"THOU
THAT TEACHEST
ANOTHER
TEACHEST THOU NOT
THYSELF?"

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NUMBER AND ITS ALGEBRA

SYLLABUS OF LECTURES ON THE THEORY OF NUMBER AND ITS ALGEBRA

INTRODUCTORY TO A

COLLEGIATE COURSE IN ALGEBRA

BY

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DEDICATED

TO

THE TEACHERS OF MATHEMATICS

IN THE

COMMON SCHOOLS.

CONTENTS.

INTRODUCTION.

"The scientific part of Arithmetic and Geometry would be of more use for regulating the thoughts and opinions of men than all the great advantage which Society receives from the practical application of them: and this use cannot be spread through the Society by the practice; for the Practitioners, however dextrous, have no more knowledge of the Science than the very instruments with which they work. They have taken up the Rules as they found them delivered down to them by scientific men, without the least inquiry after the Principles from which they are derived: and the more accurate the Rules, the less occasion there is for inquiring after the Principles, and consequently, the more difficult it is to make them turn their attention to the First Principles; and, therefore, a Nation ought to have both Scientific and Practical Mathematicians." — JAMES WILLIAMSON, *Elements of Euclid with Dissertations*, Oxford, 1781.

THE preceding arraignment is nearly as pertinent to-day in this country as it was in England more than a century ago. But so far as Geometry is concerned blame no longer rests with the scientific mathematicians. Their investigations of First Principles have not only furnished us with Euclid in his purity, but have developed entirely new and equally consistent geometries, under postulates alternate to Euclid's *petition* of the angle-sum of a rectilineal triangle. Thus has been fulfilled what must at least have opened up as dim vistas to Euclid's mind when he discerned the necessity for assuming, or petitioning as the old geometers called it, his indemonstrable postulate.*

* Called variously the 5th postulate, or the 11th or 12th axiom.

Still further, scientific mathematicians, besides offering the true Euclid *in available text-books* with desirable additions and extensions, have corrected several errors in definitions and demonstrations which constituted the sole blemishes in the most perfect work ever performed by a single man. There is no longer good excuse for teachers choosing texts which present the postulate as a common notion or axiom; to say nothing of such as baldly omit the whole doctrine of ratios and proportionality. There is a momentous difference between ratios and fractions, and text-books which present a proportion simply as an equality of fractions have set up a miserable cause of stumbling. They consider "merely a special case of no importance, whose only excuse for existence lies in the general case omitted."* Incommensurability is the rule, commensurability the exception.

On the other hand, when we consider Arithmetic and Algebra the cap of censure fits the other head. If our scientific mathematicians have furnished satisfactory text-books in these subjects, I am not acquainted with them. All of us who are teaching mathematics must agree with good old Williamson when he complains, in the dissertation already quoted, that he found it more difficult "to make a rational arithmetician than an enlightened geometer."

Let me hasten to say that the apparently controversial tone of this preface springs from no polemical spirit. I approach the task I have set myself with utmost modesty; nay, oppressed by a sense almost of presumption in attempting to clarify what so many have left confused. But so sorely needed is a successful accomplishment of what I

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* Catalogue Univ. of Texas, 1891–1892.

attempt, that an honest effort needs no apology. I wish also to explain that the present treatment takes its form from the immediate practical aim in view; viz., that of a syllabus for a rapid review of such ground of arithmetic and algebra as will best prepare for the study of what goes in our curricula by the name of "higher algebra," with special adaptation to the needs of that large portion of my classes who are taking the course in order to qualify as teachers in the public schools.

I write this Introduction, and dedicate the little work to the teachers in the common schools, however, in the hope of attaining a wider usefulness, in the way of awakening in some Practical Mathematician a desire "to make rational arithmeticians" of the youths whose studies he is directing. It is proper to explain still further that, working away from any great library, I have been compelled to prepare this matter for printing without having time to procure a few published works which I would like to see before committing myself to publication.

I must not be understood as advancing anything new to mathematicians, though I know of no English text-book which consistently expounds and maintains the theories of number and algebra here presented. The work is addressed, not to mathematicians, but to inquiring students and teachers. A sound doctrine of number and its algebra seems to be left by our text-books to chance inference, or deferred to stages seldom reached in undergraduate courses of study. A straightforward development, comprehensible by beginners, of the number concept would be of immense service in mathematical instruction.

For six years I have given my classes the substance of this syllabus as the best explanation I could offer of difficulties which could not honestly be avoided. In July, 1894, I read in the current issue of the *Monist* an article by Hermann Schubert, writing in Hamburg, on *Monism in Arithmetic*, enunciating a unifying principle which he called the Principle of No Exception, referring it originally to Hankel. Of course some such principle is more or less clearly in the mind of every student of mathematics, but having never read Hankel's own statement, I cannot say whether his *Principle of Permanence* is substantially identical with the developing principle I set forth, or rather in line with the notion of algebra as "the science which treats of the combinations of arbitrary signs and symbols, by means of defined, though arbitrary, laws," * — the view of the famous Dean of Ely, and the long line of algebraists of whom he is the prototype. The bare statements of such a principle from radically different standpoints might be confusingly similar to one not fully alive to the fundamental variance. For example, in Schubert's statement of his Principle of No Exception, I recognized what I conceive to be a somewhat inadequate expression of the postulate I had called the Principle of Continuity (I still prefer this name as pointing with direct emphasis to its cardinal outgrowth — the conception of number as continuous), whereas in the next preceding issue of the same journal he is at utter variance with me in declaring that, "all numbers, excepting the results of counting, are and remain mere symbols, nothing but artificial inventions of mathematicians."

* Peacock's *Report on the Recent Progress and Present State of certain branches of Analysis*, in the British Association Report for 1833, p. 195. *Cf.* also Peacock's *Treatise on Algebra*, 1830, republished and enlarged in 1842.

In the article above referred to, Schubert claims that in his *System of Arithmetic*, Potsdam, 1885, he "was the first to work out the idea referred to fully and logically, and in a form comprehensible for beginners;" although it had been previously expressed by Grassman, Hankel, E. Schroeder, and Kronecker. Such is the bibliography *of this special presentation* of the subject, so far as I am aware, not to mention Dr. Halsted's *Number, Discrete and Continuous*,* whose title promises a treatment of this subject, but which remains a fragment, dealing only with discrete number — what I have called Primary Number. Should I be able to spur Dr. Halsted to a completion of this work I shall not have written in vain.

Of course, Hankel's principle must be expounded in his *Theorie der complexen Zahlansysteme*, Leipzig, 1867; but I am yet ignorant of the specific publications of the other authors named, except that in Zeller's jubilee work the matter is referred to in an essay by Kronecker.

On the other hand, the theory here advocated must not be deemed retrogressive, and referred to such writers as Frend,† who, though he very philosophically maintains that, since algebra has its origin and termination in arithmetic, it cannot be considered independent, and fairly enough regards algebra as "the science which teaches the general properties and relations of numbers," yet ends by practically throwing the greater part of the science of number overboard, in rejecting all algebraic forms which do not agree with his undeveloped concept of number.

My theme may be regarded as the *underlying* harmony

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* Preface and four chapters (22 pp.) in *Scientiae Baccalaureus*, June, 1891.

† *Algebra*, 1796.

of the great makers of analytical mathematics, — and my purpose, as an attempt to present to beginners fundamental theory commonly left for the speculations of the most advanced.

Number is such a perfect and typical abstraction that it is difficult to see how a man who has, to use Newton's phrase, "in philosophical matters a competent faculty of thinking," could ever associate the terms *concrete* and *number;* nevertheless this confusion muddles many popular text-books. The question hardly requires or admits of argument. Since it is a vicious habit rather than an illogical deduction which is to be combated, good-tempered ridicule is perhaps the only fit rejoinder. In this spirit may I be permitted to relate an anecdote? Some years ago at the University of Virginia the Professor of Mathematics assigned several problems to be worked upon the blackboards by members of the Junior Class. To one he gave a problem concerning the number of oranges in a pyramidal pile of stated proportions. After expounding the error or propriety of the solutions of some of the other problems, the turn of the orange problem came. The student stood proudly beside his mechanically correct solution. "Well, Mr. Blank," exclaimed the Professor, "how many apples did you find?" A look of consternation overspread the youth's countenance. With a gesture of impatient annoyance he swept the erasing brush over the figures his chalk pencil had traced: "Oh," said he, "I thought you said *oranges!*" In all seriousness, the text-books we have all been abused by, expounding "concrete numbers," solemnly cautioning against confusion of multiplicand and multiplier, divisor and quotient, and unallowable combinations of the terms of a *numerical* proportion, are quite as ridiculous as our hero of the oranges. He displayed at least one virtue, — consistency. Such questions, however, though of great practical importance to the efficiency of our elementary schools, present no real difficulties. A little knowledge of psychology and mathematics will, *if attention be called to the question*, correct mistaken, and develop inchoate concepts of Primary Number. A far more difficult matter remains — to attain for ourselves, and to lead our pupils to attain, a rational concept of number as continuous, a concept absolutely essential to modern mathematics, and now universally assumed as a fact, — implicitly so assumed, even when explicitly denied. It is also necessary to pass beyond the great step already made by Newton, who discerns the continuity of number, but leaves it only "triplex": "Estque (*Numerus*) triplex; integer, fractus, et surdus: Integer quem unitas metitur, Fractus quem unitatis pars submultiplex metitur, et Surdus cui unitas est incommensurabilis" *, with the implied limits zero and infinity. Newton also recognized qualitative distinctions, positive and negative, but the consequent neomonic (so-called "imaginary") and complex numbers remain to be assimilated. I must return, however, to notice a uniquely erroneous view of primary number presented in the last issue of the INTERNATIONAL EDUCATION SERIES, *The Psychology of Number*, by James A. McLellan, A.M., LL.D., Principal of the Ontario School of Pedagogy, Toronto, and John Dewey, Ph.D., Head Professor of Philosophy in the University of Chicago, edited like all of the series by W. T. Harris, U. S. Commissioner of Education.

The astounding thesis is maintained that number is not a

* *Arithmetica universalis:* quoted from Halsted's *Number, Discrete and Continuous.*

magnitude, does not possess quantity at all, and that "no number can be multiplied or divided into parts." * The authors vehemently assert that we might as well talk of any absurdity "as to talk of multiplying a number." † It is much to be regretted that a work of such prestige should merely shift the misconception of concreteness from numbers to the subjects of calculation, which we are told to believe are never numbers at all. Number is most emphatically shown to be "purely abstract," ‡ yet multiplication is claimed to be only of concretes. It is nonsense, we are told, to think of multiplying six by four; you can only multiply six inches, six oranges, by four. Of course, that numbers are multiplied is a *fact*, a fact that psychology may explain, but can in no wise question. After repeatedly insisting upon "the absurdity of multiplying pure number or dividing it into parts," § the authors admit without comment, and in seeming hesitation, "of course, in all mathematical calculations we ultimately operate with pure symbols." ‖ What are these "pure symbols"? What can they be in arithmetic but the pure numbers themselves? It would be an error, shared by many algebraists, to conceive algebra as lacking specific content — as operating with "pure symbols," whatever that may mean. The chapter on the *Psychical Nature of Number* is admirable, and I gratefully invoke its corroboration of what will be found in my syllabus on the subject; but that upon the *Origin of Number*, though very acute in tracing the dependence of measurement upon "adjustment of activity," seems to me mistaken in finding the origin of number in

* *Psychology of Number*, p. 70.
† *Ib.*, p. 70.
‡ *Ib.*, p. 69.
§ *Ib.*, p. 71, foot-note.
‖ *Ib.*, p. 71.

measurement. Measurement is not the source of the concept of number, but a stimulation to clarify and develop the concept; and this is what the facts cited really show. The primary concept of number, as so correctly defined in the preceding chapter, is prerequisite to any attempt at measurement. The savage referred to needs the concept that the length of his arrow is *some* number of handbreadths before he can attempt to discover how many. And long before this he has learned to recognize a small group of objects as a "vague whole," and to "discriminate the distinct individuals," i.e., in the very terms employed to define number, the concept originated before any measurement became possible. Nor, in truth, does number originate in counting, as so commonly asserted; and for a like reason, viz., the concept of some number must precede any device for naming or anywise specializing it. The position that number has its origin in measurement cannot seek strength from the procession toward the absolute of Hegel's ascending categories, quality, quantity (including number, as "quantum in its complete specialization"), measure (*das Maass*), essence; * for Hegel's *Maass*, i.e., "qualitative quantity or measure," † is a very different matter from Dr. Dewey's measurement, in fact, it seems very nearly the same as *number* according to the growing insight of modern mathematics.

Having mentioned Hegel, it is proper to remark that we are just now being reminded on every side — Helmholtz not long ago admonished us — that students of science are frequently driven by the very logic of their subjects into

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* *The Logic of Hegel*, translation, Wallace, p. 192 ("quantum, i.e., limited quantity," p. 190).

† *Ib.*, p. 200.

the regions of philosophy. A vital service will be rendered any serious student, should he be led to consider, either at first or at second hand, the aperçus of Hegel and the power of his method. Without doubt, Hegel has pointed out the true way of logic, if he did not always follow that path. His system is far from fully elaborated; much is tentative, doubtless much mistaken; but the fundamental business of logic (in the Hegelian sense) must remain as he appointed it, a criticism of the very terms of scientific and ordinary thought; nor is a better method than his dialectic likely to be discovered. Dr. Harris (who — *pace* — writes better than he edits) has done our nation substantial service in his *Hegel's Logic*, by condensing, elucidating, and comparatively popularizing a work of prime importance in the progress of human thought. If any reader has perused even the preface of this book, I beg him to recall the suggestive and tonic way in which Dr. Harris recounts his gradual and successive attainment of various "insights" in matters philosophical, and to find encouragement therefrom should he stumble at the development of the concept of primary number which mathematics imperatively demands. D'Alembert's advice to beginners in the differential calculus was "allez en avant et la foi vous viendra."

It remains only to say that in this attempt to elucidate a unifying principle of number, and to display the nature of any algebra, I have kept in mind the capacity of "freshman" students, and have avoided all reference to ultimate categories, psychological or ontological.

I am well aware that there are other avenues of approach to the thesis here maintained, — that "various new mathematical conceptions have been employed by Weierstrass, G. Cantor, and Dedekind in establishing three independent and equally cogent theories which should prove the continuity of number *without borrowing it from space,"* * to say nothing of such theories (*e.g.*, Fine's *Number-System*) as are "content to get continuity from the line." † Something tangible for beginners is a great desideratum. My aim is practical, and it may be claimed that even if difficulties have not been surmounted, or obscurities illuminated, they have at least been reduced to one clear-cut postulate. The student may take stock of his knowledge, and rationally prosecute his studies, even though he consider a gratuitous assumption left in the rear; "la foi viendra."

To such as may condemn the occasional analogical suggestions, and references to general philosophy in this treatise, as unbecoming the proprieties of the severest of the sciences, I would beg to reply that the style of an attempt to explain how mathematics came to be, and what it is, of an effort to lead those who sit in darkness to form the concepts with which mathematics deals, ought not to be judged by the standards of the severe and self-contained procedure of the full-fledged science. My subject soon enters, but begins outside of mathematics; nor is it pedantry to be philosophical in explaining the fundamental concepts of any science. It would be impertinent to be anything else.

I would also deprecate any charge of presumption on account of several innovations in terminology. I am fully aware that reformation must come, if at all, from powerful leaders; but it seemed appropriate in a work of pedagogi-

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* *Number, Discrete and Continuous*, George Bruce Halsted, Preface. The italics are mine. So far as I know no one of these demonstrations has appeared in English.

† *Ib.*

cal intent to point out certain misnomers, and even to "practice what I preach." No confusion can arise from using *neomonic* and *protomonic* for "imaginary" and "real," etc.; and those who deem the current terms consecrated by the usage of the great geometers who have made the science, may ignore the suggestions.

My practical aim must explain the apparently arbitrary intrusion of detail, especially in the final chapter. In this final chapter, it should be said, I have drawn freely from Professor Chrystal's *Text Book of Algebra*, Adam and Charles Black, Edinburgh, 1886. Such points only are touched upon as have been shown by experience to bear directly on the preparation proper to our freshman course in algebra. Especially in freedom of arrangement and allusion, some familiarity with the subject-matter is presupposed; but the knowledge assumed need be neither great nor accurate in order to comprehend what is presented. The vague acquaintance with terms and processes possessed by the ordinary high-school graduate has sometimes warranted the projection of a particular discussion beyond the parallel development of cognate topics in a way which would not be admissible in teaching children. My classes stand upon a vantage ground whence it is permitted to look both forward and backward, and so at last to command a really comprehensive view. A teacher should never forget, however, that at every stage there should be an index pointing upward. Any period of schooling which lacks this incentive must be a barren tract in the experience of the pupil who has traversed its dull course.

I have thus here, as always, striven to avoid what may be deemed the most insidious and mischievous of all mistakes in teaching and textbook-making, — such a stooping to the fancied incapacities of pupils as requires the obscuration of pure thought, the blurring and distortion of truth by substituted analogies and illustrations. Half-truths are dangerous. Pupils nurtured on such philosophical pap too often take up the rôle of teachers without deepened insight, and the spawn of error procreates with the fecundity so characteristic of parasites. This mistake of shutting up all vistas into regions not presently under exploration stultifies the learner, and necessitates a weary process of *un*learning at each stadium. It is in the intellectual sphere the analogon of that contemptible principle of school government which, in the sphere of morals, appeals to timidity or vanity, and depends on espionage, basely ignorant that "the human character is susceptible of other incitements to correct conduct more worthy of employ and of better effect." *

In conclusion, the difficulties, practical and theoretical, of the central problem in this little work entitle it to be judged with leniency. It is submitted to my classes and to fellow-teachers for such uses as it may deserve.

ARTHUR LEFEVRE.

UNIVERSITY OF TEXAS, *January*, 1896.

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* Thomas Jefferson, quoted in "The University and the Commonwealth," an address by Professor Thornton of the University of Virginia, delivered in the University of Texas on Commencement Day, 1894.

SYLLABUS.

NUMBER AND ITS ALGEBRA.

I. Primary Number.

1. *Whole, Integral, Natural, Exact,* are all terms in vogue to designate the primary concept of number. The former two are equivalent, and objectionable as equally applicable to positive and negative numbers. They thus fail of exact designation. Though they would hardly be chosen *de novo,* they may be retained in that one of their present uses which is really proper, viz., to designate Primary Numbers, and their negatives *after* the distinction of positive and negative has been clearly made. Each of the latter two is repugnant to any concept of number adequate to the comprehension of mathematical sciences. No number must be conceived as either unnatural or inexact. The ratio of absolutely incommensurable sects, the diagonal of a square and its side, for instance, is just as exactly what it is as the ratio of an inch to a foot. By the term *primary number* no question is begged, and the very name points to the development so soon found necessary.

2. Primary Number, a normal and universal creation of the human mind, applies originally only to artificial wholes, discrete aggregates. The group whence "twelve" is abstracted must be conceived as a whole before discriminated into "twelve." Number is in nowise a sense-perception; it is purely the product of a rational process. Because the adult finds a number concept in his mind when a group of objects is attended to, he must by no means suppose any such concept in the mind of a child, though the same objects be attended to. The objects may not even be a *group* at all to the child. Adults forget how gradually any idea developed in their minds. Neither is the concept *one* necessarily in the mind of a child when a single object is attended to. The mind of the child is inclined to be absorbed in sense facts. The concept *one* is only in contrast to the concept *many*. It is not my purpose to investigate the origin and psychological processes of these concepts, *one* and *many*; nor how it comes to pass that the infant mind slowly tends to group, aggregate, make wholes of, distinct individual objects of sense-perception. It is enough to point out that from these concepts the primary concept of number springs. Various *manys* are specialized, and so distinct numbers arise in the mind. We will not enter upon the question of infant psychology concerning the stages at which the *manys* are specialized into "two," "three," etc. It may be remarked, however, that the special many "two" is recognized very early and long before "three." The concepts *one, many, two,* come almost together; and then after a long gap further specializations are attained — another distinct gap perhaps coming after *four*.

3. Definition. — Primary Number is an abstraction from a group of objects which represents their *individual* existence.

4. Each number-picture of a group is wholly abstract, in that it represents the individual existence of the elements of the group and nothing more. For use in picturing special manys a system of abstract elements is framed, where no characteristic of any element is retained beyond its simple separateness from all others.* This brings us to *Counting.*

II. Counting.

5. The fundamental concept of primary numbers is prior to, prerequisite for, not derived from, "Counting." The word is used in two senses, though its general synonyms, numeration and enumeration, seem sometimes particularly assigned to the first meaning. In the first sense, Counting is the naming of primary numbers. This naming, if carried to any great extent, must of necessity be methodical, and of course the numbers must be conceived before named. In the second sense, Counting is essentially the numerical identification, by a one-to-one correspondence, of an unfamiliar with a familiar group. In this meaning, Counting consists in assigning to each individual in a group one distinct individual in a familiar fixed series of different things — originally the fingers, usually a fixed series of different words, or different marks.

6. We must pass by many interesting facts and theories concerning word-numerals (i.e., fixed series of different words used for counting) as belonging to the domain of language, only remarking that etymology confirms what might have been surmised, that the fingers were the original series of things which mankind made use of to apply in thought to a group of objects in order to count them.

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* *Vide, Number, Discrete and Continuous,* Halsted, chap. i.

7. In all systems of numeration or counting (in the first of the senses defined in Section **5**) it soon becomes necessary, from the very limitations of human memory, to form or mark off a numerical group which the reckoner can periodically repeat. Otherwise there would be no end to the number of different words required. The number-group chosen by a majority of races at a pre-historic time, and for the reason that we possess ten fingers, is ten. As soon as any such group has been chosen, it becomes easy to express by a few number-names any number within the mental scope of the speakers.

For instance, in English with fifteen words (two of which are disfiguringly superfluous) and two significant suffixes, i.e., with seventeen words, any number whatever may be expressed.

8. The student should write out a detailed explanation and criticism of the English series of word-numerals, noting the superfluous words, also such as are not internally suggestive of their relation to the fundamental group, and what larger groups lack the simple name that would be suggested by symmetry. Let him then compare the English series with that of some other language, noting where the English is better, and where worse, than the other. All this totally irrespective of any system of notation, and purely as a question of thought and language.

According to the best practice the last of the larger groups to receive a simple name for repetition is the million. Charles W. Merrifield, F.R.S., observes, "It is worth while to remark that as regards billions there is a difference between the French and English practice; in French a *billion* (or *milliard*) is one thousand millions, in English a billion is a million millions, . . . the word is seldom used in our language. . . . The old books use a scale of this kind: A million of millions is a billion, a million of billions is a trillion, and so forth; but these names are never used in practice, and can hardly be said to belong to the language of arithmetic or to English speech."* In the late vulgar use of the word in American newspapers, *billion*, of course, signifies one thousand millions; for it is a comment upon the vastness of such numbers that even the Fifty-first Congress could not expend the thousandth part of a billion dollars in the sense of one million millions.

9. By the method just discussed a distinct name is given to each element in the series of counters; and in counting the elements, the units, the *ones* in any discrete magnitude or manifoldness, a one-to-one application is made to this series of names in a fixed order. The order being learned by rote, any word-numeral, by suggesting its definite place in the fixed series of words, recalls all those gone before; and from this comparison the mind conveys or receives an exact notion of the number of individuals in the group of objects numerically characterized by any such number-name.

10. These number-names are sometimes called cardinal numbers, to distinguish them from a series called ordinal numbers. Ordinals have little to do with arithmetic, the distinction belonging to grammar: instead of saying *the last-counted one of five* objects, we may say the *fifth*, etc. These concepts, *first*, *second*, *third*, *fourth*, etc., are the "ordinal numbers."

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* *Arithmetic and Mensuration*, p. 4, Longmans, Green, & Co., 1882.

III. Primary Number. — Some Fundamental Theory.

11. The number of objects in one group is said to be equal to the number in another when their units being counted (*vide* § **5**) come to the same finger, the same numeral-word or mark. That is to say, two primary numbers each equal to a third are equal. Also, of two such numbers one is always less than, equal to, or greater than, the other, according as in a one-to-one application to the counter-series the process ends with a prior, the same, or a subsequent, element. Also a primary number may be added to itself so as to double. (*Cf.* § **229**.) I am not concerned whether these dicta be regarded as axioms or as postulates.

Primary number is thus at once classed as a magnitude.

(Indeed it may be that the method of Hegel's dialectic of the mathematical catgeories would display *magnitude* and *number* as essentially the same; that is, as co-ordinate transitions to the same ultimate. At least, any object — as a line, a surface, a solid, a time, a temperature — is a *magnitude* or *manifoldness* only as number, in the final concept thereof, can be abstracted.)

12. There is much of prime import to be said of measurement (*vide* Chapter XIII.), but it may be remarked in this connection that the concept of measurement develops *pari passu* with that of number. To the man whose concept of number is only what has been defined as primary number, measurement is hardly to be distinguished from counting. For measurement of discrete magnitude is counting; and to the intelligence supposed there is no real measurement of continuous magnitude, but any continuous magnitude is "measured" by violently discreting it, and counting the units contained, the residue being regarded as merely a fractional redundancy. In short, he measures in what are popularly called "round numbers." True measurement of continuous magnitude is conceivable only under the developed concept of number which includes ratios.

13. Theorem. — Primary Number is independent of the order of counting.

This fact is discerned immediately from the individuality of the objects in the group. Since in counting the correspondence is one-to-one, the same extent of the counter-series is always necessary and sufficient to correspondence with any group of objects in whatever order they be applied to the counters.

The obviousness of this truth must not blind to its importance; for, as Clifford affirms, "upon this fact the whole of the science of number is based." *

IV. Notation.

14. Notation is primarily the representation of primary numbers by written symbols; but in the developed science of arithmetic it must include the symbolic representation of ways of combining numbers, and qualitative distinctions, which arise upon investigation. Notation in the primary sense is intimately blended with numeration, for it is merely the recording of the results of counting. It is of vast importance, however, and a good invention for the purpose could have been no easy task; because centuries on centuries passed after a symmetrical system of numeration had been developed in thought and language

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* *Common Sense of the Exact Sciences,* W. K. Clifford, chap. i.

before a thoroughly fitting notation was achieved. Whether the beautifully simple and perfect algorithm now so familiar to little children was perfected at a single stroke of genius on the part of a nameless Hindoo, or was a gradually consummated invention, history does not reveal.

15. Just as we passed over the etymology of numeral words, we must pretermit interesting facts and surmises as to how each written sign came to have its particular meaning in the various series of signs which mankind has in times past employed or still uses. Such signs for number are older than any other form of writing, older even than the development of language in the denary system. For an entertaining monograph on this subject, consult Professor Robertson Smith's article on "Numerals" in the ninth edition of the *Encyclopædia Britannica*, from which much of the following section has been taken.

16. The simplest representation of unity is a single stroke. The next step would be to devise a sign to represent a definite group of strokes, as it would be confusing to repeat single strokes too often. Soon a sign for a definite group of the primary groups would be required. The Babylonian inscriptions well exemplify this simplest mode of notation. The mark for unity, a vertical arrow-head, is repeated up to ten, whose symbol is a barbed sign pointing to the left. These by mere repetition serve to express primary numbers up to one hundred, for which a new sign was employed.

The most important principle of meaning-signified-by-position appears in this system. Though the symbol of the smaller number put to the right of the hundred symbol represented addition, the same symbol to the left represented a multiplier. This principle was still more significant in another system developed by the Babylonians. Strange to say, they oftened reckoned by powers of sixty, calling sixty a *soss*, and sixty times sixty a *sar*. Survivals of this sexigesimal method remain in our divisions of time, angles, and the circle. For example, the square of 59 is found recorded (translating into our symbols) 58.1, that is, 58 *soss* and one $(58 \times 60 + 1)$; but on the same tablets the cube of 30 is recorded 7.30, that is, 7 *sar* and 30 *soss*. We thus see that *because they had devised no sign for zero*, it could only be left to the judgment of the reader whether sixty or its square was intended.

After alphabets became established in a fixed order, they began to lend themselves to numerical notation. In the old Greek notation, said to go back to the time of Solon, and often called the Herodian system, after Herodian who described it in a work written about 200 A.D., 1 stood for one, Π (*πέντε*) for five, Δ (*δέκα*) for ten, H (*ἕκατον*) for hundred, X (*χίλιοι*) for thousand, M (*μυρίοι*) for ten thousand. As an artifice of condensation a great Π enclosing any symbol signified five times the number represented within. Another application of alphabets is more to my purpose. In this system (common to Greeks, Syrians, and Hebrews — in Greece displacing the Herodian), the first nine letters stood for units, the second nine for tens, the third nine for hundreds, and diacritic marks below the first nine transformed them into thousands. A great M multiplied the number after whose sign it was written by ten thousand. The notation was subsequently improved by writing the greater element always to the left, *thus dispensing with the diacritic marks.* The regular alphabet furnishing only twenty-four letters, the necessary twenty-seven were made up by calling in two old letters no longer used in phonetic writing, to signify six and ninety, and a final symbol called *sampi* represented nine hundred. Approaches still nearer to our algorithm were devised by Greek mathematicians, notably Archimedes and Apollonius of Perga; but in all the lack of the zero rendered the systems very imperfectly adapted to calculation, however perspicuous as a record.

Only one more system can be glanced at before surveying our own. This, known as the Roman, we are still familiar with. It more resembles the clumsy Herodian than the later Greek notation. The symbols were, I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, M = 1,000. Some older forms were afterwards discarded. To the extent of a few subtractive forms (IV = 4, IX = 9, XL = 40, XC = 90, and occasionally IIX = 8, XXC = 80) some meaning is attached to position, but in a way rather to hinder calculation.

In a mechanical contrivance, used in Europe from a very early date, was attained the nearest approach to our own system. The abacus (which could be ruled on waxen tablets or roughly drawn on the ground), in a permanent form, consisted of a frame in which by one means or another sets of counters were kept in separate rows or columns. These columns might represent various denominations of money value, or weight, or units, tens, hundreds, thousands, etc. In the latter case there should be only nine counters in a column. From such an abacus there are but two steps to our notation: first, to establish marks to represent respectively one, two, . . . or nine counters in any column; second, to conceive a sign for a vacant column. The invention of our nine digits and zero came slowly. The history is very obscure. Our "Arabic" system is of Indian origin, but appears to have been introduced into Europe by the

Arabs. It has been traced as far back as the fifth century of the Christian era in India, but does not seem then to have been a novelty. Hindoo writers nowhere lay claim to its invention. It was probably brought to Baghdad in the eighth century. In the ninth century Abú Jáfar Mohammed al-Kharismi published a work on the subject, and by the tenth it had spread into general use throughout the Arabian world. About the twelfth century it began to be received by Christian Europe. Arithmetic using this system was called by the barbarous name *Algoritmus* (our algorithm), probably a derivation from al-Kharizmi. Leonardo of Pisa promulgated the matter in the West, and Maximus Planudes in the East. The word zero is perhaps derived from the Arabic *sifr*, through *zephyro*, used by Leonardo. The algoritmus was at first used chiefly in astronomical tables, etc. (e.g., those published about 1252 by Alfonso the Wise). Gradually the immense superiority of the system above all others became apparent, and it has long been used by all civilized nations. In winning its way there was some confusion with prevailing notations: e.g., such forms as X2 = 12 and 504 = 54 are found, where the very essence of the method is lost sight of.

17. Our notation exactly conforms to our system of numeration. The symmetry or regularity of the notation, however, is perfect. No such anomaly as is found in the word "eleven" or "twelve" is tolerated.

The familiar symbols always mean one, two, . . . or nine; but they signify units, tens, ten-tens, ten ten-tens, etc., according to their position in the first, second, third, fourth, etc., place, counting from right to left. That is to say, they represent in definite positions corresponding powers of ten.

Under the generalization of "powers" the notation at once lends itself to the expression of fractions, the exponents becoming negative. For example, 4072.605 means —

$$(10)^3 \times 4 + (10)^2 \times 0 + (10)^1 \times 7 + (10)^0 \times 2 + (10)^{-1} \times 6 + (10)^{-2} \times 0 + (10)^{-3} \times 5.$$

It is in this regular use of a base-number that the merit of the system consists, and by no means in the choice of the base ten. Our decimal system is a perfect instrument, exciting the grateful admiration of every enlightened student of science, not because it is decimal, but because the digit figures by means of the zero always express, in their orderly position, to left or right of a point, ascending or descending powers of one basal number.

In regard to the particular base, ten, it may be remarked that, while it were idle to think of changing the confirmed habits of language, it is clear that ten is an inconvenient base. Twelve would be better. To see this, it is enough to express decimally and duodecimally a few simple fractions.

Decimally	$1/3 = 0.3333333\ldots$	Duodecimally	$1/3 = 0.4$
	$1/4 = 0.25$		$1/4 = 0.3$
	$1/6 = 0.1666666\ldots$		$1/6 = 0.2$
	$1/8 = 0.125$		$1/8 = 0.16$
	$1/9 = 0.1111111\ldots$		$1/9 = 0.14$

18. A thoughtful consideration of our notation will enable the student to adapt its essential principles to any base. So long as he feels hesitation in doing this, he may be sure he does *not* understand what he has deemed so familiar.

It is obvious that a number is the same, whether expressed in tens, or dozens, or scores,—to take, for example, two numbers, other than ten, familiar as bases to English minds, but which have never been developed into symmetrical systems of counting.

Plainly the number of digit figures required is one less than the base; since 10 must represent the base, whatever it may be.

19. The student should express various numbers, integral and fractional, on various bases, employing, say, the letters of the English alphabet in order, for additional digit figures when more than nine are required. He should also perform additions, subtractions, multiplications, divisions, with numbers so expressed.

There is no other way to test or gain a thorough comprehension of the notation so glibly used. Such an exercise will remedy many defects, and will be found to repay amply the slight cost in time and labor. (*Vide* § **277** *et seq.*)

V. Algebra.

20. An algebra is an artificial language. Its symbols have laws of combination; but these laws are the expression of actual properties and relations of the subject-matter, not laws of the algebra in any immediate sense. There is no such thing as an algebraic law; there are only algebraic conventions. The peculiar advantage of an algebra is that actual relations are given manifestations which can be experimented upon, according to organized processes, to give new knowledge. The first algebra was slowly formed through centuries to investigate the properties of number.*

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* *Vide* Halsted's *Number, Discrete and Continuous*, § 1.

21. Algebras of formal logic, of physics, of geometry, have been developed to more or less usefulness. In these the subject-matters immediately discoursed of are respectively logical, physical, geometrical entities and their relations and combinations; just as the algebra with which we have to do discourses of number and its relations and combinations. The specific entities, relations, and combinations in any case require definition, and yield laws upon their own merits.

22. A geometrical algebra must be clearly distinguished from that supremely powerful and distinctively modern branch of analysis (from whose establishment by Descartes, 1637, dates the modern, the scientific era) commonly called Analytical Geometry. In this discipline the algebra is still of number, the immediate subject of discourse is ever number; but under systematic conventions the algebra talks *in* numbers *about* geometry, just as it might be made to talk about money or temperatures. In a true and proper algebra of geometry, a and b might represent sects,* and ab be defined as the definite plane surface known as the rectangle of a and b. In this case there could be no ratio between ab and a. Also a^2 would mean the actual surface, the square on a; a^3, the actual solid, the cube on a; and a^4, etc., would be devoid of meaning in tri-dimensional space.

23. However mechanically we may at times use the symbols, it cannot be too much emphasized that in the algebra of number each expression must be a rational discourse upon number to any mind, or to that mind it is nonsense, or rather a blank, like a sentence in an unknown

* Definite pieces of straight lines.

tongue. Clifford maintains, "We may always depend upon it, that algebra which cannot be translated into good English and sound common sense, is bad algebra." *

24. Although of immense utility, the algebra of number must not be conceived as theoretically necessary to the investigations it has so signally served. The instrument has been practically prerequisite to the results that have been attained on account of the limitations of mankind's power of attention to complex details without symbolic expression, but its essentially derivative nature must not be lost sight of. Under any concept subversive of this relation — the fallacy being even more baneful when implied than when explicit — the study of an algebra becomes abusive of the noblest qualities of mind; and no irrational skill in the use of the tool can compensate for the intellectual debasement which is the price of contentment in its use and study upon such terms. It is as if one conceived the vocabulary of a spoken language as independent of the constructive thought; back of any mode of symbolic expression must lie the substantial thought.

To understand our algebra of number, we must understand number. However difficult the task, it cannot honestly be shirked.

25. Many eminent mathematicians, to say nothing of popular text-books, persist in seeking explanation of the algebra of number in the facts of geometry. They seem blind to the view that it is only adaptations of number that they thus discover; that it is numbers, not lines, surfaces, solids, that they deal with, even when they so

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* *Common Sense of the Exact Sciences*, p. 21.

usefully make the algebra of number "talk geometry."* The individual symbol in trigonometry or analytical geometry, for example, never means the geometrical concept. An equation may under a proper system of interpretation describe a line; but no x or y in it ever means a line, but the length of a line, which is a ratio, a number. Would it be less sophisticated to try to discover the nature and properties of number by studying temperatures, because, forsooth, an algebraic equation may under appropriate conditions talk temperatures as well as geometry? The confusion arising from such misconceptions is well exemplified in the following quotation from an essay by E. W. Hyde in the *American Journal of Mathematics* for September, 1883, p. 3:—

"If, in the equation $1/1 = 1 \times 1$, 1 be taken as a unit of length, then the members of the equation have evidently not the same meaning, $1/1$ being merely a numerical quantity, while 1×1 is a unit of area; it being a fundamental geometric conception that the product of a length by a length is an area, that of a length by an area a volume, while the ratio of two quantities of the same order as that of a length to a length is a mere number of the order zero."

So far from just are these observations that one would suppose it clear to any student of the subject that the physical fact is the line, the surface, the solid, and that the length, the area, the volume, *are* numbers, viz., the ratios of the line, surface, and solid respectively to other magnitudes of like kind chosen arbitrarily as units. It is a theorem which we have established geometrically, that

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* A felicitous phrase of Dr. Halsted's.

the ratio of the rectangle of two sects to the square on any third sect equals the product of the ratios of the two given sects to the third sect. That is to say, the area of a rectangle equals the product of the lengths of two adjacent sides, it being distinctly understood that the unit-surface is the square on the unit-line. This truth having been established, consistent numerical statements may be referred to such spatial entities. It is only and always in some such way that the algebra of number "talks geometry."

26. Objections to the mistake of explaining number geometrically are often made at a fatally late stage. There are writers who protest against geometric definitions of the so-called imaginary numbers after having supinely ignored a geometric, or some unnumerical, definition of -1. Their alertness comes too late when they refuse a like definition of $\sqrt{-1}$. In this, as in many other cases, it is the first principles that have been neglected. It is futile to begin inquiry with $\sqrt{-1}$. With beclouded concepts of prior phases of number, how can it be anything but vain to attempt to be critical at the final stage of that development of number which has forced itself alike upon the most practical and the most theoretical? (*Cf.* § **192**.)

27. Concerning other extant or possible algebras than that of number, I will only add that I have grave doubts of the propriety of Professor Macfarlane's aspirations towards a final and comprehensive algebra,* "which will apply *directly* to physical quantities, will include and unify

* "Principles of the Algebra of Physics," by A. Macfarlane, M.A., D.Sc., LL.D., in *Proceedings of the American Association for the Advancement of Science*, vol. xl., 1891, p. 65. The italics are mine.

the several branches of analysis, and when specialized will become ordinary algebra." Far from being the "specialized" form, the algebra of number appears to me to be the very generalization sought by Dr. Macfarlane; and it is algebras of physics, vector algebras, etc., which are the specializations. Number itself in its full development appears to my mind the very ultimate common property of all quantity, magnitude, manifoldness (*vide* § **229**) whatsoever. Search for further generalization seems mistaken. I set forth this opinion tentatively and in all modesty, fully recognizing Dr. Macfarlane's profound learning and skill in mathematics.

