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Nonclassical Optimal Growth

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Abstract

The classical optimal growth model consists of a concave production function satisfying the Inada conditions and a concave utlity function having infinite marginal utility at zero. Under these assumptions, the second order condition is always satisfied, hence consumption is a unique interior solution with marginal consumption between zero and one. In this paper a nonclassical optimal growth model, i.e., the production function is convex for small inputs and concave for large inputs, is studied. It is shown, by example, that a second order condition is needed to distinguish between maxima and minima and that marginal consumption may be negative over an interval. The second order condition is that marginal consumption is less than one. The dynamic properties of the model is also studied.

1. Introduction

The classical optimal growth model consists of a production function which is concave and satisfies the Inada conditions, and a concave utility function having infinite marginal utility at zero. It is also assumed that the planner maximizes the sum of discounted utilities. These assumptions yield a simple and elegant model explaining many aspects of capital accumulation. In fact this model leads to many intuitive and appealing results. In particular the existence of a unique optimal consumption (and investment) function yields a unique, stable optimal growth path which, independent of the initial stock of capital, converges to the unique steady state equilibrium. Other properties are also implied by this model. For example, the consumption function, and the investment function, is always an interior solution with marginal consumption positive and less than one. Many more mathematical results are also implied, and used to generate the results described above, such as the concavity and differentiability of the value function. This leads also to the fact that no second order conditions are needed, or put another way, the second order conditions are already built in to the concavity assumptions. Moreover the necessary first order conditions, i.e., the Euler equation and the transversality condition are therefore also sufficient.

While this model is attractive in view of the simplicity of its assumption and the level of generality of its conclusions, it is accepted as a suitable approximation of economic reality only for economies at an advanced state of development. In recent years, attention has been focused on what is now known as the non-classical optimal growth model, in which the preferences of the planning authority are still assumed to be convex, but the production function is convex for small input levels and concave for large ones. This assumption is a better approximation in economies which are underdeveloped but hope to develop over time. Moreover resource problems, which use optimal growth models to determine the optimal extraction of resources, are also better described by a convex initial portion. In the non-classical model it is still assumed that the production function has at least one fixed point, although it cannot satisfy the Inada condition at zero since it is convex near the origin.

The model thus formulated is obviously a problem in non-concave programming. While it has been recognized in the recent literature ([2], [3], and [5]) that the first order conditions may not be sufficient, no attempt was made to devise a systematic approach to such problems. Rather, the authors resorted to isolated arguments, often complex and sometimes elegant, in trying to establish additional characterization beyond the first order conditions for optimal paths. For example, Dechert and Nishimura [2] used the principle of optimality to prove the monotonicity property of optimal paths.

The major contribution of this paper to the study of normative dynamic models is to provide a simple unified approach to non-concave programming problems, through the use (in addition to the Euler equation and the transversality condition) of a second-order condition for optimality. It will be shown, using a simple example, that this second order condition is actually needed to distinguish maxima of the objective function from other solutions of the Euler equation. The method proposed to derive sufficiency conditions in the nonclassical case is applicable to any dynamic programming problem. However, the concise expression of the second order condition in terms of the optimal consumption function, in our case, is achieved only because our objective function does not depend on the state variable and because of the semi-linearity of the one sector model.

The characterization of optimal paths in the literature is limited to the study of the properties of the (optimal) value function and optimal state or investment functions. However omitted in these studies are the properties of the optimal consumption function. The second order condition derived in this paper reduces to the condition that the marginal consumption function is uniformly bounded from above by one. In the classical case marginal consumption being less than one is a general result of the model while in the nonclassical case this result is the second order condition for the maximization. Another important implication of the nonclassical case is that the intuitive result that marginal consumption is positive does not follow in the nonclassical model. In fact we present an example which shows that marginal consumption may be negative over an interval, i.e., that consumption declines with increasing input levels for an optimal path!

Moreover our analysis provides a rigorous mathematical treatment of the nonclassical optimal growth problem, in that it establishes the needed differentiability properties of the value function and then actually derives the Euler equation rigorously. These differentiability properties turn out to be significant in terms of the induced continuity properties of the optimal state function for the steady state analysis of optimal paths.

Finally, using the second order condition and the differentiability properties of the value function, the convergence properties of the optimal paths are studied. It is seen that, unlike the classical case, two distinct cases are possible, depending on the level of interest rate.

With mild discounting, all optimal paths converge to a unique stable steady state equilibrium regardless of the initial capital level, as in the classical case.

With strong discounting, there may be zero, one or two equilibria (not counting the origin):

i) No equilibria: The origin is then a stable equilibrium to which any optimal path converges.

ii) One equilibrium: It must be stable and on the concave part of the production function and there is an upward jump through the 45° line of the graph of the optimal state function.

iii) Two equilibria: One must be stable and the other unstable.

For both ii) and iii) there exists a critical level of capital below which it is optimal to deplete the capital, and above which it is optimal to accumulate capital up to the stable equilibrium level.

2. The Model

With the exception of the properties of the production function f, the formulation of the problem in the non-classical optimal growth model is precisely the same as that for the classical case. It is assumed that the preferences of the central planner are expressed by a strictly concave utility function u with the following properties:

P1: $u \in C^2(0,+\infty)$

P2: $u' > 0,\ u'' < 0$

P3: $\lim_{c \to 0} u'(c) = +\infty$

The objective of the planner, whose time rate of preference is given by a discount factor δ such that $0 < \delta < 1$, is to maximize the present discounted value of the utility of consumption over an infinite horizon, i.e.,

$$\sum_{t=0}^{\infty} \delta^t u(c_t) .$$

Here c_t is consumption at time t.

At any time t, aggregate output x_t may either be consumed (c_t) or invested ($x_t - c_t$). Letting f denote the aggregate production function, then,

$$x_{t+1} = f(x_t - c_t) \ , \ t = 0, 1, \ldots$$

We assume that initially $x_0 = s$, the properties of f are (refer to Figure 1).

P4: $f \in c^2[0,+\infty)$, $f' \geq 0$

P5: There exists an inflection point $x_I > 0$ such that: $f''(x) \gtrless 0$ if $x \lessgtr x_I$

P6: There exists a point $x_c > x_I$ such that $f(x_c) = x_c$ and $f(x) < x$ if $x > x_c$

Property P5 means that the production function is convex on the interval $[0,x_I]$ and concave on the interval $[x_I,+\infty)$. Thus the production process exhibits increasing returns to scale for low input levels and decreasing returns to scale for high input levels. It is straightforward to show for this one-sector production function that the average product curve $f(x)/x$ has the traditional upside down U-shape, and crosses the marginal product curve $f'(x)$ at the maximum average product x_{max}. Moreover, the marginal product curve lies above the average product curve for stocks smaller than x_{max} and below it for stocks larger than x_{max} (see Figure 2). It is always the case that $x_I < x_{max} \leq x_c$. Also, the region of increasing returns to scale is $[0,x_{max}]$ and that of decreasing returns to scale is $[x_{max},+\infty)$.

The optimization problem is

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \delta^t u(c_t) = V(s) \qquad (1)$$

$$\text{subject to} \quad x_{t+1} = f(x_t - c_t), \ t = 0, 1, \ldots \qquad (2)$$

$$x_0 = s$$

and,

$$0 \leq c_t \leq x_t \quad , \quad t = 0, 1, \ldots \tag{3}$$

We shall refer to a maximizer c_t^* as an interior solution if constraint (3) holds with strict inequality, and as a corner or boundary solution if either $c_t^* = x_t$, or $c_t^* = 0$.

