Notes on Algebraic Potentials

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1. Let $X_1 = \Pi(x-\alpha)$ and $X_n = \Pi(x^n - \alpha^n)$. I propose to call X_n the "nth potential" of X_1, and X_1 the "radical factor" of X_n. Since X_{mn} is divisible by X_m and X_n, the problem of deducing the coefficients of X_n from those of X_1 is connected with the theory of divisibility of polynomials.

2. If p is a prime number, $(\Sigma\alpha)^p - \Sigma\alpha^p$ is divisible by p. Hence, if the coefficients of X_1 are integers, they differ from those of X_p by multiples of p.

3. If the indices of x in X_1 are all divisible by n, then $X_n = X_1{}^n$.

Ex. 1. If $X_1 = x^2+1$, then $X_2 = (x^2+1)^2$, $X_3 = x^6+1$, $X_4 = (x^4-1)^2$, $X_6 = (x^6+1)^2$ . . . .

Ex. 2. If $X_1 = x^3-1$, then $X_{3m} = (x^{3m}-1)^3$.

4. If n is a prime number, and $Y_1 = X_n/X_1$, then $Y_n = X_n{}^{n-1}$. If m is prime to n, then $Y_m = X_{mn}/X_m$.

Ex. 3. Let $X_1 = x-1$, and $Y_1 = X_3/X_1 = x^2+x+1$, then $Y_3 = (x^3-1)^2$.

Ex. 4. Let $X_1 = x-1$, and $Y_1 = X_5/X_1 = x^4+x^3+x^2+x+1$, then $Y_5 = (x^5-1)^4$.

Ex. 5.—Let $X_1 = x^2+x+1$, $Y_1 = x^4+x^3+x^2+x+1$, $Z_1 = X_5/X_1 = Y_3/Y_1 = x^8-x^7+x^5-x^4+x^3-x+1$, then, if n is prime to 3 and 5, $Z_n = x^{8n}-x^{7n}+x^{5n}-x^{4n}+x^{3n}-x^n+1$. If n is divisible by 3, but not by 5, then

$$Z_n = (x^{4n}+x^{3n}+x^{2n}+x^n+1)^2.$$

If n is divisible by 5, but not by 3, then $Z_n = (x^{2n}+x^n+1)^4$. If n is divisible by 15, then $Z_n = (x^n-1)^8$.

5. If A and B are sums of alternate terms of X_1, then $X_2 = A^2 - B^2$.

If u, v, w, are the sums of every third term of X_1, then $X_3 = u^3 + v^3 + w^3 - 3uvw$.

By these formulae X_n can be determined when $n = 2^\mu 3^\nu$.

6. Let X_1 be of degree m, and let $P_1 = \Pi\alpha$ (the "absolute term"), then $P_n = P_1{}^n$, and $X_n = x^{mn} - \Sigma\alpha^n . x^{(m-1)n} + \Sigma(\alpha\beta)^n x^{(m-2n)} + \ldots$
$= \pm P_n(1 - \Sigma\alpha^{-n}.x^n + \Sigma(\alpha\beta)^{-n}x^{2n} + \ldots.)$.

$\Sigma\alpha^n$ and $\Sigma\alpha^{-n}$ may be obtained by the process of dividing X_1 into its first differentials with respect to x and x^{-1}; hence X_n is completely determined when X_1 is of 2nd or 3rd degree in x.

Ex. 6. If $X_1 = x^3 + x + 1$, the successive values of $\Sigma\alpha^n$ are derived from the expansion of $(3+0+1)/(1+0+1+1)$, and those of $\Sigma(\beta\gamma)^n$ from that of $(3+2)/(1+1+0+1)$.

Hence $x^{33} + 67x^{11} + 1$ is divisible by $x^3 + x + 1$.

Ex. 7. If $X_1 = x^3 - 2x^2 - 2$, then $X_8 = x^{24} - 960x^{16} - 2^8$, which is divisible by X_1, X_2/X_1 and X_4/X_2.

