On the Curve $y^m - G(x) = 0$, and its Associated Abelian Integrals.

DISSERTATION

PRESENTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

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Baltimore.

1894.

PRESS OF THE FRIEDENWALD COMPANY,
BALTIMORE, MD.

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INTRODUCTION.

The well-known work of Clebsch and Gordon on the Abelian Functions assumes that the fundamental curve

$$f(xy) = 0$$

is such that only two of the values of y permute at each branch point, and that the multiple points are either cusps or multiple points with distinct tangents. There exists, however, a large class of curves included in the form

$$y^m - G(x) = 0$$

(where $G(x)$ is a rational function of x) which violate both these hypotheses. They may, it is true, be made to satisfy these hypotheses by subjecting them to a set of bi-rational transformations, but in the process they are deprived of all their simplicity.

The purpose of this paper is to make a direct investigation of the integrals associated with these curves and at the same time retain their characteristic simplicity of form.

I desire here to express my gratitude and my sincere appreciation of the great kindness and assistance of Professor Craig and of Professor Franklin, not only in the preparation of this paper, but throughout my entire residence at the Johns Hopkins University.

I.

Bi-rational Transformation of the Curve $y^m - G(x) = 0$.

Consider the algebraic equation

$$y^m - G(x) = 0,$$

and write it in the form

$$y^m - \frac{R_q(x)}{R_r(x)} = 0, \tag{1}$$

where $R_q(x)$ and $R\ (x)$ are rational entire functions of x of degree q and r

respectively. On this form Osgood * has made the following reduction. Put

$$y = \frac{y'}{t}, \quad x = \frac{x'}{t},$$

and (1) becomes

$$y'^m R_r(x't) - t^{m+r-q} R_q(x't) = 0.$$

Put now

$$t = ax' + bt' \qquad [t' = 1]$$

and we have

$$y'^m R_r(x') - (ax' + b)^{m+r-q} R_q(x') = 0. \tag{2}$$

Again, let

$$y' = \frac{y''}{R_r(x')},$$

and we have

$$y''^m - \{R_r(x')\}^{m-1} R_{m+r}(x') = 0;$$

or, grouping together such factors of the product

$$\{R_r(x')\}^{m-1} R_{m+r}(x')$$

as occur to the mth degree,

$$y''^m - \{R_i(x')\}^m R_{m(r+1)-mi}(x') = 0.$$

If finally we put

$$y'' = y''' R_i(x'),$$

and drop the accents, we have

$$y^m - R_{m(\nu+1)}(x) = 0, \tag{3}$$

where ν is an integer > 0.

This is Osgood's reduction. Examination of (2) will show, however, that when

$$m + r < q$$

the function $R_{m+r}(x')$ is no longer entire, and a different reduction must be made for this form. The successive transformations

$$y' = \frac{1}{y''}, \quad y'' = \frac{y'''}{R_q(x')},$$

give us the form

$$y'''^m - \{R_i(x')\}^m R_{m(q-1)-mi}(x') = 0;$$

or as before, if

$$y''' = y^{\text{IV}} R_i(x'),$$

$$y^m - R_{m(\mu-1)}(x) = 0, \tag{4}$$

* Dissertation, Göttingen, 1890.

where μ is an integer >1. All possible curves of the form (1) are therefore reduced by bi-rational transformation to the form

$$y^m - R_{mn}(x) = 0. \qquad n > 0. \tag{5}$$

The important things to note are: 1. That the degree of the polynomial R is a multiple of the exponent of y, and 2. that not more than $m-1$ of the linear factors of R can coincide.

II.

The Curve $y^m - R_{mn}(x) = 0$. Its Genus. Its Multiple Points and their Equivalent Number of Double Points and Cusps.

