Chamberlain

APPLETONS' MATHEMATICAL SERIES

NUMBERS UNIVERSALIZED

AN

ADVANCED ALGEBRA

BY

DAVID M. SENSENIG, M. S.

PROFESSOR OF MATHEMATICS, STATE NORMAL SCHOOL, WEST CHESTER, PA.

PART SECOND

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PREFACE.

NUMBERS UNIVERSALIZED is believed to embrace all algebraic subjects usually taught in the preparatory and scientific schools and colleges of this country. For convenience, it is divided into two parts, which are bound separately and together, to accommodate all kinds and grades of schools sufficiently advanced to adopt its use.

Part Second is treated in five chapters, as follows: One embracing serial functions, including development of functions into series, convergency and divergency of infinite series, the binomial formula, the binomial theorem, the exponential and logarithmic series, summation of series, reversion of series, recurring series, and decomposition of rational fractional functions; one treating of complex numbers, graphically and analytically, including fundamental operations with complex numbers, general principles of modulii and norms, and the development and representation of sine, cosine, and tangent; one embodying a discussion on the theory of functions, including graphical representations of the meaning of the terms independent and dependent variables, continuous and discontinuous functions, increasing and decreasing functions, and turning values and limits of functions, and also a treatment of differentials and derivatives, and maxima and minima values of functions; one treating of the theory of equations, including a discussion of the properties of the roots, real and imaginary, of an equation, methods of determining the commensurable roots of a numerical equation, Sturm's theorem for detecting the number and situation of real roots, Horner's method of root extension, Cardan's formula for solving cubic equations, and a short treatment of reciprocal and binomial equations; one treating of determinants and probabilities, so far as these subjects are of interest and value to the general student. The volume closes with a supplementary discussion of continued fractions and theory of numbers.

The aim of the author in preparing this part of his work has not been so much to give completeness to the various subjects treated as to lead the student to a comprehension of the fundamentals of a wider range of subjects, and to cultivate in him a taste for mathematical investigation. It is believed that the plan adopted will give the general student a broader and more practical knowledge of algebra, and will lead to better results in a preparatory course of study for the university than would a completer treatment of fewer subjects requiring an equal amount of space in their development and more time in their mastery. While a sufficient number of examples have been placed under each head to offer opportunity for the application of the principles and laws developed, there will not be found an unnecessary multiplicity of them to retard the progress of the pupil in his onward course.

In conclusion, the author desires to acknowledge his indebtedness to the English authors, Hall and Knight, Chrystal, Aldis, Whitworth, and C. S. Smith, whose works he frequently consulted, and from which he obtained many new and valuable ideas.

DAVID M. SENSENIG.

NORMAL SCHOOL, WEST CHESTER, PA.,
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CHAPTER IX.

SERIAL FUNCTIONS.

1. Definitions.

562. Any expression containing a variable is called a *function* of the variable.

Thus, $ax + b$, ax^{-3}, $\sqrt{a + x^2}$, a^x, $\log. (a + x)$, and $a + bx + cx^2 + dx^3 +$ etc., are functions of x.

563. Any series containing variable terms is called a *serial function.*

564. The expression $f(x)$ represents any function of x, and is read *function* x.

565. When two or more functions of the same variable are used in a discussion, modified forms are used for distinction; as,

1. $f(x)$, $F(x)$, $\phi(x)$; read, *f minor, f major, phi* functions of x.

2. $f'(x)$, $f''(x)$, $f'''(x)$; read, *f prime, f second, f third* functions of x.

3. $f_1(x)$, $f_2(x)$, $f_3(x)$; read, *f one, f two, f three* functions of x.

Development of Functions into Series.

Theorem of Indeterminate Coefficients.

566. *If* $A + Bx + Cx^2 + Dx^3$ *etc.* $= A_1 + B_1x + C_1x^2 + D_1x^3 +$ *etc., for any assigned value of* x *from* $-\infty$ *to* $+\infty$, *and* $A, A_1, B, B_1, C, C_1, D, D_1$, *etc., are independent of* x, *then will* $A = A_1$, $B = B_1$, $C = C_1$, $D = D_1$, *etc.*

Demonstration: Given

$A + Bx + Cx^2 + Dx^3 +$ etc. $= A_1 + B_1x + C_1x^2 + D_1x^3 +$ etc., (A)

for any assigned value of x. Let $x = 0$; then $A = A_1$.

Therefore, $A = A_1$ for every value of x. (1)

Subtract (1) from (A),

$Bx + Cx^2 + Dx^3 +$ etc. $= B_1x + C_1x^2 + D_1x^3 +$ etc. (B)

Divide (B) by x,

$B + Cx + Dx^2 +$ etc. $= B_1 + C_1x + D_1x^2 +$ etc. (C)

Let $x = 0$, $B = B_1$.

Therefore, $B = B_1$ for every value of x. (2)

etc., etc., etc.

567. ***Corollary 1.***—*If* $A + Bx + Cx^2 + Dx^3 +$ *etc.* $= 0$, *for any assigned value of* x, *then will* $A = 0$, $B = 0$, $C = 0$, $D = 0$, *etc.*

568. ***Cor. 2.***—*A function of a single variable can be developed into a series of the ascending powers of the variable in only one way.*

For, if possible, let

$f(x) = a + bx + cx^2 +$ etc.; and

$f(x) = a_1 + b_1x + c_1x^2 +$ etc.; then will

$a + bx + cx^2 +$ etc. $= a_1 + b_1x + c_1x^2 +$ etc.;

whence $a = a_1$, $b = b_1$, $c = c_1$, etc., and the two developments will be identical.

2. Applications.

1. Expansion of Rational Fractions.

569. A rational fraction of a single variable may generally be developed into a series by dividing the numerator by the denominator, but a more expeditious method consists in the application of the principle of indeterminate coefficients.

Illustrations.—1. Develop $\frac{1-x}{1+x}$ into a series of the ascending powers of x.

Let $\frac{1-x}{1+x} = A + Bx + Cx^2 + Dx^3 + \text{etc.}$ (A)

Clear of fractions, and arrange the coefficients of the like powers of x into columns,

$$1 - x = \begin{array}{c|c|c|l} A + B & x + C & x^2 + D & x^3 + \text{etc.} \\ + A & + B & + C & \end{array} \quad \text{(B)}$$

Equate the coefficients of the like powers of x [566],

$$A = 1;\ A + B = -1,\ B + C = 0;\ C + D = 0, \text{ etc.}$$

$$\therefore \quad A = 1,\ B = -2,\ C = 2,\ D = -2, \text{ etc.}$$

Substitute these values of the coefficients in (A),

$$\frac{1-x}{1+x} = 1 - 2x + 2x^2 - 2x^3 + \text{etc.}$$

Let the student divide the numerator by the denominator, and show that the same result will follow.

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570. The first term of the expansion may be obtained by dividing the first term of the numerator by the first term of the denominator, and the remaining terms by indeterminate coefficients.

2. Develop $\frac{a}{x + bx^2}$ in the ascending powers of x.

Put $\frac{a}{x + bx^2} = ax^{-1} + Bx^0 + Cx + Dx^2 + \text{etc.}$ (A)

Clear of fractions and column coefficients,

$$a = \begin{array}{c|c|c|l} a + B & x + C & x^2 + D & x^3 + \text{etc.} \\ + ab & + Bb & + Cb & \end{array} \quad \text{(B)}$$

Equate coefficients,

(1) $B + ab = 0.$ (2) $C + Bb = 0.$ (3) $D + Cb = 0$, etc.

$$\therefore \quad B = -ab,\ C = ab^2,\ D = -ab^3, \text{ etc.}$$

Substitute these values in (A),

$$\frac{a}{x + bx^2} = ax^{-1} - ab + ab^2x - ab^3x^2 + \text{etc.}$$

EXERCISE 87.

Develop to four terms :

1. $\dfrac{1}{1-x}$	4. $\dfrac{1+x}{1+x^2}$	7. $\dfrac{a}{a+x}$
2. $\dfrac{1}{3-2x}$	5. $\dfrac{1-x^2}{1+x+x^2}$	8. $\dfrac{5x+2x^2}{1-5x+x^2}$
3. $\dfrac{1}{1-x+x^2}$	6. $\dfrac{2x-3}{x+x^2+1}$	9. $\dfrac{1+x+x^2}{1+x^3}$

2. *Expansion of Irrational Functions.*

Illustrations.—1. Expand to four terms $\sqrt{1-x+x^2}$.

Put $\sqrt{1-x+x^2} = 1 + Bx + Cx^2 + Dx^3 + \text{etc.}$ (A)

Square both members and column the coefficients,

$$1-x+x^2 = 1 + 2B \left|\begin{array}{l} x + B^2 \\ \quad + 2C \end{array}\right| \begin{array}{l} x^2 + \ 2D \\ \quad + 2BC \end{array} \left| \ x^3 + \text{etc.} \right. \qquad \text{(B)}$$

Equate the coefficients,

(1) $2B = -1$. (2) $B^2 + 2C = 1$. (3) $2D + 2BC = 0$.

$$\therefore \quad B = -\frac{1}{2},\ C = \frac{3}{8},\ D = \frac{3}{16},\ \text{etc.}$$

Substitute these values in (A),

$$\sqrt{1-x+x^2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \frac{3}{16}x^3 + \text{etc.}$$

2. Expand to three terms $\sqrt[3]{8-x^2}$.

Put $\sqrt[3]{8-x^2} = 2 + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$ (A)

Cube both members and column coefficients,

$$8 - x^2 = 8 + 12B \left|\begin{array}{l} x + 12C \\ \ \ + \ 6B^2 \end{array}\right| \begin{array}{l} x^2 + B^3 \\ \quad + 12D \\ \quad + 12BC \end{array} \left|\begin{array}{l} x^3 + 12E \\ \quad + \ 3B^2C \\ \quad + \ 6C^2 \\ \quad + 12BD \end{array}\right| x^4 + \text{etc.}$$

Equate the coefficients,

(1) $12B = 0$. (2) $12C + 6B^2 = -1$.

(3) $12BC + 12D + B^3 = 0$.

(4) $12E + 3B^2C + 6C^2 + 12BD = 0$.

$$\therefore \quad B = 0,\ C = -\frac{1}{12},\ D = 0,\ E = -\frac{1}{288},\ \text{etc.}$$

Substitute these values in (A),

$$\sqrt[3]{8-x^3} = 2 - \frac{1}{12}x^2 - \frac{1}{288}x^4 - \text{etc.}$$

EXERCISE 88.

Expand to four terms:

1. $\sqrt{4-x}$
2. $\sqrt{1+x-x^2}$
3. $\sqrt{9+x-3x^2}$
4. $\sqrt[3]{1+x}$
5. $\sqrt[3]{27+x^2}$
6. $\sqrt[3]{8+x+x^2}$
7. $\sqrt{a+x}$
8. $\sqrt[3]{a-x}$
9. $\sqrt[3]{a^3+x^3}$

Convergency and Divergency of Infinite Series.

General Definitions.

571. The *limit of a series* is the limit of the sum of n terms of the series, when n is indefinitely increased; that is, when lim. $n=\infty$.

572. A series is *convergent* when its limit is a finite constant, including *zero*.

573. A series is *divergent* when its limit is infinity.

574. A series is *indeterminate* when the sum of n terms is finite but does not approach any definite value as n is indefinitely increased.

Thus, $1-1+1-1+1-1+\ldots.$ is indeterminate, since, when n is even the sum is 0, and when n is odd the sum is 1, however great n be taken.

575. For convenience of discussion, the following notation will be adopted:

1. The terms of a series will be represented in order by $u_1, u_2, u_3 \ldots. u_n, u_{n+1} \ldots.$

2. The sum of n terms will be represented by U_n, so that $U_n = u_1 + u_2 + u_3 + \ldots. + u_n$.

3. The limit of the series will be represented by U, so that $U = u_1 + u_2 + u_3 + \ldots. + u_n + u_{n+1} + \ldots.$

Fundamental Principles.

576. *1. No series whose terms are all of the same sign can be indeterminate.*

For either the sum of n terms increases numerically without limit as n is increased indefinitely, or else it can never exceed some fixed value which it approaches as a limit. Such a series is, therefore, either convergent or divergent.

577. *2. A series of finite terms whose signs are all alike is divergent.*

For, if we let a represent the numerical value of the smallest term, then, numerically, $U > na$, whose limit is ∞, when lim. $n = \infty$ and a is a finite quantity.

Thus, the series $1 + 2 + 4 + 8 + 16 + \ldots.$ is divergent.

578. *3. If a series is convergent it will remain convergent, and if divergent it will remain divergent, if any finite number of terms be added to or subtracted from the series.*

For, the sum of any finite number of terms is finite, and, therefore, can not change the nature of the limit of the series when combined with the series by addition or subtraction.

579. *4. If a series is convergent when its terms are all positive, it is also convergent when its terms are all negative, or some positive and some negative.*

For its limit will have the same numerical value when its terms are all negative as when they are all positive, and will be numerically less when the terms do not all have the same sign as when they do.

It must not be inferred from this principle that a series is necessarily divergent when its terms are not all of the same sign, if it is divergent when they are alike in sign. Such may or may not be the case.

Theorems.

580. I. *In order that a series may be convergent, the limit of the* $(n+1)$*th term, and the limit of the sum of any number of terms beginning with the* $(n+1)$*th term must be zero, and conversely.*

Demonstration: If a series is convergent, then ultimately, if n is indefinitely increased,

(1) $U - U_n = \circ$ [498] (2) $U - U_{n+1} = \circ$

(3) $U - U_{n+2} = \circ$ (4) $U - U_{n+3} = \circ$

. . . . . .

Subtract (1) from (2); (1) from (3); (1) from (4), etc.; then,

(*a*) $U_n - U_{n+1} = \circ$; or, $u_{n+1} = \circ$; whence, lim. $u_{n+1} = 0$

(*b*) $U_n - U_{n+2} = \circ$; or, $u_{n+1} + u_{n+2} = \circ$; whence, lim. $(u_{n+1} + u_{n+2}) = 0$

(*c*) $U_n - U_{n+3} = \circ$; or, $u_{n+1} + u_{n+2} + u_{n+3} = \circ$; whence, lim. $(u_{n+1} + u_{n+2} + u_{n+3}) = 0$

etc., etc., etc., etc.

581. II. *If each term of a series whose terms are alternately positive and negative is numerically greater than the following term, the series is convergent.*

Demonstration: Let $U = u_1 - u_2 + u_3 - u_4 + \ldots \pm u_n \mp u_{n+1} \ldots$, in which $u_1 > u_2 > u_3 > u_4 \ldots$, be the given series.

(1) $U = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) +$ etc.

(2) $U = u_1 - (u_2 - u_3) - (u_4 - u_5) -$ etc.

From (1) it is evident that U is positive.

From (2) it is evident that, since U is positive, $U < u_1$.

$\therefore$ U approaches u_1 or some quantity less than u_1 as a limit, and the series is, therefore, convergent.

582. III. *A series is convergent if after some particular term the ratio of each term to the preceding term is less than unity.*

Demonstration: The most unfavorable case to convergency supposable, under the conditions given, is evidently the one in which all the terms have the same sign (say plus) and all the ratios described are equal and each equal to the greatest of them. This is, therefore, the only case that needs proof.

Let r be the greatest ratio after the nth term, but < 1; then,

$$u_n + u_{n+1} + u_{n+2} + u_{n+3} + \text{etc.} = u_n + u_n r + u_n r^2 + \text{etc.} = \frac{u_n}{1-r}$$

[499, P.] = a finite quantity. Therefore, the whole series is convergent [578].

583. IV. *A series of all positive or all negative terms is divergent, if after some particular term the ratio of each term to the preceding term is equal to or greater than unity.*

Demonstration: The most unfavorable case to divergency, and the only one that needs investigation, is the one in which all the ratios described are equal and each equal to the least of them.

Let r be the least ratio after the nth term, but = or > 1; then, $u_n + u_{n+1} + u_{n+2} + u_{n+3} +$ etc. is divergent [577]; and hence the whole series is divergent [578].

584. V. *A series of positive terms is convergent if each term is less than the corresponding term of a given convergent series of positive terms.*

Demonstration: Let $U = u_1 + u_2 + \ldots + u_n + u_{n+1} + \ldots$ be a given convergent series; and $V = v_1 + v_2 + \ldots + v_n + v_{n+1} + \ldots$ a series in which $v_1 < u_1$, $v_2 < u_2$, ... $v_n < u_n$, $v_{n+1} < u_{n+1}$, ...

From the nature of addition, it is evident that $V_n < U_n$; and hence, too, lim. $V_n <$ lim. U_n, or $V < U$; therefore, if U is convergent V is convergent.

585. The foregoing principles and theorems will serve to test the convergency and divergency of a very large number of series, but are not of universal application, inasmuch as they do not apply to all classes of series.

Note.—If the terms of a series are not all of the same sign no general method can be obtained for testing their convergency or divergency.

586. The convergency or divergency of a series may often be determined by grouping terms, as follows:

$$U = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \dots$$

$$= 1 + \left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \left(\frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3}\right) + \left(\frac{1}{8^3} + \dots + \frac{1}{15^3}\right) + \text{etc.}$$

$$\therefore\ U < 1 + \frac{2}{2^3} + \frac{4}{4^3} + \frac{8}{8^3} + \text{etc.; or } U < \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}.$$

$\therefore$ The series is convergent.

EXERCISE 89.

1. Is the series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ etc. convergent?

2. Test the series: $1 + 3x + 5x^2 + 7x^3 + \dots$ for convergency

1. When $x < 1$. 2. When $x > 1$. 3. When $x = 1$.

3. Test the series: $\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \dots$ for convergency.

4. Is $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} +$ etc. convergent, when $x < 1$?

5. Test the series: $\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} +$ etc. for convergency.

6. Test the series:

$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \text{etc.}$$

for convergency

1. When $x < 1$. 2. When $x > 1$. 3. When $x = 1$.

Suggestion.—Lim. $\frac{U_n}{U_{n+1}} = \frac{1}{x^2}$. Why?

7. Test the series: $1 + x + x^2 + x^3 +$ etc. for convergency

1. When $x = 1$. 2. When $x < 1$. 3. When $x > 1$.

8. Test $1 + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \dots$ for convergency.

The Binomial Formula.

587. The binomial formula is used to find the product of any number of binomial functions of the form of $x+a$.

Development.

$$(x+a)(x+b) = x^2+(a+b)x+ab$$

Multiply both members by $x+c$,

$$(x+a)(x+b)(x+c) = x^3+(a+b)x^2+abx+cx^2+(ac+bc)x+abc$$
$$= x^3+(a+b+c)x^2+(ab+ac+bc)x \quad +abc$$

Multiply both members by $x+d$,

$$(x+a)(x+b)(x+c)(x+d) =$$
$$x^4+(a+b+c)x^3+(ab+ac+bc)x^2+abcx$$
$$+dx^3+(ad+bd+cd)x^2+(abd+acd+bcd)x+abcd$$
$$= x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2+$$
$$(abc+abd+acd+bcd)x+abcd$$

Observe the following laws in these products:

1. The number of terms is one greater than the number of binomial factors.

2. The exponent of x in the first term equals the number of binomial factors, and decreases by unity in each succeeding term.

3. The coefficient in the first term is unity; in the second term the sum of the second terms of the binomial factors; in the third term the sum of the products of the second terms taken two together; in the fourth term the sum of the products of the second terms taken three together, etc.

4. The last term equals the product of all the second terms.

Are these laws true for any number of factors?

Assume them true for r factors, so that

$(x+a)(x+b)\ldots\ldots(x+m) = x^r+p_1x^{r-1}+p_2x^{r-2}+\ldots\ldots$ $p_{r-1}x+p_r$, in which

$$p_1 = a + b + \dots + m$$
$$p_2 = ab + ac + \dots am + bc + bd + \dots + bm + \text{etc.}$$
$$p_3 = abc + abd + \dots + abm + \dots$$
$$p_r = abc \dots m. \qquad \text{(A)}$$

Multiply by $(x+n)$, the $(r+1)$th factor, then

$$(x+a)(x+b)\dots(x+n) =$$
$$\begin{aligned} & x^{r+1} + p_1 x^r + p_2 x^{r-1} + \dots + p_r x \\ & \quad + n x^r + n p_1 x^{r-1} + \dots + n p_{r-1} x \quad + n p_r \\ \hline & = x^{r+1} + (p_1 + n) x^r + (p_2 + n p_1) x^{r-1} + \dots + n p_r \end{aligned}$$

Laws 1 and 2 are evidently still true.

$$p_1 + n \quad = a + b + c + \dots n.$$
$$p_2 + n p_1 = (ab + ac + \dots am + bc + bd + \dots$$
$$+ bm + \text{etc.}) + (an + bn + \dots + mn),$$

which is still the product of the second terms taken two and two.

. . . . . .

$np_r = abcd \dots n$. Therefore, all the laws still hold true. Hence, if they are true for r factors, they are true for $r+1$ factors. But we found them true for four factors by multiplication; hence, they are true for five factors; and, if so, for six factors; and so on. Therefore, formula (A) is general.

Note.—The number of products that enter into each coefficient may be determined by the principles of combination.

Applications.

Illustrations.—

1. Expand $(x+1)(x+2)(x-3)(x+4)$.

Solution:

$$p_1 = 1+2-3+4 = 4$$
$$p_2 = (1\times 2) + (1\times -3) + (1\times 4) + (2\times -3) + (2\times 4) + (-3\times 4) = -7$$
$$p_3 = (1\times 2\times -3) + (1\times 2\times 4) + (1\times -3\times 4) + (2\times -3\times 4) = -34$$
$$p_4 = 1\times 2\times -3\times 4 = -24$$
$$\therefore (x+1)(x+2)(x-3)(x+4) = x^4 + 4x^3 - 7x^2 - 34x - 24.$$

2. Factor $x^4 + 14x^3 + 71x^2 + 154x + 120$, if possible.

Let $(x+a)(x+b)(x+c)(x+d) = x^4+14x^3+71x^2+154x+120$.

Then, 1. $a+b+c+d = +14$

2. $ab+ac+ad+bc+bd+cd = +71$

3. $abc+abd+acd+bcd = +154$

4. $abcd = +120$

Resolve if possible $+120$ into four factors whose sum is $+14$. These we find to be 2, 3, 4, 5.

$\therefore a=2,\ b=3,\ c=4,$ and $d=5$.

Will these values satisfy 2 and 3?

$ab+ac+ad+bc+bd+cd = 6+8+10+12+15+20 = 71$, correct.

$abc+abd+acd+bcd = 24+30+40+60 = 154$, correct.

$\therefore x^4+14x^3+71x^2+154x+120 = (x+2)(x+3)(x+4)(x+5)$.

EXERCISE 90.

1. Expand $(x+2)(x+3)(x+1)$
2. Expand $(x+3)(x-2)(x-3)$
3. Expand $(x+2)(x+3)(x-1)(x-2)$
4. Expand $(x+3)(x+5)(x-2)(x-6)$
5. Expand $(x+2)(x+2)(x+2)(x+2)$
6. Expand $(x-5)(x-5)(x-5)(x-5)$
7. Expand $(2x+1)(2x+3)(2x-5)(2x-1)$

Suggestion.—Put y for $2x$.

8. Factor $x^3 + 9x^2 + 26x + 24$
9. Factor $x^3 - 2x^2 - 23x + 60$
10. Factor $x^4 + 5x^3 + 5x^2 - 5x - 6$
11. Factor $x^4 - 2x^3 - 25x^2 + 36x + 120$
12. Factor $x^5 + 4x^4 - 13x^3 - 52x^2 + 36x + 144$

The Binomial Theorem.

1. For Positive Exponents.

588. If, in the binomial formula [587, A], we assume $a = b = c = d$, etc., and $r = n$, then will

1. $(x+a)(x+b)(x+c)\ldots\ldots = (x+a)^n$.

2. $x^r = x^n$; $x^{r-1} = x^{n-1}$; $x^{r-2} = x^{n-2}$; etc.

3. $p_1 = a + a + a + \ldots\ldots$ to n terms $= n\,a$.

4. $p_2 = aa + aa + aa + \ldots\ldots = a^2$ taken as many times as there are combinations of 2 in n; or,

$$p_2 = \frac{n(n-1)}{\lfloor 2} a^2.$$

5. $p_3 = aaa + aaa + aaa + \ldots\ldots = a^3$ taken as many times as there are combinations of 3 in n; or,

$$p_3 = \frac{n(n-1)(n-2)}{\lfloor 3} a^3.$$

. . . .

6. $p^r = a \times a \times a \times a \ldots\ldots$ to n factors $= a^n$.

$\therefore\ (x+a)^n =$

$$x^n + n\,x^{n-1}a + \frac{n(n-1)}{\lfloor 2} x^{n-2}a^2 + \frac{n(n-1)(n-2)}{\lfloor 3} x^{n-3}a^3$$
$$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} x^{n-4}a^4 + \ldots\ldots a^n. \qquad \text{(B)}$$

589. ***Cor. 1.***—*If a and x be interchanged,* $(a+x)^n =$

$$a^n + n\,a^{n-1}x + \frac{n(n-1)}{\lfloor 2} a^{n-2}x^2 + \ldots\ldots + x^n. \qquad \text{(C)}$$

From (B) and (C) it will be seen that the coefficients of any two terms equidistant from the first and last terms are numerically equal.

590. ***Cor. 2.***—*If x be made negative in* (C), $(a-x)^n =$

$$a^n - n\,a^{n-1}x + \frac{n(n-1)}{\lfloor 2} a^{n-2}x^2 - \ldots\ldots \pm x^n.$$

591. ***Cor. 3.***—*The sum of the coefficients in* (C) *equals zero.*

For, put $a = 1$ and $x = 1$; then

$$(1-1)^n, \text{ or } 0 = 1 - n + \frac{n(n-1)}{\lfloor 2} - \frac{n(n-1)(n-2)}{\lfloor 3} + \ldots\ldots \pm 1.$$

2. For any Rational Exponents.

592. *Lemma.* — *Lim.* $\left(\frac{x^n - r^n}{x - r}\right)_{x=r} = n r^{n-1}$ *for any rational value of* n.

Demonstration: I. *Let* $n =$ *any positive integer.*

Now, $\frac{x^n - r^n}{x - r} = x^{n-1} + r x^{n-2} + r^2 x^{n-3} + \ldots + r^{n-1}$ [134].

$\therefore$ Lim. $\left(\frac{x^n - r^n}{x - r}\right)_{x=r} =$ lim. $x^{n-1} +$ lim. $r x^{n-2} + \ldots$

$+$ lim. r^{n-1} [401, 413] $= r^{n-1} + r^{n-1} + r^{n-1} + \ldots$ to n terms $= n r^{n-1}$.

II. *Let* $n = \frac{p}{q}$, *a positive fraction.*

Now, $\frac{x^n - r^n}{x - r} = \frac{x^{\frac{p}{q}} - r^{\frac{p}{q}}}{x - r}$. (1)

Put $x^{\frac{1}{q}} = y$, or $x = y^q$; and $r^{\frac{1}{q}} = s$, or $r = s^q$; then

$$\frac{x^n - r^n}{x - r} = \frac{y^p - s^p}{y^q - s^q} = \frac{y^p - s^p}{y - s} \div \frac{y^q - s^q}{y - s}.$$

Since $x = y^q$ and $r = s^q$, lim. $y = s$ when lim. $x = r$.

$\therefore$ Lim. $\left(\frac{x^n - r^n}{x - r}\right)_{x=r} =$ lim. $\left\{\frac{y^p - s^p}{y - s} \div \frac{y^q - s^q}{y - s}\right\}_{y=s} =$

$p y^{p-1} \div q y^{q-1}$ [I, 416] $= \frac{p}{q} y^{p-q} =$

$\frac{p}{q} \left(x^{\frac{1}{q}}\right)^{p-q} = \frac{p}{q} x^{\frac{p}{q} - 1} = n x^{n-1}$.

III. *Let* $n = -p$, *a negative integer or fraction.*

Now, $\frac{x^n - r^n}{x - r} = \frac{x^{-p} - r^{-p}}{x - r} = -x^{-p} r^{-p} \left(\frac{x^p - r^p}{x - r}\right)$.

$\therefore$ Lim. $\left(\frac{x^n - r^n}{x - r}\right)_{x=r} =$ lim. $\left\{-x^{-p} r^{-p} \left(\frac{x^p - r^p}{x - r}\right)\right\}_{x=r}$

$= -r^{-2p} \cdot p r^{p-1}$ [415] $= -r^{2n}(-n r^{-n-1}) = n r^{n-1}$.

General Demonstration of Binomial Theorem.

593. Let it be required to develop $(a + x)^n$, for any rational value of n, into a series of the descending powers of x.

$$(a+x)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n\left(1+\frac{x}{a}\right)^n \qquad \text{(A)}$$

Put $\dfrac{x}{a} = z$, or $\left(1+\dfrac{x}{a}\right)^n = (1+z)^n$ (B)

Put $(1+z)^n = 1 + Az + Bz^2 + Cz^3 + Dz^4 + \ldots.$ (C)

Since z may have any finite value, put $z = r$; then,

$$(1+r)^n = 1 + Ar + Br^2 + Cr^3 + Dr^4 + \ldots. \qquad \text{(D)}$$

Subtract (D) from (C),

$$(1+z)^n - (1+r)^n = A(z-r) + B(z^2-r^2) + C(z^3-r^3) + \ldots. \qquad \text{(E)}$$

Put Z for $1+z$ and R for $1+r$; or $Z-R$ for $z-r$; then,

$$Z^n - R^n = A(z-r) + B(z^2-r^2) + C(z^3-r^3) + \ldots. \qquad \text{(F)}$$

Divide by $Z - R = z - r$,

$$\frac{Z^n - R^n}{Z-R} = A + B\left(\frac{z^2-r^2}{z-r}\right) + C\left(\frac{z^3-r^3}{z-r}\right) + \ldots. \qquad \text{(G)}$$

Let lim. $z = r$, then lim. $Z = R$, since $Z = 1 + z$, and $R = 1 + r$; and lim. $\left(\dfrac{Z^n-R^n}{Z-R}\right)_{Z=R} =$

$$\text{lim.}\left\{A + B\left(\frac{z^2-r^2}{z-r}\right) + C\left(\frac{z^3-r^3}{z-r}\right) + \ldots.\right\}_{z=r} \qquad \text{(H)}$$

$$\therefore\ nR^{n-1}\,[592] = A + 2Br + 3Cr^2 + 4Dr^3 + \ldots. \qquad \text{(I)}$$

$$\text{or, } n(1+r)^{n-1} = A + 2Br + 3Cr^2 + 4Dr^3 + \ldots. \qquad \text{(J)}$$

Multiply by $1 + r$, and column coefficients,

$$n(1+r)^n = \begin{array}{l|l|l|l} A + \ A & r + 2B & r^2 + 3C & r^3 + \ldots. \\ \quad\ + 2B & \ \ + 3C & \quad + 4D & \end{array} \qquad \text{(K)}$$

Multiply (D) by n,

$$n(1+r)^n = n + Anr + Bnr^2 + Cnr^3 + Dnr^4 + \ldots. \qquad \text{(L)}$$

Equating the coefficients of the second members in (K) and (L), we have

(1) $A = n$ (2) $A + 2B = An$

(3) $2B + 3C = Bn$ (4) $3C + 4D = Dn$; etc.

$$\therefore\ A = n,\quad B = \frac{n(n-1)}{\lfloor\underline{2}},\quad C = \frac{n(n-1)(n-2)}{\lfloor\underline{3}},$$

$$D = \frac{n(n-1)(n-2)(n-3)}{\lfloor\underline{4}}, \text{ etc.}$$

Substituting the values in (C),

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{\lfloor\underline{2}}z^2 + \frac{n(n-1)(n-2)}{\lfloor\underline{3}}z^3 + \frac{n(n-1)(n-2)(n-3)}{\lfloor\underline{4}}z^4 + \text{etc.} \qquad \text{(M)}$$

Substitute $\frac{x}{a}$ for z (B), and multiply by a^n (A),

$$(a+x)^n = a^n + n\,a^{n-1}x + \frac{n(n-1)}{\lfloor 2}x^2 +$$
$$\frac{n(n-1)(n-2)\,a^{n-3}}{\lfloor 3}x^3 + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}a^{n-4}x^4 + \ldots. \quad \text{(N)}$$

---

594. ***Cor. 1.*** $(x+y)^n = x^n\left(1+\frac{y}{x}\right)^n =$

$$x^n\left\{1 + n\cdot\frac{y}{x} + \frac{n(n-1)}{\lfloor 2}\cdot\frac{y^2}{x^2} + \frac{n(n-1)(n-2)}{\lfloor 3}\cdot\frac{y^3}{x^3} + \text{etc.}\right\}$$
$$= x^n + n\,x^{n-1}y + \frac{n(n-1)}{\lfloor 2}x^{n-2}y^2 +$$
$$\frac{n(n-1)(n-2)}{\lfloor 3}x^{n-3}y^3 + \text{etc.} \quad \text{(P)}$$

This is the most general form of the binomial theorem, inasmuch as x and y may be both variables.

595. ***Cor. 2.***—*By inspection it will be seen that*

1. The rth term of the development of $(x+y)^n =$

$$\frac{n(n-1)(n-2)\ldots.(n-r+2)}{\lfloor r-1}\cdot x^{n-r+1}y^{r-1}$$

2. The $(r+1)$*th term* $=$

$$\frac{n(n-1)(n-2)\ldots.(n-r+2)(n-r+1)}{\lfloor r-1 \times r = \lfloor r}\cdot x^{n-r}y^r$$

3. The ratio of the $(r+1)$*th term to the rth term* $=$

$$\frac{n-r+1}{r}\cdot\frac{y}{x};\ \text{or,}\ \left\{\frac{n+1}{r}-1\right\}\frac{y}{x}$$

596. ***Cor. 3.***—*1. If n is a positive integer equal to* $r-1$, *the coefficient of the* $(r+1)$*th term, which is also the coefficient of the* $(n+2)$*th term, will reduce to zero. Therefore, the series will terminate with the* $(n+1)$*th term, which will be* y^n.

2. If n is negative or fractional no factor of the rth *term (r being a positive integer) will reduce to zero, however great r be taken. Therefore, the series will be infinite.*

597. Cor. 4.—

Since $\lim. \left\{\left(\frac{n+1}{r}-1\right)\frac{y}{x}\right\}_{r=\infty} = -1 \cdot \frac{y}{x}$, *it follows:*

1. That the coefficients of all terms in the binomial theorem are finite however far the theorem be expanded.

2. That if $y < x$ *the expanded form is convergent* [582].

3. That if $y > x$, *the literal part (not coefficient) of the* $(r+1)$*th term will increase indefinitely as r increases and will ultimately become infinitely great, and as the coefficient remains finite the whole term will become infinitely great. Therefore the expanded form will be divergent* [580].

4. If $y = \pm x$ *the expansion will be indeterminate; but* $(x+y)^n = (2x)^n$ *or* $(0)^n = 2^n x^n$ *or* 0.

5. The expansions of $(x+y)^n$ *and* $(y+x)^n$ *can not both be convergent for particular values of x and y; only the one that has the greater first term.*

598. Cor. 5.—*1. The coefficient of the rth term will evidently be greatest when* $\frac{n-r+1}{r}$ *is first* < 1; *or when* $n-r+1$ *is first* $< r$; *or when* $2r$ *is first* $> n+1$, *or when r is first* $> \frac{n+1}{2}$.

2. The rth term when the expansion is convergent, or when $x > y$, *is evidently greatest when* $\frac{n-r+1}{r} \cdot \frac{y}{x}$ *is first* < 1; *or when* $(n-r+1)\,y$ *is first* $< rx$; *or when* $(n+1)\,y$ *is first* $< (x+y)\,r$; *or when r is first* $> \left(\frac{n+1}{x+y}\right) y$.

Illustration.—In the expansion of $\left(8+\frac{2}{3}\right)^{-\frac{4}{5}}$, the greatest coefficient belongs to the term whose number is first greater than $\frac{-\frac{4}{5}+1}{2}$, or $\frac{1}{10}$; which is the first term.

The greatest term in the expansion of $\left(8+\frac{2}{3}\right)^{-\frac{4}{5}}$ is the one which immediately follows in number, $\frac{-\frac{4}{5}+1}{8\frac{2}{3}} \times \frac{2}{3}$, or $\frac{1}{65}$, which is again the first term.

EXERCISE 91.

Expand:

1. $(a-3x^2)^4$
2. $(2+5x)^5$
3. $(x-3a)^6$
4. $(2x^2+5)^7$
5. $(x^{\frac{1}{2}}-5)^8$
6. $(3x^{\frac{2}{3}}+a^{\frac{3}{2}})^7$

Expand to four terms:

7. $(1-x)^{\frac{1}{2}}$
8. $(a-x)^{\frac{2}{3}}$
9. $(x^2-1)^{\frac{3}{4}}$
10. $(x^{\frac{1}{2}}+a)^{-\frac{1}{2}}$
11. $(ax+b)^{-\frac{5}{6}}$
12. $(x^{\frac{2}{3}}-a^{\frac{2}{3}})^{-\frac{2}{3}}$

13. Extract the cube root of 126 to six decimal places.

Suggestion:

$$\sqrt[3]{126}=\sqrt[3]{125+1}=(125+1)^{\frac{1}{3}}=\left\{5^3\left(1+\frac{1}{125}\right)\right\}^{\frac{1}{3}}=5\left(1+\frac{1}{125}\right)^{\frac{1}{3}}$$

$$=5\left\{1+\frac{1}{3}\times\frac{1}{125}+\frac{\frac{1}{3}(\frac{1}{3}-1)}{\underline{|2}}\times\left(\frac{1}{125}\right)^2+\frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{\underline{|3}}\times\left(\frac{1}{125}\right)^3+\text{etc.}\right\}=5(1+0026666-{\cdot}0000071+{\cdot}0000001)=5{\cdot}0132975.$$

14. Find to 5 decimal places: $\sqrt{65}$, $\sqrt{80}$, $\sqrt[3]{344}$, $\sqrt[5]{3128}$

15. Find the 7th term of $(2x+3)^{10}$

16. Find the 5th term of $\sqrt{4+x}$

17. Find the 6th term of $\frac{a}{\sqrt{a+x}}$

18. Find the rth term of $(a-x)^{-5}$

19. Find the greatest coefficient of $(2+x)^{\frac{5}{6}}$

20. Find the coefficient of the 5th term of $(a-x^3)^{-\frac{2}{3}}$

21. Find the numerical value of the 10th term of

$$(7-5y^{\frac{2}{3}})^n, \text{ when } y=27 \text{ and } n=8$$

The Exponential Theorem.

599. The exponential theorem is the expansion of a^x in ascending powers of x, and is derived as follows:

$$\left(1+\frac{1}{n}\right)^{nx} = \left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x} \qquad \text{(A)}$$

But $$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx \cdot \frac{1}{n} + \frac{nx(nx-1)}{\underline{|2}} \cdot \frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{\underline{|3}} \cdot \frac{1}{n^3} + \text{etc.} \qquad \text{(B)}$$

And $$\left(1+\frac{1}{n}\right)^{n} = 1 + 1 + \frac{1-\frac{1}{n}}{\underline{|2}} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}} + \text{etc.} \qquad \text{(C)}$$

Substitute (B) and (C) in (A),

$$1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}} + \text{etc.}$$

$$= \left\{1 + 1 + \frac{1-\frac{1}{n}}{\underline{|2}} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}} + \text{etc.}\right\}^{x} \qquad \text{(D)}$$

Suppose lim. $n = \infty$, then

$$1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \text{etc.} = \left(1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \text{etc.}\right)^{x} \qquad \text{(E)}$$

Put e for $1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \text{etc.}$; then,

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \text{etc.} \qquad \text{(F)}$$

Put cx for x, then $e^{cx} = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \text{etc.}$ (G)

Let $e^c = a$, and assume e as the base of a system of logarithms [465], then $c = \log_e a$, read logarithm a to the base e. Substitute these values in (G),

$$a^x = 1 + x\log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \frac{x^3(\log_e a)^3}{\underline{|3}} + \text{etc.,} \qquad \text{(H)}$$

which is convergent for all finite values of x [582].

This is the Exponential Theorem.

600. Scholium. $e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \text{etc.} = 2{\cdot}7182818\ldots.$

is the base of the Napierian or natural system of logarithms, a system universally used in theoretical work instead of the system based on 10, which is used in practical work only.

The Logarithmic Series.

601. The logarithmic series is the expansion of $\log_e (1+x)$ in the ascending powers of x, and is derived as follows:

$$a^y = 1 + y\log_e a + \frac{y^2(\log_e a)^2}{\lfloor 2} + \frac{y^3(\log_e a)^3}{\lfloor 3} + \text{etc. [599, H].} \quad \text{(A)}$$

Transpose 1 and divide by y,

$$\frac{a^y - 1}{y} = \log_e a + y\left\{\frac{(\log_e a)^2}{\lfloor 2} + \frac{y(\log_e a)^3}{\lfloor 3} + \text{etc.}\right\} \quad \text{(B)}$$

Let lim. $y = 0$, then

$$\log_e a = \text{lim.}\left(\frac{a^y - 1}{y}\right)_{y=0}$$

Put $1 + x$ for a, then

$$\log_e(1+x) = \text{lim.}\frac{1}{y}\left\{(1+x)^y - 1\right\}_{y=0}$$

$$= \text{lim.}\frac{1}{y}\left\{yx + \frac{y(y-1)}{\lfloor 2}x^2 + \frac{y(y-1)(y-2)}{\lfloor 3}x^3 + \text{etc.}\right\}_{y=0}$$

$$= \text{lim.}\left\{x + \frac{y-1}{\lfloor 2}x^2 + \frac{(y-1)(y-2)}{\lfloor 3}x^3 + \text{etc.}\right\}_{y=0}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \text{etc.} \quad \text{Therefore,}$$

$$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{etc.} \quad \text{(C)}$$

This is known as the *Logarithmic Series.*

602. The ratio of the $(n+1)$th term to the nth term is $\frac{x^{n+1}}{n+1} : \frac{x^n}{n} = \frac{n}{n+1} \cdot x = \frac{1}{1+\frac{1}{n}} \cdot x$.

Now, $\text{lim.}\left(\frac{1}{1+\frac{1}{n}} \cdot x\right)_{n=\infty} = x$. Therefore, if $x < 1$, numerically the series is convergent. It is, therefore, convergent for all values of x between -1 and $+1$.

When $x = 1$, $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{etc.}$, which is convergent [582].

When $x = -1$, $\log_e 0 = -1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}\right)$, which is divergent, since $\text{lim.}\left(\frac{u_{n+1}}{u_n}\right)_{n=\infty} = \text{lim.}\left\{-1\left(\frac{1}{1+\frac{1}{n}}\right)\right\}_{n=\infty} = -1$, and all the terms have the same sign [583].

603. Resume

$$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^2 + \frac{1}{5}x^3 - \ldots \quad (1)$$

Put $-x$ for x, then

$$\log_e(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \ldots \quad (2)$$

Subtract (2) from (1), then

$$\log_e(1+x) - \log_e(1-x), \text{ or } \log_e\left(\frac{1+x}{1-x}\right) [467, \text{P. } 3] =$$

$$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right) \quad (3)$$

Put $\dfrac{m-n}{m+n}$ for x, then.