Ex. 8. If $X_1 = x^3 + x^2 + 3$, then $X_9 = x^{27} + 271x^{18} + 3^9$, which is divisible by X_1 and X_3/X_1.

Ex. 9. If $X_1 = x^3 - 5x + 5$, then $X_5 = x^{15} + 5^3x^{10} + 5^5$.

7. When X_1 is of the 4th degree, the middle term of X_n may be obtained in the following manner.

Let $X_1 = x^4 + px^3 + qx^2 + rx + s$; then $\alpha\beta + \gamma\delta$ is a root of

$$y^3 - qy^2 + (pr - 4s)y - (p^2s - 4qs + r^2) = 0.$$

Let $(x^2 + s)^3 - qx(x^2 + s)^2 + \ldots. = x^6 + a_1x^5 + a_2x^4 + \ldots.$, then the successive values of $\Sigma(\alpha\beta)^n$ are the terms of the quotient of $6 + 5a_1 + 4a_2 + 3a_3 + 2a_4 + a_5$ by $1 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6$.

Ex. 10. Let $q = 0$; then $\alpha\beta$ is a root of

$$x^6 + (pr - s)x^4 - (p^2 + r^2)\, x^3 + (pr - s)sx^2 + s^3 = 0.$$

Hence $\Sigma(\alpha\beta)^2 = 2(s - pr)$, $\Sigma(\alpha\beta)^3 = 3(p^2 + r^2)$, and

$$\Sigma(\alpha\beta)^5 = 5(s - pr)(p^2 + r^2).$$

Ex. 11. Let $X_1=x^4+x^3-1$, then $\alpha\beta$ is a root of

$$x^6+x^4+x^3-x^2-1=0,$$

and $(6+0+4+3-2)/(1+0+1+1-1+0-1)$

$$=6,\ 0-2-3,\ 6,\ 5,\ 1-14-2,\ 15,\ 23-22-39,\ 0,\ \ldots.$$

Hence $\Sigma(\alpha\beta)^{13}=0$, and $X_{13}=x^{52}+66x^{39}+13x^{13}-1$.

Ex. 12. If $X_1=x^4+x+1$, then $X_5=x^{20}+5x^{10}-4x^5+1$, and $X_{19}=x^{76}+608x^{38}-37x^{19}+1$.

Also, $X_5/X_1=u+v+w$, where $u=x^{16}-x^{13}+x^{10}-x^7-x$, $v=-x^{12}+2x^9+2x^6-x^3+1$, $w=x^8-3x^5+x^2$.

Hence $\Sigma(u^2-vw)=X_1X_{15}/X_3X_5$.

8. By means of the formulae

$$2\,\Sigma(\alpha\beta)^n=(\Sigma\alpha^n)^2-\Sigma\alpha^{2n},$$
$$2\,\Sigma(\alpha\beta)^{-n}=(\Sigma\alpha^{-n})^2-\Sigma\alpha^{-2n},$$
$$6\,\Sigma(\alpha\beta\gamma)^n=(\Sigma\alpha^n)^3-3\Sigma\alpha^n\Sigma\alpha^{2n}+2\Sigma\alpha^{3n}, \text{ etc.}$$

the third term of X_n may be deduced from the 2nd term of X_{2n}, the fourth term of X_n from the 2nd term of X_{3n}, and so on. The process is, however, a tedious one; in many cases a more direct method may be employed, as I shall proceed to show.

9. Let $X_1=x^\mu-px^\nu-q$; then, if μ is divisible by n, $X_n=(x^\mu-q)^n-(px^\nu)^n$.

If $\mu-\nu$ is divisible by n, then $X_n=(x^\mu-px^\nu)^n-q^n$.

