real

21

part of the scholium quoted from the *Principia.*
value,   It is there affirmed: "This is the ultimate velo-
city. And there is a *like limit* in all *quantities*
and proportions which *begin* and *cease* to be.
And since such limits are *certain* and *definite,*
ultimately. to determine the same is a problem *strictly geo-
metrical.* But whatever is geometrical we may
be allowed to use in determining and demon-
strating any other thing that is likewise geome-
Newton   trical."* Hence, the theory of Newton conforms
and
Leibnitz. to the second demand in the theory of Leibnitz.

DIFFERENT DEFINITIONS OF A LIMIT.

§ 342. The common impression that mathe-
Different   matics is an exact science, founded on axioms
definitions
of limits. too obvious to be disputed, and carried forward
by a logic too luminous to admit of error, is
certainly erroneous in regard to the Infinitesimal
Calculus. From its very birth, about two hun-
dred years ago, to the present time, there have
Different   been very great differences of opinion among the
views of the
Calculus.   best informed and acutest minds of each gen-
eration, both in regard to its fundamental prin-
ciples and to the forms of logic to be employed
in their development. The conflicting opinions

---

* NOTE. - The italics are added; they are not in the text.

appear, at last, to have arranged themselves into two classes; and these differ, mainly, on this question: What is the correct apprehension and *Differences.* right definition of the word *limit?* All seem to agree that the methods of treating the Calculus must be governed by a right interpretation of this word. The two definitions which involve this conflict of opinion, are these:

1. *The limit of a variable quantity is a quan-*   *Limit.* *tity towards which it may be made to approach nearer than any given quantity and which it reaches under a particular supposition.*

And the following definition, from a work on the Infinitesimal Calculus by M. Duhamel, a *M. Duhamel.* French author of recent date:

2. The limit of a variable is *the constant quan-*   *2d Defini-* *tity which the variable indefinitely approaches,* *tion.* *but never reaches.*

This definition finds its necessary complement in the following definition by the same author:

"We call," says he, "an infinitely *small quan-*   *Comple-* *tity, or simply, an infinitesimal, every variable* *ment.* *magnitude of which the limit is zero."*

The difference between the two definitions is   *Difference* simply this: by the first, the variable, ultimately,   *between* *definitions* reaches its limit; by the second, it approaches   *considered.* the limit, but never reaches it. This apparently slight difference in the definitions, is the divid-

ing line between classes of profound thinkers; and whoever writes a Calculus or attempts to teach the subject, must adopt one or the other of these theories. The first is in harmony with the theories of Leibnitz and Newton, which do not differ from each other in any important particular. It seems also to be in harmony with the great laws of quantity. In discontinous quantity, especially, we certainly include the limits in our thoughts, and in the forms of our language. When we speak of the quadrant of a circle, we include the arc zero and the arc of ninety degrees. Of its functions, the limits of the sine, are zero and radius; zero for the arc zero, and radius for the arc of ninety degrees. For the tangents, the limits are zero and infinity; zero for the arc zero, and infinity for the arc of ninety degrees; and similarly, for all the other functions. For all numbers, the limits are zero and infinity; and for all algebraic quantities, minus infinity and plus infinity.

When we consider continuous quantity, we find the second definition in direct conflict with the first Lemma of Newton, which has been well called, "the corner-stone and foundation of the *Principia*." It is very difficult to comprehend that two quantities may approach each other in value, and in any given time become

Difference.

General limits.

What we mean by them.

For all quantities.

For continuous quantity.

nearer equal than any given quantity, and yet never become equal; not even when the approach can be continued to infinity, and when the law of change imposes no limit to the decrease of their difference. This, certainly, is contrary to the theory of Newton.

Take, for example, the tangent line to a curve, at a given point, and through the point of tangency draw any secant, intersecting the curve, in a second point. If now, the second point be made to approach the point of tangency, both definitions recognize the angle which the tangent line makes with the axis of abscissas as the limit of the angles which the secants make with the same axis, as the second point of secancy approaches the tangent point. By the first defini- tion, the supposition of consecutive points causes the secant line to coincide with, and become the tangent. But by the second definition, the secant line can never become the tangent, though it may approach to it as near as we please. This is in contradiction to all the analytical methods of determining the equations of tangent lines to curves. See corollaries 1, 2, 3, and 4 of Lemma III., in which all the quantities referred to are supposed to reach their limits.

By the second definition, there would seem to be an impassable barrier placed between a vari-

what it   able quantity and its limit. If these two quan-
tities are thus to be forever separated, how can
they be brought under the dominion of a com-
does.   mon law, and enter together into the same equa-
tion? And if they cannot, how can any prop-
erty of the one be used to establish a property of
the other? The mere fact of approach, though
Result.   infinitely near, would not seem to furnish the
necessary conditions.

The difficulty of treating the subject in this
Difficulty. way is strikingly manifested in the supplement-
ary definition of an infinitesimal. It is defined,
simply, as " *every variable magnitude whose limit
is zero.*"

Now, may not zero be a limit of every variable
Not   which has not a special law of change? Is not
definite.   this definition too general to give a DEFINITE
idea of the individual thing defined—an infini-
tesimal? We have no crystallized notions of a
class, till we apprehend, distinctly, the individu-
Should be.   als of the class—their marked characteristics—
their harmonies and their differences; and also,
their laws of relation and connection.

Having given and illustrated these definitions,
M. Duha-   M. Duhamel explains the methods by which we
mel's meth- can pass from the infinitesimals to their limits;
ods not
satisfactory. and when, and under what circumstances, those
limits may be substituted and used for the quan-

tities themselves.   Those methods have not
seemed to me as clear and practical as those of
Newton and Leibnitz.

It is essential to the unity of mathematical   Unity in
science, that all the definitions, should, as far as
possible, harmonize with each other.   In all dis-
continuous quantities, the boundaries are in- Mathematics
cluded, and are the proper limits.   In the hyper-
bola, for example, we say that the asymptote is
the limit of all tangent lines to the curve.   But   absolutely
the asymptote is the tangent, when the point of
contact is at an infinite distance from the ver-
tex: and any tangent will become the asymptote,   necessary.
under that hypothesis.

If s denotes any portion of a plane surface, y Differential.
the ordinate and x the abscissa, we have the
known formula:

$$ds = ydx.$$

If we integrate between the limits of $x = 0$,
and $x = a$, we have, by the language of the   Surface.
Calculus

$$\int_{0}^{a} ds = \int ydx,$$

which is read, "integral of the surface between How read.
limits of $x = 0$, and $x = a$," in which both bound-
aries enter into the result.

Limits
of
Area.

The area, actually obtained, begins where $x = 0$, and terminates where $x = a$, and not at values *infinitely* near those limits.

WHAT QUANTITIES ARE DENOTED BY 0.

§ 343. Our acquaintance with the character 0, What quan- begins in Arithmetic, where it is used as a necessary element of the arithmetical language, and tities are where it is entirely without value, meaning, absolutely nothing. Used in this sense, the largest denoted finite number multiplied by it, gives a product by 0. equal to zero; and the smallest finite number divided by it, gives a quotient of infinity.

When we come to consider variable and continuous quantity, the infinitesimal, or element of change from one consecutive value to another, is be the 0 of not the zero of Arithmetic, though it is smaller than any number which can be expressed in Arithmetic. terms of one, the base of the arithmetical system. New Hence, the necessity of a new language. If the language variable is denoted by x, we express the infinitesimal by dx; if by y, then by dy; and similarly, necessary. for other variables.

Now, the expressions dx and dy, have no exact How it is synonyms in the language of numbers. As compared with the unit 1, neither of them can be expressed by the smallest finite part of it. Hence,

May not

New

language

necessary.

How it is

framed.

when it becomes necessary to express such quantities in the language of number, they can be denoted only by 0. Therefore, this 0, besides its first function in Arithmetic, where it is an element of language, and where the value it denotes is absolutely nothing, is used, also, to denote the numerical values of the infinitesimals. Hence, it is correctly defined as a character which sometimes denotes absolutely nothing, and sometimes an infinitely small quantity. We now see, clearly, what appears obscure in Elementary Algebra, that the quotient of zero divided by zero, may be zero, a finite quantity, or infinity.

INSCRIBED AND CIRCUMSCRIBED POLYGONS UNITE ON THE CIRCLE.

§ 344. The theory of limits, developed by Newton, is not only the foundation of the higher mathematics, but indicates the methods of using the Infinitesimal Calculus in the elementary branches. This Calculus being unknown to the ancients, their Geometry was encumbered by the tedious methods of the *reductio ad absurdum*. Newton says in the scholium : " These lemmas are premised to avoid the tediousness of deducing perplexed demonstrations *ad absurdum*, according to the method of the ancient geometers."

Lemma I., which is the " corner-stone and
Lemma I. foundation of the *Principia,*" is also the golden
link which connects geometry with the higher
mathematics.

It is demonstrated in Euclid's Elements, and
What is also in Davies' *Legendre,* Book V., Proposition
X., that " *Two regular polygons of the same num-
ber of sides can be constructed, the one circum-*
demon- *scribed about the circle and the other inscribed
within it, which shall differ from each other by*
strated. *less than any given surface.*" ˙

The nfoment it is proved that the exterior
How the and interior polygons may be made to differ
Proof is
made. from each other by less than any given surface,
Lemma I. steps in and affirms an ultimate equal-
ity between them. And when does that ultimate
equality take place, and when and where do they
What become coincident ? Newton, in substance af-
Newton
affirms, firms, in his lemmas, " on their common limit,
the circle," and under the same hypothesis as
causes the inscribed and circumscribed parallelo-
grams to become equal to their common limits,
the curvilinear area. If Lemma I. is true, the
perimeters of the two polygons will ultimately
coincide on the circumference of the circle, and
A side of the become equal to it. But what then is the side of
polygon. each polygon ? We answer, the distance between
two consecutive points of the circumference of

the circle. And what is that value? We answer, the $\sqrt{dx^2 + dy^2}$.

