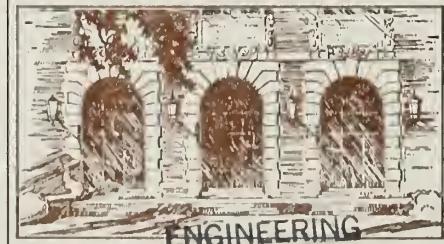


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A NUMERICAL SOLUTION OF THE MATRIX RICCATI EQUATIONS

By
Killion Noh

January 20, 1972

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MATRIX RICCATI EQUATIONS

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January 20, 1972

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ABSTRACT

The eigenvector solution of the time-invariant matrix Riccati equation is discussed. The coefficient matrix of the canonical equation is allowed to have multiple eigenvalues, namely, the matrix could be either derogatory or defective. The solution of matrix Riccati equation is then calculated from a part of similarity transformation which should reduce the coefficient matrix to the Jordan canonical form.

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1. INTRODUCTION

It is well known [1,2] that the linear regulator problem with quadratic cost functional is reduced to the problem of the matrix Riccati equation.

Although the solution of the matrix Riccati equation exists and is unique [2,3], it is not easy to obtain a numerical solution for the high-order system.

One of the numerical methods is the eigenvector solution which has been studied by several authors [4,5,6]. In their studies, however, the coefficient matrix of the canonical equation is assumed to have distinct eigenvalues, which is a restriction to the method of eigenvector solutions.

Recently, Martensson [7] discussed the case in which the coefficient matrix is nondiagonable. Similarly, in this paper, the assumption of distinct eigenvalues will also be removed so that the coefficient matrix is allowed to have the multiple eigenvalues, namely, the matrix is either derogatory or defective. To do this, we explicitly construct the similarity transformation which should reduce the coefficient matrix to the Jordan canonical form, either diagonal or nondiagonal, and then the solution of the matrix Riccati equation is calculated from a part of the similarity transformation matrix.

Since matrix Riccati equations of fairly large sizes appear in many engineering applications and since the solution algorithm can be easily constructed such that it becomes suitable for a parallel computer [8], we provide in Appendix II the basic codes written in ILLIAC IV language, GLYPNIR.

2. FORMULATION OF THE PROBLEM

In this section, the source of the matrix Riccati equation and its relation to optimal control problems are summarized in short for our further discussions [1,2,3].

Let us consider the linear time-invariant dynamical system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ x(t_0) &= x_0 \end{aligned} \tag{2.1}$$

where A , B , and C are $n \times n$, $n \times r$, and $m \times n$ constant matrices, respectively, and $x(t)$ is an n -dimensional state vector, $u(t)$ is an r -dimensional control vector, and $y(t)$ is an m -dimensional output vector, respectively. Further, we assume that the system (2.1) is completely observable. The optimal output regulator problem is then to determine the control $u(t)$ which minimizes the quadratic cost functional,

$$V_1 = \frac{1}{2} y^T(t_f) P y(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (y^T(t) Q y(t) + u^T(t) R u(t)) dt \tag{2.2}$$

where P and Q are $m \times m$ symmetric positive semidefinite matrices and R is an $r \times r$ symmetric positive definite matrix. Substituting $y(t) = Cx(t)$ into (2.2), V_1 is rewritten as

$$V_1 = \frac{1}{2} x^T(t_f) C^T P C x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (x^T(t) C^T Q C x(t) + u^T(t) R u(t)) dt \tag{2.3}$$

Since the system (2.1) is completely observable, $C^T P C$ and $C^T Q C$ are symmetric and positive semidefinite [1].

The optimal feedback control, therefore, is given by

$$u^*(t) = -R^{-1} B^T K(t) x(t) \tag{2.4}$$

where $K(t)$ is an $n \times n$ symmetric positive definite matrix which is the

solution of the matrix Riccati equation

$$\dot{K}(t) + K(t) A + A^T K(t) - K(t) B R^{-1} B^T K(t) + C^T Q C = 0 \quad (2.5)$$

with the boundary condition

$$K(t_f) = C^T P C \quad (2.6)$$

Furthermore, the optimal trajectory is the solution of the system of differential equations

$$\dot{x}(t) = (A - B R^{-1} B^T K(t)) x(t) \quad (2.7)$$

$$x(t_0) = x_0$$

The minimum cost is given by

$$V_1^* = \frac{1}{2} x^T(t) K(t) x(t) \quad (2.8)$$

In addition to the assumption of complete observability we further assume that the system is completely controllable. Setting $P = 0$ and $t_f = \infty$, the output regulator problem turns out to be the following:

minimize

$$V = \frac{1}{2} \int_0^\infty (y^T(t) Q y(t) + u^T(t) R u(t)) dt \quad (2.9)$$

subject to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ x(0) &= x_0 \end{aligned} \quad (2.10)$$

Then the optimal control is given by

$$u(t) = -R^{-1} B^T K x(t) \quad (2.11)$$

where K is the $n \times n$ symmetric positive definite matrix which is the solution of algebraic Riccati equation

$$KA + A^T K - KBR^{-1}B^T K + C^T QC = 0 \quad (2.12)$$

The optimal trajectory is the solution of the system

$$\dot{x}(t) = Gx(t)$$

$$G = A - BR^{-1}B^T K \quad (2.13)$$

$$x(0) = x_0$$

Note that for an optimal control system the matrix G is diagonalable and has eigenvalues with negative real parts. Therefore, the symmetric positive definite solution K of (2.12) must be such that G has eigenvalues with negative real parts.

Kalman [2] has shown that if the system (2.1) is completely observable and completely controllable, then

$$\lim_{t_f \rightarrow \infty} K(t_f) = K \quad (2.14)$$

We shall observe this equation in the following.

$$\text{Let } K(t) = Z(t)X^{-1}(t) \quad (2.15)$$

where $X(t)$ and $Z(t)$ are $n \times n$ matrices and $X(t)$ is non-singular. Substituting

$$\dot{K}(t) = (\dot{Z}(t) - K(t)\dot{X}(t))X^{-1}(t) \quad (2.16)$$

into (2.5) and setting

$$\dot{X}(t) = AX(t) - BR^{-1}B^T Z(t) \quad (2.17)$$

we obtain

$$\dot{Z}(t) = -C^T QCX(t) - A^T Z(t) \quad (2.18)$$

Therefore (2.5) becomes a system of $2n$ -dimensional linear homogeneous differential equations

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Z}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -C^T QC & -A^T \end{bmatrix} \begin{bmatrix} X(t) \\ Z(t) \end{bmatrix} \quad (2.19)$$

Since we are interested in increasing the time t , let

$$\tau = t_f - t \quad (2.20)$$

Then

$$\begin{bmatrix} \dot{X}(\tau) \\ \dot{Z}(\tau) \end{bmatrix} = W \begin{bmatrix} X(\tau) \\ Z(\tau) \end{bmatrix} \quad (2.21)$$

where

$$W = \begin{bmatrix} -A & BR^{-1}B^T \\ C^T Q C & A^T \end{bmatrix}$$

with

$$\begin{aligned} X(0) &= I \\ Z(0) &= C^T Q C \end{aligned} \quad (2.22)$$

Therefore, the solution will be given by the following matrix exponential:

$$\begin{bmatrix} X(\tau) \\ Z(\tau) \end{bmatrix} = e^{W\tau} \begin{bmatrix} I \\ C^T Q C \end{bmatrix} \quad (2.23)$$

Consequently, the solution of the matrix Riccati equation will be obtained by (2.15) as $\tau \rightarrow \infty$. In the next section, we shall show that two matrices $X(\tau)$ and $Z(\tau)$ are obtained from the similarity transformation which reduces W into the Jordan canonical form. Throughout the remaining discussions, the increasing time variable t will be used instead of τ .

3. SOLUTION OF THE MATRIX RICCATI EQUATION

We begin with two lemmas for our main result.