10. If $Y_1=ay^2+by+c$ & n is odd, then $Y_n=(ay^2)^n+u_ny^n+c^n$, where

$$u_n=b^n-nacb^{n-2}+\tfrac{1}{2}n(n-3)(ac)^2b^{n-4}-\tfrac{1}{6}n(n-4)(n-5)(ac)^3b^{n-6}+\ldots$$

Making $y=1$, we find that $a+b+c$ is a factor of

$$\Sigma a^3-3abc,$$
$$\Sigma a^5-5abc(b^2-ac),$$
$$\Sigma a^7-7abc(b^2-ac)^2,$$
$$\Sigma a^{11}-11abc(b^2-ac)\{(b^2-ac)^3+(abc)^2\},$$
$$\Sigma a^{13}-13abc(b^2-ac)^2\{(b^2-ac)^3+2(abc)^2\}, \text{ etc,}$$

Let $X_1 = a+b+c$ (a trinomial in x); then X_3, X_5, X_7.... are equivalent to the above expressions, provided that the indices of x in the result are all divisible by n.

When $n=5$, it is always possible to equate the terms of X_1 to a, b, c, so that $X_5 = \Sigma a^5 - 5abc(b^2-ac)$.

Also, $X_5/X_1 = AB - C^2$, where $A = a^2+b^2+c^2-ac$, $B = a^2+b^2+c^2-b(a+c)$, and $C = b(a+c)-ac$.

Ex. 13.—Let $X_1 = x^7+x+1$, then

$$X_5 = x^{35}+x^5+1+5x^{10}(x^5-1) \quad \text{and}$$

$$X_5/X_1 = (x^{14}-x^7+x^2+1)(x^{14}-x^8+x^2-x+1)-(x^8-x^7+x)^2.$$

Let $X_1 = x^7+x^2+1$, then $X_5 = (x^7+x^2)^5+1$.

Let $X_1 = x^7+x^3+1$, then $X_5 = x^{35}+x^{15}+1+5x^{10}(x^{10}-1)$.

Let $X_1 = x^7+x^4+1$, then $X_5 = x^{35}+x^{20}+1+5x^{15}(x^{10}-1)$.

Let $X_1 = x^7+x^5+1$, then $X_5 = x^{35}+(x^5+1)^5$.

Let $X_1 = x^7+x^6+1$, then $X_5 = x^{35}+x^{30}+1-5x^{20}(x^5-1)$.

Ex. 14. Let $X_1 = x^8+x^3+1$, then

$X_5 = (x^8+x^3)^5+1$.

$X_{11} = x^{88}+x^{33}+1-11x^{11}(x^{11}-1)(x^{33}-4x^{22}+3x^{11}-1)$.

$X_{13} = x^{104}+x^{39}+1-13x^{26}(x^{13}-1)^2(x^{39}-3x^{26}+5x^{13}-1)$.

X_7 cannot be found by this method; its value is probably $x^{56}+x^{21}+1+7(x^{35}+x^{28}+x^{14})$.

Ex. 15. Let $X_1 = x^9+x+1$, then X_7, X_{11}, X_{13} are intractable, but

$$X_{17} = x^{153}+x^{17}+1-17x^{17}(x^{17}-1)$$
$$(x^{102}-11x^{85}+31x^{68}-35x^{51}+20x^{34}-6x^{17}+1).$$

11. This method can be applied when X_1 has more than three terms, provided that all the indices except two, and the sum of those two, are divisible by n.

Ex. 16. Let $X_1 = x^5+px^3+qx^2+r$,
then

$$X_5 = (x^5+r)^5+(px^3)^5+(qx^2)^5-5pqx^5(x^5+r)\{(x^5+r)^2-pqx^5\}.$$

12. If $X_1 = Y_1Z_1$, then $X_n = Y_nZ_n$.

Ex. 17. Let $X_1=x^6+x-2$, $Y_1=x-1$,

$Z_1=x^5+x^4+x^3+x^2+x+2$,

then $Z_5=x^{25}+6x^{20}+16x^{15}+26x^{10}+31x^5+32$,

and $Z_7=x^{35}+x^{28}+x^{21}+15x^{14}-97x^7+128$.

