Chambers

Saml. A. Chambers

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APPLETONS' MATHEMATICAL SERIES

NUMBERS SYMBOLIZED

AN

ELEMENTARY ALGEBRA

BY

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PREFACE.

The aim of this volume is to lay the foundation for a more extensive and philosophical treatise soon to follow, and to aid in supplying the needs of the common, high, normal, and other preparatory schools and academies, where the time allotted to this department of knowledge is necessarily limited to an elementary treatise.

In scope it includes all subjects essential to a study of higher arithmetic, elementary geometry, and the elements of physics. All matter, however, is treated in an elementary manner, so that any ordinarily intelligent student, with a fair knowledge of the principles of common-school arithmetic, may master it. All broad generalizations and discussion of general problems have been purposely excluded.

In the earlier lessons, fundamental ideas and principles are developed inductively, and then formulated into as simple and concise statements as is consistent with truth. Further on, definitions appear at the beginning of subjects, and principles are deduced from the solutions of characteristic examples. And still later, noticeably in proportion, propositions are first enunciated and then logically proved. Thus, the pupil is led by easy transition from the more elementary forms of reasoning to pure mathematical demonstration.

In numerous instances, after deducing one or more principles, I have introduced selections of easy examples to be worked at sight. These are intended to give opportunity for the application of the principles under which they appear, and to cultivate in the student a quick perception of letter, exponent, sign, and factor.

An unusually large number of examples for written work are distributed throughout the book. These have been selected with special reference to the class of pupils for whom the work is intended. They are arranged for two readings. At the first reading it is recommended that all miscellaneous examples, which are generally more difficult than the others, shall be omitted. These, in connection with a review of the definitions and principles, will form a good second reading. Long, pointless examples, requiring much time and labor in their solution, have been generally avoided.

The rather extensive treatment of factoring, and the preparation provided for it by the introduction of a partial treatment of involution, a treatise on composition, and one on exact division, it is believed will be commended by teachers generally. No one can expect to make much progress in the study of algebra who is not somewhat of an adept in factoring.

The early introduction of the equation, and the frequent return to it, are features so well adapted to practical work that comment upon their merits is unnecessary.

The simplicity of the treatment of generalization and specialization, negative solutions, inequalities, binomial surds, and limiting ratios, is a sufficient excuse for their introduction into an elementary treatise on algebra. These subjects may, however, be omitted where a shorter course is desirable, without doing violence to the logic of other parts.

In conclusion, I desire to express my deep obligations to my wife, Annie M. Sensenig, whose experience as teacher has been nearly coextensive with mine, and from whom I have received many practical helps and encouragements in the preparation of this work.

I am also greatly indebted to Prof. A. J. Rickoff, of New York, for a careful examination of the manuscript before publication, and for many practical hints obtained through his criticisms.

David M. Sensenig.

Normal School, West Chester, Pa.,
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INTRODUCTION.

LITERAL QUANTITIES—IDEAS AND EXPRESSION.

EXERCISE 1.

1. What is the sum of 2 units, 3 units, and 4 units? 2 tens, 3 tens, and 4 tens? 2 fives, 3 fives, and 4 fives? What, then, is the sum of 2 times any number, 3 times that number, and 4 times that number?

2. If we let a stand for any number, what will be the sum of 2 times a, 3 times a, and 4 times a? 3 times a, 4 times a, and 6 times a?

Two times a is written $2a$, and is read *two a*; three times a is written $3a$; etc.

3. What is the sum of $4a$, $5a$, and $6a$? $8a$, $4a$, and $7a$?

4. If we let b stand for any number, what will be the sum of $4b$, $3b$, and $2b$? $5b$, $4b$, and $6b$?

In algebra, any letter may stand for any number.

5. What is the sum of $3b$, $4b$, and $2b$, if b stands for 3? If b stands for 4?

The symbol of addition is $+$, read *plus*.

6. What is the sum of $2m + 3m + 5m$? What when m equals 2? When m equals 5?

The symbol $=$ is read *equals*.

7. What is the value of $3x + 4x + 6x$? What when $x = 3$? When $x = 6$?

8. $5x + 2x + 3x =$ what? $4n + 5n + n =$ what?

---

9. What is the difference between 8 tens and 3 tens? 8 20's and 3 20's? 8 times any number and 3 times that number? What, then, is the difference between $8a$ and $3a$? $8m$ and $3m$?

The symbol of subtraction is $-$, read *minus.*

10. What is the value of $15x - 7x$? $12y - 5y$?

11. What is the value of $12a - 7a$? What when $a = 3$? When $a = 7$?

12. What is the value of $6a + 5a - 7a$? What when $a = 5$? When $a = 8$?

---

13. What is the value of a times b when $a = 3$ and $b = 4$? When $a = 5$ and $b = 7$?

The symbol of multiplication is $\times$, read *times.*

14. What is the value of $x \times y$ when $x = 6$ and $y = 8$? When $x = 10$ and $y = 9$?

The product of two or more letters is expressed by writing them together without any symbol between them. Thus, $a \times b = ab$, and $a \times b \times c = abc$.

15. What is the product of $m \times n$? $p \times q$? $x \times y \times z$?

16. What is the product of $p \times q \times r$? What when $p = 2$, $q = 3$, and $r = 4$? When $p = 3$, $q = 4$, and $r = 5$?

17. What is the value of $2a \times 3b$, when $a = 5$ and $b = 7$? When $a = 6$ and $b = 3$?

---

18. What is the quotient of x divided by 5 when $x = 10$? When $x = 15$? When $x = 30$?

19. What is the value of a divided by b when $a = 15$ and $b = 5$? When $a = 24$ and $b = 6$?

The symbol of division is $\div$, read *divided by.* Division is also expressed by writing the dividend over the divisor with a line between them. Thus, a divided by b is written $a \div b$, or $\frac{a}{b}$.

20. What is the value of $x \div y$ when $x = 15$ and $y = 5$? When $x = 63$ and $y = 9$?

21. What is the value of $\frac{m}{n}$ when $m = 12$ and $n = 3$? When $m = 18$ and $n = 6$?

22. What is the value of $\frac{ab}{c}$ when $a = 12$, $b = 6$, and $c = 9$? When $a = 10$, $b = 7$, and $c = 5$?

---

23. What two numbers multiplied together will produce 10? 15? 21? $3a$? $5x$? ay? xz?

24. What three numbers multiplied together will produce 12? 18? 30? $10a$? $5ab$? xyz?

The numbers multiplied together to produce a given number are the *factors* of that number.

25. Name the two factors of 14. 21. $5m$. cd.

26. Name the three factors of $10x$. $5ay$. pqr.

---

27. What number is produced by using 2 *twice* as a factor? Three times? Four times?

The result obtained by using a number two or more times as a factor is a *power* of the number. When the number is used twice as a factor the result is called the *square* of the number. When used three times, the *cube* of the number. When used four times, the *fourth* power of the number, etc.

28. What is the square of 3? The cube of 4? The fourth power of 2?

29. What is the square of a when $a = 4$? The cube of x when $x = 3$?

The symbol of power is a number called an *exponent*, written on the right hand above the number whose power is to be obtained. Thus, a squared is written a^2; a cubed, a^3; a fourth power, a^4, etc.

30. What is the value of x^3 when $x = 2$? When $x = 3$? When $x = 4$?

31. What is the value of $x^2 y^2$ when $x = 2$ and $y = 3$? When $x = 1$ and $y = 4$?

32. What are the factors of a^2? x^3? m^4? $x^2 y^3$?

---

33. What is one of the two equal factors of 4? 9? 16? a^2? x^2?

34. What is one of the three equal factors of 8? x^3? $27 a^3$?

One of the equal factors of which a number is composed is a *root* of the number. One of the two equal factors is the *square root;* one of the three equal factors, the *cube root;* one of the four equal factors, the *fourth root*, etc.

35. What is the square root of 16? 25? a^2? $a^2 x^2$?

36. What is the cube root of 27? 64? a^3? $a^3 x^3$?

The symbol of root is $\sqrt{\ }$, called the *radical sign.* A number called the *index* is written in the angle of the sign to show the kind of root. When no index is used the square root is expressed. Thus, $\sqrt{x}$ is the square root of x, and $\sqrt[3]{y}$ is the cube root of y.

37. What is the value of $\sqrt{x}$ when $x = 16$? When $x = 49$?

38. What is the value of $\sqrt[3]{a}$ when $a = 27$? When $a = 64$?

39. What is the value of $\sqrt{x^2}$? $\sqrt[3]{a^3}$? $\sqrt[4]{c^4}$?

40. What is the value of $\sqrt{a x}$ when $a = 4$ and $x = 9$? When $a = 2$ and $x = 8$?

41. Write the square of m; the cube of n; the product of m and n; the quotient of m and n; the square of m divided by the square of n.

In algebra, numbers expressed by figures only are called *numerical quantities;* and those expressed by letters only, or by both figures and letters, *literal quantities.* Thus, 24 is a numerical quantity, and a, x^2, and $3 b$ are literal quantities.

Kinds of Literal Quantities.

EXERCISE 2.

1. What is the sum of a and b when $a = 3$ and $b = 5$? When $a = 8$ and $b = 9$?

The sum of different literal quantities, when their values are not given, is expressed by simply writing plus between them. Thus, the sum of a and b is $a + b$, and of a, b, and c is $a + b + c$.

2. What is the sum of x and y? x, y, and z? $2a$ and $3b$? $4x$, $5y$, and $6z$?

3. What is the difference of x and y when $x = 10$ and $y = 5$? When $x = 12$ and $y = 6$?

The difference of different literal quantities, when their values are not given, is expressed by simply writing *minus* between them. Thus, the difference of a and b is written $a - b$.

4. What is the difference of m and n? $2a$ and $3b$? $5x^2$ and $7y^2$? x^3 and y^3?

The different parts of which a sum or difference is composed are called *terms*. Thus, $2a$, $3b$, and $4c$ are the terms of $2a + 3b + 4c$.

5. Name the terms in $x + y + z$. $2a + 3b + 4z$. $x - y$. $x + y - z$. $x - 2y + 3z$.

When a quantity consists of only one term it is called a *monomial;* when of two or more terms, a *polynomial.* Thus, $3ab$ is a monomial, and $2a + 3b$ and $4a - 2b + c - d$ are *polynomials.*

A polynomial of two terms is a *binomial,* and one of three terms a *trinomial.*

6. What are the values of the following binomials, when $a = 6$ and $b = 3$?

1. $a + b$	3. $a + ab$	5. $ab - b^2$	7. $a^2b - ab^2$
2. $a - b$	4. $a^2 - b^2$	6. $a^3 - b^3$	8. $2a + 3b$

7. What are the values of the following trinomials, when $x = 10$, $y = 6$, and $z = 4$?

1. $x + y + z$	4. $x - y - z$	7. $2x - y + z$
2. $x - y + z$	5. $x + 2y - z$	8. $2x + 3y + 2z$
3. $x + y - z$	6. $3x - 2y + z$	9. $5x - 3y + 2z$

When terms have the same letters affected by the same exponents they are *similar*. Thus, $3a^2b^3$, $5a^2b^3$, and $6a^2b^3$ are similar terms.

8. Arrange the following terms into groups, placing similar terms into one group:

$2ab^2$, $3a^2b$, $4ab^2$, $6a^2b$, $5a^2b^2$, $8a^2b$, $7ab^2$, $6a^2b^2$, $9a^2b$, $9a^2b^2$.

The numerical factor in a term is generally called the *coefficient* of the term, but any factor may be taken as the coefficient. When no numerical coefficient is expressed the factor 1 is understood to be the numerical coefficient.

9. Name the numerical coefficients of $3ax$, $5bc$, $6mn$, ax, $4cd$, $5x^2y^2$, a^3x^3, $25xy$, m^2n.

Sometimes terms have factors that are alike and some that are unlike; then the unlike ones are taken as the coefficients of the terms, and the terms are considered similar with respect to the like terms. Thus, axy, bxy, and cxy are similar with respect to xy. a, b, and c are the coefficients.

10. With respect to what letters are the following terms similar, and what are the coefficients of the terms?

1. ax, bx, and cx
2. cxy, dxy, and exy
3. $2ax$, $3bx$, and $4cx$
4. $2my$, $3ny$, and $4sy$

The symbol ( ), called a *parenthesis*, is used to inclose two or more terms that are to be taken together as one factor or one term. Thus, $a + b$ multiplied by c is written $(a + b)c$, and $b + c$ subtracted from a is written $a - (b + c)$.

11. Find the value of $(4a - 2b)c$ when $a = 3$, $b = 2$, and $c = 4$.

If a and b represent two numbers, what will represent

12. Their sum?

13. Their difference?

14. Their product?

15. Their quotient?

16. The square of their sum?

17. The sum of their squares?

18. The cube of their sum?

19. The sum of their cubes?

20. The product of their sum and difference?

21. Their product times their difference?

22. The quotient of their sum and difference?

Concrete Examples involving Literal Quantities.

EXERCISE 3.

1. A boy paid a cents for a slate and b cents for a book. What did he pay for both ?

Solution.—He paid for both the sum of a cents and b cents, which is $a + b$ cents.

2. I paid $2a$ cents for an apple and $3a$ cents for an orange. What was the cost of both ?

3. A man had $5a$ dollars and spent $2a$ dollars. How much money had he left ?

4. Mary bought a lemon for $3a$ cents and an orange for b cents. What did she pay for both ?

5. Thomas rode $6x$ miles and then walked $4x$ miles. How far did he go in all ?

6. Mary had $15a$ quarts of berries and sold $9a$ quarts. How many quarts had she remaining ?

7. A boy bought an apple for c cents and handed over a 10-cent piece. How much change should he receive ?

8. I bought a horse for $60 and sold it for y dollars. How much did I gain ?

9. What will be the cost of 3 chairs at x dollars apiece, and 4 tables at y dollars apiece ?

10. I bought m sheep at $6 apiece and sold them at $9 apiece. What did I gain ?

11. If 8 ropes, each b feet long, be cut from a coil containing a yards, how many feet will remain ?

12. If x acres of land are worth $1000, what is the value per acre ?

13. If 6 horses are worth $5y$ dollars, what are 10 horses worth at the same price per head ?

14. At b dollars a head, how many horses will c sheep at d dollars apiece buy ?

15. At m cents apiece, how many apples will \$1 buy?

16. At \$6 apiece, how many pigs will x dollars buy?

17. A bought a farm of m acres at n dollars an acre, and sold it at r dollars an acre. What was his gain?

18. If a bushels of wheat cost \$60, what will x bushels cost at the same price?

19. If a men can do a piece of work in m days, in how many days can b men do it?

20. A man bought a farm of a acres at x dollars an acre, and sold it at y dollars an acre. How much did he gain?

21. At m cents a pound, how many pounds of sugar are worth as much as c pounds of coffee at d cents a pound?

22. A is a years old and B is twice as old. What will be B's age 20 years hence? What was it 10 years ago?

23. A and B start from the same place at the same time and travel in the same direction. If A travels m miles a day and B n miles a day, how far apart will they be in c days?

24. What will be the cost of a rectangular piece of land x rods long and y rods wide at c dollars an acre?

25. What is the interest of a dollars for t years at r per cent?

26. A man bought a horse for p dollars and sold him at a gain of r per cent. What did he receive for him?

27. What will it cost to plaster a room a feet long, b feet wide, and c feet high at d cents a square yard?

28. A bought $4x$ bushels of clover-seed at c dollars a bushel, and sold one half of it at d dollars a bushel and the rest at cost. What did he gain?

29. A miller mixed a bushels of corn worth m cents a bushel with c bushels of oats worth n cents a bushel. What was the value per bushel of the mixture?

30. In what time will p dollars at r per cent amount to a dollars?

31. How many board feet in a plank m feet long, n inches wide, and c inches thick?

32. If A can do a piece of work in a days, what part of it can he do in c days?

33. I bought some goods at a cents a yard and sold them at b cents a yard. What was my gain or loss per cent?

34. A is a rods ahead of B, and goes c rods while B goes d rods. How many rods must B go to overtake A?

35. If A can go a mile in a minutes and B a mile in b minutes, how much will A gain on B in one hour? In c hours?

36. What is the value of a square field x rods long at m dollars an acre?

37. How much larger is a rectangular tract of land x rods long and y rods wide than a square tract z rods long?

38. What is the weight of a cubical stone a feet long if c cubic feet weigh a ton?

39. A has a garden m feet long and n feet wide. What would be the side of a square garden of equal area?

40. How many cubical blocks x inches long are equivalent to one block p feet long, q feet wide, and r feet high?

41. A rectangular field is a yards long and b yards wide. How far is it across it from corner to corner?

42. A ladder c feet long reaches to the top of a tower a feet high. How far is the foot of the ladder from the base of the tower?

43. A bought a horse for x dollars and sold him to B at a gain of x per cent, who again sold him to C at a gain of x per cent. What did B gain?

Positive and Negative Quantities.

EXERCISE 4.

1. Does money gained in business increase or diminish one's capital? Money lost has what effect?

2. Distance traveled in the direction of one's destination has what effect upon one's journey? Distance traveled in the opposite direction has what effect?

3. Power applied to assist a moving cart has what effect upon the moving force of the cart? Power applied to retard it has what effect?

Quantities that have directly opposite tendencies in a mathematical calculation are called *positive* and *negative*.

Illustrations.—1. If gains be considered positive, then losses will be negative. If losses be considered positive, then gains will be negative.

2. If past time be considered positive, then future time will be negative. If future time be considered positive, then past time will be negative.

3. If distance in any direction be considered positive, distance in the opposite direction will be negative.

It is customary, but not essential, to consider quantities that express favorable conditions in an example *positive*, and those that express unfavorable conditions *negative*.

4. Tell which of the following quantities are positive and which negative: John earns \$10, spends \$8, finds \$9, loses \$12, gives a poor man \$5, receives a reward of \$6.

5. Tell which of the following quantities are positive and which negative: A man deposits \$50 in bank, then "checks out" \$30, then deposits \$20, then deposits \$40, then "checks out" \$50, then deposits \$10, then "checks out" \$12.

A quantity is marked positive by writing the symbol + (plus) before it, and negative by writing the symbol — (minus) before it.

6. Write the following quantities with their proper signs: Thomas buys 8 sheep, sells 7, buys 9, sells 6, buys 5, kills 10, buys 12.

7. Write 12 positive units, 3 negative units, $5a$ positive units, $x+y$ negative units, $a+b$ positive units.

8. Write the following quantities with their proper signs: A Philadelphian bound for California travels west $2a$ miles on Monday, east $3a$ miles on Tuesday, west $5a$ miles on Wednesday, west $4a$ miles on Thursday, east $6a$ miles on Friday, west $7a$ miles on Saturday, and rests on Sunday.

9. If a man walks 10 miles in the direction of his destination, and then walks 5 miles in the opposite direction, what effect do the last 5 miles have upon the first 10?

10. If one boy pulls at a cart with a force of 20 pounds, and another holds back with a force of 12 pounds, what effect does the 12-pound force which the second boy exerts have upon the 20-pound force exerted by the first boy?

11. If a man gains \$15 in one transaction and loses \$25 in another, what effect does the gain have upon the loss?

Positive and negative quantities tend to destroy each other when combined in an operation, and hence are said to be *opposed to each other in character.*

12. Which is the more favorable condition, to be merely penniless or to be in debt \$10? To rest or to go 6 miles in the opposite direction from one's destination? To be idle or to lose \$20 in business?

A negative quantity is sometimes regarded as less than zero.

13. Which is the more favorable condition, to owe \$5 or to owe \$10? To lose 10 sheep or to lose 20 sheep? To go 8 miles or 15 miles in a wrong direction?

Of two negative quantities, that is considered the greater which has the less number of units.

14. One boy helps a cart along with a force of 12 pounds and another retards it with a force of 8 pounds. Write the combined effect of these forces upon that of the cart.

15. How many and what kind of units are there in $+7$? -6? $+a$? $-b$? $+3a$? $-2b$? $+a^3$? $-b^3$?

16. A miller bought 80 bushels of oats and sold 95 bushels in one day. Write the combined effect of these transactions upon the amount of oats on hand.

17. A earns a dollars and spends b dollars. Write the combined effect of these transactions upon his finances

1. When a is greater than b.
2. When a is less than b.

18. A man earned x dollars one day and y dollars another. Write the combined effect of the two days' wages upon his finances.

19. A man spent a dollars at one time and b dollars at another time. Write the combined effect of these transactions upon his finances.

20. A land-holder buys a tract of land a rods long and b rods wide. Write the effect of this transaction upon the amount of land he owns.

Definitions.

1. A ***Unit*** is a single thing.

2. One or more units of a kind is a ***Number.***

3. A definite number of units is a ***Specific Quantity;*** as, *seven* birds.

4. An indefinite number of units is a ***General Quantity;*** as, a *flock* of birds.

5. A number expressed by figures only is a ***Numerical Quantity;*** as, 125.

6. A number expressed by letters, or figures and letters, is a ***Literal Quantity;*** as, x and $5x$.

7. Numbers opposed to each other in character are distinguished by the symbols + (plus) and − (minus), and are called ***Positive*** and ***Negative Quantities.***

Note.—For complete definitions, see pages 298 and 299.

CHAPTER I.

INTEGRAL QUANTITIES.

Algebraic Addition.

EXERCISE 5.

1. A man earned $5 one day, $4 the next, and $7 the next. What was the combined effect of these earnings upon his finances?

Solution.—Since earnings increase his money, we mark each earning positive. The whole increase is evidently the sum of $5, $4, and $7, which is $16, which we mark positive.

Form.

+	$5
+	4
+	7
+	$16

2. One boy helps a cart along with a force of 16 pounds, another with a force of 20 pounds, and another with a force of 25 pounds. What is the combined effect of these forces upon that of the cart?

3. A miller sold 5 bushels of oats to one man, 6 bushels to another, and 9 bushels to another. What was the combined effect of these transactions upon the amount of oats on hand?

Solution.—Since oats sold diminishes the amount on hand, we mark each quantity negative. The whole decrease is evidently the sum of 5 bu., 6 bu., and 9 bu., or 20 bu., which we mark negative.

Form.

−	5 bu.
−	6 "
−	9 "
−	20 bu.

4. A man sold 10 cows one day, 15 the next, and 20 the next. What was the combined effect of these transactions upon the number in his herd?

5. A man earns \$5, then spends \$3, then earns \$7, then spends \$6, then earns \$8, then spends \$9. What is the combined effect of these transactions upon his finances?

Solution.—We mark all incomes positive, and all outlays negative. The sum of the incomes is \$20, which we mark positive. The sum of the outlays is \$18, which we mark negative. Now, an outlay of \$18 will destroy an income of \$18, or − \$18 will destroy + \$18, and there will remain an income of \$2, which we mark positive.

Form.

$$\begin{array}{rr} + \$5 & - \$3 \\ + 7 & - 6 \\ + 8 & - 9 \\ \hline + \$20 & - \$18 \\ - 18 & \\ \hline + \$2 & \end{array}$$

Remark.—A negative quantity will destroy a positive quantity of the same number of units when combined with it.

6. Six men push at a moving car. A pushes forward 80 pounds, B backward 90 pounds, C forward 100 pounds, D backward 95 pounds, E backward 110 pounds, and F forward 85 pounds. What is the combined effect of these forces upon that of the car?

7. A drover adds $2a$ sheep to his flock, then sells $3a$, then buys $4a$, then sells $3a$, then buys $5a$, then sells $3a$, then buys $6a$. What is the combined effect of these transactions upon the number in his flock?

Combining algebraic quantities is called *adding* them.

8. Find the sum of $+3a$, $-4a$, $+6a$, $-5a$, $-3a$, and $+2a$.

Solution.—The sum of the positive quantities is $+11a$, and the sum of the negative quantities is $-12a$. If we combine $11a$ positive units with $12a$ negative units, they will destroy $11a$ negative units, and a negative units, or $-a$, will remain.

Remark.—Equal positive and negative quantities may be omitted in addition, since they *destroy* each other.

Form.

$$\begin{array}{rr} +3a & -4a \\ +6a & -5a \\ +2a & -3a \\ \hline +11a & -12a \\ & +11a \\ \hline & -a \end{array}$$

9. Find the sum of $+2$, -3, $+4$, -5, and $+7$.

10. Find the sum of $+2a$, $+3a$, $-4a$, $-5a$, $+7a$, $-6a$, and $+3a$.

When no sign is written before an algebraic quantity, + is understood.

11. Find the sum of $3x$, $-4x$, $7x$, $-5x$, and $3x$.

12. Find the value of $+2m + (+3m) + (-2m) + (-5m) + (-7m) + (+6m) + (-m)$.

Definitions.

8. The result obtained by combining two or more quantities without regard to their character as positive or negative, is the ***Arithmetical Sum*** of the quantities.

9. The result obtained by combining two or more quantities with regard to their character as positive or negative, is the ***Algebraic Sum*** of the quantities.

Illustration.—If a man goes 10 miles in the direction of his destination and 4 miles in the opposite direction, the entire distance traveled, the *arithmetical sum,* is 14 miles; but the distance he advanced on his journey, the *algebraic sum,* is only 6 miles.

10. The process of finding the algebraic sum of two or more quantities is ***Algebraic Addition.***

Principles and Applications.

1. Find the sum of $+2a$, $+3a$, and $+4a$; also the sum of $-2a$, $-3a$, and $-4a$.

Solution.—1. The sum of $2a$ positive units, $3a$ positive units, and $4a$ positive units is evidently $9a$ positive units. Therefore, the sum of $+2a$, $+3a$, and $+4a$ is $+9a$.

2. The sum of $2a$ negative units, $3a$ negative units, and $4a$ negative units is $9a$ negative units. Therefore, the sum of $-2a$, $-3a$, and $-4a$ is $-9a$.

Forms.

$$\begin{array}{r} +2a \\ +3a \\ +4a \\ \hline +9a \end{array} \qquad \begin{array}{r} -2a \\ -3a \\ -4a \\ \hline -9a \end{array}$$

Therefore,

Principle 1.—*The algebraic sum of two or more similar terms with like signs equals their arithmetical sum with the same sign.*

SIGHT EXERCISES.

Name at sight the sum of the following quantities:

1.	**2.**	**3.**	**4.**	**5.**	**6.**
$+2a$	$-5x$	$-6y$	$+9z$	$+11a^2$	$-9ab$
$+3a$	$-7x$	$-8y$	$+8z$	$+\ 7a^2$	$-6ab$

7.	**8.**	**9.**	**10.**	**11.**	**12.**
$+2a^2$	$-6x^3$	$-4ab$	$+\ 6b$	$-5ax^2$	$+6bx$
$+3a^2$	$-5x^3$	$-6ab$	$+\ 8b$	$-7ax^2$	$+8bx$
$+5a^2$	$-7x^3$	$-7ab$	$+10b$	$-9ax^2$	$+7bx$

13. $+3a+(+4a)+(+6a)$

14. $-6x+(-5x)+(-2x)$

15. $+5x^2+(+6x^2)+(+3x^2)$

16. $-5m^3+(-7m^3)+(-3m^3)$

---

2. Find the sum of $+5a$ and $-2a$; also the sum of $-5a$ and $+2a$.

Forms.

$$\begin{array}{r} +5a \\ -2a \\ \hline +3a \end{array} \qquad \begin{array}{r} -5a \\ +2a \\ \hline -3a \end{array}$$

Solution.—1. If $2a$ negative units be combined with $5a$ positive units, they will destroy $2a$ positive units, and $3a$ positive units will remain. Therefore, the sum of $+5a$ and $-2a$ is $+3a$.

2. If $2a$ positive units be combined with $5a$ negative units, they will destroy $2a$ negative units, and $3a$ negative units will remain. Therefore, the sum of $-5a$ and $+2a$ is $-3a$.

Therefore,

Prin. 2.—*The algebraic sum of two similar terms with unlike signs equals their arithmetical difference with the sign of the greater.*

SIGHT EXERCISES.

Name the sum of the following quantities:

1.	2.	3.	4.	5.	6.
$+2a$	$+6a$	$-5a$	$-7x$	$-9x^2$	$+3ab$
$-5a$	$-3a$	$+6a$	$+2x$	$+7x^2$	$-8ab$

7.	8.	9.	10.	11.	12.
$-2x^3$	$-5xy$	$-10x^4$	$+12x^3$	$-5mn$	$+3(a+b)$
$+8x^3$	$+8xy$	$+10x^4$	$-7x^3$	$+12mn$	$-6(a+b)$

13. $-5x^2+(+2x^2)$
14. $+7xy+(-3xy)$
15. $+x^2y^3+(-x^2y^3)$
16. $-3p^2q^2+(+6p^2q^2)$

---

3. Find the sum of $+a$, $+b$, and $-c$.

Solution.—If b positive units be added to a positive units, the sum will be $a+b$ positive units; if now to $a+b$ positive units c negative units be added, they will destroy c positive units, and $a+b-c$ positive units will remain. Therefore, the sum of $+a$, $+b$, and $-c$ is $+(a+b-c)$, or simply $a+b-c$, the positive sign being understood.

Form.

$$\begin{array}{c} +a \\ +b \\ -c \\ \hline +(a+b-c) \text{ or} \\ a+b-c \end{array}$$

Remark.—If c were numerically greater than $a+b$, the sum would be $c-(a+b)$ negative units, which, as will be learned in subtraction, would still be $a+b-c$.

Therefore,

Prin. 3.—*The algebraic sum of two or more dissimilar terms equals a polynomial composed of those terms.*

SIGHT EXERCISES.

Name the sum of the following quantities:

1. a, $+3b$, and $-2c$
2. $2x$, $-4y$, and $+3z$
3. $5z^2$, $-7y^2$, and $-6x^2$
4. $2x+(-3y)+(-4z)$
5. $7z^2+(+2z)+(-5)$
6. $8p^3+(+9q^3)+(-7r^3)$

Problem 1. To add similar monomials.

Illustration.—Find the sum of $+3a$, $-4a$, $+6a$, $-5a$, $-3a$, and $+2a$.

Solution.—The sum of the positive quantities is $+11a$ [P. 1], and the sum of the negative quantities is $-2a$ [P. 1]. Now, the sum of $+11a$ and $-12a$ is $-a$ [P. 2].

Remark.—If preferred, explain as on page 14.

Suggestion to Teacher.—Require pupils to recite principles whenever reference is made to them in solutions. Do not demand the numbers of principles.

Form.

$+3a$	$-4a$
$+6a$	$-5a$
$+2a$	$-3a$
$+11a$	$-12a$
	$+11a$
	$-a$

EXERCISE 6.

Find the sum of the following columns:

1.	2.	3.	4.
$+2a$	$-6x$	$+2xy$	$-3ab$
$+5a$	$-7x$	$-7xy$	$+4ab$
$+6a$	$-8x$	$+5xy$	$-6ab$
$+7a$	$-4x$	$-9xy$	$+ab$

5.	6.	7.	8.
$3ax$	$-5x^2y$	$-5mn$	$-8pq$
$-5ax$	$7x^2y$	$-6mn$	$7pq$
$-4ax$	$-6x^2y$	$7mn$	$8pq$
$+6ax$	$-9x^2y$	$9mn$	$-9pq$
$+2ax$	$+7x^2y$	$-8mn$	$-pq$

9.	10.	11.	12.
$3(a+b)$	$-3(m-n)$	$6(x^2+y^2)$	$-5(x+y)^2$
$-4(a+b)$	$7(m-n)$	$8(x^2+y^2)$	$-7(x+y)^2$
$6(a+b)$	$-6(m-n)$	$-7(x^2+y^2)$	$+6(x+y)^2$
$2(a+b)$	$8(m-n)$	$-5(x^2+y^2)$	$+8(x+y)^2$
$-5(a+b)$	$-4(m-n)$	$-3(x^2+y^2)$	$-10(x+y)^2$

13. Add $4a^2b^2$, $-7a^2b^2$, $8a^2b^2$, $-6a^2b^2$, and $12a^2b^2$.

14. Add $-6xyz$, $-9xyz$, $7xyz$, $-4xyz$, and $8xyz$.

15. Add $3(a-m)$, $-7(a-m)$, $6(a-m)$, and $-8(a-m)$.

Note.—When quantities are written in succession, separated by positive and negative signs, their sum is intended. Thus, $3a-5a+7a-4a=(3a)+(-5a)+(+7a)+(-4a)$.

16. Collect into one quantity $7a-4a+5a-6a+7a-a$.

17. Collect $9b-7b+6b-9b+8b-b-2b+3b-7b$.

18. Collect $-6ab+7ab-10ab+3ab-5ab+7ab-3ab$.

19. Collect $9m^3n^3-8m^3n^3-12m^3n^3+7m^3n^3-6m^3n^3+4m^3n^3$.

20. Collect $3(a+b)-5(a+b)+7(a+b)-6(a+b)+4(a+b)-8(a+b)+6(a+b)-5(a+b)$.

Problem 2. To add dissimilar monomials.

Illustration.—Find the sum of $3a$, $-5b$, and $+2c$.

Solution.—Since the algebraic sum of dissimilar terms equals a polynomial composed of those terms [P. 3], the sum of $3a$, $-5b$, and $+2c$ is $3a-5b+2c$.

Form.

$$\begin{array}{r} 3a \\ -5b \\ +2c \\ \hline 3a-5b+2c \end{array}$$

EXERCISE 7.

Add the following columns:

1.	2.	3.	4.	5.
x	$2x$	$5a$	$-5m$	$-ab$
y	$-3y$	$-3b$	$+6n$	$+cd$
z	$4z$	$-2c$	$+7r$	$+4h$

6. Add $7ab$, $-4cd$, $5ac$, $-6bd$, and $4am$.

7. Add $-xy$, $+7yz$, $-4xz$, $+9my$, and $-6nx$.

8. Collect $-4ab+7xy-3ab-2xy+4ab$ and $-xy$.

Suggestion.—Collect first the similar quantities, then combine the dissimilar sums.

9. Collect $6a+4b-3a+2b-5a+7b+6a-5b$.

10. Collect $5x+4y-3z+7z-4x+5y-8y+10z$.

11. Collect $3m^2+4n^2-5mn+7m^2-n^2+6mn-4m^2-6n^2$.

12. Collect $6ax-4by+7ax-3ax+4by+5ax-15ax$.

Problem 3. To add monomials having a common factor.

Illustration.—Find the sum of axy, bxy, and $-cxy$.

Solution.—The common factor is xy. a times xy plus b times xy is $(a+b)$ times xy; and $(a+b)$ times xy added to $-c$ times xy is $(a+b-c)$ times xy [P. 3].

Form.

$$\begin{array}{r|l} a & xy \\ b & xy \\ -c & xy \\ \hline \multicolumn{2}{c}{(a+b-c)xy} \end{array}$$

EXERCISE 8.

Add the following columns:

1.	2.	3.	4.
ax	ayz	$2ay$	$2xz$
bx	$-dyz$	$3by$	$-axz$
cx	$+myz$	$-4cy$	$-bxz$

5. Add $2axy$, $3bxy$, $4cxy$, and $-dxy$.

6. Collect $anb-amb+apb-aqb+arb$.

7. Collect $axy-bxy+cxy-2axy+3bxy-4cxy$.

8. Collect $3abmy-4bcmy+6cdmy-5admy$.

9. Collect $a(c+d)+b(c+d)$.

10. Collect $a(x+y+z)+b(x+y+z)-c(x+y+z)$.

Algebraic Subtraction.

EXERCISE 9.

1. A gain of how many dollars must be added to a gain of 3 dollars to make a gain of 7 dollars? What then must be added to $+\$3$ to make $+\$7$?

2. A loss of how many dollars must be added to a loss of 3 dollars to make a loss of 7 dollars? What then must be added to $-\$3$ to make $-\$7$?

3. A loss of how many dollars must be added to a gain of 7 dollars to make a gain of only 3 dollars? What then must be added to $+\$7$ to make $+\$3$?

4. A gain of how many dollars must be added to a loss of 7 dollars to make a loss of only 3 dollars? What then must be added to $-\$7$ to make $-\$3$?

5. A gain of how many dollars must be added to a loss of 3 dollars to make a gain of 7 dollars? What then must be added to $-\$3$ to make $+\$7$?

6. A loss of how many dollars must be added to a gain of 7 dollars to make a loss of 3 dollars? What then must be added to $+\$7$ to make $-\$3$?

7. A loss of how many dollars must be added to a gain of 3 dollars to make a loss of 7 dollars? What then must be added to $+\$3$ to make $-\$7$?

8. A gain of how many dollars must be added to a loss of 7 dollars to make a gain of 3 dollars? What then must be added to $-\$7$ to make $+\$3$?

9. A loss of how many dollars must be added to a gain of 5 dollars to make neither a gain nor a loss? What then must be added to $+\$5$ to make 0?

10. A gain of how many dollars must be added to a loss of 5 dollars to make neither a gain nor a loss? What then must be added to $-\$5$ to make 0?

The quantity that must be added to one of two given quantities to make the other is the *difference* of the quantities. The process of finding the difference is *subtraction.* The quantity formed of the difference and one of the given quantities is the *minuend.* The quantity added to the difference to form the minuend is the *subtrahend.*

11. What must be added to $+3a$ to make $+7a$? What then is the difference of $+7a$ and $+3a$? Which quantity is the minuend, and which the subtrahend?

12. What must be added to $-3a$ to make $-7a$? What then is the difference between $-7a$ and $-3a$? Which quantity is the minuend, and which the subtrahend?

13. What must be added to $+4a$ to make 0? What then is the difference between 0 and $+4a$? Which quantity is the minuend, and which the subtrahend?

14. What must be added to $-5a$ to make 0? What then is the difference between 0 and $-5a$? Which quantity is the minuend, and which the subtrahend?

Definitions.

11. The ***Difference*** of two quantities is such a quantity as added to one of them will produce the other.

12. The difference of two quantities without regard to their character as positive or negative is their ***Arithmetical Difference.***

13. The difference of two quantities when regard is had to their character as positive or negative is their ***Algebraic Difference.***

Illustration.—The difference between traveling 7 miles and 4 miles, irrespective of direction, is 3 miles. This is the arithmetical difference. But the difference made in one's journey between traveling 7 miles in the direction of one's destination and 4 miles in the opposite direction, is *an increase of* 11 miles. This is the algebraic difference.

14. The process of finding the algebraic difference of two quantities is ***Algebraic Subtraction.***

15. The general problem of algebraic subtraction is: "*Given the algebraic sum of two quantities and one of them, to find the other.*"

Principle and Applications.

16. Since the difference of two quantities is such a quantity as added to the subtrahend will produce the minuend, it may readily be found in three steps, as follows:

1. Find what quantity added to the subtrahend will produce *zero.* This is evidently the subtrahend with the sign changed.

2. Find what quantity added to *zero* will produce the minuend. This is evidently the minuend.

3. The sum of the two quantities thus added is evidently the difference. Therefore,

Prin. 4. — *The algebraic difference of two quantities equals the algebraic sum obtained by adding to the minuend the subtrahend with the sign changed.*

Illustration.—Find the difference of $+5a$ and $+8a$; that is, find what quantity added to $+8a$ will produce $+5a$.

Solution.—1. If we add $-8a$ to $+8a$, we will have *zero.*

2. If we add $+5a$ to zero, we will have $+5a$.

3. Therefore, if we add the sum of $-8a$ and $+5a$, or $-3a$, to $+8a$, we will have $+5a$. Hence,

$$\text{Difference of}\left\{\begin{array}{c}+5a\\+8a\end{array}\right\}=\text{sum of}\left\{\begin{array}{c}+5a\\-8a\\\hline -3a\end{array}\right\}$$

Exercise.—Prove as in the illustration the truth of the following examples:

	1.	2.	3.	4.	5.
Minuend,	$+8a$	$-8a$	$-5a$	$+8a$	$-8a$
Subtrahend,	$+5a$	$-5a$	$-8a$	$-3a$	$+3a$
Difference,	$+3a$	$-3a$	$+3a$	$-11a$	$-11a$

17. The principle of algebraic subtraction may also be illustrated as follows :

1. Arrange positive and negative numbers as in the following scale :

−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5 +6 +7 +8 +9 +10

2. Consider the difference of two numbers, the number of units passed over in going on the scale from one of them to the other.

3. Consider units passed over in going from left to right *positive*, and from right to left *negative.*

4. To find the difference, pass from the subtrahend to *zero*, then from *zero* to the minuend, and show that the algebraic sum of these distances equals the number of units that must be passed over in going directly from the subtrahend to the minuend.

Illustration.—1. Find the difference of $+3$ and $+8$.

Solution.—From $+8$ to 0 is -8, and from 0 to $+3$ is $+3$; hence, from $+8$ to $+3$ would seem to be the sum of -8 and $+3$, or -5. This we see on the scale is true.

2. Find the difference of -3 and $+8$.

Solution.—From $+8$ to 0 is -8, and from 0 to -3 is -3; hence, from $+8$ to -3 would seem to be the sum of -8 and -3, or -11. This we see on the scale is true.

SIGHT EXERCISE.

Name the difference of the following quantities :

1.	**2.**	**3.**	**4.**	**5.**	**6.**
$+5a$	$+3a$	$-5a$	$-3a$	$+5a$	$-3a$
$+3a$	$+5a$	$-3a$	$-5a$	$-3a$	$+5a$

7.	**8.**	**9.**	**10.**	**11.**	**12.**
$-5a$	$+3a$	$-xy$	$-7x^3$	$-9xy$	$-9x^3$
$+3a$	$-5a$	$+3xy$	$+9x^3$	$-7xy$	$+9x^3$

13. $7b - (+3b)$
14. $-6x - (-8x)$
15. $8x^2 - (-9x^2)$
16. $3y^2 - (+6y^2)$
17. $-4z^2 - (-2z^2)$
18. $-6xy - (+7xy)$
19. $-2x - (+8x)$
20. $+5x - (-8x)$
21. $-5x^3 - (-10x^3)$

Problem 1. To subtract monomials.

Illustration.—

1. Find the difference of $+3ab$ and $-5ab$.

Solution:

Difference of $\left\{\begin{matrix} +3ab \\ -5ab \end{matrix}\right\}$ = sum of $\left\{\begin{matrix} +3ab \\ +5ab \end{matrix}\right\}$ [P. 4] = $+8ab$.

Form.			
2. From $-3a$	**Solution:** Minuend	=	$-3a$
take $-2b$	Subtrahend with sign changed	=	$+2b$
$2b - 3a$	Difference [P. 4]	=	$2b - 3a$

3. From axy	**Solution:** Minuend	=	$a \mid xy$
take $-bxy$	Subtrahend with sign changed	=	$+b \mid xy$
$(a+b)xy$	Difference [P. 4]	=	$(a+b)xy$

EXERCISE 10.

1. From $+7a$ take $+3a$
2. From $+6a$ take $+9a$
3. From $-9a$ take $-5a$
4. From $-3x$ take $-8x$
5. From $+7b$ take $-6b$
6. From $+3b$ take $-11b$
7. From $-9ab$ take $+7ab$
8. From $-6xy$ take $+12xy$
9. From $+5a$ take $-12b$
10. From $+b$ take $-a$
11. From $-xy$ take $-3mn$
12. Find the value of $3a^2x^2 - (4a^2x^2)$
13. Find the value of $-3m^3n^3 - (-2m^3n^3)$
14. $7x^2y^2 - (-6x^2y^2)$ = what?
15. $3xy - (+7mn) = ?$
16. $-m^2n - (-6mx) = ?$
17. $+c^2 - (-d^2) = ?$
18. $-n^2 - (-m^2) = ?$

19. From $3(a+x)$ take $4(a+x)$

20. From $5(x^2-y^2)$ take $-4(x^2-y^2)$

21. From $8(x-y)^2$ take $-8(x-y)^2$

22. From ay^2 take $-by^2$

23. $-cx^2-(-dx^2)=$?

24. From mx take n^2x

25. $2ax-(+3bx)=$?

Algebraic Multiplication.

Principles of Signs.

EXERCISE 11.

1. Five times $4a$ positive units are how many positive units? Then, $5(+4a)=$ what?

2. Five times $4a$ negative units are how many negative units? Then, $5(-4a)=$ what?

What is the value of

3. $3(+5a)$?

4. $4(-2x)$?

5. $3(+6x)$?

6. $5(-7y)$?

7. $8(+5y)$?

8. $6(-3z)$?

---

9. What is the meaning of the expression $x+3(+2b)$?

Solution: $x+3(+2b)$ denotes that 3 times $2b$ positive units are to be added to x.

10. What is the meaning of $x+3(-2b)$?

11. What is the meaning of $x-3(+2b)$?

12. What is the meaning of $x-3(-2b)$?

---

13. What is the value of $x+3(+2b)$?

Solution: $3(+2b)=+6b$; hence,
$$x+3(+2b)=x+(+6b)=x+6b \text{ [P. 3].}$$

14. Since $x+3(+2b)=x+6b$ [Ex. 8], what is the value of $+3(+2b)$? Then a positive quantity multiplied by a positive quantity will give what kind of quantity?

15. What is the value of $x-3(-2b)$?

Solution: $3(-2b)=-6b$; hence,
$x-3(-2b)=x-(-6b)=x+6b$ [P. 4].

16. Since $x-3(-2b)=x+6b$ [Ex. 15], what is the value of $-3(-2b)$? Then a negative quantity multiplied by a negative quantity gives what kind of quantity?

18. Since a positive quantity multiplied by a positive quantity gives a positive quantity [Ex. 14], and a negative quantity multiplied by a negative quantity gives a positive quantity [Ex. 16], we have,

Prin. 5.—*The product of two quantities with like signs is positive.*

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17. What is the value of $x+3(-2b)$?

Solution: $3(-2b)=-6b$; hence,
$x+3(-2b)=x+(-6b)=x-6b$ [P. 3].

18. Since $x+3(-2b)=x-6b$ [Ex. 17], what is the value of $+3(-2b)$? Then a negative quantity multiplied by a positive quantity gives what kind of quantity?

19. What is the value of $x-3(+2b)$?

Solution: $3(+2b)=+6b$; hence,
$x-3(+2b)=x-(+6b)=x-6b$ [P. 4].

20. Since $x-3(+2b)=x-6b$ [Ex. 19], what is the value of $-3(+2b)$? Then a positive quantity multiplied by a negative quantity gives what kind of quantity?

19. Since a negative quantity multiplied by a positive quantity gives a negative quantity [Ex. 18], and a positive quantity multiplied by a negative quantity gives a negative quantity [Ex. 20], we have,

Prin. 6.—*The product of two quantities with unlike signs is negative.*

Note.—Principles 5 and 6 may be stated in one, as follows: In multiplication, like signs give *plus*, and unlike signs *minus*.

SIGHT EXERCISE.

Name the products of the following quantities, reciting in each case the proper principle of signs:

1. $(+3) \times (+4)$
2. $(-3) \times (+4)$
3. $(-3) \times (-4)$
4. $(+3) \times (-4)$
5. $(+a) \times (-x)$
6. $(-a) \times (+x)$
7. $(-3) \times (+y)$
8. $(-5) \times (-7)$
9. $(-1/6) \times (-1/3)$
10. $(-2/3) \times (+3/2)$
11. $(+7/8) \times (-4/7)$
12. $(-x^2) \times (-y^2)$

Principle of Exponents, etc.

EXERCISE 12.

1. Find the product of a^4 times a^5.

Solution: $a^4 = a \times a \times a \times a$

$a^5 = a \times a \times a \times a \times a$

$\therefore a^4 \times a^5 = a \times a \times a \times a \times a \times a \times a \times a \times a = a^9.$

Find the product of

2. $a^2 \times a^3$
3. $x^2 \times x^4$
4. $x^4 \times x^3$
5. $m^5 \times m^3$
6. $a^8 \times a$
7. $a^7 \times a^2$
8. $r^5 \times r^5$
9. $y^4 \times y^8$

20. Since $a^4 \times a^5 = a^9 = a^{4+5}$ [Ex. 1], we have,

Prin. 7.—*The exponent of a factor in the product equals the sum of its exponents in the multiplicand and multiplier.*

4. Which is the greatest, $a \times bc$, $(a \times b) \times c$, or $b \times (a \times c)$

1. When $a = +3$, $b = -2$, and $c = -3$?
2. When $a = +2/3$, $b = -3/2$, and $c = -1$?

Since $a \times bc = (a \times b) \times c = b \times (a \times c)$ for any values of a, b, and c [Ex. 4], we have,

Prin. 8.—*Multiplying one factor of a quantity multiplies the quantity.*

SIGHT EXERCISE.

Name the products in the following examples in accordance with the principles of multiplication :

1. $a \times x \times y$
2. $m \times a \times b$
3. $z \times x \times n$
4. $m \times c \times a$
5. $a \times by$
6. $(+3) \times (+2y)$
7. $(+5) \times (-3x)$
8. $(-4) \times (-2c)$
9. $(-a) \times (-bc)$
10. $(-a) \times (+xy)$
11. $a \times ax$
12. $a^3 \times a^2 y$
13. $(-x^2) \times (+mx^3)$
14. $(+y^2) \times (-ny^4)$
15. $(-x^3) \times (-x^2 y^5)$
16. $3x \times 2y$
17. $(2x) \times (-3y)$
18. $(-2x) \times (+3y)$
19. $(-2x) \times (-3y)$
20. $xy \times xy$
21. $(-xy) \times (x^2)$
22. $(-x^2 y) \times (-xy^2)$

Definitions.

21. The process of taking one algebraic quantity as many times, and in such a manner, as is indicated by another, is ***Algebraic Multiplication.***

22. The quantity taken is the ***Multiplicand.***

23. The quantity which shows how many times and in what manner the multiplicand is taken is the ***Multiplier.***

Remark.—The sign of the multiplier shows in what manner the multiplicand is taken, whether additively or subtractively.

24. The result obtained by algebraic multiplication is the ***Algebraic Product.***

25. The ***Arithmetical Product*** of two quantities is their product irrespective of sign. It is the result obtained by taking one arithmetical quantity as many times as there are units in another.

Problem 1. To multiply a monomial by a monomial.

Illustration.—Multiply $+5a^3b^4c^2d$ by $-3a^2b^3c$.

Solution: Since multiplying one factor of a quantity multiplies the quantity [P. 8], $+5\times a^3\times b^4\times c^2\times d$ is multiplied by $-3\times a^2\times b^3\times c$, if $+5$ is multiplied by -3, a^3 by a^2, b^4 by b^3, c^2 by c, and d by 1. -3 times $+5$ is -15 [P. 6]; a^2 times a^3 is a^5, b^3 times b^4 is b^7, and c times c^2 is c^3 [P. 7], and 1 times d is d; hence, the product is $-15a^5b^7c^3d$.

Form.

$$\begin{array}{r} +5a^3b^4c^2d \\ -3a^2b^3c \\ \hline -15a^5b^7c^3d \end{array}$$

From the above explanation we derive the following

Rule 1.—*Take the product of the numerical coefficients, annex to it all the different literal factors used, giving each an exponent equal to the sum of its exponents in the multiplicand and multiplier.*

EXERCISE 13.

Multiply

1. $+2a$ by $+3a$

2. $-2x$ by $+4x$

3. $-5y$ by $-2y$

4. $+6m$ by $-3m$

5. $+4x^2$ by $+3x^2$

6. $-3a^3$ by $+4a^4$

7. $3a$ by $2b$

8. $5xy$ by $3xy^2$

9. $-2x^2z$ by $5xy^2z$

10. $7ab^3$ by $-3a^2b^4$

11. $-4mn^3$ by $-7am^2$

12. $3x^2y^2z$ by $-6xz^3$

13. $(a+b)^2$ by $(a+b)^3$

14. $(m-n)^4$ by $(m-n)^5$

15. $3(a-b)^3$ by $4(a-b)^2$

16. $a^2(a+b)^2$ by $a^3(a+b)^2$

Find the value of

17. $3pq\times 4mp^2\times -2rq^2$

18. $-4r^2s\times 5s^2z\times 7rz^2$

19. $7a^3b\times -2ab^3\times -a^2b^2$

20. $(2ab)(-3ab)(-ab)$

21. $(-x)(-x^2)(-x^3)$

22. $mn^2\times 3m^2n^2\times 4m^3n$

23. $(7x^2y^2)(3x^3y)(4x^3y^3)$

24. $(a+b)^3(a+b)^2(a+b)$

25. $(a-3b)^4(a-3b)^2(a-3b)^4$

26. $3a(x-y)^2(x-y)^3(x-y)^4$

Problem 2. To multiply a polynomial by a monomial.

EXERCISE 14.

1. Which is the greater, $a(b+c-d)$ or $ab+ac-ad$

1. When $a=3$, $b=4$, $c=5$, and $d=6$?

2. When $a=+4$, $b=+5$, $c=-3$, and $d=+5$?

3. When $a=+2/3$, $b=+3/4$, $c=-1\frac{1}{2}$, and $d=+6\frac{1}{4}$?

26. Since $a(b+c-d)=ab+ac-ad$ for any values of a, b, c, and d [Ex. 1], we have,

Prin. 9.—*Multiplying every term of a quantity multiplies the quantity.*

SIGHT EXERCISE.

Name the products in the following examples:

1. $x(a+b+c)$
2. $y(x+y+z)$
3. $z(x-y-z)$
4. $m^2(a^2-b^2+c^2)$
5. $-c(a-b+c)$
6. $+x^2(x^3-x^2+1)$
7. $x^3(ax^2+bx+c)$
8. $m^2(am^2-bm-m)$
9. $+4(+3x-2y+3z)$
10. $-2(-3a+6b-4z)$
11. $-5(+2x-3y+4z)$
12. $-8(-x^2-2y^2-3z^2)$
13. $ab(x+y-z)$
14. $xy(x-y+z)$
15. $2x(x^2+x-1)$
16. $-4x(x^3-x^2-x)$

WRITTEN EXERCISES.

Illustration.—Multiply $2a^2-3ab+6b^2$ by $-3ab$.

Solution: Since multiplying every term of a quantity multiplies the quantity [P. 9], we multiply each term of the multiplicand by $-3ab$, and obtain $-6a^3b+9a^2b^2-18ab^3$. Therefore,

Form.

$$\begin{array}{r} 2a^2-3ab+6b^2 \\ -3ab \\ \hline -6a^3b+9a^2b^2-18ab^3 \end{array}$$

Rule 2.—*Multiply each term of the polynomial by the monomial, bearing in mind the principles of signs.*

EXERCISE 15.

Multiply

1. $2a - 3b + 4c$ by $5a$
2. $a^2 - 2ab + b^2$ by $-3ab$
3. $5x^2 + 3xy - 2y^2$ by $5x^2y^3$
4. $7a^2x^2 - 6axy + 9ay^2$ by $2axy$
5. $x^2 - xy - y^2$ by $-2x^3y^2$
6. $3a^3b^2x + 5a^2b^3xy - 7ab^2y^2$ by $5x^3y^2$
7. $6m^4 + 5m^3 - 4m^2 + 3m - 5$ by $5m^4$
8. $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5$ by a^3b^3
9. $6x^4y - 5x^3y^2 + 7x^2y^3 - 5xy^2 + 5y^4$ by $-5a^2xy$
10. $a(x+y) + b(m+n) - c(p+q)$ by abc
11. $(a+b)x^2 - (a-b)xy + aby^2$ by $2x^2y^2$
12. $3a(x+y) - 4b(x+y)^2 - 2c(x+y)^3$ by $(x+y)^3$
13. $2x(a+b) - 3y(a+b)^2 + 4z(a+b)^3$ by $2(a+b)^2$
14. $a^2(x-y)^2 - ab(x-y)^3 + b^2(x-y)^4$ by $ab(x-y)^3$
15. $px(p+q) - qx(p+q)^2 + pq(p+q)^3$ by $pqx(p+q)^4$

Algebraic Division.

Principles of Signs.

EXERCISE 16.

1. By what algebraic number must $+4$ be multiplied to produce $+12$? Then, $+12$ divided by $+4$ will give what algebraic number? Then, the quotient of two positive quantities is what kind of quantity?

2. By what algebraic number must -4 be multiplied to produce -12? Then, -12 divided by -4 equals what algebraic number? Then, the quotient of two negative quantities is what kind of quantity?

27. Since the quotient of two positive quantities is a positive quantity [Ex. 1], and the quotient of two negative quantities is a positive quantity [Ex. 2], we have,

Prin. 10.*—The quotient of two quantities with like signs is positive.*

3. By what algebraic number must $+4$ be multiplied to produce -12? Then, -12 divided by $+4$ equals what algebraic number? Then, the quotient of a negative quantity divided by a positive quantity is what kind of quantity?

4. By what algebraic number must $+4$ be multiplied to produce $+12$? Then, $+12$ divided by -4 equals what algebraic number? Then, the quotient of a positive quantity divided by a negative quantity is what kind of quantity?

28. Since a negative quantity divided by a positive quantity gives a negative quantity [Ex. 3], and a positive quantity divided by a negative quantity gives a negative quantity [Ex. 4], we have,

Prin. 11.*—The quotient of two quantities with unlike signs is negative.*

Note.—Principles 10 and 11 may be stated in one, as follows: *In division, like signs give plus and unlike signs minus.*

SIGHT EXERCISE.

Name the quotients in the following examples:

1. $(+12) \div (+3)$

2. $(-18) \div (+6)$

3. $(+24) \div (-4)$

4. $(-36) \div (-6)$

5. $(+xy) \div (+x)$

6. $(-xy) \div (-y)$

7. $(-xy) \div (+x)$

8. $(+xy) \div (-y)$

9. $(+2/3) \div (-1/2)$

10. $(-2\,1/2) \div (+3/4)$

11. $(+6\,1/4) \div (+3\,1/8)$

12. $(-37\,1/2) \div (-6\,1/4)$

13. $(+8x) \div (+4)$

14. $(-8x) \div (-2)$

15. $(-9x^2) \div (+3)$

16. $(-6x^2) \div (-3)$

Principles of Exponents.

EXERCISE 17.

1. By what quantity must a^5 be multiplied to produce a^9? Then, a^9 divided by a^5 equals what quantity?

2. By what quantity must x^8 be multiplied to produce x^{12}? Then, x^{12} divided by x^8 equals what quantity?

29. Since $a^9 \div a^5 = a^4$ [Ex. 1], and $x^{12} \div x^8 = x^4$ [Ex. 2], we have,

Prin. 12.*—The exponent of a factor in the quotient equals the difference of the exponents of the factor in the dividend and divisor.*

30. $a^5 \div a^5 = a^0$ [P. 12].

But $a^5 \div a^5 = 1$, since $1 \times a^5 = a^5$.

$\therefore \quad a^0 = 1$. Therefore,

Prin. 13.*—Any quantity with an exponent of zero equals unity.*

SIGHT EXERCISE.

Name the quotients in the following examples:

1. $a^{12} \div a^8$
2. $a^{15} \div a^7$
3. $x^8 \div x^2$
4. $m^{20} \div m^{10}$
5. $y^8 \div y$
6. $z^3 \div z^3$
7. $(+x^9) \div (+x^3)$
8. $(-x^{10}) \div (+x^5)$
9. $(+x^{18}) \div (-x^{12})$
10. $(-x^{12}) \div (-x^6)$
11. $(+3^5) \div (-3^2)$
12. $(+5^4) \div (+5^2)$

EXERCISE 18.

1. What is the value of $54 \div 3$? What, then, is the value of $(9 \times 6) \div 3$? Is $(9 \times 6) \div 3 = (9 \div 3) \times (6 \div 3)$? Is $(9 \times 6) \div 3 = (9 \div 3) \times 6$? Is $(9 \times 6) \div 3 = 9 \times (6 \div 3)$?

2. Which is the greatest, $ab \div c$, $(a \div c) \times b$, or $a \times (b \div c)$

1. If $a = 8$, $b = 6$, and $c = 2$?
2. If $a = +12$, $b = -8$, and $c = -4$?

SIGHT EXERCISE.

31. Since $ab \div c = (a \div c) \times b = a \times (b \div c)$ for any values of a, b, and c [Ex. 2], we have,

Prin. 14.—*Dividing one factor of a quantity divides the quantity.*

SIGHT EXERCISE.

Name the quotients in the following examples :

1. $abc \div a$
2. $xyz \div y$
3. $pqr \div r$
4. $a^2x \div a$
5. $xy^3 \div y$
6. $m^3n^2 \div n^2$
7. $x^2z \div x$
8. $ax^4 \div x^2$
9. $(+6a) \div (+3)$
10. $(-12x^2) \div (-4)$
11. $(+15z) \div (-5)$
12. $(-18m) \div (+6)$
13. $(-x^2y) \div (-x)$
14. $(+x^3z) \div (-z)$
15. $(-xy^4) \div (+y^2)$
16. $(+rs^2) \div (-s)$

Definitions.

32. The process of finding how many times, and in what manner, one of two algebraic quantities must be taken to produce the other is ***Algebraic Division.***

33. The general problem of division is: "*Given the product of two factors and one of them, to find the other.*"

34. The quantity to be produced, and corresponding to the product, is the ***Dividend.***

35. The quantity taken to produce the dividend, and corresponding to the given factor, is the ***Divisor.***

36. The quantity which shows how many times, and in what manner, the divisor must be taken to produce the dividend, and corresponding to the required factor, is the ***Quotient.***

Problem 1. To divide a monomial by a monomial.

Illustration.—Divide $-32a^4b^5c$ by $+8a^2b^2$.

Solution: Since dividing one factor of a quantity divides the quantity [P. 14], $-32 \times a^4 \times b^5 \times c$ is divided by $+8 \times a^2 \times b^2$, if -32 is divided by $+8$, a^4 by a^2, b^5 by b^2, and c by 1. -32 divided by $+8$ is -4 [P. 11]; a^4 divided by a^2 is a^2, and b^5 divided by b^2 is b^3 [P. 12]; and c divided by 1 is c; hence, the quotient is $-4a^2b^3c$. Therefore,

Form.

$$+8a^2b^2\overline{)\,-32a^4b^5c\,}$$
$$-\;4a^2b^3c$$

Rule 1.*—Divide the numerical coefficient of the dividend by that of the divisor; annex to the quotient all the different literal factors that occur in the dividend, and give to each an exponent equal to its exponent in the dividend diminished by its exponent in the divisor.*

Note.—Two equal literal factors, one in the dividend and one in the divisor, may be canceled, since their quotient is *one*.

EXERCISE 19.

Divide

1. $6a^3$ by $3a^2$
2. $12a^2b$ by $-4a$
3. $-18a^3b^2c$ by $6abc$
4. $12x^4y^3z$ by $4x^2y^2z$
5. $-21m^8n^6y^2$ by $-7m^4n^6y$
6. $63a^5b^8x^3y^4$ by $-9a^2x^2y^2$
7. $-96x^5y^4z^3$ by $-12x^5yz^3$
8. $-75ax^4y^3z$ by $-25ax^2yz$
9. $(a+b)^5$ by $(a+b)^2$
10. $-(m-n)^5$ by $(m-n)^3$
11. $6a^2(a+x)^8$ by $2a(a+x)^4$
12. $21xy(a-b)^9$ by $7x(a-b)^5$
13. $-25z^2(x+y)^7$ by $-5z(x+y)^3$
14. $36x^4(x^2-y^2)^{10}$ by $-9x^2(x^2-y^2)^7$
15. $-60m^5(a^3+b^3)^8$ by $10m^2(a^3+b^3)^5$

Find the value of

16. $(4x^3y^2z \times 6xy^3z^2) \div 8x^2y^3z$
17. $(-9x^4z^5 \div 3x^3z) \times 2xyz^2$
18. $4a^3b^2c \times (-8a^4b^5c^4 \div 2a^2b^3c^2)$
19. $(4x^3y^2z \div 2x^2y) - (12x^5y^3z^2 \div 3x^4y^2z)$
20. $(-4a^3b^2c \times 3a^2b^4c^2d) + (18a^7b^8c^5d \div 6a^2b^2c^2)$
21. $(-12a^7b^3c^3d \div 4a^4bc^2d) + (21a^5b^5c^2d \div 7a^2b^3cd)$

Problem 2. To divide a polynomial by a monomial.

EXERCISE 20.

1. Which is the greater, $(ab + bc - bd) \div b$ or $a + c - d$

1. When $a = 8$, $b = 6$, $c = 5$, and $d = 2$?
2. When $a = +9$, $b = -6$, $c = +3$, and $d = +4$?
3. When $a = 2/3$, $b = 3/4$, $c = 5/6$, and $d = 1/2$?

37. Since $(ab + bc - bd) \div b = a + c - d$ for any values of a, b, c, and d [Ex. 1], we have,

Prin. 15.*—Dividing every term of a quantity divides the quantity.*

SIGHT EXERCISE.

Divide at sight:

1. $(3x + 6y - 9z) \div 3$
2. $(4x^2 - 8xy + 6y^2) \div 2$
3. $(10z + 20y - 30x) \div 10$
4. $(-25x + 30y - 15z) \div (-5)$
5. $(16x^2 - 8xy + 20y^2) \div (+4)$
6. $(x^3 + x^2 + x) \div x$
7. $(m^5 - m^4 + m^3) \div m^2$
8. $(x^8 - x^6 + x^4 - x^3) \div (+x^3)$
9. $(-y^5 + y^4 - 3y^3 + 2y^2) \div (-y^2)$
10. $(-mn^2 - n^2r - n^2s) \div (-n^2)$

WRITTEN EXERCISE.

Illustration.—Divide $8a^2b - 4a^2b^2 + 6ab^3$ by $2ab$.

Solution: Since dividing every term of a quantity divides the quantity [P. 15], we divide each term of the dividend by $2ab$ and obtain for the quotient $4a - 2ab + 3b^2$. Therefore,

Form.

$$2ab\,\overline{)\,8a^2b - 4a^2b^2 + 6ab^3}$$
$$4a - 2ab + 3b^2$$

***Rule 2.**—Divide each term of the polynomial by the monomial, bearing in mind the principles of signs.*

EXERCISE 21.

Divide

1. $a^3 + a^2$ by a
2. $3x^4 + 6x^2$ by $3x$
3. $4a^2 - 6ab$ by $2a$
4. $4x^2y^2 + 2x^3y^3$ by $2xy$
5. $4a^2b - 6ab^2$ by $2ab$
6. $a^4 + a^3 - a^2 - a$ by a
7. $4x^3 + 6x^2 + 8x$ by $2x$
8. $6a^3 - 9a^3b + 3a^2b$ by $3a^2$
9. $12ab - 18b^2 + 30b$ by $-6b$
10. $8a^2b^4 - 12a^3b^3 + 28a^4b^2$ by $4a^2b^2$
11. $-3ax^3y + 6ax^2y^2 + 9axy^3$ by $-3axy$
12. $a^4x^4 - a^3x^3 + a^2x^2 - ax$ by $-ax$
13. $14m^3n^4 - 21m^2n^3 + 28mn^2 - 35m^2n$ by $7mn$
14. $6a^3(a+b) + 9a^2(m+n) - 12a(p+q)$ by $3a$
15. $(a+x)^4 - (a+x)^3 + (a+x)^2$ by $(a+x)^2$
16. $2a(a+b)^4 - 3b(a+b)^4 + 4c(a+b)^4$ by $(a+b)^4$
17. $x(x^2+y^2) + y(x^2+y^2) - z(x^2+y^2)$ by $-(x^2+y^2)$
18. $x^2y^2(x-y)^3 - x^3y(x-y)^3 + xy^3(x-y)^3$ by $xy(x-y)^3$
19. $a^2(a+b)^5 - a^2x^2(a+b)^4 + a^2y^2(a+b)^3$ by $a^2(a+b)^3$

Simple Numerical Equations.

EXERCISE 22.

1. $4x + 3x =$ what, when $x = 2$?

2. $6x - 2x =$ what, when $x = 3$?

Expressions like $4x + 3x = 14$ and $6x - 2x = 12$ are called *equations.*

3. Complete the following equations when $x = 3$:

1. $x + 4x - 3x =$
2. $3x - 4x + 2x =$
3. $8x - 3x = x +$
4. $3x + 7 = 8x -$
5. $3x + (\ ) = 5x - 2$
6. $5x + 6 = (\ )x$

4. Complete the following equations when $x = -4$:

1. $7x - 6x + 2 =$
2. $6x - 8 + 5 = 2x -$
3. $-5x = 3x - 4x +$
4. $10x - 8x + 7 = 2x +$

The sign of equality separates an equation into two parts, called the *first* and *second members.*

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5. Complete the equation $5x + 7 = 3x +$ when $x = 4$. If we add 3 to each member, will the members still be equal? If we add x to one member and 4 to the other? Therefore,

Prin. 16.—*If the same quantity or equal quantities be added to equal quantities, the results will be equal.*

6. Complete the equation $3x - 5x + 7 = 4x -$ when $x = 3$. Will the equality of the members be destroyed if we subtract 5 from each member? If we subtract x from one member and 3 from the other? Therefore,

Prin. 17.—*If the same quantity or equal quantities be subtracted from equal quantities, the results will be equal.*

7. Complete the equation $5x - 3 = 7x -$ when $x = 5$. Will the members remain equal if both be multiplied by 3? If one be multiplied by x and the other by 5? Therefore,

Prin. 18.—*If equal quantities be multiplied by the same quantity or equal quantities, the results will be equal.*

8. Complete the equation $3x+6x=12x-$ when $x=3$. Will the members remain equal if both be divided by 3? If one be divided by x and the other by 3? Therefore,

Prin. 19.—*If equal quantities be divided by the same quantity or equal quantities, the results will be equal.*

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9. What quantity must be added to both members of the equation $5x-8=3x$ to make it $5x=3x+8$? Now, what quantity must be subtracted from both members of $5x=3x+8$ to make it $5x-3x=+8$? Are the members still equal? Why? Therefore,

Prin. 20.—*A term may be taken from one member of an equation to the other, if its sign be changed.*

Taking a term from one member of an equation to the other is called *transposing* it.

SIGHT EXERCISE.

Transpose the terms containing x to the first and the others to the second member in the following equations:

1. $3x+5=2x$
2. $3x-5=x+7$
3. $9x-8=4x-6$
4. $8-4x=7-5x$
5. $2x+8=7x-3$
6. $9x-5=8x+2$
7. $-5+2x=8-3x$
8. $x=5x-7$
9. $x+3-5=4x$
10. $6x-7=9x-5$
11. $x=5x+7-2x$
12. $10x=5-8x+3$

EXERCISE 23.

1. If both members of the equation $\frac{x}{3}+\frac{5}{6}=\frac{7x}{12}-\frac{2}{3}$ be multiplied by 12, a common denominator of the fractions, what will the resulting equation be? Will the members still be equal? Why? Therefore,

Prin. 21.—*If both members of a fractional equation be multiplied by a common denominator of its terms, it will be cleared of fractions.*

SIGHT EXERCISE.

Clear the following equations of fractions:

1. $\frac{x}{2}=\frac{x}{3}-2$
2. $\frac{x}{4}=\frac{x}{5}+\frac{1}{2}$
3. $\frac{1}{2}x+\frac{1}{3}x=\frac{1}{4}$
4. $\frac{3}{4}x-\frac{2}{3}x=5$
5. $\frac{7}{8}x-\frac{3}{4}=\frac{5}{2}x$
6. $\frac{3x}{5}-\frac{7x}{10}=\frac{3}{5}$
7. $\frac{5x}{12}=\frac{7x}{4}-\frac{5}{6}$
8. $\frac{3x}{7}-\frac{1}{14}=2$
9. $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=\frac{1}{6}$
10. $\frac{5}{6}x-\frac{2}{3}=\frac{7}{9}x-\frac{1}{18}$
11. $\frac{3}{4}x-\frac{5}{6}=\frac{7}{8}x+2$
12. $\frac{1}{5}x+3=\frac{3}{10}x-\frac{2}{5}$

EXERCISE 24.

1. If both members of the equation $-x+3=-5x-7$ be multiplied by -1, what will the resulting equation be? What change has been made in the terms? Are the members still equal? Why? Therefore,

Prin. 22.*—If the sign of every term of an equation be changed, the members will still be equal.*

Definitions.

38. An ***Equation*** is an expression of equality between two equal quantities.

39. The ***Members*** of an equation are the quantities which are placed equal to each other.

40. There are generally two kinds of quantities in an equation, ***known*** and ***unknown.***

41. A ***Known Quantity*** is one whose value is given.

42. An ***Unknown Quantity*** is one whose value is to be determined.

43. An ***Identical Equation*** is one that is true for any value of the unknown quantity; as, $2x = x + x$.

44. An ***Equation of Condition*** is one that is true only for particular values of the unknown quantity; as, $5x = 30$, in which $x = 6$.

Axioms.

45. An ***Axiom*** is a self-evident truth.

46. The following are the principal axioms of algebra:

1. Things which are equal to the same thing are equal to each other.

2. If equals be added to equals the sums will be equal.

3. If equals be subtracted from equals the remainders will be equal.

4. If equals be multiplied by equals the products will be equal.

5. If equals be divided by equals the quotients will be equal.

6. Equal powers of equal quantities are equal.

7. Equal roots of equal quantities are equal; positive equal to positive, and negative equal to negative.

8. A quantity equally increased and diminished equals the quantity itself.

9. The whole is equal to the sum of all its parts.

Solution of Simple Numerical Equations.

Illustration.—1. Find the value of x in the equation $5x - 8 = 3x - 4$.

Solution: Given $5x - 8 = 3x - 4$ (A)

Transpose $3x$ to the first member and -8 to the second and change their signs [P. 20], then

$$5x - 3x = -4 + 8 \quad (1)$$

Collect terms, $2x = 4$ (2)

Divide by 2 [P. 19], $x = 2$

2. Find the value of x in $\frac{2}{3}x - 7 = \frac{7}{6}x - 9$.

Solution: Given $\frac{2}{3}x - 7 = \frac{7}{6}x - 9$ (A)

Multiply both members by 6 to clear of fractions [P. 21],

$$4x - 42 = 7x - 54 \quad (1)$$

Transpose $7x$ to the first member and -42 to the second member and change their signs [P. 20],

$$4x - 7x = -54 + 42 \quad (2)$$

Collect terms, $-3x = -12$ (3)

Divide by -3 [P. 19], $x = 4$

Proof: Put 4 for x in equation (A),

$$\frac{2}{3} \times 4 - 7 = \frac{7}{6} \times 4 - 9, \text{ or}$$

$$\frac{8}{3} - 7 = \frac{14}{3} - 9, \text{ or}$$

$$-4\tfrac{1}{3} = -4\tfrac{1}{3}$$

47. Finding the value of the unknown quantity in an equation is called *solving* or *reducing the equation.*

48. Proving an answer obtained by solving an equation is called *verifying* the answer.

EXERCISE 25.

Solve the following equations and verify the answers :

1. $2x - 7 = x + 1$

2. $5x - 3x = 4x - 6$

3. $7x + 3 = 9x - 9$

4. $6x - 4x = 10x - 24$

5. $-8x + 12 = 14 - 12x$

6. $5x - 8 = 7x + 2x$

7. $10 + 5x - x = 7x + 2$

8. $8x - 8 + 7 = 6x + 10$

9. $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x = 13$

10. $\frac{2}{3}x - \frac{3}{4}x + \frac{7}{8}x = 19$

11. $\frac{5}{6}x = \frac{3}{4}x + 20$

12. $\frac{3}{4}x - \frac{5}{6}x = \frac{7}{8}x - \frac{3}{4}$

13. $\frac{4}{5}x - \frac{3}{10}x = \frac{1}{4} + \frac{1}{20}x$

14. $x - \frac{1}{8} = \frac{5}{6}x + \frac{1}{12}$

15. $\frac{5}{6}x - \frac{4}{3} = \frac{2}{3}x + 1\frac{2}{3}$

16. $\frac{7}{12}x - \frac{5}{9}x = \frac{x}{3} - 11$

Concrete Examples involving Equations.

EXERCISE 26.

1. A, B, and C together own 98 sheep. B owns *twice* as many as A, and C *twice* as many as B. How many does each own?

Solution: Let $x =$ A's number;

Since B owns twice as many as A,

$2x =$ B's number; and

Since C owns twice as many as B,

$4x =$ C's number.

Since they together own 98,

$$4x + 2x + x = 98.$$

Collect terms, $7x = 98.$

Divide by 7, $x = 14,$ A's number;

$2x = 28,$ B's number;

$4x = 56,$ C's number.

2. A and B together own 100 acres of land, and A owns 4 times as much as B. How many acres has each?

3. The sum of three numbers is 360. The first is 4 times, and the second 5 times, the third; required the numbers.

4. A has $^2/_3$ as much money as B, and they together have \$50. How much has each?

5. My house cost $^3/_4$, and my barn $^2/_3$, as much as my farm, and they all cost \$5800; required the cost of each.

6. Divide 108 into three such parts that the second shall be 3 times as great as the first, and the third twice as great as the second.

7. Divide 760 acres of land into three farms, such that the first shall be $^2/_3$ as large as the second, and the second $^2/_3$ as large as the third.

8. John has 5 times as much money as William, and the difference of their amounts is \$2000. How much has each?

Suggestion.—Let $x =$ William's sum, then will $5x =$ John's sum, and $5x - x = 2000.$

9. A number increased by its one half, its one third, and its one fourth is 100. What is the number?

10. My horse cost 6 times as much as my harness, and it cost $150 more. What was the cost of each?

11. Jane's age is $^2/_3$ of Mary's, and the difference of their ages is 8 years. What is the age of each?

12. The difference of two numbers is 49, and the one is 8 times the other. What are the numbers?

13. The difference of two fractions is $^5/_{13}$, and the one is $^3/_{26}$ of the other. What are the fractions?

14. If a number is diminished by the sum of its $^1/_3$ and its $^1/_4$ it will be 60. What is the number?

15. Nine times a certain number exceeds 5 times the number by 90. What is the number?

16. Five sixths of my age exceeds three fourths of it by 4 years. What is my age?

17. Two fields contain 48 acres of land, and the one is twice as large as the other, lacking 15 acres. How many acres in each field?

Suggestion.—Let $x =$ the number of acres in the smaller field, then will $2x - 15$ equal the number in the larger field, and $3x - 15 = 48$, the number in both fields.

18. John and William together have $500, and John has $50 more than William. How much has each?

19. Twice John's age, plus 15 years, equals his father's age, and the sum of their ages is 75 years. What is the age of each?

20. The difference of two numbers is 20, and their sum is 100. What are the numbers?

21. An agent bought three houses for $3500. The second was worth twice as much as the first, *plus* $150, and the third twice as much as the second, *minus* $100. What was the value of each?

22. A man divided \$4800 among his three sons: to the eldest he gave \$250 more than to the second, and to the second \$125 less than to the third. What did each receive?

23. Mr. Jones sold his horse for twice the cost, plus \$40, and gained \$190. What was the cost?

24. Mr. Smith sold his horse for $\frac{1}{3}$ of the cost, + \$10, and lost \$50. What was the cost?

25. A man spent $\frac{1}{3}$ of his money, + \$20, at one time; $\frac{1}{4}$ of it, + \$30, at another time; and $\frac{1}{5}$ of it, + \$40, at another time, and had nothing remaining. How much had he at first?

Suggestion.—Let $x =$ what he had at first, then will $\frac{1}{3}x + 20$ represent what he spent the first time, $\frac{1}{4}x + 30$ what he spent the second time, and $\frac{1}{5}x + 40$ what he spent the third time, and $\frac{1}{3}x + 20 + \frac{1}{4}x + 30 + \frac{1}{5}x + 40 = x$, the entire sum.

26. A merchant increased his capital by $\frac{4}{5}$ of itself and \$200 more, and found that he had doubled it. What was his capital?

27. If to three times a certain number you add $\frac{5}{6}$ of the number, and increase the result by 20, it will be $4\frac{1}{6}$ times the number. What is the number?

28. The difference between 9 times a number and 20 equals the difference between 140 and the number. What is the number?

29. Twice a certain number is as much below 100 as 3 times the number is above 100. What is the number?

30. Three times Mary's age, increased by 10 years, will be her age 30 years hence. What is her age?

31. Twice my age 10 years ago will be my age 10 years hence. What is my age now?

32. The sum of two numbers is 160, and 3 times the first equals 5 times the second. What are the numbers?

Suggestion.—Let $x =$ the first, then will $\frac{3}{5}x =$ the second. Or, let $5x =$ the first, then will $3x =$ the second.

33. A man sold $^2/_3$ of his land, and then bought $52^3/_4$ acres, and then had $^3/_4$ as much as he had at first. How much had he at first?

34. Six times John's age equals 4 times William's age, and the sum of their ages is 25 years. What are their ages?

35. Three times the cost of my house equals 4 times the cost of my barn, and the barn cost $1200 less than the house. What was the cost of each?

36. Two thirds of one number equals $^3/_4$ of another, and the difference of the numbers is 45. What are the numbers?

Suggestion.—Since $^2/_3$ of one number $= {}^3/_4$ of another, then 12 times $^2/_3$ of the first $=$ 12 times $^3/_4$ of the second, or 8 times the first $=$ 9 times the second, or the first $= {}^9/_8$ of the second. Let $8x =$ the second, then will $9x =$ the first.

37. A farmer raised 4900 bushels of corn and wheat; $^4/_5$ of the quantity of corn equals $^5/_6$ of the quantity of wheat. How much of each kind did he raise?

38. Five sixths of the cost of a horse equals $^8/_9$ of the selling-price, and the loss is $15. What was the cost?

39. A merchant sold an article for 60 cents, and found that $^2/_3$ of his gain was equivalent to $^2/_{15}$ of the cost. What was the cost?

40. A man agreed to work one year for $96 and a cow. At the end of 9 months he left, receiving $65 and the cow. What was the value of the cow?

41. The sum of two numbers is 28. Their difference is $^3/_5$ of the larger number. What are the numbers?

42. John sold a cow to William for $^1/_5$ more than it cost him. William sold it to Thomas for $36, which was $^1/_4$ less than it cost him. What did it cost John?

43. A, B, and C have $270. One fourth of A's share equals $^1/_5$ of B's, and $^4/_5$ of B's share equals $^8/_9$ of C's. How much has each?

Addition of Polynomials.

49. Illustration.—Since the whole equals the sum of all the parts, we arrange the terms so that similar ones stand in the same column, then add the columns separately and combine the results by Rule 2. Thus,

1.

$$\begin{array}{r} 2a^2 - 3ab + 5b^2 \\ 6a^2 + 7ab - 7b^2 \\ -9a^2 - 6ab + 6b^2 \\ 7a^2 + 4ab - 3b^2 \\ \hline 6a^2 + 2ab + b^2 \end{array}$$

2.

$$\begin{array}{r} 4(a+b) + 6(x-y) \\ 7(a+b) - 5(x-y) \\ -5(a+b) + 5(x-y) \\ -2(a+b) - 8(x-y) \\ \hline 4(a+b) - 2(x-y) \end{array}$$

EXERCISE 27.

Add the following:

1.

$$\begin{array}{r} 3a - 4b \\ 5a + 9b \\ -6a + 7b \\ 4a - 8b \\ \hline \end{array}$$

2.

$$\begin{array}{r} a + b - c \\ 2a - b + 3c \\ 7a + 6b - 5c \\ -3a + 5b - 6c \\ \hline \end{array}$$

3.

$$\begin{array}{r} x - y + z \\ 2x + 6y - 7z \\ 9x - 5y + 6z \\ 9x - 7y + 5z \\ \hline \end{array}$$

4.

$$\begin{array}{r} 2x^2 - 5y^2 + 7xy \\ -5x^2 + 9y^2 - 6xy \\ 3x^2 - 7y^2 - 5xy \\ 9x^2 + 5y^2 - 6xy \\ 2x^2 - 9y^2 + 7xy \\ \hline \end{array}$$

5.

$$\begin{array}{r} 2ab + 3cd - 4ad + e \\ 5ab - 6cd + 7ad + 2e \\ 5ab - 5cd - 6ad - 3e \\ -8ab + 7cd - 3ad + 5e \\ -4ab + 2cd + 5ad - 4e \\ \hline \end{array}$$

6. Add $5xy + 2z^2$, $3xy - 5z^2$, $-7xy + 2z^2$, $4xy + 6z^2$, and $3xy - 7z^2$

7. Add $3m^2 + 2mn + 5n^2$, $7m^2 - 3mn + 6n^2$, $5m^2 - 6mn + 7n^2$, and $-8m^2 - 6mn + 5n^2$

8. Add $3ax^2 - 5b^2y^2 + 3abxy$, $6b^2y^2 - 4ax^2 - 5abxy$, $7abxy - 5ax^2 - 7b^2y^2$, $3b^2y^2 - 5abxy + 9ax^2$

Add: **9.** $3(x+y)+5(m+n)$, $-2(x+y)-5(m+n)$, $7(x+y)-8(m+n)$, $9(m+n)-5(x+y)$

10. $7a(p+q)+6b(p-q)$, $6a(p+q)-9b(p-q)$, $5b(p-q)+7a(p+q)$, $3a(p+q)-5b(p-q)$, $6b(p-q)-8a(p+q)$

Subtraction of Polynomials.

50. ***Rule.***—*Arrange the terms of the subtrahend so that they stand under like terms of the minuend; then change the sign of each term of the subtrahend, or conceive it changed, and proceed as in addition of polynomials.*

Illustrations.—

Examples.	Solutions.
From $3x^2+7xy-2y^2$	$3x^2+7xy-2y^2$
take $5x^2-9xy+5y^2$	$-5x^2+9xy-5y^2$
Difference $=$	$-2x^2+16xy-7y^2$
From $3x-4y$	$3x-4y+0$
take $7x+6y-4z$	$-7x-6y+4z$
Difference $=$	$-4x-10y+4z$

EXERCISE 28.

1. From $3x^2+2y^2$ take $4x^2-7y^2$

2. From $9x^2-7xy+3y^2$ take $5x^2-8xy+9y^2$

3. From $9a+6b-7c$ take $12a-7b+9c$

4. From $10m^2n^2-7mn+6n^2$ take $4m^2n^2+8mn+9n^2$

5. From $8x^3-7x^2y-9y^2$ take $3x^3+5x^2y-12y^2$

6. From $9y^4-7y^3+6y+5$ take $8y^4+5y^3-7y+12$

7. From $9x^2+6x+5$ take $8x^2+7x-10$

8. From $25x^3-5y^3$ take $8x^3+7xy+5y^3$

9. From $10m^2-25n^2$ take $8m^2+7mn-9n^2$

10. From x^3+x+9 take $5x^3-7x^2+2x-6$

11. From $8(x+y)+5(x-y)$ take $7(x+y)-9(x-y)$

12. From the sum of $3x+5y-4z$ and $5x-7y+5z$ take $9x-7y+5z$

13. From $9x^2-3xy+7y^2$ take the sum of $3x^2-5xy+7y^2$ and $2x^2+4xy-5y^2$

14. From the sum of $9a^3+7a^2-3a+5$ and $6a^3-5a^2+7a-3$ take the difference of $9a^3+5a^2-7a+6$ and $3a^3+5a^2-5a+9$

Symbols of Aggregation.

Definitions.

51. The symbols of aggregation are the parenthesis ( ); the braces { }; the brackets [ ]; and the vinculum —. They signify that the quantities inclosed by them shall be considered together as one quantity.

52. In a more general sense, the term parenthesis is made to include all symbols of aggregation.

Principles.

53. The expression $a+(b-c)$ denotes that the quantity $b-c$ is to be added to a. If the addition be performed the result will be $a+b-c$. Therefore,

Prin. 23.—*If a number of terms are inclosed by a parenthesis preceded by plus, the symbol and the sign before it may be removed without altering the value of the expression.*

54. The expression $a-(b-c)$ denotes that the quantity $b-c$ is to be subtracted from a. If the subtraction be performed, the result will be $a-b+c$. Therefore,

Prin. 24.—*If a number of terms are inclosed by a parenthesis preceded by minus, the symbol and the sign before it may be removed, if the sign of every term inclosed be changed.*

SIGHT EXERCISE.

Name at sight the equivalents of the following expressions, without parentheses :

1. $x+(y-z)$ **5.** $2+(5-3)$ **9.** $a-(-a)$

2. $3x-(2y+z)$ **6.** $4-(6-4)$ **10.** $3a-(2a-a)$

3. $a+(-b-c)$ **7.** $3-(4-8)$ **11.** $4b+(6b-3b)$

4. $a-(-b-c)$ **8.** $6-(-4)$ **12.** $2y-(-y-3y)$

---

55. $a+(b-c)=a+b-c$ [P. 23],

$\therefore\ a+b-c=a+(b-c)$. Hence,

Prin. 25.—*Any number of terms may be inclosed by a parenthesis and preceded by plus, without changing the value of the expression.*

56. $a-(b-c)=a-b+c$ [P. 24],

$\therefore\ a-b+c=a-(b-c)$. Hence,

Prin. 26.—*Any number of terms may be inclosed by a parenthesis and preceded by minus, if the sign of every term inclosed be changed.*

SIGHT EXERCISE.

Inclose the last two terms of the following trinomials by parentheses preceded by plus when the middle term is positive, and by minus when it is negative :

1. $a+b-c$ **5.** $m+n+p$ **9.** $m-2n+3t$

2. $a-b+c$ **6.** $m-n+p$ **10.** $-x-3y-3z$

3. $a-b-c$ **7.** $m-n-p$ **11.** $y-2x+3z$

4. $x+2y-z$ **8.** $3x-2y+2z$ **12.** $3z-2x+y$

Problem 1. To simplify a parenthetical expression.

Illustrations.—1. Simplify $3x + (4x + 2x - 3x)$.

Solution: $3x + (4x + 2x - 3x) = 3x + 4x + 2x - 3x$ [P. 23] $= 6x$.

2. Simplify $5ab - (3ab - 2ab + 7ab)$.

Solution: $5ab - (3ab - 2ab + 7ab) = 5ab - 3ab + 2ab - 7ab$ [P. 24] $= -3ab$.

3. Simplify $2a - [3a + 2b + \{4a - 3b - (2a - 5b)\}]$

Suggestion.—Remove the inner symbols continuously until all the symbols are removed. Thus,

$$2a - [3a + 2b + \{4a - 3b - (2a - 5b)\}] =$$
$$2a - [3a + 2b + \{4a - 3b - 2a + 5b\}] =$$
$$2a - [3a + 2b + 4a - 3b - 2a + 5b] =$$
$$2a - 3a - 2b - 4a + 3b + 2a - 5b = -3a - 4b$$

Note.—The operation may often be simplified by collecting the terms inclosed by a symbol at the time of removing it.

EXERCISE 29.

Simplify

1. $3x + (4x - 3x + 5x)$
2. $4a - (3a - 7a + 6a)$
3. $a + 2b + (6a - 3b)$
4. $3x - 2y - (5x - 7y)$
5. $5m - (6m + 2m - m)$
6. $2x + \{3x - (4x - 2x)\}$
7. $2a - \{3a + (2a - b)\}$
8. $2xy - (-4xy + 7xy - 2xy)$
9. $2x + 3y - \{4x - (3x + 2y)\}$
10. $3x + (x + y) - (2x - 4y)$
11. $5xy - \{3xy + (-2xy - xy)\}$
12. $2 + [2 - \{2 + (2) - 2\} + 2]$
13. $(6 - 5) - \{6 - (5 - 6) - 5\} + \{(5 - 6) - (5 - 6) - 5\}$
14. $x - y - [x - \{y - (z - x) - y\} - z] + (x - y - z)$
15. $[a + \{x + (a + x) + a\} + x] + \{a + (x + a) + x\}$
16. $1 - [-1 + \{-1 - (-1 - \overline{1 + 1} - 1) - 1\} - 1]$
17. $x - [-x - (x + x) - \overline{x - x} - \{x + (x - x)\}]$

Problem 2. To inclose terms by symbols of aggregation.

Illustrations.—

1. Express in binomial terms $a - b - c + d$.

Solution: $a - b - c + d = (a - b) - (c - d)$ [P. 25, 26].

Note.—It is customary to place before the symbol the sign of the first term to be inclosed, and if this is negative the signs of the terms inclosed must be changed.

2. Express in trinomial terms $a + b - c - d - e + f$.

Solution: $a + b - c - d - e + f = (a + b - c) - (d + e - f)$ [P. 25, 26].

3. Express in trinomial terms having the last two terms of each inclosed by a vinculum,

$$3a - 2b + 5c - 6d + 5e - 4f.$$

Solution: $3a - 2b + 5c - 6d + 5e - 4f = (3a - 2b + 5c) - (6d - 5e + 4f) = (3a - \overline{2b - 5c}) - (6d - \overline{5e - 4f})$.

EXERCISE 30.

Express in binomial terms:

1. $2a + 3b + 5c - 2d$

2. $a - 2b + c - 2d$

3. $m + n - p + q$

4. $3m - 2n - 4p + 2q$

5. $a - b + c - d - e + f$

6. $2a - 3b - 4c + 2d - 5e + 6f$

7. $x - y + 2z - 3v - 6u + 4w$

8. $5p - 3q + 5z - 4m + 2n - 6r$

Express in trinomial terms:

9. Examples 5, 6, 7, and 8.

10. $2m - 3n + 4a - 6b + 7c - 2d - 4e + g - 2h$

11. $4a - 2b - 3c - 4d + 5e + 6f + 7g - 2h + 4l$

12. $2p - 3q + 4r - 2s + 5t + 6u - 7v + 2w - 6y$

Express in trinomial terms, having the last two terms of each inclosed by parentheses:

13. Examples 10, 11 and 12.

14. $x - y + z - m + n - p + q - r - s$

Multiplication by Polynomials.

Illustration.—Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.

