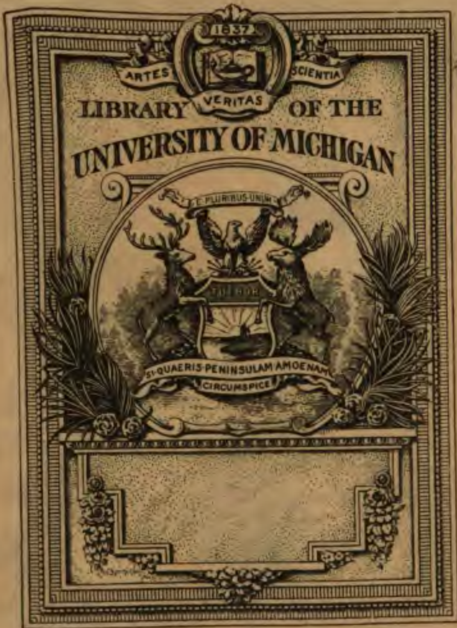
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NEWTON'S LAWS OF MOTION

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LAWS OF MOTION

BY

P. G. TAIT, M.A., SEC. R.S.E.

HON. FELLOW OF PETERHOUSE, CAMBRIDGE,
PROF. OF NAT. PHIL. IN THE UNIV. OF EDIN.

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PREFACE.

No really intelligent conception of any one of the numerous branches of Natural Philosophy, still less of their intimate interdependence, can possibly be formed by a student until he has an accurate acquaintance with its unique basis, *The Laws of Motion.*

Yet, if the Teacher have but six months given him to discuss such an enormous subject, and be expected to deal impartially with its various branches, the time which he can devote to this indispensable auxiliary must be altogether inadequate. And in the three months' course, which is now required of medical students, it cannot fail to be almost ludicrously insufficient. Hence the imperative necessity that the student should to some extent be his own teacher in this all-important special region :—that he should be assisted in the endeavour to prepare himself, by previous efforts of his own, to follow intelligently all that his Teacher has time to say about it :—and that he should have the means of refreshing his recollections of it throughout the whole of the course. This forms *one* of my reasons for producing the present little book.

But there is another reason, at least equally strong; and, as it is practically independent of the first, their effect is strictly cumulative. There is, perhaps, no more striking example of the development of misdirected zeal into pernicious habit than that which is lavishly furnished by the practice of "taking notes." Of course I do not refer to mere jottings as to the proper division of a subject, the order in which its parts must be treated, &c. Nor do I refer to the booking of isolated numerical data; such as the number of particles in a cubic inch of air, the Centigrade temperature of Absolute Zero, the number of foot-pounds equivalent to a unit of heat, &c. This may or may not be laudable, but it is harmless at least, and sometimes useful. I refer to the attempts made, often by the most eager and diligent students, to record *part* (for it cannot be more than a part, usually a small part) of a chain of reasoning, while *necessarily* depriving themselves of what may be their only chance of listening to a connected exposition of the whole. None but a fully trained stenographer could take notes of any real value under such circumstances; and even *he* would lose the unquestioned benefit of the spoken lecture; while his record would probably not be so useful for future reference as would the corresponding paragraphs of some well-written text-book. Such chains of reasoning occur, of course, in all parts of our subject, but they are specially important in connection with its fundamental principles.

I have, therefore, endeavoured to prepare a short and pointed summary of the more important features of what I have called the basis of the subject:—and I have indicated by asterisks the portions which are not indispensable. Brevity has been persistently aimed at:—but, even in the present compressed form, each part of the subject treated has been dealt with far more completely than would be possible in the very best of "note-taking" by one who tries really to *follow the Teacher*, and is not a trained shorthand-writer. To such a man the present digest may be useful in two ways, besides saving him from his note-book and its insidious but fatal allurements:

—it may not merely help him to recall to his memory what he has already heard:—it may enable him so to prepare himself beforehand as to profit fully by what he is about to hear.

In the pursuit of brevity much has inevitably been omitted:—a single example (always, if possible, of a really representative character) being often all that is given for some special branch of the subject. But, while insisting on the fundamental principles, I have selected, as examples of their development, those consequences especially which have the widest applications throughout the whole range of Physics.

This does not pretend to be a Text-book, though its composition has involved more labour than has sometimes been required for a much bulkier volume:—still less can it be called a cram-book, for its contents are not presented in a form adapted to parrot-like repetition:—it is an attempt, on the one hand, to supply an acknowledged want; on the other, to remedy what is felt by many competent authorities to be little short of a crying evil.

It is now issued, after long delay, almost as much for the sake of profiting by some unintended hint from the sarcastic criticism which experience warrants my expecting, as for the help in teaching which I confidently anticipate from it.

P. G. TAIT.

COLLEGE, EDINBURGH,
September 1899.

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INTRODUCTORY. MATTER AND ENERGY.

MATTER AND ENERGY.

Reason and experience force on all who rightly employ them the conviction of the objective reality of the Physical Universe.

It exists altogether independently of the senses and subjective impressions by which alone a conception of it can reach our minds.

Denial of this statement lands us at once in hopeless inconsistency. It is scientifically certain that the physical universe existed before there were any senses to perceive it; and that, during these ages, it would have produced sensuous impressions if organs of sense had existed. Although therefore it can only be conceived of as related to the senses, it has an existence altogether independent of the senses.

Acceptance of the statement leads to such difficulties only as exercise the ingenuity of Metaphysicians. The more reckless of the class have denied that the physical world is real; the more cautious of them have been striving to determine precisely what its objective reality means. Wishing the latter more success than they seem hitherto to have had, we leave the problem in their hands.

The objective realities in the physical world are of two kinds only:—MATTER and ENERGY. Our conviction of their objectivity is based on the experimental fact that we cannot alter the quantity of either. In technical language we therefore speak of two great General Laws:—

CONSERVATION OF MATTER, and
CONSERVATION OF ENERGY.

In their other characteristics these reals differ as widely as possible from one another. For matter (gold, lead, wood, water, air, &c.) is found to be absolutely passive (technically, *inert*). Hence all the changes, and the varieties of activity, in the physical world (winds, waves, sounds, torrents, earthquakes, lightning, the sun's light and heat, &c.) must be due to energy.

Energy cannot exist except in association with matter:—and it is not perceived by our senses except while it is being transformed, or while it is being transferred from one portion of matter to another. It is already more than probable that energy, in all its various forms (even those which at present we associate with mere *grouping* or relative position) depends upon motion of matter. One very strong argument in support of this view is afforded by the fact that while matter is found to consist of parts which preserve their identity, energy cannot be identified. For this is precisely what we should expect to find if energy always depends upon motion of matter.

We thus have other two experimental laws:—

INERTIA OF MATTER, and
TRANSFORMATION OF ENERGY.

And, as it is experimentally ascertained that some forms of energy are less capable of transformation than others, constant transformation involves gradual degradation to these less useful, and therefore lower, forms. Hence we have

DISSIPATION OF ENERGY;

which implies that the stock, though unaltered in amount, constantly deteriorates in average quality.

Since in all its known forms, energy depends upon the grouping, or upon the motion, of portions of matter, we naturally commence with some general notions as to position and motion in the abstract. These form the object of a special science, called KINEMATICS:—a mixed science of space and time. To the elements of this auxiliary science the first of the following chapters is devoted.

Though, strictly speaking, the above statements comprehend the whole of Physics, it has been found that their treatment may in very many cases be much facilitated by the use of a term, FORCE, suggested to us by the so-called muscular sense:—as, for instance, when we pull, or push, a piece of matter.

We have absolutely no reason for looking upon force as a term for anything objective:—we can, if we choose, entirely dispense with the use of it. But we continue to employ it:—partly because of its undoubted convenience, mainly because it is essentially involved in the terminology of Newton's *Laws of Motion*, which still form the simplest foundation of our subject. [It must be remembered that even in strict science we use such obvious anthropomorphisms as "the sun rises," "the wind blows," &c.] Hence, so long as we adhere to Newton's system, our study of its direct consequences is logically called DYNAMICS.

CHAPTER I

KINEMATICS.

1. Here the fundamental ideas are those of *Time* and *Space.* None but a Metaphysician could even attempt to discover *what* these are. Mathematicians and Natural Philosophers are content to recognise them as, in some sense at least, capable of being measured.

2. Each event, of whatever nature, has its position in time; each point, or particle, its position in space. But, though a point or a particle can be displaced in space, an event is fixed in time.

And in both cases the position, so far as we can treat it, is *relative* merely. We date events from some assumed Epoch or Era; and, as we have no fixed landmarks in space, from which to measure, we must assume some starting-point, or Origin. Hence our measurements, in either, are of *intervals.*

3. Absolute, or true, time (as conceived by Mathematicians) flows on, or grows, uniformly :—altogether independent of material things. It is *the* independent variable. Our sole resources for its measurement depend ultimately on the *Inertia* of matter.