Lemma 1. The eigenvalues of the matrix W of (2.21) are symmetric with respect to the imaginary axis.

Proof. Let I_1 be a $2n \times 2n$ matrix given by

$$I_1 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \quad (3.1)$$

where I is an $n \times n$ identity matrix. Then it is obvious that

$$I_1^{-1} = I_1^T \quad (3.2)$$

$$I_1 W I_1^T = -W^T \quad (3.3)$$

Since the eigenvalues of a matrix are unchanged by either similarity transformations or transposing, the eigenvalues of W occur in pairs with opposite signs.

Lemma 2. Let

$$T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (3.4)$$

where A_{11} , A_{22} , A_{12} , and A_{21} are $n \times n$, $m \times m$, $n \times m$, and $m \times n$ matrices, respectively. Assume that A_{11} is non-singular. Then

$$T^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \quad (3.5)$$

exists iff $A_{22} - A_{21} A_{11}^{-1} A_{12}$ is non-singular, and is given by

$$\begin{aligned} T_{11} &= A_{11}^{-1} + A_{11}^{-1} A_{12} T_{22} A_{21} A_{11}^{-1} \\ T_{12} &= -A_{11}^{-1} A_{12} T_{22} \\ T_{21} &= -T_{22} A_{21} A_{11}^{-1} \\ T_{22} &= (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} \end{aligned} \quad (3.6)$$

where submatrices T_{11} , T_{22} , T_{12} , and T_{21} are the same sizes as A_{11} , A_{22} , A_{12} , and A_{21} , respectively.

Proof. Suppose $A_{22} - A_{21} A_{11}^{-1} A_{12}$ is non-singular. From $TT^{-1} = I$,

$$A_{11} T_{11} + A_{12} T_{21} = I_n$$

$$A_{11} T_{12} + A_{12} T_{22} = 0$$

$$A_{21} T_{11} + A_{22} T_{21} = 0$$

$$A_{21} T_{12} + A_{22} T_{22} = I_m$$

Premultiplying the second equation by $A_{21} A_{11}^{-1}$ and subtracting from the fourth,

$$T_{22} = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}$$

And hence

$$T_{12} = -A_{11}^{-1} A_{12} T_{22}$$

Similarly, from the first and third,

$$T_{21} = -T_{22} A_{21} A_{11}^{-1}$$

$$T_{11} = A_{11}^{-1} + A_{11}^{-1} A_{12} T_{22} A_{21} A_{11}^{-1}$$

Hence T^{-1} is a right inverse of T . In a similar way we can show that T^{-1} is also a left inverse of T . Conversely, suppose T is non-singular. Then

$$\begin{aligned} 0 \neq \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &= \det \left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & I_m \end{bmatrix} \begin{bmatrix} I_n & A_{11}^{-1} A_{12} \\ 0 & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{bmatrix} \right) \\ &= \det(A_{11}) \det(A_{22} - A_{21} A_{11}^{-1} A_{12}) \end{aligned} \quad (3.8)$$

Therefore, $A_{22} - A_{21} A_{11}^{-1} A_{12}$ is nonsingular.

Assuming that W has distinct eigenvalues, we now turn to constructing the similarity transformation with which W of (2.21) is reduced to the

diagonal form. By lemma 1, W has n eigenvalues with positive real parts and n eigenvalues with negative real parts. If we construct the similarity transformation S , whose columns are the right eigenvectors of W , such that the first n columns are the right eigenvectors of W corresponding to the n eigenvalues of W with positive real parts, then

$$S^{-1}WS = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \ddots & \lambda_{2n} \end{bmatrix} \quad (3.9)$$

where, of course, the rows of S^{-1} are the left eigenvectors of W .

O'donnell [5] and Potter [6] have shown that if we partition S into four $n \times n$ submatrices,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad (3.10)$$

then the solution of the matrix Riccati equation is given by

$$K = S_{21} S_{11}^{-1} \quad (3.11)$$

provided that S_{11} is non-singular.

We shall now remove the restriction that W has distinct eigenvalues and present an alternative proof to that of Martensson [7] to show that (3.11) holds for the case in which W has nondiagonal Jordan form.

Let us consider the general Jordan form J with the similarity transformation S , which will be constructed later,

$$S^{-1}WS = J$$

where, $J = \begin{bmatrix} J_{m_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & J_{m_\ell} \end{bmatrix}$ (3.12)

and in which

$$J_{m_i} = \begin{bmatrix} \lambda_i & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & \ddots \\ & & & & \ddots & \lambda_i \end{bmatrix}, \quad (m_i \times m_i) \quad (3.13)$$

If we denote S by its column vectors s_1, \dots, s_{2n} , then

$$W [s_1, s_2, \dots, s_{2n}] = [s_1, s_2, \dots, s_{2n}] J \quad (3.14)$$

For any Jordan block J_{m_i} , if $m_i > 1$ then, dropping subscripts of s for convenience,

$$\begin{aligned} Ws^{(1)} &= s^{(1)} \lambda_i \\ Ws^{(2)} &= s^{(1)} + \lambda_i s^{(2)} \\ &\dots \\ Ws^{(m_i)} &= s^{(m_i-1)} + \lambda_i s^{(m_i)} \end{aligned} \quad (3.15)$$

or

$$\begin{aligned} (W - \lambda_i I) s^{(1)} &= 0 \\ (W - \lambda_i I)^2 s^{(2)} &= 0 \\ &\dots \\ (W - \lambda_i I)^{m_i} s^{(m_i)} &= 0 \end{aligned} \quad (3.16)$$

Thus, $s^{(j)}$, $1 \leq j \leq m_i$, is a principal vector of degree j associated with λ_i . Since S is nonsingular, the m_i principal vectors $s^{(1)}, \dots, s^{(m_i)}$ are independent. If we consider all the Jordan blocks J_{m_i} , $1 \leq i \leq \ell$, we have the $2n$ principal vectors s_1, s_2, \dots, s_{2n} which constitute the similarity transformation S in (3.12). Further, we rearrange the columns of S such that the first n vectors are the principal vectors corresponding to the n eigenvalues with positive real parts of W .

Let us now split J of (3.12) into

$$J = \begin{bmatrix} J_1 & \vdots \\ \vdots & \ddots \\ \vdots & J_2 \end{bmatrix} + \begin{bmatrix} E_1 & \vdots \\ \vdots & \ddots \\ \vdots & E_2 \end{bmatrix} \quad (3.17)$$

where J_1 and J_2 are diagonal with the eigenvalues of W with positive and negative real parts respectively, and E_1 and E_2 are matrices with one's on the super-diagonals and zeros elsewhere. To associate with (3.6), let us partition S and S^{-1} as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad S^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \quad (3.18)$$

where all submatrices are $n \times n$. Note that, by lemma 2, it is necessary to assume that S_{11} and $S_{22} - S_{21} S_{11}^{-1} S_{12}$ are nonsingular.