Solution: $a^2 - ab + b^2 = a^2 + (-ab) + (+b^2)$; therefore $a^2 - ab + b^2$ times $a^2 + ab + b^2$ equals the sum of

1. a^2 times $a^2 + ab + b^2 =$
2. $-ab$ times $a^2 + ab + b^2 =$
3. $+\ b^2$ times $a^2 + ab + b^2 =$

which $=$

Form.

$$\begin{array}{lllll}
a^2 + ab & + b^2 & & & \\
a^2 - ab & + b^2 & & & \\
\hline
a^4 + a^3 b & + a^2 b^2 & & & \\
\quad - a^3 b & - a^2 b^2 & - a b^3 & & \\
 & + a^2 b^2 & + a b^3 & + b^4 & \\
\hline
a^4 & + a^2 b^2 & & + b^4 &
\end{array}$$

***Rule 3.**—Multiply the multiplicand by each term of the multiplier and take the algebraic sum of the products.*

Note.—For convenience, arrange both multiplicand and multiplier according to the ascending or descending powers of some letter assumed as a leading letter. Thus, a better order of

$(x^2 - 5 + 7x)(3x + 2x^2 - 5)$ is $(x^2 + 7x - 5)(2x^2 + 3x - 5)$.

EXERCISE 31.

Multiply

1. $a + b$ by $a - b$
2. $a + b$ by $a + b$
3. $a - b$ by $a - b$
4. $2a + 3b$ by $2a - 3b$
5. $xy + 12$ by $xy - 6$
6. $x - a$ by $x - b$
7. $b - x$ by $c + x$
8. $m^3 + n^3$ by $m^2 + n^2$
9. $3a^2 + 2b$ by $4a^2 - 6b$
10. $7a^2 - 8b^2$ by $9a^2 + 7b^2$
11. $ac - bd$ by $by - dx$
12. $x^4 - x^2 + 1$ by $x^2 + 1$
13. $a^2 + ab + b^2$ by $a - b$
14. $x^2 + 2x + 4$ by $x - 2$
15. $a^8 - a^4 + 1$ by $a^4 + 1$
16. $81 - 9c^2 + c^4$ by $9 + c^2$
17. $b^4 - 4b^2c^2 + 16c^4$ by $b^2 + 4c^2$
18. $3m^2 + 2mn$ by $2m^2 - 3n^2$
19. $6x^2y^2 - 7y^2z^2$ by $6x^2y^2 + 7y^2z^2$
20. $9a^2 + 36a + 144$ by $3a - 12$
21. $x^3 - x^2y + xy^2 - y^3$ by $x + y$
22. $a^4 + a^3b + a^2b^2 + ab^3 + b^4$ by $a - b$

Find the value of

23. $(3x^2 + 3xy + 2y^2)(3x^2 - 3xy + 2y^2)$

24. $(5a^2 + 7ab + 4b^2)(5a^2 - 7ab + 4b^2)$

25. $(9x^2 + 4xy - y^2)(9x^2 - 4xy + y^2)$

26. $(2x^2 - 4x + 6)(x^2 + 2x + 3)(2x^4 - 4x^2 + 18)$

27. $(a^4 + 2a^3b + 4a^2b^2 + 8ab^3 + 16b^4)(a - 2b)$

28. $81x^4 - 54x^3y + 36x^2y^2 - 24xy^3 + 16y^4)(3x + 2y)$

29. $(x^4 - x^2y^2 + y^4)(x^2 + xy + y^2)(x^2 - xy + y^2)$

30. $(2x^5 + 3x^4y - 2x^3y^2 + 4x^2y^3 - 5xy^4 + 3y^5)$
$(3x^2 + 2xy + 3y^2)$

Division by Polynomials.

Illustration.—Divide $a^3 + b^3$ by $a + b$.

Form.

$$\begin{array}{r l}
a + b\,) & a^3 + b^3\,(\,a^2 - ab + b^2 \\
 & \underline{a^3 + a^2b} \\
 & -a^2b + b^3 \\
 & \underline{-a^2b - ab^2} \\
 & ab^2 + b^3 \\
 & \underline{ab^2 + b^3}
\end{array}$$

Solution: $a + b$ is contained in $a^3 + b^3$ as many times as it can be taken out of it. a is contained in a^3 a^2 times; taking a^2 times $(a + b)$, or $a^3 + a^2b$ out of $a^3 + b^3$ by subtracting it, there remains $-a^2b + b^3$. a is contained in $-a^2b$ $(-ab)$ times; taking $(-ab)$ times $(a + b)$, or $-a^2b - ab^2$ out of $-a^2b + b^3$ by subtracting it, there remains $ab + b^3$. a is contained in ab^2, $(+b)$ times; taking $(+b)$ times $(a + b)$, or $ab^2 + b^3$ out of $ab^2 + b^3$ by subtracting it, nothing remains. Therefore, $(a + b)$ is contained in $(a^3 + b^3)$, $(a^2 - ab + b^2)$ times.

Suggestions.—For convenience, arrange the terms of the dividend, divisor, and the several remainders, according to the ascending or descending powers of some letter assumed as the leading letter. After each subtraction do not bring down any more terms than will be needed for the next operation. Always divide the first term of the dividend, or partial dividend, by the first term of the divisor to obtain the next term of the quotient.

EXERCISE 32.

Divide

1. $a^2 + 9a + 18$ by $a + 3$
2. $x^2 - x - 20$ by $x - 5$
3. $x^2 - 12x + 35$ by $x - 7$
4. $x^4 + 4x^2 - 12$ by $x^2 - 2$
5. $x^2 + 2xy + y^2$ by $x + y$
6. $x^3 - y^3$ by $x - y$
7. $x^3 + 27y^3$ by $x + 3y$
8. $a^6 + b^6$ by $a^2 + b^2$
9. $a^4 - b^4$ by $a - b$
10. $8x^3 + 27y^3$ by $2x + 3y$
11. $4x^2 - 4x - 24$ by $2x + 4$
12. $4x^2 - 4ax - 3a^2$ by $2x + a$
13. $8x^2 - 14ax - 15a^2$ by $4x + 3a$
14. $64x^6 - 125y^6$ by $4x^2 - 5y^2$
15. $32x^5 + y^5$ by $2x + y$
16. $8m^9 - 27n^6$ by $2m^3 - 3n^2$
17. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$
18. $x^4 + 4x^2 + 16$ by $x^2 - 2x + 4$
19. $a^8 + a^4b^4 + b^8$ by $a^4 + a^2b^2 + b^4$
20. $16m^4n^8 + 36m^6n^6 + 81m^8n^4$ by $4m^2n^4 - 6m^3n^3 + 9m^4n^2$
21. $x^4 + 4x^3 + 6x^2 + 5x + 2$ by $x^2 + 3x + 2$
22. $6x^4 + 19x^3 + 10x^2 + 2x + 5$ by $2x + 5$
23. $5x^4 + 2x^3 - 20x^2 - 23x - 6$ by $5x^2 + 7x + 2$
24. $8a^8 - 16a^6 - 34a^4 + 32a^2 - 6$ by $2a^4 - 7a^2 + 3$
25. $2x^2 - 3xy + y^2 + xz - yz$ by $x - y$
26. $x^2 - 4xy + 4y^2 - 9z^2$ by $x - 2y + 3z$
27. $4x^2 - 9y^2 - 6yz - z^2$ by $2x + 3y + z$
28. $4x^2 + 12xy + 9y^2 - z^2$ by $2x + 3y - z$
29. $x^4 - x^2y^2 - 2xy^3 - y^4$ by $x^2 + xy + y^2$
30. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$ by $a + b + c + d$

Simultaneous Numerical Equations of Two Unknown Quantities.

EXERCISE 33.

1. What is the value of x in the equation $2x + 3y = 24$, if $y = 4$?

Solution: Put 4 for y in the equation,

$$2x + 12 = 24 \qquad (1)$$

Transpose 12, $2x = 12$ (2)

Divide by 2, $x = 6$

2. What is the value of x in the equation $4x - 3y = 26$

1. If $y = 1$? 3. If $y = 3$? 5. If $y = 5$?
2. If $y = 2$? 4. If $y = 4$? 6. If $y = 6$?

3. What is the value of x in $7x + 5y = 66$

1. If $y = 3$? 3. If $y = 5$? 5. If $y = 7$?
2. If $y = 4$? 4. If $y = 6$? 6. If $y = 8$?

57. A single equation containing two unknown quantities can be satisfied with any number of pairs of values of the unknown quantities.

4. What values of x and y will satisfy

both $4x - 3y = 26$ (A)
and $7x + 5y = 66$? (B)

Solution: Multiply (A) by 5 and (B) by 3 to make the coefficients of y numerically equal [P. 18],

$$20x - 15y = 130 \qquad (1)$$
$$21x + 15y = 198 \qquad (2)$$

Add (1) and (2) [P. 16], $41x = 328$ (3)

Divide by 41, $x = 8$

Put 8 for x in (B),

$$56 + 5y = 66$$

Transpose 56, $5y = 10$

Divide by 5, $y = 2$

Verify by putting 8 for x and 2 for y in (A) and (B),

$32 - 6 = 26$, which is true.

$56 + 10 = 66$, which is true.

$x = 8$ and $y = 2$ are the only values of the unknown quantities that will satisfy both equations.

5. If $2x + 3y = 18$

and $3x + 2y = 17$

$6x + 9y = 54$ Why?

$6x + 4y = 34$ "

$5y = 20$ "

$y = 4$ "

$2x + 12 = 18$ "

$2x = 6$ "

$x = 3$ "

Verify: Are $\left.\begin{array}{l} 2x + 3y = 18 \\ \text{and } 3x + 2y = 17 \end{array}\right\}$ if $x = 3$ and $y = 4$?

58. Two equations of two unknown quantities can be satisfied only by particular values of those quantities.

Definitions.

59. When two equations express such relations between two or more unknown quantities that neither of them can be reduced to the form of the other, they are called ***Independent Equations.***

Illustration.—1. $3x + y = 12$ and $2x - 3y = 5$ are independent equations.

2. $x + y = 6$ and $5x + 5y = 30$ are not independent of each other, since the first multiplied by 5 will give the second.

60. If two or more independent equations are to be satisfied by the same values of the unknown quantities, they are called ***Simultaneous Equations.***

61. To solve two simultaneous equations of two unknown quantities, we first deduce from them a single equation containing only one of the unknown quantities. That is, we perform such operations upon the given equations as are necessary to get rid of one of the unknown quantities. This process is called ***Elimination.***

Elimination by Addition or Subtraction.

Illustration.—Solve $2x + 3y = 12$ (A)
and $3x + 5y = 19$ (B)

Solution: Multiply (A) by 3 and (B) by 2 to make the coefficients of x alike [P. 18],

$$6x + 9y = 36 \quad (1)$$
$$6x + 10y = 38 \quad (2)$$

Subtract (1) from (2) [P. 17],

$$y = 2$$

Put 2 for y in (A),

$$2x + 6 = 12 \quad (3)$$

Transpose 6, $2x = 6$ (4)

Divide by 2, $x = 3$

Verify by putting 3 for x and 2 for y in (A) and (B),

$$\left.\begin{array}{r} 6 + 6 = 12 \\ 9 + 10 = 19 \end{array}\right\} \text{both of which are true.}$$

Note.—If the signs of the like terms in (1) and (2) were unlike, the equations would have to be added to eliminate x.

EXERCISE 34.

Solve:

1. $\left.\begin{array}{l} 3x + 4y = 29 \\ 2x + 8y = 46 \end{array}\right\}$

2. $\left.\begin{array}{l} 5x - 3y = 22 \\ 2x + 9y = 19 \end{array}\right\}$

3. $\left.\begin{array}{l} 4x - 7y = -1 \\ 3x + 11y = 48 \end{array}\right\}$

4. $\left.\begin{array}{l} 7x + 8y = 2 \\ 6x - 2y = -16 \end{array}\right\}$

5. $\left.\begin{array}{l} 3x + 7y = 83 \\ 7x + 3y = 87 \end{array}\right\}$

6. $\left.\begin{array}{l} 5x - 7y = 13 \\ 7x - 5y = -1 \end{array}\right\}$

7. $\left.\begin{array}{l} x + 3y = 10 \\ 3x - 5y = 30 \end{array}\right\}$

8. $\left.\begin{array}{l} 3x - 4y = 3 \\ 5x - 3y = 16 \end{array}\right\}$

9. $\left.\begin{array}{l} 5x + 7y = 0 \\ 8x + 6y = 0 \end{array}\right\}$

10. $\left.\begin{array}{l} 6x + 9y = 45 \\ 8x + 15y = 70 \end{array}\right\}$

11. $\left.\begin{array}{l} 14x + 19y = 25 \\ 21x - 17y = -190 \end{array}\right\}$

12. $\left.\begin{array}{l} 15x + 10y = 195 \\ 10x - 15y = -65 \end{array}\right\}$

* 13. $\left.\begin{array}{l} \frac{2}{3}x + \frac{3}{4}y = 26 \\ \frac{3}{4}x + \frac{2}{3}y = 25 \end{array}\right\}$

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* Clear of fractions first.

Concrete Examples involving Simultaneous Equations.

EXERCISE 35.

1. A and B together have $500; if A had three times and B four times as much as now, A would have $800 more than B. How much has each?

Suggestion.— Let $x =$ the number of dollars A has,
and $y =$ the number of dollars B has.

Now, since they together have $500,

$$x + y = 500 \qquad (A)$$

Since 3 times A's sum exceeds 4 times B's by $800,

$$3x - 4y = 800 \qquad (B)$$

Solve (A) and (B) to obtain results.

2. Three times A's age added to twice B's equals 85 years, and twice A's added to three times B's equals 90 years. What is the age of each?

3. If 3 apples and 4 peaches are together worth 10 cents, and 5 apples and 2 peaches 12 cents, what are they worth apiece?

4. If 4 bushels of corn and 5 bushels of oats together weigh 374 pounds, and 3 bushels of corn weigh 48 pounds more than 4 bushels of oats, what are their respective weights per bushel?

5. A house and barn together cost $3000, and three times the cost of the house exceeds five times the cost of the barn by $1000. What is the cost of each?

6. If A's money were increased by $36, he would have three times as much as B; and if B's money were diminished by $5, A would have twice as much as B. Find the amount each has.

7. The sum of two numbers is 38, and twice the less is 18 times their difference. What are the numbers?

8. Five coins of one kind and six of another are worth $4.25, but four of the first kind and seven of the second are worth $4.50. Required the value of each coin.

9. If 7 bushels of corn and 10 bushels of oats are worth \$8.20, and 6 bushels of corn and 8 bushels of oats \$6.80, what is the price of each per bushel?

10. If 8 men and 12 boys earn \$168 a week, and 9 men and 7 boys \$150 in the same time, what are the daily wages of each man and boy?

11. A drover sold 12 sheep and 8 cows for \$392; had he sold 3 more cows and 5 less sheep, he would have received \$482. What was the price of each sheep and cow?

12. A merchant bought 40 grammars and 50 readers for \$77; had he bought 50 grammars and 40 readers, they would have cost \$1 less. What was the price of each book?

13. If 8 men and 6 boys earn as much per day as 9 men and 4 boys, and the difference between the daily wages of a man and a boy is \$1, how much does each receive per day?

14. A grocer has two kinds of coffee: if he mixes 12 pounds of the first kind with 18 pounds of the second, the mixture will be worth 20 cents a pound; but if he mixes 24 pounds of the first kind with 6 pounds of the second, the mixture will be worth 16 cents a pound. What is the value per pound of each grade?

15. If A buys 40 acres of land from B, B will have twice as much as A; but if he buys 80 acres, they will have the same amount. How much has each?

16. C has 60 acres of land: if A buys C's land, A will have as much as B; but if B buys it, B will have three times as much as A. How many acres has each?

17. The sum of two numbers exceeds twice their difference by 30, and twice the first equals three times the second; required the numbers.

18. If B were to give A \$25, they would have equal sums of money; if A were to give B \$22, B would have twice as much as A. How much has each?

Partial Treatment of Algebraic Involution.

Definitions.

62. The result obtained by using a quantity two or more times as a factor is a ***Power*** of the quantity.

63. The number of times a quantity is used as a factor to produce a power is the ***Degree*** of the power.

Illustration.—Thus, a^8 is a power of the *fourth* degree when derived from a^2, since $a^2 \times a^2 \times a^2 \times a^2 = a^8$.

64. The degree of a power is expressed by a quantity called an ***Exponent,*** written on the right hand above the quantity.

Illustration.—The 4th power of a^2 is written $(a^2)^4$. 4 is the exponent, and denotes the degree of the power.

65. The process of raising an algebraic quantity to any power is ***Algebraic Involution.***

Principles.

66.
$$(+a)^2 = (+a) \times (+a) = +a^2$$
$$(-a)^2 = (-a) \times (-a) = +a^2$$
$$(+a)^4 = (+a) \times (+a) \times (+a) \times (+a) = +a^4$$
$$(-a)^4 = (-a) \times (-a) \times (-a) \times (-a) = +a^4$$

In the same way it may be shown that * $(\pm a)^6 = +a^6$; $(\pm a)^8 = +a^8$; $(\pm a)^{10} = +a^{10}$; etc. Therefore,

Prin. 27.*—An even power of a positive or a negative quantity is positive.*

67.
$$(+a)^3 = (+a) \times (+a) \times (+a) = +a^3$$
$$(-a)^3 = (-a) \times (-a) \times (-a) = -a^3$$
$$(+a)^5 = (+a) \times (+a) \times (+a) \times (+a) \times (+a) = +a^5$$
$$(-a)^5 = (-a) \times (-a) \times (-a) \times (-a) \times (-a) = -a^5$$

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* $\pm a$ is read plus or minus a.

In a similar manner it may be shown that

$(\pm a)^7 = \pm a^7$; $(\pm a)^9 = \pm a^9$; etc. Therefore,

Prin. 28.—*An odd power of a quantity has the same sign as the quantity.*

SIGHT EXERCISE.

Give the true values of the following expressions:

1. $(+2)^2$	6. $(+3)^3$	11. $(\pm a)^6$
2. $(-2)^2$	7. $(-3)^3$	12. $(-x)^{10}$
3. $(+2)^3$	8. $(-x)^4$	13. $(+5)^3$
4. $(-2)^3$	9. $(+x)^6$	14. $(-5)^3$
5. $(\pm 2)^4$	10. $(+a)^8$	15. $(\pm 5)^4$
16. $\left(\pm \frac{1}{2}\right)^2$	17. $\left(-\frac{1}{3}\right)^3$	18. $\left(+\frac{2}{3}\right)^3$

---

68. $(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{12} = a^{3\times 4}$. Therefore,

Prin. 29.—*Multiplying the exponent of a factor by the exponent of a power raises the factor to that power.*

SIGHT EXERCISE.

Give the true values of the following expressions:

1. $(a^2)^4$	5. $(-a^3)^3$	9. $(\pm a^3)^6$
2. $(x^3)^2$	6. $(-a^4)^2$	10. $(-y^4)^5$
3. $(a^4)^5$	7. $(+x^3)^5$	11. $(-2^2)^2$
4. $(+a^2)^4$	8. $(\pm a^5)^4$	12. $(-1^3)^3$

13. $\left\{+\left(\frac{2}{3}\right)^2\right\}^2$ 14. $\left\{-\left(\frac{1}{2}\right)^3\right\}^2$

69. $(a^2 b^3 c)^4 = a^2 b^3 c \times a^2 b^3 c \times a^2 b^3 c \times a^2 b^3 c =$ $a^2 \times a^2 \times a^2 \times a^2 \times b^3 \times b^3 \times b^3 \times b^3 \times c \times c \times c \times c$ [P. 8] $=$ $(a^2)^4 \times (b^3)^4 \times (c)^4$. Therefore,

Prin. 30.—*Raising every factor of a quantity to a given power raises the quantity to that power.*

SIGHT EXERCISE.

Give the true values of the following expressions:

1. $(2a^2b)^2$	5. $(+a^2b^3c^4)^3$	9. $(\pm 2a^3b^4)^4$
2. $(2a^3b^2)^3$	6. $(-x^3y^2z^3)^4$	10. $(-a^{10}b^8c^6)^{10}$
3. $(3a^4b^3)^2$	7. $(\pm m^2n^3p)^6$	11. $(+m^{10}n^{20}r^2)^3$
4. $(-2a^2x^5)^2$	8. $(+x^4y^3z^5)^5$	12. $(\pm 3m^3x^2)^3$

13. $\left(\frac{1}{2}x^2y^3\right)^4$ 15. $\left(\frac{3}{5}r^4y^7\right)^2$

14. $\left(\frac{2}{3}x^3y^4\right)^2$ 16. $\left(\pm\frac{1}{3}a^7b^9\right)^4$

Problem 1. To raise a monomial to any power.

Illustration.—Raise $-3a^3b^2c^4$ to the third power.

Form.

$$(-3a^3b^2c^4)^3 = -27a^9b^6c^{12}$$

Solution: Since raising every factor of a quantity to a power raises the quantity to that power [P. 30], $(-3a^3b^2c^4)^3 = (-3)^3 \times (a^3)^3 \times (b^2)^3 \times (c^4)^3$. $(-3)^3 = -27$ [P. 28]; $(a^3)^3 = a^9$, $(b^2)^3 = b^6$, and $(c^4)^3 = c^{12}$ [P. 29]; hence, the result is $-27a^9b^6c^{12}$. Therefore,

***Rule.**—Raise the numerical coefficient to the required power, and multiply the exponent of each literal factor by the exponent of the power.*

EXERCISE 36.

Find the value of

1. $(a^2bc^2)^5$	7. $(2a^3b^2cd)^5$
2. $(2ab^2c)^4$	8. $(-3x^4y^3z)^4$
3. $(-3a^3b^3c^2)^3$	9. $(2x^4y^3z^2)^5$
4. $(-2xyz)^4$	10. $(5mn^3z^2)^4$
5. $(+3x^2y^3z)^4$	11. $\{(a+b)\ (c+d)\}^2$
6. $(-4m^2n^3x)^3$	12. $\{mx^2(a+b)^3\}^4$

13. $\{3(a+b)^2 (x-y)^3\}^3$

14. $\{a^3 b^2 c (m+n)^2\}^4$

15. $\{2 a^2 b^3 (x^2+y^2)^2\}^3$

16. $\{m^2 (x-y)^3 (x+y)^2\}^4$

17. $(3 a^3 b^2 c)^3 \times (-2 a b c^2)^4 \times \left(\frac{1}{2} a^3 b^2 c^2\right)^2$

18. $(9 x^2 y^3 z - 3 x^2 y^3 z)^3 \div (6 x^3 y^2 z)^2$

19. $\{(8 x^3 y^6 z^9)^2 - (2 x^2 y^4 z^6)^3\} \times 3 x^2 y^2 z^2$

20. $\{(5 x^4 y^8 z^{10})^2 + (x^2 y^4 z^5)^4\} \div 2 x^4 y^{12} z^5$

21. $(a^2 b^2 c^2)^{12} + 2 (a^4 b^4 c^4)^6 - 3 (a^8 b^8 c^8)^3$

22. $(4 x^2 y^3 z)^3 \times (2 x y^2 z^3)^2 \div 8 (x^3 y^2 z)^2$

23. $(2 x^4 y^3 z^5)^4 \div (-x^3 y^2 z^2)^3 \times (-2 x y^2 z)^4$

24. $\{(3 x^3 y^2 z^2)^2 \times (-2 x^2 y^4 z^3)^2\}^2 \div (-3 x^2 y^3 z)^3$

Problem 2. To square a binomial.

Principles.

70. The square of the sum of a and b, or $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$. Therefore,

Prin. 31.*—The square of the sum of two quantities equals the square of the first, plus twice their product, plus the square of the second.*

71. The square of the difference of a and b, or $(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$. Therefore,

Prin. 32.*—The square of the difference of two quantities equals the square of the first, minus twice their product, plus the square of the second.*

Note.—These two principles may be stated together as follows: "*The square of a binomial equals the sum of the squares of its terms, and twice their algebraic product.*"

Illustrations.—Square $2a + 3b$ and $3x^2 - 5y^2$.

Solutions: $(2a+3b)^2 = (2a)^2 + 2 \times 2a \times 3b + (3b)^2$ [P. 31] $= 4a^2 + 12ab + 9b^2$.

$(3x^2 - 5y^2)^2 = (3x^2)^2 - 2 \times 3x^2 \times 5y^2 + (5y^2)^2$ [P. 32] $= 9x^4 - 30x^2y^2 + 25y^4$.

EXERCISE 37.

Find the value of

1. $(x+y)^2$
2. $(x-y)^2$
3. $(m+n)^2$
4. $(m-n)^2$
5. $(x+4)^2$
6. $(x-7)^2$
7. $(x+a)^2$
8. $(x-a)^2$
9. $(2x+y)^2$
10. $(3x-y)^2$
11. $(2a+3b)^2$
12. $(5a-3b)^2$
13. $(2x+8)^2$
14. $(3x-5)^2$
15. $(5+2x)^2$
16. $(6-3x)^2$
17. $(xy+1)^2$
18. $(xy+5)^2$
19. $(1-cd)^2$
20. $(1+xy)^2$
21. $(ab-cd)^2$
22. $(xy+yz)^2$
23. $(2pq-3r)^2$
24. $(a^2b+ab^2)^2$
25. $(2x^2y-3xy^2)^2$
26. $(\overline{a+b}-1)^2$
27. $(\overline{a-b}+1)^2$
28. $(\overline{x+y}+z)^2$
29. $(\overline{x+y}-z)^2$
30. $(a-\overline{b+c})^2$

Problem 3. To cube a binomial.

Principles.

72. The cube of the sum of a and b, or $(a+b)^3 = (a+b)(a+b)(a+b) = a^3+3a^2b+3ab^2+b^3$. Therefore,

Prin. 33.—*The cube of the sum of two quantities equals the cube of the first, plus three times the square of the first into the second, plus three times the first into the square of the second, plus the cube of the second.*

73. The cube of the difference of a and b, or $(a-b)^3 = (a-b)(a-b)(a-b) = a^3-3a^2b+3ab^2-b^3$. Therefore,

Prin. 34.—*The cube of the difference of two quantities equals the cube of the first, minus three times the square of the first into the second, plus three times the first into the square of the second, minus the cube of the second.*

Illustration.—Cube $2x+3y$.

Solutions: $(2x+3y)^3 = (2x)^3 + 3\times(2x)^2\times(3y) + 3\times(2x)\times(3y)^2 + (3y)^3$ [P. 33] $= 8x^3+36x^2y+54xy^2+27y^3$.

EXERCISE 38.

Find the value of

1. $(x+y)^3$
2. $(x-y)^3$
3. $(y+2)^3$
4. $(y-2)^3$
5. $(1+x)^3$
6. $(x-1)^3$
7. $(xy+1)^3$
8. $(2+z)^3$
9. $(x+2z)^3$
10. $(x-2y)^3$
11. $(ax+by)^3$
12. $(2x+xy)^3$
13. $(a^2-x^2)^3$
14. $(a^2+xy)^3$
15. $(2x^3+1)^3$
16. $(x^3-2y^3)^3$
17. $(2x^2-3y^2)^3$
18. $(x^2-3xy)^3$
19. $(x^2y+xy^2)^3$
20. $(3x^2-4y^2)^3$
21. $(x+\overline{y+z})^3$

Composition.

1. Definitions and General Principles.

74. A quantity composed of two or more factors other than *one* is a ***Composite*** quantity.

75. A quantity composed of no other factors than itself and *one* is a ***Prime*** quantity.

76. The process of forming a composite quantity is ***Composition.***

Note.—Involution is a special kind of composition in which the factors used are all alike.

77. Since the product of two factors with like signs is positive [P. 5], and with unlike signs negative [P. 6], show that

Prin. 35.—*The product of any even number of factors with like signs is positive.*

Prin. 36.—*The product of any odd number of factors with like signs has the same sign as the factors.*

Prin. 37.—*If the signs of an even number of factors be changed, the sign of their product will remain unchanged.*

Prin. 38.—*If the signs of an odd number of factors be changed, the sign of their product will be changed.*

SIGHT EXERCISE.

Name the signs of the products of the following sets of factors:

1. $(+a) \times (+a) \times (+a) \times (+a)$
2. $(-a) \times (-b) \times (-c) \times (-d)$
3. $(+2) \times (+3) \times (+4) \times (+5) \times (+5)$
4. $(-2) \times (-2) \times (-2) \times (-2) \times (-2)$
5. $(-3) \times (-4) \times (-5) \times (-6) \times (-7) \times (-8)$
6. $(-2) \times (-4) \times (-5) \times (-6) \times (-7) \times (-8) \times (-9)$
7. $(-a) \times (-b) \times (-c) \times (-d) \times (-e) \times (-f) \times (-g)$
8. $(-x) \times (-y) \times (-z) \times (-m) \times (-n) \times (-p) \times (-q)$

Tell which of the following expressions are true, which false, and why:

9. $1(x-y) = -1(y-x)$
10. $-1(z-y) = 1(y-z)$
11. $(z-x) = -(x-z)$
12. $2(n-m) = 2(m-n)$
13. $x \times (-y) \times (-z) = x \times y \times z$
14. $(-x) \times (-y) \times z = x \times y \times (-z)$
15. $(y-x)(z-y) = (x-y)(y-z)$
16. $(x-y)(y-z)(z-x) = (x-y)(y-z)(x-z)$
17. $(n-m)(q-p)(s-r) = (m-n)(p-q)(r-s)$
18. $(v-z)(z-v)(v+z) = (z-v)(z-v)(z+v)$
19. $(x-y)(y-z)(z-u) = (x-y)(z-y)(u-z)$
20. $(x-y)(y-x)(y-x) = (y-x)^3$
21. $(a-b)^2(b-a)^2 = (a-b)^4$
22. $(m-n)^3(n-m)^2 = (m-n)^5$
23. $(x-y)^2(y-x)^2(x-z)^2(z-x)^2 = (x-y)^4(x-z)^4$

2. Special Principles and Applications.

78. The sum of a and b multiplied by their difference, or $(a - b)\ (a + b) = a^2 - b^2$. Therefore,

***Prin. 39.**—The product of the sum and difference of two quantities equals the square of the first minus the square of the second.*

Applications.

Illustration. — Find the product of $2x^2 + 3y^3$ and $2x^2 - 3y^3$.

Solution: $(2x^2 + 3y^3)(2x^2 - 3y^3) = (2x^2)^2 - (3y^3)^2$ [P. 39] $= 4x^4 - 9y^6$.

EXERCISE 39.

Find the value of:

1. $(x + y)(x - y)$
2. $(x^3 + y^3)(x^3 - y^3)$
3. $(ax + 1)(ax - 1)$
4. $(3m + 5n)(3m - 5n)$
5. $(4ax + 3b)(4ax - 3b)$
6. $(2x^2y + 3yz)(2x^2y - 3yz)$
7. $(x^2y^3 + 4xy)(x^2y^3 - 4xy)$
8. $(5x^3y - 7z^2)(5x^3y + 7z^2)$
9. $(x^2y^3z - 7)(x^2y^3z + 7)$
10. $(12x^4 - 5y^2)(12x^4 + 5y^2)$
11. $\{(a + b) + 1\}\{(a + b) - 1\}$
12. $\{(x^2 + y^2) + z^2\}\{(x^2 + y^2) - z^2\}$
13. $(x + y)(x - y)(x^2 + y^2)$
14. $(2x + 4)(2x - 4)(4x^2 + 16)$
15. $(3x + 5y)(3x - 5y)(9x^2 + 25y^2)$
16. $(a^2x^2 + b^2y^2)(a^2x^2 - b^2y^2)(a^4x^4 + b^4y^4)$
17. $(x + 2)(x - 2)(x + 2)(x - 2)$
18. $(2x - y)^2(2x + y)(2x + y)$
19. $(ax + by)^2(ax - by)^2$

79.

(1)	(2)	(3)
$x+4$	$x+4$	$x-4$
$x+3$	$x-3$	$x-3$
x^2+4x	x^2+4x	x^2-4x
$+3x+12$	$-3x-12$	$-3x+12$
$x^2+7x+12$	$x^2+\quad x-12$	$x^2-7x+12$

Notice.—1. That, in each of the above examples, we have found the product of two binomials having a like term (x), and two unlike terms (4 and 3), the latter being both positive in (1), one positive and the other negative in (2), and both negative in (3).

2. That the first term of each product is the square of the like term; the second term is the algebraic sum of the unlike terms times the like term; and the third term is the algebraic product of the unlike terms. Therefore,

***Prin. 40.**—The product of two binomials having a common term equals the square of the common term, and the algebraic sum of the unlike terms times the common term, and the algebraic product of the unlike terms.*

Application.

Illustration.—Find the product of x^2+8 and x^2-3

Solution: $(x^2+8)(x^2-3)=(x^2)^2+(8-3)x^2+(8\times-3)$ [P. 40] $=$ $x^4+5x^2-24.$

EXERCISE 40.

Find the value of:

1. $(x+4)(x+5)$
2. $(x+5)(x+2)$
3. $(x+5)(x+6)$
4. $(x+7)(x+1)$
5. $(x+8)(x+3)$
6. $(x+2)(x+9)$
7. $(x+6)(x+8)$
8. $(x+5)(x+10)$
9. $(2x+4)(2x+3)$
10. $(3x+3)(3x+1)$
11. $(5x+2)(5x+3)$
12. $(x+2y)(x+3y)$
13. $(x+6)(x-5)$
14. $(x+7)(x-3)$
15. $(x+9)(x-3)$
16. $(x-7)(x+2)$

17. $(x-9)(x+8)$

18. $(x-6)(x+12)$

19. $(x-3)(x-7)$

20. $(x-8)(x-6)$

21. $(x-7)(x-10)$

22. $(2x+3)(2x-7)$

23. $(3x-2y)(3x+4y)$

24. $(ax+b)(ax-3b)$

25. $(x+a)(x+b)$

26. $(x-a)(x+b)$

27. $(x+a)(x-b)$

28. $(x-a)(x-b)$

---

80. Cross-Multiplication.

(1)	(2)	(3)
$2x+3$	$2x-3$	$2x-3$
$3x+2$	$3x+2$	$3x-2$
$6x^2+9x$	$6x^2-9x$	$6x^2-9x$
$+4x+6$	$+4x-6$	$-4x+6$
$6x^2+13x+6$	$6x^2-5x-6$	$6x^2-13x+6$

Notice.—1. That $6x^2$ in the three examples is the product of the first terms.

2. That $+13x$, $-5x$, and $-13x$, are respectively the algebraic sum of the products obtained by a cross-multiplication of the first and last terms; $+13x=(3\times 3x)+(2\times 2x)$; $-5x=(-3\times 3x)+(2\times 2x)$; and $-13x=(-3\times 3x)+(-2\times 2x)$.

3. That $+6$, -6, and $+6$ are respectively the algebraic products of the last terms; $+2\times +3=+6$; $+2+-3=-6$; and $-2\times -3=+6$. Therefore,

***Prin. 41.**—The product of any two binomials equals the product of the first terms, and the algebraic sum of the products obtained by a cross-multiplication of the first and second terms, and the algebraic product of the second terms.*

Application.

Illustration.—Find the product of $3x-5$ and $2x+3$.

Solution: $(3x-5)(2x+3)=3x\times 2x+(+3\times 3x-5\times 2x)+(3\times -5)$ [P. 41] $=6x^2-x-15$.

EXERCISE 41.

Find the value of:

1. $(x+3)(2x+1)$
2. $(2x+4)(x+5)$
3. $(a+2)(3a+4)$
4. $(4a+3)(2a+2)$
5. $(3a+5)(2a+4)$
6. $(x+7)(5x+3)$
7. $(2x+y)(3x+2y)$
8. $(3a+2b)(4a+b)$
9. $(2x-3)(x-5)$
10. $(4x-1)(2x-7)$
11. $(x-2y)(3x-5y)$
12. $(2x-3y)(3x-2y)$
13. $(m+2n)(2m-3n)$
14. $(x^2+5)(2x^2-6)$
15. $(3x^2-4)(5x^2+3)$
16. $(ax+2)(2ax-5)$
17. $(4x-7)(5x-9)$
18. $(x^3-y^3)(2x^3-2y^3)$
19. $(3x^2-2y)(4x^2-5y)$
20. $(3x+5)(5x-3)$
21. $(5xy-6)(4xy+5)$
22. $(3m^2+5)(5m^2-7)$

Exact Division.

Definitions.

81. Any quantity that divides a given quantity without a remainder is a ***divisor*** of the quantity.

82. All the different quantities that divide a given quantity without a remainder are ***the divisors*** of the quantity.

Illustration.—The divisors of $a^2 b$ are 1, a, a^2, b, ab, and $a^2 b$.

83. The quantities that successively divide a given quantity and the resulting quotients, excepting unity, are the ***continued divisors*** of the quantity. They are the same as the factors of the quantity.

84. The process of finding one or more divisors of a quantity is ***Exact Division.***

Special Principles.

85. If we let a and b represent any two quantities, then will $a+b$ represent their sum, $a-b$ their difference, and a^2-b^2, a^4-b^4, a^6-b^6, etc., differences of equal even powers of them. Now, we may learn by actual division,

1. That a^2-b^2, a^4-b^4, and a^6-b^6, are divisible by $a-b$.
2. That a^2-b^2, a^4-b^4, and a^6-b^6, are divisible by $a+b$.

Therefore,

Prin. 42.*—The difference of the equal even powers of two quantities is divisible by both the sum and the difference of the quantities.*

86. a^2+b^2, a^4+b^4, and a^6+b^6, are not divisible by $a+b$.
a^2+b^2, a^4+b^4, and a^6+b^6, are not divisible by $a-b$.

Therefore,

Caution 1.*—The sum of the equal even powers of two quantities is not divisible by either the sum or the difference of the quantities.*

SIGHT EXERCISE.

Tell at sight which of the following examples will give rise to entire quotients, and why:

1. $(a^4-b^4)\div(a-b)$	**11.** $(4a^2-9b^2)\div(2a-3b)$
2. $(a^4-b^4)\div(a+b)$	**12.** $(4a^2-9b^2)\div(2a+3b)$
3. $(a^6-b^6)\div(a-b)$	**13.** $(4a^2+9b^2)\div(2a-3b)$
4. $(a^6-b^6)\div(a+b)$	**14.** $(16a^4-81b^4)\div(2a+3b)$
5. $(a^6+b^6)\div(a-b)$	**15.** $(16a^4+81b^4)\div(2a+3b)$
6. $(a^6+b^6)\div(a+b)$	**16.** $(16a^4-81b^4)\div(2a-3b)$
7. $(a^6-b^6)\div(a^3-b^3)$	**17.** $(16a^4+81b^4)\div(2a+3b)$
8. $(a^6-b^6)\div(a^3+b^3)$	**18.** $(a^{12}-b^{12})\div(a^2-b^2)$
9. $(a^6+b^6)\div(a^3+b^3)$	**19.** $(a^{12}+b^{12})\div(a^2-b^2)$
10. $(a^8-b^8)\div(a^2+b^2)$	**20.** $(a^4x^4-b^4y^4)\div(ax-by)$

87. If we let a and b represent any two quantities, then will $a+b$ represent their sum, and a^3+b^3, a^5+b^5, a^7+b^7, etc., sums of equal odd powers of them. Now, we may learn by actual division:

That a^3+b^3, a^5+b^5, and a^7+b^7 are divisible by $a+b$. Therefore,

Prin. 43.—*The sum of the equal odd powers of two quantities is divisible by the sum of the quantities.*

88. a^3+b^3, a^5+b^5, and a^7+b^7 are not divisible by $a-b$. Therefore,

Caution 2.—*The sum of the equal odd powers of two quantities is not divisible by the difference of the quantities.*

SIGHT EXERCISE.

Tell at sight which of the following examples will give rise to entire quotients, and why:

1. $(a^3+x^3)\div(a+x)$
2. $(a^3+x^3)\div(a-x)$
3. $(a^5+y^5)\div(a+y)$
4. $(x^5+y^5)\div(x-y)$
5. $(8a^3+27b^3)\div(2a+3b)$
6. $(8a^3+b^3)\div(2a-b)$
7. $(a^5+32b^5)\div(a+2b)$
8. $(a^6+b^6)\div(a^2+b^2)$
9. $(1+x^5)\div(1+x)$
10. $(x^7+1)\div(x+1)$
11. $(8+x^3)\div(2+x)$
12. $(8x^3+27)\div(2x+3)$
13. $(a^6+b^6)\div(a^2-b^2)$
14. $(a^9+b^9)\div(a^3+b^3)$
15. $(a^9+b^9)\div(a^3-b^3)$
16. $(a^{10}+x^{10})\div(a^2+b^2)$
17. $(a^{10}+x^{10})\div(a^2-b^2)$
18. $(a^3+b^6)\div(a+b^2)$
19. $(a^3+27b^9)\div(a+3b^3)$
20. $(a^3+27b^9)\div(a-3b^3)$
21. $(a^{12}+b^{12})\div(a+b)$
22. $(a^{12}+b^{12})\div(a^2+b^2)$
23. $(a^{12}+b^{12})\div(a^3+b^3)$
24. $(a^{12}+b^{12})\div(a^4+b^4)$
25. $(a^{10}+32b^{10})\div(a^2+2b^2)$
26. $(8a^9+27b^6)\div(2a^3+3b^2)$

89. If we let a and b represent any two quantities, then will $a-b$ represent their difference, and a^3-b^3, a^5-b^5, a^7-b^7, etc., differences of equal odd powers of them. Now, by actual division we learn:

That a^3-b^3, a^5-b^5, and a^7-b^7 are divisible by $a-b$. Therefore,

***Prin. 44.**—The difference of the equal odd powers of two quantities is divisible by the difference of the quantities.*

90. a^3-b^3, a^5-b^5, and a^7-b^7 are not divisible by $a+b$. Therefore,

***Caution 3.**—The difference of the equal odd powers of two quantities is not divisible by the sum of the quantities.*

SIGHT EXERCISE.

Tell at sight which of the following examples will give rise to entire quotients, and why:

1. $(a^5-x^5)\div(a-x)$
2. $(a^7-b^7)\div(a+b)$
3. $(a^6-b^6)\div(a^2-b^2)$
4. $(a^6-b^6)\div(a^2+b^2)$
5. $(a^9-b^9)\div(a^3-b^3)$
6. $(a^{10}-b^{10})\div(a^2-b^2)$
7. $(8x^3-1)\div(2x-1)$
8. $(a^{10}-b^{10})\div(a^2+b^2)$
9. $(a^{11}-b^{11})\div(a-b)$
10. $(8a^3-27b^3)\div(2a-3b)$
11. $(8a^3-y^6)\div(2a-y^2)$
12. $(x^9-27y^6)\div(x^3-3y^2)$
13. $(x^9-27y^6)\div(x^3+3y^2)$
14. $(32-x^{10})\div(2-x^2)$

91. By actual division we learn that:

$$(16a^4-81b^4)\div(2a-3b)=8a^3+12a^2b+18ab^2+27b^3$$

$$(16a^4-81b^4)\div(2a+3b)=8a^3-12a^2b+18ab^2-27b^3$$

$$(a^5-32b^{10})\div(a-2b^2)=a^4+2a^3b^2+4a^2b^4+8ab^6+16b^8$$

$$(a^5+32b^{10})\div(a+2b^2)=a^4-2a^3b^2+4a^2b^4-8ab^6+16b^8$$

By careful inspection we may observe the following laws of the quotient:

***Prin. 45.**—1. The number of terms equals the exponent of the power involved in the terms of the dividend.*

2. The terms are all positive when the divisor is the difference of two quantities, and alternately positive and negative when it is the sum.

3. The first term is found by dividing the first term of the dividend by the first term of the divisor.

4. Each succeeding term may be found by dividing the preceding term by the first term of the divisor, and multiplying the quotient by the second term of the divisor, disregarding the signs.

5. The last term may also be found by dividing the last term of the dividend by the last term of the divisor.

Note.—The fifth law may be used as a check upon the fourth to discover errors in work.

2. Applications.

EXERCISE 42.

Tell which of the following expressions will give rise to entire quotients, and according to what principle. Write the quotients according to the laws in [P. 45].

1. $(x^2 - y^2) \div (x - y)$
2. $(x^3 - y^3) \div (x - y)$
3. $(x^4 - y^4) \div (x - y)$
4. $(x^5 - y^5) \div (x - y)$
5. $(x^4 - y^8) \div (x - y^2)$
6. $(8x^3 - 27) \div (2x - 3)$
7. $(8x^3 - 1) \div (2x + 1)$
8. $(a^3 - 27b^3) \div (a - 3b)$
9. $(1 + x^5) \div (1 + x)$
10. $(81x^4 - 16) \div (3x + 2)$
11. $(a^5 + 32) \div (a - 2)$
12. $(x^6 - y^6) \div (x - y)$
13. $(x^6 - y^6) \div (x + y)$
14. $(x^6 - y^6) \div (x^2 - y^2)$
15. $(x^6 + y^6) \div (x^2 + y^2)$
16. $(x^6 + y^6) \div (x^2 - y^2)$
17. $(x^6 - 8y^6) \div (x^2 - 2y^2)$
18. $(x^6 - 8y^6) \div (x^2 + 2y^2)$

19. $(32x^{25}+1)\div(2x^5+1)$

20. $(a^{12}+b^{12})\div(a^3-b^3)$

21. $(1+729x^6)\div(1+9x^2)$

22. $(64-a^6)\div(4+a^2)$

23. $(625a^4-1)\div(5a+1)$

24. $(a^{15}+b^{30})\div(a^3+b^6)$

25. $(a^{15}+b^{30})\div(a^5+b^{10})$

26. $(a^6-729y^6)\div(a-3y)$

27. $(a^6-729y^6)\div(a+3y)$

28. $(a^6+729y^6)\div(a+3y)$

29. $(16a^4-81b^4)\div(2a+3b)$

30. $(x^6+64y^6)\div(x^2+4y^2)$

31. $(8x^6+27y^9)\div(2x^2+3y^3)$

32. $(x^{10}+32y^{20})\div(x^2+2y^4)$

33. $(x^8y^4-x^4y^8)\div(x^2y-xy^2)$

34. $(a^3x^{15}+b^3y^6)\div(ax^5+by^2)$

35. $(256x^4+10{,}000)\div(4a+10)$

36. $(a^6+729y^6)\div(a^2+9y^2)$

37. $(512a^3b^3+c^3)\div(8ab+c)$

38. $(8x^6+27y^9)\div(2x^2+3y^3)$

39. $(x^{10}+32y^{20})\div(x^2+2y^4)$

Factoring.

1. Definitions and Principles.

92. The quantities multiplied together to produce a given quantity are the ***Factors*** of the quantity.

93. The prime quantities multiplied together to produce a given quantity are the ***Prime Factors*** of the quantity.

94. A composite quantity may have two or more sets of factors, but it can have only one set of prime factors.

Thus, $a^2b^2=a^2\times b^2=a^2b\times b=ab^2\times a=ab\times ab=a\times a\times b^2=a^2\times b\times b=ab\times a\times b=a\times a\times b\times b.$

The last is the only set of prime factors.

95. The process of finding the factors of a quantity is ***Factoring.***

96. $ab \div a = b$; but, a and b are the factors of ab; therefore,

Prin. 46.*—A divisor of a quantity is one of the two factors of the quantity, and the quotient is the other.*

97. Since a divisor of every term of a quantity is a divisor of the quantity [P. 15], and a divisor of a quantity is a factor of the quantity [P. 46], it follows that,

Prin. 47.*—A factor of every term of a quantity is a factor of the quantity.*

2. Problems.

1. To factor a polynomial having a common factor in its terms.

Illustration.—Factor $a^2c^2 - ac^2 + a^2c$.

Form.

$$(a^2c^2 - ac^2 + a^2c) = ac(ac - c + a)$$

Solution: We see by inspection that ac is a factor of each term of the polynomial; it is therefore a factor of the polynomial [P. 47]. Dividing by ac, the quotient $ac - c + a$ is the other factor [P. 46]. Therefore, $(a^2c^2 - ac^2 + a^2c) = ac(ac - c + a)$.

EXERCISE 43.

Factor:

1. $a^2 + ab$
2. $ab - bc$
3. $x^2 + axy$
4. $x^3 + 3x^2 - 2x$
5. $3a^2 - 6ab + 9ab^2$
6. $2a^2x + 4a^3x^2 - 6a^4x^3$
7. $6x^3y^2 - 12x^2y^3 - 18xy$
8. $10x^4 + 15x^3 - 20x^2$
9. $7r^2 - 14r^3 + 21r^4$
10. $2a(a+b) + 3b(a+b)$
11. $a(a-x) - b(a-x)$
12. $c(m+n) + d(m+n)$
13. $12a^4b^3c^2 - 24a^3b^4c^4 + 36a^4b^2c$

14. $10p^2q^3 + 15p^3q^3 - 20pqr$

15. $24x^8y^6 - 36x^5y^9 + 48x^4y^{10}$

16. $4c(a^2 + b^2) - 5d(a^2 + b^2)$

---

2. To factor the difference of two squares.

Illustrations.—Factor $a^2 - b^2$, $x^4 - y^4$, and $x^6y^2 - x^4y^4$.

Solutions: 1. $a^2 - b^2 = (a + b)(a - b)$ [P. 39].

2. $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$ [P. 39] $= (x^2 + y^2)(x + y)(x - y)$ [P. 39].

3. $x^6y^2 - x^4y^4 = x^4y^2(x^2 - y^2)$ [P. 47 and 46] $= x^4y^2(x + y)(x - y)$ [P. 39].

EXERCISE 44.

Factor:

1. $a^2 - 4b^2$
2. $4a^2 - 25b^2$
3. $9x^2 - 49y^2$
4. $a^2y^2 - 4$
5. $16 - z^2$
6. $x^2 - 64$
7. $x^2y^2 - 100$
8. $81 - z^2$
9. $a^2b^2c^2 - 36$
10. $x^2y^2 - y^2z^2$
11. $a^4 - z^4$
12. $a^8 - b^4$
13. $16a^4 - 81z^4$
14. $81y^4 - 256z^4$
15. $x^4y^2 - x^2y^4$
16. $x^6 - y^4$
17. $625 - z^4$
18. $x^8 - y^8$
19. $x^{12} - y^4$
20. $m^8 - n^{16}$
21. $(a + b)^2 - c^2$
22. $(a - x)^2 - y^2$
23. $(m - n)^2 - 1$
24. $4 - (x + y)^2$
25. $c^2 - (a + b)^2$
26. $c^2 - (a - b)^2$
27. $25a^2 - (x - y)^2$
28. $16 - (z - x)^2$
29. $1 - (x - y)^2$
30. $49 - 4(x + y)^2$

---

3. To factor the sum or difference of the equal odd powers of two quantities.

Illustration.—Factor $a^3 - b^3$, $a^3 + b^3$, and $a^6 - b^6$.

Solutions: 1. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ [P. 44 and 45].

2. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ [P. 43 and 45].

3. $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$ [P. 39] $= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2)$ [P. 43, 44, 45].

EXERCISE 45.

Factor the following binomials. Three can not be factored:

1. $x^3 - y^3$
2. $x^3 + y^3$
3. $a^3 - 1$
4. $a^3 + 1$
5. $x^3 - 8$
6. $x^3 + 8$
7. $8a^3 + b^3$
8. $8a^3 - b^3$
9. $27a^3 - 8b^3$
10. $27a^3 + 8b^3$
11. $a^6 + 1$
12. $a^6 - 1$
13. $x^6 + 8$
14. $x^6 - 8$
15. $x^5 + y^5$
16. $x^5 - y^5$
17. $x^7 + y^7$
18. $32x^5 - y^{10}$
19. $16x^5 + 81y^5$
20. $8x^6 - 27y^9$
21. $x^8 + y^8$
22. $x^9 - y^9$
23. $x^4 + x$
24. $x^5 - 8x^2$
25. $x^3y^3 + z^3$
26. $m^5 + n^3$
27. $64m^6 + 125n^6$
28. $x^4y^4 - xy$
29. $x^{10} + xy^9$
30. $(x+y)^3 - z^3$
31. $(x+y)^3 + z^3$
32. $x^3 - (y+z)^3$
33. $x^3 + (y+z)^3$
34. $(a+b)^3 - (c+d)^3$

---

4. To factor a trinomial that is a perfect square.

Illustrations.—

1. $a^2 + 2ab + b^2 = (a+b)(a+b)$, since $(a+b)^2 = a^2 + 2ab + b^2$ [P. 31].
2. $a^2 - 2ab + b^2 = (a-b)(a-b)$, since $(a-b)^2 = a^2 - 2ab + b^2$ [P. 32].
3. $4a^2 + 12ab + 9b^2 = (2a+3b)(2a+3b)$, since $(2a+3b)^2 = 4a^2 + 12ab + 9b^2$ [P. 31].
4. $4a^2 - 12ab + 9b^2 = (2a-3b)(2a-3b)$, since $(2a-3b)^2 = 4a^2 - 12ab + 9b^2$ [P. 32].

98. A trinomial is a perfect square when two of its terms are perfect squares, and the other term is $\pm$ twice the square root of their product.

EXERCISE 46.

Factor the following trinomials. Three can not be factored:

1. $x^2 + 2xy + y^2$
2. $x^2 - 2xz + z^2$
3. $x^2 + 2x + 1$
4. $x^2 - 4x + 4$
5. $x^2 + 18x + 81$
6. $4x^2 - 12x + 9$
7. $9x^2 + 12xy + 4y^2$
8. $25x^2 + 10x + 1$
9. $x^4 - 12x^2 + 36$
10. $x^4 - 8x^2 + 16$
11. $x^6 + 2x^3y^3 + y^6$
12. $a^2b^2 - 2abcd + c^2d^2$
13. $16x^4 - 72x^2y^2 + 81y^4$
14. $x^4 + 2y^2 + 4$
15. $x^{10} + 2x^5 + 1$
16. $x^6y^2 + 4x^4y^2 + 4x^2y^2$
17. $x^8 - 2x^4y^4 + y^8$
18. $4x^4 + 14x^2 + 49$
19. $4x^6 + 12x^3 + 9$
20. $9x^6 - 36x^4y^2 + 36x^2y^4$
21. $100x^2 - 110xy + 121y^2$
22. $x^4y^2 + 2x^2y^3z + y^4z^2$
23. $x^6 + 4y^6 - 4x^3y^3$
24. $16x^8 + 256y^8 - 128x^4y^4$

---

5. To factor a trinomial that is the product of two binomials having a like term.

Illustrations.—

1. $a^2 + 3a + 2 = (a + 2)(a + 1)$, since $(a + 2)(a + 1)$
$= a^2 + 3a + 2$ [P. 40].
2. $a^2 - 5a + 6 = (a - 2)(a - 3)$, since $(a - 2)(a - 3)$
$= a^2 - 5a + 6$ [P. 40].
3. $a^2 + 2a - 8 = (a + 4)(a - 2)$, since $(a + 4)(a - 2)$
$= a^2 + 2a - 8$ [P. 40].
4. $(4a^2 - 4a - 15) = (2a + 3)(2a - 5)$, since
$(2a + 3)(2a - 5) = 4a^2 - 4a - 15$ [P. 40].

99. A trinomial is the product of two binomials having a like term, when the first term is a square, and the last term is the algebraic product of two factors whose sum, multiplied by the square root of the first term, will give the middle term. The square root of the first term is the like term of the binomials, and the factors of the third term are the two unlike terms.

EXERCISE 47.

Factor the following trinomials. Three can not be factored :

1. $x^2 + 8x + 15$

2. $x^2 + 5x + 4$

3. $x^2 + 6x + 8$

4. $a^2 - 7a + 12$

5. $a^2 - 9a + 14$

6. $a^2 - 13a + 40$

7. $x^2 + 2x - 15$

8. $x^2 + 3x - 28$

9. $x^2 + 6x - 16$

10. $x^2 - 4x - 5$

11. $x^2 - 4x - 21$

12. $x^2 - 2x - 80$

13. $4x^2 + 8x + 3$

14. $4x^2 + 14x + 12$

15. $9x^2 + 9x + 2$

16. $x^2 + 4ax + 3a^2$

17. $x^2 - 2ax - 15a^2$

18. $4x^2 - 8ax - 3a^2$

19. $9y^2 + 3yz - 2z^2$

20. $36x^2 + 24bx - 5b^2$

21. $4a^2x^2 - 4ax - 15$

22. $x^4 + 7x^2 - 12$

23. $x^4 - 7ax^2 + 12a^2$

24. $4x^4 + 8x^2 - 3$

---

6. To factor a trinomial that is the product of any two binomial factors.

Illustrations.—

1. $2x^2 + 5x + 2 = (x + 2)(2x + 1)$, since
$$(x + 2)(2x + 1) = 2x^2 + 5x + 2 \quad \text{[P. 41]}.$$

2. $6x^2 - 13x + 6 = (2x - 3)(3x - 2)$, since
$$(2x - 3)(3x - 2) = 6x^2 - 13x + 6 \quad \text{[P. 41]}.$$

3. $2x^2 + x - 15 = (x + 3)(2x - 5)$, since
$$(x + 3)(2x - 5) = 2x^2 + x - 15 \quad \text{[P. 41]}.$$

4. $6x^2 - 11x - 10 = (2x - 5)(3x + 2)$, since
$$(2x - 5)(3x + 2) = 6x^2 - 11x - 10 \quad \text{[P. 41]}.$$

100. The first terms of the factors are the factors of the first term of the trinomial, the last terms of the factors are the factors of the last term of the trinomial, and the last terms of the factors are so arranged with the first terms that the algebraic sum of the products obtained by multiplying the first term of each factor by the second term of the other will give the middle term of the trinomial.

Note.—This and the following problem may be omitted until the class reaches page 192, if desirable.

EXERCISE 48.

Factor the following trinomials. Two can not be factored. Why?

1. $2x^2 + 5x + 3$	**10.** $15a^2 + 20a - 35$
2. $2x^2 + 11x + 12$	**11.** $2x^2 + 19x - 35$
3. $6x^2 + 7x + 2$	**12.** $2a^2 + ab - b^2$
4. $6x^2 + 11x + 3$	**13.** $2a^2 + 5ab + 2b^2$
5. $2x^2 - 7x + 6$	**14.** $6x^2 - 11xy + 3y^2$
6. $2x^2 + x - 6$	**15.** $6u^2 + 5uv - 6v^2$
7. $2x^2 - x - 15$	**16.** $2x^2 - 7xy - 3y^2$
8. $12x^2 + 11x - 15$	**17.** $6x^4 - 11x^2 - 35$
9. $6a^2 + a - 15$	**18.** $2a^2b^2 + ab - 6$

---

7. To factor a trinomial that is the product of two trinomials of the form of $x^2 + xy + y^2$ and $x^2 - xy + y^2$.

Solution: The product of $x^2 + xy + y^2$ and $x^2 - xy + y^2$ is $x^4 + x^2y^2 + y^4$; therefore, a trinomial is the product of two trinomials of the form of $x^2 + xy + y^2$ and $x^2 - xy + y^2$ when all its terms are positive squares and the middle term is the square root of the product of the other two.

***Rule.**—The factors may be obtained by extracting the square root of each term and making the middle term of one factor positive and that of the other negative.*

Illustrations.—

1. $a^4 + a^2 + 1 = (a^2 + a + 1)(a^2 - a + 1)$.
2. $x^4 + a^2x^2 + a^4 = (x^2 + ax + a^2)(x^2 - ax + a^2)$.

EXERCISE 49.

Factor the following trinomials. One can not be factored:

1. $x^4 + x^2 + 1$
2. $x^4 + 4x^2 + 16$
3. $a^4 + a^2b^2 + b^4$
4. $a^8 + a^4c^4 + c^8$
5. $16a^4 + 4a^2 + 1$
6. $a^8 + 4a^4b^2 + 16b^4$
7. $x^8 + x^4y^2 + y^4$
8. $x^8 + x^4y^6 + y^{12}$
9. $x^8 - x^4y^4 + y^8$
10. $81x^4 + 36x^2y^2 + 16y^4$
11. $x^8 + x^4y^8 + y^{16}$
12. $a^4x^4 + a^2x^2y^2 + y^4$
13. $a^4y^8 + a^6y^6 + a^8y^4$
14. $a^8b^8 + 4a^4b^4 + 16$
15. $81 + 9b^2 + b^4$
16. $625 + 25x^2y^2 + x^4y^4$
17. $256 + 16z^4 + z^8$
18. $x^4y^4 + x^2y^4z^2 + y^4z^4$

---

8. To factor polynomials.

Illustrations.—

1. Factor $ax + by - bx - ay$.

Solution:

$$ax + by - bx - ay =$$
$$ax - bx - ay + by =$$
$$x(a - b) - y(a - b) =$$
$$(a - b)(x - y)\ \text{[P. 46, 47].}$$

2. Factor $x^2 - 2xy + y^2 - z^2$.

Solution:

$$x^2 - 2xy + y^2 - z^2 =$$
$$(x^2 - 2xy + y^2) - z^2 =$$
$$(x - y)^2 - z^2 =$$
$$(x - y - z)(x - y + z)\ \text{[P. 39].}$$

3. Factor $x^2 - y^2 - 2yz - z^2$.

Solution:

$$x^2 - y^2 - 2yz - z^2 =$$
$$x^2 - (y^2 + 2yz + z^2) =$$
$$x^2 - (y + z)^2 =$$
$$(x + \overline{y + z})(x - \overline{y + z})\ \text{[P. 39]} =$$
$$(x + y + z)(x - y - z)\ \text{[P. 23, 24].}$$

EXERCISE 50.

Factor:

1. $ax + ay + bx + by$
2. $bx - by + cx - cy$
3. $ax - az - bx + bz$
4. $ab + 2b + 3a + 6$
5. $9 + 3x - 3y - xy$
6. $2ax + 3ay + 4bx + 6by$
7. $6ax + 4ay - 9bx - 6by$
8. $abx^2 + 2ax + 3bx + 6$
9. $axy + 6a - bxy - 6b$
10. $a^2x^2 + a^2y^2 - b^2x^2 - b^2y^2$
11. $a^2 + 2ab + b^2 - c^2$
12. $x^2 - 2x + 1 - y^2$
13. $x^2 - y^2 - 2y - 1$
14. $4x^2 + 4xy + y^2 - z^2$
15. $4z^2 - 4x^2 - 4x - 1$
16. $a^2 + 2ab + b^2 - 16$
17. $25 - x^2 - 2ax - a^2$
18. $x^4 - x^2 + 2x - 1$
19. $x^4 + 2x^2y^2 + y^4 - y^2$
20. $m^4 - p^4 - 2p^2q - q^2$

Miscellaneous Examples.

EXERCISE 51.

(Take out monomial factors first.)

Factor:

1. $a^3b - ab^3$
2. $3a^3 - 12a$
3. $2a^4 - 2ab^3$
4. $3a^3b + 3b^4$
5. $x^5y - xy^5$
6. $ax^7 + ax^2$
7. $a^7b - ab^7$
8. $2a^8b^2 + 2a^2b^8$
9. $5a^2 + 10ay + 5y^2$
10. $2a^3c + 12a^2c + 18ac$
11. $x^3y^2z - 2x^2yz + xz$
12. $3ax^5 - 6ax^3y^2 + 3axy^4$
13. $4y^7 + 8y^4 + 4y$
14. $2x^3 + 10x^2 + 12x$
15. $x^3y - 9x^2y + 20xy$
16. $4a^3b + 4a^2b - 168ab$
17. $a^3 - a(b + c)^2$
18. $a^5 + a^2(b + c)^3$
19. $a^3c - a^2c^2 - 2ac^3$
20. $4ab + 8a + 12b + 24$
21. $axy - bxy - ay^2 + by^2$
22. $x^2 - y^2 - x + y$

23. $(x+y)^2 - x - y$
24. $(a+b)^2 x^2 - c^2 x^2$
25. $x^3 - 2x^2 y + x y^2 - x z^2$
26. $x^3 - 3x^2 y + 3 x y^2 - y^3$
27. $x^4 + 6x^3 + 12 x^2 + 8x$
28. $a^4 y^4 - 1$
29. $x^3 y^3 - z^3$
30. $x^{12} + y^{12}$
31. $x^{12} - y^{12}$
32. $m^5 n^8 - m$
33. $121 a^4 + 144 b^4 + 264 a^2 b^2$
34. $16 a^2 + 8ab - 3b^2$
35. $3mn - amn + 2a - 6$
36. $a^2 + 2a + 1 - b^2$
37. $a^2 b^2 - a^2 - 2ab - b^2$
38. $1 - x^2 - 2xy - y^2$
39. $a^2 - m^2 + 2mn - n^2$
40. $1 - (a+b)^3$
41. $a^8 + a^4 b^4 + b^8$
42. $(x+y)^2 - (x-y)^2$
43. $a^9 + a^2 y^7$
44. $(a+b)^2 - 2(a+b) + 1$
45. $x^2 - (x+y+z)^2$
46. $x^4 - (y+z)^4$
47. $x^3 + 3x^2 y + 3 x y^2 + y^3$
48. $1 - (a-b)^2$
49. $a^2 bc + ab^2 c + a^2 bd + ab^2 d$
50. $3x^2 - 15xy - 2bx + 10by$

Highest Common Divisor.

1. Definitions and Principles.

101. A divisor of each of two or more quantities is a ***Common Divisor*** of the quantities.

102. The common divisor that contains the greatest number of prime factors is the ***Highest Common Divisor.***

103. Quantities that have no common divisor except *one* are ***prime to each other.***

104. Since a quantity equals the product of its prime factors, it is divisible by the product of any two or more of them; hence, too, each of two or more quantities is divisible by the product of any two or more of their common prime factors; and therefore,

***Prin. 48.**—The highest common divisor is the product of all the common prime factors.*

105. The abbreviation H. C. D. stands for highest common divisor.

2. Problems.

1. To find the highest common divisor of monomials.

Illustration.—Find the H. C. D. of $16a^2b^3c^4$, $24a^3b^2c^2$, and $36a^4b^3d^2$.

Solution:

$$16a^2b^3c^4 = 2\times2\times2\times2\times a^2\times b^3\times c^4$$
$$24a^3b^2c^2 = 2\times2\times2\times3\times a^3\times b^2\times c^2$$
$$36a^4b^3d^2 = 2\times2\times3\times3\times a^4\times b^3\times d^2$$
$$\therefore \text{ H. C. D.} = 2\times2\times a^2\times b^2 = 4a^2b^2 \text{ [P. 48].}$$

Another Solution: The H. C. D. of the coefficients is 4; of the a's is a^2; of the b's is b^2; and c and d are not common to the three quantities. Therefore, the H. C. D. is $4a^2b^2$.

***Rule.**—Find the highest common divisor of the numerical coefficients, annex to it the different common literal factors, giving each the lowest exponent it has in any one of the quantities.*

EXERCISE 52.

Find the H. C. D. of:

1. $8x^2y$, $12xy^2$, and $24x^2y^2$
2. $15a^2b^2c^2$, $25a^2b^2$, and $30a^2c^2$
3. $20xy^2z$, $30x^2yz$, and $40xyz^2$
4. $20a^2b^3$, $25a^3b^2c$, and $35a^4b^3c^2$
5. $18m^3n^2$, $24am^2n^3$, and $36bm^3n^3$
6. $(a+b)^2$, $(a+b)^3$, and $(a+b)^4$
7. $3(x+y)^3$, $6(x+y)^4$, and $9(x+y)^5$
8. $m(m+n)^2$, $m^2(m+n)^3$, and $m^3(m+n)$

9. $2ax(x^2+y^2)$, $4ax^2(x^2+y^2)^2$, and $6a^2x^2(x^2+y^2)^3$

10. $4(m-n)^2$, $6(n-m)^2$, and $8(m-n)^3$

11. $xy(a-b)^3$, $x^2y(a-b)^4$, and $y^2(a-b)^2$

12. $(a+b)(a-b)$, $(a+b)^2(a-b)^2$, and $(a+b)^2(a-b)^3$

13. $2(x-y)^2$, $4(x-y)^3$, $6(x-y)^4$, and $8(x-y)^5$

14. $3a(m-n)^2$, $6ab(m-n)^3$, $9a^2(m-n)^4$, and $12ab^2(m-n)^5$

15. $(m+n)(m-n)$, $(m+n)(n-m)$, and $(m+n)^2(m-n)^2$

16. $(p-q)(a-b)$, $(q-p)(a-b)$, and $(p-q)(b-a)$

17. $(a-b)^2$, $(b-a)^2$, and $(a-b)(b-a)$

18. $ab(a-b)$, $-b(b-a)$, and $a(a-b)^2$

---

2. To find the highest common divisor of polynomials.

Illustration.—Find the H. C. D. of x^4-y^4, x^3y-xy^3, and $x^3-x^2y-xy^2+y^3$.

Form.

$$x^4-y^4=(x^2-y^2)(x^2+y^2)=(x-y)(x+y)(x^2+y^2)$$

$$x^3y-xy^3=xy(x^2-y^2)=xy(x-y)(x+y)$$

$$x^3-x^2y-xy^2+y^3=x^2(x-y)-y^2(x-y)=$$
$$(x^2-y^2)(x-y)=(x+y)(x-y)(x-y)$$

$$\text{H. C. D.}=(x+y)(x-y)=x^2-y^2 \text{ [P. 48].}$$

Solution: We resolve the quantities into their prime factors, and observe that $a+b$ and $a-b$ are the only common factors; therefore, $(x+y)(x-y)$, or x^2-y^2, is the H. C. D. [P. 48].

EXERCISE 53.

Find the H. C. D. of:

1. $a+b$ and a^2-b^2

2. $(a+b)^2$ and a^4-b^4

3. $(a-b)^2$ and a^2-b^2

4. $(a+b)^3$ and a^3+b^3

5. x^2-xy, x^2-y^2, and $x^2-2xy+y^2$

6. $(x+y)^2$, x^2-y^2, and x^2+xy

7. x^2-y^2, $x^2-2xy+y^2$, and x^2y-xy^2

8. $(x+y)^3$, x^3+x^2y, and x^3+y^3

9. x^2-4x, $x^2-8x+16$, and x^2-2x-8

10. $a^4+4a^3+4a^2$, a^3b-4ab, and $a^4b+5a^3b+6a^2b$

11. x^3-8x^2+15x, $x^2y-8xy+15y$, and $x^2z-8xz+15z$

12. $3x^2-3y^2$, $x^4-2x^2y^2+y^4$, and x^2y-y^3

13. $x^2+xy+xz$, $xy+y^2+yz$, and $x^2-(y+z)^2$

14. x^2+x-6, $x^2+7x+12$, and $x^2-2x-15$

15. x^3+27, x^2+5x+6, and x^2+6x+9

16. $x^2+ax+bx+ab$ and $x^2+ax+cx+ac$

17. m^2-n^2, $am+an+bm+bn$, and $m^2+2mn+n^2$

18. x^4+xy^3, $x^6+2x^3y^3+y^6$, and ax^3+ay^3

19. $x^4+4x^2y+4y^2$, x^4-4y^2, and x^3y+2xy^2

20. $x^2-(y+z)^2$, $y^2-(x+z)^2$, and $z^2-(x+y)^2$

21. $x^2-(y+2)^2$, $y^2-(x+2)^2$, and $4-(x+y)^2$

22. x^3+y^3, $x^3y+2x^2y^2+xy^3$, and $x^3+3x^2y+3xy^2+y^3$

23. $x^5y+12x^3y+35xy$, $x^6+3x^4-28x^2$, and $x^5z-x^3z-56xz$

24. $6x^2+10x-24$, $2x^2-2x-24$, and $8x^2+22x-6$

25. x^3-y^3, $x^4+x^2y^2+y^4$, and $x^3y+x^2y^2+xy^3$

26. x^3+5x^2+6x, x^3y+x^2y-6xy, $x^4-x^3-12x^2$

27. x^4-16, x^4+8x^2+16, x^4+2x^2-8

28. x^4-a^4, $x^4-2a^2x^2+a^4$, $x^3-ax^2-a^2x+a^3$

29. x^6-y^6, x^2+xy+y^2, $x^4+x^2y^2+y^4$

For highest common divisor by successive division, see Appendix.

The Lowest Common Multiple.

1. Definitions.

106. A quantity that exactly contains a given quantity is a ***Multiple*** of the quantity.

107. A quantity that exactly contains each of two or more given quantities is a ***Common Multiple*** of those quantities.

108. The common multiple that contains the least number of prime factors is the ***Lowest Common Multiple.***

109. The abbreviation L. C. M. stands for lowest common multiple.

110. The L. C. M. of $a^3 b^2 c$, $a b^3 c^2$, and $a^2 b^2 c^3$ must contain each of these quantities [Art. 107] :

$$a^3 b^2 c = a \times a \times a \times b \times b \times c$$
$$a b^3 c^2 = a \times b \times b \times b \times c \times c$$
$$a^2 b^2 c^3 = a \times a \times b \times b \times c \times c \times c$$

To contain $a^3 b^2 c$, the L. C. M. must contain the prime factors a, a, a, b, b, c.

To contain $a b^3 c^2$, it must contain the additional prime factors b and c.

To contain $a^2 b^2 c^3$, it must contain the still additional prime factor c.

Since these are the only factors required to contain each of the quantities, and all are necessary, the

L. C. M. $= a \times a \times a \times b \times b \times b \times c \times c \times c = a^3 b^3 c^3$.

Therefore,

Prin. 49.—*The lowest common multiple of two or more quantities equals the product of all their different prime factors, each taken the greatest number of times it occurs in any one of them.*

2. Problems.

1. To find the lowest common multiple of monomials.

Illustration.—Find the L. C. M. of $24a^5b^3c$, $36a^3b^2c^3$, and $56a^2b^4c^2$.

Solution:
$$24a^5b^3c = 2\times2\times2\times3\times a^5\times b^3\times c$$
$$36a^3b^2c^3 = 2\times2\times3\times3\times a^3\times b^2\times c^3$$
$$56a^2b^4c^2 = 2\times2\times2\times7\times a^2\times b^4\times c^2$$
$$\therefore \text{ L. C. M.} = 2\times2\times2\times3\times3\times7\times a^5\times b^4\times c^3 = 504a^5b^4c^3 \text{ [P. 49].}$$

Another Solution: The L. C. M. of the coefficients is 504; of the a's is a^5; of the b's is b^4; and of the c's is c^3. Therefore, the L. C. M. of the quantities is $504a^5b^4c^3$.

Rule.—*Find the least common multiple of the numerical coefficients; annex to it all the different literal factors found in the quantities, giving each the highest exponent it contains in the quantities.*

EXERCISE 34.

Find the L. C. M. of:

1. $10x^2y$, $15xy^2$, and $20x^2y^2$
2. $12a^2b^3$, $18ab^2c$, and $24a^2c^3$
3. $24ax^3$, $32bxy$, and $48cy^2$
4. $22x^2y^3$, $33x^2z^2$, and $44y^2z^2$
5. $24ab^2x^2$, $36a^2x^3z$, and $48b^3z^3$
6. $48m^3n^2$, $56m^2nx$, and $63n^2x^3$
7. $(a+b)^2$, $(a+b)^3$, and $(a+b)^4$
8. $25(x+y)^2$, $50(x+y)^3$, and $100(x+y)^5$
9. $a^3(x-y)$, $a^2(x-y)^2$, and $a^4(x-y)^3$
10. $8a^2(x+z)^4$, $12ab(x+z)^3$, and $24b^2(x+z)^2$
11. $9x^2z^2(x^2+y^2)^4$, $18xz^3(x^2+y^2)^5$, and $36x^3z(x^2+y^2)^3$
12. $(a+b)(a-b)$, $(a^2+b^2)(a+b)$, and $(a^2+b^2)(a-b)$

2. To find the lowest common multiple of polynomials.

Illustration.—Find the L. C. M. of a^2-b^2, a^3-b^3, and a^2+ab.

Form.

$$a^2-b^2 = (a+b)(a-b)$$
$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$
$$a^2+ab = a(a+b)$$
$$\text{L. C. M.} = a(a+b)(a-b)(a^2+ab+b^2)$$

Solution: To contain a^2-b^2, the L. C. M. must contain the prime factors $a+b$ and $a-b$, and to contain a^3-b^3 it must contain the additional factor a^2+ab+b^2, and to contain a^2+ab it must contain the additional factor a. Therefore, the L. C. M. is $a(a+b)(a-b)(a^2+ab+b^2)$.

EXERCISE 53.

Find the L. C. M. of:

1. $(a+b)^2$ and a^2-b^2
2. $(a-b)^2$ and a^4-b^4
3. x^2-y^2 and x^4-y^4
4. x^2-y^2 and x^3-y^3
5. $x+y$, x^2-y^2, and $x^2+2xy+y^2$
6. $x-y$, x^2-y^2, and $x^2-2xy+y^2$
7. x^2-y^2, x^3-y^3, and $x^2-2xy+y^2$
8. $(x+a)(x+b)$, $(x+a)(x+c)$, and $(x+b)(x+c)$
9. $a(a-b)$, $b(b-a)$, and $-c(a-b)$
10. $(a-b)(b-c)$, $(b-a)(b-c)$, and $(b-a)(c-b)$
11. $(x+y)^3$, x^3+y^3, and x^2-y^2
12. x^2+5x+6, x^2-2x-8, and x^2-x-12
13. x^2+3x-4, x^2-6x+5, and x^2-x-20
14. $am+an+bm+bn$ and $ap+aq+bp+bq$
15. $ax-bx-ay+by$ and $ax-ay+bx-by$
16. $a^2-(b+c)^2$, $b^2-(a+c)^2$, and $c^2-(a+b)^2$
17. $a^3+3a^2b+3ab^2+b^3$ and $a^3-ab^2+a^2b-b^3$

18. $x^3 + y^3$, $x^4 + x^2y^2 + y^4$, and $x^3 - y^3$

19. $2x^2 + 11x + 15$, $2x^2 + x - 10$, and $x^2 + x - 6$

20. $6x^2 + 13x + 6$, $6x^2 - 5x - 6$, and $4x^2 - 9$

21. $ax^2 - ay^2$, $x^4 - xy^3$, and $x^3y + y^4$

22. $x^3 + 5x^2 + 6x$, $x^3y - x^2y - 6xy$, and $x^2y - 9y$

23. $x^2 + xy + y^2$, $x^2 - xy + y^2$, and $x^4 + x^2y^2 + y^4$

24. $x^2 + y^2$, $x^4 - x^2y^2 + y^4$, and $x^6 + y^6$

Cancellation.

1. Definitions and Principles.

111. Multiplying a quantity by a factor is called ***inserting*** a factor.

112. Dividing a quantity by a factor is called ***eliminating*** a factor.

113. Crossing out a quantity and writing in its stead the result obtained by inserting or eliminating a factor is ***Cancellation.***

114. $ab \times ac = a^2bc$. If we now eliminate a from ab and insert it in ac, we have $b \times a^2c$, which also equals a^2bc. Therefore,

Prin. 50.—*Dividing one quantity and multiplying another by the same factor does not alter their product.*

115. $abcd \div ab = cd$. But if we insert b in $abcd$ we have $ab^2cd \div ab$, which equals bcd. Also, if we eliminate b from ab, we have $abcd \div a$, which equals bcd. Therefore,

Prin. 51.—*Multiplying the dividend or dividing the divisor multiplies the quotient.*

116. $abcd \div ab = cd$. But if we eliminate c from $abcd$, we have $abd \div ab$, which is d. Also, if we insert c in ab, we have $abcd \div abc$, which is d. Therefore,

***Prin. 52.**—Dividing the dividend or multiplying the divisor divides the quotient.*

117. $abcd \div ab = cd$. If we now insert b in both $abcd$ and ab, we have $ab^2cd \div ab^2$, which is cd. Also, if we eliminate b from both $abcd$ and ab, we have $acd \div a$, which is cd. Therefore,

***Prin. 53.**—Multiplying or dividing both dividend and divisor by the same quantity does not alter the quotient.*

2. Problem.

To multiply or divide by cancellation.

Illustrations.—1. Multiply 36 by 25.

Solution: Since dividing one quantity and multiplying another by the same factor does not alter their product [P. 50], we divide 36 by 4 and multiply 25 by 4, and obtain 9×100, which is 900.

Form.

$$\overset{9}{\cancel{36}} \times \overset{100}{\cancel{25}} = 900$$

2. Multiply $(a+b)^2$ by $(a-b)$.

Form.

$$\overset{a+b}{\cancel{(a+b)^2}} \times \overset{a^2-b^2}{\cancel{(a-b)}} = a^3 - ab^2 + a^2b - b^3$$

Solution: Dividing $(a+b)^2$ by $(a+b)$, and multiplying $(a-b)$ by $(a+b)$ [P. 50], we have $(a+b) \times (a^2-b^2)$, which is $a^3 - ab^2 + a^2b - b^3$.

3. Divide $(a^3+b^3)(a^2-b^2)$ by $(a+b)^2$.

Form.

$$\frac{(a^3+b^3)(a^2-b^2)}{(a+b)^2} = \frac{\cancel{(a+b)}(a^2-ab+b^2)\cancel{(a+b)}(a-b)}{\cancel{(a+b)}\cancel{(a+b)}} = a^3 - 2a^2b + 2ab^2 - b^3.$$

Solution: Dividing both dividend and divisor by $(a+b)(a+b)$ [P. 53], we have $(a^2-ab+b^2)(a-b)$, which is $a^3 - 2a^2b + 2ab^2 - b^3$.

EXERCISE 56.

Solve by cancellation:

1. 44×25
2. 36×15
3. $27 \times 33\frac{1}{3}$
4. $42 \times 16\frac{2}{3}$
5. $56 \times 12\frac{1}{2}$
6. 48×36
7. $26\frac{2}{3} \div 6\frac{2}{3}$
8. $35\frac{5}{7} \div 7\frac{1}{7}$
9. $144 \div 36$

10. $\dfrac{18 \times 24}{16}$

11. $\dfrac{25 \times 36}{6\frac{1}{4} \times 7\frac{1}{5}}$

12. $\dfrac{8\frac{1}{3} + 6\frac{1}{4}}{2\frac{1}{12}}$

13. $\dfrac{x+y}{x-y} \times (x^2 - y^2)$

14. $\dfrac{(x+y)^2 (x-y)^2}{x^2 - y^2}$

15. $\dfrac{(x^3+y^3)(x^3-y^3)}{x^2-y^2}$

16. $\dfrac{(x^2-4)(x^2+6x+8)}{(x^2+2x-8)}$

17. $\dfrac{(x^2+7x+12)(x^2+11x+30)}{(x+6)(x^2+8x+15)}$

18. $\dfrac{(a^2+ac+ab+bc)(b^2+bc+bd+cd)}{(ab+ac+b^2+bc)(ab+ad+bc+cd)}$

19. $\dfrac{\{a^2-(b+c)^2\}\{a^2-(b-c)^2\}}{(a+b+c)(a+b-c)}$

Find the value of the following expressions, when $a = 10$, $b = 8$, $c = 6$, and $d = 4$.

20. $\dfrac{(a^2-2ab+b^2)(a^2+2ab+b^2)}{(a^2-b^2)}$

21. $\dfrac{a^2-(b+c)^2}{c^2-(a+b)^2} \times \dfrac{a+b+c}{a-b-c}$

22. $\dfrac{(a-b)^2-c^2}{(b-c)^2-d^2} \times \dfrac{b-c-d}{a-b+c}$

23. $\dfrac{(a^3-b^3)(c^3+d^3)}{(a-b)(c+d)} \times \dfrac{c+d}{a^2+ab+b^2}$

24. $\dfrac{\{(a+b)^2-(c+d)^2\}\{(a-b)^2-(c-d)^2\}}{(a+b+c+d)(a-b+c-d)}$

Simultaneous Numerical Equations of Three Unknown Quantities.

Elimination by Addition and Subtraction.

Direction to Pupil.—To solve three equations of three unknown quantities, select one of the unknown quantities to be eliminated. Combine any two of the equations so as to eliminate this quantity. Then combine either one of these two with the third in like manner. You will then have two equations having only two unknown quantities, which you already know how to solve.

Illustration.—Solve :

$$\left.\begin{array}{ll} 5x - 4y + 2z = 6 & \text{(A)} \\ 2x + 3y - 4z = 11 & \text{(B)} \\ 3x + 2y + 5z = 37 & \text{(C)} \end{array}\right\}.$$

Solution: Multiply (A) by 2 and bring down (B),

$$10x - 8y + 4z = 12 \qquad (1)$$
$$2x + 3y - 4z = 11 \qquad (B)$$

Add (1) and (B),

$$12x - 5y = 23 \qquad (2)$$

Multiply (A) by 5 and (C) by 2,

$$25x - 20y + 10z = 30 \qquad (3)$$
$$6x + 4y + 10z = 74 \qquad (4)$$

Subtract (4) from (3),

$$19x - 24y = -44 \qquad (5)$$

Multiply (2) by 24 and (5) by 5,

$$288x - 120y = 552 \qquad (6)$$
$$95x - 120y = -220 \qquad (7)$$

Subtract (7) from (6),

$$193x = 772 \qquad (8)$$
$$x = 4$$

* Substitute the value of x in (2) and reduce,

$$y = 5$$

Substitute the values of x and y in (B) and reduce,

$$z = 3$$

Verification: Put 4, 5, and 3 for x, y, and z, in (A), (B), and (C),

$20 - 20 + 6 = 6$; which is true.
$8 + 15 - 12 = 11$; " "
$12 + 10 + 15 = 37$; " "

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* Substitute means *put in place of.*

EXERCISE 57.

Solve:

1. $\left.\begin{aligned} x+y+z&=15\\ x-y+z&=5\\ x+y-z&=3\end{aligned}\right\}$

2. $\left.\begin{aligned} 2x+3y-z&=12\\ 4x-2y+z&=3\\ 3x+3y+z&=16\end{aligned}\right\}$

3. $\left.\begin{aligned} 5x-7y+6z&=4\\ 2x+4y-5z&=6\\ x+3y-2z&=10\end{aligned}\right\}$

4. $\left.\begin{aligned} 2x-3y+4z&=10\\ 3x+2y-2z&=17\\ x+5y-z&=22\end{aligned}\right\}$

5. $\left.\begin{aligned} -3x+2y-5z&=-26\\ 3x-2y+4z&=21\\ 4x-2y-5z&=-21\end{aligned}\right\}$

6. $\left.\begin{aligned} x+2y-3z&=-1\\ 4x-4y-z&=8\\ 3x+8y+2z&=-5\end{aligned}\right\}$

7. $\left.\begin{aligned} x+2y-3z&=-7\\ 3x-4y+2z&=4\\ 5x+3y-7z&=-16\end{aligned}\right\}$

8. $\left.\begin{aligned} 5x-6y+2z&=7\\ 3x+5y-5z&=20\\ 2x-3y+4z&=10\end{aligned}\right\}$

9. $\left.\begin{aligned} 2x-5y-3z&=-14\\ 3x-4y-5z&=-4\\ 7x-6y+8z&=27\end{aligned}\right\}$

*10. $\left.\begin{aligned} x+y&=9\\ x+z&=10\\ y+z&=11\end{aligned}\right\}$

*11. $\left.\begin{aligned} x+y-z&=3\\ x-y+z&=9\\ -x+y+z&=11\end{aligned}\right\}$

*12. $\left.\begin{aligned} x+y+z&=6\\ x+y+u&=7\\ x+z+u&=8\\ y+z+u&=9\end{aligned}\right\}$

Concrete Examples involving Simultaneous Equations of Three Unknown Quantities.

EXERCISE 58.

1. The sum of three numbers is 90; twice the first, minus three times the second, plus four times the third, is 200; and three times the first, plus twice the second, minus the third, is 10. What are the numbers?

Suggestion.—Let $x=$ the first, $y=$ the second, and $z=$ the third.

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* The 10th, 11th, and 12th are most readily solved by comparing each equation with the sum of the three.

2. The sum of three numbers is 90; twice the first, plus three times the second, is 30 less than four times the third; and the third is 10 less than the sum of the other two. Required the numbers.

3. A, B, and C together have $3500. If A had twice as much, B three times as much, and C four times as much as now, they together would have $9900; and twice C's amount exceeds the sum of A's and B's amounts by $400. How much has each?

4. A, B, and C together have 500 acres of land; if A buys 25 acres from each of the others, he will have 50 acres more than B and 25 acres less than C. How many acres has each?

5. The cost of two bushels of corn, three bushels of oats, and four bushels of rye is $5.60; of three bushels of corn, two bushels of oats, and one bushel of rye, $3.40; of four bushels of corn, one bushel of oats, and five bushels of rye, $6.80. Required the price per bushel of each kind of grain.

6. If a horse and cow together are worth $160, a horse and sheep $108, and a cow and sheep $68, what is the value of each?

7. A has as many horses as cows and sheep together; twice the number of cows is 12 less than the number of horses and sheep together, and the number of horses and sheep together equals four times the number of cows. How many of each has he?

8. The sum of A's, B's, and C's ages is 60 years; the sum of A's and C's is twice B's; and C's alone equals the sum of A's and B's. Required the age of each.

9. A man has two horses and a saddle, together worth $180. The first horse saddled is worth twice as much as the second horse; the second horse saddled is worth $20 less than the first horse. What is the value of each horse and the saddle?

CHAPTER II.

ALGEBRAIC FRACTIONS.

Preliminary Definitions.

118. An algebraic fraction, in the most general sense, is an expression denoting that one algebraic quantity is to be divided by another. The dividend written above a horizontal line is called the ***Numerator***, and the divisor written below the line is called the ***Denominator***.

Illustration.—Thus, $\frac{a+b}{c-d}$, read $a+b$ divided by $c-d$, in which a, b, c, and d may have any values, positive or negative, integral or fractional, is an algebraic fraction.

119. The numerator and denominator are called the ***Terms*** of the fraction.

120. In a very limited sense, in which the terms are restricted to arithmetical integers, an algebraic fraction may be defined as "a number of equal parts of a unit."

Illustration.—Thus, $\frac{a}{b}$, read $a-b$th, denotes a of the b equal parts of a unit.

121. Since a of the b equal parts of a unit is equivalent to one of the b equal parts of a units, or $\frac{a}{b}$ of $1 = \frac{1}{b}$ of a, $\frac{a}{b}$ in the more restricted sense may still be regarded as an expression of division.

122. An algebraic fraction is usually preceded by the positive or negative sign to indicate whether it is to be used *additively* or *subtractively.*

123. The *value of a fraction* is the result obtained by performing all the operations indicated.

Illustration.—Thus, the value of $-\frac{a}{b}$ when $a=-6$ and $b=+2$ is $-\frac{-6}{+2}=-(-3)=+3.$

124. The *apparent* sign of a fraction is the sign preceding it, the *real* sign the sign of its value.

125. An ***Integral Quantity*** in the literal notation is a quantity that is not fractional in form. It may be integral or fractional in value; as, $a=8$, or $a=\frac{2}{3}$.

126. A ***Mixed Quantity*** in the literal notation is one that is partly integral and partly fractional in form; as, $a \pm \frac{b}{c}$.

127. A ***Proper Fraction*** in the literal notation is one that can not be reduced to the integral or the mixed form; as, $\frac{a+b}{c}$.

128. An ***Improper Fraction*** in the literal notation is one that can be reduced to either the integral or mixed form; as, $\frac{a^2-b^2}{a+b}=a-b$, or $\frac{a^2+b^2}{a}=a+\frac{b^2}{a}$.

129. A ***Compound Fraction*** is a fractional part of an integral or fractional quantity; as, $\frac{a}{b}$ of c, read the $a-b$th part of c; or $\frac{a}{b}$ of $\frac{c}{d}$, read the $a-b$th part of c divided by d.

130. A ***Complex Fraction*** is a fraction one or both of whose terms are fractional in form.

131. The *inverse of a fraction* is the fraction resulting from an interchange of its terms.

Illustration.—Thus, $\frac{b}{a}$ is the inverse of $\frac{a}{b}$.

132. The *reciprocal of a quantity* is unity divided by the quantity.

Reduction of Fractions.

Definition.

133. Reduction is the process of changing the form of a quantity without altering its value.

Principles.

134. Since multiplying the dividend or dividing the divisor multiplies the quotient [P. 51], it follows that,

***Prin. 54.**—Multiplying the numerator or dividing the denominator multiplies the value of a fraction.*

Remark.—If $\frac{a}{b}$ be regarded as a of the b equal parts of a unit, it is evident that multiplying a by n and leaving b unchanged will multiply the number of equal parts taken by n without altering their size, and therefore will multiply the value of the fraction by n.

And, dividing b by n and leaving a unchanged will divide the number of equal parts into which the unit is divided by n; and, hence, make each part n times as great without changing the number of parts taken, which will also multiply the value of the fraction by n.

SIGHT EXERCISE.

Name at sight the products in the following examples:

1. $\frac{a}{b} \times c$ **4.** $\frac{a}{c\,d} \times c$ **7.** $\frac{a+b}{a^2-b^2} \times (a+b)$

2. $\frac{x}{y} \times x$ **5.** $\frac{x\,y}{z^2} \times z^2$ **8.** $\frac{x}{a\,(b+c)} \times (b+c)$

3. $\frac{a}{b^3} \times b$ **6.** $\frac{a+b}{c+d} \times (a-b)$ **9.** $\frac{a+b}{a\,x-x^2} \times x$

10. $\frac{(a+b)^2}{(a-b)^2} \times (a-b)$

11. $\frac{x}{(x-y)^3} \times (x-y)^2$

12. $\frac{x-a}{a^2-x^2} \times (a+x)$

13. $\frac{m}{a^3+b^3} \times (a+b)$

135. Since dividing the dividend or multiplying the divisor divides the quotient [P. 52], it follows that,

Prin. 55.*—Dividing the numerator or multiplying the denominator divides the value of a fraction.*

Remark.—If $\frac{a}{b}$ be regarded as a of the b equal parts of a unit, dividing a by n and leaving b unchanged divides the number of equal parts taken by n without changing their size, which evidently divides the value of the fraction by n.

And, multiplying b by n and leaving a unchanged makes the number of equal parts into which the unit is divided n times as great, and, therefore, each part only $\frac{1}{n}$ as great, without altering the number of parts taken, which divides the value of the fraction by n.

SIGHT EXERCISE.