As regards time we say *Now*, or *Then*. *Then* includes only Past and Future in its reference. There is no such idea as "sideways," or "up and down," so far as time is concerned. And the ordinary algebraic use of the signs

− and + enables us fully to distinguish between Past and Future. Hence mathematicians say time has *one dimension* only, or is a single manifoldness. An interval, or duration, or a date (which is merely an interval *from* the era), is thus fully expressed as a numerical multiple of some suitable unit, with its proper sign.

4. As regards space we say *Here*, or *There*. But *There* includes not merely Forward and Backward (analogous to Future and Past in the case of time) but, quite as essentially and necessarily, Right and Left, and Up and Down. These three pairs of opposites are obviously independent of one another.

Hence space is essentially *tridimensional*, or is a triple manifoldness. Its three dimensions are, however, of the same nature; and all intervals of space can be measured in terms of one common unit. Thus any complete description of the position of a point or a particle (relative, of course, to some assumed origin) involves necessarily *three* numbers. And one, at least, of these must be a multiple of the unit of length. Displacement, or change of position, is of course also to be fully described by three numbers.

5. We may, if we please, limit space artificially by confining our attention to one, or to two, dimensions only.

Thus we may confine ourselves to a line (straight or curved), *i.e.* "length, without breadth or thickness"; and then one number, with its proper sign, suffices as in the case of time. Thus we deal with a road, or railway; the known orbit of a planet; the path of a pendulum; &c.

Or we may limit our consideration to a surface, plane or curved:—*i.e.* "length and breadth, without thickness." Here two numbers are essential. Thus, on a map, we represent the position of a place by its latitude and longitude; or by its being (say) so much south, and so much east, of some other place assumed as origin. For these last data we may substitute the direction, and the length, of the line joining the two places. On a globe (terrestrial or celestial) we have, again, latitude and longitude, or Right Ascension and Declination. And we have the Altitude and Azimuth of a celestial body, or the Variation and Dip of a magnetized needle. These pairs of angles essentially give *direction* only; so that for complete information in three dimensions we must have *distance* also. The representation of the third dimension on a map, or other essentially two-dimensional space, is most completely given by *Contour Lines*. The simplest definition of these is that they are successive sea-margins, traced after successive equal rises of sea-level. It is obvious that data of this kind enable us to construct a model of the country, more and more exact as they are more numerous.

6. A straight line, considered as having *direction* as well as length, is called a VECTOR. [Its length may represent distance, speed, force, rate of rotation about it, intensity of magnetization, flux of heat, &c.]

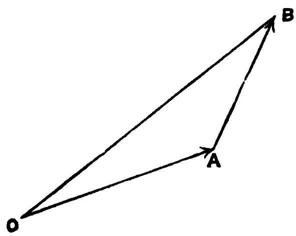
When its length is taken as so many inches, yards, miles, &c., the vector is regarded as the instrument by which a point is carried, or displaced, from one position to another, and its value depends *only* on the relative, and not on the absolute, position of its ends. This is one of the most important elementary conceptions in our subject; and in its application presents but *one* real difficulty to students, so far at least as I can gather from nearly fifty years' experience in teaching. *That* once surmounted, the rest is very plain sailing indeed. It concerns the statement that displacement is a question of space only, not of time so that:—

7. *The final result of any number of given displacements is independent of their order, and of the time occupied,* i.e. *it is the same whether they are successive or simultaneous, gradual or instantaneous.*

To accept this statement in full, we have only to realize the obvious fact that southward, eastward, and upward displacements (whether individually + or −) are entirely independent of one another, while every other displacement must in general involve more or less of each of these. For then we see that to get the final result we have to deal separately with the southward, the eastward, and the upward parts of the various displacements :—and, in so dealing with them, simultaneity or succession obviously does not matter.

[*N.B.* We have referred to southward, eastward, and upward directions as at once easily intelligible to all, and sufficient for our purpose :—but *any* three directions at right angles to one another would obviously do equally well for the selection of independent displacements.]

8. It will be readily seen that all is included in the simple statement that *the resultant of any three vectors or displacements (whether simultaneous or successive), repre-*



sented by the sides of a triangle taken all round in the same order, is nil :—so that the resultant of any two sides, *taken in the same order, is represented by the third side taken in the opposite order.*

This is otherwise expressed by saying that the sum of the vectors $\overline{OA}$ and $\overline{AB}$ is the vector $\overline{OB}$; or, what comes to the same thing:—To change the vector $\overline{OA}$ into $\overline{OB}$ we must compound it with $\overline{AB}$. In this simple statement we have the *whole* of the geometry of the composition and resolution of velocities, accelerations, and angular velocities; from which, by the *Laws of Motion*, we proceed to those of impulses, forces, fluxes, couples, &c., &c. For we may compound any two, then their resultant with a third, and so on.

9. The most important special case of this statement is that in which the angle at A is a right angle. In this case the vector $\overline{OB}$ is resolved into two components, one of which is in any assigned direction (as OA) and the other is wholly perpendicular to it. The magnitudes of these components are found from that of $\overline{OB}$ by employing as factors the fractions called respectively the *cosine* and the *sine* of the angle at O.

Other special cases arise when the angle at A is either two right angles or vanishes. For then the points O, A, B are in one straight line, and the displacements are to be compounded by ordinary algebraic addition.

10. When we pass from the consideration of displacement to that of motion, the idea of time necessarily comes in. For motion essentially consists in *continued* displacement. In the kinematics of a point, all sorts of motion are conceivable:—but we limit ourselves to such as are possible in the case of a *particle of matter.*

11. These limitations are simple, but very important.

(*a*) The path of a material particle must be a *continuous* line. [A gap in it would imply that a particle could be annihilated at one place and reproduced at another.]

(β) There can be no instantaneous finite change in the direction, or in the speed, of the motion. [*Inertia* prevents these, unless we introduce the idea of finite transformations of energy for infinitely small displacements, or (in the Newtonian system) infinite forces.]

12. The complex idea of direction *and* speed is conveniently expressed by the term *Velocity*. It is, of course (§ 6), to be treated as a vector.

When we consider the motion of a point in a definite line (*e.g.* the orbit of a planet) the direction is, at each instant, that from one point of it to the next:—*i.e.* the tangent to the curve. Thus, if the path be straight, the direction of motion is constant; if curved, it is variable.

Speed, or the rate of motion, need scarcely be defined, as everyone knows what it means. We all speak of 50 miles an hour, 10 feet per second, &c. But it is essential to remark that such statements do *not* imply more than that, *if the speed were maintained for an hour*, 50 miles would be passed over, &c. The numerical expression does not necessarily describe the nature of the motion for more than an instant. In the next instant it may not be the same. Thus we speak of *uniform*, and of *varying*, speed.

13. Whether speed be uniform or not, its *average* value during any period is a perfectly definite and intelligible conception:—whose measure is obviously the quotient of the whole space described by the whole time employed. If the speed be uniform this average value is the actual value, and can be calculated from *any one* statement as to the space actually described in an assigned time. [For a million-fold that space would be described

in a million-fold the time, or a millionth part of it in a millionth of the time.]

14. If the speed be variable its value, during any period, must sometimes exceed and sometimes fall short of the average value. But (by 11, β above, and therefore *solely* in consequence of inertia) the shorter the period considered, the more closely will the actual speed of a material particle agree with the average value :—and this without limit.

15. The most simple case of variable speed is that of a stone moving vertically, under gravity. For in this case the speed increases or diminishes at a uniform rate. [Such a rate of change of speed is a particular case of what will, in § 19 below, be defined (generally) as *Acceleration.*] This rate of gain or loss, usually denoted by the letter g, is roughly about 32·2 feet per second, per second. Hence after t seconds, a stone, originally thrown upwards with speed V, will be moving with speed

$$v = V - gt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And *because, in this special case, the speed alters at a uniform rate*, its average value for any interval lies half-way between its first and last values. Thus the average value for the first t seconds is

$$\bar{v} = \tfrac{1}{2}(V + v) = \tfrac{1}{2}(V + V - gt) = V - \tfrac{1}{2}gt \quad . \quad . \quad (2)$$

But, as above (§ 13), the space described in any time is found by multiplying the time into the corresponding average speed. Thus

$$s = \bar{v}t = Vt - \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To these we may add (as the result of eliminating t between (1) and (3))

$$\frac{V^2}{2} - \frac{v^2}{2} = gs \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

16. The expressions (1) and (3) contain full information as to the motion in any case, as soon as V is given. In the interpretation, of course, the signs + and − continue to signify upwards and downwards :—having been introduced with these significations.