Negative, neomonic, and complex numbers afford the qualitative distinctions under which, it seems, the algebra of number might be made to talk physics and geometry to our full satisfaction. Should it be found inadequate to the needs of the physicist, of course, a true and proper algebra of physics may be fashioned. The physicists must decide this question. Might not better results, however, be attained by seeking perfectly satisfactory means for interpreting, physically or geometrically, numerical statements, the algebra for which is ready to hand, than by attempting to construct any real algebra of physics to "apply directly to physical quantities"?

I may invoke here the authority of no less a physicist than James Clerk Maxwell. After pointing out the contradictions which would otherwise occur in calculation, he says: "We shall therefore consider all the symbols as mere numerical quantities, and therefore subject to all the operations of arithmetic. But in the original equations and the final equations in which every term has to be interpreted in a physical sense, we must convert every numerical expression into a concrete quantity by multiplying it by the unit of that kind of quantity."

28. If we will regard the algebra of number from the standpoint of recognition of its true nature, we may take up its natural use without more ado. (*Vide* § **156**.) There is nothing mysterious about the algebraic vocabulary, or even recondite in the algebra, until we reach more advanced investigations concerning algebraic *form*. The original obscurities and difficulties are in the arithmetic; that is, in the theory and import of number itself. For the most part I shall consider what is algebraical already familiar, and bend all energy to expounding the numerical content, as distinguished from the algebraic form. (But see § **236**.)

VI. Calculation.

29. Calculation, or computation, is primarily counting. As its methods gradually become organized, it involves a thorough investigation of the laws of thought, which, upon consideration by any normal mind, will be seen to govern the various possible combinations of numbers and the processes of these combinations.

30. In solving particular problems, whether concerning numbers or the application of numbers to concrete magnitudes, it is to be borne constantly in mind that all that can be taught in general terms is how to conceive and perform numerical operations; that, knowing this, all that remains is to understand the terms of a particular problem and the properties, real or conventional, of these terms, under which they yield numerical relations; and that until one recognizes this fact he cannot take the first rational step.

Although it is proper and necessary in teaching to make constant applications of pure or theoretical arithmetic, yet the way in which these applications are presented in ordinary text-books is grossly misleading. There is no arithmetical distinction whatever between such topics as "percentage," "interest," "discount," "commission," "brokerage," "partial payments," etc.; yet from flaring chapter-headings the distinctions appear co-ordinate with those between numeration, numerical operations, and general devices, such as methods for finding the greatest common submultiple, or the least common multiple. No new *arithmetical* lore is required in order to calculate about these mercantile transactions; the task for the pupil is merely to comprehend a few technical terms, and the numerical relations subsisting, or in practice assumed to subsist, among them. Nevertheless, it is a common result of the misconceived method of presentation that a pupil fancies he is advancing to a new development of the arithmetic when he passes from "commission" to "brokerage," for instance, as if there were the faintest arithmetical distinction between calculating a percentage on the value of a barrel of apples and the value of a block of capital stock. In like manner, pupils often make pathetic attempts to excogitate the conventional method of calculating a balance due on an account with partial payments, being blinded by incompetent teaching to the fact that the data do not afford numerical relations sufficient to a definite theoretical solution. In this matter (as in many others in Applied Arithmetic), arbitrary convention is necessary to a solution; the "rule" varies with the practice of individuals, enactments of legislatures, and rulings of courts of law and equity. No act of pope or parliament could affect the proper decision of any truly arithmetical question. Of course there is more or less numerical propriety in the substantial justice between man and man which is sought in each of the various rules for calculating a balance after partial payments, but there are questions involved whose decision is not afforded by inherent numerical relations of the facts.

31. In teaching, it is supremely helpful always to emphasize the difference between Pure Arithmetic and Applied Arithmetic. The pupil's knowledge is surely in confusion unless he sees the fundamental difference between, for example, studying how to find a least common multiple (a matter of insight), and a broker's commission (a matter of empiric information so far as it is anything new when met in a systematic course). Applied Arithmetic should be presented in text-books as merely selected specimens of many other practical applications which could be made, and as problems, not as new arithmetical topics. The necessary information should be set forth in an entirely different tone from that in which arithmetical matters proper are expounded. The particular applications usually made are sufficiently well chosen; viz., calculations concerning lines, surfaces, solids, times, weights, temperatures, and money values, with special reference to the transactions of mercantile and banking business.

But no candid criticism, even the most cursory, could avoid complaint on account of the usual results of teaching the metric system of units. Text-books are at fault here, rather negatively than positively; though some are found to write 1m3dm8cm, when the system was devised expressly to avoid this — it is as if one should write 1 dollar, 3 dimes, 8 cents. They might be expected, however, to put the matter in an appreciative and tonic way, instead of leaving the discovery of its perfections to the chance alertness of the pupil. For, because a matter is perfectly clear and simple, it does not follow that it will be so esteemed. The case in question demonstrates this paradox. The metric system of units was invented for its perfect simplicity; yet, pitiful to say, it remains a bugbear to the average teacher and pupil in the common schools. Nothing could be more blind and irrational — it is exactly as if an Englishman could not be made to see that decimal money units are simpler for all calculation than pounds, shillings, and pence. Any student may be sure that, unless he regards the metric system as perfectly clear, and vastly easier than our barbarous English units, he has entirely failed to understand it — nor could one fail to appreciate it who really understood anything of arithmetic. Its essential merit is twofold: it is decimal, and therefore fits our numerical notation; its units for lines, surfaces, solids, masses, and weights are all symmetrically dependent on one unit, the linear.

The advantages of the second property ought to be as manifest as those of the first; but I will briefly illustrate. If the volume of some homogeneous material is given as 2.76 cu. m. and its sp. gr. 3.5, the weight may be found by multiplying the numbers: $2.76 \times 3.5 = 9.66$ *tonneaux*, or, pointing off to reduce to kilograms, 9660 kg.

Now, in comparison let the student calculate the weight, given volume 2 cu. yd., 7 cu. ft., 6 cu. in., and sp. gr. 3.5. In the first place, exact calculation is impossible in the English units; for the pound and the weight of a cu. yd. of water are, of course, incommensurable. The first task is to look up in some compendium of useful information the approximate weight of a cubic yard of standard water, or of a cu. ft., or of a cu. in., as may be vouchsafed. Then reduce the volumetric terms accordingly, then multiply, and finally (to be thoroughly English) reduce the approximate decimal fraction of the pound to ounces and grains. One who has stupidly despised the metric system ought to perform, as a penance, this calculation *à l'Anglaise.*

32. It may be helpful to state explicitly that I always use the term *arithmetic* in the sense of the Science of Number.

There is no difference either in subject-matter or in scope between arithmetic and the algebra of number. The distinction made by the term algebra refers to the mode of expression (*vide* § **20**), and in a special sense to the profoundly important subject of algebraic *forms* of numerical expression. Any arithmetical statement is of particular numbers; while, from the very nature of the conventions, algebraic statements are general. It was to this end that algebra was invented; but it must never be forgotten that any algebraic expression may be made particular (*vide* § **23**), and that the form then becomes arithmetical.

"Arithmetic" is too often limited (very illogically and contrary to the best practice) to denote merely some primitive developments of the science of number. Even the distinction *positive* and *negative* is often expressly set forth as peculiarly a matter of algebra. In our view this, of course, is utterly subversive.

Arithmetic needs and uses the same symbols of operation and qualitative distinctions as the algebra of number; indeed, logically, the statement should be made the other way, viz., the algebra uses the same symbols of operation and quality as arithmetic. The symbols $+$, $-$ (in both senses of each), the exponential notation, etc., belong equally to the notation of arithmetic and to number's algebra.

Newton preferred to call the algebra Universal Arithmetic.

33. It would be a very good exercise for the student (especially those who are taking this course in order to qualify as teachers in the public schools) to critically examine some text-book on arithmetic which he has heard extolled.*

VII. Primary Number. — Numerical Operations.

34. There are seven distinct numerical operations. Three of these are direct, and four inverses of these three. The three direct arise from three different modes of combining two numbers, and the four inverse, from inverse problems, viz., given one of the two numbers in the former combination and the resulting number, to find the second of the two originals. Inasmuch as two of the direct operations are commutative (*vide* § **38**), they give rise each to only one inverse; that is, it is the same problem, given either and the result, to find the other. But the third of the direct operations is not commutative; and it therefore gives rise to two inverses, it being a very different problem, whether the first or the second of the originals be given, to find the other.

These seven operations are, by name, Addition and its inverse, Subtraction; Multiplication and its inverse, Divis-

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* Upon a reperusal, this expression seems almost satirical, since it would be impossible to find one which has not been extolled. I let the phrase stand, however, in all its innocent irony.

ion; Involution and its two inverses, Evolution and Finding the Logarithm.

No numerical operations have been developed showing characteristics essentially different from these modes of operational combination. (For fuller discussion of this point, *vide* § **104**.)

35. It would be a sad comment on previous instruction if any one is surprised to hear of the seventh of these fundamental operations; for when we find $a^b = c$, the two inverse problems are equally obvious; we may have given b and c, to find a (this is familiar as evolution), or we may have a and c given, to find b. Scientific mathematicians (*Cf.* Introduction, p. 5) are to blame that no single name denoting this operation is current. We must use the accepted phrase, finding the logarithm. Because the process of this last operation is comparatively recondite, is no excuse for not calling attention to the problem in the very beginning of any systematic teaching of arithmetic. Indeed, under any rational instruction its existence could not be concealed. It should be said to the pupil, "Evidently such a result as $a^b = c$ presents two inverse problems. At this stage you will investigate only how a few very simple roots may be extracted. The question how the exponent or logarithm may be determined must be deferred until you have acquired more knowledge and greater skill."

36. In this chapter the fundamental operations will be tentatively considered for primary number. It will become apparent that for any two primary numbers the direct operations are always possible, but that the inverse operations have meaning only in particular cases. Equally obvious will become the urgent need and propriety of extending the concept of number, both for the theoretical science, and its application to the measurement of concrete magnitudes.

The numerical symbols of the algebra employed in this chapter are general only for primary number; + and − have only their operational meanings. This strict limitation must be distinctly recognized.

37. To add one primary number to another is to so combine the former with the latter that in the resulting number each unit of the components shall retain independence and precisely the same functional relation to the result (the *sum*) that it fulfilled in its original group. The concept is so immediate to that of primary number itself (*Cf.* the specialization of various *manys*, § **2** *et seq.*), that, while definition is appropriate for the purposes of scientific discourse, it hardly admits of explanation. The numbers are aggregated, just as objects now thought in two groups may be thought in one group. Also, the addition of any two primary numbers is always possible. (But note that the definition is only for primary numbers, *vide* § **45**.)

38. It is an immediate corollary from the absolutely primary theorem of number (*vide* § **13**), and the definition of addition, that

$$a + b = b + a,$$

that is to say, addition is a commutative operation. The fact is called the *Commutative law of Addition.* It obviously extends to the sum of any number of numbers.

39. In like manner the addition of three or more primary numbers is associative, that is,

$$(a + b) + c = a + (b + c).$$

This fact is the *Associative law of Addition.*

40. An algebraic statement like the foregoing, the truth of which depends on the very nature of operations, may be called a *formula*, as distinguished from a *synthetic equation.* In a formula any numerical symbol may be made particular without restricting the generality of any other; in a synthetic equation (e.g., $a + b = c$), on the contrary, to particularize any symbol more or less restricts the meaning of every other. To *solve* a synthetic equation *for* any symbol, means to find a definite number which, supposing the significance of every other symbol known, substituted for the unknown symbol will satisfy the equations; that is to say, make of it a formula in terms of the other symbols. The name *identity* is often used for *formula* as here defined. When it is necessary to distinguish between a formula or identity and a synthetic equation, the sign $\equiv$ designates the former, and $=$ the latter.

41. If the sum of two primary numbers and one of them be given, the other may be formed by pairing off every unit of the given part with a unit of the sum, and counting the unpaired units of the sum. Since addition is commutative, the operation, as just defined, is the same, whichever of the two parts of a sum be given. Addition has therefore only one inverse, called Subtraction, and represented by the minus sign ($-$).

42. The problem is to solve for x the synthetic equation —

$$a + x = b.$$

Counting off a from the number represented by each member of the equation, we obtain $x = b - a$; * that is to say,

* Of course the common notion or axiom, "if equals be taken from equals the remainders are equal" is here involved. But truly common notions can be doubted by no sane man, and explicit statement of universal axioms is hardly required anywhere except in systematic treatises on logic or epistemology.

$(b-a)$ is the number which added to a gives b. This number $(b-a)$ is called the remainder or difference resulting from the subtraction of a from b. Substituting $(b-a)$ for x gives the formula —

$$a+(b-a)=b;$$

or, by the commutative law of addition,

$$b-a+a=b,$$

which is the formula of definition of subtraction.

43. Under the developed concept of number, any chain of additions and subtractions enjoys perfect freedom of commutation; but the first thing to strike the thoughtful student in subtraction under the primary concept, is the futility of seeking general laws, because the operation is possible only in special cases. If b is less than a, $b-a$ makes no sense.

We may observe, moreover, that provided the expressions mean anything, association may take place as follows: $a-m-n=a-(m+n)$; for, adding $m+n$ to each member of the equation, we obtain $a=a$. (*Cf.* footnote to § **42.**)

In like manner, $a+b-m-n=a-(m+n-b)=a-m-(n-b)$, etc.

Also $a+b-m-n=a+(b-m-n)$.

This is the ground of the familiar rule about "signs" and parentheses. Of course, the rule applies only under great restrictions to primary numbers.

44. In practice the problem often occurs to find the sum of a number of equal numbers; e.g., how many shoes are required to shoe twelve horses? With primary number this is only a special case of addition. It was a true instinct, however, which recognized a distinct operation. But the instinct was too often disavowed in the next breath by defining multiplication as "repeated addition." Multiplication with primary numbers is repeated addition; but this concept is incapable of development without doing great violence to the word "repeated." How can one so define multiplication, and then say $\sqrt{2} \times \sqrt{3} = \sqrt{6}$? No repeated addition can attain this result. This is anticipating; but nevertheless we may, in the expectation of development, at least be careful not to prejudice opinion. If possible, let us try to say enough to define multiplication for primary number without saying so much that the way of development is barred.

45. The gradual extension of the meaning of terms is perhaps the most powerful instrument for that ordering and simplification of knowledge, that transformation of chaos into cosmos, which is the vocation of science. The procedure should take place with the caution befitting its importance, and demands at every stadium a consummate restraint of judgment in order not to say too much. The severest self-criticism alone can repress the tendency of tyros in every science to set delimitations which confine development, and entomb thought in empiricism.

Note carefully that even addition must not be declared as necessarily increasing a number. With primary numbers a number is increased by addition; but to so define would bar development. Neither in general does multiplication increase a number, nor division decrease it, and to so define would *hide-bind* mathematics.

46. The operation of multiplication can hardly be defined for primary number without prejudice to the development so necessary to mathematics, pure and applied. A satisfactory definition has never been framed; nor must it be supposed that I consider the feat achieved in the following definition, which (or something like it) was first offered, I believe, by De Morgan. The matter is one of paramount importance; for all rational views of number have been developed under the principle of the persistence of the laws of the operations, addition, multiplication, and involution. (Their inverses would be adequately defined merely as such.) Nor, until satisfactory definitions of these operations in their utter generality are attained, can the fundamental theory of the subject be regarded as perfect or completely established. I make an effort, not in contentment with the result, but to display the difficulties.

The Multiplication of any number by another consists in affecting the former (multiplicand) in precisely the same way as *one* is affected in the other (multiplier).

Or, in Multiplication one number is so combined with another that one of them shall fulfil in the result the same functional relation that the number *one* fulfils in the other.

The result is called the product.

The multiplication of any two primary numbers is always possible.

With primary numbers the foregoing tentative definition amounts to "repeated addition," nor is it claimed that it is much better as a scientific achievement. The difference is rather pedagogical: if you tell a pupil that "multiplication is repeated addition," he is disposed to think he fully understands the nature of the operation; but if you tell him that in this operation the multiplicand is affected in the same way as *one* is affected in the multiplier, although he will not at first receive more information than before, he is in a position to widen his concept of the "ways" in which *one* may be affected in a number. And when he recognizes ratios as numbers, and that any number is its own ratio to *one*, the composition of ratios at once falls into his definition of multiplication. (*Vide* §§ **80**, **81**, **82**, **83**.) In other words, if ultimate development is not prejudiced by the definition suggested, it is only on the score of its vagueness; since in each new extension it is from the principles of multiplication itself that the way in which *one* is affected, or its functional relation to the multiplier, can be comprehended. Nevertheless, it may be considered that (when ratios have been recognized as numbers, and therefore necessarily a ratio of any two numbers, and any number as its own ratio to *one*), in the light of the independently discovered operation, the "composition of ratios,"* the definition might be read in a fuller sense than it could convey to a beginner. This may seem a pitiful plight for a definition; but I can only point out that many things have to be seen to be understood, that before such vision they must in their very nature be "unto the Jews a stumbing-block, and unto the Greeks foolishness." The extended meanings of multiplication are undreamed of to the man whose only notion of number is his abstraction from a flock of sheep, or a pile of coins.

It might be better to leave the numerical operations undefined in words, and in the case of multiplication to rest upon its commutative and associative laws (which alone would not distinguish it from addition), *and* its law of distribution with addition, as at once governing and defining the operation. (For these laws, *vide infra*.)

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* *Cf. Euclid*, Book VI., 23; and VI., def. 5; or, better, Halsted's *Elements of Geometry*, §§ 540, 544.

It may be objected that even addition has been defined only for primary number; but when number has been seen to be a continuous magnitude, and the qualitative difference, *positive* and *negative*, revealed, the "common notion" of addition immediately applies. And, as suggested in Section **37**, to define addition is like defining such terms as *more* or *less*.

The only point at all recondite is that, from the very clean-contrary nature of *positive* and *negative*, the addition of a negative number to another decreases the latter. Attention is called candidly to this generalization of addition, as well as to the application of *greater* and *less* to negative numbers (*Cf.* §§ **116, 117, 198**); but the propriety of these concepts is left to be justified of their fruits.

47. The multiplication of primary numbers is commutative, i.e., —

$$ab = ba.$$

This is the Commutative Law of Multiplication.

Its truth is obvious, for three rows of four dots in each row is the same group as four columns of three dots in each column, thus —

. . . .
. . . .
. . . .

Also, commutative freedom is shown to extend to the factors in a series of successive multiplication, i.e., —

$$abcde = abedc, \text{ etc.}$$

Multiplication with equal generality is associative; that is, any group of factors may be replaced by their product, i.e, $abcde = a(bcd)e$.

This is the Associative Law of Multiplication.

48. From the commutative nature of multiplication, it follows that when a problem is discerned as requiring the multiplication of one number by another, it never makes the slightest difference theoretically which is taken as the multiplicand. All the talk about carefully distinguishing multiplier and multiplicand so prevalent in text-books is sheer nonsense. If you wish to find how many oranges you must provide to give 3 to each of 278 children, it is utterly indifferent whether you multiply 3 by 278, or 278 by 3. As a matter of convenience in this particular example the latter process, absurdly decried as it is, is the sensible course. The problem requires the combination in multiplication of the *number* of children and the *number* of oranges. The product is interpreted as a number of oranges. In neither case have oranges or children been multiplied; processes of horticulture or procreation would be necessary in such a performance.

Concrete magnitudes can be multiplied by numbers, but such processes are not purely arithmetical. For example, a sect of a straight line can be really multiplied; but the process is a geometrical construction. Thus, to multiply a sect by 3, lay off the given sect three times in a straight, so that one of the three shall lie end-point to end-point with the other two, but no other points in common. The sect between the non-coincident end-points is the required product. It would be an easy construction to multiply a sect by $\sqrt{2}$, for this would be accomplished by the familiar process of "altering" it in the ratio of the diagonal of any square to its side.

Of course, it is perfectly legitimate to speak of the multiplication of concretes in the sense merely of an interpretation of a numerical process. Thus, there can be no objection to saying 8 pounds multiplied by 152 make 1,216 pounds; but it is utterly mistaken to protest against multiplying 152 by 8 in performing the calculation.

49. Since multiplication is commutative, there is only one inverse problem; viz., given a product and one factor, to find the other. The operation is called Division. Division requires the solution for x of the synthetic equation —

$$ax = b.$$

The formula of definition of division is —

$$a\,(b \,/\, a) = b.$$

Notationally a line *laterally* * presented to the number symbols (— or /), a colon (:), or a combination of both (÷) represents division. The first is generally to be preferred.

50. With primary number division amounts to repeated subtraction, but it is only safely defined as the inverse of multiplication. (*Vide* § 45.)

51. Under the developed concept of number, if a number is to be combined with a series of others which operate successively in multiplication and division, there is free commutation and association in using the operators in the manner displayed in the following: —

(1) $(a \times b) \div c = (a \div c) \times b = a \times b \,/\, c = a \div c \,/\, b$;

(2) $(a \div b) \div c = a \,/\, bc = a \,/\, c \div b$;

(3) $(a \div b) \times (c \div d) = ac \,/\, bd = a \,/\, d \div b \,/\, c = ac \,/\, b \div d$, etc.;

(4) $(a \div b) \div (c \div d) = ad \,/\, bc = a \,/\, c \div b - d = ad \,/\, b \div c$, etc.;

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* The "minus" sign is presented *endwise* to the number symbols.

as may easily be proved from the laws of multiplication and the definition of division, to be true for primary number, *if the operations have any meaning at all.*

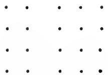
But as in the case of subtraction (§ **43**), it is vain to attempt generalizations with division under the primary concept of number, for division is possible only in particular cases.

Thus, considering that the primary numbers represented are such that the statement $(a \times b) \div c$ makes sense, $(a \div c) \times b$ may, or may not, have meaning; e.g. $(3 \times 4) \div 6$ makes sense, for there is a primary number which multiplied by 6 gives 12; but $(3 \div 6) \times 4$ is meaningless in terms of primary number, for there is no primary number which multiplied by 6 gives 3. Again $(15 \times 4) \div 6$ is intelligible, but not $(15 \div 6) \times 4$; since no primary number multiplied by 6 gives 15.

52. If a sum of two primary numbers is to be multiplied by a primary number, the product is the same as the sum of the products of each summand by the multiplier, i.e., —

$$(a + b)\, c = ac + bc.$$

For 4 rows of 5 in a row is the same group as the sum of two groups each of 4 rows, 2 and 3 respectively in a row, thus: —



The principle evidently extends to the sum of any number of summands, and is called the Distributive Law of Multiplication and Addition.

53. If the multiplier be a sum, of course redistribution will display the final result as a sum of simple products, e.g.,

$$(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd.$$

54. If each one of a number of factors be a result of mixed addition and subtraction, the Distributive Law applies, but with primary numbers only, under the miserable restrictions inherent in inverse operations.

55. Also a series of additions and subtractions is distributable with a divisor. It is sufficient to give formal proof in one instance: —

To prove $(a+b) \div c = (a \div c) + (b \div c)$.

Now, $\{(a+b) \div c\} c = a + b$ by definition of division.

Again, $\{(a \div c) + (b \div c)\} c = (a \div c) c + (b \div c) c$ by distribution of multiplication and addition; but this last also $= a + b$ by definition of division,

$\therefore (a+b) \div c = (a \div c) + (b \div c)$. (*Cf.* foot-note to § **42.**)

56. If factors be *sums*, redistribution is possible, since the original case merely recurs (*vide* § **53**); but if a divisor be a sum, it cannot be distributed.

$$(a+b) \div (c+d) = a \div (c+d) + b \div (c+d),$$

but $a \div (c+d)$ does not equal $(a \div c) + (a \div d)$, as the student may easily satisfy himself.

Let this truth emphasize the principle that all such questions are matters of fact, and not to be conventionally decided.

57. It frequently occurs in practice that it is required to repeatedly multiply a number by itself. Given the basal number and the number of times it is to occur as a factor, the process is completely determined. The original number is called the base; the number of times it is to occur as a factor is called, according to the point of view, the *exponent* of the base, or the *logarithm of* the result *to* the specific base. The result is called the *power*. The exponent is sometimes called an *index*.

Numbers in this relation are notationally represented thus: $4^3 = 64$, or $a^b = c$, where a is base; b, the exponent; c, the power. The phrase logarithm of c to base a is written in algebraic shorthand thus, $\log_a c$.

58. When the exponent is two, the power is commonly called the "square;" and when the exponent is three, the "cube." These names refer to true and proper geometrical applications of number, but have no doubt had their share in postponing general recognition of number's real nature. (*Cf.* § **25**, and §§ **230, 231.**)

59. The operation of combining two numbers in the sense represented notationally, as above explained, by a^b, is called Involution. But just as we restrained ourselves from prematurely regarding multiplication as repeated addition, we must prejudice no subsequent questions by regarding involution as repeated multiplication. It is repeated multiplication for primary numbers; but when we discern other modes of number we shall see that such is by no means the essential nature of the operation.

60. It is impossible (for me) to frame a definition of involution in terms of primary number which will satisfactorily connote the simplest and the general meaning of the operation. (*Cf.* §§ **45, 46.**) In lieu of something more satisfactory I make the following attempt: Involution is a combination of two numbers such that the base shall appear factorially in the result in a mode corresponding to that in which unity exists additively in the exponent. While this definition expresses primary involution, it is not inconsistent with ultimate meanings. For example, if unity exists three times additively in the exponent, the base must appear three times factorially in the power; yet when numbers are conceived in which unity fulfils a relation the inverse of primary addition, we need not be surprised to discover that the base appears in the power in a relation the inverse of that of a direct factor.

$$\left(a^3 = aaa, \text{ and } a^{-3} = \frac{1}{a} \cdot \frac{1}{a} \cdot \frac{1}{a}\right).$$

Again, when a number such as the ratio 1 / 3 is discerned, it becomes a development, not a recantation of former opinion, to discover that the exponent 1 / 3 imposes upon a base an operation which shall cause it to appear in the result as one of three equal factors of itself, since 1 / 3 is one of three equal summands of 1. $(a^{\frac{1}{3}} = \sqrt[3]{a}.)$

It would be anticipating too much to carry testing any further. I set forth the definition merely as the best that I can offer. Perhaps the most scientific attitude in the dilemma is merely to note the sense of involution for primary number, alertly waiting to discover what its nature may be as deepening insight reveals other modes of number, and surmising upon general grounds that if a^b means repeated multiplication when b is a primary number, it will not have this meaning if b is not a primary number.

61. I have dwelt upon this matter because it is an exceedingly important point. The application here of the PRINCIPLE OF CONTINUITY (*vide* § **103**) has led to undreamed-of advances, not only in the mathematics, but in the physical sciences.

62. Involution is evidently not commutative: a^b is not b^a. A unique case is commutative; $2^4 = 4^2$.

Neither are successive involutions associative: $a^{(b^c)}$ is not equal to $(a^b)^c$.

63. Let the student find the difference between $2^{(2^{2^2})}$ and $(2^2)^{(2^2)}$.

64. "Law of Indices." — For primary numbers it follows immediately from the definition (let the student deduce the forms, however) that $a^b a^c a^d = a^{b+c+d}$; $(a^b)^c = a^{bc}$, and $a^c b^c = (ab)^c$.

Also if $b > c$, $a^b \div a^c = a^{b-c}$. (See also §§ **158, 191.**)

65. Because involution is not commutative there are two inverse operations, requiring respectively the solution for x of the synthetic equations (1) $x^a = b$, and (2) $a^x = b$.

66. Operation (1) is called Evolution, or finding the ath root of b. In algebraic shorthand the ath root of b is written $\sqrt[a]{b}$. The radical sign is derived from the letter r (*radix*). In actual computation (arithmetical or algebraical) after the theory of exponents has been generalized, it is far better to employ indices than radical signs.

67. Operation (2) is called Finding the Logarithm (*vide* § **35**).

68. The Formula of Definition of Evolution is $(\sqrt[a]{b})^a = b$.

69. The Formula of Definition of finding the Logarithm is $a^{\log_a b} = b$.

70. As has been seen to be the case with all inverse operations in terms of primary number, these inverses of involution are evidently possible only in very special cases.

71. With the discovery of the seven operations, and their laws, Commutative, Associative, Distributive, and the Law of Indices or Exponents, the foundation of arithmetic and the algebra of number is complete. I repeat (*Cf.* §§ **20, 23**) these laws could never have originated arbitrarily, or as springing essentially from the algebra. As "algebraic laws" they must be merely the expression of actual properties and relations of number.

VIII. Devices of Computation.

72. Various devices of computation, of more or less practical utility, are familiar to all; but it will be clear upon any thoughtful consideration that they possess none of the fundamental importance suggested by the prominent rôle they play in ordinary text-books. What is usually set forth as a general exhibition of addition must be seen to be several partial additions and a convenient association of resulting summands. The same numbers would have their parts differently associated to suit different notations, e.g., XXXVII + XXXVIII = LXXV; or $37 + 38 = 75$.

The average high-school graduate labors under the impression that his fashion of "multiplying" is essential to the matter, and arises from the very nature of things. In "division" he learns what he sometimes regards as two ways, "Short" and "Long." The names are, in truth, appropriate enough, for the sole difference is that more of the necessary thought is actually written down in the Long than in the Short way. Yet the abbreviated form is taught first, and the pupil fancies he is learning something new and more difficult when he learns "Long division."

The rational method would be to teach first an expression still longer than the "Long"; then, as skill and power of retaining conclusions in mind increase, convenient abbreviations should be explained and recommended.

73. Let the student critically examine his habitual ways of "adding," "subtracting," "multiplying," and "dividing" primary numbers, both in the common algorithm of arithmetic, and algebraically. Let him denote every act of his mind in each process as an addition, subtraction, multiplication, division, commutation, association, or distribution of numbers, under the definitions and laws set forth in the preceding chapters. To take a very simple example: $(a^3 b^2 c^5)(a^5 b^6 c^{11}) \div (a^4 b^3 c^{15})$

$= (a^3 a^5 b^2 b^6 c^5 c^{11}) \div (a^4 b^3 c^{15})$ . . . by association and commutation.

$= (a^8 b^8 c^{16}) \div (a^4 b^3 c^{15})$ . . . by three partial multiplications by law of indices.

$= (a^8 / a^4)(b^8 / b^3)(c^{16} / c^{15})$ . . . by association and commutation.

$= a^4 b^5 c$ . . . by three divisions by law of indices.

74. Explain how a multiplying machine, which can do no more at one time than multiply a number of ten places by another of ten places, may be used to multiply 13693456783231 by 46381239245932.

75. The involution of primary numbers may be accomplished merely by repeated multiplication. As soon, however, as one investigates logarithmic series, and the construction and use of Tables of Logarithms, he learns command of a more facile way of performing this laborious operation. Before learning the use of logarithms, one ought to demand good wages for the toil it would cost him to find 9^{82}; afterwards it becomes the work of a few minutes.

76. Evolution, as we have seen, is only occasionally possible under the primary concept of number; but even in the simplest of these possible cases the device of calculation familiarly used by the high-school pupil is rarely understood, else he would be able to find (however laboriously) the fifth root as well as the third. Of course evolution is too laborious to be carried to any extent until Logarithmic

Tables are comprehended, when it becomes easy. But if one understood how his device for extracting a second or third root was invented, he could on occasion make his own rule for finding a fifth root. Let us investigate. Properly distributing and associating, it is seen that —

$$(a + b)^2 = a^2 + b\,(2\,a + b).$$

Also $(a + b + c)^2 = (a + b)^2 + c\,\{2\,(a + b) + c\}$, etc.

Here is declared a rule for the evolution of a second root of a number; for a specific composition of the power is displayed in a way to make decomposition easy. Likewise the formulæ for the evolution of a cube root are

$$(a + b)^3 = a^3 + b\,(3\,a^2 + 3\,ab + b^2),$$

and $(a + b + c)^3 = (a + b)^3 + c\,\{3\,(a + b)^2 + 3\,(a + b)\,c + c^2\}$, etc.

In exactly the same way the formula for the evolution of a fifth root is

$$(a + b)^5 = a^5 + b\,(5\,a^4 + 10\,a^3b + 10\,a^2b^2 + 5\,ab^3 + b^4), \text{ etc.}$$

Suppose the fifth root of 33554432 is required.

Now the preceding formulæ show that, if the root be considered as the sum of three numbers, the corresponding power of the sum of the first two is to be taken away, and the remainder decomposed to reveal the third summand of the root, and so on. Therefore we could not go wrong even by choosing parts of the root at random. But a consideration of the arithmetical notation may save much trouble; for it is plain that a fifth root of the number before us has two digit figures, that is, it is to be regarded as the sum of a number of tens and a number of ones. We compute as follows: —

			$a \quad b$
		33554432	$30 + 2$
$a^5 =$		24300000	
		9254432 . . .	(Here we guess our b; the calculation will test accuracy.)
$5\,a^4 =$	4050000		
$10\,a^3b =$	540000		
$10\,a^2b^2 =$	36000		
$5\,ab^3 =$	1200		
$b^4 =$	16		
	4627216	9254432 . . .	(Got by multiplying 4627216 by b, as the formula directs.)

77. Now let the student compute again, taking 20 for a and 12 for b. Also let him prove 12 a cube root of 1728, taking 6, then 4, then 2, as summands of the root.

IX. First Extension of the Number-Concept. Ratio. — Fractions. — Surds.

78. The first extension of the concept of number is the identification of the ratio of any two magnitudes of the same kind, and without qualitative distinction for the purposes of the comparison, as a *number*.

79. This step was taken long ago (*Cf.* Introduction, p. 11), and is now universally accepted as a dictum, even where not clearly discerned as a matter of insight.

80. This development of the number-concept was no doubt occasioned in the history of human experience by problems of practical measurement. (*Cf.* Introduction, p. 13.)

Thought must have operated as follows: If the numerical relation of a yard to a foot is 3, surely there is a number denoting the relation of a yard to two feet, and of a foot to a yard. That is, numbers which are fractions (*vide* § **83**) of primary number were discerned. This advance still leaves number discrete, that is, increasing *per saltum.* But again, as a second step, if there is a numerical relation between two magnitudes, one of which is a fraction of the other, surely there must be a numerical relation between any two magnitudes of the same kind, even though neither be a fraction (*vide* § **83**) of the other. Thus, when it is proved that the diagonal and side of a square are absolutely incommensurable (*Euclid,* Book X, 117), the mind cannot tolerate the thought that a numerical relation would exist, provided the diagonal were just the least bit shorter, yet, *de facto,* does not exist. This thought, I repeat, is intolerable. Moreover, since the ratio of a yard to a foot is an exact number, surely the ratio of a metre to a foot is *exactly* whatever it is. It is, of course, well known that the metre and foot are incommensurable

81. The connotation of all ratio (fractional and surd) as number evidently makes number continuous *one way,* to use a space metaphor on account of the exigencies of language. Thus, under this concept, number begins with a ratio smaller than any assignable fraction of 1, increases *continuously,* passing through all the discrete stages of primary number, to a ratio greater than any assignable primary number.

82. To illustrate: Start with the ratio of the weight of these pages to the weight of a granite bowlder. We begin either with a very small fraction of 1, or a surd smaller than a very small fraction of 1 (as the weights are commensurable or not, probability being vastly in favor of the latter case). Now, by gradual abrasion of the bowlder, decrease its mass; the ratios of the weights increase continuously until they reach 1. Continue the abrasion, and the ratios increase continuously, passing through 2, 3, 4, etc. At length when the bowlder has been reduced to a grain of sand, the ratio will be greater than some high primary number.

83. The foregoing discourse presumes sufficient familiarity with the subject to insure the reception of the terms employed in their precise meaning; yet it may be serviceable to set forth the following definitions (*Cf.* § **205**): —

(1) MULTIPLE. — One magnitude is a multiple of another when the former may be separated into equal parts, each equal to the latter. (Of course "multiple" includes the limiting case where the "part" is the whole, i.e., multiplication by 1. It is merely an imperfection of language which might seem to exclude this case.)

(2) SUBMULTIPLE. — In (1) the "latter" is a submultiple of the "former."

(3) FRACTION. — Any multiple of a submultiple is a fraction. (Of course if a is a fraction of b, b is a fraction of a; also a multiple of a submultiple may reduce either to submultiple or multiple.)

(4) COMMENSURABLE. — Two magnitudes are commensurable if either is a fraction of the other;

(5) INCOMMENSURABLE. — if neither is a fraction of the other.

(6) RATIO. — That definite (exact) numerical relation (*Cf.* § **80**) of two magnitudes of the same kind, in virtue of which one is either a fraction of the other, or greater than one and less than another fraction of the other, which differ as little as we please, is called the ratio of the former to the latter.

Of course, from the very concept of ratios, and the continuity of possible ratios, the ratio of the first of two magnitudes to the second is greater than the ratio to the second of any magnitude less than the first. Also two ratios are equal if every numerical fraction greater than either is greater than the other, and less than either is less than the other.

A ratio is often spoken of as "incommensurable," of course as an abbreviated expression, since it takes two things to be incommensurable. You might as well say, "x is equal," as to say "x is incommensurable." The abbreviation is for *incommensurable with 1.* Incommensurable ratios may be called surds.

Let it be clearly noted that a multiple, a submultiple, or a fraction of any magnitude, is another of the same kind; but that the ratio of two is a number. Thus a fraction of a time is a time, of a surface a surface, of a solid a solid. But the ratio of one solid to another is a number, — in this case called the *volume* of the former with respect to the latter.

Note also, any number may be regarded as its ratio to 1, and that all numerical fractions are ratios, but not all ratios are numerical fractions.

In illustration of the definition of a ratio, and its notation, if of incommensurables, consider the yard and the metre. Measurement (*vide* § **203**) not excessively refined, gives the number 0.9143 + for the ratio of a yard to a metre. This is to be understood to mean that a yard is greater than $\frac{9143}{10000}$ of a metre and less than $\frac{9144}{10000}$. Measurement more refined would yield a numerical fraction still more closely approximating the ratio. The ratio in question has been found to be greater than 0.914392, and less than 0.914393.

(7) SURD. — Of the *one-way* continuous Number, the concept of which we have now attained, those numbers which are incommensurable with 1 may be called surds. It is matter of discovery that the $\sqrt{2}$ is incommensurable or a surd.

The term surd is sometimes exclusively referred to the results of such operations as $\sqrt{2}$; but Newton's use is a philosophical one. For the $\sqrt{2}$ is found out to be 1.41421 +, that is a number, no fraction of one, but greater than 1.41421, and less than 1.41422, which is surely a number of precisely the same kind as the ratio of a yard to a metre, or of a circle to its diameter (0.914392 + and 3.14159 + respectively). Incommensurable numbers resulting from evolution may be distinguished as radical-surds, or simply radicals. (*Vide* § **145**.)

X. SIGNIFICANCE OF OPERATIONS, AND SPECIAL OPERATIONAL DEVICES, APPROPRIATE TO THE FIRST EXTENSION OF THE NUMBER-CONCEPT.

84. Euclid probably never clearly unified his concepts of ratio and number; but following Euclid (*q.v.*, and *cf.* Halsted's *Elements of Geometry*), it may be shown that there is a combination of ratios which obeys the same laws that govern the addition of primary numbers, or of fractions of concrete magnitudes, an inverse operation corresponding exactly to subtraction; another operation ("composition of ratios"), which obeys the same laws as the multiplication of primary numbers, and an inverse ("altering" a magnitude in a given ratio), corresponding to division.