But it is objected, that this introduces us to the infinitely small. True, it does; but we cannot reach a continuous quantity without it. The sides of the polygons, *so long as their number* *is finite*, will be straight lines, each diminishing in value as their number is increased. While this is so, the perimeter of each will be a discontinuous quantity, made up of the equal sides, each having a finite value, and each being the unit of change, as we go around the perimeter. As each of these sides is diminished in value, and their number increased, the discontinuous quantity approaches the law of continuity, which it reaches, under the hypothesis, that each side becomes infinitely small and their number infinitely great. Behold the polygons embracing each other on their common limit, the circle, and the perimeter of each coinciding with the circumference. Thus, the principles of the Infinitesimal Calculus take their appropriate place in Elementary Geometry, to the exclusion of the cumbrous methods of the *reductio ad absurdum* of the ancients, and the whole science of Mathematics is brought into closer harmonies and nearer relations.

DIFFERENTIAL AND INTEGRAL CALCULUS.

§ 345. We have seen that the Differential and Integral Calculus is conversant about continuous quantity. We have also seen, that such quantities are developed by considering their laws of change. We have further seen, that these laws of change are traced by means of the differences of consecutive values, taken two and two, as the variables pass from one state of value to another. Indeed, those differences are but the foot-steps of these laws.

<div style="margin-left:0">Differential and Integral Calculus defined.</div>

LANGUAGE OF THE CALCULUS.

§ 346. We are now to explain the language by which the quantities are represented, by which their changes are indicated, and by which their laws of change are traced. The constant quantities which enter into the Calculus are represented by the first letters of the alphabet, a, b, c, etc., and the variables, by the final letters, x, y, z, etc.

<div style="margin-left:0">Language of the Calculus.</div>

When two variable quantities, y and x, are connected in an equation, either of them may be supposed to increase or decrease *uniformly;* such variable is called the *independent variable,* because the *law of change is arbitrary,* and *in-*

<div style="margin-left:0">Variable quantities.</div>

dependent of the form of the equation. This variable is generally denoted by x, and called simply, the *variable*. Under this hypothesis, the change in the variable y will depend on the *form* of the equation : hence, y is called the *dependent* variable, or *function*. When such relations exist between y and x, they are expressed by an equation of the form

$$y = F(x), \quad y = f(x), \text{ or,} \quad f(x, y) = 0,$$

which is read, y a function of x. The letter F, or f, is a mere symbol, and stands for the word *function*. If y is a function of x, that is, changes with it, x may, if we please, be regarded as a function of y; hence,

One quantity is a function of another, when the two are so connected that any change of value, in either, produces a corresponding change in the other.

It has been already stated (Art. 328), that the difference between two consecutive values of a variable quantity, is indicated by simply writing the letter d as a symbol, before the letter denoting that variable ; so that dx denotes the difference between two consecutive values of the variable quantity denoted by x, and dy the difference between the *corresponding* consecutive values of

[margin notes:] Function of each other.

How they

may be

expressed.

Function.

Difference of states.

Form of language. the variable quantity denoted by y. These are mere forms of language, expressing laws of change.

Standard of Measure. How are the changes in these variable quantities, expressed by the infinitesimals, to be measured ? Only by taking one of them as a standard —and finding how many times it is contained in the other.

Independent variable.

Change uniform. The independent variable is always supposed to *increase uniformly ;* hence, the difference between any two of its consecutive values, taken at pleasure, is the same : therefore, this difference, which does not vary in the same equation, or under the same law of change, affords a convenient standard, or unit of measure, and in the Calculus, is always used as such.

Correspond-

ing change

in the

function,

not uniform. The change in the function y, denoted by dy, is always compared with the *corresponding* change of the independent variable, denoted by dx, as a standard, or unit of measure. But the change in any quantity, divided by the unit of measure, gives the *rate* of change : hence, $\dfrac{dy}{dx}$ is the rate of change of the function y. This rate of change is called the *differential coefficient* of y regarded as a function of x, and performs a very important part in the Calculus. The quantities dy and dx, being both infinitesimals, are

of the same species : hence, their quotient is an *abstract number.* Therefore, the differential co-efficient is a connecting link between the infinitesimals and numbers.

If any quantity whatever be divided by its unit of measure, the quotient will be an abstract number; and if this quotient be multiplied by the unit of measure, the product will be the concrete quantity itself. Hence, if we multiply $\frac{dy}{dx}$, by the unit of measure dx, we have $\frac{dy}{dx}\ dx$, which always denotes the difference between two consecutive values of y; and therefore, is the differential of y. Hence, *the differential of a variable function is equal to the differential coefficient multiplied by the differential of the independent variable.*

Quotient by unit of measure.

Differential of a quantity.

The method, therefore, of dealing with infinitesimals, is precisely the same as that employed for discontinuous quantities.

Same method.

We assume a unit of measure which is as arbitrary as *one,* in numbers, or, as the foot, yard, or rod, in linear measure, and then we compare all other infinitesimals with this standard. We thus obtain a ratio which is an abstract number, and if this be multiplied by the unit of measure, we go back to the concrete quantity from which the ratio was derived.

Unit of measure.

We have thus sketched an outline of the Infinitesimal Calculus. We have named the quantities about which it is conversant, the laws which govern their changes of value, and the language by which these laws are expressed. We have found here, as in the other branches of mathematics, that an arbitrary quantity, assumed as a unit of measure, is the base of the entire system; and that the system itself is made up of the various processes employed in finding the ratio of this standard, to the quantities which it measures.

Sketch of the Infinitesimal Calculus.

APPENDIX.

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§ 347. A COURSE of mathematics should present the outlines of the science, so arranged, explained, and illustrated as to indicate all those general methods of application, which render it effective and useful. This can best be done by a series of works embracing all the topics, and in which each topic is separately treated.

A course of Mathematics.

§ 348. Such a series should be formed in accordance with a fixed plan; should adopt and use the same terms in all the branches; should be written throughout in the same style; and present that entire unity which belongs to the subject itself.

How it should be. formed.

Unity of the subject.

§ 349. The reasonings of mathematics and the processes of investigation, are the same in

Reasonings

22

the same in every branch, and have to be learned but once,
all branches. if the same system be studied throughout.   The
Different   different kinds of notation, though somewhat un-
kinds of no-
tation.   like in the different subjects of the science, are,
in fact, but dialects of a common language.

Language   § 350. If, then, the language is, or may be
need be
learned but made essentially the same in all the branches of
once.   mathematical science ; and if there is, as has
been fully shown, no difference in the processes
In what   of reasoning, wherein lies that difficulty in the
consists the acquisition of mathematical knowledge which is
difficulty ?
often experienced by students, and whence the
origin of that opinion that the subject itself is
dry and difficult ?

A   § 351. Just in proportion as a branch of know-
general law,
if known, ledge is compactly united by a common law, is
renders a
subject easy. the facility of acquiring that knowledge, if we
observe the law, and the difficulty of acquiring
Faculties   it, if we pay no attention to the law.   The study
required in
mathematics. of mathematics demands, at every step, close
attention, nice discrimination, and certain judg-
ment.   These faculties can only be developed
How first   by culture.   They must, like other faculties, pass
cultivated :
through the states of infancy, growth, and ma-
turity.   They must be first exercised on sensible
and simple objects ;   then on elementary ab-

stract ideas ; and finally, on generalizations and
the higher combinations of thought in the pure
ideal.

§ 352. Have educators fully realized that the
first lessons in numbers impress the first elements
of mathematical science ? that the first con-
nections of thought which are there formed be-
come the first threads of that intellectual warp
which gives tone and strength to the mind ?
Have they yet realized that every process is, or
should be, like the stone of an arch, formed to
fill, in the entire structure, the exact place for
which it is designed ? and that the unity, beauty,
and strength of the whole depend on the adapta-
tion of the parts to each other ?   Have they
sufficiently reflected on the confusion which must
arise from attempting to put together and har-
monize different parts of discordant systems ?
to blend portions that are fragmentary, and to
unite into a placid and tranquil stream trains of
thought which have not a common source ?

§ 353. Some have supposed that Arithmetic
may be well taught and learned without the aid
of a text-book ; or, if studied from a book, that
the teacher may advantageously substitute his
own methods for those of the author, inasmuch

to be fol- as such substitution is calculated to widen the
lowed. field of investigation, and excite the mind of the
pupil to new inquiries.

Reasons. Admitting that every teacher of reasonable
intelligence, will discover methods of communi-
cating instruction better adapted to the peculiar-
ities of his own mind, than all the methods em-
Even a bet- ployed by the author he may use ; will it be safe,
ter method,
when substi- as a general rule, to *substitute* extemporaneous
tuted, may
not harmo- methods for those which have been subjected
nize with the
other parts to the analysis of science and the tests of expe-
of the work. rience ? Is it safe to substitute the results of
conjectural judgments for known laws? But if
they are as good, or better even, as isolated pro-
cesses, will they answer as well, in their new
places and connections, as the parts rejected ?
Illustration. Will the balance-wheel of a chronometer give
as steady a motion to a common watch as the
more simple and less perfect contrivance to
which all the other parts are adapted ?

§ 354. If these questions have significance, we
One of the have found at least one of the causes that have
reasons why
mathematics impeded the advancement of mathematical sci-
is difficult. ence, viz. the attempt to unite in the same course
of instruction fragments of different systems ;
thus presenting to the mind of the learner the
same terms differently defined, and the same

principles differently explained, illustrated, and applied. It is mutual relation and *connection* which bring sets of facts under general laws ; it is mutual relation and connection of ideas which form a process of science ; it is the mutual connection and relation of such processes which constitute science itself.