We write the curve (5) in the form

$$y^m - (x-a_1)(x-a_2) \quad . \quad . \quad . \quad (x-a_{mn}) = 0, \tag{6}$$

where the a's are real or imaginary constants not more than $m-1$ of which can be equal. These a's are all branch points of the function, and around each of them all the m values of y permute in one or more cycles, according as the number of coincident a's is or is not prime to m. Moreover, since the a's are mn in number, it follows that the point infinity is not a branch point of the function.*

Introducing homogeneous co-ordinates, we have

$$f \equiv z^{mn-m} y^m - \prod_{i=1}^{i=mn} (x - a_i z) = 0; \tag{7}$$

and from this form we see at once that no term is of degree, in x and z jointly, less than $m(n-1)$; *i. e.*, the curve has at $x=z=0$ an $m(n-1)$-ple point, whose tangents, as may easily be seen, are all coincident.

Again, the curve (7) has no other multiple points except such as arise from the coincidence of two or more a's. For the necessary and sufficient conditions that the point $x'\ y'\ z'$ shall be a double point, and therefore the necessary conditions that it be a multiple point of higher order, are

$$\frac{\partial}{\partial x} f(x'\, y'\, z') = 0,$$
$$\frac{\partial}{\partial y} f(x'\, y'\, z') = 0,$$
$$\frac{\partial}{\partial z} f(x'\, y'\, z') = 0.$$

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*Forsyth. Theory of Functions, p. 160.

These three conditions are equivalent to the two following,

$$z'^{m(n-1)-1}\,y'^{m}=0\,,$$

$$\sum_{i=1}^{i=mn}(x-a_1z)(x-a_2z)\ldots\ldots(x-a_{i-1}z)(x-a_{i+1}z)\ldots\ldots(x-a_{mn}z)=0\,,$$

which can be satisfied only by $x'=z'=0$ (the multiple point at infinity), or when, in connection with an identity of one or more other a's with a_i, we have $x'=a_iz'$.

If then the a's are all distinct, the curve will have only the one multiple point $x=z=0$, and its genus* will be

$$p=\frac{(mn-1)(mn-2)}{2}-E, \tag{8}$$

where E denotes the number of double points and cusps to which the $m(n-1)$-ple point is equivalent. The coincidence of the tangents at this multiple point makes the ordinary relation, "A k-ple point is equivalent to $\frac{1}{2}k(k-1)$ double points," invalid; and we must seek other means for the evaluation of E.

Assume all the a's to be distinct. The function has then mn branch points at each of which all the values of y permute in a single cycle, and these are the only branch points. The genus of the curve is therefore

$$p=\frac{(m-1)(mn)-2m+2}{2}=\frac{(m-1)(mn-2)}{2}, \tag{9}$$

and we have

$$E=\frac{m(mn-2)(n-1)}{2}. \tag{10}$$

When, however, some of the a's coincide, the resulting multiple points have common tangents, the formula $\frac{1}{2}k(k-1)$ again fails, and the method above employed gives us only the total number of double points and cusps to which the two multiple points are equivalent. We must accordingly make a direct investigation of these points, and determine whether the number of double points and cusps to which they are equivalent is variable with the a's.

Making $y=0$ the line at infinity, and returning to the non-homogeneous form, the equation of the curve becomes

$$z^{m(n-1)}-\prod_{i=1}^{i=mn}(x-a_iz)=0\,. \tag{11}$$

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*The mathematical faculty of the Johns Hopkins University have agreed to use the term "genus" in place of "deficiency."

The multiple point is now at the origin, and the line $z=0$ is the common tangent. For the evaluation of the multiple point we will follow the method introduced by Cayley,* and form by Newton's† method the expansion for each branch in the neighborhood of the origin,

$$z = Ax^{\alpha} + Bx^{\beta} + \ldots,$$

where $A, B, \ldots$ are constants and

$$1 \overline{\overline{<}} \alpha < \beta < \gamma < \ldots$$

To determine A and α we substitute in (11)

$$z = Ax^{\alpha},$$

which gives us the form,

$$A^{m(n-1)} x^{\alpha m(n-1)} - x^{mn} + A\Sigma(a_i)\, x^{mn-1+\alpha} - A^2 \Sigma(a_i a_j)\, x^{mn-2+2\alpha} + \ldots = 0.$$