3. Optimality Conditions

A. First-order Conditions

From the properties of f, it follows that the value function V is bounded since for any given s,

$$V(s) \leq \frac{1}{1-\delta} u(\max\{s, x_c\}) \ ,$$

By the principle of optimality:

$$V(x) = \max_c M(c;x) \quad , \quad 0 \leq c \leq x \quad , \quad x \geq 0 \ , \tag{4}$$

where

$$M(c;x) = u(c) + \delta V[f(x-c)] \tag{5}$$

The finite-horizon formulation of the above problem is also given, as it will prove useful in obtaining some subsequent results and illustrations. The one-period horizon (or two-period maximiation) problem is given by:

$$V_1(x) = \max_{0 \leq c_1 \leq x} M_1(c_1;x) \ , \tag{6}$$

where

$$M_1(c_1;x) = u(c_1) + \delta u[f(x-c_1)] \ .$$

Thereafter, the (n-1) period-horizon problem is given recursively by:

$$V_n(x) = \max_{0 \leq c_n \leq x} M_n(c_n;x) \ , \quad (7)$$

where

$$M_n(c_n;x) = u(c_n) + \delta V_n[f(x-c_n)]$$

Our object is to find, for the parametric optimization in equation (4), the absolute maximum of M within the given range of values of c, since the principle of optimality does not hold for local maxima. It will be shown, by way of a simple example, that an interior maximum of M may exist on an interval of x and not be the optimal solution, as it is dominated by a corner solution on part of that interval. In such a case, the local interior maximum of M does not satisfy the principal of optimality, and thus does not satisfy equations (1) and (4). Sufficient conditions on u and f which guarantee that this absolute maximum is interior, i.e., that the maximizer c* (for each positive x) is not a boundary solution of the form c* = 0 or c* = x, will be given below.

Denote this absolute maximizer by g(x) and the resulting optimal state function by H(x), i.e., $H(x) = f(x - g(x))$, $x \geq 0$. Then equations (4) and (5) imply that for any $x \geq 0$:

$$V(x) = u[g(x)] + \delta V[H(x)] \quad (8)$$

From the formulation of the problem, as given by relations (1), (2) and (3), the theorem of the maximum ([1], [6]), the concavity of the objective function and the continuity of the constraints, it follows that V is a continuous function. Therefore, M is a continuous function of c, and thus achieves its maximum on the interval [0,x] for any x.

Now, in order to characterize an interior solution, we must study the differentiability properties of the value function V. These properties turn our to be significant in determining the properties of the function H. It was shown in [2] that V has left and right derivatives on $(0,+\infty)$, and, moreover, that $V'(x^-) \leq V'(x^+)$, for all $x > 0$. The proof of this fact is the same as that for the differentiability of the value function in the classical case, and can also be found in [4].

We now show that M is differentiable at $(c^*;x)$, for all $x > 0$ or in other words that:

<u>Lemma 3.1</u>: <u>For any</u> $x > 0$, V <u>is differentiable at</u> $H(x)$.

<u>Proof</u>: Suppose that V is not differentiable at $H(x_0)$. Then,

$$\frac{\partial M(g(x_0)^-,x_0)}{\partial c} > 0 > \frac{\partial M(g(x_0)^+,x_0)}{\partial c}$$

or

$$u'[g(x_0)] - \delta V'[H(x_0)^+]f'(x_0-g(x_0)) > 0 > u'[g(x_0)] - \delta V'[H(x_0)^-]f'(x_0-g(x_0))$$

Hence $V'[H(x_0)^-] > V'[H(x_0)^+]$, a contradiction, whence the conclusion.

This lemma is a very important result, not only because it allows a rigorous treatment of the necessary and sufficient conditions for optimality, but also in view of its direct relevance to the steady-state analysis, as will be seen.

With this result in hand, the first-order condition for the maximization in (4) is, for any $x > 0$:

$$u'[g(x)] = \delta V'[H(x)]f'(x-g(x)) \ . \tag{9}$$

Differentiating both sides of equation (8) yields,

$$V'(x) = u'[g(x)]g'(x) + \delta V'[H(x)]f'(x-g(x))(1-g'(x))^{1/} . \quad (10)$$

Substituting equation (9) into (10) yields the envelope result:

$$V'(x) = u'[g(x)] , \quad (11)$$

which implies that the value of an extra unit of capital is the marginal utility of consumption derived from it.

Since equation (11) holds for one-sided derivatives at every value of x, it is clear that to a jump discontinuity in g (and thus in H) corresponds a kink in V.

Substituting equation (11) back into (9) yields the Euler equation

$$u'[g(x)] = \delta u'\{g[H(x)]\}f'(x-g(x)) , \text{ for any } x > 0 . \quad (12)$$

This equation will be used repeatedly in the sequel as the generalized first-order condition for an interior maximum of (1).

Along with equation (12), any optimal path must also satisfy the transversality condition (see [4], [2]) given by:

$$\lim_{t\to\infty} \delta^t u(c_t)(x_t-c_t) = 0$$

The existence of an absolute maximum is ensured by three simple assumptions as shown in Lemma 3.2. The proof of Lemma 3.2 requires the derivation of the finite-horizon Euler equation, which is straightforward from the formulation given by equations (6) and (7).

Lemma 3.2: The Conditions $u'(0) = +\infty$, $f(0) = 0$ and $f'(0) > 0$ are sufficient to ensure the existence of an interior solution as the global maximum to the optimization problem described by equations (1), (2) and (3).

1/ Strictly speaking, we are considering one-sided derivatives, which are taken to be $+\infty$ at upward jumps of g and $-\infty$ at downward jumps.

Proof: This result, which is illustrated in Figure 2, follows by induction on n. From

$$\frac{\partial M_1(c_1;x)}{\partial c_1} = u'(c_1) - \delta u'[f(x-c_1)]f'(x-c_1) \ ,$$

it follows that

$$\frac{\partial M_1(0;x)}{\partial c_1} = u'(0) - \delta u'[f(x)]f'(x) = +\infty \ , \text{ for } x > 0 \ .$$

and

$$\frac{\partial M_1(x;x)}{\partial c_1} = u'(x) - \delta u'[f(0)]f'(0) = -\infty \ , \text{ for } x > 0 \ .$$

Hence since M is continuous in c there exists an interior maximizer c_1^* ($0 < c_1^* < x$) for the one-period horizon problem.

Assuming the result holds for the (n-1) period horizon problem, we thus have $0 < g_n(x) < x$ and $g_n(0) = 0$ (and $H_n(0) = 0$). Then, for the n-th period horizon problem,

$$\frac{\partial M_{n+1}(c_{n+1},x)}{\partial c_{n+1}} = u'(c_{n+1}) - \delta V_n'[H_{n+1}(x)]f'(x-c_{n+1})$$
$$= u'(c_{n+1}) - \delta u'\{g_n[H_{n+1}(x)]\}f'(x-c_{n+1}) \ .$$

In particular,

$$\frac{\partial M_{n+1}(0;x)}{\partial c_{n+1}} = u'(0) - \delta u'\{g_n[f(x)]\}f'(x) = +\infty \ , \text{ for } x > 0$$

and

$$\frac{\partial M_{n+1}(x;x)}{\partial c_{n+1}} = u'(x) - \delta u'(0)f'(0) = -\infty \ , \text{ for } x > 0 \ .$$

So $0 < c_n^* < x$, $n = 2, 3, \ldots$. Taking limits as $n \to \infty$ on both sides of the equation for $\frac{\partial M_{n+1}}{\partial c_{n+1}}$, and using the fact that V_n is differentiable at $H_{n+1}(x)$ or equivalently that g_n is continuous at $H_{n+1}(x)$, for all x, the conclusion follows for the infinite-horizon problem.

Although not necessary, the three conditions, $u'(0) = +\infty$, $f(0) = 0$, $f'(0) > 0$ cannot be relaxed in general if strictly interior solutions are sought. In fact, if any one of them is omitted, there exists a u and f for which the optimal solutions will not be an interior one throughout $[0,\infty)$. In particular, see the example given below.

B. Second-Order Conditions

The purpose of the introduction of convexity (or concavity) assumptions in modern optimization theory is essentially two-fold. First, it allows the relaxation of smoothness assumptions. Second, it transforms local extrema into global ones in addition to doing away with the need for second-order conditions.

In the classical optimal growth problem, the concavity of u and f implies that of V (see [4]), thus making M a concave function of c. The optimal solution g(x) is thus unique for each x, and thus the first-order conditions for maximizations are sufficient as well as necessary [4]. Furthermore, the concavity of V also implies that the marginal consumption function g'(x) is between 0 and 1 for all x. This result coincides with economic intuition.