Connection very important.

§ 355. I would by no means be understood as expressing the opinion that a student or teacher of mathematics should limit his researches to a single author ; for, he must necessarily read and study many. I speak of the pupil alone, who *must be taught one method at a time, and taught that well, before he is able to compare different methods with each other.*

A teacher should read many books, and teach one system.

ORDER OF THE SUBJECTS—ARITHMETIC.

§ 356. Arithmetic is the most useful and simple branch of mathematical science, and is the first to be taught. If, however, the pupil has time for a full course, I would by no means recommend him to finish his Arithmetic before studying a portion of Algebra.

Arithmetic

Connection with Algebra.

ALGEBRA.

Algebra: § 357. Algebra is but a universal Arithmetic,
with a more comprehensive notation. Its ele-
ments are acquired more readily than the higher
and hidden properties of numbers ; and indeed
the elements of any branch of mathematics are
more simple than the higher principles of the
How   preceding subject ; so that all the subjects can
it should be
studied: best be studied in connection with those which
preceue and follow.

Should   § 358. Algebra, in a regular course of instruc-
precede
Geometry: tion, should precede Geometry, because the ele-
mentary processes do not require, in so high a
Why.   degree, the exercise of the faculties of abstrac-
tion and generalization. But when we have
When   completed the equation of the second degree,
Geometry
should be   the processes become more difficult, the abstrac-
commenced. tions more perfect, and the generalizations more
extended. Here then I would pause and com-
mence Geometry.

GEOMETRY.

Geometry.   § 359. Geometry, as one of the subjects of
mathematical science, has been fully considered
in Book II.   It is referred to here merely to mark
its place in a regular course of instruction.

TRIGONOMETRY—PLANE AND SPHERICAL.

§ 360. The next subject in order, after Geom- Trigonome-
try:
etry, is Trigonometry : a mere application of the
principles of Arithmetic, Algebra, and Geometry What it is.
to the determination of the sides and angles of
triangles. As triangles are of two kinds, viz.
those formed by straight lines and those formed
by the arcs of great circles on the surface of a
sphere ; so Trigonometry is divided into two Two kinds.
parts : Plane and Spherical. Plane Trigonom-
etry explains the methods, and lays down the Plane.
necessary rules for finding the remaining sides
and angles of a plane triangle, when a sufficient
number are known or given. Spherical Trigo- Spherical.
nometry explains like processes, and lays down
similar rules for spherical triangles.

SURVEYING AND LEVELLING.

§ 361. The application of the principles of
Trigonometry to the measurement of portions
of the earth's surface, is called Surveying ; and Surveying.
similar applications of the same principles to the
determination of the difference between the dis-
tances of any two points from the centre of the
earth, is called Levelling. These subjects, which Levelling.
follow Trigonometry, not only embrace the va-

rious methods of calculation, but also a descrip-
tion of the necessary Instruments and Tables.
They should be studied immediately after Trigo-
nometry; of which, indeed, they are but appli-
cations.

DESCRIPTIVE GEOMETRY.

Descriptive
Geometry: § 362. Descriptive Geometry is that branch
of mathematics which considers the positions of
the geometrical magnitudes, as they may exist in
space, and determines these positions by refer-
ring the magnitudes to two planes called the
Planes of Projection.

Its nature. It is, indeed, but a development of those gen-
eral methods, by which lines, surfaces, and vol-
umes may be presented to the mind by means
of drawings made upon paper. The processes of
What its this development require the constant exercise of
study accom-
plishes. the conceptive faculty. All geometrical mag-
nitudes may be referred to two planes of pro-
jection, and their representations on these planes
will express to the mind, their forms, extent, and
also their positions or places in space. From
How. these representations, the mind perceives, as it
were, at a single view, the magnitudes them-
selves, as they exist in space ; traces their boun-
daries, measures their extent, and sees all their
parts separately and in their connection.

In France, Descriptive Geometry is an impor- How regarded in France. tant element of education. It is taught in most of the public schools, and is regarded as indispensable to the architect and engineer. It is, indeed, the only means of so reducing to paper, and presenting at a single view, all the complicated parts of a structure, that the drawing or representation of it can be read at a glance, and all the parts be at once referred to their appropriate places. It is to the engineer or architect not Its value as a practical branch. only a general language by which he can record and express to others all his conceptions, but is also the most powerful means of extending those conceptions, and subjecting them to the laws of exact science.

SHADES, SHADOWS, AND PERSPECTIVE.

§ 363. The application of Descriptive Geometry to the determination of shades and shadows, Shades, Shadows, and Perspective. as they are found to exist on the surfaces of bodies, is one of the most striking and useful applications of science ; and when it is further extended to the subject of Perspective, we have all that is necessary to the exact representation of objects as they appear in nature. An accurate perspective and the right distribution of light and shade are the basis of every work of

Their use.   the fine arts.   Without them, the sculptor and
the painter would labor in vain : the chisel of
Canova would give no life to the marble, nor the
touches of Raphael to the canvas.

ANALYTICAL GEOMETRY.

Analytical   § 364. Analytical Geometry is the next sub-
Geometry.   ject in a regular course of mathematical study,
though it may be studied before Descriptive Ge-
ometry.   The importance of this subject cannot
Its   be exaggerated.   In Algebra, the symbols of
Importance :   quantity have generally so close a connection
with numbers, that the mind scarcely realizes
Valuable as   the extent of the generalization ; and the power
a study.   of analysis, arising from the changes that may
take place among the quantities which the sym-
bols represent, cannot be fully explained and de-
veloped.

But in Analytical Geometry, where all the
Reasons.   magnitudes are brought under the power of anal-
ysis, and all their properties developed by the
combined processes of Algebra and Geometry, we
are brought to feel the extent and potency of
Generaliza-   those methods which combine in a single equa-
tion.   tion every discovered and undiscovered property
of every line, straight or curved, which can be
formed by the intersection of a cone and plane.

To develop every property of the Conic Sec- Its extent.
tions from a single equation, and that an equa-
tion only of the second degree, by the known
processes of Algebra, and thus interpret the re-
sults, is a far different exercise of the mind from
that which arises from searching them out by
the tedious and disconnected methods of separate
propositions.   The first traces all from an inex- Its methods
haustible fountain, by the known laws of analyti- of operation
cal investigation, applicable to all similar cases,
while the latter adopts particular processes ap-
plicable to special cases only, without any gen-
eral law of connection.

DIFFERENTIAL AND INTEGRAL CALCULUS.

§ 365. The Differential and Integral Calculus Differential
presents a new view of the power, extent, and and
Integral
applications of mathematical science.   It should Calculus.
be carefully studied by all who seek to make What per-
high attainments in mathematical knowledge, or sons should
study it.
who desire to read the best works on Natural
and Experimental Philosophy.   It is that field of
mathematical investigation, where genius may
exert its highest powers and find its most certain
rewards.   It reaches, with a microscopic certain-
ty the most hidden laws of quantity, and brings
them within the range of Mathematical Analysis.

Continuous quantity.   Continuous Quantity, under all its forms, and with all its infinite laws of change, can be examined and analyzed only by the Calculus.

Language of discontinuous quantity inapplicable.   The language constructed for the development of the laws and properties of quantities composed of ascertained and definite parts, is inapplicable to quantity changing according to the law of continuity. Here, the changes can only be expressed by infinitesimals, which are mere links in the law of change, and which form no appreciable part of the quantity itself. We are thus introduced to a new form of Mathematical Science. It is this science which deals with

What the language deals with.   Time, and Space, and Force, and Motion, and Velocity, and indeed, with all Continuous Quantity. The elements of this science are infinitesimal; but the science itself reaches through all time and all space, revealing the mysteries and the omnipotence of universal law.

BOOK III.

UTILITY OF MATHEMATICS.

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CHAPTER I.

THE UTILITY OF MATHEMATICS CONSIDERED AS A MEANS OF INTELLECTUAL
TRAINING AND CULTURE.

§ 366. THE first efforts in mathematical sci- First efforts.
ence are made by the child in the process of
counting. He counts his fingers, and repeats
the words one, two, three, four, five, six, seven, Counting of
sensible ob-
eight, nine, ten, until he associates with these jects.
words the ideas of one or more, and thus ac-
quires his first notions of number. Hence, the
idea of number is first presented to the mind by
means of sensible objects; but when once clear-
ly apprehended, the perception of the sensible
objects fades away, and the mind retains only
the abstract idea. Thus, the child, after count- Generaliza-
tion.
ing for a time with the aid of his fingers or his
marbles, dispenses with these cumbrous helps, and

Abstraction. employs only the abstract ideas, which his mind embraces with clearness and uses with facility.

Analytical method: § 367. In the first stages of the analytical methods, where the quantities considered are Uses sensible objects at first. represented by the letters of the alphabet, sensible objects again lend their aid to enable the mind to gain exact and distinct ideas of the things considered; but no sooner are these ideas obtained than the mind loses sight of the things themselves, and operates entirely through the instrumentality of symbols.

Geometry. § 368. So, also, in Geometry. The right line may first be presented to the mind, as a black First impressions by sensible objects. mark on paper, or a chalk mark on a blackboard, to impress the geometrical definition, that "A straight line does not change its direction between any two of its points." When this definition is clearly apprehended, the mind needs no further aid from the eye, for the image is forever imprinted.

A plane. § 369. The idea of a plane surface may be Definition: impressed by exhibiting the surface of a polished mirror; and thus the mind may be aided in How illustrated. apprehending the definition, that "a plane surface is one in which, if any two points be taken

the straight line which joins them will lie wholly in the surface." But when the definition is understood, the mind requires no sensible object to aid its conception. The ideal alone fills the mind, and the image lives there without any connection with sensible objects.