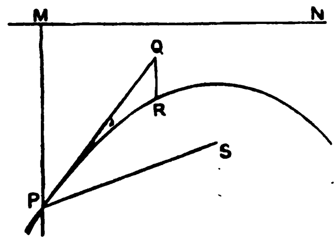
But it is clear from their form that the separate parts of the expressions for v and s, which are due respectively to V and to g, are quite independent of one another. And we see by §§ 7, 8 that this remark would still apply were the projection to take place in *any* direction.

[In fact, in *these* equations the quantities $\dot{V}$, g, v, s may be treated as vectors, of which the first two are constant. This remark does not apply to (4), for the quantities there involved are essentially non-directional.]

17. Thus if a stone be thrown with speed V in any given direction PQ, its displacement at the end of t seconds may be regarded as the resultant of $\overline{PQ} = Vt$ in the direction of projection and $\overline{QR} = \frac{1}{2}gt^2$ vertically.

Thus the path is such that, generally,

$$PQ^2 = \frac{2V^2}{g} QR.$$



But a parabola is *defined* as the set of points which are equidistant from the focus, S, and the directrix, MN, [in this case a horizontal line]. And geometry shows that if R be any point on the parabola, and RQ (drawn perpendicular to the directrix) meet a fixed tangent PQ in Q, we have

$$PQ^2 = 4PS \, . \, QR \, .$$

Thus, by comparing the two last-written formulæ, we see that the path of the stone is a parabola, in which $PM = PS = V^2/2g$ = height due to upward speed V. (§ 15 (4).)

M is thus found, and S is given at once by the fact that, in a parabola, the tangent PQ bisects the angle MPS.

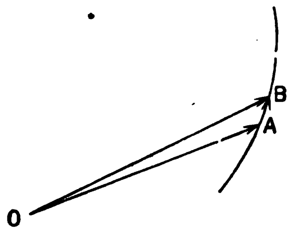
In these three sections we have the whole theory of the motion of an unresisted projectile.

18. Just as *two* consecutive points of a path determine a straight line (the tangent, or the direction of motion), so *three* consecutive points determine a circle, and with it the curvature of the path. For the essence of curvature is change of direction, and its measure is the *rate of change of direction per unit of length along the curve.* In a circle this is simply the reciprocal of the radius.

* [*Four* consecutive points of a curve do not necessarily lie in one plane. Hence the essence of *Tortuosity* in a curve is that successive planes of bending do not coincide. They have in common the second and third of the four consecutive points, so that the *osculating* plane (as it is called) rotates round the tangent.]

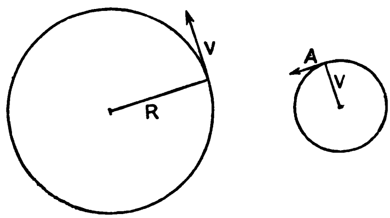
19. If, for a moment, we confine our attention to the *Velocity* of a moving point, without thinking of its position, we see that a diagram (called the *Hodograph*) can be constructed, such that its radius-vector, OA, represents at every instant the velocity of the moving point. And it is clear, from § 8, that the *Acceleration* of the point,

(*i.e.* the rate at which its velocity changes), is represented by the corresponding velocity of A:—*i.e.* by the rate of



growth of the vector $\overline{OA}$, that is, by the velocity of A along AB. When the original path is a plane curve the hodograph is in the same, or in a parallel, plane.

20. When the path is a circle, of radius R, uniformly described with speed V, it is clear that the hodograph is a circle of radius V described uniformly in the same *time* and in the same *sense* as is the first circle. The speeds in these two circles must therefore be as their radii, while the direc-



tion of motion in the second is at every instant perpendicular to that in the first. Hence the magnitude of the acceleration in the path (being the speed in the hodograph) is V^2/R; and its direction is of course *inwards* along the radius. We may write it as

$$\frac{V^2}{R^2} \cdot R, \quad \text{or } \omega^2 R,$$

where ω (the angle described in one second by the radius vector) is the so-called *Angular Velocity*. This may be expressed in terms of the periodic time T, by the obvious relation

$$\omega T = 2\pi .$$

21. The results of last section may be thus stated:—*In circles uniformly described, all in the same period, about a point, the acceleration is directed to the centre, and is proportional to the radius.* Here the radius of each circle describes equal areas, as well as equal angles, in equal times.

But these circles, if in one plane, may be projected orthogonally into similar ellipses of any form, all having the same centre. The acceleration will, in each, be inwards along the radius-vector, and *proportional to it.* And the radius-vector in each describes equal areas in equal times. Thus, with acceleration directed towards, and in a definite proportion to the distance from, a centre, the path may be *any* ellipse about that centre, but the period is the same in all.

22. The resultant of two (and therefore of any number of) uniform circular motions, of one period and in one plane, and in the same sense, is obviously a single uniform circular motion of the same period. Its radius is found, as in § 7, at any instant, by compounding the separate vector radii, and obviously depends upon their relative inclinations. This relative inclination of any two of them is called the Difference of *Phase* in the corresponding circular motions. After the lapse of any time, all parts of the figure have turned (in the same sense) through equal angles in the same plane; so that the whole turns as if rigid, and the extremity of the resultant has uniform circular motion.

23. The component of uniform circular motion parallel to any diameter (which is of course its projection on a plane perpendicular to that of the circle, so that the ellipse collapses into a finite straight line) is an alternating motion in that line, called *Simple Harmonic Motion.* And here we have still acceleration, always directed to the middle point of the range, and proportional to the distance from it.

This is, approximately, the motion of Sun-spots, Jupiter's satellites, &c., as seen from the earth. Their actual motion is nearly circular and uniform, and from a distant point in the plane of the path we see only the component of the motion which is perpendicular to the line of sight.

It is also the simplest form of vibration, as in tuning-forks, violin-strings, organ pipes, &c., and it would be that of the tide-level were there only one tide-producing body.

The two rectangular components of uniform circular motion are obviously equal in every respect:—only that the phases differ by a quarter period. For, when the moving point is at the mid-range of one of its component motions, it is at an extremity of the other. And it depends solely on which has the quarter period in advance of the other, whether the motion in the circle is with, or against, the sun:—*i.e.* negative or positive. We have only to look at the motion from the other side of its plane. Hence the resultant of two equal uniform circular motions, of one period and in one plane, but one positive and the other negative, is simple harmonic motion with double range. The direction depends on the difference of phase in the two circular motions. Again, the resultant of any two simple harmonic motions whatever, of the same period, is elliptic motion as in § 21. All possible resultants, corresponding to difference of phase in the components, are obviously inscribed in a parallelogram having the ranges for its sides.

24. From the statements of § 22 regarding the vector *figure* (which contain the whole matter), combined with the definition of § 23, it follows at once that the resultant of any number of simple harmonic motions, of the same period and in one line, is also a simple harmonic motion of the same period. Its range or Amplitude depends upon the relative phases of the components, as well as on their several amplitudes.

Should the periods of the components be slightly different, their relative phase will continually alter, and with it the phase and the amplitude of their resultant. [An excellent illustration of this is furnished by the "priming and lagging" of the tide, and by "spring and neap" tides, respectively. For the actual tide is the resultant of the lunar and solar tides, each of which (separately) would give a simple harmonic rise and fall of definite amplitude:—but their periods are (roughly) $12^h 27^m$ and 12^h.]

A very clear conception of the nature of any rectilineal motion can be formed from the diagram obtained by compounding it with a uniform motion in some other direction than its own. Thus the vertical fall of a stone was exhibited as a parabôla, by giving the stone an initial velocity which was not vertical (§ 17), and was preserved constant in virtue of inertia. Usually the uniform motion is in a direction perpendicular to the other. This is the principle upon which self-registering instruments work:—thermometers, barometers, tide-gauges, &c., as well as Lord Kelvin's syphon-recorder for submarine telegraphy:—in all of which a band of paper is drawn uniformly by clock-work in a direction perpendicular to that in which the tracing-point (or its equivalent) travels.

When we apply this process to a simple harmonic motion, the resulting trace is an undulating curve (the *Curve of Sines*) which records the various circumstances of the motion. Such a curve represents also, approximately, at any moment, a vertical section of the surface

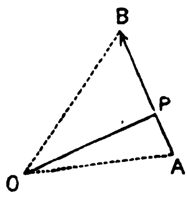
of the sea, taken transverse to the crests of a train of "rollers." It may be regarded, therefore, either as representing *for all time* the vertical disturbance of a definite point of the water-surface; or, as representing *at any instant* the simultaneous vertical disturbances of all parts of the surface.

When two sets of such waves, equal in every respect, are moving in opposite directions, there are obviously definite lines on the water-surfaoe, at which (alone) crests constantly meet one another. Troughs meet one another at these lines also, and no-where else. At such lines (called *Loops*) the disturbance is simply *double* of that which either set of waves would produce. But midway between these lines we have others (called *Nodes*) at which the crests of either series perpetually meet the troughs of the other. And these lines remain undisturbed, for it is clear that whatever elevation reaches one of them, at any moment, from one series of waves, an equal depression is simultaneously sent by the other.