Then

$$\begin{aligned} e^{Wt} &= e^{SJ S^{-1} t} \\ &= S e^{Jt} S^{-1} \\ &= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} e^{J_1 t} & 0 \\ 0 & e^{J_2 t} \end{bmatrix} \begin{bmatrix} e^{E_1 t} & 0 \\ 0 & e^{E_2 t} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \\ &= \begin{bmatrix} S_{11} e^{J_1 t} e^{E_1 t} T_{11} + S_{12} e^{J_2 t} e^{E_2 t} T_{21} & S_{11} e^{J_1 t} e^{E_1 t} T_{12} + S_{12} e^{J_2 t} e^{E_2 t} T_{22} \\ S_{21} e^{J_1 t} e^{E_1 t} T_{11} + S_{22} e^{J_2 t} e^{E_2 t} T_{21} & S_{21} e^{J_1 t} e^{E_1 t} T_{12} + S_{22} e^{J_2 t} e^{E_2 t} T_{22} \end{bmatrix} \quad (3.19) \end{aligned}$$

Since the elements of J_2 have negative real parts, $e^{J_2 t} \rightarrow 0$ as $t \rightarrow \infty$ and thus the terms having $e^{J_2 t}$ in (3.19) vanish as $t \rightarrow \infty$. Therefore,

$$\begin{aligned}
 \begin{bmatrix} X(t) \\ Z(t) \end{bmatrix} &= e^{Wt} \begin{bmatrix} I \\ C^T Q C \end{bmatrix} \\
 &= \begin{bmatrix} S_{11} e^{J_1 t} e^{E_1 t} T_{11} & S_{11} e^{J_1 t} e^{E_1 t} T_{12} \\ S_{21} e^{J_1 t} e^{E_1 t} T_{11} & S_{21} e^{J_1 t} e^{E_1 t} T_{12} \end{bmatrix} \begin{bmatrix} I \\ C^T Q C \end{bmatrix} \\
 &= \begin{bmatrix} S_{11} e^{J_1 t} e^{E_1 t} (T_{11} + T_{12}) C^T Q C \\ S_{21} e^{J_1 t} e^{E_1 t} (T_{11} + T_{12}) C^T Q C \end{bmatrix} \quad (3.20)
 \end{aligned}$$

Therefore the solution of the matrix Riccati equation is given by

$$\begin{aligned}
 K &= \lim_{t \rightarrow \infty} Z(t) X^{-1}(t) \\
 &= \lim_{t \rightarrow \infty} [S_{21} e^{J_1 t} e^{E_1 t} (T_{11} + T_{12}) C^T Q C] [S_{11} e^{J_1 t} e^{E_1 t} (T_{11} + T_{12}) C^T Q C]^{-1} \\
 &= S_{21} S_{11}^{-1} \quad (3.21)
 \end{aligned}$$

provided that the indicated inverse exists. However, we shall show that only the nonsingularity of S_{11} and $S_{22} - S_{21} S_{11}^{-1} S_{12}$ is needed for the stationary solution of the matrix Riccati equation. From (3.12),

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} -A & B R^{-1} B^T \\ C^T Q C & A^T \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = J \quad (3.22)$$

Equating the (2,1) elements on each side of (3.22),

$$(-T_{21} A + T_{22} C^T Q C) S_{11} + (T_{21} B R^{-1} B^T + T_{22} A^T) S_{21} = 0$$

or

$$-T_{21} A S_{11} + T_{22} A^T S_{21} + T_{21} B R^{-1} B^T S_{21} + T_{22} C^T Q C S_{11} = 0 \quad (3.23)$$

Substituting $T_{21} = -T_{22} S_{21} S_{11}^{-1}$ into (3.23),

$$T_{22} S_{21} S_{11}^{-1} + T_{22} A^T S_{21} - T_{22} S_{21} S_{11}^{-1} B R^{-1} B^T S_{21} + T_{22} C^T Q C S_{11} = 0 \quad (3.24)$$

Premultiplying by T_{22}^{-1} and postmultiplying by S_{11}^{-1} ,

$$S_{21} S_{11}^{-1} A + A^T S_{21} S_{11}^{-1} - S_{21} S_{11}^{-1} B R^{-1} B^T S_{21} S_{11}^{-1} + C^T Q C = 0 \quad (3.25)$$

which is to be proved.

4. COMPUTATIONAL METHODS AND NUMERICAL RESULTS

The major three steps for computing the solution of the matrix Riccati equation are as follows:

- 1) Compute eigenvalues and eigenvectors of W ,
- 2) Compute principal vectors of W if W is defective,
- 3) Compute $S_{21} S_{11}^{-1}$ from S .

For computing the eigensystem, we used two methods: (i) the QR algorithm [9] with an inverse iteration routine [10], and (ii) Jacobi-like method [11] for finding the eigenvalues and eigenvectors. Their performances are satisfactory and two numerical test examples are given in Test Result 1 and Test Result 3. When the matrix W is nondiagonable, Test Result 2, the first method requires the evaluation of the principal vectors as well [12]. This poses some numerical difficulties as we are essentially trying to solve the ill-conditioned system of equation

$$(W - \tilde{\lambda}I)^k x_{j+1} = x_j, \quad k > 1$$

where $\tilde{\lambda}$ is an approximation of true eigenvalues λ . Thus, the Jacobi-like method is more attractive in this case for providing an approximate eigen-system that is suitable for engineering purposes.

Appendix I contains the listings of the ALGOL programs used on the B-6500. Appendix II contains the listings of two ILLIAC IV programs, namely, the inverse iteration routine and a routine for solving a system of complex linear equations, both written in GLYPNIR. ILLIAC IV programs for the QR algorithm [13, 14] and the Jacobi-like algorithms [15, 16] are already available.

Test Result 1.

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The numerical example was tested by the QR algorithm and the inverse iteration method. The ten distinct eigenvalues of W are

$$\begin{aligned} & \pm 1, \\ & \pm 1.728760477185 \pm 1.577533767573i, \\ & \pm 1.353195784046 \pm 1.153749899284i. \end{aligned}$$

We choose the five eigenvectors corresponding to the five eigenvalues with positive real parts and computing $K = S_{21} S_{11}^{-1}$ from them gives the solution K as

1.262782609	2.494009759	-0.819173651	0.668267901	-0.443608958
2.494009759	7.435451164	-1.825741858	1.122910432	-0.668267901
-0.819173651	-1.825741858	1.638347303	1.825741858	-0.819173651
0.668267901	1.122910432	1.825741858	7.435451164	-2.494009759
-0.443608958	-0.668267901	-0.819173651	-2.494009759	1.262782609

All the eigenvalues of $G = A - BR^{-1}B^T K$ have negative real parts and hence the above solution K is acceptable.

Test Result 2.

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P=Q=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 1 \end{bmatrix}$$

The matrix

$$W = \begin{bmatrix} 3 & -2 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

is defective, that is, W has four eigenvalues ± 1 (double) and each ± 1 and -1 correspond to a quadratic elementary divisor, respectively. This example was tested both by Jacobi-like method and by a combination of the QR algorithm and the inverse iteration method.

The Jacobi-like method gives K as

$$-10.0000084 \quad 6.0000043$$

$$6.0000034 \quad -4.0000027$$

and the QR and inverse iteration method give K as

$$-9.9999917 \quad 5.9999947$$

$$5.9999953 \quad -3.9999983$$

The exact solution is

$$\begin{bmatrix} -10 & 6 \\ 6 & -4 \end{bmatrix}$$

For computing the principal vector by the QR and inverse iteration, we chose the perturbation for the true eigenvalues as much as 10^{-6} .

Test Result 3.

$$A = \begin{bmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C=R= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad P=Q= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The exact eigenvalues of W are

$$\pm 15, \pm 5, \pm 5, \pm 1$$

Since the matrix W is derogatory, +5 and -5 correspond to a two-dimensional subspace, respectively. The computed eigenvalues by Jacobi-like method are

14.99999999930	-14.999999984
4.99999999949	4.99999999981
-4.99999999982	-4.99999999948
0.99999999997	-0.99999999995

The computed solution K is given by

-1.999999999811	1.99999999942	1.999999999913	-2.000000000015
1.99999999984	-2.000000000015	-2.0000000000000	2.000000000087
2.000000000276	-2.00000000407	-2.000000000378	2.000000000480
-2.000000000364	2.000000000509	2.000000000480	-2.000000000568

in which the absolute error is less than 10^{-9} .