Its true conception.

§ 370. Space is indefinite extension, in which all bodies are situated. A volume is any limited portion of space embracing the three dimensions of length, breadth, and thickness. To give to the mind the true conception of a volume, the aid of the eye may at first be necessary; but the idea being once impressed, that a volume, in a strictly mathematical sense, means only a portion of space, and has no reference to the matter with which the space may be filled, the mind turns away from the material object, and dwells alone on the ideal.

Space.
Volume:

How conceived.

What it really is.

§ 371. Although quantity, in its general sense, is the subject of mathematical inquiry, yet the language of mathematics is so constructed, that the investigations are pursued without the slightest reference to quantity as a material substance. We have seen that a system of symbols, by which quantities may be represented, has been adopted, forming a language for the expression

Quantity:

Language:
How constructed.

Symbols:

of ideas entirely disconnected from material ob-
jects, and yet capable of expressing and repre-
senting such objects.   This symbolical language,
at once copious and exact, not only enables us
to express our known thoughts, in every depart-
ment of mathematical science, but is a potent
means of pushing our inquiries into unexplored
regions, and conducting the mind with certainty
to new and valuable truths.

Nature of the language:

What it accomplishes.

§ 372. The nature of that culture, which the
mind undergoes by being trained in the use of
an exact language, in which the connection be-
tween the *sign* and the *thing signified* is unmis-
takable, has been well set forth by a living
author, greatly distinguished for his scientific
attainments.*   Of the pure sciences, he says

Advantages of an exact language.

"Their objects are so definite, and our no-
tions of them so distinct, that we can reason
about them with an assurance that the words and
signs of our reasonings are full and true repre-
sentatives of the things signified ; and, conse-
quently, that when we use language or signs in
argument, we neither by their use introduce
extraneous notions, nor exclude any part of the
case before us from consideration.   For exam-

Herschel's views.

Exact language prevents error.

* Sir John Herschel, Discourse on the study of Natural
Philosophy.

ple : the words space, square, circle, a hundred, Mathematical terms exact
&c., convey to the mind notions so complete
in themselves, and so distinct from every thing
else, that we are sure when we use them we
know and have in view the whole of our own
meaning.   It is widely different with words ex- Different in regard to other terms
pressing natural objects and mixed relations.
Take, for instance, *Iron.*   Different persons at-
tach very different ideas to this word.   One who
has never heard of magnetism has a widely dif-
ferent notion of iron from one in the contrary
predicament.   The vulgar who regard this metal How iron is regarded by the chemist
as incombustible, and the chemist, who sees it
burn with the utmost fury, and who has other
reasons for regarding it as one of the most com-
bustible bodies in nature ; the poet, who uses The poet
it as an emblem of rigidity ; and the smith and
engineer, in whose hands it is plastic, and mould-
ed like wax into every form ; the jailer, who prizes The jailer:
it as an obstruction, and the electrician, who The electri-cian.
sees in it only a channel of open communication
by which that most impassable of obstacles, the
air, may be traversed by his imprisoned fluid,—
have all different, and all imperfect notions of
the same word.   The meaning of such a term Final illus-tration.
is like the rainbow—everybody sees a different
one, and all maintain it to be the same."

"It is, in fact, in this double or incomplete

Incomplete
meaning the
source of
error.

sense of words, that we must look for the origin of a very large portion of the errors into which we fall. Now, the study of the abstract sciences,

Mathematics
free from
such errors.

such as Arithmetic, Geometry, Algebra, &c., while they afford scope for the exercise of reasoning about objects that are, or, at least, may be conceived to be, external to us; yet, being free from these sources of error and mistake,

Requires a
strict use of
language.

accustom us to the strict use of language as an instrument of reason, and by familiarizing us in our progress towards truth, to walk uprightly and straightforward, on firm ground, give us that proper and dignified carriage of mind which

Results.

could never be acquired by having always to pick our steps among obstructions and loose fragments, or to steady them in the reeling tempests of conflicting meanings."

Two ways of
acquiring
knowledge.

§ 373. Mr. Locke lays down two ways of increasing our knowledge :

1st. " Clear and distinct ideas with settled names ; and,

2d. " The finding of those which show their agreement or disagreement ;" that is, the searching out of new ideas which result from the combination of those that are known.

First.

In regard to the first of these ways, Mr. Locke says : " The first is to get and settle in our minds

determined ideas of those things, whereof we Ideas of things must be distinct. have general or specific names ; at least, of so many of them as we would consider and improve our knowledge in, or reason about." * * * "For, it being evident, that our knowledge cannot exceed our ideas, as far as they are either imperfect, confused, or obscure, we cannot expect to have certain, perfect, or clear knowledge." Reason.

§ 374. Now, the ideas which make up our knowledge of mathematical science, fulfil exactly these requirements.   They are all impressed on the mind by a fixed, definite, and certain language, and the mind embraces them as so many images or pictures, clear and distinct in their outlines, with names which suggest at once their characteristics and properties. Why it is so in mathematics.

§ 375. In the second method of increasing our knowledge, pointed out by Mr. Locke, mathematical science offers the most ample and the surest means.   The reasonings are all based on self-evident truths, and are conducted by means of the most striking relations between the known and the unknown.   The things reasoned about, and the methods of reasoning, are so clearly apprehended, that the mind never hesitates or doubts.   It comprehends, or it does not compre- Second. Why mathematics offer the surest means.

hend, and the line which separates the known from the unknown, is always well defined. These characteristics give to this system of reasoning a superiority over every other, arising, not from any difference in the logic, but from a difference in the things to which the logic is applied. Observation may deceive, experiment may fail, and experience prove treacherous, but demonstration never.

Characteristics of the reasoning.

Its advantages.

Demonstration certain.

"If it be true, then, that mathematics include a perfect system of reasoning, whose premises are self-evident, and whose conclusions are irresistible, can there be any branch of science or knowledge better adapted to the improvement of the understanding? It is in this capacity, as a strong and natural adjunct and instrument of reason, that this science becomes the fit subject of education with all conditions of society, whatever may be their ultimate pursuits. Most sciences, as, indeed, most branches of knowledge, address themselves to some particular taste, or subsequent avocation; but this, while it is before all, as a useful attainment, especially adapts itself to the cultivation and improvement of the thinking faculty, and is alike necessary to all who would be governed by reason, or live for usefulness."*

Mathematics includes a certain system.

An adjunct and instrument of reason;

and necessary to all.

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* Mansfield's Discourse on the Mathematics.

§ 376. The following, among other consider- Considera
tions of the
value of
mathematics ations, may serve to point out and illustrate the value of mathematical studies, as a means of mental improvement and development.

1. We readily conceive and clearly appre- First.
They give
clear concep
tions of
things. hend the things of which the science treats; they being things simple in themselves and readily presented to the mind by plain and familiar language. For example: the idea of number, of one or more, is among the first ideas implanted Example. in the mind; and the child who counts his fingers or his marbles, understands the art of numbering them as perfectly as he can know any thing. So, likewise, when he learns the definition They estab
lish clear
relations be
tween defini
tions and
things. of a straight line, of a triangle, of a square, of a circle, or of a parallelogram, he conceives the idea of each perfectly, and the name and the image are inseparably connected. These ideas, so distinct and satisfactory, are expressed in the simplest and fewest terms, and may, if necessary, be illustrated by the aid of sensible objects.

2. The words employed in the definitions Second.
Words are
always used
in the same
sense. are always used in the same sense—each expressing at all times the same idea; so that when a definition is apprehended, the conception of the thing, whose name is defined, is perfect in the mind.

There is, therefore, no doubt or ambiguity

either in the language, or in regard to what is affirmed or denied of the things spoken of; but all is certainty, both in the language employed and in the ideas which it expresses.

3. The science of mathematics employs no definition which may not be clearly comprehended—lays down no axioms not universally true, and to which the mind, by the very laws of its nature, readily assents; and because, also, in the process of the reasoning, no principle or truth is taken for granted, but every link in the chain of the argument is immediately connected with a definition or axiom, or with some principle previously established.

4. The order established in presenting the subject to the mind, aids the memory at the same time that it strengthens and improves the reasoning powers. For example: first, there are the definitions of the names of the things which are the subjects of the reasoning; then the axioms, or self-evident truths, which, together with the definitions, form the basis of the science. From these the simplest propositions are deduced, and then follow others of greater difficulty; the whole connected together by rigorous logic—each part receiving strength and light from all the others. Whence, it follows, that any proposition may be traced to first prin-

ciples ; its dependence upon and connection *Propositions traced to their sources* with those principles made obvious ; and its truth established by certain and infallible argument.

5. The demonstrative argument of mathe- *Fifth. Argument the most certain.* matics produces the most certain knowledge of which the mind is susceptible. It establishes truth so clearly, that none can doubt or deny. For, if the premises are certain—that is, *Reasons.* such that all minds admit their truth without hesitation or doubt, and if the method of drawing the conclusions be lawful—that is, in accordance with the infallible rules of logic, the inferences must also be true. Truths thus established may be relied on for their verity ; and the knowl- *Such knowledge science.* edge thus gained may well be denominated SCIENCE.

§ 377. There are, as we have seen, in mathe- *Two systems:* matics, two systems of investigation quite different from each other: the Synthetical and the *Synthesis, Analysis.* Analytical ; the synthetical beginning with the definitions and axioms, and terminating in the highest truth reached by Geometry.

" This science presents the very method by *Synthetical.* which the human mind, in its progress from childhood to age, develops its faculties. What first meets the observation of a child ? Upon *First notions* what are his earliest investigations employed ?