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \ldots\right\} \quad (4)$$

Put $m = n + 1$, then

$$\log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \ldots\right\} \quad (5)$$

or,

$$\log_e(n+1) = \log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \ldots\right\} \quad (D)$$

As this formula converges very rapidly for all values of n, it may be used to find the Napierian logarithm of any number from that of the next preceding number, n being regarded an integer.

Computation of Logarithms.

604. The logarithms of composite numbers may be readily found, when the logarithms of primes are known, by Art. 467, P. 2.

The logarithms of prime numbers are found by formula D, Art. 603.

Illustrations.—

$\log_e 1 = 0$ [466, P.]

$\log_e 2 = \log_e (1+1) =$

$$0 + 2\left(\frac{1}{3} + \frac{1}{3 \times 3^3} + \frac{1}{5 \times 3^5} + \frac{1}{7 \times 3^7} + \ldots\right)$$

$= 0{\cdot}69314718\ldots$ (by actual reduction).

$\log_e 3 = \log_e (2+1) =$

$$\log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3 \times 5^3} + \frac{1}{5 \times 5^5} + \frac{1}{7 \times 5^7} + \ldots\right)$$

$= 1{\cdot}09861228\ldots$

$\log_e 4 = 2 \log_e 2 = 1{\cdot}38629436\ldots$

$\log_e 5 = \log_e (4+1) =$

$$\log_e 4 + 2\left(\frac{1}{9} + \frac{1}{3 \times 9^3} + \frac{1}{5 \times 9^5} + \ldots\right)$$

$= 1{\cdot}60943791\ldots$

$\log_e 6 = \log_e 3 + \log_e 2 = 1{\cdot}79175946\ldots$

$\log_e 7 = \log_e (6+1) =$

$$\log_e 6 + 2\left(\frac{1}{13} + \frac{1}{3 \times 13^3} + \frac{1}{5 \times 13^5} + \ldots\right)$$

$= 1{\cdot}94591\ldots$

$\log_e 8 = 3 \log_e 2 = 2{\cdot}07944$

$\log_e 9 = 2 \log_e 3 = 2{\cdot}19722$

$\log_e 10 = \log_e 5 + \log_e 2 = 2{\cdot}30258509$

etc., etc., etc.

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605. Let a and b represent the bases of two systems of logarithms and n any number.

Let $\log_b n = x$, then $b^x = n$ (1)

Let $\log_a b = m$, then $a^m = b$ (2)

$\therefore \quad a^{mx} = b^x = n$, or $\log_a n = m\,x$ (3)

$$\therefore \frac{\log_a n}{\log_b n} = \frac{m\,x}{x} = m; \text{ or}$$

$\log_a n = m \log_b n$. Therefore,

***Principle.**—Multiplying the* $\log_a$ *of a number by the* $\log_b$ *of b gives the* $\log_b$ *of the number.*

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606. $\text{Log}_e\, n = \log_{10} n \times \log_e 10$ [605, P.]

$$\therefore \log_{10} n = \log_e n \times \frac{1}{\log_e 10} = \log_e n \times \frac{1}{2{\cdot}30258509}$$
$$= \log_e n \times 0{\cdot}4342944\ldots.$$

607. The number $0{\cdot}4342944\ldots.$ is called the modulus of the common system, and is represented by m.

Therefore,

***Prin. 2.**—$\text{Log}_{10}\, n = m \log_e n$.*

By means of this principle the Briggean or common logarithms may be derived from the Napierian or natural logarithms.

---

608. Since $\log_e (n+1) =$

$$\log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \ldots.\right\},$$

$$\log_{10}(n+1) =$$

$$\log_{10} n + 2m\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \ldots.\right\} \quad \text{(H)}$$

By means of this formula the common logarithms may be computed directly.

Summation of Series.

609. No general method of summing series can be given. Series of special types may sometimes be summed by special methods. The student has already learned how to sum an arithmetical progression [486], a geometrical progression [493], an infinite series of the geometrical type [499], an arithmetico-geometrical progression [500], a series of square numbers [506], a series of cubic numbers [507], and series dependent upon or resolvable into these. A few additional methods will be given here.

610. I. Method by Indeterminate Coefficients.

This method is applicable when the nth term is a rational integral function of n.

Illustration.—Find the sum S_n of n terms of the series: $1\times 2^2+2\times 3^2+3\times 4^2+\ldots\ldots+n(n+1)^2$.

Solution:

Put $S_n=1\times 2^2+2\times 3^2+3\times 4^2+4\times 5^2+\ldots\ldots+n(n+1)^2$
$= A+Bn+Cn^2+Dn^3+En^4+\ldots\ldots$

and, $S_{n+1}=1\times 2^2+2\times 3^2+3\times 4^2+4\times 5^2+\ldots\ldots+(n+1)(n+2)^2$
$= A+B(n+1)+C(n+1)^2+D(n+1)^3+E(n+1)^4+\ldots\ldots$

Then, by subtraction,

$(n+1)(n+2)^2=B+(2n+1)C+(3n^2+3n+1)D$
$+(4n^3+6n^2+4n+1)E+\ldots\ldots$, or $n^3+5n^2+8n+4=$
$(B+C+D+E)+(2C+3D+4E)n+(3D+6E)n^2+4En^3$,

since all coefficients after E are zero, there being no more than four terms in the expansion.

Equating the coefficients of the like powers of n,

1. $4E=1$,
2. $3D+6E=5$,
3. $2C+3D+4E=8$,
4. $B+C+D+E=4$;

whence, $E=\frac{1}{4}$, $D=\frac{7}{6}$, $C=\frac{7}{4}$, and $B=\frac{5}{6}$.

$$\therefore \quad S_n=A+\frac{5}{6}n+\frac{7}{4}n^2+\frac{7}{6}n^3+\frac{1}{4}n^4.$$

To find A, put $n=1$, then $S_n=S_1=$ the first term $=1\times 2^2$.

$$\therefore (1\times 2)^2=A+\frac{5}{6}+\frac{7}{4}+\frac{7}{6}+\frac{1}{4}=A+4;$$

whence $A=0$; and

$$S_n=\frac{5}{6}n+\frac{7}{4}n^2+\frac{7}{6}n^3+\frac{1}{4}n^4$$

$$=\frac{1}{12}(3n^4+14n^3+21n^2+10n)$$

$$=\frac{n}{12}(3n^3+14n^2+21n+10)=\frac{n}{12}(n+1)(n+2)(3n+5)$$

611. 2. Method by Decomposition.

This method is sometimes applicable when the nth term is a rational fractional function of n, and is resolvable into the algebraic sum of the nth terms of two or more other series of the same nature.

Illustration.—

Find the sum S_n of n terms of the series: $\frac{4}{2\times 3\times 4}+\frac{7}{3\times 4\times 5}+\frac{10}{4\times 5\times 6}+\dots+\frac{3n+1}{(n+1)(n+2)(n+3)}$.

Solution:

Let $\frac{3n+1}{(n+1)(n+2)(n+3)}=\frac{A}{n+1}+\frac{B}{n+2}+\frac{C}{n+3}$; then $A=-1$, $B=5$, and $C=-4$; whence

$$\frac{3n+1}{(n+1)(n+2)(n+3)}=\left(-\frac{1}{n+1}+\frac{5}{n+2}-\frac{4}{n+3}\right).$$

$$\therefore S_n=\Sigma\left\{\frac{3n+1}{(n+1)(n+2)(n+3)}\right\}=$$

$$\Sigma\left(-\frac{1}{n+1}\right)+\Sigma\left(\frac{5}{n+2}\right)+\Sigma\left(-\frac{4}{n+3}\right).$$

$$\Sigma\left(-\frac{1}{n+1}\right)=-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\dots-\frac{1}{n+1}$$

$$\Sigma\left(\frac{5}{n+2}\right)=\frac{5}{3}+\frac{5}{4}+\dots+\frac{5}{n+1}+\frac{5}{n+2}$$

$$\Sigma\left(-\frac{4}{n+3}\right)=-\frac{4}{4}-\dots-\frac{4}{n+1}-\frac{4}{n+2}-\frac{4}{n+3}$$

Adding the last three series, we have

$$S_n=\left(-\frac{1}{2}+\frac{4}{3}+\frac{1}{n+2}-\frac{4}{n+3}\right)=\frac{5}{6}+\frac{1}{n+2}-\frac{4}{n+3}$$

If lim. $n=\infty$, then $S_\infty=\frac{5}{6}$.

The Differential Method.

612. If the first term of any series be taken from the second, the second from the third, the third from the fourth, and so on, a new series will be formed which is called the *first order of differences.* If the first order of differences be treated in the same manner as the original series, a *second order of differences* will be formed, and so on.

Thus, if we let $a, b, c, d, e, \ldots$ be any series, then

$b-a,\ c-b,\ d-c,\ e-d, \ldots$ will be the first order of differences;

$c-2b+a,\ d-2c+b,\ e-2d+c, \ldots$ the second order of differences;

$d-3c+3b-a,\ e-3d+3c-b, \ldots$ the third order of differences, and so on.

613. If we let $a_1, b_1, c_1, d_1, \ldots$ represent the first order of differences;

$a_2, b_2, c_2, d_2, \ldots$ the second order of differences;

$a_3, b_3, c_3, d_3, \ldots$ the third order of differences;

and so on, we have the following scheme:

Series:	$a,$	$b,$	$c,$	$d,$	$e,$	....
1st Differences:	$a_1,$	$b_1,$	$c_1,$	$d_1,$		....
2d Differences:		$a_2,$	$b_2,$	$c_2,$		....
3d Differences:			$a_3,$	$b_3,$		....
4th Differences:				$a_4,$		...., and so on.

$a_1, a_2, a_3, a_4, \ldots$ are the *first terms* of the successive order of differences.

614. Problem 1. To find the $(n+1)$**th term of a series.**

Solution: Take the series $a, b, c, d, e, \ldots$ then from the above scheme,

1. $b-a=a_1$, whence $b=a+a_1$ (1)
   $b_1-a_1=a_2$, " $b_1=a_1+a_2$ (2)
   $b_2-a_2=a_3$, " $b_2=a_2+a_3$ (3)
   $b_3-a_3=a_4$, " $b_3=a_3+a_4$ (4)
2. $c=b+b_1=a+2a_1+a_2$, from (1 and 2) (5)
   $c_1=b_1+b_2=a_1+2a_2+a_3$, from (2 and 3) (6)
   $c_2=b_2+b_3=a_2+2a_3+a_4$, from (3 and 4) (7)

3. $d = c + c_1 = a + 3a_1 + 3a_2 + a_3$, from (5 and 6) (8)
$d_1 = c_1 + c_2 = a_1 + 3a_2 + 3a_3 + a_4$, from (6 and 7) (9)

4. $e = d + d_1 = a + 4a_1 + 6a_2 + 4a_3 + a_4$, from (8 and 9) (10)

Now, since

$b = a + a_1$, $\quad c = a + 2a_1 + a_2$,

$d = a + 3a_1 + 3a_2 + a_3$, $\quad e = a + 4a_1 + 6a_2 + 4a_3 + a_4$,

it will be observed that the coefficients of any term are the same as the coefficients of a power of a binomial, whose index is one less than the number of the term. Hence, the

$$(n+1)\text{th term} = a + na_1 + \frac{n(n-1)}{\lfloor 2}a_2 + \frac{n(n-1)(n-2)}{\lfloor 3}a_3 + \dots \quad [A]$$

615. ***Cor.***—*The nth term* =

$$a + (n-1)a_1 + \frac{(n-1)(n-2)}{\lfloor 2}a_2 + \dots \quad [B]$$

Illustrative Example.—Find the 7th term, also the nth term, of the series: 1, 3, 6, 10, ...

Solution: 1st differences = 2, 3, 4, ...
2d " = 1, 1, ...
3d " = 0, ...

∴ 1st. $n = 7$, $a = 1$, $a_1 = 2$, $a_2 = 1$, $a_3 = 0$

Substitute these values in [B],

$$7\text{th term} = 1 + 6 \times 2 + \frac{6 \times 5}{2} \times 1 = 28$$

2d. Put $n = n$, $a = 1$, $a_1 = 2$, $a_2 = 1$, and $a_3 = 0$,

then $n\text{th term} = 1 + (n-1) \times 2 + \dfrac{(n-1)(n-2)}{2} \times 1 = \dfrac{n(n-1)}{2}$.

616. Problem 2. To find the sum of n terms of a series.

Solution: Let it be required to find the sum of n terms of the series $a, b, c, d, e, \dots$

Assume the series $0, a, a+b, a+b+c, a+b+c+d, \dots$, then,

1st. The first order of differences of the assumed series is the given series.

2d. The second, third, and nth orders of differences of the assumed series are the first, second, and $(n-1)$th orders of differences of the given series.

3d. The $(n+1)$th term of the assumed series is the sum of n terms of the given series.

Hence, if in formula [A] we put 0 for a, a for a_1, a_1 for a_2, etc., we shall have the sum of n terms of the given series. Doing so, we shall have

$$S_n = na + \frac{n(n-1)}{\underline{|2}} a_1 + \frac{n(n-1)(n-2)}{\underline{|3}} a_2 + \dots \qquad [C]$$

Note.—This method is applicable only when some finite order of differences will reduce to zero.

Example.—Find the sum of 10 terms of the series: $1 + 4 + 10 + 20 + 35 + \dots$.

Solution: First Differences = 3, 6, 10, 15, ...
Second Differences = 3, 4, 5, ...
Third Differences = 1, 1, ...
Fourth Differences = 0, ...

∴ Put $n = 10$, $a = 1$, $a_1 = 3$, $a_2 = 3$, $a_3 = 1$, and $a_4 = 0$ in formula [C]; then,

$$S_n = 10 + \frac{10 \times 9}{2} \times 3 + \frac{10 \times 9 \times 8}{2 \times 3} \times 3 + \frac{10 \times 9 \times 8 \times 7}{2 \times 3 \times 4} \times 1 = 715$$

617. Problem 3. To interpolate terms at regular intervals between the terms of a given series.

Formula [B] may sometimes be used with advantage to interpolate terms at regular intervals between the terms of a given series.

Illustrations.—1. Given

$$(651)^3 = 423801, \qquad (653)^3 = 426409,$$
$$(655)^3 = 429025, \text{ and } (657)^3 = 431645,$$

to find the value of $(652)^3$, $(654)^3$, and $(656)^3$.

Solution: Series = 423801, 426409, 429025, 431649
First Differences = 2608, 2616, 2624
Second Differences = 8, 8
Third Differences = 0

Take formula $a_n = a + (n-1)a_1 + \frac{(n-1)(n-2)}{\underline{|2}} a_2 + \dots$

Put $a = 423801$, $a_1 = 2608$, $a_2 = 8$, and $n = 1\frac{1}{2}$, $2\frac{1}{2}$, and $3\frac{1}{2}$ successively; then,

1. $(652)^3 = 423801 + \frac{1}{2} \times 2608 - \frac{1}{8} \times 8 = 425104$
2. $(654)^3 = 423801 + \frac{3}{2} \times 2608 + \frac{3}{8} \times 8 = 427716$
3. $(656)^3 = 423801 + \frac{5}{2} \times 2608 + \frac{15}{8} \times 8 = 430336$

2. $\sqrt[3]{651}=8{\cdot}666831$, $\sqrt[3]{652}=8{\cdot}671266$, $\sqrt[3]{653}=8{\cdot}675697$, $\sqrt[3]{654}=8{\cdot}680123$, and $\sqrt[3]{655}=8{\cdot}684545$, find $\sqrt[3]{653{\cdot}75}$.

Solution: Here $a=8{\cdot}666831$, $a_1={\cdot}004435$, $a_2=-0{\cdot}000004$, $a_3=-0{\cdot}000001$, $a_4=0$, and $n=3\frac{3}{4}$. Substitute these values in

$$a_n=a+(n-1)a_1+\frac{(n-1)(n-2)}{\lfloor 2}a_2+\frac{(n-1)(n-2)(n-3)}{\lfloor 3}a_3;$$ then,

$$\sqrt[3]{653{\cdot}75}=8{\cdot}666831+\frac{11}{4}\times{\cdot}004435-\frac{77}{32}\times{\cdot}000004-\frac{231}{384}\times{\cdot}000001$$
$$=8{\cdot}666831+{\cdot}012196-{\cdot}000009-{\cdot}000001=8{\cdot}679017.$$

EXERCISE 92.

Find the nth term and the sum of n terms of the following series:

1. $2+6+12+20+\ldots.$
2. $1+9+25+49+\ldots.$
3. $1+3+6+10+\ldots.$
4. $3+8+15+24+\ldots.$
5. $1+4+9+16+\ldots.$
6. $2+12+30+56+\ldots.$
7. $6+24+60+120+210+\ldots.$
8. $45+120+231+384+585+\ldots.$
9. $1.3.2^2+2.4.3^2+3.5.4^2+\ldots.$
10. $3.5.7+5.7.9+7.9.11+\ldots.$

Sum to n terms and to infinity:

11. $\frac{1}{1\times 4}+\frac{1}{4\times 7}+\frac{1}{7\times 10}+\ldots.$
12. $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\ldots.$
13. $\frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}+\ldots.$
14. $\frac{1}{1.3.7}+\frac{1}{3.5.9}+\frac{1}{5.7.11}+\frac{1}{7.9.13}+\ldots.$
15. $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+\frac{7}{4.5.6}$

Find the value of:

16. $\Sigma\{n^2(3n-2)\}$

17. $\Sigma\left\{\frac{n}{6}(n+1)(n+2)\right\}$

18. $\Sigma\left(\frac{1}{n(n+1)}\right)$

19. $\Sigma\left(\frac{n^4+n^2+1}{n^4+n}\right)$

20. The log. $950 = 2·977724$, log. $951 = 2·978181$, log. $952 = 2·978637$, log. $953 = 2·979093$, find log. $952·375$.

Reversion of Series.

618. If y is a serial function of x, then may x be developed into a serial function of y, and the process is called *Reversion of Series.*

Example.—Revert $y = a + bx + cx^2 + dx^3 + ex^4 + \ldots$ into a serial function of y.

Solution: Put the series in the form

$$y-a = bx+cx^2+dx^3+ex^4\ldots$$

Put $x = A(y-a)+B(y-a)^2+C(y-a)^3+D(y-a)^4+\ldots$

Now,

$$bx = bA(y-a)+bB(y-a)^2+bC(y-a)^3+bD(y-a)^4+\ldots$$
$$cx^2 = cA^2(y-a)^2+2cAB(y-a)^3+(cB^2+2cAC)(y-a)^4+\ldots$$
$$dx^3 = dA^3(y-a)^3+3dA^2B(y-a)^4+\ldots$$
$$ex^4 = eA^4(y-a)^4+\ldots$$

$$\therefore\ y-a = bA(y-a)+(bB+cA^2)(y-a)^2 +(bC+2cAB+dA^3)(y-a)^3 +(bD+cB^2+2cAC+3dA^2B+eA^4)(y-a)^4+\ldots$$

Equating the coefficients,

1. $bA=1$; whence $A=\frac{1}{b}$
2. $bB+cA^2=0$; whence $B=-\frac{c}{b^3}$
3. $bC+2cAB+dA^3=0$; whence $C=\frac{2c^2-bd}{b^5}$
4. $bD+cB^2+2cAC+3dA^2B+eA^4=0$;
   whence $D=-\frac{b^2e-5bcd+5c^3}{b^7}$

$$\therefore\ x = \frac{1}{b}(y-a) - \frac{c}{b^3}(y-a)^2 + \frac{2c^2-bd}{b^5}(y-a)^3 - \frac{b^2e-5bcd+5c^3}{b^7}(y-a)^4 + \ldots$$

619. ***Cor.***—If $a=0$, then

$$x=\frac{1}{b}y-\frac{c}{b^3}y^2+\frac{2c^2-bd}{b^5}y^3-\frac{b^2e-5bcd+5c^3}{b^7}y^4+\ldots.$$

EXERCISE 93.

Revert the following serial functions of x into serial functions of y:

1. $y=x+x^2+x^3+x^4+\ldots.$
2. $y=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\ldots.$
3. $y=x+2x^2+5x^3+14x^4+\ldots.$
4. $y=x+x^3+2x^5+5x^7+\ldots.$

   Suggestion.—Let $x=Ay+By^3+Cy^5+Dy^7+\ldots.$
5. $y=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\ldots.$
6. $y=1+x-2x^2+x^3$
7. Find one value of x in the equation $x^3+4x^2+5x=1$

   Suggestion.—
   Put $1=y$, and assume $x=Ay+By^2+Cy^3+Dy^4+\ldots.$
8. Find one valuc of x in $x^3+100x=1$

Recurring Series.

620. A series in the ascending powers of x, in which each term, after one or more fixed terms, is px times the preceding term, or px times the preceding $+qx^2$ times the next preceding term, or px times the preceding $+qx^2$ times the next preceding $+rx^3$ times the next preceding term, or and so on, is a *Recurring Series.*

621. A recurring series is of the *first, second,* or *nth order,* accordingly as each term, after the law begins, is derived from *one, two,* or *n* preceding terms.

622. The forms px, $px + qx^2$, $px + qx^2 + rx^3$, and so on, are called the *order scales of the series;* and p, $p + q$, $p + q + r$, and so on, the *order scales of the coefficients.*

Illustrations.—If we put $p = 3$, $q = 4$, and $r = -2$, then

1. $2 + 6x + 18x^2 + 54x^3 + \ldots.$ is a series of the first order.

2. $2 + 6x + 26x^2 + 102x^3 + \ldots.$ is a series of the second order.

3. $2 + 6x + 26x^2 + 98x^3 + 386x^4 + \ldots.$ is a series of the third order.

623. Problem 1. To determine the scale of coefficients.

1. Let $a + bx + cx^2 + dx^3 + \ldots.$ be a recurring series of the first order.

Then, $bx = apx$; $cx^2 = pbx^2$; $dx^3 = pcx^3$; and so on.

$$\therefore \quad p = \frac{b}{a}; \qquad p = \frac{c}{b}; \qquad p = \frac{d}{c}; \text{ and so on.}$$

2. Let $a + bx + cx^2 + dx^3 + \ldots.$ be a series of the second order.

Then, $aqx^2 + bpx^2 = cx^2$ (1); $bqx^3 + cpx^3 = dx^3$ (2); whence, $aq + bp = c$ (3); $bq + cp = d$ (4).

Then, by elimination,

$$p = \frac{bc - ad}{b^2 - ac} \text{ and } q = \frac{bd - c^2}{b^2 - ac}$$

3. Let $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \ldots.$ be a recurring series of the third order. Then,

1. $ar + bq + cp = d$ 2. $br + cq + dp = e$
3. $cr + dq + ep = f$

By elimination the values of p, q and r may be found. In the same manner the scale of coefficients of a recurring series of any order may be found.

624. *Scholium.*—*In order that the scale of coefficients of any recurring series may be found, there must be given at least* twice *as many terms of the series as there are terms in the scale. In the exercises concluding this subject just twice as many terms of each series will be given as are contained in the scale of the series. This will enable the student to determine at a glance the order of the series.*

When this law of notation is not followed, as when the nth term of a series only is given, it is usually best to expand the series and make trial for a scale of two terms, and, if the results thus obtained will not satisfy the series, then make trial for a scale of three terms, and so on until the proper scale is determined.

Example.—Find the scale of coefficients in the series $1 + 2x + 3x^2 + 4x^3 + \ldots.$ and expand the series.

Solution: This is a series of the second order, since four terms are given. Hence,

1. $q + 2p = 3$ 2. $2q + 3p = 4$

whence, $p = 2$ and $q = -1$, and the scale is $2 - 1$. The series is $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots. nx^{n-1}$.

625. Problem 2. To find the sum of n terms of a recurring series.

The method of finding the sum of a recurring series is the same, whatever be the scale of the series. For the sake of simplicity we will here assume a series of the second order, whose scale is $p + q$, for illustration.

Let $a_0 + a_1 x + a_2 x^2 + \ldots. a_{n-1} x^{n-1}$ be a series of the second order. Then,

$$S_n = a_0 + a_1 x + a_2 x^2 + \ldots. + a_{n-1} x^{n-1}$$

$$-px\,S_n = -p\,a_0 x - p\,a_1 x^2 - \ldots. - p\,a_{n-2} x^{n-1} - p\,a_{n-1} x^n$$

$$-qx^2 S_n = -q\,a_0 x^2 - \ldots. - q\,a_{n-3} x^{n-1} - q\,a_{n-2} x^n - q\,a_{n-1} x^{n+1}$$

Adding and remembering that the sum of the coefficients of each term from the third to the nth inclusive is *zero*,

$$(1-px-qx^2)S_n = a_0+(a_1-pa_0)x$$
$$-(pa_{n-1}+qa_{n-2})x^n - qa_{n-1}x^{n+1};$$

whence,
$$S_n = \frac{a_0+(a_1-pa_0)x}{1-px-qx^2}$$
$$-\frac{(pa_{n-1}+qa_{n-2})x^n+qa_{n-1}x^{n+1}}{1-px-qx^2}.$$

626. ***Cor.***—*If* $x<1$ *and lim.* $n=\infty$, *then*

$$S_\infty = \frac{a_0+(a_1-pa_0)x}{1-px-qx^2}.$$

Example.—Sum $1+3x+5x^2+18x^3+48x^4+145x^5+\ldots.$ to 7 terms and to infinity, when $x<1$.

Solution: 1. $5p+3q+r=18$ 2. $18p+5q+3r=48$
3. $48p+18q+5r=145$

$\therefore\ p=2,\ q=3,$ and $r=-1$.

$$S_7 = 1+3x+5x^2+18x^3+48x^4+145x^5+416x^6$$
$$-2xS_7 = -2x-6x^2-10x^3-36x^4-96x^5-290x^6-832x^7$$
$$-3x^2S_7 = -3x^2-9x^3-15x^4-54x^5-144x^6-435x^7-1248x^8$$
$$+x^3S_7 = +x^3+3x^4+5x^5+18x^6+48x^7+145x^8+416x^9$$

$$\therefore\ (1-2x-3x^2+x^3)S_7 = 1+x-4x^2-1219x^7-1103x^8+416x^9$$

$$S_7 = \frac{1+x-4x^2-1219x^7-1103x^8+416x^9}{1-2x-3x^2+x^3}$$

$$S_\infty = \frac{1+x-4x^2}{1-2x-3x^2+x^3}$$

627. If S_∞ is developed into a series by the method of indeterminate coefficients, the original series may be reproduced to any number of terms desired. Therefore, S_∞ is often called the *generatrix* of the series.

628. If the *generatrix* can be decomposed into partial fractions, the general, or nth term, of the series may easily be obtained, and hence, too, the sum of n terms.

Illustration.—Let it be required to find the nth term and the sum of n terms of the series

$$1 + 5x + 13x^2 + 41x^3 + \dots.$$

Solution: It may readily be determined that $p = 2$, $q = 3$, and the generatrix $= \dfrac{1+3x}{1-2x-3x^2} = -\dfrac{1}{2} \cdot \dfrac{1}{1+x} + \dfrac{3}{2} \cdot \dfrac{1}{1-3x}$.

Now, $-\dfrac{1}{2} \cdot \dfrac{1}{1+x} = -\dfrac{1}{2} + \dfrac{1}{2}x - \dfrac{1}{2}x^2 + \dots.$

$$(-1)^n \cdot \frac{x^{n-1}}{2} = \frac{(-1)^n x^n - 1}{2x+2} \quad [493, B];$$

and $\dfrac{3}{2} \cdot \dfrac{1}{1-3x} = \dfrac{3}{2} + \dfrac{9}{2}x + \dfrac{27}{2}x^2 + \dots.$

$$\frac{3^n}{2} \cdot x^{n-1} = \frac{3^{n+1}x^n - 3}{6x-2} \quad [493, B].$$

$$\therefore \qquad S_n = \frac{(-1)^n x^n - 1}{2x+2} + \frac{3^{n+1}x^n - 3}{6x-2};$$

$$n\text{th term} = (-1)^n \cdot \frac{x^{n-1}}{2} + \frac{3^n}{2} \cdot x^{n-1}.$$

EXERCISE 94.

Sum to infinity:

1. $1 + 2x + 5x^2 + 13x^3 + \dots.$
2. $1 + 3x + 8x^2 + 22x^3 + \dots.$
3. $2 + 2x + 4x^2 + 14x^3 + \dots.$
4. $3 + 2x - 7x^2 - 38x^3 - \dots.$
5. $1 + 2x + 3x^2 + 11x^3 + 35x^4 + 121x^5 + \dots.$
6. $1 - 3x + 5x^2 + 5x^3 + 13x^4 + 61x^5 + \dots.$
7. Sum $1 + 2x + 9x^2 + 33x^3 + \dots.$ to 6 terms.
8. Sum $1 - 2x - 7x^2 - 8x^3 + \dots.$ to 7 terms.
9. Sum $2 - x + 6x^2 - 14x^3 + \dots.$ to 8 terms.
10. Sum $1 - 2x + 3x^2 + 6x^3 + x^4 - 2x^5 + \dots.$ to 9 terms.

Find the nth term and the sum of n terms of

11. $1 + 2x - 8x^2 + 20x^3 - \dots.$
12. $1 + 5x + 9x^2 + 13x^3 + \dots.$

Decomposition of Rational Fractional Functions.

629. To decompose a rational fraction is to find two or more other fractions whose sum equals the rational fraction.

630. It will be only necessary to show how to decompose proper fractions, as all improper fractions may be reduced to mixed numbers, which process will already lead to a partial decomposition.

Principles.

631. *1. Any rational fraction of the form of*

$\frac{P}{(x+a)(x+b)\ldots\ldots(x+n)}$ *may be decomposed so that*

$$\frac{P}{(x+a)(x+b)\ldots\ldots(x+n)} = \frac{A}{x+a} + \frac{B}{x+b} + \ldots\ldots + \frac{N}{x+n}$$

Illustration.—Put

$$\frac{3x^2+14x-29}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3} \qquad \text{(A)}$$

Clear of fractions and arrange the terms according to the descending powers of x,

$$3x^2 + 14x - 29 = (A + B + C)x^2 - (A + 4B - C)x - (6A - 3B + 2C).$$

Equating the coefficients [566], we have

$$(1)\ A + B + C = 3 \qquad (2)\ A + 4B - C = -14$$
$$(3)\ 6A - 3B + 2C = 29$$

Finding the values of A, B, and C by elimination, and substituting them in (A), we obtain

$$\frac{3x^2+14x-29}{(x-1)(x+2)(x-3)} = \frac{2}{x-1} - \frac{3}{x+2} + \frac{4}{x-3}.$$

632. In a similar manner it may be shown that

$$\frac{P}{(ax+b)(cx+d)\ldots\ldots(mx+n)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \ldots\ldots + \frac{M}{mx+n}.$$

633. *2. Any rational fraction of the form of*

$$\frac{P}{(x^2+ax+b)(x^2+cx+d)\ldots\ldots(x^2+mx+n)}$$

may be decomposed so that

$$\frac{P}{(x^2+ax+b)(x^2+cx+d)\ldots\ldots(x^2+mx+n)}=$$
$$\frac{Ax+B}{x^2+ax+b}+\frac{Cx+D}{x^2+cx+d}+\ldots\ldots+\frac{Mx+N}{x^2+mx+n}.$$

Illustration.—Put $\frac{4x^4-8x^3-5x^2-15x+3}{(x^2+x+1)(x^2-x+1)(x^2+x+2)}$

$$=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}+\frac{Ex+F}{x^2+x+2} \qquad \text{(A)}$$

Clear of fractions and arrange the terms according to the descending powers of x,

$$4x^4-8x^3-5x^2-15x+3=$$
$$(A+C+E)x^5+(B+2C+D+F)x^4$$
$$+(2A+4C+2D+E)x^3+(-A+2B+3C+4D+F)x^2$$
$$+(2A-B+2C+3D+E)x+(2B+2D+F) \qquad (1)$$

Equating the coefficients [566],

(1) $A+C+E=0$
(2) $B+2C+D+F=4$
(3) $2A+4C+2D+E=-8$
(4) $A-2B-3C-4D-F=5$
(5) $2A-B+2C+3D+E=-15$
(6) $2B+2D+F=3$

Finding by elimination the values of A, B, C, D, E, and F, and substituting them in (A), we have

$$\frac{4x^4-8x^3-5x^2-15x+3}{(x^2+x+1)(x^2-x+1)(x^2+x+2)}=$$
$$\frac{3}{x^2+x+1}-\frac{4}{x^2-x+1}+\frac{5}{x^2+x+2}.$$

634. In a similar manner it may be shown that

$$\frac{P}{(ax^2+bx+c)(dx^2+ex+f)\ldots\ldots(mx^2+nx+p)}$$
$$=\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{dx^2+ex+f}+\ldots\ldots+\frac{Mx+N}{mx^2+nx+p}.$$

635. *3. A rational fraction of the form of* $\frac{P}{(x+a)^n}$ *may be decomposed so that*

$$\frac{P}{(x+a)^n}=\frac{A}{x+a}+\frac{B}{(x+a)^2}+\ldots+\frac{N}{(x+a)^n}.$$

Illustration.—Put $\frac{3x^3-14x^2+19x-5}{(x-2)^4}$

$$=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}+\frac{D}{(x-2)^4} \qquad \text{(A)}$$

Clear of fractions and arrange the terms according to the descending powers of x; then,

$$3x^3-14x^2+19x-5=Ax^3-(6A-B)x^2+$$
$$(12A-4B+C)x-(8A-4B+2C-D).$$

Equating the coefficients, we have

1. $A=3$ 2. $6A-B=14$
3. $12A-4B+C=19$ 4. $8A-4B+2C-D=5$

Solving these equations, and substituting the values of A, B, C, and D in (A), we obtain

$$\frac{3x^3-14x^2+19x-5}{(x-2)^4}=\frac{3}{x-2}+\frac{4}{(x-2)^2}-\frac{1}{(x-2)^3}+\frac{1}{(x-2)^4}.$$

636. In a similar manner it may be shown that

1. $\frac{P}{(ax+b)^n}=\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\ldots+\frac{N}{(ax+b)^n}$

2. $\frac{P}{(x^2+ax+b)^n}=\frac{Ax+B}{x^2+ax+b}+$

$$\frac{Cx+D}{(x^2+ax+b)^2}+\ldots+\frac{Mx+N}{(x^2+ax+b)^n}$$

3. $\frac{P}{(ax^2+bx+c)^n}=\frac{Ax+B}{ax^2+bx+c}+$

$$\frac{Cx+D}{(ax^2+bx+c)^2}+\ldots+\frac{Mx+N}{(ax^2+bx+c)^n}$$

4. Any rational fraction whose denominator may be resolved into linear and quadratic factors may be decomposed by a combination of the above methods.

Illustration.—

To decompose $\dfrac{P}{(x-a)(x-b)^m(x^2+px+q)^n}$, put

$$\frac{P}{(x-a)(x-b)^m(x^2+px+q)^n}=\frac{A}{x-a}+\frac{B}{x-b}+\cdots$$

$$+\frac{M}{(x-b)^n}+\frac{Px+Q}{x^2+px+q}+\cdots+\frac{P'x+Q'}{(x^2+px+q)^n}$$

EXERCISE 95.

Decompose into partial fractions:

1. $\dfrac{5x+2}{x^2-4}$

2. $\dfrac{2x-5}{4x^2-1}=\dfrac{A}{2x+1}+\dfrac{B}{2x-1}$

3. $\dfrac{6x^2+10x+2}{x(x+1)(x+2)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x+2}$

4. $\dfrac{3x-2}{(x+1)^2}$

5. $\dfrac{x^2-x+1}{(x+3)^3}$

6. $\dfrac{3x^2-14x+25}{(x-3)(x^2-x+6)}$

7. $\dfrac{(a-b)x+2ab}{ab+(a-b)x-x^2}$

8. $\dfrac{p^2x+pq+q}{(px+q)^2}$

9. $\dfrac{5x^2+x+5}{x^4+x^2+1}$

10. $\dfrac{2}{x(1-4x^2)}$

11. $\dfrac{3x^3-8x^2+10}{(x-1)^4}$

12. $\dfrac{2x+1}{(x-1)(x^2+1)}$

13. $\dfrac{x^2-1}{x^2-5x+6}$

14. $\dfrac{2x-3}{(x-1)(x^2+1)^2}$

15. $\dfrac{x^2+x+1}{(x+1)(x^2+1)}$

16. $\dfrac{1}{x^8+x^7-x^4-x^3}$

17. $\dfrac{2x^2-11x+5}{(x-3)(x^2+2x-5)}$

18. $\dfrac{5x^3+6x^2+5x}{(x^2-1)(x+1)^3}$

19. $\dfrac{1}{x^8+x^7-x^4-x^3}$

20. $\dfrac{x^2+x+1}{(x+1)(x^2+1)}$

21. $\dfrac{1+x+x^2}{1-x-x^4+x^5}$

22. $\dfrac{x^2+4}{(x+1)^2(x-2)(x+3)}$

CHAPTER X.

COMPLEX NUMBERS.

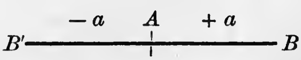
Graphical Treatment.

637. If a straight line of any assumed length be taken to represent the number *one,* then will a straight line twice as long represent the number *two,* one three times as long the number *three,* and so on. Thus, we see that any number may be represented by a line.

638. A line representing a number is called a *graph number,* or a *vector.* The point where a vector is supposed to begin is called the *origin,* and the point where it ends the *extremity.*

639. A vector is fully determined when both its length and direction are given. In a system of graphical representation of numbers, a vector running *rightward* from its origin represents a positive number and is *positive,* and one running *leftward* from its origin represents a negative number and is *negative.*

640. If the vector $+a$ be made to revolve about its origin A, through an angle of 180°, or π, it will become the vector $-a$, or will be multiplied by -1; and if the vector $-a$ be revolved about its origin A through an angle of 180°, or π, it will become the vector $+a$, or will be multiplied by -1.



Therefore,

1. Revolving a vector through an angle of 180°, or π, is equivalent to multiplying it by -1.

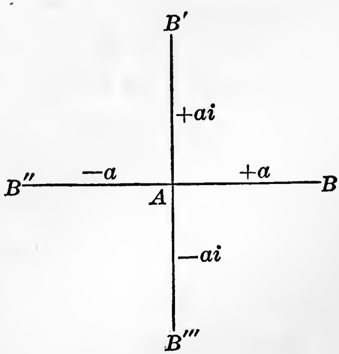
2. Revolving a vector about its origin through an angle of 360°, or 2π, is equivalent to multiplying it twice by -1, or once by $+1$, which does not affect its length or direction.

3. Since $-1 = \sqrt{-1} \times \sqrt{-1}$, revolving a vector about its origin through an angle of 90°, or $\frac{1}{2}\pi$, is equivalent to multiplying it by $\sqrt{-1}$, or i [v. P. I. 299].

641. Motion about the origin of a vector in the direction the hands of a clock go is considered negative, and counter-motion positive. The factor $+i$ may, therefore, be taken to represent circular motion about an origin through an angle of 90°, or $\frac{1}{2}\pi$, *counter-clock-wise*; and $-i$ through an angle of 90°, or $\frac{1}{2}\pi$, *clock-wise.*

642. Since $(+i) \times (+i) \times (+i)$, or $(+i)^3 = -i$, the factor symbol $-i$ may also denote circular motion about an origin through an angle of 270°, or $\frac{3}{2}\pi$, in a positive direction.

643. Since $(\pm i) \times (\pm i)$, or $(\pm i)^2 = -1$, the factor symbol -1 may denote circular motion about an origin through an angle of 180°, or π, in either direction.



Illustrations.—1. The vector $+a$ multiplied by $+i = AB$ revolved about A in the positive direction through an angle of $90° = AB'$.

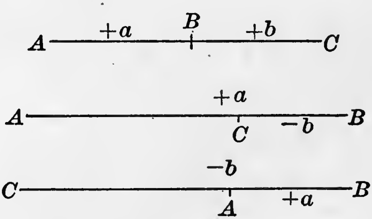
2. The vector $+a$ multiplied by $-1 = A\,B$ revolved about A in the positive or the negative direction through an angle of $180° = A\,B''$.

3. The vector $+a$ multiplied by $-i = A\,B$ revolved about A in the negative direction through an angle of $90°$, or in the positive direction through an angle of $270° = A\,B'''$.

4. In a similar manner it may be shown that $(-a) \times (+i) = A\,B'''$; $(-a) \times (-1) = A\,B$, and $(-a) \times (-i) = A\,B'$.

644. From what has been explained thus far it will be seen that, if a vector a units long running *rightward* from its origin represents $+a$, running *leftward* from its origin, it will represent $-a$; running *upward* from its origin, $+a\,i$; and running *downward*, $-a\,i$.

645. One vector is added to another by placing its origin to the extremity of the other and giving it the direction indicated by its factor symbol. The vector of their sum is the length and direction of the line joining the origin of the vector to which addition is made with the extremity of the vector added.



Illustrations.—

1. The vector $B\,C$ $(= +b)$ added to the vector $A\,B\,(= +a)$ gives the vector $A\,C$ $(= +(a+b))$.

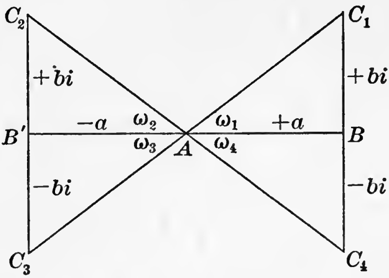
2. The vector $B\,C\,(= -b)$ added to the vector $A\,B$ $(= +a)$ gives the vector $A\,C\,\{= +(a-b)\}$, when $a > b$.

3. The vector $B\,C\,(= -b)$ added to the vector $A\,B$ $(= +a)$ gives the vector $A\,C\,\{= -(b-a)\}$, when $a < b$.

Note.—To subtract a vector is to add the vector with contrary sign.

Representation of Complex Numbers.

646. Let it be required to represent graphically the complex numbers $a+bi$, $a-bi$, $-a+bi$, and $-a-bi$.



1. $a+bi = +a+(+bi)$.

   $\therefore$ Vector $(a+bi) = AC_1$ [645].

2. $a-bi = +a+(-bi)$.

   $\therefore$ Vector $(a-bi) = AC_4$ [645].

3. $-a+bi = -a+(+bi)$.

   $\therefore$ Vector $(-a+bi) = AC_2$ [645].

4. $-a-bi = -a+(-bi)$.

   $\therefore$ Vector $(-a-bi) = AC_3$ [645].

647. The vectors of the two terms of a complex number are called the *components* of the vector of the number.

Thus, AB and BC_1 are the components of AC_1.

648. The length of the vector of a complex number is called the *modulus* of the vector, or the *modulus* of the complex number, and is equal to the square root of the sum of the squares of the lengths of the components.

Thus, mod. $(a+bi) =$ mod. $(a-bi) =$ mod. $(-a+bi)$ $=$ mod. $(-a-bi) = \sqrt{a^2+b^2}$.

649. The direction of the vector of a complex number is determined by the angle which the vector makes with its horizontal component, which angle is called the *amplitude* of the vector.

Thus, the amplitude of the vector $A C_1$ is the angle $C_1 A B$, which for distinction will be represented by ω_1; the amplitude of $A C_2$ is $C_2 A B'$, represented by ω_2; the amplitude of $A C_3$ is $C_3 A B'$, represented by ω_3; and the amplitude of $A C_4$ is $C_4 A B$, represented by ω_4.