Ex. 18. Let $X_1=x^7-2x^3+1$, $Y_1=x-1$,

$Z_1=x^6+x^5+x^4+x^3-x^2-x-1$,

then $Z_5=x^{30}+x^{25}+x^{20}+21x^{15}-11x^{10}-x^5-1$,

and $Z_7=x^{42}+8x^{35}+29x^{28}+64x^{21}-29x^{14}-8x^7-1$.

Ex. 19. Let $X_1=x^{11}+x+1$, $Y_1=x^2+x+1$,

$Z_1=x^9-x^8+x^6-x^5+x^3-x^2+1$,

then $Z_5=x^{45}-x^{40}+5x^{35}-4x^{30}+9x^{25}-5x^{20}+6x^{15}-x^{10}+1$

and $Z_{11}=x^{99}+10x^{88}+44x^{77}+111x^{66}+175x^{55}+176x^{44}$
$+111x^{33}+43x^{22}+11x^{11}+1$.

Ex. 20. Let $X_1=x^{11}+x^7+1$, $Y_1=x^2+x+1$,

$Z_1=x^9-x^8+x^6-x^4+x^3-x+1$,

then $Z_7=x^{63}-x^{56}+x^{42}+6x^{28}+15x^{21}+14x^{14}+6x^7+1$.

Ex. 21. Let $X_1=x^7-7x+10$, $Y_1=x^2-x+2$,

$Z_1=x^5+x^4-x^3-3x^2-x+5$,

then $Z_5=x^{25}+11x^{20}+89x^{15}+627x^{10}+549x^5+5^5$

and $Z_7=x^{35}+57x^{28}+1231x^{21}+11701x^{14}+40319x^7+5^7$.

Ex. 22. Let $X_1=x^7+7x^2+4$, $Y_1=x^2+x+2$,

$Z_1=x^5-x^4-x^3+3x^2-x+2$,

then $Z_5=x^{25}+24x^{20}+194x^{15}+528x^{10}-11x^5+32$.

Ex. 23. Let $X_1=x^9+17x-6$, $Y_1=x^2-x+2$,

$Z_1=x^7+x^6-x^5-3x^4-x^3+5x^2+7x-3$,

then $Z_5=x^{35}+11x^{30}+89x^{25}+627x^{20}+4049x^{15}$
$+15805x^{10}+44287x^5-3^5$.

Ex. 24. Let $X_1=x^7+13x^2-9$, $Y_1=x^2+x+3$,

$Z_1=x^5-x^4-2x^3+5x^2+x-3$,

then $Z_5=x^{25}+34x^{20}+393x^{15}+1525x^{10}+31x^5-3^5$.

Ex. 25. Let $X_1=x^7-4x^2+8$, $Y_1=x^2+2x+2$,

$Z_1=x^5-2x^4+2x^3-4x+4$,

then $Z_5=x^{25}-12x^{20}+32x^{15}+2^8x^5+2^{10}$.

Ex. 26. Let $X_1=x^8+7x-4$, $Y_1=x^3-x^2+2x-1$,

$Z_1=x^5+x^4-x^3-2x^2+x+4$,

then $Z_5=x^{25}+6x^{20}+19x^{15}+153x^{10}+601x^5+4^5$,

and $Z_7=x^{35}+36x^{28}+510x^{21}+3540x^{14}+12041x^7+4^7$.

Ex. 27. Let $X_1=x^{1}\ +3x^4-1$, $Y_1=x^3+x^2-1$.

$Z_1=x^{10}-x^9+x^8-x^6+2x^5-2x^4+x^3+x^2+1$,

then $Z_5=x^{50}+4x^{45}+11x^{40}+25x^{35}+49x^{30}+82x^{25}$

$+108x^{20}+86x^{15}+21x^{10}+5x+1$,

and $Z_{13}=x^{130}-x^{117}+27x^{104}+78x^{91}+597x^{78}+2862x^{65}$

$+12855x^{52}+41523x^{39}+1080x^{26}+26x^{13}+1$.