Thus we easily see that the resultant of two sets of equal waves, travelling in opposite directions, is a wave-pattern *fixed in position* but going through simple harmonic changes of amplitude simultaneous at all its parts; what is called a set of *Stationary Waves.*

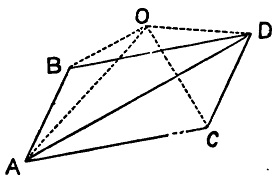
A very interesting particular case of them is well exhibited by throwing a long rope, one of whose ends is fixed and the other held in the hand, into rotation about an axis. With a little practice the experimenter can adjust the tension of the rope to the rate of rotation, so that his hand may remain nearly steady while the rope rotates as a whole, as two halves, as three third parts, &c. The outline presented, when the rope is looked at from the side, is that of the stationary waves above mentioned. We now see what these indicate, in a vibrating body such as a violin-string, a piano-forté wire, the air in an organ-pipe, &c.:—viz. two series of equal waves running in opposite directions, and perpetually reflected, with the loss of half an undulation, at the ends of the wire or air-column.

25. The *Moment* of a Vector $\overline{AB}$ (displacement, velocity, &c.) about a point O is, numerically, the product of the length of the vector into its least distance, OP, from the point: *i.e.* it is double the area of the triangle OAB, whose base is the vector and whose vertex is the point.



Reversal of the direction of the vector changes the sign of the moment. Hence the moment itself is properly a vector:—and its direction is obviously perpendicular to the plane of the triangle OAB; and, as the figure is drawn, upwards from the plane of the paper. [If we draw a line through the point O, and perpendicular to the plane of the triangle, we may represent the moment by the length of this line, considered as a vector.]

26. Elementary Geometry shows at once that if any two vectors $\overline{AB}$, $\overline{AC}$ be drawn from one point, the moment of their resultant about any point O *in their plane* is the (algebraic) sum of their separate moments. In fact, the area OAD is the (algebraic) sum of OAB and OAC; whatever point O be, in the plane of AB, AC. For the



(algebraic) sum of the distances of B and C from OA is obviously the distance of D from it.

27.* Obviously a vector, whose direction passes through the point O, has no moment about it. Hence if the moment of the velocity (double the rate at which areas are described by the radius-vector, or double the product of its square by the angular velocity) be constant, the acceleration, if any, is in the direction of the radius-vector; and conversely.

If this acceleration be inversely as the square of the radius vector it will be proportional to the angular velocity in the path; and thus the hodograph (§ 19) will be a curve of constant curvature:—*i.e.* a circle.

Also the resultant of the moments of parallel, equal, and oppositely directed velocities is the same about all points in their plane:—being, in fact, the moment of either velocity about any point in the line of the other.

28.* If the vector AB of § 25 represent *Angular Velocity*, it must be regarded as a portion of the axis about which the rotation takes place, the length being the angular speed; and its moment about a point O (as defined in § 25) may now be regarded as the linear velocity which the point has in consequence of the rotation. Thus the statements of §§ 25, 26 admit of a new interpretation; as follows.

29.* Vectors representing angular velocities about axes which intersect, are to be compounded according to the usual rule. And when the angular acceleration is about an axis perpendicular to that of rotation, it produces change of the direction of that axis, but not of the angular speed about it.

30.* Simultaneous angular velocities about parallel axes give, as resultant, angular velocity equal to their (algebraic) sum, about another parallel axis in the same plane:—except in the special case when this sum becomes infinitely small, and then the resultant axis is infinitely distant. In this case the resultant is a definite rate of *translation* perpendicular to the plane of the component axes.

31.* When one plane curvilinear figure rolls on another in the same plane, the angular speed of the rolling figure is evidently the product of the linear speed of the point of contact (on *either* boundary) by the (algebraic) difference of curvature. But, though the point of contact is thus the instantaneous centre of rotation, the student should be warned that the various points of the moving figure do *not* describe even small arcs of circles with this point as centre. [The construction gives us only *two*, not *three*, consecutive points on the path of each.]

32.* Simultaneous angular velocity about an axis, and translation perpendicular to it, are equivalent to the same angular velocity about a definite axis parallel to the first. And conversely. This follows at once from § 30.

Hence any angular velocities, accompanied by any translation, are equivalent to a definite *Screw-motion*, *i.e.* rotation accompanied by translation parallel to the axis. Such is, therefore, the most general instantaneous motion which a rigid figure can have. And it follows that any number of simultaneous screw-motions can be compounded into a single one.

33. *Strain*, so far as we need consider it, is most intelligibly to be treated as any change of form or dimensions in a plastic *solid*. [The idea of matter, thus introduced, renders our work not strictly kinematical:—but it is very convenient.] Strain is homogeneous when all originally similar, equal, and similarly situated parts of the solid remain similar, equal, and similarly situated (however they may be distorted or displaced) when the strain is applied. This is the only form of strain we can consider.

It follows at once from this that straight lines remain straight, equal parallel straight lines equal and parallel; planes remain planes, parallel planes parallel, &c. &c.

34. Hence we gather, by the aid of geometry, that circles become ellipses, spheres become ellipsoids, &c.

And, in particular, any cube circumscribing a sphere becomes a parallelepiped circumscribing an ellipsoid and *touching it at the ends of conjugate diameters.* [Conjugate diameters are such that the tangent plane at the extremities of any one of them is parallel to the plane of the other two.] But the chief axes of the ellipsoid (which are perpendicular to one another) are conjugate diameters. Hence there is always one set of three mutually perpendicular planes which remain mutually perpendicular when the strain is applied. If the unstrained body be supposed to be made up of equal cubes, whose faces are parallel to these planes, it will be made up of equal rectangular parallelepipeds when the strain is applied. Thus each cube suffers (1) displacement as a whole relatively to the others, (2) change of length of each set of its edges, (3) rotation as a whole. The third of these can be got rid of by a reverse rotation of the strained body as a whole, and the first is a mere consequence of the second. Hence the strain is equivalent to mere uniform extensions or compressions in three directions at right angles with one another:—what is called a *Pure* Strain:—together with a rotation about a definite axis. And there is always at least *one* set of parallel lines whose direction is not changed by a strain. When the strain is pure, there are obviously three sets. But the converse of this proposition is not true, unless the three sets be mutually perpendicular.

35. Two very simple cases of homogeneous strain are of special importance in the elementary treatment of elasticity:—(*a*) change of volume without change of form, (*b*) change of form without change of volume.

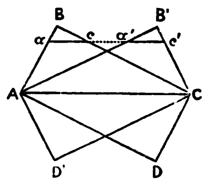
(*a*), of course, involves uniform expansion or compression:—mere magnifying, as it were:—every line in the body being extended or shortened in the same ratio, without change of direction. If unit length becomes $1+e$, unit volume becomes $(1+e)^3$. When e is small, this is nearly $1+3e$. Thus we may say that the cubical dilatation, of an isotropic body, by heat is three times its linear dilatation.

36. The case of § 35 (*b*) occurs when a spherical portion of a solid becomes by strain an ellipsoid of an equal

volume: *i.e.* when the continued product of the chief semi-axes of the ellipsoid is equal to the cube of the radius of the sphere. The simplest example of this is when a brick-shaped portion of the solid has one set of parallel edges increased, and another set diminished, in any one ratio:—the third set being unaltered, and the angles remaining right angles.

37.* Suppose the longer edges of one face to be compressed to the length of the shorter ones, and the shorter to be extended to the original length of the longer ones, the thickness being unaltered:—the volume also will be unaltered.

If the old and new faces, ABCD and AB′CD′, be superposed, so as to have the diagonal AC in common, it



is seen at a glance that the one figure can be changed into the other by the sliding of layer over layer in a direction parallel to that diagonal, by amounts proportional to their distances from it; so that *ac* (for example) is shifted to *a′c′*. It is moved through the space *aa′*, or *cc′*, obviously proportional to its distance from AC.

This differential sliding is called a simple *Shear*. And it is obvious that the same strain might have been looked on as produced by sliding parallel to the other diagonal of the face of the brick.

The pure part of this strain has been given in § 36.

If $AB/AD = 1 + f = AD'/AB'$, where f is small, the change of angles of a square whose diagonals are parallel to the axes of the pure part of the strain is easily seen to be $2f$. The rotation is evidently from AB to AB′, or from AD to AD′; *i.e.* through an angle equal to the *difference* of the parts into which the diagonal divides the right angle of the face, or half the difference of contiguous angles of the rhombus inscribed in the face.