The term x^{mn} has its exponent independent of α, and none of the following exponents can be made equal to it so long as we have $\alpha > 1$. Therefore we must have

$$\alpha m(n-1) = mn,$$

$$\alpha = \frac{n}{n-1},$$

and

$$A^{m(n-1)} = 1, \quad A = e^{\frac{2\rho\pi i}{m(n-1)}}. \qquad \rho = 1, 2, 3, \ldots m(n-1).$$

The curve has evidently $m(n-1)$ branches corresponding to the $m(n-1)$ values of A. For the farther development of any branch we assume

$$z = e^{\frac{2\rho\pi i}{m(n-1)}} x^{\frac{n}{n-1}} + Bx^{\beta},$$

and again substitute in (11). The result will be

$$x^{mn} + m(n-1)\, Be^{\frac{-2\rho\pi i}{m(n-1)}} x^{\frac{n}{n-1}[m(n-1)-1]+\beta} + \ldots + B^{m(n-1)} x^{\beta m(n-1)} - x^{mn}$$
$$+ \Sigma(a_i)\Big[e^{\frac{2\rho\pi i}{m(n-1)}} x^{mn-1+\frac{n}{n-1}} + Bx^{mn-1+\beta}\Big] \ldots = 0.$$

Now the term

$$\Sigma(a_i)\, e^{\frac{2\rho\pi i}{m(n-1)}} x^{mn-1+\frac{n}{n-1}}$$

has its exponent independent of β, and none of the following exponents can be made equal to it so long as $\beta > \frac{n}{n-1} > 1$; and the least of the preceding expo-

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* Collected Mathematical Papers, Vol. V, p. 520.

† As given by Salmon, Higher Plane Curves, p. 44.

nents is $\frac{n}{n-1}[m(n-1)-1]+\beta$. We have therefore

$$\beta=\frac{n+1}{n-1}, \quad B=\frac{-\Sigma(a_i)\,e^{\frac{4\rho\pi i}{m(n-1)}}}{m(n-1)}.$$

If, as can easily be shown, the terms of the series all have $n-1$ as the common denominator of their exponents, then each of the branches of the curve is an $(n-1)$-tic branch; *i. e.*, consists of $n-1$ partial branches given by the developments

$$z_1^{(\rho)}=e^{2\pi i\left[\frac{\rho}{m(n-1)}+\frac{1}{n-1}\right]}x^{\frac{n}{n-1}}-\frac{\Sigma(a_i)}{m(n-1)}e^{2\pi i\left[\frac{2\rho}{m(n-1)}+\frac{2}{n-1}\right]}x^{\frac{n+1}{n-1}}+\ldots,$$

$$z_2^{(\rho)}=e^{2\pi i\left[\frac{\rho}{m(n-1)}+\frac{2}{n-1}\right]}x^{\frac{n}{n-1}}-\frac{\Sigma(a_i)}{m(n-1)}e^{2\pi i\left[\frac{2\rho}{m(n-1)}+\frac{4}{n-1}\right]}x^{\frac{n+1}{n-1}}+\ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$z_{n-1}^{(\rho)}=e^{2\pi i\left[\frac{\rho}{m(n-1)}+\frac{n-1}{n-1}\right]}x^{\frac{n}{n-1}}-\frac{\Sigma(a_i)}{m(n-1)}e^{2\pi i\left[\frac{2\rho}{m(n-1)}+\frac{2(n-1)}{n-1}\right]}x^{\frac{n+1}{n-1}}+\ldots.$$

The ensemble of partial branches belonging to the $m(n-1)$ total branches will be obtained by giving to ρ in the above system the values $1, 2, 3 \ldots\ldots m(n-1)$. If now we form all possible differences

$$z_s^{(r)}-z_{s'}^{(r')},$$

and denote by M the sum of the exponents of their first terms, we may, following Cayley, say that our $m(n-1)$-ple point is equivalent to $\frac{1}{2}[M-3m(n-1)(n-2)]$ double points and $m(n-1)(n-2)$ cusps.