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APPENDIX I

ALGOL PROGRAMS

```

% TWO ALGOL PROCEDURES, RICCATI1 AND RICCATI2, FOR COMPUTING THE          00000100?
% SOLUTION OF THE MATRIX RICCATI EQUATION ARE LISTED BELOW. FOR          00000200?
% "RICCATI1", THE PROCEDURE "EIGEN" (BY EBERLEIN AND BUOTHROYD) WAS      00000300?
% USED WHILE A COMBINATION OF "HQR" (BY MARTIN, ET AL) AND          00000400?
% "INVITERATION" WAS APPLIED TO "RICCATI2". LINEAR EQUATION SOLVER      00000500?
% (BY BOWDLER, ET AL, WAS ALSO USED TO PROVIDE THE INVERSE OF A MATRIX 00000600?
% AND TO CALCULATE THE PRINCIPAL VECTORS FROM ((W-LAMBDA*I)**K)*X=0.    00000700?
%                                                               00000800?
%                                                               00000900?
%                                                               00001000?
%                                                               00001100?
%                                                               00001200?
%                                                               00001300?
%                                                               00001400?
%                                                               00001500?
%                                                               00001600?
%                                                               00001700?
%                                                               00001800?
%                                                               00001900?
%                                                               00002000?
% SO THAT THE FIRST N COLUMNS CORRESPOND TO THE EIGENVALUES            00002100?
% WITH POSITIVE REAL PARTS; DENOTING THE UPPER HALF OF N COLUMN        00002200?
% VECTORS BY SMTX1 AND THE LOWER HALF BY SMTX2, THE SOLUTION           00002300?
% K IS THEN GIVEN BY K=(SMTX2)(SMTX1INV). THE PROCEDURE                 00002400?
% "EIGEN" IS USED FOR EIGENPROBLEM OF W.                                00002500?
%                                                               00002600?
%                                                               00002700?
%                                                               00002800?
%                                                               00002900?
%                                                               00003000?
%                                                               00003100?
%                                                               00003200?
%                                                               00003300?
%                                                               00003400?
%                                                               00003500?
%                                                               00003600?
%                                                               00003700?
%                                                               00003800?
%                                                               00003900?
%                                                               00004000?
%                                                               00004100?
%                                                               00004200?
%                                                               00004300?
%                                                               00004400?
%                                                               00004500?
%                                                               00004600?
%                                                               00004700?
%                                                               00004800?
%                                                               00004900?
%                                                               00005000?
%                                                               00005100?
%                                                               00005200?
%                                                               00005300?
%                                                               00005400?
%                                                               00005500?
%                                                               00005600?
%                                                               00005700?
%
```

PROCEDURE RICCATI1(N,L,M,A,B,C,P,Q,R,TMX,W,K);
 VALUE N,L,M,TMX;
 INTEGER N,L,M,TMX;
 ARRAY A,B,C,P,Q,R,W,K[1:1];
 COMMENT SOLVES THE MATRIX RICCATI EQUATION
 KA+(AT)K-KA(RINV)(BT)K+(CT)QC=0.
 A IS N*N, B IS N*L, C IS M*N, P AND Q ARE M*M, R IS L*L
 MATRICES, RESPECTIVELY, FORM 2N*2N MATRIX W AND COMPUTE
 EIGENVALUES AND EIGENVECTORS OF W, WHERE FOUR N*N BLOCKS
 OF W ARE (1,1)=A, (1,2)=B(RINV)(BT), (2,1)=(C)QC, AND
 (2,2)=(AT), REARRANGE EIGENVECTOR(PRINCIPAL VECTOR) MATRIX
 SO THAT THE FIRST N COLUMNS CORRESPOND TO THE EIGENVALUES
 WITH POSITIVE REAL PARTS; DENOTING THE UPPER HALF OF N COLUMN
 VECTORS BY SMTX1 AND THE LOWER HALF BY SMTX2, THE SOLUTION
 K IS THEN GIVEN BY K=(SMTX2)(SMTX1INV). THE PROCEDURE
 "EIGEN" IS USED FOR EIGENPROBLEM OF W;

REGIN COMMENT FIRST DECLARE PROCEDURES;

PROCEDURE SETUP(N,L,M,A,B,C,P,Q,RINV,W);
 VALUE N,L,M; INTEGER N,L,M;
 REAL ARRAY A,B,C,P,Q,RINV,W[1:1];
 COMMENT THE PROCEDURE SETS UP THE 2N*2N MATRIX W, RINV IS INVERSE OF R;
 REGIN
 REAL ARRAY RINVBT[1:L,1:N],QC[1:M,1:N];
 REAL SUM;
 INTEGER I,J,K;
 FOR I:=1 STEP 1 UNTIL N DO
 FOR J:=1 STEP 1 UNTIL N DO
 BEGIN
 W[I,J]:=A[I,J];
 W[N+I,N+J]:=A[J,I];
 END;
 FOR I:=1 STEP 1 UNTIL L DO
 FOR J:=1 STEP 1 UNTIL N DO
 BEGIN SUM:=0.0;
 FOR K:=1 STEP 1 UNTIL L DO
 SUM:=SUM+RINV[I,K]*B[J,K];
 RINVBT[I,J]:=SUM;
 END;
 FOR I:=1 STEP 1 UNTIL N DO
 FOR J:=1 STEP 1 UNTIL N DO
 BEGIN SUM:=0.0;
 FOR K:=1 STEP 1 UNTIL L DO
 SUM:=SUM+B[I,K]*RINVBT[K,J];
 W[I,N+J]:=SUM;
 END;
 FOR I:=1 STEP 1 UNTIL M DO

```

FOR J:=1 STEP 1 UNTIL N DO          00005800?
 BEGIN SUM:=0.0;                     00005900?
   FOR K:=1 STEP 1 UNTIL M DO        00006000?
     SUM:=SUM+Q[I,K]*C[K,J];       00006100?
     QC[I,J]:=SUM;                 00006200?
   END;
 FOR I:=1 STEP 1 UNTIL N DO          00006300?
   FOR J:=1 STEP 1 UNTIL N DO        00006400?
     BEGIN SUM:=0.0;                 00006500?
       FOR K:=1 STEP 1 UNTIL M DO    00006600?
         SUM:=SUM+C[K,I]*QC[K,J];
         W[N+I,J]:=SUM;
       END;
     END;
END OF SETUP;

COMMENT INSERT THE PROCEDURES INNPROD,DECOMPOSE, AND ACCSOLVE;

PROCEDURE INVERSE(N:A,X);
VALUE N; INTEGER N;
DOUBLE ARRAY A,X[1,1];
COMMENT THE PROCEDURE COMPUTES THE INVERSE OF AN N*N REAL
      UNSYMMETRIC MATRIX A, USING THE PROCEDURES "DECOMPOSE" AND
      "ACCSOLVE", WRITING ON X;
BEGIN
  INTEGER I,J,IT,D2;
  REAL D1;
  ARRAY AA,BB,B[1:N,1:N],P[1:N];
  FOR I:=1 STEP 1 UNTIL N DO          00008500?
    FOR J:=1 STEP 1 UNTIL N DO        00008600?
      IF J=I THEN B[I,J]:=1.0 ELSE B[I,J]:=0.0;
    FOR I:=1 STEP 1 UNTIL N DO        00008800?
      FOR J:=1 STEP 1 UNTIL N DO      00008900?
        AA[I,J]:=A[I,J];
      DECOMPOSE(N,AA,D1,D2,P);      00009100?
      ACCSOLVE(N,N,A,AA,P,B,X,BB,IT); 00009200?
    END INVERSE;

COMMENT INSERT THE PROCEDURE EIGEN;

PROCEDURE MULT(N1,N2,N3,A,B,C);
VALUE N1,N2,N3; INTEGER N1,N2,N3;
DOUBLE ARRAY A,B,C[1,1];
BEGIN
  INTEGER I,J,K;
  REAL SUM;
  FOR I:=1 STEP 1 UNTIL N1 DO        00010300?
    FOR J:=1 STEP 1 UNTIL N3 DO      00010400?
      BEGIN SUM:=0.0;
        FOR K:=1 STEP 1 UNTIL N2 DO  00010500?
          SUM:=SUM+A[I,K]*B[K,J];
          C[I,J]:=SUM;
        END;
      END;
END MULT;

ARRAY RINV=SMTX1,SMTX2[1:N,1:N],WW,TEMP,VECT[1:N+N,1:N+N];
INTEGER ARRAY SEARCH[1:N];
INTEGER N2,I,J,INDEX;