38.* The resultant of two pure strains is not a pure strain, unless the *directions* of the axes be the same for each of the components. Thus we see how non-rotatory fluid motion is, in general, only *differentially irrotational.* Just, in fact, as motion in a curve is *differentially* in a straight line; or as continuously varying speed is *differentially* uniform.

[But two pure strains, in succession, leave a set of three *oblique* directions undisturbed.]

39. When a point is free to move in space, it has three *Degrees of Freedom;* for it may be displaced in any of three rectangular directions, and these displacements are entirely independent of one another.

We subject it to one *Degree of Constraint* by compelling it to move on some surface. It has then only two degrees of freedom.

A second degree of constraint limits it to some other surface, so that it must be somewhere on the line of intersection of the two surfaces. On this, it has but one degree of freedom.

40. Any displacement of a plane figure in its own plane, or of a spherical cap on a sphere, can be effected by a mere rotation about one of its points. [This point, in the plane figure, may be infinitely distant, and then the displacement is a mere translation.] The figure has,

in fact, only three degrees of freedom, and when one point is fixed, two of these are lost.

A rigid figure has six degrees of freedom, three for any one of its points, and three more for rotations about three rectangular axes passing through that point. Each degree of constraint removes one degree of freedom. Fix one point of it, and three degrees of freedom are lost. Fix a second, and two more (those which were left to that second point) are lost. Fix a third point (not in the same line with the first two) and the single remaining degree of freedom is lost.

Thus it requires infinitely perfect fitting to make one hard body stand on another by more than three legs at a time. This obvious principle is ingeniously applied in the very useful instrument of precision called the *Spherometer*.

CHAPTER II

DYNAMICS.

A. Introductory.

41. Before we give Newton's Laws of Motion it is indispensable that we should have definitions of the novel terms which they require for their enuntiation or for that of their more immediate consequences.

α. By the ***Mass*** of a body we mean the quantity of matter in it. Its true measure is the Inertia of the body, but a convenient practical measure is its Weight. [That weight and mass are proportional, is not a mere truism but an essential part of the law of gravitation.] The British unit of mass is the *Standard Pound.*

β. The *Moment of Inertia* of a body, about any axis, is defined as the sum of the products of each particle of the mass into the *square* of its distance from the axis. This is, of course, not a vector.

γ. The *Momentum* of a particle, or of a non-rotating rigid body, is the product of its mass, and its velocity. Change of momentum is the product of mass and change of velocity. Similarly with acceleration of momentum, moment of momentum, &c. These are all essentially *vectors* or directed quantities. The moment of momentum of a body which rotates about an axis is easily seen to be a vector whose direction is that of the axis, and whose length is directly as the moment of inertia, and as the angular velocity, about it.

δ. The term *Mass-vector*, used to denote the product of a mass by its vector drawn from any origin, is very useful for one special form of application. For it is easy to see that there is always *one*, and only one, point in every system of masses, which is such that with reference to it the sum of the mass-vectors is *nil*. This point is called the *Centre of Inertia* of the system. And with reference to any other point as origin the sum of the several mass-vectors is equal to the mass-vector of the sum of the masses collected at the centre of inertia (§ 8). As the rate of change of a mass-vector is momentum, we see that the momentum of the whole mass collected at the centre of inertia is the sum of the vector momenta of the several masses.

[The term *Centre of Gravity* is frequently used as equivalent to Centre of Inertia, and this involves practically very slight errors. But it is only a limited class of bodies which possess the power of attracting all other bodies *exactly* as they would do if their whole mass were concentrated in one definite point. Such bodies are called *Centrobaric* (§ 51).]

ε. The *Kinetic Energy* of a body is conveniently taken as *half* the product of its mass and the square of its speed. If the body have rotation as well as translation, it may be regarded as made up of small parts, each with its own velocity of translation:—and the kinetic energy of the whole is the sum of the kinetic energies of the several parts. The kinetic energy of rotation about any axis is easily seen to be half the product of the moment of inertia by the square of the angular velocity. Energy, in all its forms, is essentially non-directional.

ζ. *Force* (whence the term *Dynamics*) is defined as any cause which alters the momentum of a body; and is measured, if constant, by the change of momentum which it produces in one second: if variable, by the rate of production of momentum, *i.e.* by the vector-acceleration of momentum (γ). In fact (as shown in our introductory statements) it is merely another name for this quantity. Unit force is that which, acting for one second on a mass of one pound, produces in it a change of velocity of one foot per second. When, as in the collision of very hard bodies, we cannot ascertain either the magnitude of the force or the time of its action, we take the change of momentum as the measure of what is called an *Impulse*.

η. *Work* is the product of a force by the component (in the direction of the force) of the displacement of its point of application. [Thus if the displacement of the point of application is always perpendicular to the force, no work is done.] When, for the displacement, we substitute the rate of displacement, the product is called *Rate of doing Work*, or *Activity*, or (in engineering phrase) *Power*. This, also, is non-directional. When work is done on a body its equivalent is produced in the form of kinetic, or of *Potential*, energy. The simplest typical form of doing work is that of raising a weight, and thus the simplest form of potential energy is that of a raised weight. When it is water which is raised, we speak of its *Head;* and the potential energy is the product of the weight of the water into its head.

B. Laws of Motion.

42. First Law of Motion:—

Every body perseveres in its state of rest, or of uniform motion in a straight line, except in so far as it is compelled by forces to change that state.

This is the statement of the *Inertia* of matter. It furnishes us with a mode of measuring time (§ 3), and it shows that, in equal times as given by the motion of any one body free from the action of force, any other body free from the action of force will also describe equal spaces. As all motion which we can observe is relative, this may be more conveniently put in the form:—Either of two bodies, each free from the action of force, moves uniformly in a straight line relatively to the other. Clerk-Maxwell, in his little work *Matter and Motion*, has shown that a denial of this Law would be equivalent to an assertion that we could define *absolute rest.*

But Newton states further, though not in this form of words, that moment of momentum (§ 41, γ) also is preserved, in virtue of inertia, by a free rotating body. [From the point of view of *force*, however, this properly belongs to the Third Law.]

43. Second Law:—

Change of momentum is proportional to force, and takes place in the direction of that force.

See the statements of § 41, ζ, again. Nothing is said in the Law as to other simultaneous forces, nor as to the original momentum. Hence it is with the changes alone that we have to deal. When, therefore, two forces act simultaneously on the same body, the changes of *velocity* will be in the directions of the forces and proportional to their magnitudes. Thus the *Resultant* of two forces is to be found by the same rule as that of two velocities. Hence (as in § 8) the parallelogram and the triangle of forces. But it is particularly to be noticed that Newton says nothing about forces which *tend* to produce change of momentum. According to him there may be balancing of the effects of forces, but there is no balancing of forces.

[The whole of this trouble is introduced by the anthropomorphic notion, Force. All that we can say is that no transference, or transformation, of energy takes place.]

Again, if equal forces act for equal times on two bodies, the changes of momentum are equal; and therefore the masses are inversely as the respective changes of velocity.

If different forces act for equal times on equal masses, the forces are as the respective changes of velocity.

If a number of masses, each acted on by a force, acquire in one second the same change of velocity, the forces must be proportional to the respective masses. This is found to be the case with bodies falling side by side *in vacuo;* so that, in any one locality, the weights of bodies are proportional to their masses (41, a).

These two Laws enable us to determine the motion of a body under the action of given forces; or to find, from the given motion of a body, the force which is acting on it:—but they do not suffice for the investigation of a case in which two bodies influence one another's motion, as for instance the components of a chain-shot. Hence the necessity for the

44. Third Law:—

To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed.

The word *Action*, in this statement, Newton says may be taken in either of two perfectly distinct senses:—it may mean either *Force*, or *Activity*, as defined respectively in § 41, ζ and § 41, η, above. Thus the form of words above represents two different Laws:—which may be stated thus:—

Momentum is transferred (without change) from one body to another.

Energy is transferred (without change of amount) from one body to another.

The first is called the *Conservation of Momentum*, the second the *Conservation of Energy*.

As energy is essentially non-directional, Newton's Latin should be translated, for the second interpretation, "*equal, in the sense of gain and loss*"; instead of "*equal, and oppositely directed.*"

Thus, in the example at the end of § 43, the chain-shot, *as a whole*, suffers no change either of momentum, of moment of momentum (41, γ), or of energy, in virtue of the mutual action and reaction of its components.

45. A pair of equal and oppositely directed forces, acting in one line, is a particular case of what is now called a *Stress*. The stress along a stiff rod (necessarily the same across every transverse section) may be either a *Thrust* or a *Tension*, that along a string or a chain can be *Tension* only. [But the term stress, in its widest signification, means *any* system of equilibrating forces.]