However, in the nonclassical case, with the properties of the production function given by assumption P4, P5 and P6, the optimal solution g(x) is not unique. Hence a second-order condition is necessary to distinguish between maxima and minima. Moreover, g(x) may have downward jump discontinuities and g'(x) can be negative over an interval! These somewhat surprising results will be illustrated below with a simple example.

We start by deriving a concise expression for the second-order condition, assuming the conditions of Lemma 3.2 hold.

Proposition 3.3:

For each $x > 0$, $g(x)$ is the maximizer in (1) if:

i) $0 \leq g(x) \leq x$,

ii) $g(x)$ satisfies the first-order conditions, and

iii) $g'(x) < 1$.

Proof of iii):

The second-order condition for the maximization in (4) is:

$$\left.\frac{\partial^2 M(c;x)}{\partial c^2}\right|_{c = g(x)} < 0 \quad , \ \forall x > 0$$

In other words, for all $x > 0$,

$$u''[g(x)] + \delta\{V''[H(x)]f'^2(x-g(x)) + V'[H(x)]f''(x-g(x))\} < 0^{2/} \qquad (13)$$

Differentiation of equation (9) yields,

2/ Here V" is taken to mean a one-sided derivative of V' with the jumps taken as in footnote 1.

$$u''[g(x)]\, g'(x) = \delta\{V''[H(x)]f'^2(x-g(x)) + V'[H(x)]f''(x-g(x))\}(1-g'(x)) \;.$$ [3/]

(14)

Substitution of equation (14) into (13) yields

$$u''[g(x)] + \frac{u''[g(x)]g'(x)}{1-g'(x)} < 0 \;,\; \forall_x > 0 \;.$$

This reduces to

$$1 + \frac{g'(x)}{1-g'(x)} > 0 \quad \text{or} \quad \frac{1}{1-g'(x)} > 0 \quad \text{or} \quad g'(x) < 1 \;.$$

Since upward jumps violate the second order condition, it follows from the second-order condition that any jump discontinuity of g must be downward; i.e., such that $\lim_{x \uparrow x_0} g(x) > \lim_{x \downarrow x_0} g(x)$, where x_0 is a jump point of g. Consequently, it can be seen from equation (11) that $V'(x_0^-) < V'(x_0^+)$ at x_0, a jump point of g.

The following property of H, which plays a major role in the upcoming stability analysis, is elegantly proved in [2]. However using the second-order conditions it can be shown quite simply.

Corollary 3.4:

The function H is strictly increasing.

Proof: $H'(x) = f'(x-g(x))(1-g'(x)) > 0$, by the second-order condition.

[3/] The chain rule for one-sided differentiation for any two functions f and g, $[f \circ g(x)]' = f'[g(x)]g'(x)$ may not hold if $g'(x) = \pm\infty$ and $f'[g(x)] = 0$.

In our case, with x a jump point of g, if either $V''[H(x)] = 0$ or $f''(x-g(x)) = 0$, equation (14) may not hold.

C. Example

An extremely simple example in a one-period horizon context, and with the convex portion of f only, is given here. This example serves as a perfect illustration of four essential points of interest. It shows:

i) How the absolute maximum function g(x) may switch from a boundary solution to an interior solution thus generating a downward jump.

ii) That the second-order condition may be necessary.

iii) That the marginal consumption function g'(x) may be negative over an interval of x.

iv) That a relative maximum of M does not satisfy equations (1) and (4) on a certain interval of x, where it is dominated by a boundary solution.

Consider the one-period horizon problem with $u(c) = \sqrt{c}$, $c \geq 0$ and $f(x) = x^3$, $x \geq 0$,

$$V(x) = \max_{0 \leq c \leq x} \{\sqrt{c} + \sqrt{(x-c)^3}\} \tag{15}$$

Letting $M_1(c;x) = \sqrt{c} + \sqrt{(x-c)^3}$, it is easy to see that

$$\frac{\partial M(0;x)}{\partial c} = +\infty$$

and

$$\frac{\partial M(x;x)}{\partial c} = \frac{1}{2\sqrt{x}} > 0 \ .$$

All three possible representations of M(c;x) are depicted in Figure 3. The first order condition for the maximization in (15), for an interior solution is,

$$\frac{1}{2\sqrt{c}} = \frac{3}{2}\sqrt{x-c}$$

which reduces to the quadratic equation in c:

$$9c^2 - 9xc + 1 = 0 \qquad (16)$$

The two solutions to this equation are given by

$$g_+(x) = \frac{1}{6}(3x + \sqrt{9x^2-4}) \text{ and } g_-(x) = \frac{1}{6}(3x - \sqrt{9x^2-4}) \text{ for } x \geq 2/3 . \qquad (17)$$

The marginal consumption functions are given by:

$$g_+'(x) = \frac{1}{6}(3 + \frac{9x}{\sqrt{9x^2-4}}) \quad \text{and} \quad g_-'(x) = \frac{1}{6}(3 - \frac{9x}{\sqrt{9x^2-4}}) \text{ , for } x \geq 2/3 .$$

If $0 \leq x < 2/3$, the values g_+ and g_- are imaginary hence no interior solution exists, and $M(c;x)$ is maximized by $g_c(x) = x$, where the subscript c stands for a corner solution. It can be verified that for all $x \geq 2/3$, $0 < g_-(x) \leq g_+(x) < x$, i.e., both solutions are interior and feasible. Moreover, $g_+'(x) > 1 > 0 > g_-'(x)$.

Invoking the second-order condition, it follows that $g_-(x)$ is an interior maximum, while $g_+(x)$ is an interior minimum of $M(c;x)$, for $x \geq 2/3$. Moreover the implications of the second order conditions are consistent with Figure 3.

Note that the interior maximum $g_-(x)$ has the property that marginal consumption is always negative, i.e., optimal consumption is always declining, as a function of output.

The usefulness of convexity assumptions, from a purely mathematical standpoint, is mentioned in the introduction of this section.

While the relevance of convexity to recent economic theory is obvious, its impact on intuitive economic thinking is not easily isolated. Thus, if declining optimal consumption seems counter-intuitive, it is only because economic intuition is implicitly conditioned by convexity assumptions, as found in the vast literature on classical optimal growth.

Notice also that since $V''(x) = u''[g(x)]g'(x)$, the value function is convex on the interval $[\hat{x},+\infty)$ while it is concave on the interval $[0,\hat{x}]$.

It comes as no surprise that this problem has a corner solution on some interval, since in view of Lemma 3.2, the assumption $f'(0) \neq 0$ is violated. This interval will be calculated below. However, first we consider the value function. To the corner solution $g_c(x) = x$ corresponds a value function,

$$V_c(x) = \sqrt{x} ,$$

and to the interior solution $g_-(x)$ given in (17) corresponds the value function $V_I(x)$,

$$V_I(x) = \sqrt{\frac{1}{6}(3x - \sqrt{9x^2-4})} + [\frac{1}{6}(3x + \sqrt{9x^2-4})]^{3/2}$$

It is easy to check that at $x = 2/3$,

$$V_c(\frac{2}{3}) = \sqrt{\frac{2}{3}} > V_I(\frac{2}{3}) = \frac{4}{3}\sqrt{\frac{1}{3}}$$

This implies that the corner solution dominates the interior solution at $x = 2/3$. It can, in fact, be shown that there exists a unique $\hat{x}$ with the property that

$$V_c(\hat{x}) = V_I(\hat{x}) \quad \text{and} \quad V_c(x) \gtrless V_I(x) \quad \text{if} \quad x \lessgtr \hat{x} .$$

Hence, the overall optimal consumption policy, which is the absolute maximum of M, is given by:

$$g(x) = \begin{cases} x , & \text{if } x \leq \hat{x} \\ \frac{1}{6}[3x - \sqrt{9x^2-4}], & \text{if } x \geq \hat{x} , \end{cases}$$

where $\hat{x}$ is the solution of $\sqrt{x} = \sqrt{\frac{1}{6}(3x - \sqrt{9x^2-4})} + [\frac{1}{6}(3x + \sqrt{9x^2-4})]^{3/2}$.

It is readily seen that g has a jump discontinuity at $\hat{x}$ where it is double-valued, and where V is continuous but not differentiable (Figure 4).