But, from the very definition of a submultiple of any magnitude, the finding of a submultiple is identified as an operation of division, since the problem is to find a magnitude which *multiplied* produces the given magnitude. Now, when number has been discerned as a magnitude, these reflections make it plain that a fraction of a number is the number resulting from the division of that number by another, that one-half of 1 is $1 \div 2$, etc.* Also, when ratios have been identified as numbers, and number thus becomes one-way continuous, the operational significance of the principles, established in Chapter VII. for Addition and Multiplication and their inverses, extends to all number (primary, fractional, and surd) thus far conceived.

Finally, inasmuch as a fractional number is the result of dividing one primary number by another, it may be represented most conveniently by the notation already established for division. (*Vide* § **49**.)

It would be impracticable to invent individual symbols, since an unending number of different symbols would be demanded to designate even the fractional numbers lying between two consecutive primary numbers; nor could any such symbol be used otherwise than as a record, since in any calculation with fractions it is the generating numbers which are utilized, and not the fractional number itself.

85. It seems to me that there is no way substantially different from the lines of thought I have followed, whereby one can really understand what he is doing in the operation $7/8 \times 9/5$ for instance. Teachers of arithmetic would do well to ponder their methods at this point.

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* The only explanation (?) of such conclusions to be found even in the splendid *Text Book of Algebra* by Professor Chrystal, is "the statement that $\frac{2}{3} \times \frac{1}{2}$ is $\frac{1}{2}$ of $\frac{2}{3}$ is merely a matter of some interpretation, arithmetical or other, that is given to a symbolical result demonstrably in accordance with the laws of symbolical operation." Vol. i., p. 13.

86. It remains to investigate devices for performing the seven numerical operations in this extended region of number, and in two cases to discover the effect, the meaning, of an operational combination; viz., in involution, if the exponent be a fraction or a surd. In the first place, it is to be borne in mind that it is one thing to conceive an operation, and another to perform it. For example, at the conclusion of these introductory lectures, it will be plain to all (if now obscure) that such operations as involving 10 under the exponent π, or finding the logarithm of 5 to the base 12, are perfectly intelligible, even though ignorance of logarithmic series, or of the use of a table of logarithms, should leave one without devices adequate to the performance of the calculations.

87. It should be observed that the terms *numerator* and *denominator* applied to the numbers involved in a numerical fraction, or even to the "terms" of a ratio of incommensurables (e.g., $\sqrt{2}/6$) may be used as convenience suggests; but conceived operationally they are to be thought as dividend and divisor. *The numerical symbols in the algebra of this chapter are still to be understood as representing primary numbers.*

88. The "rules" for the operations of addition, multiplication, and division of fractions follow immediately from the definition of a fractional number, which is merely the recognition that the inverse of multiplication is always possible, that the result of the division of any primary number by any other is a number.

Substraction remains refractory, and meaningless unless the minuend be greater than the subtrahend.

The rules are only the generalization of Sections **51** and **55**, *q.v.*, yet it may be serviceable to discuss them.

89. By the distributive law —

$$a / d + b / d + c / d = \frac{a + b + c}{d};$$

therefore the common rule.

Also $a / d - b / d = \dfrac{a - b}{d}$ by the distributive law, *if* $a > b$; therefore the common rule.

But how shall we perform $a + b / c$, or $a / b + c / d$, if a / b, b / c, and c / d are fractions? The operation is distinctly conceivable; but the device for performing it requires an intermediary step of multiplication, which must therefore be investigated. Consider —

(1) $a / b \times c = ac / b = a \div b / c = c \div b / a.$
(2) $a / b \div c = a / bc = ac \div b.$
(3) $a / b \times c / d = ac / bd$, etc. (*Cf.* § **51**, (3)).
(4) $a / b \div c / d = ad / bc$, etc. (*Cf.* § **51**, (4)).
(5) $a / b = a / b \times c / c = ac / bc$, also $a / b = (a / b \div c) \times c = \dfrac{a / c}{b / c}.$
(6) $a \times b / c = ab / c = a \div c / b,$

all by the laws of division and multiplication (*vide* § **51**). Therefore the common rules: From (1), To multiply a fraction, multiply the numerator or divide the denominator; from (1), to multiply by a fraction, multiply by the numerator and divide by the denominator; from (2), to divide a fraction multiply the denominator or divide the numerator; from (1), to divide by a fraction divide by the numerator and multiply by the denominator, etc.; from (3) and (4) for cases where both terms of the operation are fractions. Also from (5) it is obvious that to multiply or divide both terms of a fraction by the same number neither increases nor diminishes it; and from (6), the result is indifferent whether we multiply by a fraction, or divide by its reciprocal.

90. It may be remarked that there is a distinction between dividing by a fraction and multiplying by its reciprocal, though the results are indifferent, as declared in Section **89** (1). The operations are not identical. The results of 4^3, 4×16, $4 + 60$ are the same, but the operations are by no means identical. b / a is called the *reciprocal* of a / b, and may be obtained operationally from the latter by dividing 1 by a / b; for $1 \div a / b = b / a$. Moreover "invert" is a short-cut term which may be used among those whose knowledge of first principles is assured; but it should never be used in explanation, as designating an operation — one can as little turn a number upside-down as inside-out.

In the United States of America the custom is almost universal, never to divide by a fraction, but to choose instead the equivalent operation of multiplying by its reciprocal. In Europe this is not so commonly felt to be more convenient. As a question of practical calculation the matter is of no importance; but it is surely lamentable if pupils are led to think that they are dividing by a number when they are actually multiplying by a different number of such relative value that the results are equivalent. Notationally a fraction *expressly* represents an unperformed operation. The unexpressed result is the definite number: thus, $7 / 6$ means 7 divided by 6; and the result is a number greater than 1 and less than 2, a definite value of the continuous magnitude we call Number.

A fraction in operation is to be employed as a composite term consisting of a dividend and a divisor. Now, it can be reasonably explained even to a very young student of arithmetic that to divide by a quotient is equivalent to dividing by the dividend and multiplying by the divisor. This having been established, he can see that the problem to divide by a / b resolves itself into dividing by a and multiplying by b. If the dividend is an integer, he has simply to do this. If the dividend is a fraction, he must first have been led to see that a fraction *is multiplied* by multiplying its numerator, or dividing its denominator; and *divided* by dividing its numerator, or multiplying its denominator.

If these principles are discerned, he can proceed in any manner he prefers. It is of no theoretical consequence how he sets down on paper mental conclusions. There is no obstacle to performing the division, under the principles stated, just as the symbols stand: $7 / 6 \div 3 / 5 = 35 / 18$.

A very low order of convenience is subserved by making a different problem of identical result: $7 / 6 \times 5 / 3 = 35 / 18$.

This discussion may seem almost trifling; but if one will reflect that the average common-school pupil thinks he *must* transform any such problem of division into a problem of multiplication, some deficiency in the usual instruction at this point will be apparent. I am convinced that our schools require systematic instruction in arithmetic of children entirely too young to be capable of the reasoning and insight demanded.

In such cases the best one can do is never to leave anything totally unreasonable to the child. Even to a young child very recondite matters can be a little explained — brought within a dim light of reason, if not clearly illuminated. One thing is certain, — bad history, bad grammar, bad chemistry, or bad mathematics, is always bad pedagogy as well.

If instruction in so-called arithmetic is always to have reference to concrete magnitudes, as recommended * by the latest "psychology of number" (*Cf.* Introduction), the simplest method for division by a fraction would be to reduce to common denominator: thus, $7/6 \div 3/5 = 35/30 \div 18/30 = 35/18$. Indeed, it may well be, when arithmetic has to be taught to children too young for the subject, that this method is the best *as a first presentation* of the matter. Because the crudest notion of numerical fractions, and blindness to the true significance of our notation of fractions, is not incompatible with some rational comprehension of this process.

91. We may now return to our problems, $a + b/c$ and $a/b + c/d$. By Section **89** (5) they may be brought under the case of $a/d + b/d$. For $a + b/c = ac/c + b/c = \frac{ac+b}{c}$. And $a/b + c/d = ad/bd + cb/db = \frac{ad+cb}{bd}$.

There is often a better way of solving the second problem. Evidently if b and d have a common multiple, m, less than their product, it would be advantageous, especially if several fractions were to be added, to reduce to a common denominator by multiplying both terms of each fraction by m-divided-by-the-denominator. No doubt all are familiar with a device for finding the least common multiple of two or more numbers. (*Vide* § **242**.)

92. Inasmuch as an exponent of involution when a primary number requires merely repeated multiplication, we see —

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* *Psychology of Number*, McLellan and Dewey, p. 116.

$$(a/b)^p = a/b \cdot a/b \cdot a/b \cdots = a^p/b^p.$$

Also, since $\qquad \sqrt{a}/\sqrt{b} \cdot \sqrt{a}/\sqrt{b} = a/b,$

therefore, $\qquad \sqrt{a/b} = \sqrt{a}/\sqrt{b}$, etc.

93. If a fraction is expressed as a sum of decimal fractions, e.g., 41.2164, evolution is apparently performed precisely as in Section **76**. This is permissible, because —

$41.2164 = \frac{412164}{10000}$ and $\sqrt{\frac{412164}{10000}} = \sqrt{412164} \div \sqrt{10000}$.

Our notation renders it easy to perform a portion of this calculation at a glance by "pointing off;" but the operation must be understood as finding the $\sqrt{412164}$, and then dividing it by $\sqrt{10000}$.

Let the student perform the calculation, not losing sight of *what* he is doing in *how* he does it.

Let him also fully express the operations involved in the conclusion, $41.2164 = \frac{412164}{10000}$. Our notation is so perfect that it may almost be said to work automatically, and for this very reason it often blindfolds teacher and pupil.

It would richly repay the student to perform just once in his life such a calculation as $\sqrt{41.2164}$ under an imperfect notation. Let him do this, expressing everything in the Roman characters.

94. As has been said (§ **83** (7)), it is a matter of discovery whether or not, in any particular case, $\sqrt{a}$ is a surd. (*Cf.* § **156**.) For example, if in the process displayed in Section **76**, it appears that no primary number is the root in question, we may go on in the process of Section **93**, and find a fraction approximating as near as we please the surd number which is the true root. Under such conditions the root is a surd, and the process described interminable; but it would carry us too far afield to investigate just now general criteria for deciding whether the result of given combinations of given numbers is a commensurable or incommensurable number. (*Vide* § **249**.) Let the student critically examine his familiar process in "finding $\sqrt{2}$."

95. In general an incommensurable number cannot operate, or be operated upon, in ultimate calculation in combinations with primary or fractional numbers. In lieu of using the surd itself, we must use a fraction differing from it by as little as we please; e.g., if the ratio of a circle to its diameter enter into the calculation, we employ some approximate fraction, such as 3.14159. Surds which are roots of primary numbers or of fractions may operate with their exact force in special cases, and in a partial way; e.g.,

$$(\sqrt[3]{2})^3 = 2; \quad \sqrt{2}\sqrt{3} = \sqrt{6}; \quad 2\sqrt{12} = 4\sqrt{3}; \quad \sqrt{2/3} = 1/3\sqrt{6}, \text{ etc.};$$

but investigations into such combinations must be postponed to the next chapter, as well as the interpretation of a^s, if s is a fraction or a surd.

96. Finally, let it be distinctly recognized that the great stumbling-block which confronts us at every turn is the wretched limitation to special cases of the operation the inverse of addition, that $a - b$ is meaningless if $a < b$.

XI. Final Extension of the Number-Concept. Principle of Continuity.

97. Primary number is a discrete magnitude. The first extension of the number-concept (the connotation of ratios as number) made number one way continuous. (*Vide* § **81**.)

The conception of number as continuous in a far more general sense grew from the application of a principle, at first presented as an assumption, but which is so incessantly and overwhelmingly corroborated that its rank as a genuine and compulsory theory is perhaps as firmly established as that of any scientific principle whatsoever.

98. As has been repeatedly shown in the foregoing chapters, the combination of numbers in the inverse operations is meaningless under the primary concept except in special cases. For example, $5-5$, $5-6$, $5 \div 6$, $\sqrt{5}$, $\log_5 6$, etc., result in no primary numbers at all.

The "first extension" gives meaning to the last three of the cases just cited; for, although in the treatment here presented, a^s, where s is fractional or surd, was not interpreted from the recognition of all ratio as number, the true meaning might have been developed at that point, and $\log_5 6$ thereby rendered intelligible.* All this, be it noted, without understanding $5-5$, or $5-6$ as a number, or even imagining the development yet to come after this insight is attained.

99. For centuries science rested here, either not regarding such combinations as intelligible, and their results as numbers; or only in a halting fashion, regarding the combinations as symbolic jugglery, and the results as "imaginary numbers." And at the present day it is only by the enlightened van among men of science that this stage has been passed.

Negative numbers were in this way long called "imaginary;" but, as they gradually forced themselves into reluctant minds, the appellation was narrowed to denote $\sqrt{-1}$.

100. It was only after a long struggle that negative numbers gained recognition. I have not the erudition to

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* It was deemed more convenient to take the final step at once; since the principle which displays ratio as number, and the general principle to which the whole treatment converges, are really one and the same.

furnish exact dates, but I know that Cardan in 1545 in his *Ars Magna* calls them "*numeri ficti;*" and it is commonly asserted that Descartes in the seventeenth century was the first to rend this portion of the veil: and I suppose that those who half-heartedly follow in the wake of science continued long afterward to regard negative numbers as "imaginary," and all operation therewith as somehow a trick of algebraic signs empty of numerical meaning. Certain it is that such is the attitude even to-day, not, it is true, of those who follow in the wake, but of those who do not follow science at all, though engaging a large share of public attention as teachers thereof. For certain also it is, that at the close of the seventeenth century, Newton withdrew negative numbers (and therefore, as will duly appear, zero, and positive and negative infinity) from the befogged region of "*numeri ficti*," and revealed them as "*numeri veri*."

The last stage of this gradual process of enlightenment, in which $\sqrt{-1}$ is still regarded as "imaginary," is yet the stronghold of ignorance of fact, of prejudice, and of color-blindness to philosophic evidence.

101. I would have no war of words over the appellation "imaginary." The term in this connection historically has meant, and yet baldly means, "impossible," or *incomprehensible.* Of course it has no such meaning among the best mathematicians of to-day; but that it is so received by the unscientific, by many teachers of mathematics, and by the vast majority of undergraduate students, cannot be disputed.

The matter of a change in terminology is not of prime importance, for terms may be disassociated in technical use from their general meaning. It is a question of expediency. While sympathizing with the conservative who object to all innovations as tending to confuse the vast literature of the science, *neomonic* is so much more appropriate, and "imaginary" or "impossible" so misleading, that the benefits of the change appear to outweigh the inconveniences. A reformation in terminology is not nearly so confusing as changes in notation, such as have often been brought about; for example, the famous propaganda of "d-ism *versus* dot-age" (dy/dx *versus* $\dot{y}$), which Dr. Peacock began while yet an undergraduate, in league with Herschel, Babbage, and Maule. The reform was finally adopted at Cambridge, and Newton's notation soon became entirely excluded. Nowadays mathematicians find no confusion in using both notations.

102. The principle which has so fruitfully widened the concept of number, yielding perfect self-consistency of number, and ever deepening adaptation to Nature, I call the Principle of Continuity, in emphasis of its most important outgrowth, the unlimited, twofold continuity of number.

This principle may be stated as follows: —

103. Principle of Continuity. — *The combination of two numbers in any defined operation is always possible, the result real, and a* NUMBER; *and the precise efficacy in any operation of a number thus revealed is determined by, and may be discovered from, the formula and laws of definition of the operation in question.*

104. Before considering details, a glance at the results which have more than justified the postulation of this principle may be useful in giving the student the proper perspective of the subject.

The principle at once makes negative numbers, zero, infinities, fractions, surds, neomonic and complex numbers, all equally numbers. Also Number thus becomes unlimitedly continuous in a double sense, whereby undreamed of adaptability to Nature is revealed, and all numerical operations proceed untrammelled by particularity.

One who will logically apply the *Principle of Continuity* will arrive at all classes of numbers — or divisions of Number — with equal necessity and facility. Negative or fractional numbers will appear as much derived, as little original, or primary, as those numbers still commonly called "irrational" or "imaginary." One of these classes is as foreign as any other to the primary concept of number; that is, the concept of number as discrete, the concept which knows only one number between, say, 5 and 7.

If the symbol i be set apart to represent the neomon $(\sqrt{-1})$, we seem to have in the expression $x + yi$ the most general numerical form to which the laws of number lead.* For it has appeared upon investigation that no combination of numbers in any conceived operation can result in a form essentially different. Neither has any operation essentially different from the seven fundamental operations developed from them. It might be surmised that investigation would reveal some fourth direct operation growing out of involution, as involution grew out of multiplication, and multiplication out of addition; but such does not seem to be the case. No ground of distinction is furnished for a new species of operation. That is to say, the operation, if assumed to be distinct, would show itself not essentially so, by failing to lead to new modes of *Number*. In other words, if the investigations referred

* For this expression, a complex number in algebraic form, is numerically neomonic if $x = 0$, and numerically whatever x is, if $y = 0$.

to are trustworthy (as is no doubt the case), there can arise no new opportunity to apply the Principle of Continuity, so as to still further widen the meaning of Number. Number in its ultimate sense is therefore seen to form (what primary numbers do not, nor any curtailed concept) a universe complete in itself, such that starting in it we are never led out of it. Cayley says, whether with sound philosophy or essential contradiction of terms I will not attempt to discuss, "There may very well be, and perhaps are, numbers in a more general sense of the term (quaternions are not a case in point, as the ordinary laws of combination are not adhered to); but, in order to have to do with such numbers (if any), we must start with them." *

105. I believe that very few, even among students of mathematics, are aware of the chaos of their conception of number, in spite of long and familiar use.

The difficulty here, as everywhere, is the attainment of true concepts, insight into the principles involved.

I believe that the present condition is due to the fact that successive generations of students have not had the difficulties honestly presented to them, and have seldom even considered fundamental theory. They have been entrapped into an unwarranted complacency; they have juggled with symbols which are meaningless to them, and for the most part without even noticing that no concept

* From note made long ago; exact reference lost. In regard to quaternions it may be observed, that though in their ordinary presentation certainly not numbers, it is possible that they may yet be divested of extra-numerical properties. Speaking of the anomaly according to which quaternions in the common interpretation would make $\frac{1}{2}mv^2$ negative, whereas $\frac{1}{2}m$ is positive and the whole positive, Dr. Macfarlane, in his *Algebra of Physics*, remarks, "If this is a matter of convention merely, then the convention in quaternions ought to conform to the established convention of analysis; if it is a matter of truth, which is true?"

rises with the words they utter, the symbols they write, — that their discourse upon number is *vox et praeterea nihil.* I believe that an opposite result would be prevalent, had an opposite course been pursued by teachers and authors, and that we would now be reaping harvests instead of sowing seed.

106. Some one ignorant of trigonometry, of the analytical treatment of geometry, of the Calculus, of the varied fields of applied mathematics, and to whom the boundless realms of pure mathematics loom misty and fantastic — some such one, I say, may ask, Why all this striving to make number continuous, this travail to produce concepts of number and numerical operations, which shall be perfectly general and unrestricted? The answer is, the need, intellectual and practical, is urgent, imperative. Establish the Principle of Continuity, and Arithmetic becomes a logically perfect universe, and besides, all Nature becomes harmoniously numerical; number and its laws pervading it as an essential principle. Emerson's noble lines, in which, with the poet's seer gift, he speaks truer than he knew, then become literal fact: —

> "For Nature beats in perfect tune,
> And rounds with rhyme her every rune;
> Whether she work in land or sea,
> Or hide underground her alchemy.
> Thou canst not wave thy staff in air,
> Or dip thy paddle in the lake,
> But it carves the bow of beauty there,
> And the ripples in rhymes the oar forsake . . .
> Not unrelated, unaffied,
> But to each thought and thing allied
> Is perfect Nature's every part,
> Rooted in the mighty heart."

Besides, the assumption has been made, and its first fruits are the attainments of the physical sciences during the last two centuries. The progress in exact physical science and the dependent arts has been due to the power and freedom conferred upon analysis by this postulate; for, as I have said, it is implicit in all modern analysis, even when denied with the mouth of the calculator. (See also §§ **110, 117**.)

107. Like all profound principles, this one of the continuity and qualitative distinctions of number *is a matter of insight*, and does not admit of easy demonstration. One man cannot think for another any more than he can eat for him; but if a student will fix alert and intelligent attention upon the inherent development of the idea, and upon the manifold witness borne by almost every phenomenon, he will at last *behold* the Principle, manifest in ten thousand undreamed-of relations.

108. It is not practicable to give more than one example of the mental attitude I desire to excite. I choose one which affords a double illustration: in the first place, yielding a geometrical instance of the way in which concepts in every science are extended to conform to deepening insight, an extension analogous to the development of the primary number-concept; and in the second place, displaying (as a consequence of this attainment of an adequate geometric definition) an impressive discovery of supreme law — provided ratios are numbers, and number positive and negative — in what seems, to naïve observation, utter fortuity.

109. Illustration. — (1) The primary concept of the division of a sect by a point is, of course, that the point is on the sect; but investigation shows that a widening of the concept is required to fit *facts* presented by Nature. It is discovered that if any point, P, in the straight of a sect, AB (on or out of the sect), shall *divide* it into the segments PA and PB, then innumerable theorems only partially true, and therefore none of their inverses true (*vide infra*), under the primary concept, become universally true, and therefore their inverse propositions true, under the extended concept.

(2) The same term, *division*, is necessarily retained for this new relation; for it is the very essence of the dialectic to display the inherent identity of the two relations. To conceive (or name) the relations in contradistinction would be to miss the very truth revealed by the connotation. It is everywhere discovered that the process of philosophical advance is in great part the identification of old ideas, long in use by the mind in its experience, with ideas which to brute or naïve observation appear irrelevant or distinct. Reflection upon the pure thought brings out the implicit identity with the category already named.

(3) In particular the case of external and internal division in equal ratios is discovered to be a harmony very prevalent in nature. Such division of a sect is styled "harmonic division." (*Cf.* any *Geometry* and any scientific treatise on physics.)

(4) Now consider the two plane figures (A), a triangle and *any* straight (cutting the triangle or not); and (B), a triangle and straights joining *any* point (in or out of the triangle) to the vertices of the triangle. Under the extended conception of the division of a sect by a point, the straight in A divides each side of the triangle; and of the straights in B, each divides the side of the triangle opposite to the vertex through which it passes, in such wise that the product of the three ratios of the segments of the

sides is 1 (provided that, of adjacent segments in different sides, if one be the antecedent, then the other shall be the consequent of its respective ratio). This is assuredly a most impressive exhibition of unsuspected *lawfulness* in a fact seemingly a very type of haphazard. But, be it noted, the inverse of neither A nor B is true. Now, it is an established principle that when the inverse of any proposition is not true, it is because the subject of the direct statement has been more closely limited than truth required. It is clear that the inverses of A and B are flat contradictions.

But if the ratio of sects from the same point be considered positive if one sect is part of the other, and negative if extending oppositely, then the easily demonstrated conclusion of A is that the product of the said ratios is $+1$; and of B that the product is -1. The inverse of each now holds; that is, if three points divide the sides of a triangle so that the product of the ratios taken as stated is $+1$, then the points are co-straight; and if the product of the ratios is -1, then the joins of the points with the vertices concur.

(5) The student should fully realize what is here asserted; and to this end let him draw a triangle and then dash straights at random, cutting the triangle or not: Every one of them divides the sides of the triangle in precisely the same way; *and if number be positive and negative*, given three points so dividing the sides, they are co-straight. Again, draw a triangle, dot at random points, in or out of the triangle: Any one of these points joined to the vertices gives straights which divide the opposite sides in precisely the same way; *and if number be positive and negative*, given this way of division of the sides by three

points, the straights joining the point to the vertices come together in one point.

110. When it is considered that the preceding illustration recites merely one of ten thousand examples, number is *proved* to be positive and negative, — not, be it understood, as a convention, but as a necessity of thought. Men who represent this qualitative distinction as arbitrary, or as purely a matter of algebraic symbols, do not appreciate the evidence, or do not understand what proof in such premises means. Moreover, it must never be overlooked that a still higher order of proof is afforded in the development of the pure idea, regardless of any adaptations to external facts. When this or that development of the pure science of number is to find application to facts of other sciences is a secondary matter. (*Cf.* § **117**).

111. Similar illustrations might be given to show the adaptability of number to facts presented by nature, if the other modes of number resulting from the application of the Principle of Continuity are recognized. Presentation of such evidence must be postponed for the most part to subsequent mathematical studies; and I shall in this connection only ask you to observe that the Principle of Continuity, as enunciated in Section **103**, unifies all the partial explanations of number which you will find advanced, or implied, in various treatises; and to reflect that the man who in his own opinion discovers the entirely New is probably on the pathway, not of truth, but of estrangement. If his system refutes, in utter antagonism, preceding systems, it is likely to be refuted by a successor. In all philosophy and science, advance has been genuine only in systems which have been synthetic, and unifying of previous efforts in a harmony of thought. No development of thought must be regarded as a disjoined succession of dead results, but as living insights in one line, each piercing deeper and deeper.

112. In this light, note that the revelation in antiquity of fractional and surd numbers, and the recognition of number as positive and negative which has prevailed for two centuries (these may be regarded as the "first" (§ **78**) and second extensions of the number-concept), are both merely special cases of the universal principle here advanced.

113. To generalize is to see in a multiplicity of objects similar relations to one form of mental activity that knows those objects. But until one sees the need of a deeper principle than that which he has hitherto employed, he does not seek a way leading from what is known to him to knowledge beyond. Any idea is at first bare of manifold essential relations, external and internal. By reflection such relations are slowly revealed. During the process the idea may seem derivative from the relations (*Cf.* geometric definitions of number, § **25**); but finally this looseness must be reduced to order, and then all its belongings are seen to unfold from the idea itself, — "first the blade, then the ear, after that the full corn in the ear."

114. What has just been said would do for a description of the famous dialectic which Hegel describes as "the self-movement of the notion (*Begriff*)." Indeed, it is not much more than a paraphrase of its description by Dr. Harris, "Seize an imperfect idea and it will show up its imperfection by leading to and implying another idea as a more perfect or complete form of it. *Its imperfection will show itself as dependence on another.*" (Italics mine.)

115. I know no other method by which the teacher can lead a student to attain for himself a concept of number adequate to any comprehension of modern mathematical analysis. Each tentative idea of number must pass over into the next deeper as the result of further and further insight into the subject.

It remains to apply in characteristic cases the Principle of Continuity, discovering from the formula of definition of any operation the nature of the resulting number, as well as the efficacy of any such new phase of number in any combination in the defined operations.

XII. Significance and Efficacy of Numerical Operations Under the Ultimate Concept.

116. The very first application of the Principle of Continuity to the generalization of the operation Subtraction, displays a number *sui generis*, which is of immense importance in analysis. The formula of definition of subtraction is (*vide* § **42**) $b - a + a = b$. Then $a - a =$ what number? The formula declares that it is a number which, added to a, makes a; that is, it is a number which has *no efficacy* in addition, and therefore none in subtraction. The best and only unprejudicial name for this number is *zero*. Its symbol in arithmetic and in the algebra of number is 0.

I trust that at least it has been made clear to the student that it is only the very primary and crudest concept of number which would consider zero "nothing;" for although of no efficacy in addition or subtraction, it will presently be seen to exert extraordinary effect in every other operation. I entreat the student not to slip at this point; for the human mind, once made sensible of its powers, will never afterwards suffer its conception to be clogged by the tyranny of material categories. Moreover, it may quite commonly be found necessary to translate into correct terms much discourse in mathematical treatises, even when written by men eminent for skill and learning, to say nothing of inadequate or erroneous presentations in works on physics and applied mathematics in general. For example, you may read a *Trigonometry* which defines the trigonometric ratios not as numbers, but as sects (pieces of straight lines); yet you can often catch the author adding one of his bits of straight lines to 2 or 3^2, and in a context where he really means the number 2 or 3^2, etc. Occasionally you will meet denial or even ridicule of all that I endeavor to lead you to see, and perhaps by a man of world-wide fame. For example, in a didactic treatise on *Mathematics* by De Morgan, published in a serial *Library of Useful Knowledge*, London, 1836, zero is conceived to be "nothing"; for on page 23 one reads, "Above all, he must reject the definition, still sometimes given of the quantity $-a$, that it is less than nothing. It is astonishing that the human intellect should ever have tolerated such an absurdity as the idea of a quantity less than nothing; above all, that the notion should have outlived the belief in judicial astrology and the existence of witches, either of which is ten thousand times more possible." The truly astonishing thing concerning the human intellect is that such a man as De Morgan could have written this sentence, familiar as he must have been with Newton's distinction, "Quantitates vel Affirmativæ sunt seu majores nihilo, vel Negativæ seu nihilo minores." But, although deficiency is quite as quantitative as excess, the whole remark is impertinent; for zero is not "nothing."

Negative numbers are unquestionably less than zero. Yet, taking him at his own word, De Morgan should have hesitated before ridiculing as crazy the careful dictum of as powerful and piercing an intellect as has ever served man's will.

117. Before investigating the efficacy of zero in other operations, let us look into further results of the generalization of subtraction.

What are the properties of the resulting number in the operation $b - a$, if $b < a$? Consider the results of the following series of operations, $1 + 2$; $1 + 1$; 1; $1 - 1$; $1 - 2$; $1 - 3$; $1 - 4$, etc.

Here we have a series of numbers which at first decrease by 1, viz., 3; 2; 1; 0. The subsequent numbers respectively answer the questions, what number added to 2 makes 1, added to 3 makes 1, added to 4 makes 1? Now, in these operations the sums remain the same, and the given summands in each case increase by 1; it is clear, therefore, that the required summands must decrease by 1. Moreover, these numbers in additive combination nullify 1, 2, 3, etc.; that is, make the sum in each case zero. Thus, $1 + (1 - 2) = (1 + 1) - 2 = 0$; $2 + (1 - 3) = (2 + 1) - 3 = 0$; $3 + (1 - 4) = (3 + 1) - 4 = 0$. Such reflections reveal an unending series of discrete numbers decreasing from zero, each less than the preceding by 1. Their effect in nullifying 1, 2, 3, etc., in addition, renders appropriate the appellations *positive* and *negative* to primary numbers and these now discerned. These terms are established terms in logic, and are expressive of just such a relation of clean-contradictory as has been discovered in these modes of number. On this score, either might be called positive and the other negative; but every propriety commends the course adopted — primary numbers are *positive*, and such results as we have just considered, *negative*.

That negative number finds unlimited corroboration in adaptation to the facts of other sciences, has been amply illustrated (§ **109**); but its existence for pure mathematics is nowise dependent upon such circumstances. Negative number should never be defined or explained by such oppositions as *right* and *left*, *up* and *down*, *forward* and *backward*, *north* and *south*, *past* and *future*, *capital* and *debt*; but always in its essential character as *number*.

118. Writing pos. for *positive*, and neg. for *negative*, it is evident that pos. $a +$ neg. $b =$ pos. $a -$ pos. b; for pos. 1 $-$ pos. $2 =$ neg. 1, therefore, by definition of subtraction, pos. $2 +$ neg. $1 =$ pos. 1; but pos. $2 -$ pos. $1 =$ pos. 1, etc.

Also, since subtraction is the inverse of addition,

$$\text{pos. } a - \text{neg. } b = \text{pos. } a + \text{pos. } b.$$

119. Hereby Section **42** is completely generalized, and the common rule about "signs" and parentheses for additions and subtractions established without restriction.

120. We are arrived now at a matter of extreme importance, viz., the dual significance of the signs $+$ and $-$. One of the most salient imperfections of ordinary text-books is their failure to make a clear-cut distinction between the essentially double meaning of $+$, and of $-$. Too often the operational significance alone is defined, although on the next page you may find a complacent statement $a + (-a) = 0$; whereas, if $+$ means add, and $-$ means subtract, $a + (-a)$ means, "starting with a, add and then subtract a," of course, with the result a. And under a purely operational definition such an expression as $a / -b$ is like a "sentence" made by writing words on dice and rolling them out of a box. Clifford, in his zeal against this abomination, goes too far, and gives three totally distinct meanings to each of the signs.* His first two for each are all that are needed or justifiable.

The names of the signs are respectively "plus" and "minus;" their meanings respectively *add* or *positive*, and *subtract* or *negative*.

It is, perhaps, to be regretted that beginners are not taught to use at first different symbols for these wholly distinct thoughts, and afterwards led to observe that the notation would be simplified if one symbol were used in both meanings; because the context always makes it clear which is meant, if the simple convention be established, that, if nothing is expressed, "*positive*" is understood, and if one is omitted, it is the qualitative, and not the operational, symbol. Thus, in $(2)(-3)$, $2/-3$, 3^{-2}, $\sqrt{-1}$, etc., the meaning *subtract* would not make sense, and ambiguity is impossible; and in $2+3-4$ the convention makes it clear that the meaning is pos. 2 + pos. 3 – pos. 4. It is true that $2+3-4=$ pos. 2 + pos. 3 + neg. 4, and although less consistent than the notational convention I recite, the expression might be understood in this sense; for the result, as we have seen in Section **118**, is indifferent. But see clearly that the sign cannot have both meanings at one time; for $7-9=$ pos. 7 – pos. 9 = pos. 7 + neg. 9 = neg. 2, whereas pos. 7 – neg. 9 = pos. 16.

Note, as in accordance with the convention stated, that in solving a synthetic equation for an unknown number, its qualitative nature is unknown, and no sign is to be understood after the sign meaning *add* or *subtract*.

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* *Common Sense of the Exact Sciences*, p. 34 *et seq.*

If for any purpose it is desirable to be quite explicit, the qualitative sign may be put in parentheses with the number-symbol, with the operational sign preceding. Parentheses would hardly be used for the first term, or for a term standing alone; e.g., $+7-(+8)+(+6)-(+9)=-4$, is the full expression of what is meant by $7-8+6-9=-4$. Of course, if occasion rose, write $+7+(-8)-(-6)-(+9)=-4$. In short, write what you mean, if you express fully, but remember that abbreviations must be doubly conventional. (*Vide* § **162**.)

121. What is the product of $(-a)(+b)$?

Consider $\{+m-(+a)\}(+b)$ where $m > a$.

By the distributive law, this equals $+bm-(+ba)$; but by Section **118** it equals $\{+m+(-a)\}(+b)$; but $\{+m+(-a)\}(+b)=+bm+(-a)(+b)$ by distributive law; therefore, since $+bm-(+ba)=+bm+(-ba)$, $+bm+(-ba)=+bm+(-a)(+b)$; therefore $(-a)(+b)=-ba$.

Hence the common rule of signs.

122. What is the product of $(-a)(-b)$?

Consider $\{+m-(+a)\}(-b)$. Distributing and applying Section **121** gives $(+m)(-b)-(+a)(-b)=-bm-(-ba)=-bm+(+ba)$; but by Section **118**, $\{+m-(+a)\}(-b)=\{+m+(-a)\}(-b)=-bm+(-a)(-b)$, therefore $(-a)(-b)=+ba=+ab$.

Hence the common rule.

123. Division's definition as the inverse of multiplication, of course, establishes the rule of signs for division.

124. Sections **121**, **122**, and **123** render complete under the common "rule of signs" the freedom of distribution and commutation referred to in Sections **54**, **55**, and **56**.

125. We are now prepared to investigate still further the properties of zero. We have seen that it has no efficacy in combination with other numbers in addition and in subtraction. What is its efficacy in multiplication, in division, in involution, in evolution, and in finding the logarithm ?

126. What is the product $(a)(0)$?

Consider $ba - ba = 0$. By distributive and commutative laws, and Section **122**, $ba - ba = (b - b)(+ a) = (+ a)(b - b) = (b - b)(- a) = (- a)(b - b)$; whence $(0)(+ a) = (+ a)(0) = (0)(- a) = (- a)(0) = 0$, or briefly, regardless of positive or negative quality of a,

$$(a)(0) = (0)(a) = 0.$$

127. Note that as an independent number zero is without qualitive distinction of positive and negative; for $a + 0 = a - 0$, hence $+ 0 = - 0$.

128. It may be a profitable comparison to call attention expressly to a unique property of 1 in multiplication and division; thus, —

$(a)(+ 1) = a$, and $a / + 1 = a$, that is to say, $\times (+ 1) = \div (+ 1)$.

Also $(a)(- 1) = - a$, and $a / - 1 = - a$, i.e., $\times (- 1) = \div (- 1)$.

129. $0 \times 0 = 0$, for $(a - a)(b - b) = 0 \times 0 = ab - ab - ab + ab = 0$.

130. But what is the result $0 / 0$?

This case is of extreme importance. Failure to comprehend it when it comes into systematic use in the Calculus has put a veil of irrational mystery over that whole discipline. Thousands of students, although they have met and slightly used this indeterminate form before, yet inasmuch as they have regarded it a matter of special convention that $0/0$ should represent any number, are dumfounded to find a discipline where a number is, they say, made zero in one member of an equation, and something else in the other. After a pathetic struggle to see reason in their procedure, they commonly give over, and accept the outrageous extravagance that a concatenation of deductions to be valid need not have meaning in every link; that a compulsory conclusion of an argument does not require intelligibility of its several steps; or that results may be thoroughly made out true for reasons nowise understood.

131. When the ratio $0/0$ is first presented for consideration, one may be disposed to jump to the decision that $0/0 = 0$ or $0/0 = 1$; but it is clear, from the definition of division, that in the synthetic equation $0/0 = y$, any number (0, $3/5$, 1, $\sqrt{2}$, 2, 3, etc.) substituted for y will make a formula (§ **40**), an identity. That is to say, $0/0 =$ any number.

Indeed, this statement is merely another way of saying, "any number multiplied by zero gives zero," which is commonly accepted without objection. And both of these statements are only particular applications of the postulate expressed in the Principle of Continuity.

The ratio $0/0$, then, may be any number; but in particular instances it is often a number which may be determined by independent considerations.

132. If two numbers (or any other two magnitudes of the same kind) vary, their ratio varies; but the ratio at any assigned limits of the variables is the same as at values of the variables only infinitesimally (*vide* § **222**) removed from such limits. In fact, the original definition of *equality* of ratios contains this doctrine. (*Vide* § **83** (6).)

Consider the functions of x, $x^2 - 1$, and $x - 1$.

What is the ratio $\frac{x^2 - 1}{x - 1}$ when $x = 1$?

The ratio may be evaluated without hesitation, as x assumes various values, until $x = 1$ is reached, when both functions vanish, and the ratio assumes the indeterminate form, $0 / 0$. But when x differed only infinitesimally from 1, beyond objection, $\frac{x^2 - 1}{x - 1} = x + 1$, which differs only infinitesimally from 2. Therefore, when $x = 1$, the ratio $\frac{0}{0} = \frac{x^2 - 1}{x - 1} = x + 1 = 2$, absolutely.

There is no trickery here. The Calculus, with its astonishingly powerful algorithm, applies such numerical interpretations to concrete magnitudes; nor would it, in my opinion, be out of place in this connection to give illustrations of the wonderful propriety, and accordance with independent facts, of this method, but out of deference to established custom — *usus tyrannus* — I leave such corroborations to future studies, with the simple assurance of their cogency. I shall only set forth one more very simple illustration of an evaluation of a ratio $0 / 0$. Consider the following two functions of y, $2y + 3y^2 + 4y^3$, and $3y + 9y^2 + 27y^3$. Their ratio would be easily evaluated for particular finite values of y; but suppose the variable y becomes zero, what then is the ratio of the functions?