What is first observed. Next to color, which exists only to the sight, figure, extension, dimension, are the first objects which he meets, and the first which he examines. He ascertains and acknowledges their existence; then he perceives plurality, and begins to enu- *Progress of inquiry.* merate; finally he begins to draw conclusions from the parts to the whole, and makes a law from the individual to the species. Thus, he has obtained figure, extension, dimension, enu- meration, and generalization. This is the teach- ing of nature; and hence, when this process *Process de- veloped in the system of Geometry.* becomes embodied in a perfect system, as it is in Geometry, that system becomes the easiest and most natural means of strengthening the mind in its early progress through the fields of knowledge."

First neces- sity for Analysis: "Long after the child has thus begun to gen- eralize and deduce laws, he notices objects and events, whose exterior relations afford no con- clusion upon the subject of his contemplation. Machinery is in motion—effects are produced. *Its method.* He is surprised; examines and inquires. He reasons backward from effect to cause. This is *Analysis*, the metaphysics of mathematics; and *What the science is:* through all its varieties—in Arithmetic—in Alge- bra—and in the Differential and Integral Calcu- lus, it furnishes a grand armory of weapons for acute philosophical investigation. But analysis

advances one step further by its peculiar nota-
tion; it exercises, in the highest degree, the fac-
ulty of abstraction, which, whether morally or
intellectually considered, is always connected
with the loftiest efforts of the mind.   Thus this
science comes in to assist the faculties in their
progress to the ultimate stages of reasoning;
and the more these analytical processes are cul-
tivated, the more the mind looks in upon itself,
estimates justly and directs rightly those vast
powers which are to buoy it up in an eternity
of future being."*

§ 378. To the quotations, which have already
been so ample, we will add but two more.

" In the mathematics, I can report no defi-
cience, except it be that men do not suffi-
ciently understand the excellent use of the pure
mathematics, in that they do remedy and cure
many defects in the wit and faculties intellectual.
For, if the wit be too dull, they sharpen it; if
too wandering, they fix it; if too inherent in the
sense, they abstract it."†   Again :

" Mathematics serve to inure and corroborate
the mind to a constant diligence in study, to

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* Mansfield's Discourses on Mathematics
† Lord Bacon.

mathematics affects the mind. undergo the trouble of an attentive meditation, and cheerfully contend with such difficulties as lie in the way. They wholly deliver us from credulous simplicity, most strongly fortify us against the vanity of skepticism, effectually re- Its influences. strain us from a rash presumption, most easily incline us to due assent, perfectly subjugate us to the government and weight of reason, and inspire us with resolution to wrestle against the injurious tyranny of false prejudices.

How they are exerted. " If the fancy be unstable and fluctuating, it is, as it were, poised by this ballast, and steadied by this anchor ; if the wit be blunt, it is sharpened by this whetstone ; if it be luxuriant, it is pruned by this knife ; if it be headstrong, it is restrained by this bridle ; and if it be dull, it is roused by this spur."*

§ 379. Mathematics, in all its branches, is, in fact, a science of ideas alone, unmixed with mat- Mathematics a pure science. ter or material things ; and hence, is properly termed a Pure Science. It is, indeed, a fairy land of the pure ideal, through which the mind is conducted by conventional symbols, as thought is conveyed along wires constructed by the hand of man.

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* Dr. Barrow

§ 380. In conclusion, therefore, we may claim What may fairly be claimed for mathematics for the study of Mathematics, that it impresses the mind with clear and distinct ideas; cultivates habits of close and accurate discrimination; gives, in an eminent degree, the power of abstraction; sharpens and strengthens all the faculties, and develops, to their highest range, the reasoning powers.   The tendency of this *Its tendency.* study is to raise the mind from the servile habit of imitation to the dignity of self-reliance and self-action.   It arms it with the inherent energies of its own elastic nature, and urges it out *The reasons.* on the great ocean of thought, to make new discoveries, and enlarge the boundaries of mental effort.

CHAPTER II.

THE UTILITY OF MATHEMATICS REGARDED AS A MEANS OF ACQUIRING
KNOWLEDGE—BACONIAN PHILOSOPHY.

Mathematics: § 381. In the preceding chapter, we considered the effects of mathematical studies on the mind, merely as a means of discipline and training. How consid- We regarded the study in a single point ered hereto- of view, viz. as the drill-master of the intel- fore: lectual faculties — the power best adapted to bring them all into order—to impart strength, and to give to them organization. In the How now present chapter we shall consider the study un- considered. der a more enlarged aspect—as furnishing to man the keys of hidden and precious knowledge, and as opening to his mind the whole volume of nature.

Material § 382. The material universe, which is spread Universe. out before us, is the first object of our rational

regards.   Material things are the first with which
we have to do.   The child plays with his toys   Elements of
                                                  knowledge
in the nursery, paddles in the limpid water,
twirls his top, and strikes with the hammer.
At a maturer age a higher class of ideas are
embraced.   The earth is surveyed, teeming with
its products, and filled with life.   Man looks
around him with wondering and delighted eyes.   Obtained by
                                                  observation.
The earth he stands upon appears to be made
of firm soil and liquid waters.   The land is
broken into an irregular surface by abrupt hills
and frowning mountains.   The rivers pursue
their courses through the valleys, without any   Course of
                                                  nature :
apparent cause, and finally seem to lose them-
selves in their own expansion.   He notes the
return of day and night, at regular intervals,
turns his eyes to the starry heavens, and in-
quires how far those sentinels of the night may
be from the world they look down upon.   He
is yet to learn that all is governed by general   Governed
                                                   by general
laws imparted by the fiat of Him who created   laws:
all things ; that matter, in all its forms, is sub-
ject to those laws; and that man possesses the   Man pos-
                                                   sesses the
capacity to investigate, develop, and understand   faculty to in-
them.   It is of the essence of law that it in-   vestigate and
                                                   understand
cludes all possible contingencies, and insures   them
implicit obedience ; and such are the laws of
nature.

§ 383. To the man of chance, nothing is more mysterious than the developments of science.

Uniformity: He does not see how so great a uniformity can

Variety: consist with the infinite variety which pervades every department of nature. While no two individuals of a species are exactly alike, the resemblance and conformity are so close, that the naturalist, from the examination of a single bone, finds no difficulty in determining the species, size, and structure of the animal. So,

They appear in all things. also, in the vegetable and mineral kingdoms: all the structures of growth or formation, although infinitely varied, are yet conformable to like general laws.

Science necessary to the development of law. This wonderful mechanism, displayed in the structure of animals, was but imperfectly understood, until touched by the magic wand of science. Then, a general law was found to pervade the whole. Every bone is of that length

What science shows. and diameter best adapted to its use; every muscle is inserted at the right point, and works about the right centre; the feathers of every bird are shaped in the right form, and the curves in which they cleave the air are best adapted

What may be demonstrated. to velocity. It is demonstrable, that in every case, and in all the variety of forms in which forces are applied, either to increase power or gain velocity, the very best means have been

adopted to produce the desired result.   And why Why it is so.
should it not be so, since they are employed
by the all-wise Architect ?

§ 384. It is in the investigations of the laws Applications
of nature that mathematics finds its widest of
Mathematics.
range and its most striking applications.

Experience, aided by observation and enlight-
ened by experiment, is the recognised fountain   Bacon's
Philosophy.
of all knowledge of nature.   On this foundation
Bacon rested his Philosophy.   He saw that the
Deductive process of Aristotle, in which the
conclusions do not reach beyond the premises, Aristotle's :
was not progressive.   It might, indeed, improve
the reasoning powers, cultivate habits of nice
discrimination, and give great proficiency in
verbal dialectics ; but the basis was too narrow
for that expansive philosophy, which was to Its defects.
unfold and harmonize all the laws of nature.
Hence, he suggested a careful examination of What Bacon
suggested.
nature in every department, and laid the foun-
dations of a new philosophy.   Nature was to
be interrogated by experiment, observation was
to note the results, and gather the facts into the
storehouse of knowledge.   Facts, so obtained, The means to
be employed.
were subjected to analysis and collation, and
general laws inferred from such classification by

Bacon's system Inductive.   a reasoning process called Induction. Hence, the system of Bacon is said to be Inductive.

New Philosophy: § 385. This new philosophy gave a startling impulse to the human mind. Its subject was Nature—material and immaterial; its object, the discovery and analysis of those general laws What it did. which pervade, regulate, and impart uniformity to all things; its processes, experience, experiment, and observation for the ascertainment of Its nature. facts; analysis and comparison for their classification; and reasoning, for the establishment of What aided general laws. But the work would have been it. incomplete without the aid of deductive science. General laws deduced from many separate cases, What it by Induction, needed additional proof; for, they needed. might have been inferred from resemblances too slight, or coincidences too few. Mathematical science affords such proofs.

The truths of Induction: § 386. Regarding general laws, established by Induction, as fundamental truths, expressing these by means of the analytical formulas, and then operating on these formulas by the known processes How verified cesses of mathematical science, we are enabled, by Analysis. not only to verify the truths of induction, but often to establish new truths, which were hidden from experiment and observation. As the in-

ductive process may involve error, while the deductive cannot, there are weighty scientific reasons, for giving to every science as much of the character of a Deductive Science as possible. Every science, therefore, should be constructed with the fewest and simplest possible inductions. These should be made the basis of deductive processes, by which every truth, however complex, should be proved, even if we chose to verify the same by induction, based on specific experiments.

As far as possible, all sciences should be made Deductive.