Consumption in the second period is given by

$$H(x) = \begin{cases} 0 , & \text{if } x \leq \hat{x} \\ \{\frac{1}{6}[3x + \sqrt{9x^2-4}]\}^3 , & \text{if } x \geq \hat{x} \end{cases}$$

Notice that although $u'(0) = +\infty$, the optimal consumption in the second period is zero.

It is only in the two-period case that one may hope to compute (nonlinear) solutions. However, the distinctive features illustrated in the above example can be shown to extend to problems with an arbitrary horizon.

4. Steady State Analysis

The questions of existence, uniqueness and stability of steady state equilibria of the optimal paths are examined in light of the results obtained in the previous section. Before doing this some more preliminary results will be presented.

In the classical optimal growth model, it is straightforward to show that the continuity of the optimal state function H and the assumption of the Inada conditions on f (i.e., $f'(0) = +\infty$ and $f'(\infty) = 0$) are sufficient to guarantee the existence, uniqueness and global stability of a steady state equilibrium level of capital. Thus, all optimal paths, regardless of where they started, converge to this unique level. However, with the nonclassical production function as described in P4-P6, there are no such simple results on the properties of the steady state or the stability properties of the optimal paths. It will be seen that the stability and steady state properties depend on the rate of interest (or the discount factor δ) and the initial marginal product ($f'(0)$). Since only interior solutions are of interest, in the following analysis the conditions $u'(0) = +\infty$, $f(0) = 0$ and $f'(0) \neq 0$ are assumed throughout.

Let us first formally define a steady state equilibrium.

<u>Definition 4.1</u>:

A steady state equilibrium $\bar{x} > 0$ is a fixed point of the function H, i.e., $\bar{x} = H(\bar{x})$.

In other words, at $\bar{x}$,

$$x_{t+1} = x_t = \bar{x} \text{ , for all } t \text{ .}$$

The following lemma provides a convenient characterization of such points.

<u>Lemma 4.2</u>: <u>A necessary condition for</u> $\bar{x}$ <u>to be a steady state equilibrium is that</u> $\delta f'(\bar{x}-g(\bar{x})) = 1$.

Proof: This follows directly upon substitution of $\bar{x} = H(\bar{x})$ in equation (12).

Notice that as in the classical case the location of the steady state equilibria depend only on the function f.

From the properties of f it is clear (see Figure 1) that the equation $\delta f'(y) = 1$ has either zero, one or two solutions. To each such solution $\bar{y}$ corresponds at most one $\bar{x}$ such that $\bar{x} - g(\bar{x}) = \bar{y}$. This follows from the second-order conditions, since $h(x) \equiv x - g(x)$ is a strictly increasing, and thus injective, function of x, possibly with upward jump discontinuities.

In order to see that the equation $\delta f'(\bar{x} - g(\bar{x})) = 1$ is not sufficient to guarantee that $\bar{x}$ is steady state equilibrium (for which $H(\bar{x}) = \bar{x}$), consider a production function f such that $f(x) \leq x$ for all x but $f'(x_I) > 1/\delta$. In this case, there will be two solutions to the equation $\delta f'(x - g\ (x)) = 1$, none of which will actually yield a steady state equilibrium since $H(x) < f(x) \leq x$, for all x.

The following lemma provides information on the behavior of the H function near 0.

Lemma 4.3: For x sufficiently small $H(x) > x$ if and only if $\delta f'(0) \geq 1$.

Proof: Consider the Euler equation, (12), in the form:

$$\frac{u'[g(x)]}{u'\{g[H(x)]\}} = \delta f'(x-g(x)) \ , \text{ for all } x > 0 \ .$$

For sufficiently small x, $H(x) > x$ implies that $g[H(x)] > g(x)$ since, g being an interior solution, $g(0) = 0$ and g must be continuous and increasing at zero. Thus $\delta f'(x-g(x)) > 1$. Taking the limit as $x \downarrow 0$, it follows that $\delta f'(0) \geq 1$. For similar reasons, $H(x) < x$ for small enough x implies that $\delta f'(x-g(x)) < 1$. Taking the limit as $x \downarrow 0$ and keeping in mind the fact that $h(x) = x - g(x)$ is an increasing function of x and that f' is an increasing function of x, it follows that $\delta f'(0) < 1$.

Before considering the question of stability of a given steady state equilibrium $\bar{x}$, we need the following fact:

Corollary 4.4 (to Lemma 3.1): V is differentiable at any steady state equilibrium $\bar{x}$.

Proof: $\forall x > 0$, V is differentiable at H(x), and $\bar{x} = H(\bar{x})$.

Also,

Lemma 4.5: A steady-state equilibrium $\bar{x}$ is stable if and only if $g'(\bar{x}) > 1-\delta$.

Proof: Clearly, $\bar{x}$ is stable if and only if $H'(\bar{x}) < 1$, so that $H'(\bar{x}) = f'(\bar{x}-g(\bar{x}))(1-g'(\bar{x})) < 1$ of $\frac{1}{\delta}(1-g'(\bar{x})) < 1$.

Lemma 4.6: A steady-state equilibrium $\bar{x}$ is locally stable if and only if $\bar{x} - g(\bar{x}) > x_I$.

Proof: Differentiation of the Euler equation (12) at $\bar{x}$ yields (in view of Corollary 3.4):

$$u''[g(\bar{x})]g'(\bar{x}) = \delta\{u''[g(\bar{x})]g'(\bar{x})f'^2(\bar{x}-g(\bar{x})) + u'[g(\bar{x})]\,f''(\bar{x}-g(\bar{x}))\}(1-g'(\bar{x}))$$

which can be rewritten, since $\delta f'(\bar{x}-g(\bar{x})) = 1$, as:

$$\frac{u''[g(\bar{x})]g'(\bar{x})}{1-g'(\bar{x})} = \frac{1}{\delta}\ u''[g(\bar{x})]g'(\bar{x}) + \delta u'[g(\bar{x})]\ f''(\bar{x}-g(\bar{x})) \ . \qquad (18)$$

Clearly if $f''(\bar{x}-g(\bar{x})) > 0$,

$$\frac{u''[g(\bar{x})]g'(\bar{x})}{1-g'(\bar{x})} > \frac{1}{\delta}\ u''[g(\bar{x})]g'(\bar{x})$$

or

$$g'(\bar{x})[\delta - 1 + g'(\bar{x})] < 0 \ .$$

This implies either $g'(\bar{x}) < 0$ and $g'(\bar{x}) > 1 - \delta$, which is a contradiction, or $g'(\bar{x}) > 0$ and

$$g'(\bar{x}) < 1 - \delta \ .$$

It can be shown that for any x such that $x - g(x) > x_I$, in view of the concavity of f, the desirable results from the classical one-sector optimal-growth case hold. In particular, V is concave and g (and thus H) are continuous in that region, with $0 < g'(x) < 1$. Thus, if $f''(\bar{x}-g(\bar{x})) < 0$, equation (18) yields $g'(\bar{x})\ (\delta - 1 + g'(\bar{x})) > 0$. Since $g'(\bar{x}) > 0$, it follows that $g'(\bar{x}) < 1 - \delta$.

In other words, this lemma implies that any fixed point $\bar{x}$ of H, is stable if $\bar{x} - g(\bar{x})$ is on the concave part of f (i.e., $\bar{x} - g(\bar{x}) > x_I$), and unstable if $\bar{x} - g(\bar{x})$ is on the convex part of f (i.e., $\bar{x} - g(\bar{x}) < x_I$).

Geometrically, one can think of a fixed point $\bar{x}$ of H as being stable if H crosses the 45° line from above to below, and unstable if H crosses the 45° line from below to above.

We are now in a position to put together all the preliminary results and use them to study the existence, uniqueness and stability of steady-state equilibria. However, as is clear from Lemma 4.3, two separate cases, depending on the level of discounting, must be considered.

I. Mild Discounting: $\delta f'(0) \geq 1$:

In this case the following general result holds.

Lemma 4.7: There exists a unique steady-state equilibrium $\bar{x} > x_I$ which is globally stable.