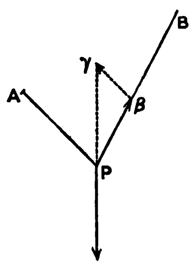
In a fluid the stress at any point is generally what is called *Hydrostatic Pressure*, whose characteristic is that the stress is the same across a small given plane area, drawn in any aspect whatever through the point. It may, in certain cases, be a *Tension*. In the motion of viscous fluids, however, there is usually tangential stress also on any plane area. In all these cases the stress is measured by the amount per unit area of the surface on which it is exerted.

46. We proceed to particular deductions from the laws of motion. We will take them in the order of their

simplicity, commencing naturally with cases in which the effects of the applied forces are such as to compensate one another; so that the motion of the system on which they act is not altered. These form a special branch of Dynamics, which is logically called *Statics*, the remaining part being called *Kinetics*. [Statics, in consequence mainly of an erroneous idea of the action of force (alluded to in § 43) has been very generally contrasted with, instead of being included in, Dynamics.]

C. Statics of a Particle.

47. The forces, equilibrating on a particle, must, as vectors, form the sides, taken in the same order round, of a closed polygon (§§ 8, 43). This statement contains the complete solution, whenever the problem is determinate: —and its application, in general, is a simple matter of geometry.

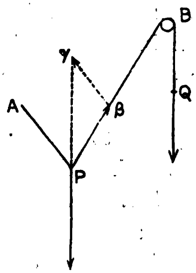


Thus, if a pellet P be supported by two strings AP, BP, their other ends may be either

1. Fixed at A and B.
2. Fixed at A. Passed over a pulley at B, and carrying a pellet Q.

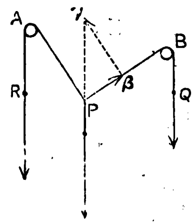
3. Passed over pullies at A and B, and carrying pellets R and Q.

In the vector triangle $P\beta\gamma$, whose sides represent the forces, the side $P\gamma$, representing the weight of P, is given in magnitude and direction for all three cases.



In case (1) the *form* of the triangle is determined by the positions of A and B, and the lengths of the strings.

In case (2) $P\beta$ is given in magnitude, so that the geometrical form of the problem of finding the position of equilibrium is:—to find a point P, at a given distance



from A, such that a triangle, two of whose sides are given, may have these sides respectively vertical and directed to B, while the third side is parallel to AP.

In case (3) we have a far simpler problem; for all three of the sides are known, besides the direction of one and the plane of the other two. Here, however, there may be no solution, for one of the three sides may be greater than the sum of the other two, and then the triangle cannot be constructed.

These different forms of a particularly simple case will, when carefully considered, be found to illustrate very fully almost every ordinary question in this part of the subject.

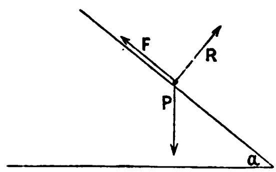
But cases of indeterminateness, such as may often occur when friction is one of the forces at work, require some special notice.

48. *Friction* is tangential to the surfaces of two bodies in contact, and it is always in such a direction as to *oppose* sliding of one on the other. Its amount adjusts itself so as just to prevent motion:—and its *utmost* amount is proportional to the normal pressure between the surfaces. The coefficient of proportionality, *i.e.* the *Coefficient of Statical Friction*, depends upon the nature of the bodies and the amount of polish, of lubricant, &c.:—but, except in extreme cases (sharp edges, points, &c.), not upon the area of the surfaces in contact.

When sliding takes place, the friction is still opposed to the motion, and proportional to the normal pressure. But the *Coefficient of Kinetic Friction* is less than that of Statical friction.

49. From last section we see that the direction of statical friction is determinate, but the amount called into play in each special case is only what is required for equilibrium. As a simple example of a question which may involve absolute indeterminateness, take a body placed on a rough inclined plane, and suppose at first that there is no other support. Here three forces, only, are involved:—one of which, the weight of the body, is fully known. The directions of the other two are known, so that the vector triangle can at once be drawn: and therefore its

sides can be calculated. If the inclination of the plane to the horizon be such that the value of F must bear to that of R a ratio greater than the coefficient of friction, it is impossible to obtain equilibrium. But suppose that the body is attached, by a string, to a point of the plane above it and in the line of steepest slope. How much of its support is due to tension of the string, and how much to friction, is a matter of chance adjustment:—and thus the indeterminateness comes in: not, of course, in the physical question itself, but in our attempts at mathematical treatment of it. The two forces have the same direction, and all that we have the means of knowing is the *sum* of their magnitudes. If the string



were passed over a smooth pulley, and supported a given mass at its other end, the indeterminateness would at once disappear:—but, if there were friction at the pulley it would again come in.

50. The process described in § 47 is not confined to the equilibrium of a *single* particle. We may apply it, in succession, to every link of a chain which hangs in equilibrium. And thus, by the employment of the Third Law (in its *first* interpretation, § 44), we may show that the horizontal component of the tension is of the same value throughout; while the vertical component varies from point to point by the weight of the portion of the chain between them. And, from these results, proper mathematical methods enable us to find the form of the

chain. [A very simple consideration (§ 55) enables us to deal at once with finite portions of the chain.]

Another simple, but extremely useful, result of such considerations is that the force, per unit of length, required to keep a tended chain in a curved form, is in the direction of the radius of curvature, and its magnitude is the product of the tension and the curvature.

D. Attraction, and the Potential.

51. The most varied examples of this part of our subject are furnished by the question of *Attraction* (gravitational, electric, magnetic, &c.) where the problem presented may be looked on as that of finding the resultant of an infinite number of infinitely minute forces, acting in different directions at one point, or upon one particle. In what follows, we assume the attraction between two particles to be inversely as the square of their mutual distance.

The most important single instance we can choose is the attraction of a uniform spherical shell of gravitating matter for a material particle. It is easily shown (*Properties of Matter*, §§ 140, 141) that such a shell exerts no attraction on a particle within it :—while it attracts any external particle precisely as if its whole mass were condensed at its centre; which is thus a true centre of gravity (§ 41, δ). Also that the part of the shell which is visible from the external point exerts half of the whole attraction. By supposing the sphere to grow indefinitely in diameter, retaining the same quantity of matter per unit of surface, and keeping its nearest point at a constant distance from the attracted particle, the attraction gradually attains a constant value $4\pi\rho$, independent of the distance from the surface. But the visible portion becomes virtually an infinite plate of the same surface density, so that the attraction of such a plate is perpendicular to its surface, and of constant amount $2\pi\rho$ wherever the (unit) attracted particle is situated.

This special case of the theorem is of great use in elementary problems of Statical Electricity, as are also the following; which apply at once to common questions of distribution of a charge, and of an induced charge, on a conductor.

Starting from the fact that a uniform mass, bounded by concentric spherical surfaces, exerts no attraction on a particle within it:—we can at once (§ 34) extend the proposition to the case of a homogeneous shell bounded by similar, similarly situated, and concentric ellipsoids.

Geometry shows that the locus of a point, the ratio of whose distances from two given points is constant, is a sphere surrounding one of these points. It is shown below that, if matter be distributed over this sphere with a surface-density (*i.e.* mass per unit of surface) inversely as the cube of the distance from either of these points, it will act upon all external masses as if it were collected at the interior point:—and upon all internal masses as if a definite multiple of its mass were concentrated at the exterior point. Such a distribution has therefore, for all external bodies, a true *Centre of Gravity* (§ 41, δ).

52. Space, surrounding any group of fixed attracting bodies, may be regarded as traversed by *Lines of Force*, whose property is that a particle, placed at any point on one of them, is attracted in the direction of the tangent. Hence what is called a *Level*, or *Equipotential*, surface, *i.e.* one which cuts the lines of force at right angles, is such that there is no work done, by the attraction, on a

particle which moves anyhow upon it. Hence each such surface can be characterised by the amount of energy (*Potential*) which unit of matter possesses when resting on it:—and a definite amount of work is thus required to transfer this unit particle by any path whatever from one given level surface to another. For, if the unit particle change its distance from a particle, m, of the attracting mass, from r to s (s being nearly equal to r) the work done may be written as

$$m/rs.(s-r) = m/r - m/s.$$

(For the force overcome is m/r^2 at one end of the short path $s-r$, and m/s^2 at the other:—so that its average value is m/rs.) The *form* of this expression shows that it is true for *all* values of s, whatever be r. Thus the potential, at distance r from a mass m, is m/r.

Hence the potential of any system is, at any point, the sum of the masses of its constituent particles, each divided by its distance from the point.

When electricity is in equilibrium, the potential must have a constant value throughout the interior of any conductor:—*i.e.* there can be no electric force there. Inside a sphere of radius r, and surface-density ρ, the potential is $4\pi r\rho$; for this is obviously its value at the centre. Outside the sphere, it is (§ 51) inversely as the distance from the centre.