Name at sight the quotients in the following examples:

1. $\frac{a^2}{b^2} \div a$

2. $\frac{a\,b^2}{c} \div a\,b$

3. $\frac{m}{n} \div p$

4. $\frac{x}{y} \div y$

5. $\frac{m^3}{n^2} \div m^2$

6. $\frac{a^2-b^2}{a\,b} \div (a+b)$

7. $\frac{m^2+n^2}{m+n} \div (m-n)$

8. $\frac{x^4-y^4}{x^4+y^4} \div (x^2-y^2)$

9. $\frac{a\,b+a^2}{c} \div (a+b)$

10. $\frac{x^3+y^3}{x\,y} \div (x+y)$

11. $\frac{x\,(x-y)}{x+y} \div (x-y)$

12. $\frac{x}{x^2+y^2} \div (x^2-y^2)$

13. $\frac{m^4-n^4}{a\,b} \div (m-n)$

14. $\frac{a\,x}{x+2} \div (x+3)$

15. $\frac{x+2}{x+3} \div (x+1)$

136. Multiplying the numerator multiplies the value of a fraction [P. 54].

Multiplying the denominator divides the value of a fraction [P. 55]. Therefore,

Prin. 56.*—Multiplying both terms of a fraction by the same quantity does not alter its value.*

137. Dividing the numerator divides the value of a fraction [P. 55].

Dividing the denominator multiples the value of a fraction [P. 54]. Therefore,

Prin. 57.*—Dividing both terms of a fraction by the same quantity does not alter its value.*

SIGHT EXERCISE.

Tell at sight which of the following equations are true, and why:

1. $\frac{a}{b} = \frac{a^2}{a\,b}$

2. $\frac{a\,b}{a\,c} = \frac{b}{c}$

3. $\frac{a\,x^2}{b\,x^2} = \frac{a}{b}$

4. $\frac{x}{y} = \frac{x\,(x+y)}{y\,(x+y)}$

5. $\frac{x-a}{x+a} = \frac{x^2-a^2}{(x+a)^2}$

6. $\frac{x^3+y^3}{x+y} = x^2+y^2$

7. $\frac{m}{n} = \frac{m\,n+m}{n^2+n}$

8. $\frac{a^2-x^2}{a^2+x^2} = \frac{a^4-x^4}{(a^2+x^2)^2}$

9. $\frac{p\,q}{r\,s} = \frac{p^2\,r\,q}{p\,r^2\,s}$

Complete at sight the following equations:

10. $\frac{b\,(a+x)}{a\,(a+x)} = \frac{\cdot}{a}$

11. $\frac{x^2+y^2}{x^2-y^2} = \frac{x^4-y^4}{\cdot}$

12. $\frac{m}{n} = \frac{m^2\,n^2}{\cdot}$

13. $\frac{a^2-b^2}{a^3-b^3} = \frac{\cdot}{a^2+a\,b+b^2}$

14. $\frac{(m+n)\,a}{a\,(a-b)} = \frac{m+n}{\cdot}$

15. $\frac{r^2\,s^2\,q^3}{r\,s^3\,q^5} = \frac{\cdot}{s\,q^2}$

138. Multiplying both terms of a fraction by -1 will change the signs of both terms. And multiplying both terms of a fraction by the same quantity does not alter its value [P. 56]. Therefore,

Prin. 58.—*Changing the signs of both terms of a fraction does not alter its value.*

139. Let $+\frac{a}{b}=+q$

then $\frac{-a}{b}$ or $\frac{a}{-b}=-q$ [P. 54]

and $-\frac{-a}{b}$ or $-\frac{a}{-b}=-(-q)=+q=+\frac{a}{b}$.

Therefore,

Prin. 59.—*Changing the apparent sign and the sign of either term of a fraction does not change the value of the fraction.*

Remarks.—1. Changing the sign of one factor of either term of a fraction changes the sign of that term. Why?

2. Changing the sign of every term of either numerator or denominator changes the sign of that term of the fraction. Why?

SIGHT EXERCISE.

Change at sight the following fractions to equivalent ones having apparently positive terms:

1. $\frac{-a}{-b}$

2. $-\frac{-a}{b}$

3. $\frac{-(a+b)}{c}$

4. $\frac{a}{-b}$

5. $\frac{-a+b}{-c+d}$

6. $\frac{a-b}{-c}$

7. $-\frac{-x^2}{y^2}$

8. $-\frac{-x(a+b)}{y(a-b)}$

9. $\frac{-(a^2+b^2)}{c^2+d^2}$

10. $\frac{x+y}{-m-n}$

11. $-\frac{x-y}{-z}$

12. $-\frac{-x-y-z}{xyz}$

13. $+\frac{m^2+n^2}{-(m-n)}$

14. $-\frac{p+q}{-(p^2+q^2)}$

15. $-\frac{-m-n}{-x-y}$

Change at sight the following fractions to equivalent ones having positive apparent signs :

16. $-\frac{a-x}{a+x}$

17. $-\frac{d-c}{c+d}$

18. $-\frac{(a-b)(c-d)}{(a+b)(c+d)}$

19. $-\frac{a(x+y)}{y-x}$

20. $-\frac{x^2+y^2}{y^2-x^2}$

21. $-\frac{(m+n)(m-n)}{m^2+n^2}$

Problem 1. To reduce a fraction to its lowest terms.

140. A fraction is in its lowest terms when its terms are prime to each other.

Illustration.—Reduce $\frac{a^2-b^2}{a^3+b^3}$ to its lowest terms.

Form.

$$\frac{a^2-b^2}{a^3+b^3}=\frac{(a+b)(a-b)}{(a+b)(a^2-ab+b^2)}=\frac{a-b}{a^2-ab+b^2}$$

Solution: Since dividing both terms of a fraction by the same quantity does not alter its value [P. 57], we divide both terms by their H. C. D. $(a+b)$. The resulting fraction is in its lowest terms, since the terms are prime to each other. Therefore,

***Rule.**—Divide both terms by their highest common divisor.*

EXERCISE 59.

Reduce to lowest terms :

1. $\frac{4a^2b^2}{6a^3b}$

2. $\frac{9a^3b^4c^2}{12a^4b^3c^4}$

3. $\frac{10x^4y^5z}{15x^6y^2z^3}$

4. $\frac{8x^7y^2z^5}{10x^9y\ z^2}$

5. $\frac{6(x+y)^2}{8(x+y)^3}$

6. $\frac{ab^2(x^3-y^3)^2}{a^2b(x^3-y^3)}$

7. $\frac{a+b}{a^2-b^2}$

8. $\frac{(a+x)^2}{a^2-x^2}$

9. $\frac{x^3+8}{(x+2)^2}$

10. $\frac{x^2-4y^2}{(x+2y)^2}$

11. $\frac{4x^2-9y^2}{(2x-3y)^2}$

12. $\dfrac{x^4 - y^4}{x^4 + 2x^2y^2 + y^4}$

13. $\dfrac{x^3 + y^3}{x^2 + 2xy + y^2}$

14. $\dfrac{4x^2 - 25y^2}{(2x + 5y)^2}$

15. $\dfrac{x^2 - (y + z)^2}{x^2 + xy + xz}$

16. $\dfrac{(a + b)^2 - c^2}{ac + bc - c^2}$

17. $\dfrac{x^2 + 5x + 6}{x^2 + 7x + 12}$

18. $\dfrac{x^2 - 9x + 20}{x^2 + x - 20}$

19. $\dfrac{x^2 - 12x + 35}{x^2 - 15x + 56}$

20. $\dfrac{8a^3 - 27b^3}{(2a - 3b)^3}$

21. $\dfrac{(x^3 - a^3)(x^3 + a^3)}{(x^2 - a^2)^2}$

22. $\dfrac{x^6 + y^6}{x^4 - y^4}$

23. $\dfrac{x^4 - 2x^2y^2 + y^4}{x^8 - y^8}$

24. $\dfrac{a^3 + 3a^2b + 3ab^2 + b^3}{(a - b)(a + b)^2}$

Problem 2. To reduce a mixed quantity to an improper fraction.

Illustration.—

Reduce $a + x - \dfrac{a^2 + x^2}{a - x}$ to an improper fraction.

Form and Solution.

$$a + x - \frac{a^2 + x^2}{a - x} = \frac{a + x}{1} - \frac{a^2 + x^2}{a - x} = \frac{a^2 - x^2}{a - x} - \frac{a^2 + x^2}{a - x}$$

$$[\text{P. } 56] = \frac{1}{a - x}(\overline{a^2 - x^2} - \overline{a^2 + x^2})\ [\text{P. } 47] = \frac{1}{a - x}(-2x^2)$$

$$= \frac{-2x^2}{a - x} = -\frac{2x^2}{a - x}\ [\text{P. } 59].$$

EXERCISE 60.

Reduce to improper fractions:

1. $a + \dfrac{a}{x}$

2. $c - \dfrac{c}{y}$

3. $a + \dfrac{x + a}{x}$

4. $m - \dfrac{mx + n}{x}$

5. $a + x + \dfrac{a^2}{a + x}$

6. $1 + \dfrac{x + a}{x - a}$

7. $a-x-\frac{x^2}{a+x}$

8. $x+a-\frac{x^2+a^2}{x+a}$

9. $\frac{x^3-y^3}{x^3+y^3}-1$

10. $2a+7-\frac{3a-5}{4a+3}$

11. $x^2+xy+y^2+\frac{a^3-x^3}{x-y}$

12. $x+7-\frac{3x+5}{x-4}$

Problem 3. To reduce improper fractions to integral or mixed quantities.

Illustrations.—

1. Reduce $\frac{x^2-x+1}{x+1}$ to a mixed quantity.

Form.

$$\frac{x^2-x+1}{x+1}=(x^2-x+1)\div(x+1)=x-2+\frac{3}{x+1}$$

Solution: Since a fraction indicates division, and the numerator is partly divisible by the denominator, we perform the division and obtain a quotient of $x-2$ and a remainder of 3. As 3 is not divisible by $x+1$, we simply indicate the division and add the result to $x-2$, which produces the mixed quantity $x-2+\frac{3}{x+1}$.

2. Reduce $\frac{x^2+x-4}{x-1}$ to a mixed quantity.

Method.

$$\frac{x^2+x-4}{x-1}=(x^2+x-4)\div(x-1)=x+2+\frac{-2}{x-1}=$$

$$x+2-\frac{2}{x-1}\text{ [P. 59].}$$

EXERCISE 61.

Reduce to whole or mixed quantities:

1. $\frac{ac+b}{c}$

2. $\frac{ax-a}{x}$

3. $\frac{ax+x^2+1}{x}$

4. $\frac{x^2}{x-1}$

5. $\frac{x^2+y^2}{x+y}$

6. $\frac{x^3-y^3}{x+y}$

7. $\frac{x^2+xy+y^2}{x+y}$

8. $\frac{3x^2+8x-3}{x+1}$

9. $\frac{x^4+x^2y^2+y^4}{x^2+xy+y^2}$

10. $\dfrac{x^3+y^3}{x-y}$ 11. $\dfrac{4x^2-4}{2x-1}$ 12. $\dfrac{25x^2}{5x-6}$

13. $\dfrac{4x^3+7x^2-3x+1}{2x}$ 14. $\dfrac{6x^3-5x^2+7x-5}{2x+1}$

Problem 4. To reduce fractions to similar forms.

1. Definitions and Principles.

141. Fractions having a common denominator are *similar*.

142. Dissimilar fractions in their lowest terms must be reduced to higher terms to have a common denominator. This is done by mutiplying both terms by the same quantity [P. 56]. Therefore, the common denominator must contain each of the given denominators. Hence,

Prin. 60.*—Any common multiple of the denominators of two or more fractions is a common denominator of the fractions.*

Prin. 61.*—The lowest common multiple of the denominators of two or more fractions in their lowest terms is the lowest common denominator.*

Note.—L. C. D. stands for lowest common denominator.

2. Examples.

Illustration.—

Reduce $\dfrac{a}{bc}$, $\dfrac{b}{ac}$, and $\dfrac{c}{ab}$ to similar fractions.

Solution: The L. C. M. of the denominators is abc, which is, therefore, the L. C. D. [P. 60]:

$$\frac{a}{bc}=\frac{a\times a}{bc\times a}\text{ [P. 55]}=\frac{a^2}{abc}$$

$$\frac{b}{ac}=\frac{b\times b}{ac\times b}\text{ [P. 55]}=\frac{b^2}{abc}$$

$$\frac{c}{ab}=\frac{c\times c}{ab\times c}\text{ [P. 55]}=\frac{c^2}{abc}$$

Note.—To determine the factor to be inserted in both terms of any fraction, divide the L. C. D. by the denominator of that fraction.

EXERCISE 62.

Reduce to similar fractions having the L. C. D.:

1. $\frac{x}{ab}, \frac{y}{ac}$, and $\frac{z}{bc}$

2. $\frac{a+b}{xy}, \frac{a-b}{xz}$, and $\frac{b}{yz}$

3. $\frac{ax}{by}, \frac{bx}{ay}$, and $\frac{cx}{ab}$

4. $\frac{a}{a+b}, \frac{b}{a-b}$, and $\frac{c}{a^2-b^2}$

5. $\frac{a+x}{a-x}, \frac{a-x}{a+x}$, and $\frac{a^2+x^2}{a^2-x^2}$

6. $\frac{a}{a+x}, \frac{b}{a^3+x^3}$, and $\frac{c}{a^2-ax+x^2}$

7. $\frac{3}{x^2+5x+6}$ and $\frac{5}{x^2-3x-10}$

8. $\frac{1}{(x+a)^2-b^2}$ and $\frac{1}{(x+b)^2-a^2}$

9. $\frac{2}{(x-1)(x-2)}, \frac{3}{(x-2)(x-3)}$, and $\frac{4}{(x-1)(x-3)}$

10. $\frac{x}{4x^2-1}, \frac{2x-1}{2x+1}$, and $\frac{2x+1}{2x-1}$

11. $\frac{a}{(a-c)(b-c)}, \frac{b}{(a-c)(c-b)}$, and $\frac{c}{(c-a)(c-b)}$

Solution:

$$\frac{b}{(a-c)(c-b)} = \frac{-b}{(a-c)(b-c)} \text{ [P. 58]} = -\frac{b}{(a-c)(b-c)} \text{ [P. 59]}$$

$$\frac{c}{(c-a)(c-b)} = \frac{c}{-(a-c) \times -(b-c)} = \frac{c}{(a-c)(b-c)}$$

$$\frac{a}{(a-c)(b-c)} = \frac{a}{(a-c)(b-c)}$$

12. $\frac{a}{1-x}, \frac{b}{x-1}, \frac{c}{1-x^2}$

13. $\frac{3}{2-x}, \frac{4}{x-2}, \frac{5}{(x-2)^2}$

14. $\frac{1}{(x-1)(2-x)}, \frac{2}{(x-2)(3-x)}$, and $\frac{3}{(1-x)(x-3)}$

15. $\frac{a}{(a-x)(x-c)}, \frac{b}{(x-a)(b-x)}$, and $\frac{c}{(c-x)(x-b)}$

Addition and Subtraction of Fractions.

1. Principles.

143. $\frac{a}{d}+\frac{b}{d}+\frac{c}{d}=\frac{1}{d}(a+b+c)$ [P. 47] $=\frac{a+b+c}{d}$.

Therefore,

Prin. 62.*—The sum of two or more similar fractions equals the sum of their numerators divided by their common denominator.*

144. $\frac{a}{c}-\frac{b}{c}=\frac{1}{c}(a-b)$ [P. 47] $=\frac{a-b}{c}$. Therefore,

Prin. 63.*—The difference of two similar fractions equals the difference of their numerators divided by their common denominator.*

2. Problem.

To add or subtract fractions.

Illustrations.—1. Find the sum of $\frac{a+x}{a-x}$ and $\frac{a-x}{a+x}$.

Solution: The L. C. D. $= a^2-x^2$

$$\frac{a+x}{a-x}=\frac{(a+x)(a+x)}{(a-x)(a+x)}=\frac{a^2+2ax+x^2}{a^2-x^2}$$

$$\frac{a-x}{a+x}=\frac{(a-x)(a-x)}{(a+x)(a-x)}=\frac{a^2-2ax+x^2}{a^2-x^2}$$

$$\frac{a^2+2ax+x^2}{a^2-x^2}+\frac{a^2-2ax+x^2}{a^2-x^2}=\frac{2a^2+2x^2}{a^2-x^2}\quad \text{[P. 62]}$$

2. Find the value of $\frac{a}{a-b}-\frac{b}{b-a}-\frac{a^2+b^2}{a^2-b^2}$.

Solution: $\frac{a}{a-b}=\frac{a(a+b)}{(a-b)(a+b)}=\frac{a^2+ab}{a^2-b^2}$

$$-\frac{b}{b-a}=+\frac{b}{a-b}=\frac{b(a+b)}{(a-b)(a+b)}=\frac{ab+b^2}{a^2-b^2}$$

$$-\frac{a^2+b^2}{a^2-b^2}=+\frac{-a^2-b^2}{a^2-b^2}$$

$$\frac{a^2+ab}{a^2-b^2}+\frac{ab+b^2}{a^2-b^2}+\frac{-a^2-b^2}{a^2-b^2}=\frac{2ab}{a^2-b^2}$$

Note.—Sometimes it is better not to combine all the fractions at one time.

3. Find the value of $\frac{1}{1+x}+\frac{1}{1-x}-\frac{1}{1+x^2}$.

Solution: $\frac{1}{1+x}+\frac{1}{1-x}=\frac{1-x}{1-x^2}+\frac{1+x}{1-x^2}=\frac{2}{1-x^2}$

$$\frac{2}{1-x^2}-\frac{1}{1+x^2}=\frac{2+2x^2}{1-x^4}-\frac{1-x^2}{1-x^4}=\frac{1+3x^2}{1-x^4}$$

4. Find the sum of $2a-3b+\frac{2a-b}{c}$ and $3a+2b-\frac{3a-b}{c}$.

Method.

$$2a-3b+\frac{2a-b}{c}=2a-3b+\frac{2a-b}{c}$$

$$3a+2b-\frac{3a-b}{c}=3a+2b+\frac{-3a+b}{c}$$

$$\text{Sum}\quad =5a-b-\frac{a}{c}$$

EXERCISE 63.

Find the value of:

1. $\frac{1}{1+a}+\frac{1}{1-a}$

2. $\frac{a}{a+x}-\frac{a}{a-x}$

3. $\frac{ab}{c}+\frac{ac}{b}+\frac{bc}{a}$

4. $\frac{1}{ab}+\frac{1}{ac}-\frac{1}{bc}$

5. $\frac{a+x}{x}-\frac{a+y}{y}$

6. $\frac{a}{x^2+y^2}+\frac{a}{x^2-y^2}$

7. $\frac{m}{m-n}-\frac{n}{n-m}$

8. $\frac{p}{p-q}+\frac{p^2}{q^2-p^2}$

9. $\frac{a}{a-x}+\frac{a}{a+x}-\frac{a^2}{x^2-a^2}$

10. $\frac{a+b}{p-q}-\frac{a-b}{q-p}$

11. $\left(\frac{3x}{4}-\frac{5y}{3}\right)+\left(\frac{7x}{3}+\frac{4y}{5}\right)$

12. $\left(\frac{5ax}{3}+\frac{2by}{4}\right)-\left(\frac{3ax}{4}-\frac{3by}{2}\right)$

13. $\left(\frac{a+b}{a}+\frac{c+d}{b}\right)-\left(\frac{a-b}{2a}+\frac{c-d}{3b}\right)$

14. $\left(\frac{2}{x}-\frac{3}{y}+\frac{5}{z}\right)+\left(\frac{5}{x}-\frac{7}{y}-\frac{3}{z}\right)$

15. $\left(a+\frac{a}{x}\right)+\left(2a-\frac{b}{x}\right)-\left(3a+\frac{c}{x}\right)$

16. $\left(a+b-\frac{b}{ax}\right)-\left(2a-3b+\frac{a}{bx}\right)$

17. $\left(\frac{2}{3x}+\frac{3}{4y}-\frac{5}{2z}\right)+\left(\frac{3}{2x}-\frac{4}{3y}+\frac{3}{4z}\right)$

18. $\left(\frac{5}{3x}-\frac{7}{5y}+\frac{4}{6z}\right)-\left(\frac{8}{5x}+\frac{3}{8y}-\frac{5}{4z}\right)$

19. $\frac{1}{a+x}+\frac{1}{a-x}-\frac{x}{(a+x)^2}+\frac{x}{(a-x)^2}$

20. $\frac{1}{x+y}+\frac{xy}{x^3+y^3}+\frac{x-y}{x^2-xy+y^2}$

21. $\frac{x}{x+y}-\frac{xy}{(x+y)^2}-\frac{x^3}{(x+y)^3}$

22. $\frac{1}{x^2-3x+2}+\frac{1}{x^2-4x+3}+\frac{1}{x^2-5x+6}$

23. $\frac{2}{x^2-7x+12}-\frac{4}{x^2-6x+8}+\frac{2}{x^2-5x+6}$

24. $\frac{1}{x^2-(y+z)^2}+\frac{1}{z^2-(x+y)^2}$

25. $\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}$

26. $\frac{3+2x}{2-x}-\frac{2-3x}{2+x}+\frac{16x-x^2}{x^2-4}$

27. $\frac{3}{1-2x}-\frac{7}{1+2x}-\frac{4-20x}{4x^2-1}$

Multiplication and Division of Fractions.

Problems.

1. To multiply or divide a fraction.

Illustrations.—1. Multiply $\frac{a+b}{ab}$ by $a-b$.

Solution: $\frac{a+b}{ab} \times (a-b) = \frac{(a+b)(a-b)}{ab}$ [P. 54] $= \frac{a^2-b^2}{ab}$

2. Multiply $\frac{ab}{a^2-b^2}$ by $a+b$.

Solution: $\frac{ab}{a^2-b^2} \times (a+b) = \frac{ab}{(a^2-b^2) \div (a+b)}$ [P. 54] $= \frac{ab}{a-b}$

3. Multiply $\frac{a+b}{a-b}$ by a^2-b^2.

Solution: $\frac{a+b}{a-b} \times (a^2-b^2) = \frac{(a+b)\overset{a+b}{\cancel{(a^2-b^2)}}}{\cancel{a-b}}$ [P. 57] $= (a+b)^2$

4. Divide $\frac{a^2-b^2}{ab}$ by $a+b$.

Solution: $\frac{a^2-b^2}{ab} \div (a+b) = \frac{(a^2-b^2) \div (a+b)}{ab}$ [P. 55] $= \frac{a-b}{ab}$

5. Divide $\frac{ab}{a+b}$ by $a-b$.

Solution: $\frac{ab}{a+b} \div (a-b) = \frac{ab}{(a+b)(a-b)}$ [P. 55] $= \frac{ab}{a^2-b^2}$

6. Divide $\frac{a^2-ab}{b^2+ab}$ by a^2-b^2.

Solution: $\frac{a^2-ab}{b^2+ab} \div (a^2-b^2) = \frac{a^2-ab}{(b^2+ab)(a^2-b^2)} =$

$$\frac{a\cancel{(a-b)}}{b(a+b)\cancel{(a-b)}(a+b)} = \frac{a}{b(a+b)^2}$$

7. $\frac{a^2-b^2}{c^2-d^2} \div (a+b) \times (c+d) =$

$$\frac{a-b}{c^2-d^2} \times (c+d) = \frac{a-b}{c-d}.$$

EXERCISE 64.

Multiply:

1. $\frac{ab}{c}$ by ax
2. $\frac{abc}{d^2 e^2}$ by de^2
3. $\frac{a^2 c^2}{b^2 d^2}$ by $c^2 b^2$
4. $\frac{x^2 + y^2}{x^2 y^2}$ by xy
5. $\frac{x + y}{x^2 + y^2}$ by $x - y$
6. $\frac{a - b}{a^3 + b^3}$ by $a + b$
7. $\frac{a + b}{a^2 - b^2}$ by $(a + b)^2$
8. $\frac{x - 5}{x^2 + 5x + 6}$ by $x + 3$

Divide:

9. $\frac{a^2 bc}{x}$ by ac
10. $\frac{ax}{c}$ by cd
11. $\frac{ax^2 y^3}{mn}$ by $3x^2 y$
12. $\frac{x^2 - y^2}{xy}$ by $x + y$
13. $\frac{x^3 + y^3}{x - y}$ by $x + y$
14. $\frac{m^4 - n^4}{3mn}$ by $4(m^2 + n^2)$
15. $\frac{x^2 + 7x + 10}{x + 3}$ by $x + 5$
16. $\frac{1}{1 - x}$ by $1 - x^2$

17. Multiply $\frac{(x - 5)(x - 6)}{(x + 5)(x + 6)}$ by $(x + 5)^2$

18. Multiply $\frac{x^2 - (a + b)^2}{x^2 - (a - b)^2}$ by $x - a + b$

19. Divide $\frac{3x^2 y(y + z)}{x + z}$ by $6x(x - z)$

20. Divide $\frac{a^2 - b^2 - 2bc - c^2}{abc}$ by $a + b + c$

Simplify:

21. $\frac{x^3 - y^3}{(x + y)^2} \times (x + y) \div (x - y)$

22. $\frac{x^3 + y^3}{(x - y)^2} \div (x + y) \times (x - y)$

2. To multiply or divide by a fraction.

1. Definitions and Principles.

145. To multiply or divide by a fraction is to multiply or divide by the quotient of two quantities.

Illustrations.—1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

Solution: Since multiplying one quantity and dividing another by the same factor does not alter their product [P. 50],

$$\frac{a}{b} \times \frac{c}{d} = \left(\frac{a}{b} \div d\right) \times \left(\frac{c}{d} \times d\right) = \frac{a}{b\,d} \times c \text{ [P. 55 and 54]} = \frac{a\,c}{b\,d}$$

[P. 54]. Therefore,

Prin. 64.—*The product of two fractions equals the product of their numerators divided by the product of their denominators.*

SIGHT EXERCISE.

Name at sight the product in the following examples:

1. $\frac{x}{y} \times \frac{m}{n}$
2. $\frac{x^3}{y^2} \times \frac{x^2}{y^3}$
3. $-\frac{x}{y^2} \times \frac{m^2}{n^3}$
4. $\frac{a+x}{a+b} \times \frac{a-x}{a-b}$
5. $\frac{x+2}{x+3} \times \frac{x+1}{x-3}$
6. $\frac{x+4}{x-5} \times \frac{x+3}{x+2}$
7. $\left(-\frac{m\,n}{x\,y}\right) \times \left(-\frac{p\,q}{r\,s}\right)$
8. $-\frac{a\,(m-n)}{b\,(m+n)} \times \frac{(m-n)^2}{(m+n)^2}$
9. $\left(-\frac{x-y}{c}\right) \times \left(-\frac{x+y}{b}\right)$

2. Multiply $\frac{a^5}{b^4} \times \frac{b^2}{c^2}$.

Solution: Since multiplying one quantity and dividing another by the same factor does not alter their product [P. 50], we multiply the first fraction by b^2 by dividing its denominator, and divide the second fraction by a^2 by dividing its numerator, and have remaining $\frac{a^5}{b^2} \times \frac{1}{c^2}$, which equals $\frac{a^5}{b^2\,c^2}$. Therefore,

Prin. 65.—*Canceling a factor common to the numerator of one fraction and the denominator of another does not alter the product of the fractions.*

SIGHT EXERCISE.

Multiply at sight the following fractions:

1. $\frac{a^3}{b^2} \times \frac{b}{a^2}$ 2. $\frac{m^2 p}{n^2 q} \times \frac{n^2}{m^2}$ 3. $\frac{x^2 y^2}{p q} \times \frac{p y}{q x}$

4. $\frac{a-b}{x+y} \times \frac{x^2-y^2}{a^2-b^2}$

5. $\frac{a(x+y)}{b(x-y)} \times \frac{b(x^2-y^2)}{a}$

6. $\frac{(a+2)^2}{(a-3)^2} \times \frac{(a-3)^4}{(a+2)^3}$

7. $\frac{(x+y)^3}{(x-y)^3} \times \frac{x-y}{x+y}$

8. $-\frac{x^2}{y^2} \times \frac{y^3}{x^3}$

9. $\frac{y-x}{x-z} \times \frac{z-x}{x-y}$

3. Divide $\frac{a}{b} \div \frac{c}{d}$.

Solution: Since multiplying both dividend and divisor by the same factor does not alter the quotient [P. 53],

$$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \times d\right) \div \left(\frac{c}{d} \times d\right) = \left(\frac{a}{b} \times d\right) \div c \text{ [P. 54]} = \frac{a}{b} \times \frac{d}{c}$$

[P. 14]. Therefore,

Prin. 66.*—The quotient of two fractions equals the dividend multiplied by the inverse of the divisor.*

SIGHT EXERCISE.

Name at sight the quotient of the following fractions:

1. $\frac{x}{y} \div \frac{m}{n}$ 2. $\frac{x^3}{y^2} \div \frac{y}{x}$ 3. $\frac{a b}{c d} \div \frac{c}{d}$

4. $\left(-\frac{x^2}{y^3}\right) \div \left(-\frac{y}{x}\right)$

5. $\frac{a^2-x^2}{x} \div \frac{a-x}{x^2}$

6. $\frac{m n p}{q r s} \div \frac{q r s}{m n p}$

7. $\frac{2}{x y} \div \frac{x y}{2}$

8. $\left(-\frac{1}{x^2 y^2}\right) \div \frac{x^2 y^2}{2}$

9. $\left(\frac{m^2}{n^2} \times \frac{p}{q}\right) \div \frac{m p}{n q}$

10. $\frac{a(x-y)}{1} \div \frac{1}{x-y}$

11. $\frac{a^2-b^2}{x^2-y^2} \div \frac{a-b}{x-y}$

4. Divide $\frac{a^2 - b^2}{x}$ by $\frac{a - b}{x^2}$.

Solution: Since dividing both dividend and divisor by the same quantity does not alter the value of the quotient, we divide both fractions by $a - b$ by dividing their numerators by $a - b$, and obtain $\frac{a + b}{x} \div \frac{1}{x^2}$. Since multiplying both dividend and divisor by the same quantity does not alter the value of the quotient, we multiply both fractions by x by dividing both denominators by x, and obtain $\frac{a + b}{1} \div \frac{1}{x}$, which equals $(a + b)x$ [P. 66]. Therefore,

***Prin. 67.**—Canceling a factor common to the numerators or the denominators of two fractions does not alter their quotient.*

SIGHT EXERCISE.

Name at sight the quotients of the following fractions:

1. $\frac{a^3}{b^2} \div \frac{a}{b}$.
2. $\frac{x^4}{y^4} \div \frac{x^2}{y^2}$
3. $\frac{a^2 b^3}{x^2 y^2} \div \frac{a b^2}{x y}$
4. $\frac{a^2 - x^2}{x y} \div \frac{a - x}{x}$
5. $\frac{(x + 2)^2}{x + 3} \div \frac{x + 2}{x + 3}$
6. $\frac{a(p + q)}{b(p - q)} \div \frac{p + q}{p - q}$
7. $\frac{r^2 (a + x)^2}{s^2 (a - x)^2} \div \frac{(a + x)^2}{(a - x)^2}$
8. $\frac{m^3 + n^3}{m^2 - n^2} \div \frac{m + n}{m - n}$
9. $\frac{(x + 2)(x - 3)}{(x - 2)(x + 3)} \div \frac{x - 3}{x - 2}$

WRITTEN EXAMPLES.

Illustrations.—1. Multiply $\frac{a b}{a^2 - b^2}$ by $\frac{a + b}{b c}$.

Solution: $\frac{a b}{a^2 - b^2} \times \frac{a + b}{b c} = \frac{a \cancel{b}}{\underset{a - b}{\cancel{a^2 - b^2}}} \times \frac{\overset{1}{\cancel{a + b}}}{\cancel{b} c}$ [P. 65] $= \frac{a}{c(a - b)}$.

2. Divide $\frac{a^2 - b^2}{c d}$ by $\frac{a c - b c}{d e}$.

Solution: $\frac{a^2 - b^2}{c d} \div \frac{a c - b c}{d e} = \frac{\overset{a + b}{\cancel{a^2 - b^2}}}{c \cancel{d}} \div \frac{c \cancel{(a - b)}}{\cancel{d} e}$ [P. 67] $=$

$\frac{a + b}{c} \times \frac{e}{c}$ [P. 66] $= \frac{e(a + b)}{c^2}$.

3. Divide $1+\frac{x}{y}$ by $1-\frac{x}{y}$.

Solution: $\left(1+\frac{x}{y}\right) \div \left(1-\frac{x}{y}\right) = \frac{1+\frac{x}{y}}{1-\frac{x}{y}} = \frac{\left(1+\frac{x}{y}\right)\times y}{\left(1-\frac{x}{y}\right)\times y} = \frac{y+x}{y-x}.$

Or,

$$\left(1+\frac{x}{y}\right) \div \left(1-\frac{x}{y}\right) = \frac{y+x}{y} \div \frac{y-x}{y} = \frac{y+x}{y} \times \frac{y}{y-x} = \frac{y+x}{y-x}.$$

EXERCISE 65.

Find the value of:

1. $\frac{a\,x}{b\,y} \times \frac{c\,x}{d\,y}$

2. $\frac{a^2\,b^3\,c}{x\,y^2\,z^3} \times \frac{x^2\,y^3\,z}{a^2\,b^4\,c^3}$

3. $\frac{a\,x}{a-x} \times \frac{b\,x}{a+x}$

4. $\frac{a^2-x^2}{a\,x} \times \frac{a^2\,x^2}{a+x}$

5. $\frac{a\,x}{b\,y} \times \frac{c\,y}{d\,x} \times \frac{x\,y}{a\,d}$

6. $\left(1+\frac{c}{d}\right) \times \left(1+\frac{d}{c}\right)$

7. $\frac{a\,x^2}{b\,y^3} \div \frac{c^2\,x}{d\,y^2}$

8. $\frac{a+x}{c\,d} \div \frac{a^2-x^2}{d\,e}$

9. $\frac{a\,x^3}{a^3+x^3} \div \frac{a^2\,x^2}{a^2-x^2}$

10. $\frac{a^2+x^2}{a+y} \div \frac{a^4-x^4}{a^2-y^2}$

11. $\frac{p+q}{p-q} \div \frac{p-q}{p+q}$

12. $\frac{(a+b)^2}{x-y} \times \frac{x^2-y^2}{a^2+a\,b}$

13. $\left(x^2+\frac{x^2}{y^2}\right) \div \left(x+\frac{y^2}{x^2}\right)$

14. $\left(x+\frac{1}{x}\right) \div \left(x^2-\frac{1}{x^2}\right)$

15. $\frac{x^3+y^3}{(a+b)^3} \times \frac{a^2+2\,a\,b+b^2}{(x+y)^2}$

16. $\frac{x^2-2\,x-24}{x^2+7\,x+10} \div \frac{x^2+6\,x+8}{x^2-x-30}$

17. $\frac{a\,c+a\,d+b\,c+b\,d}{a\,c-a\,d-b\,c+b\,d} \times \frac{a-b}{c+d}$

18. $\frac{x^3+27\,y^3}{(x+3\,y)^3} \div \frac{x^2-3\,x\,y+9\,y^2}{(x+3\,y)^2}$

Complex Fractions.

To reduce a complex fraction to a simple one.

Illustration.—1. Reduce $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ to a simple fraction.

Solution: $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{\frac{a}{b} \times bd}{\frac{c}{d} \times bd} = \dfrac{ad}{bc}$; Or, $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}$.

2. Reduce $\dfrac{x^2 - \frac{1}{x^2}}{x^3 - \frac{1}{x^3}}$ to a simple fraction.

Solution:

$$\frac{x^2 - \frac{1}{x^2}}{x^3 - \frac{1}{x^3}} = \frac{\left(x^2 - \frac{1}{x^2}\right)x^3}{\left(x^3 - \frac{1}{x^3}\right)x^3} = \frac{x^5 - x}{x^6 - 1} = \frac{x(x^2+1)(x+1)(x-1)}{(x+1)(x^2-x+1)(x-1)(x^2+x+1)} = \frac{x(x^2+1)}{x^4+x^2+1} \quad \textit{Ans.}$$

Or,

$$\frac{x^2 - \frac{1}{x^2}}{x^3 - \frac{1}{x^3}} = \frac{\frac{x^4-1}{x^2}}{\frac{x^6-1}{x^3}} = \frac{x^4-1}{x^2} \times \frac{x^3}{x^6-1} = \frac{x(x^2+1)(x+1)(x-1)}{(x+1)(x^2-x+1)(x-1)(x^2+x+1)} =$$

EXERCISE 66.

Simplify:

1. $\dfrac{\frac{a}{x}}{\frac{b}{y}}$

2. $\dfrac{\frac{ax}{by}}{\frac{xy}{ab}}$

3. $\dfrac{\frac{x+1}{x-1}}{\frac{x-1}{x+1}}$

4. $\dfrac{a + \frac{b}{c}}{a - \frac{b}{c}}$

5. $\dfrac{1 + \frac{a}{x}}{1 - \frac{a^2}{x^2}}$

6. $\dfrac{\frac{x+2}{x-3}}{\frac{x-2}{x+3}}$

7. $\dfrac{\frac{a+x}{x}}{a - \frac{x^2}{a}}$

8. $\dfrac{1}{1 + \frac{1}{x}}$

9. $\dfrac{x}{x+\dfrac{a}{x}}$

11. $\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x}}}$

10. $\dfrac{x+6+\dfrac{1}{x-6}}{x-6+\dfrac{1}{x+6}}$

12. $\dfrac{a+\dfrac{2a}{a-3}}{a-\dfrac{2a}{a-3}}$

Involution of Fractions.

146. Involution of fractions is the process of raising a fraction to any power.

Problem.

To raise a fraction to any power.

Illustration.—1. Raise $\frac{a}{b}$ to the fifth power.

Solution: $\left(\frac{a}{b}\right)^5 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^5}{b^5}$. Therefore,

***Prin. 68.**—Raising both terms of a fraction to any power raises the fraction to that power.*

SIGHT EXERCISE.

Name at sight the indicated powers of the following fractions:

1. $\left(\frac{x}{y}\right)^4$

2. $\left(\frac{ax^2}{by}\right)^3$

3. $\left(-\frac{x^2}{y^3}\right)^4$

4. $\left(\frac{2a^2}{3b^2}\right)^4$

5. $\left(-\frac{2x}{3y}\right)^3$

6. $\left(-\frac{m^2 n}{pq}\right)^5$

7. $\left(\frac{a-b}{a+b}\right)^2$

8. $\left(\frac{x-y}{x+y}\right)^3$

9. $\left\{\frac{a(p-q)}{b(p+q)}\right\}^2$

10. $\left(-\frac{a^2 b^3}{m^2 p^3}\right)^5$

11. $\left(-\frac{r^2 s^{10}}{q^4 n^8}\right)^7$

12. $\left(-\frac{x^4 y^5}{m^2 z^3}\right)^6$

WRITTEN EXERCISES.

EXERCISE 67.

Find the value of:

1. $\left(\frac{1}{2} \times \frac{1}{3}\right)^3$

2. $\left(\frac{a}{c} \times \frac{c}{d}\right)^4$

3. $\left(\frac{2p^3q^2}{3r^2}\right)^4$

4. $\left\{\left(\frac{2}{3}\right)^3 \times \left(\frac{2}{3}\right)^2\right\}^2$

5. $\left(-\frac{a^2b^2}{c^3}\right)^5$

6. $\left\{\frac{(a+b)^2}{c^2}\right\}^3$

7. $\left\{\frac{(m+n)^2}{(m-n)^2}\right\}^4$

8. $\left\{\frac{c(c-x)^2}{x^2}\right\}^5$

9. $\left\{\frac{(x-y)^2(x-y)^3}{(x+y)^2}\right\}^4$

10. $\left\{\frac{p(p+1)}{q(q-1)}\right\}^3$

11. $\left\{-\frac{(x+z)(x-z)}{x+y}\right\}^2$

12. $\left(-\left\{\frac{(m-n)}{(m+n)^2}\right\}^2\right)^2$

Miscellaneous Examples.

EXERCISE 68.

1. Reduce $\dfrac{x^2-x-30}{x^2+11x+30}$ to its lowest terms.

2. Reduce $\dfrac{x^4-y^4}{x^6-y^6}$ to its lowest terms.

3. Reduce $\dfrac{ac-3ad+5bc-15bd}{ac+3ad+5bc+15bd}$ to its lowest terms.

4. Reduce $1-\dfrac{(x-y)^2}{x^2+y^2}$ to an improper fraction.

5. Reduce $2x-y+\dfrac{(x-y)^2}{y}$ to an improper fraction.

6. Reduce $\dfrac{1-2x^2}{1+x}$ to a mixed quantity.

7. Simplify $\dfrac{3}{1-2a} - \dfrac{7}{1+2a} - \dfrac{4-20a}{4a^2-1}$.

8. Simplify $\dfrac{a^2+b^2}{ab} - \dfrac{a^2}{ab+b^2} - \dfrac{b^2}{a^2+ab}$

9. Simplify $\dfrac{x^2+y^2}{x^2-y^2} + \dfrac{2x^2y}{x^3+y^3} + \dfrac{2xy^2}{y^3-x^3}$

10. Simplify $\dfrac{a^2+ab}{a-b} \times \dfrac{(a-b)^2}{a^4-b^4}$

11. Simplify $\left\{1+\left(\dfrac{a-b}{a+b}\right)^2\right\} \div \left\{1-\left(\dfrac{a-b}{a+b}\right)^2\right\}$

12. Simplify $\dfrac{y^2-7y+6}{y^2+3y-4} \times \dfrac{y^2+10y+24}{y^2-14y+48} \div \dfrac{y^2+6y}{y^3-8y^2}$

13. Multiply $\dfrac{a}{b}+\dfrac{c}{d}$ by $\dfrac{c}{a}+\dfrac{d}{b}$

Operation.

$$
\begin{array}{rrr}
\dfrac{a}{b} + \dfrac{c}{d} & & \\
\dfrac{c}{a} + \dfrac{d}{b} & & \\
\hline
\dfrac{c}{b} + \dfrac{c^2}{ad} & & \\
+\dfrac{ad}{b^2} + \dfrac{c}{b} & & \\
\hline
\dfrac{2c}{b} + \dfrac{c^2}{ad} + \dfrac{ad}{b^2} & &
\end{array}
$$

14. Multiply $\dfrac{a}{b}-\dfrac{c}{d}$ by $\dfrac{b}{a}+\dfrac{d}{c}$

15. Multiply $\dfrac{x}{y}+\dfrac{y}{x}$ by $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}$

16. Square $\dfrac{x}{a}+\dfrac{y}{b}$; also $\dfrac{a}{y}-\dfrac{y}{a}$ [See P. 31, 32.]

17. Cube $1+\dfrac{a}{x}$; also $\dfrac{x}{y}-b$ [See P. 33, 34.]

18. Find the value of $\left(\dfrac{a}{x}+\dfrac{c}{y}\right)\left(\dfrac{a}{x}-\dfrac{c}{y}\right)$ [See P. 39.]

19. Find the value of $\left(x^3+\frac{1}{x^3}\right)\left(x^3-\frac{1}{x^3}\right)$

20. Find the value of $\left(x+1+\frac{1}{x}\right)^2$; $\left(x-1-\frac{1}{x}\right)^2$

21. Find the value of $\left(\frac{a+x}{a-x}+\frac{a-x}{a+x}\right)\left(\frac{a+x}{a-x}-\frac{a-x}{a+x}\right)$

22. Divide $\frac{a^3}{x^3}-y^3$ by $\frac{a}{x}-y$ [See P. 44.]

23. Divide $\frac{a^3}{b^3}+\frac{b^3}{a^3}$ by $\frac{a}{b}+\frac{b}{a}$

24. Divide $\frac{a^4}{x^4}-\frac{b^4}{y^4}$ by $\frac{a}{x}-\frac{b}{y}$; also by $\frac{a}{x}+\frac{b}{y}$

25. Factor $a^2-\frac{b^2}{c^2}$; $x^2y^2-\frac{x^2}{y^2}$; $\frac{a^2}{b^2}-c^2d^2$

26. Factor $1-\frac{x^4}{y^4}$; $a^4-\frac{b^4}{c^4}$; $\frac{x^2}{a^2}-\frac{a^2}{x^2}$

27. Factor $x^3+\frac{1}{x^3}$; $x^3-\frac{a^3}{x^3}$; $a^6-\frac{x^6}{y^6}$

28. Factor $\frac{x^3}{y^3}+\frac{y^3}{x^3}$; $\frac{x^3}{m^6}-\frac{y^3}{n^6}$; $1-\left(\frac{x+y}{x-y}\right)^2$

29. Divide $x^5+\frac{1}{x^5}$ by $x+\frac{1}{x}$

30. Find the value of :

$$\frac{1+\frac{x-y}{x+y}}{1-\frac{x-y}{x+y}}\div\frac{1+\frac{x^2-y^2}{x^2+y^2}}{1-\frac{x^2-y^2}{x^2+y^2}}$$

31. Multiply $\frac{x}{a}-\frac{a}{x}+\frac{y}{b}-\frac{b}{y}$ by $\frac{x}{a}-\frac{a}{x}-\frac{y}{b}+\frac{b}{y}$ as in multiplication of entire polynomials.

32. Find the value of $\frac{a-z}{b-z}$ when $z=\frac{ab}{a+b}$

CHAPTER III.

GENERAL TREATMENT OF SIMPLE EQUATIONS.

General Definitions.

147. An equation in which the known quantities are numerical is a ***Numerical*** equation; as,

$$3x + 2 = 5x - 4.$$

148. An equation in which some or all of the known quantities are literal is a ***Literal*** equation; as,

$$2ax - 4bx = c.$$

149. The degree of a term of an equation is determined by the number of unknown prime factors it contains.

Thus: In $ax^2 + bxy + cy^2 + dx + ey + f = 0$, ax^2, bxy, and cy^2 are of the second degree; dx and ey of the first degree; and f of no degree.

150. A term in an equation that does not contain an unknown quantity is an ***Absolute*** term.

151. The degree of an equation is the same as the degree of its highest term.

152. An equation of the first degree is a ***Simple*** equation; as, $2x - 3x + 5x = 12$; or, $ax + by = c$.

153. An equation of the second degree is a ***Quadratic*** equation; as, $x^2 + 4x = 6$; or, $x^2 + xy = 12$.

154. An equation of the third degree is a ***Cubic*** equation; as, $x^3 + 3x^2 + 2x + 5 = 0$; or, $x^3 + x^2y + x + y = 12$.

Transformation of Equations.

Definition and Principles.

155. The process of changing the form of an equation without destroying the equality of its members is *transformation* of equations.

156. An equation may be transformed:

***Prin. 69.**—1. By adding the same or equal quantities to both members.*

2. By subtracting the same or equal quantities from both members.

3. By multiplying both members by the same or equal quantities.

4. By dividing both members by the same or equal quantities.

5. By raising both members to the same power.

6. By taking the same root of both members.

157. If we take the equation

$$ax - b = cx + d, \qquad (1)$$

and add b to both members [P. 69, 1], we obtain

$$ax = cx + d + b. \qquad (2)$$

If we now subtract cx from both members, we obtain

$$ax - cx = d + b. \qquad (3)$$

If we now compare (3) with (1), we observe that:

***Prin. 70.**—Any term of an equation may be transposed from one member to the other if its sign be changed.*

158. If we take the equation

$$\frac{x}{3} + \frac{2x}{9} = \frac{5x}{6} - 2, \qquad (1)$$

and reduce all the terms to a common denominator, we obtain

$$\frac{6x}{18} + \frac{4x}{18} = \frac{15x}{18} - \frac{36}{18}. \qquad (2)$$

If we now multiply both members by 18 [P. 69, 3], we obtain $6x+4x=15x-36$, an equation without fractional terms. But 18, the common denominator of the fractional terms, is a common multiple of the denominators of the fractions. Therefore,

***Prin. 71.**—An equation with fractional terms may be cleared of fractions by multiplying both members by a common multiple of the denominators of the fractions.*

Simple Equations of One Unknown Quantity.

1. Solution of Numerical Equations.

Illustrations.—

1. Given $5x+7=3x-5+6x$ to find the value of x.

Solution: $5x+7=3x-5+6x$ (A)

Transposing 7, $3x$, and $6x$ [P. 70],

$5x-3x-6x=-5-7$ (1)

Uniting terms, $-4x=-12$ (2)

Dividing by -4, $x=3$ [P. 69, 4].

Proof: Substituting $x=3$ in equation (A)

$15+7=9-5+18$; whence

$22=22$.

2. Given $\frac{3x}{2}-4+\frac{7x}{3}=\frac{5x}{4}+\frac{7}{6}$ to find the value of x.

Solution: $\frac{3x}{2}-4+\frac{7x}{3}=\frac{5x}{4}+\frac{7}{6}$ (A)

Clearing of fractions by multiplying by 12 [P. 71]

$18x-48+28x=15x+14$ (1)

Transposing -48 and $15x$ [P. 70]

$18x+28x-15x=14+48$ (2)

Uniting terms, $31x=62$.

Dividing by 31, $x=2$.

Proof: Substituting $x=2$ in equation (A),

$3-4+\frac{14}{3}=\frac{5}{2}+\frac{7}{6}$; whence

$\frac{11}{3}=\frac{11}{3}$.

3. Solve $x - \frac{3x-2}{6} = 2 - \frac{x+6}{12}$.

Solution: Given $$x - \frac{3x-2}{6} = 2 - \frac{x+6}{12} \qquad \text{(A)}$$

Clearing of fractions by multiplying by 12,

$$12x - 2(3x-2) = 24 - (x+6) \qquad (1)$$

Expanding, $$12x - 6x + 4 = 24 - x - 6 \qquad (2)$$

Transposing and uniting terms, $$7x = 14 \qquad (3)$$

Dividing by 7, $$x = 2.$$

Equation (1) may be omitted if the following principle be heeded:

159. *If a fraction is preceded by minus, the sign of every term in the numerator must be changed when its denominator is removed.* Why?

EXERCISE 69.

Solve the following equations:

1. $5x + 3 = 7x - 3$
2. $3x + 5x + 14 = 9x + 10$
3. $x + 7 - 3x = 5 - 6(x-1)$
4. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 7$
6. $\frac{x+5}{3} - \frac{x-6}{4} = \frac{x}{6} + 2$
5. $\frac{x}{4} - \frac{x}{6} + \frac{x}{12} = x - 20$
7. $3x - \frac{2x-3}{2} = 12 - \frac{2x}{5}$
8. $3(x+5) - 4(x+6) = x - 14$
9. $x - \frac{2}{3}(3x+6) = \frac{2}{5}(5x-10)$
10. $3x - (2x - \overline{x-2}) = 3x - 9$
11. $3x - \frac{5}{3}x + 7 = \frac{13}{3}x - 2$
12. $\frac{15x - 6x}{7} - 8 = 7 - \frac{18x}{21}$
13. $\frac{5x-6}{5} + \frac{7x-3}{10} = \frac{2x+8}{15}$
14. $\frac{1-x}{3} - \frac{x-1}{6} = \frac{x+1}{12}$

160. It is sometimes better to unite some terms before clearing of fractions. Thus,

Given $\frac{x+6}{4}-\frac{2x-5}{8}=\frac{4x+7}{10}-\frac{7x+4}{10}-\frac{3}{5}$

Reducing to common denominators,

$$\frac{2x+12}{8}-\frac{2x-5}{8}=\frac{8x+14}{10}-\frac{7x+4}{10}-\frac{6}{10}$$

Uniting terms, $\frac{17}{8}=\frac{x+4}{10}$

Clearing of fractions, $8x+32=170$,

whence $8x=138$

and $x=17\frac{1}{4}$

15. Solve $\frac{3x-5}{4}-\frac{5}{7}=\frac{x+7}{4}+\frac{2}{7}$

16. Solve $\frac{3x-2}{4}+\frac{18}{19}=\frac{5x+3}{2}-\frac{1}{19}$

17. Solve $\frac{x+3}{5}+\frac{5x+6}{8}=\frac{4x+5}{5}+\frac{3x-2}{8}$

18. Solve $\frac{3-x}{3+x}-\frac{3+x}{3-x}=\frac{5}{x^2-9}$

19. Solve $\frac{x+2}{x+3}-\frac{x-3}{x-2}=\frac{x-3}{x^2+x-6}$

20. Solve $\frac{1}{x-2}-\frac{1}{x+2}=\frac{x+1}{x^2-4}$

21. Solve $\frac{2x+2\frac{1}{3}}{5}-\frac{2x-2}{7x-6}=\frac{4x-2}{10}$

22. Solve $\frac{6x+7}{18}=\frac{x-5}{x-6}+\frac{2x-3}{6}$

23. Solve $\frac{7x-5}{21}-\frac{x-2}{3}=\frac{5x+6}{x-3}$

24. Solve $\frac{3x-2}{3}+\frac{5-9x}{9}=\frac{x-7}{x+7}$

2. Solution of Simple Literal Equations.

Illustrations.—1. Solve $\frac{x}{a}+\frac{x}{b}=\frac{a}{c}$.

Clearing of fractions,

$$bcx+acx=a^2b$$

Factoring, $(bc+ac)x=a^2b$

Dividing by coefficient of x, $x=\frac{a^2b}{bc+ac}$

2. Solve $\frac{a+x}{a}+\frac{b+x}{ab}=\frac{c+x}{b}$.

Clearing of fractions,

$$ab+bx+b+x=ac+ax$$

Transposing, $bx-ax+x=ac-ab-b$

Factoring, $(b-a+1)x=ac-ab-b$

Dividing by the coefficient of x, $x=\frac{ac-ab-b}{b-a+1}$

EXERCISE 70.

Solve:

1. $\frac{x}{a}+\frac{x}{b}=c$
2. $ax-bx=a-b$
3. $cx+dx=c^2-d^2$
4. $m^2x-n^2x=m^4-n^4$
5. $\frac{a+x}{b}+\frac{a-x}{c}=\frac{a}{c}$
6. $\frac{a-x}{m}-\frac{b-x}{n}=\frac{c}{mn}$
7. $3x-\frac{ax}{2}=\frac{7ax}{3}+2$
8. $x-\frac{a-x}{5}=2x+\frac{b+x}{10}$
9. $\frac{mx-nx}{m+n}=\frac{m}{m-n}$
10. $\frac{x}{a+b}-\frac{x}{a-b}=1$
11. $\frac{ab-cx}{d}=\frac{cd+ax}{a}$
12. $\frac{a-mx}{c}=\frac{c-nx}{d}$
13. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}$
14. $\frac{a}{b}+\frac{b}{c}-\frac{c}{d}=\frac{d}{x}$
15. $\frac{a}{1-bx}=\frac{b}{1-ax}$
16. $\frac{ac}{bx}-\frac{bc}{ax}=a+b$
17. $(a+x)(b+x)=(c+x)(d+x)$

Miscellaneous Examples.

EXERCISE 71.

Solve:

1. $\frac{x-1}{6}+\frac{x-2}{9}-\frac{x-3}{12}=2$

2. $\frac{10x-14}{3}-\frac{6x-4}{7}=\frac{x-5}{2}$

3. $\frac{x-a}{9}-\frac{{}^2/_3 x-b}{5}-\frac{a-x}{6}=\frac{10a+11b}{3}$

4. $\frac{6x+13}{5}-\frac{9x+15}{5}=\frac{2x}{1\,{}^2/_3}$

5. $\frac{10x+17}{6}-\frac{36x+6}{11x-8}=\frac{5x-4}{3}$

6. $\frac{18x+7}{2}-\left(2x-\frac{2x-1}{7}\right)=36$

7. $\frac{ab}{2x}=\frac{bm+n}{2}+\frac{1}{2x}$

8. $(2x-1)(3x+2)=(3x-5)(2x+20)$

9. $(3-x)^2-(x-5)^2=-4x$

10. $\frac{2x+1\,{}^1/_2}{5}-\frac{(2\,{}^2/_5)x-1}{x-{}^1/_5}=\frac{x-{}^1/_2}{2\,{}^1/_2}$

11. $\frac{10-4x}{x+1}=\frac{3-2x}{{}^1/_2 x+2}$

12. $\frac{x^2-x}{x^2-2x}-\frac{x^2+x}{x^2+2x}=\frac{3x}{x^3-4x}$

13. $(6a-2x)(3a+6x)=(10a+2x)(3a-6x)$

14. $x=\frac{1+2ax}{a}-\frac{4x-2}{a^2}$

15. $\cdot07(10x+1\cdot3)=\cdot3(\cdot4x-\cdot01)+\cdot5$

16. $\cdot66x-\frac{\cdot144x-\cdot11}{\cdot5}=\cdot02x+\cdot198$

17. $\frac{9}{1-x} - \frac{6}{1+x} = -\frac{3}{x^2-1}$

18. $(cx-b)^2 = b^2 - ab + (a-cx)^2$

19. $\frac{4}{5}x + 1\frac{11}{25} + \frac{5x+20}{9x-16} = \frac{12}{15}x + 3\frac{11}{25}$

20. $x - 1 + \frac{m}{a}(x-2) = \frac{m^2}{a^2}$

21. $\frac{x-a}{b} = \frac{a^2+b^2}{ab} - \frac{x-b}{a}$

22. $\frac{1{\cdot}2x - {\cdot}06}{1{\cdot}4} - \frac{{\cdot}32x - {\cdot}024}{{\cdot}35} = -\frac{2{\cdot}3x - 1{\cdot}8}{2}$

23. $x - \frac{a^2-3bx}{a^2} - b^2 = \frac{bx}{a} + \frac{6bx-5a^2}{2a^2} - \frac{bx+4a}{4a}$

24. $\frac{18x+10}{42} + \frac{16x-14}{18x+6} = \frac{72x+30}{168} + \frac{20\frac{1}{2}}{42}$

25. $\frac{1}{x-2} + \frac{1}{x-5} = \frac{1}{x-4} + \frac{1}{x-3}$

Examples involving Simple Equations of One Unknown Quantity.

EXERCISE 72.

A has twice as much money as B, and C has three times as much as B, and they together have $1200. How much has each?

Let $x =$ the number of dollars B has,
then $2x =$ the number A has,
and $3x =$ the number C has.

$\therefore\ x + 2x + 3x = 1200$
$6x = 1200$
$x = 200$, B's number of dollars;
$2x = 400$, A's number of dollars;
$3x = 600$, C's number of dollars.

1. A number increased by 3 times itself and 4 times itself equals 80. What is the number?

2. A man paid $255 for a horse, cow, and pig; the cow cost 10 times as much as the pig, and the horse 4 times as much as the cow. Required the cost of each.

3. If a certain number be increased by the sum of $^2/_3$ and $^5/_6$ of itself it will be 45. What is the number?

4. Grandfather's age diminished by its $^1/_4$, and the remainder diminished by *its* $^1/_3$, is 40 years. What is his age?

5. A farmer raised 660 bushels of wheat in three fields; he raised $^2/_3$ as much in the second field as in the first, and $^4/_5$ as much in the third as in the second. How much did he raise in each?

6. A number increased by its $^5/_6$, and the result diminished by *its* $^2/_3$, leaves a remainder of 48. What is the number?

The sum of two numbers is 70, and one is 16 more than the other. Required the numbers.

Let $x =$ the smaller number,
then $x + 16 =$ the greater number,
and $2x + 16 =$ their sum.

$$\therefore 2x + 16 = 70$$
$$2x = 54$$
$$x = 27, \text{ the smaller number;}$$
$$x + 16 = 43, \text{ the greater number.}$$

7. The sum of two numbers is 84, and the greater is 12 more than double the smaller. What are the numbers?

8. A horse and carriage cost $252, and the cost of the horse was $12 more than twice the cost of the carriage. Required the cost of each.

9. If $^7/_8$ of a number increased by 10 more than $^2/_3$ of the number is 190, what is the number?

10. An estate of $16,000 is to be divided among A, B, and C. B is to have $1000 more than 4 times as much as A, and C is to have $500 more than $^1/_2$ as much as B. What is the share of each?

11. The sum of two numbers is 163, and their difference is 19. What are the numbers?

12. A man has three coils of rope containing 200 yards: the second contains $\frac{2}{3}$ as much as the first, $+ 15$ yards; and the third $\frac{1}{3}$ as much as the second, $+ 10$ yards. How many yards are there in each coil?

If a certain number be doubled, and then increased by 84, it will be five times the number. What is the number?

Let $x =$ the number,

then $2x + 84 =$ twice the number increased by 84,

and $5x =$ five times the number.

$$\therefore \quad 2x + 84 = 5x \text{ (Ax. 1)}$$
$$3x = 84$$
$$x = 28$$

13. In a certain orchard one half the trees bear apples, one fourth plums, one fifth peaches, and the remaining 20 cherries. How many trees in the orchard?

14. A man spent $\frac{2}{5}$ of his money and then earned \$12, after which he had $\frac{4}{5}$ as much as he had at first. How much had he at first?

15. Three men purchased a ship. A paid $\frac{7}{20}$ of it; B, $\frac{8}{25}$ of it; and C, the remainder, which was \$6600. Required the value of the ship.

16. A baker uses $\frac{1}{2}$ of a barrel of flour, $- 2$ pounds, at one time; $\frac{3}{4}$ of the remainder, $+ 5$ pounds, at another time; and $\frac{4}{5}$ of what then remains, $+ 4$ pounds, at another time, and finds the barrel empty. How many pounds were in the barrel at first?

17. If a certain number be halved, and then diminished by 14, it will be $\frac{1}{3}$ of its original self. Find the number.

18. A certain sum is to be divided among A, B, and C. A is to have \$30 less than $\frac{1}{2}$ of it, B \$10 less than $\frac{1}{3}$ of it, and C \$8 more than $\frac{1}{4}$ of it. What will each receive?

The sum of two numbers is 121, and 4 times the greater equals 7 times the less. Required the numbers.

Since 4 times the greater equals 7 times the less, the greater equals $^7/_4$ of the less.

$$\begin{aligned} \text{Let} \quad 4x &= \text{the less,} \\ \text{then} \quad 7x &= \text{the greater,} \\ \text{and} \quad 11x &= 121, \text{ their sum.} \\ x &= 11 \\ 4x &= 44, \text{ the less;} \\ 7x &= 77, \text{ the greater.} \end{aligned}$$

19. Four times John's age equals 5 times William's, and the difference of their ages is 4 years. What are their ages?

20. Two thirds of A's money equals $^4/_5$ of B's, and they together have $5500. How much has each?

21. A farmer raised 500 bushels of corn and oats; $^3/_5$ of the quantity of corn equaled $^9/_{10}$ of the quantity of oats. How much of each kind did he raise?

22. A and B together husked 180 shocks of corn; $^3/_4$ of the number A husked equals 7 more than $^7/_{12}$ of the number B husked. How many did each husk?

A sold a horse for $160.50, and thereby gained 7%. What did he pay for the horse?

$$\text{Let} \qquad x = \text{the cost of the horse.}$$

Since he gained 7%, he gained $\frac{7}{100}$ of the cost, which is $\frac{7}{100}$ of x, or $\frac{7x}{100}$.

$$\text{Then} \quad x + \frac{7x}{100} = \text{the selling-price.}$$

$$\therefore \quad x + \frac{7x}{100} = \$160.50$$

$$\text{whence} \qquad x = \$150$$

23. A miller sells corn at 66 cents a bushel, and thereby gains 10%. What is the cost of the corn?

24. If a number be increased by $33\frac{1}{3}\%$ of itself, and that sum increased by 25% of itself, it will be 110. What is the number?

25. A farmer lost 8% by selling a cow for \$69. What was the cost of the cow?

26. A and B together have \$22,500, and A has 25% more than B. How much has each?

27. If A spends 25% of his money, and then earns 40% of what he has remaining, and then has \$140, how much money has he at first?

28. A's house cost \$300 more than B's, and 24% of the cost of A's equals $34\frac{2}{7}$% of the cost of B's. What was the cost of each?

29. A merchant bought cloth 20% below marked price and sold it at a gain of 30%, and gained 70 cents a yard. What was the marked price?

A jeweler bought watches for \$40 apiece and sold them at an advance of \$16 apiece. What was his gain per cent?

Let $x =$ his gain per cent;

then $\frac{x}{100}$ of 40, or $\frac{2x}{5} =$ the gain in dollars;

whence $\frac{2x}{5} = 16$

and $x = 40$

30. Bought a cow for \$50, and sold her for \$65. What was the gain per cent?

31. Sold a horse for \$150, and thereby lost \$25. What was the loss per cent?

32. By what per cent must the fraction $\frac{7}{8}$ be increased to make the fraction $\frac{15}{16}$?

33. A man doubled his capital each year for three years, and the fourth year lost all he had previously gained. What per cent did he lose the fourth year?

34. A sold B an acre of land for \$150, and gained 25% of the cost; B sold it to C for the same price that A paid. What per cent did B lose?

What principal will in 5 years at 6% amount to \$624?

Let $x =$ the principal.

For 5 years at 6%, $\frac{30}{100}$ or $\frac{3}{10}$ of the principal equals the interest;

hence, $\frac{3}{10}$ of x, or $\frac{3x}{10} =$ the interest;

whence $x + \frac{3x}{10} =$ the amount in dollars,

and $x + \frac{3x}{10} = 624$

$$x = 480$$

35. What sum of money must be put at simple interest for 8 years at $4\frac{1}{2}$% to amount to \$6800?

36. The interest on a certain principal for 7 years at 6% is \$464 less than the principal. Required the principal.

37. A has \$200 more than B, and the sum of their interests for 5 years at 4% is \$160. What is the principal of each?

38. A gives B \$800 for 3 years at 6% per annum. How many dollars must B give A for 4 years at 5% per annum to yield the same amount of interest?

In how many years will \$700 at 6%, simple interest, amount to \$910?

Let $x =$ the number of years.

At 6% for 1 year, $\frac{6}{100}$ or $\frac{3}{50}$ of the principal equals the interest, and

for x years the interest is $\frac{3x}{50}$ of the principal;

hence $\frac{3x}{50}$ of $700 =$ the interest in dollars;

and $910 - 700 =$ the interest in dollars.

$$\therefore \quad \frac{3x}{50} \times 700 = 910 - 700 \text{ (Ax. 1)}$$

whence $x = 5$

39. In what time will \$800 at $4\frac{1}{2}$% amount to \$1004, simple interest?

40. In what time will \$750, at $6\frac{2}{3}\%$ per annum, double itself at simple interest?

41. In what time will a dollars at r per cent, simple interest, treble itself?

42. At what rate will \$750 in 6 years, at simple interest, amount to \$1020?

43. At what rate will m dollars in n years, at simple interest, double itself?

I bought a \$100 bond, bearing 5% interest, for \$80. What per cent of my investment did I gain annually?

Let $x =$ the annual gain per cent,

then $\frac{x}{100}$ of \$80 = the entire gain;

but $\frac{5}{100}$ of \$100 = the entire gain.

$$\therefore \quad \frac{x}{100} \text{ of } 80 = \frac{5}{100} \text{ of } 100,$$

whence $x = 6\frac{1}{4}$

44. Bought railroad stock, par value \$50 a share, for \$45 a share; the company declared a dividend of 6%. What per cent did I receive on my investment?

45. At what price must I buy railroad stock, par value \$100 a share, in order that a 6% dividend will bring me an income of 8% on my investment?

46. I bought a \$50 share for \$40; the company declared a dividend which I found was $7\frac{1}{2}\%$ of my investment. What per cent of the par value was it?

47. If 25% of the par value of stock equals 40% of the market value, what is the par value of stock that is selling at $\$62\frac{1}{2}$ a share?

48. If stock bought at 90 yields an income of 5%, at what price would it yield 6%?

49. What capital invested in 5's at 80 will yield the same income as \$4500 invested in 6's at 90?

How far may a person ride in a coach, going at the rate of 5 miles an hour, that he may walk back at the rate of 2 miles an hour and be gone 5 hours?

Let $x =$ the number of miles,

then will $\frac{x}{5} =$ the time going;

and $\frac{x}{2} =$ the time returning;

whence $\frac{x}{5} + \frac{x}{2} = 5$

and $x = 7\,^1/_7$

50. If a boat sailed down a stream at the rate of 10 miles an hour and returned at the rate of 6 miles an hour, and was gone 6 hours, how far did it sail down the stream?

51. A boat whose rate of sailing in still water is 10 miles an hour, goes down a stream whose rate is two miles an hour, and returns, making the round trip in 5 hours. How far does it go down the stream?

52. A boat whose rate of sailing in still water is 6 miles an hour, goes a miles down the stream in one half the time it requires to return. What is the rate of the current?

A can do a piece of work in 5 days and B can do it in 8 days. In what time can they do it working together?

Let $x =$ the number of days required,

then $\frac{1}{x} =$ the part they can do in 1 day,

$\frac{1}{5} =$ the part A can do in 1 day,

$\frac{1}{8} =$ the part B can do in 1 day,

hence $\frac{1}{x} = \frac{1}{5} + \frac{1}{8}$

$x = 3\,^1/_{13}$

53. A can do a piece of work in 4 days, B in 5 days, and C in 6 days. In what time can they do it working together?

54. Two pipes can fill a cistern in 5 hours, and one alone can fill it in 8 hours. In what time can the other fill it?

55. There are 3 pipes connected with a reservoir: the first can fill it in 10 hours, the second in 8 hours, and the third can empty it in 6 hours. In what time will it be filled if all run together?

56. A can do a piece of work in $2\frac{1}{2}$ days, working 8 hours a day, and B can do it in $3\frac{1}{3}$ days, working 9 hours a day. In how many days, working 6 hours a day, can they together do it?

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57. A has \$800 and B has $\frac{5}{8}$ as much. How much must A give to B in order that A may have $\frac{5}{8}$ as much as B?

58. B has \$300 more than A, and earns \$5 a day; A earns \$8 a day. How much must each earn in order that they may have the same sum?

59. A man has two horses, and a saddle worth \$10. The first horse and saddle are worth $\frac{3}{4}$ as much as the second horse, and the second horse and saddle $\frac{29}{20}$ as much as the first horse. Required the value of each horse.

60. A general draws up his army in the form of a square, and has 140 men over; he then endeavors to increase each side by 2 men, and finds he lacks 24 men to complete the square. How many men has he?

61. A is 15 years old and B is 30. In how many years will $\frac{2}{3}$ of A's age equal $\frac{4}{9}$ of B's?

62. A man loaned \$1500, a part at 5% and the rest at 6%; his annual interest was \$81. How much did he loan at 5%?

63. How many pounds of sugar at 10 cents a pound must be mixed with 25 pounds worth 8 cents a pound to make a sugar worth $8\frac{3}{4}$ cents a pound?

64. A man agreed to work one year for \$180 and house-rent free. At the expiration of 9 months he was deprived of work by sickness for the rest of the year, but retained the house; he was paid \$120 in money for his services. What was the house-rent valued at?

65. What time of day is it when $^2/_3$ of the time past noon equals $^3/_4$ of the time to midnight?

Suggestion.—Let $x =$ the number of hours past noon.

66. At what time of day is the time past noon $^1/_7$ of the time past midnight?

67. At what time between 4 and 5 o'clock are the minute- and hour-hands of a clock together? At right angles? Opposite each other?

Suggestion.—At 4 o'clock the minute-hand must gain 20 minute-spaces, 5 or 35 minute-spaces, and 50 minute-spaces respectively.

68. A son's age is $^2/_5$ that of his father's, but in 16 years it will be $^4/_7$ that of the father's. What are the ages now?

69. A and B together can do a piece of work in 24 days, A and C in 30 days, B and C in 40 days. In what time can they do it all working together?

70. A boy spent $^1/_2$ his money and $^1/_2$ a cent; then, $^1/_2$ of the remainder and $^1/_2$ a cent; then, $^1/_2$ of what then remained and $^1/_2$ a cent, and had 9 cents remaining. How much money had he at first?

Simple Equations of Two Unknown Quantities.

Definitions and Principles.

161. A single equation of two unknown quantities may be satisfied by any number of values of the unknown quantities, and is therefore said to be ***Indeterminate.***

Thus, $2x - y = 10$ is true when $x = 6$ and $y = 2$; when $x = 7$ and $y = 4$; when $x = 8$ and $y = 6$; etc.

162. Two simultaneous simple equations of two unknown quantities can be satisfied by only one pair of values of the unknown quantities.

Thus, $x + 2y = 7$ and $5x - 3y = 9$ are satisfied only by $x = 3$ and $y = 2$.

163. Generally, when there are as many independent simultaneous equations given as there are unknown quantities involved, their solution can be effected by elimination. (See page 96.)

164. There are three easy methods of elimination :

1. By addition and subtraction. 2. By substitution. 3. By comparison.

Note.—For elimination by addition and subtraction, see pages 59 and 96.

Elimination by Substitution.

Illustration.—

Given $\left\{\begin{array}{ll} 5x - 2y = 4 & \text{(A)} \\ 3x + y = 9 & \text{(B)} \end{array}\right\}$ to find the values of x and y.

Solution: Transpose $3x$ in (B),

$$y = 9 - 3x \qquad (1)$$

Substitute (1) in (A),

$$5x - 2(9 - 3x) = 4 \qquad (2)$$

Solve (2), $x = 2$

Substitute 2 for x in (1), and reduce,

$$y = 3$$

EXERCISE 73.

Solve :

1. $\left\{\begin{array}{l} 4x + 7y = 19 \\ 3x + 2y = 11 \end{array}\right\}$

2. $\left\{\begin{array}{l} 3x - 5y = 7 \\ 4x + 7y = 23 \end{array}\right\}$

3. $\left\{\begin{array}{l} 3x - 2y = 5 \\ 2x + 3y = 25 \end{array}\right\}$

4. $\left\{\begin{array}{l} 6x - 9y = 0 \\ 4x + 3y = 3 \end{array}\right\}$

5. $\left\{\begin{array}{l} 8x + 7y = 83 \\ 5x - 3y = 15 \end{array}\right\}$

6. $\left\{\begin{array}{l} 7x - 2y = 27 \\ 9x + 6y = 69 \end{array}\right\}$

7. $\begin{cases} 8x - 5y = -8 \\ 10x + 7y = 96 \end{cases}$

8. $\begin{cases} 11x - 7y = -1 \\ 12x + 9y = -51 \end{cases}$

9. $\begin{cases} 13x - 15y = -13 \\ 14x + 17y = -14 \end{cases}$

10. $\begin{cases} 24x - 18y = 48 \\ 14x + 24y = 166 \end{cases}$

11. $\begin{cases} x + y = 0 \\ 25x + 24y = 4 \end{cases}$

12. $\begin{cases} 18x - 15y = 7 \\ 15x + 24y = 91 \end{cases}$

Elimination by Comparison.

Illustration.—Solve $\begin{cases} 3x + 5y = 21 & \text{(A)} \\ 4x - 3y = -1 & \text{(B)} \end{cases}$

Solution: Transpose $5y$ in (A) and divide by 3,

$$x = \frac{21 - 5y}{3} \qquad (1)$$

Transpose $-3y$ in (B) and divide by 4,

$$x = \frac{3y - 1}{4} \qquad (2)$$

Compare (1) and (2),

$$\frac{21 - 5y}{3} = \frac{3y - 1}{4}$$

Reduce (3), $y = 3$

Substitute 3 for y in (1) and reduce,

$$x = 2$$

EXERCISE 74.

Solve:

1. $\begin{cases} x + 2y = 10 \\ 3x + 4y = 24 \end{cases}$

2. $\begin{cases} 4x - 2y = 12 \\ 3x + 2y = 2 \end{cases}$

3. $\begin{cases} 6x - 5y = 4 \\ 4x + 7y = 44 \end{cases}$

4. $\begin{cases} 3x + 5y = -5 \\ 5x + 3y = -19 \end{cases}$

5. $\begin{cases} 3x - y = 19 \\ x - 3y = 1 \end{cases}$

6. $\begin{cases} 8x - 9y = -66 \\ 7x + 10y = 121 \end{cases}$

7. $\begin{cases} 2x + 3y = 5 \\ 8x + 9y = 18 \end{cases}$

8. $\begin{cases} 12x - 25y = 1 \\ 22x + 15y = 14 \end{cases}$

9. $\begin{cases} 12x - 7y = 1 \\ 11x - 5y = 8 \end{cases}$

10. $\begin{cases} 6x + 9y = 5 \\ 9x + 6y = 5 \end{cases}$

Solution of Numerical Equations.

Illustration.—Solve
$$\begin{cases} \dfrac{x+4}{2}+\dfrac{y-2}{3}=7 & \text{(A)} \\ \dfrac{x-3}{3}+\dfrac{y+4}{4}=4 & \text{(B)} \end{cases}$$

Solution: Clearing (A) and (B) of fractions,

$$3x+12+2y-4=42 \qquad (1)$$
$$4x-12+3y+12=48 \qquad (2)$$

Transposing and uniting terms in (1) and (2),

$$3x+2y=34 \qquad (3)$$
$$4x+3y=48 \qquad (4)$$

Multiplying (3) by 3 and (4) by 2,

$$9x+6y=102 \qquad (5)$$
$$8x+6y=96 \qquad (6)$$

Subtracting (6) from (5),

$$x=6$$

Substituting 6 for x in (3) and reducing,

$$y=8$$

EXERCISE 75.

Solve:

1. $$\begin{cases} x+\frac{5}{7}y=24 \\ \frac{5}{7}x+y=24 \end{cases}$$

2. $$\begin{cases} \frac{2}{5}x-\frac{5}{6}y=-6 \\ \frac{3}{10}x+\frac{2}{3}y=11 \end{cases}$$

3. $$\begin{cases} \dfrac{x+2y}{5}+\dfrac{3x-2y}{6}=\dfrac{36}{5} \\ \dfrac{2x+4y}{16}-\dfrac{5x+4y}{31}=0 \end{cases}$$

4. $$\begin{cases} \dfrac{x}{12}+\dfrac{y}{18}=8 \\ \dfrac{x}{9}+\dfrac{y}{15}=10 \end{cases}$$

5. $$\begin{cases} 7x-5y=6 \\ \frac{2}{5}(x+7)-\frac{3}{4}(y-9)=2 \end{cases}$$

6. $$\begin{cases} \frac{1}{2}(x-y)+\frac{1}{3}(x+y)=\frac{23}{6} \\ \frac{1}{2}(x+y)-\frac{1}{3}(x-y)=\frac{29}{6} \end{cases}$$

7. $\begin{cases} \dfrac{3x-y}{6} - \dfrac{4x+2y}{11} = -4 \\ \dfrac{5x+3y}{10} + \dfrac{2x-7y}{3} = -20 \end{cases}$

8. $\begin{cases} \dfrac{2}{x+2y} = \dfrac{3}{x-3y} \\ x+20y=40 \end{cases}$

9. $\begin{cases} \dfrac{x+y}{x-y} = \dfrac{5}{3} \\ 3x-4y=20 \end{cases}$

10. $\begin{cases} \dfrac{5x+6y}{5} - \dfrac{8x-4y}{3} = 12 \\ \dfrac{7x+9y}{11} = \dfrac{9x+8y}{12} \end{cases}$

11. $\begin{cases} 1 - \dfrac{x+y}{x-y} = \dfrac{3x}{x-y} \\ \dfrac{7x-3y}{23} = 3 \end{cases}$

12. $\begin{cases} \dfrac{12y+7x}{5x-3y} = 1 \\ \dfrac{18y-7x}{6x+10} = 2 \end{cases}$

13. $\begin{cases} \dfrac{x-3y}{9} - \dfrac{7x+3y}{43} = -4 \\ \dfrac{5y-7x}{13} = -5 \end{cases}$

14. $\begin{cases} \dfrac{3x-8y}{7} - \dfrac{5x-3y}{14} = -14 \\ \dfrac{x+8y}{11} = 2x+y+48 \end{cases}$

15. $\begin{cases} \dfrac{8x-5y}{7} + \dfrac{11y-4x}{5} = 4 \\ \dfrac{17x-13y}{5} + \dfrac{2x}{3} = 7 \end{cases}$

16. $\begin{cases} \dfrac{x-y}{2} = 4\dfrac{1}{6} - \dfrac{1}{3}(x+y) \\ x+y-5 = \dfrac{2}{3}(y-x) \end{cases}$

165. It is sometimes easier to eliminate one of the unknown quantities without clearing of fractions.

Illustration.—Solve $\begin{cases} \dfrac{3}{2x} + \dfrac{4}{3y} = \dfrac{7}{6} & \text{(A)} \\ \dfrac{5}{3x} - \dfrac{3}{4y} = \dfrac{13}{72} & \text{(B)} \end{cases}$

Solution: Reduce the corresponding terms in (A) and (B) to a common denominator,

$$\frac{9}{6x} + \frac{16}{12y} = \frac{84}{72} \qquad (1)$$

$$\frac{10}{6x} - \frac{9}{12y} = \frac{13}{72} \qquad (2)$$

Multiply (1) by 10 and (2) by 9,

$$\frac{90}{6x} + \frac{160}{12y} = \frac{840}{72} \qquad (3)$$

$$\frac{90}{6x} - \frac{81}{12y} = \frac{117}{72} \qquad (4)$$

Subtract (4) from (3), $\dfrac{241}{12y} = \dfrac{723}{72}$ (5)

Clear of fractions, $723 \times 12y = 72 \times 241$

$$y = \frac{72 \times 241}{723 \times 12} = 2$$

Substitute 2 for y in (A),

$$\frac{3}{2x} + \frac{4}{6} = \frac{7}{6}$$

$$\frac{3}{2x} = \frac{3}{6}$$

$$6x = 18$$

$$x = 3$$

EXERCISE 76.

Solve:

1. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5 \\ \dfrac{1}{x} - \dfrac{1}{y} = -1 \end{cases}$

2. $\begin{cases} \dfrac{1}{x+1} + \dfrac{2}{y+1} = \dfrac{5}{6} \\ \dfrac{3}{x+1} + \dfrac{4}{y+1} = 2 \end{cases}$

3. $\begin{cases} \dfrac{3}{x} + \dfrac{2}{y} = \dfrac{13}{12} \\ \dfrac{5}{x} + \dfrac{7}{y} = \dfrac{29}{12} \end{cases}$

4. $\begin{cases} \dfrac{5}{2x} + \dfrac{3}{4y} = 7\dfrac{1}{4} \\ \dfrac{5}{3x} + \dfrac{3}{2y} = 7\dfrac{5}{6} \end{cases}$

5. $\left\{\begin{array}{l}\dfrac{2}{x+2}+\dfrac{5}{y-2}=6 \\ \dfrac{3}{x+2}-\dfrac{1}{y-2}=\dfrac{1}{2}\end{array}\right\}$

6. $\left\{\begin{array}{l}\dfrac{1}{2(x+1)}+\dfrac{4}{3(y+1)}=5 \\ \dfrac{2}{2(x+1)}-\dfrac{1}{3(y+1)}=1\end{array}\right\}$

7. $\left\{\begin{array}{l}\dfrac{5}{x}+\dfrac{7}{2y}=37\dfrac{1}{2} \\ \dfrac{3}{2x}-\dfrac{5}{2y}=-6\dfrac{1}{2}\end{array}\right\}$

8. $\left\{\begin{array}{l}\dfrac{3}{2x}+\dfrac{5}{3y}=34 \\ \dfrac{2}{3x}-\dfrac{4}{5y}=-8\end{array}\right\}$

Solution of Literal Equations.

Illustrations.—Solve $\left\{\begin{array}{ll} ax+by=c & \text{(A)} \\ mx+ny=d & \text{(B)}\end{array}\right\}$

Solution: Multiply (A) by m and (B) by a,

$$amx+bmy=cm \qquad (1)$$
$$amx+any=ad \qquad (2)$$

Subtract (2) from (1),

$$(bm-an)y=cm-ad \qquad (3)$$
$$y=\frac{cm-ad}{bm-an}$$

Multiply (A) by n and (B) by b,

$$anx+bny=cn \qquad (4)$$
$$bmx+bny=bd \qquad (5)$$

Subtract (5) from (4),

$$(an-bm)x=cn-bd$$
$$x=\frac{cn-bd}{an-bm}$$

Solve $\left\{\begin{array}{ll}\dfrac{a}{x}+\dfrac{b}{y}=c & \text{(A)} \\ \dfrac{a}{x}-\dfrac{c}{y}=d & \text{(B)}\end{array}\right\}$

Solution: Subtract (B) from (A),

$$\frac{b+c}{y}=c-d \qquad (1)$$
$$b+c=(c-d)y \qquad (2)$$
$$y=\frac{b+c}{c-d}$$

EXERCISE 77.

Solve:

1. $\begin{cases} ax + by = c \\ ax - by = d \end{cases}$

2. $\begin{cases} x + y = m \\ ax + by = n \end{cases}$

3. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = a \\ \dfrac{x}{a} - \dfrac{y}{b} = b \end{cases}$

4. $\begin{cases} mx + ny = p \\ rx + sy = q \end{cases}$

5. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = c \\ \dfrac{m}{x} + \dfrac{n}{y} = d \end{cases}$

6. $\begin{cases} ax + my = n \\ bx - my = c \end{cases}$

7. $\begin{cases} x - y = a \\ mx + ny = b \end{cases}$

8. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{a^2} = \dfrac{1}{a} \\ \dfrac{x}{b} - \dfrac{y}{b^2} = \dfrac{1}{b} \end{cases}$

9. $\begin{cases} cx - dy = b \\ mx - ny = b \end{cases}$

10. $\begin{cases} \dfrac{a}{mx} + \dfrac{b}{ny} = c \\ \dfrac{a}{nx} + \dfrac{b}{my} = d \end{cases}$

Examples involving Simple Equations of Two Unknown Quantities.

EXERCISE 78.

1. A man bought two farms, one of 80 acres and one of 50 acres, for $22190. Had the first contained 70 acres and the second 60 acres, the second would have cost $^{25}/_{28}$ as much as the first. What was the price of each farm per acre?

2. Two thirds of A's fortune added to three fourths of B's is $700, and B's increased by $100 is five sixths of A's. What is the fortune of each?

3. The sum of two numbers is 7, and if the larger be added to the numerator and the smaller to the denominator of the fraction $^5/_9$, the result will equal $^3/_4$. What are the numbers?

4. The sum of two fractions is $^{15}/_{16}$, and their difference is $^{13}/_{16}$. What are the fractions?

5. A man has a certain quantity of oats and corn. If he mixes two thirds of his oats with one half of his corn, he will have a mixture of 60 bushels; but if he mixes all his corn with four fifths of his oats, the oats in the mixture will exceed the corn by 8 bushels. How many bushels of each kind has he?

6. A man has two watches and a chain. The first watch is worth \$60. If the chain be put on the first watch they together will be worth $^{4}/_{5}$ as much as the second; but if the chain be put on the second watch they together will be worth twice as much as the first. Required the value of the second watch and chain respectively.

7. If 2 be added to the numerator of a fraction it will be $^{1}/_{2}$, but if 3 be added to the denominator it will be $^{1}/_{3}$. What is the fraction?

Suggestion.—

Let $\frac{x}{y}$ = the fraction; then, $\frac{x+2}{y}=\frac{1}{2}$, and $\frac{x}{y+3}=\frac{1}{3}$.

8. If 3 be added to both terms of a fraction it will be $^{5}/_{6}$, but if 3 be subtracted from both terms it will be $^{2}/_{3}$. What is the fraction?

9. The difference of two numbers is 5, and if the greater be subtracted from the numerator, and the less from the denominator of $^{12}/_{17}$, the result will be $^{2}/_{7}$. What are the numbers?

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10. There is a number consisting of two figures, such that if 9 be added to the number the figures will change places, and the sum of the figures is 7. Required the number.

Suggestion.—Let x = the ten's figure and y the unit's figure. Then, $10x + y$ = the number, and $10y + x$, the number with the figures interchanged; whence,

$$10x + y + 9 = 10y + x \qquad \text{(A)}$$
$$x + y = 7 \qquad \text{(B)}$$

11. The sum of the two digits of a number is 12, and if 54 be added to the number the digits will change places. What is the number?

12. A certain number is four times the sum of its two digits, and if 9 be added to the number its digits will change places. Required the number.

13. The difference of the two digits of a number is $^1/_{12}$ of the number, and if 6 be added to the number its value will be five times the sum of the digits of the original number. Required the original number.

---

14. A and B together can do a piece of work in 8 days, and A can do as much in 3 days as B can do in 5 days. In how many days can each alone do it?

Suggestion.—Let $x =$ the time in which A can do it,
and $y =$ the time in which B can do it.

$$\frac{1}{x} = \text{the part A can do in 1 day.}$$

$$\frac{1}{y} = \text{the part B can do in 1 day.}$$

$$\frac{1}{8} = \text{the part both can do in 1 day.}$$

$$\therefore \quad \frac{1}{x} + \frac{1}{y} = \frac{1}{8} \qquad \text{(A)}$$

Since A can do as much in 3 days as B can do in 5 days,

$$\frac{3}{x} = \frac{5}{y} \qquad \text{(B)}$$

15. If A works 3 days and B 5 days, they can accomplish a piece of work; but if A works 2 days and B 3 days, they will accomplish only $^5/_8$ of it. In what time can each alone do it?

16. Two thirds of what A can do in a day equals three fourths of what B can do, and they together can do a job in 8 days. How long would it take each alone to do it?

17. Five men and 3 boys can do a piece of work in 6 days, and 4 men can do as much as 6 boys. How long would it take 1 man and 1 boy each to do it?

18. A field may be divided into 8 lots of one size and 9 lots of another size; but 4 lots of the first size and 10 of the second size together will occupy only $^{29}/_{36}$ of the field. Into how many lots of each size may the field be divided?

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19. The distance around a room is 52 feet, and if 4 feet be added to the length it will be twice the width. Required the length and width respectively.

Suggestion.—Let $x =$ the length and y the width, then
$2x + 2y = 52$, the number of feet around the room;
also, $x + 4 = 2y$, twice the width.

20. A man has two square fields, one of which is 6 rods longer than the other, and the sum of the distances around them is 96 rods. What is the length of each field?

21. A man has a rectangular lot, such that twice the length increased by 6 yards equals four times the width diminished by 4 yards, and the distance around it is 50 yards. Required the length and width respectively.

22. A rectangular field has a perimeter of 52 rods, and if its width be increased by 6 rods and its length by 8 rods, the width will be $^2/_3$ of the length. Required the dimensions of the rectangle.

23. A certain fishing-rod consists of two parts: the length of the upper part is $^5/_7$ of the length of the lower part, and 9 times the upper part together with 13 times the lower part exceed 11 times the whole rod by 36 inches. Find the length of the two parts.

24. A and B ran a race which lasted 5 minutes: B had a start of 20 yards, but A ran 3 yards while B ran 2, and won by 30 yards. Find the length of the course and the speed of each.

25. A man having worked 20 days and been idle 8 days, saved \$50. Had he worked 24 days and been idle 12 days, he would have saved \$57. What were his daily wages, provided he maintained himself?

26. If the length of a rectangle be increased by 2 feet and the width by 3 feet, the area will be increased by 42 square feet; but, if the length be diminished by 2 feet and the width be increased by 4 feet, the area will be increased by 12 feet. Required the length and width of the rectangle.

Suggestion.—Let x equal the length and y equal the width.

27. If a farmer had planted 5 more hills of corn in one row, and had planted 5 more rows, he would have had 700 hills of corn more; but, had he planted 5 hills less in one row, and 4 rows less, he would have had 620 hills less. How many hills did he plant?

28. If there had been two more persons at a dinner-party, and each person had paid one shilling less, the entire bill would have been 4 shillings more; but if there had been two persons less, and each person would have paid two shillings more, the bill would have been 2 shillings less. Required the bill and number of persons.

29. If the length of a rectangle were diminished by 5 feet and the width increased by 4 feet, the area would remain unchanged; but, if the length were to remain unchanged and the width increased by 7 feet, the area would be increased by 224 square feet. Required the dimensions and area of the rectangle.

30. A certain sum of money, put out at simple interest, amounts in 6 years to $780, and in 10 years to $900. Required the sum and rate per cent.

31. A certain principal in a given time at 3 per cent amounts to $920, and at 5 per cent for the same time to $1000. Required the principal and time.

32. If two trains start from two stations 40 miles apart at the same time, and approach each other, they will meet in one hour; but if they run in the same direction it will require the faster train 4 hours to overtake the slower. What are their respective rates of running?

33. A passenger-train 200 feet long will pass a freight-train 680 feet long in 30 seconds, if they run in opposite directions; but if they run in the same direction it will require 1 minute to pass it. How many miles per hour does each train run?

34. A and B jointly loan C a sum of money which in five years at 6 per cent amounts to $1170; 60 per cent of A's share of the principal equals 75 per cent of B's share. How much of the amount belongs to each?

Simple Equations of Three or more Unknown Quantities.