Proof: H must be a continuous function in this case. Otherwise, if x_0 is a jump discontinuity of H, consider the following possible cases (refer to Figure 5):

i) $x_0 < H(x_0)$: then there exists $y < x_0$ such that $H(y) = x_0$, and thus V has a kink at H(y) for some y, which is impossible since V is differentiable at H.

ii) $x_0 > H(x_0)$: then the only way not to have a point y such that $H(y) = x_0$ is to have another jump discontinuity of H, say at x_1, such that x_0 is between $\lim_{x \downarrow x_1} H(x)$ and $\lim_{x \uparrow x_1} H(x)$, and so on. However, since beyond a certain point, H is a continuous function (as in the classical case), this sequence of jumps cannot take place.

From Lemma 4.3, H(x) is above x near 0, which together with continuity, and the fact that the equation $\delta f'(\bar{x}-g(\bar{x})) = 1$ has exactly one solution in this case, implies the conclusion.

Note that in this case, from Lemma 4.5, $\bar{x}$ must be on the concave part of f, i.e., $\bar{x} > x_I$. Thus, any optimal path converges to $\bar{x}$ irrespective of the initial capital stock.

II. Strong Discounting: $\delta f'(0) < 1$

The only definite conclusion in this case is that if $f(x) \leq x$ for all x, then H is a continuous function (the proof of this is already given in (I ii)) such that $H(x) < f(x) \leq x$, and 0 is the only (globally stable) steady-state equilibrium. Hence, since all optimal paths converge to 0, extinction of capital is optimal. However, because of the infinite marginal utility at 0, actual depletion of the capital stock never occurs.

For production functions with $\max \frac{f(x)}{x} > 1$, one cannot determine a priori whether H will have steady-state equilibria (one or two are possible), or if $H(x) < x$ as described above. If H is ever above the 45° line, two cases are possible:

Case 1: If H has only one steady-state equilibrium $\bar{x}$, it readily follows from the properties of f (see Figure 1) that $\bar{x} - g(\bar{x}) > x_I$, and thus $\bar{x}$ is in the concave portion of f. Moreover, since H is below the 45° line initially, there exists a point $\hat{x}$, with $0 < \hat{x} < \bar{x}$, where H has an upward jump that skips the 45° line, i.e., such that $\lim_{x \uparrow \hat{x}} H(x) < \hat{x} < \lim_{x \downarrow \hat{x}} H(x)$. Thus all optimal paths with initial capital stock below $\hat{x}$ converge to the origin, and all those with initial capital stock above $\hat{x}$ converge to $\bar{x}$. Furthermore, if the initial output is $\hat{x}$, then two optimal paths are possible, both yielding the same value $V(\hat{x})$, such that the one with higher initial consumption converges to the origin, and the other one to $\bar{x}$.

In this case, it can easily be seen (referring to Figure 6) that there exists no y with $H(y) = \hat{x}$, and thus no violation of the fact that V is differentiable at H(x), for all x, occurs at the jump discontinuity $\hat{x}$ of H (and g).

Also, the equation $\delta f'(y) = 1$ has two solutions in this case, say $\bar{y}_1$ and $\bar{y}_2$ with $\bar{y}_1 < \bar{y}_2$. However, the equation $h(x) = x - g(x) = \bar{y}_i$, $i = 1, 2$, does not have a solution $\bar{x}_1$ if $y_i = y_1$ because $\lim_{x \uparrow \hat{x}} h(x) < \bar{y}_1 < \lim_{x \downarrow \hat{x}} h(x)$. In other words, the horizontal line $x = \bar{y}$, does not intersect the graph of $h(x)$ due to the upward jump of $h(x)$ at $\hat{x}$. On the other hand, $\bar{x} = H(\bar{x})$ is the solution to $x - g(x) = \bar{y}_2$.

Nothing rules out the possibility that H may have several jump points, in which case all of them must be confined to the open interval $(0,\bar{x})$. Such a jump, say at point $\underline{x}$, must be such that there is no $\underline{y}$ with $H(\underline{y}) = \underline{x}$, for otherwise it would violate Lemma 3.1. Geometrically, this means that the horizontal line at $\underline{x}$ must not cross the graph of H, as illustrated in Figure 6. It can easily be seen, however, that extra jump points confined to the interval $(0,\bar{x})$ do not change the behavior of optimal paths in any way as far as convergence is concerned.

Case 2: It is also possible that H has two steady-state equilibria $\bar{x}_1$ and $\bar{x}_2$ (say with $\bar{x}_1 < \bar{x}_2$), in which case, according to Lemma 4.6, $\bar{x}_1 - g(\bar{x}_1) < x_I$ and $\bar{x}_1$ is an unstable fixed point of H, while $\bar{x}_2 - g(\bar{x}_2) > x_I$ and $\bar{x}_2$ is a stable fixed point of H. In this case the equation $\delta f'(\bar{y}) = 1$ has two solutions $\bar{y}_1$ and $\bar{y}_2$, each of which has a corresponding solution $\bar{x}_1$ and $\bar{x}_2$ from the equation $x - g(x) = \bar{y}_i$, $i = 1, 2$. Although it is clear that $\bar{x}_2$ is in the concave region of f (i.e., $\bar{x}_2 > x_I$), nothing can be said in general about the position of $\bar{x}_2$ relative to that of x_I (both possibilities may occur).

Both cases actually yield the same economic interpretation: there exists a threshold level of capital ($\hat{x}$ in Case 1 and $\bar{x}_1$ in Case 2) below which it is not optimal to start accumulating in view of the extremely low marginal product of capital, and above which it pays to accumulate capital up to the level of the stable steady-state equilibrium.

If the initial capital stock is at this threshold level, optimality dictates remaining indefinitely at that level in Case 2, and a choice between accumulation of capital up to $\bar{x}$ and depletion of capital without ever reaching zero because of the three conditions of Lemma 3.2 in Case 1.

Example:

This example will serve to illustrate the following possibilities

i) the existence of an unstable steady-state equilibrium if $\delta f'(0) < 1$.

ii) the existence of an interior optimal solution even with a strictly convex production function, for appropriate values of the discount factor, δ.

The limiting case of a linear production function is studied in [4]. It was shown there that when coupled with an isoelastic utility function, i.e., $u(c) = \ln c$ or $u(c) = c^{\alpha}$, $0 < \alpha < 1$, this production function yields linear consumption policies. Moreover, the only other choice of f and u which produces linear policies is $u(c) = \ln c$ and $f(x) = x^{\beta}$, $0 < \beta < 1$.

It can be shown that if $u(c) = \ln c$ and $f(x) = x^{\beta}$, $\beta > 1$ (i.e., only the convex portion of f in the nonclassical case), the optimal consumption policy is linear, provided the discount rate is small enough. This is true independent of the length of the horizon.

If the discount rate is large the optimal policy g_n is linear for all n but the infinite horizon optimal policy g does not exist.

Specifically, consider:

$$u(c) = \ln c \ , \ c > 0 \quad \text{and} \quad f(x) = x^2 \ , \ x > 0 \ .$$

The Euler equation reduces to:

$$\frac{1}{g(x)} = \frac{2\delta(x-g(x))}{g[(x-g(x))^2]} \ , \ \forall x > 0 \tag{19}$$

Substitution $g(x) = \lambda x$[4/] in the above equation and solving for λ, yields $g(x) = (1- 2\delta)x$ if $\delta < 1/2$. If $\delta \geq 1/2$, no solution exists for the infinite horizon problem. In the infinite horizon problem with $\delta \geq \frac{1}{2}$ optimality requires that the output be built up indefinitely with no consumption. This is clearly the case since only if the future utilities are strongly discounted will it be optimal, with the increasing marginal product of f, to consume a positive quantity in each period instead of letting all of the output grow indefinitely.

Thus, if $\delta < 1/2$, the optimal stock function H is (see Figure 7)

$$H(x) = [x - (1-2\delta)x]^2 = (2\delta x)^2 = 4\delta^2 x^2 \ .$$

H has a fixed point $\bar{x} = \frac{1}{4\delta^2}$, which is an unstable steady-state equilibrium.

The transversality condition is satisfied by this solution since the exponential decay of δ^t overcomes the parabolic increase in H.