The evaluation of M, while possible in any particular example, is entirely too complicated to be attempted in the general case. The important thing to notice is that since the expansions for the partial branches differ only in the exponents of e, the exponents of the first terms of the differences $z_s^{(r)}-z_{s'}^{(r')}$ will be independent of the relative values of the a's; and therefore that the number of the double points and cusps to which the $m(n-1)$-ple point is equivalent is unchanged when two or more of the a's coincide. If then this number can be found when all the a's are distinct, it is found for all cases. But this has already been done. We can therefore now affirm that the $m(n-1)$-ple point is equivalent to $\frac{m(mn-2)(n-1)}{2}$ ordinary double points and cusps.

If $k\,(k < m)$ of the a's coincide, the method of page 4 gives us the combined equivalence of the two multiple points; and the permanence of the equivalence of the first enables us to find at once the equivalence of the second.

In particular, if $R_{mn}(x)$ has a factor of the form $(x-a_i)^k$, where k is prime to m, we will denote its equivalence by E_i. Then

$$p=\frac{(mn-1)(mn-2)}{2}-\frac{m(n-1)(mn-2)}{2}-E_i.$$

But the function has now $mn-k+1$ branch points, around each of which all the values of y permute in a single cycle. Its genus is therefore

$$p=\frac{(m-1)(mn-k+1)-2m+2}{2},$$

$$\therefore\ E_i=\frac{(m-1)(k-1)}{2}. \tag{12}$$

On the other hand, if k is not prime to m, let $k=l\rho$, $m=\lambda\rho$, where l is prime to λ. Then the function will have $mn-k$ branch points at which all the values of y permute in a single cycle, and one where they permute in ρ cycles. Its genus will therefore be

$$p=\frac{(m-1)(mn-k)+m-\rho-2m+2}{2},$$

and we have

$$E_i=\frac{(m-1)(k-1)+\rho-1}{2}. \tag{13}$$

Moreover, a second Newton expansion will show that the equivalence of a_i is unaffected by the relative value of the remaining a's. We are therefore able to find the equivalence of all the multiple points of

$$y^m-(x-a_1)^{k_1}(x-a_2)^{k_2}\,\ldots\,(x-a_\alpha)^{k_\alpha}=0, \tag{14}$$

where

$$\sum_{i=1}^{i=\alpha}k_i=mn.$$

III.

The Most General Rational Function of x and y.

The most general rational function of x and y, when they are connected by the relation (5), is of the form

$$\frac{A'y^{m-1}+B'y^{m-2}+\ldots\ldots+L'y+M'}{Ay^{m-1}+By^{m-2}+\ldots\ldots+Ly+M}, \qquad (15)$$

where $A', B', \ldots\ldots M', A, B, \ldots\ldots M$, are arbitrary rational entire functions of x. The first reduction to be made is to render the denominator a function of x alone.

When $m=2$ we have the hyperelliptic case. When $m=3$, Thomae* makes the reduction by multiplying both numerator and denominator by

$$(Ay^2\tau^2+By\tau+C)(Ay^2\tau+By\tau^2+C),$$

where

$$\tau=e^{\frac{2\pi i}{3}}.$$

In dealing with the general case we may either extend this method of Thomae's, or follow the method used in the general theory of Abelian Functions. For this denote $y^m-R_{mn}(x)$ by f and

$$Ay^{m-1}+By^{m-2}+\ldots\ldots+Ly+M \text{ by } \varphi.$$

Then the product of φ by a factor which renders it rational in x alone is, as is well known,

$$\begin{vmatrix}
1 & 0 & 0 & 0 & 0\ldots\ldots 0 & -R & 0 & 0 & 0\ldots\ldots fy^{m-2} \\
0 & 1 & 0 & 0 & 0\ldots\ldots 0 & 0 & -R & 0 & 0\ldots\ldots fy^{m-3} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A & B & C & D & E\ldots\ldots M & 0 & 0 & 0 & 0\ldots\ldots;\varphi y^{m-1} \\
0 & A & B & C & D\ldots\ldots L & M & 0 & 0 & 0\ldots\ldots \varphi y^{m-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0\ldots\ldots A & B & C & D & E\ldots\ldots \varphi
\end{vmatrix}$$

* Ueber eine specielle Klasse Abelscher Functionen. Halle, 1877.