§ 387. Every branch of Natural Philosophy was originally experimental; each generalization rested on a special induction, and was derived from its own distinct set of observations and experiments. From being sciences of pure experiment, as the phrase is, or, to speak more correctly, sciences in which the reasonings consist of no more than one step, and that a step of induction; all these sciences have become, to some extent, and some of them in nearly their whole extent, sciences of pure reasoning : thus, multitudes of truths, already known by induction, from as many different sets of experiments, have come to be exhibited as deductions, or corollaries from inductive propositions of a simpler and more universal character. Thus, mechan-

Natural Philosophy was experimental :

Is now deductive.

Mathematical or

24

Deductive Sciences: ics, hydrostatics, optics, and acoustics, have successively been rendered mathematical ; and astronomy was brought by Newton within the laws of general mechanics.

Their advantages: The substitution of this circuitous mode of proceeding for a process apparently much easier and more natural, is held, and justly too, to be the greatest triumph in the investigation of nature. They rest on Inductions. But, it is necessary to remark, that although, by this progressive transformation, all sciences tend to become more and more deductive, they are not, therefore, the less inductive ; for, every step in the deduction rests upon an antecedent in- Sciences deductive or experimental. duction. The opposition is, perhaps, not so much between the terms Deductive and Inductive as between Deductive and Experimental.

Experimental Science: § 388. A science is experimental, in proportion as every new case, which presents any peculiar features, stands in need of a new set of observations and experiments, and a fresh induction. It is deductive, in proportion as it can When deductive. draw conclusions, respecting cases of a new kind, by processes which bring these cases under old inductions, or show them to possess known marks of certain attributes.

§ 389. We can now, therefore, perceive, what

is the generic distinction between sciences that can be made deductive and those which must, as yet, remain experimental.   The difference consists in our having been able, or not yet able, to draw from first inductions as from a general law, a series of connected and depend ent truths.   When this can be done, the de ductive process can be applied, and the sci ence becomes deductive.   For example : when Newton, by observing and comparing the mo tions of several of the heavenly bodies, discov ered that all the motions, whether regular or apparently anomalous, of all the observed bodies of the Solar System, conformed to the law of moving around a common centre, urged by a centripetal force, varying directly as the mass, and inversely as the square of the distance from the centre, he *inferred the existence of such a law for all the bodies of the system*, and then de monstrated, by the aid of mathematics, that no other law could produce the motions.   This is the greatest example which has yet occurred of the transformation, at one stroke, of a science which was in a great degree purely experimen tal, into a deductive science.

§ 390. How far the study of mathematics pre pares the mind for such contemplations and

prepares the | such knowledge, is well set forth by an old wri-
mind. | ter, himself a distinguished mathematician.   He
says :

Dr. Barrow's | " The steps are guided by no lamp more clear-
opinion. | ly through the dark mazes of nature, by no thread
more surely through the infinite turnings of the
labyrinth of philosophy ; nor lastly, is the bottom
of truth sounded more happily by any other line.

How | I will not mention with how plentiful a stock
mathematics | of knowledge the mind is furnished from these ;
furnish the | with what wholesome food it is nourished, and
mind. | what sincere pleasure it enjoys.   But if I speak
further, I shall neither be the only person nor
the first, who affirms it, that while the mind is

Abstract | abstracted, and elevated from sensible matter,
and elevate | distinctly views pure forms, conceives the beau-
it : | ty of ideas, and investigates the harmony of pro-
portions, the manners themselves are sensibly
corrected and improved, the affections composed
and rectified, the fancy calmed and settled, and
the understanding raised and excited to more

Confirmed by | divine contemplations : all of which I might de-
philosophers. | fend by the authority and confirm by the suf-
frages of the greatest philosophers."*

Herschel's | § 391. Sir John Herschel, in his Introduction

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* Dr. Barrow.

to his admirable Treatise on Astronomy, very justly remarks, that,

" Admission to its sanctuary [the science of Astronomy], and to the privileges and feelings of a votary, is only to be gained by one means— *sound and sufficient knowledge of mathematics, the great instrument of all exact inquiry, without which no man can ever make such advances in this or any other of the higher departments of science as can entitle him to form an independent opinion on any subject of discussion within their range.*

" It is not without an effort that those who possess this knowledge can communicate on such subjects with those who do not, and adapt their language and their illustrations to the necessities of such an intercourse. Propositions which to the one are almost identical, are theorems of import and difficulty to the other ; nor is their evidence presented in the same way to the mind of each. In treating such propositions, under such circumstances, the appeal has to be made, not to the pure and abstract reason, but to the sense of analogy—to practice and experience : principles and modes of action have to be established, not by direct argument from acknowledged axioms, but by continually referring to the sources from which the axioms them-

Must begin with the simplest elements:

selves have been drawn, viz. examples; that is to say, by bringing forward and dwelling on simple and familiar instances in which the same principles and the same or similar modes of action take place; thus erecting, as it were, in each particular case, a separate induction, and constructing at each step a little body of science to

Illustration of the difference between instruction by scientific and unscientific methods.

meet its exigencies. The difference is that of pioneering a road through an untraversed country, and advancing at ease along a broad and beaten highway; that is to say, if we are determined to make ourselves distinctly understood, and will appeal to reason at all." Again:

Mathematics necessary to physics:

"A certain moderate degree of acquaintance with abstract science is highly desirable to every one who would make any considerable progress in physics. As the universe exists in time and place; and as motion, velocity, quantity, number, and order, are main elements of our knowledge of external things and their changes, an acquaintance with these, abstractedly consid-

Why it is so necessary.

ered (that is to say, independent of any consideration of particular things moved, measured, counted, or arranged), must evidently be a useful preparation for the more complex study of nature."*

* Sir John Herschel on the study of Natural Philosophy.

§ 392. If we consider the department of chemistry,—which analyzes matter, examines the elements of which it is composed, develops the laws which unite these elements, and also the agencies which will separate and reunite them,—we shall find that no intelligent and philosophical analysis can be made without the aid of mathematics.

§ 393. The mechanism of the physical universe, and the laws which govern and regulate its motions, were long unknown. As late as the 17th century, Galileo was imprisoned for promulgating the theory that the earth revolves on its axis; and to escape the fury of persecution, renounced the deductions of science. Now, every student of a college, and every ambitious boy of the academy, may, by the aid of his Algebra and Geometry, demonstrate the existence and operation of those general laws which enable him to trace with certainty the path and motions of every body which circles the heavens.

§ 394. What knowledge is more precious, or more elevating to the mind, than that which assures us that the solar system, of which the sun is the centre, and our earth one of the smaller bodies, is governed by the general law of gravitation; that is, that each body is retained in its orbit by attracting, and being at-

Necessary in chemistry.

Laws long unknown.

Galileo.

His theory.

Now known to all:

By what means demonstrated.

Value of scientific knowledge:

What
it teaches.

tracted by, all the others? This power of attrac.
tion, by which matter operates on matter, is the
great governing principle of the material world.
The motion of each body in the heavens de-

The things
not easy.

pends on the forces of attraction of all the
others ; hence, to estimate such forces—varying
as they do with the quantity of matter in each
body, and inversely as the squares of their dis-
tances apart—is no easy problem ; yet analy-

Analysis:

sis has solved it, and with such certainty, that
the exact spot in the heavens may be marked
at which each body will appear at the expiration

What it has
done :

of any definite period of time. Indeed, a tele-
scope may be so arranged, that at the end of

How a
result might
be verified
by experi-
ment.

that time either one of the heavenly bodies
would present itself to the field of view ; and
if the instrument could remain fixed, though the
time were a thousand years, the precise mo-
ment would discover the planet to the eye of the
observer, and thus attest the certainty of science.

§ 395. But analysis has done yet more. It
has not only measured the attractive power of

Analysis
determines
balancing
forces.

each of the heavenly bodies ; determined their
distances from a common point and from each
other ; ascertained their specific gravities and
traced their orbits through the heavens ; but
has also discovered the existence of balancing

and conservative forces, evincing the highest
evidence of contrivance and design.

§ 396. A superficial view of the architecture
of the heavens might inspire a doubt of the sta-
bility of the entire system. The mutual action
of the bodies on each other produces what is
called· an irregularity in their motions. The
earth, for example, in her annual course around.
the sun, is affected by the attraction of the
moon and of all the planets which compose the
solar system ; and these attracting forces appear
to give an irregularity to her motions. The
moon in her revolutions around the earth is also
influenced by the attraction of the sun, the earth,
and of all the other planets, and yields to each a
motion exactly proportionate to the force exert-
ed ; and the same is equally true of all the bodies
which belong to the system. It was reserved
for analysis to demonstrate that every supposed
irregularity of motion is but the consequence of
a general law ; that every change is constancy,
and every diversity uniformity. Thus, mathe-
matical science assures us that our system has
not been abandoned to blind chance, but that
a superintending Providence is ever exerted
through those general laws, which are so minute
as to govern the motions of the feather as it is

Generality of wafted along on the passing breeze, and yet so
laws.
omnipotent as to preserve the stability of worlds.

§ 397. But analysis goes yet another step.
That class of wandering bodies, known to us by
Comets: the name of comets, although apparently escaped
from their own spheres, and straying heedlessly
What   through illimitable space, have yet been pursued
mathematics
proves in re- by the telescope of the observer until sufficient
gard to them.
data have been obtained to apply the process
of analysis. This done, a few lines written upon
paper indicate the precise times of their reap-
Results stri- pearance. These results, when first obtained,
king.
were so striking, and apparently so far beyond
the reach of science itself, as almost to need
Verification. the verification of experience. At the appointed
times, however, the comets reappear, and sci-
ence is thus verified by observation.

Nature   § 398. The great temple of nature is only to
cannot be in-
vestigated be opened by the keys of mathematical science.
without
mathematics. We may perhaps reach the vestibule, and gaze
with wonder on its gorgeous exterior and its
exact proportions, but we cannot open the por-
Illustration. tal and explore the apartments unless we use
the appointed means. Those means are the
exact sciences, which can only be acquired by
discipline and severe mental labor.