Ex. 28. Let $X_1=x^{11}-23x+22$, $Y_1=x^3+x-2$,

$Z_1=x^8-x^6+2x^5+x^4-4x^3+3x^2+6x-11$,

then $X_5=(x^{11}-23x)^5+22^5$, $Y_5=(x^5-1)(x^{10}+11x^5+32)$

and $Z_5=x^{40}-10x^{35}-36x^{30}+602x^{25}-294x^{20}$

$-10854x^{15}+12308x^{10}+95446x^5-11^5$.

Ex. 29. Let $X_1=x^8+3x^3-1$, $Y_1=x^3+x-1$,

$Z_1=x^5-x^3+x^2+x+1$,

then $Z_{11}=x^{55}-67x^{33}+2674x^{22}+34x^{11}+1$.

13. A function which is expressible in the forms $A+mB^2$ and C^2+nD^2 can also be expressed in the form E^2-mnF^2, and, when $m=n$, it has two rational factors. (The exceptional case, where $n=3$, will be dealt with later.)

Ex. 30. $x^4+4x^2+1=(x^2+1)^2+2x^2$

$=(x^2-1)^2+6x^2=(x^2+2)^2-3$.

Ex. 31. $x^4-6x^2+1=(x^2-3)^2-8=(x^2+1)^2-8x^2$

$=(x^2-1)^2-4x^2$.

Ex. 32. Let α and β be roots of $x^2+x+1=0$,

then $x^2+3=(x+2\alpha+1)(x+2\beta+1)$, and $(x^6+27)/(x^2+3)$

$=\{x+\alpha(2\beta+1)\}\{x+\beta(2\alpha+1)\}\{x+\beta(2\beta+1)\}\{x+\alpha(2\alpha+1)\}$

$=(x^2+3x+3)(x^2-3x+3)$.

14. If n is a prime number of form $4m+1$, and $X_1=A^2-nB^2$, or if n is a prime of form $4m-1$, and $X=A^2+nB^2$, then X_n/X_1 has two factors Y_1, Z_1, such that $Y_n=Z_n=X_n^{\frac{1}{2}(n-1)}$.

Ex. 33. Let $X_1=x^2+x+1$, $Y_1=x^2-2x+1$,
then $$X_3=Y_3=X_1^2\,Y_1=(x^3-1)^2.$$

Ex. 34. Let $X_1=x^2+x+7$, $Y_1=x^2+4x+7$, $Z_1=x^2-5x+7$,
then $$X_3=Y_3=Z_3=X_1Y_1Z_1.$$

Ex. 35. Let $X_1=x^4+x^3-2x+1$, $Y_1=x^4-2x^3+x+1$, $Z_1=x^4+x^3+3x^2+x+1$, then $X_3=Y_3=Z_3=X_1Y_1Z_1$.

Ex. 36. Let $X_1=x^2-5$, $Y_1, Z_1=x^4\pm 5x^3+15x^2\pm 25x+25$, then $X_1Y_1Z_1=x^{10}-5^5$ and $Y_5=Z_5=(x^{10}-5^5)^2$.

Ex. 37. Let $X_1=x^2+x-1$, $Y_1=\{(x-1)^5-1\}/(x-2)$, $Z_1=\{x^5+(x+1)^5\}/(2x+1)$, then $X_{5m}=X_mY_mZ_m$ (unless m is divisible by 5).

Ex. 38. Let $X_1=x^4+5x^2+5$, Y_1, $Z_1=x^4\pm 5x+5$,
$$V_1,\ W_1=x^4\pm 5^{\frac{1}{4}}x^3+5,$$
then $$X_5=Y_5=Z_5=V_5=W_5=x^{20}+5^4x^{10}+5^5.$$

Ex. 39. Let $X_1=x^2+x+2$, $X_7=x^{14}-13x^7+2^7$
$$Y_1=x^6+3x^5+2x^4-x^3+4x^2+12x+8$$
$$Z_1=x^6-4x^5+9x^4-15x^3+18x^2-16x+8,$$
then $X_7=X_1Y_1Z_1$ and $Y_7=Z_7=X_7^3$.