650. It is equally accurate, and sometimes more convenient, to define the amplitude of a vector as the angle included between the vector and the vector $+a$, measured in a positive direction from the vector $+a$.

Thus, the amplitude of $A C_2$ is $C_2 A B$. The amplitude of $A C_3$ is the reflex angle $B A C_3$, described by revolving $A B$ about A in a positive direction until it coincides with $A C_3$. The amplitude of $A C_4$ is the reflex angle $B A C_4$.

651. A vector and its components may be constructed from its modulus and amplitude as follows:

1. Draw an indefinite horizontal line, and select some point in this line for the origin of the vector.

2. At the origin, deflect an angle with a protractor equal to the amplitude of the vector, and in the proper position.

3. Lay off from the origin, on the deflected side of the angle, from a scale of equal parts, the modulus of the vector. The vector is then determined.

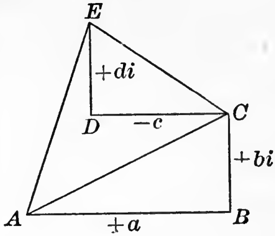
4. At the extremity of the vector let fall a perpendicular to the horizontal line. This perpendicular and the part of the horizontal line intercepted between the origin and the foot of the perpendicular will be the components of the vector. Their lengths may be obtained by actual measurement on the scale of equal parts.

Problems.

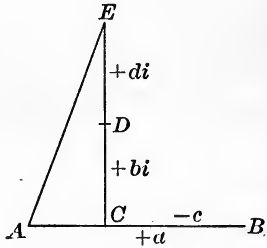
652. To find the sum of two complex numbers, graphically.

Let it be required to find the sum of $a+bi$ and $-c+di$.

1. Add $-c+di$ to $a+bi$, graphically.



Solution: Construct $a+bi$. Its vector is AC. At its extremity, C, construct $-c+di$. Its vector is CE. Join A with E. AE is the vector of the sum [645]. $\sqrt{(b+d)^2+(a-c)^2}$ is its modulus, and the angle EAB its amplitude.

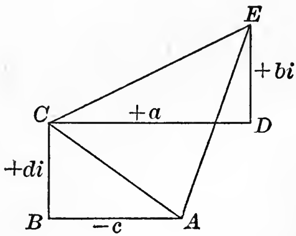


Proof: Add $-c+di$ to $a+bi$ algebraically, and construct the sum. Thus,

$$(a+bi)+(-c+di) = (a-c)+(b+d)i.$$

Construct $AB = +a$, $BC = -c$; then $AC = a-c$. At C erect $CE = (b+d)i$. Join A with E. The vector AE will be identical with AE in the preceding diagram; therefore, the solution is correct.

2. Add $a+bi$ to $-c+di$.



Solution: Construct $-c+di$. Its vector is AC. At its extremity, C, construct $+a+bi$. Its vector is CE. Join A with E. The vector AE will be identical with the vector AE in the first case, *which proves the commutative law of addition graphically as applied to complex numbers.*

Exercise.—Find graphically the sum of:

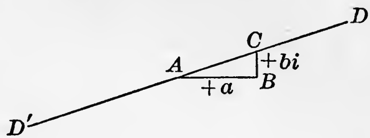
1. $a+bi$ and $c+di$
2. $a-bi$ and $c-di$
3. $a+bi$ and $c-di$
4. $-a-bi$ and $-c-di$
5. $a+bi$ and $a-bi$
6. $0+bi$ and $0-di$

653. To multiply a complex number by a rational number, graphically.

Let it be required to multiply $a+bi$ by $-c$.

Solution:

Construct $a+bi$. Prolong its vector AC to D, making $AD=C\times AC$. Revolve AD about A through 180°, then AD' is the vector of $-c$ times $a+bi$.



Exercise.—Construct the vectors of:

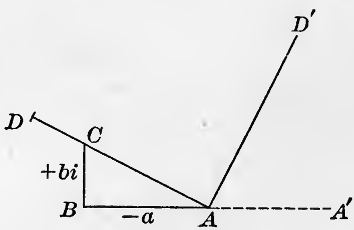
1. $(a+bi)\times c$
2. $(a-bi)\times c$
3. $(-a+bi)\times c$
4. $(-a-bi)\times c$
5. $(a-bi)\times(-c)$
6. $(-a+bi)\times(-c)$

---

654. To multiply a complex number by a simple imaginary number, graphically.

Let it be required to multiply $-a+bi$ by $-ci$.

Solution: Construct $-a+bi$. Prolong its vector AC to D, making $AD=c\times AC$. AD is the vector of c times $-a+bi$. Revolve AD about A through an angle of 90° clock-wise [641], or, which is the same, draw $AD'=AD$ and perpendicular to AD. AD' is the vector of $-ci$ times $-a+bi$. $D'AA'$ is its amplitude.

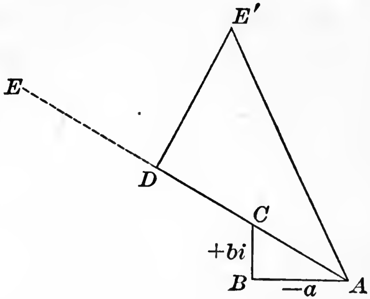


Exercise.—Construct the vectors of:

1. $(a+bi)\times ci$
2. $(-a-bi)\times ci$
3. $(a-bi)\times(-ci)$
4. $(a+bi)\times(-ci)$
5. $(-a+bi)\times ci$
6. $(a-bi)\times ci$
7. $(-a-bi)\times(-ci)$
8. $(0-bi)\times(-ci)$
9. $(0+bi)\times ci$
10. $(0+bi)\times(-ci)$

655. To multiply a complex number by a complex number, graphically.

Let it be required to multiply $-a+bi$ by $c-di$.



Solution: The vector of the sum of c times $-a+bi$ and $-di$ times $-a+bi$ is required. Construct $-a+bi$. Its vector is AC. Prolong AC to D, making $AD=c$ times AC; then AD is the vector of c times $-a+bi$. Prolong AD to E, making $DE=d$ times AC, and revolve DE about D through an angle of 90° clock-wise, then is DE' the vector of $-di$ times $-a+bi$ constructed at the extremity of AD. Join A and E'. AE' is the vector of the sum of c times $-a+bi$ and $-di$ times $-a+bi$.

Exercise.—Multiply graphically :

1. $a+bi$ by $c+di$
2. $a-bi$ by $c+di$
3. $a-bi$ by $c-di$
4. $-a+bi$ by $c+di$
5. $-a-bi$ by $-c+di$
6. $-a-bi$ by $-c-di$

General Principles.

656. *1. The sum, the difference, the product, and the quotient of two complex numbers are, in general, complex numbers.*

For, 1. $(a+bi)+(c+di)=(a+c)+(b+d)i.$

2. $(a+bi)-(c+di)=(a-c)+(b-d)i.$

3. $(a+bi)(c+di)=ac+bci+adi-bd$

$$=(ac-bd)+(bc+ad)i.$$

4. $$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

$$=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+\left(\frac{bc-ad}{c^2+d^2}\right)i.$$

657. *2. The sum and the product of two conjugate complex numbers are real.*

For, 1. $(a + bi) + (a - bi) = 2a.$

2. $(a + bi)(a - bi) \quad = a^2 + b^2.$

Scholium. *$a^2 + b^2$ is the square of the modulus of $\pm a + bi$ and of $\pm a - bi$, and is called the* **norm** *of each.* Therefore,

Cor.—*The product of two conjugate complex numbers equals their norm.*

---

658. *3. The norm of the product of two complex numbers equals the product of their norms.*

For, norm $(a + bi)(c + di)$

$$= \text{norm } \{(ac - bd) + (ad + bc)i\}$$
$$= (ac - bd)^2 + (ad + bc)^2 \quad [657, \text{Sch.}]$$
$$= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$$
$$= (a^2 + b^2) \times (c^2 + d^2)$$
$$= \text{norm } (a + bi) \text{ multiplied by norm } (c + di).$$

Cor.—*The modulus of the product of two complex numbers equals the product of their moduli.*

---

659. *4. If $a + bi = 0$, then $a = 0$ and $b = 0$.*

For, if $a + bi = 0$, $bi = -a$ and $-b^2 = a^2$; whence, $a^2 + b^2 = 0$, which is possible only when $a = 0$ and $b = 0$.

Cor.—*If a complex number vanishes, its modulus vanishes; and conversely, if the modulus vanishes, the complex number vanishes.*

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660. *5. If $a + bi = c + di$, then $a = c$ and $b = d$.*

For, if $a + bi = c + di$, $(a - c) + (b - d)i = 0$; whence, $a - c = 0$ and $b - d = 0$ [P. 4],

and $\qquad a = c$ and $b = d$.

661. Problem. To find the value of e^{x+yi}.

Solution: Assuming the exponential law of multiplication [275, P.], and Formula (G), Art. 599, sufficiently general to include imaginary exponents; then

$$e^{x+yi} = e^x \times e^{yi} = e^x \left\{1 + yi + \frac{y^2 i^2}{2!} + \frac{y^3 i^3}{3!} + \frac{y^4 i^4}{4!} + \text{etc.}\right\}$$

$$= e^x \left\{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \text{etc.}\right) + \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \text{etc.}\right) i\right\} \quad \text{(A)}$$

---

662. The expression $1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \text{etc.}$ is called *cosine y*, and is written *cos. y*.

The expression $y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \text{etc.}$ is called *sine y*, and is written *sin. y*. Therefore,

$$e^{yi} = \cos. y + i \sin. y. \quad \text{(B)}$$

---

663. Resume the equations

$$\cos. y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \text{etc.} \quad (1)$$

$$\sin. y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \text{etc.} \quad (2)$$

$$e^{yi} = \cos. y + i \sin. y. \quad (3)$$

Put $-y$ for y in (1), (2), and (3), then

$$\cos. (-y) = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \text{etc.} = \cos. y \quad (4)$$

$$\sin. (-y) = -y + \frac{y^3}{3!} - \frac{y^5}{5!} + \frac{y^7}{7!} - \text{etc.} = -\sin. y \quad (5)$$

$$e^{-yi} = \cos. (-y) + i \sin. (-y) = \cos. y - i \sin. y \quad (6)$$

Multiply (3) by (6),

$$1 = (\cos. y)^2 - i^2 (\sin. y)^2 = \cos.^2 y + \sin.^2 y \quad \text{(C)}$$

Note. $\cos.^2 y$ denotes $(\cos. y)^2$ and $\sin.^2 y$ denotes $(\sin. y)^2$.

Cor. $\sin. y = \sqrt{1 - \cos.^2 y}$;
$\cos. y = \sqrt{1 - \sin.^2 y}$.

664. Put ny for y in (B); then

$$e^{nyi} = \cos. ny + i \sin. ny.$$

Raise (B) to the nth power; then

$$e^{nyi} = (\cos. y + i \sin. y)^n.$$

$$\therefore \cos. ny + i \sin. ny = (\cos. y + i \sin. y)^n \qquad \text{(D)}$$

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665. Let $n = 2$ in (D); then

$$\cos. 2y + i \sin. 2y = (\cos. y + i \sin. y)^2$$

$$= \cos.^2 y - \sin.^2 y + 2 (\sin. y \cos. y) i.$$

$$\therefore \cos. 2y = \cos.^2 y - \sin.^2 y \ [660] \qquad \text{(E)}$$

$$\sin. 2y = 2 \sin. y \cos. y \ [660] \qquad \text{(F)}$$

---

666. Put $x + y$ for y in (B); then

$$e^{(x+y)i} = e^{xi} . e^{yi} = \cos. (x + y) + i \sin. (x + y).$$

But $e^{xi} . y^{xi} = (\cos. x + i \sin. x)(\cos. y - i \sin. y)$ (B)

$$= \cos. x \cos. y - \sin. x \sin. y + i (\sin. x \cos. y + \cos. x \sin. y)$$

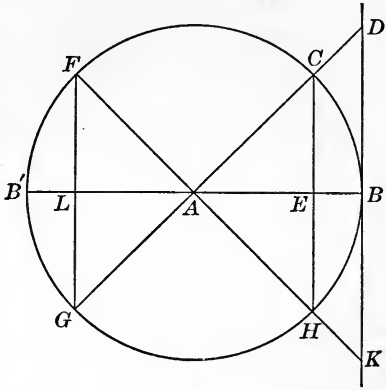
$$\therefore \cos. (x + y) = \cos. x \cos. y - \sin. x \sin. y \ [660] \qquad \text{(G)}$$

$$\sin. (x + y) = \sin. x \cos. y + \cos. x \sin. y \ [660] \qquad \text{(H)}$$

Graphical Representation of sin. y and cos. y.

667. It is evident that all the conditions expressed in the equation $\sin.^2 y + \cos.^2 y = 1$ will be satisfied by assuming 1 as the modulus of a vector whose amplitude is the variable angle y and whose components are sin. y and cos. y. But, to make this expression conform to the numerical values of sin. y and cos. y as expressed in Art. 662, y must be taken to represent the number of vector units in the arc which measures the amplitude, and sin. y as the vertical and cos. y as the horizontal component of the vector; for in this way only would $\sin. y = 0$ and $\cos. y = 1$ when $y = 0$.

668. The ratio of sin. y to cos. y is called *tangent* y, and is written tan. y. It may be expressed graphically as follows:



Let $BC = y$, CE = sin. y, and AE = cos. y. At B draw an indefinite tangent to the circle. Prolong the vector AC until it meets the indefinite tangent at D. BD will be tan. y. For, from the similar triangles DAB and CAE we have $BD : CE :: AB : AE$; or,

$$BD : \text{sin. } y :: 1 : \text{cos. } y\text{; whence,}$$

$$BD = \frac{\text{sin. } y}{\text{cos. } y} = \text{tan. } y.$$

If $BF = y$, FL = sin. y, and AL = cos. y. The triangles AFL and BAK will be similar, and BK = tan. y.

If $BFG = y$, then GL = sin. y, and AL = cos. y. The triangles GAL and DAB will be similar, and DB = tan. y.

If $BFH = y$, then HE = sin. y, and AE = cos. y. The triangles KAB and HAE will be similar, and KB = tan. y.

***Scholium.**—So long as $y < \frac{1}{2}\pi$ [90°], sin. y and cos. y are positive; hence, tan. y (BD) is positive. When $y > \frac{1}{2}\pi$ but $< \pi$, sin. y is positive and cos. y negative; hence, tan. y (BK) is negative. When $y > \pi$ but $< \frac{3}{2}\pi$, sin. y is negative and cos. y negative; hence, tan y (DB) is positive. When $y > \frac{3}{2}\pi$ but $< 2\pi$, sin. y is negative and cos. y is positive; hence, tan. y (KB) is negative.*

CHAPTER XI.

THEORY OF FUNCTIONS.

Definitions.

669. A quantity whose value changes, or is supposed to change, according to a definable law, is a *definite variable,* or simply *a variable.*

670. A variable whose law of change is not dependent upon that of another variable is an *independent variable.*

671. A variable whose law of change is dependent upon that of another variable is a *dependent variable,* and is called a *function* of that variable.

Hence it is, that any expression containing a variable is a function of that variable [562].

672. Any law of change may be imposed upon an independent variable; but, when it is once imposed, the law of change of any function of the variable becomes determined.

673. The simplest treatment of functions of a single variable is that in which the variable is supposed to increase or decrease uniformly by equal increments, finite or infinitely small.

674. A function is said to be *continuous* so long as an infinitely small change in the independent variable produces an infinitely small change in the function, and *dis-*

continuous when an infinitely small change in the independent variable produces a finite or infinitely great change in the function.

Illustration.—Thus, the function $\frac{1}{1-x}$ assumes all values between $+1$ and $+\infty$ as x assumes all values between 0 and $+1$, and is, therefore, *continuous* from 0 to $+\infty$; but, as the value of x continues to increase from a quantity infinitesimally less than $+1$ to a quantity infinitesimally greater than $+1$, or takes an infinitely small step across $+1$, the function takes a leap through the whole gamut of numbers from $+\infty$ to $-\infty$, and is, therefore, *discontinuous* between these values.

675. So long as a function increases in value as the independent variable increases in value, and hence, too, decreases in value as the independent variable decreases in value, it is an *increasing function;* but when it decreases in value as the independent variable increases in value, and, hence, increases in value as the independent variable decreases in value, it is a *decreasing function.*

Illustration.—Let $y = f(x) = x^2 - 4x + 3$.

Assign values to x and calculate the corresponding values of y by synthetic division [106], you will obtain results as follows:

For $x = -3, -2, -1, \quad 0, +1, +2, +3, +4, +5$

$y = +24, +15, +8, +3, \quad 0, -1, \quad 0, +3, +8$

Here y decreases from $+24$ to -1 as x increases from -3 to $+2$, and is, therefore, a decreasing function between these values of x; and it increases from -1 to $+8$ as x increases from $+2$ to $+5$, and is, therefore, an increasing function between these values of x.

676. The *maximum value* of a function is the value at which the function changes from an increasing to a decreasing function.

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677. The *minimum value* of a function is the value at which the function changes from a decreasing to an increasing function.

678. The maxima and minima values of a function are often called the *turning values* of the function.

679. A turning value of a function may be a finite constant, zero, or infinity.

Illustrations.—1. Take $y = f(x) = 3 + (4 - x)^2$.

As x increases from 0 to 4, y decreases from 19 to 3; and as x continues to increase from 4 to ∞, y increases from 3 to ∞. Therefore, 3 is a turning value (a minimum) of y.

2. Take $y = (a - x)^2$.

As x increases from 0 to a, y decreases from a^2 to 0; and as x continues to increase from a to ∞, y increases from 0 to ∞. Therefore, 0 is a turning value of y.

3. Take $y = f(x) = \dfrac{1}{(1 - x)^2}$.

As x increases in value from 0 to $+1$, $(1 - x)^2$ decreases from 1 to 0, and y increases from 1 to ∞; and as x continues to increase from 1 to ∞, $(1 - x)^2$ increases from 0 to ∞, and y decreases from ∞ to 0. Therefore, ∞ is a turning value (a maximum) of y.

680. The *limit* of a function is the value of the function at which it ceases to be continuous.

Note.—Notice the distinction between the meaning of the word *limit* as here used and as used in Art. 398. In the latter sense, ∞ would be the limit of y in illustration 3, Art. 679, instead of a maximum.

681. The limit of a function may be a finite constant, zero, or infinity.

Illustrations.—1. Take $y = f(x) = 2 - \dfrac{2}{2^x}$.

As x increases from 0 to ∞, $\dfrac{2}{2^x}$ decreases from 2 to 0, and y increases from 0 to 2; and, as x can not be supposed greater than ∞, y can not become greater than 2, neither can y begin to decrease at 2. Therefore, 2 is the limit of y.

2. Take $y = f(x) = x^2(1 + x^{\frac{1}{2}})$.

As x decreases from 1 to 0, y decreases from 2 to 0; and as x can not be taken less than 0 (negative) without making y imaginary, y can not become less than 0, neither can y change from an increasing to a decreasing function at 0. Therefore, 0 is the limit of y.

3. We have already seen [674] that $y = f(x) = \frac{1}{1-x}$ increases from 1 to ∞ as x increases from 0 to $+1$, and thereafter becomes discontinuous. Therefore, ∞ is the limit of y.

682. A function may have two sets of values approaching the same or different limits for the same set of values of the independent variable.

Illustrations.—1. Take $y^2 = f(x) = 16 - x^2$;

$$\text{then } y = \pm\sqrt{16 - x^2}.$$

Here are two values of y for each value of x, numerically equal but opposed in sign. As x increases from 0 to 4 one value of y decreases from 4 to 0, and the other increases from -4 to 0. If x becomes infinitesimally greater than 4 both values of y become imaginary. Therefore, 0 is the limit of both values of y.

2. Take $y^2 = f(x) = 4x$;

$$\text{then } y = \pm 2\sqrt{x}.$$

Here, again, are two values of y for each value of x. As x increases from 0 to $+\infty$, one value of y increases from 0 to $+\infty$ and the other decreases from 0 to $-\infty$; and as x can not be supposed greater than $+\infty$, $+\infty$ is the limit of one value of y and $-\infty$ the limit of the other value.

683. The limit of an increasing function is a *superior* or maximum limit; that of a decreasing function an *inferior* or minimum limit.

Graphical Representation of Functions of a Single Variable.

684. Every function of a single variable may be approximately represented by a line, straight or curved, called the *graph* of the function.

Method.—Let $y = f(x)$. Assign successive values to x and calculate the corresponding values of y. Construct two indefinite straight lines intersecting each other at right angles, one running right and left and the other up and down from their intersection. These are the *axes of reference.* The first is the x-axis and the second the y-axis, and their intersection the *origin.* Regard distance rightward from the y-axis *positive,* and distance leftward *negative;* distance upward from the x-axis *positive,* and distance downward *negative.*

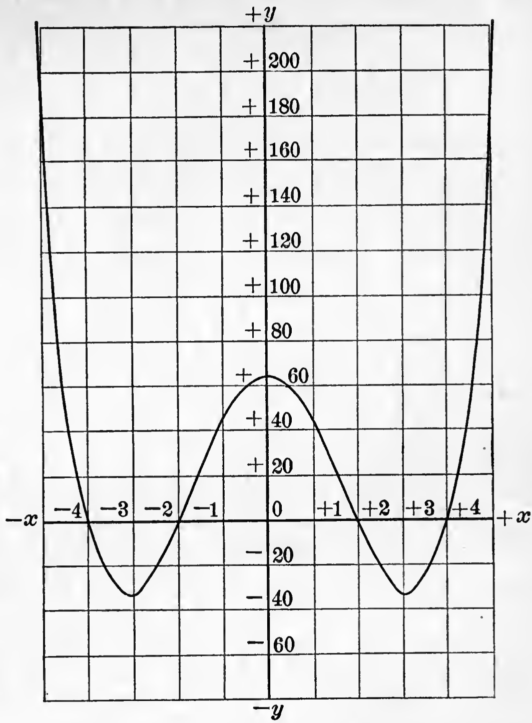
Assume a fixed length as a unit of scale, and lay off on the x-axis from the origin the successive values of x based on this scale, and at the extremity of each x value, and on a line parallel to the y-axis, lay off the corresponding values of y. Thus will be located a series of successive points; draw a continuous line through these points; it will be the graph of the function, and its accuracy will depend upon the nearness to each other of the successive values of x taken, the relation of the unit of scale to that of x and y, and the correctness of the instruments used in plotting.

Illustrations.—1. Take $y = f(x) = x^4 - 20x^2 + 64$.

Assign special values to x and calculate the corresponding values of y by synthetic division [106]. You will readily derive the following table of values and make the following plot:

x	y
0	64
1	44
2	0
3	−35
4	0
5	189
∞	∞
−1	44
−2	0
−3	−35
−4	0
−5	189
−∞	∞

PLOT.



Observations.—1. The curved line on the plot is the graph of the function.

2. The unit of scale used in plotting the graph represents 20 units of y to one unit of x.

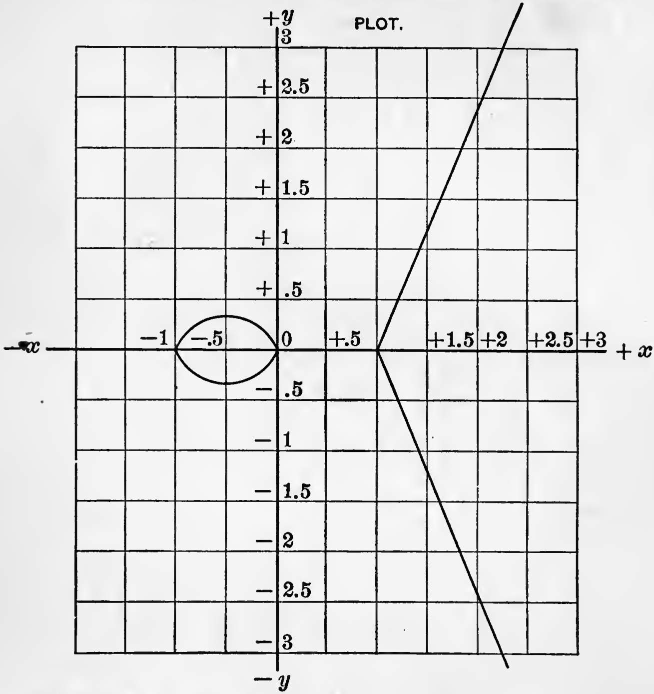
3. The graph exhibits *three* turning values of the function; two *minima* at the points ($x = 3$, $y = -35$) and (-3, -35), and one *maximum* at the point (0, 64).

4. When $x = +\infty$, $y = +\infty$, and when $x = -\infty$, $y = +\infty$. The graph, like the function, descends from ($-\infty$, $+\infty$) to (-3, -35), then ascends from (-3, -35) to (0, 64), then descends from (0, 64) to (3, -35), then again ascends from (3, -35) to ($+\infty$, $+\infty$). It is a continuous graph from beginning to end.

5. At $x = 2$, 4, -2, and -4, the graph crosses the x-axis, exhibiting the fact that for these values of x, $y = f(x) = x^4 - 20x^2 + 64 = 0$. The values of x that render $f(x) = 0$ are, however, the roots of the equation $f(x) = 0$; therefore, the values of the roots of $f(x) = 0$ may be approximately found even if incommensurable, by plotting $f(x) = y$ and determining with a scale of equal parts where the graph crosses the x-axis.

2. Plot $y = \pm\sqrt{x^3 - x} = \pm\sqrt{x(x+1)(x-1)}$.

The following table of values may readily be obtained :



Observations.—1. The graph consists of two branches between the points (− 1, 0) and (0, 0), symmetrical with respect to the x-axis. These branches are *confluent* at the points mentioned.

2. The graph is discontinuous for all values of x antecedent to − 1, counting from $x = -\infty$, and also for all values of x between 0 and + 1.

3. The graph again consists of two branches, symmetrical with respect to the x-axis, for all positive values of x greater than + 1.

4. The limits of the first branch are at (−1, 0) and (0, 0); the limits of the second branch are also at (−1, 0) and (0, 0); the limits of the third branch are at (+ 1, 0) and (+ ∞, + ∞); and the limits of the fourth branch are at (+ 1, 0) and (+ ∞, − ∞).

x	y
0	0
$+ < +1$	$\sqrt{-}$
1	0
1·5	± 1·4
2	± 2·4
2·5	± 3·6
3	± 4·9
− 1	0
$- < -1$	$\sqrt{-}$
− ·2	± ·44
− ·4	± ·58
− ·5	± ·61
− ·6	± ·62
− ·8	± ·53

5. The first branch has a turning-point (maximum) somewhere between (— ·5, + ·61) and (— ·8, + ·53). The second branch also has a turning-point (minimum) between (— ·5, — ·61) and (— ·8, — ·53).

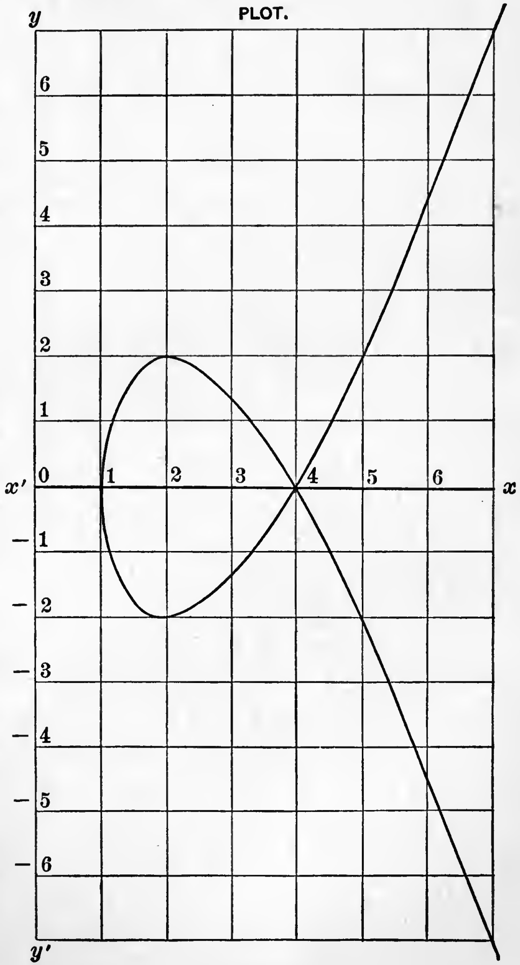
6. The branches of the graph meet the x-axis when $x = 0$, $+1$, and -1. These values are, therefore, the roots of $f(x) = x^3 - x = 0$.

3. Plot $y^2 = x^3 - 9x^2 + 24x - 16$.

or $y = \pm\sqrt{x^3 - 9x^2 + 24x - 16}$.

x	y
< 1	$\sqrt{-}$
1	0
1·5	± 1·76
2	± 2
3	± 1·41
4	0
5	± 2
6	± 4·47
7	± 7·34
+ ∞	± ∞

PLOT.



Questions.—

1. How many branches has this graph?

2. How many turning-points? Locate and name them.

3. What are the roots of the function $x^3 - 9x^2 + 24x - 16 = 0$?

4. Between what limits are the branches of the graph continuous?

5. Where is the function an increasing function and where a decreasing?

EXERCISE 96.

Plot and discuss the following functions:

(Use paper ruled in squares, called plotting-paper.)

1. $y = 4x + 6$
2. $y = 8x$
3. $y = 81x^{-3}$
4. $y^2 = 4x$
5. $y^2 = 16 - x^2$
6. $y^2 = x^3$
7. $y^2 = x^2(x - 1)$
8. $y = x^3 - 8x^2 + 20x - 10$
9. $y = 3x + 18x^2 - 2x^3$
10. $y^2 = x^3 + 3x^2 - 5x - 20$

Differentials and Derivatives of Functions.

Definitions.

685. The limit of the ratio of the increment of a function to the increment of the independent variable producing the increment of the function, when the limit of the increment of the independent variable is zero, is called the *derivative* of the function.

Thus, if we let $y = f(x)$, and represent the increment of x by Δx and the corresponding increment of y by Δy, then will lim. $\left(\dfrac{\Delta y}{\Delta x}\right)_{\Delta x = 0}$ = the derivative of the function.

686. The limit of the increment of the independent variable is called the *differential of the independent variable,* and is represented by dx; and the limit of the increment of the function is called the *differential of the function,* and is represented by dy.

Therefore, lim. $\left(\dfrac{\Delta y}{\Delta x}\right)_{\Delta x = 0} = \dfrac{dy}{dx}$.

Notice. dy and dx represent single quantities (differentials) and are not equivalent to $d \times y$ and $d \times x$.

Illustration.—

Let $y = x^2$ (1)

then, $y + \triangle y = (x + \triangle x)^2 = x^2 + 2x(\triangle x) + (\triangle x)^2$ (2)

Subtract (1) from (2), $\triangle y = 2x(\triangle x) + (\triangle x)^2$ (3)

Divide by $\triangle x$, $\frac{\triangle y}{\triangle x} = 2x + \triangle x$ (4)

$\therefore$ Lim. $\left(\frac{\triangle y}{\triangle x}\right)_{\triangle x = 0} =$ lim. $(2x + \triangle x)_{\triangle x = 0}$ [401, P.] (5)

$\therefore$ $\frac{dy}{dx} = 2x$ = the derivative of x^2,

and $dy = 2x\,dx$ = the differential of x^2.

687. *The differential of a function equals the derivative of the function multiplied by the differential of the independent variable.*

688. *The derivative of a function equals the differential of the function divided by the differential of the independent variable.*

689. *If the differential of a function, and hence, too, the derivative of a function, is positive, the function is an increasing one; if negative, a decreasing one.*

Principles.

690. Let $y = f(x) = x^n$, (1)

then, $y + \triangle y = (x + \triangle x)^n$

$= x^n + n x^{n-1} \,.\, \triangle x + A \,.\, (\triangle x)^2$ (2)

in which $A =$

$\frac{n(n-1)}{\lfloor 2} x^{n-2} + \frac{n(n-1)(n-2)}{\lfloor 3} x^{n-3} \,.\, \triangle x +$ etc. [593].

Subtracting (1) from (2), $\triangle y = n x^{n-1} \,.\, \triangle x + B \,.\, (\triangle x)^2$ (3)

Dividing by $\triangle x$, $\frac{\triangle y}{\triangle x} = n x^{n-1} + B \,.\, \triangle x$ (4)

$\therefore$ Lim. $\left(\frac{\triangle y}{\triangle x}\right)_{\triangle x = 0} =$ lim. $(n x^{n-1} + B \,.\, \triangle x)_{\triangle x = 0}$

$\therefore$ $\frac{dy}{dx} = n x^{n-1}$, since lim. B = a finite constant [582], and lim. $\triangle x = 0$. (5)

$\therefore$ $dy = n x^{n-1} dx$. Therefore,

Prin. 1.—*The differential of a variable with a constant exponent equals the continued product of the exponent, the variable with its exponent diminished by unity, and the differential of the independent variable.*

Illustrations.—

1. $d(x^4) = 4x^3 dx$ 2. $d(x)^{-\frac{2}{3}} = -\frac{2}{3}x^{-\frac{5}{3}}dx$

3. $d(a+bx)^p = p(a+bx)^{p-1}d(a+bx)$

---

691. Let $y = ax$ (1)

then, $y + \Delta y = a(x + \Delta x) = ax + a(\Delta x)$ (2)

Subtracting, $\Delta y = a(\Delta x)$ (3)

Dividing, $\frac{\Delta y}{\Delta x} = a$

$\therefore$ Lim. $\frac{\Delta y}{\Delta x} = a$

whence, $\frac{dy}{dx} = a$, and $dy = a\,dx$. Therefore,

Prin. 2.—*The differential of a constant times a variable equals the constant times the differential of the variable.*

Thus, $d(3x^5) = 3 \cdot d(x^5) = 3 \times 5x^4 dx = 15x^4 dx$.

---

692. Let $y = ax + b$ (1)

then, $y + \Delta y = a(x + \Delta x) + b =$
$ax + a(\Delta x) + b$ (2)

Subtracting, $\Delta y = a(\Delta x)$

Dividing, $\frac{\Delta y}{\Delta x} = a$; whence, $\frac{dy}{dx} = a$,

and $dy = a\,dx$. Therefore,

Prin. 3.—*The differential of a constant term is zero.*

---

693. Let $v = f(x)$, $w = f'(x)$, and $z = f''(x)$; and let $y = v + w - z$ (1)

then, $y + \Delta y = v + \Delta v + w + \Delta w - (z + \Delta z)$
$= v + w - z + \Delta v + \Delta w - \Delta z$ (2)

Subtracting, $\Delta y = \Delta v + \Delta w - \Delta z$ (3)

Dividing by Δx, $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta v}{\Delta x} + \dfrac{\Delta w}{\Delta x} - \dfrac{\Delta z}{\Delta x}$ (4)

whence, $\lim. \dfrac{\Delta y}{\Delta x} = \lim. \dfrac{\Delta v}{\Delta x} + \lim. \dfrac{\Delta w}{\Delta x} - \lim. \dfrac{\Delta z}{\Delta x}$ (5)

[413, P.].

$\therefore$ $\dfrac{dy}{dx} = \dfrac{dv}{dx} + \dfrac{dw}{dx} - \dfrac{dz}{dx}$ (6)

whence, $dy = dv + dw - dz$. Therefore,

***Prin. 4.**—The differential of a polynomial whose terms are functions of the same independent variable equals the algebraic sum of the differentials of its terms.*

Illustration. $d(x^3 + 3x^2 - 2x + 5)$

$= d(x^3) + d(3x^2) + d(-2x) + d(5)$

$= 3x^2 dx + 6x dx - 2 dx = (3x^2 + 6x - 2) dx.$

---

694. Let $v = f(x)$ and $z = f'(x)$,

and $y = vz$ (1)

then, $y + \Delta y = (v + \Delta v)(z + \Delta z)$

$= vz + v \,.\, \Delta z + z \,.\, \Delta v + \Delta v \,.\, \Delta z$ (2)

Subtracting, $\Delta y = v \,.\, \Delta z + z \,.\, \Delta v + \Delta v \,.\, \Delta z$ (3)

Dividing by Δx, $\dfrac{\Delta y}{\Delta x} = v \,.\, \dfrac{\Delta z}{\Delta x} + z \,.\, \dfrac{\Delta v}{\Delta x} + \dfrac{\Delta v}{\Delta x} \,.\, \Delta z$

$\therefore$ $\text{Lim.} \dfrac{\Delta y}{\Delta x} = \lim. \left(v \,.\, \dfrac{\Delta z}{\Delta x}\right) + \lim. \left(z \,.\, \dfrac{\Delta v}{\Delta x}\right)$

$+ \lim. \left(\dfrac{\Delta v}{\Delta x} \,.\, \Delta z\right)$

$\therefore$ $\dfrac{dy}{dx} = v \,.\, \dfrac{dz}{dx} + z \,.\, \dfrac{dv}{dx} + 0$ [413, P.]

whence, $dy = v\,dz + z\,dv$. Therefore,

***Prin. 5.**—The differential of the product of two continuous functions of the same independent variable equals the sum of the products obtained by multiplying each function by the differential of the other.*

Illustration.—

$d(-3x^3 \times 5x^{\frac{2}{3}}) = -3x^3 \times d(5x^{\frac{2}{3}}) + 5x^{\frac{2}{3}} \times d(-3x^3)$

$= \{-3x^3 \times \frac{2}{3} \times 5x^{-\frac{1}{3}} + 5x^{\frac{2}{3}} \times 3 \times (-3x^2)\}\, dx$

$= -55\, x^{\frac{8}{3}}\, dx.$

Cor. $d(vwz) = v \,.\, d(wz) + wz \,.\, dv$ [694, P.]
$= vw\,dz + vz\,dw + wz\,dv.$ [694, P.]; etc.

---

695. Let $v = f(x)$, and $z = f'(x)$; and

$$y = \frac{v}{z} = vz^{-1};$$

then $dy = v \,.\, d(z^{-1}) + z^{-1}dv$ [694, P.]

$$= -vz^{-2}dz + z^{-1}dv$$

$$= \frac{dv}{z} - \frac{v\,dz}{z^2} = \frac{z\,dv - v\,dz}{z^2}.$$

Therefore,

Prin. 6.—*The differential of a fraction whose terms are continuous functions of the same independent variable equals the denominator into the differential of the numerator minus the numerator into the differential of the denominator, all divided by the square of the denominator.*

Thus, $d\left(\frac{x^2}{y^3}\right) = \frac{y^3 \,.\, d(x^2) - x^2 \,.\, d(y^3)}{y^6}$

$$= \frac{2xy^3dx - 3x^2y^2dy}{y^6}.$$

---

696. Let $y = \log_e x$

then $y + \Delta y = \log_e(x + \Delta x) = \log_e x\left(1 + \frac{\Delta x}{x}\right)$

$= \log_e x + \log_e\left(1 + \frac{\Delta x}{x}\right)$ [467, P. 2]

$= \log_e x + \frac{\Delta x}{x} - \frac{1}{2} \,.\, \frac{(\Delta x)^2}{x^2} + \frac{1}{3} \,.\, \frac{(\Delta x)^3}{x^3} - \text{etc.}$ [601, C.]

$$\Delta y = \frac{\Delta x}{x} - \frac{1}{2} \,.\, \frac{(\Delta x)^2}{x^2} + \frac{1}{3} \,.\, \frac{(\Delta x)^3}{x^3} - \text{etc.}$$

$\frac{\Delta y}{\Delta x} = \frac{1}{x} + B \,.\, \Delta x$; in which $B = -\frac{1}{2} \,.\, \frac{1}{x^2} + \frac{1}{3} \,.\, \frac{\Delta x}{x^3} - \text{etc.}$;
which for very small values of Δx is convergent [582].

$$\therefore \text{ Lim.}\left(\frac{\Delta y}{\Delta x}\right)_{\Delta x = 0} = \text{lim.}\left(\frac{1}{x} + B \,.\, \Delta x\right)_{\Delta x = 0}$$

whence, $\frac{dy}{dx} = \frac{1}{x}$; and $dy = \frac{dx}{x}$. Therefore,

Prin. 7.—*The differential of the $\log_e$ of a quantity equals the differential of the quantity divided by the quantity itself.*

Cor.—*Since* $\log_{10} x = m \log_e x$ [607, P.]

$$d\,.\,(\log_{10} x) = \frac{m\,dx}{x}.$$

Illustration.—

Thus, $d\left(\log_e \frac{x+a}{x}\right) = d\left(\frac{x+a}{x}\right) \div \frac{x+a}{x}$

$$= \frac{x}{x+a}\left\{\frac{x\,dx + (x+a)\,dx}{x^2}\right\} = \frac{1}{x+a}\left(\frac{2x+a}{x}\right)dx.$$

---

697. Let $y = a^x$, in which a is a constant.

then, $\log. y = x \log. a$ [468, P.]

$$d(\log. y) = \log. a\,.\,dx$$

or, $$\frac{dy}{a^x} = \log. a\,.\,dx$$

whence, $dy = a^x \log. a\,.\,dx$. Therefore,

Prin. 8.—*The differential of a constant with a variable exponent equals the continued product of the original quantity, the logarithm of the constant, and the differential of the variable exponent.*

Thus, $d\,(a+b)^{\sqrt{x}} = d\,(a+b)^{x^{\frac{1}{2}}} = (a+b)^{x^{\frac{1}{2}}} \log. (a+b) \times d\,(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}(a+b)^{x^{\frac{1}{2}}} \log. (a+b)\,dx.$

698. Problem. Find the differential of x^x.

Let $y = x^x$

then, $\log. y = x \log. x$

and $d(\log. y) = x\,.\,d(\log. x) + \log. x\,.\,dx$

or, $$\frac{dy}{x^x} = x\,.\,\frac{dx}{x} + \log. x\,.\,dx$$

whence, $dy = x^x(1 + \log. x)\,dx.$

EXERCISE 97.

Differentiate:

1. $y = 5ax^3 - 3bx^2 + 2cx - d$
2. $y = 5x^3z^2 + z$
3. $y = x^3 + 3x^2 + 4x + 5$
4. $y = (a + bx)^{m^2}$
5. $y = \dfrac{x^3 + 3x^2 + 2}{2x}$
6. $y = \sqrt{\dfrac{ax + b}{c}}$
7. $y = \sqrt[3]{5x + 6}$
8. $y^2 = 2px$
9. $y^3 = 3ax^2$
10. $a^2y^2 + b^2x^2 = a^2b^2$
11. $y = ax^{\frac{3}{2}} + bx^{\frac{1}{2}} + c$
12. $y = \dfrac{2x^4}{(a + x)^3}$
13. $y = \dfrac{x}{\sqrt{x + a}}$
14. $y = \dfrac{a}{(b^2 + x^2)^3}$
15. $y = (x + a)^2(x + b)^3$
16. $y = (x + a)^n(x - b)^p$
17. $y = \log_e(x^2 + x)$
18. $y = (\log_e x)^3$
19. $y = 3x^{3x}$
20. $y = ax^x$
21. $y = x^4(a + x^2)^{-\frac{3}{2}}$
22. $y = \log_e(a + x)^n$
23. $y = c^x \div d^x$
24. $y = \dfrac{\sqrt{1 + x}}{\sqrt{1 - x}}$
25. $y = a^{\log. x}$

Applications.

EXERCISE 98.

1. At what rate is the area of a circle increasing when the radius is 6 inches and is increasing at the rate of 3 inches per second?