If $\quad y = 0, \quad \frac{2y + 3y^2 + 4y^3}{3y + 9y^2 + 27y^3} = 0 / 0.$

And if each term be divided by y we have

$$\frac{2 + 3y + 4y^2}{3 + 9y + 27y^2} = 2 / 3, \text{ when } y = 0.$$

Now, it might well be objected that in dividing by y, if $y = 0$, the most we could say would be

$$\frac{2\,(y\,/\,y) + 3\,y\,(y\,/\,y) + 4\,y^2\,(y\,/\,y)}{3\,(y\,/\,y) + 9\,y\,(y\,/\,y) + 27\,y^2\,(y\,/\,y)},$$

and that this remains as obscure as the original if $0\,/\,0$ is any number. The explanation is, that the ratio of one and the same variable to itself is constantly 1; $a\,/\,a = 1$ always. Therefore, even when $y = 0$, $y\,/\,y = 1$, and if so,

$$\frac{2\,(y\,/\,y) + 3\,y\,(y\,/\,y) + 4\,y^2\,(y\,/\,y)}{3\,(y\,/\,y) + 9\,y\,(y\,/\,y) + 27\,y^2\,(y\,/\,y)} = \frac{2 + 3\,y + 4\,y^2}{3 + 9\,y + 27\,y^2} = 2\,/\,3.$$

It would take us too far afield to go further into the doctrine of limits of variable magnitudes and infinitesimals, and the appropriate application of number. The whole question of the use of this indeterminate form $0\,/\,0$ may not improperly be postponed by the student, who for the present might content himself with the discernment that, whether it be possible to evaluate $0\,/\,0$ or not in particular problems, $0\,/\,0$ may be any number.

133. What is the result of the operation $0\,/\,a$? and what of $a\,/\,0$?

The first asks the question, what number multiplied by a gives zero; and from the formula of definition and Section **126**, the answer is evidently *zero*.

Also $0/+a = 0 = 0\,/-a$.

The second asks the question, what number multiplied by zero gives a?

From Section **126** it is evident that no number yet discerned answers this question.

But a consideration of the continuously increasing ratios (*vide* §§ **81, 82**) of the same number to a decreasing series of numbers, reveals that, if the ratio $+ a / 0$ is a number, it is one greater than any primary number, and of peculiar efficacy in operational combination. This number, whose reality is requisite for untrammelled numerical analysis, is called *positive infinity*, and notationally expressed as $+\infty$.

Similar generalization under the Principle of Continuity makes $- a / 0$ *negative infinity*, written $-\infty$.

134. The discovery of many properties of infinity, positive and negative, must be left to future studies; as well as the principles of evaluation of ratios of infinities differently derived, analogous to evaluations of ratios $0 / 0$. (*Vide* § **132**.)

It will be found that in the application of Number to certain magnitudes (e.g., straight lines in Euclidean Geometry) that for them it appears the points at infinity coincide. Other "one-dimensional" (*vide* § **232**) magnitudes show a double absolute: for example, *Probability* ranges from absolute certainty *for*, to absolute certainty *against*.

Without going too deeply into philosophical questions, it may be remarked that Hegel, in discussing the mathematical infinite, "points out that the mathematical infinite . . . uses the idea of the true infinite, and therefore stands higher than the so-called metaphysical infinite. The latter opposes the infinite to the finite as the mere negative of the latter, and thereby makes two finites, the former the void of the latter; whereas the mathematical infinite expresses self-relation as its true form."* Much might be said also of how important to philosophy is the mathematical concept of continuity. Indeed, many of Hegel's

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* *Hegel's Logic*, Harris, p. 278.

conceptions are true only as glimmerings of what mathematicians had before made clear, or have since illuminated.

135. I present in tabular form * the possible meanings of the ratio x / y, as x and y independently vary from 0 to ∞. The student can readily verify the statements, and extend them to cover distinctions of positive and negative in x and y: —

(1)	x / y is finite		if x is finite and y finite.
(2)	may be finite		if $x = 0$ and $y = 0$,
(3)			or if $x = \infty$ and $y = \infty$.
(4)	$x / y = 0$		if $x = 0$ and y not 0,
(5)			or if x not ∞ and $y = \infty$.
(6)	may $= 0$		if $x = 0$ and $y = 0$,
(7)			or if $x = \infty$ and $y = \infty$.
(8)	$x / y = \infty$		if $x = \infty$ and y not ∞,
(9)			or if x not 0 and $y = 0$.
(10)	may $= \infty$		if $x = 0$ and $y = 0$,
(11)			or if $x = \infty$ and $y = \infty$.

136. Of course $\infty + \infty = \infty$. But $\infty - \infty$ is indeterminate; since any number (0, finite, or infinite) substituted for x satisfies the synthetic equation $\infty + x = \infty$.

137. Of course $\infty \times \infty = \infty$. But $0 \times \infty = \infty \times 0$ is indeterminate; since the multiplications of which Section **135** (5), (7) are the inverses, show $0 \times \infty =$ any number.

138. Various considerations dependent upon the continuity of number confirm the interpretation that $x^0 = 1$, if x is finite. But it may suffice to consider that if x, y, and z are finite, $x^y \div x^z = x^{y-z}$; and if $y = z$, $1 = x^y \div x^y = x^{y-y} = x^0$.

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* A similar table occurs in Chrystal's *Text Book of Algebra*, Part I., p. 317.

139. Evidently (1) $0^x = 0$ if x is finite.
(2) $x^{+\infty} = \infty$ if x is finite and > 1.
(3) $= 0$ if $x < 1$ and > 0.
(4) $x^{-\infty} = 0$ if x is finite and > 1.
(5) $= \infty$ if $x < 1$ and > 0.
(6) $0^{+\infty} = 0$. (8) $\infty^{+\infty} = \infty$.
(7) $0^{-\infty} = \infty$. (9) $\infty^{-\infty} = 0$.

As the student may convince himself. (§ **143** is anticipated.)

140. But the results (1) 0^0, (2) ∞^0, (3) $1^{+\infty}$, (4) $1^{-\infty}$ are indeterminate; as may be seen most readily by considering that $x^y = m^{y \log_m x}$, where m is finite and greater than $+ 1$. x^y is accordingly determinate when $y \log_m x$ is determinate, and indeterminate when $y \log_m x$ is indeterminate. The cases when $y \log_m x$ is indeterminate are, by Section **137** :—

(1) When $y = 0$, $\log_m x = -\infty$; i.e., when $y = 0$, $x = 0$.
(2) When $y = 0$, $\log_m x = +\infty$; i.e., when $y = 0$, $x = \infty$.
(3) When $y = +\infty$, $\log_m x = 0$; i.e., when $y = +\infty$, $x = 1$.
(4) When $y = -\infty$, $\log_m x = 0$; i.e., when $y = -\infty$, $x = 1$.

141. Every indeterminate form may be reduced to $0/0$, and in this sense it may be said that the one fundamental case of indetermination is $0/0$. For example :—

$$\infty - \infty = 1/0 - 1/0 = \frac{1-1}{0} = 0/0;$$

or,

$$\infty/\infty = \frac{1/0}{1/0} = 0/0.$$

142. Let the student tabulate from the foregoing sections all the indeterminate operations.

He must be content to postpone investigation into the evaluation of these indeterminate results as they arise from particular functions of variables, regarding Section **132** as a simple example of the general principle.

143. It remains to investigate the efficacy, as exponents of evolution, of fractional, surd, and negative numbers.

What is the meaning of the operation a^x, if a is positive and finite, and x a fraction?

The conclusion is corroborated by the continuity of number, by countless correspondences, and by perfect consistency with all other laws; but regarding the Law of Indices as the essential definition of the operation, the meaning is immediately revealed. Thus: let $a^{m/n}$, where m and n are positive integers, equal z. Then, since $a^{m/n}$ is subject to the Law of Indices, $z^n = zzz \ldots n$ factors $= a^{m/n} a^{m/n} \ldots n$ factors $= a^{m/n + m/n \ldots n \text{ terms}} = a^m$. That is to say, z is a number whose nth power is a^m; or z is an nth root of a^m; i.e., $a^{m/n} = \sqrt[n]{a^m}$.

In particular, if $m = 1$, $a^{1/n} = \sqrt[n]{a}$.

As we saw in Section **94**, the operation $\sqrt[n]{a}$ (where a is positive) is always possible, in the sense that, if the result be a surd number, it can be determined to any degree of approximation. (But see § **153**, *et seq.*)

144. It will appear in the studies to which these lectures are introductory that there are n nth roots of a, where n is a primary number; but the student may observe now, that when n is even there are at least two roots of a, one the negative of the other; e.g., $4^{1/2} = \pm 2$. But note that the law of indices has regard only to the corresponding roots of numbers, simply because $\sqrt{a}\sqrt{a}$ does not equal a, if one positive and one negative root be taken. (*Vide* §§ **191** and **146**.)

145. It is necessary to say at this point that we must either use the terms "rational," "irrational," "real," and "imaginary," or invent equivalents. (*Vide* Introduction, p. 16, and § **101**.) The terms are unquestionably abusive, and perhaps the time is ripe for a protest. No number is irrational, and all numbers are real. Therefore, if merely as an experiment, I shall be consistent in calling numbers *commensurable* (*with 1*, understood) where the text-books say "rational;" either *incommensurable* or *surd*, where they say "irrational;" *radical-surd*, where they say "surd" (*Cf.* § **83**); *protomonic*, where they say "real"; *neomonic*, where they say "imaginary" (unless they say "imaginary" when they mean *complex*); and when functions or operations are spoken of as "rational" or "irrational," in substituting *stirpal* and *radical* respectively. These words, except *protomonic* and *stirpal*, are in good usage either exactly or approximately in the senses defined. *Protomonic* and *stirpal* I coin; reluctantly, but unavoidably. I hope they justify themselves as antitheses of *neomonic* and *radical*. Of course, *surd* is not much better etymologically than "irrational;" but the metaphor is dead, and consequently harmless. Concerning *commensurable*, see Section **205**. (See also § **156**.)

146. Before passing to other cases of the exponential function a^x, it is proper to call attention to certain paradoxes which may arise in the interpretation of such functions. (*Cf.* § **191**.) For example, $a^{4/2} = a^2$. But as a fractional index, $a^{4/2}$ means $\sqrt{a^4} = \pm a^2$, which at first sight might seem to assert that $a^2 = \pm a^2$. Likewise, one might be led to say, since $(a^m)^n = a^{mn} = (a^n)^m$, $(4^{1/2})^2 = (4^2)^{1/2}$, and so $(\pm 2)^2 = \pm 4$, that is, $+4 = \pm 4$. (*Cf.* § **144**.)

Such difficulties will arise in $a^{m/n}$, etc., when m/n is not in its lowest terms. $a^{4/2} = a^2$ is not a radical function at all; though it is quite true that the second roots of

a^4 are $+ a^2$ and $- a^2$. The law of indices is not a matter of arbitrary or meaningless symbols, but of *facts*. If algebraic expressions are not regarded as logical statements, and full account taken of the nature of the derivation of one equation from another, apparent contradictions will often arise. (*Cf.* § **319** *et. seq.*)

147. What is the effect of a negative exponent of involution?

Consider $\qquad a^{-m} = a^{-m} \times a^m / a^m.$

By law of indices,

$$a^{-m} \times a^m / a^m = a^{-m+m} \div a^m = a^0 / a^m = 1 / a^m,$$

by Section **138**; therefore,

$$a^{-m} = 1 / a^m.$$

That is to say, a^{-m} is the reciprocal of a^m.

148. The continuity of number at once extends all that has been shown to be true of integral and fractional exponents to surd exponents.

Thus in the function a^x, whether x be commensurable or surd, we can always find two fractions, m / n and $\frac{m+1}{n}$, between which x lies, and which differ by as little as we please. As stated in Section **95**, in calculation we use $a^{m/n}$ instead of a^x, where m / n is a fraction closely approximating the surd x.

149. When a is positive and > 1, and regarding only protomonic positive roots, a^x is a continuous function of x, passing through all values from 0 to $+\infty$, as x varies from $-\infty$ to $+\infty$. Thus,—

	a^x is	0,	<1,	$1/a$,	1,	>1,	a,	$+\infty$
when	x is	$-\infty$,	$-$,	-1,	0,	$+$,	$+1$,	$+\infty$.

When a is positive and < 1, the values of a^x are $+\infty$, > 1 $1/a$, 1, < 1, a, 0, corresponding to the same values of x.

150. As has been explained, $b = a^x$ and $x = \log_a b$, are merely different ways of writing the same functional relation. Thus all laws and properties of logarithms are derivable from the principles of involution, in brief, from the law of indices. Until the uses of logarithms and the construction of logarithmic tables are investigated, it is enough to say that for the *same base* the following are the leading properties of logarithms, — as the student may easily discover from the law of indices: —

(1) The log. of a product of positive numbers is the sum of the logs. of the factors.

(2) The log. of the quotient (ratio) of two positive numbers is the log. of dividend minus log. of divisor.

(3) The log. of any power of a positive number is the log. of the number multiplied by the exponent. (Power is used in the general sense; for the statement is true for all exponents, and therefore inclusive of the commonly separated rule for roots.)

(4) $(\log_a b)(\log_b a) = 1$, and $\log_a m = \dfrac{\log_b m}{\log_b a}$.

The base of "common" logarithms for purposes of final calculation is 10; but the base discovered to be primarily appropriate to mathematical investigations is an incommensurable number, called ϵ.

$$\epsilon = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \text{ to } \infty \text{ terms *}$$
$$= 2 \cdot 7182818284 +.$$

* 1×2 may be abbreviated $2!$
$1 \times 2 \times 3$ may be abbreviated $3!$
$1 \times 2 \times 3 \times 4$ may be abbreviated $4!$
Read "factorial two," "factorial three," etc.

The base 10 gives logarithms vastly more convenient in calculation; the base ϵ, in analysis.

Formulæ (4) yield the simple process for deducing from a given table of logarithms to any base, the logarithms to any other base. Thus, to deduce $\log_a m$ from $\log_b m$, multiply by $\frac{1}{\log_b a}$.

The constant multiplier, $\frac{1}{\log_b a}$, is called the *modulus* of the system whose base is a with respect to the system whose base is b.

The modulus of the system whose base is 10 with respect to the system whose base is ϵ, is $\frac{1}{\log_\epsilon 10} = 0 \cdot 4342944819 +$.

The modulus of the system, base ϵ, with respect to the common system, is $\frac{1}{\log_{10} \epsilon} = 2 \cdot 7182818284 +$.

Of course, $\frac{1}{\log_\epsilon 10} = \log_{10} \epsilon$; that is to say, the reciprocal moduli of two systems are reciprocals in the numerical sense.

151. For interesting historical sketches, the student is referred to the articles, "Logarithms" and "John Napier of Merchiston," by J. W. L. Glaisher, in the *Encyclopædia Britannica*, ninth edition. A perusal of these monographs will lead him to appreciate the brilliancy of Napier's invention, and the merit of Briggs and Vlacq, as well as the claims of Byrgius, a Swiss contemporary of Napier, as an independent but crude inventor. He should bear in mind that this achievement came prior to the exponential notation, or any clear idea among mathematicians of exponential functions. An attempt to prove — to say nothing of discovering — the laws of logarithms, after divesting one's self of knowledge of the generalization of involution and all modern advantages from the correspondences of two series of numbers, one in "arithmetic" and the other in "geometric" progression, would afford a very high estimate of Napier's genius and acumen.

152. The student can easily discern the laws of the function a^x if a is negative, and x zero or integral. If x is zero or a multiple of 2, the power is positive; if x is odd, the power is negative.

153. Also, if a is negative, and x fractional, with odd denominator, the power is protomonic (*vide* § **145**). In other words, if a is negative and n odd, there is always a protomonic nth root of a. For consider the function y^n, where n is an odd positive integer. The function passes through all values from $-\infty$ to 0, as y passes from $-\infty$ to 0. Therefore, there must be some protomonic negative number, y, for any negative number, a, such that $y^n = a$. That is to say, there is always a protomonic odd root of a negative protomonic number.

154. But if in the operation a^x, a is negative and x a fraction in its lowest terms, with even denominator, there is no result whatever, nor is the operation intelligible within protomonic number. For the function y^n, where n is an even integer, is always positive. Therefore there is no protomonic number, y, such that y^n is negative when n is even. That is to say, there is no protomonic even root of a negative protomonic number.*

155. Evidently, then, unless the Principle of Continuity

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* Also in terms of protomonic number there is no logarithm of a negative number to a positive base. At this stage we cannot investigate such functions; but $\log_{(+a)}(-b)$ has been shown to be indeterminately any member of an infinite series of complex numbers. Thus in no case are we led out of complex number as the ultimate generalization. (*Cf.* § **202.**)

shall widen our concept of number, the generality of numerical operations abruptly fails at this point. But the Principle of Continuity does apply here as everywhere else; and the power of analysis is enhanced, and the applicability of Number to the relations of concrete magnitudes perfected beyond the dreams of mathematical science prior to this development.

Before taking this step, however, we must investigate a few fundamental properties of radical-surds.

156. In all algebraic expression of number the student must avoid confusion on account of any possible value of a function for particular numbers in place of the algebraic symbols. Thus $(p^{1/2})^m$ is a radical function of p in algebraic form; although, of course, in cases where $m = 2n$, and n an integer $(p^{1/2})^{2n} = p^n$, the nature of which again depends on the character of p. Or, the $\sqrt{x}$ is algebraically a radical-surd; although if $x = 4$, it is commensurable, and so forth.

It is not necessary to be constantly "providing" obvious conditions. Intelligent attention will always secure comprehension of the algebraic statements in the sense intended, whenever explicit provision is omitted.

157. A radical-surd number, or multiple, or fraction thereof, is called a simple, monomial radical-surd. The sum or difference of two such, or of one such and a commensurate number, is called a simple binomial radical-surd.

It will be seen that every stirpal function of a radical-surd can be expressed as a simple radical-surd.

Two radical-surds are called similar when they can be expressed as multiples or fractions of the same radical-surd: e.g., $\sqrt{3/4}$ and $\sqrt{12}$ are similar; for $\sqrt{3/4} = 1/2\sqrt{3}$, and $\sqrt{12} = 2\sqrt{3}$.

Radical-surds with the same base and same root-index are called equiradical; e.g., $a^{1/5}$, $a^{7/5}$, $a^{n/5}$.

Radical-surds with the same root-index are called of the same order — quadratic, cubic, biquadratic, quintic, . . . n-tic; e.g., $\sqrt{3}$, $5^{3/2}$, $x^{n/2}$ are quadratic surds; $\sqrt[5]{3}$, $5^{3/5}$, $x^{n/5}$ are quintic surds.

158. From the Law of Indices ($a^m a^n = a^{m+n}$) it is easily proved for protomonic numbers (but see § **191**) that

$a^m a^m = (aa)^m$, or $a^m b^m = (ab)^m$. Thus, if m is integral
$a^m b^m = (aaa \ . \ . \ . \ m$ factors$) \times (bbb \ . \ . \ . \ m$ factors) by definition,
$= (ab.ab.ab \ . \ . \ . \ m$ factors) by laws of association and commutation,
$= (ab)^m$ by definition.

And if m is fractional, say $m = 1/n$ where n is a positive integer, $(a^{1/n} a^{1/n} a^{1/n} \ . \ . \ . \ n$ factors$) \times (b^{1/n} b^{1/n} b^{1/n} \ . \ . \ . \ n$ factors$) = (a^{1/n} b^{1/n}) (a^{1/n} b^{1/n}) \ . \ . \ . \ n$ factors; but the left-hand member equals ab; therefore $(a^{1/n} b^{1/n}) (a^{1/n} b^{1/n}) \ . \ . \ . \ n$ factors $= ab$, therefore $a^{1/n} b^{1/n} (ab)^{1/n}$, *if* positive roots of a, b, and ab are alone considered (*vide* §§ **144, 146**).

159. A special case, $\sqrt[n]{a^n b} = a\sqrt[n]{b}$, is important in reducing radical-surds to similarity.

160. Note also $\sqrt[n]{a} = \sqrt[np]{a^p}$; for $a^{1/n} = a^{p/np} = \sqrt[np]{a^p}$.

161. Also, $\sqrt[n]{a^{pn+q}} = a^p \sqrt[n]{a^q}$; for $a^{\frac{pn+q}{n}} = a^p a^{q/n}$.

162. Similar radical-surds are "added" or "subtracted" by distributing the radical-surd factor with the coefficients. (*Vide* § **73**.) If possible, first reduce by the principle of Sections **159, 160**, e.g., $1/3 \sqrt{32} - \sqrt{18} + 3 \sqrt[4]{64} = 1/3 \sqrt{(16)(2)} - \sqrt{(9)(2)} + 3 \sqrt{(4)(2)} = 4/3 \sqrt{2} - 3 \sqrt{2} + 6 \sqrt{2} = (4/3 - 3 + 6) \sqrt{2} = 13/3 \sqrt{2}$.

Statements involving radicals are usually intended to concern only positive roots; but in abstract operation such statements are necessarily various, including the roots in every combination. The whole truth about the result in the example is

$$\{\pm 4/3 - (\pm 3) + (\pm 6)\} \sqrt{2} = \pm 13/3 \sqrt{2}, \text{ or }$$
$$\pm 5/3 \sqrt{2}, \text{ or } \pm 23/3 \sqrt{2}, \text{ or } \pm 31/3 \sqrt{2}.$$

The $\sqrt{2}$ is also both positive and negative; but since each commensurable factor has already occurred with both signs, no new value would be obtained from the double value of the $\sqrt{2}$. But if all this is to be signified, it would be better to be explicit, and write $1/3\,(\pm\sqrt{32}) - (\pm\sqrt{18}) + 3\,(\pm\sqrt[4]{64})$. (*Vide* § **120.**)

163. Section **158** affords the rule for the multiplication or division of similar radical-surds, or of radical-surds of the same order.

If radical-surds are not of the same order they may be made so by Section **160.**

The Law of Indices immediately furnishes the rule for the involution or evolution of radical-surds.

164. The student should exercise himself in these operations.

165. Two simple binomial quadratic surds are called conjugate when one is the sum and the other the difference of the same two terms: e.g., $a + \sqrt{b}$ and $a - \sqrt{b}$, or $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$.

166. Theorem. — The product of conjugate binomial quadratic surds is a stirpal function of their bases (a commensurate number if the bases are commensurate numbers), namely, the difference of the squares of the terms.

Proof: $(\sqrt{a} + \sqrt{b})\ (\sqrt{a} - \sqrt{b}) = a + \sqrt{a}\ \sqrt{b} - \sqrt{b}\ \sqrt{a} - b = a - b.$

167. It is usually preferable in the division of one radical-surd by another, or of a commensurable number or non-radical surd by a radical-surd, to stirpalize * the denominator.

This is accomplished when the divisor is a monomial radical-surd, as $\sqrt[n]{a^m}$, by multiplying both dividend and divisor by $\sqrt[n]{a^{n-m}}$. For example, —

$$\frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{4\sqrt{2}\sqrt{2}} = 3/8\sqrt{2}; \quad \frac{c}{b\sqrt[5]{a^3}} = \frac{c\sqrt[5]{a^2}}{b\sqrt[5]{a^3}\sqrt[5]{a^2}} =$$

$$c/ba\sqrt[5]{a^2}; \quad \frac{c}{b\sqrt[3]{a^5}} = \frac{c}{b\sqrt[3]{a^3}\sqrt[3]{a^2}} = \frac{c}{ba\sqrt[3]{a^2}} = c/ba^2\sqrt[3]{a}.$$

When the divisor is a binomial quadratic surd, multiply both dividend and divisor by the conjugate quadratic surd; when a trinomial, make it a binomial by association, and apply the principle twice.

168. Let the student find a stirpalizing multiplier for $a^{m/n}\, b^{p/q}\, c^{r/s} \ldots$

This is the most general case of a monomial.

169. A *stirpal integral term* with respect to any numbers, means the product of positive integral powers of those numbers.

A *stirpal integral function* of any numbers is a series (one or more) of stirpal integral terms combined in addition or subtraction.

Where no ambiguity is to be feared we may say merely "integral function." $x/a + y/b + z/c - 1$ is an integral function of x, y, z; but is not an integral function of a, b, c.

In integral functions the *degree* of any term is the sum

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* The common term is "rationalize;" but having eschewed this, we must say *stirpalize*.

of the exponents of the numbers considered (commonly called *variables*); and the *degree of the function* is the highest of the degrees of its terms.

An integral function of the 1st degree is often called a *linear* function.

The term *degree* applies only to integral functions. Thus, $\frac{1}{x} + \frac{1}{x^2} + 1$ is of no degree at all; the term does not apply.

Functions in which the variables are affected by positive, but not integral, exponents are called radical functions. For example, $a + \sqrt{b + x}$, or $a + (b + x)^{1/2}$, is a radical function of x (also of b); and $\sqrt[a]{x + \sqrt[b]{y}}$, or $(x - y^{1/b})^{1/a}$, is a radical function of x and y.

Functions in which the variable occurs with negative index are called fractional functions, and distinguished as stirpal or radical fractional functions, according as the negative index is integral or not. Thus, $\frac{a}{x}$, or ax^{-1}, is a stirpal fractional function of x; and $\frac{a}{\sqrt{x}}$, or $ax^{-1/2}$, is a radical fractional function of x.

Integral, radical, and fractional functions are classed, not very felicitously, as "algebraical" functions, in distinction from others equally algebraical, called "transcendental." I shall have no occasion to use these objectionable terms, since the other functions are all particularly named upon their own merits.

Functions in which the number considered occurs as an exponent are called exponential functions; e.g., a^x, $a^{x^{-2}}$ are exponential functions of x.

The foregoing classes of functions are those organically involved in numerical operations. Others, less essentially connected with organic laws, are named from their several points of view; e.g., log x, logarithmic function; sin x, cos x, tan x, etc., trigonometric functions, etc.

Numerical functions (*Cf.* §§ **230**, **234**) of every variety are termed analytical functions (*Cf.* §§ **145** and **156**.)

170. THEOREM. — Every integral function of quadratic surds ($\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ . . .) can be expressed as a sum of a non-radical term and multiples or fractions of the radicals and their products —

($\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ . . . $\sqrt{ab}$, $\sqrt{ac}$, $\sqrt{bc}$ . . . $\sqrt{abc}$ . . .).

Proof: Consider any integral function of one quadratic surd, say $\phi(\sqrt{a})$. Terms of even degree are non-radical, and terms of odd degree can all be reduced to the form $na^m\sqrt{a}$. Collecting the even and odd degree terms, we have $\phi(\sqrt{a}) = k + h\sqrt{a}$, where k and h are stirpal.

If we have $\phi(\sqrt{a}, \sqrt{b})$, proceeding as before, we get $\phi(\sqrt{a}, \sqrt{b}) = K + H\sqrt{a}$, where K and H are stirpal so far as $\sqrt{a}$ is concerned, and each an integral function of $\sqrt{b}$. These can be reduced, and will yield only terms such that $\phi(\sqrt{a}, \sqrt{b}) = k + h\sqrt{a} + m\sqrt{b} + n\sqrt{ab}$.

171. COROLLARY. — It follows that $\phi(-\sqrt{a}) = k - h\sqrt{a}$; and therefore if $\phi\sqrt{a}$ be any integral function of $\sqrt{a}$, then, $\phi(-\sqrt{a})$ is a stirpalizing factor of $\phi\sqrt{a}$. (*Cf.* § **166**.)

Also if in $\phi(\sqrt{a}, \sqrt{b}, \sqrt{c}, \ldots)$ we change the sign of any one, say, $\sqrt{b}$, then $\phi(\sqrt{a}, \sqrt{b}, \sqrt{c}, \ldots) \times \phi(\sqrt{a} - \sqrt{b}, \sqrt{c}, \ldots)$ is stirpal so far as $\sqrt{b}$ is concerned.

172. Extension of the theorem to all stirpal functions, integral or not, of quadratic surds — and of the corollary to the entire stirpalization of $\phi(\sqrt{a}, \sqrt{b}, \sqrt{c}, \ldots)$ — is left as an exercise to the student.

173. As a very simple example of the utility of these principles, suppose one had to calculate to five decimal places,

$\frac{1}{1+\sqrt{2}+\sqrt{3}}$. Time and labor would be saved by reducing to the equivalent integral function of the radicals, $1/2+1/4\sqrt{2}-1/4\sqrt{6}$, before calculating.

174. Theorem. — "If p, q, A, B, be all commensurable, and $\sqrt{p}$ and $\sqrt{q}$ incommensurable, then we cannot have $\sqrt{p}=A+B\sqrt{q}$.

"For, squaring, we should have, as a consequence, $p=A^2+B^2q+2AB\sqrt{q}$; whence, $\sqrt{q}=\frac{p-A^2-B^2q}{2AB}$, which asserts, contrary to our hypothesis, that $\sqrt{q}$ is commensurable."

The proof of this theorem, which is copied verbatim from Chrystal's *Text Book of Algebra*, Vol. I, p. 200, establishes what may seem at first sight a contradiction of the doctrine of the Continuity of Number. Especially so, under the somewhat ambiguous title of the section in Professor Chrystal's work (perhaps the best yet written in English), the "*Independence of Surd Numbers*." Radical-surds are definite parts of the continuous magnitude, Number; nor does the theorem contradict this; nor are radical-surds "independent" in any other sense than that there are no *commensurable* numbers such that $\sqrt{p}=A+B\sqrt{q}$.

175. Since, by Section **170**, any integral function of a quadratic surd can be expressed as in the form, $A+B\sqrt{q}$, it follows from Section **174** that one quadratic surd cannot be expressed as an integral function of a dissimilar surd.

176. It is an obvious corollary of Section **174** that if $k+h\sqrt{a}+m\sqrt{b}+n\sqrt{ab}=0$, where neither a nor b is zero, then $k=0$, $h=0$, $m=0$, and $n=0$.

177. One case, whose utility is experienced very early in algebraic studies deserves special mention. If $a+\sqrt{x}$

$= b + \sqrt{y}$, then $a = b$ and $x = y$, provided a, b, x, and y are all commensurable, and $\sqrt{x}$ and $\sqrt{y}$ surds.

178. Let the student prove that the product or quotient of two similar quadratic surds is commensurable; and inversely.

The like is not true for radical-surds of higher orders; but let him show that the product of two similar, or of two equiradical, surds is either commensurable or an equiradical surd.

179. We are now prepared to take up the consideration of the problem presented in $a^{1/n}$ where a is negative, and n an even, positive integer.

As we saw in Section **154**, the operation is unintelligible under the concept of Number thus far attained. But if the Principle of Continuity is valid, the result must be a number; and if not any number hitherto conceived, we must investigate the characteristics of this unknown number, x in the synthetic equation $(-1)^{1/2} = x$.

180. Whether fortunately or unfortunately, this problem confronts pupil and teacher at a very elementary stage of numerical analysis. In every high school the solution of quadratic equations is attempted; and these, even in the simplest form, are in general solvable only in terms of neomonic and complex numbers. The question, therefore, cannot be postponed; and it behooves every teacher to clear up his ideas on this subject.

181. Mathematicians of to-day have left the point of view of the sixteenth century, from which numbers were characterized as "rational" and "irrational," "real" and "imaginary;" they use $\sqrt{-1}$ as naturally as -1. Neomonic *one*, and negative *one*, bear a similar relation to Primary Number.

The conception of neomonic number is not essentially more difficult than that of negative number. He who can conceive the one, can conceive the other. The $\sqrt{-1}$ is no more an impossible and meaningless operation in terms of protomonic number, than $1 - 2$ is impossible and unintelligible in terms of primary number. Terms are often babbled in unconscious vacuity of thought. Many speak quite familiarly of negative number, who nevertheless regard neomonic number as some irrational and meaningless trick of handwriting. As suggested in Chapter XII, I lament *imperfect concepts of Number on the part of us all*, but let no man pigeon-hole in his mind contradictory opinions. It seems to me something to put neomonic numbers on the same footing as negative numbers, or even numerical fractions.

When this point of view is attained, I think we stand in the dawn; or rather that the sun has risen upon Arithmetic, even as it has risen upon Geometry. Perhaps we shall not have long to wait for still fuller and more satisfying interpretations of number than have been expounded hitherto; because not one man, but hundreds, have reached some such standpoint as that from which I have endeavored to present the subject. During two thousand years after Euclid saw that he must assume the "parallel postulate" it was universally regarded either as an axiom, or as a theorem capable of demonstration. But finally the true insight was gained (regained) by many minds about the same time; and then the Non-Euclidean Geometry, and daylight became, indeed, "inevitable." *

* The *Monist*, July, 1894, '*Non-Euclidean Geometry Inevitable*, by George Bruce Halsted. Of course the majority of text-books still present Geometry at this crucial place from the mediæval standpoint; but

182. If the $\sqrt{-1}$ is a number, we have by definition

$$\sqrt{-1}\,\sqrt{-1} = -1,$$

also $\sqrt{a}\,\sqrt{a} = a$, where a is any positive protomonic number;

therefore $(\sqrt{a}\,\sqrt{-1})\,(\sqrt{a}\,\sqrt{-1}) = -a$, multiplying member by member;

therefore $\sqrt{a}\,\sqrt{-1} = \sqrt{-a}$, taking square root of each member.

Consequently it appears that the square root of any negative number is the product of the square root of the corresponding positive number and $\sqrt{-1}$. Considering also all multiples and fractions of $\sqrt{-1}$, and the negatives of each, we discern a continuous Number whose unit is $\sqrt{-1}$, and which has, therefore, been called Neomonic Number. The Number whose unit is 1 may be called Protomonic in contradistinction.

Writing i for $\sqrt{-1}$, this continuous series may be represented —

$$-\infty i \;..\; -2i \;..\; -\sqrt{2}\,i \;..\; -i \;..\; -\tfrac{1}{2}i \;..\; 0\,(i) \;..\; +\tfrac{1}{2}i \;...\; +i \;..\; +\sqrt{2}\,i \;..\; +2i \;..\; +\infty i.$$

The protomonic series may be represented —

$$-\infty \;...\; -2 \;...\; -\sqrt{2} \;...\; -1 \;..\; -1/2 \;..\; 0 \;...\; +1/2 \;..\; +1 \;..\; +\sqrt{2} \;...\; +2 \;...\; +\infty.$$

183. No neomonic number can equal any protomonic number except $0\,i = 0$. For it is deducible from various

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this is probably as much due to the mercantile rule of using up a stock-on-hand before advancing to something better, as to ignorance of recent developments. No doubt hundreds of teachers put the "axiom" in its right place in their expositions of the text; and so, as it were by a note, bring their text-books "up to date."

premises that $0\,i = 0$. Thus, if $xi = 0$, then $(xi)\,(xi) = 0$, that is, $-x^2 = 0$; therefore $x = 0$, and therefore $0\,i = 0$.

184. Most laws of operation with neomonic numbers are evident from familiar principles. Thus:—

$ai + bi = (a + b)\,i$ . . . hence the sum of two neomonic numbers is neomonic.

$ai - bi = (a - b)\,i$ . . . hence the difference of two neomonic numbers is neomonic.

$ai \times b = abi$ . . . hence the product of a neomonic and a protomonic number is neomonic.

$ai \times bi = -ab$ . . . hence the product of two neomonic numbers is protomonic.

$ai \div b = (a/b)\,i$, $b \div ai = (-b/a)\,i$ . . . hence ratios of protomonic and neomonic numbers are neomonic.

$ai \div bi = a/b$ . . . hence ratios of neomonic numbers are protomonic.

$i^2 = -1$; $i^3 = -1\sqrt{-1} = -i$; $i^4 = i^2 i^2 = +1$; and where n is a positive integer,

$i^{4n} = +1$; $i^{4n+1} = +i$; $i^{4n+2} = -1$; $i^{4n+3} = -i$; and

$(ai)^n = (ai\,.\,ai \ldots n \text{ factors}) = (aaa \ldots n \text{ factors})\,(iii \ldots n \text{ factors}) = a^n i^n$,

that is, the positive integral power of a neomonic number is protomonic or neomonic according as the same power of i is protomonic or neomonic. Moreover, the integral powers of i are seen to recur in a period or cycle of four different values. Negative exponents result as always in the reciprocal of the same number with like positive exponent.

185. Discussion of radical functions of i, and the interpretation of neomonic exponents, is postponed to more advanced studies; but we are not led to any new application of the principle of Continuity, and therefore to no new mode of Number, beyond the result of combining protomonic and neomonic numbers in addition and subtraction.

186. The extension of the number-concept reaches its own essential terminus in the operation $a + bi$, where a and b are protomonic.

In $a + bi$ we have the most general expression of number; for it is protomonic, neomonic, or complex, according as $b = 0$, $a = 0$, or neither equals 0.

187. The result of the operation $a + bi$, is called a complex number; and is seen to be really a new mode of Number by considering the series of complex numbers formed in $a + bi$, as a and b pass independently through all protomonic values.

188. It is highly important to note this two-fold, two-dimensional (*vide* § **229**, *et seq.*), character of complex number, and its consequent contrast with protomonic and neomonic number. There is only one way of varying x continuously (without repetition of intermediate values) from -2 to $+3$, if it remains protomonic. Likewise, only one way for continuous passage of x from $-2i$ to $+3i$, if it is to be always neomonic. But in utter contrast, there is an infinite variety of ways for x to pass continuously from $-2 + 3i$ to $+2 + 3i$, remaining always a complex number. (*Vide* § **197**.)

189. If $a = 0$ and $b = 0$, $a + bi = 0$; and inversely.

190. Complex number contains all protomonic and all neomonic number as special cases, and is therefore Number in its final generalization.

191. The student should everywhere carefully avoid confusion in dealing with the alternate square roots of any number; but especially is this the case with neomonic numbers. Having been accustomed to write (*vide* §§ **64, 158**) $\sqrt{a}\sqrt{b} = \sqrt{ab}$, he may fall into the error of writing $\sqrt{-a}\sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab}$. I call this an error because we must be consistent in algebraic conventions; and in such contexts the positive root is understood by $\sqrt{ab}$.*

It is not a true statement that $\sqrt{a}\sqrt{b} = \sqrt{ab}$, if the square roots are to be taken at random. One cannot make various assertions in the same sentence. Therefore, in $\sqrt{a}\sqrt{b} = \sqrt{ab}$, we evidently mean only the positive square roots to be considered. If negative roots are to be taken into account, we must say what we mean. Thus (writing $+\sqrt{a}$ for *positive square root of* a, and $-\sqrt{a}$ for *negative square root of* a) $(-\sqrt{a})(-\sqrt{b}) = +\sqrt{ab}$; or $(-\sqrt{a})(+\sqrt{b}) = -\sqrt{ab}$, etc.

Now, if in accordance with the algebraic convention plainly exhibited above, we consider only positive square

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* In a translation just published of Durège's *Theory of Functions of a Complex Variable*, by Professors Fischer and Schwatt of the University of Pennsylvania, Philadelphia, 1896, it is stated on Page 10 of the Introduction: "Euler himself taught, as now generally accepted, that, if a and b denote two positive quantities, $\sqrt{-a}\sqrt{-b} = \sqrt{ab}$; i.e., that the product of two imaginary quantities is equal to a real quantity."

The omission of the minus sign before $\sqrt{ab}$ may be a typographical error; for the authors, like all others, use $\sqrt{-a}\sqrt{-b} = -\sqrt{ab}$.