Illustrative Examples.—

1. Solve $$\left\{\begin{aligned} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}&=62 \qquad \text{A}\\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}&=47 \qquad \text{B}\\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}&=38 \qquad \text{C}\end{aligned}\right.$$

Solution: Clear equations A, B, and C of fractions,

$$6x+4y+3z=744 \qquad (1)$$
$$20x+15y+12z=2820 \qquad (2)$$
$$15x+12y+10z=2280 \qquad (3)$$

Multiply (1) by 4 and bring down (2),

$$24x+16y+12z=2976 \qquad (4)$$
$$20x+15y+12z=2820 \qquad (2)$$

Subtract (2) from (4), $4x+y=156$ (5)

Multiply (1) by 10 and (3) by 3,

$$60x+40y+30z=7440 \qquad (6)$$
$$45x+36y+30z=6840 \qquad (7)$$

Subtract (7) from (6), $15x+4y=600$ (8)

Multiply (5) by 4, $16x+4y=624$ (9)

Subtract (8) from (9), $x=24$

Substitute value of x in (8), and reduce,

$$y=60$$

Substitute values of x and y in (1), and reduce,

$$z=120$$

2. Solve
$$\begin{cases} \frac{2}{x}+\frac{3}{y}+\frac{4}{z}=29 & \text{(A)} \\ \frac{5}{x}-\frac{3}{y}+\frac{2}{z}=9 & \text{(B)} \\ \frac{3}{x}+\frac{4}{y}-\frac{5}{z}=-2 & \text{(C)} \end{cases}$$

Solution: Add (A) and (B),

$$\frac{7}{x}+\frac{6}{z}=38 \qquad (1)$$

Multiply (A) by 4 and (C) by 3,

$$\frac{8}{x}+\frac{12}{y}+\frac{16}{z}=116 \qquad (2)$$

$$\frac{9}{x}+\frac{12}{y}-\frac{15}{z}=-6 \qquad (3)$$

Subtract (3) from (2), $-\frac{1}{x}+\frac{31}{z}=122 \qquad (4)$

Multiply (4) by 7 and bring down (1),

$$-\frac{7}{x}+\frac{217}{z}=854 \qquad (5)$$

$$\frac{7}{x}+\frac{6}{z}=38 \qquad (1)$$

Add (5) and (1), $\frac{223}{z}=892 \qquad (6)$

Divide by 223, $\frac{1}{z}=4$; whence $z=\frac{1}{4}$

Substitute the value of z in (1), and reduce,

$$x=\frac{1}{2}$$

Substitute the values of x and z in (A), and reduce,

$$y=\frac{1}{3}$$

---

3. Solve
$$\begin{cases} x+y=14 & \text{(A)} \\ x+z=16 & \text{(B)} \\ y+z=18 & \text{(C)} \end{cases}$$

Solution: Take the sum of (A), (B), and (C),

$$2x+2y+2z=48 \qquad (1)$$

Divide (1) by 2, $x+y+z=24 \qquad (2)$

Subtract (A) from (2), $z=10$

Subtract (B) from (2), $y=8$

Subtract (C) from (2), $x=6$

EXERCISE 79.

Solve:

1. $\begin{cases} -3x + 2y - 5z = -16 \\ 3x - 2y + 4z = -10 \\ 4x + 2y - 5z = -2 \end{cases}$

2. $\begin{cases} 5x - 7y + 6z = 4 \\ 2x + 4y - 5z = 6 \\ x + 3y - 2z = 10 \end{cases}$

3. $\begin{cases} 2x + 3y - z = 12 \\ 4x - 2y + z = 3 \\ 3x + 3y + z = 16 \end{cases}$

4. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 23 \\ \frac{1}{3}x + \frac{1}{2}y + \frac{1}{4}z = 25 \\ \frac{1}{4}x + \frac{1}{2}y + \frac{1}{3}z = 27 \end{cases}$

5. $\begin{cases} x + y = \frac{5}{6} \\ x + z = \frac{2}{3} \\ y + z = \frac{1}{2} \end{cases}$

6. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 7 \\ \frac{1}{x} + \frac{1}{z} = 11 \\ \frac{1}{y} + \frac{1}{z} = 14 \end{cases}$

7. $\begin{cases} ax + by = m \\ ax + cz = n \\ by + cz = r \end{cases}$

8. $\begin{cases} x + y + z = 15 \\ x - y + z = 5 \\ x + y - z = 3 \end{cases}$

9. $\begin{cases} 2x - 3y + 4z = 20 \\ 3x + 2y - 2z = 34 \\ x + 5y - z = 44 \end{cases}$

10. $\begin{cases} \frac{1}{4}x - \frac{1}{8}y + \frac{1}{2}z = 12 \\ \frac{1}{8}x + \frac{1}{2}y - \frac{1}{4}z = 3 \\ \frac{1}{2}x - \frac{1}{4}y + \frac{1}{8}z = 3 \end{cases}$

11. $\begin{cases} x + 2y - 3z = -1 \\ 4x - 4y - z = 8 \\ 3x + 8y + 2z = -5 \end{cases}$

12. $\begin{cases} x + y + z = 12 \\ x + y + u = 11 \\ x + z + u = 10 \\ y + z + u = 9 \end{cases}$

13. $\begin{cases} \frac{2}{x} + \frac{3}{y} - \frac{4}{z} = 8 \\ \frac{3}{x} - \frac{4}{y} + \frac{5}{z} = 26 \\ \frac{5}{x} - \frac{3}{y} + \frac{2}{z} = 23 \end{cases}$

14. $x + 2y = 5z - 10x = y + z = 60$

15. $\begin{cases} ax + by - cz = a^2 + b^2 - c^2 \\ ax - by + cz = a^2 - b^2 + c^2 \\ -ax + by + cz = b^2 + c^2 - a^2 \end{cases}$

Examples involving Simple Equations of Three or More Unknown Quantities.

EXERCISE 80.

1. If 5 bushels of corn, 6 bushels of oats, and 8 bushels of rye together are worth \$10.30; 3 bushels of corn, 5 bushels of oats, and 8 bushels of rye, \$8.75; and 1 bushel of oats mixed with 1 bushel of rye is worth as much as $1\frac{2}{3}$ bushel of corn—what is the value of each per bushel?

2. A's farm, plus $\frac{1}{3}$ of B's and C's, equals 230 acres; B's, plus $\frac{1}{4}$ of A's and C's, equals A's; and C's, plus $\frac{1}{5}$ of A's and B's, equals 170 acres. How many acres are there in each farm?

3. If A should give B one half of his money, and then B give C one half of his, C would have \$550; if B should give C one half of his money, and then C give A one half of his, A would have \$800; if C should give A one half of his money, and then A give B one half of his, B would have \$750. How much has each?

4. A, B, and C together can do a piece of work in $5\frac{5}{11}$ days; A can do twice as much as B or three times as much as C in a day. How long will it take each alone to do it?

5. The sum of A's and B's ages is 55 years; the sum of A's and C's is 62 years; and the sum of B's and C's is 77 years. Required the age of each.

6. A and B can do a piece of work in 4 days, A and C in 5 days, and B and C in 6 days. In what time can each alone do it?

7. Two supply-pipes, A and B, and one discharge-pipe, C, are connected with a cistern. If the three pipes run together for 2 hours, the cistern will be $\frac{7}{60}$ full; if A runs 3 hours, B, 4 hours, and C, 2 hours, it will be $\frac{7}{30}$ full; and if A runs 5 hours, B, 3 hours, and C, 2 hours, it will be $\frac{3}{10}$ full. How long will it take A and B each to fill it, and C to empty it?

8. A man bought a horse, carriage, and harness for \$500. The horse cost \$5 more than the carriage and harness, and the carriage cost $^2/_3$ as much as the horse and harness. Required the cost of each.

9. There is a number consisting of three digits: the sum of the digits is 13; the middle digit is $^4/_9$ of the other two; and if 297 be added to the number the unit's and hundred's digits will change places. Required the number.

10. A's money in 9 years at 6% will produce as much interest as the sum of B's and C's in $4\,^1/_2$ years at 4%; B's in 8 years at 5% as much as A's and C's in $3\,^1/_3$ years at 6%; and C's in 7 years at 3%, \$42 more than A's and B's in 3 years at 4%. Required the principal of each.

11. A, B, C, and D received \$1000. B got half as much as A. The excess of C's share over D's was $^1/_3$ of A's share, and B's share, increased by \$100, was equal to the sum of C's and D's shares. How much did each receive?

12. If 40 peaches are worth as much as 20 quinces and 4 oranges; and 40 quinces are worth as much as 30 peaches and 12 oranges; and 40 oranges, 70 peaches, and 20 quinces are worth \$4—what is the price of each apiece?

13. A man has \$180 in three parcels. If he takes \$20 from the first parcel and puts it with the second, and then takes one half of the second and puts it with the third, the third will be worth twice as much as the other two; but if he takes \$20 from the third parcel and puts it with the second, and then takes one half of the second and puts it with the first, the value of the first will be $^4/_3$ of the value of the third. Required the value of each parcel.

14. If 5 casks, 3 cans, and 2 jugs of oil be drawn from a barrel containing 60 gallons, it will remain $^{11}/_{30}$ full; if 4 casks, 5 cans, and 8 jugs be drawn, it will remain $^3/_{20}$ full; and if 3 casks, 5 cans, and 10 jugs be drawn, it will remain $^1/_6$ full. What is the capacity of a cask, a can, and a jug respectively?

Generalization and Specialization.

1. Definitions.

166. Any question proposed for solution is a ***Problem.***

167. A problem whose given quantities are literal, or general, is a *general problem.*

168. A problem whose given quantities are numerical, or special, is a *special problem,* or *an example.*

169. A number of examples with different given quantities but like conditions and requirements constitute *a class.*

170. A general problem involves a whole class of examples. It is the type of a class, and its solution the solution of a class.

171. The solution of a general problem gives rise to a *formula,* which, interpreted, gives a *rule* for the solution of every example of a class.

172. The process of converting a special problem into a general one, by substituting literal for numerical quantities, is ***Generalization.***

173. The process of converting a general problem into a special one, by substituting numerical for literal quantities, is ***Specialization.***

2. Examples.

Illustrations.—1. If A can do a piece of work in 4 days and B can do it in 5 days, in what time can they do it working together? Generalize this question and solve it.

Solution: Put a for 4 and b for 5. Let x equal the time required for both to do it.

$$\text{Then} \quad \frac{1}{a} + \frac{1}{b} = \frac{1}{x} \qquad \text{(A)}$$

$$\text{whence,} \quad x = \frac{ab}{a+b} = \frac{4 \times 5}{4+5} = \frac{20}{9} = 2\frac{2}{9} \text{ days.}$$

2. A and B can do a piece of work in a days, A and C in b days, and B and C in c days; in what time can each alone do it? Solve this problem and specialize for $a=10$, $b=8$, and $c=6$.

Solution: Let x equal the time required by A, y the time required by B, and z the time required by C; then

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{a} \qquad \text{(A)}$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b} \qquad \text{(B)}$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{c} \qquad \text{(C)}$$

Adding (A), (B), and (C), and subtracting from the sum twice (A), twice (B), and twice (C) respectively, we have

$$\frac{2}{z}=\frac{ac+ab-bc}{abc} \qquad (1)$$

$$\frac{2}{y}=\frac{ab+bc-ac}{abc} \qquad (2)$$

$$\frac{2}{x}=\frac{bc+ac-ab}{abc} \qquad (3)$$

whence

$$x=\frac{2abc}{bc+ac-ab} \qquad (a)$$

$$y=\frac{2abc}{ab+bc-ac} \qquad (b)$$

$$z=\frac{2abc}{ac+ab-ac} \qquad (c)$$

Put $a=10$, $b=8$, and $c=6$,

$$x=\frac{2\times 10\times 8\times 6}{48+60-80}=34\frac{2}{7}$$

$$y=\frac{2\times 10\times 8\times 6}{80+48-60}=14\frac{2}{17}$$

$$z=\frac{2\times 10\times 8\times 6}{60+80-48}=10\frac{10}{23}$$

EXERCISE 81.

1. The sum of two numbers is 20, and their difference is 8. Find the numbers.

Suggestion.—Generalize by putting a for 20 and b for 8 in the problem, then 20 for a and 8 for b in the result.

2. A's age is three times B's, but in 12 years it will be only twice B's. Required the age of each.

Suggestion.—Put m for 3, n for 2, and t for 12 in the problem, and 3 for m, 2 for n, and 12 for t in the result.

3. A and B have $170, and $^2/_3$ of A's share equals $^3/_4$ of B's. How much has each?

Suggestion.—Put m for $^2/_3$, n for $^3/_4$, and c for 170, etc.

4. A can do a piece of work in 6 days and B can do it in 8 days. In what time can they do it working together? Generalize and solve.

5. A has $400 more than B, and B has $500 less than C, and they together have $1800. How much has each? Generalize and solve.

6. If a certain number be increased by 20, the result will be twice as great as when the number is diminished by 10. Required the number. Generalize and solve.

7. What number added to both terms of the fraction $^4/_7$ will give the fraction $^3/_4$? Generalize and solve.

Suggestion.—Put $\frac{p}{q}$ for $\frac{4}{7}$ and $\frac{r}{s}$ for $\frac{3}{4}$.

8. B has 40 acres more land than A, but if A buys 60 acres from B, A will have $1^4/_5$ times as much as B. How many acres has each? Generalize and solve.

9. If a man work 5 days and a boy 3 days, they together earn $23, but if the man and boy each work 4 days they together earn $20. Required the daily wages of each. Generalize and solve.

10. The sum of A's and B's ages is c years, and A is d years older than B. Required the age of each. Specialize by making $c = 36$ and $d = 8$ in the result.

11. Mr. Jones has a coins worth a dollar; some of them are c-cent pieces, and the rest are d-cent pieces. How many of each are there? Specialize by making $a = 14$, $c = 10$, and $d = 5$.

12. The sum of three consecutive numbers is 18. Required the numbers. Generalize and solve.

13. James is a years younger than William; but if m times James's age be subtracted from n times William's, the remainder will be d years. How old is each? Specialize by making $a = 4$, $m = 2$, $n = 3$, and $d = 22$.

14. If a cows and b oxen are worth m dollars, and c cows and d oxen, n dollars, required the value of a cow and of an ox. Specialize by putting 5 for a, 7 for b, 10 for c, 3 for d, 370 for m, and 355 for n.

15. A and B can do a piece of work in d days. After working together c days, B leaves, and A does the balance in a days. In what time could each do it alone? Specialize by putting 30 for d, 18 for c, and 20 for a.

16. If a certain rectangle had been a feet broader and b feet longer, it would have been c square feet larger. But, if it had been b feet wider and a feet longer, it would have been d square feet larger. Required its dimensions. Specialize by making $a = 2$, $b = 3$, $c = 64$, and $d = 68$.

17. There is a number consisting of two digits whose sum is a, and if b be subtracted from the number, the digits will change places. Required the number. Specialize by putting 13 for a and 27 for b.

18. The wages of a men and b women in one week amount to c dollars, and b men receive d dollars more than e women. What does each receive per week? Put 5 for a, 7 for b, 170 for c, 80 for d, and 6 for e.

19. Three children, taken two at a time, weighed a pounds, b pounds, and c pounds. What was the weight of each? Put $a = 94$, $b = 76$, and $c = 90$.

20. A purse holds a crowns and b guineas; c crowns and d guineas fill $^{17}/_{63}$ of it. How many will it hold of each? Put 19 for a, 6 for b, 4 for c, and 5 for d. Enunciate the special problem thus formed.

CHAPTER IV.

POWERS AND ROOTS.

Involution of Binomials.

1. Principle

174. We may learn by actual multiplication that:

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

By a careful inspection of the above results the following laws will appear:

The Binomial Theorem.

Prin. 72.—*1. The number of terms in each result is one greater than the exponent of the binomial.*

2. When the binomial is the sum of two quantities, all the terms of the power are positive; when the difference of two quantities, the terms are alternately positive and negative.

3. The first letter occurs in all the terms but the last, and the second letter in all the terms but the first.

4. The exponent of the leading letter in the first term is the same as the exponent of the binomial, and decreases by *unity in each succeeding term. The exponent of the second letter is* **one** *in the second term, and increases by unity in each succeeding term.*

5. The coefficient of the first term is **one**; *of the second term, the exponent of the binomial; and that of each succeeding term may be found by multiplying the coefficient of the preceding term by the exponent of the leading letter in that term, and dividing the product by the number of that term from the beginning.*

Note.—The coefficients after the middle term are the same, in an inverse order, as those before it. When the exponent of the binomial is odd, there are two middle terms with like coefficients.

2. Applications.

EXERCISE 82.

Expand:

1. $(c+d)^4$	**5.** $(m+n)^6$	**9.** $(x-y)^8$
2. $(a-d)^4$	**6.** $(m-n)^6$	**10.** $(c+z)^9$
3. $(x+y)^5$	**7.** $(c-x)^7$	**11.** $(y-x)^{10}$
4. $(x-z)^5$	**8.** $(x+z)^8$	**12.** $(z+y)^{11}$

13. The fourth power of the sum of two quantities equals what?

Suggestion.—Since $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, the fourth power of the sum of two quantities equals the fourth power of the first + 4 times the cube of the first into the second + 6 times the square of the first into the square of the second, etc.

14. The 4th power of the difference of two quantities equals what?

15. The 5th power of the sum of two quantities equals what?

16. The 5th power of the difference of two quantities equals what?

17. $(x+1)^4 = ?$	**19.** $(x+1)^5 = ?$	**21.** $(1+z)^6 = ?$
18. $(x-1)^4 = ?$	**20.** $(x-1)^5 = ?$	**22.** $(1-z)^6 = ?$

Expand $(3x^2 - 2y^3)^4$.

Solution: Let $m = 3x^2$ and $n = 2y^3$, then $(3x^2 - 2y^3)^4 = (m-n)^4 = m^4 - 4m^3n + 6m^2n^2 - 4mn^3 + n^4 = (3x^2)^4 - 4 \times (3x^2)^3 \times 2y^3 + 6 \times (3x^2)^2 \times (2y^3)^2 - 4 \times 3x^2 \times (2y^3)^3 + (2y^3)^4 = 81x^8 - 216x^6y^3 + 216x^4y^6 - 96x^2y^9 + 16y^{12}$.

EXERCISE 83.

Expand :

1. $(a+2b)^4$
2. $(3a-2b)^4$
3. $(2x+1)^5$
4. $\left(a - \frac{x}{y}\right)^5$
5. $\left(\frac{a}{b} + \frac{c}{d}\right)^5$
6. $\left(1 - \frac{1}{x^5}\right)^5$
7. $(1-2x^2)^5$
8. $(x^3+y^3)^6$
9. $(a^3-b^4)^6$
10. $\left(x^3 - \frac{2}{x^3}\right)^6$
11. $\left(\frac{2a}{3b} - \frac{3b}{2a}\right)^4$
12. $\left(\frac{2}{3} - \frac{5}{6}x^2\right)^5$
13. $(2a^2+3x^3)^6$
14. $\{-(x^2-y^2)\}^6$
15. $(-2x-3y)^5$
16. $\left(\frac{2}{3}x - \frac{3}{2}y\right)^4$
17. $\left(a^2x^2 - \frac{1}{a^2x^2}\right)^5$
18. $\left(2x^2 - \frac{3a^2}{x}\right)^5$

Involution of Polynomials.

I. Principles.

175. By actual multiplication :

1. $(a+b+c)^2 = (a+b+c)(a+b+c) = a^2+b^2+c^2+2ab+2ac+2bc.$
2. $(a-b+c)^2 = (a-b+c)(a-b+c) = a^2+b^2+c^2-2ab+2ac-2bc.$
3. $(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$
4. $(a-b+c-d)^2 = a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd.$

Therefore,

Prin. 73.*—The square of a polynomial equals the sum of the squares of its terms, and twice the product of each term into all the following terms.*

176. By actual multiplication:

1. $(a+b+c)^3 = (a+b+c)(a+b+c)(a+b+c) =$
$a^3+b^3+c^3+3a^2b+3a^2c+3b^2a+3b^2c+$
$3c^2a+3c^2b+6abc.$

2. $(a-b+c)^3 = (a-b+c)(a-b+c)(a-b+c) =$
$a^3-b^3+c^3-3a^2b+3a^2c+3b^2a+3b^2c+$
$3c^2a-3c^2b-6abc.$

Therefore,

***Prin. 74.**—The cube of any trinomial equals the sum of the cubes of its terms, and three times the square of each term into all the other terms, and six times the product of the three terms.*

2. Applications.

EXERCISE 84.

Expand:

1. $(x+y+z)^2$
2. $(x-y-z)^2$
3. $(x+y+1)^2$
4. $(a-b+2)^2$
5. $(a^2+ab+b^2)^2$
6. $(2a+3b-c)^2$
7. $\left(x^2+1+\frac{1}{x^2}\right)^2$
8. $(2x+5y-3c^2)^2$
9. $\left(\frac{x}{y}+2+\frac{y}{x}\right)^2$
10. $(x-y+z-2)^2$
11. $\left(\frac{x^2}{y}-xy+\frac{y^2}{x}\right)^2$
12. $(m^3+m^2+m+1)^2$
13. $(a+b+1)^3$
14. $(x-y-z)^3$
15. $(x+2+y)^3$
16. $(2x-3y+5)^3$
17. $(x^2+xy+y^2)^3$
18. $\left(\frac{2}{3}a+\frac{3}{4}b+\frac{5}{6}c\right)^3$
19. $(x+2y-3z)^3$
20. $\left(x^3+1+\frac{1}{x^3}\right)^3$
21. $\left(x^2+2-\frac{1}{x^2}\right)^3$
22. $(1+5x+3x^2)^3$
23. $\left(y^2-\frac{1}{2}y+\frac{2}{3}\right)^3$
24. $(x^2+x-5)^3$

Algebraic Evolution.

Definitions.

177. One of the equal factors of which a quantity is composed is a ***Root*** of the quantity.

Thus, since $a^3 = a \times a \times a$, a is the cube root of a^3.

178. The number of equal factors into which a quantity is resolved is the *degree* of the root.

179. The symbol of root is $\sqrt{\ }$, called the *radical sign.*

180. The degree of a root is expressed by an ***Index*** written in the angle of the radical sign. Thus, the fourth root is expressed $\sqrt[4]{\ }$; $\sqrt{\ } = \sqrt[2]{\ }$ is the square root.

181. The process of obtaining a root of a quantity is evolution.

Principles.

182. Since $(a^2)^3 = a^{2 \times 3}$ [P. 29] $= a^6$, $\sqrt[3]{a^6} = a^{6 \div 3} = a^2$. Therefore,

Prin. 75.*—Dividing the exponent of any factor by the index of a root takes that root of the factor.*

SIGHT EXERCISE.

Name at sight:

1. $\sqrt{a^6}$	4. $\sqrt[7]{a^{21}}$	7. $\sqrt[12]{z^{36}}$	10. $\sqrt[15]{c^{45}}$
2. $\sqrt[3]{a^9}$	5. $\sqrt[8]{x^{24}}$	8. $\sqrt[6]{a^{36}}$	11. $\sqrt{a^{22}}$
3. $\sqrt[5]{a^{20}}$	6. $\sqrt[10]{y^{30}}$	9. $\sqrt[9]{m^{36}}$	12. $\sqrt[20]{a^{100}}$

183. Since $(a^2 b^3 c^4)^4 = (a^2)^4 \times (b^3)^4 \times (c^4)^4$ [P. 30] $=$
$$a^8 b^{12} c^{16},\ \sqrt[4]{a^8 b^{12} c^{16}} = \sqrt[4]{a^8} \times \sqrt[4]{b^{12}} \times \sqrt[4]{c^{16}} = a^2 b^3 c^4.$$
Therefore,

Prin. 76.*—Taking any root of every factor of a quantity takes that root of the quantity.*

SIGHT EXERCISE.

Name at sight:

1. $\sqrt{a^2 b^4}$	5. $\sqrt[4]{x^8 y^{12} z^4}$	9. $\sqrt{4 \times 9 \times 16}$
2. $\sqrt{4 a^4 b^6}$	6. $\sqrt[6]{a^{12} y^{24} z^{18}}$	10. $\sqrt[3]{8 \times 27 \times 64}$
3. $\sqrt[3]{a^9 c^5}$	7. $\sqrt[7]{x^{14} y^{21} z^{28}}$	11. $\sqrt[4]{16 a^4 b^8}$
4. $\sqrt[5]{a^{10} b^{20}}$	8. $\sqrt[8]{a^{16} b^{24} z^8}$	12. $\sqrt[5]{a^{10} (a+b)^{25}}$

184. Any even power of a positive or a negative quantity is positive [P. 27]. Therefore,

***Prin. 77.**—Any even root of a positive quantity may be either positive or negative.*

Illustration: $\sqrt{+64} = \pm 8$, since $(\pm 8)^2 = +64$.

185. Any odd power of a quantity has the same sign as the quantity [P. 28]. Therefore,

***Prin. 78.**—Any odd root of a quantity has the same sign as the quantity.*

Illustration: $\sqrt[3]{+27} = +3$, since $(+3)^3 = +27$.
$\sqrt[3]{-27} = -3$, since $(-3)^3 = -27$.

186. Since no even power is negative [P. 27],

***Prin. 79.**—An even root of a negative quantity is impossible.*

Illustration: $\sqrt{-16}$ is neither ± 4, since $(\pm 4)^2 = 16$.

Note.—The indicated even root of a negative quantity, as $\sqrt{-16}$, is called an *imaginary* quantity.

SIGHT EXERCISE.

Name at sight, giving the proper signs:

1. $\sqrt{16}$	4. $\sqrt[4]{81}$	7. $\sqrt{a^4}$	10. $\sqrt[6]{a^6 b^{12}}$
2. $\sqrt[3]{8}$	5. $\sqrt[3]{-x^6}$	8. $\sqrt[3]{-x^3 y^6}$	11. $\sqrt{-4}$
3. $\sqrt[3]{-27}$	6. $\sqrt[5]{-32}$	9. $\sqrt[4]{x^{12}}$	12. $\sqrt[4]{-16}$

187. Since raising both terms of a fraction to any power raises the fraction to that power [P. 68],

***Prin. 80.**—Extracting any root of both terms of a fraction extracts that root of the fraction.*

Illustration.—

$$\sqrt{\frac{4}{9}} = \pm\frac{\sqrt{4}}{\sqrt{9}} = \pm\frac{2}{3}, \text{ since } \left(\pm\frac{2}{3}\right)^2 = +\frac{(2)^2}{(3)^2} = +\frac{4}{9}.$$

SIGHT EXERCISE.

Name at sight:

1. $\sqrt{\frac{25}{36}}$	**5.** $\sqrt[4]{\frac{a^{12}}{b^{16}}}$	**9.** $\sqrt[5]{\frac{a^{10}b^{15}}{32}}$
2. $\sqrt[3]{\frac{8}{27}}$	**6.** $\sqrt[3]{\frac{x^9}{y^{12}}}$	**10.** $\sqrt{\frac{1}{4}\times\frac{1}{9}}$
3. $\sqrt{\frac{a^4}{b^6}}$	**7.** $\sqrt{\frac{a^4b^2}{x^2y^4}}$	**11.** $\sqrt[3]{\frac{1}{27}\times\frac{1}{64}}$
4. $\sqrt[3]{-\frac{a^6}{b^9}}$	**8.** $\sqrt[3]{-\frac{m^9x^{12}}{x^6y^3}}$	**12.** $\sqrt[3]{-\frac{1}{8}\times\frac{1}{27}}$

Problem 1. To find a root of a numerical quantity by factoring.

Illustration.—

Find the cube root of 1728.

Solution: Since the cube root of a number is one of the three equal factors of the number, we resolve 1728 into its prime factors, and take one of every three equal ones, and find their product.

Note.—To find the square root, take one of every *two* equal factors; to find the fourth root, one of every four equal factors, etc.

Form.

2	1728
2	864
*2	432
2	216
2	108
*2	54
3	27
3	9
	*3

$$\sqrt[3]{1728} = 2 \times 2 \times 3 = 12$$

EXERCISE 85.

Find the value of:

1. $\sqrt{324}$
2. $\sqrt{1296}$
3. $\sqrt{2304}$
4. $\sqrt[3]{512}$
5. $\sqrt[3]{3375}$
6. $\sqrt[3]{5832}$
7. $\sqrt[4]{4096}$
8. $\sqrt[4]{20736}$
9. $\sqrt[5]{248832}$

Problem 2. To find a root of a monomial.

Illustration.—

Find the 5th root of $-32\,a^{10}\,b^5\,c^{20}$.

Form.

$$\sqrt[5]{-32\,a^{10}\,b^5\,c^{20}} = -2\,a^2\,b\,c^4$$

Solution: Since taking a root of every factor of a quantity takes the root of the quantity [P. 76], $\sqrt[5]{-32\,a^{10}\,b^5\,c^{20}} = \sqrt[5]{-32} \times \sqrt[5]{a^{10}} \times \sqrt[5]{b^5} \times \sqrt[5]{c^{20}}$. $\sqrt[5]{-32} = -2$ [P. 78]; $\sqrt[5]{a^{10}} = a^2$, $\sqrt[5]{b^5} = b$, and $\sqrt[5]{c^{20}} = c^4$ [P. 75]; hence the result is $-2\,a^2\,b\,c^4$. Therefore,

Rule.*—Take the required root of the numerical coefficient and divide the exponent of each literal factor by the index of the root.*

EXERCISE 86.

Find the value of:

1. $\sqrt{a^2\,b^4\,c^6}$
2. $\sqrt{4\,a^8\,b^6\,c^{10}}$
3. $\sqrt[3]{x^3\,y^9\,z^{12}}$
4. $\sqrt[3]{8\,m^3\,n^{15}}$
5. $\sqrt[3]{-27\,x^6\,y^{12}}$
6. $\sqrt[4]{m^4\,n^8\,p^{16}}$
7. $\sqrt[5]{-32\,a^{10}\,b^{15}\,c^{20}}$
8. $\sqrt{144\,a^8\,x^6\,y^{10}}$
9. $\sqrt[3]{-729\,(a+x)^3}$
10. $\sqrt[4]{256\,(a-x)^8}$
11. $\sqrt[5]{-(a+b)^5\,c^{10}}$
12. $\sqrt[4]{10000\,x^8\,(x+y)^{16}}$
13. $\sqrt[5]{-243\,(m+n)^{10}}$
14. $\sqrt[6]{64\,(x^2-y^2)^{12}}$
15. $\sqrt{x^4\,y^8\,(x-y)^4}$, when $x=6$ and $y=3$
16. $\sqrt[4]{16\,a^8\,b^{12}\,(a^2-b^2)^4}$, when $a=5$ and $b=3$

17. $\sqrt{(a^2-b^2)^4} \div \sqrt{(a+b)^4}$ when $a=3$ and $b=2$

18. $\sqrt[3]{(a+x)^6}+\sqrt[4]{(a-x)^8}+\sqrt{(a^2-x^2)^4}$, when $a=4$ and $x=2$

19. $\sqrt{\sqrt[3]{a^6 b^{12}}}+\sqrt[3]{\sqrt{a^{12} b^{18}}}-\sqrt[5]{a^5 b^{10}}$, when $a=-3$ and $b=5$

Find the value of :

20. $\sqrt{\frac{1}{16}}$, $\sqrt{\frac{4}{9}}$, $\sqrt{\frac{9}{16}}$, $\sqrt{\frac{4}{25}}$, $\sqrt{\frac{64}{81}}$

21. $\sqrt{\frac{a^2}{b^6}}$, $\sqrt{\frac{a^4 b^6}{c^8}}$, $\sqrt{\frac{16 x^2 y^6}{25 a^4 b^2}}$, $\sqrt{\frac{x^4 y^6 z^{12}}{9 m^4 n^6}}$

22. $\sqrt[3]{\frac{8}{27}}$, $\sqrt[3]{-\frac{1}{8}\times\frac{x^3}{y^3}}$, $\sqrt[3]{-\frac{8 x^6 y^9}{27 a^3 b^{12}}}$, $\sqrt[3]{\frac{x^6 (a+x)^3}{c^9 d^{18}}}$

23. $\sqrt{\frac{(a+b)^2}{(a-b)^4}}$, $\sqrt[3]{-\frac{(x-y)^6}{(x+y)^9}}$, $\sqrt[4]{\frac{a^4 b^8 c^{12}}{(a-b)^8}}$, $\sqrt[5]{-\frac{x^5 y^{10}}{32 a^{15}}}$

24. $\sqrt{\frac{12}{27}}$, $\sqrt[3]{\frac{2}{3}\times\frac{4}{9}}$, $\sqrt[3]{\frac{54}{128}}$, $\sqrt[6]{\frac{a^6 x^{12}}{(a-x)^{18}}}$, $\sqrt[6]{\frac{3}{192}}$

Problem 3. To extract the square root of a polynomial.

1. Method.

188. *Since the square of a polynomial is the sum of the squares of its terms and twice the product of each term into all the following terms* [P. 73], the square root of a polynomial that is a perfect square may be obtained by inspection, if no terms of the power have disappeared by collection.

Illustration.—

$\sqrt{(a^2+2ab+b^2+2ac+2bc+c^2)}=$

$\sqrt{(a^2+b^2+c^2+2ab+2ac+2bc)}=a+b+c$, since

$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$

EXERCISE 87.

Extract the square root of :

1. $a^2 + b^2 + 2ab$
2. $x^2 + y^2 - 2xy$
3. $x^2 + 16 + 8x$
4. $x^2 - 6x + 9$
5. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$
6. $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$
7. $a^2 + 4b^2 + 9c^2 + 4ab + 6ac + 12bc$
8. $4x^2 + y^2 + 9 - 4xy + 12x - 6y$
9. $x^4 + y^2 + 2x^2y + 2x^2 + 2y + 1$
10. $9x^2 - 24xy + 16y^2$
11. $\frac{1}{4}x^4 - \frac{1}{3}x^2y^3 + \frac{1}{9}y^6$
12. $4x^4 + \frac{1}{4}y^2 + 9 + 2x^2y + 12x^2 + 3y$

189. When the law of development does not appear by inspection, the following method must be resorted to.

Illustration.—Extract the square root of

$$x^4 + y^4 + 3x^2y^2 - 2x^3y - 2xy^3.$$

Form.

$$(m+n)^2 = m^2 + (2m+n)n$$

$$\begin{array}{r|l|l}
 & x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4 & \overbrace{x^2 - xy}^{m+n} + y^2 \quad (m+n) \\
 & x^4 & \\
\hline
2x^2 - xy & -2x^3y + 3x^2y^2 & \\
 & -2x^3y + x^2y^2 & \\
\hline
2x^2 - 2xy + y^2 & 2x^2y^2 - 2xy^3 + y^4 & \\
 & 2x^2y^2 - 2xy^3 + y^4 & \\
\hline
\end{array}$$

Solution: Having arranged the terms according to the descending powers of x assumed as the leading letter, we will proceed to take out of the polynomial the square of the first two terms of the root. For this purpose we let $m + n$ represent the first two terms of the root. Now $(m+n)^2 = m^2 + 2mn + n^2$, or $m^2 + (2m+n)n$. m^2 obviously equals x^4, or $m = x^2$. Subtracting x^4 from the polynomial and bringing down the next two terms, we have $-2x^3y + 3x^2y^2$. This remainder consists mainly of $(2m + n)n$; hence if we use $2m$, or $2x^2$, as a *trial divisor*, we will obtain the value of n, which is $-2x^3y \div 2x^2$, or $-xy$; adding this value of n to that of $2m$, we have $2x^2 - xy$, the *complete divisor*; multiplying the value of $2m + n$, or $x^2 - xy$, by the value of n, or $-xy$, we have $-2x^3y + x^2y^2$. Subtracting this product from $-2x^3y + 3x^2y^2$ and bringing down the remaining terms, we have $2x^2y^2 - 2xy^3 + y^4$.

We now let m represent $x^2 - xy$ and n the next term of the root, and proceed as before to take out of the polynomial the square of $m + n$, or $m^2 + (2m + n)n$. m^2 or $(x^2 - xy)^2$ has already been removed, hence the remainder $2x^2y - 2xy^3 + y^4$ is composed of $(2m + n)n$. Using $2m$, or $2x^2 - 2xy$, as a *trial divisor*, we obtain y^2 for n; adding this to the value of $2m$, or $2x^2 - xy$, we have $2x^2 - 2xy + y^2$ for the *complete divisor*. Multiplying the complete divisor by y^2 and subtracting, nothing remains. Therefore the given polynomial is the square of $x^2 - xy + y^2$, or $x^2 - xy + y^2$ is the square root required.

From an inspection of the above solution the following rule will appear:

1. Arrange the terms of the polynomial according to the ascending or descending powers of some letter assumed as a leading letter.

2. Take the square root of the first term of the polynomial for the first term of the root. Subtract the square of this term of the root from the polynomial.

3. Double the root found for a trial divisor. Divide the first term of the remainder by the trial divisor for the next term of the root.

4. Add the last term of the root found to the trial divisor for the complete divisor. Multiply the complete divisor by the last term of the root found, and subtract the product from the remainder, and bring down such terms as are needed.

5. If the root has more than two terms, double the root already found for a new trial divisor, and proceed as before to obtain the next term of the root and the complete divisor. Continue this process until all the terms of the polynomial have been used.

Note.—In the formula $m^2 + (2m + n)n$, m represents the root as far as found, $2m$ the trial divisor, n the next term of the root and also the correction, and $2m + n$ the complete divisor.

EXERCISE 88.

Extract the square root of:

1. $x^4 + 2x^3 + 3x^2 + 2x + 1$
2. $x^4 - 4x^3 + 6x^2 - 4x + 1$
3. $x^4 + 4x^3 + 10x^2 + 12x + 9$
4. $x^4 - 6x^3y + 13x^2y^2 - 12xy^3 + 4y^4$
5. $x^6 - 4x^5 + 10x^4 - 12x^3 + 9x^2$
6. $4x^4 - 20x^3y + 37x^2y^2 - 30xy^3 + 9y^4$
7. $x^4 + x^3 + \frac{3}{4}x^2 + \frac{1}{4}x + \frac{1}{16}$
8. $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$
9. $x^6 + 2x^5 - x^4 + 3x^2 - 2x + 1$

Problem 4. To extract the square root of numerical quantities.

1. Definition and Principle.

190. Every numerical quantity of two or more figures may be considered a polynomial. Thus,

123456 = 12 ten-thousands + 34 hundreds + 56 units.

191. The square of a *unit* is a *unit*, the square of a *ten* is a *hundred*, the square of a *hundred* is a *ten-thousand*, the square of a *thousand* is a *million*, etc.; hence, the square denominations in order are the *unit*, the *hundred*, the *ten-thousand*, the *million*, etc. Therefore,

Prin. 81.—*If a number be pointed off into terms of two figures each, beginning at the units, the unit of each term will be a perfect square.*

2. Examples.

Illustration.—Extract the square root of 105625.

Solution: We point the number off into terms of two figures each to keep the unit of each term a perfect square [P. 81]. 105625 equals 10 ten-thousands + 56 hundreds + 25 units.

The square root of 10 ten-thousands is 3 hundreds, the first term of the root. Squaring 3 hundreds, we have 9 ten-thousands; subtracting 9 ten-thousands from 10 ten-thousands and bringing down the next term, we have 156 hundreds. Doubling the root already found for a trial divisor, we have 6 hundreds; dividing 15 thousands by 6 hundreds, we have 2 tens for the next term of the root; adding 2 tens to the trial divisor, we have 62 tens for the complete divisor; multiplying 62 tens by 2 tens, we have 124 hundreds; subtracting 124 hundreds from 156 hundreds and bringing down the next term, we have 3225 units. Doubling 32 tens, we have 64 tens for a new trial divisor; dividing 322 tens by 64 tens, we have 5 units for the next term of the root; adding 5 units to 64 tens, we have 645 units for the complete divisor; multiplying 645 units by 5, we have 3225 units; subtracting this product from 3225, nothing remains. Therefore the square root of 105625 is 325.

Form.

```
      10'56'25 | 325
       9
    -------
62  | 156
    | 124
     ------
645  | 3225
     | 3225
```

Note.—The square root may also be obtained by means of the formula $(m + n)^2 = m^2 + (2m + n)n$, as in example (Art. 189).

EXERCISE 89.

Extract the square root of:

1. 289

2. 676

3. 1225

4. 1849

5. 2704

6. 4761

7. 5041

8. 7056

9. 16129

10. 60025

11. 104976

12. 166464

13. What is the value of:

$(\cdot 1)^2$? $(\cdot 01)^2$? $(\cdot 001)^2$? $(\cdot 0001)^2$?

$\sqrt{\cdot 01}$? $\sqrt{\cdot 0001}$? $\sqrt{\cdot 000001}$? $\sqrt{\cdot 00000001}$?

192. The square decimal units below *one* are the *hundredth,* the *ten-thousandth,* the *millionth,* the *hundred-millionth,* etc. Therefore,

193. If a decimal contain, or be made to contain by annexing ciphers, an even number of figures, its unit will be a perfect square.

Illustration.—

$$\cdot 24 = 24 \times \cdot 01 \qquad \cdot 536420 = 536420 \times \cdot 000001$$

$$\cdot 3645 = 3645 \times \cdot 0001 \qquad 5\cdot 000000 = 5000000 \times \cdot 000001$$

14. Extract the square root of 5 to thousandths.

Solution : Since the square root is to be expressed in thousandths, the number must be reduced to millionths. 5 = 5,000,000 millionths. The square root of 5,000,000 is 2236 +, and the square root of a millionth is a thousandth. Therefore the square root of 5,000,000 millionths is 2236 + thousandths, or 2·236 + .

Form.

```
      5·'00'00'00 | 2·236+
      4
  42 | 100
     |  84
 443 |  1600
     |  1329
4466 |  27100
     |  26796
```

Extract the square root of :

15. ·0049 **19.** ·00000016 **23.** 108·5764

16. ·0625 **20.** ·104976 **24.** 1024·6401

17. ·000144 **21.** 33·8724 **25.** 99·980001

18. 882·09 **22.** 11·4244 **26.** 8010·25

27. Find the square root of 10, 11, 12, and 13 to within one ten-thousandth.

28. Find the value of $\sqrt{2}$, $\sqrt{3\cdot 3}$, $\sqrt{\frac{7}{8}}$, $\sqrt{\frac{5}{16}}$, to 4 decimal places.

29. Find the value of $\sqrt{40}$, $\sqrt{41}$, $\sqrt{42}$, and $\sqrt{43}$ to within one thousandth.

Problem 5. To extract the cube root of a polynomial.

Illustration.—Extract the cube root of

$$x^6 + 3x^5y + 6x^4y^2 + 7x^3y^3 + 6x^2y^4 + 3xy^5 + y^6.$$

Form.

$$(m+n)^3 = m^3 + (3m^2 + 3mn + n^2)\,n$$

Root: $m + n$; $\overbrace{m + n}$; $x^2 + xy + y^2$

		$x^6+3x^5y+6x^4y^2+7x^3y^3+6x^2y^4+3xy^5+y^6$
		x^6
T. D.	$=3x^4$	$3x^5y+6x^4y^2+7x^3y^3$
1st Cor. =	$3x^3y$	
2d Cor. =	x^2y^2	
C. D.	$=3x^4+3x^3y+x^2y^2$	$3x^5y+3x^4y^2+x^3y^3$
T. D.	$=3x^4+6x^3y+3x^2y^2$	$3x^4y^2+6x^3y^3+6x^2y^4+3xy^5+y^6$
1st Cor. =	$3x^2y^2+3xy^3$	
2d Cor. =	y^4	
C. D.	$=3x^4+6x^3y+6x^2y^2+3xy^3+y^4$	$3x^4y^2+6x^3y^3+6x^2y^4+3xy^5+y^6$

Solution: Having arranged the terms according to the descending powers of x, assumed as the leading letter, we proceed to take out of the polynomial the cube of the first two terms of the root. For this purpose we let $m + n$ represent these terms. Now, $(m + n)^3 = m^3 + 3m^2n + 3mn^2 + n^3$, or $m^3 + (3m^2 + 3mn + n^2)\,n$. m^3 obviously equals x^6, or m equals x^2. Subtracting x^6 from the polynomial and bringing down three terms, we have for the first remainder, $3x^5y + 6x^4y^2 + 7x^3y^3$. This remainder consists mainly of $(3m^2 + 3mn + n^2)\,n$; hence, if we use $3m^2$, or $3x^4$, for a trial divisor, we will obtain the value of n, which is $3x^5y \div 3x^4$, or xy. Substituting the values of m and n in $3mn$ and n^2, we have $3x^3y$ for the *first* and x^2y^2 for the *second correction;* adding the two corrections to the trial divisor, we have $3x^4 + 3x^3y + x^2y^2$ for $3m^2 + 3mn + n^2$, the *complete divisor;* multiplying the complete divisor by xy, the value of n, we have $3x^5y + 3x^4y^2 + x^3y^3$; subtracting this from the first remainder, and bringing down the remaining terms, we have $3x^4y^2 + 6x^3y^3 + 6x^2y^4 + 3xy^5 + y^6$.

We now let m stand for $x^2 + xy$, the root already found, and n for the next term of the root, and proceed as before to take from the polynomial the cube of $m + n$, or $m^3 + (3m^2 + 3mn + n^2)\,n$. m^3, or $(x^2 + xy)^3$, has already been subtracted; hence, the remainder consists of $(3m^2 + 3mn + n^2)\,n$. Using as before $3m^2$, or $3x^4 + 6x^3y + 3x^2y^2$, as a *trial divisor*, we obtain y^2 for n; substituting the values of m and n in $3mn$ and n^2, we have $3x^2y^2 + 3xy^3$ for the *first* and y^4 for the *second correction*, and $3x^4 + 6x^3y + 6x^2y^2 + 3xy^3 + y^4$ for the complete divisor. Multiplying the complete divisor by y^2, and subtracting the product from the last remainder, nothing remains. Therefore, $x^2 + xy + y^2$ is the cube root required.

Note.—In the formula $m^3 + (3m^2 + 3mn + n^2)n$, m stands for the root already found, $3m^2$ for the trial divisor, $3mn$ for the first correction, n^2 for the second correction, $3m^2 + 3mn + n^2$ for the complete divisor, and n for the next term of the root.

From an inspection of the above solution the following rule will appear:

1. Arrange the terms of the polynomial according to the ascending or descending powers of some letter assumed as a leading letter.

2. Take the cube root of the first term of the polynomial for the first term of the root. Subtract the cube of this term of the root from the polynomial.

3. Take three times the square of the root already found for a trial divisor. Divide the first term of the remainder by the trial divisor for the next term of the root.

4. Add to the trial divisor three times the last term of the root found multiplied by the preceding part of the root, and the square of the last term found, for a complete divisor. Multiply the complete divisor by the last term of the root found, and subtract the result from the remainder, bringing down only such terms as are needed.

5. If the root has more than two terms, take three times the square of the root already found for a new trial divisor, and proceed as before to obtain the next term of the root, the new corrections, and the new complete divisor. Continue the process until all the terms have been used.

EXERCISE 90.

Extract the cube root of:

1. $x^3 + 3x^2 + 3x + 1$
2. $a^3 - 3a^2b + 3ab^2 - b^3$
3. $x^3 + 12x^2 + 48x + 64$
4. $8x^6 + 36x^4 + 54x^2 + 27$
5. $x^6 + 3x^5 - 5x^3 + 3x - 1$
6. $y^6 - 3y^5 + 5y^3 - 3y - 1$
7. $a^3x^3 - 3a^2bx^2 + 3ab^2x - b^3$
8. $8a^3x^3 - 36a^2bx^2y + 54ab^2xy^2 - 27b^3y^3$

9. $x^6 - 6x^5 + 21x^4 - 44x^3 + 63x^2 - 54x + 27$

10. $x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$ 11. $a^3x^3 - 6ax + \frac{12}{ax} - \frac{8}{a^3x^3}$

12. $x^6 + 3x^4 + 6x^2 + 7 + \frac{6}{x^2} + \frac{3}{x^4} + \frac{1}{x^6}$

13. $a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3b^2c +$
$3ac^2 + 3bc^2 + 6abc$

Problem 6. To extract the cube root of numerical quantities.

1. Principle.

194. The cube of a *unit* is a *unit,* the cube of a *ten* is a *thousand,* the cube of a *hundred* is a *million,* the cube of a *thousand* is a *trillion,* etc.; hence, the cubic denominations in order are the *unit,* the *thousand,* the *million,* the *trillion,* etc. Therefore,

***Prin. 82.**—If a number be pointed off into terms of three figures each, beginning at the units, the unit of each term will be a perfect cube.*

2. Examples.

Illustration.—Extract the cube root of 16387064.

Form.

$$(m+n)^3 = m^3 + (3m^2 + 3mn + n^2)n$$

$$\begin{array}{llrr|l}
 & & & & \overset{m+n}{\overbrace{m+n}} \\
 & & & 16'387'064 & 2\ 5\ \ 4 \\
 & & m^3 = & 8 & \\
3m^2 & = 3\times(2..)^2 & = 12\ldots. & 8387 & \\
3mn & = 3\times 2..\times 5. = & 30\ldots & & \\
n^2 & = (5.)^2 = & 25.. & & \\
3m^2+3mn+n^2 & & = 1525.. & 7625 & \\
3m^2 & = 3\times(25.)^2 = & 1875.. & 762064 & \\
3mn & = 3\times 25.\times 4 = & 300. & & \\
n^2 & = 4^2 = & 16 & & \\
3m^2+3mn+n^2 & = & 190516 & 762064 &
\end{array}$$

Solution : We point off the number into terms of three figures each to make the unit of each term a perfect cube [P. 82]. We find thus that 16,387,064 = 16 million, + 387 thousand, + 64 units. The cube root of 16 million is 2 hundred +. Cubing 2 hundred, we have 8 million; subtracting 8 million from 16 million and bringing down the next term, we have 8387 thousand. Taking 3 times the square of the root already found ($3m^2$) for a trial divisor, we have 12 ten-thousands; dividing 83 hundred-thousands by 12 ten-thousands, we have 5 tens (n), the next term of the root; taking 3 times the root previously found ($3m$) and multiplying it by the last term found (n), we have 30 thousand for the first correction, and squaring the last term of the root (n^2), we have 25 hundred for the second correction; adding the trial divisor and the two corrections, we have 1525 hundred for the complete divisor ($3m^2 + 3mn + n^2$); multiplying the complete divisor by the last term of the root (n), we have 7625 hundred; subtracting 7625 hundred from 8387 hundred, and bringing down the next term, we have 762064 units.

Taking 3 times the square of 25 tens (the new value of m), we have 1875 hundred (the new trial divisor); dividing 7620 hundred by 1875 hundred, we have 4, the next term of the root (n); finding as before the values of $3mn$ and n^2, we have for the two corrections 300 tens and 16 units, and for the complete divisor ($3m^2 + 3mn + n^2$) 190516; multiplying by 4, or n, we have 762064, which subtracted from 762064 leaves nothing. Therefore the cube root of 16,387,064 is 254.

Abbreviated Rule.

1. Point off the number into terms of three figures each.

2. The cube root of the first term gives the first figure of the root.

3. Three times the square of the root already found always gives the trial divisor.

4. The remainder, exclusive of the two right-hand figures, divided by the trial divisor, gives the next figure of the root.

5. Three times the root previously found multiplied by the last figure found gives the first correction, and the square of the last figure found, the second correction.

6. The right-hand figure of the first correction is placed one order to the right of the trial divisor, and that of the second correction one order to the right of the first correction.

7. The sum of the trial divisor and the two corrections gives the complete divisor.

EXERCISE 91.

Extract the cube root of :

1. 2744	**5.** 373248	**9.** 1815848
2. 19683	**6.** 592704	**10.** 10941048
3. 42875	**7.** 681472	**11.** 28372625
4. 300763	**8.** 941192	**12.** 74088000

13. What is the value of :

$(\cdot 1)^3$?	$(\cdot 01)^3$?	$(\cdot 001)^3$?
$\sqrt[3]{\cdot 001}$?	$\sqrt[3]{\cdot 000001}$?	$\sqrt[3]{\cdot 000000001}$?

195. The cubic decimal units below *one* are the *thousandth*, the *millionth*, the *billionth*, the *trillionth*, etc. Therefore,

196. If a decimal contain, or be made to contain by annexing ciphers, a whole number of times three figures, its unit will be a perfect cube.

Illustration : $\cdot 325 = 325 \times \cdot 001$;

$4 \cdot 25 = 4 \cdot 250000 = 4250000 \times \cdot 000001$

14. Extract the cube root of 3·25 to hundredths.

Form.

	3·'250'000 | 1·47
	1
$3 \times 1^2 = 3$	2250
$3 \times 1 \times 4 = 12$	
$4^2 = 16$	
436	1744
$3 \times 14^2 = 588$	506000
$3 \times 14 \times 7 = 294$	
$7^2 = 49$	
61789	432523

Solution : Since the cube root is to be expressed in hundredths, the number must be reduced to millionths. $3 \cdot 25 = 3{,}250{,}000$ millionths. The cube root of 3,250,000 is 147+, and the cube root of 1 millionth is 1 hundredth. Therefore the cube root of 3,250,000 millionths is 147+ hundredths, or 1·47+.

Extract the cube root of:

15. ·008 **16.** ·001728 **17.** ·000027 **18.** ·015625 **19.** ·029791 **20.** ·125000 **21.** 39·304 **22.** 13·824 **23.** 81·37 1/27

24. Find the value to thousandths of $\sqrt[3]{3}$, $\sqrt[3]{\frac{5}{8}}$, $\sqrt[3]{\cdot 27}$

Problem 7. To extract higher roots of quantities.

197. Since $(a^2)^2 = a^4$, $\sqrt[4]{a^4} = \sqrt{\sqrt{a^4}}$; also, since $(a^2)^3 = a^6$, $\sqrt[6]{a^6} = \sqrt[3]{\sqrt{a^6}}$; again, since $(a^3)^3 = a^9$, $\sqrt[9]{a^9} = \sqrt[3]{\sqrt[3]{a^9}}$.

Therefore,

To extract the fourth root of a quantity, extract the square root of the square root; to extract the sixth root, extract the cube root of the square root; and to extract the ninth root, extract the cube root of the cube root.

EXERCISE 92.

1. Extract the fourth root of

$$x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8$$

2. Extract the sixth root of

$$x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

3. Find the value of $\sqrt[4]{256}$, $\sqrt[6]{729}$, $\sqrt[9]{1953125}$, $\sqrt[4]{\cdot 0016}$

Factoring with the aid of Evolution.

198. If a polynomial is the difference of the squares, the difference of the cubes, or the sum of the cubes of two quantities, it may be factored by P. 39, 44, 43.

Illustrations.—1. Factor $x^4 + 2x^3 + 5x^2 + 4x + 3$.

Solution: By extracting the square root of the given polynomial we find that it lacks 1 of being the square of $x^2 + x + 2$,

$\therefore\ x^4 + 2x^3 + 5x^2 + 4x + 3 = (x^2 + x + 2)^2 - 1 =$
$(x^2 + x + 2 + 1)(x^2 + x + 2 - 1)$ [P. 39] $= (x^2 + x + 3)(x^2 + x + 1)$.

2. Factor $x^3 + 6x^2 + 12x + 7$.

Solution: By extracting the cube root of $x^3 + 6x^2 + 12x + 7$, we find that it lacks 1 of being the cube of $x + 2$,

$\therefore\ x^3 + 6x^2 + 12x + 7 = (x + 2)^3 - 1;$

$\therefore\ x^3 + 6x^2 + 12x + 7$ is divisible by $x + 2 - 1$ or $x + 1$ [P. 44];

$\therefore\ x^3 + 6x^2 + 12x + 7 = (x + 1)(x^2 + 5x + 7).$

3. Factor $x^6 + 3ax^4 + 3a^2x^2 + 9a^3$.

Solution: By extracting the cube root, or by inspection, we see that $x^6 + 3ax^4 + 3a^2x^2 + 9a^3$ is $8a^3$ more than the cube of $x^2 + a$,

$\therefore\ x^6 + 3ax^4 + 3a^2x^2 + 9a^3 = (x^2 + a)^3 + 8a^3;$

$\therefore\ x^6 + 3ax^4 + 3a^2x^2 + 9a^3$ is divisible by $x^2 + a + 2a$, or $x^2 + 3a$ [P. 43];

$\therefore\ x^6 + 3ax^4 + 3a^2x^2 + 9a^3 = (x^2 + 3a)(x^4 + 3a^2).$

EXERCISE 93.

Factor:

1. $x^4 + a^2x^2 + a^4$
2. $x^2 + 6x + 5$
3. $4x^2 + 12xy + 5y^2$
4. $25x^4 - 9x^2y^2 + 16y^4$
5. $4p^4 - 37p^2q^2 + 9q^4$
6. $64a^4 + 128a^2b^2 + 81b^4$
7. $x^4 + 2x^3 + 3x^2 + 2x - 3$
8. $4x^4 + 20x^3y + 29x^2y^2 + 10xy^3 - 3y^4$
9. $8x^3 + 60x^2 + 150x + 61$
10. $27x^3 - 54x^2y + 36xy^2 - 7y^3$
11. $x^6 + 6x^4y^2 + 12x^2y^4 - 19y^6$
12. $8a^6 - 12a^4b^2 + 6a^2b^4 - 9b^6$
13. $x^3 + 30x^2 + 300x + 875$
14. $8a^9 - 12a^6 + 6a^3 + 7$
15. $a^9 + 3a^6b^3 + 3a^3b^6 + 9b^9$
16. $a^{12} - 3a^8b^4 + 3a^4b^8 - 28b^{12}$
17. $4a^2 + 12ab + 8b^2 + 16ac + 22bc + 15c^2$
18. $27a^3 - 135a^2b + 225ab^2 - 61b^3$
19. $a^6x^3 + 3a^4b^2x^2y + 3a^2b^4xy^2 - 7b^6y^3$

CHAPTER V.

QUADRATIC EQUATIONS.

Quadratic Equations of One Unknown Quantity. Pure Quadratics.

1. Definitions and Principles.

199. A quadratic equation containing only the second power of an unknown quantity is a *pure* or *incomplete* quadratic equation; as, $3x^2 = 8$, or $\frac{x^2}{3} + \frac{7x^2}{5} = \frac{4}{3}$.

200. Every equation containing fractional terms may be cleared of fractions [P. 71]. All the unknown terms in the second member may be transposed to the first, and all the known terms in the first member to the second [P. 70]. All the unknown terms in the first member may be united into one and the coefficient represented by a, and all the known terms in the second member may be represented by b. If a is negative, the equation may be divided by -1. Therefore,

Prin. 83.*—Every pure quadratic equation of one unknown quantity may be reduced to the form of* $ax^2 = b$, *in which* a *and* b *are integral and* a *positive.*

201. Any value of the unknown quantity that will satisfy an equation—that is, will make the two members equal—is a *root* of the equation.

202. Take the equation $a x^2 = b$.

Divide by a, $x^2 = \frac{b}{a}$.

Take the square root of both members [P. 69, 6],

$x = \pm \sqrt{\frac{b}{a}}$. Therefore,

Prin. 84.—Every pure quadratic equation of one unknown quantity has two roots, numerically equal, but opposed in sign.

2. Solution of Pure Quadratics.

Illustrations.—1. Solve $\frac{3x^2}{2} + 5 = 2x^2 - 3$.

Solution: Given $\frac{3x^2}{2} + 5 = 2x^2 - 3$ (A)

Clearing of fractions, $3x^2 + 10 = 4x^2 - 6$ (1)

Transposing, $-x^2 = -16$ (2)

Dividing by -1, $x^2 = 16$ (3)

Taking $\sqrt{}$, $x = \pm 4$.

203. Sometimes equations have the pure quadratic form, and may be solved as such when they are not really such. Their roots may not be numerically equal.

2. Solve $\frac{(x+a)^2}{4} + 3a^2 = \frac{3(x+a)^2}{2} - 2a^2$. (A)

Suggestion.—

Clear of fractions, $(x+a)^2 + 12a^2 = 6(x+a)^2 - 8a^2$ (1)

Transpose terms, $-5(x+a)^2 = -20a^2$ (2)

Divide by -5, $(x+a)^2 = 4a^2$ (3)

Take $\sqrt{}$, $x + a = \pm 2a$ (4)

Transpose, $x = +2a - a$; or $-2a - a$ (5)

Collect terms, $x = a$ or $-3a$.

EXERCISE 94.

Solve:

1. $\frac{3x^2}{2} + \frac{5x^2}{3} = 114$

2. $\frac{x^2-4}{2} - \frac{3-2x^2}{4} = \frac{5}{4}$

3. $x + \frac{2}{x} = \frac{3x}{2}$

4. $\frac{3x^2-5}{x^2} = 1$

5. $\frac{x}{x+2}=\frac{x-2}{2x}$

6. $\left(x+\frac{1}{x}\right)^2=4x^2$

7. $x+\frac{a}{x}=\frac{x}{c}$

8. $\frac{a+bx^2}{x}=\frac{x}{c}$

9. $\frac{x}{a-x}=\frac{a+x}{x}$

10. $(3x^2-9)^2=\frac{9}{4}x^4$

11. $\frac{x^2+5}{x}-\frac{x}{3}=\frac{3x}{4}+\frac{17}{4x}$

12. $\left(ax-\frac{a}{x}\right)^2=\frac{1}{4}a^2x^2$

13. $\frac{1}{2}(x+4)^2=\frac{2}{3}(x+4)^2-6$

14. $\frac{16}{25}(5x^2+36)^2=\frac{9}{16}(8x^2-4)^2$

15. If the side of a square be doubled, its area will be increased 75 square rods. What is the side of the square?

16. Four times one number equals five times another, and the difference of their squares is 81. Find the numbers.

17. A man bought a tract of land for \$5000, paying *twice* as many dollars per acre as there were acres in the tract. How many acres were there?

18. I sold a horse at a gain of \$81, and thereby gained as many per cent as there were dollars in the cost. What was the cost?

19. Three times the sum of two numbers equals 10 times the smaller number, and if the sum be multiplied by the greater, the product will be 630. Required the numbers.

20. If the dimensions of a certain cube be quadrupled, the entire surface will be increased by 7290 square inches; what are its dimensions?

21. A man received $^4/_5$ as many dollars per day as he worked days; had he worked only $^3/_5$ as many days and received $^9/_8$ as many dollars per day, he would have received \$26 less. How many days did he work?

Affected Quadratics.

1. Definition and Principles.

204. A quadratic equation which contains both the first and second powers of an unknown quantity is an *affected* or *complete* quadratic equation; as,

$$3x^2 + 7x = 15, \text{ or } \frac{5x^2}{3} - 5 = \frac{7x}{4} + 6.$$

205. It may be shown, as in Art. 200, that:

Prin. 85.—*Every complete quadratic equation of one unknown quantity may be reduced to the form of* $ax^2 + bx = c$, *in which* a, b, *and* c *are integral, and* a *positive.*

206. Take the equation $ax^2 + bx = c$.

Divide by a, $$x^2 + \frac{b}{a}x = \frac{c}{a}.$$

Put p for $\frac{b}{a}$ and q for $\frac{c}{a}$, $x^2 + px = q$. Therefore,

Prin. 86.—*Every complete quadratic equation of one unknown quantity may be reduced to the form of* $x^2 + px = q$, *in which* p *and* q *may be integral or fractional, positive or negative.*

207. Since p and q may be either positive or negative in the equation $x^2 + px = q$, it follows that:

Every complete quadratic equation may be reduced to one of the four following special forms:

1. $x^2 + px = +q$
2. $x^2 - px = +q$
3. $x^2 + px = -q$
4. $x^2 - px = -q$

---

2. Solution of Numerical Affected Quadratics.

208. The first member of an affected quadratic equation can always be made the square of a binomial by a process called "*completing the square,*" then the square root of both members may be taken, and the resulting simple equation solved.

Illustration.—Solve $8x^2 - 3x = 26.$ (A)

Solution: Multiply by 2 to make the first term a perfect square,

$$16x^2 - 6x = 52 \qquad (1)$$

Regard $16x^2 - 6x$ as the first two terms of the square of a binomial, then $\sqrt{16x^2}$, or $4x$, is the first term of the binomial, and $-6x$ is twice the product of the two terms; therefore $(-6x) \div (2 \times 4x)$, or $(-6) \div (2 \times 4)$, which is $-\frac{3}{4}$, is the second term of the binomial, and $\left(-\frac{3}{4}\right)^2$, or $\frac{9}{16}$, is the third term of the square of the binomial. Add this to both members,

$$16x^2 - 6x + \frac{9}{16} = 52 + \frac{9}{16} = \frac{841}{16} \qquad (2)$$

Extract $\sqrt{\ }$, $$4x - \frac{3}{4} = \pm\frac{29}{4} \qquad (3)$$

Transpose, $$4x = 8 \text{ or } -\frac{13}{2} \qquad (4)$$

Divide, $$x = 2 \text{ or } -1\frac{5}{8}$$

209. Hence we have the following rule:

1. Reduce the equation to one of the typical forms.

2. Multiply or divide both members of the equation by any quantity that will render the first term a perfect square.

3. Add to both members the square of the quotient obtained by dividing the coefficient of x by twice the square root of the coefficient of x^2, to complete the square.

4. Extract the square root of both members and solve the resulting simple equation.

Illustrations.—1. Solve $8x^2 - 12x = 8.$ (A)

Solution: Divide by 2, $$4x^2 - 6x = 4 \qquad (1)$$

Add $\left(\frac{6}{2\sqrt{4}}\right)^2$, or $\frac{9}{4}$, $$4x^2 - 6x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4} \qquad (2)$$

Extract the $\sqrt{\ }$, $$2x - \frac{3}{2} = \pm\frac{5}{2} \qquad (3)$$

Transpose, $$2x = 4 \text{ or } -1 \qquad (4)$$

Divide, $$x = 2 \text{ or } -\frac{1}{2}$$

2. Solve $3x^2 + 2x = 33$. (A)

Solution: Divide by 3, $x^2 + \frac{2}{3}x = 11$ (1)

Add $\left(\frac{2}{3} \div 2\sqrt{1}\right)^2$, or $\left(\frac{1}{3}\right)^2$, $x^2 + \frac{2}{3}x + \frac{1}{9} = 11 + \frac{1}{9} = \frac{100}{9}$ (2)

Extract $\sqrt{}$, $x + \frac{1}{3} = \pm\frac{10}{3}$ (3)

Transpose,

Therefore, $x = 3$ or $-3\frac{2}{3}$.

Scholium 1.—*When the coefficient of x^2 is 1, the quantity to be added to both members to complete the square is the square of half the coefficient of x.*

---

3. Solve $3x^2 - 5x = 28$. (A)

Solution: Multiply by 3, $9x^2 - 15x = 84$ (1)

Add $\left(\frac{15}{2\sqrt{9}}\right)^2$, or $\left(\frac{5}{2}\right)^2$, $9x^2 - 15x + \frac{25}{4} = 84 + \frac{25}{4} = \frac{361}{4}$ (2)

Extract the $\sqrt{}$, $3x - \frac{5}{2} = \pm\frac{19}{2}$ (3)

Transpose, $3x = 12$ or -7 (4)

Divide,

Therefore, $x = 4$ or $-2\frac{1}{3}$.

Scholium 2.—*When the equation is multiplied through by the coefficient of x^2, the quantity to be added to both members is the square of half the coefficient of x in the typical equation. This method is generally the best, as it avoids all fractions above and below* ***fourths.***

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4. Solve $2x^2 + 3x = 14$. (A)

Solution: Multiply by 4×2, $16x^2 + 24x = 112$ (1)

Add $\left(\frac{24}{2\sqrt{16}}\right)^2$, or 3^2, $16x^2 + 24x + 9 = 121$ (2)

Extract the $\sqrt{}$, $4x + 3 = \pm 11$ (3)

Transpose, $4x = 8$ or -14 (4)

Divide,

Therefore, $x = 2$ or $-3\frac{1}{2}$.

Scholium 3.—*When the equation is multiplied through by four times the coefficient of x^2, the quantity to be added to both members is the square of the coefficient of x in the typical equation. This is called the Hindoo method of completing the square. It avoids all fractions, but often gives rise to very large whole numbers.*

EXERCISE 95.

Solve :

1. $x^2 + 2x = 8$
2. $x^2 - 2x = 24$
3. $x^2 + 5x = -6$
4. $x^2 - 9x = -20$
5. $x^2 - 3x = 18$
6. $x^2 - x = 20$
7. $x^2 - 11x = -28$
8. $x^2 + 4x = 60$
9. $x^2 + x = 56$
10. $x^2 - x = 110$
11. $4x^2 + 10x = -6$
12. $3x^2 - 7x = 6$
13. $2x^2 - 7x = 30$
14. $6x^2 + 9x = 81$
15. $5x^2 + 39x = -28$
16. $6x^2 - 47x = 63$
17. $6x^2 + 19x = -15$
18. $2x^2 - 9x = 35$
19. $9x^2 + 27x = -14$
20. $16x^2 - 16x = 45$
21. $x^2 + \frac{7}{3}x = -\frac{10}{9}$
22. $x^2 + x = 8\frac{3}{4}$
23. $\frac{x^2+1}{5} + \frac{x+2}{3} = 2\frac{1}{3}$
24. $\frac{2x^2-3}{7} - \frac{x-6}{14} = 2\frac{5}{14}$
25. $x^2 - 4x = -1$
26. $x^2 - 6x = 11$
27. $x + \frac{1}{x} = 4\frac{1}{4}$
28. $\frac{x+1}{x} - \frac{x}{x+1} = \frac{13}{42}$
29. $x^2 - 2x = -\frac{1}{2}$
30. $\frac{x}{x+4} + \frac{x-1}{x-4} = 2\frac{5}{12}$
31. $\frac{x}{x+1} - \frac{1}{x-1} = \frac{x}{x^2-1}$
32. $\frac{x^2+x+1}{6} - \frac{x^2-x+1}{5} = \frac{23}{30}$

3. Solution of Literal Affected Quadratics.

Illustrations.—1. Solve $x^2+ax=b$.

Solution: Complete the square,

$$x^2+ax+\frac{a^2}{4}=b+\frac{a^2}{4}=\frac{4b+a^2}{4}$$

Extract the √, $\quad x+\frac{a}{2}=\pm\frac{1}{2}\sqrt{4b+a^2}$

Transpose, $\quad x=-\frac{a}{2}\pm\frac{1}{2}\sqrt{4b+a^2}=-\frac{1}{2}(a\mp\sqrt{4b+a^2})$

2. Solve $ax^2+bx=c$. (A)

Solution: Multiply by a,

$a^2x^2+abx=ac$ (1)

Complete the square,

$a^2x^2+abx+\frac{b^2}{4}=ac+\frac{b^2}{4}=\frac{4ac+b^2}{4}$ (2)

Extract the √,

$ax+\frac{b}{2}=\pm\frac{1}{2}\sqrt{4ac+b^2}$ (3)

Transpose, $ax=-\frac{b}{2}\pm\frac{1}{2}\sqrt{4ac+b^2}=-\frac{1}{2}(b\mp\sqrt{4ac+b^2})$ (4)

Divide by a, $x=-\frac{1}{2a}(b\mp\sqrt{4ac+b^2})$

EXERCISE 96.

Solve:

1. $x^2+2ax=3a^2$
2. $x^2-3bx=-2b^2$
3. $x^2+mx=6m^2$
4. $a^2x^2-acx=2c^2$
5. $x^2+(ab+b)x=-ab^2$
6. $x^2+a^2=ax$
7. $\frac{x}{a}-\frac{3a}{x}=2$
8. $\frac{a+x}{a}+\frac{a}{a-x}=1$
9. $\frac{x-b}{a}=\frac{b}{x-b}$
10. $\frac{x}{p}+\frac{x}{q}=\frac{p+q}{x}$
11. $\frac{a+x}{x}=\frac{a+x}{a-x}$
12. $x+\frac{1}{x}=a+\frac{1}{a}$
13. $(x+b)(x+c)=\frac{4cx+3b^2}{4}$

14. $\frac{(x+m)(x-n)}{2}=\frac{4mx-n^2}{8}$

15. $acx^2-bcx+adx-bd=0$

16. $abx^2-(a^2-b^2)x-ab=0$

Equations in the Quadratic Form.

Definitions.

210. When an equation contains two and only two exponents of the unknown terms, and one of them is twice the other, it is said to have the quadratic form; as,

$$x^4+6x^2=16,\quad ax^6+bx^3=c,\ \text{or}$$
$$(a+bx)^6+p(a+bx)^3=c.$$

211. Any equation having the quadratic form, whatever its degree, may be solved by any of the methods employed to solve an affected quadratic.

Illustrations.—1. Solve $x^4+6x^2=16.$ (A)

Solution: Complete the square,

$x^4+6x^2+9=25$ (1)

Extract $\sqrt{\ }$, $x^2+3=\pm 5$ (2)

Transpose, $x^2=2$ or -8 (3)

Extract $\sqrt{\ }$, $x=\pm\sqrt{2}$, or $\pm 2\sqrt{-2}$

2. $(x+4)^8+(x+4)^4=3\frac{3}{4}.$ (A)

Solution: Complete the square,

$(x+4)^8+(\quad)+\frac{1}{4}=4$ (1)

Extract $\sqrt{\ }$, $(x+4)^4+\frac{1}{2}=\pm 2$ (2)

Transpose, $(x+4)^4=1\frac{1}{2}$ or $-2\frac{1}{2}$ (3)

Extract $\sqrt[4]{\ }$, $x+4=\pm\sqrt[4]{1\frac{1}{2}}$ or $\pm\sqrt[4]{-2\frac{1}{2}}$ (4)

Transpose, $x=4\pm\sqrt[4]{1\frac{1}{2}}$ or $4\pm\sqrt[4]{-2\frac{1}{2}}$

EXERCISE 97.

Solve:

1. $x^4 + 2x^2 = 24$

2. $x^6 - 9x^3 = -8$

3. $x^8 + x^4 = 6$

4. $2x^4 - 10x^2 = 72$

5. $x^3 + \frac{1}{x^3} = 8\frac{1}{8}$

6. $x^{10} + 31x^5 = 32$

7. $81x + \frac{1}{x^3} = \frac{82}{x}$

8. $x^3 + \frac{1225}{x} = 74x$

9. $x^4 + \frac{1}{x^4} = 2$

10. $ax^3 + \frac{a}{x^3} = b$

11. $(x+2)^4 + 4(x+2)^2 = 21$

12. $(2x+1)^2 + 3(2x+1) = 70$

13. $\left(x + \frac{1}{2}\right)^2 - \left(x + \frac{1}{2}\right) = 6$

14. $(x^2 + x + 2)^2 + x^2 + x + 2 = 6$

15. $\left(x + \frac{1}{x}\right)^2 + 4\left(x + \frac{1}{x}\right) = 16\frac{1}{4}$

16. $(x^2 + 2x)^2 - 23(x^2 + 2x) + 120 = 0$

17. $(x^2 - 5x)^2 - 8x^2 + 40x = 84$

18. $\left(2 - \frac{1}{x}\right)^2 - \frac{2}{x}\left(2 - \frac{1}{x}\right) = \frac{15}{x^2}$

19. $2\left(\frac{3}{4}x^2 - \frac{1}{2}\right)^2 - 11(3x^2 - 2) = -10$

20. $(a + bx)^2 - \frac{c}{(a+bx)^2} = 2m$

21. $x^{2n} + 4x^n = 12$

22. $x^2 + x + \frac{2}{x^2 + x} = 3$

23. $x^2 + \frac{1}{x^2} + x + \frac{1}{x} = \frac{27}{4}$

Solution of Equations by Factoring.

Illustrations.—1. Solve $4x^2 = 1$. (A)

Solution: Transpose 1, $4x^2 - 1 = 0$ (1)
Factor, $(2x+1)(2x-1) = 0$ [P. 39] (2)
Divide by $(2x+1)$, $2x - 1 = 0$
Transpose and divide, $x = \frac{1}{2}$
Divide (2) by $(2x-1)$, $2x + 1 = 0$
Transpose and divide, $x = -\frac{1}{2}$

2. Solve $x^2 + 5x + 6 = 0$. (A)

Solution: Factor, $(x+2)(x+3) = 0$ [P. 40] (1)
Divide by $x + 2$, $x + 3 = 0$
Transpose, $x = 3$
Divide (1) by $(x+3)$, $x - 2 = 0$
Transpose, $x = 2$

3. Solve $6x^2 + 11x - 10 = 0$. (A)

Solution: Factor, $(3x-2)(2x+5) = 0$ [P. 41] (1)
Divide by $(3x-2)$, $2x + 5 = 0$ (2)
Transpose and divide, $x = -2\frac{1}{2}$
Divide (1) by $(2x+5)$, $3x - 2 = 0$ (3)
Transpose and divide, $x = \frac{2}{3}$

4. Solve $x^3 - 1 = 0$. (A)

Solution: Factor, $(x-1)(x^2+x+1) = 0$ [P. 44] (1)
Divide by (x^2+x+1), $x - 1 = 0$
Transpose, $x = 1$
Divide (1) by $(x-1)$, $x^2 + x + 1 = 0$ (2)
Solve (2), $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$

5. Solve $x^4 + x^2 + 1 = 0$. (A)

Solution: Factor, $(x^2+x+1)(x^2-x+1) = 0$ [page 84] (1)
Divide by (x^2+x+1), $x^2 - x + 1 = 0$ (2)
Solve (2), $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$
Divide (1) by (x^2-x+1), $(x^2+x+1) = 0$ (3)
Solve (3), $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$

6. Solve $x^3 - ax^2 - a^2x + a^3 = 0$. (A)

Solution: Factor (A), $x^2(x-a) - a^2(x-a) = 0$ (1)

Factor (1), $(x^2 - a^2)(x-a) = 0$ (2)

Factor (2), $(x-a)(x+a)(x-a) = 0$ (3)

Divide (3) by $(x-a)(x+a)$, $x - a = 0$ (4)

Transpose, $x = a$

Divide (3) by $(x-a)(x-a)$, $x + a = 0$ (5)

Transpose, $x = -a$

Divide (3) by $(x+a)(x-a)$, $x - a = 0$ (6)

Transpose, $x = a$

7. Solve $4x^4 + 12x^3 + 29x^2 + 30x + 21$. (A)

Solution: By extracting the square root of the first member we find that it lacks 4 of being $(2x^2 + 3x + 5)^2$;

$\therefore$ $(2x^2 + 3x + 5)^2 - 4 = 0$ (1)

Factor, $(\overline{2x^2 + 3x + 5} + 2)(\overline{2x^2 + 3x + 5} - 2) = 0$ [P. 39] (2)

Collect terms, $(2x^2 + 3x + 7)(2x^2 + 3x + 3) = 0$ (3)

Divide (3) by $2x^2 + 3x + 7$, $2x^2 + 3x + 3 = 0$ (4)

Solve (4), $x = -\frac{1}{4}(3 \pm \sqrt{-15})$

Divide (3) by $2x^2 + 3x + 3$, $2x^2 + 3x + 7 = 0$ (5)

Solve (5), $x = -\frac{1}{4}(3 \pm \sqrt{-47})$

Note.—This equation may also be solved by adding 4 to both members and extracting $\sqrt{\ }$.

EXERCISE 98.

Solve by factoring:

1. $x^2 - 4 = 0$
2. $4x^2 - 9 = 0$
3. $9x^2 - 4a^2 = 0$
4. $x^2 - 1/16 = 0$
5. $x^2 + 7x + 10 = 0$
6. $x^2 - 6x + 8 = 0$
7. $x^2 + 3x - 18 = 0$
8. $4x^2 - 2x - 12 = 0$
9. $4x^2 + 12x + 9 = 0$
10. $9x^2 - 24x + 16 = 0$
11. $2x^2 - 7x - 15 = 0$
12. $6x^2 - 13x + 6 = 0$
13. $8x^2 + 14x - 15 = 0$
14. $3x^2 + 8x - 35 = 0$
15. $x^3 - 8 = 0$
16. $x^3 + 1 = 0$
17. $x^3 + 8 = 0$
18. $x^3 - 27 = 0$

19. $x^3 + 27 = 0$

20. $x^4 - a^4 = 0$

21. $x^4 - 81 = 0$

22. $x^6 - 1 = 0$

23. $x^6 - a^6 = 0$

24. $8x^3 - 27a^3 = 0$

25. $x^3 + 6x^2 - 4x - 24 = 0$

26. $x^3 - x^2 - x + 1 = 0$

27. $x^4 + x^2 + 1 = 0$

28. $x^3 + 3x^2 + 3x + 1 = 0$

29. $x^4 - 13x^2 + 36 = 0$

30. $x^4 - 2x^3 + 3x^2 - 2x + 1 = 0$

31. $x^3 + 6x^2 + 12x + 8 = 0$

32. $x^4 - 8x^3 + 24x^2 - 32x + 7 = 0$

33. $(x - a)(x - b)^2 + x^2 - (a + b)x + ab = 0$

Formation of Quadratic Equations.

1. Principles.

212. If we solve the general equation $x^2 + px = q$, we will find the roots to be :

$$-\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q} \text{ and}$$

$$-\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q}$$

The sum of these roots is $-p$;

Their product is $-\frac{1}{4}p^2 - \left(\frac{1}{4}p^2 + q\right) = -q.$

Therefore,

Prin. 87.—*The sum of the two roots of an equation of the form of $x^2 + px = q$ equals the coefficient of **x**, with the sign changed.*

Prin. 88.—*The product of the two roots of an equation of the form of $x^2 + px = q$ equals the absolute term with the sign changed.*

2. Examples.

Illustrations.—1. Find the equation whose roots are $+4$ and -6.

Solution: The coefficient of $x = -(4-6)$ [P. 87] $= +2$;
The absolute term $= -(+4 \times -6)$ [P. 88] $= +24$;
Therefore the equation is $x^2 + 2x = 24$.

2. Find the equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

Solution:
The coefficient of $x = -\{(2+\sqrt{3}) + (2-\sqrt{3})\}$ [P. 87] $= -4$;
The absolute term $= -(2+\sqrt{3})(2-\sqrt{3})$ [P. 88] $= 1$;
Therefore the equation is $x^2 - 4x = 1$.

EXERCISE 99.

Form the equations whose roots are :

1. $+2$ and $+4$
2. -3 and $+5$
3. -8 and $+3$
4. -5 and -4
5. $2a$ and a
6. $3p$ and $-2p$
7. a and $-8a$
8. $a+b$ and $a-b$
9. a^2+b^2 and a^2-b^2
10. $1+\sqrt{2}$ and $1-\sqrt{2}$
11. $3+\sqrt{2}$ and $3-\sqrt{2}$
12. $2a-b$ and $2a+b$
13. $a+\sqrt{b}$ and $a-\sqrt{b}$
14. $\frac{3}{4}$ and $\frac{5}{6}$
15. $\frac{4}{3}$ and $\frac{3}{4}$
16. $a+2m$ and $a-2m$

Formation of Equations by Composition.

Illustrations.—

1. Form the equation whose roots are $+2$ and -2.

Solution: If $x = +2$, $x - 2 = 0$ (1)
If $x = -2$, $x + 2 = 0$ (2)

Multiplying together (1) and (2),

$$(x-2)(x+2) = 0 \qquad (3)$$

Expanding, $x^2 - 4 = 0$

2. Form the equation whose roots are -4 and $+7$.

Solution:

$$\text{If } x=-4, \quad x+4=0 \qquad (1)$$
$$\text{If } x=+7, \quad x-7=0 \qquad (2)$$
$$\therefore \quad (x+4)(x-7)=0 \qquad (3)$$

Expanding, $$x^2-3x-28=0$$

3. Form the equation whose roots are $-\frac{3}{4}$ and $\frac{2}{3}$.

Solution:

$$\text{If } x=-\frac{3}{4}, \quad 4x=-3, \text{ and } 4x+3=0 \qquad (1)$$
$$\text{If } x=\frac{2}{3}, \quad 3x=2, \quad \text{and } 3x-2=0 \qquad (2)$$
$$\therefore \quad (4x+3)(3x-2)=0 \qquad (3)$$
$$\text{or,} \quad 12x^2+x-6=0$$

4. Form the equation whose roots are $+2$, -2, and $+3$.

Solution:

$$\text{If } x=+2, \quad (x-2)=0 \qquad (1)$$
$$\text{If } x=-2, \quad (x+2)=0 \qquad (2)$$
$$\text{If } x=+3, \quad (x-3)=0 \qquad (3)$$
$$\therefore \quad (x-2)(x+2)(x-3)=0 \qquad (4)$$

Expanding, $$x^3-3x^2-4x+12=0 \qquad (5)$$

Observe that an equation always has as many roots as there are units in its degree.

EXERCISE 100.

Form the equations whose roots are:

1. $+3$ and -3
2. $+5$ and -7
3. -5 and $+7$
4. -5 and -7
5. $+5$ and $+7$
6. $\frac{2}{3}$ and $\frac{3}{4}$
7. 5 and $-\frac{5}{2}$
8. $-\frac{3}{2}$ and $-\frac{2}{3}$
9. $+\frac{5}{6}$ and -3
10. a and $-2a$
11. 2, 3, and -3
12. 3, -3, and 4
13. 1, 2, and 2
14. 3, 0, and 1
15. $-\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{2}$
16. $\frac{2}{5}$, $\frac{5}{2}$, and 1
17. 0, $\frac{3}{4}$, and $\frac{1}{4}$
18. $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$

Miscellaneous Examples.

EXERCISE 101.

Solve:

1. $x^2 - a^2 = a^4 - (x - a)^2$

2. $\frac{1}{2}(x-3)^2 + 13\frac{1}{2} = 3x$

3. $\frac{2x}{3x+180} = \frac{14}{9x-15}$

4. $\frac{x-5}{x+3} = 1 - \frac{3}{x} + \frac{1}{x^2}$

5. $\frac{3}{2x+12} + \frac{6}{x} = \frac{3}{2x+4}$

6. $\frac{10}{12+3x} - \frac{10}{3x-12} = \frac{16}{9}$

7. $\frac{6x^2+10}{40} - \frac{2x^2+58}{15} = \frac{234}{5} - 2x^2$

8. $\frac{2x+14}{3x^2-21x} - \frac{2x-14}{3x^2+21x} = \frac{14}{3x^2-219}$

9. $x + 4 - \frac{4x+7}{3} = \frac{21-3x}{x-3} - 3$

10. $x(x-1) = \frac{8}{x}(x^2+2x)$

11. $\frac{2}{3} \cdot \frac{2x-10}{8-x} + \frac{2x+6}{6-3x} = 1\frac{1}{3}$

12. $\frac{x+3}{x+2} = \frac{3-2x}{1-x} + \frac{x-3}{2-x}$

13. $\frac{x}{m} + \frac{m}{x} = \frac{x}{n} + \frac{n}{x}$

14. $ax - \frac{acx^2}{d} = \frac{bcx}{d} - b$

15. $ax^2 - 2a^2x + a^3 = \frac{1}{a}$

16. $\frac{1}{a-b+x} = \frac{1}{a} - \frac{1}{b} + \frac{1}{x}$

Find the approximate values of x in the following equations:

17. $\frac{3x^2}{2} - 12x = 21$

18. $\frac{6x^2+3x}{5} = 6{\cdot}6$

19. $1 - \frac{6}{x} = -\frac{4}{x^2}$

20. $\frac{7x}{x+1} - 5x = -2$

21. $\frac{39x^2}{28} - \frac{26x^2}{42} = \frac{13}{14}$

22. $\frac{(5-x)(x+5)}{x^2-16} = \frac{11}{12}$

23. $\frac{21}{10-2x} = \frac{25}{14} - \frac{x}{14}$

24. $\frac{5}{x} - \frac{3x+1}{x^2} = \frac{1}{4}$

Solve :

25. $x^2 + \frac{1}{x^2} = a^2 + \frac{1}{a^2}$

26. $x^6 + 1 = 0$

27. $4x^4 + 16x^3 + 16x + 4 = 57x^2$

28. $x^2 - 2x - 2 + \frac{3}{x} - \frac{108}{x^2} = 0$

29. Find the three cube roots of 1.

Suggestion.—Let $x = \sqrt[3]{1}$, then $x^3 = 1$, or $x^3 - 1 = 0$.

30. Find the three cube roots of -1.

31. Find the four fourth roots of 1.

32. Form the equation whose roots are

$a, -a, b, -b, c, -c.$

Examples involving Quadratic Equations of One Unknown Quantity.

EXERCISE 102.

1. The product of two consecutive numbers is 156. What are the numbers?

2. The sum of two numbers is 17, and their product is 42. Required the numbers.

3. The difference of two numbers is 8, and their product is 105. What are the numbers?

4. There are 900 trees in an orchard, and the number in one row exceeds twice the number of rows by 5. How many rows are there?

5. A man bought some cloth for \$90; had he bought 15 yards more for the same money, he would have paid \$1 a yard less. How many yards did he buy?

6. The sum of two numbers is 30, and their quotient is the less number. Required the numbers.

7. A lot that is 2 rods longer than wide contains 48 square rods. What are its dimensions?

8. If a train would increase its speed 5 miles an hour, it would go 360 miles one hour sooner. What is the rate of the train?

9. A can do a piece of work in 2 days less than B, and they together can do it in $2\frac{2}{5}$ days. In what time can each alone do it?

10. One pipe can fill a cistern 3 hours sooner than another can empty it, and if they run together the cistern will be filled in $13\frac{1}{3}$ hours. In what time could the first fill it?

11. A certain number exceeds its square root by 30. Required the number.

Suggestion.—Let x^2 equal the number.

12. If the circumference of a wheel were increased by 4 feet, the wheel would make 110 revolutions less in going a mile. What is the circumference of the wheel?

13. A man sold a horse for \$75, and thereby gained as many per cent as there were dollars in the cost. Required the cost.

14. A sold his farm at \$48 an acre, and thereby lost one half as many per cent as there were dollars in the cost. Required the cost per acre.

15. A man increased his capital stock by \$500 without increasing his gain, which was \$500, in consequence of which his rate of gain was lowered 5%. What was his original stock?

16. How much must be added to both the length and the width of a rectangle, 18 by 20 inches, to make it contain 483 square inches?

17. Eggs rose 5 cents a dozen, in consequence of which 3 eggs less could be purchased for 25 cents. What was the price per dozen before the rise?

18. A and B together earned \$432. B earned \$6 a month more than A, and the number of months they worked was one third of the number of dollars A earned in a month. How much did each earn per month, and how many months did they labor?

19. A boy rowed 3 miles down a river and back again in $1\frac{1}{2}$ hour. The rate of the current was 2 miles an hour. Determine his rate of rowing in still water.

20. A and B are 320 miles apart. If A travels 8 miles a day more than B, they will meet in one half as many days as B travels miles per day. How far does each travel per day?

21. Twice the length of a rectangle equals three times the width, and if 2 feet be added to the length and 3 feet to the width, the area will be 56 square feet. What are the dimensions?

22. A and B have each a debt of \$150 to pay. A pays \$3 a week more than B, and pays his debt $8\frac{1}{3}$ weeks sooner. How much does A pay per week?

23. A and B undertook to earn \$640. A earned \$8 a week more than B, and the number of weeks required was one fourth of the number of dollars that B earned in a week. What were the weekly wages of each?

24. Around a flower-bed, 18 feet by 12, is a gravel-walk whose area equals that of the flower-bed. What is the width of the walk?

25. A farmer sold 7 pigs and 12 lambs for \$50, and found that he had sold 3 more pigs for \$10 than he sold lambs for \$6. Required the price of each.

26. One person husked 48 shocks of corn in a day; another husked the same number two hours sooner, and husked 2 shocks per hour more than the first. How many shocks per hour does each husk ?

27. A and B were engaged at different rates of wages. A worked a certain number of days and received \$24, and B, who worked 6 days fewer, received \$13½. If A had worked 6 days fewer and B 6 days more, they would have received the same sum. How many days did each work ?

Quadratic Equations of Two Unknown Quantities.

Definitions.

213. A quadratic equation of two unknown quantities is *complete* when it contains all the second degree and all the first degree terms possible ; as,

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

214. A quadratic equation of two unknown quantities is *pure*, or *homogeneous*, if all the terms containing unknown quantities are of the second degree ; as,

$$ax^2 + bxy + cy^2 = d.$$

215. Any equation containing two unknown quantities is *symmetrical* if the unknown quantities may change places without destroying the equation ; as,

$$2x^2 + 3xy + 2y^2 = 12 \text{ and } 2y^2 + 3yx + 2x^2 = 12.$$

216. The solution of two simultaneous quadratic equations of two unknown quantities often involves the solution of a bi-quadratic equation.

Illustration.—

Given $\left\{\begin{array}{ll} x^2 - y = 2 & \text{(A)} \\ x - y^2 = 3 & \text{(B)} \end{array}\right\}$ to find x and y.

Solution : Transpose (A), $\quad y = x^2 - 2 \quad (1)$

Square (1), $\quad y^2 = x^4 - 4x^2 + 4 \quad (2)$

Substitute (2) in (B), $\quad x^4 - 4x^2 - x = -7$, a bi-quadratic.

Solvable Classes.

I. When one equation is of the first degree and the other of the second degree, they are solvable as quadratics.

Illustrations.—1. Solve $\begin{cases} 2x + y = 8 & \text{(A)} \\ xy - y^2 = 2 & \text{(B)} \end{cases}$

Solution: Transpose (A), $2x = 8 - y$ (1)

Divide (1) by 2, $x = \frac{8-y}{2}$ (2)

Substitute (2) in (B), $\frac{8y - y^2}{2} - 2y^2 = 2$ (3)

Clear of fractions, $8y - y^2 - 2y^2 = 4$ (4)

Rearrange terms, $3y^2 - 8y = -4$ (5)

Complete the square, $9y^2 - 24y + 16 = -12 + 16 = 4$ (6)

Extract the $\sqrt{\ }$, $3y - 4 = \pm 2$ (7)

Transpose and divide, $y = 2 \text{ or } \frac{2}{3}$ (8)

Substitute (8) in (2), $x = \frac{1}{2}(8-2) \text{ or } \frac{1}{2}\left(8 - \frac{2}{3}\right)$ (9)

Reduce (9), $x = 3 \text{ or } 3\frac{2}{3}$

217. Many equations of this class may be solved by more elegant methods.

2. Solve $\begin{cases} x + y = 7 & \text{(A)} \\ xy = 12 & \text{(B)} \end{cases}$

Square (A), $x^2 + 2xy + y^2 = 49$ (1)
Multiply (B) by 4, $4xy = 48$ (2)
Subtract (2) from (1), $x^2 - 2xy + y^2 = 1$ (3)
Extract $\sqrt{\ }$, $x - y = \pm 1$ (4)
Add (4) to (A), $2x = 8 \text{ or } 6$ (5)
Divide by 2, $x = 4 \text{ or } 3$
Subtract (4) from (A), $2y = 6 \text{ or } 8$ (6)
Divide by 2, $y = 3 \text{ or } 4$
Therefore $y = 3$ when $x = 4$, and $y = 4$ when $x = 3$.