Solution: Let $y =$ the area, and $x =$ the radius; then,

$$y = \pi x^2$$

and $dy = 2\pi x\,dx.$

This denotes that at any instant the rate of increase of the area is $2\pi x$ times as great as the rate of increase of the radius at the same instant. But when the radius is 6 inches, it increases at the rate of 3 inches per second; or, when $x = 6$ inches, $dx = 3$ inches.

$\therefore dy = 2\pi \times 6$ inches $\times$ 3 inches $= 36\pi$ square inches; that is, the area is increasing at such a rate that, if kept uniform for one second, the increase would amount to 36π square inches.

2. At what rate is the area of a square increasing when the side of the square is 4 inches and is increasing at the rate of 2 inches per second?

3. The volume of a sphere increases how many times as fast as its radius? When its radius is 6 inches and increases at the rate of 1 inch per second, at what rate is the volume increasing?

4. At what rate is the diagonal of a square increasing when the side of the square is 8 inches and is increasing at the rate of 2 inches per second?

5. The radius of a circle is 4 inches and its circumference is increasing at the rate of 2π inches per second. At what rate is the radius increasing at the same instant?

6. A boy approaches a tree 90 feet high standing on a level road at the rate of 3 miles an hour. At what rate is he approaching the top of the tree when he is 220 feet from the base?

7. The diagonal of a cube is increasing at the rate of 36 inches per second, when the side of the cube is 5 inches long. At what rate is the side increasing at the same time?

8. If x increases at the rate of ·5 per instant, at what rate is $\log_{10} x$ increasing when $x = 42$?

9. The $\log_{10} 42 = 1·62325$. What, then, would be the $\log_{10} 42·5$, if the increase were uniform? How does the result compare with $\log_{10} 42·5$ as found in the table?

Successive Derivatives.

699. If the derivative of $f(x)$ be treated as a new function of x $[f_1(x)]$, there may be found from it a second derivative of $f(x)$ $[f_2(x)]$ in the same way as $f_1(x)$ was derived from $f(x)$, and so on, until a derivative is found that is independent of x $[f(x_0)]$.

Illustration.—

Let $f(x) = x^5 + 4x^4 - 3x^3 + 2x^2 - 5x + 7$; then,

$$f_1(x) = 5x^4 + 16x^3 - 9x^2 + 4x - 5 \quad [693, \text{P. } 688]$$
$$f_2(x) = 20x^3 + 48x^2 - 18x + 4$$
$$f_3(x) = 60x^2 + 96x - 18$$
$$f_4(x) = 120x + 96$$
$$f(x_0) = 120$$

EXERCISE 99.

Find the first derivative of:

1. $x^3 - 4x^2 + 7x + 2$
2. $(x+2)^3(x-2)^4$
3. $x(x+2) + x^2(x+3)$
4. $(a+x)^5(a-x)^3$
5. $(a+x^5)(a-x^3)$
6. $(a+x)^5 \div (a-x)^5$

Factorization of Polynomials containing Equal Factors.

700. Let $f(x) = (x+a_1)(x+a_2)(x+a_3) \dots (x+a_n)$ = any polynomial composed of binomial factors of the form of $x+a$; then

$$\begin{aligned} f_1(x) = {} & (x+a_2)(x+a_3)(x+a_4) \dots (x+a_n) + \\ & (x+a_1)(x+a_3)(x+a_4) \dots (x+a_n) + \\ & (x+a_1)(x+a_2)(x+a_4) \dots (x+a_n) + \\ & (x+a_1)(x+a_2)(x+a_3) \dots \; x+a_n \; + \\ & \quad \vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \\ & (x+a_1)(x+a_2)(x+a_3) \dots (x+a_{n-1}) \end{aligned}$$

[694, Cor. 688].

Observations.—1. *If no two factors of $f(x)$ are alike, $f(x)$ and $f_1(x)$ have no common factor.*

2. If two, three, or r factors of $f(x)$ are equal, and all equal to $x+a$, then will $x+a$, $(x+a)^2$, or $(x+a)^{r-1}$ be a common factor of $f(x)$ and $f_1(x)$.

3. In general, if $f(x)$ contains the factor $x+a$ p times, $(x+b)$ q times, $(x+c)$ r times .... then will $(x+a)^{p-1}(x+b)^{q-1}(x+c)^{r-1}$ .... be the H. C. D. of $f(x)$ and $f_1(x)$.

4. The H. C. D. of $f(x)$ and $f_1(x)$ contains one factor less of each kind than does $f(x)$.

701. ***Theorem.***—*Every polynomial composed of binomial factors of the first degree, some of which are equal, may be decomposed into factors containing no equal binomial factors of the first degree.*

For, let $f(x)$ be a polynomial composed of binomial factors of the first degree, some of which are equal, $f_1(x)$ its first derivative, $f'(x)$ the H. C. D. of $f(x)$ and $f_1(x)$, and $\phi(x)$ the other factor of $f(x)$; then,

1. $\phi(x)$ will be devoid of equal factors of the first degree [700, 4].

2. If $f'(x)$ still contains equal factors of the first degree it may be resolved into two factors, $f''(x)$ and $\phi'(x)$, in which $\phi'(x)$ is devoid of equal factors [700, 4].

3. This process may be continued until no factor is left that contains equal factors of the first degree, which will be when the last H. C. D. found is unity.

Illustration.—Resolve $x^7+x^6-12x^5-12x^4+48x^3+48x^2-64x-64$ into factors devoid of equal binomial factors of the first degree.

Solution:

$f(x) = x^7+x^6-12x^5-12x^4+48x^3+48x^2-64x-64$

$f_1(x) = 7x^6+6x^5-60x^4-48x^3+144x^2+96x-64$

$f^1(x) = x^4-8x^2+16$ [158] $= (x^2-4)^2 = (x+2)(x+2)(x-2)(x-2)$

$\phi(x) = f(x) \div f^1(x) = x+1$

$\therefore\ f(x) = (x+2)(x+2)(x-2)(x-2)(x+1).$

EXERCISE 100.

Factor:

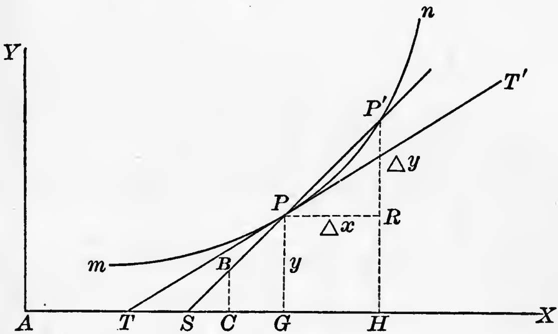
1. $x^4+2x^3-11x^2-12x-36$

2. $x^6-5x^5+x^4+37x^3-86x^2+76x-24$

3. $x^8+x^7-26x^6-27x^5+216x^4+243x^3-486x^2-729x-729$
4. $x^{10}-30x^8+345x^6-1900x^4+5040x^2-5184$
5. $x^{10}-13x^8+42x^6-58x^4+37x^2-9$

Graphical Significance of $f_1(x)$.

702. Let mn be the graph of $y=f(x)$.



Let P be a point on the graph whose co-ordinates are x and y. Let $GH=PR=\Delta x$; then will $P'R=\Delta y$.

Draw the secant line $P'PS$, also the tangent line $T'PT$.

Take $SB=1$, and draw $BC=\sin. S$ and $SC=\cos. S$.

Now, the triangles $P'PR$ and BSC are similar.

$$\therefore \quad \frac{P'R}{PR}=\frac{BC}{SC}=\tan. S \ [668] \qquad (1)$$

$$\therefore \quad \frac{\Delta y}{\Delta x}=\tan. S \qquad (2)$$

Let the point P' approach the point P on the graph so as to make Δx diminish uniformly; then will the secant line $P'PS$ revolve about P and approach the tangent line $T'PT$ as its limit, and the angle S will approach the angle T as its limit.

$$\therefore \quad \text{Lim.}\left(\frac{\Delta y}{\Delta x}\right)_{\Delta x=0}=\text{lim.}\,(\tan. S)_{\Delta x=0}$$

$$\text{or,} \quad \frac{dy}{dx}=\tan. T. \quad \text{Therefore,}$$

The first derivative of a function is equivalent to the tangent of the angle which a tangent line to the graph of the function makes with the axis of abscissas.

Maxima and Minima of Functions.

703. The maximum or minimum value of a quadratic function may readily be found, as follows:

Example 1.—What is the maximum or minimum value of $x^2 + 8x + 6$, and what value of x will render it a maximum or minimum?

Solution: Let $f(x) = x^2 + 8x + 6 = m$

Complete the square, $x^2 + 8x + 16 = m + 10$

Extract the $\sqrt{\ }$, $x + 4 = \pm\sqrt{m + 10}$

Transpose, $x = -4 \pm \sqrt{m + 10}$

Now, $m \nless -10$, else would x be imaginary.

$\therefore$ $m = -10$ is the minimum value of $f(x)$.

But when $m = -10$, $x = -4$; then, $x = -4$ renders $f(x) = x^2 + 8x + 6 = -10$, a minimum.

Example 2.—What is the maximum or minimum value of $8x - 3x^2 + 9$, and what value of x will render it a maximum or a minimum?

Solution: Let $f(x) = 8x - 3x^2 + 9 = m$

Complete the square, $9x^2 - 24x + 16 = 43 - 3m$

Extract the $\sqrt{\ }$, $3x - 4 = \pm\sqrt{43 - 3m}$

Transpose and divide, $x = \frac{4}{3} \pm \frac{1}{3}\sqrt{43 - 3m}$

Now, $3m \ngtr 43$, or $m \ngtr 14\frac{1}{3}$, else would x be imaginary.

$\therefore$ $m = 14\frac{1}{3}$ is the maximum value of $f(x)$.

But, when $m = 14\frac{1}{3}$, $x = 1\frac{1}{3}$; therefore, $x = 1\frac{1}{3}$ renders $f(x) = 8x - 3x^2 + 9 = 14\frac{1}{3}$, a maximum.

Example 3.—Divide 36 into two parts whose product shall be the greatest possible.

Solution: Let x and $36 - x =$ the two parts,

and $x(36 - x)$, or $36x - x^2 = m$.

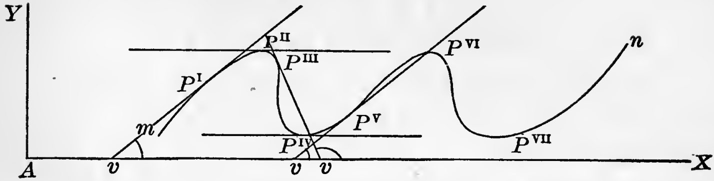
Then, $x = 18 \pm \sqrt{324 - m}$.

Now, $m = 324$ is a maximum;

$\therefore$ $x = 18$ and $36 - x = 18$.

704. *General Method.*—

Let mn be the graph of $y = f(x)$.



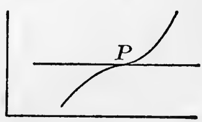
Conceive a point, P, to move along the graph, carrying with it a tangent line to the graph, in such a manner as to cause the abscissa (x) of the point to increase uniformly. Let v be the value of the variable angle which the tangent line makes with the x-axis. At P^{I} $v < 90$; hence, tan. v, or $f_1(x)$, is positive [668, Sch.]. This is true, however near P^{I} is to P^{II}. At P^{II}, the tangent line is parallel to the x-axis; hence, $v = 0$, and tan. v, or $f_1(x) = 0$. At P^{III}, $v > 90$; hence, tan. v, or $f_1(x)$, is negative [668, Sch.]. This is true, however near P^{III} is to P^{II}. Again, just before P arrives at P^{IV}, $v > 90°$, and tan. v is negative; when P is at P^{IV}, $v = 0$ and tan. $v = 0$; when P has just passed P^{IV}, $v < 90$ and tan. v is positive. Therefore,

705. ***Prin. 1.*** $f_1(x) = 0$ *at turning values of* $f(x)$.

Prin. 2. *Immediately before a maximum value of* $f(x)$, $f_1(x)$ *is positive, and immediately after, negative.*

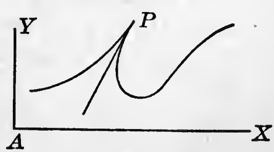
Prin. 3. *Immediately before a minimum value of* $f(x)$, $f_1(x)$ *is negative, and immediately after, positive.*

706. Caution 1.—A root of $f_1(x) = 0$ is not necessarily the abscissa of a turning point. For a tangent line to a graph may be parallel to the x-axis where there is no turning point, as where two branches tangent to the same line coalesce at the point of tangency. (See diagram.)



It is only when Prin. 2 or Prin. 3 is satisfied, as well as $f_1(x) = 0$, that a turning point is established.

Caution 2.—There may be turning points under peculiar conditions when $f_1(x) \neq 0$. For there may be turning points where the tangent line to the graph is not parallel to the x-axis; as where two branches coalesce and cease. (See diagram.)



707. Observations.—1. So long as $f(x)$ remains continuous, its maxima and minima values succeed each other alternately.

2. If two successive turning values of $f(x)$ have the same sign, the graph of $f(x)$ between these values can not cross the x-axis, or $f(x) \neq 0$ between these values.

3. If two successive turning values of $f(x)$ have opposite signs, the graph of $f(x)$ must cross the x-axis between these values, or $f(x) = 0$ somewhere between these values.

4. If $x = a$ and $x = b$ render $f(x) = 0$, and $a \neq b$, there must be a turning value of $f(x)$ between $x = a$ and $x = b$.

Example.—Find the turning values of

$$f(x) = x^3 - 9x^2 + 24x + 16.$$

Solution: $f(x) = x^3 - 9x^2 + 24x + 16$

$f_1(x) = 3x^2 - 18x + 24 = 0;$

or, $f_1(x) = x^2 - 6x + 8 = 0;$

whence, $x = 4$ or 2, critical values.

$$f_1(x - \Delta x)\left\{\begin{matrix} x=4 \\ \Delta x = \circ \end{matrix}\right\} = (4 - \Delta x)^2 - 6(4 - \Delta x) + 8 = -$$

$$f_1(x + \Delta x)\left\{\begin{matrix} x=4 \\ \Delta x = \circ \end{matrix}\right\} = (4 + \Delta x)^2 - 6(4 + \Delta x) + 8 = +$$

$\therefore$ $f(x)$ is a minimum when $x = 4$

But $f(x)_{x=4} = 4^3 - 9 \times 4^2 + 24 \times 4 + 16 = 32.$

$\therefore$ Minimum value of $f(x) = 32$

$$f_1(x - \Delta x)\left\{\begin{matrix} x=2 \\ \Delta x = \circ \end{matrix}\right\} = (2 - \Delta x)^2 - 6(2 - \Delta x) + 8 = +$$

$$f_1(x + \Delta x)\left\{\begin{matrix} x=2 \\ \Delta x = \circ \end{matrix}\right\} = (2 + \Delta x)^2 - 6(2 + \Delta x) + 8 = -$$

$\therefore$ $f(x)_{x=2}$ is a maximum

But $f(x)_{x=2} = 2^3 - 9 \times 2^2 + 24 \times 2 + 16 = 36$

$\therefore$ Maximum value of $f(x) = 36.$

The value of $f(x)_{x=a}$ is best obtained by synthetic division, as in Art. 106.

EXERCISE 101.

Find the maxima and minima values of:

1. $4x^3 - 15x^2 + 12x - 1$
2. $2x^3 - 21x^2 + 36x - 20$
3. $x^2 + 6x + 5$
4. $x^2 - 6x + 5$
5. $x^3 - 3x^2 - 9x + 5$
6. $(x - 1)^4(x + 2)^3$
7. $(x - a)^2(x + b)^2$
8. $x^3 - 3x^2 + 3x + 7$

9. $x^2 + 6x - 5$

10. $x^2 - 6x - 5$

11. $x^4 - 8x^2 + 16$

12. $x^4 + x^3 + x^2 - 16$

13. Show where a line a feet long must be divided so that the rectangle of the two parts may be the greatest possible.

14. Find the altitude of the maximum cylinder that can be inscribed in a sphere whose radius is r.

Suggestion.—Let $BC = x$, $BD = r - x$, and $AB = y$,

then, $y^2 = (r + x)(r - x) = r^2 - x^2$, and

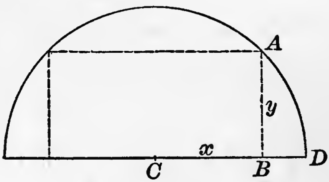
$$f(x) = V = \pi y^2 \times 2x = 2\pi x(r^2 - x^2)$$

$$f_1(x) = 2\pi x \times (-2x) + (r^2 - x^2) \times 2\pi = 0$$

whence, $x = \frac{r}{3}\sqrt{3}$

and, $y^2 = r^2 - x^2 = \frac{2}{3}r^2$

$$y = \frac{2}{3}r\sqrt{3}$$



15. Find the altitude of the maximum cylinder that can be inscribed in a cone whose altitude is a and whose radius is b.

16. Find the volume of the maximum cone that can be inscribed in a given sphere.

17. Find the area of the maximum rectangle that can be inscribed in a square whose side is a.

18. What is the maximum convex surface of a cylinder the sum of whose altitude and diameter is a constant a?

19. Find the altitude of the maximum cylinder that can be inscribed in a right cone whose altitude is a and the radius of whose base is b.

20. Required the area of the maximum rectangle that can be inscribed in a given circle.

21. Required the greatest right triangle which can be constructed upon a given line as hypotenuse.

CHAPTER XII.

THEORY OF EQUATIONS.

Introduction.

708. Equations of the first and second degree have already been treated, and need no further attention here.

709. Jerome Cardan, an Italian mathematician (1501–1576), published in 1545 a method of solving cubic equations, now known as "Cardan's Formula." But, as this formula is not finally reducible when the roots of an equation are real and unequal, it is not of much practical value.

710. René Descartes, a French mathematician (1596–1650), transformed the general bi-quadratic equation so as to make its solution depend upon that of the cubic equation; but, as he invented no new method of solving the latter, the same difficulties are encountered in the application of his rule as are met in Cardan's.

711. Nicholas Henry Abel, a Norwegian mathematician (1802–1829), demonstrated, in 1825, the impossibility of a general solution of an equation of a higher degree than the fourth. Previous to that date many such solutions were attempted.

712. The real roots of *numerical equations* of any degree are, however, attainable through laws and principles to be developed in this chapter.

Normal Forms.

713. ***Theorem I.***—*Every equation of one unknown quantity with real and rational coefficients can be transformed into an equation of the form of*

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + L = 0,$$

in which A and all the exponents of x are positive integers, and each of the remaining coefficients, including L, is either an integer or zero.

Note. L may be regarded the coefficient of x^0.

Demonstration.—1. If the equation contains fractional terms, it may be cleared of fractions.

2. If there are any terms in the second member, they may be transposed to the first member.

3. All terms containing like exponents of x may be collected into one term by addition.

4. If A is negative, both members may be divided by -1.

5. If x contains negative exponents, both members may be multiplied by x with a positive exponent numerically equal to the greatest negative exponent.

6. If x contains fractional exponents, x^m may be substituted for x, in which m is the L. C. M. of the denominators of the fractional exponents.

The roots of the transformed equation will be the mth root of the roots of the original equation.

7. The terms may now be arranged according to the descending powers of x.

714. The equation

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + L = 0,$$

is known as the *first normal form* of an equation of one unknown quantity, and will hereafter be represented by $F_n(x) = 0$.

Example.—Transform $3x^{\frac{2}{3}} + \frac{4}{x^{\frac{1}{3}}} - 8 + 7x^{-\frac{5}{6}} = \frac{4}{3} + \frac{3}{x^{\frac{1}{6}}}$ into the first normal form, and compare the corresponding roots of the two equations.

Solution: Given $3x^{\frac{2}{3}} + \frac{4}{x^{\frac{1}{3}}} - 8 + 7x^{-\frac{5}{6}} = \frac{4}{3} + \frac{3}{x^{\frac{1}{6}}}$. (A)

Clear of fractions, $9x + 12 - 24x^{\frac{1}{3}} + 21x^{-\frac{1}{2}} = 4x^{\frac{1}{3}} + 9x^{\frac{1}{6}}$ (B)

Transpose and collect terms,

$$9x + 21x^{-\frac{1}{2}} - 28x^{\frac{1}{3}} - 9x^{\frac{1}{6}} + 12 = 0 \quad \text{(C)}$$

Multiply by $x^{\frac{1}{2}}$,

$$9x^{\frac{3}{2}} + 21 - 28x^{\frac{5}{6}} - 9x^{\frac{2}{3}} + 12x^{\frac{1}{2}} = 0 \quad \text{(D)}$$

Put $x = x^6$, $9x^9 + 21 - 28x^5 - 9x^4 + 12x^3 = 0$ (E)

Rearrange terms,

$9x^9 + 0x^8 + 0x^7 + 0x^6 - 28x^5 - 9x^4 + 12x^3 + 0x^2 + 0x + 21 = 0$ (F)

The roots of (A) $= \sqrt[6]{\text{of the roots of (F)}}$.

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715. An equation that contains all the powers of x, from the highest to the lowest, is called a *Complete Equation.* An incomplete equation may be written in the form of a complete equation by supplying the wanting terms with coefficients of *zero.*

Thus, $x^5 - 4x^3 + 2x - 5 = 0$ may be written $x^5 \pm 0x^4 - 4x^3 \pm 0x^2 + 2x - 5 = 0$.

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716. ***Theorem II.***—*The equation* $F_n(x) = 0$ *may be transformed into an equation of the form of*

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0,$$

in which the coefficient of x^n *is unity, and each of the remaining coefficients is either an integer or zero.*

Demonstration.—

Take $F_n(x) = Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + L = 0$

Put $x = \frac{x}{A}$, $\frac{Ax^n}{A^n} + \frac{Bx^{n-1}}{A^{n-1}} + \frac{Cx^{n-2}}{A^{n-2}} + \ldots + L = 0$

Multiply by A_{n-1},

$$x^n + Bx^{n-1} + ACx^{n-2} + \ldots + A^{n-1}L = 0$$

Put p_1 for B, p_2 for AC, ... p_n for $A^{n-1}L$,

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$$

This is the *second normal form* of an equation of one unknown quantity, and will hereafter be represented by $f_n(x) = 0$.

717. Cor. 1.—*Each root of $f_n(x) = 0$ is A times as great as the corresponding root of $F_n(x) = 0$.*

718. Cor. 2.—*The coefficient of the second term of $f_n(x)$ is the same as the coefficient of the second term of $F_n(x)$, and the succeeding coefficients of $f_n(x)$ are obtained by multiplying the succeeding coefficients of $F_n(x)$, in order, by A, A^2, A^3, ... A^{n-1}.*

Note.—If terms are wanting, supply them with coefficients of 0.

Example.—Transform the equation $4x^5 - 3x^4 + 2x^2 - 7 = 0$ into an equation of the form of $f(x) = 0$.

Solution:

Given $F(x) = 4x^5 - 3x^4 + 0x^3 + 2x^2 + 0x - 7 = 0$,

then will $f(x) = x^5 - 3x^4 + 4 \times 0x^3 + 4^2 \times 2x^2 + 4^3 \times 0x - 4^4 \times 7 = 0$ [718]

or, $f(x) = x^5 - 3x^4 + 32x^2 - 1792 = 0$.

The roots of $f(x) = 0$ are 4 times as great as those of $F(x) = 0$.

EXERCISE 102.

Transform the following equations into equations of the form of $f(x) = 0$. Compare the roots of the transformed equation with the roots of the original equation.

1. $3x^4 + 2x^3 - 3x^2 + 7x - 5 = 0$
2. $2x^5 + 4x^4 - x^3 + x^2 - 7 = 0$
3. $4x^6 + 3x^4 - 5x^2 + 7x - 1 = 0$
4. $3x^{\frac{5}{6}} - 2x^{\frac{2}{3}} + 3x^{\frac{1}{2}} - 2x^{\frac{1}{3}} + 4x^{\frac{1}{6}} - 2 = 0$
5. $x^{-\frac{6}{7}} + 2x^{-\frac{5}{7}} + 3x^{-\frac{4}{7}} - x^{-\frac{3}{7}} + x^{-\frac{2}{7}} - 2x^{-\frac{1}{7}} + 2 = 0$
6. $\frac{3}{4}x + \frac{5}{6}x^{-\frac{2}{3}} + \frac{2}{3}x^{\frac{3}{4}} - \frac{1}{3}x^{-2} + 3 = 0$
7. $x^{\frac{2}{3}} - 3x^{\frac{5}{6}} - 2x^{\frac{1}{3}} + 3x^{-\frac{1}{3}} + 2 = 0$
8. $\frac{5}{6}x^{\frac{5}{6}} + \frac{3}{4}x^{\frac{2}{3}} - \frac{1}{3}x + 1 = 0$
9. $\frac{2}{3} + \frac{5}{6}x - \frac{3}{4}x^{-\frac{1}{3}} + 5 = 0$

Divisibility of Equations.

719. ***Theorem III.***—*If* **a** *is a root of* $F_n(x) = 0$, *then* $x - a$ *is a factor of* $F_n(x)$.

For, let $F_n(x) \div (x - a) = F_{n-1}(x) + \dfrac{r}{x - a}$

then, $\{F_{n-1}(x)\}(x - a) + r = F_n(x) = 0$

but, $x - a = 0$, since $x = a$.

$\therefore$ $r = 0$;

whence, $F_n(x) \div (x - a) = F_{n-1}(x)$.

720. ***Cor. 1.***—*If* a *is an integral root of* $F_n(x) = 0$, *it is a divisor of the absolute term of* $F_n(x)$ [163].

721. ***Cor. 2.***—*If* $x - a$ *is a factor of* $F_n(x)$, *then* **a** *is a root of* $F_n(x) = 0$.

For, $F_n(x) = \{F_{n-1}(x)\}(x - a) = 0$;

whence, $x - a = 0$, and $x = a$.

722. ***Cor. 3.***—*If* x *is a factor of* $F_n(x)$, *then zero is a root of* $F_n(x) = 0$.

Number of Roots.

723. ***Theorem IV.*** $F_n(x) = 0$ *has at least one root.*

The demonstration of this theorem may be found in special treatises on the Theory of Equations. It is too long and tedious to be introduced here.

724. ***Theorem V.*** $F_n(x) = 0$ *has* n *roots and only* n.

For, $F_n(x) = 0$ has at least one root. [T. IV.]

Let $a =$ one root of $F_n(x) = 0$;

then, $F_n(x) = \{F_{n-1}(x)\}\{x - a\} = 0$ [T. III.]

$\therefore$ $F_{n-1}(x) = 0$.

Let $b =$ one root of $F_{n-1}(x) = 0$; [T. IV.]

then, $F_{n-1}(x) = \{F_{n-2}(x)\}\{x - b\} = 0$ [T. III.]

$\therefore$ $F_{n-2}(x) = 0$.

Now, as $F_n(x)=0$ is of the nth degree, and each time a root is removed by division the degree is lowered by unity, it follows that n roots and only n can be removed before $F_n(x)$ reduces to an absolute factor. Therefore, $F_n(x)=0$ has n roots and only n.

725. *Cor.* *$F_n(x)=0$ may be written*

$$A(x-a)(x-b)(x-c)\ldots\ldots(x-l)=0;$$

or simply $(x-a)(x-b)(x-c)\ldots\ldots(x-l)=0$, in which there are n factors of the form of $x-r$, the second terms of which are the roots of $F_n(x)=0$ with their signs changed, and may be positive or negative, fractional or integral, rational, irrational, or imaginary, subject only to restrictive conditions explained hereafter.

Relation of Roots to Coefficients.

726. *Theorem VI.*—*If $F_n(x)=0$ be put in the form of $x_n+B_1x^{n-1}+C_1x^{n-2}+\ldots\ldots+L_1=0$, by dividing both members of the equation by A, the coefficient of x^n, then will*

1. $B_1=$ the sum of the roots with their signs changed.

2. $C_1=$ the sum of the products of the roots taken two together.

3. $D_1=$ the sum of the products of the roots with their signs changed, taken three together.

4. $E_1=$ the sum of the products of the roots taken four together. And so on to

5. $L_1=$ the product of all the roots with their signs changed.

Demonstration: Let the n roots of the equation be $a, b, c, \ldots\ldots l$; then, $F_n(x)=x^n+B_1x^{n-1}+C_1x^{n-2}+\ldots\ldots+L_1$

$$=(x-a)(x-b)(x-c)\ldots\ldots(x-l) \quad [725].$$

After which the theorem is a direct inference from the binomial formula [587], and the principle that "changing the signs of an even number of factors does not change the sign of their product" [page 26, Ex. 3].

Cor.—*Changing the signs of the alternate terms of* $F_n(x) = 0$ *changes the signs of its roots.*

Imaginary Roots.

727. Theorem VII.—*Imaginary roots can enter* $F_n(x) = 0$ *only in conjugate pairs.*

For in this way only will their sum and the sum of their products be real [657], as they must be [713].

728. Cor. 1.—*The product of the imaginary roots of* $F_n(x) = 0$ *is positive.*

For the product of each pair is positive.

Thus, $(a + b\,i)(a - b\,i) = a^2 + b^2$.

729. Cor. 2.—*When all the roots of* $F_n(x) = 0$ *are imaginary the absolute term is positive.*

Suggestion.—For the equation is then of an even degree.

730. Cor. 3. $F_n(x) = 0$ *has at least one real root opposite in sign to the absolute term, when n is odd.*

731. Cor. 4. $F_n(x) = 0$ *has at least two real roots, one positive and the other negative, if n is even and the absolute term is negative.*

732. Cor. 5.—*The sign of* $F_n(x)$ *for any real value of x depends on the real roots of* $F_n(x) = 0$.

For the product of $x - (a + b\,i)$ and $x - (a - b\,i) = (x - a)^2 + b^2$, a positive quantity ; and this is true of every pair of factors containing conjugate imaginary terms.

733. Cor. 6.—*Every entire function of x with real and rational coefficients may be divided into real factors of the first or second degree.*

Fractional Roots.

734. *Theorem VIII.*—*No root of* $f_n(x)=0$ *can be a rational fraction.*

Take $f_n(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ [716]. If possible, let $x = \frac{a}{b}$, a rational fraction in its lowest terms. Then, by substitution,

$$\frac{a^n}{b^n} + \frac{p_1 a^{n-1}}{b^{n-1}} + \frac{p_2 a^{n-2}}{b^{n-2}} + \ldots + p_n = 0.$$

Multiplying by b^{n-1}, and transposing terms, we have

$$\frac{a^n}{b} = -p_1 a^{n-1} - p_2 a^{n-2} b - \ldots - p_n b^{n-1} =$$

an integer, which is impossible.

Scholium.—*From this theorem it follows that the rational fractional roots of* $F_n(x)=0$ *may be obtained by transforming* $F_n(x)=0$ *into* $f_n(x)=0$ *and dividing the roots of the latter equation by* A*, the coefficient of* x^{n-1} *in the former.*

Relations of Roots to Signs of Equation.

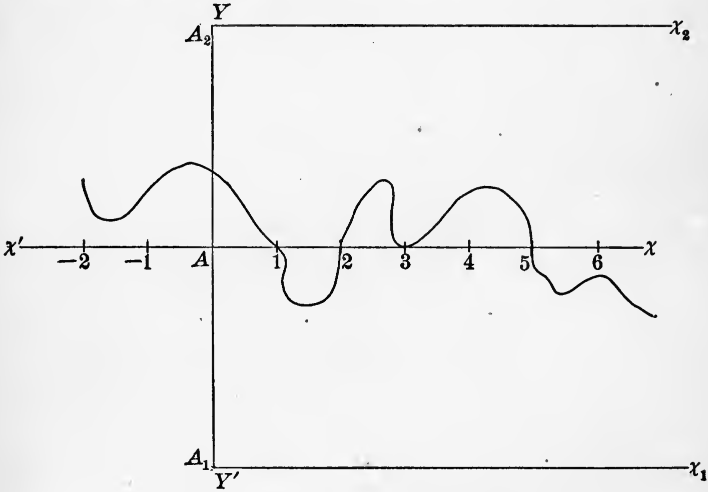
735. *Theorem IX.*—*If* $F_n(x)=0$ *has no equal roots, then* $F_n(x)$ *will change sign if* x *passes through a real root.*

For, take $F_n(x) = (x-a)(x-b)(x-c)\ldots(x-l) = 0$ [725]; conceive x to start with a value less than the least root and continually increase until it becomes greater than the greatest root. At first, every factor of $F_n(x)$ is negative, but, at the instant it becomes greater than the least root, the sign of the factor containing that root will become plus, while the others remain minus; whence, $F_n(x)$ will change sign. It will, moreover, retain its new sign until it passes over the next greater root, when it will again change sign, and so on.

736. Cor. 1.—*If for any two assigned values of x, $F_n(x)$ has different signs, one, or, if more than one, an odd number of roots of $F_n(x) = 0$ lie between these values.*

737. Cor. 2.—*If for any two assigned values of x, $F_n(x)$ has the same sign, either no root or an even number of roots of $F_n(x) = 0$ lie between these values.*

738. Some of the properties of $F_n(x) = 0$, already discussed, are beautifully illustrated by the following graph.



1. It is seen that $y = F_n(x) = 0$ when $x = 1$, 2, 3, and 5. Therefore, these values of x are roots of $F_n(x) = 0$.

2. Immediately before $x = 1$, y is positive, and immediately after $x = 1$, y is negative; immediately before $x = 2$, y is negative, and immediately after $x = 2$, y is positive, etc.; illustrating that when x passes over a real root, $F_n(x)$ changes sign.

3. At $x = 3$ two values of y become zero; therefore, two roots become identical, or, in other words, 3 is twice a root. Were the absolute term of $F_n(x)$ so changed as to make y somewhat less, the x-axis would cross the graph twice between $x = 2$ and $x = 4$, once before $x = 3$, and once after, thus proving conclusively the duality of the root 3, when $y = 0$.

4. Immediately before $x = -2$ and $x = 6$, the graph approaches

the x-axis, but in each case makes a turn before reaching it, preventing, thereby, equal roots or unequal real roots. These turns locate the position of imaginary roots. The truth of this statement becomes manifest when we suppose the absolute term of $F_n(x)$ to so change as to cause y to gradually decrease, the x-axis will gradually arise and finally touch the graph at $x = -2$, thereby making two equal roots, and, if y continues to decrease, the x-axis will cross both branches above the turn at $x = -2$, making two unequal real roots.

The student will be interested in observing the changes in the roots if the absolute term of the equation so changes as to cause the x-axis to gradually move from the position $A_1\chi_1$ to the position $A_2\chi_2$.

5. It must not be assumed, however, that imaginary roots always denote a turning point in the graph of the equation. Such may or may not be the case.

739. If any two successive terms in a complete equation have like signs, there is a *permanence* of sign; if unlike signs, a *variation* of sign. Thus, in the equation

$$x^6 - 5x^5 + 8x^4 + 7x^3 - 3x^2 + 2x - 5 = 0$$

there are five variations and one permanence.

740. ***Theorem X.***—*No complete equation has a greater number of positive roots than there are variations of sign, nor a greater number of negative roots than there are permanences of sign.*

Demonstration: Let the following be the successive signs of a complete equation:

$$+ \quad - \quad - \quad + \quad + \quad -$$

There are here two permanences and three variations. To introduce another positive root, the equation must be multiplied by $x - a$.

The signs of the product will readily appear from the following work:

$$\begin{array}{ccccccc} + & - & - & + & + & - & \\ + & - & & & & & \\ \hline + & - & - & + & + & - & \\ & - & + & + & - & - & + \\ \hline + & - & \pm & + & \pm & - & + \end{array}$$

The double sign denotes a doubt, growing out of an ignorance of the relative numerical magnitudes of the terms added.

Now, a careful inspection will show that, whether we regard both doubtful signs negative, both positive, or one negative and the other positive, the number of permanences will not be increased, but the number of terms is increased by one; therefore, the number of variations must be increased by *at least one*. Since the introduction of a positive root introduces at least one variation, it follows that the number of positive roots can not exceed the number of variations.

In a similar manner, by introducing the factor $x + a$, it may be shown that the number of negative roots can not exceed the number of permanences of sign.

This is Descartes' celebrated rule of signs.

741. ***Cor. 1.***—*If all the roots of an equation are real, the number of variations equals the number of positive roots, and the number of permanences equals the number of negative roots.*

742. ***Cor. 2.***—*An equation whose terms are all positive can have no positive roots.*

743. ***Cor. 3.***—*An equation whose terms are alternately positive and negative can have no negative roots.*

Limits of Roots.

744. A number known to be equal to or larger than the largest root of an equation is called a *superior limit* to the roots of the equation.

745. A number known to be equal to or smaller than the smallest root of an equation is called an *inferior limit* to the roots of the equation.

746. ***Theorem XI.***—*If the first h coefficients of $F_n(x)$ are positive, and P is the smallest of them, then, if Q is numerically the largest subsequent coefficient, $\sqrt[h]{\frac{Q}{P}}+1$ is a superior limit to the roots of $F_n(x) = 0$.*

Demonstration: It is evident that the case in which x must have the greatest value to make $F_n(x) = 0$ when the first h coefficients are positive, is the one in which these coefficients are all equal to the least one of them (P), and the remaining $n + 1 - h$ coefficients are all negative and each equal to the greatest among them (Q). Therefore, the value of x is a superior limit to the roots of $F_n(x) = 0$, if

$$Px^n + Px^{n-1} + \dots + Px^{n+1-h} = Qx^{n-h} + Qx^{n-h-1} + \dots + Q;$$

or, $$\frac{Px^{n+1} - Px^{n+1-h}}{x-1} = \frac{Qx^{n+1-h} - Q}{x-1}$$

or, $$Px^{n+1-h}(x^h - 1) = Q(x^{n+1-h} - 1)$$

or, $$P(x^h - 1) = Q\left(1 - \frac{1}{x^{n+1-h}}\right)$$

or if, $$P(x^h - 1) = Q, \text{ since } 1 > \left(1 - \frac{1}{x^{n+1-h}}\right)$$

or $$x = \sqrt[h]{\frac{Q}{P} + 1}.$$

747. ***Cor.***—*If the signs of the alternate terms of an equation be changed, then will the superior limit to the roots of the transformed equation, with its sign changed, be the inferior limit to the roots of the original equation* [726, Cor.].

Equal Roots.

748. ***Theorem XII.***—*If $F_n(x) = 0$ has equal roots, it may be separated into two or more equations with unequal roots.*

This is a direct inference from Art. 701.

Commensurable Roots.

749. The integral and rational fractional roots of $F_n(x) = 0$ are called its *commensurable roots.*

750. Problem 1. To find the commensurable roots of $F_n(x) = 0$.

Solution: Pursue the following line of investigation:

1. Determine the number of roots the equation has [724].

2. Determine how many roots may be positive and how many negative [739].

3. Determine the limit to the positive and the negative roots [746, 747].

4. Determine what integral numbers may be roots [720].

5. Find and remove the integral roots by synthetic division [719, 105].

6. Determine whether there are any equal roots [701], and if so, remove them by synthetic division.

7. Find the rational fractional roots from the equation resulting from the removal of the integral roots, and according to Theorem VIII, Scholium.

Illustrations.—1. Find the commensurable roots of

$$F_4(x) = 24x^4 + 122x^3 + 5x^2 - 26x - 5 = 0.$$

Solution:

1. This equation has four roots, all real, or two real [724, 731].

2. There are one variation and three permanences of sign; therefore, there can not be more than one positive nor more than three negative roots [739].

3. The only integral roots possible are $+1$, -1, $+5$, and -5 [720].

4. The largest positive root $< \sqrt[3]{\frac{26}{5}} + 1$, or < 2 [746].

5. Neither $+1$ nor -1 is a root, since $F_4(x)$ is not divisible by either $x - 1$ or $x + 1$, as witness:

$$\begin{array}{r|rrrrr} -1) & 24 & +122 & +5 & -26 & -5 \\ & & -24 & -98 & +93 & -67 \\ \hline & & 98 & -93 & +67 & -72^* \end{array}$$

$$\begin{array}{r|rrrrr} +1) & 24 & +122 & +5 & -26 & -5 \\ & & 24 & +146 & +151 & +125 \\ \hline & & 146 & +151 & +125 & +120^* \end{array}$$

Note.—It is evident that when $+1$ is a root of $F_n(x) = 0$, the sum of the positive coefficients must equal the sum of the negative coefficients; and, if -1 is a root, $+1$ is a root if the signs of the alternate terms are changed. These facts determine a more expeditious method of testing whether either $+1$ or -1 is a root.

6. -5 is a root, as witness:

$$\begin{array}{r|rrrrr} -5) & 24 & +122 & +5 & -26 & -5 \\ & & -120 & -10 & +25 & +5 \\ \hline & & +2 & -5 & -1 & \end{array}$$

7. The resulting equation, after removing the root -5, is $F_3(x) = 24x^3 + 2x^2 - 5x - 1 = 0$, which has no integral roots.

Transform $F_3(x) = 0$ into an equation of the form of $f_3(x) = 0$, $f_3(x) = x^3 + 2x^2 - 120x - 576 = 0$ [733, Sch.].

8. $f_3(x) = 0$ has three roots [724], only one of which can be positive, and the largest positive root possible is $\sqrt{577} = 24$.

9. The divisors of 576 not exceeding 24 are 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24. From the relative values of the positive and negative coefficients it will be seen at a glance that $x > 6$.

10. $+8$ and $+9$ are not roots, but $+12$ is a root, as witness:

$$\begin{array}{r|rrrr} +8) & 1 + & 2 - & 120 - & 576 \\ & & +\ 8 + & 80 - & 320 \\ \hline & & +\ 10 - & 40 - & 896^* \end{array}$$

$$\begin{array}{r|rrrr} +9) & 1 + & 2 - & 120 - & 576 \\ & & +\ 9 + & 99 - & 189 \\ \hline & & +\ 11 - & 21 - & 765^* \end{array}$$

$$\begin{array}{r|rrrr} +12) & 1 + & 2 - & 120 - & 576 \\ & & +\ 12 + & 168 + & 576 \\ \hline & & 14 + & 48 & \end{array}$$

11. The resulting equation, after removing the root $+12$ from $f_3(x) = 0$, is $f_2(x) = x^2 + 14x + 48 = 0$; whose roots are found to be -8 and -6 [331].

12. The four roots of $F_4(x) = 0$ are, therefore, -5, $+\frac{12}{24}$, $-\frac{8}{24}$, and $-\frac{6}{24}$ [733, Sch.], or -5, $+\frac{1}{2}$, $-\frac{1}{3}$, and $-\frac{1}{4}$.

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2. Find the commensurable roots of

$$f_7(x) = x^7 + 3x^6 - 12x^5 - 36x^4 + 48x^3 + 144x^2 - 64x - 192 = 0.$$

Solution: It may readily be found by synthetic division that $+2$, -2, and -3 are the only integral roots of this equation.

The resulting equation, after removing these roots, is $f_4(x) = x^4 - 8x^2 + 16 = 0$. If any of the roots of this equation are integral, they must be equal to one or more of the roots already found. They may, however, all be incommensurable or imaginary.

Factoring $f_4(x)$, we have $(x^2 - 4)(x^2 - 4) = 0$;

whence, $x = +2, \ -2, \ +2, \ -2$.

Therefore, all the roots of $f_7(x) = 0$ are ± 2, ± 2, ± 2, and -3.

EXERCISE 103.