```

```

N2=N+N;
INVERSE(L,R=RINV);
SETUP(N,L,M,A,B,C,Q,RINV,W);
FOR I=1 STEP 1 UNTIL N2 DO
FOR J=1 STEP 1 UNTIL N2 DO
  TEMP[I,J]:=W[I,J];
EIGEN(N2,TMX,TEMP,VECT);
COMMENT SEARCH N COLUMNS OF VECT CORRESPONDING TO N EIGENVALUES
      WITH POSITIVE REAL PARTS, SMTX1 AND SMTX2 ARE ITS UPPER
      AND LOWER HALVES, RESPECTIVELY;
INDEX:=0;
FOR I=1 STEP 1 UNTIL N2 DO
  IF TEMP[I,I] GTR 0 THEN
    BEGIN
      INDEX:=INDEX+1;
      SEARCH[INDEX]:=I;
    END;
FOR J=1 STEP 1 UNTIL N DO
FOR I=1 STEP 1 UNTIL N DO
BEGIN
  SMTX1[I,J]:=VECT[I+SEARCH[J]];
  SMTX2[I,J]:=VECT[N+I+SEARCH[J]];
END;
INVERSE(N,SMTX1,TEMP);
MULT(N,N,N,SMTX2,TEMP,K);
END RICCATI;

PROCEDURE RICCATI2(N,L,M,A,B,C,P,Q,R,W,K,DEFECT,DEROGAIE);
VALUE N,L,M;
INTEGER N,L,M;
ARRAY A,B,C,P,Q,R,W,K[1:1];
INTEGER ARRAY DEFECT,DERUGATE[1];
COMMENT SOLVES THE MATRIX RICCATI EQUATION
  KA+(AT)K=KR(RINV)(BT)K+(CT)QC=0,
  A IS N*N, B IS N*L, C IS M*N, P AND Q ARE M*M, R IS L*L
  MATRICES, RESPECTIVELY. FORM 2N*2N MATRIX W AND COMPUTE
  EIGENVALUES AND EIGENVECTORS OF W, WHERE FOUR N*N BLOCKS
  OF W ARE (1,1)=A, (1,2)=B(RINV)(BT), (2,1)=(C)QC AND
  (2,2)=(AT). REARRANGE EIGENVECTOR(PRINCIPAL VECTOR) MATRIX
  SO THAT THE FIRST N COLUMNS CORRESPOND TO THE EIGENVALUES
  WITH POSITIVE REAL PARTS, DENOTING THE UPPER HALF OF N COLUMN
  VECTORS BY SMTX1 AND THE LOWER HALF BY SMTX2. THE SOLUTION
  K IS THEN GIVEN BY K=(SMTX2)(SMTX1INV). THE PROCEDURES
  "HSHLD", "HQR", AND "INVITERATION" ARE USED FOR EIGENPROBLEM
  OF W;
BEGIN COMMENT FIRST DECLARE PROCEDURES;

PROCEDURE SETUP(N,L,M,A,B,C,Q,RINV,W);
VALUE N,L,M; INTEGER N,L,M;
REAL ARRAY A,B,C,Q,RINV,W[1:1];
COMMENT THE PROCEDURE SETS UP THE 2N*2N MATRIX W. RINV IS INVERSE OF R;
BEGIN
  REAL ARRAY RINVBT[1:L,1:N],QC[1:M,1:N];
  REAL SUM;
  INTEGER I,J,K;
  00011500?
  00011600?
  00011700?
  00011800?
  00011900?
  00012000?
  00012100?
  00012200?
  00012300?
  00012400?
  00012500?
  00012600?
  00012700?
  00012800?
  00012900?
  00013000?
  00013100?
  00013200?
  00013300?
  00013400?
  00013500?
  00013600?
  00013700?
  00013800?
  00013900?
  00014000?
  00014100?
  00014200?
  00014300?
  00014400?
  00014500?
  00014600?
  00014700?
  00014800?
  00014900?
  00015000?
  00015100?
  00015200?
  00015300?
  00015400?
  00015500?
  00015600?
  00015700?
  00015800?
  00015900?
  00016000?
  00016100?
  00016200?
  00016300?
  00016400?
  00016500?
  00016600?
  00016700?
  00016800?
  00016900?
  00017000?
  00017100?

```

```

FOR I:=1 STEP 1 UNTIL N DO          00017200?
FOR J:=1 STEP 1 UNTIL N DO          00017300?
BEGIN                               00017400?
  W[I,J]:=A[I,J];
  W[N+I,N+J]:=A[J,I];
END;
FOR I:=1 STEP 1 UNTIL L DO          00017500?
FOR J:=1 STEP 1 UNTIL N DO          00017600?
BEGIN SUM:=0.0;
  FOR K:=1 STEP 1 UNTIL L DO        00017700?
    SUM:=SUM+RINV[I,K]*B[J,K];
    RINVBT[I,J]:=SUM;
  END;
FOR I:=1 STEP 1 UNTIL N DO          00017800?
FOR J:=1 STEP 1 UNTIL N DO          00017900?
BEGIN SUM:=0.0;
  FOR K:=1 STEP 1 UNTIL L DO        00018000?
    SUM:=SUM+B[I,K]*RINVBT[K,J];
    W[I,N+J]:=SUM;
  END;
FOR I:=1 STEP 1 UNTIL N DO          00018100?
FOR J:=1 STEP 1 UNTIL N DO          00018200?
BEGIN SUM:=0.0;
  FOR K:=1 STEP 1 UNTIL L DO        00018300?
    SUM:=SUM+B[I,K]*RINVBT[K,J];
    RINVBT[I,J]:=SUM;
  END;
FOR I:=1 STEP 1 UNTIL M DO          00018400?
FOR J:=1 STEP 1 UNTIL N DO          00018500?
BEGIN SUM:=0.0;
  FOR K:=1 STEP 1 UNTIL M DO        00018600?
    SUM:=SUM+Q[I,K]*C[K,J];
    QC[I,J]:=SUM;
  END;
FOR I:=1 STEP 1 UNTIL N DO          00018700?
FOR J:=1 STEP 1 UNTIL N DO          00018800?
BEGIN SUM:=0.0;
  FOR K:=1 STEP 1 UNTIL M DO        00018900?
    SUM:=SUM+C[K,I]*QC[K,J];
    W[N+I,J]:=SUM;
  END;
END;
END OF SETUP;

COMMENT INSERT THE PROCEDURES INNPROD,DECOMPOSE, AND ACCSOLVE;

PROCEDURE INVERSE(N,A,X);
VALUE N; INTEGER N;
DOUBLE ARRAY A,X[1..1];
COMMENT THE PROCEDURE COMPUTES THE INVERSE OF AN N*N REAL
UNSYMMETRIC MATRIX A, USING THE PROCEDURES "DECOMPOSE" AND
"ACCSOLVE", WRITING ON X;
REGIN
  INTEGER I,J,IT,D2;
  REAL D1;
  ARRAY AA,BB,B[1..N,1..N],P[1..N];
  FOR I:=1 STEP 1 UNTIL N DO
    FOR J:=1 STEP 1 UNTIL N DO
      IF J=I THEN B[I,J]:=1.0 ELSE B[I,J]:=0.0;
  FOR I:=1 STEP 1 UNTIL N DO
    FOR J:=1 STEP 1 UNTIL N DO
      AA[I,J]:=A[I,J];
  DECOMPOSE(N,AA,D1,D2,P);
  ACCSOLVE(N,N,A,AA,P,B,X,BB,IT);
END INVERSE;