In a similar manner may be solved equations of the form of:

$$\begin{Bmatrix} x - y = a \\ xy = b \end{Bmatrix}, \quad \begin{Bmatrix} x^2 + y^2 = a \\ xy = b \end{Bmatrix}, \quad \begin{Bmatrix} x^2 + y^2 = a \\ x \pm y = b \end{Bmatrix}$$

EXERCISE 103.

Solve:

1. $\begin{cases} x+y=9 \\ xy=20 \end{cases}$

2. $\begin{cases} x-y=6 \\ xy=16 \end{cases}$

3. $\begin{cases} x^2+y^2=52 \\ xy=-24 \end{cases}$

4. $\begin{cases} x^2+y^2=72 \\ x+y=12 \end{cases}$

5. $\begin{cases} x^2+y^2=50 \\ x-y=10 \end{cases}$

6. $\begin{cases} x^2-y^2=-7 \\ x+y=7 \end{cases}$

7. $\begin{cases} x^2-y^2=a^2-b^2 \\ x+y=a+b \end{cases}$

8. $\begin{cases} x^2-y^2=16 \\ x-y=4 \end{cases}$

9. $\begin{cases} \frac{1}{x^2}-\frac{1}{y^2}=\frac{5}{36} \\ \frac{1}{x}-\frac{1}{y}=\frac{1}{6} \end{cases}$

10. $\begin{cases} x^2+4y^2=25 \\ xy=6 \end{cases}$

11. $\begin{cases} x^2-9y^2=16 \\ x-3y=2 \end{cases}$

12. $\begin{cases} 2x+3y=40 \\ xy=50 \end{cases}$

13. $\begin{cases} 4x^2-9y^2=108 \\ 2x-3y=-6 \end{cases}$

14. $\begin{cases} x+2y=17 \\ x^2+y^2=61 \end{cases}$

15. $\begin{cases} x=y+2 \\ 3x^2=4y^2+48 \end{cases}$

16. $\begin{cases} x+3y=2 \\ x^2+xy=48 \end{cases}$

17. $\begin{cases} 3x-7y=-5 \\ xy-y^2=25 \end{cases}$

18. $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{7}{12} \\ \frac{1}{x^2}+\frac{1}{y^2}=\frac{25}{144} \end{cases}$

218. Some equations of a higher degree may be reduced to this class by division.

Illustration.—Solve $\begin{cases} x^3-y^3=56 & \text{(A)} \\ x-y=2 & \text{(B)} \end{cases}$

Solution: Divide (A) by (B),	$x^2+xy+y^2=28$	(1)
Square (B),	$x^2-2xy+y^2=4$	(2)
Subtract (2) from (1),	$3xy=24$	(3)
Divide,	$xy=8$	(4)
Add (4) to (1),	$x^2+2xy+y^2=36$	(5)
Extract the $\sqrt{\ }$,	$x+y=\pm 6$	(6)
Add (B) and (6), and divide,	$x=4$ or -2	(7)
Substitute (7) in (4), and divide,	$y=2$ or -4	(8)

Therefore, when $x=4$, $y=2$, and when $x=-2$, $y=-4$.

In a similar manner may be solved equations of the form of:

$$\left\{\begin{matrix} x^3+y^3=a \\ x+y=b \end{matrix}\right\}, \quad \left\{\begin{matrix} x^4-y^4=a \\ x^2\pm y^2=b \end{matrix}\right\}, \quad \left\{\begin{matrix} x^4+x^2y^2+y^4=a \\ x^2\pm xy+y^2=b \end{matrix}\right\}$$

EXERCISE 104.

Solve:

1. $\left\{\begin{matrix} x^3+y^3=35 \\ x+y=5 \end{matrix}\right\}$

2. $\left\{\begin{matrix} x^3-y^3=61 \\ x-y=1 \end{matrix}\right\}$

3. $\left\{\begin{matrix} x^4-y^4=80 \\ x^2-y^2=8 \end{matrix}\right\}$

4. $\left\{\begin{matrix} x^4-y^4=-65 \\ x^2+y^2=13 \end{matrix}\right\}$

5. $\left\{\begin{matrix} x^4+x^2y^2+y^4=481 \\ x^2-xy+y^2=13 \end{matrix}\right\}$

6. $\left\{\begin{matrix} x^4+x^2y^2+y^4=21 \\ x^2+xy+y^2=3 \end{matrix}\right\}$

7. $\left\{\begin{matrix} x^3+27y^3=243 \\ x+3y=9 \end{matrix}\right\}$

8. $\left\{\begin{matrix} 8x^3-y^3=98 \\ 2x-y=2 \end{matrix}\right\}$

9. $\left\{\begin{matrix} 8x^3+27y^3=35 \\ 2x+3y=5 \end{matrix}\right\}$

10. $\left\{\begin{matrix} x^4-16y^4=80 \\ x^2-4y^2=8 \end{matrix}\right\}$

11. $\left\{\begin{matrix} 81x^4-16y^4=175 \\ 9x^2+4y^2=25 \end{matrix}\right\}$

12. $\left\{\begin{matrix} 16x^4+4x^2y^2+y^4=91 \\ 4x^2-2xy+y^2=7 \end{matrix}\right\}$

219. Sometimes there is a common factor in the first members of two equations that may be removed by division.

Illustration.—Solve $\left\{\begin{matrix} x^2-y^2=5 & \text{(A)} \\ xy-y^2=2 & \text{(B)} \end{matrix}\right\}$

Solution: Factor (A) and (B), $(x+y)(x-y)=5$ (1)

$y(x-y)=2$ (2)

Divide (1) by (2), $\dfrac{x+y}{y}=\dfrac{5}{2}$ (3)

Clear of fractions, $2x+2y=5y$ (4)

Transpose, $2x=3y$ (5)

Divide, $x=\dfrac{3}{2}y$ (6)

Substitute (6) in (2), $\dfrac{1}{2}y^2=2$ (7)

Reduce, $y=\pm 2$ (8)

Substitute (8) in (6), $x=\pm 3$ (9)

EXERCISE 105.

Solve :

1. $\begin{cases} x^2 - y^2 = 12 \\ xy + y^2 = 12 \end{cases}$

2. $\begin{cases} x^2 + 2xy = 27 \\ x^2 + 3xy + 2y^2 = 54 \end{cases}$

3. $\begin{cases} x^2 + 3xy + 2y^2 = 12 \\ x^2 + 4xy + 3y^2 = 15 \end{cases}$

4. $\begin{cases} 4x^2 - 9y^2 = -108 \\ 2xy - 3y^2 = -24 \end{cases}$

5. $\begin{cases} 4x^2 + 8xy + 3y^2 = 96 \\ 2x^2 + xy = 48 \end{cases}$

6. $\begin{cases} 16x^2 - 9y^2 = 319 \\ 8x^2 + 6xy = 290 \end{cases}$

7. $\begin{cases} 4x^2 + 4xy + y^2 = 169 \\ 2xy + y^2 = 39 \end{cases}$

8. $\begin{cases} x^2 - 3xy - 4y^2 = -150 \\ 2x^2 - 8xy = -150 \end{cases}$

9. $\begin{cases} 6x^2 - 13xy + 6y^2 = 50 \\ 9xy - 6y^2 = 30 \end{cases}$

10. $\begin{cases} 6x^2 + 19xy + 15y^2 = 40 \\ 6x^2 - xy - 15y^2 = -10 \end{cases}$

220. Sometimes one or both equations have the quadratic form, or may be reduced to the quadratic form.

Illustrations.—

1. Solve $\begin{cases} (x+y)^2 + (x+y) = 20 & \text{(A)} \\ x - y = 4 & \text{(B)} \end{cases}$

Solution : Complete the square in (A),

$$(x+y)^2 + (x+y) + \frac{1}{4} = 20 + \frac{1}{4} = \frac{81}{4} \qquad (1)$$

Extract the $\sqrt{\ }$, $\quad x + y + \frac{1}{2} = \pm\frac{9}{2} \qquad (2)$

Transpose, $\quad x + y = 4 \text{ or } -5 \qquad (3)$

Add (B) to (3), $\quad 2x = 8 \text{ or } -1 \qquad (4)$

Divide, $\quad x = 4 \text{ or } -\frac{1}{2} \qquad (5)$

Subtract (B) from (3), $\quad 2y = 0 \text{ or } -9 \qquad (6)$

Divide, $\quad y = 0 \text{ or } -4\frac{1}{2}$

Therefore, when $x = 4$, $y = 0$, and when $x = -\frac{1}{2}$, $y = -4\frac{1}{2}$.

2. Solve $\begin{cases} x^2 + y^2 + x + y = 32 & \text{(A)} \\ 2xy + x + y = 31 & \text{(B)} \end{cases}$

Solution: Add (B) to (A),

$$x^2 + 2xy + y^2 + 2x + 2y = 63 \quad (1)$$

Factor, $(x + y)^2 + 2(x + y) = 63$ (2)

Complete the square,

$$(x + y)^2 + 2(x + y) + 1 = 64 \quad (3)$$

Extract the $\sqrt{\ }$, $x + y + 1 = \pm 8$ (4)

Transpose, $x + y = 7 \text{ or } -9$ (5)

Substitute (5) in (A) and (B), and transpose,

$$x^2 + y^2 = 25 \text{ or } 41 \quad (6)$$

$$2xy = 24 \text{ or } 40 \quad (7)$$

Subtract (7) from (6), $x^2 - 2xy + y^2 = 1 \text{ or } 1$ (8)

Extract the $\sqrt{\ }$, $x - y = \pm 1, \text{ or } \pm 1$ (9)

Add (9) and (5), $2x = 8 \text{ or } 6, -8 \text{ or } -10$ (10)

Divide, $x = 4, 3, -4, \text{ or } -5$ (11)

Subtract (9) from (5), $2y = 6 \text{ or } 8, -10 \text{ or } -8$ (12)

Divide, $y = 3, 4, -5, \text{ or } -4$

EXERCISE 106.

Solve:

1. $\begin{cases} x^2y^2 + xy = 42 \\ x + y = 5 \end{cases}$

2. $\begin{cases} \dfrac{x^2}{y^2} + \dfrac{2x}{y} = 4\dfrac{4}{9} \\ x - y = 1 \end{cases}$

3. $\begin{cases} (x + y)^2 + x + y = 56 \\ xy = 10 \end{cases}$

4. $\begin{cases} (x^2 + y^2)^2 + x^2 + y^2 = 30 \\ x^2 - y^2 = 3 \end{cases}$

5. $\begin{cases} (x + y)^2 + 2x + 2y = 80 \\ (x - y)^2 - (x - y) = 6 \end{cases}$

6. $\begin{cases} (x + y)^2 - 2(x + y) = 575 \\ xy = 150 \end{cases}$

7. $\begin{cases} x^2 + y^2 + x + y = 152 \\ xy + x + y = 76 \end{cases}$

8. $\begin{cases} x^2 + y^2 = 100 \\ 2xy + x + y = 110 \end{cases}$

9. $\begin{cases} \dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} + \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{27}{4} \\ x^2 + y^2 = 20 \end{cases}$

10. $\begin{cases} x^2 + y^2 = 58 \\ xy - x + y = 25 \end{cases}$

II. Two homogeneous equations of the second degree may be solved by putting $y = vx$ when they can not be more easily solved otherwise.

Illustration.—Solve $\left\{\begin{array}{ll} x^2 + xy + y^2 = 28 & \text{(A)} \\ x^2 - 2xy - y^2 = -28 & \text{(B)} \end{array}\right\}$

Solution: Substitute vx for y in (A) and (B),

$$x^2 + vx^2 + v^2x^2 = 28 \quad (1)$$

$$x^2 - 2vx^2 - v^2x^2 = -28 \quad (2)$$

Factor (1) and (2) and divide,

$$\frac{x^2(1 + v + v^2) = 28}{x^2(1 - 2v - v^2) = -28} \quad \begin{array}{l}(3)\\(4)\end{array}$$

or, $$\frac{1 + v + v^2}{1 - 2v - v^2} = -1 \quad (5)$$

Clear of fractions, $1 + v + v^2 = -1 + 2v + v^2$ (6)

Transpose, $v = 2$ (7)

Substitute in (3), $7x^2 = 28$ (8)

Divide, $x^2 = 4$ (9)

Extract the $\sqrt{\ }$, $x = \pm 2$ (10)

Substitute (10) in $y = vx$, $y = 2 \times (\pm 2) = \pm 4.$

EXERCISE 107.

Solve:

1. $\left\{\begin{array}{r} x^2 + y^2 = 5 \\ 2x^2 + xy + y^2 = 8 \end{array}\right\}$

2. $\left\{\begin{array}{r} x^2 - xy + y^2 = 12 \\ x^2 + 2xy + 2y^2 = 52 \end{array}\right\}$

3. $\left\{\begin{array}{r} x^2 + xy + y^2 = 28 \\ 2x^2 + 3y^2 = 44 \end{array}\right\}$

4. $\left\{\begin{array}{r} 5x^2 - 3y^2 = -63 \\ x^2 + xy = 27 \end{array}\right\}$

5. $\left\{\begin{array}{r} x^2 - xy = -\frac{1}{16} \\ xy + y^2 = \frac{3}{8} \end{array}\right\}$

6. $\left\{\begin{array}{r} x^2 + 2xy + 2y^2 = 5 \\ 2x^2 + 5y^2 = 7 \end{array}\right\}$

7. $\left\{\begin{array}{r} x^2 + 3xy = 10 \\ y^2 - x^2 = 8 \end{array}\right\}$

8. $\left\{\begin{array}{r} 2x^2 - xy - y^2 = -40 \\ x^2 + y^2 = 40 \end{array}\right\}$

9. $\left\{\begin{array}{r} x^2 - xy - y^2 = -125 \\ x^2 + 2xy = 125 \end{array}\right\}$

10. $\left\{\begin{array}{r} x^2 + 3xy + 2y^2 = 40 \\ 2x^2 - 2xy + y^2 = 5 \end{array}\right\}$

Miscellaneous Examples.

EXERCISE 108.

Solve:

1. $\begin{cases} x - y = 2 \\ xy - y^2 = 6 \end{cases}$

2. $\begin{cases} x + 2y = 11 \\ x^2 + xy = 63 \end{cases}$

3. $\begin{cases} x - 3y = 2 \\ x^2 + xy + y^2 = 84 \end{cases}$

4. $\begin{cases} x + y = a - b \\ xy = -ab \end{cases}$

5. $\begin{cases} x^2 + y^2 = a^2 \\ xy = b \end{cases}$

6. $\begin{cases} x^3 + y^3 = 126 \\ x + y = 6 \end{cases}$

7. $\begin{cases} x^4 + x^2y^2 + y^4 = 651 \\ x^2 - xy + y^2 = 21 \end{cases}$

8. $\begin{cases} x^4 + x^2y^2 + y^4 = 133 \\ x^2 + xy + y^2 = 19 \end{cases}$

9. $\begin{cases} x^3 + y^3 = 7(x + y) \\ x - y = 1 \end{cases}$

10. $\begin{cases} x^2 + xy = 48 \\ y^2 + xy = 16 \end{cases}$

11. $\begin{cases} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = 3\dfrac{1}{3} \\ x^2 + y^2 = 80 \end{cases}$

12. $\begin{cases} \dfrac{x}{x-y} - \dfrac{x-y}{x+y} = 1 \\ 2 + 3xy = 3x \end{cases}$

13. $\begin{cases} x^2 + y^2 - 1 = 2xy \\ xy(xy + 1) = 42 \end{cases}$

14. $\begin{cases} x^2y^2(x + y)^2 = 2916 \\ x^2 + y^2 = 2xy \end{cases}$

15. $\begin{cases} x^2y^2 + \dfrac{1}{x^2y^2} = 16\dfrac{1}{16} \\ x^2 + y^2 = 8 \end{cases}$

16. $\begin{cases} \dfrac{x^2 + y^2}{x^2 - y^2} = \dfrac{29}{21} \\ xy = 10 \end{cases}$

17. $\begin{cases} x^2 + xy = 9 \\ x + y = 1 \end{cases}$

18. $\begin{cases} \dfrac{a^2}{x^2} + \dfrac{y^2}{b^2} = 18 \\ \dfrac{ab}{xy} = 1 \end{cases}$

19. $\begin{cases} x^2 + y^2 + 2x + 2y = 50 \\ xy + x + y = 23 \end{cases}$

20. $\begin{cases} x(x + y) + y(x + y) = 81 \\ xy(x + y = 180 \end{cases}$

21. $\begin{cases} x + y + z = 6 \\ xy + xz + yz = 11 \\ x + z = y^2 \end{cases}$

Examples involving Quadratic Equations of Two Unknown Quantities.

EXERCISE 109.

1. The sum of two numbers is 13, and their product is 40. Find the numbers.

Suggestion.—Let x equal one number and y the other.

2. The difference of the squares of two numbers is 40, and their sum is 10. Find the numbers.

3. A man has two fields in the form of squares, containing 16,400 square rods, and the one is 20 rods longer than the other. Required the length of each.

4. The difference between two cubical blocks of marble is 152 cubic feet, and the difference of their lengths is 2 feet. Required the length of each.

5. A has a rectangular field containing 240 perches, and a square field containing 676 perches. If the side of the square field is equal to the diagonal of the rectangular one, what are the dimensions of each?

6. A man bought a number of horses for $3600; had he bought five more at $5 apiece less, they would have cost him $225 more. How many did he buy, and at what price?

7. A and B each worked as many days as he received dollars a day, and together received $89. Had A worked as many days as B, and B as many as A, they would have received only $80. How long did each labor, and what did he receive per day?

8. The product of two numbers exceeds the square root of the product by 30, and the quotient exceeds the square root of the quotient by $^3/_4$. What are the numbers?

9. There is a number consisting of two digits; the sum of the squares of the digits exceeds their product by 21, and if 9 be added to the number the digits will change places. Required the number.

10. A man and boy worked at one time as many weeks as the man earned dollars a week, and received $700; at another time as many weeks as the boy earned dollars per week, and received $525. How much did each earn per week?

11. A man has a rectangular lot containing 1 acre; if it were 4 rods longer and 4 rods narrower, it would contain only $^3/_4$ of an acre. What are the dimensions of his lot?

12. The fore-wheel of a carriage makes 88 revolutions more in going a mile than the hind-wheel; but if the circumference of the fore-wheel be diminished 1 foot, it will make $146\,^2/_3$ revolutions more than the hind-wheel. What is the circumference of each wheel?

13. A and B each invested $182 in wheat, A receiving 10 bushels more than B. Had A paid 5 cents a bushel more for his, and B 5 cents a bushel less for his, they would have received the same amount. At what prices did they buy?

14. A man sculled 24 miles down a river and back again. He found that it took him 8 hours longer to return than to go, and that his rate down was 3 times his rate returning. What was his rate of sculling in still water, and what was the rate of the current?

15. The area of a rectangle is 4 feet less than the area of a square of equal perimeter, and the length is $^6/_5$ of the breadth. Required the side of the square.

16. The greater of two numbers divided by their sum, added to the smaller divided by their difference, gives $3\,^4/_7$, and the difference of their cubes is 4625. Required the numbers.

17. The difference between the hypotenuse and the base of a right triangle is 6, and the difference between the hypotenuse and the perpendicular is 3. What is the length of each side?

18. A man invested equal sums of money in 6% and 7% stocks, paying \$10 a share more for the latter and receiving 10 shares less. The income of the latter was \$20 more than of the former. What was the sum invested, and the price of the shares?

19. A drover sold 10 horses and 7 cows for \$800. He sold 5 cows more for \$160 than he did horses for \$198. At what price did he sell each?

20. A and B start together on a journey of 36 miles. A travels one mile per hour faster than B, and arrives 3 hours before him. Find the rate of each.

21. Two partners gained \$140 by trade. A's money was in trade 3 months, and his gain was \$60 less than his stock; B's money was \$50 more than A's, and was in trade 5 months. What was A's stock and what was B's gain?

Negative Results.

221. Some questions, evidently intended to be taken in an arithmetical sense, give rise when solved by algebra to negative results. How shall these results be interpreted?

1. A negative result may arise from an erroneous statement of a condition.

Illustration.—A certain number increased by 5 equals $^3/_4$ of the number, diminished by 3. Required the number.

Solution: Let $x =$ the number;

then $x + 5 = \frac{3}{4}x - 3$;

or, $4x + 20 = 3x - 12$;

whence, $x = -32$.

In an algebraic sense this result may be verified, but in an arithmetical sense it is meaningless. The condition of the question as stated is erroneous. If it be modified to read, "A certain number *diminished* by 5 equals $^3/_4$ of the number, *increased* by 3," the result will be 32, which is consistent and intelligible.

2. A negative result may arise from an erroneous statement of a question.

Illustration.—A man is 40 years old, and his brother is 25. When will he be *twice* as old as his brother?

Solution: Let $x =$ the number of years hence;
then $40 + x = 2(25 + x) = 50 + 2x$;
whence, $x = -10$.

In an arithmetical sense this result is not intelligible. If the question be asked, "When was he twice as old as his brother?" the result will be 10 years, which will satisfy the question.

3. A negative result may arise from an erroneous supposition made in the solution of a question.

Illustration.—A man sold his horse for \$80; had he sold him for \$40 more, he would have gained 20%. Required his gain or loss.

Solution: Let $x =$ his gain;
then $80 - x =$ the cost,
and $\frac{1}{5}(80 - x) =$ the gain by second condition;
whence $\frac{6}{5}(80 - x) =$ the selling-price by second condition.
$\therefore \quad \frac{6}{5}(80 - x) = \$120$,
and $x = -\$20$.

In an algebraic sense, gaining $-\$20$ is equivalent to losing \$20. Had x been assumed equal to the loss, the result would have been \$20.

4. Negative results in examples involving quadratics are generally the numerical equivalents of positive results in analogous examples.

Illustrations.—1. A man has a square board, such that the number of inches in length added to the number of square inches in the area equals 12. Required the length.

Solution: Let $x =$ the length;
then $x^2 =$ the area,
and $x^2 + x = 12$;
whence $x = 3$ or -4.

$x = 3$ satisfies the question as stated. $x = -4$ indicates that the number of inches should be arithmetically subtracted from (algebraically added to) the area.

2. If to 280 more than the square of my age you add 34 times my age, the result will be *zero*. Required my age.

Solution: Let $x = \text{my age};$

then $x^2 + 280 + 34x = 0;$

or, $x^2 + 34x = -280;$

whence $x = -14$ or $-20.$

No value of x will arithmetically satisfy the question as stated. But, since x^2 is positive and $34x$ is negative for these values of x, the analogous question, "If from 280 more than the square of my age you *subtract* 34 times my age, the result will be zero; required my age," will be satisfied by $x = 14$ or 20.

EXERCISE 110.

Solve and interpret the results of the following examples:

1. What number increased by 7 equals 5?

2. 12 diminished by what number equals 20?

3. A man is 40 years old and his son is 20. In how many years will the father be $2\frac{1}{2}$ times as old as the son?

4. A line 40 feet long was cut into two parts, such that one part increased by 20 feet equaled the other part diminished by 30 feet. Required the length of each piece.

5. The numerator of a fraction is 2 greater than the denominator, and if 9 be added to both terms the result will equal 2. What is the fraction?

6. Two thirds of A's age increased by 12 years equals $\frac{2}{5}$ of his age. What is his age?

7. The square of a number diminished by the number is 12. Required the number.

8. The product of the sum and difference of two numbers is 17, and one of the numbers is 9. What is the other number?

9. A garden, 40 yards long and 30 yards wide, has a gravel-walk along its perimeter that occupies $\frac{13}{48}$ of the garden. Required its width. *Ans.*, $2\frac{1}{2}$ yd. or $32\frac{1}{2}$ yd.

Interpret the meaning of $32\frac{1}{2}$ yards.

CHAPTER VI.

EXPONENTS, RADICALS, AND INEQUALITIES.

Fractional and Negative Exponents.

Principles and Applications.

222. We learned [P. 75] that dividing the exponent of a factor by the index of a root extracts the root of the factor. If this principle be accepted as general in its character, and applied when the exponent of the factor is not divisible by the index of the root, the result will be a quantity with a fractional exponent. Thus,

$\sqrt[3]{a^2} = a^{\frac{2}{3}}$, read a, exponent *two thirds.*

223. A fractional exponent, from the nature of its origin, denotes *a root of a power,* the numerator being the exponent of the power, and the denominator the index of the root.

224. Since

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$ [223] $= \sqrt[n]{a \times a \times a \times \ldots \text{ to } m \text{ factors}} =$ $\sqrt[n]{a} \times \sqrt[n]{a} \times \sqrt[n]{a} \times \ldots$ to m factors [P. 76] $= (\sqrt[n]{a})^m$, it follows that a fractional exponent may also be regarded as denoting *a power of a root,* the numerator still being the exponent of the power, and the denominator the index of the root.

SIGHT EXERCISE.

Name the value of :

1. $8^{\frac{2}{3}}$	**3.** $(a^4)^{\frac{1}{2}}$	**5.** $(-x^3)^{\frac{2}{3}}$	**7.** $32^{\frac{2}{5}}$
2. $(-8)^{\frac{2}{3}}$	**4.** $16^{\frac{3}{4}}$	**6.** $27^{\frac{1}{3}}$	**8.** $(b^4)^{\frac{3}{2}}$
9. $\left(\frac{1}{16}\right)^{\frac{3}{4}}$	**10.** $\left(\frac{16}{81}\right)^{\frac{1}{4}}$	**11.** $\left(\frac{8}{27}\right)^{\frac{2}{3}}$	

Name the equivalents of the following quantities :

12. $\sqrt{a}$	**15.** $\sqrt[5]{a^3}$	**18.** $x^{\frac{2}{3}}$	**21.** $(\sqrt[5]{a})^3$
13. $\sqrt[3]{a}$	**16.** $(\sqrt[3]{a})^2$	**19.** $x^{\frac{m}{n}}$	**22.** $(\sqrt[4]{a})^5$
14. $7^{\frac{2}{5}}$	**17.** $4^{\frac{3}{4}}$	**20.** $3^{\frac{5}{6}}$	**23.** $(\sqrt[3]{4})^2$

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225. Since $a^{\frac{8}{12}} = \sqrt[12]{a^8} = \sqrt[3]{\sqrt[4]{a^8}} = \sqrt[3]{a^2} = a^{\frac{2}{3}}$, it follows that $a^{\frac{2}{3}}$ and $a^{\frac{8}{12}}$ are equivalent. Therefore,

Prin. 89.*—Multiplying or dividing both terms of a fractional exponent by the same quantity does not change its value.*

226. Since $a^{\frac{2}{3}} = a^{\frac{8}{12}}$ [P. 89], and $a^{\frac{3}{4}} = a^{\frac{9}{12}}$ [P. 89], $a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{8}{12}} \times a^{\frac{9}{12}} = (\sqrt[12]{a})^8 \times (\sqrt[12]{a})^9 = (\sqrt[12]{a})^{17} = a^{\frac{17}{12}} = a^{\frac{2}{3}+\frac{3}{4}}$, it follows that,

Prin. 90.*—The exponent of a factor in the product equals the sum of the exponents of the same factor in the multiplicand and multiplier, when the exponents are positive fractions.*

227. Since $a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{17}{12}}$ [P. 90], it follows that $a^{\frac{17}{12}} \div a^{\frac{3}{4}} = a^{\frac{2}{3}} = a^{\frac{17}{12}-\frac{3}{4}}$. Therefore,

Prin. 91.*—The exponent of a factor in the quotient equals the exponent of the same factor in the dividend, minus the exponent of that factor in the divisor, when the exponents are positive fractions.*

SIGHT EXERCISE.

Complete the following expressions :

1. $a^{\frac{2}{3}} = a^{\frac{}{12}}$, $a^{\frac{3}{4}} = a^{\frac{}{12}}$, $a^{\frac{1}{2}} = a^{\frac{}{12}}$, and $a^{\frac{5}{6}} = a^{\frac{}{12}}$

2. $a^{\frac{3}{4}} = a^{\frac{}{8}} = a^{\frac{}{16}} = a^{\frac{}{20}} = a^{\frac{}{24}} = a^{\frac{}{28}} = a^{\frac{}{32}} = a^{\frac{}{36}}$

3. $a^{\frac{9}{12}} = a^{\frac{}{4}}$; $a^{\frac{21}{24}} = a^{\frac{}{8}}$; $a^{\frac{25}{30}} = a^{\frac{}{6}}$; $a^{\frac{10}{25}} = a^{\frac{}{5}}$

4. $a^{\frac{1}{2}} \times a^{\frac{1}{3}} = ?$ $x^{\frac{2}{3}} \times x^{\frac{5}{6}} = ?$ $x^{\frac{1}{2}} \times x^{\frac{1}{3}} \times x^{\frac{1}{4}} = ?$ $x^{\frac{5}{9}} \times x^{\frac{2}{3}}$?

5. $a^{\frac{5}{6}} \div a^{\frac{2}{3}} = ?$ $c \div c^{\frac{2}{5}} = ?$ $c^{\frac{11}{12}} \div c^{\frac{1}{4}} = ?$ $c^{\frac{5}{2}} \div c^{2} = ?$

6. $a^{\frac{7}{8}} \div a^{\frac{3}{4}} = ?$ $(x^{\frac{2}{3}} \times x^{\frac{5}{6}}) \div x^{\frac{3}{4}} = ?$ $(x^{\frac{1}{2}} \div x^{\frac{1}{4}}) \times x^{\frac{3}{4}} = ?$

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228. We learned [P. 12, 91] that the exponent of a factor in the quotient equals the exponent of the same factor in the dividend minus the exponent of that factor in the divisor. If this principle be accepted as true when the exponent of the divisor exceeds that of the dividend, *negative exponents* will arise from its application.

Thus, $a^5 \div a^8 = a^{-3}$, read a, exponent minus 3 ; also, $a^{\frac{2}{3}} \div a^{\frac{5}{6}} = a^{-\frac{1}{6}}$, read a, exponent minus one sixth.

229. $a^{3n} \div a^{5n} = a^{-2n}$ [228] ; but

$$a^{3n} \div a^{5n} = 1 \div a^{2n} \text{ [P. 53]} = \frac{1}{a^{2n}}$$

$$\therefore \quad a^{-2n} = \frac{1}{a^{2n}} \text{ [Ax. 1]. Therefore,}$$

Prin. 92.*—A quantity affected by a negative exponent equals the reciprocal of the quantity affected by a numerically equal positive exponent.*

230. $a^{5n} \div a^{3n} = a^{2n}$ [P. 12] ; but

$$a^{5n} \div a^{3n} = 1 \div a^{-2n} \text{ [P. 53]} = \frac{1}{a^{-2n}}$$

$$\therefore \quad a^{2n} = \frac{1}{a^{-2n}} \text{ [Ax. 1]. Therefore,}$$

Prin. 93.*—A quantity affected by a positive exponent equals the reciprocal of the quantity affected by a numerically equal negative exponent.*

SIGHT EXERCISE.

1. Find the value in negative exponents of:

$$a^2 \div a^3;\ x^3 \div x^6;\ x \div x^3;\ a^{\frac{1}{2}} \div a^{\frac{5}{6}};\ c^{\frac{2}{3}} \div c^{\frac{5}{6}}$$

2. Express in positive exponents:

$$a^{-4},\ a^{-6},\ x^{-\frac{2}{3}},\ x^{-\frac{5}{6}},\ 2^{-2},\ 3^{-1},\ (x\,y)^{-5},\ \left(\frac{x}{y}\right)^{-3}$$

3. Express in the integral form:

$$\frac{1}{a^2},\ \frac{1}{x^3},\ \frac{1}{2^{-2}},\ \frac{1}{x^{-5}},\ \frac{1}{x^4},\ \frac{1}{x^{-n}}$$

---

231. Since

$$\frac{a^2\,b^{-2}}{c^{-3}\,d} = \frac{a^2 \times \frac{1}{b^2}}{\frac{1}{c^3} \times d}\ [\text{P. } 92] = \frac{\frac{a^2}{b^2}}{\frac{d}{c^3}} = \frac{a^2}{b^2} \times \frac{c^3}{d} = \frac{a^2\,c^3}{b^2\,d},$$

it follows that,

***Prin. 94.**—A factor may be transferred from either term of a fraction to the other if the sign of its exponent be changed.*

232. Since

$$a^{-4} \times a^{-5} = \frac{1}{a^4} \times \frac{1}{a^5}\ [\text{P. } 92] = \frac{1}{a^9} = a^{-9}\ [\text{P. } 94],\ \text{and}$$

$$a^{-\frac{2}{3}} \times a^{-\frac{3}{4}} = \frac{1}{a^{\frac{2}{3}}} \times \frac{1}{a^{\frac{3}{4}}}\ [\text{P. } 92] = \frac{1}{a^{\frac{17}{12}}} = a^{-\frac{17}{12}}\ [\text{P. } 94],$$

it follows that,

***Prin. 95.**—The exponent of a factor in the product equals the sum of the exponents of the same factor in the multiplicand and the multiplier when the exponents are negative.*

233. Since

$$a^{-6} \div a^{-3} = \frac{1}{a^6} \div \frac{1}{a^3}\ [\text{P. } 92] = \frac{1}{a^6} \times \frac{a^3}{1} = \frac{1}{a^3} = a^{-3};\ \text{and}$$

$$a^{-\frac{5}{6}} \div a^{-\frac{2}{3}} = \frac{1}{a^{\frac{5}{6}}} \div \frac{1}{a^{\frac{2}{3}}} = \frac{1}{a^{\frac{5}{6}}} \times \frac{a^{\frac{2}{3}}}{1} = \frac{1}{a^{\frac{1}{6}}} = a^{-\frac{1}{6}},$$

it follows that,

Prin. 96.—*The exponent of a factor in the quotient equals the exponent of the same factor in the dividend, minus the exponent of that factor in the divisor, when the exponents are negative.*

SIGHT EXERCISE.

1. Clear of negative exponents:

$$\frac{a^{-2}}{b^{-3}};\ \frac{a^2}{b^{-2}};\ \frac{a^{-3}}{b^2};\ \frac{a^{-3}b}{c^{-2}d};\ \frac{x^{-\frac{2}{3}}}{y^{-\frac{3}{4}}};\ \frac{xy^{-1}}{z^{-1}};\ \frac{a^{-2}b^{-3}}{2};\ \frac{a^{\frac{2}{3}}b}{c^{-\frac{5}{6}}}$$

2. Express in the integral form:

$$\frac{a}{b};\ \frac{a^{-2}}{b^2};\ \frac{a}{b^3};\ \frac{3a^2}{b^{-4}};\ \frac{ax}{b^{-1}c^{-1}};\ \frac{x^{-4}}{y^{-4}};\ \frac{mn^3}{p^{-5}q^{-3}}$$

3. Express in the integral form:

$$\frac{a}{c},\ \frac{a^2}{x^3},\ \frac{a^{-2}}{x^{-3}},\ \frac{a^2b}{c^{-4}},\ \frac{ab}{cd},\ \frac{1}{a^{-3}b^3},\ \frac{2}{a^{-3}},\ \frac{2}{2^{-2}}$$

4. Express in positive exponents:

$$\frac{a^{-4}}{b^{-2}},\ \frac{a^{-2}b^2}{c^{-3}},\ \frac{x^{-4}y^4}{z^2},\ \frac{2^{-1}\times 3}{5^{-1}},\ \frac{(a+b)^{-2}}{(a+b)^{-3}}$$

5. Find the numerical value of:

$$4^{-\frac{3}{2}},\ \frac{1}{4^{\frac{5}{2}}},\ \frac{2^{\frac{3}{4}}\times 2^{\frac{2}{8}}}{2^{-1}},\ 8^{-\frac{3}{4}},\ \frac{5^{\frac{5}{2}}\times 5^{-\frac{1}{2}}}{5^{-2}},\ (64)^{-\frac{1}{3}}\div 8^{\frac{2}{3}}$$

6. Find the value of:

$$a^5\times a^{-3};\ a^2\times a^{-5};\ x^{-\frac{2}{3}}\times x^{-\frac{1}{2}};\ a^{-3}\times a^{-4};$$
$$a^{-2}\times a^{-4};\ x^{-\frac{5}{6}}\times x^{-\frac{1}{6}};\ 3^{-2}\times 3^{-2};\ 4^{-2}\times 4^3$$

7. Find the value of:

$$x^2\div x^{-4};\ a^{-6}\div a^{-3};\ a^{-3}\div a^{-6};\ a^{-\frac{1}{2}}\div a^{-\frac{1}{3}};$$
$$a^{-\frac{1}{3}}\div a^{-\frac{1}{2}};\ x^{-2}\div x^{-\frac{5}{2}};\ 2^{-4}\div 2^{-6};\ 2^{-\frac{5}{6}}\div 2^{-2}$$

8. Find the value in positive exponents of:

$$a^{-3}\times a^{-5};\ a^{-3}\div a^{-5};\ a^{-\frac{1}{2}}\times a^{\frac{1}{3}};\ a^{-\frac{1}{2}}\div a^{\frac{1}{3}}$$
$$\frac{a^3}{a^{-2}};\ \frac{x^{-2}y^2}{x^2y^{-2}};\ \frac{(a-b)^{-2}}{(a-b)^2};\ \frac{(x+y)^{\frac{2}{3}}}{(x+y)^{\frac{5}{6}}};\ \frac{p^{-2}q^3}{p^{-1}q^{-1}}$$

2. General Principles.

234. 1. $a^m = a \times a \times a \times \ldots$ to m factors.

$a^n = a \times a \times a \times \ldots$ to n factors.

$\therefore\ a^m \times a^n = (a \times a \times a \ldots$ to m factors$) \times (a \times a \times a \ldots$ to n factors$) = a \times a \times a \ldots$ to $(m+n)$ factors $= a^{m+n}$

2. $a^{\frac{p}{q}} = a^{\frac{pn}{qn}}$ [P. 89] $= (\sqrt[qn]{a})^{pn}$ [224]

$a^{\frac{m}{n}} = a^{\frac{mq}{qn}}$ [P. 89] $= (\sqrt[qn]{a})^{mq}$ [224]

$\therefore\ a^{\frac{p}{q}} \times a^{\frac{m}{n}} = (\sqrt[qn]{a})^{pn} \times (\sqrt[qn]{a})^{mq} = (\sqrt[qn]{a})^{pn+mq}$ [1] $=$ $a^{\frac{pn+mq}{qn}}$ [P. 75] $= a^{\frac{p}{q}+\frac{m}{n}}$

3. $a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n}$ [P. 92] $= \frac{1}{a^{m+n}}$ [1] $=$ $a^{-(m+n)}$ [P. 94]

4. $a^{-\frac{p}{q}} \times a^{-\frac{m}{n}} = \frac{1}{a^{\frac{p}{q}}} \times \frac{1}{a^{\frac{m}{n}}} = \frac{1}{a^{\frac{pn+mq}{qn}}}$ [2] $=$ $a^{-\left(\frac{pn+mq}{qn}\right)}$ [P. 94] $= a^{-\left(\frac{p}{q}+\frac{m}{n}\right)}$

Therefore,

Prin. 97. $a^x \times a^y = a^{x+y}$ *for any positive or negative, integral or fractional, values of x and y.*

235. $a^m \div a^n = a^{m-n}$, since $a^{m-n} \times a^n = a^m$ [P. 97]

$a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}}$, since $a^{\frac{p}{q}-\frac{r}{s}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}}$ [P. 97]

$a^{-m} \div a^{-n} = a^{n-m}$, since $a^{n-m} \times a^{-n} = a^{-m}$ [P. 97]

$a^{-\frac{p}{q}} \div a^{-\frac{r}{s}} = a^{\frac{r}{s}-\frac{p}{q}}$, since $a^{\frac{r}{s}-\frac{p}{q}} \times a^{-\frac{r}{s}} = a^{-\frac{p}{q}}$ [P. 97]. Therefore,

Prin. 98. $a^x \div a^y = a^{x-y}$ *for any positive or negative, integral or fractional, values of x and y.*

Scholium.*—Principles 97 and 98 are stated in general language, although the cases in which one of the quantities considered has a positive and the other a negative exponent are omitted in the demonstrations. Let the pupil supply the omissions.*

236. If we let x be general, and let $y = n$, $\frac{p}{q}$, $-n$, and $-\frac{p}{q}$ successively, then will

1. $(a^x)^n = a^x \times a^x \times a^x \ldots.$ to n factors $= a^{xn}$ [P. 97]

2. $(a^x)^{\frac{p}{q}} = \sqrt[q]{(a^x)^p}$ [223] $= \sqrt[q]{a^{xp}}$ [1] $= a^{\frac{xp}{q}}$ [P. 75] $= a^{x \times \frac{p}{q}}$

3. $(a^x)^{-n} = \frac{1}{(a^x)^n}$ [P. 92] $= \frac{1}{a^{xn}}$ [1] $= a^{-xn}$ [P. 94] $= a^{x \times (-n)}$

4. $(a^x)^{-\frac{p}{q}} = \frac{1}{(a^x)^{\frac{p}{q}}}$ [P. 92] $= \frac{1}{a^{\frac{xp}{q}}}$ [3] $= a^{-\frac{xp}{q}}$ [P. 94] $= a^{x \times \left(-\frac{p}{q}\right)}$

Therefore,

Prin. 99. *$(a^x)^y = a^{xy}$ for any positive or negative, integral or fractional, values of x and y.*

237. 1. $(ab)^n = ab \times ab \times ab \ldots.$ to n factors $=$ $(a \times a \times a \ldots.$ to n factors$) \times (b \times b \times b \ldots.$ to n factors$) = a^n \times b^n$

2. $(ab)^{\frac{p}{q}} = \sqrt[q]{(ab)^p}$ [223] $= \sqrt[q]{a^p b^p}$ [1] $= a^{\frac{p}{q}} \times b^{\frac{p}{q}}$ [P. 76]

3. $(ab)^{-n} = \frac{1}{(ab)^n}$ [P. 92] $= \frac{1}{a^n \times b^n}$ [1] $= a^{-n} \times b^{-n}$ [P. 94]

4. $(ab)^{-\frac{p}{q}} = \frac{1}{(ab)^{\frac{p}{q}}}$ [P. 92] $= \frac{1}{a^{\frac{p}{q}} \times b^{\frac{p}{q}}}$ [2] $=$ $a^{-\frac{p}{q}} \times b^{-\frac{p}{q}}$ [P. 94]

Therefore,

Prin. 100. *$(ab)^x$ and $a^x \times b^x$ are equivalent for any positive or negative, integral or fractional, values of x.*

238. If we let x be general, as in the preceding articles, we have,

$$\left(\frac{a}{b}\right)^x = (ab^{-1})^x \text{ [P. 94]} = a^x \times b^{-x} \text{ [P. 100]} = \frac{a^x}{b^x} \text{ [P. 94]}.$$

Therefore,

Prin. 101. *$\left(\frac{a}{b}\right)^x$ and $\frac{a^x}{b^x}$ are equivalent for any positive or negative, integral or fractional, values of x.*

Miscellaneous Examples.

EXERCISE 111.

1. Find the value of $8^{\frac{2}{3}}$, $16^{\frac{3}{4}}$, $27^{\frac{5}{3}}$, and $32^{\frac{4}{5}}$

2. Find the value of $(-64)^{\frac{2}{3}}$, $(-125)^{\frac{4}{3}}$, and $(-32)^{\frac{2}{5}}$

3. Find the value of $(27)^{-\frac{2}{3}}$, $(625)^{-\frac{3}{4}}$, and $(-64)^{-\frac{4}{3}}$

4. Find the value of $4^{\frac{3}{2}} \times 4^{\frac{5}{2}}$, $2^{\frac{1}{2}} \times 8^{\frac{1}{2}}$, and $3^{\frac{2}{3}} \times 9^{\frac{2}{3}}$

5. Find the value of $(2 \times 8)^{\frac{1}{4}}$, $(3 \times 9)^{\frac{2}{3}}$, $(4 \times 8)^{\frac{2}{5}}$, and $(2 \times 2 \times 2)^{\frac{4}{3}}$

6. Find the value of $(4 \times 16 \times 25)^{\frac{3}{2}}$, $(8 \times 27 \times 64)^{\frac{2}{3}}$

7. Simplify $(x^{-3})^{\frac{4}{3}}$; $x^{-\frac{3}{4}} \times x^{\frac{4}{3}}$; $x^{-\frac{5}{6}} \times (a x^2)^{\frac{3}{4}}$

8. Divide $a^2 x^{-\frac{3}{4}}$ by $a x^{\frac{5}{6}}$; $x^{-\frac{4}{5}} y^{\frac{3}{4}}$ by $x^{\frac{2}{5}} y^{\frac{5}{6}}$

9. Express in positive exponents $a x^{-2}$; $a^{-2} b^{-3} c^4$; and $(x^{-2})^{-3}$

10. Reduce to lowest terms: $\dfrac{a^5 x^{\frac{3}{4}} y^{\frac{1}{2}}}{a^{-4} x^{\frac{1}{2}} y^{\frac{3}{4}}}$; $\dfrac{x^{-\frac{4}{5}} y^{\frac{3}{2}} z^{-\frac{2}{3}}}{x^{\frac{1}{5}} y^{-1} z^{\frac{5}{6}}}$

11. Expand $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$; $(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2$; $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$

12. Expand $(x^{\frac{2}{3}} + y^{\frac{2}{3}})^2$; $(x^{\frac{5}{6}} - y^{\frac{5}{6}})^2$; $(a^{\frac{2}{3}} + b^{-\frac{2}{3}})(a^{\frac{2}{3}} - b^{-\frac{2}{3}})$

13. Expand $(x^{\frac{1}{3}} + y^{\frac{1}{3}})^3$; $(x^{-2} - y^{-2})^3$; $\left(x^{-2} + \dfrac{1}{x^{-2}}\right)^3$

14. Multiply $x^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$

15. Multiply $x^{-2} - y^{-2} + x^{-1} y^{-1}$ by $x^2 + y^2$

16. Multiply $x^{\frac{4}{5}} - x^{\frac{3}{5}} y^{\frac{1}{5}} + x^{\frac{2}{5}} y^{\frac{2}{5}} - x^{\frac{1}{5}} y^{\frac{3}{5}} + y^{\frac{4}{5}}$ by $x^{\frac{1}{5}} + y^{\frac{1}{5}}$

17. Divide $x^{\frac{3}{4}} - y^{\frac{3}{4}}$ by $x^{\frac{1}{4}} - y^{\frac{1}{4}}$; $x^{-2} - y^{-4}$ by $x^{-1} + y^{-2}$

18. Divide $x^{\frac{4}{3}} + x + x^{\frac{1}{3}} + 1$ by $x^{\frac{2}{3}} + 2 x^{\frac{1}{3}} + 1$

19. Multiply $(a^2 + x^2)^2$ by $(a^2 - x^2)^2$; $(x^3 + 1)^{\frac{1}{2}}$ by $(x^3 - 1)^{\frac{1}{2}}$

20. Multiply $(x^2 + x + 1)^{\frac{1}{4}}$ by $(x^2 - x + 1)^{\frac{1}{4}}$;
$x^{-2} + y^{-2}$ by $x^{-1} + y^{-1}$

21. Resolve into two factors $a - b$; $x + 2 x^{\frac{1}{2}} y^{\frac{1}{2}} + y$

22. Factor $a^{\frac{2}{5}} - b^{\frac{2}{5}}$; $a^{\frac{3}{4}} + b^{\frac{3}{4}}$; $a^{\frac{3}{7}} - b^{\frac{3}{7}}$

23. Expand $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^4$; $(a^{-2} - b^{-2})^4$; $(a^{\frac{3}{2}} - a^{-\frac{3}{2}})^5$

24. Express in simplest form with positive exponents:

$$\frac{(a+b)^2(a-b)^2}{(a^2-b^2)^2}; \quad \frac{(a^3+b^3)^2(a^3-b^3)^2}{a^6-b^6}; \quad \frac{(a^{\frac{1}{2}}-b^{\frac{1}{2}})^{\frac{1}{2}}(a^{\frac{1}{2}}+b^{\frac{1}{2}})^{\frac{1}{2}}}{(a-b)^{\frac{3}{2}}}$$

Radicals.

Definitions and Principles.

239. Any quantity affected by the radical sign or a fractional exponent is a *radical;* as, $\sqrt{4}$, $\sqrt[3]{a}$, $a^{\frac{2}{3}}$.

240. A quantity without a root sign, or one whose indicated root can be exactly obtained, is a *rational quantity.*

241. A radical that can not be reduced to a rational quantity is an *irrational radical,* or a *surd;* as, $\sqrt{3}$, $\sqrt[4]{5}$, $\sqrt[3]{a}$.

242. A surd that expresses an even root of a negative quantity is an *imaginary surd,* or *imaginary quantity;* as, $\sqrt{-5}$, $\sqrt{-a^2}$.

243. Any quantity that is not imaginary is a *real quantity;* as, 5, $-a$, $\pm\sqrt{a^2}$, $\pm\sqrt{b}$.

244. A factor placed before a radical to show how many times it is taken is the *coefficient* of the radical.

245. A radical that has no factor whose indicated root may be found is a *pure radical;* as, $\sqrt{7ab}$.

246. A radical that has one or more factors whose indicated root may be found is a *mixed radical;* as, $\sqrt{4a^2b}\ (= \pm 2a\sqrt{b})$.

247. The degree of a radical is denoted by the index of the indicated root. Thus, $\sqrt[3]{5}$ is of the third degree.

248. A radical is in its simplest form when it is pure and is in the lowest degree to which it can be reduced.

249. Radicals are similar if they contain the same surd factor when they are made pure. Thus, $\sqrt{4a^2b}$, which equals $\pm 2a\sqrt{b}$, is similar to $3\sqrt{a^2b}$, which equals $3a\sqrt{b}$.

250. Since $(ab)^{\frac{1}{n}}$ and $a^{\frac{1}{n}} \times b^{\frac{1}{n}}$ are equivalent [P. 100], it follows that $\sqrt[n]{ab}$ and $\sqrt[n]{a} \times \sqrt[n]{b}$ are equivalent.

Therefore,

Prin. 102.—*Any root of the product of two quantities equals the product of the like roots of those quantities.*

Prin. 103.—*The product of the equal roots of two quantities equals the like root of their product.*

251. Since $\left(\frac{a}{b}\right)^{\frac{1}{n}}$ and $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$ are equivalent [P. 101], it follows that $\sqrt[n]{\frac{a}{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ are equivalent. Therefore,

Prin. 104.—*Any root of the quotient of two quantities equals the quotient of the like roots of those quantities.*

Prin. 105.—*The quotient of the equal roots of two quantities equals the like root of their quotient.*

252. Since $\sqrt[n]{\frac{a}{b}} = \sqrt[n]{\frac{a}{b} \times \frac{b^{n-1}}{b^{n-1}}}$ [P. 56] $= \sqrt[n]{\frac{a\,b^{n-1}}{b^n}} = \sqrt[n]{\frac{a\,b^{n-1}}{b}} = \frac{1}{b}\sqrt[n]{a\,b^{n-1}}$, it follows that,

Prin. 106.—*No fractional radical is pure.*

253. Since $a = \sqrt[n]{a^n}$, it follows that,

Prin. 107.—*Any quantity equals the nth root of the nth power of the quantity.*

254. Since $(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}}$ [P. 99], we have,

Prin. 108.—*The* $\sqrt[m]{\sqrt[n]{a}}$ *and the* $\sqrt[mn]{a}$ *are equivalent.*

Reduction of Radicals.

Problems.

1. Mixed to pure radicals.

Illustrations.—

Reduce $\sqrt{8a^3b}$ and $\sqrt[3]{\frac{24}{25}}$ to pure radicals.

Solutions:

1. $\sqrt{8a^3b} = \sqrt{4a^2 \times 2ab} = \sqrt{4a^2} \times \sqrt{2ab}$ [P. 102] $= \pm 2a\sqrt{2ab}.$

2. $\sqrt[3]{\frac{24}{25}} = \sqrt[3]{\frac{24}{25} \times \frac{5}{5}} = \sqrt[3]{\frac{120}{125}} = \sqrt[3]{\frac{8}{125} \times 15} = \sqrt[3]{\frac{8}{125}} \times \sqrt[3]{15}$ [P. 102] $= \frac{2}{5}\sqrt[3]{15}.$

EXERCISE 112.

Reduce to pure radicals:

1. $\sqrt{12}$, $\sqrt{18}$, and $\sqrt{32}$
2. $\sqrt{45}$, $\sqrt{48}$, and $\sqrt{75}$
3. $\sqrt[3]{16}$, $\sqrt[3]{-54}$, and $\sqrt[3]{128}$
4. $\sqrt[3]{-56}$, $\sqrt[3]{108}$, and $\sqrt[3]{135}$
5. $\sqrt{4a^3}$, $\sqrt{16b^5}$, and $\sqrt{8c^4}$
6. $\sqrt{16ab^2}$, $\sqrt{18x^3y}$, and $\sqrt[3]{-8a^4}$
7. $\sqrt{50a^3b^5}$, $\sqrt{45x^2y^5}$, and $\sqrt[3]{27x^4y^2}$
8. $\sqrt{16a^5b^7}$, $\sqrt[3]{-x^{10}y^8z^4}$, and $\sqrt[4]{x^9y^4z^6}$
9. $\sqrt{a^2(a+b)}$ and $\sqrt{a(a+b)^2}$
10. $\sqrt{x(x+y)^3}$ and $\sqrt[3]{x(x+y)^3}$
11. $\sqrt{(x+y)^2(x-y)^3}$ and $\sqrt[3]{x^5y^2(x+y)^4}$
12. $\sqrt{x^3+2x^2y+xy^2}$ and $\sqrt[3]{x(x+y)^5}$
13. $\sqrt{4(x^2-y^2)(x+y)}$
14. $\sqrt{a^2b^3(a^3-b^3)(a-b)}$
15. $\sqrt{\frac{2}{3}}$, $\sqrt{\frac{5}{9}}$, and $\sqrt[3]{\frac{4}{7}}$

16. $\sqrt{\frac{3}{4}}$, $\sqrt[3]{\frac{8}{3}}$, and $\sqrt[3]{\frac{4}{5}}$

17. $\sqrt{\frac{5}{8}}$, $\sqrt{\frac{3}{10}}$, and $\sqrt[3]{\frac{5}{6}}$

18. $\sqrt{\frac{8}{9}}$, $\sqrt[3]{\frac{16}{27}}$, and $\sqrt[3]{\frac{27}{16}}$

19. $\sqrt{\frac{2}{5}a^3}$, $\sqrt{\frac{3}{4}x^5}$, and $\sqrt[3]{\frac{5}{8}x^7}$

20. $\sqrt{\frac{a}{x}}$, $\sqrt{\frac{m^2}{n}}$, and $\sqrt[3]{-\frac{x}{3}}$

21. $\sqrt{\frac{4x}{5y}}$, $\sqrt{\frac{a+b}{a}}$, and $\sqrt[3]{-\frac{a}{x^2}}$

22. $\sqrt{\frac{a^2b}{c}}$, $\sqrt{\frac{5a}{3b^2}}$, and $\sqrt[3]{\frac{a}{5}}$

23. $\sqrt{\frac{a}{a+b}}$ and $\sqrt{\frac{a-b}{a+b}}$

24. $\sqrt{\frac{x^2-y^2}{x}}$ and $\sqrt{\frac{(x+y)^2}{x}}$

25. $\sqrt[3]{\frac{x^2}{(x+y)^2}}$ and $\sqrt{\frac{x^6}{(x-y)^5}}$

26. $\sqrt{\frac{(x+y)^2}{(x-y)^3}}$ and $\sqrt[3]{\frac{3}{5}\times\frac{x}{z}}$

2. To lower degree.

Illustrations.—

Reduce $\sqrt[4]{9a^2}$ and $\sqrt[6]{64a^3}$ to simplest form.

Solution: 1. $\sqrt[4]{9a^2}=\sqrt{\sqrt{9a^2}}$ [P. 108] $=\sqrt{3a}$.

2. $\sqrt[6]{64a^3}=\sqrt{\sqrt[3]{64a^3}}$ [P. 108] $=\sqrt{8a}=\sqrt{4\times 2a}=$

$\sqrt{4}\times\sqrt{2a}=\pm 2\sqrt{2a}$.

EXERCISE 113.

Reduce to simplest form:

1. $\sqrt[4]{9}$, $\sqrt[6]{64}$, and $\sqrt[9]{27}$

2. $\sqrt[4]{\dfrac{1}{4}}$, $\sqrt[6]{\dfrac{8}{27}}$, $\sqrt[10]{\dfrac{16}{9}}$

3. $\sqrt[4]{16\,x^2\,y^2}$ and $\sqrt[6]{25\,x^4\,y^2}$

4. $\sqrt[6]{x^3\,y^6\,z^9}$ and $\sqrt[8]{a^4\,x^8\,y^{12}}$

5. $\sqrt[4]{\dfrac{a^2}{x^2}}$ and $\sqrt[6]{-\dfrac{a^3}{x^3}}$

6. $\sqrt[10]{-32\,a^5\,y^{10}}$ and $\sqrt[12]{x^8\,y^{12}\,z^{16}}$

7. $\sqrt[8]{81}$, $\sqrt[9]{125}$, and $\sqrt[6]{729}$

8. $\sqrt[4]{\dfrac{9}{25}}$, $\sqrt[8]{\dfrac{16}{81}}$, $\sqrt[12]{\dfrac{81}{256}}$

9. $\sqrt[8]{x^4\,y^6\,z^2}$ and $\sqrt[9]{-8\,x^6\,y^9}$

10. $\sqrt[9]{a^3-3\,a^2\,b+3\,a\,b^2-b^3}$

11. $\sqrt[9]{-\dfrac{8\,x^6}{27\,y^9}}$ and $\sqrt[8]{\dfrac{a^4\,b^8}{c^4}}$

12. $\sqrt[4]{a^2+2\,a\,b+b^2}$

3. Rational to radical quantities.

Illustrations.—

1. Reduce $2\,a$ to a radical of the third degree.

Solution: $2\,a = \sqrt[3]{(2\,a)^3}$ [P. 107] $= \sqrt[3]{8\,a^3}$.

2. Free $2\,a\,\sqrt[3]{2\,a}$ of its coefficient.

Solution:

$2\,a\,\sqrt[3]{2\,a} = \sqrt[3]{(2\,a)^3} \times \sqrt[3]{2\,a} = \sqrt[3]{8\,a^3} \times \sqrt[3]{2\,a} = \sqrt[3]{16\,a^4}$ [P. 103].

EXERCISE 114.

1. Reduce 5, $3\,x$, and $\dfrac{2}{3}x^2$ to radicals of the second degree.

2. Reduce $3\,a$, $5\,b\,x$, and $\dfrac{1}{x}$ to radicals of the third degree.

3. Reduce $a^2\,b^3$, $x\,y^4$, and $\dfrac{a^2}{b^2}$ to radicals of the fifth degree.

4. Free $2\sqrt{5}$, $3\sqrt[3]{3}$, and $\dfrac{1}{2}\sqrt{5}$ of coefficients.

5. Free $a\sqrt{3\,a}$, $x^2\sqrt{3\,x}$, and $\dfrac{1}{x}\sqrt[3]{x^2}$ of coefficients.

6. Free $a(a+b)^{\frac{1}{2}}$ and $(a-b)\sqrt{a+b}$ of coefficients.

7. Reduce x^2, y^3, and 4 to equivalent expressions having an exponent of $\frac{2}{3}$

8. Free $x(y)^{\frac{2}{3}}$ and $a^2(z)^{\frac{5}{6}}$ of coefficients.

9. Free $x^6(y)^3$ and $16(x)^4$ of coefficients.

10. Show that $x^{\frac{4}{3}}\left(1+\frac{y^2}{x^2}\right)^{\frac{2}{3}} = (x^2+y^2)^{\frac{2}{3}}$

4. Reduction to same degree.

Illustration.—

Reduce $\sqrt{3}$, $\sqrt[3]{2}$, and $\sqrt[4]{2}$ to the same degree.

Solution:

$$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{6}{12}} \text{ [P. 89]} = \sqrt[12]{3^6} = \sqrt[12]{729}$$
$$\sqrt[3]{2} = 3^{\frac{1}{3}} = 2^{\frac{4}{12}} \text{ [P. 89]} = \sqrt[12]{2^4} = \sqrt[12]{16}$$
$$\sqrt[4]{2} = 2^{\frac{1}{4}} = 2^{\frac{3}{12}} \text{ [P. 89]} = \sqrt[12]{2^3} = \sqrt[12]{8}$$

Note.—The operation may be shortened by remembering that both index and exponent may be multiplied by the same number [P. 89].

Reduce $\sqrt[3]{a^2}$, $\sqrt[6]{a^5}$, and $\sqrt{a^4}$ to the same degree.

Solution:

$$\sqrt[3]{a^2} = \sqrt[12]{(a^2)^4} \text{ [P. 89]} = \sqrt[12]{a^8}$$
$$\sqrt[6]{a^5} = \sqrt[12]{(a^5)^2} \text{ [P. 89]} = \sqrt[12]{a^{10}}$$
$$\sqrt{a^4} = \sqrt[12]{(a^4)^6} \text{ [P. 89]} = \sqrt[12]{a^{24}}$$

Note.—The common index is the L. C. M. of the given indices. Why?

EXERCISE 115.

Reduce to the same degree :

1. $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[6]{4}$
2. $\sqrt{x^2}$, $\sqrt[4]{xy}$, and $\sqrt[8]{y^2}$
3. $a^{\frac{2}{3}}$, $b^{\frac{1}{6}}$, and $c^{\frac{3}{4}}$
4. $\sqrt{a+b}$ and $\sqrt[3]{a+b}$
5. $\sqrt[3]{\frac{1}{2}}$, $\sqrt{\frac{1}{3}}$, and $\sqrt[4]{\frac{1}{4}}$
6. a, $\sqrt[3]{a^2}$, and $\sqrt[6]{a^3}$
7. $\sqrt[3]{a^2b^2}$, $\sqrt[4]{a^3b}$, and $\sqrt[6]{ab^3}$
8. $\sqrt[4]{x(x+y)}$ and $(x+y)^{\frac{2}{3}}$

Addition and Subtraction of Radicals.

Illustrations.—

1. Find the sum of $4\sqrt{8}$, $5\sqrt{\frac{1}{8}}$, and $\frac{3}{2}\sqrt{\frac{2}{9}}$

Solution:

$$\begin{aligned} 4\sqrt{8} &= 4\sqrt{4\times 2} = 4\times\sqrt{4}\times\sqrt{2} = 8\sqrt{2} \\ 5\sqrt{\tfrac{1}{8}} &= 5\sqrt{\tfrac{2}{16}} = 5\times\sqrt{\tfrac{1}{16}}\times\sqrt{2} = \tfrac{5}{4}\sqrt{2} \\ \tfrac{3}{2}\sqrt{\tfrac{2}{9}} &= \tfrac{3}{2}\sqrt{\tfrac{1}{9}\times 2} = \tfrac{3}{2}\times\sqrt{\tfrac{1}{9}}\times\sqrt{2} = \tfrac{1}{2}\sqrt{2} \\ \text{Sum} &= 9\tfrac{3}{4}\sqrt{2} \end{aligned}$$

2. Find the value of $2\sqrt[3]{-81a^4}+8\sqrt[3]{3a^4}-2a\sqrt[3]{-24a}$.

Solution:

$$\begin{aligned} 2\sqrt[3]{-81a^4} &= 2\sqrt[3]{-27a^3\times 3a} = 2\times\sqrt[3]{-27a^3}\times\sqrt[3]{3a} = -6a\sqrt[3]{3a} \\ +8\sqrt[3]{3a^4} &= +8\sqrt[3]{a^3\times 3a} = +8\times\sqrt[3]{a^3}\times\sqrt[3]{3a} = +8a\sqrt[3]{3a} \\ -2a\sqrt[3]{-24a} &= -2a\sqrt[3]{-8\times 3a} = -2a\times\sqrt[3]{-8}\times\sqrt[3]{3a} = +4a\sqrt[3]{3a} \\ \text{Sum} &= 6a\sqrt[3]{3a} \end{aligned}$$

EXERCISE 116.

Find the value of:

1. $\sqrt{4a}+\sqrt{16a}+\sqrt{36a}$
2. $\sqrt{2}+\sqrt{8}-\sqrt{\frac{1}{2}}$
3. $\sqrt{27x}+\sqrt{12x}-\sqrt{48x}$
4. $2\sqrt[3]{40}+\sqrt[3]{\frac{5}{8}}+\frac{1}{2}\sqrt[3]{135}$
5. $\sqrt{2a^2}+\sqrt{2b^2}+\sqrt{2c^2}$
6. $\sqrt{(a+b)^2x}+\sqrt{(a-b)^2x}$
7. $\sqrt{a^2x}-\sqrt{a^2y}+\sqrt{a^2z}$
8. $\sqrt{\frac{x}{a^2}}+\sqrt{\frac{x}{b^2}}+\sqrt{\frac{x}{c^2}}$
9. $\sqrt{\frac{a+b}{a-b}}+\frac{1}{b}\sqrt{a^2-b^2}$
10. $\sqrt[3]{\frac{a}{b^2}}-\sqrt[3]{\frac{ab}{c^3}}+\sqrt[3]{\frac{b}{a^2}}$
11. $\sqrt{7a^2}+\sqrt{7b^2}-\sqrt{7c^2}$
12. $\sqrt[3]{a^3n^2}+\sqrt[3]{b^6n^2}-\sqrt[3]{c^9n^2}$
13. $2\sqrt{3}+\sqrt[6]{27}+3\sqrt[8]{81}$
14. $2\sqrt{27}+5\sqrt{\frac{3}{4}}+\frac{9}{4}\sqrt{\frac{1}{3}}$
15. $(a^3+2a^2b+ab^2)^{\frac{1}{2}}-(a^3-2a^2b+ab^2)^{\frac{1}{2}}$
16. $\sqrt{a^3-2a^2b+ab^2}\pm\sqrt{a^3+2a^2b+ab^2}$

17. $3\sqrt[3]{\frac{1}{2}}+2\sqrt[6]{\frac{1}{4}}-\sqrt[9]{\frac{1}{8}}$

18. $\sqrt[6]{\frac{1}{a^3}}+\sqrt[8]{\frac{1}{a^4}}+\sqrt[12]{\frac{1}{a^6}}$

19. $\frac{1}{a-b}\sqrt{a^2+ab}+\sqrt{\frac{a}{a+b}}$

20. $\sqrt{\frac{x}{a-x}}-x\sqrt{ax-x^2}$

Multiplication of Radicals.

Illustrations.—

Multiply $3\sqrt{\frac{1}{2}}$ by $5\sqrt{\frac{2}{3}}$, and $3\sqrt{2}$ by $2\sqrt[3]{3}$.

Solutions:

1. $5\sqrt{\frac{2}{3}}\times 3\sqrt{\frac{1}{2}}=5\times 3\times\sqrt{\frac{2}{3}}\times\sqrt{\frac{1}{2}}=15\sqrt{\frac{2}{3}\times\frac{1}{2}}$ [P. 103]

$=15\sqrt{\frac{1}{3}}=15\sqrt{\frac{3}{9}}=15\sqrt{\frac{1}{9}}\times\sqrt{3}$ [P. 102] $=5\sqrt{3}.$

2. $2\sqrt[3]{3}\times 3\sqrt{2}=2\times 3\times\sqrt[3]{3}\times\sqrt{2}=6\times\sqrt[6]{9}\times\sqrt[6]{8}$ [P. 89]

$=6\sqrt[6]{72}$ [P. 103].

EXERCISE 117.

Find the value of:

1. $\sqrt{3}\times\sqrt{6}$

2. $\sqrt[3]{3}\times\sqrt[3]{6}$

3. $2\sqrt{\frac{1}{3}}\times 3\sqrt{\frac{1}{2}}$

4. $-4\sqrt{2ab}\times 3\sqrt{ac}$

5. $\sqrt{\frac{a}{b}}\times\sqrt{\frac{b}{c}}$

6. $\sqrt[3]{\frac{1}{2}}\times\sqrt[3]{\frac{1}{4}}\times\sqrt[3]{3}$

7. $\sqrt{3a}\times\sqrt{2a}\times\sqrt{4a}$

8. $2x\sqrt[3]{x}\times 3y\sqrt[3]{x}\times z\sqrt[3]{x}$

9. $\sqrt[3]{\frac{a}{x}}\times\sqrt[3]{\frac{x}{y}}\times\sqrt[3]{\frac{y}{z}}$

10. $\sqrt{\frac{c}{a+b}}\times\sqrt{\frac{a+b}{a-b}}$

11. $\sqrt{5}\times\sqrt[3]{2}$

12. $2\sqrt{3}\times 3\sqrt[3]{3}\times\sqrt[6]{3}$

13. $2\sqrt{ab}\times -3\sqrt[3]{a^2c}$

14. $(\sqrt{2}+\sqrt{3}+\sqrt{5})\times\sqrt{2}$

15. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$

16. $\sqrt{a+b}\times\sqrt{a-b}\times\sqrt{a^2+b^2}$

17. $\sqrt[3]{\frac{a+x}{a-x}}\times\sqrt[3]{\frac{a-x}{a^2-x^2}}\times\sqrt[3]{(a-x)^2}.$

18. $a\sqrt{xy^2}\times b\sqrt[4]{x^2y^3}\times c\sqrt[6]{x^3y^4}$

19. $(3\sqrt{2}-4\sqrt{6}+5\sqrt{10})\times\sqrt{6}$

20. $(2\sqrt{x}-3\sqrt{y})(2\sqrt{x}+3\sqrt{y})$

21. $(x-\sqrt{xy}+y)(\sqrt{x}+\sqrt{y})$

22. $(x+\sqrt{xy}+y)(x-\sqrt{xy}+y)$

Division of Radicals.

Illustrations.—

Divide $3\sqrt[3]{\frac{2}{3}}$ by $2\sqrt[3]{\frac{4}{9}}$, and $\sqrt{2}$ by $\sqrt[3]{3}$.

Solutions:

1. $3\sqrt[3]{\frac{2}{3}}\div 2\sqrt[3]{\frac{4}{9}}=\frac{3}{2}\sqrt[3]{\frac{2}{3}\div\frac{4}{9}}$ [P. 105] $=\frac{3}{2}\sqrt[3]{\frac{2}{3}\times\frac{9}{4}}=\frac{3}{2}\sqrt[3]{\frac{3}{2}}=$
$\frac{3}{2}\sqrt[3]{\frac{12}{8}}=\frac{3}{2}\sqrt[3]{\frac{1}{8}}\times\sqrt[3]{12}$ [P. 102] $=\frac{3}{4}\sqrt[3]{12}.$

2. $\sqrt{2}\div\sqrt[3]{3}=\sqrt[6]{8}\div\sqrt[6]{9}=\sqrt[6]{8\div 9}$ [P. 105] $=\sqrt[6]{\frac{8}{9}}=$
$\sqrt[6]{\frac{8}{9}\times\frac{3^4}{3^4}}=\sqrt[6]{\frac{128}{3^6}}=\frac{\sqrt[6]{128}}{3}=\frac{1}{3}\sqrt[6]{128}.$

EXERCISE 118.

Find the value of:

1. $\sqrt{8}\div\sqrt{2}$
2. $4\sqrt{15}\div 2\sqrt{3}$
3. $\sqrt{\frac{2}{3}}\div 2\sqrt{\frac{5}{6}}$
4. $5\sqrt{30}\div 3\sqrt{6}$
5. $\sqrt{ab}\div\sqrt{bc}$
6. $\sqrt[3]{\frac{1}{2}}\div 2\sqrt[3]{2}$

7. $2\sqrt[3]{ab^2} \div \sqrt[3]{ab}$

8. $a^2\sqrt[3]{x^2y^2} \div a\sqrt[3]{xy^2}$

9. $\sqrt{2} \div \sqrt[3]{2}$

10. $2 \div \sqrt[3]{2}$

11. $(x-y) \div (\sqrt{x}+\sqrt{y})$

12. $2\sqrt{3} \times 3\sqrt{2} \div \sqrt{\frac{1}{2}}$

13. $6\sqrt{3} \times 2\sqrt{5} \div \sqrt{15}$

14. $(2\sqrt{30} \div \frac{1}{2}\sqrt{2}) \times \sqrt{6}$

15. $a\sqrt{a} \times b\sqrt{b} \div \frac{a}{b}\sqrt{\frac{a}{b}}$

16. $2 \div \sqrt{2} - \sqrt{2} \times \sqrt{\frac{1}{2}}$

17. $\sqrt{x+xy} \div \sqrt{x}$

18. $(\sqrt{6}+\sqrt{8}+\sqrt{10}) \div \sqrt{2}$

19. $(3\sqrt{10}+4\sqrt{5}-6\sqrt{15}) \div \sqrt{5}$

20. $(5a\sqrt{a^2-b^2}-10ab\sqrt{a+b}) \div 5a\sqrt{a+b}$

21. $(x+2\sqrt{xy}+y) \div (\sqrt{x}+\sqrt{y})$

22. $(x^2+xy+y^2) \div (x+\sqrt{xy}+y)$

Involution of Radicals.

Illustrations.—1. Raise $\sqrt[3]{5}$ to the second power.

Solution: $(\sqrt[3]{5})^2 = (5^{\frac{1}{3}})^2 = 5^{\frac{2}{3}}$ [P. 99] $= \sqrt[3]{5^2} = \sqrt[3]{25}$.

2. Raise $\sqrt[6]{a}$ to the third power.

Solution: $(\sqrt[6]{a})^3 = (a^{\frac{1}{6}})^3 = a^{\frac{3}{6}}$ [P. 99] $= a^{\frac{1}{2}}$ [P. 89] $= \sqrt{a}$.

3. Raise $\sqrt[3]{a}$ to the sixth power.

Solution: $(\sqrt[3]{a})^6 = (a^{\frac{1}{3}})^6 = a^{\frac{6}{3}}$ [P. 99] $= a^2$ [P. 89].

EXERCISE 119.

Find the value of:

1. $(\sqrt[3]{2})^2$

2. $(\sqrt{3})^5$

3. $(2\sqrt{2})^2$

4. $(3\sqrt[3]{4})^3$

5. $(2\sqrt[4]{5})^2$

6. $(a\sqrt[6]{ab})^3$

7. $(\sqrt{2ab})^4$

8. $(\sqrt[5]{a^2b})^6$

9. $\{(a+b)\sqrt{ab}\}^2$

10. $(\sqrt{a-b})^3$

11. $(\sqrt[3]{x+y})^6$

12. $(\sqrt[4]{a^2-b^2})^8$

13. $\left(\sqrt[3]{\frac{2}{3}}\right)^6$

14. $(\sqrt{a}+\sqrt{b})^2$

15. $\left(2\sqrt[4]{\frac{7}{8}}\right)^2$

16. $(\sqrt{x}-\sqrt{y})^2$

17. $\left(\sqrt[3]{\frac{a}{b}}\right)^9$

18. $(2\sqrt{3}-3\sqrt{2})^2$

19. $\left(\frac{a}{b}\sqrt{\frac{a}{b}}\right)^4$

20. $(2+\sqrt{3})^3$

21. $\left(\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}\right)^2$

22. $(\sqrt[3]{3}-1)^3$

23. $\left(\frac{1}{2}\sqrt{2}+\frac{1}{3}\sqrt{3}\right)^2$

24. $\{-a(\sqrt{a}-\sqrt{b})\}^2$

Evolution of Radicals.

Illustrations.—1. Extract the square root of $\sqrt[3]{5}$.

Solution: $\sqrt{\sqrt[3]{5}}=\sqrt[6]{5}$ [P. 108].

2. Extract the cube root of $\sqrt[5]{8}$.

Solution: $\sqrt[3]{\sqrt[5]{8}}=\sqrt[15]{8}$ [P. 108] $=\sqrt[5]{\sqrt[3]{8}}$ [P. 108] $=\sqrt[5]{2}$.

3. Extract the square root of $8\sqrt[3]{2}$.

Solution: $\sqrt{8\sqrt[3]{2}}=\sqrt{4\times 2\sqrt[3]{2}}=\sqrt{4\times\sqrt[3]{16}}=$

$\sqrt{4}\times\sqrt{\sqrt[3]{16}}=\pm 2\sqrt[3]{4}$.

EXERCISE 120.

Find the value of :

1. $\sqrt{\sqrt{3a}}$

2. $\sqrt{\sqrt[3]{4a^2}}$

3. $\sqrt[3]{2a\sqrt{2a}}$

4. $\sqrt{\sqrt[5]{16a^4x^2}}$

5. $\sqrt[3]{\sqrt{\frac{1}{8}x^3y^6}}$

6. $\sqrt[4]{\sqrt[3]{a^2b^2c^2}}$

7. $\sqrt{2\sqrt{2}}$

8. $\sqrt[3]{-\sqrt{8}}$

9. $\sqrt{\sqrt[3]{12ab}}$

10. $\sqrt{\frac{1}{2}\sqrt{2a}}$

11. $\sqrt[3]{16\sqrt{2a}}$

12. $\sqrt{\sqrt[3]{(a-b)^2}}$

13. $\sqrt{\sqrt[3]{a^2+2ab+b^2}}$

14. $\sqrt[3]{\sqrt{(a+b)^3x^3}}$

15. $\sqrt{2\sqrt{15(a+x)}}$

16. $\sqrt{\sqrt{(a+b)(a-b)}}$

17. $\sqrt[3]{\sqrt{a^3+3a^2b+3ab^2+b^3}}$

18. $\sqrt[3]{27\sqrt[3]{2ax^2}}$

19. $\sqrt{\frac{1}{a}\sqrt{\frac{1}{a}}}$

20. $\sqrt{\frac{1}{a+b}\sqrt{(a+b)}}$

21. $\sqrt{\frac{x}{y}\sqrt{\frac{y}{x}}}$

Rationalization.

255. To rationalize a quantity is to clear it of radicals.

Illustrations.—1. Rationalize the denominator of $\frac{2}{\sqrt{3}}$

Solution: $\frac{2}{\sqrt{3}} = \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$ [P. 56] $= \frac{2\sqrt{3}}{3} = \frac{2}{3}\sqrt{3}$.

2. Rationalize the denominator of $\frac{2+\sqrt{3}}{2-\sqrt{3}}$

Solution:

$$\frac{2+\sqrt{3}}{2-\sqrt{3}} = \frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{4+4\sqrt{3}+3}{4-3} = 7+4\sqrt{3}$$

EXERCISE 121.

Rationalize the denominators of:

1. $\frac{1}{\sqrt{3}}$

2. $\frac{2}{\sqrt{5}}$

3. $\frac{\sqrt{3}}{\sqrt{5}}$

4. $\frac{a}{\sqrt{b}}$

5. $\frac{\sqrt{a}}{\sqrt{b}}$

6. $\frac{1}{1-\sqrt{2}}$

7. $\frac{2}{2+\sqrt{2}}$

8. $\frac{3}{\sqrt{2}-\sqrt{3}}$

9. $\frac{a}{a-\sqrt{b}}$

10. $\dfrac{c}{\sqrt{x}+\sqrt{y}}$ 12. $\dfrac{3-\sqrt{2}}{3+\sqrt{2}}$ 14. $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$

11. $\dfrac{2+\sqrt{2}}{2-\sqrt{2}}$ 13. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$ 15. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

16. If the $\sqrt{2}=1{\cdot}4142$, what is the value of $\dfrac{1}{\sqrt{2}}$?

17. What is the numerical value of $\dfrac{2-\sqrt{2}}{2+\sqrt{2}}$?

Imaginary Quantities.

1. Principle and Definition.

256. $\sqrt{-4}=\sqrt{4\times(-1)}=\sqrt{4}\times\sqrt{-1}$ [P. 102] $=$
$\pm 2\sqrt{-1}.$

$\sqrt{-6}=\sqrt{6\times(-1)}=\sqrt{6}\times\sqrt{-1}$ [P. 102] $=$

Therefore, $\pm\sqrt{6}\,(\sqrt{-1}).$

***Prin. 109.**—Every imaginary quantity of the second degree may be reduced to the form of $\pm x\sqrt{-1}$, in which x may be rational or irrational.*

257. The factor $\sqrt{-1}$ is the imaginary unit. $-\sqrt{-1}$ is equivalent to $-1\times\sqrt{-1}.$

2. Examples.

EXERCISE 122.

Reduce to simple form :

1. $\sqrt{-9}$, $\sqrt{-4a^2}$, and $\sqrt{-16}$

2. $\sqrt{-25x^2}$, $\sqrt{-36a^2x^4}$, and $\sqrt{-49a^4y^6}$

3. $\sqrt{-8}$, $\sqrt{-12a}$, and $\sqrt{-18a^2x^3}$

4. Add $\sqrt{-4}$, $\sqrt{-9}$, and $\sqrt{-16}$

5. Find the value of $\sqrt{-4a^2}+\sqrt{-25a^2}-\sqrt{16a^2}$

6. Simplify $(\sqrt{-1})^2$, $(\sqrt{-1})^3$, $(\sqrt{-1})^4$, $(\sqrt{-1})^5$

7. Simplify $(-\sqrt{-1})^2$, $(-\sqrt{-1})^3$, $(-\sqrt{-1})^4$, and $(-\sqrt{-1})^5$

8. Multiply $\sqrt{-3}$ by $\sqrt{-6}$

Suggestion. $\sqrt{-3}=\sqrt{3}\times\sqrt{-1}$ and $\sqrt{-6}=\sqrt{6}\times\sqrt{-1}$; hence, $\sqrt{-3}\times\sqrt{-6}=\sqrt{3}\times\sqrt{6}\times\sqrt{-1}\times\sqrt{-1}=\sqrt{18}\times(\sqrt{-1})^2=\sqrt{18}\times(-1)=-\sqrt{18}=-3\sqrt{2}.$

Multiply:

9. $\sqrt{-4}$ by $\sqrt{-9}$; $\sqrt{-5}$ by $\sqrt{-20}$; $\sqrt{-3}$ by $\sqrt{-2}$

10. $2\sqrt{-3}$ by $3\sqrt{-3}$; $4\sqrt{-16}$ by $2\sqrt{-25}$; $a\sqrt{-b^2}$ by $b\sqrt{-a^2}$

11. $2+\sqrt{-3}$ by $2-\sqrt{-3}$; $\sqrt{-2}+\sqrt{-3}$ by $\sqrt{-2}-\sqrt{-3}$

12. Square $2+\sqrt{-4}$; $3+\sqrt{-1}$; $\sqrt{-2}-\sqrt{-5}$

13. Divide $\sqrt{-36}$ by $\sqrt{-4}$

Suggestion. $\sqrt{-36}=\sqrt{36\times-1}=6\sqrt{-1}$; and $\sqrt{-4}=\sqrt{4\times-1}=2\sqrt{-1}$; $\dfrac{\sqrt{-36}}{\sqrt{-4}}=\dfrac{6\sqrt{-1}}{2\sqrt{-1}}=\dfrac{6}{2}=3.$

Divide:

14. $\sqrt{-6}$ by $\sqrt{-2}$; $\sqrt{-12}$ by $\sqrt{-3}$; $\sqrt{-9}$ by $\sqrt{-4}$

15. $2\sqrt{-4x^2}$ by $\sqrt{-x^2}$; $\sqrt{-16x^4}$ by $\sqrt{-2x}$; $\sqrt{-2}$ by $\sqrt{-4}$

Rationalize the denominators in:

16. $\dfrac{2+\sqrt{-2}}{2-\sqrt{-2}}$, $\dfrac{a+\sqrt{-b}}{a-\sqrt{-b}}$, and $\dfrac{\sqrt{-a}-\sqrt{-b}}{\sqrt{-a}+\sqrt{-b}}$

Square Root of Binomial Surds.

1. Definitions and Principles.

258. A binomial one or both of whose terms are surds is a *binomial surd;* as, $a \pm \sqrt{b}$, $\sqrt{a} \pm \sqrt{b}$.

Note.—The discussion in this section will be limited to binomial surds of the second degree.

259. Since the rational term, if there be any, may be put in the form of a radical, and the coefficients of the terms, if there be any, be placed under the radical sign—

Prin. 110.—*Every binomial surd of the second degree may be reduced to the form of $\sqrt{a} \pm \sqrt{b}$, in which one of the terms may be rational.*

260. The square of $(\sqrt{a} \pm \sqrt{b})$, or $(\sqrt{a} \pm \sqrt{b})^2 = a \pm 2\sqrt{ab} + b = (a+b) \pm 2\sqrt{ab}$, a binomial surd.

Therefore,

Prin. 111.—*A binomial surd may be a perfect square, and, when it is the square of a binomial surd of the second degree, one of the terms is rational.*

261. Since

$(\sqrt{a} \pm \sqrt{b})^2 = (a+b) \pm 2\sqrt{ab}$, $(a+b) \pm 2\sqrt{ab}$ is the type of a binomial surd that is a perfect square.

Therefore,

Prin. 112.—*A binomial surd with a rational term, and the coefficient of the irrational term reduced to ± 2, is a perfect square when the quantity under the radical sign is composed of two factors whose sum equals the rational term; and its square root equals the sum or difference of the square roots of these factors.*

262. $(a+b) \pm 2\sqrt{ab}$ is the type of a square binomial surd. Now, $(a+b)^2 - (\pm 2\sqrt{ab})^2 = a^2 + 2ab + b^2 - 4ab = a^2 - 2ab + b^2 = (a-b)^2$. Therefore,

***Prin. 113.**—When a binomial surd is a perfect square, the difference of the squares of its terms is a perfect square, and is equal to the square of the difference of the two factors described in Prin. 112.*

2. Examples.

Illustrations.—1. Extract the square root of $14+6\sqrt{5}$.

Solution: $14+6\sqrt{5}=14+2\sqrt{45}$. The two factors of 45 whose sum is 14 are 9 and 5; therefore, the square root of $14+2\sqrt{45}$ is $\pm(\sqrt{9}+\sqrt{5})$ [P. 112] $=\pm(3+\sqrt{5})$.

2. Extract the square root of $2a-2\sqrt{a^2-b^2}$.

Solution: The two factors of a^2-b^2 whose sum is $2a$ are $a+b$ and $a-b$; therefore, $\sqrt{2a-2\sqrt{a^2-b^2}}=\sqrt{a+b}-\sqrt{a-b}$; or $\sqrt{a-b}-\sqrt{a+b}=\pm(\sqrt{a+b}-\sqrt{a-b})$.

3. Extract the square root of $81-36\sqrt{5}$.

Solution: $\sqrt{81-36\sqrt{5}}=\sqrt{81-2\sqrt{1620}}$. The two factors of 1620, whose sum is 81, are not readily seen. Let x equal one and y equal the other. Then,

$$(1)\quad x+y=81;\text{ and}$$
$$(2)\quad (x-y)^2=81^2-(2\sqrt{1620})^2=81\ \text{[P. 113]}$$
$$\therefore\ (3)\quad x-y=9$$

Add (3) to (1) and subtract (3) from (1),

$$2x=90 \text{ and } 2y=72$$
$$x=45 \text{ and } y=36$$

$$\therefore\ \sqrt{81-2\sqrt{1620}}=\sqrt{36}-\sqrt{45},\text{ or } \sqrt{45}-\sqrt{36}=$$
$$6-3\sqrt{5}\text{ or } 3\sqrt{5}-6=\pm(6-3\sqrt{5})$$

EXERCISE 123.

Extract the square root, when possible, of the following expressions:

1. $5+2\sqrt{6}$
2. $9-4\sqrt{5}$
3. $7+4\sqrt{3}$
4. $8-2\sqrt{15}$
5. $15+6\sqrt{6}$
6. $4+\sqrt{3}$
7. $3a-2a\sqrt{2}$
8. $6x+2x\sqrt{5}$
9. $11+2\sqrt{30}$
10. $13-2\sqrt{42}$

11. $16 - 4\sqrt{20}$
12. $2x + 2\sqrt{x^2 - y^2}$
13. $15 - 2\sqrt{56}$
14. $2x + 2\sqrt{x^2 - 4y^2}$
15. $22 - 4\sqrt{30}$
16. $(x + y) - 2\sqrt{xy}$
17. $\frac{5}{6} - 2\sqrt{\frac{1}{6}}$
18. $\frac{7}{12} + \frac{1}{3}\sqrt{3}$
19. $(1 + 2x) + 2\sqrt{x^2 + x}$
20. $9 + \sqrt{72}$
21. $23 - \sqrt{528}$
22. $16 + \sqrt{156}$
23. $15 - \sqrt{56}$
24. $(a + 1) + \sqrt{4a}$
25. $2\frac{1}{3} + 2\sqrt{1\frac{1}{9}}$
26. $1 - \frac{2}{5}\sqrt{6}$
27. $25 + 2\sqrt{156}$
28. $-3 + 4\sqrt{-1}$
29. $x - y - 2\sqrt{-xy}$
30. $42 + 36\sqrt{2}$
31. $41 - 4\sqrt{105}$
32. $0 + 2\sqrt{-1}$
33. $6 + \sqrt{35}$
34. $10 - \sqrt{100 - 4x^2}$
35. $(x + y)^2 - 4(x - y)\sqrt{xy}$
36. $(2x + 1) + 2\sqrt{x^2 + x - 2}$

Miscellaneous Examples.

EXERCISE 124.

Express with fractional exponents:

1. $\sqrt[3]{a^2 b}$
2. $\sqrt[4]{(a^2 b c^3)^5}$
3. $\sqrt[n]{a^m b^s}$
4. $\sqrt[5]{(a + b)^2}$
5. $\sqrt[3]{8(x - y)^2}$
6. $\sqrt[n-1]{(ab)^n}$
7. $\sqrt[m]{(a^2 - x^2)^n}$
8. $\sqrt[n+1]{(a^n - b^n)^m}$

Express with the radical sign:

9. $x^{\frac{5}{8}}$
10. $(ac)^{\frac{5}{2}}$
11. $a^{\frac{2}{3}} b^{\frac{1}{3}}$
12. $x^{\frac{2}{3}} y^{\frac{5}{6}}$
13. $(a + x)^{\frac{3}{4}}$
14. $(a^2 - x^2)^{\frac{2}{3}}$
15. $x^{\frac{1}{4}} (x + y)^{\frac{3}{4}}$
16. $a^{\frac{2}{3}} (a + b)^{\frac{5}{6}}$

Express as mixed surds :

17. $3\sqrt{3}$

18. $a^2\sqrt[3]{b^2}$

19. $(a+b)\sqrt{a+b}$

20. $a^3\sqrt[3]{a^2}$

21. $x^{\frac{2}{3}}\sqrt{y^{\frac{1}{3}}}$

22. $xy\sqrt{x \div y}$

23. $\frac{4}{5}\sqrt{15}$

24. $(a+x)\sqrt{a-x}$

Place the coefficients of the following expressions within the parentheses :

25. $3(3)^{\frac{1}{2}}$

26. $4(2)^{\frac{1}{3}}$

27. $4(4)^{\frac{2}{3}}$

28. $a(a)^4$

29. $x^2(x^2)^{-3}$

30. $y(y^{-2})^{-\frac{1}{3}}$

31. $-8(a-b)^{\frac{3}{2}}$

32. $a(a^{-3}b)^{\frac{1}{3}}$

33. $a^{-1}(a^{-1})^{-1}$

34. $x^{\frac{4}{3}}(a+bx^{-3})^{\frac{1}{3}}$

35. $x^{-\frac{1}{2}}(a+x)^{-\frac{1}{2}}$

36. $x^{\frac{p}{q}}(x^{-\frac{p}{q}})^{\frac{q}{p}}$

Reduce the following expressions to equivalent ones having a coefficient of 2 :

37. $6\sqrt{3}$

38. $5\sqrt{5}$

39. $\sqrt{20}$

40. $7\sqrt[3]{2}$

41. $-3\sqrt[3]{4}$

42. $3(a-b)^{\frac{1}{2}}$

43. $-8\sqrt[3]{-3}$

44. $8(a-b)^2$

45. $54(x+y)^3$

Complete the following expressions :

46. $(8ax)^{\frac{2}{3}} = 4(\quad)^{\frac{2}{3}}$

47. $(16xy)^{\frac{3}{2}} = 8(\quad)^{\frac{3}{2}}$

48. $(32ay)^{\frac{2}{5}} = 4(\quad)^{\frac{2}{5}}$

49. $\left(\frac{1}{2}\right)^{\frac{1}{2}} = 2(\quad)^{\frac{1}{2}}$

50. $(a^2x^3)^{\frac{1}{2}} = a^3(\quad)^{\frac{1}{2}}$

51. $(x^2y^3)^{\frac{2}{3}} = x^{\frac{2}{3}}(\quad)^{\frac{2}{3}}$

52. $a^{\frac{m}{n}}(a^{-1})^{\frac{n}{m}} = (\quad)^{\frac{n}{m}}$

53. $\left(\frac{1}{2}\right)^{\frac{1}{3}} = 2(\quad)^{\frac{1}{3}}$

Express as pure surds :

54. $\sqrt{a^3+2a^2b+ab^2}$

55. $\sqrt{(a-x)(a^2-x^2)}$

56. $\sqrt[n+1]{a^{n+3}}$

57. $\sqrt[n]{ax^{-2n}}$

58. $\sqrt{\frac{a}{a+b}}$

59. $\sqrt{\frac{x^2}{a-x}}$

60. $\sqrt[3]{\frac{a}{b^2}}$

61. $\sqrt{\frac{a-b}{a+b}}$

Simplify:

62. $7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{432}$

63. $(x^2y^3 - 3ay^3)^{\frac{1}{3}} - 2(x^2z^3 - 3az^3)^{\frac{1}{3}}$

64. $\sqrt{(a+b)^{-1}} \times \sqrt[3]{(a+b)^{-2}}$

65. $\left(\frac{ax}{b}\right)^{\frac{1}{2}} \times \left(\frac{by}{x}\right)^{\frac{1}{3}} \times \left(\frac{xy}{ab}\right)^{\frac{1}{4}}$

66. $(\sqrt{14} + \sqrt{21} - \sqrt{42}) \div \sqrt{7}$

67. $(\sqrt[3]{40} - \sqrt[3]{16} + \sqrt[3]{56}) \div \sqrt[3]{8}$

68. $(a - x) \div (\sqrt[6]{a} - \sqrt[6]{x})$

69. $\frac{4\sqrt{40}}{3\sqrt{108}} \times \frac{14\sqrt{12}}{5\sqrt{14}} \div \frac{2\sqrt{60}}{3\sqrt{84}}$

Reduce to equivalent forms having a rational denominator:

70. $\frac{2x\sqrt{5a}}{\sqrt{3a}}$

71. $\frac{1}{\sqrt[3]{2}}$

72. $\frac{\sqrt[3]{4}}{\sqrt[3]{5}}$

73. $\frac{a}{a + \sqrt{b}}$

74. $\frac{4}{\sqrt{3} - \sqrt{2}}$

75. $\frac{2}{1 - \sqrt{5}}$

76. $\frac{3 + 2\sqrt{2}}{\sqrt{5} - \sqrt{3}}$

77. $\frac{1 - \sqrt{-2}}{1 + \sqrt{-2}}$

78. $\frac{\sqrt{-3} - \sqrt{-5}}{\sqrt{-3} + \sqrt{-5}}$

Arrange in the increasing order of magnitude:

79. $3\sqrt{2},\ 2\sqrt{3},\ \sqrt{15}$

80. $\sqrt{5},\ \sqrt[3]{10},\ 2\sqrt[6]{2}$

81. $\sqrt{6},\ \sqrt[3]{15},\ 2\sqrt[6]{3}$

82. $\sqrt[3]{2},\ \sqrt[4]{5},\ \sqrt[12]{20}$

Resolve into two binomial factors [see P. 39]:

83. $a - b$

84. $x - 4$

85. $x^{\frac{1}{2}} - y^{\frac{1}{2}}$

86. $x^{\frac{1}{3}} - y^{\frac{1}{3}}$

87. $16 - x$

88. $x - 25$

89. $x^2 - 5$

90. $y - 2$

Write the quotients of the following examples [see P. 45]:

91. $(x - y) \div (x^{\frac{1}{3}} - y^{\frac{1}{3}})$

92. $(x + y) \div (x^{\frac{1}{3}} + y^{\frac{1}{3}})$

93. $(x - y) \div (x^{\frac{1}{5}} - y^{\frac{1}{5}})$

94. $(x + y) \div (x^{\frac{1}{5}} + y^{\frac{1}{5}})$

95. $(8x + 27y) \div (2x^{\frac{1}{3}} + 3y^{\frac{1}{3}})$

96. $(16x - 81y) \div (2x^{\frac{1}{4}} - 3y^{\frac{1}{4}})$

97. $(16x - 81y) \div (2x^{\frac{1}{4}} + 3y^{\frac{1}{4}})$

98. $(x^{\frac{1}{3}} - y^{\frac{1}{3}}) \div (x^{\frac{1}{9}} - y^{\frac{1}{9}})$

Reduce to lowest terms:

99. $\dfrac{x - y}{\sqrt{x} + \sqrt{y}}$

100. $\dfrac{a^2 - b}{a - \sqrt{b}}$

101. $\dfrac{x(x + y)}{\sqrt[3]{x} + \sqrt[3]{y}}$

102. $\dfrac{x - \sqrt{y}}{\sqrt{x} + \sqrt[4]{y}}$

103. $\dfrac{x^2 - y^2}{(\sqrt{x} - \sqrt{y})\sqrt{x + y}}$

104. $\dfrac{\sqrt{x^2 - y^2}}{x\sqrt{x + y}}$

Expand:

105. $\{x^{\frac{2}{3}} + (xy)^{\frac{1}{3}} + y^{\frac{2}{3}}\}\{x^{\frac{2}{3}} - (xy)^{\frac{1}{3}} + y^{\frac{2}{3}}\}$

106. $(a + \sqrt{a^2 - x^2})^2$

107. $(a - b\sqrt{-1})^3$

108. $(\sqrt{a} + \sqrt{b})^4$

109. $(a + b\sqrt{-1})^3$

110. $(-2 - 3\sqrt{-2})^3$

111. $(2\sqrt{2} - \sqrt{3})^4$

Simplify:

112. $\sqrt{27\sqrt[3]{135a^6b^4}}$

113. $\sqrt[4]{25x^4\sqrt{y}}$

114. $\sqrt[3]{\dfrac{a}{3}\sqrt{\dfrac{3}{a}}}$

115. $\sqrt{49 + 12\sqrt{5}}$

116. $\sqrt{2x - \sqrt{4x^2 - 4}}$

117. $\sqrt{-3 - 2\sqrt{2}}$

118. $\dfrac{x + \sqrt{-y}}{x - \sqrt{-y}} + \dfrac{x - \sqrt{-y}}{x + \sqrt{-y}} =$ what?