Find the commensurable roots of:

1. $x^3 - 3x^2 + 7x - 5 = 0$
2. $x^3 - 6x^2 + 10x - 8 = 0$
3. $x^3 - 11x^2 + 41x - 55 = 0$

4. $x^3 + 6x^2 + 14x + 12 = 0$

5. $x^4 - 3x^3 - 2x^2 + 12x - 8 = 0$

6. $12x^3 + 8x^2 - 3x - 2 = 0$

7. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$

8. $x^4 - 8x^3 + 10x^2 + 24x + 5 = 0$

9. $x^5 + 3x^4 - 3x^3 - 9x^2 - 4x - 12 = 0$

10. $x^4 - 5x^3 + 3x^2 + 2x + 8 = 0$

11. $8x^3 - 16x^2 - 8x + 21 = 0$

12. $16x^4 - 48x^3 + 32x^2 + 12x - 9 = 0$

13. $3x^5 + 2x^4 - 21x^3 - 14x^2 + 36x + 24 = 0$

14. $9x^5 + 81x^4 + 203x^3 + 99x^2 - 92x - 60 = 0$

15. $18x^5 + 9x^4 + 22x^3 + 11x^2 - 96x - 48 = 0$

16. $x^4 + 4x^3 - 13x^2 - 28x + 60 = 0$

17. $x^4 + 2x^3 - 11x^2 - 12x + 36 = 0$

18. $x^5 + 4x^4 + x^3 - 10x^2 - 4x + 8 = 0$

19. $3x^6 + 22x^5 + 8x^4 - 42x^3 - 79x^2 - 52x - 12 = 0$

Incommensurable Roots.

751. The *incommensurable roots* of an equation are best sought for after all the commensurable roots have been removed by division and the resulting equation transformed into an equation of the form of $f_n(x) = 0$.

752. The first step necessary in the search for the values of the incommensurable roots of an equation is to find the number and situation of such roots.

Jacques Charles François Sturm, a Swiss mathematician (1803–1855), discovered a method of doing this in 1829, known as *Sturm's method*.

I

753. *Sturm's Series of Functions.*—Assuming that $f_n(x)=0$ has no equal roots, this eminent mathematician formed a series of functions, as follows:

The first two terms of the series are $f_n(x)$, and its first derivative, which we will now represent by $f_{n-1}(x)$.

The other functions, and which are called Sturmian functions, are derived as follows: Divide $f_n(x)$ by $f_{n-1}(x)$, and represent the remainder *with its sign* changed by $f_{n-2}(x)$. Divide $f_{n-1}(x)$ by $f_{n-2}(x)$, and represent the remainder with its sign changed by $f_{n-3}(x)$; continue this process until the last remainder with its sign changed is an absolute term. Represent this remainder $f_0(x)$. There will then be $n+1$ of these functions, as follows:

$$f_n(x),\ f_{n-1}(x),\ f_{n-2}(x)\ \ldots\ldots f_0(x).$$

Caution.—Care must be taken in the operation of successive division not to reject any negative factors except in the remainders.

754. *Relation of the terms of Sturm's series of functions.*—If we put q_1, q_2, q_3.... as the successive quotients obtained in finding the Sturmian functions, it is evident that

$$f_n(x)=f_{n-1}(x)\,q_1-f_{n-2}(x) \qquad (1)$$
$$f_{n-1}(x)=f_{n-2}(x)\,q_2-f_{n-3}(x) \qquad (2)$$
$$f_{n-2}(x)=f_{n-3}(x)\,q_3-f_{n-4}(x) \qquad (3)$$
$$f_{n-3}(x)=f_{n-4}(x)\,q_4-f_{n-5}(x) \qquad (4)$$
$$f_{n-4}(x)=f_{n-5}(x)\,q_5-f_{n-6}(x) \qquad (5)$$

etc., etc., etc.

755. Fundamental Principles.

1. *No two consecutive functions can vanish, i. e., become* 0, *for the same value of* x.

For, if possible, let $x=a$ make $f_{n-2}=0$ and $f_{n-3}=0$; then will $f_{n-4}=0$ [754, 3], and hence, too, $f_{n-5}=0$

[754, 4], and so on until lastly $f_0(x)=0$; but $f_0(x)$ is the absolute term and can not be zero. Therefore, etc.

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2. If any one of the functions intervening between $f_n(x)$ and $f_0(x)$ vanishes for any value of x, the two adjacent functions have opposite signs for this value.

Thus, if $x=a$ causes $f_{n-3}(x)$ to vanish, $f_{n-2}(x)= -f_{n-4}(x)$ [754, 3].

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3. If any value of x, as $x=a$, causes any intervening function to vanish, then will the number of variations and the number of permanences in the signs of the functions be the same for the immediately preceding and the immediately succeeding values of x, i. e., for $x=a-\circ$ and $x=a+\circ$.

For the two adjacent functions will have opposite signs when $x=a$ [755, 2], and will not change their signs for any value of x from $x=a-\circ$ and $x=a+\circ$, since no root of either can lie between these values [755, 1]. But the function in question does change its sign, since x passes over a root of the function in going from $x=a-\circ$ to $x=a+\circ$. If the signs of the three functions for $x=a-\circ$ are $+$, $+$, $-$, for $x=a+\circ$ they will be $+$, $-$, $-$, which in either case form one permanence and one variation. Similarly, $+$, $-$, $-$ will change to $+$, $+$, $-$; $-$, $+$, $+$ will change to $-$, $-$, $+$; and $-$, $-$, $+$ will change to $-$, $+$, $+$.

---

4. If any value of x causes $f_n(x)$ to vanish, then will one variation in the signs of the functions be lost in passing from the immediately preceding value of x to the immediately succeeding value.

Let $f_n(x) =$

$$(x - a_1)(x - a_2)(x - a_3) \ldots\ldots (x - a_n);$$

then $f_{n-1}(x) =$

$$\begin{array}{ll} (x - a_2)(x - a_3)(x - a_4) \ldots\ldots (x - a_n) + & \text{(1st term)} \\ (x - a_1)(x - a_3)(x - a_4) \ldots\ldots (x - a_n) + & \text{(2d term)} \\ (x - a_1)(x - a_2)(x - a_4) \ldots\ldots (x - a_n) + & \text{(3d term)} \\ (x - a_1)(x - a_2)(x - a_4) \ldots\ldots (x - a_n) + & \text{(4th term)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \\ (x - a_1)(x - a_2)(x - a_3) \ldots\ldots x - a_{n-1}. & (n\text{th term}) \end{array}$$

Now, if x equals, say a_3, then will $f_n(x)$ and all the terms of $f_{n-1}(x)$, except the third, vanish.

Now, the third term of $f_{n-1}(x)$ contains all the factors of $f_n(x)$ except $x - a_3$. Therefore, if x is infinitesimally less than a_3, $x - a_3$ will be negative and $f_n(x)$ and $f_{n-1}(x)$ will have opposite signs or will form a variation; but, if x is infinitesimally greater than a_3, $x - a_3$ will be positive and $f_n(x)$ and $f_{n-1}(x)$ will have like signs or will form a permanence. Therefore, a variation is lost in passing from $a_3 - 0$ to $a_3 + 0$.

756. These principles are true if $f_n(x)$ contains imaginary roots as well as when all the roots are real, since the signs of the functions depend wholly upon the real factors they contain [732].

Sturm's Theorem.

757. *The number of variations of sign lost in the terms of the Sturmian series, as the value of x continuously changes from **a** to **b**, **a** being less than **b**, equals the number of real roots of $f_n(x) = 0$ lying between **a** and **b**.*

Demonstration.—For each time the value of x, in ascending from a to b, passes over a root of $f_n(x) = 0$, there is lost one variation of sign [755, 4] and only one [755, 3].

758. Cor. 1.—*The theorem is equally true for $F_n(x) = 0$, there being nothing in the demonstration of it to restrict its application to $f_n(x) = 0$.*

759. Cor. 2.—*The difference between the number of variations when $+\infty$ and $-\infty$ are substituted for x in the series is the number of real roots in the equation.*

760. Cor. 3.—*The difference between the number of variations when 0 and $+\infty$ are substituted for x is the number of positive roots, and, when 0 and $-\infty$ are substituted for x, the number of negative roots.*

761. Remark 1.—It is evident that the sign of the absolute term of a function is the sign of the value of the function, when $x = 0$.

762. Remark 2.—The sign of the first term of a function is the sign of the value of the function, when $x = \pm\infty$.

For, $Ax^n = Ax^{n-1} . x > Bx^{n-1} + Cx^{n-1} + Dx^{n-1} + \ldots . + Lx^{n-1} > Bx^{n-1} + Cx^{n-2} + Dx^{n-3} + \ldots . + L$, when $x = \pm\infty$.

763. Remark 3.—The sign of the value of a function for any integral or decimal value of x is best determined by the method explained in Art. 106.

Illustration.—Find the sign of $F_4(x) = 3x^4 - 2x^3 + 7x^2 - 3x - 8$ when $x = 1{\cdot}2$.

Solution: The value of $F_4(x)$ when $x = 1{\cdot}2$ is $+ {\cdot}3808$, as witness:

$$\begin{array}{rrrrrr} 1{\cdot}2)\,3 & -2 & +7 & -3 & -8 \\ & 3{\cdot}6 & +1{\cdot}92 & +9{\cdot}984 & +8{\cdot}3808 \\ \hline & 1{\cdot}6 & +8{\cdot}92 & +6{\cdot}984 & +{\cdot}3808\,* \end{array}$$

$\therefore$ The sign of $F_4(x)$ is $+$.

Note.—In practice it is usually not necessary to make the last multiplication and addition to determine the *sign* of the value.

764. Remark 4.—Though it is not usually best to apply Sturm's method of solution to equations before the commensurable roots have been removed by division, on account of the great labor involved in deriving and evaluating the different functions when the equation is of a high degree, yet such a course may be pursued. If there are equal roots, the fact will appear in deriving the functions, and if there are integral or fractional roots they will be discovered in evaluating the functions to determine their signs.

Example.—Determine the number and situation of the real roots in $f_3(x) = x^3 - 12x^2 + 57x - 94 = 0$.

Solution:

$$f_3(x) = x^3 - 12x^2 + 57x - 94$$
$$f_2(x) = 3x^2 - 24x + 57$$
$$f_1(x) = -x + 3$$
$$f_0(x) = -$$

Substituting in these functions as follows, we shall have:

For $x = +\infty$,	+	+	−	−	one variation.
For $x = 0$,	−	+	+	−	two variations.
For $x = -\infty$,	−	+	+	−	two variations.

There is, therefore, one real root between 0 and $+\infty$. There is no negative root. Therefore, there are two imaginary roots.

To find the situation of the real root, we proceed as follows:

For $x = 1$, we have	−	+	+	−	two variations.
For $x = 2$, we have	−	+	+	−	two variations.
For $x = 3$, we have	−	+	±	−	two variations.
For $x = 4$, we have	+	+	−	−	one variation.

Therefore, there is one real root between 3 and 4, or the first figure of the real root is 3.

To find the next figure, we proceed as follows:

For $x = 3{\cdot}1$, we have	−	+	−	−	two variations.
For $x = 3{\cdot}2$, we have	−	+	−	−	two variations.
For $x = 3{\cdot}3$, we have	−	+	−	−	two variations.
For $x = 3{\cdot}4$, we have	+	+	−	−	one variation.

Therefore, the root lies between 3·3 and 3·4, or the first two figures of the root are 3·3.

By a continuation of this process the root might be extended to any number of figures. A more expeditious method, however, is known, and will be explained hereafter, for extending a root after a sufficient number of figures have been found to distinguish the root from any other root lying near it. Thus, if an equation had the two roots 3·1256.. and 3·1234..., the first four figures of each root only would be found by Sturm's theorem.

When it is known, as in the above example, that only one real root lies between two numbers, it becomes necessary only to study the signs of $f_n(x)$, since passing over the roots of the intermediate functions does not cause a change in the number of variations.

The same conclusion will be reached by the simple application of Art. 735, since $f_3(x)$ changes sign between $x = 3$ and $x = 4$.

EXERCISE 104.

Find the number and situation of the real roots in the following equations:

1. $x^3 - 4x^2 - 6x + 8 = 0$
2. $x^3 + 6x^2 - 3x + 9 = 0$
3. $x^4 + 3x^3 - 6x + 2 = 0$
4. $x^5 - 10x^3 + 6x + 1 = 0$
5. $2x^4 - 11x^2 + 8x - 16 = 0$
6. $x^4 - 12x^2 + 12x - 3 = 0$

Horner's Method of Root Extension.

765. In 1819, W. G. Horner, an English mathematician, published an elegant method of extending a root of an equation to any desired number of places, after a sufficient number of initial figures have been found by other methods to distinguish the root from other roots of the equation. This method is based upon the following principle:

766. ***Principle.***—*If $F_n(x)$ be continuously divided by $x - a$, the successive remainders will be the coefficients in inverse order of an equation whose roots are* **a** *less than the roots $F_n(x) = 0$.*

Demonstration:

Take $F_n(x) = Ax^{n-1} + Bx^{n-2} + \dots$

$$+ Jx^2 + Kx + L = 0 \qquad \text{(A)}$$

Put $x_1 + a = x$, or $x_1 = x - a$;

$$F_n(x_1 + a) = A(x_1 + a)^{n-1} + B(x_1 + a)^{n-2} + \dots$$
$$+ J(x_1 + a)^2 + K(x_1 + a) + L = 0 \qquad \text{(B)}$$

Expand terms, bracket coefficients of like powers of x_1, and represent the coefficients of the transformed equation by $A_1, B_1, \dots J_1, K_1, L_1$; then,

$$F_n(x_1) = A_1 x_1^{n-1} + B_1 x_1^{n-2} + \dots$$
$$J_1 x_1^2 + K_1 x_1 + L_1 = 0 \qquad \text{(C)}$$

Now, the roots of (C) are evidently a less than those of (A).

Substitute $x_1 = x - a$ in (C),

$$F_n(x - a) = A_1(x - a)^n + B_1(x - a)^{n-1} + \dots$$
$$+ J_1(x - a)^2 + K_1(x - a) + L_1 = 0 \qquad \text{(D)}$$

Now, $F_n(x-a)$ is evidently equivalent to $F_n(x)$, and will leave the same remainder when divided by $x-a$ as will $F_n(x) \div (x-a)$. But, if $F_n(x-a)$ is continuously divided by $x-a$, the successive remainders will be L_1, K_1, J_1, ... B_1, and A_1, or the coefficients of $F_n(x_1)$ in inverse order. Therefore the theorem.

Applications.

1. Transform $x^4+x^3+x^2+3x-100=0$ into an equation whose roots are 2 less than those of the given equation.

Form.

1	+ 1	+ 1	+ 3	− 100	(+ 2
	+ 2	+ 6	+ 14	+ 34	
	+ 3	+ 7	+ 17	− 66*	
	+ 2	+ 10	+ 34		
	+ 5	+ 17	+ 51*		
	+ 2	+ 14			
	+ 7	+ 31*			
	+ 2				
	+ 9*				

The transformed equation is $x^4+9x^3+31x^2+51x-66=0$.

Explanation.—Dividing by $x-2$, by synthetic division, the coefficients of the first quotient are $1+3+7+17$, and the first remainder is -66, the absolute term of the transformed equation.

Dividing $1+3+7+17$ again by $+2$, the second quotient is $1+5+17$, and the second remainder, or the coefficient of x in the transformed equation, is $+51$.

Dividing $1+5+17$ again by $+2$, the third quotient is $1+7$, and the third remainder, or the coefficient of x^2 in the transformed equation, is $+31$.

Dividing $1+7$ again by $+2$, the fourth quotient is 1, and the fourth remainder, or the coefficient of x^3 in the transformed equation, is $+9$.

Therefore, the transformed equation is $x^4+9x^3+31x^2+51x-66=0$.

Query.—Could you tell by inspection that the roots of the transformed equation are less than those of the original equation?

Query.—Since $1+9+31+51>66$, can $x^4+9x^3+31x^2+51x-66=0$ have a positive root equal to or greater than unity? Why, or why not?

2. Transform $x^4 + 9x^3 + 31x^2 + 51x - 66 = 0$ into an equation whose roots are ·8 less than those of the given equation.

1	+ 9	+ 31	+ 51	− 66	(·8
	·8	7·84	31·072	65·6576	
	9·8	38·84	82·072	− 0·3424 *	
	·8	8·48	37·856		
	10·6	47·32	+ 119·928 *		
	·8	9·12			
	11·4	+ 56·44 *			
	·8				
	12·2 *				

The coefficients of the first quotient are 1 + 9·8 + 38·84 + 82·072, and the first remainder is − 0·3424, which is the absolute term of the transformed equation. The second remainder, or the coefficient of x, is 119·928. The third remainder, or the coefficient of x^2, is 56·44. The fourth remainder, or the coefficient of x^3, is 12·2.

The transformed equation is

$$x^4 + 12{\cdot}2\,x^3 + 56{\cdot}44\,x^2 + 119{\cdot}928\,x - {\cdot}3424 = 0.$$

3. Transform

$$x^4 + 12{\cdot}2\,x^3 + 56{\cdot}44\,x^2 + 119{\cdot}928\,x - {\cdot}3424 = 0$$

into an equation whose roots are ·002 less than those of the given equation.

1	12·2	56·44	119·928	− ·3434........	(·002
	·002	·024404	·112928808	+ ·240081857616	
	12·202	56·464404	120·040928808	− ·102318142384 *	
	·002	·024408	·112977624		
	12·204	56·488812	120·153906432 *		
	·002	·024412			
	12·206	56·513224 *			
	·002				
	12·208 *				

The coefficients of the first quotient are 1 + 12·202 + 56·464404 + 120·040928808, and the first remainder, or the absolute term of the transformed equation, is − ·102318142384. The second, third, and fourth remainders, or the coefficients of x, x^2, and x^3, are 120·153906432, 56·513224, and 12·208.

The transformed equation is

$$x^4 + 12{\cdot}208\,x^3 + 56{\cdot}513224\,x^2 + 120{\cdot}153906432\,x - {\cdot}102318142384 = 0.$$

4. The integral part of one of the roots of $x^4 + x^3 + x^2 + 3x - 100 = 0$ is 2. Extend the root.

		Form.		
1 +1	+ 1	+ 3	− 100	2·802
+ 2	+ 6	+ 14	+ 34	
+ 3	+ 7	+ 17	− 66 *	
+ 2	+ 10	+ 34	65·6576	
+ 5	+ 17	+ 51 *	−0·3424 *	
+ 2	+ 14	31·072		
+ 7	+ 31 *	82·072	·240081857616	
+ 2	7·84	37·856	−·102318142384 *	
+ 9 *	38·84	119·928 *		
·8	8·48			
+ 9·8	47·32	·112928808		
·8	9·12	120·040928808		
10·6	56·44 *	·112977624		
·8	·024404	120·153906432 *		
11·4	56·464404			
·8	·024408			
12·2 *	56·488812			
·002	·024412			
12·202	56·513224 *			
·002				
12·204				
·002				
12·206				
·002				
12·208 *				

Explanation.—1. Transform the equation into one whose roots are less by 2. The new equation is $x^4 + 9x^3 + 31x^2 + 51x - 66 = 0$. The roots corresponding to the one we are considering will now be a decimal.

2. Since $(\cdot 1)^2 = \cdot 01$; $(\cdot 1)^3 = \cdot 001$; and $(\cdot 1)^4 = \cdot 0001$, the first three terms are small in comparison to $51x$; therefore, $51x = 66$ nearly, whence 51 may be taken as a trial divisor to find the next figure of the root, considerable allowance being made for the omitted terms. At first we would be tempted to try ·9 for the value of x. But, upon transforming the equation into one whose roots are less by ·9, we shall find that the absolute term will become positive, which shows that ·9

is a superior limit. We therefore use ·8 for the next term of the root, and transform the equation into one whose roots are ·8 less. The transformed equation is $x^4 + 12{\cdot}2\,x^3 + 56{\cdot}44\,x^2 + 119{\cdot}928\,x - {\cdot}3424 = 0$. The root of this equation is now less than ·1.

3. Omitting the first three terms of the equation on account of their smallness, and using the coefficient of x as a trial divisor, we see that the root is less than ·01 and is about ·002. The next figure of the root is therefore 0, and the following one 2. Transform the equation into one whose roots are less by ·002; the resulting equation is $x^4 + 12{\cdot}208\,x^3 + 56{\cdot}513224\,x^2 + 120{\cdot}153906432\,x - {\cdot}102318142384 = 0$.

The work may be extended as far as we please.

767. **Remark 1.**—When the number of decimal places in the absolute term becomes equal to the number of such places desired in the root, we may begin to drop one figure in the preceding term (trial divisor), two in the next preceding term, and so on toward the left. When all the figures of the first term are exhausted, the remaining figures of the root may be found by simply dividing by the trial divisor.

768. **Remark 2.**—The absolute term after each transformation must be negative, else would the last figure of the root used be too large (a superior limit).

769. **Remark 3.**—The method may be applied with equal facility to extending an integral root after a sufficient number of initial figures have been obtained by trial or by Sturm's Theorem to distinguish the root from others of the equation. It may be used with exactness whenever there is an exact root; hence, the incorrectness of the title "Horner's Method of Approximation" given the method by most authors.

770. **Remark 4.**—The negative roots are the numerical equivalents of the positive roots of the equation resulting from changing the signs of the alternate terms, and may be found accordingly.

5. Solve $x^3 - 1728 = 0$, or find the $\sqrt[3]{1728}$.

Solution.

1	+ 0	+ 0	− 1728	(12
	10	100	1000	
	10	100	− 728 *	
	10	200	+ 728	
	20	300 *	0	
	10	64		
	30 *	364		
	32			

6. Extract the 5th root of 4312345 to thousandths, i. e., solve approximately $x^5 - 4312345 = 0$.

Solution.

1	0	0	0	0	− 4312345	21·229
	20	400	8000	160000	3200000	
	20	400	8000	160000	− 1112345 *	
	20	800	24000	640000		
	40	1200	32000	800000 *		
	20	1200	48000	84101	884101	
	60	2400	80000 *	884101	− 228244 *	
	20	1600	4101	88304		
	80	4000 *	84101	972405 *	198220·84832	
	20	101	4203		− 30023·15168 *	
	100 *	4101	88304	18699·2416		
	1	102	4306	991104·2416		
	101	4203	92610 *	18877·3264		
	1	103	886·208	1009981·5680 *		
	102	4306	93496·208			
	1	104	890·424			
	103	4410 *	94386·632			
	1	21·04	894·648			
	104	4431·04	95281·280 *			
	1	21·08				
	105 *	4452·12				
	·2	21·12				
	105·2	4473·24				
	·2	21·16				
	105·4	4494·40 *				
	·2					
	105·6					
	·2					
	105·8					
	·2					
	106·0 *					

The number of decimal places in the second remainder is greater than the number required in the root; therefore, the remaining figures may be found by dividing the remainder by 1009981·568 [767, Rem. 1].

EXERCISE 105.

Solve:

1. $x^2 - 704x - 58425 = 0$

2. $x^3 - 15348907 = 0$

3. $x^3 + 3x^2 - 3x - 7 = 0$

4. $x^4 - 4x^3 - 6x^2 + 32x - 26 = 0$

5. $x^4 - 19x^3 + 24x^2 + 712x - 40 = 0$

6. $x^5 + 12x^4 + 59x^3 + 150x^2 + 201x + 94 = 0$

7. $3x^4 + 24x^3 + 68x^2 + 82x - 964 = 0$

8. Find the cube root of 2

9. Find the fifth root of 5

10. $x^3 + 11x^2 - 102x + 181 = 0$

11. $x^4 + 9x^3 + 31x^2 + 51x - 66 = 0$

12. $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321 = 0$

13. One root of the equation $x^3 + 2x^2 + 3x - 13089030$ is 235. Find a cubic equation whose root is 225.

Cubic Equations.

771. A cubic equation containing an integral root may be readily factored.

Let $-a$ be a root of a cubic equation, then $x + a$ is a factor of the equation [719]. Let $x^2 + mx + n$ be the other factor; then,

$$(x + a)(x^2 + mx + n) = 0 \qquad \text{(A)}$$

or, $$x^3 + (a + m)x^2 + (am + n)x + an = 0 \qquad \text{(B)}$$

We now observe that if we subtract the factor a of the absolute term from the coefficient of x^2, and the factor n from the coefficient of x, the latter remainder divided by the former will give a, the root with the sign changed. This, then, is the condition under which a factor of the absolute term is a root with the sign changed.

Illustration.—Solve $x^3 - x^2 - 4x + 4 = 0$.

Solution: 1. The factors of $+4$ are $+2$ and $+2$; -2 and -2; $+4$ and $+1$; and -4 and -1.

Try whether $+4$ is a root with the sign changed.

Take $\quad -1$ and -4, the coefficients of x^2 and x.
Subtract, $\quad \underline{+4}$ and $\underline{+1}$, the factors of the absolute term.
Divide, $\quad -5) \quad -5($which $\neq 4$.

$\therefore$ 4 is not a root with the sign changed.

2. Try whether -2 is a root.

Take $\quad -1$ and -4
Subtract, $\quad \underline{+2}$ and $\underline{+2}$
Divide, $\quad -3) \quad -6(+2$

$\therefore \quad -2$ is a root.

Now, $x^3 - x^2 - 4x + 4 = (x+2)(x^2 - 3x + 2) = 0$;
whence, $\qquad x = -2$, 2, and 1.

Cardan's Formula.

772. I. The general cubic equation $x^3 + ax^2 + bx + c = 0$ may be transformed into an equation of the form of $y^3 + py + q = 0$, by putting $x = y - \frac{1}{3}a$.

Demonstration: Take $x^3 + ax^2 + bx + c = 0$. (A)

Put $\qquad x = y - \frac{1}{3}a$; then,

$$x^3 = y^3 - ay^2 + \frac{1}{3}a^2y - \frac{1}{27}a^3$$

$$ax^2 = ay^2 - \frac{2}{3}a^2y + \frac{1}{9}a^3$$

$$bx = by - \frac{1}{3}ab$$

$$c = c$$

$$\therefore x^3 + ax^2 + bx + c = y^3 + \left(b - \frac{1}{3}a^2\right)y + \left(\frac{2}{27}a^3 - \frac{1}{3}ab + c\right)$$

Put p for $b - \frac{1}{3}a^2$, and q for $\frac{2}{27}a^3 - \frac{1}{3}ab + c$; then,

$$x^3 + ax^2 + bx + c = y^3 + py + q = 0. \qquad \text{(A)}$$

773. II. The equation $y^3+py+q=0$ may be transformed into a quadratic by putting $y=z-\frac{p}{3z}$.

Demonstration: Take $y^3+py+q=0$. (B)

Put $$y=z-\frac{p}{3z}; \text{ then,}$$

$$y^3=z^3-pz+\frac{p^2}{3z}-\frac{p^3}{27z^3}$$

$$py= \quad pz-\frac{p^2}{3z}$$

$$q= \quad q$$

$$\therefore \quad y^3+py+q=z^3-\frac{p^3}{27z^3}+q=0;$$

whence, $27z^6+27qz^3-p^3=0$;

or, $z^6+qz^3-\frac{1}{27}p^3=0$. (C)

774. III. The roots of $z^6+qz^3-\frac{1}{27}p^3=0$ are

$$z=\left(-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}}; \text{ whence,}$$

$$y=z-\frac{p}{3z}=$$

$$\left(-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}}+\left(-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}} \quad \text{(D)}$$

This is Cardan's formula.

The values of x may be obtained by subtracting $\frac{1}{3}a$ from the values of y.

775. IV. Cardan's formula fails when all the roots are real and unequal.

For, let the roots be $a+\sqrt{b}$, $a-\sqrt{b}$, and c. Then, since the coefficient of y^2 is 0,

$(-a-\sqrt{b})+(-a+\sqrt{b})-c=0$ [726, 1];

whence, $c=-2a$.

The equation whose roots are $a+\sqrt{b}$, $a-\sqrt{b}$, and $-2a$, is $y^3-(3a^2+b)y-2(a^3-ab)=0$.

$\therefore p=-(3a^2+b)$, and $q=2(a^3-ab)$,

whence, $\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}=\left(a^2-\frac{1}{9}b\right)\sqrt{-3b}$, which is imaginary, and, therefore, irreducible when b is positive, or when the roots are all real and unequal.

Illustrations.—1. Solve $x^3-3x^2+4=0$.

Here $a=-3$; hence, $x=y-\left(\frac{-3}{3}\right)=y+1$.

Substitute, $(y+1)^3-3(y+1)^2+4=0$.

Reduce, $y^3-3y+2=0$. (1)

Here $p=-3$; hence, $y=z-\frac{p}{3z}=z+\frac{1}{z}$.

Substitute, $\left(z+\frac{1}{z}\right)^3-3\left(z+\frac{1}{z}\right)+2=0$.

Reduce, $z^6+2z^3=-1$.

Complete the square, $z^6+2z^3+1=0$.

Extract the $\sqrt{\ }$, $z^3+1=0$.

Factor, $(z+1)(z^2-z+1)=0$;

whence, $z=-1$ or $\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

$x=y+1=z+\frac{1}{z}+1=-1$, or 2, or 2.

---

776. *Sometimes an integral root can only be approximately found.*

2. Solve $x^3+3x^2+9x-13=0$. (A)

Here $a=3$; hence, $x=y-\left(\frac{3}{3}\right)=y-1$.

Substitute,

$(y-1)^3+3(y-1)^2+9(y-1)-13=0$.

Reduce, $y^3+6y-20=0$. (1)

Here $p=6$; hence, $y=z-\frac{p}{3z}=z-\frac{2}{z}$.

Substitute in (1),

$$\left(z-\frac{2}{z}\right)^3+6\left(z-\frac{2}{z}\right)-20=0.$$

Reduce, $z^6-20z^3=8$. (2)

Complete the square, $z^6-20z^3+100=108$.

Extract the $\sqrt{\ }$, $z^3-10=\pm 10{\cdot}392304$.

Transpose, $z^3=20{\cdot}392304$ or $-{\cdot}392304+$

Extract the $\sqrt[3]{\ }$, $z=2{\cdot}73+$ or $-{\cdot}73+$

$x=y-1=z-\frac{2}{z}+1=2{\cdot}73-{\cdot}73+1=3$, or $-{\cdot}73+2{\cdot}73+1=3$.

These two values are identical. The other two roots are found by dividing equation (A) by $x-3$.

EXERCISE 106.

Solve:

1. $x^3 - 3x^2 + 7x - 5 = 0$
2. $x^3 - 6x^2 + 10x - 8 = 0$
3. $x^3 - 11x^2 + 41x - 55 = 0$
4. $x^3 + 6x^2 + 14x + 12 = 0$
5. $x^3 - 4x^2 + 5x - 6 = 0$
6. $x^3 + 5x^2 + 6x + 8 = 0$
7. $x^3 + 7x^2 + 16x + 12 = 0$
8. $x^3 + x^2 - 8x - 12 = 0$
9. $x^3 - x^2 - 8x + 12 = 0$
10. $x^3 - 11x - 20 = 0$
11. $x^3 - 26x + 60 = 0$
12. $x^3 - 4x^2 + 3 = 0$
13. $x^3 - 4x^2 - 7x + 10 = 0$
14. $x^3 + 4x^2 - 7x - 10 = 0$

Recurring Equations.

777. A *Recurring Equation* is one in which the coefficients of the first and last terms, and of those equidistant from the first and last terms, are numerically equal, and the signs of the corresponding terms are either alike throughout or unlike throughout; as,

1. $x^5 - 4x^4 + 5x^3 + 5x^2 - 4x + 1 = 0.$
2. $x^5 + 3x^4 - 2x^3 + 2x^2 - 3x - 1 = 0.$
3. $x^6 + 4x^5 - 5x^4 + 3x^3 - 5x^2 + 4x + 1 = 0.$

778. In a recurring equation of an even degree in which the corresponding terms have unlike signs, the middle term is wanting.

For, according to definition, it is both positive and negative.

779. A *Reciprocal Equation* is one such that, if a is a root, $\frac{1}{a}$ is also a root.

780. ***Theorem I.****—A recurring equation is also a reciprocal equation.*

Demonstration: Let a be a root of

$$f_n(x) = x^n + Ax^{n-1} + Bx^{n-2} + \ldots \pm Bx^2 \pm Ax \pm 1 = 0 \qquad (A)$$

then, $$a^n + Aa^{n-1} + Ba^{n-2} + \ldots \pm Ba^2 \pm Aa \pm 1 = 0 \qquad (B)$$

Substitute $\frac{1}{a}$ for x in $f_n(x) = 0$,

$$\frac{1}{a^n} + \frac{A}{a^{n-1}} + \frac{B}{a^{n-2}} + \ldots \pm \frac{B}{a^2} \pm \frac{A}{a} \pm 1 = 0 \qquad (C)$$

whence, $$1 + Aa + Ba^2 + \ldots \pm Ba^{n-2} \pm Aa^{n-1} \pm a^n = 0 \qquad (D)$$

or, $$a^n + Aa^{n-1} + Ba^{n-2} + \ldots \pm Ba^2 \pm Aa \pm 1 = 0 \qquad (E)$$

Now, (E) is identical with (B); therefore, if a is a root of (A), $\frac{1}{a}$ is also a root.

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781. *Theorem II.*—*A recurring equation of an odd degree has* $+1$ *for a root when the signs of the corresponding terms are unlike.*

Demonstration:

Let $$x^{2n+1} + Ax^{2n} + Bx^{2n-1} + \ldots - Bx^2 - Ax - 1 = 0 \qquad (A)$$

then, $$(x^{2n+1} - 1) + Ax(x^{2n-1} - 1) + Bx^2(x^{2n-3} - 1) + \ldots = 0 \qquad (B)$$

Now, each term of (B) is divisible by $x - 1$ [134, P.];

$\therefore \quad x - 1 = 0$, or $x = 1$.

---

782. *Theorem III.*—*A recurring equation of an odd degree has* -1 *for a root when the signs of the corresponding terms are alike.*

Demonstration:

Let $$x^{2n+1} + Ax^{2n} + Bx^{2n-1} + \ldots + Bx^2 + Ax + 1 = 0 \qquad (A)$$

then, $$(x^{2n+1} + 1) + Ax(x^{2n-1} + 1) + Bx^2(x^{2n-3} + 1) + \ldots = 0 \qquad (B)$$

Now, each term of (B) is divisible by $x + 1$ [135, P.];

$\therefore \quad x + 1 = 0$, or $x = -1$.

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783. *Theorem IV.*—*A recurring equation of an even degree has* $+1$ *and* -1 *for roots when the signs of the corresponding terms are unlike.*

Demonstration:

Let $$x^{2n} + Ax^{2n-1} + Bx^{2n-2} + \ldots - Bx^2 - Ax - 1 = 0 \qquad (A)$$

then, $$(x^{2n} - 1) + Ax(x^{2n-2} - 1) + Bx^2(x^{2n-4} - 1) + \ldots = 0 \qquad (B)$$

Now, (B) is divisible by both $x - 1$ and $x + 1$ [134, 136, P.];

$\therefore \quad x - 1 = 0$ and $x + 1 = 0$; whence, $x = \pm 1$.

784. *Theorem V.*—*A recurring equation of an even degree may be transformed into an equation of one half the degree when the signs of the corresponding terms are alike.*

Demonstration:

Let $x^{2n} + Ax^{2n-1} + Bx^{2n-2} + \ldots . + Bx^2 + Ax + 1 = 0 \quad (A)$

Divide by x^n, and collect terms,

$$\left(x^n + \frac{1}{x^n}\right) + A\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + B\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \ldots . + P = 0 \quad (B)$$

Put $x + \frac{1}{x} = z$; then will

$$x^2 + \frac{1}{x^2} = z^2 - 2$$

$$x^3 + \frac{1}{x^3} = z^3 - 3z,$$

and, in general, each term of (B) may be transformed into a term of only half the degree.

Illustration.—

Take $x^6 + 4x^5 - 3x^4 + 2x^3 - 3x^2 + 4x + 1 = 0. \quad (A)$

Divide by x^3, $\quad x^3 + 4x^2 - 3x + 2 - \frac{3}{x} + \frac{4}{x^2} + \frac{1}{x^3} = 0 \quad (B)$

Rearrange the terms and factor,

$$x^3 + \frac{1}{x^3} + 4\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) + 2 = 0 \quad (C)$$

Put $\quad x + \frac{1}{x} = y; \quad (1)$

then, $\quad x^2 + 2 + \frac{1}{x^2} = y^2; \quad (2)$

or, $\quad x^2 + \frac{1}{x^2} = y^2 - 2;$

and, $\quad x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = y^3;$

or, $\quad x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = y^3;$

or, $\quad x^3 + \frac{1}{x^3} = y^3 - 3y. \quad (3)$

Substitute (1), (2), and (3) in (C),

$$(y^3 - 3y) + 4(y^2 - 2) - 3y + 2 = 0;$$

whence, $\quad y^3 + 4y^2 - 6y - 6 = 0.$

EXERCISE 107.

Solve :

1. $x^3 - 2x^2 + 2x - 1 = 0$
2. $x^3 - 3x^2 - 3x + 1 = 0$
3. $x^4 - 3x^3 + 3x - 1 = 0$
4. $x^4 + 3x^3 - 3x - 1 = 0$
5. $2x^4 - 5x^3 + 4x^2 - 5x + 2 = 0$
6. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$
7. $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$
8. $5x^5 + 11x^4 - 88x^3 - 88x^2 + 11x + 5 = 0$

Reduction of Binomial Equations.

785. A *Binomial Equation* is an equation of two terms, one of which is absolute ; as, $x^n \pm a = 0$.

786. *Every binomial equation can be reduced to the form* $y^n \pm 1 = 0$.

Demonstration: Take the general binomial equation $x^n \pm a = 0$.

Put $\frac{\sqrt[n]{a}}{y}$ for x, $\frac{a}{y^n} + a = 0$; whence, $y^n \pm 1 = 0$.

787. $y^n \pm 1 = 0$ is a recurring equation, and may be so solved.

Illustrative Solutions.—1. Solve $x^4 + 1 = 0$. (A)

Divide by x^2, $x^2 + \frac{1}{x^2} = 0$ (1)

Put $x + \frac{1}{x} = y$ (2)

Square $x^2 + 2 + \frac{1}{x^2} = y^2$

Transpose, $x^2 + \frac{1}{x^2} = y^2 - 2$

Substitute in (1), $y^2 - 2 = 0$

Factor, $(y + \sqrt{2}), (y - \sqrt{2}) = 0$;

whence, $y = \pm \sqrt{2}$.

Substitute in (2), $x + \frac{1}{x} = \pm \sqrt{2}$ (3)

whence, $x = \frac{1}{2}(\sqrt{2} \pm \sqrt{-2})$, or $-\frac{1}{2}(\sqrt{2} \mp \sqrt{-2})$.

CHAPTER XIII.

DETERMINANTS AND PROBABILITIES.

Introduction.

788. In the polynomial

$$a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1, \quad \text{(A)}$$

it will be seen :

1. That the letters a, b, and c of each term are arranged in natural order.

2. That the subscript figures, 1, 2, and 3, are distributed among the letters in the six different terms in as many ways as possible, using all in each term and making no repetitions.

3. That the first term contains no *inversions* of subscript figures, they advancing in natural order from left to right; the second term contains *one inversion*, 3 standing before 2; the third term contains *two inversions*, 2 and 3 both standing before 1; the fourth term contains *one inversion*, 2 standing before 1; the fifth term contains *two inversions*, 3 standing before 1 and 2; and the sixth term contains *three inversions*, 3 and 2 both standing before 1, and 3 standing before 2.

4. That in the positive terms there is an even number of inversions (zero being regarded an even number), and in the negative terms there is an odd number of inversions.

789. If we now arrange the nine different quantities found in (A) in a square, as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}; \qquad \text{(B)}$$

form all the possible products of them taken three together, using in each product one and only one from each row, and one and only one from each column; arrange the factors of the products in the natural literal order; consider those products positive which have an even number of inversions of subscript figures, and those negative which have an odd number; and take the algebraic sum of these products, we will have:

$$a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1. \qquad \text{(A)}$$

Therefore, form (B) may be taken as the representative of form (A), and when so taken it is called a *determinant*, and (A) its *development*.

790. Definition.—A *Determinant* is any n^2 quantities arranged in a square, as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \dots a_n \\ b_1 & b_2 & b_3 \dots b_n \\ c_1 & c_2 & c_3 \dots c_n \\ . & . & . \;\dots . \\ p_1 & p_2 & p_3 \dots p_n \end{vmatrix}, \qquad \text{(C)}$$

and interpreted to denote the algebraic sum of all the products that may be formed by taking one and only one quantity from each row, and one and only one from each column; arranging the letters of each product in the natural literal order, and regarding all products positive that have an even number of inversions of subscript figures, and all negative that have an odd number of such inversions.

791. The quantities contained in a determinant are called the *elements* of the determinant.

792. Determinants are divided into orders, named *second, third,* .... *n*th, accordingly as they contain 2^2, 3^2, ... n^2 elements.

Thus, $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is a determinant of the *second order.* Form (B) is a determinant of the *third order,* and form (C) a determinant of the *nth order.*

793. The diagonal joining the upper left-hand element with the lower right-hand element is called the *principal diagonal;* and the one joining the upper right-hand element with the lower left-hand element the *secondary diagonal.*

794. The product of all the elements along the principal diagonal is called the *principal term* of the development.

795. If the elements on the principal diagonal are known in order, the entire determinant may be written; hence it is that a determinant is often expressed by a modified form of the principal term of its development; as, $[a_1 b_2 c_3 \dots p_n]$, or $\Sigma(\pm a_1 b_2 c_3 \dots p_n)$.

796. It is evident that there are as many terms in the development of a determinant of the *n*th order as there are permutations of *n* things taken all together, or $\underline{|n}$.

Properties of Determinants.

797. If we rearrange the factors of the terms in form (A) so as to place the subscripts in natural order, we shall have

$$a_1 b_2 c_3 - a_1 c_2 b_3 + c_1 a_2 b_3 - b_1 a_2 c_3 + b_1 c_2 a_3 - c_1 b_2 a_3. \quad (A_1)$$

It will be observed that in form (A_1),

1. The value of each term and of the entire polynomial is the same as in form (A).

2. The first term contains no literal inversion; the second term contains *one*, c standing before b; the third term contains *two*, c standing before both a and b; the fourth term contains *one*, b standing before a; the fifth term contains *two*, b and c both standing before a; and the sixth term contains *three*, c and b both standing before a, and c before b.

3. The terms which contain an *even* number of literal inversions are *positive*, and those which contain an *odd* number *negative*.

798. If we now interchange the rows and columns in (B), giving us the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \qquad \text{(D)}$$

make all the possible products of three elements, using, each time, one and only one from each row, and one and only one from each column; arrange the factors of the products so that the subscripts stand in natural order; consider those products positive which have an even number of literal inversions, and those negative which have an odd number; and take the algebraic sum of these products, we shall have

$$a_1 b_2 c_3 - a_1 c_2 b_3 + c_1 a_2 b_3 - b_1 a_2 c_3 + b_1 c_2 a_3 - c_1 b_2 a_3. \qquad (A_1)$$

This shows that in a determinate of the third order an interchange of rows and columns does not change the value.

Is this law true for a determinant of the nth order?