The precious metals are not scattered pro-
fusely over the surface of the earth ; they are,
for wise purposes, buried in its bosom, and can
be disinterred only by toil and labor. So with
science : it comes not by inspiration ; it is not
borne to us on the wings of the wind ; it can
neither be extorted by power, nor purchased by
wealth ; but is the sure reward of diligent and
assiduous labor. Is it worth that labor ? What
is it not worth ? It has perforated the earth,
and she has yielded up her treasures ; it has
guided in safety the bark of commerce over dis-
tant oceans, and brought to civilized man the
treasures and choicest products of the remotest
climes. It has scaled the heavens, and searched
out the hidden laws which regulate and govern
the material universe ; it has travelled from
planet to planet, measuring their magnitudes, sur-
veying their surfaces, determining their days and
nights, and the lengths of their seasons. It has
also pushed its inquiries into regions of space,
where it was supposed that the mind of the
Omnipotent never yet had energized, and there
located unknown worlds—calculating their di-
ameters, and their times of revolution.

§ 399. Mathematical science is a magnetic
telegraph, which conducts the mind from orb

aid the
mind in its
inquiries:

to orb through the entire regions of measured space. It enables us to weigh, in the balance of universal gravitation, the most distant planet of the heavens, to measure its diameter, to determine its times of revolution about a common centre and about its own axis, and to claim it as a part of our own system.

How they
enlarge it:

In these far reachings of the mind, the imagination has full scope for its highest exercise. It is not led astray by the false ideal and fed by illusive visions, which sometimes tempt reason from her throne, but is ever guided by the de-

May be
relied on.

ductions of science ; and its ideal and the real are united by the fixed laws of eternal truth.

Mind
delights in
certainty.

§ 400. There is that within us which delights in certainty. The mists of doubt obscure the mental, as the mists of the morning do the physical vision. We love to look at nature through a medium perfectly transparent, and to see every object in its exact proportions. The science of

Why
mathematics
afford it.

mathematics is that medium through which the mind may view, and thence understand all the parts of the physical universe. It makes manifest all its laws, discovers its wonderful harmonies, and displays the wisdom and omnipotence of the Creator.

CHAPTER III.

THE UTILITY OF MATHEMATICS CONSIDERED AS FURNISHING THOSE RULES OF
ART WHICH MAKE KNOWLEDGE PRACTICALLY EFFECTIVE.

§ 401. There is perhaps no word in the English language less understood than Practical. By many it is regarded as opposed to theoretical. It has become a pert question of our day, "Whether such a branch of knowledge is practical?" "If any *practical* good arises from pursuing such a study?" "If it be not full time that old tomes be permitted to remain untouched in the alcoves of the library, and the minds of the young fed with the more stimulating food of modern progress?"

§ 402. Such inquiries are not to be answered by a taunt. They must be met as grave questions, and considered and discussed with calmness. They have possession of the public mind; they affect the foundations of education: they

Practical:
Little understood.

Its popular signification.

Questions relating to studies and books.

Inquiries: How to be considered

Their influence.

Their
importance.

influence and direct the first steps ; they control the very elements from which must spring the systems of public instruction.

Practical :

Common acceptation :

What it implies

§ 403. The term " practical," in its common acceptation, that is, in the sense in which it is often used, refers to the acquisition of useful knowledge by a short process.   It implies a substitution of natural sagacity and " mother wit " for the results of hard study and laborious effort. It implies the use of knowledge before its acquisition ; the substitution of the results of mere experiment for the deductions of science, and the placing of empiricism above philosophy.

In this sense,
it is opposed
to progress:

Conse-
quences.

§ 404. In this view, the practical is adverse to sound learning, and directly opposed to real progress.   If adopted, as a basis of national education, it would shackle the mind with the iron fetters of mere routine, and chain it down to the drudgery of unimproving labor.   Under such a system, the people would become imitators and rule-men.   Great and original principles would be lost sight of, and the spirit of investigation and inquiry would find no field for their legitimate exercise.

Right
signification.

But give to " practical " its true and right signification, and it becomes a word of the

choicest import. In its right sense, it is the best   Best means
of applying
knowledge.
means of making the true ideal the actual; that
is, the best means of carrying into the business
and practical affairs of life the conceptions and
deductions of science.   All that is truly great
in the practical, is but the actual of an antece-
dent ideal.

§ 405. It is under this view that we now pro-   Mathemati-
cal science:
pose to consider the practical advantages of
mathematical science.   In the two preceding
chapters we have pointed out its value as a
means of mental development, and as affording
facilities for the acquisition of knowledge. We
shall now show how intimately it is blended   Its practical
value.
with the every-day affairs of life, and point out
some of the agencies which it exerts in giving
practical development to the conceptions of the
mind.

§ 406. We begin with Arithmetic, as this   Arithmetic
considered
practically
branch of mathematics enters more or less into
all the others.   And what shall we say of its
practical utility ?   It is at once an evidence and
element of civilization.   By its aid the child in
the nursery numbers his toys, the housewife
keeps her daily accounts, and the merchant sums
up his daily business.   The ten little characters,

which we call figures, thus perform a very im-
portant part in human affairs. They are sleepless
sentinels watching over all the transactions of
trade and commerce, and making known their
final results. They superintend the entire busi-
ness affairs of the world. Their daily records
exhibit the results on the stock exchange, and
of enterprises reaching over distant seas. The
mechanic and artisan express the final results of
all their calculations in figures. The dimensions
of buildings, their length, breadth, and height, as
well as the proportions of their several parts, are
all expressed by figures before the foundation
stones are laid ; and indeed, all the results of
science are reduced to figures before they can
be made available in practice.

§ 407. The rules and practice of all the me-
chanic arts are but applications of mathematical
science. The mason computes the quantity of
his materials by the principles of Geometry and
the rules of Arithmetic. The carpenter frames
his building, and adjusts all its parts, each to
the others, by the rules of practical Geometry.
The millwright computes the pressure of the
water, and adjusts the driving to the driven
wheel, by rules evolved from the formulas of
analysis.

§ 408. Workshops and factories afford marked Workshops and factories exhibit applications of science. illustrations of the utility and value of practical science. Here the most difficult problems are resolved, and the power of mind over matter exhibited in the most striking light. To the uninstructed eye of a casual observer, confusion appears to reign triumphant. But all the parts Parts adjusted on a general plan. of that complicated machinery are adjusted to each other, and were indeed so arranged, and according to a general plan, before a single wheel was formed by the hand of the forger. The power necessary to do the entire work was Power calculated and distributed. first carefully calculated, and then distributed throughout the ramifications of the machinery. Each part was so arranged as to fulfil its office. Every circumference, and band, and cog, has its specific duty assigned it. The parts are Parts fit in their proper places. made at different places, after patterns formed by the rules of science, and when brought together, fit exactly. They are but formed parts of an entire whole, over which, at the source of power, an ingenious contrivance, called the Governor, presides. His function is to regulate Governor· the force which shall drive the whole according to a uniform speed. He is so intelligent, and of such delicate sensibility, that on the slightest Its functions. increase of velocity, he diminishes the force, and adds additional power the moment the speed

All is but the result of science

slackens.   All this is the result of mathematical calculation.   When the curious shall visit these exhibitions of ingenuity and skill, let them not suppose that they are the results of chance and experiment.   They are the embodiments, by intelligent labor, of the most difficult investigations of mathematical science.

§ 409. Another striking example of the application of the principles of science is found in

Steamship: the steamship.

In the first place, the formation of her hull,

How the hull is formed.

so as to divide the waters with the least resistance, and at the same time receive from them the greatest pressure as they close behind her,

Her masts: is not an easy problem.   Her masts are all

How adjusted.

to be set at the proper angle, and her sails so adjusted as to gain a maximum force.   But the complication of her machinery, unless seen through the medium of science, baffles investigation, and exhibits a startling miracle.   The burning furnace, the immense boilers, the mass-

Machinery: ive cylinders, the huge levers, the pipes, the lifting and closing valves, and all the nicely-adjusted apparatus, appear too intricate to be comprehended by the mind at a single glance.

The whole constructed

Yet in all this complication—in all this variety of principle and workmanship, science has ex-

erted its power.   There is not a cylinder, whose dimensions were not measured — not a lever, whose power was not calculated—nor a valve, which does not open and shut at the appointed moment.   There is not, in all this structure, a bolt, or screw, or rod, which was not provided for before the great shaft was forged, and which does not bear to that shaft its proper proportion. And when the workmanship is put to the test, and the power of steam is urging the vessel on her distant voyage, science alone can direct her way.

In the captain's cabin are carefully laid away, for daily use, maps and charts of the port which he leaves, of the ocean he traverses, and of the coasts and harbors to which he directs his way. On these are marked the results of much scien- tific labor.   The shoals, the channels, the points of danger and the places of security, are all in- dicated.   Near by, hangs the barometer, con- structed from the most abstruse mathematical formulas, to indicate changes in the weight of the atmosphere, and admonish him of the ap- proaching tempest.   On his table lie the sextant, and the tables of Bowditch.   These enable him, by observations on the heavenly bodies, to mark his exact place on the chart, and learn his posi- tion on the surface of the earth.   Thus, practical

Science
guides the
ship:

science, which shaped the keel of the ship to its proper form, and guided the hand of the mechanic in every workshop, is, under Providence, the means of conducting her in safety over the ocean. It is, indeed, the cloud by day and the

What
thus accom-
plishes.

pillar of fire by night. Guiding the bark of commerce over trackless waters, it brings distant lands into proximity, and into political and social relations.