119. $\dfrac{\sqrt{x + y} + \sqrt{x - y}}{\sqrt{x + y} - \sqrt{x - y}} + \dfrac{\sqrt{x + y} - \sqrt{x - y}}{\sqrt{x + y} + \sqrt{x - y}} =$ what?

120. Square $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}}$

121. Square $\sqrt{x-\frac{1}{x}}-\sqrt{x+\frac{1}{x}}$

122. Extract the square root of:

$$x^{\frac{4}{3}}+2\,x\,y^{\frac{1}{3}}+3\,x^{\frac{2}{3}}\,y^{\frac{2}{3}}+2\,x^{\frac{1}{3}}\,y+y^{\frac{4}{3}}$$

123. Extract the cube root of:

$$x^{\frac{3}{2}}+3\,x+6\,x^{\frac{1}{2}}+7+\frac{6}{x^{\frac{1}{2}}}+\frac{3}{x}+\frac{1}{x^{\frac{3}{2}}}$$

Radical Equations.

Illustrations.—1. Solve $x^{\frac{2}{3}}=4$ (A)

Solution: Extract $\sqrt{\ }$, $x^{\frac{1}{3}}=\pm 2$ (1)

Cube (1), $x=\pm 8$

2. Solve $\sqrt[3]{2\,x^2}=2$ (A)

Solution: Cube (A), $2\,x^2=8$ (1)

Divide, $x^2=4$ (2)

Extract $\sqrt{\ }$, $x=\pm 2$

3. Solve $\sqrt{x+14}+\sqrt{x-14}=14$ (A)

Solution:

Square (A), $x+14+2\sqrt{x^2-196}+x-14=196$ (1)

Transpose, $2\sqrt{x^2-196}=196-2\,x$ (2)

Divide, $\sqrt{x^2-196}=98-x$ (3)

Square, $x^2-196=9604-196\,x+x^2$ (4)

Transpose, $196\,x=9800$ (5)

Divide, $x=50$

Caution.—In squaring a radical binomial, do not simply square each term. Thus, $(\sqrt{x}\times\sqrt{a})^2$ is not $x+a$, but $x+2\sqrt{a\,x}+a$.

4. Solve $\sqrt{x^2+a\,x}-\sqrt{x}=x$ (A)

Solution: Transpose, $\sqrt{x^2+a\,x}=x+\sqrt{x}$ (1)

Square, $x^2+a\,x=x^2+2\,x\sqrt{x}+x$ (2)

Transpose, $2\,x\sqrt{x}=x\,(a-1)$ (3)

Divide by x, $2\sqrt{x}=(a-1)$ (4)

Square, $4\,x=(a-1)^2$ (5)

Divide, $x=\frac{(a-1)^2}{4}$

5. Solve $x^2 - 3x - 6\sqrt{x^2 - 3x - 3} = -2$ (A)

Solution: Subtract 3 from both members,

$$(x^2 - 3x - 3) - 6(x^2 - 3x - 3)^{\frac{1}{2}} = -5 \quad (1)$$

Complete the square,

$$(x^2 - 3x - 3) - 6(\quad) + 9 = 4 \quad (2)$$

Extract $\sqrt{\ }$, $\sqrt{x^2 - 3x - 3} - 3 = \pm 2$ (3)

Transpose, $\sqrt{x^2 - 3x - 3} = 5 \text{ or } 1$ (4)

Square, $x^2 - 3x - 3 = 25 \text{ or } 1$ (5)

Transpose, $x^2 - 3x = 28 \text{ or } 4$ (6)

Complete the square, $x^2 - 3x + \frac{9}{4} = \frac{121}{4} \text{ or } \frac{25}{4}$ (7)

Extract $\sqrt{\ }$, $x - \frac{3}{2} = \pm\frac{11}{2} \text{ or } \pm\frac{5}{2}$ (8)

Transpose, $x = 7, -4, 4, \text{ or } -1$

EXERCISE 125.

Solve:

1. $\sqrt[3]{4x^2} = 16$
2. $(x+2)^{\frac{2}{3}} = 36$
3. $(3x)^{\frac{2}{3}} = 9$
4. $\sqrt[5]{x^2} = 1$
5. $\sqrt[3]{(x-a^3)^2} = 4a^2$
6. $\sqrt[5]{(x^2+a^2)^2} = a^4$
7. $\sqrt{x-4} = 5$
8. $\sqrt{x} = 2 + \sqrt{x-12}$
9. $x - 3 = \sqrt{x^2+4}$
10. $\sqrt{x+a} = 2\sqrt{x-a}$
11. $\sqrt{4x} = 7 - \sqrt{4x-7}$
12. $\sqrt[4]{x^2+2} = \sqrt{x-4}$
13. $\sqrt{x+2} + \sqrt{x-2} = 2$
14. $\sqrt{x + \sqrt{x+2}} = \sqrt{x+3}$
15. $\sqrt{x-10} + \sqrt{x-9} = 5$
16. $\sqrt{x+3} + \sqrt{x-3} = 2\sqrt{x+\frac{1}{2}}$
17. $\frac{x+1}{\sqrt{x}} + 5 = \sqrt{x}$
18. $\frac{x+a}{\sqrt{x}} - a = \frac{x+b}{\sqrt{x}}$
19. $\frac{\sqrt{x}+2}{\sqrt{x}-2} = 2$
20. $\frac{a+\sqrt{x}}{a-\sqrt{x}} = b$

21. $\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}=5$

22. $\frac{x-4}{\sqrt{x}-2}=4$

23. $\sqrt{x}+\frac{1}{\sqrt{x}}=2$

24. $\frac{x-a}{\sqrt{x}-\sqrt{a}}=2\sqrt{a}$

25. $\sqrt{x+1}+\frac{1}{\sqrt{x+1}}=3\frac{1}{3}$

26. $\frac{x+\sqrt{x}+1}{7}=\frac{1}{x-\sqrt{x}+1}$

27. $\sqrt{x-\frac{1}{x}}=\sqrt{\frac{x+1}{x}}$

28. $\frac{a}{x^{\frac{1}{2}}}+\frac{x^{\frac{1}{2}}}{a}=x^{\frac{3}{2}}$

29. $\frac{\sqrt{ax}+a}{\sqrt{x}-\sqrt{a}}=\sqrt{x}+\sqrt{a}$

30. $\frac{x+a}{\sqrt{x}+\sqrt{a}}=\frac{\sqrt{x}-\sqrt{a}}{x-a}$

31. $\frac{x}{\sqrt{x}+\sqrt{a}}=\frac{\sqrt{x}-\sqrt{a}}{x+a}$

32. $4x-9a=2\sqrt{x}+3\sqrt{a}$

33. $x-a=\sqrt{x}+\sqrt{a}$

34. $\sqrt{1-x}=1-\sqrt{x}$

35. $x+x^{\frac{1}{2}}=6$

36. $ax^m+bx^{\frac{m}{2}}=c$

37. $x+4+\sqrt{x+4}=20$

38. $x^{\frac{2}{3}}-x^{\frac{1}{3}}=3$

39. $3x^{\frac{3}{4}}+x^{\frac{3}{8}}=2$

40. $x^{\frac{m}{2}}+x^{\frac{m}{4}}=2$

41. $2x^2+3x-4\sqrt{2x^2+3x-2}=7$

42. $2ax-x^2=2a^2-a\sqrt{2ax-x^2}$

43. $x^4-4x^2-2\sqrt{x^4-4x^2+4}=31$

44. $(x-3)^2=13-(x^2-6x+16)^{\frac{1}{2}}$

45. $x^2+5\sqrt{x^2-16x}=16x+300$

46. $x+\frac{1}{x}+\sqrt{x+\frac{1}{x}}=2$

47. $\begin{cases} 2\sqrt{x} + 3\sqrt{y} = 13 \\ 3\sqrt{x} - 2\sqrt{y} = 0 \end{cases}$

48. $\begin{cases} x + \sqrt{y} = \sqrt{x} \\ x - \sqrt{y} = \sqrt{y} \end{cases}$

49. $\begin{cases} \sqrt{x+y} + \sqrt{x-y} = 8 \\ \sqrt{x+y} - \sqrt{x-y} = 2 \end{cases}$

50. $\begin{cases} \sqrt{x} + \sqrt{y} = 1 \\ \sqrt{4x} + \sqrt{9y} = 4 \end{cases}$

51. $\begin{cases} x + \sqrt{x+y} = 11 \\ y + \sqrt{x+y} = 4 \end{cases}$

52. $\begin{cases} x^2 - y^2 + \sqrt{x^2 - y^2} = 20 \\ xy(xy+1) = 240 \end{cases}$

53. $x^2 + 2x = 6 + 4\sqrt{3}$

Character of the Roots of Equations.

Definitions.

263. A root containing one or more imaginary terms is an *imaginary root*; as, $x = a \pm b\sqrt{-1}$.

264. A root containing no imaginary terms is a *real root*; as, $x = a + b$, or $x = 3 \pm \sqrt{2}$.

265. A real root that contains one or more irrational terms is an *irrational root*; as, $x = 3 \pm \sqrt{2}$.

266. A real root that contains no irrational term is a *rational root*; as, $x = a + b$, or $x = 3 \pm \sqrt{4}$.

SIGHT EXERCISE.

Tell which of the following roots are real and which imaginary:

1. $x = \sqrt{4+5}$
2. $x = \sqrt{8-2}$
3. $x = \sqrt{5-9}$
4. $x = \sqrt{12 - 3 \times 5}$
5. $x = \sqrt{4^2 - 8}$
6. $x = \sqrt{5^3 - 7^2}$
7. $x = \sqrt{5(7-12)}$
8. $x = \sqrt{-3(6-9)}$
9. $x = \sqrt{(-2)^2 - 1}$
10. $x = \sqrt{5^2 - 4^2}$

If a and b are positive, and a is greater than b:

11. $x = \sqrt{a - b}$
12. $x = \sqrt{b - a}$
13. $x = \sqrt{a(a - b)}$
14. $x = \sqrt{-a(a - b)}$
15. $x = \sqrt{b^2 - ab}$
16. $x = \sqrt{a^3 - a^2 b}$

Tell which of the following roots are rational, which irrational, and which imaginary:

17. $x = \sqrt{18 + 7}$
18. $x = \sqrt{30 - 5}$
19. $x = \sqrt{15 - 3}$
20. $x = \sqrt{3^2 - 2 \times 5}$
21. $x = \sqrt{8^2 - 20}$
22. $x = \sqrt{8 - 3^2}$
23. $x = \sqrt{7^2 - 13}$
24. $x = \sqrt{4 \times 5 - 6 \times 3}$
25. $x = \sqrt{5^2 - 3^2}$
26. $x = \sqrt{25 - 4^2}$

Give the sign of each of the following roots:

27. $x = 4 + \sqrt{12}$
28. $x = 8 - \sqrt{96}$
29. $x = -2 - \frac{1}{2}\sqrt{40}$
30. $x = -4 + \sqrt{5}$
31. $x = 8 - \sqrt{10}$
32. $x = -10 + \frac{1}{2}\sqrt{120}$

If p and q are positive, and p^2 is greater than q:

33. $x = p + \sqrt{p^2 + q}$
34. $x = p - \sqrt{p^2 + q}$
35. $x = -p + \sqrt{p^2 + q}$
36. $x = -p - \sqrt{p^2 + q}$
37. $p + \sqrt{p^2 - q}$
38. $-p + \sqrt{p^2 - q}$

The roots of the equation $x^2 + px = q$ are

$$-\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q} \text{ and } -\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q},$$

in which $\frac{1}{4}p^2$ is the square of $\frac{1}{2}$ the coefficient of x, and q is the absolute term.

Tell the character of the above roots, whether real or imaginary:

1. If p and q are positive.
2. If p is negative and q positive.
3. If p is positive and q negative, and $q < \frac{1}{4}p^2$, numerically.
4. If p is positive and q negative, and $q > \frac{1}{4}p^2$, numerically.
5. If p is negative and q negative, and $q < \frac{1}{4}p^2$, numerically.
6. If p is negative and q negative, and $q > \frac{1}{4}p^2$, numerically.

Tell whether the above roots are rational or irrational:

7. If $\frac{1}{4}p^2 + q$ is a perfect square.
8. If $q = 0$.
9. If q is negative, and numerically equal to $\frac{1}{4}p^2$.
10. If $p = 0$.

Give the signs of the above roots, and tell which root is numerically the greater:

11. If p and q are both positive.
12. If p is negative and q positive.
13. If p is positive, q negative, and $\frac{1}{4}p^2 > q$, numerically.
14. If p and q are negative, and $\frac{1}{4}p^2 > q$, numerically.
15. If p and q are negative, and $\frac{1}{4}p^2 = q$, numerically.

What are the values of p and q in the following equations:

16. $x^2 + 4x = 7$
17. $x^2 - 9x = 5$
18. $x^2 + 6x = -4$
19. $x^2 - 8x = -5$
20. $6x^2 - 9 = 0$
21. $2x^2 + 5x - 3 = 0$

In the following equations, are the roots—

1. Real or imaginary? 2. Rational or irrational? 3. Positive or negative? 4. What are their relative values?

22. $x^2 + 6x = 7$
23. $x^2 - 4x = 5$
24. $x^2 + 5x = -6$
25. $x^2 - 3x = -2$
26. $x^2 + 7x = -16$
27. $x^2 + \frac{1}{2}x = \frac{3}{4}$
28. $x^2 + 6x = -9$
29. $x^2 - 5x = 10$
30. $x^2 + 3x = -5$
31. $x^2 - 5x = -8$
32. $4x^2 - 7x = -1$
33. $x^2 - \frac{3}{4}x = -\frac{1}{8}$

Inequalities.

1. Definitions and Principles.

267. An expression denoting that two quantities are unequal in value is an ***Inequality.***

268. The symbol of inequality is $>$, read *greater than;* or $<$, read *less than.*

269. The quantities compared in an inequality are the *members* of the inequality.

270. Two inequalities are said to subsist in the *same sense,* when the first members are both greater or both less than the second members.

271. Two inequalities are said to subsist in an *opposite* or *contrary* sense, when the first member of the one is the greater and the second member of the other.

272. A negative quantity is considered less than a positive quantity, whatever their absolute values.

273. The process of changing the form of an inequality without changing its sense is *transformation.*

274. The following principles of transformation may readily be illustrated :

Prin. 114.—*1. The same or equal quantities may be added to both members of an inequality.*

2. The same or equal quantities may be subtracted from both members of an inequality.

3. Both members of an inequality may be multiplied by the same or equal positive quantities.

4. Both members of an inequality may be divided by the same or equal positive quantities.

5. Two unequal positive members may be raised to the same power.

6. Two unequal positive members may have the same root extracted, provided the positive results only are compared.

7. The sum of two inequalities, subsisting in the same sense, may be taken member by member.

275. $(a-b)^2 > 0$ whether $a > b$ or $b > a$ [P. 27].

Expanding, $a^2 - 2ab + b^2 > 0$ (1)

Add $2ab$ to both members [P. 114, 1] $a^2 + b^2 > 2ab$.

Therefore,

Prin. 115.—*The sum of the squares of two unequal quantities is greater than twice their product.*

276.

$$a^2 + b^2 > 2ab \text{ [P. 115]} \qquad (1)$$
$$a^2 + c^2 > 2ac \quad \text{“} \qquad (2)$$
$$b^2 + c^2 > 2bc \quad \text{“} \qquad (3)$$

Adding member by member [P. 114, 7],

$$2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc \qquad (4)$$

Dividing by 2 [P. 114, 4],

$$a^2 + b^2 + c^2 > ab + ac + bc.$$

Therefore,

Prin. 116.—*The sum of the squares of three unequal quantities is greater than the sum of their products taken two and two.*

2. Examples.

Illustration.—Which is the greater, a^3+b^3 or a^2b+ab^2, for any positive values of a and b?

Solution: $a^3+b^3 >=< a^2b+ab^2$

Factoring, $(a+b)(a^2-ab+b^2) >=< ab(a+b)$

Dividing by $(a+b)$, $a^2-ab+b^2 >=< ab$

Adding ab to both members, $a^2+b^2 >=< 2ab.$

But $a^2+b^2>2ab$ [P. 115],

$\therefore$ $a^3+b^3>a^2b+ab^2$, since no operation has been performed to change the sense of the inequality.

EXERCISE 126.

Prove the following statements true for unequal positive values of the letters:

1. $a^3+ab^2>2a^2b$
2. $a^3b+ab^4>a^2b^2+ab^3$
3. $(a+b)^2>4ab$
4. $(a+b)^3>4a^2b+4ab^2$
5. $a^2+3b^2>2ab+2b^2$
6. $a^3>a^2+a-1$
7. $a^4+a^2b^2+a^2c^2>a^3b+a^3c+a^2bc$
8. $a^2>2a-1$
9. If $x^2+4x>12$, show that $x>2$
10. If $3x^2+5x>42$, show that $x>3$
11. If $7x^2-3x<160$, show that $x<5$
12. $\frac{x}{y}+\frac{y}{x}>2$
13. $\frac{x}{y^2}+\frac{y}{x^2}>\frac{1}{x}+\frac{1}{y}$
14. $x^2y+xy^2+x^2z+xz^2+y^2z+yz^2>6xyz$
15. $\frac{a+b}{c+b}>\frac{a}{c}$, when $a<c$
16. $\frac{a+b}{c+b}<\frac{a}{c}$, when $a>c$
17. $\frac{a-b}{c-b}>=<\frac{a}{c}$, when $a<=>c$
18. Which is the greater, $\frac{a+b}{a-b}$ or $\frac{a^2+b^2}{a^2-b^2}$, if a and b are positive?
19. What integral value of x will satisfy $3x^2+4x>64$ and $3x^2+4x<132$?

CHAPTER VII.

RATIO, PROPORTION, AND PROGRESSION.

Ratio.

1. Definitions and Principles.

277. A relation of values exists between two similar quantities—that is, one of them is a number of times or a part of the other.

278. The relation which the value of one quantity bears to that of another is the ***Ratio*** of the quantities, and is obtained by dividing the quantity compared by the quantity with which it is compared.

Illustration.—The ratio of 3 apples to 5 apples is $\frac{3}{5}$, since 3 apples $= \frac{3}{5}$ of 5 apples.

279. A ratio is expressed by writing a colon between the quantities compared, or by a common fraction.

Illustration.—The ratio of a to b is $a : b$, or $\frac{a}{b}$.

280. The quantity compared, or the first term of a ratio, is the *antecedent;* and the quantity with which the comparison is made, or the second term of the ratio, is the *consequent.*

281. Since the ratio is obtained by dividing the quantity compared by the quantity with which it is compared, it follows that,

Prin. 117.*—The ratio equals the antecedent divided by the consequent; or* $r = \frac{a}{c}$.

282. Since $r = \frac{a}{c}$, $a = c \times r$, and $c = \frac{a}{r}$. Therefore,

Prin. 118.*—The antecedent equals the ratio times the consequent.*

Prin. 119.*—The consequent equals the antecedent divided by the ratio.*

283. Since $\frac{a}{c} = r$, $\frac{n\,a}{c} = n\,r$, and $\frac{a}{c \div n} = n\,r$ (Ax. 4). Therefore,

Prin. 120.*—Multiplying the antecedent or dividing the consequent multiplies the ratio.*

284. Since $\frac{a}{c} = r$, $\frac{a \div n}{c} = \frac{r}{n}$, and $\frac{a}{n\,c} = \frac{r}{n}$ (Ax. 5). Therefore,

Prin. 121.*—Dividing the antecedent or multiplying the consequent divides the ratio.*

285. Since $\frac{a}{c} = r$, $\frac{n\,a}{n\,c} = r$, and $\frac{a \div n}{c \div n} = r$. Therefore,

Prin. 122.*—Multiplying or dividing both terms of a ratio by the same quantity does not alter its value.*

286. The product of two or more simple ratios is a *compound ratio*. Thus, $\left\{\begin{matrix} a : b \\ c : d \end{matrix}\right\}$ or $\frac{a}{b} \times \frac{c}{d}$ is a compound ratio.

287. To *duplicate* a ratio is to use it twice in a compound ratio. Thus, the duplicate of $a : b$ is $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$.

288. To *triplicate* a ratio is to use it three times in a compound ratio.

Thus, the triplicate of $a : b$ is $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$.

289. When the antecedent and consequent of a ratio are equal, it is called a *ratio of equality;* as, $a : a$, or $1 : 1$.

290. When the antecedent is greater than the consequent, the ratio is greater than *one,* and is called a ratio of *greater inequality.*

291. When the antecedent is less than the consequent, the ratio is less than *one,* and is called a ratio of *lesser inequality.*

292. When the ratio of two quantities can be exactly expressed by a rational number or fraction, it is said to be *commensurable.*

293. When the ratio of two quantities can not be exactly expressed by a rational number or fraction, it is called an *incommensurable* ratio; as, $\sqrt{2} : \sqrt{3} = \sqrt{\frac{2}{3}}$.

2. Examples.

EXERCISE 127.

1. Find the ratio of 4 to 20; 16 to 12; $\frac{3}{5}$ to 9; 8 to $\frac{5}{6}$; $\frac{2}{3}$ to $\frac{3}{4}$; $x : y$

2. Find the value of $a b^2 : a^2 b$; $a^2 - x^2 : a + x$;

$$(a + b)^2 : a^2 - b^2;\quad a^3 + x^3 : a^2 - a x + x^2$$

3. Find the value of:

$$\frac{m+n}{m-n} : \frac{m^2 - n^2}{(m-n)^2};\quad \frac{x^3 - y^3}{x+y} : \frac{x-y}{x^3+y^3};\quad \frac{1}{x^2 - y^2} : \frac{a}{x-y}$$

4. Reduce to their lowest terms:

$$25 : 75;\quad a^2 b^3 : a b^2;\quad a^2 + a b : a b + b^2$$

5. Clear of fractions and reduce to lowest terms:

$$2\frac{3}{4} : 7\frac{1}{2};\quad 18\frac{3}{4} : 31\frac{1}{4};\quad \frac{x}{y} : \frac{z}{x y};\quad a + \frac{a}{b} : c + \frac{c}{b}$$

6. Compound the ratios $\cdot 4 : 5$, $5 : \cdot 6$, and $\cdot 03 : 40$

7. Compound the ratios $2\cdot 5 : \cdot 32$, $\cdot 08 : 1\cdot 5$, and $\cdot 12 : \cdot 016$

8. Which is the greater, the ratio of $2\frac{1}{2} : 7\frac{1}{2}$ or the duplicate ratio of $2\frac{1}{2} : 7\frac{1}{2}$?

9. What must be subtracted from both terms of $a : b$ to make it $c : d$?

10. Compound the ratios of:

$$\frac{a^2+b^2}{a-b} : \frac{a^2-b^2}{(a+b)^2} \text{ and } \frac{a^4-b^4}{a+b} : \frac{(a^2+b^2)^2}{(a-b)^2}$$

11. If 5 horses and 8 cows cost as much as 8 horses and 2 cows, what is the relative value of a cow to a horse?

12. Find the ratio of 2 to $\sqrt{2}$ to within one thousandth; also, the ratio of 3 to $\sqrt{3}$.

13. The side of a square is 4 feet. What is the approximate ratio of the side to the diagonal?

14. If the same number be added to both terms of a ratio of lesser inequality, will it be increased or diminished? Which, if the same number be subtracted from both terms?

15. If the same number be added to both terms of a ratio of greater inequality, what will be the effect? What, if the same number be subtracted from both terms?

16. The ratio of A's money to B's is the same as the ratio of 5 to 6, and they together have \$1320. How much has each?

17. The sum of A's and B's ages bears the same relation to A's age as A's age bears to 8 years, and the difference of their ages is 10 years. Required the age of each.

18. The fore-wheel of a wagon makes 128 revolutions more in going a mile than the hind-wheel, and their circumferences are in the ratio of 5 : 6. What is the circumference of each?

Proportion.

Definitions.

294. The equality of two or more ratios may be expressed by writing between them a double colon, or the symbol of equality.

Thus, the fact that the ratio of 2 to 3 equals the ratio of 4 to 6 may be expressed:

1. $2 : 3 :: 4 : 6$
2. $\frac{2}{3} = \frac{4}{6}$

read 2 is to 3 as 4 is to 6.

295. The expression of the equality of two or more equal ratios is called a ***Proportion***.

296. A proportion of two simple ratios is a *simple proportion;* one of three or more ratios, a *multiple proportion.*

297. The ratios of a proportion are called *couplets.*

298. If, in a multiple proportion, the consequent of each couplet is the same as the antecedent of the following couplet, it is called a *continued proportion.*

Thus, $a : b :: b : c :: c : d$ is a continued proportion.

299. Every simple proportion has four terms. The first and fourth are called the *extremes;* the second and third the *means;* the first and third the *antecedents;* and the second and fourth the *consequents.*

300. A *mean proportional* between two quantities is a quantity to which the first bears the same relation that the quantity bears to the second.

Thus, b is a mean proportional between a and c, when $a : b :: b : c$.

301. A *third* proportional to two quantities is a quantity to which the second bears the same relation that the first bears to the second.

Thus, c is a third proportional to a and b, when $a:b::b:c$.

302. If a proportion contains one or more compound ratios, it is a *compound proportion.*

When the word "proportion" is used alone, it designates a simple proportion.

Propositions.

I. *In any proportion, the product of the extremes equals the product of the means.*

Given $a:b::c:d:$ (A)

Prove $a \times d = b \times c$

Demonstration: $\frac{a}{b}=\frac{c}{d}$ [another form for (A)],

Clear of fractions, $a \times d = b \times c$.

Corollary 1.—*Either extreme equals the product of the means divided by the other extreme.*

Cor. 2.—*Either mean equals the product of the extremes divided by the other mean.*

II. *If the product of two quantities equals the product of two other quantities, either pair may be made the extremes, and the other pair the means, of a proportion.*

Given $m \times n = p \times q$ (A)

Prove,

1. $m:p::q:n$	5. $p:m::n:q$
2. $m:q::p:n$	6. $p:n::m:q$
3. $n:p::q:m$	7. $q:m::n:p$
4. $n:q::p:m$	8. $q:n::m:p$

Demonstration: 1. Divide (A) by n, $m=\frac{p \times q}{n}$ (1)

Divide (1) by p, $\frac{m}{p}=\frac{q}{n}$ (2)

Write in another form, $m : p :: q : n$.

Let the pupil derive the remaining seven.

Exercise.—Write the eight proportions deducible from :

$$3 \times 4 = 2 \times 6;\ a \times d = b \times c;\ x \times z = v \times y$$

---

303. A proportion is taken by *alternation* when the means or the extremes are made to change places.

III. *If four quantities are in proportion, they are also in proportion by alternation.*

Given $a : b :: c : d$ (A)

Prove, 1. $a : c :: b : d$ 2. $d : b :: c : a$

Demonstration: $a \times d = c \times b$ [P. I],

$$\therefore \quad \left\{ \begin{array}{l} a : c :: b : d \\ d : b :: c : a \end{array} \right\} \text{ [P. II].}$$

Exercise.—Write by alternation :

$$3 : 4 :: 9 : 12;\ x : y :: m : n;\ x : a :: y : b$$

---

304. A proportion is taken by inversion when the means are made the extremes and the extremes the means.

IV. *If four quantities are in proportion, they are also in proportion by inversion.*

Given $a : b :: c : d$

Prove, 1. $b : a :: d : c$ 3. $c : a :: d : b$

2. $b : d :: c : a$ 4. $c : d :: a : b$

Demonstration: $a \times d = b \times c$,

$$\left\{ \begin{array}{l} b : a :: d : c \\ b : d :: a : c \end{array} \right\} \text{ and } \left\{ \begin{array}{l} c : a :: d : b \\ c : d :: a : b \end{array} \right\} \text{ [P. II].}$$

Exercise.—Write by inversion : $5 : 10 :: 15 : 30$;

$$x : m :: n : y;\ a + b : a - b :: c + d : c - d$$

---

305. A proportion is taken by *composition* when the sum of the two terms of each couplet is compared with either the antecedent or the consequent of that couplet.

V. *If four quantities are in proportion, they are also in proportion by composition.*

Given $a : b :: c : d$ (A)

Prove, 1. $a + b : b :: c + d : d$

2. $a + b : a :: c + d : c$

Demonstration: $\frac{a}{b} = \frac{c}{d}$ [another form of (A)]

Add 1 to both members, $\frac{a}{b} + 1 = \frac{c}{d} + 1$

Reduce to improper fractions, $\frac{a + b}{b} = \frac{c + d}{d}$; or

$$a + b : b :: c + d : d$$

Let the pupil prove the second part.

Exercise.—Write by composition:

$2 : 3 :: 6 : 9$; $8 : 2 :: 16 : 4$; $x : a :: y : b$

---

306. A proportion is taken by *division* when the difference of the two terms of each couplet is compared with either the antecedent or the consequent of that couplet.

VI. *If four quantities are in proportion, they are also in proportion by division.*

Given $a : b :: c : d$

Prove, 1. $a - b : b :: c - d : d$

2. $a - b : a :: c - d : c$

Demonstration: $\frac{a}{b} = \frac{c}{d}$ [another form of (A)]

Subtract 1 from both members, $\frac{a}{b} - 1 = \frac{c}{d} - 1$

Reduce, $\frac{a - b}{b} = \frac{c - d}{d}$; or

$$a - b : b :: c - d : d$$

Let the pupil prove the second part.

Exercise.—Write by division:

$3 : 9 :: 6 : 18$; $6 : 3 :: 12 : 6$; $a + x : x :: b + y : y$

307. A proportion is taken by *composition and division* when the sum of the two terms of each couplet is compared with the difference of these terms.

VII. *If four quantities are in proportion, they are also in proportion by composition and division.*

Given $a : b :: c : d$ (A)

Prove $a+b : a-b :: c+d : c-d$

Demonstration: Take (A) by composition, $\frac{a+b}{b} = \frac{c+d}{d}$ (1)

Take (A) by division, $\frac{a-b}{b} = \frac{c-d}{d}$ (2)

Divide (1) by (2), $\frac{a+b}{a-b} = \frac{c+d}{c-d}$; or

$$a+b : a-b :: c+d : c-d$$

Exercise.—Write by composition and division:

$3 : 8 :: 12 : 32$; $x : y :: m : n$; $x-y : x+y :: 3 : 5$

VIII. *If two proportions have a couplet in each the same, the remaining couplets form a proportion.*

Given $\begin{cases} a : b :: c : d & \text{(A)} \\ e : f :: c : d & \text{(B)} \end{cases}$

Prove $a : b :: e : f$

Demonstration: $\frac{a}{b} = \frac{c}{d}$ (A) and $\frac{e}{f} = \frac{c}{d}$ (B)

$\therefore$ $\frac{a}{b} = \frac{e}{f}$ (Ax. 1); whence

$$a : b :: e : f$$

Exercise.—Prove that,

Cor. 1.—*If two proportions have the antecedents alike, the consequents form a proportion;* or,

Given $a : b :: c : d$ and $a : x :: c : y$

Prove $b : d :: x : y$

Cor. 2.—*If two proportions have the consequents alike, the antecedents form a proportion;* or,

Given $a : b :: c : d$ and $x : b :: y : d$

Prove $a : c :: x : y$

Cor. 3.—*If two proportions have a couplet in proportion, the remaining couplets form a proportion;* or,

Given $a : b :: c : d$, $e : f :: g : h$, and $a : b :: e : f$

Prove $c : d :: g : h$

---

308. Equimultiples of two or more quantities are the products obtained by multiplying each of the quantities by the same number.

IX. *Equimultiples of two quantities are proportional to the quantities themselves.*

Given the two quantities a and b and their equimultiples ma and mb,

Prove $ma : mb :: a : b$

Demonstration: $\frac{ma}{mb} = \frac{a}{b}$

$\therefore$ $ma : mb :: a : b$

Exercise.—*Prove that equal parts of two quantities are proportional to the quantities themselves;* or that,

$$\frac{a}{m} : \frac{b}{m} :: a : b$$

X. *If four quantities are in proportion, equimultiples of the first couplet are proportional to equimultiples of the second couplet.*

Given $a : b :: c : d$ (A)

Prove $ma : mb :: nc : nd$

Demonstration: $\frac{a}{b} = \frac{c}{d}$ [another form of (A)]

$$\frac{a}{b} = \frac{ma}{mb}, \text{ and } \frac{c}{d} = \frac{nc}{nd}$$

$\therefore$ $\frac{ma}{mb} = \frac{nc}{nd}$, or

$$ma : mb :: nc : nd$$

Exercise.—Prove that,

1. Equal parts of the first couplet are proportional to equal parts of the second couplet.

2. Equimultiples of the antecedents are proportional to equimultiples of the consequents.

3. Either extreme may be multiplied and the other divided by the same quantity.

4. Either mean may be multiplied and the other divided by the same quantity.

XI. *If two quantities are increased or diminished by like parts of themselves, the results are proportional to the quantities themselves.*

Given the two quantities a and b, to prove,

1. $a + \frac{m}{n}a : b + \frac{m}{n}b :: a : b$

2. $a - \frac{m}{n}a : b - \frac{m}{n}b :: a : b$

Demonstration: $\frac{a\left(1 \pm \frac{m}{n}\right)}{b\left(1 \pm \frac{m}{n}\right)} = \frac{a}{b}$ $\quad \therefore \quad \frac{a \pm \frac{m}{n}a}{b \pm \frac{m}{n}b} = \frac{a}{b}$; or

$$a \pm \frac{m}{n}a : b \pm \frac{m}{n}b :: a : b$$

Exercise.—Given $a : b :: c : d$

Prove $a \pm \frac{m}{n}a : b \pm \frac{m}{n}b :: c \pm \frac{p}{q}c : d \pm \frac{p}{q}d$

XII. *If four quantities are in proportion, like powers and like roots of them are also in proportion.*

Given $a : b :: c : d$ (A)

Prove, 1. $a^n : b^n :: c^n : d^n$

2. $\sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}$

Demonstration: $a \times d = b \times c$ [P. I] (1)

Raise both members to the nth power,

$a^n \times d^n = b^n \times c^n$ (2)

Then, $a^n : b^n :: c^n : d^n$ [P. II] (3)

Extract the nth root of (1), $\sqrt[n]{a} \times \sqrt[n]{d} = \sqrt[n]{b} \times \sqrt[n]{c}$ (4)

Then, $\sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}$ (5)

XIII. *The corresponding members of two equations form a proportion.*

Given $a = b$ (A) and $c = d$ (B)

Prove $a : c :: b : d$

Demonstration: Divide (A) by (B), $\frac{a}{c} = \frac{b}{d}$; (1)

or $a : c :: b : d$

XIV. *The products or quotients of the corresponding terms of two proportions form a proportion.*

Given $\left\{ \begin{array}{ll} a : b :: c : d & \text{(A)} \\ e : f :: g : h & \text{(B)} \end{array} \right\}$

Prove, 1. $a \times e : b \times f :: c \times g : d \times h$

2. $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$

Demonstration: $a \times d = b \times c$ (1), and $e \times h = f \times g$ (2) [P. I]
Multiply (1) by (2), $(a \times e) \times (d \times h) = (b \times f) \times (c \times g)$ (3)
Therefore, $a \times e : b \times f :: c \times g : d \times h$ [P. II]

Let the pupil prove the second part.

XV. *If two proportions have three terms of the one equal to three terms of the other, each to each, the fourth terms are also equal.*

Given $\left\{ \begin{array}{ll} a : b :: c : d & \text{(A)} \\ a : b :: c : x & \text{(B)} \end{array} \right\}$

Prove $x = d$

Demonstration: $d = \frac{b \times c}{a}$, and $x = \frac{b \times c}{a}$ [P. I, Cor. 1]

Therefore, $x = d$

XVI. *In any multiple proportion the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Given $a : b :: c : d :: e : f$

Prove $a + c + e : b + d + f :: a : b$

Demonstration: Let r equal the ratio of each couplet,

Then, $\frac{a}{b} = r$, $\frac{c}{d} = r$, and $\frac{e}{f} = r$

Clear of fractions, $a = br$ (1), $c = dr$ (2), and $e = fr$ (3)

Add (1), (2), and (3), $a + c + e = (b + d + f)r$ (4)

Divide by $b + d + f$, $\frac{a + c + e}{b + d + f} = r = \frac{a}{b}$ (5)

Therefore, $a + c + e : b + d + f :: a : b$

XVII. *A mean proportional between two quantities equals the square root of their product.*

Given b, a mean proportional between a and c, to prove $b = \sqrt{ac}$.

Demonstration: $a : b :: b : c$ [317]

$\therefore$ $b^2 = ac$ [P. I]; whence $b = \sqrt{ac}$

Additional Propositions.

EXERCISE 128.

If $a : b :: c : d$, prove that:

1. $2a : 3b :: 2c : 3d$

2. $3a : 4b :: 6c : 8d$

3. $na : mb :: nc : md$

4. $a + c : a :: b + d : c$

5. $2a + 3b : 2c + 3d :: 2a : 3b$

6. $na + mb : nc + md :: a : c$

7. $a : d :: bc : d^2$

8. $a + ax : b + bx :: c + cy : d + dy$

9. $(a + c)x : (a - c)x :: (b + d)y : (b - d)y$

If $a : b :: b : c$, prove that:

10. $a : c :: a^2 : b^2$

11. $b^2 : c^2 :: a : c$

12. $a : a + b :: a - b : a - c$

13. $a - c : b - c :: b + c : c$

Clear of fractions:

14. $x : 2\frac{1}{2} :: 3\frac{3}{4} : 7\frac{1}{2}$

15. $\frac{1}{2}x : \frac{3}{4}y :: \frac{7}{8}a : \frac{5}{6}b$

3 Solution of Equations and Proportions.

Illustrations.—

1. Given $\frac{x+4}{x-4}=\frac{8}{7}$ to find the value of x. (A)

Solution: Write the equation in the form of a proportion,

$$x+4 : x-4 :: 8 : 7 \qquad (1)$$

Take (1) by composition and division,

$$2x : 8 :: 15 : 1 \qquad (2)$$

Divide first couplet by 2, $x : 4 :: 15 : 1$ (3)

$$\therefore \quad x = 4 \times 15 = 60.$$

---

2. Given

$$\left\{\begin{array}{lr} \sqrt{x}+\sqrt{y} : \sqrt{x}-\sqrt{y} :: a : b & (A) \\ xy = (a+b)^2 & (B) \end{array}\right\} \text{ to find } x \text{ and } y.$$

Solution: Take (A) by composition and division,

$$2\sqrt{x} : 2\sqrt{y} :: a+b : a-b \qquad (1)$$

Divide the first couplet by 2,

$$\sqrt{x} : \sqrt{y} :: a+b : a-b \text{ [P. 122]} \qquad (2)$$

Square the terms, $x : y :: (a+b)^2 : (a-b)^2$ [P. II] (3)

Multiply the first couplet by y,

$$xy : y^2 :: (a+b)^2 : (a-b)^2 \text{ [P. 122]} \qquad (4)$$

Substitute (B) in (4),

$$(a+b)^2 : y^2 :: (a+b)^2 : (a-b)^2 \qquad (5)$$

Divide the antecedents by $(a+b)^2$,

$$1 : y^2 :: 1 : (a-b)^2 \text{ [P. 122]} \qquad (6)$$

$$\therefore \quad y^2 = (a-b)^2$$

$$y = \pm(a-b)$$

Substitute the value of y in (3),

$$x : \pm(a-b) :: (a+b)^2 : (a-b)^2$$

Divide the consequents by $(a-b)$,

$$x : \pm 1 :: (a+b)^2 : (a-b)$$

$$\therefore \quad x = \pm\frac{(a+b)^2}{a-b}$$

EXERCISE 129.

Solve:

1. $x : x+6 :: 4 : 7$ **2.** $x+6 : x-6 :: 4 : 1$

3. $\sqrt{x}+\sqrt{3} : \sqrt{x} :: \sqrt{3}+5 : 5$

4. $\sqrt{x}+\sqrt{a} : \sqrt{x}-\sqrt{a} :: \sqrt{a}+\sqrt{b} : \sqrt{a}-\sqrt{b}$

5. $x^2 - a^2 : a^2 - b^2 :: x^2 + a^2 : a^2 + b^2$

6. $x^2 - a^2 : x + a :: x + a : 2$

7. $\sqrt{x^2 - a^2} : \sqrt{x - a} :: \sqrt{x + a} : x\sqrt{x}$

8. $\dfrac{\sqrt{x} + \sqrt{2}}{\sqrt{x} - \sqrt{2}} = \dfrac{\sqrt{2} + 1}{\sqrt{2} - 1}$

9. $\dfrac{\sqrt{x+3} + \sqrt{x-3}}{\sqrt{x+3} - \sqrt{x-3}} = \dfrac{1}{3}$

10. $\dfrac{\sqrt{x+a} + \sqrt{x-a}}{\sqrt{x+a} - \sqrt{x-a}} = b$

11. $\begin{cases} x + y : x :: 5 : 4 \\ x^2 - y^2 = 135 \end{cases}$

12. $\begin{cases} x + y : x - y :: 3 : 1 \\ x^3 - y^3 = 56 \end{cases}$

13. $x^3 + y^3 : x^3 - y^3 :: 35 : 19$
$x^2 + y^2 = 52$

---

4 Examples involving Proportion.

Illustration.—1. A's age is to B's as 2 to 3; but in 10 years their ages will be to each other as 3 to 4. Required the age of each.

Solution: Let $2x =$ A's age; then will
$3x =$ B's age; and
$2x + 10 =$ A's age 10 years hence; and
$3x + 10 =$ B's age 10 years hence;
then $2x + 10 : 3x + 10 :: 3 : 4$ (A)
$8x + 40 = 9x + 30$ (1)
$x = 10$
$2x = 20$, A's age;
$3x = 30$, B's age.

2. The sum of A's and B's capital is to the difference of their capitals as 9 to 5; but if A withdraws \$100 and B adds \$100, their capitals will be to each other as 4 to 3. Required the capital of each.

Solution: Let $x =$ A's capital
and $y =$ B's;
then $x + y : x - y :: 9 : 5$, (A)
and $x - 100 : y + 100 :: 4 : 3$ (B)

Solve (A) and (B), $x = \$423\frac{1}{13}$, $y = \$107\frac{9}{13}$

EXERCISE 130.

1. The length of a room is to its width as 4 to 3, and the floor contains 588 square feet of boards. What are the dimensions of the room?

2. A man's age is to his wife's as 6 to 5; but 30 years ago his age was to hers as 7 to 5. Required the age of each.

3. The difference of two numbers is to the difference of the squares of the numbers as 1 to 13, and the product of the numbers is 42. Find the numbers.

4. A and B are in partnership. A's capital is to the whole capital as 5 to 8; but if A withdraws $2000 and B adds $2000, A's capital will be to the whole capital as 3 to 5. Required each man's share of the stock.

5. The length of a rectangular field is to its width as 5 to 4; but if 4 rods be added to the length and 5 rods to the width, they will be to each other as 6 to 5. Find the area.

6. Two thirds of A's money is to $^3/_4$ of B's as 5 to 6, and $^2/_3$ of A's + $^3/_4$ of B's is $1500. Required the fortune of each.

7. The rate of a fast train is to that of a slow train as 5 to 3, and if it is 60 miles behind the slow train it will overtake it in 3 hours. What is the rate of each train?

8. I have a cubical box, such that if each of its dimensions be increased by one foot the contents will be to the entire surface as 1 to 2. Required the contents.

9. The circumferences of circles are to each other as their diameters. If the circumference of a circle whose diameter is *one* is $\pi = 3{\cdot}1416$, what is the circumference of a circle whose diameter is d?

10. The areas of circles are to each other as the squares of their diameters. If the area of a circle whose radius is

one is π, what is the area of a circle whose radius is r? What, when $r = 4$?

11. The surfaces of spheres are to each other as the squares of their diameters. If the surface of a sphere whose radius is *one* is 4π, what is the surface of a sphere whose radius is r? What, when $r = 5$?

12. The volumes of spheres are to each other as the cubes of their radii. If the volume of a sphere whose radius is *one* is $^4/_3\,\pi$, what is the volume of a sphere whose radius is r? What, when $r = 6$?

Surfaces and volumes that have the same shape are *similar.* To have the same shape, they must have their corresponding angles equal and their corresponding dimensions proportional.

13. Similar surfaces are to each other as the squares of their like dimensions. If a field a rods long contains m acres, what will a similar field c rods long contain?

14. Similar volumes are to each other as the cubes of their like dimensions. If a keg whose bung diameter is c inches holds n gallons, what will a similar keg d inches in bung diameter hold?

15. The quantities of water that flow through circular pipes are to each other as the squares of the diameters of the pipes. If c gallons flow through a pipe m inches in diameter in one minute, how many gallons will flow through a pipe n inches in diameter in the same time?

Limiting Ratios.

Definitions and Principles.

309. A quantity that retains the same value throughout an operation or discussion is a *constant.*

310. A quantity that continuously changes its value—

that is, passes from one value to another by successively assuming all values lying between them—is a *variable.*

Illustration.—A line a foot long is a constant. A line traced by a point moving according to some well-defined law is a variable.

311. A *finite unit* is a unit of comprehensible size or value.

312. A quantity that can be expressed in finite units is a *finite quantity.*

313. A quantity too small to be expressed in finite units is said to be *infinitely small.* An infinitely small variable is called an *infinitesimal,* and may be expressed by the character $\circ$, read an *infinitesimal* or *zeroid.*

314. A quantity too large to be expressed in finite units is said to be *infinitely large.* An infinitely large variable is called an *infinite,* and may be expressed by the character $\propto$, read an *infinite.*

315. The entire absence of quantity is called *zero,* and is expressed by the character 0, read *zero.*

316. The unlimited whole of quantity, or rather unlimited quantity, is called *infinity,* and is expressed by the character ∞, read *infinity.*

317. If, in the fraction $\frac{x}{a}$, x decreases by a constant ratio until it becomes an infinitesimal and a remains a finite constant, the value of the fraction decreases in the same ratio [P. 55], and becomes an infinitesimal.

Therefore,

Prin. 123. $\frac{\circ}{a} = \circ$. *An infinitesimal divided by a finite constant is an infinitesimal.*

318. Since $\frac{\circ}{a} = \circ$, it follows that,

Prin. 124. $\circ \times a = \circ$. *An infinitesimal multiplied by a finite constant is an infinitesimal.*

319. Since $\circ \times a = \circ$, it follows that,

Prin. 125. $\frac{\circ}{\circ} = a$. *An infinitesimal divided by an infinitesimal may be any finite constant.*

320. If, in the fraction $\frac{x}{a}$, x increases by a constant ratio until it becomes an infinite and a remains a finite constant, the value of the fraction increases in the same ratio [P. 54], and becomes an infinite. Therefore,

Prin. 126. $\frac{\infty}{a} = \infty$. *An infinite divided by a finite constant is an infinite.*

321. Since $\frac{\infty}{a} = \infty$, it follows that,

Prin. 127. $\infty \times a = \infty$. *An infinite multiplied by a finite constant is an infinite.*

322. Since $\infty \times a = \infty$, it follows that,

Prin. 128. $\frac{\infty}{\infty} = a$. *An infinite divided by an infinite may be any finite constant.*

323. If, in the fraction $\frac{a}{x}$, x decreases by a constant ratio until it becomes an infinitesimal and a remains a finite constant, the value of the fraction increases in the same ratio [P. 54], and becomes an infinite. Therefore,

Prin. 129. $\frac{a}{\circ} = \infty$. *A finite constant divided by an infinitesimal is an infinite.*

324. Since $\frac{a}{\circ} = \infty$, it follows that,

Prin. 130. $\circ \times \infty = a$. *The product of an infinitesimal and an infinite may be any finite constant.*

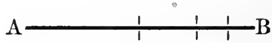
325. Since $\circ \times \propto = a$, it follows that,

Prin. 131. $\frac{a}{\propto} = \circ$. *A finite constant divided by an infinite is an infinitesimal.*

326. Since $\frac{\circ}{\circ}$, $\frac{\propto}{\propto}$, and $\propto \times \circ$ are each satisfied by any finite constant, they are *symbols of indetermination.*

327. The limit of a variable is a value which the variable continually approaches but which it can never reach, but may be made to differ from it by less than any assignable quantity.

Illustration.—If a point starts at A in the direction of B, and goes $1/2$ the distance the first second, $1/2$ the remaining distance the next, $1/2$ the remaining distance the third, and so on, the distance passed over constantly approaches the distance from A to B, and will eventually differ from this distance by an infinitesimal, but it can never equal this distance. From A to B is therefore the limit of the distance the point can go.



328. The limit of a variable that decreases by a constant ratio is zero.

Illustration.—If $1/2$ a line be cut off, then $1/2$ the remainder, and so on indefinitely, the part retained continually approaches zero, from which it will eventually differ by less than any assignable quantity. Therefore, zero is the limit of the remainder.

329. A variable quantity that increases by a constant ratio has no limit. This fact is sometimes expressed by saying that its limit is *infinity.*

330. A *function* of a variable quantity is any expression that contains the variable.

Thus, $ax^2 + b$ is a function of x.

331. A function of a variable is generally a variable also. It is then called the *dependent* variable, and the variable upon which it depends the *independent* variable.

332. The limit of a function, when the independent variable approaches its limit, may be zero, infinity, or a finite quantity.

Illustration.—1. If x approaches a as a limit, the function $\frac{x-a}{x}$ approaches $\frac{0}{a}$, or 0 as a limit.

2. If x approaches a as a limit, the function $\frac{x}{x-a}$ approaches $\frac{a}{0}$, or ∞ as a limit.

3. If x approaches a as a limit, the function $\frac{x}{x+a}$ approaches $\frac{a}{a+a}$, or $\frac{1}{2}$ as a limit.

333. Sometimes a factor whose limit is *zero* is common to both terms of a fraction. The limit of the fraction will then assume the irreducible form $\frac{0}{0}$. The true limit is then found by removing the common factor before passing to the limit.

Illustration.—If x approaches a as a limit, the function $\frac{x^3-a^3}{x^2-a^2}$ approaches $\frac{0}{0}$ as a limit. This form results, because the common factor $x-a$ has 0 for its limit. Removing this factor, we have $\frac{x^2+ax+a^2}{x+a}$, which has for its limit $\frac{a^2+a^2+a^2}{a+a}=\frac{3a^2}{2a}=\frac{3}{2}a$.

334. To express that the limit of $\frac{x^3-a^3}{x^2-a^2}$ is $\frac{3}{2}a$ when the limit of $x=a$, we write:

$$\text{Lim. } \frac{x^3-a^3}{x^2-a^2}\,(x=a)=\frac{3}{2}a.$$

Examples.

Illustrations.—1. Find Lim. $\frac{x^3 - a^3}{x - a}(x = a)$.

Solution: $\frac{x^3 - a^3}{x - a} = x^2 + ax + a^2$, $\therefore$ Lim. $\frac{x^3 - a^3}{x - a}(x = a) =$
Lim. $x^2 + ax + a^2, (x = a) = a^2 + a^2 + a^2 = 3a^2$.

2. Find Lim. $\frac{x - a}{x}(x = \infty)$.

Solution: $\frac{x - a}{x} = 1 - \frac{a}{x}$, $\therefore$ Lim. $\frac{x - a}{x}(x = \infty) =$
Lim. $1 - \frac{a}{x}, (x = \infty) = 1 - \frac{a}{\infty} = 1 - 0 = 1$.

EXERCISE 131.

Find:

1. Lim. $\frac{x^2 - a^2}{x - a}(x = a)$
2. Lim. $\frac{x - 1}{x}(x = 1)$
3. Lim. $\frac{ax + a}{x^2 + x}(x = 1)$
4. Lim. $\frac{x^2 - a^2}{(x - a)^2}(x = a)$
5. Lim. $\frac{x - x^6}{1 - x}(x = 1)$
6. Lim. $x^2 + \frac{1}{x^2}(x = 0)$
7. Lim. $\frac{x - a}{\sqrt{x} - \sqrt{a}}(x = a)$
8. Lim. $\frac{x^6 - a^6}{x + a}(x = a)$
9. Lim. $\frac{x}{x + 1}(x = \infty)$
10. Lim. $\frac{mx^2}{mx^2 + nx}(x = 0)$
11. Lim. $\frac{x^n - a^n}{x - a}(x = a)$
12. Lim. $\frac{x^m - 1}{x - 1}(x = 1)$
13. Lim. $\frac{x + a}{\sqrt{x + a}}(x = -a)$
14. Lim. $\frac{a^2}{a - x}(x = 0)$
15. Lim. $\frac{ax + bx}{x^3 - ax^2}(x = \infty)$
16. Lim. $\frac{ax + x + 1}{x}(x = \infty)$
17. Lim. $\frac{x - \sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}}(x = a)$
18. Lim. $\frac{x^4 + a^2x^2 + a^4}{x^2 + ax + a^2}(x = a)$

Arithmetical Progressions.

Definitions and Principles.

335. Any number of quantities that increase or decrease according to a law constitute a ***Series,*** or ***Progression.***

336. The quantities which compose a series, or progression, are called the *terms.*

337. A progression in which each term after the first is derived from the preceding term by the addition of a constant quantity, is an *arithmetical progression.*

338. The constant quantity added to any term of an arithmetical progression to produce the next term, is called the *common difference.*

339. If the common difference is positive, the series, or progression, is an *ascending* one; if negative, a *descending* one.

Thus, a, $3a$, $5a$, $7a$, etc., is an ascending series;

and, $7a$, $5a$, $3a$, a, etc., is a descending series.

340. In the general discussion of arithmetical progressions, a represents the first term, d the common difference, l the last term, n the number of terms, and S the sum of the terms.

341. If we represent the first term by a and the common difference by d, the

2d term $= a + d$ 4th term $= a + 3d$

3d term $= a + 2d$ 5th term $= a + 4d$

Here we observe that each term equals the first term plus the common difference multiplied by the number of terms less *one*; hence, the nth term $= a + (n-1)d$; but the nth term is the last term l; therefore,

Prin. 132. $l = a + (n-1)d$. [Formula A.]

Because $l - d + d = l$.
also $l - 2d + l = l$-

342. If we represent the sum by S, we have,

$$S = a + (a + d) + (a + 2d) + \ldots .$$
$$(l - 2d) + (l - d) + l \quad \text{(A)}$$

If we write the series in an inverse order,

$$S = l + (l - d) + (l - 2d) + \ldots .$$
$$(a + 2d) + (a + d) + a \quad \text{(B)}$$

Add (A) and (B),

$$2S = (l + a) + (l + a) + (l + a) + \ldots .$$
$$(l + a) + (l + a) + (l + a); \text{ or}$$

$2S = (l + a)$ taken n times $= (l + a)n$; therefore,

Prin. 133. $S = (l + a)\frac{n}{2}$. [Formula B.]

Examples involving Arithmetical Progressions.

Illustrations.—1. Find the last term and sum of the series: 3, 7, 11, 15, etc., to 10 terms.

Solution: Here $a = 3$, $d = 4$, and $n = 10$. Substitute these values in formula A:

$$l = a + (n - 1)d$$
$$l = 3 + (10 - 1) \times 4 = 39$$

Substitute the values of l, a, and n in formula B:

$$S = (l + a)\frac{n}{2}$$
$$S = (39 + 3) \times 5 = 210$$

---

2. The first term of an arithmetical progression is 25, the number of terms is 6, and the sum of the terms is 102. Required the last term.

Solution: Substitute the values $a = 25$, $n = 6$, and $S = 102$ in formula B:

$$S = (l + a)\frac{n}{2}$$
$$102 = (l + 25)3 = 3l + 75$$
$$3l = 27$$
$$l = 9$$

3. Given $l=31$, $d=4$, and $S=136$, to find n.

Solution: Substitute these values in formulas (A) and (B):

1. Since $l=a+(n-1)d$, $31=a+(n-1)4$, or $a+4n=35$ (A)

2. Since $S=(l+a)\frac{n}{2}$, $136=(31+a)\frac{n}{2}$, or $31n+an=272$ (B)

Transpose (A), $a=35-4n$ (1)

Substitute (1) in (B),

$$31n+35n-4n^2=272 \qquad (2)$$

Transpose (2), $4n^2-66n=-272$ (3)

Complete the square,

$$4n^2-66n+\left(\frac{33}{2}\right)^2=-\frac{1088}{4}+\frac{1089}{4}=\frac{1}{4} \qquad (4)$$

Extract the $\sqrt{}$, $2n-\frac{33}{2}=\pm\frac{1}{2}$ (5)

Transpose, $2n=16$ or 17

Divide, $n=8$ or $8\frac{1}{2}$

Note.—Since the number of terms is a whole number, 8 is the true answer.

EXERCISE 132.

1. Find the 10th term and the sum of 10 terms of the series: 4, 8, 12, etc.

2. Find the 12th term and the sum of 12 terms of the series: 27, 25, 23, etc.

3. Find the 9th term and the sum of 9 terms of the series: $\frac{1}{2}+\frac{5}{6}+\frac{7}{6}$, etc.

4. Find the nth term and the sum of the n terms of the series: 1, 2, 3, etc.

5. Find the rth term and the sum of r terms of the series: 2, 4, 6, etc.

6. Given $a=3$, $l=28$, and $n=6$, find d.

7. Given $S=112$, $n=7$, and $a=25$, find l and d.

8. Given $n=8$, $a=8$, and $d=5$, find S and l.

9. Given $d=1\frac{1}{2}$, $S=58$, and $a=2$, find l and n.

10. Show that $d = \frac{l - a}{n - 1}$ **11.** Show that $n = \frac{2S}{l + a}$

12. Show that $n = \frac{l - a}{d} + 1$

13. Show that $l = \frac{2S}{n} - a$ **14.** Show that $a = \frac{2S}{n} - l$

15. Show that $l = \frac{S}{n} + \frac{n - 1}{2} \times d$

16. Show that $S = \frac{1}{2} n [2a + (n - 1) d]$

17. Show that $S = \frac{l + a}{2} + \frac{l^2 - a^2}{2d}$

18. Show that $a = \frac{S}{n} - \frac{n - 1}{2} d$

19. Show that $d = \frac{2(S - an)}{n(n - 1)}$

20. Given $d = \frac{1}{2}$, $l = 6\frac{1}{2}$, and $S = 45$, to find a and n.

21. Given $d = 4$, $S = 190$, and $a = 1$, to find l and n.

22. Given $d = 3$, $l = 35$, and $S = 220$, to find n and a.

23. Show that $n = \frac{d - 2a \pm \sqrt{(2a - d)^2 + 8dS}}{2d}$

Concrete Examples involving Arithmetical Progressions.

Illustrations.—1. Insert three arithmetical means between 3 and 11.

Solution: Since there are to be three arithmetical means, the number of terms is 5, the first term is 3, and the last term is 11.

Take $l = a + (n - 1) d$

Substitute, $11 = 3 + 4d$

$d = 2$

$\therefore$ The means are 5, 7, and 9.

2. Find the series whose nth term is $4n - 1$.

Solution: Since the nth term may be any term,

Let $n = 1$, then nth term = 1st term = $4 - 1 = 3$
Let $n = 2$, then nth term = 2d term = $8 - 1 = 7$
Let $n = 3$, then nth term = 3d term = $12 - 1 = 11$
Etc., etc., etc.

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3. Find the series the sum of n terms of which is $n^2 + n$.

Solution:

Let $S = n^2 + n$
Let $n = 1$, then S = first term = 2
Let $n = 2$, then S = sum of two terms = $4 + 2 = 6$
Let $n = 3$, then S = sum of three terms = $9 + 3 = 12$
Let $n = 4$, then S = sum of four terms = $16 + 4 = 20$
Etc., etc., etc.

Since the sum of two terms is 6 and the first term is 2, the second term is $6 - 2 = 4$.

Since the sum of three terms is 12 and the sum of two is 6, the third term is $12 - 6 = 6$.

Similarly the fourth term is $20 - 12 = 8$.

And the series is 2, 4, 6, 8, etc.

---

4. The sum of five numbers in arithmetical progression is 30, and the difference of the squares of the extremes is 96. Required the numbers.

Solution: Let x equal the middle term and y the common difference, then will the numbers be

$$x - 2y,\ x - y,\ x,\ x + y,\ x + 2y \qquad (A)$$

Since their sum is 30, $5x = 30$ (1)

$x = 6$ (2)

Since the difference of the squares of the extremes is 96,

$$(x + 2y)^2 - (x - 2y)^2 = 96 \qquad (3)$$

Expand and collect terms, $8xy = 96$ (4)

$xy = 12$ (5)

Substitute (2) in (5) and reduce, $y = 2$

Substitute the values of x and y in (A), the series becomes 2, 4, 6, 8, 10.

---

5. The sum of four numbers in arithmetical progression is 24, and the product of the means is 35. Find the numbers.

Solution: Let $x-y$ and $x+y$ be the two means, the common difference being $2y$, then will the series be

$$x-3y,\ x-y,\ x+y,\ x+3y \qquad \text{(A)}$$

Since the sum is 24, $4x=24$ (1)

$x=6$ (2)

Since the product of the means is 35, $x^2-y^2=35$ (3)

Substitute (2) in (3) and reduce, $y=\pm 1$ (4)

Substitute the values of x and y in (A), the series becomes

3, 5, 7, 9,

or 9, 7, 5, 3

---

343. Any arithmetical series of an even number of terms may be formed by putting $x-y$ and $x+y$ for the two middle terms, making $2y$ the common difference.

344. Any series of an odd number of terms is more conveniently formed by putting x for the middle term and y for the common difference.

EXERCISE 133.

1. The sum of three numbers in arithmetical progression is 30, and their product is 910. Required the numbers.

2. The amounts of \$100 for 1, 2, and 3 years respectively are \$105, \$110, and \$115. What is the amount of the same sum for 15 years?

3. What is the amount of \$200 for 10 years at 6%, simple interest?

4. Insert three arithmetical means between 2 and 22.

5. Find the sum of all the whole numbers from 1 to 100, inclusive.

6. There are four numbers in arithmetical progression whose sum is 38, and the product of the extremes is 70. Find the numbers.

7. If the nth term of a series is $2n-1$, what is the series?

8. If a body falls through $16\frac{1}{12}$ feet the first second, three times as far the next second, five times as far the next, and so on, how far will it fall in half a minute? 14475

9. A man walks 1 mile and back the first day, 2 miles and back the second, 3 miles and back the third, and so on. In how many days will he walk 72 miles?

10. A man put out at interest $1 at the end of each month for 10 years. What did the interest amount to at 6%, simple interest?

11. The sum of five numbers in arithmetical progression is 15, and the sum of their squares is 55. Find the numbers.

12. If the sum of n terms of a series is $\frac{3n^2 + n}{2}$, find the series.

13. A travels 2 miles the first day, 4 the second, 6 the third, and so on. Five days later B starts out and travels uniformly 24 miles a day. In how many days will he overtake A?

14. The product of two numbers is 28, and the product of the two arithmetical means between them is 60. Find the numbers.

15. A man increased his capital stock $500 at the end of each year for 10 years, and then had invested $6500. What was his capital at first?

Geometrical Progressions.

Definitions and Principles.

345. A series in which each term after the first is derived from the preceding one by multiplying it by a constant quantity, called the *ratio*, is a *geometrical progression*.

346. If the ratio is greater than *one*, the series is an *ascending* one; if less than *one*, a *descending* one.

Thus, a, $3a$, $9a$, $27a$, etc., is an ascending series;
and, $27a$, $9a$, $3a$, a, etc., is a descending series.

347. If we represent the first term by a, the ratio by r, the number of terms by n, and the last term by l, the

2d term $= ar$ 4th term $= ar^3$

3d term $= ar^2$ 5th term $= ar^4$

Here we observe that each term equals the first term multiplied by the ratio raised to a power whose exponent is one less than the number of terms; hence, the nth term $= ar^{n-1}$. But the nth term is l. Therefore,

Prin. 134. $l = ar^{n-1}$. [Formula A.]

348. If we represent the sum of a geometrical series by S, we have,

$$S = a + ar + ar^2 + ar^3 + \ldots \frac{l}{r} + l \quad (1)$$

Multiply (1) by r,

$$rS = ar + ar^2 + ar^3 + \ldots l + lr \quad (2)$$

Subtract (1) from (2),

$$(r-1)S = lr - a. \quad \text{Therefore,} \quad (3)$$

Prin. 135. $S = \dfrac{lr - a}{r - 1}$. [Formula B.]

Cor. $S = \dfrac{ar^n - a}{r - 1}$, since $lr = ar^n$.

Examples in Geometrical Progression.

Illustrations.—

1. Find the 9th term of the series: 2, 4, 8, etc.

Solution: Here $a = 2$, $r = 2$, and $n = 9$. Substitute these values in formula (A),

$$l = ar^{n-1} = 2 \times 2^8 = 2^9 = 512.$$

2. Find the sum of 7 terms of the series: 3, 9, 27, etc.

Solution: Here $a=3$, $r=3$, and $n=7$, to find l and S. Substitute in formulas (A) and (B),

1. $l = ar^{n-1} = 3 \times 3^6 = 3^7 = 2187$
2. $S = \frac{lr-a}{r-1} = \frac{2187 \times 3 - 3}{2} = 3279$

3. Given $a=2$, $r=2$, and $l=256$, find n.

Solution: Substitute these values in formula (A),

$$l = ar^{n-1}$$

$$\therefore \quad 256 = 2 \times 2^{n-1} \quad (1)$$

$$\text{Divide by 2,} \quad 128 = 2^{n-1}, \quad (2)$$

$$\text{But} \quad 128 = 2^7, \quad (3)$$

$$\therefore \quad 2^{n-1} = 2^7, \quad (4)$$

$$\text{or} \quad n-1 = 7, \quad (5)$$

$$\text{and} \quad n = 8 \quad (6)$$

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4. Given $l=320$, $r=2$, and $n=7$, to find a and S.

Solution: Substitute these values in formulas (A) and (B),

1. $l = ar^{n-1}$

$\therefore \quad 320 = a \times 2^6 = 64a$

and $\quad a = 5$

2. $S = \frac{lr-a}{r-1} = \frac{320 \times 2 - 5}{1} = 635$

EXERCISE 134.

1. Find the 8th term of the series: 2, 6, 18, etc.
2. Find the 7th term of the series: 4, -12, 36, etc.
3. Find the 8th term of the series: 162, 54, 18, etc.
4. Find the 10th term of the series:

$$1, \; -\frac{1}{2}, \; \frac{1}{4}, \; -\frac{1}{8}, \text{ etc.}$$

5. Find the nth term of the series: 1, 2, 4, etc.
6. Find the sum of 6 terms of the series:

$$3 + 12 + 48, \text{ etc.}$$

7. Find the sum of 7 terms of the series: $1, \frac{1}{3}, \frac{1}{9}$, etc.
8. Find the sum of n terms of the series: $1, \frac{1}{2}, \frac{1}{4}$, etc.

9. Given $a = 3$, $r = 2$, and $n = 6$, find l and S

10. Given $a = 3$, $r = 3$, and $S = 363$, find l and n

11. Given $r = \frac{1}{2}$, $S = \frac{63}{128}$, and $n = 6$, find l and a

12. Given $a = 2r$, $r = r$, and $n = 10$, find l and S

13. Show that $a = \frac{l}{r^{n-1}}$

14. Show that $r = \sqrt[n-1]{\frac{l}{a}}$

15. Show that $S = \frac{lr^n - l}{r^n - r^{n-1}}$

16. Show that $a = (l - S)r + S$

17. Show that $l = \frac{S(r-1) + a}{r}$

18. Show that $ar^n - Sr = a - S$

19. If the first term of a geometrical progression is 2, the number of terms 4, and the sum of the terms 80, what is the series?

20. Show that the following series are in geometrical progression:

1. $x^2,\ xy,\ y^2$
2. $x,\ \sqrt{xy},\ y$
3. $\frac{x^2}{y},\ x,\ y,\ \frac{y^2}{x}$
4. $x,\ xy,\ xy^2,\ xy^3$

Concrete Examples involving Geometrical Progressions.

Illustrations.—1. Insert three geometrical means between 3 and 764.

Solution: Since there are to be three means, the number of terms in the series will be 5, the first term 3, and the last term 768.

Take $\quad l = ar^{n-1}$

Substitute, $\quad 768 = 3r^4$

$$r^4 = 256$$

$$r = 4$$

$\therefore$ The means are 12, 48, and 192.

2. Find the series whose nth term is 2^n.

Solution: Since the nth term may be any term,
Substitute $n=1$, $n=2$, $n=3$, etc., in

$$\begin{aligned} n\text{th term} &= 2^n \\ \text{1st term} &= 2^1 = 2 \\ \text{2d term} &= 2^2 = 4 \\ \text{3d term} &= 2^3 = 8 \\ \text{4th term} &= 2^4 = 16 \end{aligned}$$

Etc., etc., etc.

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3. Find the series the sum of n terms of which is $3^n - 1$.

Solution: Given $S = 3^n - 1$,

Let $n=1$, then $S=$ first term $= 3 - 1 = 2$
Let $n=2$, then $S=$ sum of two terms $= 3^2 - 1 = 8$
Let $n=3$, then $S=$ sum of three terms $= 3^3 - 1 = 26$
Let $n=4$, then $S=$ sum of four terms $= 3^4 - 1 = 80$

Since the sum of two terms is 8 and the first term is 2, the second term is $8 - 2 = 6$.

The third term is $26 - 8 = 18$.
The fourth term is $80 - 26 = 54$.
∴ The series is 2, 6, 18, 54, etc.

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4. The sum of three numbers in geometrical progression is 63, and their product is 1728. Find the numbers.

Solution: Let x^2, xy, and y^2 be the numbers,

then $x^2 + xy + y^2 = 63$ (A)
and $x^3y^3 = 1728$ (B)
Extract the $\sqrt[3]{B}$, $xy = 12$ (1)
Add (1) to (A), $x^2 + 2xy + y^2 = 75$ (2)
Extract $\sqrt{(2)}$, $x + y = \pm 5\sqrt{3}$ (3)
Subtract 3 times (1) from (A),
$x^2 - 2xy + y^2 = 27$ (4)
Extract $\sqrt{(4)}$, $x - y = \pm 3\sqrt{3}$ (5)
Add (5) and (3), $2x = \pm 8\sqrt{3}$ (6)
$x = \pm 4\sqrt{3}$ (7)
$x^2 = 48$
Subtract (5) from (3), $2y = \pm 2\sqrt{3}$ (8)
$y = \pm\sqrt{3}$
$y^2 = 3$

The numbers are 48, 12, and 3.

EXERCISE 133.

1. A man increases his capital stock at the end of each year by $\frac{1}{5}$ of itself. If he begins with \$100, what will his stock be at the end of 8 years?

2. If a rangeman begins with 256 head of cattle, and increases his herd each year by 25% of itself, in how many years will he have 625 cattle?

3. A has twice as much money as B, B twice as much as C, C twice as much as D, and D twice as much as E, and they together have \$6200. How much has each?

4. Insert two geometrical means between $\frac{1}{2}$ and $\frac{1}{16}$.

5. Insert three geometrical means between $2\frac{1}{2}$ and $\frac{405}{512}$.

6. An elastic ball is thrown up 10 feet, then falls and rebounds 5 feet, then falling rebounds $2\frac{1}{2}$ feet, and so on. How many times must it rebound to pass over $38\frac{1}{8}$ feet?

7. Show that the geometrical mean between a and b is $\sqrt{ab}$. Find the geometrical mean between 2 and 8.

8. There are three numbers in geometrical progression. The product of the first two is 75, and the product of the last two is 225. Find the numbers.

9. There are four numbers in geometrical progression. The product of the first and third is 9, and the product of the second and fourth is 81. Find the numbers.

10. The sum of three numbers in geometrical progression is 84, and the quotient of the third and first is 16. Find the numbers.

11. Find the series whose nth term is 2×3^n.

12. The sum of three numbers in geometrical progression is 42, and the sum of their squares is 1092. Find the numbers.

13. The three digits of a number are in geometrical progression; the sum of the digits is 14 and their product is 64. Find the number.

14. The sum of n terms of a series is $\frac{3}{2}(3^n - 1)$. Find the series.

15. A milkman drew a gallon of milk from a can containing 40 quarts, then put in the can a gallon of water; he then drew off a gallon of the mixture and put in its place a gallon of water. He did this five times. What part of the contents then was water?

16. Find the sum of n terms of the series $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$, etc.

17. Insert three geometrical means between 3 and 243, and find their sum.

Infinite Series.

Definitions and Principles.

349. Any series of an unlimited number of terms is called an ***Infinite*** series.

350. When the sum of n terms of a series constantly approaches some definite value as n increases indefinitely, the series is said to be *convergent*.

Thus, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$ etc., is a convergent series, since the greater the number of terms taken, the nearer will their sum approach to 2.

351. A series that is not convergent is called *divergent*.

352. The *limit* of a convergent series is the value which the sum of n terms continually approaches as n is increased indefinitely, but which it can never quite reach, though it may be made to differ from it by less than any assignable quantity.

Thus, 2 is the limit of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$ etc.

353. In any geometrical progression,

$$S=\frac{ar^n-a}{r-1}=\frac{a-ar^n}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}$$

Suppose $r<1$, then r^n approaches 0 as a limit as n is increased indefinitely, and hence the limit of $\frac{ar^n}{1-r}$ is also 0; and the limit (L) of S is $\frac{a}{1-r}$

Therefore,

Prin. 136. $L=\frac{a}{1-r}$ (C)

Examples involving Infinite Series.

Illustrations.—

1. Find the limit of the series: 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, etc.

Solution: Here $a=3$, and $r=\frac{1}{3}$. Substitute these values in formula (C),

$$L=\frac{a}{1-r}=\frac{3}{1-\frac{1}{3}}=4\frac{1}{2}$$

2. Find the value of the circulating decimal $\cdot\dot{3}\dot{6}$.

Solution: $\cdot\dot{3}\dot{6}=\cdot363636$, etc. $=\frac{36}{100}+\frac{36}{10000}+\frac{36}{1000000}+$ etc.

Here $a=\frac{36}{100}$ and $r=\frac{1}{100}$. Substitute these values in (C),

$$L=\frac{a}{1-r}=\frac{\frac{36}{100}}{1-\frac{1}{100}}=\frac{36}{99}=\frac{4}{11}$$

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3. Find the value of the circulating decimal $\cdot 2\dot{4}$.

Solution: $\cdot 2\dot{4}=\cdot 24444$, etc. $=\frac{2}{10}+\left(\frac{4}{100}+\frac{4}{1000}+\frac{4}{10000}+\text{ etc.}\right)$

The limit of $\left(\frac{4}{100}+\frac{4}{1000}+\frac{4}{10000}+\text{ etc.}\right)=\frac{a}{1-r}=\frac{\frac{4}{100}}{1-\frac{1}{10}}=\frac{4}{90}$

$$\frac{2}{10}+\frac{4}{90}=\frac{18}{90}+\frac{4}{90}=\frac{22}{90}=\frac{11}{45}$$

4. A hound is 20 rods behind a fox, and runs 2 rods to the fox's one. How far must the hound run to catch the fox?

Solution: While the hound runs the 20 rods the fox is ahead, the fox runs 10 rods; then, while the hound runs these 10 rods, the fox runs 5 rods; then, while the hound runs these 5 rods, the fox runs $2\frac{1}{2}$ rods, etc. Therefore the hound runs in all the sum of

$$20 \text{ rd.} + 10 \text{ rd.} + 5 \text{ rd.} + 2\frac{1}{2} \text{ rd.} + \text{etc.; or, since}$$

$$L = \frac{a}{1-r} = \frac{20}{1-\frac{1}{2}} = 40 \text{ rods.}$$

EXERCISE 136.

1. Find the limit of the series:

1. $2 + \frac{2}{3} + \frac{2}{9} + \text{etc.}$
2. $9 + 3 + 1 + \text{etc.}$
3. $10 + 1 + \frac{1}{10} + \text{etc.}$
4. $12\frac{1}{2} + 6\frac{1}{4} + 3\frac{1}{8} + \text{etc.}$

2. Find the limit of the series:

1. $a + \frac{a}{b} + \frac{a}{b^2} + \text{etc.}$
2. $ax + x + \frac{x}{a} + \text{etc.}$
3. $a + b + \frac{b^2}{a} + \text{etc.}$
4. $\frac{x}{y} + \frac{x^2}{y^2} + \frac{x^3}{y^3} + \text{etc.}$

3. Find the value of:

1. $\cdot\dot{4}\dot{5}$
2. $\cdot\dot{1}2\dot{4}$
3. $\dot{1}72\dot{8}$
4. $\cdot3\dot{6}$
5. $\cdot0\dot{1}\dot{2}$
6. $\cdot01\dot{2}$

4. A ball is thrown up 10 feet, falls and rebounds 5 feet, then falls and rebounds $2\frac{1}{2}$ feet, and so on. How far does it move before it stops?

5. An officer is 100 rods behind a thief, and goes 3 rods while the thief goes 2 rods. How far must the officer go to catch the thief?

6. At what time after 4 o'clock are the hour and minute hands of a watch together?

CHAPTER VIII.

Miscellaneous Examples.

EXERCISE 137.

1. If $x = 1$, $y = 3$, $z = 5$, $u = 0$, find the value of $x^2 + 2y^2 + 3z^2 + 4u^2$

2. If $x = 2$, $y = 0$, $z = -1$, $u = 1$, find the value of $(xy - uz)(yz - ux)(uy - xz)$

3. Find the value of:

$$\frac{2x^2 + 2y^2 - 2z^3 + 4xy}{3x^2 - 3y^2 - 3z^2 + 6yz} \text{ if } x = 4, \ y = \frac{1}{2}, \ z = 1$$

4. Add $\frac{2}{3}a + \frac{3}{4}b + \frac{5}{6}c$, $\frac{3}{4}a - \frac{2}{3}b + \frac{1}{4}c$, and $\frac{5}{6}b - \frac{1}{4}a + \frac{2}{3}c$

5. $3ax + 5bx - 2cx + 7dx - 4ax + 6bx + 11cx - 6dx$ equals how many times x?

6. From $\frac{3}{4}x - \frac{4}{5}y + \frac{2}{3}z$ take $\frac{1}{2}x + \frac{3}{4}y - \frac{4}{5}z$

7. Simplify $a - [\{-a - (a + a) - a\} - a - (a + a)]$

8. From $am^2 + bmn + cn^2$ take $(b - c)m^2 - (a - c)mn - (b - a)n^2$

9. What is the value of $4(mq - np) - \{(m - n) - (p - q)\}^2$, if $m = 0$, $n = 2$, $p = -3$, $q = 4$?

10. Multiply $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{-\frac{2}{3}} + x^{-\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}$

11. Expand $\left(x^{\frac{1}{2}} + \frac{1}{2}\right)\left(x^{\frac{1}{2}} - \frac{1}{2}\right)\left(x + \frac{1}{4}\right)\left(x^2 + \frac{1}{16}\right)$

12. Divide $x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc$ by $x+b$

13. Write the quotient of $a^{10}+b^{10}$ divided by a^2+b^2

14. Factor $x^{12}+y^{12}$ **15.** Factor $x^8+x^4y^4+y^8$

16. Find the H. C. D. of $ax+ay+bx+by$ and $cx+cy-dx-dy$

17. Find the L. C. M. of $9x^2+12x+4$ and $3x^2+11x+6$

18. Simplify $\dfrac{x^2+xy+y^2}{x^2-xy+y^2} \times \dfrac{x^3+y^3}{x^3-y^3}$

19. Reduce $\dfrac{x^2+y^2+2xy-z^2}{x^2-y^2+2yz-z^2}$ to its lowest terms.

20. Simplify $\dfrac{p+q}{p-q} - \dfrac{p-q}{p+q} - \dfrac{4pq}{q^2-p^2}$

21. Simplify:

$$\frac{qr}{(p-q)(p-r)} - \frac{pr}{(q-p)(r-q)} + \frac{pq}{(r-q)(r-p)}$$

22. Simplify $\dfrac{\dfrac{1}{a+1} - \dfrac{a}{a^2-1}}{\dfrac{1}{1-a^2} - \dfrac{a}{1+a}}$

23. Solve $\dfrac{4x+2\frac{1}{2}}{7} + \dfrac{7x-3}{2+6x} = \dfrac{4x+3\frac{3}{4}}{7} + \dfrac{9}{28}$

24. Solve $\left\{\begin{array}{r} x+y-z=3 \\ x-y+z=5 \\ -x+y+z=7 \end{array}\right\}$

25. Solve $\left\{\begin{array}{r} \dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{z}=1 \\ \dfrac{1}{x}-\dfrac{1}{y}+\dfrac{1}{z}=3 \\ -\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=5 \end{array}\right\}$

26. Square $\sqrt{x}+\sqrt{y}-\sqrt{z}$ **27.** Cube $1-\sqrt{2}$

28. Cube $a + \sqrt{a\,x} + x$

29. $(x^{-p})^{-q} \div (x^{pq})^{\frac{p}{q}} = ?$

30. Expand $\left(x^{-3} + \dfrac{1}{x^{-3}}\right)^4$ and express the result without negative exponents.

31. Multiply $2\sqrt{18} + 2\sqrt{288} - 3\sqrt{32} - \sqrt{128}$ by $\sqrt{2}$

32. Simplify $(x + \sqrt{-y})^2\,(x - \sqrt{-y})^2$

33. Solve $x^2 + (a + b + c)\,x = -\,b\,(a + c)$

34. Solve $\dfrac{x + \sqrt{2 - x^2}}{x - \sqrt{2 - x^2}} = \dfrac{4}{3}$

35. Solve $2\,x^2 + 3\sqrt{2\,x^2 + 3\,x} = 18 - 3\,x$

36. Solve $\left\{\begin{array}{l} 2\,x^2 + 5\,x\,y + 2\,y^2 = 56 \\ 2\,x^2 + 3\,x\,y - 2\,y^2 = 28 \end{array}\right\}$

37. Extract the square root of:

$$x - 2\,x^{\frac{3}{4}}\,y^{\frac{1}{4}} + 3\,x^{\frac{1}{2}}\,y^{\frac{1}{2}} - 2\,x^{\frac{1}{4}}\,y^{\frac{3}{4}} + y$$

38. Extract the cube root of:

$$x^2 - 3\,x^{\frac{5}{3}} + 6\,x^{\frac{4}{3}} - 7\,x + 6\,x^{\frac{2}{3}} - 3\,x^{\frac{1}{3}} + 1$$

39. Divide $x^4 + x^2\,y^2 + y^4$ by $x + \sqrt{x\,y} + y$

40. Simplify:

$$\frac{1}{x\,(x - y)\,(x - z)} - \frac{1}{y\,(y - x)\,(z - y)} + \frac{1}{z\,(x - z)\,(y - z)}$$

41. Find the value of $x^2 - x\,y + y^2$ when

$$x = \frac{a - b}{a + b} \text{ and } y = \frac{a + b}{a - b}$$

42. Multiply $m\,x^2 + n\,x - p$ by $a\,x - b$, and inclose the coefficients of the different powers of x in parentheses.

43. Factor $x^4 - \dfrac{1}{x^4}$ and $a^3 + 3\,a^2\,b - 4\,a\,b^2 - 12\,b^3$

44. Put y for $x + \dfrac{1}{x}$ in the following expressions, and simplify: $x^2 + \dfrac{1}{x^2}$; $x^3 + \dfrac{1}{x^3}$; $x^4 + \dfrac{1}{x^4}$

45. Factor $32x^5 + z^5$ and $x^2 + xy + y^2$

46. Raise $1 - \sqrt{-3}$ to the fourth power.

47. Place the monomial factors of the following expressions within the parentheses:

$$4a^2(2x^2 - 3y^2)^{\frac{2}{3}};\ x(x^3 - y^3)^{\frac{5}{6}};\ x^{-2}(a + bx)^{-\frac{2}{3}}$$

48. Simplify $1 - [-1 - \{-1 - (-1) - 1\} - 1] - 1$

49. Simplify $\left[\left\{(-2)^{-2}\right\}^{-2}\right]^{-2}$

50. Solve $x^{-1} + \dfrac{1}{x^{-1}} = x^{-2} - \dfrac{1}{x^{-2}}$

51. Solve $x = \dfrac{1}{x} - y$ and $y = \dfrac{1}{y} - x$

52. Solve $\dfrac{x+2}{x+3} - \dfrac{x-2}{x-3} = \dfrac{x+4}{x-5} - \dfrac{x-4}{x+5}$

53. Develop $\dfrac{1}{1-x}$ into a series by division. By the law of the series, what will the 10th term be? What the nth term?

54. Simplify and clear of negative exponents:

$$\frac{x^{-2} + y^{-2}}{x^{-4} - y^{-4}};\ \frac{x^{-3} + y^{-3}}{x^{-1} + y^{-1}};\ \frac{(x^2 - y^2)^{-2}}{(x - y)^{-2}}$$

55. Multiply $2\sqrt{30} - \sqrt{45} + \sqrt{75}$ by $\sqrt{8} + \sqrt{3} - \sqrt{5}$

56. Rationalize the denominators of:

$$\frac{x^{\frac{1}{2}} - y^{\frac{1}{2}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}},\ \frac{1}{\sqrt[3]{3}},\ \frac{1 + \sqrt{-1}}{1 - \sqrt{-1}},\ \text{and}\ \frac{1}{\sqrt[4]{-1}}$$

57. Solve $\dfrac{\sqrt{x} + 4}{\sqrt{x} + 6} = \dfrac{\sqrt{x} + 8}{\sqrt{x} + 9}$

58. Find the H. C. D. of $2ax + 2bx + 7a + 7b$
and $2cx - 2dx + 7c - 7d$

59. From $my^2 + nyz + rz^2$
take $(n - r)y^2 - (m - r)yz - (n - m)z^2$

60. Find the value of x in $x : a - 1 :: 1 - \sqrt{a} : 1 + \sqrt{a}$

61. Find the value of $\frac{x^3 - y^3}{x - y}$,

when $x = \sqrt[3]{2}$ and $y = -\sqrt[3]{2}$

62. Find the value of :

$1 + x + x^2 + x^3 +$ etc., *ad infinitum*, when $x = \frac{1}{2}$

63. Find the equation whose roots are

$1 + \sqrt{-1}$ and $1 - \sqrt{-1}$

64. Find the equation whose roots are a, b, c, and d.

65. Solve $3ax^2 + 2bx = 4c$

66. Solve $x - \sqrt{x} = a + \sqrt{a}$

67. Solve $\begin{Bmatrix} x + y + \sqrt{x + y} = 20 \\ x^2y^2 + xy = 3660 \end{Bmatrix}$

68. If $x = y^2 + y + 1$, what is the value of $x^2 + x + 1$?

69. Substitute $ay^2 - by$ for x in $x^2 + xy + y^2$, and bracket the coefficients of the like powers of x.

70. Solve $\frac{x + 2\sqrt{x^2 + a^2}}{x - 2\sqrt{x^2 + a^2}} = \frac{a + b}{a - b}$ by proportion.

71. Solve $\frac{x - a}{\sqrt{x} - \sqrt{a}} = a - \sqrt{a}$

72. Solve $\begin{Bmatrix} x^2 + xy + y^2 = 21 \\ x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 7 \end{Bmatrix}$

73. Solve $\begin{Bmatrix} x + y = 35 \\ x^{\frac{1}{3}} + y^{\frac{1}{3}} = 5 \end{Bmatrix}$

74. Solve $\begin{Bmatrix} x + xy + xy^2 = 26 \\ x - xy^3 = -52 \end{Bmatrix}$

75. Solve $\begin{Bmatrix} x^2 + y^2 + xy = 49 \\ xy + 2x + 2y = 31 \end{Bmatrix}$

76. If n is integral, what kind of number, odd or even, is represented by $2n + 1$? $2n - 1$? $2n$?

77. Find $\sqrt[2n-1]{+1}$, $\sqrt[2n+1]{-1}$, $\sqrt[2n]{+1}$, $\sqrt[2n]{16}$

78. Solve $\sqrt[2n-1]{x} = -1$, $\sqrt[2n]{x} = -1$

EXERCISE 138.

1. John and James together had \$6800. John spent $^1/_3$ of his money, and James $^1/_4$ of his, and each had the same sum remaining. How much had each at first?

2. John is $^1/_3$ as old as his father, but in 20 years he will be $^7/_{13}$ as old. How old is each?

3. Divide 100 into two such parts that the quotient of the smaller part divided by the difference between the parts may be 12.

4. The sum of the two digits of a number is 10, and if 36 be added to the number the order of the digits will be reversed. Find the number.

5. Find a fraction such that if 2 be added to the numerator the fraction equals 1, but if 5 be added to the denominator the fraction equals $^1/_2$.

6. If a rectangle had its width increased by 4 feet and its length diminished by 8 feet, it would become a square inclosing an equal area. Find the dimensions of the rectangle.

7. A can do a piece of work in 2 hours 45 minutes, and B can do it in 3 hours 15 minutes. In what time can they do it working together?

8. A man was engaged to work for 40 days on condition that for every day he worked he was to receive \$3, and for every day he was idle to forfeit \$$1^1/_2$. At the end of the time he received \$75. How many days was he idle?

9. At an election there were three candidates for sheriff. The whole number of votes polled was 5325. B received 662 votes more than C, and A's majority over B and C was 1 vote. How many votes had each?

10. One man rides a mile on a bicycle in $5^5/_6$ minutes; another, a mile in $6^2/_3$ minutes. If they start at the same time from two towns 18 miles apart and approach each other, in what time will they meet?