In the translators' Introduction it is very appropriately remarked:—

"To follow the gradual development of the theory of imaginary quantities is especially interesting, for the reason that we clearly perceive with what difficulties is attended the introduction of ideas, either not at all known before, or at least not sufficiently current. The times at which negative, fractional, and irrational quantities were introduced into mathematics are so far removed from us, that we can form no adequate conception of the difficulties which the introduction of those quantities may have encountered. Moreover, the knowledge of the nature of imaginary quantities has helped us to a better understanding of negative, fractional, and irrational quantities, a common bond closely uniting them all."

Of course I would have one read *numbers* in the place of "quantities."

roots of neomonic numbers, $\sqrt{-a}\,\sqrt{-b}$ does not equal $\sqrt{ab}$, but $-\sqrt{ab}$; —
for $\sqrt{-a}\,\sqrt{-b} = \sqrt{a}\,i\sqrt{b}\,i = i^2\sqrt{ab} = (-1)\sqrt{ab} = -\sqrt{ab}$.

One need find no difficulty in reconciling with the Principle of Continuity the statements that, regarding only positive roots, $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$, while $\sqrt{-a}\,\sqrt{-b}$ is not equal to $\sqrt{(-a)(-b)}$. The law of indices must be applied with due regard to other laws. The essential statement of the law of indices is $a^x a^y = a^{x+y}$. This includes all particular cases as a, x, and y assume different characters. But it has been necessary with every phase of number to understand in this statement that only corresponding roots are considered when x and y are fractional with even denominators. (*Cf.* §§ **144**, **146**.) For example, $1^{1/2} \times 1^{1/2}$ would not equal $1^{1/2+1/2}$, or 1, if one positive and one negative root were taken. Now, this fundamental statement of the law of indices holds for all number. It is the very definition of $\sqrt{-1}$, that $(-1)^{1/2}(-1)^{1/2} = (-1)^{1/2+1/2} = (-1)^1 = -1$.

It was easily proved for protomonic number that, regarding only corresponding roots when x is a fraction with even denominator, $a^x a^x = (aa)^x$, and $a^x b^x = (ab)^x$; but when a and aa differ in quality, the very conditions of the original statement are abolished (it is as if one positive and one negative root of a^x had been taken), and different conclusions might be anticipated under the *same* laws.

In fine, all this is not an anomaly of $\sqrt{-1}$ in operation, but merely an alternative statement of its existence. The difficulty lies in the origin of neomonic number, not in its operation.

On the other hand, $a^x/b^x = (a/b)^x$, established for protomonic number, does hold if a^x and b^x are neomonic, — simply because, in this case, no qualitative difference arises in the direct performance of the operations indicated by the two members of the equation, if, in accordance with the meaning of the formula, only positive roots are regarded.

For example, $(-4)^{1/2}/(-9)^{1/2} = (4/9)^{1/2}$; for $(-4)^{1/2} = 2i$, and $(-9)^{1/2} = 3i$; therefore, $(-4)^{1/2}/(-9)^{1/2} = 2i/3i = 2/3$. Also the positive square root of $4/9$ is $2/3$.

Note, also, that for a like reason $\sqrt{a}\sqrt{-b} = \sqrt{-ab}$; for $\sqrt{a}\sqrt{-b} = \sqrt{a}\sqrt{b}\,i = \sqrt{ab}\,i$, and $\sqrt{-ab} = \sqrt{ab}\,i$.

The safe practice is to express every neomonic number in its essentially proper form, as based upon a new unit.

Rules of thumb would conduct one to true results in all operations except multiplication; but for many reasons, always express $\sqrt{-a}$ as $\sqrt{a}\,i$. If you do this, correct calculation will be easy under the very definition of the neomon, $i^2 = -1$.

192. As a natural consequence of the view that Algebra is some mysterious conglomeration of "pure symbols" (*Cf.* Introduction, pp. 8, 12) without content, existing for itself, void of numerical meaning, it was long discussed, as if it were a matter to be settled by parliament, whether $\sqrt{-a}\sqrt{-b}$ should equal $\sqrt{-ab}$, or $-\sqrt{ab}$. Only one hundred years ago English mathematicians were divided on this question. One party argued that the product must be $\sqrt{-ab}$; because the product of one "impossible quantity" by another, could not possibly equal a "real quantity" — as if *a priori* deduction of what is, or is not, possible with *impossible quantities* was not *ab initio* an impossible discussion within the realm of Reason.

May not the foregoing discussion (as well as every other investigation we have pursued) serve to emphasize the cardinal thesis of these lectures; namely, that the essential nature of any algebra is as defined in Section **20**; that it is Arithmetic, as the science of Number, which everywhere underlies, shapes, and organizes our Algebra; that it is real numerical laws and operations that the algebra conventionally expresses; that, although Number is certainly a creation of the human intellect, it is not, therefore, the creature of our choice or whim; that, once formed, the Idea unfolds itself; that every numerical problem is a question of Truth; that the explanation is to be discovered; and that the verdict is nowise subject to conventional decision or parliamentary settlement.

193. It might be very helpful to illustrate the properties of complex number by the graphic representation known as Argand's diagram, which constitutes the foundation of a beautiful application to geometry; but we shall here confine ourselves to purely analytical investigations.

We have seen (§ **188**) the two-dimensional nature of complex number, and the infinite variety of ways in which it may vary continuously from $a + bi$ to $c + di$, because the protomonic and neomonic parts may vary independently.

In order that $x + yi$ shall become zero, x and y must vanish simultaneously. For, if $x + yi = 0$, $x = -yi$, and hence $x = 0$ and $y = 0$, — else would a protomonic number equal a neomonic, which is impossible except both be zero. (*Vide* § **183**.)

On the other hand, if either x or y becomes infinite, $x + yi = \infty$. (*Vide* § **198**).

194. If $a + bi = c + di$, then $a = c$ and $b = d$.

For, subtracting $c + di$ from each member of the given equation, $a - c + (b - d)\,i = 0$; therefore, by Section **193**, $a - c = 0$ and $b - d = 0$; that is, $a = c$ and $b = d$.

Of course, since $x + (-y) = x - (+y)$ or $x - y$ (§ **120**), the preceding formula includes all combinations as a, b, c, and d are positive or negative; e.g., if $a + bi = c - di$, $a = c$, $b = -d$.

195. Two complex numbers which differ only in that one is the result of the addition, the other of subtraction, of the neomonic part, are called conjugate; e.g., $-1/2 + 2\,i$ and $-1/2 - 2\,i$, or $2\,i$ and $-2\,i$, or generally $x + yi$ and $x - yi$.

Obviously the sum of conjugate complex numbers is protomonic, but so also is their product:—

$$(x + yi)\,(x - yi) = x^2 - y^2\,i^2 = x^2 + y^2.$$

196. Let the student prove the inverse proposition.

197. $x^2 + y^2$ is called the *norm* of the complex number $x + yi$, or $x - yi$; and, as seen in Section **195**, the product of conjugate complex numbers is the norm of each.

But note that also norm $(-x - yi) = (-x)^2 + y^2 = x^2 + y^2$, although $-x - yi$ is not conjugate with $x + yi$, nor is their product the norm of either; for $(-x - yi)(x + yi) = y^2 - x^2 - 2\,xyi$.

198. The positive square root of the norm of a complex number is called its *modulus:* mod $(x + yi) = +\sqrt{x^2 + y^2}$.

This modulus has extremely important properties.

The attentive student may have already discerned difficulty in applying comparisons of greater or less to complex numbers; for example, which is the greater, $3 + 4\,i$ or $2 + 5\,i$?

The *quantity* (*vide* § **229**) of a complex number is discovered to depend upon its modulus. Complex numbers with equal moduli are quantitatively equal, though not identical numbers. Any magnitude of two dimensions must exhibit this mode of equivalence without congruence. Argand's diagram would give a good illustration of this relation: the points representing (or terminating the radii which represent) complex numbers of equal moduli would all lie on a circle; points corresponding to complex numbers of less moduli would lie within the circle, and of greater moduli without.

This property of the modulus is exhibited analytically in the fact that, since mod $(x + yi) = +\sqrt{x^2 + y^2}$, which is positive regardless of the quality of x or y, if *either* x or y increases, the modulus increases, and if *either* x or y decreases, the modulus decreases. And this change is continuous, the modulus vanishing with the number, and inversely.

If two numbers are equal, their moduli are equal; for we have seen (§ **194**), if $a + bi = c + di$, $a = c$ and $b = d$. But the inverse is not true; for if $a^2 + b^2 = c^2 + d^2$, it does not follow that $a = c$ and $b = d$.

Note that if $y = 0$ in $x + yi$, that is, if the complex number be wholly protomonic, the modulus becomes $+\sqrt{x^2} = +x$, — and this whether x in the complex number be positive or negative. Thus, the mod $(+3) = +\sqrt{(+3)^2} = +3$; and mod $(-3) = +\sqrt{(-3)^2} = +3$.

For this reason, many European continental writers use the term modulus of x ("mod x") where x is a protomonic number, instead of the term "numerical value of x," employed by English writers. For example, we constantly speak of $+3$ and -3 as "numerically equal," whereas, if equal — being numbers — they could only be numerically equal; and they are not equal, for their difference, instead of being zero, is 6.

It would therefore serve accuracy and propriety to follow the practice of the writers referred to.

199. Evidently the sum of any number of complex numbers is a complex number.

Likewise the product of any number of complex numbers is a complex number.

Also the ratio of two complex numbers is a complex number. For —

$$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{c^2+d^2}=\frac{(ac+bd)-(ad-cb)i}{c^2+d^2}$$
$$=\left(\frac{ac+bd}{c^2+d^2}\right)-\left(\frac{ad-cb}{c^2+d^2}\right)i,$$

which is a complex number.

Since stirpal functions can involve only the operations of addition, multiplication, and their inverses, we have thus established the theorem: Every stirpal function of one or more complex numbers is a complex number.

200. Several other fundamental theorems concerning stirpal functions of complex numbers, and moduli of complex numbers, are deferred to the final chapter.

In regard to radical functions of complex numbers, we can consider here only the particular case of the square root: —

Assume that the square root of a complex number is a complex number, and let —

$$\sqrt{a+bi}=A+Bi,$$

where a, b, A, and B are protomonic.

Squaring each member: $a+bi=A^2-B^2+2ABi$;

therefore, by Section **194**, $a = A^2 - B^2$ (1)

and $b = 2\,AB.$ (2)

Adding the squares of (1) and (2)

$$a^2 + b^2 = (A^2 + B^2)^2. \quad (3)$$

Taking square roots of (3), and remembering that $A^2 + B^2$ is essentially positive:

$$+\sqrt{a^2 + b^2} = A^2 + B^2. \quad (4)$$

Adding (1) and (4): $+\sqrt{a^2 + b^2} + a = 2\,A^2$

therefore $$A = \pm\sqrt{\frac{+\sqrt{a^2 + b^2} + a}{2}}.$$

Subtracting (1) from (4): $+\sqrt{a^2 + b^2} - a = 2\,B^2$,

therefore $$B = \pm\sqrt{\frac{+\sqrt{a^2 + b^2} - a}{2}}$$

Since $+\sqrt{a^2 + b^2} > a$, these values of A and B are protomonic. Since $b = 2\,AB$, like signs in the values of A and B must be taken if b is positive, and unlike, if b is negative, therefore if b is positive,

$$\sqrt{a + bi} = \pm\left\{\sqrt{\frac{+\sqrt{a^2 + b^2} + a}{2}} + i\sqrt{\frac{+\sqrt{a^2 + b^2} - a}{2}}\right\} \quad \text{I.}$$

and if b is negative,

$$\sqrt{a + bi} = \pm\left\{\sqrt{\frac{+\sqrt{a^2 + b^2} + a}{2}} - i\sqrt{\frac{+\sqrt{a^2 + b^2} - a}{2}}\right\} \quad \text{II.}$$

Let the student verify by multiplication.

If the protomonic part of the complex number vanish, we have here the formula for the square root of a neomonic number.

Particularly if $a = 0$, and $b = +1$, formula I becomes,—

$$\sqrt{+i} = \pm\frac{1 + i}{\sqrt{2}}.$$

If $a = 0$, and $b = -1$, we have from formula II, —

$$\sqrt{-i} = \pm \frac{1 - i}{\sqrt{2}}.$$

By means of these results the student can readily find four 4th roots of $+1$, and of -1.

201. Since radical functions of any number involve only fractional exponents besides stirpal operations upon the number, if we show that the nth roots, when n is a primary number, of a complex number are complex numbers, we establish the theorem: All radical functions of complex numbers are complex numbers.

The investigation must be postponed to future studies; for more powerful instruments of analysis (e.g., Demoivre's theorem) are required than are at the command of the students to whom these lectures are primarily addressed. But the theorem has been established.

202. Command of the proper means of analysis (e.g., logarithmic series) would enable the student to prove that exponential functions (*vide* § **169**) of complex numbers lead to no new mode of number.

Thus, finally, it has appeared that the ultimate generalization, $(a + bi)^{x+yi}$, is still a complex number; and that therefore the Universe of Number closes, returns upon itself, is *complete.*

XIII. Measurement.

203. The *measurement* of any magnitude (concrete or abstract) is the process of finding its ratio to another magnitude of the same kind, arbitrarily chosen as a unit.

204. The measure of a magnitude is this ratio — a number.

Under the conventions of English speech, the measure of any magnitude is expressed by a phrase made up of this number and the name of the chosen unit.

205. The noun, *measure*, is commonly used in the sense of Section **204**, in the sense of *submultiple* (*vide* § **83**), and in the sense of *unit*. There may be no very good ground of choice in these terms, but consistency in the same discourse is desirable. It may be better not to say, "the greatest common measure" of two or more magnitudes, since any magnitude of the same kind would be a common measure, in another meaning of the word; for example, the yard may be the common measure of all lines, and so may the metre. It may be better to say, instead, *the greatest common submultiple.* On the other hand, *commensurable* and *incommensurable* point the same way as *common measure.*

The use of *measure* in the sense of *unit* is superfluous in the presence of the clearer term, unit, and appears to foster a confusion of concepts with commensurability; whereas it is very seldom that a unit is commensurable with the magnitude measured. Attention is merely called to this confusion in our language, and consequently in our thought. Under the necessity of some choice, I have selected *commensurable*, *submultiple*, and *unit*, and have simply avoided *measure* as the inconsistent synonym both of *submultiple* and *unit.*

206. The unit of any kind of magnitude may be any magnitude of the same kind.

Convenience, or lack of concerted choice, often establishes in common usage many units for magnitudes of the same kind.

207. Magnitudes are of the same kind when, of any two, one is necessarily greater than, equal to, or less than the other.

Magnitudes between which there is no such comparison are of different kinds; and between such there is no ratio, nor could one be added to the other.

208. The ratio of any two magnitudes is independent of any unit, or units, of measurement. Their absolute values can in no way depend upon the arbitrary standard, or standards, by which they may happen to be estimated. For example, the ratio of the time of rotation of Mars to the time of rotation of Venus is that exact numerical relation of the former to the latter, in virtue of which the former is a fraction of the latter; or greater than one, and less than another fraction of the latter, which differ as little as we please. (*Cf.* § **83**.) Evidently this ratio can nowise depend upon other comparisons of these times with any other periods of time whatsoever.

209. But the ratio of any two magnitudes equals the ratio of their respective measures in comparison with the same unit. For example, the ratio of the two periods of planetary rotation just mentioned equals the ratio of their respective ratios to any third period of time, say, the time of the earth's rotation, — the period we name a day.

210. There is such a thing as direct operation with concrete magnitudes; but it is only through their measures, that is, their ratios to some unit, that magnitudes other than Number can become subjects of genuine calculation, the proper subjects of which are numbers, and numbers alone. (*Cf.* §§ **27** and **48**.) For example, the sum of two sects is a sect, which may be found directly by placing the given sects end to end in a straight, with none but these end-points in common. The sect between the non-coincident end-points is the sum. But in calculation we add the *lengths* (i.e., the numbers which are the ratios of the sects to any unit-sect) of the sects, and obtain the *length* of their sum (i.e., the number which is the ratio of the sum-sect to the same unit).

211. As stated in Section **203**, the measurement of a magnitude consists in finding its ratio to another magnitude of the same kind, chosen as a basis of comparison. Howsoever this ratio may be found, the magnitude is *measured.*

In physical science magnitudes are commonly measured, not directly, but indirectly; that is, the direct comparison is not between the magnitude which is to be measured and a chosen unit, but between two magnitudes of a different kind which are proportional to the magnitude which is to be measured and its unit. It is highly important that this fact be recognized by all students of physical science. It also emphasizes very clearly the absurdity of omitting a sound exposition of the doctrine of proportionality from elementary instruction in mathematics.

The doctrine of proportionality is not especially difficult or recondite; but, even if it were, its thorough exposition cannot be postponed, because comprehension thereof is prerequisite for understanding ordinary measurements in the most elementary physical science, and the commonplace problems of daily life. For example, temperatures are never measured directly, but always by means of their assumed proportionality to the volumes of some body at the temperatures in question. Again, masses are usually measured by their proportionality to the corresponding weights in the same place, etc.

212. Arcs of a circle are so conveniently measured by means of their proportionality to the angles they subtend when the vertices of the angles are at the centre of the circle, that they are seldom measured directly. It must be carefully noted, also, that only angles less than a perigon are so proportional, and therefore so measurable. The indirectness of such measurement of arcs is not sufficiently emphasized in many text-books. The most faithful English translator of Euclid long ago warned teachers of the dangers lurking around this question. In the first of his introductory Dissertations, he gives good advice for leading a pupil to attain an exact and adequate concept of an angle, and especially deprecates any association of angles and arcs, averring himself at this stage "afraid to meddle with circular arches, lest we should *conjure up* a prejudice which we might want art afterwards *to lay.*"

In more than one instance — old Roger Ascham's sage counsel anent teaching Latin comes to mind — modern tyros in pedagogics would have done better to consult wise predecessors than to follow every fad of educational milliners as they vie with each other in designing latest fashions. In many of our high schools it would be difficult to find a pupil who knows exactly what an angle is; and not impossible to find some who would speak of an arc as equal to, or half of, an angle.

213. Definition. — One series of magnitudes of the same kind is proportional to another series of magnitudes of the same or of a different kind, corresponding one-to-one to the first series, when the ratio of any two of the one series equals the ratio of the corresponding two of the other series.

This is the direct meaning of the statement that one series of magnitudes is proportional to another series. Criteria sufficient to prove this relation will be discussed presently.

214. Too much prominence is commonly given to the case where each series consists of two magnitudes. Of course two magnitudes are proportional to two others, when a ratio of the one pair equals the corresponding ratio of the other pair.

215. Criteria sufficient to prove the relation of proportionality are often, and on high authority, set forth as definitions of proportionality. Of course there is no error in this; but it appears to create confusion. I believe it is the principal explanation of the not uncommon opinion that the true doctrine of proportionality is unteachable to high-school pupils.

216. Euclid's definition of equality of ratios affords the usual criterion of proportionality: Two series of magnitudes will be proportional provided that, if any equimultiples of a corresponding pair of magnitudes one in each series be taken, and any equimultiples whatsoever of any other corresponding pair be taken, then the multiple of the first magnitude in one series is greater than, equal to, or less than the multiple of the second of the same series, according as the multiple of the first taken of the other series is greater than, equal to, or less than the multiple of the second of that series.

217. That these requirements are capable of being applied as a test, may be shown by the following case: Rectangles of equal bases are proportional to their altitudes upon those bases.

Here a series of surfaces (*vide* § **229**), corresponding one-to-one to a series of lines, is declared proportional to the latter. It is so, for the ratio of any two of the surfaces equals the ratio of the corresponding two of the lines; because, if upon sects equal to the given base two rectangles be constructed whose altitudes are any multiples of the altitudes of any two of the given rectangles, the rectangles so formed are respectively the same multiples of the original rectangles. Thus equimultiples of a corresponding pair, one in each series, and any equimultiples of a second corresponding pair, have been taken. Also, if the altitude of one of these trial rectangles be greater than the altitude of the other, the first rectangle is greater than the second; and if equal, equal; and if less, less. Therefore, the ratio of any two of the series of rectangles equals the ratio of the corresponding two of the altitudes; that is to say, the rectangles are proportional to the altitudes.

Note that all this is regardless of commensurability of the altitudes or of the rectangles.

218. Euclid's criterion has been objected to because it is required that the conditions be satisfied for any, that is all, multiples; and it is impossible to try all primary numbers. This objection is not valid, though there may be cases which require a searching test in order to avoid error. For example: Consider the numbers, 4 and 3, and 5 and 4.

Multiplying the two antecedents each by 6, and the two consequents each by 9, we get 24, 27; 30, 36 — where $24 < 27$, and $30 < 36$. Making multiples in like manner with 6 and 7, we get 24, 21; 30, 28 — where $24 > 21$, and

$30 > 28$. Nevertheless, 4 and 3 are not proportional to 5 and 4. Thus we see that the criterion may be satisfied for certain multiples, and yet not satisfied; for it demands that the excess or defect be on the same side for all multiples under the stated conditions. In the example cited, if we use 10 and 13 for multipliers, we get 40, 39; 50, 52 — where $40 > 39$, but $50 < 52$.

Of course, where the question concerns the proportionality of four given numbers there is no occasion to apply any general criterion; but the relation may be tested immediately by a comparison of the ratios. Thus, if 4, 3; 5, 4 are proportional $4/3$ equals $5/4$; but $4/3$ does not equal $5/4$, because $4/3 - 5/4 = 16/12 - 15/12 = 1/12$.

219. Alternative criteria, especially adapted to test proportionality in many cases which arise in geometry and physics, are presented and their adequacy established, on page 93 of Halsted's *Synthetic Geometry* (John Wiley and Sons, New York): —

Two series of magnitudes which correspond one-to-one, are proportional (that is to say, the ratio of any two of the first series equals* the ratio of the corresponding two of the second series) provided (1) If any two of the one series are equal, so are the corresponding two of the other series; and (2) To the sum of any two of the one series corresponds the sum of the corresponding two of the other series.

For example: — The intercepts made by a system of parallel straights upon one transversal are proportional to the intercepts made upon any other transversal: for if any two intercepts on one transversal are equal, so also are

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* *Vide* Section **83** (6).

the corresponding two on another transversal; and to the sum of two on one transversal corresponds the sum of the corresponding two on the other transversal.

Again, consider arcs of the same or equal circles and their chords. Arcs are not proportional to their chords, because, although if two of the arcs are equal their chords are equal, yet to the sum of two arcs does not correspond the sum of their chords.

220. For the continuous magnitude, Space, the scientific fundamental unit is the metre, which is the sect between two marks on a metal bar preserved at Paris. The sect is to be taken when the bar is at the temperature of melting ice. This temperature has the advantage of being readily fixed; but a point so far from ordinary working temperatures requires the correction of all observations of objects not iced, and coefficients of expansion need to be accurately known for all substances employed. The original (1799) French standard metre is a platinum bar end-standard about 1 inch wide and $\frac{1}{7}$ inch thick. End-standards are objectionable because they can be observed only by contact, and attrition at the ends is inevitable. The new standard of the International Metric Commission is a line-standard of platino-iridium, about 40 inches long and 0.8 inch square, grooved on four sides so that its section is between an X and H form. This gives rigidity and a surface in the axis of the bar to bear the lines of the standard.

This standard is preserved at the International Metric Bureau at Paris, where the most refined methods of comparison are provided for, and which is supported and directed by seventeen nations.

The legal theory of the Metric System of Units is: —

(1) The standard metre, with decimal fractions and multiples thereof. (2) The litre (declared to be a cube of 0.1 metre edge), with decimal fractions and multiples. (3) The kilogram (defined as the weight in vacuum of a litre of water at 4°C.), with decimal fractions and multiples.

No standard litre exists, all liquid measures being fixed by weight.

When established in 1799 the metre was supposed to be one ten-millionth of the terrestrial quadrant through Paris. It differs from this fanciful value by about $\frac{1}{4000}$.

The merits of the metric system of units were briefly discussed in Section **31**.

221. The fundamental units for the measurement of physical magnitudes, chosen by the Units Committee of the British Association, and unquestionably the most scientific ever agreed upon, are the centimetre, gram, and second. The system is known as the C.G.S. system. For details of its application to all branches of physical science (e.g., to electricity) the student is referred to Professor Everett's *Units and Physical Constants*, Macmillan and Co.

XIV. Mathematics

222. Formal thought, consciously recognized as such, is the means of all exact knowledge; and a correct understanding of the main formal sciences, Logic and Mathematics, is the proper and only safe foundation for a scientific education.

The origin and nature of the truths of the formal sciences are not so recondite as they are often made to appear. The validity of Reason is the sole postulate. Mathematical truths are discovered as the results of rational operations upon certain elementary concepts determined by the definitions with which the science begins. The operations are not capricious, nor is their nature arbitrary. They are not empty words, but realities — not "material" realities, but all the more real. For example, numbers, as we have seen, are not concrete things; and as soon as we forget that they are the products of rational processes, we at once fall into error and confusion. Such confusion is most prominent in concepts of zero and infinity. A vague concreting of infinity is often observable, even among those who do not make a like mistake with any other number. Because Infinity as a concrete is inconceivable, the number infinity is commonly spoken of as inconceivable, and a prevalent opinion regards finite numbers as the only ones we can reason about. Charles S. Pierce, eminent as logician and mathematician (and mastery of both sciences is requisite to authority in either), says, "I long ago showed that finite collections are distinguished from infinite ones only by one circumstance and its consequences: namely, that to them *(the finite)* is applicable a peculiar and unusual mode of reasoning called by its discoverer, DeMorgan, the 'syllogism of transposed quantity.' . . . DeMorgan, as an actuary, might have argued that if an insurance company pays to its insured on an average more than they have ever paid it, including interest, it must lose money. But every modern actuary would see a fallacy in that, since the business is continually on the increase. But should war, or other cataclysm, cause the class of insured to be a finite one, the conclusion would turn out painfully correct after all. . . . If a person does not know how to reason logically, and I must say that a great many fairly good mathematicians — yea, distinguished ones — fall under this category, but simply uses a rule of thumb in blindly drawing inferences like other inferences that have turned out well, he will, of course, be continually falling into error about infinite numbers. The truth is, such people do not reason at all. But for the few who do reason, reasoning about infinite numbers is easier than about finite numbers." *

In regard to infinitesimals (the word is simply the Latin ordinal form of *infinity*), and contending opinions concerning the methods of the Infinitesimal Calculus, it may be remarked that, under the true doctrine of continuity and limits, infinitesimals are presupposed, and that there can be no reason except expediency to shun them in the differential calculus. And since they are indispensable for the integral calculus, Mr. Pierce is probably right in his view of the proper procedure of the whole discipline, when he says, in the paper quoted above, "as a mathematician, I prefer the method of infinitesimals to that of limits, as far easier and less infested with snares." At all events, any avoidance of infinitesimals as absurdities, or as offer-

* "Law of Mind," *Monist*, July, 1892.

ing obstacles to sound and lucid reasoning, is unnecessary.

223. Mathematics has often been characterized as the most conservative of all sciences. This is true in the sense of the immediate dependence of new upon old results. All the marvellous new advancements presuppose the old as indispensable steps in the ladder. It is on this account that "there is no royal road" to mathematics. This inaccessibility of special fields of mathematics, except by the regular way of logically antecedent acquirements, renders the study discouraging or hateful to weak or indolent minds. In reality similar demands are made by every science; but elsewhere they are not so imperious, so uncompromising. It is possible for one who has not mastered fundamental knowledge in the sciences of philology, history, biology, physics, chemistry, etc., to nurse the delusion of proficiency and comprehension of advanced problems; but mathematics is inviolable against such vain assaults. Instant and conscious is the curb upon her votaries of inadequate knowledge.

The modern tendency to dangerously narrow specialization within the bounds of one science is, also, more surely checked in mathematics than elsewhere. The attempt was made in mathematics as in the other sciences; but it has been restrained. Professor Felix Klein remarked at the opening of the Mathematical and Astronomical Congress at Chicago, in 1893: "When we contemplate the development of mathematics in this nineteenth century, we find something similar to what has taken place in other sciences. The famous investigators of the preceding period were all great enough to embrace all branches of mathematics. . . . With the succeeding generation, however, the tendency to specialization manifests itself. . . . Such conditions are unquestionably to be regretted. . . . I wish on the present occasion to state and to emphasize that in the last two decades a marked improvement from within has asserted itself in our science with constantly increasing success. The matter has been found simpler than was at first believed. It appears indeed that the different branches of mathematics have actually developed — not in opposite, but in parallel directions, that it is possible to combine their results into certain general conceptions. . . . A distinction between the present and the earlier period lies evidently in this: that what was formerly begun by a single master mind, we now must seek to accomplish by united efforts and co-operation."

224. Another trait of mathematics which renders it attractive to some minds and repellant to others, is its self-sufficiency, its isolation, its independence of other sciences. But it must never be forgotten that mathematics is ever at the service of other sciences; and it is for them to so formulate their problems as to make them susceptible of mathematical treatment. Indeed, in some instances, the difficulties which balk a thorough investigation of certain physical phenomena consist in the mathematical problems encountered in the solution of numerical equations, the summation of numerical series, etc., which the skill of experimenters has succeeded in deducing from the phenomena in question. Thus, on the one hand, the physicist or economist is unceasingly occupied in attempting to express the relations of the entities with which he deals, as some numerical function known to be within the reach of mathematical reduction, — for so compendious is the language of Algebra, that theoretically most quantitative and many qualitative relations are *somehow* expressible as numerical relations. On the other hand, the algebraist is constantly striving to bring more and more algebraic forms within the powers of his analysis. By their joint labors the confines of knowledge are steadily widened.

225. It may be helpful to offer a definition of Mathematics, not in the sense of final delimitation, but in order to afford a clear notion of what is meant by subjects or relations *capable of mathematical treatment*. I cannot do better than quote Professor George Chrystal in his article on "Mathematics," *Encyclopædia Britannica*, ninth edition, who makes the following definition: "Any concept * which is definitely and completely determined by means of a finite number of specifications, say by assigning a finite number of elements, is a mathematical concept. Mathematics has for its function to develop the consequences involved in the definition of a group of mathematical concepts. Interdependence and mutual logical consistency among the members of the group are postulated, otherwise the group would either have to be treated as several distinct groups, or would lie beyond the sphere of mathematics."

226. Examples of concepts completely determined by a finite number of specifications are familiar. On the other hand, *horse, tree, gold, beauty, love*, are examples of non-mathematical concepts. Of course, Number may be abstracted from these, or any other separate objects of thought or sense-perception, and Number is a mathematical concept; but the concept of a number of trees is not at all the concept of the trees. Again, the form of an irregu-

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* I have taken the liberty of changing the word "conception" to *concept* three times in this passage.

lar piece of wood cannot be determined by a finite number of specifications, and its form therefore cannot be mathematically treated (its weight of course could). But if from this irregular piece of wood a sphere be turned, its form is specified by stating that it is a sphere, and giving the length of its radius. This illustrates at once the boundaries of mathematics, and the relation of mathematics to the arts.

227. Mensuration is an important function of mathematics; but it occupies too prominent a place in some notions of the subject-matter of the science. I have already (*Cf.* Introduction, p. 13, and § **12**) referred to the mistake of assigning the origin of Number to measurement. Nor is the prevalent notion that mathematics is the "science of quantity" correct. Projective geometry in the purity of its recent development is displayed as a mathematical treatment and method well-nigh void of quantitative relations, and dealing for the most part with qualitative relations of spacial manifoldnesses.*

228. When we reach elementary concepts we always find that they cannot be defined except in cognate terms. Such elemental concepts are *quality, one, many, space, time,* and the interrelated concepts, *whole, part, more, less, equal, quantity.* All that can be done in the way of defining such concepts, is to exhibit the phenomena from which they have been abstracted, and the processes of abstraction; and then, for purposes of exact expression, make the definition in cognate terms. A good, short definition of quantity according to a standard dictionary (the *Century*) is: "The intrinsic mode by virtue of which a thing is more

* See a work on projective geometry by Dr. Halsted, just now in press.

or less than another." (*Mode* = system of relationship.) It is plain that some things exist in this mode; that is, possess quantity, which are not magnitudes, or manifoldnesses, in the mathematical sense. For example, beauty is quantitative, is *more* or *less*: but in no proper sense can it be added to itself so as to double. It is a loose figure of speech to say, the beauty of one thing is twice that of another, as is at once apparent, should we go on to say that it was eleven times that of some other.

229. Many words have been used to denote the characteristic relation of a mathematical concept to its elements. *Magnitude* and *quantity* are the familiar terms. In the preceding discourse I have employed the former term; but for reasons both of intrinsic propriety, and less ambiguity owing to irregular usage, the word *manifoldness*, which has lately come into use,* is perhaps the most fitting term; though manifoldness is also used to denote a group of correlated magnitudes differing in kind.

Quantity and *magnitude* are each used in two respectively synonymous senses. Either magnitude or quantity may be found defined for mathematical purposes as "anything which may be added to itself so as to double;" and yet the same writer may be found speaking of the *magnitude* of some such MAGNITUDE, or the *quantity* of some such QUANTITY. The ancient, and still universally current, categorical sense of *quantity* seems to me to render it the more appropriate term for the sense of the italicized words in the phrases cited; and therefore, by exclusion,

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* *Cf.* the article on "Mathematics" by Prof. Chrystal, above referred to; also the article on "Measurement" by Sir Robert Ball, Royal Astronomer for Ireland.

magnitude should be confined to the sense of the capitalized words.

Manifoldness in one sense is entirely synonymous with *magnitude* in the use I have made of the latter. But besides a single totality, *manifoldness* often means a single system of different totalities, and the difference may be in kind. Thus, a line is a manifoldness, so is a surface, so is an angle; yet the system of lines, along with the angles and surface determined, which we call a triangle, is also termed a manifoldness.

Now, a triangle, in the sense of the whole figure, is not a magnitude; its surface is a magnitude, its sides are magnitudes, and its angles are magnitudes. The word triangle often plainly means exclusively the surface of the triangle, and the abbreviation is legitimate where there is no danger of confusion (*Cf.* § **217**); but if *triangle* means the entire definite system of surface, lines, and angles, then, clearly, a triangle is not a magnitude, but a system of different magnitudes. But a triangle, in this sense of the whole figure, the system of magnitudes, three sects, three angles, and one surface, is called a manifoldness. Such manifoldnesses have been termed discrete; but this is a totally different sense of *discrete*, from its meaning in any statement that a single magnitude is discrete, e.g., Primary Number is a discrete magnitude. *Discrete* is the antithesis or antonym of *continuous*. Most magnitudes are continuous; number, time, and space are the great continuums, with which mathematics has most to do. If *manifoldness* is to be used in this double sense, it is necessary to distinguish the meanings by some adjectives; and *discrete* is not a good term for the latter sense. *Homogeneous* and *disparate* would not be abusive terms. I shall use them.

230. Number is the very web of mathematics, the manifoldness upon which are woven investigations concerning all other manifoldnesses whatsoever. All other manifoldnesses are even fundamentally determined (as will presently appear) by means of Number; but *Number determines itself.*

Geometry cannot even apparently proceed without arithmetic. Euclid makes the formal connection in his fifth book; but there is a more primary and essential connection. We have considered the error of seeking geometric definitions of number, particularly negative, neomonic, and complex number. But the tables are entirely turned when we consider that geometric or any other manifoldnesses are defined in some very fundamental properties by means of number.

231. Most text-books on stereometry set forth that all solids have three dimensions, length, breadth, and thickness. But what does this exactly mean? What is the length, breadth, and thickness of a pyramid, a rough stone, a bunch of grapes? No solids, except cubes or right parallelopipeds, clearly determine three principal directions in which length, breadth, and thickness may be discerned.

The dimensions are clearly and sharply defined only by considering the number of specifications necessary and sufficient to fully determine any element. Thus, solid space regarded as point aggregates is tri-dimensional, because, given three concurrent straights or planes, as ground of reference, three *numbers* are necessary and sufficient to determine any one point-element, distinguishing it from all others.

Note also that the space of our experience is four-dimensional if regarded as an assemblage of geodesic lines, because in that case four numbers are required to determine one element.

232. Manifoldnesses, homogeneous or disparate, are one-dimensional, two-dimensional, etc., (or one-fold, two-fold, etc.), according as in the totality or system considered, one number, or two numbers, etc., are necessary and sufficient to determine and distinguish any particular element in the homogeneous totality, or in the system.

The distinction between homogeneous and disparate manifoldnesses must not be confounded with that between continuous and discrete manifoldnesses. A homogeneous manifoldness is either continuous or discrete; a disparate manifoldness is a system of homogeneous (continuous or discrete) manifoldnesses. *Disparate* denotes a system of manifoldnesses differing in kind; that is, such as could not be compared with one another (*vide* § **207**), e.g., the surface, lines, and angles of a triangle. As already said, most homogeneous manifoldnesses are continuous. Primary Number is the conspicuous discrete magnitude with which we have to do.

According to different standpoints, the same manifoldness may be of various dimensions.

233. Examples. — A straight line regardless of position, time, temperature, probability, the totality of all spheres distinguished, not in respect of position, but solely in regard to size or quantity, are one-fold manifoldnesses. All such are homogeneous, for of course no one-fold manifoldness could be disparate.

The assemblage of points on a plane, the sphere as surface (*Cf.* latitude and longitude), are two-fold manifoldnesses.

Space as an assemblage of points is a tri-dimensional manifoldness. A triangle considered without reference to position (because it may be completely determined in various ways by assigning three elements) is a triple disparate manifoldness.

The totality of all spheres each to be completely determined is a four-fold manifoldness.

Since a plane quadrilateral is completely determined when five elements are known, it is a quintuple or five-fold disparate manifoldness.

A plane n-gon in like manner is a $(2n - 3)$-fold disparate manifoldness.

234. There are two general methods in the mathematical investigation of manifoldnesses. They are called the synthetic, or synoptic method, and the analytic method. The analytic method is mainly numerical; the synthetic deals directly with the magnitudes considered, and only unavoidable numerical relations are involved. Of course there is no sharp line of demarcation, and the two methods yield identical results.

In geometry metrical relations are in general more readily investigated by the analytic; descriptive properties by the synoptic method.

235. The synthetic method is peculiarly fitted to pure geometry, but this is not its only field. Ever since Riemann's epoch-making dissertation, *Ueber die Hypothesen welche der Geometrie zu Grunde liegen*, 1854, synoptic methods have been applicable to n-fold manifoldnesses; and the applications to Statistics and Physics are familiar.

236. In mathematics all analytic methods employ an algebra (*vide* § **20** *et seq.*); but it is the Algebra of Number which is the most highly developed and powerful instrument of such methods of research. It is to the study of this organized and compendious instrument of numerical expression that these lectures are introductory. Plainly the first step to the understanding of the algebra of number is to understand the nature and laws of number. It is hoped that these lectures have been a fairly adequate guide and stimulus to this step. After mastering what may be called the vocabulary of the language (proficiency in this matter has been assumed), the next step is to grasp the idea of algebraic *form*. In the study of Algebra this should be the main standpoint. It is only by following out the problems which arise in a systematic study of algebraic form that the modern developments of pure algebra, or its applications to geometry, can be rightly comprehended.

237. In conclusion, I may say, in reference both to this little work, and to any text-book which may engage your attention, that if a mathematical treatise is worth reading at all, it is worth re-reading, and reading backwards and forwards, and in special topics. As Professor Chrystal says in the preface to his *Text Book of Algebra*, "When you come on a hard or dreary passage, pass it over; and come back to it after you have seen its importance or found the need for it further on."