Multiply the first column by y^m and add it to the $(m+1)$th column. Multiply the second by y^m and add it to the $(m+2)$nd, etc. Then, since $f=0$ we have for our rationalizing factor

$$\Delta = \begin{vmatrix} M & Ay^m & By^m \dots & Jy^m & Ky^m & y^{m-1} \\ L & M & Ay^m \dots & Iy^m & Jy^m & y^{m-2} \\ K & L & M \dots & Hy^m & Iy^m & y^{m-3} \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ C & D & E \dots & M & Ay^m & y^2 \\ B & C & D \dots & L & M & y \\ A & B & C \dots & K & L & 1 \end{vmatrix}$$

a determinant of order m whose law of formation is evident.

The same result may be obtained by assuming a factor of the form

$$\alpha y^{m-1} + \beta y^{m-2} + \dots + \lambda y + \mu,$$

performing the multiplication, and equating to zero the coefficients of

$$y,\ y^2,\ y^3 \dots y^{m-1}$$

which will give us $m-1$ homogeneous equations for the determination of the m quantities $\alpha, \beta, \gamma, \dots \mu$.

All of the methods indicated must give (except for a possible factor in x alone) the same result. For given any two factors $F(xy)$ and $\Phi(xy)$ which will reduce $\varphi(xy)$ to a function of x alone, we have at once

$$\varphi(xy)\ F(xy) = \psi_1(x),$$
$$\varphi(xy)\ \Phi(xy) = \psi_2(x),$$

from which

$$F(xy) = \frac{\psi_1(x)}{\psi_2(x)}\ \Phi(xy).$$

The most general rational function of x and y, when they are connected by the relation $f(xy)=0$, will be of the form

$$\frac{A_1 y^{m-1} + A_2 y^{m-2} + \dots + A_m}{\Delta\varphi}$$

where $A_1, A_2, \dots A_m$, and the product $\Delta\varphi$ are functions of x alone.

IV.

Integrals Connected with the Curve $f=0$. Their Reduction to Two Standard Types.

All possible integrals connected with the curve $f=0$ may now by the use of the multiplier $\Delta \frac{\partial f}{\partial y}$ be reduced to the form

$$\int \frac{A_1 y^{m-1} + A_2 y^{m-2} + \ldots + A_m}{X \frac{\partial f}{\partial y}} dx, \tag{16}$$

where A_i and X are rational entire functions of x.

From this point we have two analogies to follow. The curve $y^m - R_{mn}(x) = 0$ evidently occupies a middle ground between the hyperelliptic curve $y^2 - R_i(x) = 0$, on the one hand, and the general Abelian curve $F(xy) = 0$ on the other.

We will follow first the analogy of the hyperelliptic integrals and reduce the general integral (16) to integrals of two more simple types.

The theory of decomposition of rational fractions enables us at once to reduce (16) to a sum of integrals of the two forms

$$\int \frac{P_1(xy)\, dx}{\frac{\partial f}{\partial y}}, \text{ and } \int \frac{\varphi_1(xy)\, dx}{(x-a)^l \frac{\partial f}{\partial y}},$$

where P_1 and φ_1 are rational entire functions. These in turn by simple separation of their terms and use of the relations $f=0$, and $\frac{\partial f}{\partial y} = my^{m-1}$, are reduced to the two forms

$$\int \frac{Q(x)\, y^a\, dx}{my^{m-1}}, \tag{17}$$

$$\int \frac{\varphi(x)\, y^a\, dx}{(x-a)^l\, my^{m-1}}, \tag{18}$$

where a is a positive integer less than m.

Consider the form (17). Let the degree of $Q(x)$ be denoted by q, and let $L(x)$ be a rational entire function of degree l. If then we subtract from

$$\int \frac{Q(x) y^a\, dx}{my^{m-1}}$$

the expression

$$d(L(x)\, y^{a+1}),$$

we shall reduce the integral to an integrated part and to the new integral

$$\int \left[\frac{Q(x)\, y^a\, dx}{my^{m-1}} - y^{a+1} L'(x)\, dx - (a+1)\, L(x)\, y^a \frac{dy}{dx}\, dx \right].$$