1. It is evident that the number of terms in the development of both forms is the same, each being $\underline{|n}$.

2. Each term in the development of either form has a corresponding term of equal numerical value in the development of the other form, because both developments contain all the possible products of n elements that can be formed from the n^2 elements by taking one and only one from each row and one and only one from each column.

3. *The signs of the corresponding terms will be the same.* For the number of literal inversions in a term of the second development is equal to the number of subscript inversions in the corresponding term of the first development, as will readily appear from the fact that, if a subscript in any term of the first development follows r subscripts greater than itself, then, in the second development, the letter containing this subscript must precede r letters antecedent to it in the natural order. Therefore,

Prin. 1.—*An interchange of rows and columns in a determinant of any order does not change the value of the determinant.*

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799. In form (A) and in form (A_1) the second term equals *minus* the first term with the subscripts of b and c interchanged; the third term equals minus the second term with the subscripts of a and c interchanged; and so on, showing that any term in the development of a determinant of the third order equals minus some other term in the development with the subscripts of two factors interchanged.

Is this law true for the development of a determinant of the nth order?

1. It is evident that, if $Pc_r k_m$ be any term in the development of a determinant of the nth order, then will $Pc_m k_r$ be numerically another term of the development; because P in both instances is the product of $n-2$ ele-

ments, none of which are taken from rows c and k, and none from columns r and m; and $c_r k_m$ and $c_m k_r$ are different elements taken from these rows and columns and combined with P. Therefore, the products are not identical.

2. *The signs of the original and the derived terms are always opposite.* For,

(1) Suppose the two subscripts interchanged to be consecutive. Let the original term be $P c_m k_n Q$, and the derived term $P c_n k_m Q$. Since m and n follow all the subscripts contained in P and precede all contained in Q, an interchange of them can not affect the number of inversions they make with the subscripts of either P or Q; but such an interchange will either change a *natural* into an *inversion* or an *inversion* into a *natural*, either of which will evidently cause a change of sign.

(2) Suppose the two subscripts interchanged to be non-consecutive. Let the original term be $P c_n Q k_m R$, and the derived term $P c_m Q k_n R$. Suppose Q to contain q subscripts. Let m, in the original term, interchange consecutively with each of the subscripts in Q and with the subscript of c, then will it make $q+1$ interchanges before it becomes the subscript of c. n will now be the subscript of the first element in Q. Let it now interchange consecutively with each of the remaining subscripts in Q and with the subscript of k; then will it make q interchanges before it becomes the subscript of k. Therefore, for the two subscripts of c and k in the original term to interchange there must be made $2q+1$, or an odd number of consecutive interchanges, each one of which will cause a change of sign (1) in the entire term. Therefore, the sign of the term will be changed. Therefore,

Prin. 2.—*If two subscripts be interchanged in any term of the development of a determinant, another term of the development will be obtained whose sign is opposite to that of the original term.*

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800. If we let $P h_m Q k_n R$ be a term in the development of a determinant, then will $P k_m Q h_n R$ be the term formed by the elements which occupy the same places, if rows h and k be interchanged, and will have the same sign. But $P k_m Q h_n R$ is also a term of the development of the original determinant, and has there an opposite sign to $P h_m Q k_n R$ [P. 2]. Therefore,

Prin. 3.—*Interchanging two rows in a determinant changes the sign of the determinant.*

Cor. 1.—*Interchanging two columns in a determinant changes the sign of the determinant.*

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801. It is evident that if two columns or two rows of a determinant are in every respect alike, an interchange of them would not affect either the form or value of the determinant. But, according to Principle 3, the sign of the value would be changed. Now, both these statements can be true only when the value of the determinant is *zero.* Therefore,

Prin. 4.—*A determinant that has two rows or two columns identical equals zero.*

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802. Since every term in the development of a determinant contains one factor and only one from each row and one and only one from each column, it follows that,

Prin. 5.—*Multiplying or dividing all the elements of one row or one column of a determinant by any quantity multiplies or divides the determinant by that quantity.*

Cor. 1.—Changing the signs of all the elements in any row or column changes the sign of the determinant.

Cor. 2.—If two rows or two columns of a determinant differ only by a common factor, the value of the determinant is zero.

803. Definitions.—If any number of rows and the same number of columns be *deleted* (stricken out) of a determinant, the remaining elements, taken in order, form a determinant called a *minor*, and the elements common to the deleted rows and columns form another *minor*. These minors are said to be *complementary*.

Thus, in the following determinant of the fourth order,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

the complementary minors are

$$\begin{vmatrix} b_1 & d_1 \\ b_3 & d_3 \end{vmatrix} \text{ and } \begin{vmatrix} a_2 & c_2 \\ a_4 & c_4 \end{vmatrix}.$$

804. If a single row and a single column be deleted, the remaining minor is called the *principal minor*, and it, together with its complementary minor, which in this instance is a single element, are called *cofactors*.

805. Problem. To develop a determinant.

Let it be required to develop

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

into a series of determinants of a lower order.

Let A_1, A_2, A_3, and A_4 represent respectively the cofactors of a_1, a_2, a_3, and a_4. Then, it is readily seen that all the terms in the development containing the factor a_1 are formed from a_1 and its cofactor A_1, and the sum of these terms is $a_1 A_1$. Similarly, the sum of all the terms containing a_2 is $a_2 A_2$, the sum of all the terms containing a_3 is $a_3 A_3$, and the sum of all the terms containing a_4 is $a_4 A_4$. Now, in each term of $a_2 A_2$ there occurs one more inversion than in each term of $a_1 A_1$, since a subscript 2 will precede a subscript 1; similarly, in each term of $a_3 A_3$ there occur two more inversions of subscripts than in $a_1 A_1$, and in each term of $a_4 A_4$ there occur three more inversions of subscripts than in $a_1 A_1$.

Therefore,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

Therefore,

Rule.*—Multiply each element of the first row by its cofactor, making the products alternately plus and minus, and take the algebraic sum of the results.*

806. ***Scholium.****—The successive application of this rule will eventually make the full development of any determinant depend upon the development of a determinant of the second order.* Thus,

$$\begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} = b_2 \begin{vmatrix} c_3 & c_4 \\ d_3 & d_4 \end{vmatrix} - b_3 \begin{vmatrix} c_2 & c_4 \\ d_2 & d_4 \end{vmatrix} + b_4 \begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix} =$$

$$b_2 (c_3 d_4 - c_4 d_3) - b_3 (c_2 d_4 - c_4 d_2) + b_4 (c_2 d_3 - c_3 d_2)$$

Illustrative Example.—Find the value of $\begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 4 \\ 4 & 3 & 2 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 4 \\ 4 & 3 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} - 3 \begin{vmatrix} 3 & 4 \\ 4 & 2 \end{vmatrix} + 4 \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} =$$

$$2(2 \times 2 - 4 \times 3) - 3(3 \times 2 - 4 \times 4) + 4(3 \times 3 - 2 \times 4) =$$

$$-16 + 30 + 4 = 18.$$

EXERCISE 108.

Find the value of :

1. $\begin{vmatrix} 3 & 1 & 5 \\ 4 & 2 & 7 \\ 1 & 6 & 4 \end{vmatrix}$ 2. $\begin{vmatrix} 4 & 3 & 2 \\ 5 & 1 & 4 \\ 2 & 4 & 5 \end{vmatrix}$ 3. $\begin{vmatrix} 3 & -1 & 4 \\ 2 & 1 & -3 \\ 4 & 2 & 7 \end{vmatrix}$

4. $\begin{vmatrix} a & 0 & b \\ -a & b & 0 \\ a & b & c \end{vmatrix}$ 5. $\begin{vmatrix} m & 0 & n \\ m & p & 0 \\ 0 & p & n \end{vmatrix}$ 6. $\begin{vmatrix} b+c & c & b \\ c & a+c & a \\ b & a & a+b \end{vmatrix}$

7. $\begin{vmatrix} 2 & 3 & 2 & 3 \\ 3 & 2 & 3 & 2 \\ 2 & 3 & 3 & 2 \\ 3 & 2 & 2 & 3 \end{vmatrix}$ 8. $\begin{vmatrix} -1 & 2 & 3 & 4 \\ 2 & -1 & 3 & 4 \\ 3 & 2 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{vmatrix}$ 9. $\begin{vmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 3 & 2 & 0 & 2 \\ 4 & 1 & 2 & 0 \end{vmatrix}$

10. $\begin{vmatrix} 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 4 \\ 2 & 3 & 1 & 2 \\ 1 & 4 & 3 & 2 \end{vmatrix}$ 11. $\begin{vmatrix} 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 \end{vmatrix}$ 12. $\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ a & b & c & d \\ i & k & l & m \end{vmatrix}$

13. $\begin{vmatrix} 3 & 4 & 1 & 2 \\ 6 & 8 & 2 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{vmatrix}$ 14. $\begin{vmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \\ 4 & 8 & 0 & 1 \end{vmatrix}$ 15. $\begin{vmatrix} a & b & c & d \\ \frac{a}{n} & \frac{b}{n} & \frac{c}{n} & \frac{d}{n} \\ a & c & b & d \\ c & a & d & b \end{vmatrix}$

Additional Properties of Determinants.

807. ***Prin. 6.***—*If every element of one row, or column, of a determinant is a binomial, the determinant can be expressed as the sum of two other determinants, one of which is derived from the original determinant by dropping the* ***second*** *terms of the binomials and the other by dropping the* ***first*** *terms.*

Demonstration.—Each term of the development contains one and only one of the binomial elements as a factor. Therefore, each term of the development can be separated into two terms, one of which is the first term of the binomial factor times the remaining factors of the term, and the other the second term of the binomial factor times the remaining factors of the term. The sum of the component parts that contain the first terms of the binomial elements will form a determinant which is independent of the second terms of the binomial elements, and the sum of the component parts that contain the second terms of the binomial elements will form a second determinant which is independent of the first terms of the binomial elements.

Thus, $$\begin{vmatrix} a & b+c & b \\ b & a+c & a \\ c & a+b & b \end{vmatrix} = \begin{vmatrix} a & b & b \\ b & a & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} a & c & b \\ b & c & a \\ c & b & b \end{vmatrix}$$

808. ***Cor. 1.***—*If every element in any row, or column, consists of* ***m*** *terms, the determinant can be expressed as the sum of* ***m*** *other determinants.*

809. ***Cor. 2.***—*If the elements of* ***r*** *rows, or columns, consist of* ***a, b, c, ... m*** *terms respectively, the determinant can be expressed as the sum of* ***abc.... m*** *determinants.*

810. ***Scholium.***—*These truths are of value in reducing a determinant when one or more of the derived determinants reduce to zero.* Thus,

$$\begin{vmatrix} a & b+a+c & c \\ d & e+d+f & f \\ g & h+g+k & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} + \begin{vmatrix} a & a & c \\ d & d & f \\ g & g & k \end{vmatrix} + \begin{vmatrix} a & c & c \\ d & f & f \\ g & k & k \end{vmatrix} =$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} + 0 + 0 \ [801, \text{P.}] = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

811. ***Prin. 7.***—*If to all the elements of any row, or column, be added equimultiples of the corresponding elements of any other row, or column, the value of the determinant will remain unchanged.*

Demonstration.—Consider a determinant of the third order. Thus,

Prove $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+p\,a_3 & a_2 & a_3 \\ b_1+p\,b_3 & b_2 & b_3 \\ c_1+p\,c_3 & c_2 & c_3 \end{vmatrix}$$

Demonstration.—

$$\begin{vmatrix} a_1 + p\,a_3 & a_2 & a_3 \\ b_1 + p\,b_3 & b_2 & b_3 \\ c_1 + p\,c_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} p\,a_3 & a_2 & a_3 \\ p\,b_3 & b_2 & b_3 \\ p\,c_3 & c_2 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 0 \text{ [802, Cor. 2]} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The method of proof employed in this example is general, and is, therefore, applicable to a determinant of any order.

812. ***Cor.***—*It may also be shown that*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + p\,a_2 + q\,a_3 & a_2 & a_3 \\ b_1 + p\,b_2 + q\,b_3 & b_2 & b_3 \\ c_1 + p\,c_2 + q\,c_3 & c_2 & c_3 \end{vmatrix}; \text{ etc.}$$

That is,

To all the elements of any row, or column, may be added equimultiples of the corresponding terms of a second row, or column, and again equimultiples of the corresponding terms of a third row, or column, etc.

813. ***Scholium.***—*This principle is of practical value in the reduction of a determinant, if, by its application, two rows, or columns, can be made identical, or one of them a multiple of the other.* Thus,

$$\begin{vmatrix} 5 & 8 & 11 \\ 6 & 9 & 12 \\ 7 & 10 & 13 \end{vmatrix} = \begin{vmatrix} 5 + 8 + 11 & 8 & 11 \\ 6 + 9 + 12 & 9 & 12 \\ 7 + 10 + 13 & 10 & 13 \end{vmatrix} = \begin{vmatrix} 24 & 8 & 11 \\ 27 & 9 & 12 \\ 30 & 10 & 13 \end{vmatrix} = 0$$

[802, Cor. 2]. *It may also be used to simplify a complex determinant.*

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814. ***Prin. 8.***—*If the elements of one column of a determinant be multiplied by the cofactors of the corresponding elements of another column, the sum of the products, taken alternately plus and minus, is zero.*

Demonstration.—Take two determinants, alike in every respect, except that, in the second, the qth column is identical with the pth column. Now, in the first, the sum of the products of the elements in the pth column and the cofactors of the corresponding elements of the qth column, when the products are taken alternately plus and minus, is $a_p A_q - b_p B_q + \ldots.$

The value of the second determinant is

$$a_q A_q - b_q B_q + \ldots. \text{ [805, R.]} = 0 \text{ [801, P.].}$$

But it is evident that $a_q, b_q, \ldots.$ are identical with $a_p, b_p, \ldots.$

Therefore, $a_p A_q - b_p B_q + \ldots. = 0.$

EXERCISE 109.

Find the value of:

1. $\begin{vmatrix} 3 & 2 & 3 \\ 5 & 3 & 4 \\ 7 & 4 & 5 \end{vmatrix}$ **2.** $\begin{vmatrix} -4 & 2 & 0 \\ 6 & -2 & 2 \\ -9 & 3 & -3 \end{vmatrix}$ **3.** $\begin{vmatrix} 12 & -5 & 2 \\ -3 & 4 & 5 \\ 7 & 0 & 7 \end{vmatrix}$

4. $\begin{vmatrix} 0 & -1 & -2 & 1 \\ 8 & 5 & 2 & 1 \\ 6 & 9 & 8 & 1 \\ -8 & 4 & 8 & 1 \end{vmatrix}$ **5.** $\begin{vmatrix} y+z & x-z & x-y \\ y-z & z+x & y-x \\ z-y & z-x & x+y \end{vmatrix}$

6. $\begin{vmatrix} a+p & b+q & c+r \\ b+p & c+q & a+r \\ c+p & a+q & b+r \end{vmatrix} + \begin{vmatrix} a-p & b-q & c-r \\ b-p & c-q & a-r \\ c-p & a-q & b-r \end{vmatrix}$

7. Prove $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$

8. Prove $\begin{vmatrix} a^2-bc & c^2-ab \\ c^2-ab & b^2-ac \end{vmatrix} = c(3abc - a^3 - b^3 - c^3)$

Multiplication of Determinants.

815. ***Lemma.***—

$$\begin{vmatrix} a_1 & b_1 & c_1 & l & m & n \\ a_2 & b_2 & c_2 & p & q & r \\ a_3 & b_3 & c_3 & s & t & u \\ 0 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Demonstration.—

$$\begin{vmatrix} a_1 & b_1 & c_1 & l & m & n \\ a_2 & b_2 & c_2 & p & q & r \\ a_3 & b_3 & c_3 & s & t & u \\ 0 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & p & q & r \\ b_3 & c_3 & s & t & u \\ 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & l & m & n \\ b_3 & c_3 & s & t & u \\ 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix}$$

$$+ a_3 \begin{vmatrix} b_1 & c_1 & l & m & n \\ b_2 & c_2 & p & q & r \\ 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} = a_1 b_2 \begin{vmatrix} c_3 & s & t & u \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} - a_1 b_3 \begin{vmatrix} c_2 & p & q & r \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix}$$

$$- a_2 b_1 \begin{vmatrix} c_3 & s & t & u \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} + a_2 b_3 \begin{vmatrix} c_1 & l & m & n \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} + a_3 b_1 \begin{vmatrix} c_2 & p & q & r \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix}$$

$$- a_3 b_2 \begin{vmatrix} c_1 & l & m & n \\ 0 & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} = a_1 b_2 c_3 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} - a_1 b_3 c_2 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$- a_2 b_1 c_3 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} + a_2 b_3 c_1 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} + a_3 b_1 c_2 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$- a_3 b_2 c_1 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad [789, (A)].$$

Scholium.—It will be seen that the elements $l, m, n, p, q, r, s, t, u$ of the first member do not appear in the second member. This is due to the fact that in the development of the first member all the terms containing these elements eventually vanish.

816. Problem. To find the product of two determinants of any order in terms of a determinant of the same order.

Example.—Find the product of:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Explanation:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} \quad [815]$$

$$= \begin{vmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ a_1x_1+a_2y_1+a_3z_1 & b_1x_1+b_2y_1+b_3z_1 & c_1x_1+c_2y_1+c_3z_1 & x_1 & y_1 & z_1 \\ a_1x_2+a_2y_2+a_3z_2 & b_1x_2+b_2y_2+b_3z_2 & c_1x_2+c_2y_2+c_3z_2 & x_2 & y_2 & z_2 \\ a_1x_3+a_2y_3+a_3z_3 & b_1x_3+b_2y_3+b_3z_3 & c_1x_3+c_2y_3+c_3z_3 & x_3 & y_3 & z_3 \end{vmatrix}$$

(This last form is derived from the preceding by adding to the first column a_1 times the 4th column + a_2 times the 5th column + a_3 times the 6th column; to the second column, b_1 times the 4th + b_2 times the 5th + b_3 times the 6th; and to the third column, c_1 times the 4th + c_2 times the 5th + c_3 times the 6th [812])

$$= \begin{vmatrix} a_1x_1 + a_2y_1 + a_3z_1 & b_1x_1 + b_2y_1 + b_3z_1 & c_1x_1 + c_2y_1 + c_3z_1 \\ a_1x_2 + a_2y_2 + a_3z_2 & b_1x_2 + b_2y_2 + b_3z_2 & c_1x_2 + c_2y_2 + c_3z_2 \\ a_1x_3 + a_2y_3 + a_3z_3 & b_1x_3 + b_2y_3 + b_3z_3 & c_1x_3 + c_2y_3 + c_3z_3 \end{vmatrix}$$

[805, R.].

Let the student observe how the elements of the first column of the product are derived from the elements of the multiplicand and multiplied; then, how the elements of the second and third columns are found. The laws which he will observe are general for determinants of any order.

Example.—Show that, according to the laws above observed,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} a_1x_1 + a_2y_1 & b_1x_1 + b_2y_1 \\ a_1x_2 + a_2y_2 & b_1x_2 + b_2y_2 \end{vmatrix}$$

EXERCISE 110.

Find the value of:

1. $\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \times \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} \times \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} \times \begin{vmatrix} 3 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{vmatrix}$

4. $\begin{vmatrix} 2 & 0 & 3 \\ 3 & 1 & 0 \\ 2 & 0 & 3 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 3 & 2 \end{vmatrix}$

Applications.

817. I. *Solution of Simultaneous Equations of the First Degree.*

1. Solve $a_1 x + b_1 y = r_1$ (A)

$a_2 x + b_2 y = r_2$ (B)

Solution: Multiply (A) by A_1 and (B) by $-A_2$, in which A_1 and A_2 are the cofactors of a_1 and a_2 in the determinant $[\, a_1 \;\; b_2 \,]$, and take the sum,

$$(a_1 A_1 - a_2 A_2)x + (b_1 A_1 - b_2 A_2)y = r_1 A_1 - r_2 A_2 \qquad \text{(C)}$$

Now, $b_1 A_1 - b_2 A_2 = 0$ [814, P.]. Therefore,

$$(a_1 A_1 - a_2 A_2)x = r_1 A_1 - r_2 A_2; \text{ whence,}$$

$$x = \frac{r_1 A_1 - r_2 A_2}{a_1 A_1 - a_2 A_2} = \frac{[\, r_1 \;\; b_2 \,]}{[\, a_1 \;\; b_2 \,]} \text{ [805, 795].}$$

Again, multiply (A) by B_1 and (B) by $-B_2$, in which B_1 and B_2 are the cofactors of b_1 and B_2, and take the sum,

$$(a_1 B_1 - a_2 B_2)x + (b_1 B_1 - b_2 B_2)y = r_1 B_1 - r_2 B_2 \qquad \text{(D)}$$

Now, $a_1 B_1 - a_2 B_2 = 0$ [814, P.]. Therefore,

$$y = \frac{r_1 B_1 - r_2 B_2}{b_1 B_1 - b_2 B_2} = \frac{[\, a_1 \;\; r_2 \,]}{[\, a_1 \;\; b_2 \,]} \text{ [805, 795].}$$

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2. Solve $a_1 x + b_1 y + c_1 z = r_1$ (A)

$a_2 x + b_2 y + c_2 z = r_2$ (B)

$a_3 x + b_3 y + c_3 z = r_3$ (C)

Solution: Multiply (A) by A_1, (B) by $-A_2$, (C) by A_3, in which A_1, A_2, and A_3 are respectively the cofactors of a_1, a_2, and a_3 in the determinant $[\, a_1 \;\; b_2 \;\; c_3 \,]$, and add the resulting equations,

$$(a_1 A_1 - a_2 A_2 + a_3 A_3)x + (b_1 A_1 - b_2 A_2 + b_3 A_3)y$$
$$+ (c_1 A_1 - c_2 A_2 + c_3 A_3)z = r_1 A_1 - r_2 A_2 + r_3 A_3$$

Now, the coefficients of y and z vanish [814, P.];

$$\therefore \quad x = \frac{r_1 A_1 - r_2 A_2 + r_3 A_3}{a_1 A_1 - a_2 A_2 + a_3 A_3} = \frac{[\, r_1 \;\; b_2 \;\; c_3 \,]}{[\, a_1 \;\; b_2 \;\; c_3 \,]} \text{ [805, 795].}$$

Then, by symmetry,

$$y = \frac{[\, a_1 \;\; r_2 \;\; c_3 \,]}{[\, a_1 \;\; b_2 \;\; c_3 \,]} \text{ and } z = \frac{[\, a_1 \;\; b_2 \;\; r_3 \,]}{[\, a_1 \;\; b_2 \;\; c_3 \,]}.$$

In a similar manner it may be shown that, if we take n equations of the first degree of the form of $a_1 x + b_1 y + c_1 z + \ldots\ldots = r_1$, multiply the first by A_1, the second by $-A_2$, the third by A_3, ..., in which A_1, A_2, A_3, ..., are the cofactors of a_1, a_2, a_3, ..., and take the sum, the coefficients of all the unknown quantities, except x, will vanish, and we shall have

$$x = \frac{[\, r_1 \ b_2 \ c_3 \ldots\ldots]}{[\, a_1 \ b_2 \ c_3 \ldots\ldots]}; \text{ and, by symmetry,}$$

$$y = \frac{[\, a_1 \ r_2 \ c_3 \ldots\ldots]}{[\, a_1 \ b_2 \ c_3 \ldots\ldots]}, \text{ etc. Therefore,}$$

Principle.—*Any unknown quantity in a complete system of simultaneous equations of the first degree equals a fraction whose denominator is a determinant formed from the coefficients of the terms of the equations taken in order, and whose numerator is formed from the denominator by replacing the coefficients of the unknown quantity by the corresponding right members of the equations.*

EXERCISE 111.

Solve :

1. $3x + 2y = 16$
$2x - 3y = 2$

2. $5x - 3y = 6$
$2x + 5y = 21$

3. $ax + by = c$
$mx + ny = d$

4. $(a+b)x - (c+d)y = m$
$(a-b)x + (c-d)y = n$

5. $2x + 3y - 2z = 5$
$3x - 2y + 4z = 16$
$4x - 3y - z = -5$

6. $x - y + z = 6$
$3x + 5y - 3z = 14$
$2x + 4y + 3z = 20$

7. $ax + by + cz = d$
$cx + by + az = e$
$bx + cy + az = h$

8. $(a+b)x + by + az = m$
$ax + (a+b)y + bz = n$
$bx + ay + (a+b)z = r$

9. $x + y + z + u = 14$
$x - y + z - u = -2$
$x + y - z - u = -4$
$x - y - z + u = 0$

10. $ax + by + cz = m$
$bx + cy + au = n$
$cx + az + bu = p$
$ay + bz + cu = q$

11. $2x - 3y + 2z + u = -12$
$3x + 2y - 3z + 2u = 12$
$x - 3y + 4z + 3u = -24$
$2x + 2y - 3z - 4u = 28$

12. $ax + by + cz + du = p$
$ax - by + cz - du = q$
$ax + by - cz - du = r$
$ax - by - cz + du = s$

13. $2x + 3y - 4z + 2u + 3v = 19$
$3x - 2y + 2z - 3u + 4v = 13$
$2x - 4y + 3z - 2u + 2v = 5$
$x + y - 3z + 2u + v = 7$
$x + 2y + 3z - 4u + 5v = 23$

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818. II. *To determine under what condition* $(n+1)$ *equations of* n *unknown quantities may be simultaneously true.*

Assume the equations

$$a_1 x + b_1 y + c_1 = 0, \qquad \text{(A)}$$
$$a_2 x + b_2 y + c_2 = 0, \qquad \text{(B)}$$
and
$$a_3 x + b_3 y + c_3 = 0 \qquad \text{(C)}$$
to be simultaneously true.

Multiply (A) by C_1, (B) by $-C_2$, and (C) by C_3, in which C_1, C_2, and C_3 are the cofactors of c_1, c_2, and c_3 in the determinant $[\,a_1\ b_2\ c_3\,]$, and add the results. Then,

$$(a_1 C - a_2 C_2 + a_3 C_3)x + (b_1 C_1 - b_2 C_2 + b_3 C_3)y$$
$$+ (c_1 C_1 - c_2 C_2 + c_3 C_3) = 0 \text{ [Ax. 2].}$$

But, $a_1 C - a_2 C_2 + a_3 C_3 = b_1 C_1 - b_2 C_2 + b_3 C_3 =$
$$c_1 C_1 - c_2 C_2 + c_3 C_3 = [\,a_1\ b_2\ c_3\,] \text{ [814, P.; 805, R.].}$$

$\therefore$ $[\,a_1\ b_2\ c_3\,] = 0$ is the condition under which (A), (B), and (C) are simultaneously true.

Note.—The equation $[\,a_1\ b_2\ c_3\,] = 0$ is called the *eliminant* of the group.

In a similar manner it may be shown that $n+1$ equations of n unknown quantities, of the form of

$$a_1 x + b_1 y + \ldots . + r_1 = 0,$$
are simultaneously true, when
$$[\,a_1\ b_2\ c_3 \ldots . r_{n+1}\,] = 0.$$

Note.—Equations that are simultaneously true are said to *consist*, that is, they are *consistent*.

EXERCISE 112.

Test the consistency of :

1. $2x+3y-13=0$
   $3x+2y-12=0$
   $x+3y-11=0$

2. $3x+2y-17=0$
   $5x-3y-3=0$
   $2x-5y+15=0$

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819. III. *To eliminate x from any two rational integral equations in x.*

Illustrations.—1. Eliminate x from the equations

$$ax^2+bx+c=0 \quad \text{(A)}$$
$$mx^2+nx+r=0 \quad \text{(B)}$$

Solution : It is evident that

$$\begin{aligned} ax^3+bx^2+cx \quad &=0\\ ax^2+bx+c&=0\\ mx^3+nx^2+rx \quad &=0\end{aligned}$$

and

$$mx^2+nx+r=0$$

are simultaneously true. Therefore,

$$\begin{vmatrix} a & b & c & 0\\ 0 & a & b & c\\ m & n & r & 0\\ 0 & m & n & r \end{vmatrix}=0$$

2. Eliminate x from the equations

$$ax^4+bx^3+cx^2+dx+e=0$$
$$mx^3+nx^2+px+q=0$$

Solution : It is evident that

$$\begin{aligned} ax^5+bx^4+cx^3+dx^2+ex \quad &=0\\ ax^4+bx^3+cx^2+dx+e&=0\\ mx^5+nx^4+px^3+qx^2 \quad &=0\\ mx^4+nx^3+px^2+qx \quad &=0\\ mx^3+nx^2+px+q&=0\end{aligned}$$

are simultaneously true. Therefore,

$$\begin{vmatrix} a & b & c & d & e & 0\\ 0 & a & b & c & d & e\\ m & n & p & q & 0 & 0\\ 0 & m & n & p & q & 0\\ 0 & 0 & m & n & p & q \end{vmatrix}=0$$

This method is known as Sylvester's Method of Elimination.

Probabilities.

Definitions and Fundamental Principles.

820. When the number of ways in which an event may occur is greater than the number of ways in which it may fail, and the ways are equally likely to happen, we say:

1. The event is probable.

2. The event is likely to happen.

3. The chance is in favor of the event.

4. The odds are in favor of the event.

821. When the number of ways in which an event may fail is greater than the number of ways in which it may occur, and the ways are equally likely to happen, we say:

1. The event is improbable.

2. The event is not likely to happen.

3. The chance is against the event.

4. The odds are against the event.

822. When the number of ways in which an event may occur is equal to the number of ways in which it may fail, and the ways are equally likely to happen, we say:

1. The occurrence and failure of the event are equally probable.

2. The event is as likely to happen as to fail.

3. There is an even chance for and against the event.

4. The odds are even for and against the event.

823. If an event can occur in a ways and fail in b ways, and the ways are equally likely to happen, we need more definite language to express the exact probability or chance of the event. Thus, we say:

1. The odds are as a to b in favor of the event.

2. The odds are as b to a against the event.

824. If we let k represent the probability of any particular way happening, a the number of ways favorable to the event, and b the number of unfavorable ways, then will ak represent the probability of the event happening, and bk the probability of its failing, and $ak + bk$, or $(a + b)k$, *certainty*, which is taken as the unit of measure, then $(a + b)k = 1$; whence, $k = \frac{1}{a+b}$;

and $ak = \frac{a}{a+b}$, the probability or chance of the event, and $bk = \frac{b}{a+b}$, the probability or chance against the event. Therefore,

Prin. 1.*—The probability or chance of an event happening equals the number of favorable ways divided by the whole number of ways.*

Prin. 2.*—The probability or chance of an event failing equals the number of unfavorable ways divided by the whole number of ways.*

825. Since an event is certain to happen or fail, and certainty is expressed by unity, it follows that,

Prin. 3.*—The probability of an event happening equals unity minus the probability that it will fail; and the probability that it will fail equals unity minus the probability that it will happen.*

Illustration.—If there are 3 black and 2 white balls in a bag containing only 5 balls, what is the chance,

1. That a black ball will be drawn on the first trial?
2. That a black ball will not be drawn on the first trial?

Solution: 1. There are 3 favorable ways out of 5 to draw a black ball; therefore, the chance is $\frac{3}{5}$ (Prin. 1).

2. There are 2 unfavorable ways out of 5 to draw a black ball, namely, the two favorable ways for drawing a white ball; therefore, the chance of failing to draw a black ball is $\frac{2}{5}$. Or,

That a black ball will be drawn or not drawn on the first trial is *certainty*. The chance for drawing a black ball is $\frac{3}{5}$; therefore, the chance of failure is $1 - \frac{3}{5} = \frac{2}{5}$.

826. *Exclusive Events.*—Two or more events are mutually exclusive when the happening of one of them precludes the possibility of any other one happening. Thus, if a coin be thrown up, it may fall either head or tail. If it fall head, or is supposed to fall head, it can not fall tail, or be supposed to fall tail, in the same throw. Falling head and falling tail are, therefore, mutually exclusive events.

827. In a bag are d balls; a of them are white, b blue, c red, and the remaining ones yellow. What is the chance of drawing, on the first trial,

1. Either a red or a white ball?
2. A red, a white, or a blue ball?

Solution: 1. The chance of drawing a red ball is $\frac{c}{d}$, and the chance of drawing a white ball is $\frac{a}{d}$; and the chance of drawing either a red or a white ball is $\frac{a+c}{d} = \frac{a}{d} + \frac{c}{d}$.

2. The chance of drawing a red ball is $\frac{c}{d}$; of drawing a white ball, $\frac{a}{d}$; of drawing a blue ball, $\frac{b}{d}$; and of drawing a red, a white, or a blue ball, $\frac{c+a+b}{d}$; which equals $\frac{c}{d} + \frac{a}{d} + \frac{b}{d}$. Therefore,

Prin. 4.—*The chance that one of several mutually exclusive events will happen equals the sum of their separate chances of happening.*

EXERCISE 118.

1. What is the chance of throwing 4 with a single die?

Suggestion.—A die has six faces, which are equally liable to turn up, but only one of these contains four dots. Therefore, the chance is $\frac{1}{6}$.

2. What is the chance of throwing an even number with a single die?

Suggestion.—Three of the faces have an even number of dots; therefore, the chance is $\frac{3}{6}$, or $\frac{1}{2}$.

3. If the odds be 4 to 3 in favor of an event, what are the respective chances of the success and failure of the event?

Suggestion.—There are 4 points out of 7 favorable and 3 out of 7 unfavorable to the happening of the event; therefore, the respective chances of success and failure are $\frac{4}{7}$ and $\frac{3}{7}$.

4. If 4 coppers are tossed, what are the odds against exactly 2 turning up head?

Suggestion.—Each coin may fall in two ways; hence, the four coins may fall in $2^4 = 16$ ways [550, Cor.]. The two coins that may turn up head can be selected from the four coins in $\frac{4 \times 3}{\underline{|2}}$, or 6 ways. Therefore, the chance of success is $\frac{6}{16}$, or $\frac{3}{8}$, and the chance of failure is $1 - \frac{3}{8} = \frac{5}{8}$. Therefore, the odds are as 5 to 3 against the event.

5. In a bag are 7 white and 5 red balls; if two are drawn, find the chance that 1 is red and 1 white.

Solution: Two balls can be selected from 12 balls in $\frac{12 \times 11}{\underline{|2}} = 66$ ways. One white ball can be selected from 7 white balls in 7 ways, and 1 red ball from 5 red balls in 5 ways. Hence, 1 white ball and 1 red ball can be selected from 7 white and 5 red balls in 7×5, or 35 ways. Therefore, 35 out of 66 ways are favorable to drawing 1 white and 1 red ball. Therefore, the chance is $\frac{35}{66}$.

6. Twenty persons take their seats at a round table. What are the odds against two persons thought of sitting together?

Solution: Let the two persons be A and B. Besides the place where A may sit, there are 19 places, two of which are adjacent to him, and the remaining 17 not adjacent. Any of these B may select. Therefore, the odds are as 17 to 2 against A and B sitting together.

828. *Expectation.* — The value of any probability of prize or property depending upon the occurrence of some uncertain event is called an *Expectation.*

7. A person holds a tickets in a lottery in which the whole number of tickets issued is n. There is only one prize offered, and this is worth $\$p$. What is the person's expectation?

Solution: It is evident that the n tickets are worth $\$p$, and that the tickets are of equal value before the drawing; therefore, the a tickets are worth $\frac{a}{n}$ of $\$p$, which is $\$p \times \frac{a}{n}$. Therefore,

829. ***Prin. 5.***—*The expectation of an event equals the product of the sum to be realized and the chance of the event.*

8. A person is allowed to draw two bank-notes from a bag containing 8 ten-dollar bills and 20 two-dollar bills. What is his expectation?

Solution: The two notes can be drawn from 28 notes in $\frac{28 \times 27}{\lfloor 2}$ $= 378$ ways. Two ten-dollar notes can be drawn from 8 ten-dollar notes in $\frac{8 \times 7}{\lfloor 2} = 28$ ways. Two two-dollar notes can be drawn from 20 two-dollar notes in $\frac{20 \times 19}{\lfloor 2} = 190$ ways.

One ten-dollar note and one two-dollar note can be drawn from 8 ten-dollar notes and 20 two-dollar notes in $8 \times 20 = 160$ ways. Therefore,

The chance of drawing \$20 is $\frac{28}{378}$, and the expectation is $\$1.48\frac{4}{27}$.

The chance of drawing \$4 is $\frac{190}{378}$, and the expectation is $\$2.01\frac{11}{189}$.

The chance of drawing \$12 is $\frac{160}{378}$, and the expectation is $\$5.07\frac{59}{63}$.

$\therefore$ The entire expectation is $\$1.48\frac{4}{27} + \$2.01\frac{11}{189} + \$5.07\frac{59}{63} = \$8.57\frac{1}{7}$.

9. A bag contains a £5 note, a £10 note, and six pieces of blank paper of the same size and texture as a bank-note. Show that the expectation of a man who is allowed to draw out one piece of paper is £1 17*s.* 6*d.*

830. *Independent Events.*—Two or more events are independent of each other when the happening of one of them does not affect the probability of any other one's happening.

10. There are b balls in one bag, a of which are white; d in another, c of which are white; and f in another, e of which are white. Show that the chance of drawing one white ball from each bag in a single trial is $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$.

Solution: One ball can be drawn from each bag in $b \times d \times f$ ways [550]. One white ball can be drawn from each bag in $a \times c \times e$ ways [550]. Therefore, the chance of drawing a white ball from each bag is $\frac{a \times c \times e}{b \times d \times f}$ [824, P. 1] $= \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$. Therefore,

831. ***Prin. 6.***—*The chance of two or more independent events happening simultaneously is the product of their several chances of happening.*

832. ***Cor. 1.***—*The chance of two or more independent events failing simultaneously is the product of their several chances of failing.*

833. ***Cor. 2.***—*The chance of one of two independent events failing and the other happening is the product of the chance that one fails and the chance that the other happens.*

11. A can solve 3 problems out of 4, B 5 out of 6, and C 7 out of 8. What is the chance that a certain problem will be solved, if all try?

Solution: Unless all fail, the problem will be solved. The chance that A will fail is $\frac{1}{4}$, that B will fail $\frac{1}{6}$, that C will fail $\frac{1}{8}$, that all will fail $\frac{1}{4} \times \frac{1}{6} \times \frac{1}{8} = \frac{1}{192}$. Therefore, the chance of success is $\frac{191}{192}$.

834. *Dependent Events.*—In a series of events, any assumed event is said to be *dependent* upon a preceding event, if the happening of the preceding event changes the probability of the happening of the assumed event.

12. Find the chance of drawing 3 white balls in succession from a bag containing 5 white and 3 red balls.

Solution: The chance of drawing a white ball on the first trial is $\frac{5}{8}$. Having drawn a white ball, there remain in the bag 7 balls, 4 of which are white. The chance of drawing a white ball on the second trial is therefore $\frac{4}{7}$. Similarly, the chance of drawing a white ball on the third trial is $\frac{3}{6}$. Therefore, the chance of drawing three white balls in succession is $\frac{5}{8} \times \frac{4}{7} \times \frac{3}{6}$ [831] $= \frac{5}{28}$. Therefore,

835. ***Prin. 7.***—*The chance that a series of events should happen is the continued product of the chance that the first should happen, the chance that the second should then happen, the chance that the third should follow, and so on.*

13. In one of two bags are 3 red and 4 white balls, and in the other 5 red and 3 white balls, and a ball is to be drawn from one or other of the bags. Find the chance that the ball drawn will be white.

Solution: The chance that the first bag will be chosen is $\frac{1}{2}$. Then, the chance of drawing a white ball from the first bag is $\frac{4}{7}$; hence, the real chance of drawing a white ball from the first bag is $\frac{1}{2}$ of $\frac{4}{7} = \frac{2}{7}$. Similarly, the chance of drawing a white ball from the second bag is $\frac{1}{2}$ of $\frac{3}{8} = \frac{3}{16}$. These events are mutually exclusive; therefore, the chance required is $\frac{2}{7} + \frac{3}{16} = \frac{53}{112}$.

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836. *Inverse Probability.*—When an event is known to have happened from one of two or more known causes, the determination of the chance that it has happened from any particular one of these causes is a problem of *inverse probability.*

14. It is known that a black ball has been drawn from one of two bags. The first of these bags contained m balls, a of which were black, and the second n balls, b of which were black. What is the chance that the ball was drawn from the first bag?

Solution: Suppose that $2N$ drawings were made. The chance is that N were made from each bag. In the N drawings from the first bag the chance is that $\frac{a}{m} \times N$ were black balls. In the drawings from the second bag the chance is that $\frac{b}{n} \times N$ were black balls. Therefore, in $2N$ drawings, the chance is that $\left(\frac{a}{m} + \frac{b}{n}\right)N$ were black balls. Therefore, the chance that a black ball was drawn from the first bag is $\left(\frac{a}{m} \times N\right) \div \left(\frac{a}{m} + \frac{b}{n}\right)N = \frac{an}{an + bm}$.

837. ***Theorem.***—*If an event is believed to have been produced by some one of the causes $P_1, P_2, P_3, \ldots. P_n$, which are mutually exclusive, and $p_1, p_2, p_3, \ldots. p_n$ represent the respective probabilities of these causes when no other causes exist, then the probability that P_r produced the event is* $\frac{P_r p_r}{P_1 p_1 + P_2 p_2 + \ldots. + P_n p_n}$.

Demonstration.—Let N be the number of trials made in producing the event. The first cause operated $N \times P_1$ times; therefore, on the supposition that no other causes operated than those named, the probability that the event was produced by the first cause is $N \times P_1 \times p_1$. Under similar restrictions, the probability that the event was produced by the second cause is $N \times P_2 \times p_2$; by the third cause, $N \times P_3 \times p_3$; by the rth cause, $N \times P_r \times p_r$; by any one of the causes, $N(P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots. + P_n p_n)$. Therefore, the real chance of its having been caused by the rth cause, or P_r, is

$$\frac{N \times P_r \times p_r}{N(P_1 p_1 + P_2 p_2 + \ldots. + P_n p_n)} = \frac{P_r \times p_r}{P_1 p_1 + P_2 p_2 + \ldots. + P_n p_n}.$$

15. Four bags were known to contain 3 red and 4 white, 4 red and 3 white, 5 red and 1 white, and 4 red and 4 white balls respectively. A white ball was drawn at random from one of the bags. Find the chance that it was drawn from the second bag.

Solution: $P_1 = P_2 = P_3 = P_4 = \frac{1}{4}$, $p_1 = \frac{4}{7}$, $p_2 = \frac{3}{7}$, $p_3 = \frac{1}{6}$, and $p_4 = \frac{1}{2}$. Therefore, the required probability is

$$\frac{\frac{1}{4} \times \frac{3}{7}}{\frac{1}{4}\left(\frac{4}{7} + \frac{3}{7} + \frac{1}{6} + \frac{1}{2}\right)} = \frac{\frac{3}{7}}{\frac{5}{3}} = \frac{9}{35}.$$

838. *Probability of Testimony.*—The following examples illustrate how to deal with questions relating to the credibility of testimony:

16. A speaks the truth a times in m, B b times in n, and C c times in r. What is the chance that a statement is true which all affirm? Which A and B affirm and C denies?