XV. Some Theorems and Problems.

238. Every primary number is a multiple (§ **83**) of *one* and of itself: if it has no other submultiple, it is called a *prime* number; if it has another submultiple, it is called *composite.*

If one primary number is a submultiple of each of two or more others, it is called a *common submultiple.*

Primary numbers (prime or composite) with no common submultiple other than unity, are said to be prime to each other.

239. Theorem. — Every composite primary number can be resolved into factors which are positive integral powers of prime numbers.

Every primary number less than a composite number either is, or is not, a submultiple of the latter: let a be the least primary number (> 1), that is a submultiple of the composite number, A.

Then $A = ax$. If x be also a multiple of a, $x = ay$, and $A = a^2 y$. Finally $A = a^m u$, where u is either 1, or prime to a, and either prime, or a multiple of some prime $> a$ and $< A$, say, b.

In like manner $u = b^n v$, where $v < u$; and $v = c^p w$, where $w < v$, and so on.

Clearly the process must end with 1; therefore

$$A = a^m b^n c^p \ . \ . \ . \ ,$$

where $a, b, c \ . \ . \ .$ are prime numbers.

It will be seen below that this resolution can be effected in only one way; also, that positive integral powers of prime numbers are prime to each other.

240. Understanding the numerical symbols as representing integers (positive or negative), and extending* the meaning of the term *multiple* to include the relation of one number to another if the ratio of the former to the latter be integral, then: —

If A is a multiple of a, any multiple of A is a multiple of a. This is obvious.

Also, if A and B have a common submultiple, m, then $Ax \pm By$ is a multiple of m.

For, say, $A = pm$ and $B = qm$;

then $Ax \pm By = xpm \pm yqm$,

and therefore, distributing the right-hand member,

$$Ax \pm By = m(xp + yq).$$

From these two theorems is deduced a means of finding the highest† common submultiple (h. c. s.) of two or more integers.

For, if $A = pB + c$, the h. c. s. of A and B is the h. c. s. of B and c. To prove this it is necessary and sufficient to show —

(1) Every submultiple of B and c is a submultiple of A and B.

(2) Every submultiple of A and B is a submultiple of B and c.

(1) As just shown, every submultiple of B and c is a submultiple of $pB + c$, that is of A; therefore, every submultiple of B and c is a submultiple of A and B.

(2) Since $A = pB + c$, $c = A - pB$. Therefore, again, every submultiple of A and B is a submultiple of $A - pB$,

---

* A violent extension (*Cf.* definition, § **83**); but custom is a tyrant, and brevity tempting. Some such term as *co-multiple* would adequately distinguish this relation, e.g., of 12 to -3.

† The term *highest* is employed in order to avoid contradictory uses of "greatest" in regard to negative numbers. (*Cf.* § **242**.)

that is of c, and consequently every submultiple of A and B is a submultiple of B and c.

Thus, to find the h. c. s. of A and B, where $A > B$, we have, by successive divisions, —

$$A = pB + c \quad \dots \quad \text{where } c < B,$$
$$B = qc + d \quad \dots \quad \text{where } d < c,$$
$$c = rd + e \quad \dots \quad \text{where } e < d,$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$k = ul + m \quad \dots \quad \text{where } m < l,$$
$$l = vm + n \quad \dots \quad \text{where } n < m, \text{ and may be } 1,$$

and, finally, $m = wn \quad \dots \quad$ where, if $n = 1$, $w = m$.

Whence, n is the h. c. s. of A and B; for the h. c. s. of A and $B =$ h. c. s. of B and $c =$ h. c. s. of c and $d = \dots =$ h. c. s. of m and n. But $m = wn$, and therefore n is the h. c. s., since n can have no submultiple higher than itself.

If $n = 1$, A and B have no common submultiple but unity, and are prime to each other.

For example, find the h. c. s. of $A = 2911$ and $B = 1763$. The calculation may be compared with the foregoing as follows: —

$$
\begin{array}{l}
B = 1763\,)\,2911 = A\,(1 = p \\
\qquad\qquad\quad 1763 \\
\qquad c = \overline{1148}\,)\,1763\,(1 = q \\
\qquad\qquad\qquad\quad 1148 \\
\qquad\qquad d = \overline{615}\,)\,1148\,(1 = r \\
\qquad\qquad\qquad\qquad\quad 615 \\
\qquad\qquad\qquad e = \overline{533}\,)\,615\,(1 = s \\
\qquad\qquad\qquad\qquad\qquad 533 \\
\qquad\qquad\qquad\quad f = \overline{82}\,)\,533\,(6 = t \\
\qquad\qquad\qquad\qquad\qquad\quad 492 \\
\qquad\qquad\qquad\qquad g = \overline{41}\,)\,82\,(2 = u \\
\qquad\qquad\qquad\qquad\qquad\qquad 82 \\
\qquad\qquad\qquad\qquad\qquad\qquad \overline{\ 0\ }
\end{array}
$$

whence, $g = 41$ is the h. c. s.

It must be discerned that the essence of this process is merely that the quotients be integral, and the moduli (*vide* § 198) of the dividends be in decreasing order, for qualitative distinctions are ignored; ± 4, for instance, being indifferently the h. c. s. of 8 and 12.

In accordance with these considerations the process may be abbreviated in various ways. If convenient, remainders may be negative, and any submultiple of a divisor evidently prime to the dividend, or submultiple of dividend prime to divisor, may be cast out. The above calculation might have been abbreviated thus: —

Since neither 3 nor 5 is a submultiple of 1763, 15 may be cast out of 615.	1763) 2911 (2 3526 − 15) − 615 41) 1763 (43 164 123 123

Every common submultiple of $A, B, C \ldots$ is a common submultiple of A and B, and therefore of m, the h. c. s. of A and B. Consequently, to find the h. c. s. of $A, B, C \ldots$, find the h. c. s. of m and C, and so on.

241. It follows from the preceding discussion, that, if a and b be prime to each other, any common submultiple of aN and b must be a submultiple of N.

Also, if a be a submultiple of bN and prime to b, it is a submultiple of N.

Also, if a be prime to $l, m, n \ldots$, it is prime to their product, lmn; and consequently if $a, b, c \ldots$, be each prime to all of $l, m, n \ldots$, the product, $abc \ldots$, is prime to the product, $lmn \ldots$.

In particular, if a be prime to b, a^n is prime to b^m. This is true, of course, when a and b are prime numbers; that is to say, positive integral powers of prime numbers are prime to each other.

Moreover, an integer can be resolved into factors which are powers of prime numbers in only one way. (*Vide* § **239**.)

For, if two resolutions be possible, let $abcd = lmnrs$.

Then $abcd$ is a multiple of l; but since l is a positive integral power of a prime number, it is prime to each of a, b, c, d, except one which is a not less power of the same prime number; and there must be such a one, or l could not be a submultiple of $abcd$; — say a is this one. Again $lmnrs$ is a multiple of a, and it follows as before that l must be a multiple of a. But if a is a multiple of l, and l of a, $a = l$. Likewise three more of $mnrs$ must respectively equal b, c, and d, and therefore the unpaired factor must be 1.

242. The lowest common multiple* of two integers equals their product divided by their highest common submultiple.

For if $A = sx$ and $B = sy$, where s is the h. c. s. of A and B, then $AB = s^2xy$. But s, x, and y are prime to each other, and therefore sxy is the l. c. m. of A and B, — and

$$sxy = \frac{AB}{s}$$

---

* In respect to primary numbers, the term *least common multiple* means exactly what it says; but in reference to both positive and negative integers a variance in the meaning of the term "least" is to be noted, such as was remarked in the foot-note of Section **240** concerning "greatest." Indifferent alternatives — one positive, the other negative — are always considered in the highest common submultiple, and the lowest common multiple of two integers.

Therefore, to find the lowest common multiple of two integers, we have the rules: —

Divide their product by their h. c. s.; or

Divide either by their h. c. s., and multiply the other by the quotient; or

Divide each by their h. c. s., and take the product of the quotients and the h. c. s.

Any one of these three rules may in a special case be the most convenient.

The l. c. m. of more than two integers is the l. c. m. of the l. c. m. of the first two and the third, and so on.

243. Plainly (symbols meaning integers) $a = xb + r$ in an infinite variety of ways; for x may be fixed arbitrarily and r found, so that $r = a - xb$. But important special cases arise if a, b, and x are positive, and r restricted: —

(1) When $r < b$.

(2) When, though r is negative, mod $r <$ mod b. (*Vide* § **198.**)

In both cases $a = xb + r$ in only one way.

(1) If xb be the greatest multiple of b, not $> a$, then $r = a - xb$, where $r < b$. Nor could there be a second resolution under the same conditions, else $xb + r$ would equal $x'b + r'$, and therefore $r - r' = (x' - x)b$, and therefore $r - r'$ would be a multiple of b, — an impossibility, since r and r', being each less than b, $r - r'$ is less than b.

(2) If xb be the least multiple of b not $< a$, then $a - xb = r$, where r is negative, but mod $r <$ mod b; and the resolution is unique as before.

In these cases r is called the least positive remainder and "least"* negative remainder of a with respect to b.

---

* *Cf.* foot-notes to Sections **240** and **242.**

Least remainder, unqualified, is to be understood in the former sense.

Obviously a is prime to b if the least remainder of a with respect to b does not vanish, and not prime if it does vanish.

244. Let the student prove, if the least remainders of x and y with respect to z be equal, $x - y$ is a multiple of z, and inversely.

245. When the ratio x / y is not integral, x / y is said to be essentially fractional, or briefly, fractional.

If $a / b = c / d$ when $a > b$ and $c > d$, prove that the fractions, reduced to form $n + r / b$, where $r < b$, must have their integral and fractional parts equal separately.

246. *Prove:* If $A / B = a / b$ and a / b is at its lowest terms (i.e., a prime to b), then $A = na$ and $B = nb$.

247. Prove that, using only positive remainders in the process of finding the h. c. s. of two positive integers, A and B, every remainder equals $\pm (Ax - By)$, where x and y are positive integers, and the upper sign goes with the 1st, 3d, etc., and the lower with the 2d, 4th, etc., remainders.

Also, if a and b be prime to each other, positive integers can always be found such that $xa - yb = \pm 1$.

It is obvious that these numbers, when determined, will be prime to each other, for by Section **240**, 1 is a multiple of every common submultiple of x and y.

248. *Prove:*

(1) If x prime to y, $(x + y)^n$ and $(x - y)^n$ have h. c. s. not $> 2^n$.

(2) If x prime to y, $x^n + y^n$ and $x^n - y^n$ are prime, or have h. c. s. $= 2$.

(3) If x prime to y, $x + y$ and $x^2 + y^2 - xy$ are prime, or have h. c. s. $= 2$ or 3.

(4) The difference of the squares of two odd integers is a multiple of 8.

(5) The difference of the squares of two consecutive integers equals their sum.

(6) The product of three consecutive even integers is a multiple of 48.

(7) The sum of the squares of three consecutive odd numbers and 1 is a multiple of 12, but never of 24.

(8) The product of the cubes of three consecutive integers is a multiple of their sum.

249. At several points in preceding chapters, it has been taken for granted that the operation of evolution upon many integers results in essentially surd or incommensurable number; that is to say, that no fraction can possibly be the required root — although fractions approximating the surd as nearly as desired can be obtained. Fractional number is still discrete, fractions are continuous through surds. (*Vide* §§ **94**, **81–82**.)

To demonstrate these propositions, it is enough merely to consider that no power of an essentially fractional number can be an integer. For, if x / y is a fraction in its lowest terms, x is prime to y, and therefore, by Section **241**, any power of x is prime to any power of y, and consequently any power of x / y is still essentially fractional.

For example: Obviously no integer is the square root of 7, but some number greater than 2 and less than 3. But this number is no fraction, for, as just shown, no power whatsoever of any essentially fractional number can be an integer. Thus, it is proved that the familiar process of approximate calculation of roots of such integers is absolutely interminable. (Moreover, the endless decimal fraction obtainable can never form a repeating period of figures — (*vide* § **284**).

In this way it is plain that no integers except the series, 1, 4, 9, 16 . . ., (the squares of 1, 2, 3, 4 . . ., and called "square numbers") can have any but incommensurable square roots; that the cube roots of all integers but 1, 8, 27, 64 . . . (1^3, 2^3, 3^3, 4^3 . . .) are incommensurable, and so on.

250. For proof of the proposition: The number of prime integers is infinite (see *Euclid*, IX, 20).

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251. Attentive perusal of the following sections will bring out a general distinction (correct apprehension of which is highly important) between the applications of a confusingly similar terminology to individual numbers and to analytical functions of such numbers, — the distinction between algebraic *form*, and particular numerical values.

For example, note the distinction between "exactly divisible" applied to algebraic forms, and *submultiple* applied to numbers. It is not even true that the highest common submultiple of two numbers which are obtained from the substitution of particular numbers for the numerical symbols in two analytical functions, is the same number that would be obtained by substituting the same values in the *highest common factor* of the two algebraic forms; nor would it be possible to make a definition of the algebraical highest common factor, so that this should be true.

The investigations immediately following apply only to integral functions.

252. If A and D be integral functions of x, and $\frac{A}{D} = Q$,

Q is a stirpal but not necessarily an integral function of x. (*Vide* § **169**.)

When Q is an integral function of the variables, A is said to be *exactly* divisible* by D.

When $\phi(x)$ cannot be transformed into an integral function, it is said to be essentially fractional, or fractional.

An essentially integral function cannot be identically (*vide* § **40**) equal to an essentially fractional function.

In $\frac{A}{D} = Q$, if all the functions are integral, the degree of Q is the degree of A minus the degree of D.

If the degree of A be less than the degree of D, Q is essentially fractional.

253. If $A = PD + R$ (all integral functions) R is exactly divisible by D or not, according as A is exactly divisible by D or not.

For, since
$$A = PD + R$$
$$\frac{A}{D} = \frac{PD + R}{D} = P + \frac{R}{D}$$
or,
$$\frac{A}{D} - P = \frac{R}{D};$$
therefore, as $\frac{A}{D}$ is integral or not, $\frac{R}{D}$ is integral or not.

254. Fundamental theorem in algebraic division: —
$$\frac{A_m}{D_n} = P_{m-n} + \frac{R}{D_n}, \text{ where } m > n,$$
and where R vanishes or is an integral function of degree $< n$.

---

* There are not the same objections to this phrase, as against terming one number exact and another inexact. *Cf.* Sections **1** and **80**.

(The subscripts represent the respective degrees of the functions.)

Arranging A_m and D_n according to descending powers of the variable, we would get by dividing the first term of A_m by the first term of D_n,

$$A_m = p\,x^{m-n}\,D_n + R_{m-1\ \text{(at utmost)}}.$$

Dividing by D_n gives

$$\frac{A_m}{D_n} = P_{m-n} + \frac{R_{r<n}}{D_n}.$$

Moreover, this result can occur in only one way; for, if $\frac{A}{D} = P + \frac{R}{D} = P' + \frac{R'}{D}$, where the functions satisfy the foregoing conditions, then would $P - P' = \frac{R'}{D} - \frac{R}{D}$ (by subtracting $P' + \frac{R}{D}$ from each member); and therefore $P - P' = \frac{R' - R}{D}$, which is impossible; since $P - P'$ is an integral function, and $\frac{R' - R}{D}$ cannot be integral, since the degrees of R' and R are less than the degree of D.

255. If $\frac{A}{D} = P + \frac{R}{D}$, the degrees and character of the functions being as stated in the preceding section, P is called the integral quotient and R the remainder (*par excellence*).

Plainly, the necessary and sufficient condition for "exact divisibility" is that the remainder vanish.

256. Example of the "long rule" for division of integral functions: —

Divide $\frac{1}{4}x^3 + \frac{1}{72}xy^2 + \frac{1}{12}y^3$ by $\frac{1}{2}x + \frac{1}{3}y$.

The work may conveniently be arranged thus: —

$$\begin{array}{l|l}
\frac{1}{4}x^3 + \frac{1}{72}xy^2 + \frac{1}{12}y^3 & \frac{1}{2}x + \frac{1}{3}y \\ \cline{2-2}
\frac{1}{4}x^3 + \frac{1}{6}x^2y & \frac{1}{2}x^2 - \frac{1}{3}xy + \frac{1}{4}y^2 \\ \cline{1-1}
-\frac{1}{6}x^2y + \frac{1}{72}xy^2 + \frac{1}{12}y^3 & \\
-\frac{1}{6}x^2y - \frac{1}{9}xy^2 & \\ \cline{1-1}
\qquad \frac{1}{8}xy^2 + \frac{1}{12}y^3 & \\
\qquad \frac{1}{8}xy^2 + \frac{1}{12}y^3 & \\ \cline{1-1}
\qquad 0 + 0 \qquad = R;
\end{array}$$

therefore the latter function is an exact divisor of the former.

257. The special case of the division of the general integral function of the nth degree by a binomial divisor of the 1st degree, of form $x - a$, is of extreme importance.

If the student will closely examine his results in the operation $\{ax^n + bx^{n-1} + cx^{n-2} + \ldots\ lx + k\} \div (x - a)$, he will discover the following general laws: —

The degrees of the terms of the integral quotient regularly descend.

The first coefficient of the integral quotient is the first coefficient of the dividend.

Each subsequent coefficient is the next preceding multiplied by a, $+$ the corresponding (in *order*, not *degree*, of term) coefficient in the dividend.

The remainder, if it does not vanish, may be obtained precisely as if it were a subsequent coefficient.

Care must be taken to supply by zeros any lacking terms in a particular case.

Example. — Divide $3x^4 + 5x^2 - 9x + 11$ by $x - 2$.

$$\begin{array}{rrrrrl}
+3 & +0 & +5 & -9 & +11 & \ldots \text{ (Coefficients of dividend).} \\
 & +6 & +12 & +34 & +50 & \ldots \text{ (Each preceding number in third line multiplied by 2).} \\ \hline
+3 & +6 & +17 & +25 & +61 &
\end{array}$$

Therefore integral quotient $= 3x^3 + 6x^2 + 17x + 25$,
and remainder $= +61$.

258. The general process which displayed the foregoing theorem proves * also the following: —

Remainder Theorem. — If any integral function of x be divided by $x - a$, the remainder is the same function of a as the dividend is of x. That is to say, the remainder may be obtained by substituting a for x in the dividend.

Thus, in the example above,

$$61 = 3(2)^4 + 5(2)^2 - 9(2) + 11.$$

If the divisor were $x + 2$, we need only consider $x + 2 = x - (-2)$, where a is -2.

For instance, $\{3x^4 + 5x^2 - 9x + 11\} \div (x + 2)$

gives

$$\begin{array}{rrrrr} +3 & +0 & +5 & -9 & +11 \\ & -6 & +12 & -34 & +86 \\ \hline +3 & -6 & +17 & -43 & +97 \end{array}$$

Therefore integral quotient $= 3x^3 - 6x^2 + 17x - 43$,
and remainder $= +97$.

And, in accordance with the *remainder theorem*,

$$97 = 3(-2)^4 + 5(-2)^2 - 9(-2) + 11.$$

259. The remainder theorem is clearly proved in the process of dividing the general function of x of the nth degree by $x - a$; but on account of its fundamental importance in the theory of equations, I transcribe an independent proof: —

Let $\phi_n(x)$ be an integral function of x of the nth degree; then

---

* Proved independently in Section **259**.

$\frac{\phi_n(x)}{x-a} = Q_{n-1} + \frac{R}{x-a}$ where R does not involve x;

therefore, $\phi_n(x) = Q_{n-1}(x-a) + R.$

But this equation, being an identity, holds when $x = a$, when, $\phi_n(a) = 0 + R,$
which, remembering the meaning of $\phi_n\ (a)$, is merely an algebraic statement of the "remainder theorem."

The full meaning of this statement must not be missed; for it at once declares the remainder when $\phi(x) \div (x-a)$, and the value of $\phi(x)$ when $x = a$: — the statement is $\phi\ (a) = R$.

Thus in the preceding examples

$3x^4 + 5x^2 - 9x + 11 = 61$ when $x = 2$, and $= 97$ when $x = -2$.

This method of calculating the value of an integral function of x for a particular value of the variable generally saves work in comparison with direct substitution.

260. *Prove:* If an integral function of x, $\phi(x)$, be divided by $ax + b$,

$$R = \phi\left(-\frac{b}{a}\right).$$

261. Note that if $\phi(x)$ vanishes for any value of x, say r, then upon division by $x - r$, $R = 0$, and inversely.

262. If $a_1, a_2, a_3 \ldots a_r$ be r different values of x, for which an integral function of x of the nth degree vanishes where $n > r$, then

$$\phi(x) = (x-a_1)(x-a_2) \ldots (x-a_r) f_{n-r}(x),$$

where $f_{n-r}(x)$ is an integral function of x of the $(n-r)$th degree. And when $n = r$,

$$\phi(x) = (x-a_1)(x-a_2) \ldots (x-a_n) f_0(x),$$

where $f_0(x)$ must be a constant. (But see § **268**.)

263. An integral function of any number of variables is called *homogeneous* when the degree of every term is the same; e.g., $ax + by$, or $ax^2 + bxy + y^2$.

264. *Prove:* If each variable in a homogeneous function of the nth degree be multiplied by m, the result is the same as if the function were multiplied by m^n.

Also: The product of two homogeneous functions of the mth and nth degrees respectively, is a homogeneous function of the $(m + n)$th degree.

Let this last theorem always be applied to test the accuracy of distribution of a product of homogeneous functions.

265. An integral function is called symmetrical with respect to its variables when their interchange leaves the function unaltered. Several approximations to symmetry have received special names; e.g., if a function be not altered except in sign by interchange of variables, it is called *alternating*. Functions are often both homogeneous and symmetrical.

266. From the definition, it follows that the sum, difference, product, or quotient of two symmetrical functions is a symmetrical function, — a useful rule in testing and abbreviating algebraic work.

Since symmetry concerns only coefficients, general forms are easily written down.

Write down the general integral symmetrical function of x, y, z of third degree.

267. Since the coefficients are independent of the variables, if two integral functions are equal as an identity (*vide* § **40**), and the coefficients of one are determined by any means, then these coefficients are determined once for all.

This theorem has been called (not very happily) the Theorem of Undetermined Coefficients.

It is most useful even in its elementary applications to integral functions, and becomes an indispensable instrument in dealing with infinite series.*

For example: Required the product

$$(x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz);$$

we can write down by symmetry

$$(x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz) = A(x^3 + y^3 + z^3) + B(x^2y + x^2z + xy^2 + xz^2 + y^2z + z^2y) + Cxyz.$$

Since this identity must hold for all values of x, y, z, taking

$$x = 1, \quad y = 0, \quad z = 0, \quad \text{gives} \quad 1 = A.$$

Putting $x = 1$, $y = 1$, and $z = 0$, and using the discovered permanent value of A, we have

---

* Since the whole matter of infinite series is postponed to subsequent studies, this subject cannot be entered upon further than to caution the student that in such an algebraic statement as $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \ldots$ it is never to be understood that $\frac{1}{1-x}$ equals, or even approximates, the infinite series unless the series be convergent; i.e., unless the sum continually approximates a definite limit. Evidently if $x > 1$, it would be absurd to take the above statement into consideration for a moment. In fine, such statements are understood as plainly concerning only such values of the variable as make the series convergent. Compare various obvious ellipses common in all expression of thought.

Let this be the student's reply to the cavilling he may sometimes hear upon this matter.

Of course if the *remainder* is added at any point, the expression is an identity, always true; e.g., $\frac{1}{1-x} = 1 + x + x^2 + \frac{x^3}{1-x}$; thus, if $x = 10$, we have $-\frac{1}{9} = 1 + 10 + 100 + 1000 +$ more and more untrue, the more numerous the terms; but if the remainder be added at any stage, we have a true equation:

$$-\frac{1}{9} = 1 + 10 + 100 + \frac{1000}{-9} = \frac{9 + 90 + 900}{9} - \frac{1000}{9} = -\frac{1}{9}.$$

$$1(1+1)(1+1-1) = 1(1+1) + B(1+1);$$

or $$2 = 2 + 2B;$$

therefore $$B = 0.$$

Using these determined values of A and B and $x = 1$, $y = 1$, $z = 1$, we get,

$$(1+1+1)0 = 1(1+1+1) + C;$$

therefore $$C = -3.$$

Therefore the required product is $x^3 + y^3 + z^3 - 3xyz$.

268. Returning now to Section **262**, it is plain from Section **267** that $f_o(x)$ must equal the coefficient of x^n in $\phi(x)$.

The "if" in Section **262** must be carefully noted. It has not been shown that n integral, 1st degree functions can be found, such that

$$\phi_n(x) = k(x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n).$$

This question is also deferred to subsequent studies in Theory of Equations, when it will be proved that every equation has a root, and that every equation of the nth degree has n roots (all of which need not be different).

By a *root* of the equation $\phi(x) = 0$ is meant a value of the variable which causes the function to vanish; that is, *satisfies* the equation $\phi(x) = 0$. We have seen (§ **261**), that when an integral function of x is exactly divisible by $x - a$, a is a root of the equation, and inversely.

The general formal proof that "every equation has a root" must be postponed; yet we might almost assume the fact as implicit in the Principle of Continuity (§ **103**). Assuming this, we can prove that every integral equation of the nth degree has n roots, and no more.

Let a be one root; then,

$$\phi(x) = (x - a) f_{n-1}(x);$$

but $f_{n-1}(x)$ must have a root, and so on for n roots (some of which might be repeated), and a constant factor, $f_o(x)$ (*vide* § **262**). Moreover, $\phi_n(x)$ cannot have more than n different roots, because if any integral function of nth degree vanish, for more than n values of the variable, it must vanish identically; that is, for all values of x (i.e., every coefficient in form $ax^n + bx^{n-1} + cx^{n-2} + \ldots + dx + k$ must be zero). For, let

$$\phi_n(x) = a(x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n) \ldots (1).$$

Now, if possible, let r be another value of the variable for which the function vanishes. Since (1) holds for all values of x, then

$$\phi(r) = a(r - r_1)(r - r_2)(r - r_3) \ldots (r - r_n) = 0;$$

and since each "r" by hypothesis is different, a must be zero.

But a is the coefficient of the x^n term in $\phi_n(x)$. In this way, step by step, each coefficient in $\phi_n(x)$ is shown to vanish if more than n values of the variable satisfy the equation $\phi_n(x) = 0$.

For example, $x^2 - (x+1)(x-1) - 1$ is of the 2d degree, yet plainly it vanishes for 0, 1, 2, — and therefore for all values of x.

269. The preceding section affords an independent proof of the theorem of undetermined coefficients, which may be re-stated as follows: —

Any function of x is transformable into an integral function in only one way. For, if possible, suppose the two following different integral functions, derived from the same function, as identities, and therefore equal for all values of x:

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots + jx + k = a_1x^n + b_1x^{n-1} + c_1x^{n-2} + \ldots + j_1x + k_1.$$

No generality is lost in regarding them as of the same degree; for, if not, it would simply mean that the coefficients concerned were zero. Subtracting the right-hand member from each, we get

$$(a - a_1)\,x^n + (b - b_1)\,x^{n-1} + (c - c_1)\,x^{n-2} + \ldots (j - j_1)\,x + k - k_1 = 0$$

for more than n values of x.

Therefore, $a - a_1 = 0,\ b - b_1 = 0, \ldots k - k_1 = 0$; that is to say, $a = a_1,\ b = b_1, \ldots k = k_1$.

270. Professor Chrystal remarks at this point in his *Text Book of Algebra*, "the danger with the theory we have just been expounding is not so much that the student may refuse his assent to the demonstration given, as that he may fail to apprehend fully the scope and generality of the conclusions." Their utility cannot fail to be more and more highly appreciated by the attentive student.

271. (1) Determine the value of k such that $2x^3 - 8x^2 + 7x + k$ shall be exactly divisible by $x + 2$. By Section **259**, the remainder to division by $x - (-2)$ is $2(-2)^3 - 8(-2)^2 + 7(-2) + k = -62 + k$.

If the function is to be exactly divisible by $x + 2$, this remainder must vanish, or $-62 + k$ must be zero; i.e., $k = 62$.

(2) In like manner the question of exact divisibility may be readily tested:

When $\dfrac{x^n - y^n}{x - y}$, $R = y^n - y^n = 0$; the division is always exact.

When $\frac{x^n - y^n}{x + y}$, $R = (-y)^n - y^n = 0$, if n be even, $= -2y^n$ if n be odd.

When $\frac{x^n + y^n}{x - y}$, $R = y^n + y^n = 2y^n$; the division is never exact.

When $\frac{x^n + y^n}{x + y}$, $R = (-y)^n + y^n = 0$, if n be odd, $= 2y^n$ if n be even.

(3) If $A \div D$ gives remainder R, and $B \div D$ remainder R', show that $AB \div D$ and $RR' \div D$ give identical remainders.

(4) Observe that, in the proposition that an equation of the nth degree has n roots and no more, we prove that any finite number has n nth roots and no more, — all of which need not be different.

To find these roots of any number, a requires the solution of the equation $x^n = a$, or $x^n - a = 0$; that is to say, the factorization of $x^n - a$ in the form,

$$(x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n).$$

(5) We are also enabled to make an integral equation of given roots. Thus, to form an equation whose roots are 0, $+1$, $-\sqrt{2}$, -1, we have simply to write,

$$Cx(x - 1)(x + \sqrt{2})(x + 1) = 0,$$

where C is any constant we please; e.g., this equation, taking $C = 1$, is

$$x^4 + \sqrt{2}\,x^3 - x^2 - \sqrt{2}\,x = 0;$$

or, taking $C = \sqrt{2}$,

$$\sqrt{2}\,x^4 + 2x^3 - \sqrt{2}\,x^2 - 2x = 0.$$

272. Having thoroughly explained the meaning of "exact" divisibility as applied to the division of one integral function by another, the sense in which one function is termed the highest common factor of two others is apparent: —

The integral function of x of highest degree which "exactly divides" each of two or more integral functions of x, is their highest common factor (h. c. f.). (But see § **251**.)

If the given functions are easily resolvable into factors which are integral functions of the first degree, the h. c. f. is readily taken by inspection; since it is simply the product of such of these first degree factors as are common, each raised to the lowest power in which it occurs in either of the given functions.

Otherwise we may proceed very much as in Section **240**, since if $A = BQ + R$, the h. c. f. of A and B is the h. c. f. of B and R: proved by considering Section **253**.

Consequently, to find the h. c. f. of two integral functions of x, A and B, where the degree of B is less than that of A, we may

divide A by B so that $A = BQ_1 + R_1$
and divide B by R_1 so that $B = R_1Q_2 + R_2$,
and divide R_1 by R_2 so that $R_1 = R_2Q_3 + R_3$, etc., until
R_{n-1}/R_n gives $R_{n-1} = R_n Q_{n+1} + R$,

where R vanishes, or is of zero degree, that is, a constant. In the latter case, there is no h. c. f.; in the former R_n is the h. c. f. For by Section **253**, A and B, B and R_1, R_1 and R_2, . . . R_{n-1} and R_n, are of descending degree, and all have the same h. c. f., and no factor of higher degree than R_n can exactly divide R_n. In case R is a constant, R_{n-1} and R_n have no common exact divisor other than R; that is to say, there is no common divisor in the sense intended, although any constant will "exactly divide" any integral function in the sense of giving an integral quotient; i.e., remainder zero. (*Vide* §§ **255**, **256**.)

It follows from the nature of this process of finding the h. c. f. that at any stage either divisor or dividend may be multiplied, or divided by any integral function of the variables (of course including any constant), provided it is certain that the factor so introduced or removed has no factor in common with the other functions. Any function which is obviously a common factor of both dividend and divisor at any stage may be removed from each, provided we multiply the h. c. f. afterwards resulting by the removed common factor. In dealing with factors which are constants, regard "factor" in the sense of *common submultiple of the coefficients*. Finally, it must be observed that the recurring operations are, on account of such modifications as have been ascribed, not divisions in the ordinary sense; for the "division" may, if convenient, be arrested at any stage (while the remainder is yet of higher degree than the divisor), to remove common, or introduce independent, factors.

273. (1) Find h. c. f. of $9x^6 - 30x^4 + 45x^2 + 24x$ and $15x^5 - 30x^4 - 90x^3 + 60x^2 + 195x + 90$. (Problem worked out on page 168.)

Of the originally removed factors, $3x$ and 15, 3 is common; therefore, $x^3 - 3x^2 + 3x + 1$ must be multiplied by 3 to obtain the h. c. f., $3x^3 + 9x^2 + 9x + 3$.

	$3x)\ 9x^6-30x^4+\qquad 45x^2+24x$	$15)\ 15x^5-30x^4-90x^3+60x^2+195x+90$	
3	$3x^5-\qquad 10x^3\qquad +15x+8$	$x^5-2x^4-6x^3+4x^2+13x+6$	
	$3x^5-6x^4-18x^3+12x^2+39x+18$		
	$2)\ 6x^4+8x^3-12x^2-24x-10$	3	
	$3x^4+4x^3-6x^2-12x-5$	$3x^5-6x^4-18x^3+12x^2+39x+18$	x
	* 10	$3x^5+4x^4-6x^3-12x^2-5x$	
-3	$30x^4+40x^3-60x^2-120x-50$	$-10x^4-12x^3+24x^2+44x+18$	$-10x+18$
	$30x^4+36x^3-72x^2-132x-54$		
	$4)\ 4x^3+12x^2+12x+4$		
	x^3+3x^2+3x+1	$-10x^4-30x^3-30x^2-10x$	
		$18x^3+54x^2+54x+18$	
		$18x^3+54x^2+54x+18$	
		0	

* We multiply by 10 in this column and reverse the division, instead of multiplying by 3 in the other column and continuing with the same divisor, — merely in illustration of the freedom of operation in this process. Quotients are set down in the side columns nearest to their *dividends*.

(2) What is the necessary relation among their coefficients in order that $ax^2 + bx + c$ and $cx^2 + bx + a$ may have an exact common divisor of the first degree?

$$
\begin{array}{l|l|l|l}
 & ax^2 + bx + c & cx^2+bx+a & c/a \\
 & & cx^2+\frac{bc}{a}x+\frac{c^2}{a} & \\
\cline{3-3}
ax+\left(b-\frac{a(a+c)}{b}\right) & & \left(b-\frac{bc}{a}\right)x+a-\frac{c^2}{a} & \\
 & ax^2+\frac{a(a+c)}{b}x & \text{(dividing by coef. of } x \text{ gives)} & \\
\cline{2-2}
 & \left\{b-\frac{a(a+c)}{b}\right\}x+c & x+\frac{a+c}{b} & \\
\cline{3-3}
 & \left\{b-\frac{a(a+c)}{b}\right\}x+a+c-\frac{a(a+c)^2}{b^2} & & \\
\cline{2-3}
 & \multicolumn{2}{r}{c-a-c+\frac{a(a+c)^2}{b^2}=R.} & \\
\end{array}
$$

Now, if the functions have an exact common divisor of the first degree, R must vanish; therefore the condition is:—

$$c - a - c + \frac{a(a+c)^2}{b^2} = 0.$$

Whence $\qquad -ab^2 + a(a+c)^2 = 0;$

or, dividing by a, $\qquad (a+c)^2 = b^2;$

or $\qquad a + c = \pm b.$

274. From Sections **253** and **272** it is plain that the h. c. f. of three integral functions is the h. c. f. of the h. c. f. of two and the third, and so on.

275. Integral functions which have no common exact divisor are said to be algebraically prime. Many conditions of algebraic primeness, more or less analogous to those established concerning absolute numerical primeness, might be investigated.

276. The precise meaning of *algebraic lowest common multiple* of two integral functions will now easily be understood as the integral function of lowest degree exactly divisible by both.

Let $A = HQ$ and $B = HQ'$, where the symbols represent integral functions, and H the h. c. f. of A and B. Let M be any common multiple of A and B; then —

$M = AE$, where E is an integral function of x.

Therefore $$M = HQE.$$

But M is an algebraic multiple of $B = HQ'$;

therefore $\dfrac{M}{HQ'} = \dfrac{HQE}{HQ'} = \dfrac{QE}{Q'}$, where $\dfrac{QE}{Q'}$ must be an integral function. But since Q and Q' are by hypothesis algebraically prime, $E/Q' = X$ or $E = Q'X$, where X is integral. Consequently

$$M = HQE = HQQ'X.$$

But this last algebraic statement (translated) declares that any common multiple of A and B is the product of H, Q, and Q', as defined, and some other integral function, X. Hence, M is of the lowest possible degree when X is of zero degree; that is, a constant. And since constants are not altogether ignored in the desired result, M is the "lowest common multiple" when $X = 1$; that is to say,— since $HQQ' = \dfrac{AB}{H}$, the l. c. m. of two integral functions is the quotient of their product divided by their h. c. f.

Alternative rules are similar to those for single numbers (*vide* § **242**). The algebraic l. c. m. has neither the practical nor the theoretical importance of the algebraic h. c. f.

277. The fundamental theorem in the expression of numbers, in a notation such as our common system, is the following: —

Any primary number may be expressed finitely, and in only one way in the form —

$$c_o + c_1(r_1) + c_2(r_1r_2) + c_3(r_1r_2r_3) + \ldots c_n(r_1r_2r_3 \ldots r_n),$$

where $r_1, r_2, r_3, \ldots r_n$ is a series of primary numbers, unrestricted except that there are as many as may be required, and $c_0 < r_1$, $c_1 < r_2$, $c_2 < r_3$, etc.

For if I be any primary number, dividing I by r_1 gives

(1) $$I = c_0 + Q_1r_1 \text{ where } c_0 < r_1;$$

and dividing Q_1 by r_2 gives

(2) $$Q_1 = c_1 + Q_2r_2 \text{ where } c_1 < r_2;$$

and dividing Q_2 by r_3 gives

(3) $$Q_2 = c_2 + Q_3r_3 \text{ where } c_2 < r_3;$$

and so on until $Q_n < r_{n+1}$ is reached.

(1) and (2) give

$$I = c_0 + (c_1 + Q_2r_2)r_1 = c_0 + c_1r_1 + Q_2r_1r_2;$$

and substituting for Q_2 from (3)

$$I = c_0 + c_1r_1 + c_2(r_1r_2) + Q_3r_1r_2r_3;$$

and so on until $Q_{n-1} = c_{n-1} + Q_nr_n$, where, writing c_n for Q_n, we have

$$I = c_0 + c_1(r_1) + c_2(r_1r_2) + c_3(r_1r_2r_3) + \ldots c_n (r_1r_2r_3 \ldots r_n).$$

Moreover, this expression is unique for the same series of r's, because, if not, let

$$c_0 + c_1r_1 + c_2(r_1r_2) + \ldots = c_0' + c_1'r_1 + c_2'(r_1r_2) + \ldots$$

dividing each member by r_1, gives

$$\frac{c_0}{r_1} + c_1 + c_2 r_2 + \ldots = \frac{c_0'}{r_1} + c_1' + c_2' r_2 + \ldots$$

But since $c_0 < r_1$, and $c_0' < r_1$, by Section **245** the essentially integral and essentially fractional parts of these numbers must be equal separately; that is to say, —

$$\frac{c_0}{r_1} = \frac{c_0'}{r_1}, \text{ or } c_0 = c_0', \text{ and so forth.}$$

EXAMPLE. — Express the number represented in our notation by 200 on the scale, 7, 3, 5, 2, etc.

$$\begin{array}{r|l} 7 & 200 \\ \hline 3 & 28 \ \ldots\ 4 \\ \hline 5 & 9 \ \ldots\ 1 \\ \hline & 1 \ \ldots\ 4 \end{array}$$

since the quotient 1 is less than $r_4 = 2$, the process terminates, and

$$200 = 4 + 1 \times 7 + 4\,(7 \times 3) + 1\,(7 \times 3 \times 5).$$

Express 100 in the scale 3, 4, 5, 6, 7, etc.