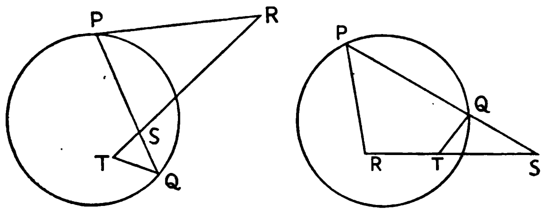
*[Suppose a cone of very small angle, whose vertex is S, to cut out small areas, P and Q, from a spherical surface. Then we have, obviously,

$$\frac{\mathrm{P}}{\mathrm{SP}^2} = \frac{\mathrm{Q}}{\mathrm{SQ}^2}.$$

And, of course, the rectangle SP.SQ is constant, say c^2.

Let R be any point, outside the sphere if S be inside, and *vice-versâ;* and take T (always *inside* the sphere) on RS so that SR.ST $= c^2$. Then, by similar triangles, we have

$$SQ.PR = SR.QT.$$



From these it follows directly that

$$\frac{\frac{P}{SP^3}}{PR} = \frac{Q}{SQ^2.SP.PR} = \frac{Q}{c^2SR.QT}.$$

Thus

$$\Sigma\left(\frac{\frac{P}{SP^3}}{PR}\right) = \frac{1}{c^2SR}\,\Sigma\left(\frac{Q}{QT}\right).$$

The first member is the potential, at R, if the surface-density be everywhere inversely as the cube of the distance from S. The second is the potential, at R, of a mass *concentrated at S;* since $\Sigma(Q/QT)$ is constant, being the potential of the (uniform) shell at an internal point.

The mass of the centrobaric shell is

$$M = \Sigma\frac{P}{SP^3} = \Sigma\frac{Q}{SQ^2.SP} = \frac{1}{c^2}\Sigma\frac{Q}{SQ}$$

so that the expression for its potential at R is

$$\frac{\Sigma\frac{Q}{QT}}{\Sigma\frac{Q}{QS}}\,\frac{M}{SR}.$$

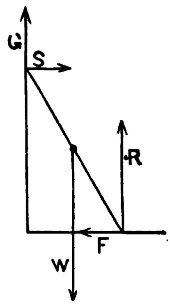
While S is inside the shell, the first factor is unity; otherwise it is directly as the ratio of the distance of S from the centre of the sphere, to the radius. Thus we prove by elementary considerations the important propositions enuntiated in § 51.]

E. Statics of a Rigid Body.

53. For equilibrium of a *Rigid* body (*i.e.* one whose shape and size are unalterable, or at least unaltered) a very simple consideration enables us to find the necessary relations among the applied forces. In fact, if they were not in equilibrium they would set it in motion:—*i.e.* do work upon it. It follows that there can be *no* indefinitely small displacement of the body in which, on the whole, work would be done by the applied forces. Hence, by supposing the displacement to be translational, we see that the forces must satisfy the polygon test, as for a single particle; and, by imagining a rotational displacement, we see that they can have no moment about any axis. These conditions are six in number. They are all we can obtain; and they are sufficient, except in cases of necessary indeterminateness.

54. As an instructive, and at the same time simple, case take that of a ladder, resting on the ground and leaning against a wall. The possible motions contemplated are confined to a vertical plane, so that our six conditions are reduced to three:—viz. there must be equilibrium for horizontal and for vertical translations, and for a rotational displacement, in that plane.

The forces involved are:—first, the weight of the ladder, acting at its centre of gravity (so-called; § 41, δ); second, forces at the ends of the ladder normal and tangential to the ground and to the wall. These are sufficiently indicated in the sketch below.



The condition of equilibrium for a horizontal displacement is

$$F - S = 0.$$

For a vertical displacement,

$$R + G - W = 0.$$

For a rotational displacement (taking moments of the forces about an axis through the lower end)

$$Sl \sin \alpha + Gl \cos \alpha - Wel \cos \alpha = 0,$$

where l is the length of the ladder, el the distance of its centre of gravity from the lower end, α its inclination to the horizontal.

If α be given, there are but three equations to enable us to find four unknown quantities, F, R, G, S; so that the problem is indeterminate. We may make it determinate in many ways. Thus, if there were no friction, but F were due to a rope fastening the lower end to the wall, we should have

$$G = 0, \ R = W, \text{ and } F = S = We\frac{\cos \alpha}{\sin \alpha}.$$

But, if F be due to friction (on the ground alone), its utmost value (§ 48) is μR, *i.e.* μW. Thus the ladder must slide down if

$$e \frac{\cos a}{\sin a} > \mu .$$

Consequently it is more likely to be stable the more nearly vertical it is, and the lower is its centre of gravity. Thus a man, standing on the lower part, may make it stable when it would otherwise fall:—and he may bring it down by climbing it, if it were otherwise stable.

Again, let the coefficient of friction on the wall be equal to that on the ground, and let the inclination be such that the whole is about to slide; we find $\tan a = \left(\mu + \frac{1}{\mu}\right)e - \mu$, from which corresponding consequences may be drawn.

F. Hydrostatics.

55. When we deal with the equilibrium of fluids (liquids or gases) we make use of another very simple principle which enables us at once to reduce any problem on the subject to a corresponding one in statics of a rigid body. For it is obvious that *additional constraints* (§ 39) *will not disturb equilibrium.* Hence, when a fluid is at rest, we may imagine any part of it whatever to become solidified, each particle retaining its mass and position, and the equilibrium will not be affected. The fluid pressures on this rigidified portion must therefore, along with its weight, satisfy the conditions of § 53. By a proper choice of the portions which we suppose to become rigid, we are at once led to the following results.

In a fluid at rest the pressure (§ 45) is the same in all directions at any one point; and, if there be no external forces such as gravity, the same at every point. In this case, practically that of a gas in a vessel of moderate dimensions, it is the same as the pressure on the containing vessel. This is at once evident if we consider the solidified portion to be a small prism, or a tetrahedron. For external force depends on its *volume;* and thus becomes (without limit) of less consequence in comparison with the pressures, which depend upon the *surface*, as the whole is made smaller.

The effect of gravity is to alter the pressure from point to point, in a vertical direction only. Hence the pressure is the same at all points in any horizontal plane:—and it increases, from one such plane to a lower one, by the weight of a vertical column of the fluid, of unit section, extending between the two planes.

The effect of fluid pressures upon a body wholly or partially immersed is, of course, the same as it would be upon a rigidified portion of the fluid which may be supposed to take the place of the body:—in so far, at least, as fluid is displaced by it. Hence a floating body is in equilibrium when it displaces its own weight of the fluid, and has its centre of gravity in the same vertical as that of the displaced fluid. For there is then no unbalanced force, or moment of force, exerted upon it.

G. Character of Equilibrium.

56. Equilibrium may be *stable*, *unstable*, or *neutral*. Though, in special cases, it may be stable or neutral for some displacements, if it be unstable for others it must be regarded as unstable. [Thus a very small pellet, at the middle point of a saddle, is in an essentially unstable

position, though there is an infinite number of directions of displacement for which it is stable, and two for which it is neutral.] Here the conservation of energy furnishes a crucial test. For potential energy is always ready to be transformed into its equivalent in a kinetic form. Thus a body, or a system, cannot be in thoroughly stable equilibrium, if there be any possible (small) displacement by which its potential energy would be diminished. This condition alone, with proper mathematical development, enables us to solve all problems connected with the equilibrium of flexible systems, &c. One very special case of it, which is of great use when gravity is the only external force, may be stated in the very simple form :—*For stable equilibrium of a system, its centre of gravity must be as low as possible.* Take, for instance, the case in which a vessel contains a number of liquids (of different densities) which do not mix or react chemically on one another.

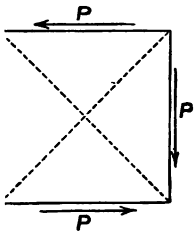
H. Statics of an Elastic Solid.

57. Here we consider those stresses only which produce the two simplest forms of strain :—viz. change of bulk without change of form, and simple shear (mere sliding of layer on layer § 36), *i.e.* the most elementary change of form without change of bulk. For all other strains, however complex, can be built up of these elements. And it is an experimental result of very great importance that, so long as the strains produced by them individually are small, any number of stresses, as well as the corresponding strains, may be superposed. This result includes the statement that, with the same limitation, the reversed stress gives the reversed strain. Of course we thus arrive at what is called *Hooke's Law* :—viz. that (within the

s they are called) *strain is propor-*

ıe cases cited above the stress is hydrostatic pressure, or it may be lirections.