```

```

COMMENT INSERT THE PROCEDURES HSHLD AND HQR1          00022900?
00023000? 00023100?
00023200? 00023300?
00023400? 00023500?
00023600? 00023700?
00023800? 00023900?
00024000? 00024100?
00024200? 00024300?
00024400? 00024500?
00024600? 00024700?
00024800? 00024900?
00025000? 00025100?
00025200? 00025300?
00025400? 00025500?
00025600? 00025700?
00025800? 00025900?
00026000? 00026100?
00026200? 00026300?
00026400? 00026500?
00026600? 00026700?
00026800? 00026900?
00027000? 00027100?
00027200? 00027300?
00027400? 00027500?
00027600? 00027700?
00027800? 00027900?
00028000? 00028100?
00028200? 00028300?
00028400? 00028500?

COMMENT INSERT THE PROCEDURES HSHLD AND HQR1          00022900?
00023000? 00023100?
00023200? 00023300?
00023400? 00023500?
00023600? 00023700?
00023800? 00023900?
00024000? 00024100?
00024200? 00024300?
00024400? 00024500?
00024600? 00024700?
00024800? 00024900?
00025000? 00025100?
00025200? 00025300?
00025400? 00025500?
00025600? 00025700?
00025800? 00025900?
00026000? 00026100?
00026200? 00026300?
00026400? 00026500?
00026600? 00026700?
00026800? 00026900?
00027000? 00027100?
00027200? 00027300?
00027400? 00027500?
00027600? 00027700?
00027800? 00027900?
00028000? 00028100?
00028200? 00028300?
00028400? 00028500?

PROCEDURE MULT(N1,N2,N3,A,B,C)           VALUE N1,N2,N3; INTEGER N1,N2,N3;
DOUBLE ARRAY A,B,C[1,1];
BEGIN
  INTEGER I,J,K;
  REAL SUM;
  FOR I:=1 STEP 1 UNTIL N1 DO
  FOR J:=1 STEP 1 UNTIL N3 DO
  BEGIN SUM:=0.0;
    FOR K:=1 STEP 1 UNTIL N2 DO
      SUM:=SUM+A[I,K]*B[K,J];
    C[I,J]:=SUM;
  END;
END MULT;

PROCEDURE INVITERATION(N,W,X,KK,P);
VALUE N; INTEGER N,KK;
DOUBLE ARRAY W,X[1,1];
ARRAY P[1];
COMMENT SOLVES WX=0 BY INVERSE ITERATION, WHERE W IS AN N*N REAL
MATRIX, WRITING SOLUTION ON X. THUS, THE PROCEDURE COMPUTES
THE EIGENVECTORS CORRESPONDING TO THE GIVEN EIGENVALUES.
N IS THE ORDER OF W AND KK IS THE NUMBER OF ITERATIONS
NEEDED FOR ITERATING EACH EIGENVECTOR;
BEGIN
  INTEGER I,J,D2,R,L,MAXINDEX;
  DOUBLE D1,MAXVAL;
  DOUBLE ARRAY WW[1:N,1:N],B,BB[1:N,1:1];
  LABEL INVIT,OUT;
  FOR I:=1 STEP 1 UNTIL N DO
  FOR J:=1 STEP 1 UNTIL N DO
    WW[I,J]:=W[I,J];
  DECOMPOSE(N,WW,D1,D2,P);
  FOR I:=1 STEP 1 UNTIL N DO
    BLI,I]:=1,0;
  KKI:=1; R:=1;
INVIT:
  ACCSOLVE(N,R,W,WW,P,B,X,BB,L);
  MAXVAL:=0;
  FOR I:=1 STEP 1 UNTIL N DO
    IF ABS(X[I,1]) GTR MAXVAL THEN
      BEGIN
        MAXVAL:=ABS(X[I,1]);
        MAXINDEX:=I;
      END;
  MAXVAL:=X[MAXINDEX+1];
  FOR I:=1 STEP 1 UNTIL N DO
    X[I,1]:=X[I,1]/MAXVAL;
  FOR I:=1 STEP 1 UNTIL N DO
    IF ABS(B[I,1]-X[I,1]) GTR 0.12 THEN
      BEGIN
        IF KK GTR 10 THEN GO TO OUT;
        FOR J:=1 STEP 1 UNTIL N DO
          B[J,1]:=X[J,1];
      END;
END;

```

```

      KK:=KK+1;
      GO TO INVITI;
    END;
  OUTI;
END INVERSEIT;

INTEGER I,J,JJ,CNT,N2,INDEX,ITCOUNT,
ARRAY TEMP,ORGW,SMTX,VECTOR[1:N+N,1:N+N],RINV,SMTX1,SMTX2[1:N,1:N],
EIGR,EIGI,INTCH[1:N+N];
REAL PERTURB,MPLIERS;
INTEGER ARRAY SEARCH[1:N];
N2=N+N;
INVERSE(L,R,RINV);
SETUP(N,L,M,A,B,C,Q,RINV,W);
FOR I:=1 STEP 1 UNTIL N2 DO
FOR J:=1 STEP 1 UNTIL N2 DO
  TEMP[I,J]:=W[I,J];
HSHLD(1,N2,TEMP);
HQR(N2,TEMP,EIGR,EIGI,CNT);
FOR I:=1 STEP 1 UNTIL N2 DO
FOR J:=1 STEP 1 UNTIL N2 DO
  ORGW[I,J]:=W[I,J];
COMMENT COMPUTE EIGENVECTORS OR PRINCIPAL VECTORS;
FOR JJ:=1 STEP 1 UNTIL N2 DO
BEGIN
  FOR I:=1 STEP 1 UNTIL N2 DO
  FOR J:=1 STEP 1 UNTIL N2 DO
    W[I,J]:=ORGW[I,J];
  FOR I:=1 STEP 1 UNTIL N2 DO
    W[I,I]:=W[I,I]-EIGR[JJ]+PERTURB;
  IF JJ=DEFECT[JJ] THEN
    BEGIN
      FOR I:=1 STEP 1 UNTIL N2 DO
        W[I,I]:=W[I,I]+PERTURB*MPLIERS;
      MULT(N2,N2,N2,W,W,TEMP);
      FOR I:=1 STEP 1 UNTIL N2 DO
      FOR J:=1 STEP 1 UNTIL N2 DO
        W[I,J]:=TEMP[I,J];
    END;
  IF JJ=DEROGATE[JJ] THEN
    FOR I:=1 STEP 1 UNTIL N2 DO
      W[I,I]:=W[I,I]+PERTURB*JJ;
  INVITERATION(N2,W,VECTOR,ITCOUNT,INTCH);
  FOR I:=1 STEP 1 UNTIL N2 DO
  FOR J:=1..2 DO
    SMTX[I,JJ]:=VECTOR[I,1];
END JJ;
COMMENT SEARCH N COLUMNS OF SMTX CORRESPONDING TO N EIGENVALUES
      WITH POSITIVE REAL PARTS. SMTX1 AND SMTX2 ARE ITS UPPER
      AND LOWER HALVES, RESPECTIVELY;
INDEX:=0;
FOR I:=1 STEP 1 UNTIL N2 DO
  IF EIGR[I] GTR 0 THEN
  BEGIN
    INDEX:=INDEX+1;
    SEARCH[INDEX]:=I;
  END;
  00028600?
  00028700?
  00028800?
  00028900?
  00029000?
  00029100?
  00029200?
  00029300?
  00029400?
  00029500?
  00029600?
  00029700?
  00029800?
  00029900?
  00030000?
  00030100?
  00030200?
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  00033500?
  00033600?
  00033700?
  00033800?
  00033900?
  00034000?
  00034100?
  00034200?