[4/] One must actually prove that there are no other (nonlinear) solutions to equation (19). This could be easily done by considering the finite-horizon version of this problem, from which it will follow that $g = \lim_n g_n$ is unique (with $g_n = \lambda_n x$).

Moreover, $g(x) = (1 - 2\delta)x$ is an interior solution, and the second-order condition is satisfied.

Notice that by inserting a concave portion to f (in a twice continuous differentiable manner) far enough to the right, one will have a valid nonclassical optimal growth problem with one unstable steady-state equilibrium and a stable one.

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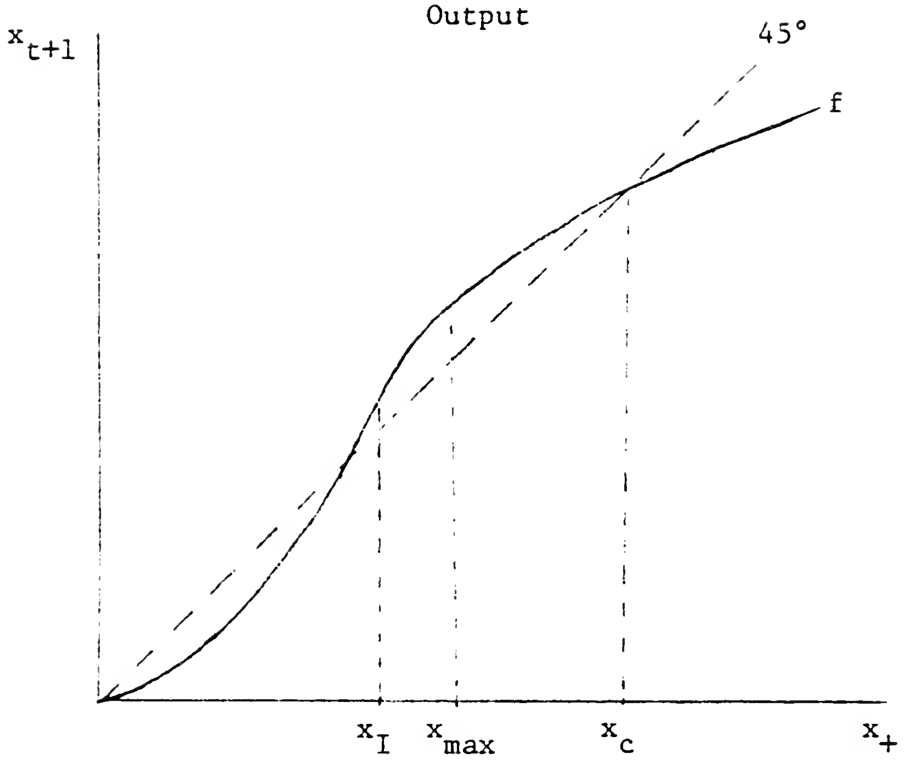
Figure 1

Output



Marginal and Average Product

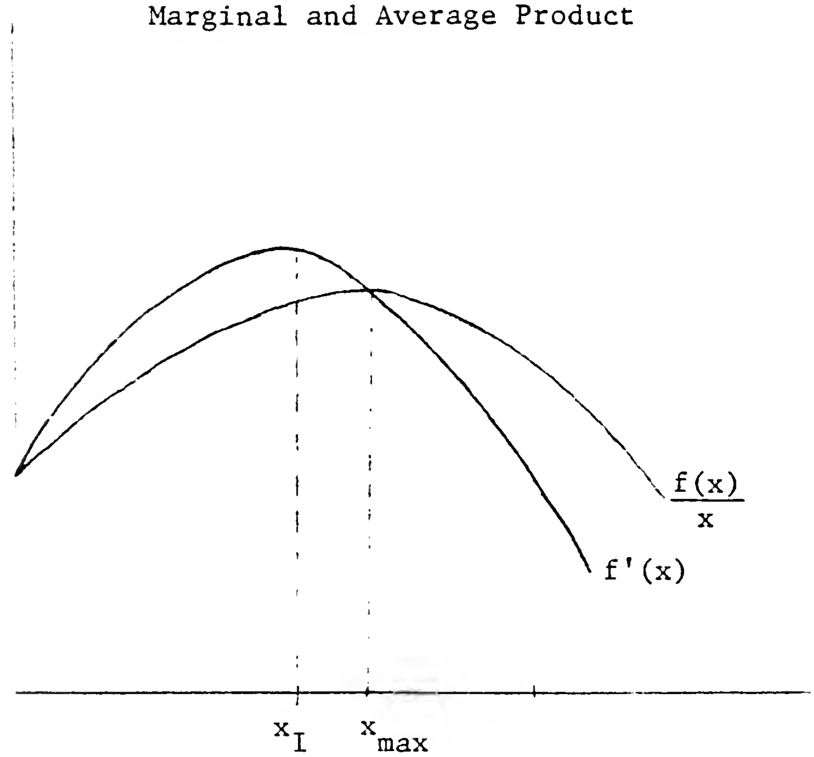


Figure 2

Graph of M(c;x) for a given x
(with $u'(0) = +\infty$, $f(0) = 0$, $f'(0) \neq 0$)