Ex. 40. Let $X_1=x^2+11$, x^2+x+3, or $x^2+7^2x+5^4$, then X_{11}/X_1 has two factors of 10th degree.

Ex. 41. Let $X_1=x^2+x-3$, x^2+3x-1, or $x^4-x^3-x^2-x+1$, then X_{18}/X_1 has two factors.

Ex. 42. Let $X_1=x^2+\frac{1}{2}x+1$, then X_1X_{15}/X_3X_5 has two factors of 8th degree.

Ex. 43. Let $X_1=x^2+\frac{1}{3}x+1$, then X_1X_{35}/X_5X_7 has two factors of 24th degree.

15. When n is an odd number, $(x^n-1)/(x-1)=\Pi(x^2-\lambda x+1)$, λ being a root of an equation of degree $\frac{1}{2}(n-1)$.

Let $X_1=\Pi(x-\lambda)$ and $Y_1=\Pi(x^2-\lambda^2x+\lambda^2)$; then Y_1 is a rational factor of X_n, and $Y_n=X^2_n$.

Ex. 44. Let $n=7$;
then $X_1=x^3+x^2-2x-1$, $X_7=x^{21}+57x^{14}-289x^7-1$, and $Y_1=\{(x-1)^7-1\}/(x-2)=x^6-5x^5+11x^4-13x^3+9x^2-3x+1$.

Let $n=11$, then $X_1=x^5+x^4-4x^3-3x^2+3x+1$ and X_{11} is divisible by $\{(x-1)^{11}-1\}/(x-2)$.

Ex. 45. Let $X_1=x^3-3x+1$; then X_9/X_3 is divisible by $(x-1)^6+(x-1)^3+1$.

16. Let u, v, w be the sum of every third term of X_1; then $4X_3/X_1=(u+v-2w)^2+3(u-v)^2$. Hence X_3/X_1 can be expressed in the form A^2+3B^2 in three ways, X_3Y_3/X_1Y_1 in six ways, $X_3Y_3Z_3/X_1Y_1Z_1$ in twelve ways, and so on.

Ex. 46.
$$\begin{aligned} 4(x^9-1)/(x-1) &= (x^4+2x^3+2x+1)^2+3(x^4-1)^2 \\ &= (2x^4+x^3+x+2)^2+3(x^3-x)^2 \\ &= (2x^4+x^3+x-1)^2+3(x^3+x+1)^2 \\ &= (x^4-x^3-x-2)^2+3(x^4+x^3+x)^2 \\ &= (x^4+2x^3-x+1)^2+3(x^4+x+1)^2 \\ &= (x^4-x^3+2x+1)^2+3(x^4+x^3+1)^2. \end{aligned}$$

17. In trinomial equations, various properties of the roots can be investigated by means of the potential coefficients.

Ex. 47. Let α, β, γ . . . be roots of $x^6+x+1=0$, then the successive values of $\Sigma(\alpha\beta)^n$ are

15, 0, 0, 3, 0, 10, 15, 0, 8, 30, 10, 44 . . . .

and those of $\Sigma(\alpha\beta)^{-n}$ or of $\Sigma(\gamma\delta\epsilon\zeta)^n$ are

15, 0, 0, 3, 4, 5, 15, 14, 12, −42, 25, 55, ...

If $xdy/ydx=15+3x^{-3}+10x^{-5}+15x^{-6}+$ . . . . , then $\alpha\beta$ is a root of $y=0$, or of

$$x^{15}-x^{12}-x^{11}-x^{10}-2x^9-x^8+2x^6+2x^5+x^3-1=0.$$

If $xdz/zdx=15+3x^{-3}+4x^{-4}+5x^{-5}+$ . . . . , then $z=0$ is the same equation reversed.

Similarly it may be shown that $\alpha\beta\gamma+\delta\epsilon\zeta$ is a root of

$$x^{10}-9x^8+27x^6+2x^5-27x^4-9x^3+6x+1=0.$$