Solution: 1. The statement is either true or false. If true, all have spoken the truth; the probability of which is $\frac{a}{m} \times \frac{b}{n} \times \frac{c}{r} = \frac{abc}{mnr}$. If false, all have lied; the probability of which is

$$\left(1 - \frac{a}{m}\right)\left(1 - \frac{b}{n}\right)\left(1 - \frac{c}{r}\right) = \frac{(m-a)(n-b)(r-c)}{mnr}.$$

Hence, the probability of the truth of the statement is,

$$\frac{abc}{mnr} \div \left\{\frac{abc}{mnr} + \frac{(m-a)(n-b)(r-c)}{mnr}\right\} = \frac{abc}{abc + (m-a)(n-b)(r-c)}.$$

2. If the statement is true, A and B have told the truth and C has lied; the probability of which is $\frac{a}{m} \times \frac{b}{n} \times \left(1 - \frac{c}{r}\right) = \frac{ab(r-c)}{mnr}$. If the statement is false, A and B have lied and C has told the truth; the probability of which is $\left(1 - \frac{a}{m}\right)\left(1 - \frac{b}{n}\right)\left(\frac{c}{r}\right) = \frac{(m-a)(n-b)c}{mnr}$. Hence, the probability of the truth of the statement is,

$$\frac{ab(r-c)}{mnr} \div \left\{\frac{ab(r-c)}{mnr} + \frac{(m-a)(n-b)c}{mnr}\right\} = \frac{ab(r-c)}{ab(r-c) + (m-a)(n-b)c}$$

17. A, B, and C tell the truth to the best of their knowledge and belief. A observes correctly 4 times out of 5, B 3 times out of 5, and C 5 times out of 7. What is the probability that a phenomenon occurred (which was just as likely to fail as to occur), provided all had equal opportunity of observing, and all report its occurrence? What if A and B report its occurrence and C its failure?

Solution: 1. The phenomenon either occurred or failed. If it occurred, A, B, and C observed correctly; the probability of which is $\frac{4}{5} \times \frac{3}{5} \times \frac{5}{7}$. The inherent probability that it would occur is $\frac{1}{2}$. Hence, the probability that the assumption that it occurred is correct is $\frac{1}{2} \times \frac{4}{5} \times \frac{3}{5} \times \frac{5}{7} = \frac{6}{35}$.

If it did not occur, all observed falsely; the probability of which is $\frac{1}{5} \times \frac{2}{5} \times \frac{2}{7}$; and the probability of the correctness of the assumption that the phenomenon failed is $\frac{1}{2} \times \frac{1}{5} \times \frac{2}{5} \times \frac{2}{7} = \frac{2}{175}$. Hence, the chance that the phenomenon occurred is $\frac{6}{35} \div \left(\frac{6}{35} + \frac{2}{175}\right) = \frac{15}{16}$.

2. The probability of the correctness of the assumption that the phenomenon occurred is $\frac{1}{2} \times \frac{4}{5} \times \frac{3}{5} \times \frac{2}{7} = \frac{12}{175}$.

The probability of the correctness of the assumption that the phenomenon failed is $\frac{1}{2} \times \frac{1}{5} \times \frac{2}{5} \times \frac{5}{7} = \frac{1}{35}$.

Hence, the chance of the event is $\frac{12}{175} \div \left(\frac{12}{175} + \frac{1}{35}\right) = \frac{12}{17}$.

Note.—For a fuller treatment of Choice and Chance than space will permit to give in this book, see Whitworth's "Choice and Chance."

EXERCISE 114.

1. If A's chance of winning a race is $\frac{1}{6}$ and B's chance $\frac{1}{8}$, show that the chance that both will fail is $\frac{17}{24}$.

2. If the odds be m to n in favor of an event, show that the chance of the event is $\frac{m}{m+n}$, and the chance against the event is $\frac{n}{m+n}$.

3. If the letters e, t, s, n be arranged in a row at random, show that the chance of having an English word is $\frac{1}{6}$.

4. Show that the chance that the year $1900 + x$, in which $x < 100$, is a leap-year, is $\frac{8}{33}$.

5. A draws 3 balls from a bag containing 3 white and 6 black balls; B draws 1 ball from another bag containing 1 white and 2 black balls. Show that A's chance of drawing a white ball is to B's chance as 16 to 7.

6. Show that when two dice are thrown the chance that the throw will amount to more than 8 is $\frac{5}{18}$.

7. Show that the chance of throwing exactly 11 in one throw with two dice is $\frac{1}{18}$.

8. One purse contains 5 sovereigns and 4 shillings; another contains 5 sovereigns and 3 shillings. Show that the chance of drawing a sovereign is $\frac{85}{144}$, if a purse is selected at random and a coin drawn from it at random. Show that the expectation of the privilege is 12*s.* $2\,{}^{7}/_{12}\,d.$

9. There are three independent events whose several chances are $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{1}{2}$. Show that the chance that one of them will happen and only one is $\frac{3}{10}$.

10. If two letters are taken at random out of *esteemed*, show that the odds against both being *e* are the same as the odds in favor of one at least being *e*.

11. A letter is taken at random out of each of the words *choice* and *chance*. Show that the chance that they are the same letter is $\frac{1}{6}$.

12. A bag contains 6 black and 1 red ball. Show that the expectation of a person who is to receive a shilling for every ball he draws out before drawing the red one is 3 shillings.

13. Two numbers are chosen at random. Show that the chance is $\frac{1}{2}$ that their sum is even.

14. An archer hits his target on an average 3 times out of 4. Show that the chance that he will hit it exactly 3 times in 4 successive trials is $\frac{27}{64}$.

15. A box contains 10 pairs of gloves. A draws out a single glove; then B draws one; then A draws a second; then B draws a second. Show that A's chance of drawing a *pair* is the same as B's; and that the chance of neither drawing a pair is $\frac{290}{323}$.

16. Show that with two dice the chance of throwing more than 7 is equal to the chance of throwing less than 7.

17. Two persons throw a die alternately, with the understanding that the first who throws 6 is to receive 11 cents. Show that the expectation of the first is to that of the second as 6 to 5.

18. A's chance of winning a single game against B is $\frac{3}{5}$. Show that his chance of winning at least 2 games out of 3 is $\frac{81}{125}$.

19. A party of n persons take their seats at random at a round table. Show that it is $n-3$ to 2 against two specified persons sitting together.

20. Show that the chance that a person with 2 dice will throw double aces exactly 3 times in 5 trials is $\left(\frac{1}{36}\right)^3 \times \left(\frac{35}{36}\right)^2 \times 10$.

21. There are 10 tickets, five of which are numbered 1, 2, 3, 4, 5, and the rest are blank. Show that the probability of drawing a total of *ten* in three trials, one ticket being drawn each time and replaced, is $\frac{33}{1000}$.

SUPPLEMENT.

CONTINUED FRACTIONS.

1. Definitions.

839. An expression in the form of

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\text{etc.},}}}$$

is a *Continued Fraction.*

840. The discussion in this section will be limited to continued fractions in the form of

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\text{etc.},}}}$$

and $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, etc., will be called *Partial Fractions.*

841. A continued fraction may be written in a more convenient form, as follows:

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots\frac{1}{p+}\,\frac{1}{q+}\,\frac{1}{r+}\cdots.$$

842. When the number of partial fractions in a continued fraction is finite, it is a *terminating* continued fraction; when infinite, an *interminate* continued fraction.

843. If at some stage in an interminate continued fraction one or more partial fractions begin to repeat in the same order, it is called a *periodic* continued fraction.

844. A periodic continued fraction is *pure* when it contains no other than repeating partial fractions, and *mixed* when it contains one or more partial fractions before the repeating ones.

Thus, $\frac{1}{a} + \frac{1}{b} + \frac{1}{a} + \frac{1}{b} + \cdots\cdots$, or $\frac{\dot{1}}{a} + \frac{\dot{1}}{b}$,

is a pure periodic fraction;

and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{b} + \frac{1}{c} + \cdots\cdots$, or $\frac{1}{a} + \frac{\dot{1}}{b} + \frac{\dot{1}}{c}$,

is a mixed periodic fraction.

845. The fraction resulting from stopping at any stage is called a *convergent*.

2. The Formative Law of Successive Convergents.

846. In the continued fraction

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \cdots\cdots + \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} + \cdots\cdots,$$

$$\frac{1}{a} = \text{the first convergent.}$$

$$\frac{1}{a} + \frac{1}{b} = \frac{b}{ab+1} = \text{the second convergent.}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc}{(ab+1)c+a} = \text{the third convergent.}$$

It will be seen that

1. The numerator of the third convergent is the numerator of the second convergent multiplied by the denominator of the third partial fraction, plus the numerator of the first convergent; and

2. The denominator of the third convergent is the denominator of the second convergent multiplied by the denominator of the third partial fraction, plus the denominator of the first convergent.

Will these laws hold true in the formation of any convergent from the two preceding convergents?

Let $\frac{P}{P_1}$, $\frac{Q}{Q_1}$, $\frac{R}{R_1}$, and $\frac{S}{S_1}$ be respectively the $(n-2)$th, $(n-1)$th, nth, and $(n+1)$th convergents; and p, q, r, and s the denominators of the $(n-2)$th, $(n-1)$th, nth, and $(n+1)$th partial fractions.

Suppose the laws to hold true in the formation of the convergent $\frac{R}{R_1}$, then will $\frac{R}{R_1}=\frac{rQ+P}{rQ_1+P_1}$. (A)

Now, from the nature of the continued fraction, $\frac{S}{S_1}$ may be formed by putting $r+\frac{1}{s}$ for r in $\frac{R}{R_1}$. Therefore,

$$\frac{S}{S_1}=\frac{\left(r+\frac{1}{s}\right)Q+P}{\left(r+\frac{1}{s}\right)Q_1+P_1}=\frac{(sr+1)Q+sP}{(sr+1)Q_1+sP_1}$$

$$=\frac{s(rQ+P)+Q}{s(rQ_1+P_1)+Q_1}=\frac{sR+Q}{sR_1+Q_1}.$$

Therefore, if the laws are applicable in the formation of the nth convergent, they are also applicable in the formation of the $(n+1)$th convergent. But we have seen that they do apply in the formation of the third convergent, and, hence, apply in the formation of the fourth convergent, and so on. Therefore, in general,

1. *The numerator of the nth convergent equals the numerator of the* $(n-1)$*th convergent multiplied by the denominator of the nth partial fraction, plus the numerator of the* $(n-2)$*th convergent;* and

2. *The denominator of the nth convergent equals the denominator of the* $(n-1)$*th convergent multiplied by the denominator of the nth partial fraction, plus the denominator of the* $(n-2)$*th convergent.*

Example.—Find the first 8 convergents of the continued fraction $\frac{1}{2}+\frac{1}{3}+\frac{1}{1}+\frac{1}{4}+\frac{1}{5}+\frac{1}{2}+\frac{1}{4}+\frac{1}{3}$.

Solution:

$$\frac{R}{R_1}=\frac{rQ+P}{rQ_1+P_1}=\frac{1}{2},\ \frac{3}{7},\ \frac{4}{9},\ \frac{19}{43},\ \frac{99}{224},\ \frac{217}{491},\ \frac{967}{2188},\ \frac{3118}{7055}.$$

Properties of Convergents.

847. Take the continued fraction

$$y=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\cdots.$$

1. $a<a+\frac{1}{b}+\frac{1}{c}+\cdots$, $\therefore \frac{1}{a}>y$

2. $b<b+\frac{1}{c}+\frac{1}{d}+\cdots$, $\therefore a+\frac{1}{b}>a+\frac{1}{b}+\frac{1}{c}+\cdots$;

   whence, $\frac{1}{a}+\frac{1}{b}<y$

3. $c<c+\frac{1}{d}+\cdots$, $\therefore b+\frac{1}{c}>b+\frac{1}{c}+\frac{1}{d}+\cdots$;

   whence, $a+\frac{1}{b}+\frac{1}{c}<a+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\cdots$;

   and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>y$, etc. Therefore,

***Prin. 1.**—The successive convergents are alternately greater and less than the continued fraction (the odd orders being too great and the even orders too small).*

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848. The difference between the first two convergents $=\frac{1}{a}-\frac{b}{ab+1}=\frac{1}{a(ab+1)}=$ *unity divided by the product of their denominators.* Is this a general law?

Let $\frac{P}{P_1}$, $\frac{Q}{Q_1}$, and $\frac{R}{R_1}$ be the $(n-1)$th, nth, and $(n+1)$th convergents, and p, q, and r the denominators of the

$(n-1)$th, nth, and $(n+1)$th partial fractions, and let $\sim$ denote *difference between.*

Assume $\frac{P}{P_1} \sim \frac{Q}{Q_1} = \frac{P\,Q_1 \sim Q\,P_1}{P_1\,Q_1} = \frac{1}{P_1\,Q_1}$; then will

$$\frac{R}{R_1} \sim \frac{Q}{Q_1} = \frac{R\,Q_1 \sim Q\,R_1}{Q_1\,R_1} = \frac{Q_1(r\,Q + P) \sim Q(r\,Q_1 + P_1)}{Q_1\,R_1}$$

$$= \frac{P\,Q_1 \sim Q\,P_1}{Q_1\,R_1} = \frac{1}{Q_1\,R_1}. \qquad \text{(A)}$$

Therefore, if the law holds good for the difference between the $(n-1)$th and nth convergents, it will also for the difference between the $(n+1)$th and the nth convergents. But we have seen that it does hold good for the difference between the first and second convergents, and, hence, it will for the difference between the next higher pair, and so on. Therefore,

Prin. 2.*—The difference between any two consecutive convergents equals unity divided by the product of their denominators.*

---

849. Since $P\,Q_1 \sim Q\,P_1 = 1$ [848, A], P and P_1 can not have a common factor, neither can Q and Q_1.

Therefore,

Prin. 3.*—Every convergent is in its lowest terms.*

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850. If we let $\frac{U}{U_1}$ represent the true value of the continued fraction; then will

$$\frac{P}{P_1} \sim \frac{U}{U_1} < \frac{P}{P_1} \sim \frac{Q}{Q_1} \text{ [P. 1]; also, } \frac{U}{U_1} \sim \frac{Q}{Q_1} < \frac{P}{P_1} \sim \frac{Q}{Q_1}$$

$$\therefore \frac{P}{P_1} \sim \frac{U}{U_1} < \frac{1}{P_1\,Q_1} \text{ [P. 2]; also, } \frac{U}{U_1} \sim \frac{Q}{Q_1} < \frac{1}{P_1\,Q_1}.$$

Hence, if either $\frac{P}{P_1}$ or $\frac{Q}{Q_1}$ be used for $\frac{U}{U_1}$, the error will be less than $\frac{1}{P_1\,Q_1}$, or less than $\frac{1}{Q_1^2}$.

851. Let $\frac{P}{P_1}$, $\frac{Q}{Q_1}$, and $\frac{R}{R_1}$ be three consecutive convergents whose terminal partial fractions are $\frac{1}{p}$, $\frac{1}{q}$, and $\frac{1}{r}$; and $\frac{U}{U_1}$, the true value of the continued fraction.

Then, $\frac{U}{U_1}$ differs from $\frac{R}{R_1}$ only in the use of $r + \frac{1}{s+}$ etc. for r. Put $r + \frac{1}{s+}$ etc. $= x$.

$$\text{Now, } \frac{R}{R_1} = \frac{rQ + P}{rQ_1 + P_1} \qquad \therefore \frac{U}{U_1} = \frac{xQ + P}{xQ_1 + P_1}$$

$$\therefore \frac{U}{U_1} \sim \frac{Q}{Q_1} = \frac{xQ + P}{xQ_1 + P_1} \sim \frac{Q}{Q_1} = \frac{PQ_1 \sim P_1 Q}{Q_1(xQ_1 + P_1)}$$
$$= \frac{1}{Q_1(xQ_1 + P_1)}$$

$$\text{and, } \frac{P}{P_1} \sim \frac{U}{U_1} = \frac{P}{P_1} \sim \frac{xQ + P}{xQ_1 + P_1} = \frac{x(PQ_1 - P_1 Q)}{P_1(xQ_1 + P_1)}$$
$$= \frac{x}{P_1(xQ_1 + P_1)}$$

Now, $x > 1$, and $P_1 < Q_1$,

$$\therefore \frac{1}{Q_1(xQ_1 + P_1)} < \frac{x}{P_1(xQ_1 + P_1)}; \text{ or,}$$

$\frac{Q}{Q_1}$ is nearer $\frac{U}{U_1}$ than is $\frac{P}{P_1}$. Therefore,

***Prin. 4.**—The higher the order of a convergent the nearer does it approach to the true value of the continued fraction.*

852. ***Cor.**—A continued fraction is the limit of its convergents; or, if y be a continued fraction and x its variable convergent, $y = lim.\ x$.*

---

853. The denominators of successive convergents increase more rapidly than their numerators [846; 1, 2]; therefore, of any two convergents, that is the greater which has the greater denominator. But may there not be some other fraction, not a convergent, with smaller de-denominator, that is a nearer approximation to a continued fraction than a given convergent?

Suppose $\frac{M}{M_1}$ not a convergent, and nearer to $\frac{U}{U_1}$ than $\frac{Q}{Q_1}$, and $M_1 < Q_1$; then $\frac{U}{U_1} \sim \frac{M}{M_1} < \frac{Q}{Q_1} \sim \frac{U}{U_1}$;

$$\therefore \frac{M}{M_1} \sim \frac{P}{P_1} < \frac{Q}{Q_1} \sim \frac{P}{P_1}; \text{ or, } \frac{MP_1 \sim M_1P}{M_1P_1} < \frac{1}{Q_1P_1}.$$

But $M_1P_1 < Q_1P_1$, since $M_1 < Q_1$.

$\therefore MP_1 \sim M_1P < 1$; which is impossible, since M, M_1, P, and P_1 are integral. Therefore,

Prin. 5.—*Any convergent is nearer the true value of a continued fraction than any fraction with smaller denominator.*

Problems.

854. 1. To reduce a common fraction to a terminating continued fraction.

Since an improper fraction is equivalent to an integer and a proper fraction, it will be necessary only to investigate a method for extending a proper fraction.

Let $\frac{b}{a}$ = a proper fraction in its lowest terms.

Divide both terms by b, and put for the improper fraction $\frac{a}{b}$, the mixed number $p + \frac{c}{b}$; then,

$$\frac{b}{a} = \cfrac{1}{p + \cfrac{c}{b}}$$

Divide both terms of $\frac{c}{b}$ by c, and put $\frac{b}{c} = q + \frac{d}{c}$;

then,
$$\frac{b}{a} = \cfrac{1}{p + \cfrac{1}{q + \cfrac{d}{c}}}$$

Again, divide both terms of $\frac{d}{c}$ by d, and put $\frac{c}{d} = r + \frac{e}{d}$;

then, $$\frac{b}{a} = \frac{1}{p+} \frac{1}{q+} \frac{1}{r + \frac{e}{d}}$$

It will now be seen that the denominators of the successive partial fractions have been obtained as follows:

$$\begin{array}{l} b)\ a\ (p \\ \quad \underline{bp} \\ \quad c)\ b\ (q \\ \qquad \underline{cq} \\ \qquad d)\ c\ (r \\ \qquad\quad \underline{rd} \\ \qquad\quad e \text{ etc.} \end{array}$$

Since a and b are integral, they have a highest common divisor, and the division will eventually terminate.

Therefore, the continued fraction will be a *terminating one.*

Rule.—*To reduce a proper fraction to a terminating continued fraction, find the highest common divisor of its terms by successive division, and use the quotients in regular order for the denominators of the partial fractions.*

855. 2. To reduce a quadratic surd to a continued fraction.

Illustrations.—1. Reduce $\sqrt{26}$ to a continued fraction.

Solution: $\sqrt{26} = 5 + \frac{1}{x}$

$$\therefore \quad x = \frac{1}{\sqrt{26} - 5} = \sqrt{26} + 5 = 10 + \frac{1}{x}$$

$$= 10 + \frac{1}{10 + \frac{1}{x}} = 10 + \frac{1}{10 + \frac{1}{10 + \frac{1}{x}}}$$

$$\therefore \quad \sqrt{26} = 5 + \frac{1}{10+} \frac{1}{10+} \frac{1}{10+} \cdots = 5 + \frac{\dot{1}}{10}$$

2. Reduce $\sqrt{19}$ to a continued fraction.

Solution: $\sqrt{19} = 4 + \frac{1}{x}$

$$\therefore \quad x = \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3} = 2 + \frac{1}{x_1}$$

$$\therefore \quad x_1 = \frac{3}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{5} = 1 + \frac{1}{x_2}$$

$$\therefore \quad x_2 = \frac{5}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{2} = 3 + \frac{1}{x_3}$$

$$\therefore \quad \sqrt{19} = 4 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\cdots$$

***Scholium.**—A quadratic surd may always be reduced to a periodic continued fraction if the expansion is carried sufficiently far.*

856. 3. To reduce a periodic continued fraction to a simple fraction.

The periodic continued fraction

$$\frac{\dot{1}}{p+}\,\frac{1}{q+}\,\frac{\dot{1}}{r} = \frac{1}{p+}\,\frac{1}{q+}\,\frac{1}{r+x} = x$$

$$\therefore \quad \frac{qr+qx+1}{pqr+pqx+p+r+x} = x$$

whence, $(pq+1)x^2 + (pqr-q+p+r)x = qr+1$

The value of x found from this equation is the value of the continued fraction.

857. 4. To approximate the ratio of two numbers.

Example.—When the diameter of a circle is 1, the circumference is $3{\cdot}1415926+$. Approximate the ratio of the diameter to the circumference.

Solution:

$$1 : 3{\cdot}1415926 = \frac{10000000}{31415926} = \frac{1}{3+}\,\frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\cdots \quad \text{[Prob. 1].}$$

The successive convergents, which are also the successive approximations of the ratio, are: $\frac{1}{3}$, $\frac{7}{22}$, $\frac{106}{333}$, $\frac{113}{355}$, etc.

EXERCISE 115.

Reduce to continued fractions:

1. $\frac{125}{317}$ 2. $\frac{140}{213}$ 3. $\frac{100}{999}$ 4. $\frac{106}{729}$

5. $\sqrt{10}$ 6. $\sqrt{12}$ 7. $\sqrt{30}$ 8. $\sqrt{57}$

9. $\cdot 3183$ 10. $3\cdot 1416$ 11. $67°, 20', 30''$

Find the successive convergents of:

12. $\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5}$ 13. $\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{9+}\,\frac{1}{1}$

14. $\frac{\dot{1}}{2+}\,\frac{\dot{1}}{3}$ 15. $\frac{1}{1+}\,\frac{\dot{1}}{2+}\,\frac{\dot{1}}{3}$ 16. $\frac{1}{2+}\,\frac{\dot{1}}{3}$

Find the true value of:

17. $\frac{\dot{1}}{4}$ 18. $\frac{\dot{1}}{2+}\,\frac{\dot{1}}{4}$ 19. $\frac{1}{2+}\,\frac{\dot{1}}{4}$ 20. $\frac{\dot{1}}{1+}\,\frac{1}{2+}\,\frac{\dot{1}}{1}$

21. Find a series of common fractions converging to $1 : \sqrt{3}$.

22. Express approximately the ratio of a liquid quart (57·75 cu. in.) to a dry quart (67·2 cu. in.).

23. The square root of 600 is 24·494897, and the cube root of 600 is 8·434327. Find a series of four common fractions approximating nearer and nearer to the ratio of the latter to the former.

24. The imperial bushel of Great Britain contains 2218·192 cu. in., and the Winchester bushel 2150·42 cu. in. Find the nearest approximation, that can be expressed by a common fraction whose denominator is less than 100, of the ratio of the latter to the former.

25. Two scales of equal length having their zero points coinciding also have the 27th gradation of the one to coincide with the 85th gradation of the other. Show that the 7th and 22d more nearly coincide than any other two gradations.

THEORY OF NUMBERS.

Systems of Notation.

1. Definitions.

858. *Notation* is the art of expressing numbers by means of characters.

859. A *system of notation* is a method of expressing numbers in a series of powers of some fixed number.

860. The order of progression on which any system of notation is founded is called the *scale* of the system, and the fixed number on which the scale is based is called the *radix.*

861. Any integral number, except unity, may be taken as the *radix.* When the radix is two, the scale and system are called *binary*; when three, *ternary*; when four, *quaternary*; when five, *quinary*; when six, *senary*; when seven, *septenary*; when eight, *octary*; when nine, *nonary*; when ten, *denary* or *decimal*; when eleven, *undenary*; when twelve, *duodenary*; etc.

862. In the decimal or denary system,

$56342 = 5 \times 10,000 + 6 \times 1000 + 3 \times 100 + 4 \times 10 + 2 = 5 \times 10^4 + 6 \times 10^3 + 3 \times 10^2 + 4 \times 10 + 2$;

or, in inverse order,

$2 + 4 \times 10 + 3 \times 10^2 + 6 \times 10^3 + 5 \times 10^4$.

In the octary system,

$34725 = 3 \times 8^4 + 4 \times 8^3 + 7 \times 8^2 + 2 \times 8 + 5$;

or, in inverse order,

$5 + 2 \times 8 + 7 \times 8^2 + 4 \times 8^3 + 3 \times 8^4$.

∴ In general, if r be taken as the radix, and

$$a_0,\ a_1,\ a_2,\ a_3 \ldots\ldots a_{n-1}$$

as the n digits of a number, reckoning in order from right to left, the number is represented by

$$a_{n-1}r^{n-1} + a_{n-2}r^{n-2} + a_{n-3}r^{n-3} + \ldots\ldots + a_2r^2 + a_1r + a_0$$

863. ***Theorem.****—Any integral number may be expressed in the form of*

$$ar^n + br^{n-1} + cr^{n-2} + \ldots\ldots + pr^2 + qr + s,$$

in which the coefficients are each less than r.

Demonstration: Let N equal the number of units in any number, and r^n the highest power of the radix less than N.

Divide N by r^n, and let the quotient be a and the remainder N'. Then $N = ar^n + N'$.

Now, a is less than r, else r^n would not be the highest power of r less than N; and N' is less than r^n.

Divide N' by r^{n-1}, and let the quotient be b and the remainder N''. Then $N' = br^{n-1} + N''$, in which $b < r$ and $N'' < r^{n-1}$.

In like manner, divide N'' by r^{n-2}, and let the quotient be c, and the remainder N'''. Then $N'' = cr^{n-2} + N'''$, in which $c < r$ and $N''' < r^{n-2}$.

If this process be continued, a remainder, s, will eventually be reached less than r. Therefore, $N = ar^n + br^{n-1} + cr^{n-2} + \ldots\ldots + pr^2 + qr + s$, in which the coefficients are each less than r.

864. ***Cor.****—In any system of notation, the number of digits including* 0 *is equal to the radix.*

865. Problem. To express a given number in any proposed scale.

Solution: Let N be the number and r the radix of the proposed scale.

Suppose $N = ar^n + br^{n-1} + cr^{n-2} + \ldots\ldots + pr^2 + qr + s$, it is required to find the values of $a, b, c, \ldots\ldots p, q, s$.

$$\frac{N}{r} = ar^{n-1} + br^{n-2} + cr^{n-3} + \ldots\ldots + pr + q + \frac{s}{r}.$$

Therefore, the remainder, after dividing N by r, is the last digit.

Suppose $N' = ar^{n-1} + br^{n-2} + cr^{n-3} + \ldots\ldots + pr + q$.

$$\frac{N'}{r} = ar^{n-2} + br^{n-3} + cr^{n-4} + \ldots\ldots + p + \frac{q}{r}.$$

Therefore, the remainder, after dividing N' by r, is the next digit.

Suppose $N'' = ar^{n-2} + br^{n-3} + cr^{n-4} + \dots + p.$

$$\frac{N''}{r} = ar^{n-3} + br^{n-4} + cr^{n-5} + \dots + \frac{p}{r}.$$

Therefore, the remainder, after dividing N'' by r, is the next digit.

Etc., etc., etc.

Therefore,

Rule.*—Divide the number by the radix, then the quotient by the radix, and so on until the quotient becomes less than the radix; the successive remainders will be the digits of the number, beginning with the units.*

Illustrative Examples.—1. Express 35432 (denary scale) in the senary scale; also, 35432 (senary scale) in the octary scale.

(1)

6)35432
6)5905 − 2
6)984 − 1
6)164 − 0
6)27 − 2
4 − 3

$\therefore 35432_{r=10} = 432012_{r=6}$

(2)

8)35432
8)2545 − 4
8)212 − 1
8)14 − 0
1 − 2

$\therefore 35432_{r=6} = 12014_{r=8}$

Explanation of (2):

$35 \div 8 = (3 \times 6 + 5) \div 8 = 23 \div 8 = 2$, and 7 over;
$74 \div 8 = (7 \times 6 + 4) \div 8 = 46 \div 8 = 5$, and 6 over;
$63 \div 8 = (6 \times 6 + 3) \div 8 = 39 \div 8 = 4$, and 7 over;
$72 \div 8 = (7 \times 6 + 2) \div 8 = 44 \div 8 = 5$, and 4 over.

2. Express 35439 (denary scale) in duodenary scale; also, 34439 (nonary scale) in undenary scale.

Note.—The undenary scale needs a character to represent *ten*, and the duodenary scale two characters to represent *ten* and *eleven*. We will represent ten by t and eleven by e.

(1)

12)34439
12)2869 − e
12)239 − 1
12)19 − e
1 − 7

$\therefore 34439_{r=10} = 17e1e_{r=12}$

(2)

11)35439
11)2852 − 5
11)236 − 8
11)18 − 8
1 − 6

$\therefore 35439_{r=9} = 16885_{r=11}$

EXERCISE 116.

1. Find the sum in senary scale of $4532_{r=6}$, $3452_{r=6}$, $5423_{r=6}$, and $3251_{r=6}$

2. Find the difference (octary scale) of $3574_{r=8}$ and $2756_{r=8}$.

3. Multiply $36425_{r=7}$ by 8; also $25436_{r=8}$ by 10

4. Divide $4765t4_{r=11}$ by 9; also $2e58t3_{r=12}$ by 11

5. Express $43250_{r=5}$ in the denary scale.

6. Express $38472_{r=9}$ in the septenary scale.

7. Express $35243_{r=6}$ in the duodenary scale.

8. Express $8et950_{r=12}$ in the quaternary scale.

9. Find the sum (denary scale) of $3472_{r=8}$ and $5842_{r=10}$

10. Find the difference (nonary scale) of $5t34_{r=11}$ and $6432_{r=7}$

11. What is the radix of the scale in which $476_{r=10} = 2112$?

Suggestion.—Let $r =$ the radix; then will $2r^3 + r^2 + r + 2 = 476$.

12. In what scale is 3 times $134 = 450$?

13. In what scale is $135_{r=6} = 43$?

14. What is the H. C. D. of $36_{r=8}$, $48_{r=8}$, and $60_{r=8}$?

15. Multiply $28_{r=9}$ by $45_{r=9}$; also square $25_{r=6}$

16. In what scale is 1552 the square of 34?

17. Show that 35, 44, and 53 are in arithmetical progression in any scale of notation.

18. Show that 1331 is a perfect cube in any system of notation.

19. Show that 14641 is a perfect fourth power in any system of notation.

20. Show that 11, 220, and 4400 are in geometrical progression in any system of notation.

Divisibility of Numbers and their Digits.

866. ***Theorem I.***—*If a number, N, be divided by any factor of r, r^2, r^3, etc., respectively (r being the radix), it will leave the same remainder as when the number expressed by the last term, the last two terms, the last three terms, etc., is divided by the same factor.*

Demonstration: Suppose x a factor of r, y a factor of r^2, and z a factor of r^3, etc.

$N = ar^{n-1} + br^{n-2} + \ldots + pr^2 + qr + s.$

Now, x is certainly a factor of every term of N, except s; y, a factor of every term, except $qr + s$; and z, a factor of every term, except $pr^2 + qr + s$, etc. Therefore,

1. $\frac{N}{x} = \text{an integer} + \frac{s}{x}$.

2. $\frac{N}{y} = \text{an integer} + \frac{qr + s}{y}$.

3. $\frac{N}{z} = \text{an integer} + \frac{pr^2 + qr + s}{z}$; which was to be proved.

867. ***Cor.***—*In the decimal system of notation,*

1. A number is divisible by any factor of 10, if the units' digit is divisible by that factor.

2. A number is divisible by any factor of 100, if the number expressed by the last two figures is divisible by that factor.

3. A number is divisible by any factor of 1000, if the number expressed by the last three figures is divisible by that factor.

868. ***Theorem II.***—*The difference between a number and the sum of its digits is divisible by the radix less* **one.**

Demonstration:

Let $N = ar^{n-1} + br^{n-2} + \ldots + pr^2 + qr + s$ = any number; then, $a + b + \ldots + p + q + s$ = the sum of the digits.

Now, $a(r^{n-1} - 1) + b(r^{n-2} - 1) + \ldots + p(r^2 - 1) + q(r - 1)$ = the difference between the number and the sum of its digits, and every term is divisible by $r - 1$.

869. ***Cor.***—*In the decimal system of notation,*

The difference between a number and the sum of its digits is divisible by 9 or 3.

870. ***Theorem III.***—*A number, N, divided by* $r-1$, *leaves the same remainder as the sum of its digits divided by* $r-1$, *r being the radix.*

Demonstration.—Put s for the sum of the digits; q and q' for the quotients; and c and c' for the remainders.

1. $N = q(r-1) + c$

2. $s = q'(r-1) + c'$

$\therefore\ N - s = (q - q')(r-1) + (c - c').$

Now, $N-s$ is divisible by $r-1$ [T. II], and $(q-q')(r-1)$ is evidently divisible by $r-1$; therefore, $c-c'$ is divisible by $r-1$. But c and c' are each less than $r-1$; hence, $c-c'=0$, or $c=c'$.

871. ***Cor.***—*In the decimal system,*

A number is divisible by 9, if the sum of its digits is divisible by 9.

872. ***Theorem IV.***—*If from a number, N, we subtract the digits of the even powers of r, and add those of the odd powers, the result will be divisible by* $r+1$.

Demonstration.—Let $N = ar^4 + br^3 + cr^2 + dr + e.$

$$\text{Add}\quad -a \quad +b \quad -c \quad +d \quad -e,$$

then, $a(r^4-1) + b(r^3+1) + c(r^2-1) + d(r+1)$, the result, is divisible by $r+1$, since every term is divisible by $r+1$.

873. ***Theorem V.***—*If a number, N, be divided by* $r+1$, *the remainder will be the same as when the difference between the sums of the digits of the even and odd powers of r is divided by* $r+1$.

Demonstration.—Put d for the difference between the sums of the digits of the even and odd powers of r; q and q' for the quotients; and c and c' for the remainders; then will

$$N = q(r+1) + c,$$

and

$$d = q'(r+1) + c'.$$

$$\therefore\ N - d = (q-q')(r+1) + c - c'.$$

Now, $N-d$ is divisible by $r+1$ [T. IV], and $(q-q')(r+1)$ is evidently divisible by $r+1$. Therefore, $c-c'$ is divisible by $r+1$. But c and c' are each less than $r+1$; hence, $c-c'=0$, or $c=c'$.

874. ***Cor.***—*In the decimal system of notation,*

A number is divisible by 11, if the difference between the sums of the digits in the even and odd places is divisible by 11.

Even and Odd Numbers.

875. An *even number* is a number that is exactly divisible by 2.

876. An *odd number* is a number that is not exactly divisible by 2.

877. If we let x represent any integral number including zero, and regard zero as an even number, it becomes evident that the general formula for an even number is $2x$, and for an odd number $2x+1$.

878. ***Theorem I.***—*The sum of any number of even numbers is even.*

Demonstration.—Let $2x_1$, $2x_2$, $2x_3, \ldots . 2x_n$ represent n even numbers; then will their sum be

$$2x_1 + 2x_2 + 2x_3 + \ldots . + 2x_n = 2(x_1 + x_2 + x_3 + \ldots . + x_n)$$

an even number.

879. ***Theorem II.***—*The sum of an even number of odd numbers is even.*

Demonstration.—Let $2x_1 + 1$, $2x_2 + 1$, $2x_3 + 1 + \ldots . + 2x_{2n} + 1$ represent 2_n odd numbers; then will their sum be

$$(2x_1 + 1) + (2x_2 + 1) + (2x_3 + 1) + \ldots . + (2x_{2n} + 1) =$$
$$2x_1 + 2x_2 + 2x_3 + \ldots . + 2x_{2n} + 2n =$$
$$2(x_1 + x_2 + x_3 + \ldots . + x_{2n} + n), \text{ an even number.}$$

880. ***Theorem III.***—*The sum of an odd number of odd numbers is odd.*

Demonstration.—Let $2x_1 + 1$, $2x_2 + 1$, $2x_3 + 1, \ldots . 2x_{2n+1} + 1$ represent $2n + 1$ odd numbers; then will their sum be

$$(2x_1 + 1) + (2x_2 + 1) + (2x_3 + 1) + \ldots . + 2x_{2n+1} + 1) =$$
$$2x_1 + 2x_2 + 2x_3 + \ldots . + 2x_{2n+1} + 2n + 1 =$$
$$2(x_1 + x_2 + x_3 + \ldots . + x_{2n+1} + n) + 1, \text{ an odd number.}$$

881. ***Theorem IV.****—The sum of an equal even number of even and odd numbers is even.*

Demonstration.—Let $(2x_1 + 1) + (2x_2 + 1) + \ldots + (2x_{2n} + 1) =$ the sum of $2n$ odd numbers;

and $2x'_1 + 2x'_2 + \ldots + 2x'_{2n} =$ the sum of $2n$ even numbers; then will their sum be

$\{2(x_1 + x'_1) + 1\} + \{2(x_2 + x'_2) + 1\} + \ldots + \{2(x_{2n} + x'_{2n}) + 1\}$, which is even [T. II].

882. ***Theorem V.****—The sum of an equal odd number of even and odd numbers is odd.*

Demonstration.—Let $(2x_1 + 1) + (2x_2 + 1) + \ldots + (2x_{2n+1} + 1) =$ the sum of $2n + 1$ odd numbers;

and $2x'_1 + 2x'_2 + \ldots + 2x'_{2n+1} =$ the sum of $2n + 1$ even numbers; then will their sum be

$\{2(x_1 + x'_1) + 1\} + \{2(x_2 + x'_2) + 1\} + \ldots + \{2(x_{2n+1} + x'_{2n+1}) + 1\}$, which is odd [T. III].

883. ***Theorem VI.****—The difference between two numbers, if both are odd or both even, is even.*

Demonstration.—1. Let $2x$ and $2x'$ be two even numbers.

Their difference is $2x - 2x' = 2(x - x')$, which is even.

2. Let $2x + 1$ and $2x' + 1$ be two odd numbers.

Their difference is $(2x + 1) - (2x' + 1) = 2x - 2x' = 2(x - x')$, which is even.

884. ***Theorem VII.****—The difference between an odd and an even number is odd.*

Demonstration.—Let $2x + 1$ be any odd number, and $2x'$ any even number.

Their difference is $(2x + 1) - 2x' = 2(x - x') + 1$, which is odd.

885. ***Theorem VIII.****—The product of any number of even numbers is even.*

Demonstration.—Let $2x_1, 2x_2, 2x_3, \ldots 2x_n$ be n even numbers.

Their product is $2(2^{n-1}x_1, x_2, x_3, \ldots x_n)$, which is even.

Cor.*—Any power of an even number is even.*

886. ***Theorem IX.****—The product of any number of odd numbers is odd.*

Demonstration.—Let $2x_1 + 1, 2x_2 + 1, \ldots . 2x_n + 1$ be n odd numbers. It is evident, from the nature of multiplication, that the product of these numbers will contain the factor 2 in every term, except the last, which will be 1. That is, the product will have the form of $2x' + 1$, which is odd.

887. ***Cor.***—*Any power of an odd number is odd.*

888. ***Theorem X.***—*The product of any number of odd and even numbers is even.*

Demonstration.—The product of the odd numbers is odd [T. IX], and may be represented by $2x + 1$.

The product of the even numbers is even [T. VIII], and may be represented by $2x'$.

$\therefore$ The entire product is $2x'(2x+1) = 2(xx' + x')$, which is even.

Example.—It is required to divide one dollar among 15 boys, giving to each boy an odd number of cents. Is this question possible?

Prime, Composite, Square, and Cubic Numbers.

1. Definitions.

889. A *Prime Number* is a number that can not be produced by multiplying together factors other than itself and unity.

A prime number is divisible only by itself and unity.

890. A *Composite Number* is a number that may be produced by multiplying together other factors than itself and unity.

A composite number is divisible by other factors than itself and unity.

891. A *Square Number* is one that may be resolved into two equal factors.

892. A *Cubic Number* is one that may be resolved into three equal factors.

893. Two or more numbers are *prime to each other* when they have no common factor, except unity.

2. Primes.

894. ***Theorem I.***—*The number of primes is unlimited.*

For, let n be the number of primes, and, if n is not unlimited, let p be the greatest prime number. Then will $2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times p$ be divisible by all primes not greater than p; and $(2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times p) + 1$ not be divisible by any prime not greater than p. Therefore, $(2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times p) + 1$ is itself a prime greater than p, or is divisible by a prime greater than p. In either case, p is not the greatest prime. Therefore, n is unlimited.

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895. ***Theorem II.***—*Every prime number, except 2 and 3, belongs to the form* $6n \pm 1$.

For, every number evidently belongs to one of the forms $6n$, $6n+1$, $6n+2$, $6n+3$, $6n+4$, or $6n+5$, in which n may be any integer including 0. Now, $6n$, $6n+2$, and $6n+4$, are each divisible by 2, and $6n+3$ by 3; hence, these forms are composite, except when $n=0$ in $6n+2$ and $6n+3$, in which case we have the primes 2 and 3.

The only forms remaining to contain primes are $6n+1$ and $6n+5$. But $6n+5=(6n+6)-1=6(n+1)-1$ $=6n'-1$. Therefore, the general form $6n \pm 1$ contains all primes, except 2 and 3.

Scholium.—*It must not be inferred from this proposition that all numbers expressed by* $6n \pm 1$ *are prime. Thus, when* $n=4$, $6n+1=25$; *and when* $n=11$, $6n-1=65$.

Cor.—Every prime above 3, increased or diminished by unity, is divisible by 6.

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896. ***Theorem III.***—*No rational formula can represent primes only.*

For, if possible, let $a + bx + cx^2 + dx^3 + \ldots$ be prime for all values of x.

When $x = m$, let $a + bx + cx^2 + dx^3 + \ldots = p$;

then, $p = a + bm + cm^2 + dm^3 + \ldots$

When $x = m + np$, let $a + bx + cx^2 + dx^3 + \ldots = q$;

then,
$$q = a + b(m + np) + c(m + np)^2 + d(m + np)^3 + \ldots$$
$$= a + bm + cm^2 + dm^3 + \ldots + rp$$
$$= p + rp = p(1 + r), \text{ a composite number.}$$

897. ***Scholium.***—*The form* $n^2 + n + 41$ *is prime for all values of* n *from 0 to 39 inclusive, and the form* $2n^2 + 29$ *for all values of* n *from 0 to 28 inclusive. These forms have been discovered by trial, and are not demonstrable.*

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898. ***Theorem IV.***—*If a number is not divisible by a factor equal to or less than its square root, it is a prime.*

For, let $N = x \times y$ be any number not prime. Then, if $x = y$, $N = y^2$, and $y = \sqrt{N}$. But, if $x > \sqrt{N}$, then $y < \sqrt{N}$, since $x \times y = N$. But N is divisible by y. Therefore, if N is not prime, it is divisible by a factor equal to or less than $\sqrt{N}$. Hence, too, if a number is not divisible by a factor equal to or less than its square root, it is prime.