```

```
FOR J:=1 STEP 1 UNTIL N DO          00034300?
FOR I:=1 STEP 1 UNTIL N DO          00034400?
BEGIN                                00034500?
  SMTX1[I,J]:=SMTX[I+SEARCH[J]];
  SMTX2[I,J]:=SMTX[N+I+SEARCH[J]];
END;
INVERSE(N,SMTX1,TEMP);
MULT(N,N,N,SMTX2,TEMP,K);
END RICCATI2;
```

APPENDIX II

ILLIAC IV PROGRAMS

% TWO ILLIAC IV GLYPNIR PROGRAMS, "CLINSYS" AND "INVITERATION" ARE
 % LISTED BELOW. "CLINSYS" IS USED TO OBTAIN THE INVERSE OF A MATRIX
 % AND THE SOLUTION OF A COMPLEX SYSTEM OF UNSYMMETRIC LINEAR EQUATIONS.
 % "INVITERATION" IS USED TO CALCULATE THE PRINCIPAL VECTORS OF DEGREE
 % K FROM $((W-\lambda\alpha\beta^*)^{**}K)^*X=0$. 00000100?
 00000200?
 00000300?
 00000400?
 00000500?
 00000600?
 00000700?
 00000800?
 00000900?
 00001000?
 00001100?
 00001200?
 00001300?
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 00004100?
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 00005000?
 00005100?
 00005200?
 00005300?
 00005400?
 00005500?
 00005600?
 00005700?

```

SUBROUTINE CLINSYS(CINT N,CINT R,PCPOINT A,PCPOINT X,PCPOINT B);
BEGIN % THE SUBROUTINE SOLVES A COMPLEX SYSTEM OF UNSYMMETRIC LINEAR
% EQUATIONS AX=R. A IS N*2N AND B IS N*2R MATRICES, RESPECTIVELY,
% THEIR GENERAL ELEMENTS ARE A[I,2J-1]+I*A[I,2J] AND B[I,2J-1]+
% I*B[I,2J], RESPECTIVELY. R IS THE NUMBER OF RIGHT-HAND SIDES. 00000100?
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00001000?
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00005000?
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00005200?
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00005500?
00005600?
00005700?

SUBROUTINE INNPROD(CREAL C1,CREAL C2,PREAL AK,PREAL BK,CREAL D1,
                    CREAL D2);
BEGIN % ACCUMULATES THE SUM OF PRODUCTS AND ADDS IT TO THE INITIAL
      % VALUE (C1,C2). 00000100?
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00000300?
00000400?
00000500?
00000600?
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00000900?
00001000?
00001100?
00001200?
00001300?
00001400?
00001500?
00001600?
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00005700?

SUBROUTINE CDECOMPOSE(CINT N,PCPOINT A,CREAL DETR,CREAL DETI,
                      CINT DETE,CNPOINT INT);
BEGIN % THE COMPLEX UNSYMMETRIC MATRIX A IS DECOMPOSED INTO LU,
      % WHERE L IS LOWER TRIANGULAR AND U IS UNIT UPPER TRIANGULAR
      % MATRICES, AND OVERWRITTEN ON A. A IS STORED IN N*2N ARRAY
      % AND ITS GENERAL ELEMENTS ARE A[I,2J-1]+I*A[I,2J]. INT[I]
      % KEEPS THE RECORD OF ANY INTERCHANGES MADE TO THE ROWS OF
      % A. THE DETERMINANT (DETR+I*DETI)*2**DETE IS ALSO COMPUTED. 00000100?
00000200?
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00000500?
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00000700?
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00001000?
00001100?
00001200?
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00005600?
00005700?
  
```

```

MODE1=TRUE AND PEN EQL P=1; A[I],@Y;
MODE1=TRUE;
MODE1=TRUE AND PEN=I;
X:= (X*X+Y*Y)/INT[I];
IF X GTR Z THEN
BEGIN Z1=X; L:=I; END;
END;
MODE1=TRUE; MODE1=TRUE AND PEN LSS N+N;
IF L NEO K THEN
BEGIN DETR:=DETR;
DETI:=DETI;
ZZ:=A[K];
A[K]:=A[L];
A[L]:=ZZ;
INT[L]:=INT[K];
END;
INT[K]:=L;
X:=GRABONE(A[K],PP=1);
Y:=GRABONE(A[K],P=1);
Z:=X*X+Y*Y;
W:=X*DETR-Y*DETI;
DETE:=X*DETI+Y*DETR;
DETR:=W;
IF ABS(DETR) GTR ABS(DETI)
THEN W:=DETR ELSE W:=DETI;
IF W = 0 THEN BEGIN DETE:=0; GO TO FAIL; END;
BEGIN LABEL L1;
IF ABS(W) GEQ 1 THEN
BEGIN W:=W*0.0625;
DETI:=DETI*0.0625;
DETE:=DETE*4;
GO TO L1;
END;
END;
BEGIN LABEL L2;
L2: IF ABS(W) LSS 0.0625 THEN
BEGIN W:=W*16;
DETR:=DETR*16;
DETI:=DETI*16;
DETE:=DETE*4;
GO TO L2;
END;
END;
LOOP JI=K+1..N DO
BEGIN
P:=J+J; PP:=P-1;
MODE1=TRUE;
MODE1=TRUE AND PEN LSS K+K-2;
INNPROD:=GRABONE(A[K],PP=1),0,A[K],AT[PP],H,HH);
INNPROD(-H,-HH,RTL(1,A[K]),AT[P],V,HH));
INNPROD(-GRABONE(A[K],P=1),0,RTL(1,A[PP]),AT[PP],H,HH));
INNPROD(H,HH,A[K],AT[P],H,HH));
W:=-H;
MODE1=TRUE;
MODE1=TRUE AND PEN=PP=1;
A[K]:=-(V*X+H*Y)/Z;
MODE1=TRUE;
00005800?
00005900?
00006000?
00006100?
00006200?
00006300?
00006400?
00006500?
00006600?
00006700?
00006800?
00006900?
00007000?
00007100?
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00011400?

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```

MODE1=TRUE AND PEN=P=1;
A[K]:= (W*X-V*Y)/Z;
END;
END;
FAIL;
END; %OF CDECOMPOSE.

SUBROUTINE CACCSOLVE(CINT N,CINT R,PCPOINT A,PCPOINT AA,CNPOINT P,
                      PCPOINT B,PCPOINT X,PCPOINT BB,CINT L);
BEGIN % SOLVES AX=B, WHERE A IS AN N*2N COMPLEX UNSYMMETRIC AND
      % B IS AN N*2R COMPLEX MATRICES, RESPECTIVELY. AA IS LU
      % DECOMPOSITION PRODUCED BY "CDECOMPOSE". THE RESIDUALS
      % BB=B-AX ARE ALSO CALCULATED AND AD=BB IS SOLVED, OVER-
      % WRITING D ON BB. L IS THE NUMBER OF ITERATIONS.
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00016900?
00017000?
00017100?

SUBROUTINE CSOLVE(CINT N,CINT R,PCPOINT A,CNPOINT INT,
                  PCPOINT B);
BEGIN
  CINT I,J,K,KK,P,PP;
  PREAL VECTOR BT[64];
  CREAL X,Y,Z,Z1,Z2,H,HH;
  PREAL VECTOR C[64];
  BOOLEAN AMODE1;
  LOOP I:=1..1..N DO
    IF INT[I] NEQ I THEN
      LOOP J:=R+R..1..1 DO
        BEGIN
          X:=GRABONE(B[I],J=1);
          MODE1=TRUE AND PEN=J=1;
          B[I]:=GRABONE(B[INT[I]],J=1);
          B[INT[I]]:=X;
        END;
    LOOP K:=R+R..2..2 DO
    BEGIN
      KK:=K-1;
      LOOP I:=1..1..N DO
        BEGIN
          MODE1=TRUE AND PEN LSS I+I=2;
          INNPROD(-GRABONE(B[I],KK=1),0,A[I],BT[KK],H,HH);
          INNPROD(-H,-HH,RTL(1..A[I]),BT[K],X,HH);
          INNPROD(-GRABONE(B[I],K=1),0,RTL(1..A[I]),BT[KK],H,HH);
          INNPROD(H,HH,A[I],BT[K],H,HH);
          Y:= -H;
          P:=I+I; PP:=P=1;
          Z1:=GRABONE(A[I],PP=1);
          Z2:=GRABONE(A[I],P=1);
          Z:=Z1+Z2+Z2;
          MODE1=TRUE AND PEN=KK;
          B[I]:= (X+Z1+Y+Z2)/Z;
          MODE1=TRUE AND PEN=K;
          B[I]:= (Y+Z1-X+Z2)/Z;
        END;
    LOOP I:=N..-1..1 DO
    BEGIN
      AMODE1= TRUE AND PEN GEO N;
      MODE1=AMODE1;
      INNPROD(-GRABONE(B[I],KK=1),0,A[I],BT[KK],H,HH);
    END;
  END;