11. A man can row 4 miles an hour in still water, and 11 miles down a river and back again in $6\frac{2}{5}$ hours. What is the velocity of the current?

12. At what time between 5 and 6 o'clock are the hour and minute hands of a watch together? At right angles? Opposite each other?

13. Find two numbers in the ratio of 2 to 3, such that if each be diminished by 12, they shall be in the ratio of 1 to 2.

14. A boat steaming $\frac{1}{2}$ a mile an hour above its ordinary rate gains $17\frac{1}{7}$ minutes in going 60 miles. What is its usual rate?

15. Six silver and 4 gold pieces are worth as much as 16 silver and 2 gold pieces, and 10 of each are together worth $30. What is the value of a gold piece? What of a silver piece?

16. Two numbers are in the ratio of p to q, and the sum of their squares is $p^4 - q^4$. What are the numbers?

17. Show that any square exceeds a rectangle of equal perimeter by the square of $\frac{1}{2}$ the difference of the length and breadth of the rectangle.

18. A man bought 12 apples. Had he bought 3 less for the same sum, they would have cost him 1 cent apiece more. What did he pay apiece?

19. A and B together have 110 sheep, A and C together 100 sheep, and B and C together 90 sheep. How many has each?

20. Twelve men agreed to do a piece of work in a given time, but 4 men did not report for work, in consequence of which the time had to be extended 4 days. What was the time agreed upon?

21. Two trains pass a station at an interval of 3 hours, moving respectively at the rate of 20 and 32 miles an hour. In what time will the fast train overtake the slow train?

22. What principal will in 8 years at 6% produce as much interest as $800 in 9 years at 8%?

23. The population of a Western city increases annually 10%; it is now 29,282. What was it 4 years ago?

24. The sum of the fourth powers of three consecutive numbers is 962. Find the numbers.

25. A number of two digits is to the number formed by interchanging the digits as 4 to 7, and the difference of the two numbers is 27. What is the number?

26. A man saved $1026 in 3 years. How much were his annual savings if they increased in geometrical progression, and he saved $18 the first year?

27. The sum of 7 numbers in arithmetical progression is 56, and the sum of their squares is 560. Required the numbers.

28. The volumes of two stones are to each other as 3 to 4, and the weights of equal volumes as $1\frac{1}{2}$ to 1. What is the weight of each if their united weight is 340 pounds?

29. The diagonal of a rectangle is 10, but, if the length be increased by 4 and the width by 3, the diagonal will be 15. Required the length and width of the rectangle.

30. A man has three horses. The value of the first is $60, the value of the second equals the value of the first and $\frac{1}{2}$ the value of the third, and the value of the third equals the value of the first two. Required the value of the three together.

31. A man owns $15,000 worth of stock, part of which is 5% and part 6% stocks; his annual income is $830. How much of each kind has he?

32. I sold a horse for as many per cent above $150 as I lost per cent on the cost, and lost $45. What was the cost and loss per cent?

33. A and B can do a piece of work in 5 days, working 10 hours a day; A and C in 6 days, working 9 hours a day; and B and C in 8 days, working 8 hours a day. In what time can each alone do it, working 8 hours a day?

34. A is 50 yards in advance of B, and goes 1 yard the first minute, 3 yards the second, 5 yards the third, and so on; B goes uniformly 15 yards a minute. In how many minutes will he overtake A?

35. A passenger-train, after running one hour, was partially disabled, and could run only at $^4/_5$ of its usual rate, which caused it to be one hour late at its destination. Had the accident occurred 15 miles farther on, the train would have been only $52\,^1/_2$ minutes late. What was the usual rate of the train?

36. The sum of the two means of a geometrical progression of four terms is 36, and the sum of the extremes is 84. Find the series.

37. The product of five numbers in arithmetical progression is 23,040, and their sum is 40. Find them.

38. The sum, the product, and the difference of the squares of two numbers are equal. Find the numbers.

39. A is a feet behind B, and goes m feet in t seconds, while B goes n feet in p seconds. In how many seconds will A overtake B?

40. If from a number whose four figures are in arithmetical progression, be subtracted the number formed by reversing the order of the digits, the remainder will be 6174; the sum of the digits is 24. Required the number.

41. Two numbers are in the ratio of 2 to 3, and if 2 be added to each of them, they will be in the ratio of 3 to 4. Find them.

42. A's age 2 years hence is to his age 3 years ago as 9 times his age 3 years ago is to 4 times his age 2 years hence. Required his age.

43. The capacities of two cubical cisterns are to each other as 1 to 8; but if 2 feet were added to each of their dimensions, their capacities would be to each other as 27 to 125. Required the capacity of each.

44. The sum of the squares of three numbers is 29, the sum of the products of them taken two together is 26, and the first is 5 less than the sum of the other two. Find the numbers.

45. A general drew up his army in the form of a square and found he had 615 men over; he then increased the side of the square by 5 men, and lacked 60 men to complete the square. How many men were in the army?

46. A man has a farm of 150 acres, in the form of a rectangle, whose length is to its breadth as 5 to 3. A road of uniform width, containing $3\,{}^{39}/_{40}$ acres, surrounds the farm, and is a part of it. How wide is the road?

47. The fore-wheel of a carriage makes 88 revolutions more in going a mile than the hind-wheel, but if the circumference of the fore-wheel were diminished 2 feet the fore-wheel would make 220 revolutions more than the hind-wheel. Required the circumference of each wheel.

48. A merchant gains annually 20 per cent of his capital; of this he spends $1000, and adds the balance to his capital for the next year; at the end of 4 years his stock is $25,736. What was his original stock?

49. A man starts at the foot of a mountain to walk to its top. During the first half of the distance he walks $^1/_2$ a mile an hour faster than during the last half, and he reaches the top in 4 hours 24 minutes. Returning, he walks $^1/_2$ a mile an hour faster than during the latter half of his ascent, and completes the descent in 4 hours. Find the distance to the top of the mountain.

50. A lump of gold 22 carats fine contains 36 ounces of alloy. How many ounces of alloy in a lump of the same weight only 16 carats fine?

51. In a mile walk, A gives B a start of 1 minute and overtakes him at the mile-post. In a second trial, A gives B a start of 60 yards, and beats him 10 seconds. At the rate of how many miles an hour does each walk?

52. If the cost of an article had been 8% less, the gain would have been 10% more. Find the gain per cent.

53. A railway-train, after traveling for 1 hour, has an accident which delays it 60 minutes, after which it proceeds at $3/5$ of its former speed, and arrives at its destination 3 hours behind time. Now, had the accident occurred 50 miles farther on, the train would have arrived $1\,3/4$ hour sooner. What is the length of the line?

54. Two men, A and B, engaged to work for a certain number of days at different rates. At the end of the time, A, who had been idle 4 days, received 75 shillings; but B, who had been idle 7 days, received only 48 shillings. Now, had B been idle only 4 days, and A 7 days, they would have received the same sum. For how many days were they engaged?

55. Three pipes, A, B, and C, can fill a cistern in one hour. B delivers twice as much water per minute as A. C alone will fill it in one hour less than B alone. How long will it take each to fill it?

General Definitions.

1. *Quantity* is anything that may be increased, diminished, and measured.

2. Quantity is estimated by assuming some definite portion of it as a standard of measure, and finding how many times it contains this standard.

3. Any definite portion of quantity assumed as a standard of measure is a *unit.*

4. *Number* is that which denotes how many units a quantity contains.

5. A quantity that contains a definite number of units is a *specific quantity*; as, *five pounds.*

6. A quantity that may contain any number of units is a *general quantity*; as, a *flock.*

7. The number of units in a specific quantity is expressed by one or more of the figures of arithmetic.

8. The number of units in a general quantity is expressed by one or more of the letters of the alphabet, or by both figures and letters.

9. By a figure of speech, the representation of the number of units in a quantity, by figures or letters, is also called a quantity.

10. When the number of units in a quantity is denoted by figures, the expression is called a *numerical quantity.*

11. When the number of units in a quantity is represented wholly or partially by letters, the expression is called a *literal quantity.*

12. Quantities which are opposed to each other in character — that is, which tend to destroy each other when combined—are *positive* and *negative* quantities.

13. Of two opposite quantities, it does not matter which is considered positive and which negative, if consistency is maintained throughout the operation or investigation into which they enter.

14. The number of units in a positive quantity is characterized by placing before it the symbol + (plus); and the number of units in a negative quantity, by placing before it the symbol − (minus). This peculiar notation gives rise to *symbolized numbers.*

15. *Arithmetic* is the science of numbers, irrespective of their character as positive or negative. Arithmetic based on the *literal* notation is *Literal Arithmetic.*

16. Algebra is the science of *symbolized numbers* as the representatives of positive and negative quantities.

Principles.

1. The algebraic sum of two or more similar terms with like signs equals their arithmetical sum with the same sign (page 16).

2. The algebraic sum of two similar terms with unlike signs equals their arithmetical difference with the sign of the greater (page 16).

3. The algebraic sum of two or more dissimilar terms equals a polynomial composed of those terms (page 17).

4. The algebraic difference of two quantities equals the algebraic sum obtained by adding to the minuend the subtrahend with the sign changed (page 23).

5. The product of two quantities with like signs is positive (page 27).

6. The product of two quantities with unlike signs is negative (page 27).

7. The exponent of a factor in the product equals the sum of its exponents in the multiplicand and multiplier (page 28).

8. Multiplying one factor of a quantity multiplies the quantity (page 28).

9. Multiplying every term of a quantity multiplies the quantity (page 31).

10. The quotient of two quantities with like signs is positive (page 33).

11. The quotient of two quantities with unlike signs is negative (page 33).

12. The exponent of a factor in the quotient equals the difference of the exponents of the factor in the dividend and divisor (page 34).

13. Any quantity with an exponent of zero equals unity (page 34).

14. Dividing one factor of a quantity divides the quantity (page 35).

15. Dividing every term of a quantity divides the quantity (page 37).

16. If the same quantity or equal quantities be added to equal quantities, the results will be equal (page 39).

17. If the same or equal quantities be subtracted from equal quantities, the results will be equal (page 39).

18. If equal quantities be multiplied by the same or equal quantities, the results will be equal (page 39).

19. If equal quantities be divided by the same quantity or equal quantities, the results will be equal (page 40).

20. A term may be taken from one member of an equation to the other, if its sign be changed (page 40).

21. If both members of a fractional equation be multiplied by a common denominator of its terms, it will be cleared of fractions (page 40).

22. If the sign of every term of an equation be changed, the members will still be equal (page 41).

23. If a number of terms are inclosed by a parenthesis preceded by plus, the symbol and the sign before it may be removed without altering the value of the expression (page 50).

24. If a number of terms are inclosed by a parenthesis preceded by minus, the symbol and the sign before it may be removed, if the sign of every term inclosed be changed (page 51).

25. Any number of terms may be inclosed by a parenthesis and preceded by plus, without changing the value of the expression (page 51).

26. Any number of terms may be inclosed by a parenthesis and preceded by minus, if the sign of every term inclosed be changed (page 51).

27. An even power of a positive or a negative quantity is positive (page 62).

28. An odd power of a quantity has the same sign as the quantity (page 63).

29. Multiplying the exponent of a factor by the exponent of a power raises the factor to that power (page 63).

30. Raising every factor of a quantity to a given power raises the quantity to that power (page 63).

31. The square of the sum of two quantities equals the square of the first, plus twice their product, plus the square of the second (page 65).

32. The square of the difference of two quantities equals the square of the first, minus twice their product, plus the square of the second (page 65).

33. The cube of the sum of two quantities equals the cube of the first, plus three times the square of the first into the second, plus three times the first into the square of the second, plus the cube of the second (page 66).

34. The cube of the difference of two quantities equals the cube of the first, minus three times the square of the first into the second, plus three times the first into the square of the second, minus the cube of the second (page 66).

35. The product of any even number of factors with like signs is positive (page 67).

36. The product of any odd number of factors with like signs has the same sign as the factors (page 67).

37. If the signs of an even number of factors be changed, the sign of their product will remain unchanged (page 67).

38. If the signs of an odd number of factors be changed, the sign of their product will be changed (page 67).

39. The product of the sum and difference of two quantities equals the square of the first minus the square of the second (page 69).

40. The product of two binomials having a common term equals the square of the common term, and the algebraic sum of the unlike terms times the common term, and the algebraic product of the unlike terms (page 70).

41. The product of any two binomials equals the product of the first terms, and the algebraic sum of the products obtained by a cross-multiplication of the first and second terms, and the algebraic product of the second terms (page 71).

42. The difference of the equal even powers of two quantities is divisible by both the sum and the difference of the quantities (page 73).

43. The sum of the equal odd powers of two quantities is divisible by the sum of the quantities (page 74).

44. The difference of the equal odd powers of two quantities is divisible by the difference of the quantities (page 75).

45. The laws of the quotient in exact division (page 76).

46. A divisor of a quantity is one of the two factors of the quantity, and the quotient is the other (page 78).

47. A factor of every term of a quantity is a factor of the quantity (page 78).

48. The highest common divisor is the product of all the common prime factors (page 87).

49. The lowest common multiple of two or more quantities equals the product of all their different prime factors, each taken the greatest number of times it occurs in any one of them (page 90).

50. Dividing one quantity and multiplying another by the same factor does not alter their product (page 93).

51. Multiplying the dividend or dividing the divisor multiplies the quotient (page 93).

52. Dividing the dividend or multiplying the divisor divides the quotient (page 94).

53. Multiplying or dividing both dividend and divisor by the same quantity does not alter the quotient (page 94).

54. Multiplying the numerator or dividing the denominator multiplies the value of a fraction (page 101).

55. Dividing the numerator or multiplying the denominator divides the value of a fraction (page 102).

56. Multiplying both terms of a fraction by the same quantity does not alter its value (page 103).

57. Dividing both terms of a fraction by the same quantity does not alter its value (page 103).

58. Changing the signs of both terms of a fraction does not alter its value (page 104).

59. Changing the apparent sign and the sign of either term of a fraction does not change the value of the fraction (page 104).

60. Any common multiple of the denominators of two or more fractions is a common denominator of the fractions (page 108).

61. The lowest common multiple of the denominators of two or more fractions in their lowest terms is the lowest common denominator (page 108).

62. The sum of two or more similar fractions equals the sum of their numerators divided by their common denominator (page 110).

63. The difference of two similar fractions equals the difference of their numerators divided by their common denominator (page 110).

64. The product of two fractions equals the product of their numerators divided by the product of their denominators (page 115).

65 Canceling a factor common to the numerator of one fraction and the denominator of another does not alter the product of the fractions (page 115).

66. The quotient of two fractions equals the dividend multiplied by the inverse of the divisor (page 116).

67. Canceling a factor common to the numerators or the denominators of two fractions does not alter their quotient (page 117).

68. Raising both terms of a fraction to any power raises the fraction to that power (page 120).

69. Principles of transformation of equations (page 125).

70. Any term of an equation may be transposed from one member to the other if its sign be changed (page 125).

71. An equation with fractional terms may be cleared of fractions by multiplying both members by a common multiple of the denominators of the fractions (page 126).

72. The binomial theorem (page 161).

73. The square of a polynomial equals the sum of the squares of its terms, and twice the product of each term into all the following terms (page 163).

74. The cube of any trinomial equals the sum of the cubes of its terms, and three times the square of each term into all the other terms, and six times the product of the three terms (page 164).

75. Dividing the exponent of any factor by the index of a root takes that root of the factor (page 165).

76. Taking a root of every factor of a quantity takes the root of the quantity (page 165).

77. Any even root of a positive quantity may be either positive or negative (page 166).

78. Any odd root of a quantity has the same sign as the quantity (page 166).

79. An even root of a negative quantity is impossible (page 166).

80. Extracting a root of both terms of a fraction extracts the root of the fraction (page 167).

81. If a number be pointed off into terms of two figures each, beginning at the units, the unit of each term will be a perfect square (page 172).

82. If a number be pointed off into terms of three figures each, beginning at the units, the unit of each term will be a perfect cube (page 177).

83. Every pure quadratic equation of one unknown quantity may be reduced to the form of $ax^2 = b$, in which a and b are integral and a positive (page 182).

84. Every pure quadratic equation of one unknown quantity has two roots, numerically equal, but opposed in sign (page 183).

85. Every complete quadratic equation of one unknown quantity may be reduced to the form of $ax^2 + bx = c$, in which a, b, and c are integral, and a positive (page 185).

86. Every complete quadratic equation of one unknown quantity may be reduced to the form of $x^2 + px = q$, in which p and q may be integral or fractional, positive or negative (page 185).

87. The sum of the two roots of an equation of the form of $x^2 + px = q$ equals the coefficient of x, with the sign changed (page 194).

88. The product of the two roots of an equation of the form of $x^2 + px = q$ equals the absolute term with the sign changed (page 194).

89. Multiplying or dividing both terms of a fractional exponent by the same quantity does not change its value (page 215).

90. The exponent of a factor in the product equals the sum of the exponents of the same factor in the multiplicand and multiplier, when the exponents are positive fractions (page 215).

91. The exponent of a factor in the quotient equals the exponent of the same factor in the dividend, minus the exponent of that factor in the divisor, when the exponents are positive fractions (page 215).

92. A quantity affected by a negative exponent equals the reciprocal of the quantity affected by a numerically equal positive exponent (page 216).

93. A quantity affected by a positive exponent equals the reciprocal of the quantity affected by a numerically equal negative exponent (page 216).

94. A factor may be transferred from either term of a fraction to the other if the sign of its exponent be changed (page 217).

95. The exponent of a factor in the product equals the sum of the exponents of the same factor in the multiplicand and the multiplier when the exponents are negative (page 217).

96. The exponents of a factor in the quotient equals the exponent of the same factor in the dividend, minus the exponent of that factor in the divisor, when the exponents are negative (page 218).

97. $a^x \times a^y = a^{x+y}$ for any positive or negative, integral or fractional, values of x and y (page 219).

98. $a^x \div a^y = a^{x-y}$ for any positive or negative, integral or fractional, values of x and y (page 219).

99. $(a^x)^y = a^{xy}$ for any positive or negative, integral or fractional, values of x and y (page 220).

100. $(a\,b)^x$ and $a^x \times b^x$ are equivalent for any positive or negative, integral or fractional, values of x (page 220).

101. $\left(\frac{a}{b}\right)^x$ and $\frac{a^x}{b^x}$ are equivalent for any positive or negative, integral or fractional, values of x (page 220).

102. Any root of the product of two quantities equals the product of the like roots of those quantities (page 223).

103. The product of the equal roots of two quantities equals the like root of their product (page 223).

104. Any root of the quotient of two quantities equals the quotient of the like roots of those quantities (page 223).

105. The quotient of the equal roots of two quantities equals the like root of their quotient (page 223).

106. No fractional radical is pure (page 223).

107. Any quantity equals the nth root of the nth power of the quantity (page 223).

108. The $\sqrt[m]{\sqrt[n]{a}}$ and the $\sqrt[mn]{a}$ are equivalent (page 223).

109. Every imaginary quantity of the second degree may be reduced to the form of $\pm x\sqrt{-1}$, in which x may be rational or irrational (page 234).

110. Every binomial surd of the second degree may be reduced to the form of $\sqrt{a} \pm \sqrt{b}$, in which one of the terms may be rational (page 236).

111. A binomial surd may be a perfect square, and, when it is the square of a binomial surd of the second degree, one of the terms is rational (page 236).

112. A binomial surd with a rational term, and the coefficient of the irrational term reduced to ± 2, is a perfect square when the quantity under the radical sign is composed of two factors whose sum equals the rational term; and its square root equals the sum or difference of the square roots of these factors (page 236).

113. When a binomial surd is a perfect square, the difference of the squares of its terms is a perfect square, and is equal to the square of the difference of the two factors described in Prin. 112 (page 237).

114. Principles of transformation of inequalities (page 249).

115. The sum of the squares of two unequal quantities is greater than twice their product (page 249).

116. The sum of the squares of three unequal quantities is greater than the sum of their products taken two and two (page 249).

117. The ratio equals the antecedent divided by the consequent (page 252).

118. The antecedent equals the ratio times the consequent (page 252).

119. The consequent equals the antecedent divided by the ratio (page 252).

120. Multiplying the antecedent or dividing the consequent multiplies the ratio (page 252).

121. Dividing the antecedent or multiplying the consequent divides the ratio (page 252).

122. Multiplying or dividing both terms of a ratio by the same quantity does not alter its value (page 252).

123. An infinitesimal divided by a finite constant is an infinitesimal (page 268).

124. An infinitesimal multiplied by a finite constant is an infinitesimal (page 269).

125. An infinitesimal divided by an infinitesimal may be any finite constant (page 269).

126. An infinite divided by a finite constant is an infinite (page 269).

127. An infinite multiplied by a finite constant is an infinite (page 269).

128. An infinite divided by an infinite may be any finite constant (page 269).

129. A finite constant divided by an infinitesimal is an infinite (page 269).

130. The product of an infinitesimal and an infinite may be any finite constant (page 269).

131. A finite constant divided by an infinite is an infinitesimal (page 270).

132. In an arithmetical progression,

$$l = a + (n - 1)\,d \quad \text{(page 273).}$$

133. In an arithmetical progression,

$$S = (l + a)\frac{n}{2} \quad \text{(page 274).}$$

134. In a geometrical progression,

$$l = a\,r^{n-1} \quad \text{(page 280).}$$

135. In a geometrical progression,

$$S = \frac{l\,r - a}{r - 1} \quad \text{(page 280).}$$

136. In an infinite series, $S = \dfrac{a}{1 - r}$ (page 286).

APPENDIX.

Highest Common Divisor by Successive Division.

Definitions and Principles.

1. If one quantity be divided by another, then the divisor by the remainder, then the next divisor by the next remainder, and so on, until the division terminates, the process is called ***Successive Division.***

2. Since a is a divisor of $a x$ and also of $n a x$, it follows that,

***Prin. 1.**—Any divisor of a quantity is also a divisor of any number of times the quantity.*

3. Since a is a common divisor of $a b$ and $a c$, and also a divisor of $a b \pm a c$, it follows that,

***Prin. 2.**—A common divisor of two quantities is also a divisor of their sum and of their difference.*

4. ***Theorem.**—The last divisor obtained by the successive division of two quantities is their highest common divisor.*

Demonstration: Let A and B represent any two quantities. Divide B by A, and let the quotient be q and the remainder R; divide A by R, and let the quotient be q' and the remainder R'; divide R by R', and let the quotient be q'' and the remainder *zero.* Prove R' the H. C. D. of A and B.

$$
\begin{array}{l}
A)\ B\ (q \\
\quad \underline{A q} \\
\quad\quad R)\ A\ (q' \\
\quad\quad\quad \underline{R q'} \\
\quad\quad\quad\quad R')\ R\ (q'' \\
\quad\quad\quad\quad\quad \underline{R' q''} \\
\quad\quad\quad\quad\quad\quad 0
\end{array}
$$

1. R' is a divisor of R, since the division has terminated; hence it is also a divisor of Rq' [P. 1] and of $R' + Rq'$, or A [P. 2]; and, therefore, of Aq [P. 1], and of $R + Aq$, or B [P. 2]; therefore, R' is a common divisor of A and B.

2. There can be no higher common divisor of A and B than R'; for if there could be it would be a divisor of A and B, and hence, too, of Aq [P. 1], and of $B - Aq$, or R [P. 2], and, therefore, of Rq' [P. 1], and of $A - Rq'$, or R' [P. 2]; that is, a quantity of a higher degree than R' would be a divisor of R', which is impossible. Therefore, R' is the H. C. D. of A and B.

5. Since the highest common divisor equals the product of all the common factors, it follows that,

Prin. 3.*—Either of two quantities may be multiplied or divided by a factor not found in the other without changing their highest common divisor.*

Problem. To find the highest common divisor by successive division.

Illustrations.—1. Find the H. C. D. of:

$$2x^4 - 9x^3 - 14x + 3 \text{ and } 3x^4 - 14x^3 - 9x + 2$$

Form.

$$2x^4 - 9x^3 - 14x + 3\,)\,6x^4 - 28x^3 - 18x + 4\,(\,3$$
$$6x^4 - 27x^3 - 42x + 9$$
$$-1\,)\,-x^3 + 24x - 5$$
$$x^3 - 24x + 5$$

$$x^3 - 24x + 5\,)\,2x^4 - 9x^3 - 14x + 3\,(\,2x - 9$$
$$2x^4 - 48x^2 + 10x$$
$$-9x^3 + 48x^2 - 24x + 3$$
$$-9x^3 + 216x - 45$$
$$48\,)\,48x^2 - 240x + 48$$
$$x^2 - 5x + 1$$

H. C. D.

$$x^2 - 5x + 1\,)\,x^3 - 24x + 5\,(\,x + 5$$
$$x^3 - 5x^2 + x$$
$$5x^2 - 25x + 5$$
$$5x^2 - 25x + 5$$

Solution: Taking the second polynomial for the dividend, we observe that the first term of it is not divisible by the first term of the divisor; we therefore multiply the dividend by 2 [P. 3], and then divide and obtain for the first remainder $-x^3 + 24x - 5$. We now divide the remainder by -1 [P. 3], and divide the divisor by the result, obtaining for the next remainder $48x^2 - 240x + 48$. We reject the factor 48 from this remainder [P. 3] and divide the previous divisor by the result, and obtain no remainder. The last divisor, $x^2 - 5x + 1$, is the H. C. D. (theorem).

6. If one polynomial can be factored, its factors may be made available in factoring the other by trial. The first term of a factor is always a divisor of the first term of the polynomial, and the last term of a factor a divisor of the last term of the polynomial.

2. Find the H. C. D. of:

$$x^4 - 3x^2 - 28 \text{ and } x^5 - 2x^4 + 7x^3 - 10x^2 + 12x - 8$$

Solution: The factors of $x^4 - 3x^2 - 28$ are $x^2 - 7$ and $x^2 + 4$; $x^2 - 7$ is not a factor of $x^5 - 2x^4 + 7x^3 - 10x^2 + 12x - 8$, since 7 is not a divisor of 8; if the two polynomials have a common divisor, it must therefore be $x^2 + 4$. By trial we find that $x^2 + 4$ is a divisor of the second, and is therefore the H. C. D. of the two.

7. Since each remainder is a number of times the H. C. D., it is sometimes more convenient to use the remainders, or a remainder and one of the quantities, than to use the polynomials themselves in the progress of the work.

3. Find the H. C. D. of $x^3 - 6x^2 - x + 30$ and $x^3 + 9x^2 + 26x + 24$.

Form.

$$\begin{array}{r} x^3 - 6x^2 - x + 30\,)\,x^3 + 9x^2 + 26x + 24\,(\,1 \\ \underline{x^3 - 6x^2 - \quad x + 30} \\ 3\,)\,\underline{15x^2 + 27x - \ 6} \\ 5x^2 + \ 9x - \ 2 = \\ (5x - 1)(x + 2) \end{array}$$

$\therefore$ H. C. D. $= x + 2$

EXERCISE 1.

Find the H. C. D. of:

1. x^3+3x^2+3x+2 and x^3-x^2-x-2

2. x^3-5x^2+3x+4 and x^3+5x^2-7x-6

3. $2x^4-x^3+2x^2+x-1$ and $2x^3-3x^2+5x-2$

4. $10x^4+5x^3+9x^2-3x-9$ and $15x^4-10x^3-4x^2+6x-3$

5. $2x^4-3x^3+4x^2+3x-6$ and $2x^4-3x^3+8x^2-3x+6$

6. $6a^4+7a^3+7a^2+3a+1$ and $14a^4+a^3+8a^2-a+2$

7. $x^2+8x+15$ and $x^3-3x^2-10x+24$

8. $6x^2+5x-6$ and $8x^3-22x^2-21x+45$

9. x^3+3x^2+3x+1 and $3x^3-x^2-11x-7$

10. $x^4-6x^3+6x^2-3x-10$ and $3x^4-13x^3-11x^2+8x-15$

11. $4y^3+8y^2+8y+4$ and $7y^3-14y^2+21$

12. $a^{12}+b^{12}$ and $a^{16}+a^8b^8+b^{16}$

13. $a^2+2ab+b^2-c^2$ and $a^2-ab-2b^2+4ac+bc+3c^2$

14. $x^4+x^2y^2+y^4$ and $x^4+3x^3y+4x^2y^2+3xy^3+y^4$

15. $3a^4+201a^2+198$ and $5a^4+10a^3+10a^2+10a+5$

16. $6ax^2-3axy-18ay^2$ and $6bx^2-16bxy+8by^2$

8. *The highest common divisor of three quantities may be obtained by finding the highest common divisor of two of them, and then the highest common divisor of that and the third quantity.*

Demonstration.—Let x, y, and z be three quantities, and m the H. C. D. of x and y, and n the H. C. D. of m and z; then will m be the product of the factors common to x and y, and n the product of the factors common to m and z, or n will be the product of the factors common to x, y, and z, which is their H. C. D.

EXERCISE 2.

Find the H. C. D. of:

1. $x^3 - xy^2$, $x^3y + y^4$, and $x^5y - xy^5$
2. $x^2 + xy + y^2$, $x^4 + x^2y^2 + y^4$, and $x^8 + x^4y^4 + y^8$
3. $2x^2 + 3x + 1$, $2x^2 + 5x + 2$, and $2x^3 + 5x^2 - 4x - 3$
4. $3x^2 - 17x + 10$, $3x^3 - 2x^2 - 3x + 2$, and $3x^4 - 2x^3 + 3x - 2$
5. $2x^3 + 7x^2 + 8x + 3$, $2x^3 - x^2 - 4x + 3$, and $2x^5 + 3x^4 + 2x^3 + 3x^2 + 2x + 3$
6. $4x^3 + 4x^2 - 36x - 36$, $4x^3 - 4x^2 - 36x + 36$, and $2x^3 + 6x^2 - 2x - 6$
7. $x^4 + x^3 - 8x^2 - 9x - 9$, $x^5 + 3x^4 + x^3 + 3x^2 + x + 3$, and $x^5 + 2x^4 + x^3 + 2x^2 + x + 2$
8. $12x^3 - 2x^2 - 3x + 2$, $18x^3 - 9x^2 - 8x + 4$, and $36x^4 - 25x^2 + 4$

Lowest Common Multiple of Quantities not readily factored.

9. To find the lowest common multiple of quantities not readily factored,

Theorem.*—The lowest common multiple of two quantities equals their product divided by their highest common divisor.*

Demonstration.—Let c equal the H. C. D. of A and B.

Suppose $\frac{A}{c} = x$, and $\frac{B}{c} = y$;

then $A = c \times x$, and $B = c \times y$.

Now x and y are prime to each other, since c is the product of all the common factors.

$\therefore$ L. C. M. $= c \times x \times y = A \times y = A \times \frac{B}{c} = \frac{A \times B}{c}$

Illustration.—Find the L. C. M. of $6x^2 + 13x + 6$ and $6x^3 + 9x^2 + 8x + 12$.

Solution: We find the H. C. D. to be $2x + 3$. Therefore, the

$$\text{L. C. M.} = \frac{\overset{3x^2+4}{\cancel{(6x^2 + 13x + 6)}}(6x^3 + 9x^2 + 8x + 12)}{\cancel{2x + 3}} = (3x^2 + 4)(6x^3 + 9x^2 + 8x + 12).$$

EXERCISE 3.

Find the L. C. M. of:

1. $6x^2 + 14x + 8$ and $8x^2 + 6x - 20$
2. $x^3 + 6x^2 + 6x + 5$ and $2x^3 + 3x^2 + 3x + 1$
3. $a^3 - 2a - 1$ and $a^3 + 2a^2 + 2a + 1$
4. $a^4 + 2a^2 + 9$ and $7a^3 - 11a^2 + 15a + 9$
5. $2x^3 - x^2y + xy^2 + y^3$ and $2x^3 + 3x^2y + 3xy^2 + y^3$
6. $x^3 + 2x^2y + 2xy^2 + y^3$ and $x^3 + 3x^2y + 3xy^2 + 2y^3$
7. $3x^3 + 5x^2 + 3x + 1$ and $3x^4 + 2x^3 + 4x^2 + 2x + 1$
8. $2x^3 + ax^2 + a^2x - a^3$ and $3x^3 + 4ax^2 + 4a^2x + a^3$
9. $3x^3 - 13x^2 + 14x - 6$ and $6x^3 + x^2 - 8x + 6$
10. $4a^6 - 4a^4 - 29a^2 - 21$ and $4a^6 + 24a^4 + 41a^2 + 21$
11. $20z^4 + z^2 - 1$ and $25z^4 + 5z^3 - z - 1$
12. $3x^4 + 5x^3 + 5x^2 + 5x + 2$ and $3x^4 - x^3 + x^2 - x - 2$
13. $6x^4 + 17x^3 - 10x + 8$ and $9x^4 + 18x^3 - x^2 - 9x + 4$
14. $2a^3 - 4a^2 - 13a - 7$ and $6a^3 - 11a^2 - 37a - 20$
15. $6m^3 + 15m^2 - 6m + 9$ and $9m^3 + 6m^2 - 51m + 36$
16. $n^4 - an^3 - a^2n^2 - a^3n - 2a^4$ and $3n^3 - 7an^2 + 3a^2n - 2a^3$

ANSWERS.

Exercise 1.

1. 9 units, 9 tens, 9 fives; 9 times the number.

2. 9 times a, 13 times a **3.** $15a$, $19a$ **4.** $9b$, $15b$

5. 27, 36 **6.** $10m$, 20, 50 **7.** $13x$, 39, 78 **8.** $10x$, $10n$

9. 5 tens, 5 twenties; 5 times the number; $5a$, $5m$

10. $8x$, $7y$ **11.** $5a$, 15, 35 **12.** $4a$, 20, 32 **13.** 12, 35

14. 48, 90 **15.** mn, pq, xyz **16.** pqr, 24, 60

17. 210, 108 **18.** 2, 3, 6 **19.** 3, 4

20. 3, 7 **21.** 4, 3 **22.** 8, 14

23. 2, 5; 3, 5; 3, 7; 3, a; 5, x; a, y; x, z

24. 2, 2, 3; 2, 3, 3; 2, 3, 5; 2, 5, a; 5, a, b; x, y, z

25. 2, 7; 3, 7; 5, m; c, d **26.** 2, 5, x; 5, a, y; p, q, r

27. 4, 8, 16 **28.** 9, 64, 16 **29.** 16, 27 **30.** 8, 27, 64

31. 36, 16 **32.** a, a; x, x, x; m, m, m, m; x, x, y, y, y

33. 2, 3, 4, a, x **34.** 2, x, $3a$ **35.** 4, 5, a, ax

36. 3, 4, a, ax **37.** 4, 7 **38.** 3, 4

39. x, a, c **40.** 6, 4 **41.** m^2, n^3, mn, $\frac{m}{n}$, $\frac{m^2}{n^2}$

Exercise 2.

1. 8, 17 **2.** $x+y$, $x+y+z$, $2a+3b$, $4x+5y+6z$

3. 5, 6 **4.** $m-n$, $2a-3b$, $5x^2-7y^2$, x^3-y^3

6. 9, 3, 24, 27, 9, 189, 54, 21 **7.** 20, 8, 12, 0, 18, 22, 18, 46, 40

8. $2ab^2$, $4ab^2$, $7ab^2$; $3a^2b$, $6a^2b$, $8a^2b$, $9a^2b$; $5a^2b^2$, $6a^2b^2$, $9a^2b^2$ **11.** 32 **12.** $a+b$ **13.** $a-b$ **14.** ab

15. $\frac{a}{b}$ **16.** $(a+b)^2$ **17.** a^2+b^2 **18.** $(a+b)^3$ **19.** a^3+b^3

20. $(a+b)(a-b)$ **21.** $ab(a-b)$ **22.** $\frac{a+b}{a-b}$

Exercise 3.

2. $5a$ ct. **3.** $\$3a$ **4.** $(3a+b)$ ct.
5. $10x$ mi. **6.** $6a$ qt. **7.** $(10-c)$ ct.
8. $\$(y-60)$ **9.** $\$(3x+4y)$ **10.** $\$3m$
11. $3a-8b$ **12.** $\frac{\$1000}{x}$ **13.** $\$\frac{25}{3}y$
14. $\frac{cd}{b}$ **15.** $\frac{100}{m}$ **16.** $\frac{x}{6}$ **17.** $\$m(r-n)$
18. $\$\frac{60x}{a}$ **19.** $\frac{am}{b}$ **20.** $\$a(y-x)$ **21.** $\frac{cd}{m}$
22. $(2a+20)$ yr., $(2a-10)$ yr. **23.** $(m-n)c$ or $(n-m)c$
24. $\$\frac{cxy}{160}$ **25.** $\frac{atr}{100}$ **26.** $\$\left(p+\frac{pr}{100}\right)$
27. $\$\left(\frac{2ac+2bc+ab}{900}\right)d$ **28.** $2dx-2cx$ **29.** $\frac{am+cn}{a+c}$ ct.
30. $\frac{a}{p\left(1+\frac{r}{100}\right)}$ yr. **31.** $\frac{mnc}{12}$ **32.** $\frac{c}{a}$
33. $\left(\frac{b-a}{a}\right)100$, $\left(\frac{a-b}{a}\right)100$ **34.** $\frac{ad}{d-c}$ rd.
35. $\left(\frac{1}{a}-\frac{1}{b}\right)60$, $\left(\frac{1}{a}-\frac{1}{b}\right)60c$ mi. **36.** $\$\frac{mx^2}{160}$
37. $(xy-z^2)$ sq. rd. **38.** $\frac{a^3}{c}$ tons.
39. $\sqrt{mn}$ ft. **40.** $\frac{1728pqr}{x^3}$
41. $\sqrt{a^2+b^2}$ yd. **42.** $\sqrt{c^2-a^2}$ **43.** $\$\left(x+\frac{x^2}{100}\right)\frac{x}{100}$

Exercise 4.

14. $+4$ lb. **16.** -15 bu. **17.** $+(a-b)$, $-(b-a)$
18. $+(x+y)$ **19.** $-(a+b)$ **20.** $+ab$ sq. rd.

Exercise 5.

2. $+61$ lb. **4.** -45 cows **6.** -30 lb. **7.** $+8a$
9. $+5$ **10.** 0 **11.** $+4x$ **12.** $-4m$

Exercise 6.

1. $+20a$ **2.** $-25x$ **3.** $-9xy$ **4.** $-4ab$
5. $+2ax$ **6.** $-6x^2y$ **7.** $-3mn$ **8.** $-3pq$

9. $2(a+b)$ 10. $2(m-n)$ 11. $-(x^2+y^2)$
12. $-8(x+y)^2$ 13. $11a^2b^2$ 14. $-4xyz$
15. $-6(a-m)$ 16. $8a$ 17. 0
18. $-7ab$ 19. $-6m^3n^3$ 20. $-4(a+b)$

Exercise 7.

1. $x+y+z$ 2. $2x-3y+4z$ 3. $5a-3b-2c$
4. $7r+6n-5m$ 5. $4h+cd-ab$ 8. $4xy-3ab$
6. $7ab-4cd+5ac-6bd+4am$ 9. $4a+8b$
7. $7yz-xy-4xz+9my-6nx$ 10. $x+y+14z$
11. $6m^2+mn-3n^2$ 12. 0

Exercise 8.

1. $(a+b+c)x$ 2. $(a-d+m)yz$
3. $(2a+3b-4c)y$ 4. $(2-a-b)xz$
5. $(2a+3b+4c-d)xy$ 6. $(n-m+p-q+r)ab$
7. $(-a+2b-3c)xy$ 8. $(3ab-4bc+6cd-5ad)my$
9. $(a+b)(c+d)$ 10. $(a+b-c)(x+y+z)$

Exercise 10.

1. $+4a$ 2. $-3a$ 3. $-4a$ 4. $5x$ 5. $13b$
6. $+14b$ 7. $-16ab$ 8. $-18xy$ 9. $5a+12b$
10. $a+b$ 11. $3mn-xy$ 12. $-a^2x^2$ 13. $-m^3n^3$
14. $13x^2y^2$ 15. $3xy-7mn$ 16. $6mx-m^2n$
17. c^2+d^2 18. m^2-n^2 19. $-(a+x)$
20. $9(x^2-y^2)$ 21. $16(x-y)^2$ 22. $(a+b)y^2$
23. $(d-c)x^2$ 24. $(m-n^2)x$ 25. $(2a-3b)x$

Exercise 13.

1. $+6a^3$ 2. $-8x^2$ 3. $+10y^2$ 4. $-18m^2$
5. $+12x^4$ 6. $-12a^7$ 7. $6ab$ 8. $15x^2y^3$
9. $-10x^3y^2z^2$ 10. $-21a^3b^7$ 11. $28am^3n^8$
12. $-18x^3y^2z^4$ 13. $(a+b)^5$ 14. $(m-n)^9$
15. $12(a-b)^5$ 16. $a^5(a+b)^4$ 17. $-24mp^3q^3r$
18. $-140r^3s^3z^3$ 19. $14a^6b^6$ 20. $6a^3b^3$
21. $-x^6$ 22. $12m^6n^5$ 23. $84x^8y^6$
24. $(a+b)^6$ 25. $(a-3b)^{10}$ 26. $3a(x-y)^9$

Exercise 15.

1. $10a^2-15ab+20ac$
2. $-3a^3b+6a^2b^2-3ab^3$
3. $25x^4y^3+15x^3y^4-10x^2y^5$
4. $14a^3x^3y-12a^2x^2y^2+18a^2xy^3$
5. $-2x^5y^2+2x^4y^3+2x^3y^4$
6. $15a^3b^2x^4y^2+25a^2b^3x^4y^3-35ab^2x^3y^4$
7. $30m^8+25m^7-20m^6+15m^5-25m^4$
8. $a^8b^3-a^7b^4+a^6b^5-a^5b^6+a^4b^7-a^3b^8$
9. $-30a^2x^5y^2+25a^2x^4y^3-35a^2x^3y^4+25a^2x^2y^3-25a^2xy^5$
10. $a^2bc(x+y)+ab^2c(m+n)-abc^2(p+q)$
11. $2(a+b)x^4y^2-2(a-b)x^3y^3+2abx^2y^4$
12. $3a(x+y)^4-4b(x+y)^5-2c(x+y)^6$
13. $4x(a+b)^3-6y(a+b)^4+8z(a+b)^5$
14. $a^3b(x-y)^5-a^2b^2(x-y)^6+ab^3(x-y)^7$
15. $p^2qx^2(p+q)^5-pq^2x^2(p+q)^6+p^2q^2x(p+q)^7$

Exercise 19.

1. $2a$
2. $-3ab$
3. $-3a^2b$
4. $3x^2y$
5. $3m^4y$
6. $-7a^3b^8xy^2$
7. $8y^3$
8. $3x^2y^2$
9. $(a+b)^3$
10. $-(m-n)^2$
11. $3a(a+x)^4$
12. $3y(a-b)^4$
13. $5z(x+y)^4$
14. $-4x^2(x^2-y^2)^3$
15. $-6m^3(a^3+b^3)^3$
16. $3x^2y^2z^2$
17. $-6x^2yz^6$
18. $-16a^5b^4c^3$
19. $-2xyz$
20. $-9a^5b^6c^3d$
21. 0

Exercise 21.

1. a^2+a
2. x^3+2x
3. $2a-3b$
4. $2xy+x^2y^2$
5. $2a-3b$
6. a^3+a^2-a-1
7. $2x^2+3x+4$
8. $2a-3ab+b$
9. $-2a+3b-5$
10. $2b^2-3ab+7a^2$
11. $x^2-2xy-3y^2$
12. $-a^3x^3+a^2x^2-ax+1$
13. $2m^2n^3-3mn^2+4n-5m$
14. $2a^2(a+b)+3a(m+n)-4(p+q)$
15. $(a+x)^2-(a+x)+1$
16. $2a-3b+4c$
17. $-x-y+z$
18. $xy-x^2+y^2$
19. $(a+b)^2-x^2(a+b)+y^2$

Exercise 25.

1. $x=8$
2. $x=3$
3. $x=6$
4. $x=3$
5. $x=\frac{1}{2}$
6. $x=-2$
7. $x=2\frac{2}{3}$
8. $x=5\frac{1}{2}$

9. $x=12$ 10. $x=24$ 11. $x=240$ 12. $x=\frac{18}{23}$

13. $x=\frac{5}{9}$ 14. $x=1\frac{1}{4}$ 15. $x=18$ 16. $x=36$

Exercise 26.

2. 80, 20 3. 144, 180, 36 4. \$20, \$30

5. \$1600, \$1800, \$2400 6. $10\frac{4}{5}$, $32\frac{2}{5}$, $64\frac{4}{5}$

7. 160, 240, 360 8. \$500, \$2500 9. 48

10. \$30, \$180 11. 16 yr., 24 yr. 12. 7, 56

13. $\frac{10}{23}$, $\frac{15}{299}$ 14. 144 15. $22\frac{1}{2}$

16. 48 yr. 17. 21 A., 27 A. 18. \$225, \$275

19. 20 yr., 55 yr. 20. 40, 60

21. \$450, \$1050, \$2000 22. \$1725, \$1475, \$1600

23. \$150 24. \$90 25. \$415$\frac{5}{13}$

26. \$1000 27. 60 28. 16

29. 40 30. 10 yr. 31. 30 yr.

32. 60, 100 33. $126\frac{3}{5}$ A. 34. 10 yr., 15 yr.

35. \$3600, \$4800 36. 360, 405 37. 2400 bu., 2500 bu.

38. \$240 39. 50 ct. 40. \$28

41. 8, 20 42. \$40 43. $78\frac{6}{23}$, $97\frac{19}{23}$, $93\frac{21}{23}$

Exercise 27.

1. $6a+4b$ 2. $7a+11b-9c$

3. $21x-7y+5z$ 4. $11x^2-7y^2-3xy$

5. $cd-ad+e$ 6. $8xy-2z^2$

7. $7m^2-13mn+23n^2$ 8. $3ax^2-3b^2y^2$

9. $3(x+y)+(m+n)$ 10. $15a(p+q)+3b(p-q)$

Exercise 28.

1. $-x^2+9y^2$ 2. $4x^2+xy-6y^2$

3. $-3a+13b-16c$ 4. $6m^2n^2-15mn-3n^2$

5. $5x^3-12x^2y+3y^2$ 6. $y^4-12y^3+13y-7$

7. x^2-x+15 8. $17x^3-7xy-10y^3$

9. $2m^2-7mn-16n^2$ 10. $-4x^3+7x^2-x+15$

11. $(x+y)+14(x-y)$, or $15x-13y$ **12.** $-x+5y-4z$
13. $4x^2-2xy+5y^2$ **14.** $9a^3+2a^2+6a+5$

Exercise 29.

1. $9x$ **2.** $2a$ **3.** $7a-b$ **4.** $-2x+5y$ **5.** $-2m$
6. $3x$ **7** $-3a+b$ **8.** xy **9.** $x+5y$ **10.** $2x+5y$
11. $5xy$ **12.** 4 **13.** -6 **14.** $2x-2y-z$
15. $5a+5x$ **16.** 1 **17.** $5x$

Exercise 30.

1. $(2a+3b)+(5c-2d)$ **2.** $(a-2b)+(c-2d)$
3. $(m+n)-(p-q)$ **4.** $(3m-2n)-(4p-2q)$
5. $(a-b)+(c-d)-(e-f)$ **6.** $(2a-3b)-(4c-2d)-(5e-6f)$
7. $(x-y)+(2z-3v)-(6u-4w)$
8. $(5p-3q)+(5z-4m)+(2n-6r)$
9. $(a-b+c)-(d+e-f)$, $(2a-3b-4c)+(2d-5e+6f)$,
$(x-y+2z)-(3v+6u-4w)$, $(5p-3q+5z)-(4m-2n+6r)$
10. $(2m-3n+4a)-(6b-7c+2d)-(4e-g+2h)$
11. $(4a-2b-3c)-(4d-5e-6f)+(7g-2h+4l)$
12. $(2p-3q+4r)-(2s-5t-6u)-(7v-2w+6y)$
13. 1. $\{2m-(3n-4a)\}-\{6b-(7c-2d)\}-\{4e-(g-2h)\}$
2. $\{4a-(2b+3c)\}-\{4d-(5e+6f)\}+\{7g-(2h-4l)\}$
3. $\{2p-(3q-4r)\}-\{2s-(5t+6u)\}-\{7v-(2w-6y)\}$
14. $\{x-(y-z)\}-\{m-(n-p)\}+\{q-(r+s)\}$

Exercise 31.

1. a^2-b^2 **2.** $a^2+2ab+b^2$ **3.** $a^2-2ab+b^2$
4. $4a^2-9b^2$ **5.** $x^2y^2+6xy-72$ **6.** $x^2-ax-bx+ab$
7. $bc+bx-cx-x^2$ **8.** $m^5+m^3n^2+m^2n^3+n^5$
9. $12a^4-10a^2b-12b^2$ **10.** $63a^4-23a^2b^2-56b^4$
11. $abcy-acdx-b^2dy+bd^2x$ **12.** x^6+1 **13.** a^3-b^3
14. x^3-8 **15.** $a^{12}+1$ **16.** $729+c^6$ **17.** b^6+64c^6
18. $6m^4+4m^3n-9m^2n^2-6mn^3$ **19.** $36x^4y^4-49y^4z^4$
20. $27a^3-1728$ **21.** x^4-y^4 **22.** a^5-b^5
23. $9x^4+3x^2y^2+4y^4$ **24.** $25a^4-9a^2b^2+16b^4$
25. $81x^4-16x^2y^2+8xy^3-y^4$ **26.** $4x^8+56x^4+324$
27. a^5-32b^5 **28.** $243x^5+32y^5$ **29.** $x^8+x^4y^4+y^8$
30. $6x^7+13x^6y+6x^5y^2+17x^4y^3-13x^3y^4+11x^2y^5-9xy^4+9y^7$

Exercise 32.

1. $a+6$ 2. $x+4$ 3. $x-5$ 4. x^2+6 5. $x+y$
6. x^2+xy+y^2 7. $x^2-3xy+9y^2$ 8. $a^4-a^2b^2+b^4$
9. $a^3+a^2b+ab^2+b^3$ 10. $4x^2-6xy+9y^2$ 11. $2x-6$
12. $2x-3a$ 13. $2x-5a$ 14. $16x^4+20x^2y^2+25y^4$
15. $16x^4-8x^3y+4x^2y^2-2xy^3+y^4$ 16. $4m^6+6m^3n^2+9n^4$
17. x^2-xy+y^2 18. x^2+2x+4 19. $a^4-a^2b^2+b^4$
20. $4m^2n^4+6m^3n^3+9m^4n^2$ 21. x^2+x+1
22. $3x^3+2x^2+1$ 23. x^2-x-3 24. $4a^4+6a^2-2$
25. $2x-y+z$ 26. $x-2y-3z$ 27. $2x-3y-z$
28. $2x+3y+z$ 29. x^2-xy-y^2 30. $a+b-c-d$

Exercise 34.

1. $x=3,\ y=5$ 2. $x=5,\ y=1$ 3. $x=5,\ y=3$
4. $x=-2,\ y=2$ 5. $x=9,\ y=8$ 6. $x=-3,\ y=-4$
7. $x=10,\ y=0$ 8. $x=5,\ y=3$ 9. $x=0,\ y=0$
10. $x=2\frac{1}{2},\ y=3\frac{1}{3}$ 11. $x=-5,\ y=5$ 12. $x=7,\ y=9$
13. $x=12,\ y=24$

Exercise 35.

1. \$400, \$100 2. 15 yr., 20 yr. 3. 2 ct., 1 ct.
4. 56 lb., 30 lb. 5. \$2000, \$1000 6. \$42, \$26
7. 18, 20 8. 25 ct., 50 ct. 9. 60 ct., 40 ct.
10. \$2, \$1 11. \$6, \$40 12. 80 ct , 90 ct.
13. \$2, \$1 14. 14 ct., 24 ct. 15. 40 A., 200 A.
16. 60 A., 120 A. 17. 30, 20 18. \$116, \$166

Exercise 36.

1. $a^{10}b^5c^{10}$ 2. $16a^4b^8c^4$ 3. $-27a^9b^9c^6$ 4. $16x^4y^4z^4$
5. $81x^8y^{12}z^4$ 6. $-64m^6n^9x^3$ 7. $32a^{15}b^{10}c^5d^5$
8. $81x^{16}y^{12}z^4$ 9. $32x^{20}y^{15}z^{10}$ 10. $625m^4n^{12}z^8$
11. $(a+b)^2(c+d)^2$ 12. $m^4x^8(a+b)^{12}$ 13. $27(a+b)^6(x-y)^9$
14. $a^{12}b^8c^4(m+n)^8$ 15. $8a^6b^9(x^2+y^2)^6$ 16. $m^8(x-y)^{12}(x+y)^8$
17. $108a^{19}b^{14}c^{15}$ 18. $6y^5z$ 19. $168x^8y^{14}z^{20}$
20. $13x^4y^4z^{15}$ 21. 0 22. $32x^2y^9z^7$
23. $-256x^{11}y^{14}z^{18}$ 24. $-48x^{14}y^{15}z^{17}$

Exercise 37.

1. $x^2+2xy+y^2$
2. $x^2-2xy+y^2$
3. $m^2+2mn+n^2$
4. $m^2-2mn+n^2$
5. $x^2+8x+16$
6. $x^2-14x+49$
7. $x^2+2ax+a^2$
8. $x^2-2ax+a^2$
9. $4x^2+4xy+y^2$
10. $9x^2-6xy+y^2$
11. $4a^2+12ab+9b^2$
12. $25a^2-30ab+9b^2$.
13. $4x^2+32x+64$
14. $9x^2-30x+25$
15. $25+20x+4x^2$
16. $36-36x+9x^2$
17. $x^2y^2+2xy+1$
18. $x^2y^2+10xy+25$
19. $1-2cd+c^2d^2$
20. $1+2xy+x^2y^2$
21. $a^2b^2-2abcd+c^2d^2$
22. $x^2y^2+2xy^2z+y^2z^2$
23. $4p^2q^2-12pqr+9r^2$
24. $a^4b^2+2a^3b^3+a^2b^4$
25. $4x^4y^2-12x^3y^3+9x^2y^4$
26. $(a+b)^2-2(a+b)+1$
27. $(a-b)^2-2(a-b)+1$
28. $(x+y)^2+2(x+y)z+z^2$
29. $(x+y)^2-2(x+y)z+z^2$
30. $a^2-2a(b+c)+(b+c)^2$

Exercise 38.

1. $x^3+3x^2y+3xy^2+y^3$
2. $x^3-3x^2y+3xy^2-y^3$
3. $y^3+6y^2+12y+8$
4. $y^3-6y^2+12y-8$
5. $1+3x+3x^2+x^3$
6. x^3-3x^2+3x-1
7. $x^3y^3+3x^2y^2+3xy+1$
8. $8+12z+6z^2+z^3$
9. $x^3+6x^2z+12xz^2+8z^3$
10. $x^3-6x^2y+12xy^2-8y^3$
11. $a^3x^3+3a^2bx^2y+3ab^2xy^2+b^3y^3$
12. $8x^3+12x^3y+6x^3y^2+x^3y^3$
13. $a^6-3a^4x^2+3a^2x^4-x^6$
14. $a^6+3a^4xy+3a^2x^2y^2+x^3y^3$
15. $8x^9+12x^6+6x^3+1$
16. $x^9-6x^6y^3+12x^3y^6-8y^9$
17. $8x^6-36x^4y^2+54x^2y^4-27y^6$
18. $x^6-9x^5y+27x^4y^2-27x^3y^3$
19. $x^6y^3-3x^5y^4+3x^4y^5+x^3y^6$
20. $27x^6-108x^4y^2+144x^2y^4-64y^6$
21. $x^3+3x^2(y+z)+3x(y+z)^2+(y+z)^3$

Exercise 39.

1. x^2-y^2
2. x^6-y^6
3. a^2x^2-1
4. $9m^2-25n^2$
5. $16a^2x^2-9b^2$
6. $4x^4y^2-9y^2z^2$
7. $x^4y^6-16x^2y^2$
8. $25x^6y^2-49z^4$
9. $x^4y^6z^2-49$
10. $144x^8-25y^4$
11. $(a+b)^2-1$
12. $(x^2+y^2)^2-z^4$
13. x^4-y^4
14. $16x^4-256$
15. $81x^4-625y^4$
16. $a^8x^8-b^8y^8$
17. x^4-8x^2+16
18. $16x^4-8x^2y^2+y^4$
19. $a^4x^4-2a^2b^2x^2y^2+b^4y^4$

Exercise 40.

1. $x^2+9x+20$
2. $x^2+7x+10$
3. $x^2+11x+30$
4. x^2+8x+7
5. $x^2+11x+24$
6. $x^2+11x+18$
7. $x^2+14x+48$
8. $x^2+15x+50$
9. $4x^2+14x+12$
10. $9x^2+12x+3$
11. $25x^2+25x+6$
12. $x^2+5xy+6y^2$
13. x^2+x-30
14. $x^2+4x-21$
15. $x^2+6x-27$
16. $x^2-5x-14$
17. x^2-x-72
18. $x^2+6x-72$
19. $x^2-10x+21$
20. $x^2-14x+48$
21. $x^2-17x+70$
22. $4x^2-8x-21$
23. $9x^2+6xy-8y^2$
24. $a^2x^2-2abx-3b^2$
25. $x^2+(a+b)x+ab$
26. $x^2-(a-b)x-ab$
27. $x^2+(a-b)x-ab$
28. $x^2-(a+b)x+ab$

Exercise 41.

1. $2x^2+7x+3$
2. $2x^2+14x+20$
3. $3a^2+10a+8$
4. $8a^2+14a+6$
5. $6a^2+22a+20$
6. $5x^2+38x+21$
7. $6x^2+7xy+2y^2$
8. $12a^2+11ab+2b^2$
9. $2x^2-13x+15$
10. $8x^2-30x+7$
11. $3x^2-11xy+10y^2$
12. $6x^2-13xy+6y^2$
13. $2m^2+mn-6n^2$
14. $2x^4+4x^2-30$
15. $15x^4-11x^2-12$
16. $2a^2x^2-ax-10$
17. $20x^2-71x+63$
18. $2x^6-4x^3y^3+2y^6$
19. $12x^4-23x^2y+10y^2$
20. $15x^2+16x-15$
21. $20x^2y^2+xy-30$
22. $15m^4+4m^2-35$

Exercise 42.

1. $x+y$
2. x^2+xy+y^2
3. $x^3+x^2y+xy^2+y^3$
4. $x^4+x^3y+x^2y^2+xy^3+y^4$
5. $x^3+x^2y^2+xy^4+y^6$
6. $4x^2+6x+9$
7. $4x^2-2x+1$
8. $a^2+3ab+9b^2$
9. $1-x+x^2-x^3+x^4$
10. $27x^3-18x^2+12x-8$
11. Not divisible. Why?
12. $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$
13. $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$
14. $x^4+x^2y^2+y^4$
15. $x^4-x^2y^2+y^4$
16. Not divisible. Why?
17. $x^4+2x^2y^2+4y^4$
18. Not divisible. Why?
19. $16x^{20}-8x^{15}+4x^{10}-2x^5+1$
20. Not divisible. Why?
21. $1-9x^2+81x^4$
22. ———
23. $125a^3-25a^2+5a-1$
24. $a^{12}-a^9b^6+a^6b^{12}-a^3b^{18}+b^{24}$
25. $a^{10}-a^5b^{10}+b^{20}$

26. $a^5+3a^4y+9a^3y^2+27a^2y^3+81ay^4+243y^5$

27. $a^5-3a^4y+9a^3y^2-27a^2y^3+81ay^4-243y^4$

29. $8a^3-12a^2b+18ab^2-27b^3$ **30.** $x^4-4x^2y^2+16y^4$

31. $4x^4-6x^2y^3+9y^6$ **32.** $x^8-2x^6y^4+4x^4y^8-8x^2y^{12}+16y^{16}$

33. $x^6y^3+x^5y^4+x^4y^5+x^3y^6$ **34.** $a^2x^{10}-abx^5y^2+b^2y^4$

36. $a^4-9a^2y^2+81y^4$ **37.** $64a^2b^2-8abc+c^2$

38. $4x^4-6x^2y^3+9y^6$

39. $x^8-2x^6y^4+4x^4y^8-8x^2y^{12}+16y^{16}$

Exercise 43.

1. $a(a+b)$ **2.** $b(a-c)$ **3.** $x(x+ay)$

4. $x(x^2+3x-2)$ **5.** $3a(a-2b+3b^2)$

6. $2a^2x(1+2ax-3a^2x^2)$ **7.** $6xy(x^2y-2xy^2-3)$

8. $5x^2(2x^2+3x-4)$ **9.** $7r^2(1-2r+3r^2)$

10. $(a+b)(2a+3b)$ **11.** $(a-x)(a-b)$

12. $(m+n)(c+d)$ **13.** $12a^3b^2c(abc-2b^2c^3+3a)$

14. $5pq(2pq^2+3p^2q^2-4r)$ **15.** $12x^4y^6(2x^4-3xy^3+4y^4)$

16. $(a^2+b^2)(4c-5d)$

Exercise 44.

1. $(a+2b)(a-2b)$ **2.** $(2a+5b)(2a-5b)$

3. $(3x+7y)(3x-7y)$ **4.** $(ay+2)(ay-2)$

5. $(4+z)(4-z)$ **6.** $(x+8)(x-8)$

7. $(xy+10)(xy-10)$ **8.** $(9+z)(9-z)$

9. $(abc+6)(abc-6)$ **10.** $y^2(x+z)(x-z)$

11. $(a^2+z^2)(a+z)(a-z)$ **12.** $(a^4+b^2)(a^2+b)(a^2-b)$

13. $(4a^2+9z^2)(2a+3z)(2a-3z)$

14. $(9y^2+16z^2)(3y+4z)(3y-4z)$

15. $x^2y^2(x+y)(x-y)$ **16.** $(x^3+y^2)(x^3-y^2)$

17. $(25+z^2)(5+z)(5-z)$ **18.** $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$

19. $(x^6+y^2)(x^3+y)(x^3-y)$

20. $(m^4+n^8)(m^2+n^4)(m+n^2)(m-n^2)$

21. $(a+b+c)(a+b-c)$ **22.** $(a-x+y)(a-x-y)$

23. $(m-n+1)(m-n-1)$ **24.** $(2+x+y)(2-x-y)$

25. $(c+a+b)(c-a-b)$ **26.** $(c+a-b)(c-a+b)$

27. $(5a+x-y)(5a-x+y)$ **28.** $(4+z-x)(4-z+x)$

29. $(1+x-y)(1-x+y)$ **30.** $(7+2x+2y)(7-2x-2y)$

Exercise 45.

1. $(x-y)(x^2+xy+y^2)$
2. $(x+y)(x^2-xy+y^2)$
3. $(a-1)(a^2+a+1)$
4. $(a+1)(a^2-a+1)$
5. $(x-2)(x^2+2x+4)$
6. $(x+2)(x^2-2x+4)$
7. $(2a+b)(4a^2-2ab+b^2)$
8. $(2a-b)(4a^2+2ab+b^2)$
9. $(3a-2b)(9a^2+6ab+4b^2)$
10. $(3a+2b)(9a^2-6ab+4b^2)$
11. $(a^2+1)(a^4-a^2+1)$
12. $(a+1)(a-1)(a^2-a+1)(a^2+a+1)$
13. $(x^2+2)(x^4-2x^2+4)$
14. $(x^2-2)(x^4+2x^2+4)$
15. $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$
16. $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$
17. $(x+y)(x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6)$
18. $(2x-y^2)(16x^4+8x^3y^2+4x^2y^4+2xy^6+y^8)$
19. ———
20. $(2x^2-3y^3)(4x^4+6x^2y^3+9y^6)$
21. ———
22. $(x-y)(x^2+xy+y^2)(x^6+x^3y^3+y^6)$
23. $x(x+1)(x^2-x+1)$
24. $x^2(x-2)(x^2+2x+4)$
25. $(xy+z)(x^2y^2-xyz+z^2)$
26. ———
27. $(4m^2+5n^2)(16m^4-20m^2n^2+25n^4)$
28. $xy(xy-1)(x^2y^2+xy+1)$
29. $x(x+y)(x^2-xy+y^2)(x^6-x^3y^3+y^6)$
30. $(x+y-z)\{(x+y)^2+(x+y)z+z^2\}$
31. $(x+y+z)\{(x+y)^2-(x+y)z+z^2\}$
32. $(x-y-z)\{x^2+x(y+z)+(y+z)^2\}$
33. $(x+y+z)\{x^2-x(y+z)+(y+z)^2\}$
34. $(a+b-c-d)\{(a+b)^2+(a+b)(c+d)+(c+d)^2\}$

Exercise 46.

1. $(x+y)^2$
2. $(x-z)^2$
3. $(x+1)^2$
4. $(x-2)^2$
5. $(x+9)^2$
6. $(2x-3)^2$
7. $(3x+2y)^2$
8. $(5x+1)^2$
9. $(x^2-6)^2$
10. $(x+2)^2(x-2)^2$
11. $(x+y)^2(x^2-xy+y^2)^2$
12. $(ab-cd)^2$
13. $(2x+3y)^2(2x-3y)^2$
14. ———
15. $(x+1)^2(x^4-x^3+x^2-x+1)^2$
16. $x^2y^2(x^2+2)^2$
17. $(x^2+y^2)^2(x+y)^2(x-y)^2$
18. ———
19. $(2x^3+3)^2$
20. $9x^2(x^2-2y^2)^2$
21. ———
22. $y^2(x^2+yz)^2$
23. $(x^3-2y^3)^2$
24. $16(x^2+2y^2)^2(x^2-2y^2)^2$

Exercise 47.

1. $(x+3)(x+5)$
2. $(x+4)(x+1)$
3. $(x+4)(x+2)$
4. $(a-4)(a-3)$
5. $(a-7)(a-2)$
6. $(a-8)(a-5)$
7. $(x+5)(x-3)$
8. $(x+7)(x-4)$
9. $(x+8)(x-2)$
10. $(x-5)(x+1)$
11. $(x-7)(x+3)$
12. $(x-10)(x+8)$
13. $(2x+3)(2x+1)$
14. $(2x+4)(2x+3)$
15. $(3x+2)(3x+1)$
16. $(x+3a)(x+a)$
17. $(x-5a)(x+3a)$
18. ———
19. $(3y+2z)(3y-z)$
20. $(6x+5b)(6x-b)$
21. $(2ax-5)(2ax+3)$
22. ———
23. $(x^2-4a)(x^2-3a)$
24. ———

Exercise 48.

1. $(2x+3)(x+1)$
2. $(2x+3)(x+4)$
3. $(3x+2)(2x+1)$
4. $(3x+1)(2x+3)$
5. $(2x-3)(x-2)$
6. $(2x-3)(x+2)$
7. $(2x+5)(x-3)$
8. $(4x-3)(3x+5)$
9. $(3a+5)(2a-3)$
10. $5(3a+7)(a-1)$
11. ———
12. $(2a-b)(a+b)$
13. $(2a+b)(a+2b)$
14. $(3x-y)(2x-3y)$
15. $(3u-2v)(2u+3v)$
16. ———
17. $(3x^2+5)(2x^2-7)$
18. $(2ab-3)(ab+2)$

Exercise 49.

1. $(x^2+x+1)(x^2-x+1)$
2. $(x^2+2x+4)(x^2-2x+4)$
3. $(a^2+ab+b^2)(a^2-ab+b^2)$
4. $(a^2+ac+c^2)(a^2-ac+c^2)(a^4-a^2c^2+c^4)$
5. $(4a^2+2a+1)(4a^2-2a+1)$
6. $(a^4+2a^2b+4b^2)(a^4-2a^2b+4b^2)$
7. $(x^4+x^2y+y^2)(x^4-x^2y+y^2)$
8. $(x^4+x^2y^3+y^6)(x^4-x^2y^3+y^6)$
9. ———
10. $(9x^2+6xy+4y^2)(9x^2-6xy+4y^2)$
11. $(x^2+xy^2+y^4)(x^2-xy^2+y^4)(x^4-x^2y^4+y^8)$
12. $(a^2x^2+axy+y^2)(a^2x^2+axy+y^2)$
13. $a^4y^4(y^2+ay+a^2)(y^2-ay+a^2)$
14. $(a^4b^4+2a^2b^2+4)(a^4b^4-2a^2b^2+4)$
15. $(9+3b+b^2)(9-3b+b^2)$
16. $(25+5xy+x^2y^2)(25-5xy+x^2y^2)$
17. $(4+2z+z^2)(4-2z+z^2)(16-4z^2+z^4)$
18. $y^4(x^2+xz+z^2)(x^2-xz+z^2)$

Exercise 50.

1. $(a+b)(x+y)$
2. $(b+c)(x-y)$
3. $(a-b)(x-z)$
4. $(b+3)(a+2)$
5. $(3-y)(3+x)$
6. $(a+2b)(2x+3y)$
7. $(2a-3b)(3x+2y)$
8. $(ax+3)(bx+2)$
9. $(a-b)(xy+6)$
10. $(a+b)(a-b)(x^2+y^2)$
11. $(a+b+c)(a+b-c)$
12. $(x+y-1)(x-y+1)$
13. $(x+y+1)(x-y-1)$
14. $(2x+y+z)(2x+y-z)$
15. $2z+2x+1)(2z-2x-1)$
16. $(a+b+4)(a+b-4)$
17. $(5+x+a)(5-x-a)$
18. $(x^2+x-1)(x^2-x+1)$
19. $(x^2+y^2+y)(x^2+y^2-y)$
20. $(m^2+p^2+q)(m^2-p^2-q)$

Exercise 51.

1. $ab(a+b)(a-b)$
2. $3a(a+2)(a-2)$
3. $2a(a-b)(a^2+ab+b^2)$
4. $3b(a+b)(a^2-ab+b^2)$
5. $xy(x^2+y^2)(x+y)(x-y)$
6. $ax^2(x+1)(x^4-x^3+x^2-x+1)$
7. $ab(a-b)(a+b)(a^2-ab+b^2)(a^2+ab+b^2)$
8. $2a^2b^2(a^2+b^2)(a^4-a^2b^2+b^4)$
9. $5(a+y)^2$
10. $2ac(a+3)^2$
11. $xz(xy-1)^2$
12. $3ax(x+y)^2(x-y)^2$
13. $4y(y+1)^2(y^2-y+1)^2$
14. $2x(x+2)(x+3)$
15. $xy(x-4)(x-5)$
16. $4ab(a+7)(a-6)$
17. $a(a+b+c)(a-b-c)$
18. $a^2(a+b+c)(a^2-ab-ac+b^2+2bc+c^2)$
19. $ac(a-2c)(a+c)$
20. $4(a+3)(b+2)$
21. $y(a-b)(x-y)$
22. $(x-y)(x+y-1)$
23. $(x+y)(x+y-1)$
24. $x^2(a+b+c)(a+b-c)$
25. $x(x-y+z)(x-y-z)$
26. $(x-y)^3$
27. $x(x+2)^3$
28. $(a^2y^2+1)(ay+1)(ay-1)$
29. $(xy-z)(x^2y^2+xyz+z^2)$
30. $(x^4+y^4)(x^8-x^4y^4+y^8)$
31. $(x+y)(x-y)(x^2+y^2)(x^2-xy+y^2)(x^2+xy+y^2)(x^4-x^2y^2+y^4)$
32. $m(mn^2+1)(mn^2-1)(m^2n^4+1)$
33. $(11a^2+12b^2)^2$
34. $(4a+3b)(4a-b)$
35. $(mn-2)(3-a)$
36. $(a+b+1)(a-b+1)$
37. $(ab+a+b)(ab-a-b)$
38. $(1+x+y)(1-x-y)$
39. $(a+m-n)(a-m+n)$
40. $(1-a-b)(1+a+b+a^2+2ab+b^2)$
41. $(a^4-a^2b^2+b^4)(a^2+ab+b^2)(a^2-ab+b^2)$
42. $2x\times 2y$

43. $a^2(a+y)(a^6-a^5y+a^4y^2-a^3y^3+a^2y^4-ay^5+y^6)$

44. $(a+b-1)^2$ **45.** $-1(2x+y+z)(y+z)$

46. $(x+y+z)(x-y-z)(x^2+y^2+2yz+z^2)$

47. $(x+y)^3$ **48.** $(1+a-b)(1-a+b)$

Exercise 52.

1. $4xy$ **2.** $5a^2$ **3.** $10xyz$

4. $5a^2b^2$ **5.** $6m^2n^2$ **6.** $(a+b)^2$

7. $3(x+y)^3$ **8.** $m(m+n)$ **9.** $2ax(x^2+y^2)$

10. $2(m-n)^2$ **11.** $y(a-b)^2$ **12.** $(a+b)(a-b)$

13. $2(x-y)^2$ **14.** $3a(m-n)^2$ **15.** $(m+n)(m-n)$

16. $(p-q)(a-b)$ **17.** $(a-b)^2$, or $(b-a)^2$ **18.** $\pm(a-b)$

Exercise 53.

1. $a+b$ **2.** $a+b$ **3.** $a-b$

4. $a+b$ **5.** $x-y$ **6.** $x+y$

7. $x-y$ **8.** $x+y$ **9.** $x-4$

10. a^2+2a **11.** $x^2-8x+15$ **12.** x^2-y^2

13. $x+y+z$ **14.** $x+3$ **15.** $x+3$

16. $x+a$ **17.** $m+n$ **18.** x^3+y^3

19. x^2+2y **20.** $x+y+z$ **21.** $x+y+2$

22. $x+y$ **23.** $x(x^2+7)$ **24.** $2x+6$

25. x^2+xy+y^2 **26.** x^2+3x **27.** x^2+4

28. x^2-a^2 **29.** x^2+xy+y^2

Exercise 54.

1. $60x^2y^2$ **2.** $72a^2b^3c^3$ **3.** $96abcx^3y^2$

4. $132x^2y^3z^2$ **5.** $144a^2b^3x^3z^3$ **6.** $1008m^3n^2x^3$

7. $(a+b)^4$ **8.** $100(x+y)^5$ **9.** $a^4(x-y)^3$

10. $24a^2b^2(x+z)^4$ **11.** $36x^3z^3(x^2+y^2)^5$

12. $(a+b)(a-b)(a^2+b^2)$

Exercise 55.

1. $(a+b)^2(a-b)$ **2.** $(a-b)^2(a^2+b^2)(a+b)$ **3.** x^4-y^4

4. $(x-y)(x+y)(x^2+xy+y^2)$ **5.** $(x-y)(x+y)^2$

6. $(x+y)(x-y)^2$ **7.** $(x+y)(x-y)^2(x^2+xy+y^2)$

8. $(x+a)(x+b)(x+c)$ **9.** $abc(a-b)$

10. $(a-b)(b-c)$
11. $(x+y)^3(x-y)(x^2-xy+y^2)$
12. $(x+3)(x-4)(x+2)$
13. $(x-1)(x+4)(x-5)$
14. $(a+b)(m+n)(p+q)$
15. $(a-b)(a+b)(x-y)$
16. $(a+b+c)(a-b-c)(b-a-c)(c-a-b)$
17. $(a+b)^3(a-b)$
18. $(x+y)(x-y)(x^4+x^2y^2+y^4)$
19. $(2x+5)(x+3)(x-2)$
20. $(3x+2)(2x+3)(2x-3)$
21. $axy(x^2-y^2)(x^4+x^2y^2+y^4)$
22. $xy(x+2)(x+3)(x-3)$
23. $x^4+x^2y^2+y^4$
24. x^6+y^6

Exercise 56.

1. 1100
2. 540
3. 900
4. 700
5. 700
6. 1728
7. 4
8. 5
9. 4
10. 27
11. 20
12. 7
13. $(x+y)^2$
14. x^2-y^2
15. $x^4+x^2y^2+y^4$
16. x^2+4x+4
17. $x+4$
18. 1
19. $(a-b)^2-c^2$
20. 36
21. -2
22. $-\frac{2}{3}$
23. 280
24. 0

Exercise 57.

1. $x=4,\ y=5,\ z=6$
2. $x=2,\ y=3,\ z=1$
3. $x=3,\ y=5,\ z=4$
4. $x=5,\ y=4,\ z=3$
5. $x=3,\ y=4,\ z=5$
6. $x=1,\ y=-1,\ z=0$
7. $x=2,\ y=3,\ z=5$
8. $x=5,\ y=4,\ z=3$
9. $x=7,\ y=5,\ z=1$
10. $x=4,\ y=5,\ z=6$
11. $x=6,\ y=7,\ z=10$
12. $x=1,\ y=2,\ z=3,\ u=4$

Exercise 58.

1. 10, 20, 60
2. 20, 30, 40
3. \$1900, \$300, \$1300
4. 125 A., 150 A., 225 A.
5. 60 ct., 40 ct., 80 ct.
6. \$100, \$60, \$8
7. 15, 6, 9
8. 10 yr., 20 yr., 30 yr.
9. \$100, \$60, \$20

Exercise 59.

1. $\frac{2b}{3a}$
2. $\frac{3b}{4ac^2}$
3. $\frac{2y^3}{3x^2z^2}$
4. $\frac{4yz^3}{5x^2}$
5. $\frac{3}{4(x+y)}$
6. $\frac{b(x^3-y^3)}{a}$
7. $\frac{1}{a-b}$
8. $\frac{a+x}{a-x}$

9. $\frac{x^2-2x+4}{x+2}$ 10. $\frac{x-2y}{x+2y}$ 11. $\frac{2x+3y}{2x-3y}$ 12. $\frac{x^2-y^2}{x^2+y^2}$

13. $\frac{x^2-xy+y^2}{x+y}$ 14. $\frac{2x-5y}{2x+5y}$ 15. $\frac{x-y-z}{x}$

16. $\frac{a+b+c}{c}$ 17. $\frac{x+2}{x+4}$ 18. $\frac{x-5}{x+5}$

19. $\frac{x-5}{x-8}$ 20. $\frac{4a^2+6ab+9b^2}{(2a-3b)^2}$ 21. $\frac{x^4+a^2x^2+a^4}{x^2-a^2}$

22. $\frac{x^4-x^2y^2+y^4}{x^2-y^2}$ 23. $\frac{x^2-y^2}{x^6+x^4y^2+x^2y^4+y^6}$ 24. $\frac{a+b}{a-b}$

Exercise 60.

1. $\frac{ax+a}{x}$ 2. $\frac{cy-c}{y}$ 3. $\frac{ax+x+a}{x}$

4. $-\frac{n}{x}$ 5. $\frac{2a^2+2ax+x^2}{a+x}$ 6. $\frac{2x}{x-a}$

7. $\frac{a^2-2x^2}{a+x}$ 8. $\frac{2ax}{x+a}$ 9. $-\frac{2y^3}{x^3+y^3}$

10. $\frac{8a^2+31a+26}{4a+3}$ 11. $\frac{a^3-y^3}{x-y}$ 12. $\frac{x^2-33}{x-4}$

Exercise 61.

1. $a+\frac{b}{c}$ 2. $a-\frac{a}{x}$ 3. $a+x+\frac{1}{x}$ 4. $x+1+\frac{1}{x-1}$

5. $x-y+\frac{2y^2}{x+y}$ 6. $x^2-xy+y^2-\frac{2y^3}{x+y}$ 7. $x+\frac{y^2}{x+y}$

8. $3x+5-\frac{8}{x+1}$ 9. x^2-xy+y^2 10. $x^2+xy+y^2+\frac{2y^3}{x-y}$

11. $2x+1-\frac{3}{2x-1}$ 12. $5x+6+\frac{36}{5x-6}$

13. $2x^2+3x-1+\frac{x^2-x+1}{2x}$ 14. $3x^2-4x+5+\frac{x-10}{2x+1}$

Exercise 62.

1. $\frac{cx}{abc}, \frac{by}{abc}, \frac{az}{abc}$ 2. $\frac{az+bz}{xyz}, \frac{ay-by}{xyz}, \frac{bx}{xyz}$

3. $\frac{a^2x}{aby}, \frac{b^2x}{aby}, \frac{cxy}{aby}$ 4. $\frac{a^2-ab}{a^2-b^2}, \frac{ab+b^2}{a^2-b^2}, \frac{c}{a^2-b^2}$

5. $\frac{(a+x)^2}{a^2-x^2}, \frac{(a-x)^2}{a^2-x^2}, \frac{a^2+x^2}{a^2-x^2}$ 6. $\frac{a^3-a^2x+ax^2}{a^3+x^3}, \frac{b}{a^3+x^3}, \frac{ac+cx}{a^3+x^3}$

7. $\frac{3x-15}{(x+2)(x+3)(x-5)}$, $\frac{5x+15}{(x+2)(x+3)(x-5)}$

8. $\frac{x-a+b}{(x+a+b)(x+a-b)(x-a+b)}$, $\frac{x+a-b}{(x+a+b)(x+a-b)(x-a+b)}$

9. $\frac{2x-6}{(x-1)(x-2)(x-3)}$, $\frac{3x-3}{(x-1)(x-2)(x-3)}$, $\frac{4x-8}{(x-1)(x-2)(x-3)}$

10. $\frac{x}{4x^2-1}$, $\frac{4x^2-4x+1}{4x^2-1}$, $\frac{4x^2+4x+1}{4x^2-1}$

12. $\frac{a+ax}{1-x^2}$, $-\frac{b+bx}{1-x^2}$, $\frac{c}{1-x^2}$ 13. $-\frac{3x-6}{(x-2)^2}$, $\frac{4x-8}{(x-2)^2}$, $\frac{5}{(x-2)^2}$

14. $\frac{3-x}{(x-1)(x-2)(x-3)}$, $\frac{2-2x}{(x-1)(x-2)(x-3)}$, $\frac{6-3x}{(x-1)(x-2)(x-3)}$

15. $-\frac{ab-ax}{(a-x)(b-x)(c-x)}$, $-\frac{bc-bx}{(a-x)(b-x)(c-x)}$, $-\frac{ac-cx}{(a-x)(b-x)(c-x)}$

Exercise 63.

1. $\frac{2}{1-a^2}$ 2. $-\frac{2ax}{a^2-x^2}$ 3. $\frac{a^2b^2+a^2c^2+b^2c^2}{abc}$

4. $\frac{b+c-a}{abc}$ 5. $\frac{ay-ax}{xy}$ 6. $\frac{2ax^2}{x^4-y^4}$ 7. $\frac{m+n}{m-n}$

8. $\frac{pq}{p^2-q^2}$ 9. $\frac{3a^2}{a^2-x^2}$ 10. $\frac{2a}{p-q}$ 11. $3x+\frac{5x-52y}{60}$

12. $\frac{11}{12}ax+2by$ 13. $\frac{a+3b}{2a}+\frac{2c+4d}{3b}$

14. $\frac{7}{x}-\frac{10}{y}+\frac{2}{z}$ 15. $\frac{a-b-c}{x}$ 16. $4b-a-\frac{a^2+b^2}{abx}$

17. $\frac{13}{6x}-\frac{7}{12y}-\frac{7}{4z}$ 18. $\frac{1}{15x}-\frac{71}{40y}-\frac{23}{12z}$

19. $\frac{2a^3+2ax^2}{(a^2-x^2)^2}$ 20. $\frac{2x^2}{x^3+y^3}$ 21. $\frac{x^2y}{(x+y)^3}$

22. $\frac{3}{(x-1)(x-3)}$ 23. 0 24. $\frac{2y}{(x+y+z)(x-y-z)(x+y-z)}$

25. $\frac{2}{x(1-4x^2)}$ 26. $\frac{1}{x+2}$ 27. 0

Exercise 64.

1. $\frac{a^2bx}{c}$ 2. $\frac{abc}{d}$ 3. $\frac{a^2c^4}{d^2}$ 4. $\frac{x^2+y^2}{xy}$

5. $\frac{x^2-y^2}{x^2+y^2}$ 6. $\frac{a-b}{a^2-ab+b^2}$ 7. $\frac{(a+b)^2}{a-b}$ 8. $\frac{x-5}{x+2}$

9. $\frac{ab}{x}$ 10. $\frac{ax}{c^2d}$ 11. $\frac{ay^2}{3mn}$ 12. $\frac{x-y}{xy}$

13. $\frac{x^2-xy+y^2}{x-y}$ 14. $\frac{m^2-n^2}{12mn}$ 15. $\frac{x+2}{x+3}$

16. $\frac{1}{1-x-x^2+x^3}$ 17. $\frac{(x^2-25)(x-6)}{x+6}$ 18. $\frac{x^2-(a+b)^2}{x+a-b}$

19. $\frac{xy(y+z)}{2(x^2-z^2)}$ 20. $\frac{a-b-c}{abc}$ 21. $\frac{x^2+xy+y^2}{x+y}$

22. $\frac{x^2-xy+y^2}{x-y}$

Exercise 65.

1. $\frac{acx^2}{bdy^2}$ 2. $\frac{xy}{bc^2z^2}$ 3. $\frac{abx^2}{a^2-x^2}$

4. $ax(a-x)$ 5. $\frac{cxy}{bd^2}$ 6. $\frac{(c+d)^2}{cd}$

7. $\frac{adx}{bc^2y}$ 8. $\frac{e}{c(a-x)}$ 9. $\frac{x(a-x)}{a(a^2-ax+x^2)}$

10. $\frac{a-y}{a^2-x^2}$ 11. $\frac{(p+q)^2}{(p-q)^2}$ 12. $\frac{(a+b)(x+y)}{a}$

13. $\frac{x^4y^2+x^4}{x^3y^2+y^4}$ 14. $\frac{x}{x^2-1}$ 15. $\frac{x^2-xy+y^2}{(a+b)(x+y)}$

16. $\frac{x^2-12x+36}{x^2+4x+4}$ 17. $\frac{a+b}{c-d}$ 18. 1

Exercise 66.

1. $\frac{ay}{bx}$ 2. $\frac{a^2}{y^2}$ 3. $\frac{(x+1)^2}{(x-1)^2}$ 4. $\frac{ac+b}{ac-b}$.

5. $\frac{x}{x-a}$ 6. $\frac{x^2+5x+6}{x^2-5x+6}$ 7. $\frac{a}{x(a-x)}$ 8. $\frac{x}{x+1}$

9. $\frac{x^2}{x^2+a}$ 10. $\frac{x+6}{x-6}$ 11. $\frac{x^2+1}{x^3+2x}$ 12. $\frac{a-1}{a-5}$

Exercise 67.

1. $\frac{1}{216}$ 2. $\frac{a^4}{d^4}$ 3. $\frac{16p^{12}q^8}{81r^8}$ 4. $\frac{1024}{59049}$

5. $-\frac{a^{10}b^{10}}{c^{15}}$ 6. $\frac{(a+b)^6}{c^6}$ 7. $\frac{(m+n)^8}{(m-n)^8}$ 8. $\frac{c^5(c-x)^{10}}{x^{10}}$

9. $\frac{(x-y)^{20}}{(x+y)^8}$ 10. $\frac{p^3(p+1)^3}{q^3(q-1)^3}$ 11. $\frac{(x^2-z^2)^2}{(x+y)^2}$ 12. $\frac{(m-n)^4}{(m+n)^8}$

Exercise 68.

1. $\frac{x-6}{x+6}$ 2. $\frac{x^2+y^2}{x^4+x^2y^2+y^4}$ 3. $\frac{c-3d}{c+3d}$ 4. $\frac{2xy}{x^2+y^2}$

5. $\frac{x^2}{y}$ 6. $1-x-\frac{x^2}{1+x}$ 7. 0 8. 1

9. $\frac{x^6+2x^5y-2xy^5+y^6}{x^6-y^6}$ 10. $\frac{a}{a^2+b^2}$ 11. $\frac{a^2+b^2}{2ab}$

12. y 14. $\frac{ad}{bc}-\frac{bc}{ad}$ 15. $\frac{x^3}{y^3}+\frac{x}{y}+\frac{y}{x}+\frac{y^3}{x^3}$

16. $\frac{x^2}{a^2}+\frac{2xy}{ab}+\frac{y^2}{b^2}$, $\frac{a^2}{y^2}-2+\frac{y^2}{a^2}$

17. $1+\frac{3a}{x}+\frac{3a^2}{x^2}+\frac{a^3}{x^3}$, $\frac{x^3}{y^3}-\frac{3bx^2}{y^2}+\frac{3b^2x}{y}-b^3$

18. $\frac{a^2}{x^2}-\frac{c^2}{y^2}$ 19. $x^6-\frac{1}{x^6}$

20. $x^2+2x+\frac{2}{x}+\frac{1}{x^2}+3$; $x^2-2x-1+\frac{2}{x}+\frac{1}{x^2}$

21. $\left(\frac{a+x}{a-x}\right)^2-\left(\frac{a-x}{a+x}\right)^2$ or $\frac{8a^3x+8ax^3}{(a^2-x^2)^2}$

22. $\frac{a^2}{x^2}+\frac{ay}{x}+y^2$ 23. $\frac{a^2}{b^2}-1+\frac{b^2}{a^2}$

24. $\frac{a^3}{x^3}+\frac{a^2b}{x^2y}+\frac{ab^2}{xy^2}+\frac{b^3}{y^3}$; $\frac{a^3}{x^3}-\frac{a^2b}{x^2y}+\frac{ab^2}{xy^2}-\frac{b^3}{y^3}$

25. $\left(a+\frac{b}{c}\right)\left(a-\frac{b}{c}\right)$; $\left(xy+\frac{x}{y}\right)\left(xy-\frac{x}{y}\right)$; $\left(\frac{a}{b}+cd\right)\left(\frac{a}{b}-cd\right)$

26. $\left(1+\frac{x^2}{y^2}\right)\left(1+\frac{x}{y}\right)\left(1-\frac{x}{y}\right)$; $\left(a^2+\frac{b^2}{c^2}\right)\left(a+\frac{b}{c}\right)\left(a-\frac{b}{c}\right)$; $\left(\frac{x}{a}+\frac{a}{x}\right)\left(\frac{x}{a}-\frac{a}{x}\right)$

27. $\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$; $\left(x-\frac{a}{x}\right)\left(x^2+a+\frac{a^2}{x^2}\right)$; $\left(a-\frac{x}{y}\right)\left(a+\frac{x}{y}\right)\left(a^2-\frac{ax}{y}+\frac{x^2}{y^2}\right)\left(a^2+\frac{ax}{y}+\frac{x^2}{y^2}\right)$

28. $\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{x^2}{y^2}-1+\frac{y^2}{x^2}\right)$; $\left(\frac{x}{m^2}-\frac{y}{n^2}\right)\left(\frac{x^2}{m^4}+\frac{xy}{m^2n^2}+\frac{y^2}{n^4}\right)$; $\left(1+\frac{x+y}{x-y}\right)\left(1-\frac{x+y}{x-y}\right)$

29. $x^4-x^2+1-\frac{1}{x^2}+\frac{1}{x^4}$ 30. $\frac{y}{x}$

31. $\left(\frac{x^2-a^2}{ax}\right)^2-\left(\frac{y^2-b^2}{by}\right)^2$ 32. $\frac{a^2}{b^2}$

Exercise 69.

1. $x=3$
2. $x=4$
3. $x=1$
4. $x=12$
5. $x=24$
6. $x=14$
7. $x=4\frac{3}{8}$
8. $x=2\frac{1}{2}$
9. $x=0$
10. $x=7$
11. $x=3$
12. $x=7$
13. $x=1\frac{14}{47}$
14. $x=\frac{5}{7}$
15. $x=8$
16. $x=-\frac{4}{7}$
17. $x=1\frac{5}{7}$
18. $x=\frac{5}{12}$
19. $x=8$
20. $x=3$
21. $x=\frac{3}{4}$
22. $x=-3$
23. $x=-1\frac{19}{32}$
24. $x=5\frac{3}{5}$

Exercise 70.

1. $x=\frac{abc}{a+b}$
2. $x=1$
3. $x=c-d$
4. $x=m^2+n^2$
5. $x=\frac{ac}{b-c}$
6. $x=\frac{c+bm-an}{m-n}$
7. $x=\frac{12}{18-17a}$
8. $x=-\frac{2a+b}{9}$
9. $x=\frac{m(m+n)}{(m-n)^2}$
10. $x=\frac{b^2-a^2}{2b}$
11. $x=\frac{a^2b-cd^2}{ad+ac}$
12. $x=\frac{ad-c^2}{md-nc}$
13. $x=\frac{abc}{ab+ac+bc}$
14. $x=\frac{bcd^2}{acd+b^2d-bc^2}$
15. $x=\frac{1}{a+b}$
16. $x=\frac{(a-b)c}{ab}$
17. $x=\frac{cd-ab}{a+b-c-d}$

Exercise 71.

1. $x=11$
2. $x=\frac{67}{83}$
3. $x=25a+24b$
4. $x=-\frac{2}{9}$
5. $x=4$
6. $x=\frac{457}{102}$
7. $x=\frac{ab-1}{bm+n}$
8. $x=2$
9. $x=2$
10. $x=\frac{9}{19}$
11. $x=4\frac{1}{4}$
12. $x=1\frac{1}{2}$
13. $x=\frac{1}{7}a$
14. $x=\frac{1}{2-a}$
15. $x=\frac{7}{10}$
16. $x=-\frac{1}{16}$
17. $x=0$
18. $x=\frac{a}{2c}$
19. $x=4$
20. $x=\frac{a+m}{a}$
21. $x=\frac{2(a^2+b^2)}{a+b}$
22. $x=.8$
23. $x=\frac{4ab^2-10a}{4a-3b}$
24. $x=2$
25. $x=3\frac{1}{2}$

Exercise 72.