: stress is represented as consisting pressures, applied as below to four ɔe. But it is easy to see, by resolv-



ɔ the diagonals of the free faces ıis stress may be regarded as made .el to one of these diagonals and an to the other. If P be the common al forces on the unit faces, we see at :ude of these new stresses is $P\sqrt{2}$. (diagonal) planes of the cube, over d, are $\sqrt{2}$. Thus P per square unit ces of the stress also.

that a tension P, parallel to one set of be, lengthens them to $1+p$, at the ; the others to $1-q$. Then hydro- l convert each edge of the cube to f k be the resistance to compression

$$3(p-2q)=P/k$$

γ e of § 35.

Similarly, we see that

$$2(p+q) = \mathrm{P}/n$$

where n is the rigidity, and $p+q$ is f of § 37.

From these relations we have at once the values of p and q, thus

$$p = \mathrm{P}\left(\frac{1}{9k} + \frac{1}{3n}\right),$$

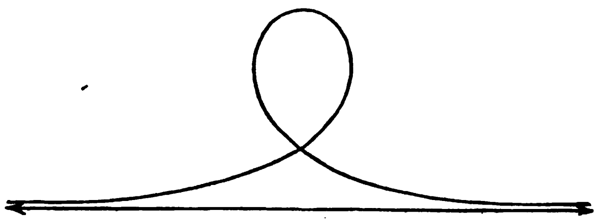
$$q = \mathrm{P}\left(\frac{1}{6n} - \frac{1}{9k}\right).$$

The ratio p/P, or

$$\frac{3k+n}{9kn},$$

is called *Young's Modulus*. It represents, in terms of the two elastic constants, the ratio of the fractional extension of a rod of any material to the tension which produces it; and, with them, is of great use in practical applications.

59. Many cases of extremely complex strain can be treated, sufficiently for important applications, in a very simple manner. Thus, in the bending of a cylindrical wire, the distortion of each element of a transverse slice



differs from that of another, those on the concave side are shortened in the direction of the length of the cylinder and variously expanded in transverse directions, &c. Yet the curvature (unless the bending be carried to an extreme) is directly as the bending moment. Thus the form of an endless wire, with one kink on it, subject to tension, is of the form sketched above:—the curvature at each point being directly proportional to the distance from the line in which the stress acts.

This gives many interesting results as to the forms of capillary surfaces, where the change of pressure (and therefore of level) is, by § 50, proportional to the curvature.

K. Kinetics of a Particle.

60. It follows at once from the second law (§ 43) that we have only to introduce the mass of the particle as a factor of both sides in order to convert the kinematical results of §§ 13–32 into corresponding results in particle kinetics.

Thus, in particular, the force required to keep a particle of mass m revolving in a circle of radius r, with uniform speed v or angular velocity ω, is

$$\frac{mv^2}{r} \text{ or } mr\omega^2$$

and is directed inwards to the centre. And in elliptic motion about the centre, or in simple harmonic motion, we have still the expression

$$mr\omega^2$$

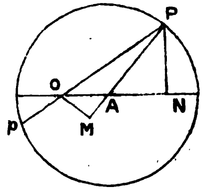
for the requisite force, provided we take ω so that

$$\omega \mathrm{T} = 2\pi$$

where T is the complete period of the motion.

61.* Again, in § 27, we showed that when the acceleration is directed to a fixed point, and is inversely as the square of the distance from it, the hodograph is a circle.

By means of this we may at once prove that the path of a particle, under the gravitational attraction of a spherical mass (*e.g.* the path of a planet or comet about the sun) is a conic section, one of whose foci is at the centre of the mass. Let O be the origin, and suppose the figure to represent the hodograph. Since OP represents the velocity, the tangent at P represents the direction of



acceleration. Thus the component velocity in the direction of acceleration is represented by OM, drawn perpendicular to the radius PA. But PN represents the component velocity in a fixed direction, perpendicular to the greatest and least values of the velocity (the parts of the diameter through O). And the similar triangles show that

$$OM : PN :: AO : AP, \text{ a constant ratio.}$$

Thus the path must be such that the velocity component along the radius vector bears a constant ratio to that in a fixed direction. Hence the length of the radius vector, *i.e.* the distance of a point in the curve from a fixed point, bears a constant ratio to its distance from a fixed line. But this is the focus and directrix property of any conic section.

L. Kinetics of a Rope or Chain.

62. The special example to which we restrict ourselves is not only of singularly extensive application but it

illustrates in a simple manner some very delicate parts of theory.

Suppose a uniform chain of indefinite length to run uniformly, under tension T, in a straight line, except in so far as concerns the part of it which at any instant is passing through a smooth tube whose interior it just fits. The tube is of finite, continuous, curvature, with its terminal portions in the line of the free parts of the chain; but, otherwise, of *any* form; and it is fixed.

By § 50 each unit of length of the chain (in virtue of its tension) presses on the *convex* side of the tube with a force T/ρ; and, by § 60 (in consequence of inertia) it presses on the *concave* side with a force mv^2/ρ; ρ being the radius of curvature, m the mass of unit length, and v the speed.

Hence if $T = mv^2$ there is no pressure on the tube, which thus becomes unnecessary when this state of things is arrived at. To a person walking along with the chain, it will be at rest except where the solitary wave, running along at speed v, is for the moment distorting it.

We thus see how any disturbance whatever is propagated along a stretched cord; and recognise, conversely, that *every* motion of such a cord is resolvable into two simultaneous disturbances running without alteration, at the same speed, but in opposite directions, along it. See the interesting particular case in § 24.

M. Kinetics of a Rigid Body.

63. Here we see that the centre of inertia moves as if it were a massive particle to which all the vector forces acting at different points of the body are applied:—while the motion relative to that centre is the same as it would have been under the given forces, had the centre been

fixed. The first part, therefore, belongs to the kinetics of a particle; while the simplest, and at the same time one of the most important, cases of the second part is rotation about a fixed axis.

If ω be the angular speed of rotation of a body, r the distance of one of its particles, m, from the axis, the momentum of m is $mr\omega$, and the moment of momentum $mr^2\omega$. The whole moment of momentum is, therefore (§ 41, β), the product of the moment of inertia by the angular speed. [Similarly the kinetic energy is half the product of the moment of inertia by the square of the angular speed.] The treatment of any problem of rotation about a fixed axis is thus reduced to that of the motion of a particle in a definite path :—the speed of the particle taking the place of the angular speed, the mass taking the place of the moment of inertia, and the force taking the place of the moments of the forces about the axis.

An interesting case is when a rope or chain, fixed at each end, rotates as a whole, whether it have (equidistant) points of contrary flexure or no. Each link of the chain describes a circle about the axis, and the resultant of the tensions at its ends must therefore be the force (§ 62) required to keep it so moving. The part of this motion, resolved in any plane through the axis, is simple harmonic. It is in fact the simplest case of motion of a violin-string or a piano-forté wire; and this mode of arriving at it shows how the details of any case can be calculated.

N. IMPACT.

64. As an illustration of the treatment of Impulse (§ 41, ζ) we take the very simple case of the direct collision of two hard spheres. Experiment shows that the operation

consists of two stages; the first in which the bodies temporarily deform one another, the second that in which they (partially, at least) recover their forms before finally separating. And it further shows that, for each given pair of bodies, the additional momentum transferred between them in the second stage bears a definite ratio, called the *Coefficient of Restitution* (a quantity never greater than 1), to that transferred during the first stage. Another way of stating this result is that the bodies separate with a relative speed which bears, to that with which they approached one another, this same definite ratio. This is often called a coefficient of elasticity; and bodies which separate with the same relative speed as that with which they collide, are thus called "perfectly elastic." The converse of this statement is frequently made. But we have only to think of a bell, in which there may be *no* rebound of the clapper, even if the materials of both are perfectly elastic, to see the error involved in such modes of expression.

Let M, m be the masses, U and u the speeds before collision, V and v those after separation. Then the whole momentum transferred is expressed by either of the quantities

$$M(U - V) = m(v - u);$$

and we have, as above, if e be the coefficient of restitution,

$$v - V = e(U - u).$$

From these two equations V and v can be found when U and u are given.

The kinetic energy lost is easily found to be

$$\frac{Mm}{2(M + m)}(1 - e^2)(U - u)^2.$$

O. Hydrokinetics.

65. In a frictionless fluid any little element usually has its form changed, from instant to instant, by a pure strain. But two pure strains, in succession, are not in general equivalent to a single pure strain (§ 34), so that the motion is only *differentially* irrotational. Vortex motion cannot be produced, or destroyed, in such a fluid :—for, to do either, friction is required.

In the special case of what is called steady motion, *i.e.* where the direction and speed of motion of a fluid particle depend only upon its position, we have the important theorem :—The whole work required to make room for a small additional element of fluid, and to give it the energy (potential as well as kinetic) corresponding to its position, is the same wherever it is inserted. This is usually stated in the incomplete form :—The pressure is least where the speed of the fluid is greatest.

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