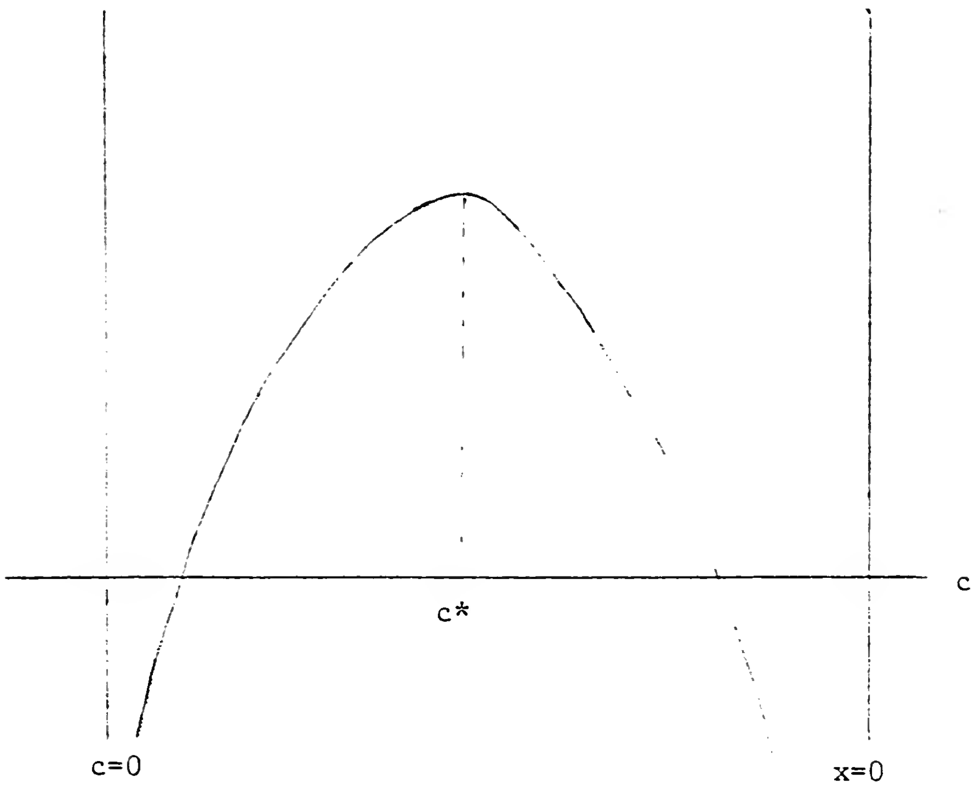


Figure 3

$\tilde{M}_1(c;x)$ for Three Different Values of x

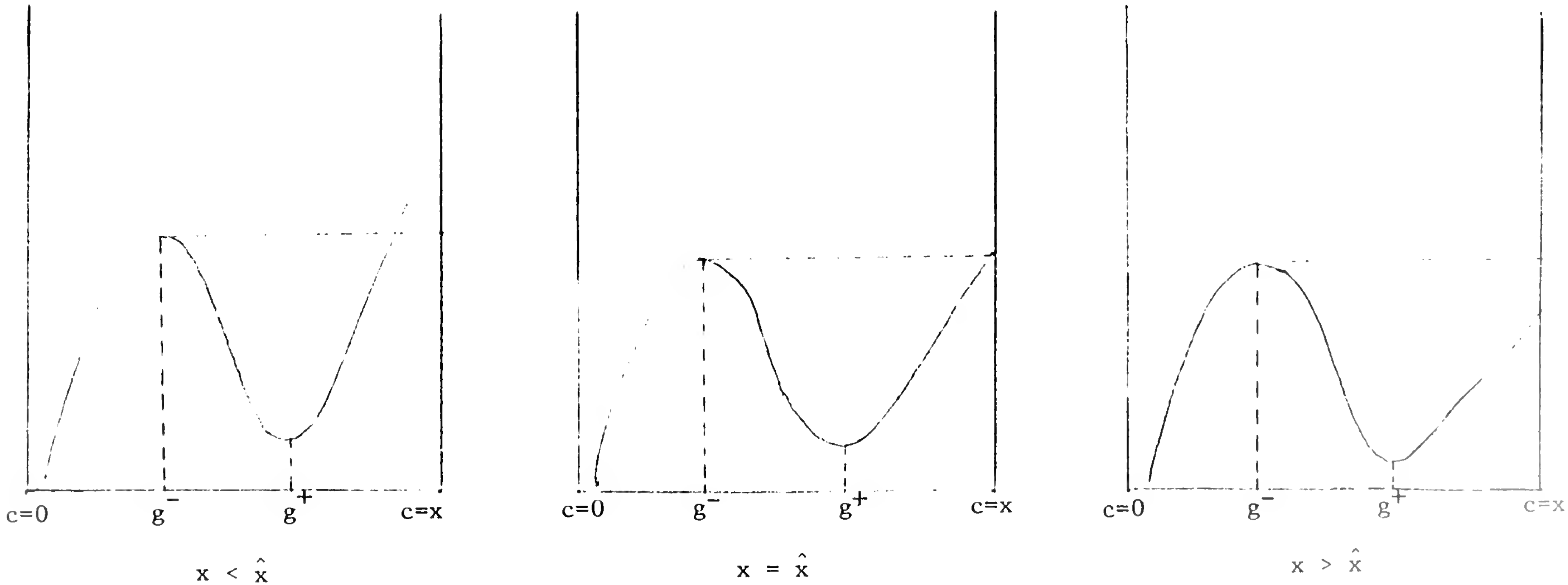
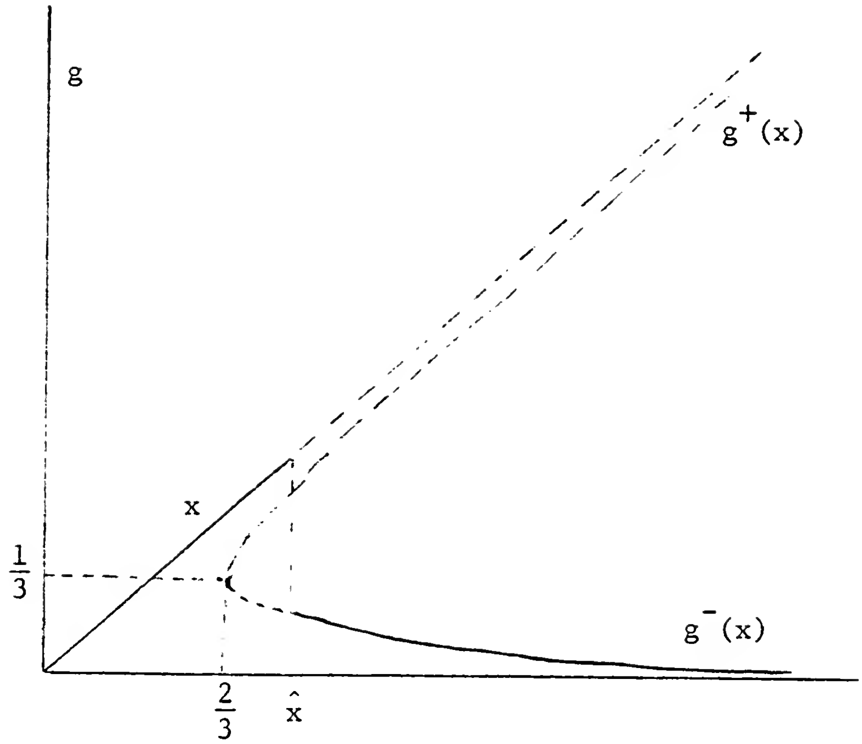


Figure 4

The Optimal Consumption Function



The Value Function

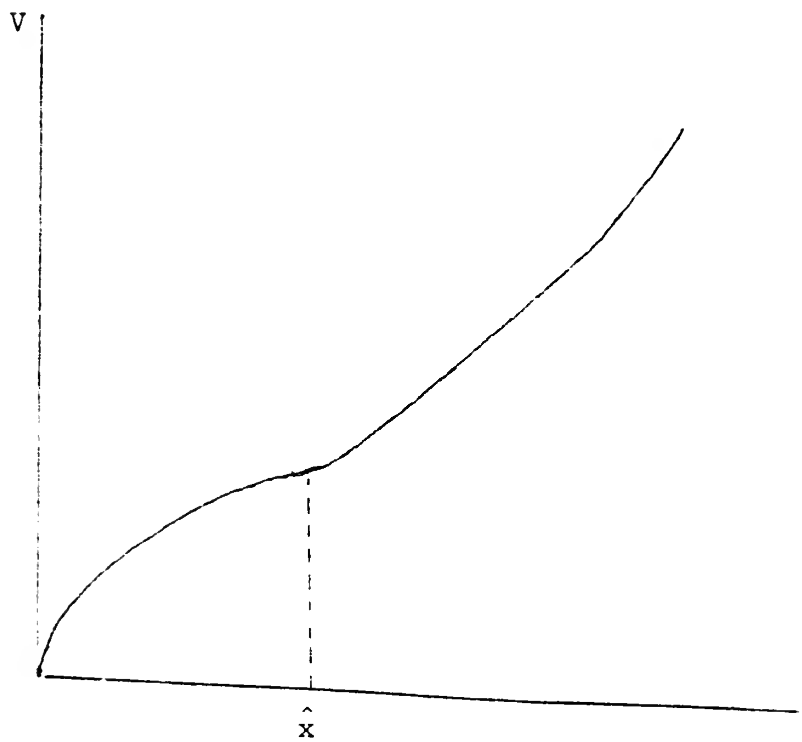
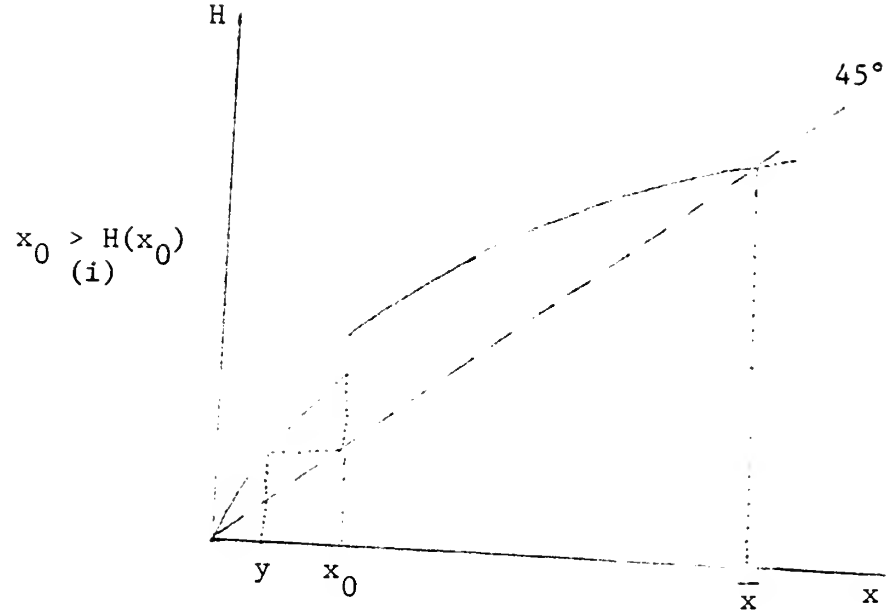
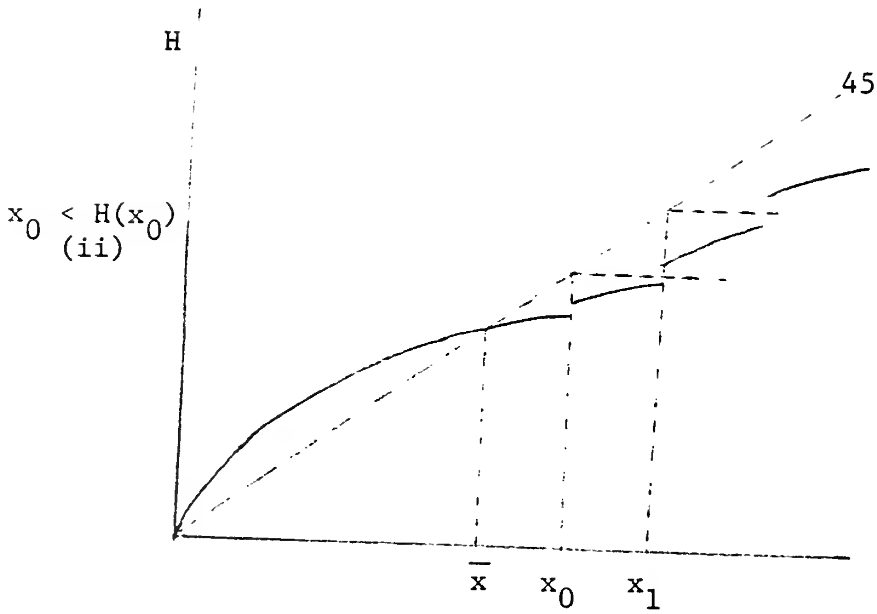