1. 10 **2.** \$5, \$50, \$200 **3.** 18 **4.** 80 yr.
5. 300 bu., 200 bu., 160 bu. **6.** $78\frac{6}{11}$ **7.** 24, 60
8. \$80, \$172 **9.** $116\frac{28}{37}$ **10.** \$2000, \$9000, \$5000
11. 72, 91 **12.** 90 yd., 75 yd., 35 yd. **13.** 400
14. \$60 **15.** \$20,000 **16.** 196 lb. **17.** 84
18. \$162, \$118, \$104 **19.** 20 yr., 16 yr.
20. \$3000, \$2500 **21.** 300 bu., 200 bu. **22.** 84, 96
23. 60 ct. **24.** 66 **25.** \$75 **26.** \$12,500, \$10,000
27. $\$133\frac{1}{3}$ **28.** \$1000, \$700 **29.** $\$2.91\frac{2}{3}$ **30.** 30%
31. $14\frac{2}{7}\%$ **32.** $7\frac{1}{7}\%$ **33.** $87\frac{1}{2}\%$ **34.** 20%
35. \$5000 **36.** \$800 **37.** \$500, \$300 **38.** \$720
39. $5\frac{2}{3}$ yr. **40.** 15 yr. **41.** $\frac{200}{r}$ yr. **42.** 6%
43. $\frac{100}{n}\%$ **44.** $6\frac{2}{3}\%$ **45.** \$75 **46.** 6%
47. \$100 **48.** 75% **49.** \$4800 **50.** $22\frac{1}{2}$ mi.
51. 24 mi. **52.** 2 mi. **53.** $1\frac{23}{37}$ da. **54.** $13\frac{1}{3}$ hr.
55. $17\frac{1}{7}$ hr. **56.** 2 da. **57.** \$300 **58.** \$800, \$500
59. \$200, \$280 **60.** 1740 **61.** 15 yr. **62.** \$900
63. 15 lb. **64.** \$60 **65.** $6\frac{6}{17}$ hr. P. M. **66.** 2 P. M.
67. $29\frac{9}{11}$ min. past 4; $5\frac{5}{11}$ min., or $38\frac{2}{11}$ min. past 4; $54\frac{6}{11}$ min. past 4. **68.** 16 yr., 40 yr. **69.** 20 da. **70.** 79 ct.

Exercise 73.

1. $x=3,\ y=1$ **2.** $x=4,\ y=1$ **3.** $x=5,\ y=5$
4. $x=\frac{1}{2},\ y=\frac{1}{3}$ **5.** $x=6,\ y=5$ **6.** $x=5,\ y=4$
7. $x=4,\ y=8$ **8.** $x=-2,\ y=-3$ **9.** $x=-1,\ y=0$
10. $x=5,\ y=4$ **11.** $x=4,\ y=-4$ **12.** $x=2\frac{1}{3},\ y=2\frac{1}{3}$

Exercise 74.

1. $x=4,\ y=3$
2. $x=2,\ y=-2$
3. $x=4,\ y=4$
4. $x=-5,\ y=2$
5. $x=7,\ y=2$
6. $x=3,\ y=10$
7. $x=1\frac{1}{2},\ y=\frac{2}{3}$
8. $x=\frac{1}{2},\ y=\frac{1}{5}$
9. $x=3,\ y=5$
10. $x=\frac{1}{3},\ y=\frac{1}{3}$

Exercise 75.

1. $x=14,\ y=14$
2. $x=10,\ y=12$
3. $x=10,\ y=3$
4. $x=36,\ y=90$
5. $x=13,\ y=17$
6. $x=5\frac{7}{13},\ y=4\frac{9}{13}$
7. $x=9,\ y=15$
8. $x=-60,\ y=5$
9. $x=10,\ y=2\frac{1}{2}$
10. $x=51\frac{3}{7},\ y=38\frac{4}{7}$
11. $x=6,\ y=-9$
12. $x=-\frac{100}{107},\ y=\frac{40}{321}$
13. $x=15,\ y=8$
14. $x=-27,\ y=13$
15. $x=3,\ y=2$
16. $x=4,\ y=-5$

Exercise 76.

1. $x=\frac{1}{2},\ y=\frac{1}{3}$
2. $x=2,\ y=3$
3. $x=4,\ y=6$
4. $x=\frac{1}{2},\ y=\frac{1}{3}$
5. $x=0,\ y=3$
6. $x=-\frac{1}{2},\ y=-\frac{2}{3}$
7. $x=\frac{1}{4},\ y=\frac{1}{5}$
8. $x=\frac{1}{6},\ y=\frac{1}{15}$

Exercise 77.

1. $x=\frac{c+d}{2a},\ y=\frac{c-d}{2b}$
2. $x=\frac{n-bm}{a-b},\ y=\frac{am-n}{a-b}$
3. $x=\frac{a}{2}(a+b),\ y=\frac{b}{2}(a-b)$
4. $x=\frac{ps-nq}{ms-nr},\ y=\frac{pr-mq}{nr-ms}$
5. $x=\frac{an-bm}{cn-bd},\ y=\frac{bm-an}{mc-ad}$
6. $x=\frac{c+n}{a+b},\ y=\frac{bn-ac}{m(a+b)}$
7. $x=\frac{an+b}{m+n},\ y=\frac{b-am}{m+n}$
8. $x=1,\ y=0$
9. $x=\frac{bn-bd}{nc-md},\ y=\frac{bm-bc}{nc-dm}$
10. $x=\frac{a(n^2-m^2)}{mn(cn-dm)},\ y=\frac{b(m^2-n^2)}{mn(cm-dn)}$

Exercise 78.

1. \$168, \$175 **2.** \$600, \$400 **3.** 4, 3

4. $\frac{1}{16}$, $\frac{7}{8}$ **5.** 60 bu., 40 bu. **6.** \$100, \$20

7. $\frac{7}{18}$ **8.** $\frac{7}{9}$ **9.** 8, 3 **10.** 34 **11.** 39

12. 12 **13.** 24 **14.** $12\frac{4}{5}$ da., $21\frac{1}{3}$ da.

15. 8 da. **16.** $15\frac{1}{9}$ da., 17 da. **17.** 42 da., 63 da.

18. 16, 18 **19.** 16 ft., 10 ft. **20.** 15 rd., 9 rd.

21. 15 yd., 10 yd. **22.** 16 rd., 10 rd. **23.** 45 in., 63 in.

24. 150 yd., 30 yd. per min., 20 yd per min. **25.** \$3

26. 8 ft., 6 ft. **27.** 3500 **28.** 8 persons, 56 s.

29. 32 ft., $21\frac{3}{5}$ ft., $691\frac{1}{5}$ sq. ft. **30.** \$600, 5%

31. \$800, 5 yr. **32.** 25 mi., 15 mi.

33. 15 mi., 5 mi. **34.** \$500, \$400

Exercise 79.

1. $x=2,\ y=60,\ z=26$ **2.** $x=3,\ y=5,\ z=4$

3. $x=2,\ y=3,\ z=1$ **4.** $x=12,\ y=24,\ z=36$

5. $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{6}$ **6.** $x=\frac{1}{2},\ y=\frac{1}{5},\ z=\frac{1}{9}$

7. $x=\frac{m+n-r}{2a},\ y=\frac{m-n+r}{2b},\ z=\frac{n+r-m}{2c}$

8. $x=4,\ y=5,\ z=6$ **9.** $x=10,\ y=8,\ z=6$

10. $x=8,\ y=16,\ z=24$ **11.** $x=1,\ y=-1,\ z=0$

12. $x=5,\ y=4,\ z=3,\ u=2$ **13.** $x=\frac{1}{7},\ y=\frac{1}{10},\ z=\frac{1}{9}$

14. $x=12,\ y=24,\ z=36$ **15.** $x=a,\ y=b,\ z=c$

Exercise 80.

1. 60 ct., 35 ct., 65 ct. **2.** 160 A., 90 A., 120 A.

3. \$600, \$400, \$200 **4.** 10 da., 20 da., 30 da.

5 20 yr., 35 yr., 42 yr. **6.** $7\frac{1}{17}$ da., $9\frac{3}{13}$ da., $17\frac{1}{7}$ da.

7. 20 hr., 30 hr., 40 hr. **8.** \$$252\frac{1}{2}$, \$200, \$$47\frac{1}{2}$

9. 346 **10.** \$600, \$800, \$1000

11. \$450, \$225, $\$237\frac{1}{2}$, $\$87\frac{1}{2}$ **12.** 2 ct., 3 ct., 5 ct.

13. \$40, \$60, \$80 **14.** 5 gal., 3 gal., 2 gal.

Exercise 81.

1. 14, 6 **2.** 36 yr., 12 yr. **3.** \$90, \$80

4. $3\frac{3}{7}$ da. **5.** \$700, \$300, \$800 **6.** 40

7. 5 **8.** 120 A., 160 A. **9.** \$4, \$1

10. $\frac{1}{2}(c+d)$, $\frac{1}{2}(c-d)$, 22 yr., 14 yr. **12.** 5, 6, 7

11. $\frac{100-ad}{c-d}$, $\frac{ac-100}{c-d}$; 6, 8 **13.** $\frac{d-am}{n-m}$, $\frac{d-an}{n-m}$; 14, 10

14. $\frac{bn-dm}{bc-ad}$, $\frac{cm-an}{bc-ad}$; \$25, \$35 **15.** $\frac{ad}{d-c}$, $\frac{ad}{a-d+c}$; 50, 75

16. $\frac{bd-ac-ab(b-a)}{b^2-a^2}$, $\frac{bc-ad-ab(b-a)}{b^2-a^2}$; 14, 10

17. $\frac{11a+b}{2}$, 85 **18.** $\frac{ce+bd}{ae+b^2}$, $\frac{bc-ad}{ae+b^2}$; 20, 10

19. $\frac{1}{2}(a+b-c)$, $\frac{1}{2}(a+c-b)$, $\frac{1}{2}(b+c-a)$; 40, 54, 36

20. $\frac{63(ad-bc)}{63d-17b}$, $\frac{63(ad-bc)}{17a-63c}$; 21, 63

Exercise 82.

1. $c^4+4c^3d+6c^2d^2+4cd^3+d^4$

2. $a^4-4a^3d+6a^2d^2-4ad^3+d^4$

3. $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

4. $x^5-5x^4z+10x^3z^2-10x^2z^3+5xz^4-z^5$

5. $m^6+6m^5n+15m^4n^2+20m^3n^3+15m^2n^4+6mn^5+n^6$

6. $m^6-6m^5n+15m^4n^2-20m^3n^3+15m^2n^4-6mn^5+n^6$

7. $c^7-7c^6x+21c^5x^2-35c^4x^3+35c^3x^4-21c^2x^5+7cx^6-x^7$

8. $x^8+8x^7z+28x^6z^2+56x^5z^3+70x^4z^4+56x^3z^5+28x^2z^6+8xz^7+z^8$

9. $x^8-8x^7y+28x^6y^2-56x^5y^3+70x^4y^4-56x^3y^5+28x^2y^6-8xy^7+y^8$

10. $c^9+9c^8z+36c^7z^2+84c^6z^3+126c^5z^4+126c^4z^5+84c^3z^6+$ $36c^2z^7+9cz^8+z^9$

11. $y^{10}-10xy^9+45x^2y^8-120x^3y^7+210x^4y^6-252x^5y^5+210x^6y^4$ $-120x^7y^3+45x^8y^2-10x^9y+x^{10}$

12. $z^{11}+11z^{10}y+55z^9y^2+165z^8y^3+330z^7y^4+462z^6y^5+462z^5y^6$ $+330z^4y^7+165z^3y^8+55z^2y^9+11zy^{10}+y^{11}$

17. $x^4+4x^3+6x^2+4x+1$ 18. $x^4-4x^3+6x^2-4x+1$

19. $x^5+5x^4+10x^3+10x^2+5x+1$

20. $x^5-5x^4+10x^3-10x^2+5x-1$

21. $1+6z+15z^2+20z^3+15z^4+6z^5+z^6$

22. $1-6z+15z^2-20z^3+15z^4-6z^5+z^6$

Exercise 83.

1. $a^4+8a^3b+24a^2b^2+32ab^3+16b^4$

2. $81a^4-216a^3b+216a^2b^2-96ab^3+16b^4$

3. $32x^5+80x^4+80x^3+40x^2+10x+1$

4. $a^5-\frac{5a^4x}{y}+\frac{10a^3x^2}{y^2}-\frac{10a^2x^3}{y^3}+\frac{5ax^4}{y^4}-\frac{x^5}{y^5}$

5. $\frac{a^5}{b^5}+\frac{5a^4c}{b^4d}+\frac{10a^3c^2}{b^3d^2}+\frac{10a^2c^3}{b^2d^3}+\frac{5ac^4}{bd^4}+\frac{c^5}{d^5}$

6. $1-\frac{5}{x^5}+\frac{10}{x^{10}}-\frac{10}{x^{15}}+\frac{5}{x^{20}}-\frac{1}{x^{25}}$

7. $1-10x^2+40x^4-80x^6+80x^8-32x^{10}$

8. $x^{18}+6x^{15}y^3+15x^{12}y^6+20x^9y^9+15x^6y^{12}+6x^3y^{15}+y^{18}$

9. $a^{18}-6a^{15}b^4+15a^{12}b^8-20a^9b^{12}+15a^6b^{16}-6a^3b^{20}+b^{24}$

10. $x^{18}-12x^{12}+60x^6-160+\frac{240}{x^6}-\frac{192}{x^{12}}+\frac{64}{x^{18}}$

11. $\frac{16a^4}{81b^4}-\frac{16a^2}{9b^2}+6-\frac{9b^2}{a^2}+\frac{81b^4}{16a^4}$

12. $\frac{32}{243}-\frac{200}{243}x^2+\frac{500}{243}x^4-\frac{625}{243}x^6+\frac{3125}{1944}x^8-\frac{3125}{7776}x^{10}$

13. $64a^{12}+576a^{10}x^3+2160a^8x^6+4320a^6x^9+4860a^4x^{12}+$
$2916a^2x^{15}+729x^{18}$

14. $x^{12}-6x^{10}y^2+15x^8y^4-20x^6y^6+15x^4y^8-6x^2y^{10}+y^{12}$

15. $-32x^5-240x^4y-720x^3y^2-1080x^2y^3-810xy^4-243y^5$

16. $\frac{16}{81}x^4-\frac{16}{9}x^3y+6x^2y^2-9xy^3+\frac{81}{16}y^4$

17. $a^{10}x^{10}-5a^6x^6+10a^2x^2-\frac{10}{a^2x^2}+\frac{5}{a^6x^6}-\frac{1}{a^{10}x^{10}}$

18. $32x^{10}-240a^2x^7+720a^4x^4-1080a^6x+\frac{810a^8}{x^2}-\frac{243}{x^5}$

Exercise 84.

1. $x^2+y^2+z^2+2xy+2xz+2yz$

2. $x^2+y^2+z^2-2xy-2xz+2yz$

3. $x^2+y^2+1+2xy+2x+2y$

4. $a^2+b^2+4-2ab+4a-4b$

5. $a^4+2a^3b+3a^2b^2+2ab^3+b^4$

6. $4a^2+9b^2+c^2+12ab-4ac-6bc$

7. $x^4+2x^2+3+\frac{2}{x^2}+\frac{1}{x^4}$

8. $4x^2+25y^2+9c^4+20xy-12c^2x-30c^2y$

9. $\frac{x^2}{y^2}+\frac{4x}{y}+6+\frac{4y}{x}+\frac{y^2}{x^2}$

10. $x^2+y^2+z^2+4-2xy+2xz-4x-2yz+4y-4z$

11. $\frac{x^4}{y^2}-2x^3+x^2y^2+2xy-2y^3+\frac{y^4}{x^2}$

12. $m^6+2m^5+3m^4+4m^3+3m^2+2m+1$

13. $a^3+b^3+1+3a^2b+3a^2+3ab^2+3b^2+3a+3b+6ab$

14. $x^3-y^3-z^3-3x^2y-3x^2z+3xy^2-3y^2z+3xz^2-3yz^2+6xyz$

15. $x^3+8+y^3+6x^2+3x^2y+12x+12y+3xy^2+6y^2+12xy$

16. $8x^3-27y^3+125-36x^2y+60x^2+54xy^2+135y^2+150x$
$-225y-180xy$

17. $x^6+3x^5y+6x^4y^2+7x^3y^3+6x^2y^4+3xy^5+y^6$

18. $\frac{8}{27}a^3+\frac{27}{64}b^3+\frac{125}{216}c^3+a^2b+\frac{10}{9}a^2c+\frac{9}{8}ab^2+\frac{45}{32}b^2c+\frac{25}{18}ac^2+$
$\frac{25}{16}bc^2+\frac{5}{2}abc$

19. $x^3+8y^3-27z^3+6x^2y-9x^2z+12xy^2-36y^2z+27xz^2+54yz^2$
$-36xyz$

20. $x^9+3x^6+6x^3+7+\frac{6}{x^3}+\frac{3}{x^6}+\frac{1}{x^9}$

21. $x^6+6x^4+9x^2-4-\frac{9}{x^2}+\frac{6}{x^4}-\frac{1}{x^6}$

22. $1+15x+84x^2+215x^3+252x^4+135x^5+27x^6$

23. $y^6-\frac{3}{2}y^5+\frac{11}{4}y^4-\frac{17}{8}y^3+\frac{11}{6}y^2-\frac{2}{3}y+\frac{8}{27}$

24. $x^6+3x^5-12x^4-29x^3+60x^2+75x-125$

Exercise 85.

1. ± 18 2. ± 36 3. ± 48 4. 8 5. 15
6. 18 7. ± 8 8. ± 12 9. 12

Exercise 86.

1. $\pm ab^2c^3$ 2. $\pm 2a^4b^3c^5$ 3. xy^3z^4
4. $2mn^5$ 5. $-3x^2y^4$ 6. $\pm mn^2p^4$
7. $-2a^2b^3c^4$ 8. $\pm 12a^4x^3y^5$ 9. $-9(a+x)$

10. $\pm 4(a-x)^2$ 11. $-(a+b)c^2$ 12. $\pm 10x^2(x+y)^4$
13. $-3(m+n)^2$ 14. $\pm 2(x^2-y^2)^2$ 15. ± 26244
16. ± 21600 17. 1 18. 184
19. 1125 20. $\pm\frac{1}{4}, \pm\frac{2}{3}, \pm\frac{3}{4}, \pm\frac{2}{5}, \pm\frac{8}{9}$
21. $\pm\frac{a}{b^3}, \pm\frac{a^2b^3}{c^4}, \pm\frac{4xy^3}{5a^2b}, \pm\frac{x^2y^3z^6}{3m^2n^3}$
22. $\frac{2}{3}, -\frac{1}{2}\times\frac{x}{y}, -\frac{2x^2y^3}{3ab^4}, \frac{x^2(a+x)}{c^3d^6}$
23. $\pm\frac{a+b}{(a-b)^2}, -\frac{(x-y)^2}{(x+y)^3}, \pm\frac{ab^2c^3}{(a-b)^2}, -\frac{xy^2}{2a^3}$
24. $\pm\frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \pm\frac{ax^2}{(a-x)^3}, \pm\frac{1}{2}$

Exercise 87.

1. $\pm(a+b)$ 2. $\pm(x-y)$ 3. $\pm(x+4)$
4. $\pm(x-3)$ 5. $\pm(x+y+z)$ 6. $\pm(x-y-z)$
7. $\pm(a+2b+3c)$ 8. $\pm(2x-y+3)$ 9. $\pm(x^2+y+1)$
10. $\pm(3x-4y)$ 11. $\pm\left(\frac{1}{2}x^2-\frac{1}{3}y^3\right)$ 12. $\pm\left(2x^2+\frac{1}{2}y+3\right)$

Exercise 88.

1. $\pm(x^2+x+1)$ 2. $\pm(x^2-2x+1)$
3. $\pm(x^2+2x+3)$ 4. $\pm(x^2-3xy+2y^2)$
5. $\pm(x^3-2x^2+3x)$ 6. $\pm(2x^2-5xy+3y^2)$
7. $\pm\left(x^2+\frac{1}{2}x+\frac{1}{4}\right)$ 8. $\pm\left(x^2+2+\frac{1}{x^2}\right)$ 9. $\pm(x^3+x^2-x+1)$

Exercise 89.

1. ± 17 2. ± 26 3. ± 35 4. ± 43
5. ± 52 6. ± 69 7. ± 71 8. ± 84
9. ± 127 10. ± 245 11. ± 324 12. ± 408
15. $\pm .07$ 16. $\pm .25$ 17. $\pm .012$ 18. ± 29.7
19. $\pm .0004$ 20. $\pm .324$ 21. ± 5.82 22. ± 3.38
23. ± 10.42 24. ± 32.01 25. ± 9.999 26. ± 89.5
27. 3.1622, 3.3166, 3.4641, 3.6055
28. 1.4142, 1.8165, .9354, .5590
29. 6.324, 6.403, 6.480, 6.557

Exercise 90.

1. $x+1$
2. $a-b$
3. $x+4$
4. $2x^2+3$
5. x^2+x-1
6. y^2-y-1
7. $ax-b$
8. $2ax-3by$
9. x^2-2x+3
10. $x+\frac{1}{x}$
11. $ax-\frac{2}{ax}$
12. $x^2+1+\frac{1}{x^2}$
13. $a+b+c$

Exercise 91.

1. 14
2. 27
3. 35
4. 67
5. 72
6. 84
7. 88
8. 98
9. 122
10. 222
11. 305
12. 420
15. .2
16. .12
17. .03
18. .25
19. .31
20. .50
21. 3.4
22. 2.4
23. $4\frac{1}{3}$
24. 1.442, .854, .646

Exercise 92.

1. $\pm(x^2+y^2)$
2. $\pm(x+y)$
3. $\pm 4, \pm 3, 5, \pm .2$

Exercise 93.

1. $(x^2+ax+a^2)(x^2-ax+a^2)$
2. $(x+5)(x+1)$
3. $(2x+5y)(2x+y)$
4. $(5x^2+7xy+4y^2)(5x^2-7xy+4y^2)$
5. $(2p+q)(p+3q)(2p-q)(p-3q)$
6. $(8a^2+4ab+9b^2)(8a^2-4ab+9b^2)$
7. $(x^2+x+3)(x^2+x-1)$
8. $(2x+3y)(x+y)(2x^2+5xy-y^2)$
9. $(2x+1)(4x^2+28x+61)$
10. $(3x-y)(9x^2-15xy+7y^2)$
11. $(x+y)(x-y)(x^4+7x^2y^2+19y^4)$
12. $(2a^2-3b^2)(4a^4+3b^4)$
13. $(x+5)(x^2+25x+175)$
14. $(2a^3+1)(4a^6-8a^3+7)$
15. $(a^3+3b^3)(a^6+3b^6)$
16. $(a^2+2b^2)(a^2-2b^2)(a^8+a^4b^4+7b^8)$
17. $(2a+4b+5c)(2a+2b+3c)$
18. $(3a-b)(9a^2-42ab+61b^2)$
19. $(a^2x-b^2y)(a^4x^2+4a^2b^2xy+7b^4y^2)$

Exercise 94.

1. $x=\pm 6$
2. $x=\pm 2$
3. $x=\pm 2$
4. $x=\pm\frac{1}{2}\sqrt{10}$
5. $x=\pm 2\sqrt{-1}$
6. $x=\pm 1, \pm\frac{1}{3}\sqrt{-3}$
7. $x=\pm\sqrt{\frac{ac}{1-c}}$

8. $x=\pm\sqrt{\frac{ac}{1-bc}}$ **9.** $x=\pm\frac{1}{2}a\sqrt{2}$ **10.** $x=\pm\sqrt{6},\ \pm\sqrt{2}$

11. $x=\pm 3$ **12.** $x=\pm\sqrt{2},\ \pm\frac{1}{3}\sqrt{6}$ **13.** $x=2,\ -10$

14. $x=\pm\sqrt{15\frac{9}{10}},\ \pm\sqrt{-2\frac{29}{50}}$ **15.** 5 rd. **16.** 15, 12

17. 50 A. **18.** $90 **19.** 9, 21 **20.** 9 in. **21.** 10 da.

Exercise 95.

1. $x=2,\ -4$ **2.** $x=6,\ -4$ **3.** $x=-2,\ -3$

4. $x=4,\ 5$ **5.** $x=6,\ -3$ **6.** $x=5,\ -4$

7. $x=4,\ 7$ **8.** $x=6,\ -10$ **9.** $x=7,\ -8$

10. $x=11,\ -10$ **11.** $x=-1,\ -1\frac{1}{2}$ **12.** $x=3,\ -\frac{2}{3}$

13. $x=6,\ -2\frac{1}{2}$ **14.** $x=3,\ -4\frac{1}{2}$ **15.** $x=-7,\ -\frac{4}{5}$

16. $x=9,\ -1\frac{1}{6}$ **17.** $x=-1\frac{2}{3},\ -1\frac{1}{2}$ **18.** $x=7,\ -2\frac{1}{2}$

19. $x=-2\frac{1}{3},\ -\frac{2}{3}$ **20.** $x=2\frac{1}{4},\ -1\frac{1}{4}$ **21.** $x=-\frac{2}{3},\ -1\frac{2}{3}$

22. $x=2\frac{1}{2},\ -3\frac{1}{2}$ **23.** $x=2,\ -3\frac{2}{3}$ **24.** $x=3,\ -2\frac{3}{4}$

25. $x=2\pm\sqrt{3}$ **26.** $x=3\pm 2\sqrt{5}$ **27.** $x=4,\ \frac{1}{4}$

28. $x=6,\ -\frac{7}{13}$ **29.** $x=1\pm\frac{1}{2}\sqrt{2}$ **30.** $x=8,\ -10\frac{2}{5}$

31. $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{13}$ **32.** $x=8,\ 3$

Exercise 96.

1. $x=a,\ -3a$ **2.** $x=2b,\ b$ **3.** $x=2m,\ -3m$

4. $x=\frac{2c}{a},\ -\frac{c}{a}$ **5.** $x=-ab,\ -b$ **6.** $x=\frac{a}{2}(1\pm\sqrt{-3})$

7. $x=3a,\ -a$ **8.** $x=\frac{a}{2}(1\pm\sqrt{5})$ **9.** $x=b\pm\sqrt{ab}$

10. $x=\pm\sqrt{pq}$ **11.** $x=\frac{1}{2}a,\ -a$ **12.** $x=a,\ \frac{1}{a}$

13. $x=\frac{1}{2}(-b\pm 2\sqrt{b^2-bc})$ **14.** $x=\frac{1}{2}(n\pm 2\sqrt{mn})$

15. $x=\frac{b}{a},\ -\frac{d}{c}$ **16.** $x=\frac{a}{b},\ -\frac{b}{a}$

Exercise 97.

1. $x=\pm 2,\ \pm\sqrt{-6}$
2. $x=1,\ 2$
3. $x=\pm\sqrt[4]{2},\ \pm\sqrt[4]{-3}$
4. $x=\pm 3,\ \pm 2\sqrt{-1}$
5. $x=2,\ \frac{1}{2}$
6. $x=1,\ -2$
7. $x=\pm 1,\ \pm\frac{1}{9}$
8. $x=\pm 7,\ \pm 5$
9. $x=\pm 1,\ \pm\sqrt{-1}$
10. $x=\sqrt[3]{\frac{1}{2a}(b\pm\sqrt{b^2-4a^2})}$
11. $x=-2\pm\sqrt{3},\ -2\pm\sqrt{-7}$
12. $x=3,\ -5\frac{1}{2}$
13. $x=\pm 2\frac{1}{2}$
14. $x=0,\ -1,\ -\frac{1}{2}(1\mp\sqrt{-19})$
15. $x=2,\ \frac{1}{2},\ \frac{1}{4}(-13\pm\sqrt{153})$
16. $x=3,\ -5,\ 2,\ -4$
17. $x=7,\ 3,\ \pm 2$
18. $x=3,\ -1$
19. $x=\pm 1,\ \pm 2$
20. $x=\frac{1}{b}(\sqrt{m\pm\sqrt{c+m^2}}-a)$
21. $x=\sqrt[n]{2},\ \sqrt[n]{-6}$
22. $x=-2,\ 1,\ -\frac{1}{2}(1\mp\sqrt{5})$
23. $x=2,\ \frac{1}{2},\ \frac{1}{4}(-7\pm\sqrt{33})$

Exercise 98.

1. $x=\pm 2$
2. $x=\pm 1\frac{1}{2}$
3. $x=\pm\frac{2}{3}a$
4. $x=\pm\frac{1}{4}$
5. $x=-2,\ -5$
6. $x=2,\ 4$
7. $x=3,\ -6$
8. $x=2,\ -1\frac{1}{2}$
9. $x=-1\frac{1}{2},\ -1\frac{1}{2}$
10. $x=1\frac{1}{3},\ 1\frac{1}{3}$
11. $x=5,\ -1\frac{1}{2}$
12. $x=\frac{2}{3},\ \frac{3}{2}$
13. $x=\frac{3}{4},\ -2\frac{1}{2}$
14. $x=2\frac{1}{3},\ -5$
15. $x=2,\ -1\pm\sqrt{-3}$
16. $x=-1,\ \frac{1}{2}(1\pm\sqrt{-3})$
17. $x=-2,\ 1\pm\sqrt{-3}$
18. $x=3,\ \frac{1}{2}(-3\pm\sqrt{-27})$
19. $x=-3,\ \frac{1}{2}(3\pm\sqrt{-27})$
20. $x=\pm a,\ \pm a\sqrt{-1}$
21. $x=\pm 3,\ \pm 3\sqrt{-1}$
22. $x=\pm 1,\ \frac{1}{2}(\mp 1\pm\sqrt{-3})$
23. $x=\pm a,\ \frac{1}{2}a(\mp 1\pm\sqrt{-3})$
24. $x=\frac{3}{2}a,\ \frac{a}{4}(-3\pm\sqrt{-27})$
25. $x=\pm 2,\ -6$
26. $x=\pm 1,\ +1$

27. $x=\frac{1}{2}(\mp 1 \pm \sqrt{-3})$
28. $x=-1, \ -1, \ -1$
29. $x=\pm 2, \ \pm 3$
30. $x=\frac{1}{2}(1 \pm \sqrt{-3})$
31. $x=-2, \ -2, \ -2$
32. $x=2 \pm \sqrt{3}, \ 2 \pm \sqrt{-3}$
33. $x=a, \ b, \ b-1$

Exercise 99.

1. $x^2-6x=-8$
2. $x^2-2x=15$
3. $x^2+5x=24$
4. $x^2+9x=-20$
5. $x^2-3ax=-2a^2$
6. $x^2-px=6p^2$
7. $x^2+7ax=8a^2$
8. $x^2-2ax=b^2-a^2$
9. $x^2-2a^2x=b^4-a^4$
10. $x^2-2x=1$
11. $x^2-6x=-7$
12. $x^2-4ax=b^2-4a^2$
13. $x^2-2ax=b-a^2$
14. $x^2-\frac{19}{12}x=-\frac{5}{8}$
15. $x^2-\frac{25}{12}x=-1$
16. $x^2-2ax=4m^2-a^2$

Exercise 100.

1. $x^2-9=0$
2. $x^2+2x-35=0$
3. $x^2-2x-35=0$
4. $x^2+12x+35=0$
5. $x^2-12x+35=0$
6. $12x^2-17x+6=0$
7. $2x^2-5x-25=0$
8. $6x^2+13x+6=0$
9. $6x^2+13x-15=0$
10. $x^2+ax-2a^2=0$
11. $x^3-2x^2-9x+18=0$
12. $x^3-4x^2-9x+36=0$
13. $x^3-5x^2+8x-4=0$
14. $x^3-4x^2+3x=0$
15. $12x^3-4x^2-3x+1=0$
16. $10x^3-39x^2+39x-10=0$
17. $16x^3-16x^2+3x=0$
18. $60x^3-133x^2+98x-24=0$

Exercise 101.

1. $x=\frac{1}{2}a(1 \pm \sqrt{2a^2+1})$
2. $x=6, \ 6$
3. $x=14, \ -10$
4. $x=1, \ \frac{3}{5}$
5. $x=-3, \ -4$
6. $x=\pm 1$
7. $x=\pm 5$
8. $x=\pm 9$
9. $x=21, \ 5$
10. $x=\frac{1}{2}(9 \pm \sqrt{145})$
11. $x=7, \ \frac{4}{5}$
12. $x=4, \ 0$
13. $x=\pm \sqrt{mn}$
14. $x=\frac{d}{c}, \ -\frac{b}{a}$
15. $x=a \pm \frac{1}{a}$

16. $x=b,\ -a$ **17.** $x=9.477,\ -1.477$ **18.** $x=2.108,\ -2.608$

19. $x=5.236,\ .764$ **20.** $x=1.148,\ -0.348$

21. $x=\pm 1.095445$ **22.** $x=\pm 4.54923$

23. $x=30.716,\ -0.716$ **24.** $x=7.464,\ 0.536$

25. $x=\pm a,\ \pm\frac{1}{a}$ **26.** $x=\pm\sqrt{-1},\ \pm\sqrt{\frac{1}{2}(1\pm\sqrt{-3})}$

27. $x=2,\ \frac{1}{2},\ \frac{1}{4}(-13\pm\sqrt{153})$ **28.** $x=4,\ -3$

29. $x=1,\ -\frac{1}{2}(1\mp\sqrt{-3})$ **30.** $x=-1,\ \frac{1}{2}(1\pm\sqrt{-3})$

31. $x=\pm 1,\ \pm\sqrt{-1}$ **32.** $(x^2-a^2)(x^2-b^2)(x^2-c^2)=0$

Exercise 102.

1. 12, 13	**2.** 3, 14	**3.** 7, 15
4. 20 rows.	**5.** 30 yd.	**6.** 5, 25
7. 8 rd., 6 rd.	**8.** 40 mi. an hr.	**9.** 4 da., 6 da.
10. 5 hr.	**11.** 36	**12.** 12 ft.
13. \$50	**14.** \$80, \$120	**15.** \$2000
16. 3 in.	**17.** 20 ct.	**18.** \$24, \$30
19. $2+\sqrt{8}$ mi.	**20.** 24 mi., 16 mi.	**21.** 6 ft., 4 ft.
22. \$9	**23.** \$41.83+, \$33.83+	**24.** 3 ft.
25. \$2, \$3	**26.** 6, 8	**27.** 24, 18

Exercise 103.

1. $\begin{cases} x=4,\ 5 \\ y=5,\ 4 \end{cases}$ **2.** $\begin{cases} x=8,\ -2 \\ y=2,\ -8 \end{cases}$ **3.** $\begin{cases} x=\pm 6,\ \pm 4 \\ y=\mp 4,\ \mp 6 \end{cases}$

4. $\begin{cases} x=6 \\ y=6 \end{cases}$ **5.** $\begin{cases} x=5 \\ y=-5 \end{cases}$ **6.** $\begin{cases} x=3 \\ y=4 \end{cases}$

7. $\begin{cases} x=a \\ y=b \end{cases}$ **8.** $\begin{cases} x=4 \\ y=0 \end{cases}$ **9.** $\begin{cases} x=2 \\ y=3 \end{cases}$

10. $\begin{cases} x=\pm 4,\ \pm 3 \\ y=\pm 1\frac{1}{2},\ \pm 2 \end{cases}$ **11.** $\begin{cases} x=5 \\ y=1 \end{cases}$ **12.** $\begin{cases} x=15,\ 5 \\ y=3\frac{1}{3},\ 10 \end{cases}$

13. $\begin{cases} x=-6 \\ y=-2 \end{cases}$ **15.** $\begin{cases} x=8 \\ y=6 \end{cases}$ **17.** $\begin{cases} x=10,\ -10\frac{5}{12} \\ y=5,\ -3\frac{3}{4} \end{cases}$

14. $\begin{cases} x=1\frac{4}{5},\ 5 \\ y=7\frac{3}{5},\ 6 \end{cases}$ **16.** $\begin{cases} x=8,\ -9 \\ y=-2,\ 3\frac{2}{3} \end{cases}$ **18.** $\begin{cases} x=3,\ 4 \\ y=4,\ 3 \end{cases}$

Exercise 104.

1. $\begin{cases} x=3, & 2 \\ y=2, & 3 \end{cases}$

2. $\begin{cases} x=5, & -4 \\ y=4, & -5 \end{cases}$

3. $\begin{cases} x=3, & 3, & -3, & -3 \\ y=1, & -1, & 1, & -1 \end{cases}$

4. $\begin{cases} x=2, & 2, & -2, & -2 \\ y=3, & -3, & 3, & -3 \end{cases}$

5. $\begin{cases} x=4, & -4, & 3, & -3 \\ y=3, & -3, & 4, & -4 \end{cases}$

6. $\begin{cases} x=2, & -2, & 1, & -1 \\ y=-1, & 1, & -2, & 2 \end{cases}$

7. $\begin{cases} x=6, & 3 \\ y=1, & 2 \end{cases}$

8. $\begin{cases} x=2\frac{1}{2}, & -1\frac{1}{2} \\ y=3, & -5 \end{cases}$

9. $\begin{cases} x=1\frac{1}{2}, & 1 \\ y=\frac{2}{3}, & 1 \end{cases}$

10. $\begin{cases} x=3, & 3, & -3, & -3 \\ y=\frac{1}{2}, & -\frac{1}{2}, & \frac{1}{2}, & -\frac{1}{2} \end{cases}$

11. $\begin{cases} x=1\frac{1}{3}, & 1\frac{1}{3}, & -1\frac{1}{3}, & -1\frac{1}{3} \\ y=1\frac{1}{2}, & -1\frac{1}{2}, & 1\frac{1}{2}, & -1\frac{1}{2} \end{cases}$

12. $\begin{cases} x=1\frac{1}{2}, & -1\frac{1}{2}, & \frac{1}{2}, & -\frac{1}{2} \\ y=1, & -1, & 3, & -3 \end{cases}$

Exercise 105.

1. $\begin{cases} x=\pm 4 \\ y=\pm 2 \end{cases}$

2. $\begin{cases} x=\pm 3 \\ y=\pm 3 \end{cases}$

3. $\begin{cases} x=\pm 2 \\ y=\pm 1 \end{cases}$

4. $\begin{cases} x=\pm 3 \\ y=\pm 4 \end{cases}$

5. $\begin{cases} x=\pm 2\sqrt{6} \\ y=0 \end{cases}$

6. $\begin{cases} x=\pm 5 \\ y=\pm 3 \end{cases}$

7. $\begin{cases} x=\pm 5 \\ y=\pm 3 \end{cases}$

8. $\begin{cases} x=\pm 5 \\ y=\pm 5 \end{cases}$

9. $\begin{cases} x=\pm 4 \\ y=\pm 1 \end{cases}$

10. $\begin{cases} x=\pm 1 \\ y=\pm 1 \end{cases}$

Exercise 106.

1. $\begin{cases} x=2, & 3, & \frac{1}{2}(5\pm\sqrt{53}) \\ y=3, & 2, & \frac{1}{2}(5\mp\sqrt{53}) \end{cases}$

2. $\begin{cases} x=4, & \frac{10}{13} \\ y=3, & -\frac{3}{13} \end{cases}$

3. $\begin{cases} x=2, & 5, & -4\pm\sqrt{6} \\ y=5, & 2, & -4\mp\sqrt{6} \end{cases}$

4. $\begin{cases} x=\pm 2 \\ y=\pm 1 \end{cases}$

5. $\begin{cases} x=3, & 5\frac{1}{2}, & -6, & -3\frac{1}{2} \\ y=5, & 2\frac{1}{2}, & -4, & -6\frac{1}{2} \end{cases}$

6. $\begin{cases} x=15, & 10, & \frac{1}{2}(-23\pm\sqrt{-71}) \\ y=10, & 15, & \frac{1}{2}(-23\mp\sqrt{-71}) \end{cases}$

7. $\begin{cases} x=10, & 6, & \frac{1}{2}(-19\pm\sqrt{-19}) \\ y=6, & 10, & \frac{1}{2}(-19\mp\sqrt{-19}) \end{cases}$

8. $\begin{cases} x=8, & 6, & \pm\frac{5}{2}(3\pm\sqrt{-1}) \\ y=6, & 8, & \pm\frac{5}{2}(3\mp\sqrt{-1}) \end{cases}$

9. $\begin{cases} x=\pm 4, & \pm 2 \\ y=\pm 2, & \pm 4 \end{cases}$

10. $\begin{cases} x=3, & -7, & 1\pm 2\sqrt{7} \\ y=7, & -3, & -1\pm 2\sqrt{7} \end{cases}$

Exercise 107.

1. $\begin{cases} x=\pm 1, & \pm\frac{3}{2}\sqrt{2} \\ y=\pm 2, & \mp\frac{1}{2}\sqrt{2} \end{cases}$

2. $\begin{cases} x=\pm 2, & \pm\frac{14}{13}\sqrt{13} \\ y=\pm 4, & \pm\frac{10}{13}\sqrt{13} \end{cases}$

3. $\begin{cases} x=\pm 4, & \pm\frac{5}{7}\sqrt{7} \\ y=\pm 2, & \pm\frac{3}{7}\sqrt{7} \end{cases}$

4. $\begin{cases} x=\pm 3, & \pm\frac{9}{2}\sqrt{-6} \\ y=\pm 6, & \mp\frac{11}{2}\sqrt{-6} \end{cases}$

5. $\begin{cases} x=\pm\frac{1}{4}, & \pm\frac{1}{8}\sqrt{2} \\ y=\pm\frac{1}{2}, & \pm\frac{3}{8}\sqrt{2}. \end{cases}$

6. $\begin{cases} x=\pm 1, & \pm 11\sqrt{\frac{1}{41}} \\ y=\pm 1, & \pm 3\sqrt{\frac{1}{41}} \end{cases}$

7. $\begin{cases} x=\pm 1, & \pm\frac{5}{2}\sqrt{-2} \\ y=\pm 3, & \mp\frac{3}{2}\sqrt{-2} \end{cases}$

8. $\begin{cases} x=0, & \pm 2 \\ y=\pm 2\sqrt{10}, & \pm 6 \end{cases}$

9. $\begin{cases} x=\pm 5, & \pm 5\sqrt{-5} \\ y=\pm 10, & \mp 5\sqrt{-5} \end{cases}$

10. $\begin{cases} x=\pm 2, & \pm\frac{3}{7}\sqrt{15} \\ y=\pm 3, & \pm\frac{5}{7}\sqrt{15} \end{cases}$

Exercise 108.

1. $\begin{cases} x=5 \\ y=3 \end{cases}$

2. $\begin{cases} x=7, & -18 \\ y=2, & 14\frac{1}{2} \end{cases}$

3. $\begin{cases} x=8, & -7\frac{3}{13} \\ y=2, & -3\frac{1}{13} \end{cases}$

4. $\begin{cases} x=a, & -b \\ y=-b, & a \end{cases}$

5. $\begin{cases} x=\pm\frac{1}{2}(\sqrt{a^2+2b}\pm\sqrt{a^2-2b}) \\ y=\pm\frac{1}{2}(\sqrt{a^2+2b}\mp\sqrt{a^2-2b}) \end{cases}$

6. $\begin{cases} x=5, & 1 \\ y=1, & 5 \end{cases}$

7. $\begin{cases} x=\pm 5, & \pm 1 \\ y=\pm 1, & \pm 5 \end{cases}$

8. $\begin{cases} x=\pm 3, & \pm 2 \\ y=\pm 2, & \pm 3 \end{cases}$

9. $\left\{\begin{array}{l} x=3,\ -2 \\ y=2,\ -3 \end{array}\right\}$ 10. $\left\{\begin{array}{l} x=\pm 6 \\ y=\pm 2 \end{array}\right\}$ 11. $\left\{\begin{array}{l} x=+8,\ -8 \\ y=\pm 4,\ \pm 4 \end{array}\right\}$

12. $\left\{\begin{array}{l} x=1,\ 2 \\ y=\frac{1}{3},\ \frac{2}{3} \end{array}\right\}$ 13. $\left\{\begin{array}{l} x=\pm 2,\ \pm 3 \\ y=\pm 3,\ \pm 2 \end{array}\right\}$ 14. $\left\{\begin{array}{l} x=\pm 3 \\ y=\pm 3 \end{array}\right\}$

15. $\left\{\begin{array}{l} x=\pm 2 \\ y=\pm 2 \end{array}\right\}$ 16. $\left\{\begin{array}{l} x=\pm 5,\ \pm 5\sqrt{-1} \\ y=\pm 2,\ \pm 2\sqrt{-1} \end{array}\right\}$ 17. $\left\{\begin{array}{l} x=9 \\ y=-8 \end{array}\right\}$

18. $\left\{\begin{array}{l} x=\pm\frac{1}{3}a \\ y=\pm 3b \end{array}\right\}$ 19. $\left\{\begin{array}{l} x=5,\ 3,\ -5,\ -7 \\ y=3,\ 5,\ -7,\ -5 \end{array}\right\}$

20. $\left\{\begin{array}{l} x=\pm 5,\ \pm 4 \\ y=\pm 4,\ \pm 5 \end{array}\right\}$ 21. $\left\{\begin{array}{l} x=1,\ 3 \\ y=2,\ 2 \\ z=3,\ 1 \end{array}\right\}$

Exercise 109.

1. 5, 8 **2.** 3, 7 **3.** 80 rd., 100 rd.
4. 4 ft., 6 ft. **5.** 24 rd., 10 rd.; 26 rd. **6.** 40, \$90.
7. 8 da. @ \$8, 5 da @ \$5 **8.** 4, 9 **9.** 45
10. \$20, \$15 **11.** 16 rd., 10 rd. **12.** 10 ft., 12 ft.
13. \$1.30, \$1.40 **14.** 4 mi., 2 mi. **15.** 22 ft.
16. 15, 20 **17.** 9, 12, 15 **18.** \$7200, \$80, \$90
19. \$66, \$20 **20.** 4 mi., 3 mi. **21.** \$100, \$100

Exercise 110.

1. -2 **2.** -8 **3.** $-6\frac{2}{3}$ yr. **4.** $-5,\ 45$
5. $\frac{-5}{-7}$ **6.** -45 yr. **7.** $4,\ -3$ **8.** ± 8

Exercise 111.

1. 4, ± 8, 243, 16 **2.** 16, 625, 4 **3.** $\frac{1}{9},\ \pm\frac{1}{125},\ \frac{1}{256}$
4. 256, ± 4, 9 **5.** ± 2, 9, 4, 16 **6.** $\pm 64{,}000$, 576
7. $x^{-4},\ x^{\frac{7}{12}},\ a^{\frac{3}{4}}x^{\frac{2}{3}}$ **8.** $ax^{-\frac{13}{12}},\ x^{-\frac{6}{5}}y^{-\frac{1}{12}}$
9. $\frac{a}{x^2},\ \frac{c^4}{a^2b^3},\ x^6$ **10.** $\frac{a^9x^{\frac{1}{4}}}{y^{\frac{1}{4}}},\ \frac{y^{\frac{5}{2}}}{xz^{\frac{3}{2}}}$
11. $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b,\ a-2a^{\frac{1}{2}}b^{\frac{1}{2}}+b,\ a-b$
12. $x^{\frac{4}{3}}+2x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}},\ x^{\frac{5}{3}}-2x^{\frac{5}{6}}y^{\frac{5}{6}}+y^{\frac{5}{3}},\ a^{\frac{4}{3}}-b^{-\frac{4}{3}}$
13. $x+3x^{\frac{2}{3}}y^{\frac{1}{3}}+3x^{\frac{1}{3}}y^{\frac{2}{3}}+y,\ x^{-6}-3x^{-4}y^{-2}+3x^{-2}y^{-4}-y^{-6},$
$x^{-6}+3x^{-2}+3x^2+x^6$

14. $x-y$ **15.** $x^{-2}y^2+x^{-1}y+xy^{-1}-x^2y^{-2}$

16. $x+y$ **17.** $x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$, $x^{-1}-y^{-2}$

18. $x^{\frac{2}{3}}-x^{\frac{1}{3}}+1$ **19.** $a^8-2a^4x^4+x^8$, $(x^6-1)^{\frac{1}{2}}$

20. $(x^4+x^2+1)^{\frac{1}{4}}$, $x^{-3}+x^{-2}y^{-1}+x^{-1}y^{-2}+y^{-3}$

21. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})$, $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2$

22. $(a^{\frac{1}{5}}+b^{\frac{1}{5}})(a^{\frac{1}{5}}-b^{\frac{1}{5}})$, $(a^{\frac{1}{4}}+b^{\frac{1}{4}})(a^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}})$,
$(a^{\frac{1}{7}}-b^{\frac{1}{7}})(a^{\frac{2}{7}}+a^{\frac{1}{7}}b^{\frac{1}{7}}+b^{\frac{2}{7}})$

23. $a^{\frac{8}{3}}+4a^2b^{\frac{2}{3}}+6a^{\frac{4}{3}}b^{\frac{4}{3}}+4a^{\frac{2}{3}}b^2+b^{\frac{8}{3}}$, $a^{-8}-4a^{-6}b^{-2}+6a^{-4}b^{-4}$
$-4a^{-2}b^{-6}+b^{-8}$, $a^{\frac{15}{2}}-5a^{\frac{9}{2}}+10a^{\frac{3}{2}}-10a^{-\frac{3}{2}}+5a^{-\frac{9}{2}}-a^{-\frac{15}{2}}$

24. 1, a^6-b^6, $\frac{1}{(a-b)^{\frac{1}{2}}}$

Exercise 112.

1. $\pm2\sqrt{3}$, $\pm3\sqrt{2}$, $\pm4\sqrt{2}$ **2.** $\pm3\sqrt{5}$, $\pm4\sqrt{3}$, $\pm5\sqrt{3}$

3. $2\sqrt[3]{2}$, $-3\sqrt[3]{2}$, $4\sqrt[3]{2}$ **4.** $-2\sqrt[3]{7}$, $3\sqrt[3]{4}$, $3\sqrt[3]{5}$

5. $\pm2a\sqrt{a}$, $\pm4b^2\sqrt{b}$, $\pm2c^2\sqrt{2}$

6. $\pm4b\sqrt{a}$, $\pm3x\sqrt{2xy}$, $-2a\sqrt[3]{a}$

7. $\pm5ab^2\sqrt{2ab}$, $\pm3xy^2\sqrt{5y}$, $3x\sqrt[3]{xy^2}$

8. $\pm4a^2b^3\sqrt{ab}$, $-x^3y^2z\sqrt[3]{xy^2z}$, $\pm x^2yz\sqrt[4]{xz^2}$

9. $\pm a\sqrt{a+b}$, $\pm(a+b)\sqrt{a}$ **10.** $\pm(x+y)\sqrt{x(x+y)}$, $(x+y)\sqrt[3]{x}$

11. $\pm(x^2-y^2)\sqrt{x-y}$, $x(x+y)\sqrt[3]{x^2y^2(x+y)}$

12. $\pm(x+y)\sqrt{x}$, $(x+y)\sqrt[3]{x(x+y)^2}$

13. $\pm2(x+y)\sqrt{x-y}$ **14.** $\pm ab(a-b)\sqrt{b(a^2+ab+b^2)}$

15. $\pm\frac{1}{3}\sqrt{6}$, $\pm\frac{1}{3}\sqrt{5}$, $\frac{1}{7}\sqrt[3]{196}$ **16.** $\pm\frac{1}{2}\sqrt{3}$, $\frac{2}{3}\sqrt[3]{9}$, $\frac{1}{5}\sqrt[3]{100}$

17. $\pm\frac{1}{4}\sqrt{10}$, $\pm\frac{1}{10}\sqrt{30}$, $\frac{1}{6}\sqrt[3]{180}$ **18.** $\pm\frac{2}{3}\sqrt{2}$, $\frac{2}{3}\sqrt[3]{2}$, $\frac{3}{4}\sqrt[3]{4}$

19. $\pm\frac{a}{5}\sqrt{10a}$, $\pm\frac{x^2}{2}\sqrt{3x}$, $\frac{x^2}{2}\sqrt[3]{5x}$

20. $\pm\frac{1}{x}\sqrt{ax}$, $\pm\frac{m}{n}\sqrt{n}$, $-\frac{1}{3}\sqrt[3]{9x}$

21. $\pm\frac{2}{5y}\sqrt{5xy}$, $\pm\frac{1}{a}\sqrt{a(a+b)}$, $-\frac{1}{x}\sqrt[3]{ax}$

22. $\pm\frac{a}{c}\sqrt{bc}$, $\pm\frac{1}{3b}\sqrt{15a}$, $\frac{1}{5}\sqrt[3]{25a}$

23. $\pm\frac{1}{a+b}\sqrt{a^2+ab}$, $\pm\frac{1}{(a+b)^2}\sqrt{a^2-b^2}$

24. $\pm\frac{1}{x}\sqrt{x^3-xy^2},\quad \pm\frac{x+y}{x}\sqrt{x}$

25. $\frac{1}{x+y}\sqrt[3]{x^2(x+y)},\quad \pm\frac{x^3}{(x-y)^3}\sqrt{x-y}$

26. $\pm\frac{x+y}{(x-y)^2}\sqrt{x-y},\quad \frac{1}{5z}\sqrt[3]{75xz^2}$

Exercise 113.

1. $\pm\sqrt{3},\ \pm 2,\ \sqrt[3]{3}$

2. $\pm\frac{1}{2}\sqrt{2},\ \pm\frac{1}{3}\sqrt{6},\ \pm\frac{1}{3}\sqrt[5]{324}$

3. $\pm 2\sqrt{xy},\ \pm\sqrt[3]{5x^2y}$

4. $\pm yz\sqrt{xz},\ \pm xy\sqrt{ay}$

5. $\pm\frac{1}{x}\sqrt{ax},\ \pm\frac{1}{x}\sqrt{-ax}$

6. $\pm y\sqrt{-2a},\ \pm yz\sqrt[3]{x^2z}$

7. $\pm\sqrt{3},\ \sqrt[3]{5},\ \pm 3$

8. $\pm\frac{1}{5}\sqrt{15},\ \pm\frac{1}{3}\sqrt{6},\ \frac{1}{4}\sqrt[3]{48}$

9. $\pm\sqrt[4]{x^2y^3z},\ -y\sqrt[3]{2x^2}$

10. $\sqrt[3]{a-b}$

11. $-\frac{1}{3y}\sqrt[3]{18x^2},\ \pm\frac{b}{c}\sqrt{ac}$

12. $\pm\sqrt{a+b}$

Exercise 114.

1. $\sqrt{25},\ \sqrt{9x^2},\ \sqrt{\frac{4}{9}x^4}$

2. $\sqrt[3]{27a^3},\ \sqrt[3]{125b^3x^3},\ \sqrt[3]{\frac{1}{x^3}}$

3. $\sqrt[5]{a^{10}b^{15}},\ \sqrt[5]{x^5y^{20}},\ \sqrt[5]{\frac{a^{10}}{b^{10}}}$

4. $\sqrt{20},\ \sqrt[3]{81},\ \sqrt{\frac{5}{4}}$

5. $\sqrt{3a^3},\ \sqrt{3x^5},\ \sqrt[3]{\frac{1}{x}}$

6. $(a^3+a^2b)^{\frac{1}{2}},\ \sqrt{(a-b)^2(a+b)}$

7. $(x^3)^{\frac{2}{3}},\ (y^{\frac{9}{2}})^{\frac{2}{3}},\ 8^{\frac{2}{3}}$

8. $(x^{\frac{3}{2}}y)^{\frac{2}{3}},\ (a^{\frac{12}{5}}z)^{\frac{5}{6}}$

9. $(x^2y)^3,\ (2x)^4$

Exercise 115.

1. $\sqrt[6]{8},\ \sqrt[6]{9},\ \sqrt[6]{4}$

2. $\sqrt[4]{x^4},\ \sqrt[4]{xy},\ \sqrt[4]{y}$

3. $a^{\frac{8}{12}},\ b^{\frac{2}{12}},\ c^{\frac{9}{12}}$

4. $\sqrt[6]{(a+b)^3},\ \sqrt[6]{(a+b)^2}$

5. $\sqrt[12]{\frac{1}{16}},\ \sqrt[12]{\frac{1}{729}},\ \sqrt[12]{\frac{1}{64}}$

6. $\sqrt[6]{a^6},\ \sqrt[6]{a^4},\ \sqrt[6]{a^3}$

7. $\sqrt[12]{a^8b^8},\ \sqrt[12]{a^9b^3},\ \sqrt[12]{a^2b^6}$

8. $\sqrt[12]{x^3(x+y)^3},\ (x+y)^{\frac{8}{12}}$

Exercise 116.

1. $12\sqrt{a}$

2. $\frac{5}{2}\sqrt{2}$

3. $\sqrt{3x}$

4. $6\sqrt[3]{5}$

5. $(a+b+c)\sqrt{2}$ 6. $2a\sqrt{x}$ 7. $a(\sqrt{x}-\sqrt{y}+\sqrt{z})$

8. $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\sqrt{x}$ 9. $\frac{a}{b(a-b)}\sqrt{a^2-b^2}$

10. $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\sqrt[3]{ab}$ 11. $(a+b-c)\sqrt{7}$

12. $(a+b^2-c^3)\sqrt[3]{n^2}$ 13. $\sqrt[6]{3}$ 14. $\frac{37}{4}\sqrt{3}$

15. $2a^{\frac{1}{2}}b$ 16. $2a\sqrt{a}$ or $-2b\sqrt{a}$ 17. $2\sqrt[3]{4}$

18. $\frac{3}{a}\sqrt{a}$ 19. $\frac{2a}{a^2-b^2}\sqrt{a^2+ab}$ 20. $\frac{1-ax+x^2}{a-x}\sqrt{ax-x^2}$

Exercise 117.

1. $\pm 3\sqrt{2}$ 2. $\sqrt[3]{18}$ 3. $\pm\sqrt{6}$ 4. $\mp 12a\sqrt{2bc}$

5. $\pm\frac{1}{c}\sqrt{ac}$ 6. $\frac{1}{2}\sqrt[3]{3}$ 7. $\pm 2a\sqrt{6a}$ 8. $6x^2yz$

9. $\frac{1}{z}\sqrt[3]{az^2}$ 10. $\pm\frac{1}{a-b}\sqrt{c(a-b)}$ 11. $\sqrt[6]{500}$

12. 18 13. $\mp 6a\sqrt[6]{ab^3c^2}$ 14. $2+\sqrt{6}+\sqrt{10}$

15. $a-b$ 16. $\sqrt{a^4-b^4}$ 17. $\sqrt[3]{a-x}$

18. $\pm abcxy^2\sqrt[12]{x^6y^5}$ 19. $6\sqrt{3}-24+10\sqrt{15}$

20. $4x-9y$ 21. $x\sqrt{x}+y\sqrt{y}$ 22. x^2+xy+y^2

Exercise 118.

1. ± 2 2. $2\sqrt{5}$ 3. $\frac{1}{5}\sqrt{5}$ 4. $\frac{5}{3}\sqrt{5}$

5. $\frac{1}{c}\sqrt{ac}$ 6. $\frac{1}{4}\sqrt[3]{2}$ 7. $2\sqrt[3]{b}$ 8. $a\sqrt[3]{x}$

9. $\sqrt[6]{2}$ 10. $\sqrt[3]{4}$ 11. $\sqrt{x}-\sqrt{y}$ 12. $12\sqrt{3}$

13. ± 12 14. $12\sqrt{10}$ 15. $\pm b^3$ 16. $\sqrt{2}-1$

17. $\sqrt{1+y}$ 18. $\sqrt{3}+2+\sqrt{5}$ 19. $3\sqrt{2}+4-6\sqrt{3}$

20. $\sqrt{a-b}-2b$ 21. $\sqrt{x}+\sqrt{y}$ 22. $x-\sqrt{xy}+y$

Exercise 119.

1. $\sqrt[3]{4}$ 2. $9\sqrt{3}$ 3. 8 4. 108

5. $4\sqrt{5}$ 6. $a^3\sqrt{ab}$ 7. $4a^2b^2$ 8. $a^2b\sqrt[5]{a^2b}$

9. $ab(a+b)^2$ 10. $(a-b)\sqrt{a-b}$ 11. $(x+y)^2$

12. $(a^2-b^2)^2$ 13. $\frac{4}{9}$ 14. $a+2\sqrt{ab}+b$

15. $\sqrt{14}$ 16. $x-2\sqrt{xy}+y$ 17. $\frac{a^3}{b^3}$

18. $6(5-2\sqrt{6})$ 19. $\frac{a^6}{b^6}$ 20. $26+15\sqrt{3}$

21. $\frac{5}{6}+\frac{1}{3}\sqrt{6}$ 22. $2+3\sqrt[3]{3}-3\sqrt[3]{9}$

23. $\frac{5}{6}+\frac{1}{3}\sqrt{6}$ 24. $a^2(a-2\sqrt{ab}+b)$

Exercise 120.

1. $\sqrt[4]{3a}$ 2. $\sqrt[3]{2a}$ 3. $\sqrt{2a}$ 4. $\sqrt[5]{4a^2x}$

5. $\frac{1}{2}y\sqrt{2x}$ 6. $\sqrt[6]{abc}$ 7. $\sqrt[4]{8}$ 8. $-\sqrt{2}$

9. $\sqrt[6]{12ab}$ 10. $\frac{1}{2}\sqrt[4]{8a}$ 11. $2\sqrt[6]{8a}$ 12. $\sqrt[3]{a-b}$

13. $\sqrt[3]{a+b}$ 14. $\sqrt{(a+b)x}$ 15. $\sqrt[4]{60(a+x)}$

16. $\sqrt[4]{a^2-b^2}$ 17. $\sqrt{a+b}$ 18. $3\sqrt[9]{2ax^2}$

19. $\frac{1}{a}\sqrt[4]{a}$ 20. $\frac{1}{a+b}\sqrt[4]{(a+b)^3}$ 21. $\frac{1}{y}\sqrt[4]{xy^3}$

Exercise 121.

1. $\frac{1}{3}\sqrt{3}$ 2. $\frac{2}{5}\sqrt{5}$ 3. $\frac{1}{5}\sqrt{15}$

4. $\frac{a}{b}\sqrt{b}$ 5. $\frac{1}{b}\sqrt{ab}$ 6. $-1-\sqrt{2}$

7. $2-\sqrt{2}$ 8. $-3(\sqrt{2}+\sqrt{3})$ 9. $\frac{a^2+a\sqrt{b}}{a^2-b}$

10. $\frac{c}{x-y}(\sqrt{x}-\sqrt{y})$ 11. $3+2\sqrt{2}$

12. $\frac{1}{7}(11-6\sqrt{2})$ 13. $2\sqrt{6}-5$ 14. $\frac{a^2-2a\sqrt{b}+b}{a^2-b}$

15. $\frac{x+2\sqrt{xy}+y}{x-y}$ 16. .7071 17. .1716

Exercise 122.

1. $\pm 3\sqrt{-1}$, $\pm 2a\sqrt{-1}$, $\pm 4\sqrt{-1}$

2. $\pm 5x\sqrt{-1}$, $\pm 6ax^2\sqrt{-1}$, $\pm 7a^2y^3\sqrt{-1}$

3. $\pm 2\sqrt{2}\times\sqrt{-1}$, $\pm 2\sqrt{3a}\times\sqrt{-1}$, $\pm 3ax\sqrt{2x}\times\sqrt{-1}$

4. $9\sqrt{-1}$ 5. $3a\sqrt{-1}$ 6. -1, $-\sqrt{-1}$, $+1$, $+\sqrt{-1}$

7. $-1,\ +\sqrt{-1},\ +1,\ -\sqrt{-1}$

9. $-6,\ -10,\ -\sqrt{6}$

10. $-18,\ -160,\ -a^2b^2$

11. 7, 1

12. $8\sqrt{-1},\ 8+6\sqrt{-1},\ -7+2\sqrt{10}$

14. $\sqrt{3},\ \pm 2,\ \pm\frac{3}{2}$

15. $\pm 4,\ \pm 2x\sqrt{2x},\ \pm\frac{1}{2}\sqrt{2}$

16. $\frac{1}{3}(1+2\sqrt{-2}),\ \frac{a^2-b+2a\sqrt{-b}}{a^2+b},\ \frac{2\sqrt{ab}-a-b}{b-a}$

Exercise 123.

1. $\pm(\sqrt{3}+\sqrt{2})$
2. $\pm(2-\sqrt{5})$
3. $\pm(2+\sqrt{3})$
4. $\pm(\sqrt{3}-\sqrt{5})$
5. $\pm(3+\sqrt{6})$
6. Not a square.
7. $\pm(\sqrt{2a}-\sqrt{a})$
8. $\pm(\sqrt{5x}+\sqrt{x})$
9. $\pm(\sqrt{6}+\sqrt{5})$
10. $\pm(\sqrt{7}-\sqrt{6})$
11. ———
12. $\pm(\sqrt{x+y}+\sqrt{x-y})$
13. $\pm(2\sqrt{2}-\sqrt{7})$
14. $\pm(\sqrt{x+2y}+\sqrt{x-2y})$
15. $\pm(2\sqrt{3}-\sqrt{10})$
16. $\pm(\sqrt{x}-\sqrt{y})$
17. $\pm\left(\frac{1}{3}\sqrt{3}-\frac{1}{2}\sqrt{2}\right)$
18. $\pm\left(\frac{1}{2}+\frac{1}{3}\sqrt{3}\right)$
19. $\pm(\sqrt{x+1}+\sqrt{x})$
20. $\pm(\sqrt{6}+\sqrt{3})$
21. $\pm(2\sqrt{3}-\sqrt{11})$
22. $\pm(\sqrt{13}+\sqrt{3})$
23. $\pm(1-\sqrt{14})$
24. $\pm(1+\sqrt{a})$
25. $\pm\left(\frac{1}{3}\sqrt{15}+\frac{1}{3}\sqrt{6}\right)$
26. $\pm\frac{1}{5}(\sqrt{15}-\sqrt{10})$
27. $\pm(2\sqrt{3}+\sqrt{13})$
28. $\pm(1+2\sqrt{-1})$
29. $\pm(\sqrt{x}-\sqrt{-y})$
30. ———
31. $\pm(2\sqrt{5}-\sqrt{21})$
32. $\pm(1+\sqrt{-1})$
33. $\pm\frac{1}{2}(\sqrt{14}+\sqrt{10})$
34. $\pm(\sqrt{5+x}-\sqrt{5-x})$
35. $\pm\{2\sqrt{xy}-(x-y)\}$
36. $\pm(\sqrt{x+2}+\sqrt{x-1})$

Exercise 124.

1. $a^{\frac{2}{3}}b^{\frac{1}{3}}$
2. $a^{\frac{5}{2}}b^{\frac{5}{4}}c^{\frac{15}{4}}$
3. $a^{\frac{m}{n}}b^{\frac{s}{n}}$
4. $(a+b)^{\frac{2}{5}}$
5. $2(x-y)^{\frac{2}{3}}$
6. $a^{\frac{n}{n-1}}b^{\frac{n}{n-1}}$
7. $(a^2-x^2)^{\frac{n}{m}}$
8. $(a^n-b^n)^{\frac{m}{n+1}}$
9. $\sqrt[8]{x^5}$
10. $\sqrt{a^5c^5}$
11. $\sqrt[3]{a^2b}$
12. $\sqrt[6]{x^4y^5}$
13. $\sqrt[4]{(a+x)^3}$
14. $\sqrt[3]{(a^2-x^2)^2}$
15. $\sqrt[4]{x(x+y)^3}$
16. $\sqrt[6]{a^4(a+b)^5}$
17. $\sqrt{27}$
18. $\sqrt[3]{a^6b^2}$

19. $\sqrt{(a+b)^3}$ **20.** $\sqrt[3]{a^{11}}$ **21.** $\sqrt[6]{x^4 y}$

22. $\sqrt{x^3 y}$ **23.** $\sqrt{9\frac{3}{5}}$ **24.** $\sqrt{(a+x)^2 (a-x)}$

25. $(27)^{\frac{1}{2}}$ **26.** $(128)^{\frac{1}{8}}$ **27.** $(32)^{\frac{2}{3}}$

28. $(a^{\frac{5}{4}})^4$ **29.** $(x^{\frac{4}{3}})^{-3}$ **30.** $(y^{-\frac{7}{2}})^{-\frac{2}{3}}$

31. $(4a-4b)^{\frac{3}{2}}$ **32.** $(b)^{\frac{1}{8}}$ **33.** $(1)^{-1}$

34. $(ax^4+bx)^{\frac{1}{8}}$ **35.** $(ax+x^2)^{-\frac{1}{2}}$ **36.** $(x^{\frac{p^2-pq}{q^2}})^{\frac{q}{p}}$

37. $2\sqrt{27}$ **38.** $2\sqrt{31\frac{1}{4}}$ **39.** $2\sqrt{5}$

40. $2\sqrt[3]{85\frac{3}{4}}$ **41.** $2\sqrt{-13\frac{1}{2}}$ **42.** $2\left(\frac{9}{4}a-\frac{9}{4}b\right)^{\frac{1}{2}}$

43. $2\sqrt[3]{192}$ **44.** $2(2a-2b)^2$ **45.** $2(3x+3y)^3$

46. $4(ax)^{\frac{2}{3}}$ **47.** $8(4xy)^{\frac{3}{2}}$ **48.** $4(ay)^{\frac{2}{5}}$

49. $2\left(\frac{1}{8}\right)^{\frac{1}{2}}$ **50.** $a^3(a^{-4}x^3)^{\frac{1}{2}}$ **51.** $x^{\frac{2}{3}}(xy^3)^{\frac{2}{3}}$

52. $(a^{\frac{m^2-n^2}{n^2}})^{\frac{n}{m}}$ **53.** $2\left(\frac{1}{16}\right)^{\frac{1}{8}}$ **54.** $\pm(a+b)\sqrt{a}$

55. $\pm(a-x)\sqrt{a+x}$ **56.** $a\sqrt[n+1]{a^2}$ **57.** $\frac{1}{x^2}\sqrt[n]{a}$

58. $\pm\frac{1}{a+b}\sqrt{a^2+ab}$ **59.** $\pm\frac{x}{a-x}\sqrt{a-x}$

60. $\frac{1}{b}\sqrt[3]{ab}$ **61.** $\pm\frac{1}{a+b}\sqrt{a^2-b^2}$ **62.** $33\sqrt[3]{2}$

63. $(y-2z)(x^2-3a)^{\frac{1}{3}}$ **64.** $(a+b)^{-2}\sqrt[6]{(a+b)^5}$

65. $\pm\frac{1}{b}\sqrt[12]{a^3b^7x^5y^7}$ **66.** $\sqrt{2}+\sqrt{3}-\sqrt{6}$

67. $\sqrt[3]{5}-\sqrt[3]{2}+\sqrt[3]{7}$ **68.** $a^{\frac{5}{6}}+a^{\frac{2}{3}}b^{\frac{1}{6}}+a^{\frac{1}{2}}b^{\frac{1}{3}}+a^{\frac{1}{3}}b^{\frac{1}{2}}+a^{\frac{1}{6}}b^{\frac{2}{3}}+b^{\frac{5}{6}}$

69. $3\frac{11}{15}$ **70.** $\frac{2}{3}x\sqrt{15}$ **71.** $\frac{1}{2}\sqrt[3]{4}$ **72.** $\frac{1}{5}\sqrt[3]{100}$

73. $\frac{a^2-a\sqrt{b}}{a^2-b}$ **74.** $4\sqrt{3}+4\sqrt{2}$ **75.** $-\frac{1}{2}(1+\sqrt{5})$

76. $\frac{1}{2}(3\sqrt{5}+3\sqrt{3}+2\sqrt{10}+2\sqrt{6})$ **77.** $-\frac{1}{3}(1+2\sqrt{-2})$

78. $-4+\sqrt{15}$ **79.** $2\sqrt{3},\ \sqrt{15},\ 3\sqrt{2}$

80. $\sqrt[3]{10},\ \sqrt{5},\ 2\sqrt[6]{2}$ **81.** $2\sqrt[6]{3},\ \sqrt{6},\ \sqrt[3]{15}$

82. $\sqrt[3]{2},\ \sqrt[12]{20},\ \sqrt[4]{5}$ **83.** $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$

84. $(\sqrt{x}+2)(\sqrt{x}-2)$ **85.** $(x^{\frac{1}{4}}+y^{\frac{1}{4}})(x^{\frac{1}{4}}-y^{\frac{1}{4}})$

86. $(x^{\frac{1}{6}}+y^{\frac{1}{6}})(x^{\frac{1}{6}}-y^{\frac{1}{6}})$ **87.** $(4+\sqrt{x})(4-\sqrt{x})$

88. $(\sqrt{x}+5)(\sqrt{x}-5)$ **89.** $(x+\sqrt{5})(x-\sqrt{5})$

90. $(\sqrt{y}+\sqrt{2})(\sqrt{y}-\sqrt{2})$ **91.** $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$

92. $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ **93.** $x^{\frac{4}{5}}+x^{\frac{3}{5}}y^{\frac{1}{5}}+x^{\frac{2}{5}}y^{\frac{2}{5}}+x^{\frac{1}{5}}y^{\frac{3}{5}}+y^{\frac{4}{5}}$

94. $x^{\frac{4}{5}}-x^{\frac{3}{5}}y^{\frac{1}{5}}+x^{\frac{2}{5}}y^{\frac{2}{5}}-x^{\frac{1}{5}}y^{\frac{3}{5}}+y^{\frac{4}{5}}$ **95.** $4x^{\frac{2}{3}}-6x^{\frac{1}{3}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}$

96. $8x^{\frac{3}{4}}+12x^{\frac{1}{2}}y^{\frac{1}{4}}+18x^{\frac{1}{4}}y^{\frac{1}{2}}+27y^{\frac{3}{4}}$

97. $8x^{\frac{3}{4}}-12x^{\frac{1}{2}}y^{\frac{1}{4}}+18x^{\frac{1}{4}}y^{\frac{1}{2}}-27y^{\frac{3}{4}}$ **98.** $x^{\frac{2}{9}}+x^{\frac{1}{9}}y^{\frac{1}{9}}+y^{\frac{2}{9}}$

99. $\sqrt{x}-\sqrt{y}$ **100.** $a+\sqrt{b}$ **101.** $x(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})$

102. $\sqrt{x}-\sqrt[4]{y}$ **103.** $(\sqrt{x+y})(\sqrt{x}+\sqrt{y})$ **104.** $\frac{1}{x}\sqrt{x-y}$

105. $x^{\frac{4}{3}}+(xy)^{\frac{2}{3}}+y^{\frac{4}{3}}$ **106.** $2a^2-x^2+2a\sqrt{a^2-x^2}$

107. $a^3-3ab^2-(3a^2b-b^3)\sqrt{-1}$

108. $a^2+(4a+4b)\sqrt{ab}+6ab+b^2$

109. $a^3-3ab^2-(b^3-3a^2b)\sqrt{-1}$ **110.** $(100+18\sqrt{-2})$

111. $217-88\sqrt{6}$ **112.** $9a\sqrt[6]{5b^4}$ **113.** $x\sqrt[8]{625y}$

114. $\frac{1}{3}\sqrt[6]{243a}$ **115.** $\pm(2+3\sqrt{5})$

116. $\pm(\sqrt{x+1}-\sqrt{x-1})$ **117.** $\pm(\sqrt{-1}+\sqrt{-2})$

118. $\frac{2(x^2-y)}{x^2+y}$ **119.** $\frac{2x}{y}$ **120.** $\frac{x}{y}+2+\frac{y}{x}$

121. $2x-\frac{2}{x}\sqrt{x^4-1}$ **122.** $\pm(x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})$

123. $x^{\frac{1}{2}}+1+\frac{1}{x^{\frac{1}{2}}}$

Exercise 125.

1. $x=\pm 32$ **2.** $x=214,\ -218$ **3.** $x=\pm 9$ **4.** $x=\pm 1$

5. $x=9a^3,\ -7a^3$ **6.** $x=\pm a\sqrt{\pm a^5-1}$ **7.** $x=29$

8. $x=16$ **9.** $x=\frac{5}{6}$ **10.** $x=\frac{5}{3}a$

11. $x=4$ **12.** $x=1\frac{3}{4}$ **13.** $x=2$ **14.** $x=7$

15. $x=15\frac{19}{25}$ **16.** $x=-5$ **17.** $x=\frac{1}{25}$

18. $x=\left(\frac{a-b}{a}\right)^2$ **19.** $x=36$ **20.** $x=a^2\left(\frac{b-1}{b+1}\right)^2$

21. $x=\frac{5}{9}$ **22.** $x=4$ **23.** $x=1$ **24.** $x=a$

25. $x=8,\ -\frac{8}{9}$ **26.** $x=2,\ -3$ **27.** $x=2,\ -1$

28. $x=\frac{1}{2a}(1\pm\sqrt{4a^3+1})$ **29.** $x=4a,\ a$ **30.** $x=a,\ 1-a$

31. $x=\frac{1}{2}(1-a\pm\sqrt{1-6a+a^2})$ **32.** $x=\frac{1}{4}(3\sqrt{a}+1)^2$

33. $x=a$ **34.** $x=1,\ 0$ **35.** $x=4,\ 9$

36. $x=\left\{\frac{1}{2a}(-b\pm\sqrt{4ac+b^2})\right\}^{\frac{2}{m}}$ **37.** $x=21,\ 12$

38. $x=5\pm2\sqrt{13}$ **39.** $x=1,\ \frac{4}{27}\sqrt[3]{12}$ **40.** $x=\sqrt[m]{1},\ \sqrt[m]{16}$

41. $x=3,\ -4\frac{1}{2},\ \frac{1}{4}(-3\pm\sqrt{33})$ **42.** $x=a,\ a(1\pm\sqrt{-3})$

43. $x=\pm3,\ \pm\sqrt{7},\ \pm\sqrt{-5},\ \pm\sqrt{-3}$ **44.** $x=0,\ 6,\ 3(1\pm\sqrt{2})$

45. $x=25,\ -9,\ 8\pm4\sqrt{29}$ **46.** $x=2\pm\sqrt{3},\ \frac{1}{2}(1\pm\sqrt{-3})$

47. $x=4,\ y=9$ **48.** $x=0,\ \frac{4}{9};\ y=0,\ \frac{4}{81}$

49. $x=17,\ y=8$ **50.** $x=1,\ y=4$ **51.** $x=8,\ 16;\ y=1,\ 9$

52. $\begin{cases} x=\pm5,\ \pm3\sqrt{-1}, \text{ etc.} \\ y=\pm3,\ \mp5\sqrt{-1}, \text{ etc.}\end{cases}$ **53.** $x=1+\sqrt{3},\ -(3+\sqrt{3})$

Exercise 126.

18. $\frac{a+b}{a-b}$ **19.** $x=5$

Exercise 127.

1. $\frac{1}{5},\ 1\frac{1}{3},\ \frac{1}{15},\ 9\frac{3}{5},\ \frac{8}{9},\ \frac{x}{y}$ **2.** $\frac{b}{a},\ a-x,\ \frac{a+b}{a-b},\ a+x$

3. $1,\ x^4+x^2y^2+y^4,\ \frac{.1}{ax+ay}$ **4.** $1:3;\ ab:1;\ a:b$

5. $11:30;\ 3:5;\ x^2:z;\ a:c$ **6.** $\frac{1}{2000}$

7. $3\frac{1}{8}$ **8.** The first. **9.** $\frac{bc-ad}{c-d}$ **10.** a^2-b^2

11. 1 : 2 **12.** 1.414, 1.732 **13.** .707

14. Increased, diminished. **15.** Diminished, increased.

16. \$600, \$720 **17.** 20 yr., 30 yr. **18.** $6\frac{7}{8}$ ft., $8\frac{1}{4}$ ft.

Exercise 128.

14. $2x:5::1:2$ **15.** $2x:3y::21a:20b$

Exercise 129.

1. $x=8$ **2.** $x=10$ **3.** $x=25$ **4.** $x=\frac{a^2}{b}$

5. $x=\pm\frac{a^2}{b}$ **6.** $x=3a$ **7.** $x=1$ **8.** $x=4$

9. $x=5$ **10.** $x=\frac{a(b^2+1)}{2b}$

11. $\begin{cases} x=\pm 12 \\ y=\pm 3 \end{cases}$ **12.** $\begin{cases} x=4 \\ y=2 \end{cases}$ **13.** $\begin{cases} x=\pm 6 \\ y=\pm 4 \end{cases}$

Exercise 130.

1. 28 ft., 21 ft. **2.** 72 yr., 60 yr. **3.** 7, 6

4. \$50,000, \$30,000 **5.** 2000 sq. rd.

6. $\$1022\frac{8}{11}$, $\$1090\frac{10}{11}$ **7.** 50 mi., 30 mi. **8.** 8 cu. ft.

9. πd, or $3.1416\times d$ **10.** πr^2, 16π, 16×3.1416

11. $4\pi r^2$, 100π **12.** $\frac{4}{3}\pi r^3$, 288π **13.** $m\times\frac{c^2}{a^2}$ A.

14. $n\times\frac{d^3}{c^3}$ gal. **15.** $c\times\frac{n^2}{m^2}$

Exercise 131.

1. $2a$ **2.** 0 **3.** a **4.** ∞ **5.** 5 **6.** ∞

7. $2\sqrt{a}$ **8.** 0 **9.** 1 **10.** 0

11. na^{n-1} **12.** m **13.** 0 **14.** a

15. 0 **16.** $a+1$ **17.** 1 **18.** a^2

Exercise 132.

1. $l=40$, $S=220$ **2.** $l=5$, $S=192$ **3.** $l=3\frac{1}{6}$, $S=16\frac{1}{2}$

4. $l=n$, $S=\frac{1}{2}(n^2+n)$ **5.** $l=2r$, $S=r^2+r$

6. $d=5$ **7.** $l=7$, $d=-3$ **8.** $l=43$, $S=204$

9. $l=12\frac{1}{2}$, $n=8$ **20.** $a=1$, $-\frac{1}{2}$; $n=12$, 15

21. $l=37$, $n=10$ **22.** $n=11$, $a=5$

Exercise 133.

1. 7, 10, 13 **2.** \$175 **3.** \$320 **4.** 7, 12, 17

5. 5050 **6.** 5, 8, 11, 14 **7.** 1, 3, 5, 7, etc.

8. 14475 ft. **9.** 8 da. **10.** \$35.70 **11.** 1, 2, 3, 4, 5

12. 2, 5, 8, 11, etc. **13.** 3 da., 10 da. **14.** 2, 14 **15.** \$1500

Exercise 134.

1. 4374 **2.** 2916 **3.** $\frac{2}{27}$ **4.** $-\frac{1}{512}$

5. 2^{n-1} **6.** 4095 **7.** $1\frac{364}{729}$ **8.** $\frac{2^n-1}{2^{n-1}}$

9. 96, 189 **10.** 243, 5 **11.** $\frac{1}{128}=l$, $\frac{1}{4}=a$

12. $l=2r^{10}$; $S=2(r^{10}+r^9+r^8+\dots r)$ **19.** 2, 6, 18, 54, etc.

Exercise 135.

1. \$429.98 **2.** 4 yr. **3.** \$3200, \$1600, \$800, \$400, \$200

4. $\frac{1}{4}$, $\frac{1}{8}$ **5.** $1\frac{7}{8}$, $1\frac{13}{32}$, $1\frac{7}{128}$ **6.** 4 **7.** ± 4

8. $5\sqrt[4]{3}$, $5\sqrt[4]{27}$, $15\sqrt[4]{3}$ **9.** 1, 3, 9, 27

10. 4, 16, 64 **11.** 6, 18, 54, etc. **12.** 2, 8, 32

13. 248, 842 **14.** 3, 9, 27, etc. **15.** $\left(\frac{9}{10}\right)^5$

16. $\frac{1}{x^{n-3}}y(x^{n-1}+x^{n-2}y+\dots y^{n-1})$ **17.** 9, 27, 81; 117

Exercise 136.

1. 3, $13\frac{1}{2}$, $11\frac{1}{9}$, 25 **2.** $\frac{ab}{b-1}$, $\frac{a^2x}{a-1}$, $\frac{a^2}{a-b}$, $\frac{x}{y-x}$

3. $\frac{5}{11}$, $\frac{124}{999}$, $\frac{192}{1111}$, $\frac{11}{30}$, $\frac{2}{165}$, $\frac{11}{900}$ **4.** 40 ft.

5. 300 rd. **6.** $21\frac{9}{11}$ min.

Exercise 137.

1. 94 **2.** -4 **3.** $\frac{22}{27}$ **4.** $\frac{7}{6}a+\frac{11}{12}b+\frac{21}{12}c$

5. $11b+9c+d-a$ **6.** $\frac{1}{4}x-\frac{31}{20}y+\frac{22}{15}z$ **7.** $8a$

8. $(a-b+c)m^2+(a+b-c)mn+(b+c-a)n^2$ **9.** -1

10. $3+2x^{-\frac{1}{3}}y^{\frac{1}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{-\frac{1}{3}}+x^{\frac{2}{3}}y^{-\frac{2}{3}}$ **11.** $x^4-\frac{1}{256}$

12. $x^2+(a+c)x+ac$ **13.** $a^8-a^6b^2+a^4b^4-a^2b^6+b^8$

14. $(x^4+y^4)(x^8-x^4y^4+y^8)$

15. $(x^4-x^2y^2+y^4)(x^2+xy+y^2)(x^2-xy+y^2)$ **16.** $x+y$

17. $(3x+2)^2(x+3)$ **18.** $\frac{x+y}{x-y}$ **19.** $\frac{x+y+z}{x-y+z}$

20. $\frac{8pq}{p^2-q^2}$ **21.** 1 **22.** $\frac{1}{a^2-a+1}$ **23.** $x=1$

24. $x=4,\ y=5,\ z=6$ **25.** $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{4}$

26. $x+y+z+2(\sqrt{xy}-\sqrt{xz}-\sqrt{yz})$ **27.** $7-5\sqrt{2}$

28. $a^3+6a^2x+6ax^2+x^3+(3a^2+7ax+3x^2)\sqrt{ax}$

29. $x^{p(q-p)}$ **30.** $x^{12}+4x^6+6+\frac{4}{x^6}+\frac{1}{x^{12}}$ **31.** 20

32. $x^4+2x^2y+y^2$ **33.** $x=-b,\ -(a+c)$ **34.** $x=\pm 1\frac{2}{5}$

35. $x=1\frac{1}{2},\ -3,\ -\frac{3}{4}(1\mp\sqrt{33})$ **36.** $x=\pm 3,\ y=\pm 2$

37. $x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ **38.** $x^{\frac{2}{3}}-x^{\frac{1}{3}}+1$

39. $x^3-x^{\frac{5}{2}}y^{\frac{1}{2}}+x^{\frac{3}{2}}y^{\frac{3}{2}}-x^{\frac{1}{2}}y^{\frac{5}{2}}+y^3$ **40.** $\frac{1}{xyz}$ **41.** $\frac{a^4+14a^2b^2+b^4}{(a^2-b^2)^2}$

42. $amx^3+(an-bm)x^2-(ap+bn)x+bp$

43. $\left(x^2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$; $(a+2b)(a-2b)(a+3b)$

44. $y^2-2,\ y^3-3y,\ y^4-4y^2+2$

45. $(2x+z)(16x^4-8x^3z+4x^2z^2-2xz^3+z^4)$; $(x+\sqrt{xy}+y)$

46. $8(\sqrt{-3}-1)$ $(x-\sqrt{xy}+y)$

47. $(16a^3x^2-24a^3y^2)^{\frac{2}{3}},\ (x^{\frac{21}{5}}-x^{\frac{6}{5}}y^3)^{\frac{5}{6}},\ (ax^3+bx^4)^{-\frac{2}{3}}$

48. 1 **49.** $\frac{1}{256}$ **50.** $x=\pm\sqrt{-1},\ -\frac{1}{2}(1\mp\sqrt{5})$

51. $x=\pm\frac{1}{2}\sqrt{2},\ y=\pm\frac{1}{2}\sqrt{2}$ **52.** $x=\pm\frac{1}{5}\sqrt{265}$

53. $1+x+x^2+x^3+$ etc., $x^9,\ x^{n-1}$

54. $\frac{x^2y^2}{y^2-x^2},\ \frac{1}{x^2}-\frac{1}{xy}+\frac{1}{y^2},\ \frac{1}{(x+y)^2}$

55. 30 **56.** $\frac{x+y-2\sqrt{xy}}{x-y},\ \frac{1}{3}\sqrt[3]{9},\ \sqrt{-1},\ -\sqrt[4]{-1}$

57. $x=144$ **58.** $2x+7$

59. $(m-n+r)y^2+(m+n-r)yz+(r+n-m)z^2$

60. $x=-(\sqrt{a}-1)^2$ **61.** $\sqrt[3]{4}$ **62.** 2 **63.** $x^2-2x=-2$

64. $x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2-(abc+abd+acd+bcd)x+abcd=0$

65. $x=\frac{1}{3a}(-b\pm\sqrt{12ac+b^2})$ **66.** $x=a,\ (1+\sqrt{a})^2$

67. $x=10,\ 6,\ 8\pm5\sqrt{5},\ \frac{1}{2}(25\pm\sqrt{385}),\ \frac{1}{2}(25\pm\sqrt{869}),$

$y=6,\ 10,\ 8\mp5\sqrt{5},\ \frac{1}{2}(25\mp\sqrt{385}),\ \frac{1}{2}(25\mp\sqrt{869})$

68. $y^4+2y^3+4y^2+3y+3$

69. $a^2y^4+(a-2ab)y^3+(b^2-b+1)y^2$ **70.** $x=\pm\frac{2a^2}{\sqrt{b^2-4a^2}}$

71. $x=(a-2\sqrt{a})^2$ **72.** $x=4,\ 1;\ y=1,\ 4$

73. $x=27,\ 8;\ y=8,\ 27$ **74.** $x=2,\ y=3$

75. $x=3,\ 5,\ -5\pm\sqrt{-26};\ y=5,\ 3,\ -5\mp\sqrt{-26}$

76. Odd, odd, even. **77.** $+1,\ -1,\ \pm1,\ \sqrt[n]{\pm4}$

78. $x=-1,\ x=1$

Exercise 138.

1. \$3600, \$3200 **2.** 15 yr., 45 yr. **3.** 48, 52

4. 37 **5.** $\frac{7}{9}$ **6.** 16 ft., 4 ft. **7.** 1 hr. 29 min. $22\frac{1}{2}$ sec.

8. 10 da. **9.** 2663, 1662, 1000 **10.** 56 min.

11. $1\frac{1}{2}$ mi. **12.** $27\frac{3}{11}$ min., $10\frac{10}{11}$ min., or $43\frac{7}{11}$ min., 60 min.

13. 24, 36 **14.** 10 mi. **15.** \$$2\frac{1}{2}$, 50 ct.

16. $p\sqrt{p^2-q^2},\ q\sqrt{p^2-q^2}$

18. 3 ct. **19.** 60, 50, 40 **20.** 8 da. **21.** 5 hr.

22. \$1200 **23.** 20,000 **24.** 3, 4, 5 **25.** 36

26. \$126, \$882 **27.** 2, 4, 6, 8, 10, 12, 14

28. 180 lb., 160 lb. **29.** 8, 6 **30.** \$480

31. \$7000, \$8000 **32.** \$225, 20%

33. $10\frac{910}{989}$ da., $14\frac{454}{739}$ da., $17\frac{413}{611}$ da. **34.** 10 min.

35. 30 mi. **36.** 3, 9, 27, 81 **37.** 4, 6, 8, 10, 12

38. $\frac{1}{2}(1+\sqrt{5}),\ \frac{1}{2}(3+\sqrt{5})$ **39.** $\frac{apt}{pm-tn}$ seconds. **40.** 9753

41. 4, 6 **42.** 13 yr. **43.** 64 cu. ft., 512 cu. ft.

44. 2, 3, 4 **45.** 4840 **46.** 1 rd. **47.** 10 ft., 12 ft.

48. \$15,000 **49.** 12 mi. **50.** 144 oz.

51. $2\frac{118}{211}$ mi., $2\frac{5}{11}$ mi. **52.** 15% **53.** $106\frac{2}{3}$ mi.

54. 19 da. **55.** 6 hr., 3 hr., 2 hr.

APPENDIX.

Exercise 1.

1. x^2+x+1 **2.** x^2-x-1 **3.** $2x-1$
4. $5x^2-3$ **5.** $2x^2-3x+6$ **6.** $2a^2+a+1$
7. $x+3$ **8.** $2x+3$ **9.** x^2+2x+1
10. $x-5$ **11.** $y+1$ **12.** $a^8-a^4b^4+b^8$
13. $a+b+c$ **14.** x^2+xy+y^2 **15.** a^2+1
16. $x-2y$

Exercise 2.

1. $x+y$ **2.** x^2+xy+y^2 **3.** $2x+1$
4. $3x-2$ **5.** $2x+3$ **6.** $2x+6$
7. x^2+x+1 **8.** $3x-2$

Exercise 3.

1. $2(3x^2+7x+4)(4x^2+3x-10)$ **2.** $(x^2+x+1)(x+5)(2x+1)$
3. $(a+1)(a^2-a-1)(a^2+a+1)$
4. $(a^2-2a+3)(a^2+2a-3)(7a+3)$
5. $(2x+y)(x^2+xy+y^2)(x^2-xy+y^2)$
6. $(x^2+xy+y^2)(x+y)(x+2y)$ **7.** $(3x^2+2x+1)(x+1)(x^2+1)$
8. $(x^2+ax+a^2)(2x-a)(3x+a)$ **9.** $(3x^2-4x+2)(x-3)(2x+3)$
10. $(2a^4+5a^2+3)(2a^2-7)(2a^2+7)$
11. $(5z^2-1)(4z^2+1)(5z^2+z+1)$
12. $(3x+2)(x^3-x^2+x-1)(x^3+x^2+x+1)$
13. $(3x+4)(2x^3+3x^2-4x+2)(3x^3+2x^2-3x+1)$
14. $(2a^2-6a-7)(6a^3-11a^2-37a-20)$
15. $3(m+3)(2m^2-m+1)(3m^2-7m+4)$
16. $(n-2a)(n^3+an^2+a^2n+a^3)(3n^2-an+a^2)$

THE END.