Illustration.

"We have before us an anecdote communicated to us by a naval officer,* distinguished for the extent and variety of his attainments, which shows how impressive such results may

Capt. Hall's
voyage.

become in practice. He sailed from San Blas, on the west coast of Mexico, and after a voyage

Its length:

of eight thousand miles, occupying eighty-nine days, arrived off Rio de Janeiro; having in this interval passed through the Pacific Ocean, round-

and
incidents.

ed Cape Horn, and crossed the South Atlantic, without making any land, or even seeing a single sail, with the exception of an American whaler off Cape Horn. Arrived within a week's sail of Rio, he set seriously about determining, by

Observations
taken.

lunar observations, the precise line of the ship's course, and its situation in it, at a determinate moment; and having ascertained this within

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* Captain Basil Hall.

from five to ten miles, ran the rest of the way by those more ready and compendious methods, known to navigators, which can be safely employed for short trips between one known point and another, but which cannot be trusted in long voyages, where the moon is the only sure guide.

" The rest of the tale, we are enabled, by his kindness, to state in his own words : ' We steered towards Rio de Janeiro for some days after taking the lunars above described, and having arrived within fifteen or twenty miles of the coast, I hove-to at four in the morning, till the day should break, and then bore up : for although it was very hazy, we could see before us a couple of miles or so. About eight o'clock it became so foggy, that I did not like to stand in further, and was just bringing the ship to the wind again, before sending the people to breakfast, when it suddenly cleared off, and I had the satisfaction of seeing the great Sugar-Loaf Rock, which stands on one side of the harbor's mouth, so nearly right ahead that we had not to alter our course above a point in order to hit the entrance of Rio. This was the first land we had seen for three months, after crossing so many seas, and being set backwards and forwards by innumerable currents and foul winds.' The effect on all on board might well be conceived to have been electric :

on the crew. and it is needless to remark how essentially the
authority of a commanding officer over his crew
may be strengthened by the occurrence of such
incidents, indicative of a degree of knowledge
and consequent power beyond their reach."*

Surveying.   § 410. A useful application of mathematical
science is found in the laying out and measure-
Measure-   ment of land. The necessity of such measure-
ment of land.
ment, and of dividing the surface of the earth
into portions, gave rise to the science of Geom-
Ownership: etry. The ownership of land could not be de-
How   termined without some means of running boun
determined.
dary lines, and ascertaining limits.   Levelling
is also connected with this branch of practical
mathematics.

By the aid of these two branches of practical
science, we measure and determine the area or
Contents of contents of ground; make maps of its surface;
ground.
measure the heights of hills and mountains;
Rivers.   find the directions of rivers; measure their vol-
umes, and ascertain the rapidity of their cur-
rents. So certain and exact are the results, that
entire countries are divided into tracts of con-
venient size, and the rights of ownership fully
Certainty   secured. The rules for mapping, and the con-

* Sir John Herschel, on the study of Natural Philosophy

ventional methods of representing the surface   Mapping.
of ground, the courses of rivers, and the heights
of mountains, are so well defined, that the nat-
ural features of a country may be all indicated   Features of the ground.
on paper.   Thus, the topographical features of
all the known parts of the earth may be cor-   Their representation.
rectly and vividly impressed on the mind, by a
map, drawn according to the rules of art, by the
human hand.

§ 411. Our own age has been marked by a   Railways.
striking application of science, in the construc-
tion of railways.   Let us contemplate for a mo-   The problem presented.
ment the elements of the problem which is pre-
sented in the enterprise of constructing a railroad
between two given points.

In the first place, the route must be carefully   Examination of their routes.
examined to ascertain its general practicability.
The surveyor, with his instruments, then ascer-   Surveys.
tains all the levels and grades.   The engineer
examines these results to determine whether the   Office of the engineer.
power of steam, in connection with the best
combination of machinery, will enable him to
overcome the elevations and descend the decliv-
ities in safety.   He then calculates the curves   Calculations of curves.
of the road, the excavations and fillings, the
cost of the bridges and the tunnels, if there are
any ; and then adjusts the steam-power to meet

Completion and use.

the conditions.   In a few months after the enter-
prise is undertaken, the locomotive, with its long
train of passenger and freight cars, rushes over
the tract with a superhuman power, and fulfils
the office of uniting distant places in commer-
cial and social relations.

The striking fact.

But that which is most striking in all this, is
the fact, that before a stump is grubbed, or a
spade put into the ground, the entire plan of the
work, having been subjected to careful analysis,
is fully developed in all its parts.   The construc-

The whole the result of science.

tion is but the actual of that perfect ideal which
the mind forms within itself, and which can
spring only from the far-reaching and immuta-
ble principles of abstract science.

§ 412. Among the most useful applications of
practical science, in the present century, is the

Croton aqueduct.

introduction of the Croton water into the city
of New York.

In the Highlands of the Hudson, about fifty
miles from the city, the gushing springs of the

Sources of the river.

mountains indicate the sources of the Croton
river, which enters the Hudson a few miles
below Peekskill.   At a short distance from the

Principal reservoir.

mouth, a dam fifty-five feet in height is thrown
across the river, creating an artificial lake for
the permanent supply of water.   The area of this

lake is equal to about four hundred acres.   The   Its area.
aqueduct commences at the Croton dam, on a   Aqueduct.
line forty feet above the level of the Hudson
river, and runs, as near as the nature of the
ground will permit, along the east bank, till it
reaches its final destination in the reservoirs
of the city.   There are on the line sixteen tun-   Its tunnels
nels, varying in length from 160 to 1,263 feet,
making an aggregate length of 6,841 feet.   The
heights of the ridges above the grade level of the   Their
heights.
tunnels range from 25 to 75 feet. * Twenty-five
streams are crossed by the aqueduct in West-   Streams
crossed.
chester county, varying from 12 to 70 feet below
the grade line, and from 25 to 83 feet below the
top covering of the aqueduct.   The Harlem   Harlem river
river is passed at an elevation of 120 feet above
the surface of the water.   The average dimen-
sions of the interior of the aqueduct, are about
seven feet in width and eight feet in height.

The width of the Harlem river, at the point   Its width.
where the aqueduct crosses it, is six hundred
and twenty feet, and the general plan of the
bridge is as follows : There are eight arches,   Bridge ·
each of 80 feet span, and seven smaller arches,
each of 50 feet span, the whole resting on piers
and abutments.   The length of the bridge is   Its length:
1,450 feet.   The height of the river piers from
the lowest foundation is 96 feet.   The arches

Its height: are semi-circular, and the height from the low-
est foundation of the piers to the top of the
Its width. parapet is 149 feet. The width across, on the
top, is 21 feet.

To afford a constant supply of water for dis-
tribution in the city two large reservoirs have
Receiving been constructed, called the receiving reservoir
Reservoir: and the distributing reservoir. The surface of
the receiving reservoir, at the water-line, is equal
Its extent. to thirty-one acres. It is divided into two parts
by a wall running east and west. The depth of
Depth of water in the northern part is twenty feet, and
water. in the southern part thirty feet.

Distributing The distributing reservoir is located on the
Reservoir: highest ground which adjoins the city, known
Its capacity. as Murray Hill. The capacity of this reservoir
is equal to 20,000,000 of gallons, which is about
one-seventh that of the receiving reservoir, and
the depth of water is thirty-six feet.

Power The full power of science has not yet been
of science. illustrated. A perfect plan of this majestic
structure was arranged, or should have been,
before a stone was shaped, or a pickaxe put into
the ground. The complete conception, by a
single mind, of its general plan and minutest
details, was necessary to its successful prosecu-
What it ac- tion. It was within the range and power of
complished. science to have given the form and dimensions

of every stone, so that each could have been shaped at the quarry. The parts are so connected by the laws of the geometrical forms, that the dimensions and shape of each stone was exactly determined by the nature of that portion of the structure to which it belonged.

Connections of the parts.

§ 413. We have presented this outline of the Croton aqueduct mainly for the purpose of illustrating the power and celebrating the triumphs of mathematical science. High intellect, it is true, can alone use the means in a work so complicated, and embracing so great a variety of intricate details. But genius, even of the highest order, could not accomplish, without continued trial and laborious experiment, such an undertaking, unless strengthened and guided by the immutable truths of mathematical science.

View of the Croton aqueduct: Why given

Little accomplished without science.

§ 414. The examination of this work cannot but fill the mind with a proud consciousness of the power and skill of man. The struggling brooks of the mountains are collected together—accumulated—conducted for forty miles through a subterranean channel, to form small lakes in the vicinity of a populous city.

What science has done.

From these sources, by an unseen process, the

pure water is carried to every dwelling in the
large metropolis.   The turning of a faucet de-
livers it from a spring at the distance of fifty
miles, as pure as when it gushes from its granite
hills.   That unseen power of pressure, which
resides in the fluid as an organic law, exerts its
force with unceasing and untiring energy.   To
minds enlightened by science, and skill directed
by its rules, we are indebted for one of the no-
blest works of the present century.   May we
not, therefore, conclude that science is the only
sure means of giving practical development to
those great conceptions which confer lasting
benefits on mankind?   "All that is truly great
in the practical, is but the result of an antece-
dent ideal."

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$$x = 7$$
$$y = 6$$

$1+$

$$\left(1+\frac{0.5}{n}\right)^{n} =$$

$$\left(1+\frac{a}{n}\right)^{n} = 1 + \left(\frac{a}{n}\right)n - \frac{\frac{a}{n}\left(\frac{a}{n}-1\right)}{1 \cdot 2}n^2$$

$$\left(1+a+\frac{a^2}{2} - \right)$$

$$\varepsilon^{a}$$

$$\left(\varepsilon^{a}\right)$$

Date Due