Figure 5





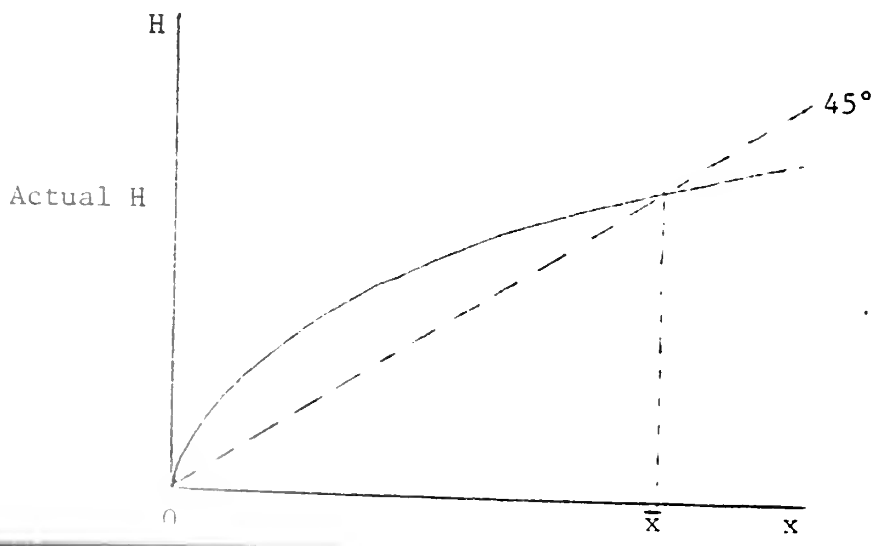
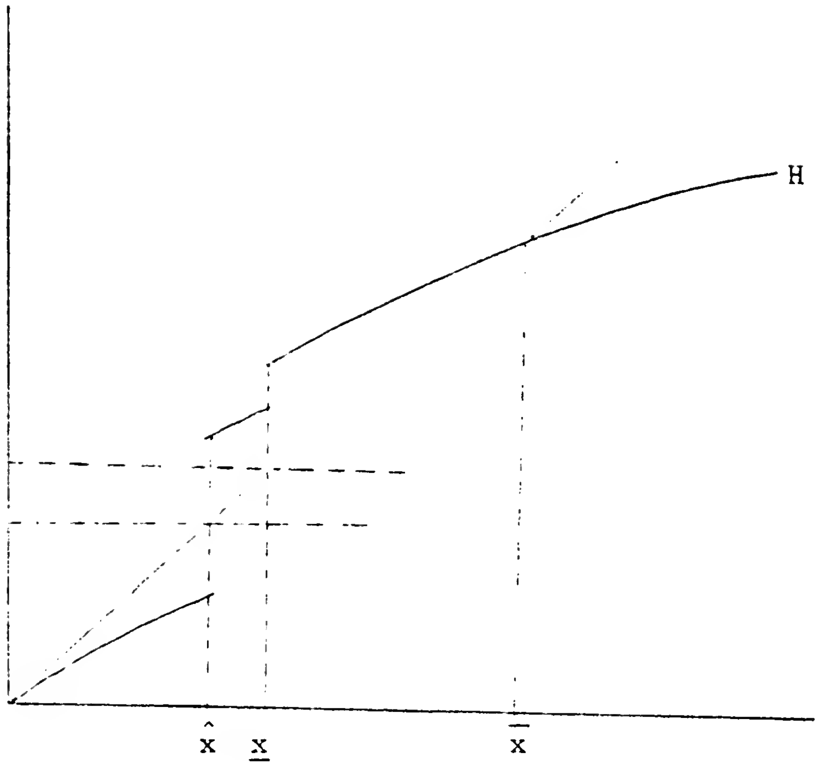


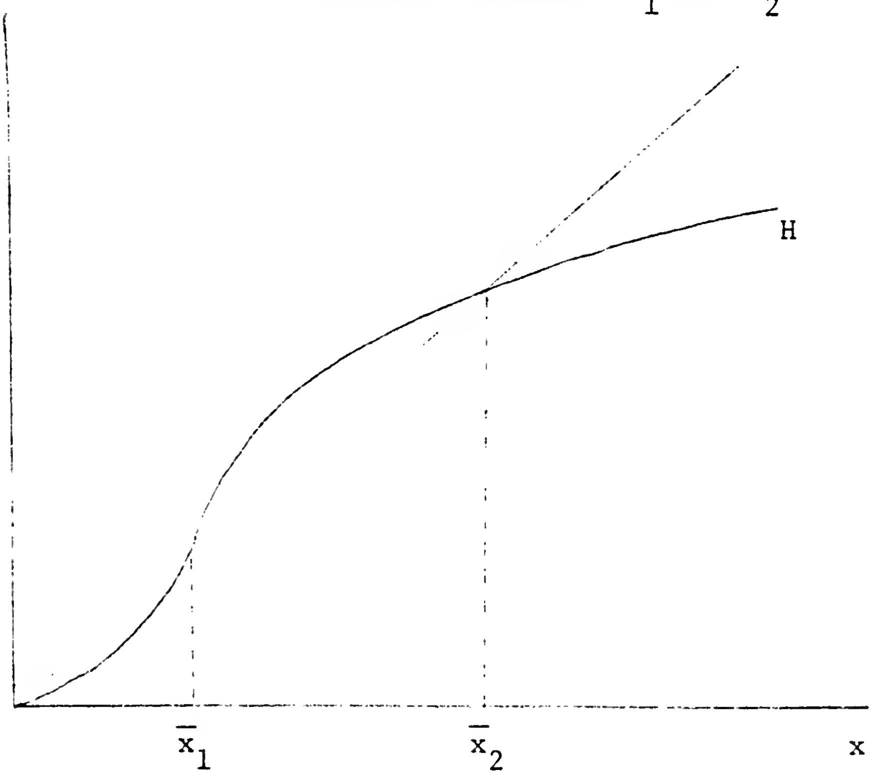
Figure 6

H With One Fixed Point $\bar{x}$



Case 1

H With Two Fixed Points $\bar{x}_1$ and $\bar{x}_2$



Case 2

Figure 7