278. A corresponding theorem for the expression of numbers essentially fractional (§ **245**) where numerator < denominator (" proper " fractions), may be proved; that is to say: —

$$\frac{N}{D} = \frac{d_1}{r_1} + \frac{d_2}{r_1 r_2} + \frac{d_3}{r_1 r_2 r_3} + \ldots \frac{d_n}{r_1 r_2 r_3 \ldots r_n} + \frac{f}{D'},$$

where $$D' = (r_1 r_2 r_3 \ldots r_n)\, D,$$

where $d_1 < r_1$, $d_2 < r_2$, etc., and where f may vanish. The general proof, and demonstration that $f = 0$, when $r_1 r_2 r_3 \ldots r_n$ is a multiple of D, is left as an exercise for the student.

EXAMPLE.—In this way express 7/10 in scale 5, 7, 9, 11, 13, 15, 17, . . .

```
           7
           5
       10)35
           3 . . . 5
                   7
               10)35
                   3 . . . 5
                           9
                       10)45
                           4 . . . 5
                                  11
                              10)55
                                   5 . . . 5
```

evidently this cannot terminate, because $r_1, r_2, r_3 \ldots r_n$ can never be a multiple of 10.

Therefore, $7/10 = \frac{3}{5} + \frac{3}{5 \cdot 7} + \frac{4}{5 \cdot 7 \cdot 9} + \frac{5}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 10}$.

279. Since the scale of r's may be arbitrary, it may be chosen so that all the d's shall be one. Decompose 7/10 as a sum of fractions with unit numerators.

280. When the scale of notation is constant; that is, when all the r's are equal is the important special case of the foregoing Theorem.

In this case

$$I = c_0 r^0 + c_1 r + c_2 r^2 + c_3 r^3 + \ldots c_n r^n;$$

and $$\frac{N}{D} = d_1 r^{-1} + d_2 r^{-2} + d_3 r^{-3} + \ldots d_n r^{-n} + \frac{f}{r^n D};$$

where f may be zero.

Here, of course, is recognized our system of notation, where $r = 10$.

It is our custom to omit the r's, whose powers are understood from the order of the c's and d's, the proper place being displayed by never failing to express the zero when any c or d has this value. Also we omit the signs of addition, and write the integral series from right to left, so that it may be regularly continued by the fractional series, a mere point amply serving to separate the two. In this way the powers of the radix or base decrease by ones from left to right, thus:

$c_n r^n + c_{n-1} r^{n-1} + \ldots c_2 r^2 + c_1 r^1 + c_0 r^0 + d_1 r^{-1} + d_2 r^{-2} + d_3 r^{-3}$, etc.; or, omitting $+$'s and r's and pointing off the d's: $c_n c_{n-1} \ldots c_2 c_1 c_0 \cdot d_1 d_2 d_3$.

281. From the condition that the c's and d's must all be less than r, it is obvious that in any such notation $r - 1$ figures are required to uniquely designate the possible values of c's and d's.

It is also plain that all the rules of the decimal algorithm apply to any other base, say 12, except that the "carriages" would go by 12's instead of 10's. Of course for radix 12, two new digit figures would be required; and for radix 2, symbols for 1 and 0 only could be used. Thus *ten* on the binary scale would be 1010; that is,

$$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0.$$

282. Example. — Express 102305 (radix ten) on base twelve

$$\begin{array}{r|l}
12 & 102305 \\
12 & 8525 \;\ldots\; 5 \\
12 & 710 \;\ldots\; 5 \\
12 & 59 \;\ldots\; 2 \\
 & 4 \;\ldots\; b
\end{array}$$

(using a and b as digits for *ten* and *eleven*).

Therefore 102305 ($r =$ ten) is $4\,b\,255$ ($r =$ twelve).

Inversely, to express 4 *b* 255 (radix twelve) on decimal base.

Consider the expression means (using our common notation for calculation),

$$4\,(12)^4 + 4\,(12)^3 + 2\,(12)^2 + 5\,(12) + 5\,;$$

or, performing the indicated operations,

$$\begin{array}{r@{\;=\;}l} 5 & 5 \\ 60 & 5\,(12) \\ 288 & 2\,(12)^2 \\ 19008 & 11\,(12)^3 \\ \underline{82944} & 4\,(12)^4 \\ 102305 & \end{array}$$

If the student will refer to Sections **257–259**, he will notice that the remainder theorem yields an easier way for this calculation. (1) The problem is merely the evaluation of the given function of twelve. We may therefore write: —

$$\begin{array}{rrrrr} 4, & 11, & 2, & 5, & 5, \\ & 48, & 708, & 8520, & 102300, \\ \hline 4, & 59, & 710, & 8525, & \underline{\underline{102305.}} \end{array}$$

It would be good practice to work out as follows — in the duodecimal algorithm, on its own merits, "carrying" *twelve* — but from our fixed habit of thought this is much more difficult: —

$$\begin{array}{rl} a \,|\, \underline{4\,b\,255} & \\ a \,|\, \underline{5\,b\,06} & \ldots\; 5 \\ a \,|\, \underline{713} & \ldots\; 0 \\ a \,|\, \underline{86} & \ldots\; 3 \\ a \,|\, \underline{a} & \ldots\; 2 \\ 1 & \ldots\; 0 \end{array}$$

that is, 4 *b* 255 (radix twelve) is 102305 (radix ten).

283. Fractions expressed in such notations as are under discussion are called radix fractions, *decimal* if the base is ten, *duodecimal* if the base is twelve, etc.

A fraction $\frac{N}{D}$ expressed as a radix fraction cannot terminate unless Nr^n is a multiple of D; for

$$\frac{N}{D} = \cdot d_1 d_2 d_3 \ . \ . \ . + \frac{f}{r^n D}.$$

Multiplying each member by r^n reduces the radix fraction part of the right-hand member to an integer, giving

$$\frac{Nr^n}{D} = d_1 d_2 d_3 \ . \ . \ . \ d_n + \frac{f}{D},$$

where $d_1 \, d_2 \ . \ . \ .$ is the integral part of the quotient; and there must be a fractional part unless Nr^n is a multiple of D. Also if N/D is in "lowest terms," i.e., if N be prime to D, it is plain that a radix fraction cannot terminate unless r^n is a multiple of D. Nor can r^n be a multiple of D unless it be resolvable into powers of primes which are prime factors of r. For example, to express N/D (in its lowest terms) as a decimal fraction, we must have $D = 2^x \, 5^y$, where either x or y may be zero.

284. If, when the proper fraction $\frac{N}{D}$ in its lowest terms is expressed as a radix fraction, the latter does not terminate, its digit figures must repeat in a cycle of not more than $D - 1$ figures. For, evidently only $D - 1$ different remainders can occur, and when one recurs, the figures of the quotient must repeat. Such radix fractions are called *repeating*, *recurring*, or *circulating*.

The repeating period may begin at once, or may begin after figures which do not repeat, — commonly distinguished as *pure* and *mixed* circulates. The repeating period is sometimes called *perfect* when it consists of the full complement $(D-1)$ of figures. The repeating period is denoted by dotting its first and last figures.

This subject could be better discussed in connection with infinite series, and "geometrical" progression; but repeating decimals occur so frequently in practice that their reduction to simple fractions cannot be left in the dark.

Consider —

$$3\,\underline{|1\cdot\qquad}$$
$$0\cdot333333 + \frac{1}{3 \times 10^6} \quad \ldots\ 1/3 = 0\cdot\dot{3}$$

$$7\,\underline{|1\cdot\qquad}$$
$$0\cdot142857142857 + \frac{1}{7 \times 10^{12}} \quad \ldots\ 1/7 = 0\cdot\dot{1}4285\dot{7}$$

$$11\,\underline{|1\cdot\qquad}$$
$$0\cdot0909 + \frac{1}{11 \times 10^4} \quad \ldots\ 1/11 = 0\cdot\dot{0}\dot{9}$$

$$24\,\underline{|1\cdot\qquad}$$
$$0\cdot04133 + \frac{8}{24 \times 10^5} \quad \ldots\ 1/24 = 0\cdot041\dot{3}$$

The remainders, inexpressible as a radix fraction, may be introduced at any point. I express them to avoid discussion of infinitesimals; and if regarded as implicit in the notation of the repeating decimals, the reasoning in this section is exact in terms of thought familiar to beginners.

Now

$$\tfrac{1}{9} = 0\cdot111 +$$
$$\tfrac{1}{99} = 0\cdot010101 +$$
$$\tfrac{1}{999} = 0\cdot001001 +$$
$$\tfrac{1}{9999} = 0\cdot00010001 +$$

The law is plain, and furnishes a way to transform repeating decimals into simple fractions.

For example: Express $0{\cdot}\dot{3}2\dot{4}$ as a common fraction in its lowest terms. Evidently $0{\cdot}\dot{3}2\dot{4} = 324 \times 0{\cdot}\dot{0}0\dot{1}$. But $0{\cdot}\dot{0}0\dot{1} = \frac{1}{999}$; therefore $0{\cdot}\dot{3}2\dot{4} = \frac{324}{999} = \frac{36}{111} = \frac{12}{37}$.

Hence the rule: To express any pure circulating decimal as a common fraction, write the repeating period for numerator, and for denominator as many nines as there are decimal places in the repeating period.

To find the rule for mixed circulates, consider:—

$$F = 0{\cdot}3\dot{1}\dot{8}$$

$$\left.\begin{aligned} 1000\,F &= 318{\cdot}1818 + f \\ 10\,F &= 3{\cdot}1818 + f \end{aligned}\right\}$$ where these remainder fractions, f, are absolutely the same.

therefore $990\,F = 315$

and $F = \frac{315}{990} = \frac{7}{22}$.

Again consider:*—

$$F = 0{\cdot}03693\dot{1}\dot{8}$$

$$\left.\begin{aligned} 10{,}000{,}000\,F &= 369318{\cdot}18 + f \\ 100{,}000\,F &= 3693{\cdot}18 + f \end{aligned}\right\}$$ the remainder fractions are absolutely the same.

therefore $9900000\,F = 365625$

and $F = \frac{365625}{9900000} = \frac{13}{352}$.

Hence the rule: To express a mixed circulating decimal as a common fraction, subtract the non-repeating part from the whole circulate for the numerator, and for the denominator write as many nines as there are decimal

* The same result may be obtained thus:

$$\begin{aligned} 0{\cdot}03693\dot{1}\dot{8} &= 0{\cdot}03693 + 0{\cdot}00000\dot{1}\dot{8} \\ &= \tfrac{3693}{100000} + \tfrac{1}{100000} \times 0{\cdot}\dot{1}\dot{8} \\ &= \tfrac{3693}{100000} + \tfrac{1}{100000} \times \tfrac{18}{99} \\ = \tfrac{3693}{100000} + \tfrac{2}{1100000} &= \tfrac{40625}{1100000} = \tfrac{13}{352}. \end{aligned}$$

places in the repeating period, followed by as many zeros as there are places in the non-repeating part.

285. Inasmuch as we have seen that any integer is expressible in only one way in any radix scale, it is clear that a common fraction in any scale of notation is expressible as a common fraction in any other scale. Consequently any terminating or repeating radix fraction in any scale transforms into a common fraction, and therefore into a terminating *or* repeating fraction in any other scale.

Note carefully that a terminating radix fraction in one scale need not transform into a terminating radix fraction in another scale, but into a terminating *or* repeating radix fraction.

286. To transform a fraction from one scale to a radix fraction in another, simply multiply by the new base, and the fractional part of the product again by the new base, and so on. The integral parts of these products in due order are the figures of the transformation. For example, to express $\frac{3}{8}$ as a duodecimal fraction: —

$$\tfrac{3}{8} \times 12 = 4\tfrac{1}{2}$$
$$\tfrac{1}{2} \times 12 = 6,$$

therefore $\frac{3}{8} = 0.46$ (radix twelve).

Or again, to express 0.13 as a radix fraction in the seven scale: —

$$\begin{array}{r} 0.13 \\ 7 \\ \hline 0.91 \\ 7 \\ \hline 6.37 \\ 7 \\ \hline 2.59 \\ 7 \\ \hline 4.13 \end{array}$$

here 13 recurs, and the fraction repeats this period, so, —

$$0.13 \text{ (radix ten)} = 0\cdot\dot{0}62\dot{4} \text{ (radix 7)}.$$

To prove the propriety of this process, consider a proper fraction, F, and let —

$$F = \frac{x_1}{r} + \frac{x_2}{r^2} + \frac{x_3}{r^3} + \ldots$$

in some new scale of notation whose base is r.

Then $$rF = x_1 + \frac{x_2}{r} + \frac{x_3}{r^2} + \ldots$$

say $rF = x_1 + E$, where E must be a proper fraction; therefore x_1 is the integral part of rF.

Again $$rE = x_2 + \frac{x_3}{r} + \frac{x_4}{r^2} +$$

And in like manner x_2 is the integral part of rE; and so on.

287. If I be any integer, and s the sum of its digits, and r the radix of the scale of notation, then the remainder of

$$\frac{I}{r-1} = \text{the remainder of } \frac{s}{r-1}.$$

For, let $I = c_0 + c_1 r + c_2 r^2 + c_3 r^3 + \ldots c_n r^n$.

Subtracting the sum of the digits from each member of the equation gives

$$I - s = c_1(r-1) + c_2(r^2-1) + c_3(r^3-1) + \ldots c_n(r^n-1).$$

Since, by Section **271**, each term of the right-hand member is a multiple of $(r-1)$, if we divide each member by $(r-1)$ (or any submultiple) we get

$$\frac{I}{r-1} - \frac{s}{r-1} = \text{some integer.}$$

Therefore, by Section **245**, the essentially fractional parts of $I/r-1$ and $s/r-1$ must be equal.

288. From this theorem follows the special corollary that in our decimal notation any integer and the sum of its digits give the same remainders to 9 or 3.

This is the reason of the familiar rule for "casting out 9's," in order to test the accuracy of calculations.

If $P = MN = 9x + p = (9y + m)(9z + n)$, where p, m, and n are the respective remainders to 9 of P, M, and N, it follows that p and mn give the same remainders to 9,

$$\text{since} \quad 9x + p = (9y + m)(9z + n)$$
$$= 9^2yz + 9(ny + mz) + mn$$
$$= 9(9yz + ny + mz) + mn.$$

In practice, find p, m, and n, not by dividing P, M, and N, by 9, but, in accordance with the theorem, by dividing their digit-sums by 9; "cast out" the nines. It is plain also that the remainder to 9 (or 3) of $A + B + C$ equals the like remainder to $a + b + c$, where a, b, and c are the respective remainders to A, B, and C.

Therefore to test addition: —

(1)	8277	remainder to 9 . . .	6
	3485	remainder to 9 . . .	2
	7146	remainder to 9 . . .	0
	8036	remainder to 9 . . .	8
	26944		16

The sums, 26944, and 16, each, give the same remainder, 7; consequently the addition* is probably correct, — only probably, because this check could not take note of an error of any multiple of 9, or compensating errors, or transposition of figures.

* Strictly, partial additions, and associations to suit our notation. (*Vide* §§ 72, 73.)

(2) To test subtraction: —

87235	remainder to 9 . . .	7
14565	remainder to 9 . . .	3
72670		4

The difference, 72670, and 4 give same remainder, 4, to 9.

(3) To test multiplication: —

349751	remainder to 9 . . .	2
28637	remainder to 9 . . .	8
10015819387		16

The product and 16 give the same remainder, 7.

(4) To test division, let the student prove that if $P = QD + R$,

The remainder to nine of P = remainder to nine of $(qd + r)$, where q, d, and r are the respective remainders to nine of Q, D, and R. Thus to test the division,

$$\frac{27220662}{47923} = 568 + \frac{398}{47923}: —$$

remainder to	47923 (divisor)	= 7
remainder to	568 (quotient)	= 1
remainder to	398	= 2
remainder to $7 \times 1 + 2$		= 0
remainder to	dividend	= 0

289. PROBLEMS: —

(1) Expressed in a certain scale *seventy-nine* becomes 142, what is the radix?

(2) In what scale of notation does 301 express the second power of an integer?

(3) Deduce a test of multiplication by "casting out elevens."

(4) Prove that any integer of four digits in the scale of

ten is a multiple of 7, if its first and last digits be equal, and the hundreds digit twice the tens digit.

(5) In ten scale a number of 6 digits whose 1st and 4th, 2d and 5th, 3d and 6th digits are respectively the same, is a multiple of 7, 11, and 13.

290. The common process of finding algebraic square roots, cube roots, etc., is familiar to all, most text-books making far too much of it. The method has little interest, theoretical or practical.* Even the analogous numerical calculations are better dispensed with, if a table of logarithms is at hand; and the method for the algebraic problem is rendered superfluous by the simpler method of "undetermined coefficients." We consider only cases where the function is a perfect square, because further discussion would take us into the question of infinite series.

Example. — Required the algebraic square root of —

$$x^4 + x^3 - \frac{5x^2}{12} - \frac{x}{3} + \frac{1}{9}.$$

If a "perfect square," the root must be of the form, $ax^2 + bx + c$, the square of which is $a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2$. The corresponding coefficients must be equal; therefore, $a = 1$. $2ab = 1 \therefore b = 1/2$. $2bc = -1/3 \therefore c = -1/3$; therefore the required square root is

$$x^2 + \frac{x}{2} - 1/3.$$

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* Professor Chrystal remarks: "The method was probably obtained by analogy from the arithmetical process. It was first given by Recorde in *The Whetstone of Witte* (black letter, 1557) the earliest English work on algebra." It would be serviceable to the student to compare the difference between the numerical and the algebraic problems.

To find c we might have taken either of the last three coefficients.

A similar method would yield the cube root of a function which is a "perfect cube," etc.

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291. Without going too far into the subject, it is proper to add here several fundamental theorems concerning complex numbers, postponed from Chapter XII.

If $\phi(x+yi)$ be an integral function of a complex number, we saw in Chapter XII. that it is reducible to a complex number, say $A+Bi$. Now, if all the coefficients of $\phi(x+yi)$ are protomonic, A and B are protomonic, and A can contain only even, and B only odd, powers of y; therefore, if $x+yi$ be changed to $x-yi$, A will remain unaltered, and B changed to $-B$. That is to say, if $\phi(x+yi)=A+Bi$, $\phi(x-yi)=A-Bi$.

The theorem is readily extended to include all stirpal functions, integral or fractional, of a complex number, and generalized for such functions of more than one complex number.

292. As a corollary, if all the coefficients of the stirpal function $\phi(u)$ be protomonic, and if $\phi(u)=0$, when $u=a+bi$, then $\phi(u)=0$, when $u=a-bi$; for if $A+Bi=0$, $A=0$ and $B=0$ (§ **193**).

State the corollary for $\phi(u, v, w \ldots)$.

293. Since $\phi(x+yi)=A+Bi$ and $\phi(x-yi)=A-Bi$, when all the coefficients in the functions are protomonic; and since

$$\text{norm}\,\phi(x+yi) \doteq \text{norm}\,(A+Bi)=A^2+B^2=(A+Bi)(A-Bi);$$ therefore

$$\text{norm}\,\phi(x+yi)=\text{norm}\,\phi(x-yi)=\phi(x+yi)\,\phi(x-yi);$$

and therefore :—

$\mathrm{mod}\,\phi(x+yi) = \mathrm{mod}\,\phi(x-yi) = +\sqrt{\phi(x+yi)\,\phi(x-yi)}$;
and in general

$$\mathrm{mod}\,\phi\{x+yi,\ u+vi,\ \ldots\} = \mathrm{mod}\,\phi\{x-yi,\ u-vi,\ \ldots\}$$
$$= +\sqrt{\{\phi(x+yi,\ u+vi,\ \ldots)\,\phi(x-yi,\ u-vi\ \ldots)\}}.$$

294. If the function be the *product* of several complex numbers, this theorem gives

$$\mathrm{mod}\{(r+si)(t+ui)(v+wi)\} = \sqrt{\{(r+si)(t+ui)(v+wi)(r-si)(t-ui)(v-wi)\}} = +\sqrt{\{(r^2+s^2)(t^2+u^2)(v^2+w^2)\}}$$
$$= \sqrt{r^2+s^2}\,\sqrt{t^2+u^2}\,\sqrt{v^2+w^2} = \mathrm{mod}\,(r+si)\,\mathrm{mod}\,(t+ui)\,\mathrm{mod}\,(v+wi);$$

that is to say, the modulus of the product of any number of complex numbers equals the product of their moduli.

It might plausibly be taken for granted (since we have seen that if $x+yi=0$, $x=0$, and $y=0$); but it is better to prove distinctly that the product of two complex numbers cannot be zero, unless one of the complex numbers is zero:

If $yz=0$, where y and z are complex numbers, mod $(yz) = 0$. But mod $(yz) =$ mod y mod z; therefore, mod y mod $z = 0$.

But mod y and mod z are protomonic. Therefore, either mod $y=0$, or mod $z=0$; and consequently, by Section **198**, either $y=0$, or $z=0$.

295. Again, as a special case of the general theorem in Section **293**, if the particular function be the *quotient* of two complex numbers, we have

$$\mathrm{mod}\left\{\frac{t+ui}{v+wi}\right\} = +\sqrt{\left\{\frac{t+ui}{v+wi}\cdot\frac{t-ui}{v-wi}\right\}}$$
$$= +\sqrt{\left\{\frac{t^2+u^2}{v^2+w^2}\right\}} = \frac{\sqrt{t^2+u^2}}{\sqrt{u^2+w^2}} = \frac{\mathrm{mod}\,(t+ui)}{\mathrm{mod}\,(v+wi)};$$

that is to say, the modulus of the quotient of two complex numbers is the quotient of their moduli.

296. The modulus of the sum of complex numbers may equal the sum of their moduli, cannot be greater, and is in general less. For consider two complex numbers, $t + ui$ and $v + wi$.

By Section **293**, mod $(t + ui + v + wi) =$

$$+\sqrt{\{(t + ui + v + wi)(t - ui + v - wi)\}} =$$

$+\sqrt{\{(t + v)^2 + (u + w)^2\}}$, therefore we desire to prove, $+\sqrt{\{(t + v)^2 + (u + w)^2\}}$ not $> +\sqrt{(t^2 + u^2)} + \sqrt{(v^2 + w^2)}$, or, since only positive roots are concerned, that $(t + v)^2 + (u + w)^2$ not $> t^2 + u^2 + v^2 + w^2 + 2\sqrt{(b^2 + u^2)(v^2 + w^2)}$. Subtracting $t^2 + u^2 + v^2 + w^2$ from both members of this inequality gives, $2\,tv + 2\,uw$ not $> 2\sqrt{(t^2 + u^2)(v^2 + w^2)}$, and dividing by 2, $tv + uw$ not $> \sqrt{(t^2 + u^2)(v^2 + w^2)}$.

The right-hand member is essentially positive, and therefore not less than the left, if the latter is negative (as might be on account of the original quality of t, u, v, or w); and the theorem is consequently proved for that case.

If the left-hand member is not negative, by squaring both sides we get

$$t^2v^2 + 2\,tuvw + u^2w^2 \text{ not } > t^2v^2 + u^2w^2 + t^2w^2 + v^2u^2,$$

or $$2\,tvwu \text{ not } > t^2w^2 + v^2u^2,$$

or $$0 \text{ not } > t^2w^2 + v^2u^2 - 2\,tvwu,$$

or $$0 \text{ not } > (tw - vu)^2.$$

But this is true, since the right-hand member is essentially positive.

297. Argand's diagram beautifully applies to geometrical relations these properties of complex numbers, thus analytically displayed.

298. It would be interesting and instructive to follow a great many very curious and useful investigations of various properties of primary or discrete (to say nothing of complex, or continuous) number, of which no mention ever has been made. But to do so would carry us into ideas and notations equally strange, and would be deemed a transgression of appropriate bounds for such an elementary treatise as is this little work. For instance, Gauss makes the notion of congruence fundamental in his *Disquisitiones Arithmeticae*, *Congruence* meaning the relation of I and J, if $I = a\mu + r$, and $J = b\mu + r$, where μ is termed the modulus of I and J, and I and J are called congruent with respect to modulus μ. Some astonishing facts are directly deducible from this simple mode of classification. It is not from the difficulties of the more elementary portion of the Theory of Numbers * that the field lies fallow for our undergraduate courses in mathematics, and I believe the interest of students would be less disposed to flag if the firmer grasp of thought were commanded which such studies would infallibly encourage.

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299. If the equation $\phi(x, y, z) = \psi(x, y, z)$ is satisfied for all values of the variables, it is called an identical equation, or an identity, or a formula. (*Vide* § **40**.)

In this case the equation is formally true, under the very laws of numerical operation, regardless of particular values of the variables.

If, on the other hand, an equation is satisfied only for special values of the variables, it is called a synthetic, or conditional, equation. From this point of view, the con-

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* For bibliography of the interesting and important subject which bears this name, see *Numbers, Theory of*, Cayley, Ency. Brit., 9th ed.

stants are commonly spoken of as *known*, and the variables as *unknown* "quantities," — numbers, in the algebra of number.

Synthetic equations are classified and named with reference to their unknown numbers, precisely as functions are characterized in regard to their variables. (*Vide* § **169**.)

Synthetic equations involving only stirpal and radical functions (exponential, etc., equations are deferred to future studies) can always be made to depend upon an equation of the form

$$\phi(x, y, z, \ldots) = 0,$$

where ϕ is an integral function.

This form, therefore, is of prime importance in the theory of equations.

300. Synthetic equations concerning the same variables may occur in sets, or systems. In this case they are called *simultaneous*, and the problem is to find the sets of values of the variables which render every equation of the system an identity.

Such a set of values is said to satisfy the system, and is called a solution of the system.

Such solutions are to be distinguished in many ways from the solutions of one integral equation in one variable, where a solution is called a *root*.

301. It is important to distinguish between two different kinds of solution: — (1) *Numerical solution*, exact or approximate, which can often be obtained where formal algebraical solution would be out of the question; and (2) What may be called *formal solution*, that is, a solution in which the variables are expressed as definite analytical functions of the constants. Such solutions of equations of degree higher than the fourth cannot, in general, be found.

302. The final test of any solution is the satisfaction of the equation, upon substitution therein of the values obtained for the unknown numbers. No matter how the solution has been obtained, if it does not stand this test, it is no solution; and no matter how obtained, if it does stand this test, it is a solution. It is often a good way to guess a solution, and make the test.

303. Fundamental Proposition in the Theory of Equations. — If in the equation $\phi_n(x) = 0$, $\phi_n(x)$ be an integral function of x of the nth degree (the coefficients, in general complex, in particular, protomonic, numbers) where the coefficient of the x^n term is not zero, then $\phi_n(x)$ is the product of n factors, each of the first degree.

With one provision we proved this proposition in Section **268**, and it has also been shown that these factors can always be in the form

$$C(x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n),$$

where C is the coefficient of x^n in $\phi_n(x)$, and r_1, r_2, r_3, $\ldots$ r_n are the roots of the equation. Consequently, the problem of solving an integral equation with one unknown number, is identical with the problem of resolving the general function of one variable, of like degree, into factors of form

$$C(x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n).$$

304. It is worth while to call attention to the fact that

$$x^2 + x + 1 = (x + 1 + \sqrt{x})(x + 1 - \sqrt{x}),$$

often given by beginners when required to factor $x^2 + x + 1$, although a true identity, is no factorization in the sense intended, because the factors are not integral functions.

305. Nothing need be said of the solution of integral equations of the first degree: properly associating the terms, and reducing by the distributive law to the form $Cx = N$, gives $x = \frac{N}{C}$.

306. Recurring to Section **271** (4), we know that

$$(x - a)(x - b)(x - c) \ldots (x - n) = 0$$

is an equation whose roots are $a, b, c, \ldots n$.

Performing the multiplications, we have the form:

$$x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_{n-1} x + c_n = 0,$$

where, $c_1 = -(a + b + c + \ldots + n)$

$c_2 = ab + ac + bc + \ldots + mn$

$c_3 = -(abc + abd + acd + \ldots + lmn)$

$c_n = \pm\, abcd \ldots n.$

(*Plus* or *minus*, as n is even or odd).

Hence, if an integral equation of the nth degree is in the above general form:

The coefficient of the second term is minus the sum of the roots.

The coefficient of the third term is the sum of their products, taken two at a time.

The coefficient of the fourth term is minus the sum of their products, taken three at a time, etc.

The last term (the constant) is plus or minus the product of all the roots, according as n is even or odd.

307. It follows: In every equation of the nth degree in the general form,

If the second term is wanting, the sum of the roots is

If the last term is wanting, at least one root is zero.

If all the roots are integral, they are submultiples of the last term, which must be integral. But the inverse does not follow; since the last term may be integral, yet roots be fractional. But if the last term is not integral, some of the roots are not integral.

If all but one of the roots are known, the remaining one may be found by adding the sum of the known roots to the coefficient of the second term, and changing the qualitative sign of the result. Or, by dividing the last term by plus or minus the product of the known roots, according as n is even or odd.

If m roots are known, the equation may be depressed to another of the $(n \dot{-} m)$th degree, by dividing by the product of m factors of the form,

$(x - r_1)\ (x - r_2)\ \ldots\ (x - r_m)$, and therefore:—

If all but two roots are known, the coefficient of the depressed equation is the sum of the known roots and the coefficient of the second term of the given equation. And the last term of the depressed equation is the last term of the given equation, divided by plus or minus the product of the known roots, according as n is even or odd.

308. From the process of multiplication required in Section **306**, it is evident that if all the r's are positive, the quality of the terms is alternately $+$ and $-$. Hence, if the roots of an equation are all positive, the signs of its terms (supplying missing terms by zeros) are alternately $+$ and $-$, and inversely.

Again, if all the r's be negative, there is no change in the signs of the terms.

It would not be difficult to deduce here *Descarte's Rule of Signs:* An integral equation cannot have more positive roots than it has changes of signs, nor more negative roots than it has continuations of the same sign.

309. *Prove:* Any integral equation may be transformed into another whose roots are the negatives of the original roots, by changing the signs of alternate terms, beginning with the second.

310. To transform an integral equation into another, whose roots are the roots of the original equation multiplied by a given number, k: —

In the general form substitute y / k for x, obtaining, —

$$\left(\frac{y}{k}\right)^n + c_1\left(\frac{y}{k}\right)^{n-1} + c_2\left(\frac{y}{k}\right)^{n-2} + \cdots + c_{n-1}\left(\frac{y}{k}\right) + c_n = 0 \quad (1)$$

Multiplying by k^n gives

$$y^n + c_1 k y^{n-1} + c_2 k^2 y^{n-2} + \ldots c_{n-1} k^{n-1} y + c_n k^n = 0 \quad (2)$$

The roots of (2) are the values of y that satisfy it; but $y = kx$; therefore, noting the coefficients in (2), to effect the desired transformation, multiply the second term by k, the third by k^2, and so on.

311. Equations may be transformed in many other useful ways; for example, so that the roots shall be the original roots $\pm$ some constant. This mode of transformation is most serviceable in the special case of making the exact increment which will cause the second term to vanish, — a device for preparing cubic and biquadratic equations for solution. For a simple illustration see Section **316**.

312. Seeing that we have the unique resolution: —

$$\phi_n(x) = 0 = c(x - r_1)(x - r_2) \ldots (x - r_n),$$

it follows from Section **292** that if $\phi_n(x)$ has all its coefficients protomonic, and vanishes when $x = a + bi$, it must vanish when $x = a - bi$.

This is to say, that in any integral equation whose coefficients are protomonic, roots which are complex numbers must occur in conjugate pairs.

In like manner (*vide* § **170**) surd roots can enter equations with commensurable coefficients only in conjugate pairs.

Thus, all such equations, if of an odd degree, must have, in the former case at least one protomonic, and in the latter at least one commensurable, root.

313. The general equation of the second degree in one variable is $ax^2 + bx + c = 0$. The general theory of solution is already in our hands, and in this case the *formal solution* (*vide* § **301**) is always obtainable. Various methods may be followed.

The general equation,

$$ax^2 + bx + c = 0,$$

may be reduced without altering the roots (§ **303**) to

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

or

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

From consideration of the formula $(x + y)^2 = x^2 + 2\,xy + y^2$, it is plain that the left-hand member may be made a "complete square" in x by adding $\left(\frac{b}{2\,a}\right)^2$ to each member, which gives —

$$x^2 + \frac{b}{a}x + \frac{b^2}{4\,a^2} = -\frac{c}{a} + \frac{b^2}{4\,a^2} = \frac{b^2 - 4\,ac}{4\,a^2}.$$

Taking the square root of each member, *

* The double sign before the left-hand member would be superfluous, since nothing more would be said than is expressed as the statement stands; e.g.: —

$\pm\,(a + b) = \pm\,(c + d)$ says no more than

$a + b = \pm\,(c + d)$, as one may readily satisfy himself.

See also Section **325**.

$$x+\frac{b}{2a}=\pm\frac{1}{2a}\sqrt{b^2-4ac},$$

or

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

We have here a *formal solution* of the general quadratic equation.

Also the quadratic function, ax^2+bx+c, has been factored. For, by the principles clearly exhibited in Section **303**, —

$$ax^2+bx+c=a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right).$$

314. In solving a particular quadratic in one variable, we may give this process of "completing the square" its particular application; or we may employ the formal solution as a *rule; that is, after reducing the given equation to the form* $ax^2+bx+c=0$, simply write down the particular values in

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Of course, if the given equation in form $ax^2+bx+c=0$, affords a function readily factorable by inspection, it would be absurd to feign an investigation for what is already known. For instance, one with any skill in the algebra cannot fail to see that in $x^2+5x+6=0$, we have $(x+3)(x+2)=0$; which is to say, that $x=-3$, and $x=-2$.

The device of reducing the given equation to the form, $4a^2x^2+4abx+4ac=0$, before "completing the square" (known as the *Hindoo Method*), is hardly worth mentioning, since it merely avoids fractions which offer no obstacle to calculation. It is doubtless a relic of the times when fractional number was regarded with suspicion.

315. If the *formal solution* of $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

be considered, it will be seen that, when the coefficients are protomonic, the roots are: —

(1) Protomonic and unequal, if $b^2 - 4ac$ is positive.
(2) Protomonic and equal, if $b^2 - 4ac = 0$.
(3) Commensurable, if $\sqrt{b^2 - 4ac}$ is commensurable.
(4) Conjugate surds, if $\sqrt{b^2 - 4ac}$ is incommensurable.
(5) Conjugate complex numbers, if $b^2 - 4ac$ is negative.
(6) Equal, if $b^2 = 4ac$.
(7) Equal moduli, but one positive, other negative, if $b = 0$,

$$\left(r_1 = +\sqrt{-\frac{c}{a}},\ r_2 = -\sqrt{-\frac{c}{a}},\ \text{protomonic or neomonic, as } -\frac{c}{a} \text{ is } + \text{ or } -\right).$$

(8) One zero, other $= -b/a$, if $c = 0$.
(9) Both zero, if $b = 0$ and $c = 0$.

It may be profitable to find, from a different standpoint, more or less the same criteria: —

From Section **306**. the equation, $ax^2 + bx + c = 0$, gives the following relations of roots and coefficients, —

$$r_1 + r_2 = -\frac{b}{a}, \text{ and } r_1 r_2 = \frac{c}{a}.$$

Consequently r_1 and r_2 are

positive if $\frac{b}{a}$ is negative and $\frac{c}{a}$ positive;

negative if $\frac{b}{a}$ is positive and $\frac{c}{a}$ positive;

of opposite quality if $\frac{c}{a}$ is negative.

These statements presuppose (1) above, $b^2 - 4ac > 0$.

$$r_1 = -r_2 \text{ if } \frac{b}{a} = 0.$$

$$r_1 = 0 \text{ or } r_2 = 0 \text{ if } \frac{c}{a} = 0.$$

$$r_1 = 0 \text{ and } r_2 = 0 \text{ if } \frac{b}{a} = 0 \text{ and } \frac{c}{a} = 0.$$

If $ax^2 + bx + c = 0$ be still regarded as a quadratic when $a = 0$, then one root is ∞. If b also is zero, both roots become infinite.

These criteria may be tabulated :—

ROOTS.	CRITERION.	ROOTS.	CRITERION.
Protomonic . . .	$b^2 - 4ac > 0$.	Positive . .	$\frac{c}{a}+$, and $\frac{b}{a}-$.
Commensurable .	$\sqrt{b^2 - 4ac}$, commensurable.	Negative . .	$\frac{c}{a}+$, and $\frac{b}{a}+$.
Surd	$\sqrt{b^2 - 4ac}$, surd.	One +, One —	$\frac{c}{a}-$.
Complex	$b^2 - 4ac < 0$.	One, 0 . . .	$c = 0$.
Equal	$b^2 - 4ac = 0$.	Both, 0 . .	$b = 0$ and $c = 0$.
Equal moduli, but one +, other — .	$b = 0$.	One, ∞ . . .	$a = 0$.
		Both, ∞ . .	$a = 0$ and $b = 0$.

316. Another method of solving a quadratic equation is important from its bearing on the solution of cubic equations.

The general equation, $ax^2 + bx + c = 0$ . . . (1), may be reduced by a change in the variable to the form $ay^2 + d = 0$ . . . (2), from the immediate solution of which the original variable is recovered. To discover what change must be made in the original to serve this purpose (*vide* § **311**), let $x = y + e$.

If $x = y + e$, (1) is equivalent to

$$a(y+e)^2 + b(y+e) + c = 0;$$

or $$ay^2 + (2ae + b)y + ae^2 + be + c = 0. \quad (3)$$

To make the second term vanish, $2ae + b$ must be zero,

or $$e = -\frac{b}{2a}.$$

With this value of e, (3) becomes

$$ay^2 - \frac{b^2 - 4ac}{4a} = 0;$$

whence $$y = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

But $$x = y + e = y - \frac{b}{2a};$$

therefore $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ as before.}$$

317. If an equation contains two (*a fortiori*, more than two) unknown numbers, it is obviously indeterminate. An extraneous condition (e.g., that the variables shall be integers) sometimes affords a basis for a determinate solution.

A system of simultaneous equations is in general determinate when the number of equations equals the number of the variables.

If the number of equations is less than the number of variables, the solution is in general indeterminate.

If the number of equations is greater than the number of variables, there is in general no solution, the system being inconsistent, contradictory.

These are ultimate logical principles; special limitations of the statements are needed rather than proof.

It must suffice here to point out that a system may be apparently determinate, yet indeterminate by reason of one being analytically derivable from the others. Also it may happen that a system of analytically independent equations may have more equations than variables, yet not be contradictory.

Let the student frame examples of such conditions.

318. A determinate system of integral equations involving the variables, $x, y, z, \ldots$, cannot have more than, and in general has exactly $abc \ldots$ solutions, where $a, b, c, \ldots$, are the degrees of the system in the respective variables.

Proof of this proposition must await future studies; but it is useful to know the theorem, and the *question* presents itself at once, and should not be ignored by the teacher.

319. Two systems of equations, each of which may consist of only one, are termed *equivalent* when every solution of each is a solution of the other.

From any system we may, in an infinite variety of ways, deduce another system; but the derived system is not generally equivalent to the original.

This matter is of fundamental importance, even at the most elementary stages. It is commonly (with several notable exceptions) left in the dark by our text-books, though "there are few parts of algebra more important than the logic of the derivation of equations, and few, unhappily, that are treated in more slovenly fashion in elementary teaching. No mere blind adherence to set rules will avail in this matter; while a little attention to a few simple principles will readily remove all difficulty." *

320. If A and B are two functions, which do not become infinite for any finite values of the variables (such cases must be considered separately), the only values of the vari-

* *Text Book of Algebra*, Chrystal, vol. i., p. 285.

ables which make $A \times B = 0$ are such as make $A = 0$, or $B = 0$, according to laws already fully demonstrated.