3. Composites.

899. ***Theorem I.***—*If a number is a factor of the product of two numbers and is not a factor of one of them, it is a factor of the other.*

Thus, let x be a factor of ab, and not a factor of a; then will it be a factor of b.

For, $\frac{a}{x}$ may be reduced to a terminating continued fraction [854]. Let $\frac{p}{q}$ be the convergent next in value to $\frac{a}{x}$. Then, $\frac{p}{q} \sim \frac{a}{x} = \frac{1}{qx}$ [848, P.]; whence,

$$px \sim aq = 1; \text{ and } bpx \sim abq = b.$$

Now, bpx and abq are each divisible by x; therefore, their difference, b, is divisible by x.

900. ***Cor.***—*If a number is prime to each of two or more other numbers, it is prime to their product.*

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901. ***Theorem II.***—*Every composite number may be resolved into one set of prime factors and into only one set.*

1. Any composite number (N) is the product of two or more factors each less than N, which are all composite, all prime, or some composite and some prime. As many of these as are composite are again resolvable into other factors less than themselves, and so on, until no factor is further resolvable into factors less than itself and greater than unity, at which stage all the factors are prime.

2. Let one set of prime factors of N be $a, b, c, \ldots$, and, if possible, let another set be $p, q, r, \ldots$; then will $a \times b \times c \times \ldots = p \times q \times r \times \ldots$

Now, suppose a different from $q, r, \ldots$, then it is not contained in $q \times r \times \ldots$ [900]; it must, therefore, be contained in p, but this can only be when $a = p$, since p is a prime. But, if $a = p$, $b \times c \times \ldots = q \times r \times \ldots$; from which it follows as before that b is identical with one of the factors in $q \times r \times \ldots$; etc.

902. ***Theorem III.***—*The product of any r consecutive numbers is divisible by* $\underline{|r}$.

For, $\dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}}$ is the product of r consecutive numbers divided by $\underline{|r}$, and it is also the number of combinations of n things taken r together, which is evidently a whole number.

903. ***Cor. 1.***—The coefficient of the $(r+1)$th term of the binomial theorem is $\dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}}$ [595]; therefore,

The coefficient of every term of the binomial theorem is integral when n is a positive integer.

904. ***Cor. 2.***—If we represent $\dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}}$ by q, it follows that,

All factors of the numerator that are prime and are greater than r are divisors of q.

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905. ***Theorem IV.***—*Fermat's Theorem. If* p *be any prime number, and* a *be a number prime to* p, *then* $a^{p-1}-1$ *will be divisible by* p.

Demonstration : $a^p = [1+(a-1)]^p$

$$= 1 + p(a-1) + \frac{p(p-1)}{\underline{|2}}(a-1)^2 + \ldots\ldots + (a-1)^p \qquad \text{(A)}$$

$$\therefore \quad a^p - (a-1)^p - 1 = p(a-1) + \frac{p(p-1)}{\underline{|2}} + \text{etc.}$$

$$= \text{a multiple of } p \text{ [901. 140, P.].} \qquad \text{(B)}$$

Let $a = 2$, then

$$a^p - (a-1)^p - 1 = 2^p - 2 = \text{a multiple of } p.$$

Let $a = 3$, then

$$a^p - (a-1)^p - 1 = 3^p - 2^p - 1 = (3^p - 3) - (2^p - 2)$$

$$= \text{a multiple of } p.$$

$\therefore \quad 3^p - 3$ is a multiple of p [157, P.].

By continuing this process, it may be shown by induction that $a^p - a$ is a multiple of p.

But $a^p - a = a(a^{p-1} - 1)$ and a is prime to p; therefore, $a^{p-1} - 1$ is divisible by p.

Perfect Squares.

906. ***Theorem I.***—*Every square number is of the form* $3m$ *or* $3m + 1$.

For, every number is of the form of $3x$ or $3x \pm 1$.

Now, $(3x)^2 = 9x^2 = 3(3x^2) = 3m$; and

$(3x \pm 1)^2 = (9x^2 \pm 6x + 1) = 3(3x \pm 2) + 1 = 3m + 1$.

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907. ***Theorem II.***—*Every square number is of the form* $4m$ *or* $4m + 1$.

For, every number is of the form of $4x$, $4x + 1$, $4x + 2$, or $4x + 3$.

Now, $(4x)^2 = 16x^2 = 4(4x^2) = 4m$;

$$(4x + 1)^2 = 16x^2 + 8x + 1 = 4(4x^2 + 2x) + 1 = 4m + 1;$$

$$(4x + 2)^2 = 16x^2 + 16x + 4 = 4(4x^2 + 4x + 1) = 4m;$$

and $(4x + 3)^2 = 16x^2 + 24x + 9 = 4(4x^2 + 6x + 2) + 1 = 4m + 1$.

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908. ***Theorem III.***—*Every square number is of the form* $5m$ *or* $5m \pm 1$.

For, every number is of the form $5x$, $5x \pm 1$, or $5x \pm 2$.

Now, $(5x)^2 = 25x^2 = 5(5x^2) = 5m$;

$$(5x \pm 1)^2 = 25x^2 \pm 10x + 1 = 5(5x^2 \pm 2x) + 1 = 5m + 1;$$

and $(5x \pm 2)^2 = 25x^2 \pm 20x + 4 = 5(5x^2 \pm 4x + 1) - 1 = 5m - 1$.

909. ***Theorem IV.***—*If $a^2+b^2=c^2$ when a, b, and c are integers, then will abc be a multiple of 60.*

For, 1. a^2 and b^2 can not both be of the form $3m+1$, else would c^2 be of the form $3m+2$, which is not a square. Therefore, either a or b is a multiple of 3 [906].

2. a^2 and b^2 can not both be of the form of $4m+1$, else would c^2 be of the form $4m+2$, which is not a square. Therefore, either a or b must be a multiple of 4, or each of them a multiple of 2 [907]. In either case, abc is a multiple of 4.

3. a^2 and b^2 can not both be of the form $5m+1$ or $5m-1$, else would c^2 be of the form $5m\pm 2$, which is not a square. Therefore, either a^2 or b^2 must be of the form $5m$, or one of the form $5m+1$ and the other of $5m-1$ [908]. In the former case, either a or b is a multiple of 5, and in the latter, c is a multiple of 5, and in either case, abc is a multiple of 5.

4. Since abc is a multiple of 3, 4, and 5, and these numbers are prime to each other, abc is a multiple of 60.

Scholium.—*By means of this theorem and the formula $a=\sqrt{(c+b)(c-b)}$, rational values of a, b, and c may be determined by inspection that will satisfy the equation $a^2+b^2=c^2$.*

910. Problem. To determine the rational value of x that will render x^2+px+q a perfect square.

Solution: Let $x^2+px+q=(x+m)^2$

then, $$x^2+px+q=x^2+2mx+m^2$$

whence, $x=\dfrac{m^2-q}{p-2m}$, in which m may have any rational value from $-\infty$ to $+\infty$.

Illustration.—What value of x will render x^2-7x+2 a perfect square?

Solution: Here $p = -7$, $q = 2$, and let $m = 5$,

then, $$x = \frac{25 - 2}{-7 - 10} = \frac{23}{-17} = -1\frac{6}{17}$$

and, $$x^2 - 7x + 2 = \frac{529}{289} + \frac{161}{17} + 2 = \frac{3844}{289} = \left(\frac{62}{17}\right)^2.$$

911. ***Cor. 1.****—For $m > \sqrt{q}$ and $2m < p$, or $m < \sqrt{q}$ and $2m > p$, m being positive, x will be positive.*

912. ***Cor. 2.****—Put $m^2 - q = n(p - 2m)$; then*

$$q = m^2 - n(p - 2m);$$

$x = n$, an integer; and

$$x^2 + px + q = x^2 + px + m^2 - n(p - 2m)$$

$$= (n + m)^2, \text{ an integer.}$$

913. ***Cor. 3.****—Put $x = \dfrac{m^2 - q}{p - 2m} = -m$; or*

$$m = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}, \text{ then,}$$

$$x^2 + px + q = 0, \text{ and } x = -\frac{p}{2} \mp \sqrt{\frac{p^2}{4} - q},$$

which conforms to Art. 337.

Perfect Cubes.

914. ***Theorem I.****—Every cube is of the form $4m$ or $4m \pm 1$.*

For, every number is of the form $4x$, $4x + 1$, $4x + 2$, or $4x + 3$.

Now, $(4x)^3 = 64x^3 = 4(16x^3) = 4m$;

$$(4x + 1)^3 = 64x^3 + 48x^2 + 12x + 1$$
$$= 4(16x^3 + 12x^2 + 3x) + 1 = 4m + 1$$
$$(4x + 2)^3 = 64x^3 + 96x^2 + 48x + 8$$
$$= 4(16x^3 + 24x^2 + 12x + 2) = 4m$$
$$(4x + 3)^3 = 64x^3 + 144x^2 + 108x + 27$$
$$= 4(16x^3 + 36x^2 + 27x + 7) - 1 = 4m - 1$$

915. Problem. To determine rational values of x that will render $x^3 + px^2 + qx + r$ a perfect cube.

Solution: Put $x^3 + px^2 + qx + r = (x + m)^3$; then,

$$(p - 3m)x^2 + (q - 3m^2)x + (r - m^3) = 0. \qquad \text{(A)}$$

1. Put $3m = p$, or $m = \frac{1}{3}p$; then

$$x = \frac{m^3 - r}{q - 3m^2} = \frac{p^3 - 27r}{27q - 9p^2}; \text{ and}$$

$$x^3 + px^2 + qx + r = (x + m)^3 = \left(\frac{p^3 - 27r}{27q - 9p^2} + \frac{p}{3}\right)^3$$

$$= \left(\frac{9pq - 27r - 2p^3}{27q - 9p^2}\right)^3.$$

Cor.—If $9pq - 27r - 2p^3 = 0$, *or* $r = \dfrac{9pq - 2p^3}{27}$, $x^3 + px^2 + qx + r = 0$, *and* $x = \dfrac{p^3 - 3pq}{9q - 3p^2}$.

2. Put $m^3 = r$, and suppose $r = r_1^3$, then $m = r_1$; and (A) will become $(p - 3r_1)x^2 + (q - 3r_1^2)x = 0$; whence,

$$x = \frac{3r_1^2 - q}{p - 3r_1}; \text{ and } x^3 + px^2 + qx + r = x^3 + px^2 + qx + r_1^3;$$

and $(x + m)^3 = \left(\dfrac{pr_1 - q}{p - 3r_1}\right)^3$. Therefore,

$$x = \frac{3r_1^2 - q}{p - 3r_1} \text{ will render } x^3 + px^2 + qx + r_1^3 = \left(\frac{pr_1 - q}{p - 3r_1}\right)^3$$

Cor.—If $q = pr_1$, $x = -r_1$, *and*

$$x^3 + px^2 + qx + r_1^3 = 0.$$

Scholium.*—Other values under particular suppositions may be obtained by putting* $3m^2 = q = 3q_1^2$.

EXERCISE 117.

1. Find which of the following numbers are prime: 197, 251, 313, 281, 461, 829, 957.

2. Find the least multiplier that will render 3174 a perfect square.

3. Find the least multiplier that will render 13168 a perfect cube.

4. Find which of the following numbers are divisible by 9, which by 11, and which by both 9 and 11: 11205, 24530, 342738, 25916, 558657.

5. Show that, if $p+q$ is an even number, then is $p-q$ also an even number, provided p and q are integral.

6. Show that every cube number is of the form $7n$ or $7n \pm 1$.

7. Find such a value of x as will render the $\sqrt{ax}$ rational.

Suggestion.—Put $\sqrt{ax} = p$.

8. Find such values of x as will render $\sqrt{ax+b}$ rational.

9. Prove that $2^{4n} - 1$ is a multiple of 15.

10. Show that no square number is of the form $3n-1$.

11. Show that $n(n+1)(2n+1)$ is divisible by 6.

12. Show that $(n^2+3)(n^2+7)$ is divisible by 32, when n is odd.

13. Show that $n^5 - n$ is a multiple of 30.

14. Show that the fourth power of any number is of the form $5m$ or $5m+1$.

15. Every even power of every odd number is of the form $8n+1$.

16. Show that every square can be expressed as the difference between two squares.

17. Show that $a^x + a$ and $a^x - a$ are even numbers.

18. Show that every number and its cube leave the same remainder when divided by 6.

19. If $n > 2$, show that $n^5 - 5n^3 + 4n$ is divisible by 120.

20. If n is a prime number greater than 3, show that $n^2 - 1$ is divisible by 24.

21. Find such a value of x as will render $\sqrt{ax}$ rational.

Suggestion.—

Let $\sqrt{ax} = p$, any rational quantity, then will $x = \frac{p^2}{a}$.

22. Find such a value of x as will rationalize $\sqrt{ax+b}$.

Suggestion.—Put $\sqrt{ax+b} = p$, and prove $x = \frac{p^2-b}{a}$.

23. Find such a value of x as will rationalize

$$\sqrt{ax^2+bx}.$$

Suggestion.—Put $\sqrt{ax^2+bx} = px$, and prove $x = \frac{b}{p^2-a}$.

24. Find such a value of x as will rationalize

$$\sqrt{ax^2+bx+c^2}.$$

Suggestion.—

Put $\sqrt{ax^2+bx+c^2} = px+c$, and prove $x = \frac{2pc-b}{a-p^2}$.

25. Find such a value of x as will rationalize

$$\sqrt{a^2x^2+bx+c}.$$

Suggestion.—

Put $\sqrt{a^2x^2+bx+c} = ax+p$, and prove $x = \frac{p^2-c}{b-2ap}$.

26. Find such a value of x as will render $\sqrt{ax^2+bx+c}$ rational when b^2-4ac is a perfect square.

Suggestion.—Put $\sqrt{ax^2+bx+c} = 0$, and $b^2-4ac = q^2$, and prove $x = \frac{-b \pm q}{2a}$.

27. Find such a value of x as will rationalize

$$\sqrt{ax^3+bx^2}.$$

Suggestion.—Put $\sqrt{ax^3+bx^2} = px$, and prove $x = \frac{p^2-b}{a}$.

28. Find such a value of x as will rationalize

$$\sqrt{ax^3+bx^2+cx+d^2}.$$

Suggestion.—

Put $\sqrt{ax^3+bx^2+cx+d^2} = \frac{c}{2d}x+d$, and prove $x = \frac{c^2-4bd}{4ad^2}$.

ANSWERS.

Exercise 87.

1. $1+x+x^2+x^3$

2. $\frac{1}{3}+\frac{2}{9}x+\frac{4}{27}x^2+\frac{8}{81}x^3$

3. $1+x-x^3-x^4$

4. $1+x-x^2-x^3$

5. $1-x-x^2+2x^3$

6. $-3+5x-2x^2-3x^3$

7. $1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}$

8. $5x+27x^2+130x^3+623x^4$

9. $1+x+x^2-x^3$

Exercise 88.

1. $2-\frac{1}{4}x+\frac{1}{64}x^2+\frac{1}{256}x^3$

2. $1+\frac{1}{2}x-\frac{5}{8}x^2-\frac{1}{2}x^3$

3. $3+\frac{1}{6}x-\frac{109}{216}x^2+\frac{109}{3888}x^3$

4. $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3$

5. $3+\frac{1}{27}x^2-\frac{1}{3.27^2}x^4+\frac{15}{3.27^4}x^6$

6. $2+\frac{1}{12}x+\frac{23}{2.12^2}-\frac{25}{2.12^4}$

7. $a^{\frac{1}{2}}+\frac{x}{2a^{\frac{1}{2}}}-\frac{x^2}{8a^{\frac{3}{2}}}+\frac{x^3}{16a^{\frac{5}{2}}}$

8. $a^{\frac{1}{3}}-\frac{x}{3a^{\frac{2}{3}}}+\frac{x^2}{9a^{\frac{5}{3}}}+\frac{7}{81a^{\frac{8}{3}}}$

9. $a+\frac{x^3}{3a^2}-\frac{x^6}{9a^5}+\frac{5x^9}{81a^8}$

Exercise 89.

1. Divergent.
2. Convergent; divergent; divergent.
3. Convergent.
4. Convergent.
5. Divergent.
6. Convergent; divergent; divergent.
7. Divergent; convergent; divergent.
8. Convergent.

Exercise 90.

1. $x^3+6x^2+11x+6$

2. $x^3-2x^2-9x+18$

3. $x^4+2x^3-7x^2-8x+12$

4. $x^4-37x^2-24x+180$

5. $x^4+8x^3+24x^2+32x+16$

6. $x^4-20x^3+150x^2-500x+625$

7. $16x^4-16x^3-64x^2+4x+15$

8. $(x+2)(x+3)(x+4)$

9. $(x-3)(x-4)(x+5)$

10. $(x+2)(x+3)(x+1)(x-1)$

11. $(x+2)(x-3)(x+4)(x-5)$

12. $(x+2)(x-2)(x+3)(x-3)(x+4)$

Exercise 91.

1. $a^4-12a^3x^2+54a^2x^4-108ax^6+81x^4$

2. $32+400x+2000x^2+5000x^3+6250x^4+3125x^5$

3. $x^6-18ax^5+135a^2x^4-540a^3x^3+1215a^4x^2-1458a^5x+729a^6$

4. $128x^{14}+2240x^{12}+16800x^{10}+70000x^8+175000x^6+262500x^4+218750x^2+78125$

5. $x^4-40x^{\frac{7}{2}}+700x^3-7000x^{\frac{5}{2}}+43750x^2-175000x^{\frac{3}{2}}+437500x-625000x^{\frac{1}{2}}+390625$

6. $2187x^{\frac{14}{3}}+5103a^{\frac{3}{2}}x^4+5103a^3x^{\frac{10}{3}}+2835a^{\frac{9}{2}}x^{\frac{8}{3}}+945a^6x^2+189a^{\frac{15}{2}}x^{\frac{4}{3}}+21a^9x^{\frac{2}{3}}+a^{\frac{21}{2}}$

7. $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3$

8. $a^{\frac{2}{3}}-\frac{2}{3}a^{-\frac{1}{3}}x-\frac{1}{9}a^{-\frac{4}{3}}x^2-\frac{4}{81}a^{-\frac{7}{3}}x^3$

9. $x^{\frac{3}{2}}-\frac{3}{4}x^{-\frac{1}{4}}-\frac{3}{32}x^{-\frac{5}{4}}-\frac{5}{128}x^{-\frac{9}{4}}$

10. $x^{-\frac{1}{4}}-\frac{1}{2}ax^{-\frac{3}{4}}+\frac{3}{8}a^2x^{-\frac{5}{4}}-\frac{5}{16}a^3x^{-\frac{7}{4}}$

11. $a^{-\frac{5}{6}}x^{-\frac{5}{6}}-\frac{5}{6}a^{-\frac{11}{6}}bx^{-\frac{11}{6}}+\frac{55}{72}a^{-\frac{17}{6}}b^2x^{-\frac{17}{6}}-\frac{935}{1296}a^{-\frac{23}{6}}b^3x^{-\frac{23}{6}}$

12. $x^{-\frac{4}{9}}+\frac{2}{3}a^{\frac{2}{3}}x^{-\frac{10}{9}}+\frac{5}{9}a^{\frac{4}{3}}x^{-\frac{16}{9}}+\frac{40}{27}a^2x^{-\frac{22}{9}}$

14. 8·06225; 8·94427; 7·006796; 5·000980

15. $2449440x^4$

16. $-\frac{5}{128^2}x^4$

17. $-\frac{63}{256}a^{-\frac{11}{2}}x^5$

18. $\pm\frac{r(r+1)(r+2)(r+3)}{\lfloor 4}a^{-(r+4)}x^{r-1}$

19. $2\frac{5}{6}$

20. $\frac{110}{243}$

21. 0

Exercise 92.

1. $n(n+1)$; $\frac{1}{3}n(n+1)(n+2)$

2. $(2n-1)^2$; $\frac{1}{3}n(4n^2-1)$

3. $\frac{1}{2}n(n+1)$; $\frac{1}{6}n(n+1)(n+2)$

4. $n(n+2)$; $\frac{1}{6}n(n+1)(2n+7)$

5. n^2; $\frac{1}{6}n(n+1)(2n+1)$

6. $2n(2n-1)$; $\frac{1}{3}n(n+1)(4n-1)$

7. $n(n+1)(n+2)$; $\frac{1}{4}n(n+1)(n+2)(n+3)$

8. $n(n+4)(n+8)$; $\frac{1}{4}n(n+1)(n+8)(n+9)$

9. $n(n+2)(n+1)^2$; $\frac{1}{10}n(n+1)(n+2)(n+3)(2n+3)$

10. $(2n+1)(2n+3)(2n+5)$; $\frac{1}{8}(2n+1)(2n+3)(2n+5)(2n+7)-13\frac{1}{8}$

11. $\frac{n}{3n+1}$; $\frac{1}{3}$

12. $\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right)$; $\frac{11}{18}$

13. $\frac{1}{8}\left(\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\right)$; $\frac{1}{32}$

14. $\frac{11}{180}-\frac{6n+11}{12(2n+1)(2n+3)(2n+5)}$; $\frac{11}{180}$

15. $\frac{5}{4}-\frac{2n+5}{2(n+1)(n+2)}$; $\frac{5}{4}$

16. $\frac{n}{12}(9n^3+10n^2-3n-4)$

17. $\frac{n(n+1)(n+2)(n+3)}{\underline{|4}}$

18. $\frac{n}{n+1}$

19. $n+1-\frac{1}{n+1}$

20. 2·978809

Exercise 93.

1. $x=y-y^2+y^3-y^4+\dots\dots$

2. $x=y+\frac{1}{2}y^2+\frac{1}{6}y^3+\frac{1}{24}y^4+\dots\dots$

3. $x=y-2y^2+3y^3-4y^4+\dots\dots$

4. $x=y-y^3+y^5-y^7+\dots\dots$

5. $x=(y-1)-\frac{1}{2}(y-1)^2+\frac{1}{3}(y-1)^3-\frac{1}{4}(y-1)^4+\dots\dots$

6. $x=(y-1)+2(y-1)^2+7(y-1)^3+30(y-1)^4+\dots\dots$

7. $x=\frac{1}{5}-\frac{4}{5^2}+\frac{27}{5^5}-\frac{58}{5^7}=\cdot17590144$

8. $x=\cdot00999999$

Exercise 94.

1. $p=3,\ q=-1$; $\frac{1-x}{1-3x+x^2}$

2. $p=2,\ q=2$; $\frac{1+x}{1-2x-2x^2}$

3. $p=5,\ q=-3$; $\frac{2+8x}{1-5x+3x^2}$

4. $p=4,\ q=-5$; $\frac{3-10x}{1-4x+5x^2}$

5. $p=3,\ q=2,\ r=-2;\ \dfrac{1-x-5x^2}{1-3x-2x^2-2x^3}$

6. $p=2,\ q=3,\ r=4;\ \dfrac{1-5x+8x^2}{1-2x-3x^2-4x^4}$

7. $p=3,\ q=3;\ \dfrac{1-x-1809x^6-1431x^7}{1-3x-3x^2}$

8. $p=2,\ q=-3;\ \dfrac{1-4x-24x^7+189x^8}{1-2x+3x^2}$

9. $p=-2,\ q=2;\ \dfrac{2+3x-2208x^8+1616x^9}{1+2x-2x^2}$

10. $p=1,\ q=-1,\ r=1;\ \dfrac{1-3x+6x^2+2x^9-5x^{10}+x^{11}}{1-x+x^2-x^3}$

11. $(-1)^{n-1}\{4-3\cdot 2^{n-1}\}x^{n-1};\ \dfrac{4\{1-(-1)^n x^n\}}{1+x}-\dfrac{3\{1-(-2)^n x^n\}}{1+2x}$

12. $(4n-3)x^{n-1};\ \dfrac{4(1-x^n)}{(1-x)^2}-\dfrac{(4n-3)x^n+3}{1-x}$

Exercise 95.

1. $\dfrac{2}{x+2}+\dfrac{3}{x-2}$

2. $\dfrac{3}{2x+1}-\dfrac{2}{2x-1}$

3. $\dfrac{1}{x}+\dfrac{2}{x+1}+\dfrac{3}{x+2}$

4. $\dfrac{3}{x+1}-\dfrac{5}{(x+1)^2}$

5. $\dfrac{1}{x+3}-\dfrac{5}{(x+3)^2}+\dfrac{7}{(x+3)^3}$

6. $\dfrac{3}{x+2}+\dfrac{4}{(x-3)^2}$

7. $\dfrac{a}{a-x}+\dfrac{b}{b+x}$

8. $\dfrac{p}{px+q}+\dfrac{q}{(px+q)^2}$

9. $\dfrac{2}{x^2+x+1}+\dfrac{3}{x^2-x+1}$

10. $\dfrac{2}{x}-\dfrac{2}{1+2x}+\dfrac{2}{1-2x}$

11. $\dfrac{5}{(x-1)^4}-\dfrac{7}{(x-1)^3}+\dfrac{1}{(x-1)^2}+\dfrac{3}{x-1}$

12. $\dfrac{3}{2(x-1)}+\dfrac{1-3x}{2(1+x^2)}$

13. $1-\dfrac{3}{x-2}+\dfrac{8}{x-3}$

14. $-\dfrac{1}{4(x-1)}+\dfrac{x+1}{4(x^2+1)}+\dfrac{x+5}{2(x^2+1)^2}$

15. $\dfrac{1}{2(x+1)}+\dfrac{x+1}{2(x^2+1)}$

16. $-\dfrac{1}{x}+\dfrac{1}{x^2}-\dfrac{1}{x^3}+\dfrac{1}{8(x-1)}+\dfrac{9}{8(x+1)}+\dfrac{1}{4(x+1)^2}-\dfrac{x+1}{4(x^2+1)}$

17. $\dfrac{3x}{x^2+2x-5}-\dfrac{1}{x-3}$

18. $\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{3}{(x+1)^2}-\dfrac{3}{(x+1)^3}+\dfrac{2}{(x+1)^4}$

19. $-\dfrac{1}{x}+\dfrac{1}{x^2}-\dfrac{1}{x^3}+\dfrac{1}{8(x-1)}+\dfrac{9}{8(x+1)}+\dfrac{1}{4(x+1)^2}-\dfrac{x+1}{4(x^2+1)}$

20. $\frac{1}{2(x+1)}+\frac{x+1}{2(x^2+1)}$ 21. $\frac{3}{4(x-1)^2}-\frac{3}{8(x-1)}+\frac{1}{8(x+1)}+\frac{x-1}{4(x^2+1)}$

22. $\frac{17}{36(x+1)}-\frac{5}{6(x+1)^2}+\frac{8}{45(x-2)}-\frac{13}{20(x+3)}$

Exercise 97.

1. $dy=(15ax^2-6bx+2c)\,dx$
2. $dy=15z^2x^2\,dx+10x^3z\,dz+dz$ 3. $dy=(3x^2+6x+4)\,dx$
4. $dy=bm^2(a+bx)^{m^2-1}\,dx$ 5. $dy=(x+\frac{3}{2}-x^{-2})\,dx$
6. $dy=\frac{a\,dx}{2c^{\frac{1}{2}}}\cdot\frac{1}{(ax+b)^{\frac{1}{2}}}$ 7. $dy=\frac{5\,dx}{3}\cdot\frac{1}{(5x+6)^{\frac{2}{3}}}$
8. $dy=\sqrt{\frac{2p}{x}}\,.\,dx$ 9. $dy=\frac{2}{3}(3a^{\frac{1}{3}})\,.\,x^{-\frac{1}{3}}\,dx$
10. $dy=-\frac{2b}{a}\cdot\frac{x\,dx}{\sqrt{a^2-x^2}}$ 11. $dy=\left(\frac{3}{2}ax^{\frac{1}{2}}+bx^{-\frac{1}{2}}\right)dx$
12. $dy=\frac{2x^3(4a+x)}{(a+x)^4}\,.\,dx$ 13. $dy=\frac{(2x+a)\,dx}{(x+a)^{\frac{3}{2}}}$
14. $dy=-\frac{6ax\,dx}{(b^2+x^2)^4}$ 15. $dy=(x+a)(x+b)^2(4x+3a+b)\,dx$
16. $dy=(x+a)^{n-1}(x-b)^{p-1}(2x+a-b)\,dx$ 17. $dy=\frac{2x+1}{x^2+x}\,.\,dx$
18. $dy=(\log_e x)^2\,.\,\frac{3\,dx}{x}$ 19. $dy=9x^{3x}(1+\log_e 3+\log_e x)\,dx$
20. $dy=2a^{x^2}\,.\,x\log_e a\,dx$ 21. $dy=\frac{2x^3\,dx}{(a+x^2)^{\frac{5}{2}}}(2a+3x^2)$
22. $dy=\frac{n\,dx}{a+x}$ 23. $dy=\left(\frac{c}{d}\right)^x.\,(\log_e c-\log_e d)\,dx$
24. $dy=\frac{2\,dx}{(1-x)(1-x^2)^{\frac{1}{2}}}$ 25. $dy=a^{\log_e x}\,.\,\log_e a\,.\,\frac{dx}{x}$

Exercise 98.

2. 16 sq. in. per second. 3. $4\pi r^2$; 144π cu. in. per second.
4. $2\sqrt{2}$ in. 5. 1 in. per second. 6. About 6 mi. an hr.
7. $12\sqrt{3}$ in. per sec. 8. ·00517 9. 1·62842; ·00003

Exercise 99.

1. $3x^2-8x+7$ 2. $(x+2)^2(x-2)^3(7x+2)$
3. $3x^2+8x+2$ 4. $(a+x)^4(a-x)^2(8a-2x)$
5. $-8x^7+5ax^4-3ax^2$ 6. $-10x\,.\,\frac{(a+x)^4}{(a-x)^6}$

Exercise 100.

1. $(x+3)^2(x-2)^2$
2. $(x-2)^3(x-1)^2(x+3)$
3. $(x+3)^3(x-3)^3(x^2+x+1)$
4. $(x-2)^3(x+2)^3(x-3)^2(x+3)^2$
5. $(x-1)^4(x+1)^4(x+3)(x-3)$

Exercise 101.

1. Max. $1\frac{3}{4}$; min. -5
2. Min. -3; max. -128
3. Min. -4
4. Min. -4
5. Max. 10; min. -22
6. Min. 0; max. 18+
7. Max. $\left(\frac{a+b}{2}\right)^4$
8. No turning values.
9. Min. -14
10. Min. -14
11. Min. 0
12. Min. -16
13. At the middle point.
15. $\frac{1}{3}a$
16. $\frac{32}{81}\pi r^3$, or $\frac{8}{27}$ of the vol. of the sphere.
17. $\frac{1}{2}a^2$, or an inscribed square.
18. $\frac{1}{4}\pi a^2$
19. $\frac{1}{3}a$
20. Square $= 2r^2$
21. An isosceles triangle.

Exercise 102.

1. $x^4+2x^3-9x^2+63x-135=0$; roots of $f_n(x) = 3x$ roots of $F_n(x)$
2. $x^5+4x^4-2x^3+4x^2-112=0$; roots of $f_n(x) = 2x$ roots of $F_n(x)$
3. $x^6+12x^4-320x^2+1792x-1024=0$; roots of $f_n(x) =$ $4x$ roots of $F_n(x)$
4. $x^5-2x^4+9x^3-18x^2+108x-162=0$; roots of $f_n(x) =$ $3\sqrt[6]{\text{roots of } (A)}$
5. $x^6-2x^5+2x^4-4x^3+24x^2+32x+32=0$; roots $= 2\sqrt[7]{\text{roots of } (A)}$
6. $x^{36}+8 \,.\, 9^2x^{33}+36 \,.\, 9^{11}x^{24}+10 \,.\, 9^{19}x^{16}-4 \,.\, 9^{34}=0$; roots $=$ $9\sqrt[12]{\text{roots of } (A)}$
7. $x^7-x^6+18x^4-162x^2-2187=0$; roots $= 3\sqrt[6]{\text{roots of } (A)}$
8. $x^6-10x^5-36x^4-12288=0$; roots $= 4\sqrt[6]{\text{roots of } (A)}$
9. $x^4+6800x-9000=0$; roots $= 10\sqrt[3]{\text{roots of } (A)}$

Exercise 103.

1. $x=1$
2. $x=4$
3. $x=5$
4. $x=-2$
5. $x=1,\ 2,\ 2,\ -2$
6. $x=\frac{1}{2},\ -\frac{1}{2},\ -\frac{2}{3}$
7. $x=1,\ 2,\ -2,\ -3$
8. $x=5,\ -1$
9. $x=+2,\ -2,\ -3$
10. $x=2,\ 4$

11. $x = 1\frac{1}{2}$

12. $\frac{1}{2}, -\frac{1}{2}, 1\frac{1}{2}, 1\frac{1}{2}$

13. $x = 2, -2, -\frac{1}{3}$

14. $x = -3, -5, -1, \pm\frac{2}{3}$

15. $x = -\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}$

16. $x = 2, -3, -5, 2$

17. $x = 2, 2, -3, -3$

18. $x = 1, 1, -2, -2, -2$

19. $x = -1, -1, -1, 2, -3, -\frac{2}{3}$

Exercise 104.

1. $-1+, 4+, \cdot 9+$ **2.** $-6\cdot 6+$ **3.** $0\cdot 3+$

4. $0\cdot 8+, 3+, -3+, -0\cdot 1+, -0\cdot 6+$

5. $2+, -2+$ **6.** $2+, 0\cdot 6+, 0\cdot 4+, -3+$

Exercise 105.

1. 775 **2.** 240 **3.** ·87938, −2·53208, −1·34729

4. 3·85808, 1·60601, 1·44327, −2·90737

5. −5, 4·05607, 11·15306, 12·79085

6. −1·12579 **7.** 2·34244 **8.** 1·25992 **9.** 1·3797

10. 3·2131, 3·2295, −17·4426 **11.** ·80285, −5·4335

12. 8·41445 **13.** $x^3 + 32x^2 + 343x - 13087800 = 0$

Exercise 106.

1. $x = 1, 1 \pm 2\sqrt{-1}$

2. $x = 4, 1 \pm \sqrt{-1}$

3. $x = 5, 3 \pm \sqrt{-2}$

4. $x = -2 \pm \sqrt{-2}$

5. $x = 3, \frac{1}{2}(1 \pm \sqrt{-7})$

6. $x = -4, -\frac{1}{2}(1 \pm \sqrt{-7})$

7. $x = -2, -2, -3$

8. $x = 3, -2, -2$

9. $x = 2, 2, -3$

10. $x = 4, -2 \pm \sqrt{-1}$

11. $x = -6, 3 \pm \sqrt{-1}$

12. $x = 1, \frac{1}{2}(3 \pm \sqrt{21})$

13. $x = 5, 1, -2$

14. $x = -5, -1, 2$

Exercise 107.

1. $x = 1, \frac{1}{2}(1 \pm \sqrt{-3})$

2. $x = -1, 2 \pm \sqrt{3}$

3. $x = 1, -1, \frac{1}{2}(3 \pm \sqrt{5})$

4. $x = 1, -1, \frac{1}{2}(-3 \pm \sqrt{5})$

5. $x = \frac{1}{2}(5 \pm \sqrt{21}, \pm\sqrt{-1}$

6. $x = -1, -1, -1, -1, -1$

7. $x = 1, 2, \frac{1}{2}, -3, -\frac{1}{3}$

8. $x = -1, -5, -\frac{1}{5}, 2 \pm \sqrt{3}$

Exercise 108.

1. -9 2. -59 3. 65 4. $abc-2ab^2$
5. $2mnp$ 6. $4abc$ 7. 28 8. -82
9. -108 10. 0 11. 0 12. 0
13. 0 14. 0 15. 0

Exercise 109.

1. 0 2. 0 3. 0 4. 0
5. $8xyz$ 6. $6abc-2a^3-2b^3-2c^3$

Exercise 110.

1. 10 2. -48 3. -199 4. 0

Exercise 111.

1. $x=4,\ y=2$ 2. $x=3,\ y=3$

3. $x=\dfrac{cn-bd}{an-bm},\ y=\dfrac{ad-cm}{an-bm}$

4. $x=\dfrac{cm-dm+cn+dn}{2(ac-bd)},\ y=\dfrac{an+bn-am+bm}{2(ac-bd)}$

5. $x=2,\ y=3,\ z=4$ 6. $x=5,\ y=1,\ z=2$

7. $x=\dfrac{abd-acd-abe+ec^2+abh-bch}{a^2b-a^2c-abc+c^3+ab^2-b^2c}$

$y=\dfrac{a^2e-a^2h-acd+c^2h+abd-bce}{a^2b-a^2c-abc+c^3+ab^2-b^2c}$

$z=\dfrac{abh-ace-bch+c^2d+b^2e-b^2d}{a^2b-a^2c-abc+c^3+ab^2-b^2c}$

8. $x=\dfrac{a^2(m+n-r)+ab(m-n-r)+b^2(m-n+r)}{2a^3+2b^3}$

$y=\dfrac{a^2(n-m+r)+ab(2an-r-m-n)+b^2(n-r+m)}{2a^3+2b^3}$

$z=\dfrac{a^2(r-n+m)+ab(2r-n-r-m)+b^2(r+n-m)}{2a^3+2b^3}$

9. $x=2,\ y=3,\ z=4,\ u=5$

10. $$x=\frac{m\begin{vmatrix} c & 0 & a \\ 0 & a & b \\ a & b & c \end{vmatrix}-n\begin{vmatrix} b & c & 0 \\ 0 & a & b \\ a & b & c \end{vmatrix}+p\begin{vmatrix} b & c & 0 \\ c & 0 & a \\ a & b & c \end{vmatrix}-q\begin{vmatrix} b & c & 0 \\ c & 0 & a \\ 0 & a & b \end{vmatrix}}{a\begin{vmatrix} c & 0 & a \\ 0 & a & b \\ a & b & c \end{vmatrix}-b\begin{vmatrix} b & c & 0 \\ 0 & a & b \\ a & b & c \end{vmatrix}+c\begin{vmatrix} b & c & 0 \\ c & 0 & a \\ a & b & c \end{vmatrix}}.$$

$$y=\left\{a\begin{vmatrix} n & 0 & a \\ p & a & b \\ q & b & c \end{vmatrix} - b\begin{vmatrix} m & c & 0 \\ p & a & b \\ q & b & c \end{vmatrix} + c\begin{vmatrix} m & c & 0 \\ n & 0 & a \\ q & b & c \end{vmatrix}\right\} \div c\,.\,d$$

$$z=\left\{a\begin{vmatrix} c & n & a \\ 0 & p & b \\ a & q & c \end{vmatrix} - b\begin{vmatrix} b & m & 0 \\ 0 & p & b \\ a & q & c \end{vmatrix} + c\begin{vmatrix} b & m & 0 \\ c & n & a \\ a & q & c \end{vmatrix}\right\} \div c\,.\,d$$

$$u=\left\{a\begin{vmatrix} c & 0 & n \\ 0 & a & p \\ a & b & q \end{vmatrix} - b\begin{vmatrix} b & c & m \\ 0 & a & p \\ a & b & q \end{vmatrix} + c\begin{vmatrix} b & c & m \\ c & 0 & n \\ a & b & q \end{vmatrix}\right\} \div c\,.\,d$$

11. $x=2,\ y=3,\ z=-2,\ u=-3$

12. $x=\dfrac{p+q+r+s}{4a},\ y=\dfrac{p-q+r-s}{4b}$

$z=\dfrac{p+q-r-s}{4c},\ u=\dfrac{p-q-r+s}{4d}$

13. $x=1,\ y=2,\ z=3,\ u=4,\ v=5$

Exercise 112.

1. Consistent. **2.** Inconsistent.

Exercise 115.

1. $\dfrac{1}{2+}\dfrac{1}{1+}\dfrac{1}{1+}\dfrac{1}{6+}\dfrac{1}{2+}\dfrac{1}{4}$ **2.** $\dfrac{1}{1+}\dfrac{1}{1+}\dfrac{1}{1+}\dfrac{1}{11+}\dfrac{1}{6}$

3. $\dfrac{1}{9+}\dfrac{1}{1+}\dfrac{1}{99}$ **4.** $\dfrac{1}{6+}\dfrac{1}{1+}\dfrac{1}{7+}\dfrac{1}{6+}\dfrac{1}{2}$ **5.** $3+\dfrac{\dot{1}}{6}$

6. $3+\dfrac{\dot{1}}{2+}\dfrac{\dot{1}}{6}$ **7.** $5+\dfrac{\dot{1}}{2+}\dfrac{\dot{1}}{10}$ **8.** $7+\dfrac{1}{1+}\dfrac{\dot{1}}{4}$

9. $\dfrac{1}{3+}\dfrac{1}{7+}\dfrac{1}{17+}\dfrac{1}{2+}\dfrac{1}{1+}\dfrac{1}{8}$ **10.** $3+\dfrac{1}{7+}\dfrac{1}{16+}\dfrac{1}{11}$

11. $67+\dfrac{1}{2+}\dfrac{1}{1+}\dfrac{1}{12+}\dfrac{1}{1+}\dfrac{1}{1}$ deg. **12.** $\dfrac{1}{2},\ \dfrac{3}{7},\ \dfrac{13}{80},\ \dfrac{68}{157}$

13. $\dfrac{1}{3},\ \dfrac{1}{4},\ \dfrac{10}{39},\ \dfrac{11}{43}$ **14.** $\dfrac{1}{2},\ \dfrac{3}{7},\ \dfrac{7}{16},\ \dfrac{24}{55},\ \dfrac{55}{126},\ \dfrac{189}{433}$, etc.

15. $\dfrac{1}{1},\ \dfrac{2}{3},\ \dfrac{7}{10},\ \dfrac{16}{23},\ \dfrac{55}{79}$, etc. **16.** $\dfrac{1}{2},\ \dfrac{3}{7},\ \dfrac{10}{23},\ \dfrac{33}{76},\ \dfrac{109}{251}$, etc.

17. $\sqrt{5}-2$ **18.** $\sqrt{6}-2$ **19.** $\sqrt{5}$ **20.** $\dfrac{1}{3}(\sqrt{10}-1)$

21. $\dfrac{2}{3},\ \dfrac{5}{9},\ \dfrac{7}{12},\ \dfrac{19}{33},\ \dfrac{26}{45},\ \dfrac{71}{123}$, etc. **22.** $\dfrac{6}{7}$

23. $\dfrac{1}{2},\ \dfrac{1}{3},\ \dfrac{10}{29},\ \dfrac{21}{61}$ **24.** $\dfrac{95}{98}$

Exercise 116.

1. 25542
2. 616
3. 434005, 327454
4. $58072\frac{8}{9}$, $32853\frac{6}{11}$
5. 2950
6. 125344
7. 2 e 23
8. 202033030
9. 7692
10. 14860
11. $r=6$
12. $r=6$
13. $r=14$
14. 2
15. 1414, 1201
16. $r=7$

Exercise 117.

1. 197, 251, 313, 281, 461, 829
2. 6
3. 1646
4. By 9: 11205, 342738, 558657
   By 11: 24530, 342738, 25916, 558657
   By 9 and 11: 342738, 558657

7. $x=\frac{p^2}{a}$
8. $x=\frac{p^2-b}{a}$

THE END.