```

```

INNPROD(=H,=HH,RTL(1,A[I]),BT[K],H,HH);          00017200?
MODEI=TRUE AND PEN=KK=1; C[I]:=H;                  00017300?
MODEI=AMODE;                                         00017400?
INNPROD(=GRABONE(B[I],K=1),0,RTL(1,A[I]),BT[KK],H,HH); 00017500?
INNPROD(H,HH,A[I],BT[K],H,HH);                     00017600?
MODEI=TRUE AND PEN=K=1; C[I]:=H;                  00017700?
MODEI=TRUE AND (PEN=KK=1 OR PEN=K=1); B[I]:=C[I];  00017800?
END;
END;
END; % OF CSOLVE.

CINT I=J=K=D2=C,CC;
CREAL E,D0,D1=XMAX,BBMAX,EPS,E1,E2,H,HH;
PREAL VECTOR XT[64];
BOOLEAN AMODE;
LABEL L3,LILL;
EPS:=1.0E-10;
AMODE;#PEN GEQ N;
MUDEI=AMODE;
LUOP II=1..1..N DO
BEGIN X[I]:=0.0;
BB[I]:=B[I];
END;
LI=0; D0:=0;
MUDEI=TRUE;
L3: CSOLVE(N,R,AA,P,BB);
LI=L+1; D2:=0; D1:=0;
MUDEI=AMODE;
LUOP II=1..1..N DO
X[I]:=X[I]+BB[I];
LUOP JI=1..1..R DO
BEGIN
XMAX:=0; BBMAX:=0;
C:=J+JI; CC:=C-1;
LOOP II=1..1..N DO
BEGIN
E1:=GRABONE(X[I],CC-1); E2:=GRABONE(X[I],C-1);
E:=E1+E2+E2;
IF E GTR XMAX THEN XMAX:=E;
E1:=GRABONE(BB[I],CC); E2:=GRABONE(BB[I],C);
E:=E1+E2+E2;
IF E GTR BBMAX THEN BBMAX:=E;
MODEI=TRUE;
INNPROD(=GRABONE(B[I],CC),0,A[I],XT[CC],H,HH);
INNPROD(=H,-HH,RTL(1,A[I]),XT[C],H,HH);
MODEI=TRUE AND PEN=CC=1; BB[I]:=H;
MODEI=TRUE;
INNPROD(=GRABONE(B[I],C),0,RTL(1,A[I]),XT[C],H,HH);
INNPROD(H,HH,A[I],XT[C],H,HH);
MODEI=TRUE AND PEN=C=1; BB[I]:=H;
END;
IF BBMAX/XMAX GTR D1 THEN D1:=BBMAX/XMAX;
IF BBMAX GTR (2*EPS)*(2*EPS)+XMAX THEN D2:=1;
END;
IF D1 GTR D0*0.25 AND L NEQ 1 THEN GO TO LILL;
D0:=D1;
IF D2=1 THEN GO TO L3;

```

```

ILLI
END; % OF CACCSOLVE.

PREAL VECTOR AA,BB[64];
CREAL VECTOR INTCH[64];
CINT I=N=R=DETE=IT;
CREAL DETR,DETI;
LOOP II=1,1,N DO
  AA[I]:=AA[I];
  CDECOMPOSE(N,AA,DETR,DETI,DETE,INTCH);
  CACCSOLVE(N,R,A,AA,INTCH,B,X,BB,IT);
END; % OF CLINSYS.

SUBROUTINE INVITERATION(CINT N,PCPOINT W,PCPOINT X,CINT KK);
REGIN % THE SUBROUTINE COMPUTES THE EIGENVECTORS AND PRINCIPAL
      % VECTORS BY SOLVING ((W=LAMBDA*I)**K)*X=0. THE SUBRUITINES
      % "CDECOMPOSE" AND "CACCSOLVE" IN "CLINSYS" ARE USED.
PE REAL SUBROUTINE SQRT AS RGA(PE REAL ARG AS RGA);
BEGIN
S+ CALL SQRT64();
END; % OF SQRT.
CINT I=J=II=DETE,R=L;
CREAL DETR,DETI,MAXVAL,XX1,XX2;
PREAL VECTOR W2=B,B2=BB=XX[64];
CREAL VECTOR P,ROOT[64];
PREAL TR1,TR2;
LABEL INVIT,NEXT,FINIS;
MODEI=OPEN LSS N+N;
LOOP II=1,1,N DO W2[I]:=W[I];
  CDECOMPOSE(N,W2,DETR,DETI,DETE,P);
  MODEI=OPEN LSS 2;
  LOOP II=1,1,N DO
    BEGIN B2[I]:=1.0; B[I]:=B2[I]; END;
    KK:=1;
    RI=1;
INVIT:
  MODEI=OPEN LSS N+N;
  CACCSOLVE(N,R,W,W2,P,B,X,BB,L);
  MODEI=OPEN LSS 2;
  LOOP II=1,1,N DO
    BEGIN
      ROOT[I]:=ROWSUM(X[I]*X[I]);
      ROOT[I]:=SQRT(ROOT[I]);
    END;
    MAXVAL:=ROOT[1];
    LOOP II=2,1,N DO
      IF MAXVAL GEQ ROOT[I] THEN MAXVAL:=MAXVAL ELSE MAXVAL:=ROOT[I];
    LOOP II=1,1,N DO
      IF ROOT[I]=MAXVAL THEN BEGIN II:=II; GO TO NEXT; END;
NEXT:
  XX1:=GRABONE(X[II],0);
  XX2:=-GRABONE(X[II],1);
  IF -XX2 NEQ 0 THEN
    BEGIN
      LOOP II=1,1,N DO
        BEGIN

```

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MODE1=PEN LSS 2;
  TR1:=GRABONE(X[I],0)*XX1:=GRABONE(X[I],1)*XX2;
  TR2:=GRABONE(X[I],0)*XX2+GRABONE(X[I],1)*XX1;
  MODE1:=PEN=0;  XX[I]:=TR1;
  MODE1:=PEN=1;  XX[I]:=TR2;
END;
MODE1=PEN LSS 2;
  XX1:=GRABONE(XX[I],0);
LOOP I:=1..N DO X[I]:=XX[I]/XX1;
END;
ELSE LOOP I:=1..N DO X[I]:=X[I]/XX1;
LOOP I:=1..N DO
BEGIN
  IF ABS(B2[I]-X[I]) GTR P=10 THEN
  BEGIN
    IF KK GTR 10 THEN GO TO FINIS;
    LOOP J:=1..N DO BEGIN B[I]:=X[I];
      B2[I]:=X[I];
    END;
    KK:=KK+1;
    GO TO INVIT;
  END;
  ELSE GO TO FINIS;
END;
FINIS;
END; UF INVITERATION.

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 00031000?
 00031100?

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13. ABSTRACT

The eigenvector solution of the time-invariant matrix Riccati equation is discussed. The coefficient matrix of the canonical equation is allowed to have multiple eigenvalues, namely, the matrix could be either derogatory or defective. The solution of matrix Riccati equation is then calculated from a part of similarity transformation which should reduce the coefficient matrix to the Jordan canonical form.

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