321. Axiomatically, if $A = B$, (1)

then $$A \pm C = B \pm C. \quad (2)$$

Also, (1) and (2) are equivalent, for neither can be true unless the other is true.

Note the corollaries whereby we "transpose a term with changed signs," or "change all signs," or reduce any equation to the form $Q = 0$, without destroying equivalence.

322. On the other hand, although, if

$$A = B, \quad (1)$$

then $$AC = BC, \quad (2)$$

the derivation being perfectly legitimate, and the resulting equation true, yet (2) is *not* equivalent to (1), unless C is a constant not zero; for, by Section **321** (2) is equivalent to

$$AC - BC = 0$$

that is to $$C(A - B) = 0 \quad (3)$$

Now, if C is a constant not zero, (3) is equivalent to (1) by Section **320**; but not otherwise, for if C is a function of the variables, (3) is satisfied by all values of the variables that satisfy the equation, $C = 0$ . . . (4), which in general will not satisfy (1). Therefore (2) is not equivalent to (1), but to (1) and (4).

In this way it is plain that multiplying both members of an integral equation by an integral function introduces roots, and dividing the members of such an equation by an integral function loses roots.

Also, from any integral equation another equivalent equation can always be derived in which the coefficient of any term shall be as desired, say $+1$ for the highest term; for this is obtainable by multiplying by a constant.

323. Fractional equations must never be confounded, in the matter of *degree* and number of *roots*, with integral equations. The very term *degree* does not apply to fractional equations. Fractional functions of x may sometimes be integral functions of some function of $x\left(\text{e.g.}, \frac{1}{x}\right)$; but in general no such relations as obtain between degree and number of roots in integral equations subsist for fractional equations. The latter must be solved under the logic of the equivalence of derived integral equations.

From any fractional equation an integral equation may be deduced, which may or may not be equivalent. If $E = F$, where E and F are fractional functions, and $M =$ l. c. m. of the denominators in E and F, then $EM = FM$ is integral.

Here extraneous solutions of $M = 0$ may be introduced, but not necessarily or generally. E and F contain fractions whose denominators are factors in M, and in general roots of $M = 0$ would make E or F infinite, and consequently $M(E - F)$ not necessarily zero.

See examples below for clear understanding of this point.

324. If both members of an equation be raised to the same power, in general the resulting equation is not equivalent. Thus $A = B$; then $A^2 = B^2$, or $A^2 - B^2 = 0$. But the last is equivalent to $(A + B)(A - B) = 0$; hence the solutions of $A + B = 0$ would in general be introduced.

It may be noted that in squaring $A = B$ the result is the same as if the members of the equivalent equation, $A - B = 0$, were multiplied by $A + B$. (*Vide* § **322**.)

325. Neither the equation between the positive, nor that between the negative, square roots of the members of the equation $A = B$, is an equivalent equation; but the two equations (generally written together with double signs) between the positive root of one, and both roots of the other, constitute an equivalent system. (*Vide* § **313**.)

$$+\sqrt{A} = +\sqrt{B} \qquad (1)$$

$$\text{nd} + \sqrt{A} = -\sqrt{B} \qquad (2)$$

is a system equivalent to $A = B$.

For $A = B$ is equivalent to $A - B = 0$, which is equivalent to $(\sqrt{A} + \sqrt{B})(\sqrt{A} - \sqrt{B}) = 0$, which is equivalent to the system (1) and (2).

326. If $A = B$ be a radical equation, repeated involutions will deduce an integral equation which may or may not be equivalent. Extraneous solutions may be introduced; and, if like roots in the original equation alone be regarded, often no solution of the derived equation will satisfy the original.

327. Two equations which are not equivalent are called *independent*. Two or more independent equations involving a corresponding number of variables may be capable of coincident solution; if so, they are termed *simultaneous*, that is, consistent, or involving variables which, though unknown, are the same. Contradictory statements, no matter how artfully veiled the contradiction, can lead only to nonsense in algebra, as elsewhere.

Compare again Sections **300, 317, 318**.

The devices of elimination, whereby an equation in one variable is deduced from a system of simultaneous equations in several variables, are familiar; but the logic of such derivations, and the paramount question of the equivalence of the derived and original systems may have been overlooked.

The present discussion must be concluded with two propositions specially concerning the equivalence of simultaneous systems. The subject will have been by no means exhausted; but my purpose of stimulating alert and intelligent observation in the important matter of solving algebraic equations will probably be fulfilled. The student's skill and knowledge will steadily increase, if strict attention be always paid to the question of equivalence.

328. The system,

$$\left.\begin{array}{l} A = 0 \ (1) \\ B = 0 \ (2) \end{array}\right\} \text{I is equivalent to} \quad \left.\begin{array}{r} A = 0 \ (1) \\ pA + qB = 0 \ (2) \end{array}\right\} \text{II};$$

for any solution of I makes A zero, and B zero, and therefore satisfies II; and any solution of II makes A zero. and therefore reduces II (2) to $q B = 0$, or $B = 0$.

Consequently any solution of either satisfies both.

It may be suggestive to state this proposition again in the form

$$\left.\begin{array}{l} A = B \\ C = D \end{array}\right\} \text{is equivalent to} \quad \left.\begin{array}{r} A = B \\ A + C = B + D \end{array}\right\}.$$

On the other hand, —

$$\left.\begin{array}{l} A = B \\ C = D \end{array}\right\} \text{I is not equivalent to} \quad \left.\begin{array}{r} A = B \\ AC = BD \end{array}\right\} \text{II}.$$

For, though all the solutions of I are solutions of II, II has in addition all the solutions of $C = 0$, and $D = 0$. Let the student satisfy himself of the truth of this proposition. It explains many "answers" which may have been incomprehensible to him.

The following examples may serve to impress what has been said concerning the equivalence of derived equations with their originals, although at every point the student should have found specific illustrations.

329. (1) Solve

$$\frac{6}{x-3} - \frac{2x}{x+6} = 1. \qquad (1)$$

Multiplying each member by $(x-3)(x+6)$ gives

$$x^2 - 3x - 18 = 0,$$

or, $$(x-6)(x+3) = 0, \qquad (2)$$

whence $$x = 6 \text{ and } x = -3.$$

Both of these are solutions of (1). No roots of $(x-3)(x+6) = 0$ were introduced, because $x = 3$ or $x = -6$ would make the left-hand member of (1) infinite, and therefore $M(E - F)$ not zero. (*Cf.* § **323**.)

(2) Solve $$1 - \frac{x^2}{x-1} = \frac{1}{1-x} - 6. \qquad (1)$$

Transposing, and adding the fractions, gives

$$1 - \frac{x^2 - 1}{x - 1} = -6,$$

or $$1 - (x+1) = -6,$$

or $$x = 6 \quad . \; . \; . \text{ equivalent to (1).}$$

But a beginner might multiply by $x - 1$, deriving

$$x - 1 - x^2 = -1 - 6x + 6 \qquad (2)$$

whence $$x = 1 \text{ and } x = 6,$$

where 1 is no solution of the original, and therefore (2) is not equivalent to (1).

Multiplying by any integral function, not necessary to clear of fractions, will derive an equation not equivalent. Accordingly, every device for identical simplification should be employed before multiplying by the *lowest* common multiple.

(3) Solve $$1 + \frac{x^2 + x - 6}{x - 2} = \frac{x^2 - 3x + 2}{x + 2}. \qquad (1)$$

Multiplying each member by $(x - 2)(x + 2)$, and reducing identically, gives

$$3x^2 - 4x - 4 = 0, \qquad (2)$$

whence $x = \dfrac{4 \pm \sqrt{16 + 48}}{6} = \dfrac{4 \pm 8}{6} = 2$ or $-2/3$.

Equation (2) is not equivalent to (1), the root, 2, of $(x - 2)(x + 2) = 0$ having been introduced, because the fraction in the left-hand member of (1) is not in its lowest terms. If (1) be reduced before clearing of fractions we obtain

$$1 + x + 3 = \frac{x^2 - 3x + 2}{x + 2};$$

whence, multiplying by $x + 2$,

$$x^2 + 6x + 8 = x^2 - 3x + 2, \qquad (3)$$

or

$$x = -2/3,$$

where (3) is equivalent to (1).

(4) Solve $\sqrt{4 - x} = x - 4. \qquad (1)$

Squaring $4 - x = x^2 - 8x + 16,$

or $x^2 - 7x + 12 = 0,$

or $(x - 3)(x - 4) = 0, \qquad (2)$

whence $x = 3$ and $x = 4$.

Of these solutions of (2), 4 is a solution of (1) if the positive square root be taken, and 3 is not a solution; whereas, if the negative root be taken, 3 is a solution and 4 is not. Thus (2) is equivalent to

$$+\sqrt{4 - x} = x - 4 \text{ and } -\sqrt{4 - x} = x - 4.$$

(5) Solve $\sqrt{3x + 1} = \sqrt{9x + 4} - \sqrt{2x - 1} \qquad (1)$

Squaring twice, and reducing identically, gives

$$x^2 - \tfrac{9}{2}x - \tfrac{5}{2} = 0 \qquad (2)$$

whence $x = 5$ and $x = -\frac{1}{2}$.

Using only positive roots of the radicals, 5 is a solution of (1); but $-\frac{1}{2}$ substituted in (1) gives

$$\sqrt{-\tfrac{1}{2}} = \sqrt{-\tfrac{1}{2}} - \sqrt{-2},$$

or

$$\tfrac{1}{2}\sqrt{2}\,i = \tfrac{1}{2}\sqrt{2}\,i - \sqrt{2}\,i, \qquad (3)$$

an absurdity if the statement be restricted to positive roots; but if the negative root of the left-hand member be taken with the positive roots of the terms in the right-hand member, (3) is an identity.

Therefore (2) is equivalent to

$$+\sqrt{3x+1} = +\sqrt{9x+4} - (+\sqrt{2x-1}),$$

and

$$-\sqrt{3x+1} = +\sqrt{9x+4} - (+\sqrt{2x-1}).$$

(6) Solve $2 - \sqrt{2x+8} + 2\sqrt{x+5} = 0$ (1)

Squaring twice, we deduce

$$x^2 = 16 \qquad (2)$$

whence

$$x = \pm 4.$$

In this case, using the positive roots of the radicals in (1), neither $+4$ nor -4 is a solution.

So far as I am acquainted with them, treatises upon algebra, if they notice such cases, merely declare that the original equation is impossible and has no solution. Professor Chrystal states the theorem: —

"From every algebraical equation we can derive a rational integral equation, *which will be satisfied by any solution of the given equation*; but it does not follow that every solution, or even that *any solution*, of the derived equation will satisfy the original one."

The italics are mine, and would mark logical contradictions if there *is* "any solution" of the original. Professor Chrystal's example is: —

$$\sqrt{x+1} + \sqrt{x-1} = 1.$$

The derived equation yields the single solution, $x = \frac{5}{4}$. The only remark is, "it happens here that $x = \frac{5}{4}$ is not a solution."

Note, $\frac{5}{4}$ is a solution of $+\sqrt{x+1} + (-\sqrt{x-1}) = 1$.

Professor Taylor, in his *Academic Algebra*, Boston, 1893, which deserves rare praise for emphasizing from the beginning the question of equivalence of equations, uses example (6) above, concluding "$2 - \sqrt{2x+8} + 2\sqrt{x+5} = 0$ is an impossible equation, for it has no solution."

Now, I must not be understood as disputing these statements; they are true, taking the numerical statements to be restricted to positive roots. But it seems to me that the student stands in need of further explanation: he should be directed to observe, that though one may write down what he pleases, as an isolated statement, no restrictions can be put upon the *operational effect* of such numerical relations. The square root of 4, as an inexorable fact, is $+2$ *or* -2. In general operation, radical surds necessarily include all their roots. If one says, $\sqrt[6]{x}$, he has expressed six distinct subjects of affirmation, nor can the logical consequences of these alternatives be avoided in numerical analysis.

The conclusion of the particular problem under consideration is, that no *finite* number satisfies the equation, taking positive square roots; but by reason of the perfect generality and freedom of numerical operations, if there is a number such that either of the square roots concerned fulfils the conditions, it must be yielded as a solution of the equation. We had occasion to notice in Section **323**, and in example (1) above, that indeterminate infinite solutions do not obliterate or interfere with finite solutions, if there be any such.

And in general, the complete analysis of any radical equation would seem to require the investigation of all the alternative equations arising from the indifferent roots of radical surds. Some of these may be impossible, in the sense of having no finite solution; but if a finite number will satisfy any one in the system, it will certainly discover itself in the attempted solution of any other, — and simply because the choice of particular roots is arbitrary, and an equation cannot be made to yield nonsense, or contradiction, so long as there is possible consistency of its terms.

(7) Solve the simultaneous system

$$\left.\begin{array}{lr} x + 2y = 5 & (1) \\ x^2 + 2y^2 = 9 & (2) \end{array}\right\} A$$

Solving (1) for x $\qquad x = 5 - 2y \qquad (3)$

Substituting in (2) from (3) $(5 - 2y)^2 + 2y^2 = 9.$

or $\qquad 3y^2 - 10y + 8 = 0$

or $\qquad (3y - 4)(y - 2) = 0 \qquad (4)$

System A is equivalent to system B (calling (3) and (4) system B). But system B is equivalent to the double system

$$\left.\begin{array}{l} x = 5 - 2y \\ 3y - 4 = 0 \end{array}\right\} c, \quad \text{and} \quad \left.\begin{array}{l} x = 5 - 2y \\ y - 2 = 0 \end{array}\right\} d.$$

The solution of c is $x = \frac{7}{3}$, $y = \frac{4}{3}$.

The solution of d is $x = 1$, $y = 2$.

Hence these are the two solutions of A. (*Vide* § **318**.)

(8) Solve the simultaneous system

$$\left.\begin{array}{ll} x^2 - 2\,xy = 0 & (1) \\ 4x^2 + 9y^2 = 225 & (2) \end{array}\right\} A$$

Factor (1) $\qquad x\,(x - 2y) = 0.$

Hence A is equivalent to the double system

$$\left.\begin{array}{r} 4x^2 + 9y^2 = 225 \\ x = 0 \end{array}\right\} b, \quad \text{and} \quad \left.\begin{array}{r} 4x^2 + 9y^2 = 225 \\ x - 2y = 0 \end{array}\right\} c.$$

The solutions of b are obviously $x = 0$, $y = 5$; and $x = 0$, $y = -5$. Substituting $x = 2y$ in the first equation of c gives

$$y^2 = 9, \text{ or } y = \pm 3.$$

Substituting in $x - 2y = 0$ we have for the solutions of c, $x = 6$, $y = 3$; and $x = -6$, $y = -3$.

Hence the four (*vide* § **318**) solutions of A are $x = 0$, $y = 5$; $x = 0$, $y = -5$; $x = 6$, $y = 3$; $x = -6$, $y = -3$. In the solution of simultaneous systems, attention must always be given to the correct association of values of the variables.

330. When a simultaneous system has its equations of the second degree, its solution demands in general the solution of a biquadratic equation in one variable. Inasmuch as the studies to which these lectures are introductory may be regarded as beginning about at this point, I bring these discussions to a close, without treating of the solution of simultaneous quadratic systems, or of cubic equations, or of biquadratic equations, to say nothing of equations of higher degree, except in so far as the general fundamental theory may suffice in particular instances.

Such matters are to be studied in detail; but it may be remarked in closing that, if a simultaneous quadratic system has only one of its equations of the second degree, or if the equations are homogeneous or symmetrical (*vide*

§ 263), means are offered for the deduction of equivalent equations in one variable of the second degree, and the system may in these cases be solved by the methods for quadratics. Indeed, it is often the case that, on account of symmetry, this is true for a system of simultaneous equations of degree higher than the second. Again, any equation of form, $ax^{2n} + bx^n + c = 0$, may be solved as a quadratic in x^n, and the two unaffected equations, $x^n = \pm k$, which result may then be solved by factoring the functions $x^n + k$ and $x^n - k$, if n be integral, or by involution of the members of $x^n = \pm k$ if n be fractional with numerator 1, or by both devices if n be fractional with numerator > 1. For it must never be overlooked that the solution of an integral equation in one variable, in form $A = 0$, is identical with the problem of factoring the function A into the form $c\,(x - r_1)\,(x - r_2)\;.\;.\;.\;(x - r_n)$.

EXAMPLE. — Find the six sixth roots of $+ 1$, and of $- 1$.

(1) Let $x^6 = 1$;

then $$x^6 - 1 = 0 = (x^3 + 1)\,(x^3 - 1),$$

or $$(x + 1)\,(x^2 - x + 1)\,(x - 1)\,(x^2 + x + 1) = 0.$$

This equation is satisfied when any factor $= 0$. Taking the factors in order, and equating to zero, gives the following six roots: —

$$x = -1;\; x = \frac{1 \pm \sqrt{3}\,i}{2};\; x = 1;\; x = \frac{-1 \pm \sqrt{3}\,i}{2};$$

any one of which, of course, taken six times as a factor, makes $+ 1$.

(2) Let $x^6 = -1$;

then $$x^6 + 1 = 0 = (x^3 + i)\,(x^3 - i),$$

or $$(x - i)\,(x^2 + ix - 1)\,(x + i)\,(x^2 - ix - 1) = 0;$$

whence, as before,

$$x = i;\ x = \frac{-i \pm \sqrt{3}}{2};\ x = -i;\ x = \frac{i \pm \sqrt{3}}{2}$$

any one of which, taken six times as a factor, makes -1.

331. In the application of Number to concrete problems, the logic of the connection of the numerical statements with the particular concrete conditions must be thoroughly comprehended. It should constitute one of the most important parts of mathematical studies and training. It ought to be no matter for surprise that numerical results are often obtained, totally meaningless in regard to the particular problem. On the contrary, such results should be generally expected, alertly watched for, in order to reject them from the problem in question.

Number is a twofold continuous magnitude, and therefore its thoroughgoing application is possible only to twofold continuous magnitudes. (*Cf.* § **188**). In reference to time, a one-dimensional continuum, all protomonic number (positive and negative, fractional and surd) has intelligible application; but neomonic and complex number could have no application to temporal relations. To space, all number, protomonic, neomonic, and complex, may have due application. Space, in fact, being a threefold continuum, in a manner transcends Number, in the sense of permitting an infinite reapplication of number. We have seen, however, that, given three planes of reference, it is possible to uniquely determine any point in solid space by means of three protomonic numbers, and that it is this circumstance which constitutes the ultimate meaning of the statement that space is tri-dimensional.

On the other hand, if a problem require a *number of men*, it is limited in its very terms to primary number; since $\frac{7}{8}$ *men*, or $\sqrt{3}$ *men*, would be as inapplicable as $2 + 7i$

men, unless, indeed, implicit reference to some continuous magnitude afforded ground for the application of such results; e.g., if a problem concerns the number of men in a regiment, applicable results are exclusively in primary number, and if such are not found, there is contradiction in the problem as given; whereas, if a problem concerns the number of men required to dig a ditch, any positive protomonic number might be interpretable.

Not only must the student expect to find solutions of his equations which have no bearing on a particular problem, but it may be that no solution of a correct algebraic translation of the numerical conditions of a problem is applicable. The interpretation of such results is that the problem is self-contradictory, the required conditions impossible.

The clear logical principle is, that, if the problem have any solution, it must be yielded among the solutions of any system of algebraic equations which correctly state the numerical conditions of the problem, no matter how many inapplicable solutions may also be yielded. If no numerical solution is applicable, the problem is impossible, that is to say, its conditions constitute an absolute contradiction of any such outcome as was contemplated.

In many minor ways, also, it is impossible to restrict the perfect generality of numerical operations, and the numerical symbols of the algebra. For example, an unknown number may be *added* to another; but whether the addition increases or decreases a given number, it is rash to say before the quality of the unknown is discovered.

Thus it is ill-considered to demand that 15 be divided "into two such parts that the greater shall *exceed* 3 times the less by as much as half the less *exceeds* three." For (representing the greater by x, and the less by $15 - x$) the numerical conditions are plainly intended to be

$$x - 3\,(15 - x) = \tfrac{1}{2}\,(15 - x) - 3,$$

whence $$x = 11, \text{ and } 15 - x = 4.$$

But on turning to the requirement it is seen that 11 *falls short of*, not "exceeds" 3 times 4 by as much as half of 4 falls short 3. In fine, one cannot choose the issue of absolute facts according to his whim, and the problem as given is presumptuous; all that could have been safely required were numbers which would give equal *differences* for the intended subtractions.

The indeterminate result $\frac{0}{0}$ has already been referred to; it may mean that *any* number answers the requirement, or it may be susceptible of evaluation.

332. Very often all that is required may be discovered from equations without solving them, by transformations into various equivalent forms. Consequently the principles governing the equivalence of derived and original systems, and the study of functions, as distinguished from equations, have, besides their theoretical importance, a practical usefulness quite apart from their bearing upon solution. Indeed, the whole subject of the solution of equations has widened into that of the *variation of functions.* For a long time equations have been losing, and functions gaining, prominence, both in analytical importance and practical utility. Nowadays, instead of seeking merely the values of the variables which cause the function to vanish, that is, solving the equation $\phi\,(x) = 0$, all values of the variable, as it varies continuously, and *the corresponding values of the function,* are considered. The function is calculated for enough specific values of the variable to give a clear idea of its variation. Especial attention must be given to such values of the variable as cause the function to pass through critical values, — zero among others.

Independently of the analytical treatment of geometry (where the purpose of geometrical investigation is so powerfully served by the numerical analysis), this modern way of regarding analytical functions receives reciprocal assistance — if not theoretically, at least as affording the bodily eye a clear representation — by drawing what is called the *graph* of the function.

The graph of a function of one variable is plotted by laying off, to any scale, sects proportional to (*vide* § **213**) arbitrarily chosen values of the variable, in a straight line, to the right or left of a point, according as the chosen value is positive or negative; and at the points so determined, laying off perpendicularly (one way for $+$, the other for $-$) sects proportional to the corresponding values of the function, plotting the end points of these sects. By sketching a curve through such points, a representation of the corresponding variations of function and variable is afforded. The curve so obtained will generally give warning of critical values of the function, at which stages closely consecutive values of the variable must be taken to insure a correct graph of the function.

It is usual to write $y = \phi(x)$, and find the values of y corresponding to selected values of x.

For example, let the student plot the graphs of the following functions, also tabulating the chosen values of x with the corresponding y's.

(1) $y = 1 - x^2$. (2) $y = \dfrac{1}{(1-x)^2}$. (3) $y = \dfrac{1}{1-x}$.

(4) $y = \sqrt{x^2 - 1}$.

At first one may be disposed to examine far more values than necessary. Always plot first the y's corresponding to x's which allow evaluation by inspection, — often these will suffice.

A systematic study of the variations of functions would be surprisingly interesting, even to students who have hitherto found their mathematics dull. The subject could be introduced profitably, even at very elementary stages of algebraic studies, and, while stimulating interest and sustaining attention, would give a better preparation, both for continued study of pure mathematics, and for the manifold practical uses of mathematics in other sciences, than do the methods at present in vogue.

333. The general theory of Inequalities, and of Maxima and Minima values of functions also, deserves a more thorough and independent treatment than it commonly receives in our elementary text-books. The fundamental principles are of so simple and instructive a character, and form so valuable an introduction to the methods of analysis employed in more advanced studies, that our usual elementary courses need in this matter thoroughgoing reformation. The theory of inequalities is the best introduction to that of infinite series, and the latter is indispensable in the study of logarithms and many other subjects which are at once entered upon in the first-year courses of our colleges and universities.

For the most part, the logic of inequalities, and the derivation of equivalent inequalities, runs parallel to the analogous theory for equations, except where restrictions intervene in regard to inequalities, owing to the fact that the members of an inequality cannot, like the members of an equation, be interchanged.

The student may be reminded (*vide* § **198**), in this connection, that there is no comparison in the ordinary sense of *greater* and *less* between complex numbers, because such numbers are in terms of heterogeneous units. Of course this general statement includes particular cases where one of the numbers is either protomonic or neomonic, and the other complex, or where one is protomonic and the other neomonic. With complex numbers, as we have seen, the comparison must be between their moduli.

A fruitful source of error with beginners (on account of the prevailing inadequacy of number concepts) is neglect of the fact that any negative number is less than zero ($-\infty < 0$), and that $x > y$, or $x < y$, according as $x - y$ is positive or negative.

The freedom of transposition of terms with changed signs, in an inequality, is quite as immediate a corollary of axiomatic judgments, and the significance of the symbols, as the like freedom in equations. For it is the same axiom that, if equals be added to unequals, the results are correspondingly unequal, as that, "if equals be added to equals, the results are equal." (*Vide* § **42**, foot-note.)

EXAMPLES.

(I.) *Prove:* $x^2 + y^2 > 2xy$, if x and y are protomonic numbers. $(x - y)^2$ is positive whether $x > y$ or $x < y$; but $(x - y)^2 = x^2 - 2xy + y^2$, therefore $x^2 - 2xy + y^2$ is positive, and therefore $x^2 + y^2 > 2xy$.

In order to emphasize the extreme importance of limiting values, I have allowed a fallacy to pass unchallenged in this argument. It is not true that $x^2 + y^2 > 2xy$. For, although $(x - y)^2$ is positive, it may, if $x = y$, be zero, when $x^2 + y^2 = 2xy$; consequently the true statement is

$$x^2 + y^2 \text{ not } < 2xy.$$

(II.) *Prove:* The sum of a positive fraction and its reciprocal is not less than 2.

Consider $$\frac{x}{y} + \frac{y}{x} \text{ not } < 2. \quad (1)$$

Multiplying each member by xy gives

$$x^2 + y^2 \text{ not } < 2\,xy \quad (2)$$

But (2) has just been proved; therefore its equivalent inequality, (1), is true.

(III.) *Prove:* Half the sum of two positive numbers is not less than the square root of their product.

Consider $$\frac{x+y}{2} \text{ not } < (xy)^{\frac{1}{2}} \quad (1)$$

Squaring gives $$\frac{x^2 + 2\,xy + y^2}{4} \text{ not } < xy. \quad (2)$$

But, by Ex. I, $x^2 + y^2$ not $< 2\,xy$;
therefore $x^2 + 2\,xy + y^2$ not $< 4\,xy$;

therefore (2), and therefore its equivalent inequality (1), is true.

This proposition is readily generalized by the reasoning called "mathematical induction,"* by showing that if it is true for any number of numbers, it is true for one more: — but it is true for two, therefore for three, and so on. Thus we prove for n numbers†: —

$$\frac{a+b+c+\ .\ .\ .}{n} \text{ not } < (abc\ .\ .\ .)^{1/n}.$$

---

* Not true and proper induction, but absolutely cogent deduction, involving no assumption except the *validity of reason*, the postulate of all thought.

† The left-hand member is called the "arithmetic mean," and the right, the "geometric mean," of the n numbers.

A *maximum* of a function does not mean its *greatest possible,* nor a minimum its *least possible,* value. A maximum value of a function is a value toward which it increases, and from which it decreases as the variable continuously varies, whether by increasing or decreasing. And a minimum value of a function is a value before which the function decreases, and after which it increases as the variable varies continuously, whether by increasing or decreasing. Maxima and minima for a function may repeat, definitely or indefinitely; or there may be only one maximum or one minimum for a function, in which case the maximum is the greatest possible, or the minimum the least possible value.

The general connection between inequalities and the theory of maxima and minima values of functions is exemplified in the principle, that if $\phi\,(x, y, z, \ldots)$ and $\psi\,(x, y, z, \ldots)$ be two functions of the same variables such that

$$\phi\,(x, y, z, \ldots) = N, \tag{1}$$

and $$\psi\,(x, y, z, \ldots) \text{ not } > \phi\,(x, y, z, \ldots); \tag{2}$$

and if any values of $x, y, z, \ldots$, say, $a, b, c, \ldots$, can be found which satisfy (1) and at the same time make (2) an *equation*, then $\psi\,(a, b, c, \ldots)$ is a maximum value of $\psi\,(x, y, z, \ldots)$.

Also, if $$\psi\,(x, y, z, \ldots) = N, \tag{3}$$

and $$\phi\,(x, y, z, \ldots) \text{ not } < \psi\,(x, y, z, \ldots); \tag{4}$$

and if any values of $x, y, z, \ldots$, say, $a, b, c, \ldots$, can be found which satisfy (3) and simultaneously make (4) an equation, then $\phi\,(a, b, c, \ldots)$ is a minimum value of $\phi\,(x, y, z, \ldots)$.

Example. — Find the maximum volume of a rectangular parallelopiped of given surface, and minimum surface for given volume.

Let x, y, and z be the *lengths* of three adjacent edges; then the geometrical data of the problem are, the *area* of the surface is $2(xy + xz + yz)$, and the *volume* of the solid is xyz. (*Vide* § **25**.) Writing $u = xy$, $v = xz$, $w = yz$, the area becomes $2(u + v + w)$, and the volume, $\sqrt{uvw}$.

Hence the analytical problem is to find the maximum (or maxima) of the function $\sqrt{uvw}$, given the function $2(u + v + w) =$ a constant. Since the meaning of the problem excludes negative number (*vide* § **331**), the problem as assigned is equivalent to finding the maxima of $\sqrt[3]{uvw}$ given, $\frac{u+v+w}{3} = k$; for (considering only positive protomonic numbers, all that apply to the problem) $\sqrt{uvw}$ is maximum for the same values of the variables that $\sqrt[3]{uvw}$ is maximum; and given $2(u + v + w) =$ a constant, we have $\frac{u+v+w}{3} = k$. But this transformation was adopted because we know (Ex. III, above) that

$$\frac{u+v+w}{3} \text{ not } < (uvw)^{\frac{1}{3}},$$

which is to say that

$$(uvw)^{\frac{1}{3}} \text{ not } > \frac{u+v+w}{3},$$

Consequently we have

$$\frac{u+v+w}{3} = k, \qquad (1)$$

and

$$(uvw)^{\frac{1}{3}} \text{ not } > \frac{u+v+w}{3}. \qquad (2)$$

It only remains to find values of u, v, w which satisfy (1) and make (2) an equation. But (2) cannot be an equation unless $u = v = w$. This, therefore, is the condition, and $(uvw)^{\frac{1}{3}}$ is uniquely maximum when $u = v = w$; and (remembering the meaning of u, v, and w), if $u = v = w$, then $x = y = z = \left(\frac{K}{6}\right)^{\frac{1}{2}}$, where $K =$ the given area.

The reciprocity, implicit in the theorem immediately preceding this example, gives the same condition ($x = y = z$) for the solution of the second part of this problem; but the beginner may have failed to note the reciprocal relations of the conditions for maxima and minima of two functions displayed in the general investigation.

In like manner, then, the second part of the problem gives

$$(uvw)^{\frac{1}{3}} = l, \tag{1}$$

and

$$\frac{u + v + w}{3} \text{ not } < (uvw)^{\frac{1}{3}};$$

whence, $\frac{u + v + w}{3}$ is uniquely minimum when $u = v = w$, and therefore, as before, the area is minimum when $x = y = z = (L)^{\frac{1}{3}}$, where L is the given volume.

APPENDIX.

PEDAGOGICAL NOTE.

THE primary concept of number is the same in all men, and the conception could not be obstructed, even if teachers set themselves to thwart it. As an original question, therefore, there is little pedagogical import in discussion of methods of stimulating the infant mind to definite specialization of various *manys*. (*Vide* § 2.)

It would be enough to point out to the inexperienced teacher that when the time for definite and systematic specialization of *manys* comes, a child can learn the general system as a whole better than he can learn it piecemeal; that the so-called arithmetic of the first two or three grades in our schools is properly a matter of language, a matter of naming, in the manner of the child's linguistic environment, universal concepts already attained by the young innocents when committed to the mercies of the primary school. There is no more sense in attempting to explain *what* "twelve" is, than in making a like effort in regard to "time" or "space," or such concepts as "more," "less," "greater," "equal." The child really knows these things as well as his teacher. Even if a child lived eight years in an English-speaking society without learning the English name for the special *many*, "twelve," or even without having definitely recognized it, the substance of the thought, as distinguished from the symbolism of a particular language, would nevertheless be familiar to him, and nearly as well known as it can be until one gives profound study to epistomology.

The simple and easily taught subjects of counting, and the elementary phases of numerical operations, have been confused by the inane verbosity of pedagogical writers. In his admirable *Philosophy of Education* * (one of the best books ever written on the

* Translated in the *International Education Series.*

subject) Rosenkranz justly remarks, "Treatises written upon it [*education*] abound more in shallowness than any other literature. Shortsightedness and arrogance find in it a most congenial atmosphere, and uncritical methods and declamatory bombast flourish as nowhere else."

It is enough to point out one example of injurious methods of dealing with imaginary difficulties. Ignorance of psychology and lack of common-sense have led many superintendents, even where the minimum school age is eight years, to prohibit all mention of numbers greater than *ten* in the "first grade," and greater than *twenty* in the "second." This makes both the teaching and the learning a sham, and the nemesis of all dishonesty dogs it. It is benumbing to honest, depraving to vain or deceitful, pupils. I know a city whose school superintendent has instituted such methods with fatuous braggadocio, where a visitor, after witnessing an hour's counterfeit teaching, — What is one and two? one and three? two and three? If you had five apples, and gave one to Mary and one to John, how many would you have left? and so forth, with occasional introduction of such prodigious numbers as nine or ten, — followed the class to the playground. It was the season of *hully-gull*. Each urchin knew well the score of the treasures in his bulging pockets. "Hully-gull, hand-full, how many?" challenged one young plunger. "Twenty-two," guessed his opponent. One second for the count and the subtraction, and back came the triumphant cry, "Give me seven to make it twenty-two!"

On the other hand, there seem to be peculiar difficulties, even for adults, in attaining the concept of number absolutely essential to comprehension of arithmetic, — the discernment of number as a continuous magnitude with fractional parts and qualitative distinctions termed positive and negative. Here pedagògical devices are sorely needed. It is not enough to warn against mistaken interference; the teacher's skill will be taxed to the utmost to stimulate the minds he is guiding to develop concepts of a high order of abstraction, and such as, left to himself, the pupil would never form at all.

As "object-lessons" to young children — the aim being to clear up normal and universal concepts of quantity — presentation of yard-sticks and foot-rules, gallon and quart measures, etc., may be a useful practice, and it does teach about fractions; but it does *not* at all immediately suggest fractions of numbers. A fraction of a line is a line, of a solid is a solid ; and these can be and universally are discerned under the primary concept of number, and without discernment of numerical fractions. Every savage knows that a quart is a fraction of a gallon. The "object-lessons" mentioned really constitute an elementary discipline in geometry (if every primary school exercise must be labelled with the name of some science). Lines and solids are spacial entities, and contemplation of their relations is primarily a geometrical exercise. I say *primarily*, because any two magnitudes of the same kind have an absolute numerical relation ; but to see that a quart is one-fourth of a gallon (only another way of saying that four quarts equal a gallon) is not at all to see the number called one-fourth in the systematic terminology of arithmetic. Every child sees the former, an obvious geometric fact, — too many of his teachers have never discerned the latter. (*Vide* §§ 78–90.) The primary concept of number is universal and normal to the human mind, just as the concept of space is common and original to all men. Systematic development of the latter gives geometry, of the former gives arithmetic. The developments of the one are quite as much matters of fact as the developments of the other.

Ontological definition of number is as little to be required of arithmetic as like definitions of space of geometry, or of matter and force of physics. Each science simply takes its respective common notions, which it develops according to inherent characteristics. The developed science always casts light back upon primary notions (*Cf.* the effect of Non-Euclidean geometries upon native ideas of space, or the exigencies of dilemmas in physics upon naïve concepts of matter); but no such questions are to be raised for young students beginning to study arithmetic, geometry, or physics. The most important maxim for wise teaching in any science is never to set delimitations which confine development and entomb thought in empiricism, — never to clip the growing tree at the top.

Now every man (and every dog) knows that one side of a triangle is less than the sum of the other two sides ; but no one would suppose that this circumstance entitled every man to opinions concerning the conclusions of geometry ; yet similar presumptions are rife among teachers of arithmetic. Men possessing (in common with their most savage brethren) only the primary concept with which arithmetic begins, often misrepresent as matter of convention or symbolic jugglery the arithmetical conclusions that number is a continuous magnitude, with fractional parts, and qualitative distinctions — as much matters of fact as any conclusions of geometry.

The developments of the number concept are undreamed of to the man whose only thought thereof is his abstraction from a flock of sheep or pile of coins. As soon as man's energetic and organizing thought develops this concept, the insight is infallibly attained that number is a continuous magnitude, not concrete, not material, but none the less real.

The concept which appears to me most like the first development of primary number, which includes all ratio (including fractions), and the qualitative distinctions, positive and negative, is Time. Even children recognize time as a continuous magnitude, — as more or less; that of two times one must be definitely greater than, equal to, or less than the other, — and the qualitative distinctions of past and future. The analogy of *present* and *zero* is also perfect.

I suggest that teachers, called upon as they always are to teach arithmetic to children somewhat too young for the reasoning and insight required, would do well, in attempting to stimulate the conceptual energies of their pupils, to use definite times rather than lines, surfaces, solids, etc., in illustrating numerical relations subsisting between any two magnitudes of the same kind. Although no better success can be assured in this way (for any fraction or part of a time is a time and not a number); yet from the very fact that times cannot be seen or handled, the abstracting functions of the mind are brought into play, and there is better ground of hope that the desired conception will take place than if objects of sense-perception had been presented. It may be well to remark in this connection that in all illustration great care is demanded lest the analog hide instead of revealing. Rosenkranz, in his valuable *Philosophy of Education*, already referred to, wisely cautions : "Our age inclines at present to the superstition that man is able, by means of simple sense-perception, to attain a knowledge of the essence of things, and thereby dispense with the trouble of thinking. It is vain to try to get behind things, or to comprehend them, except by thinking."

I am not aware that the suggestion has been made hitherto ; but, in the light of the above warning against abuse, I am convinced that teachers of arithmetic would do well to contemplate the similarity of the concepts Time and Number, as the latter is conceived, not in the savage stadium of thought, but in its first scientific development.

There are many subsidiary advantages also in choosing the universally conceived magnitude, time, for such illustrations. The mind is unconsciously but directly led from the tyranny of material categories of thought; and the human mind, once made sensible of its powers, will never again suffer its conceptions to be shackled in this native slavery of the race.

Rightly employed, arithmetic might be used with more efficacy in the intellectual emancipation, which is one of the chief ends of education, than any subject in the curricula of common schools. There is no other field where one pure idea is developed in such unbroken consistency, and such freedom from involvement in complex relations with foreign elements.

In conclusion, no matter whether the pupil at a given stage be in a position to see the end of his studies or not, it is evident that the teacher, with no notion of the end, will be a faulty guide, since he leads he knows not whither.

INDEX.

The numbers refer to sections.

ERRATA.

PAGE 52.

2d line from bottom: instead of $b - d$ read b / d.

PAGE 83.

Top line: instead of "point to the vertices" read *points to the opposite vertices.*

PAGE 105.

13th line from bottom: sign of equality is omitted in latter portion of the line.

PAGE 115.

7th line from bottom: read § **198**.

PAGE 165.

3d line: read $x - y$ in denominator.

PAGE 174.

11th line: read $d_3 r^{-3}$.

PAGE 180.

7th line: read $\frac{x_2}{r} + \frac{x_3}{r^2}$.

PAGE 203.

11th line: instead of $1 - 6$ in second denominator, read $1 - x$.