The relation $f=0$ gives us

$$\frac{dy}{dx}=\frac{R'(x)}{my^{m-1}},$$

and by means of this and $f=0$ we can put the integral in the form

$$\int\frac{[Q(x)-mL'(x)R(x)-(\alpha+1)L(x)R'(x)]\,y^{\alpha}}{my^{m-1}}\,dx.$$

If now we take $l=q-mn+1$, the last two terms in the bracket become a polynomial in x of degree q. We can now so determine the $q-mn+2$ arbitrary constants in this polynomial that the entire expression in the brackets becomes a polynomial in x of degree $q-q+mn-2=mn-2$. If now we separate the terms of this polynomial, we shall reduce all the integrals of the form (17) to the form

$$\int\frac{x^{\beta}\,y^{\alpha}\,dx}{my^{m-1}},\qquad \begin{array}{l}\beta\overline{\gtrless}\,mn-2\\ \alpha\overline{\gtrless}\,mn-1\end{array} \tag{19}$$

and these we will call integrals of the first type.

The form (18) divides into two classes according as a is or is not a root of $R_{mn}(x)=0$. When $R_{mn}\,a\neq 0$, we subtract from the integral (18) an expression of the form

$$d\left(\frac{Cy^{\alpha+1}}{(x-a)^{l-1}}\right),$$

where C is an undetermined constant. The form (18) then reduces to an integrated part and to the integral

$$\int\left[\frac{\varphi(x)\,y^{\alpha}\,dx}{(x-a)^{l}my^{m-1}}-\frac{C(\alpha+1)\,y^{\alpha}(x-a)\dfrac{dy}{dx}\,dx-(l-1)\,Cy^{\alpha+1}\,dx}{(x-a)^{l}}\right]$$

$$=\int\frac{[\varphi(x)-C(\alpha+1)R'(x)(x-a)+(l-1)\,CmR(x)]\,y^{\alpha}}{(x-a)^{l}my^{m-1}}\,dx.$$

If now C be determined by the relation

$$\varphi(a)+(l-1)\,CmR(a)=0,$$

the above integral reduces to one of the form

$$\int\frac{\Phi(x)\,y^{\alpha}\,dx}{(x-a)^{l-1}\,my^{m-1}};$$

and this process may evidently be continued till $l=1$, *i. e.* all the integrals of the form (18) reduce, when $R_{mn}(a)\neq 0$, to the form,

$$\int\frac{F(x)\,y^{\alpha}\,dx}{(x-a)\,my^{m-1}}.\quad \alpha\overline{\gtrless}\,m-1. \tag{20}$$

These we shall call integrals of the second type.

If, however, a is a root of $R(x)=0$ of order k, we subtract from the integral (18) an expression of the form

$$d\left(\frac{Cy^{\alpha+1}}{(x-a)^{l+k-1}}\right)$$

$$=\frac{C(\alpha+1)(x-a)\,y^{\alpha}\,\dfrac{dy}{dx}\,dx-C(l+k-1)\,y^{\alpha+1}\,dx}{(x-a)^{l+k}}$$

$$=y^{\alpha}\frac{[C(\alpha+1)(x-a)\,R'(x)-C(l+k-1)\,mR(x)]}{(x-a)^{l+k}\,my^{m-1}}\,dx\,.$$

Put now

$$R(x)=(x-a)^{k}\,G_1(x)\,,$$
$$R'(x)=(x-a)^{k-1}\,G_2(x)\,,$$

and this fraction becomes

$$y^{\alpha}\frac{[C(\alpha+1)\,G_2(x)-C(l+k-1)\,mG_1(x)]}{(x-a)^{l}\,my^{m-1}}\,dx\,.$$

We are therefore able to reduce the form (18) to an integrated part and to the integral

$$\int y^{\alpha}\frac{[\varphi(x)-C\{(\alpha+1)\,G_2(x)+(l+k-1)\,mG_1(x)\}]}{(x-a)^{l}\,my^{m-1}}\,dx\,.$$

Since $G_1(a)\neq 0$, and $G_2(a)\neq 0$, we can so determine C by the relation

$$\varphi(a)-C\{(\alpha+1)\,G_2(a)+(l+k-1)\,m\,G_1(a)\}=0$$

that this integral shall reduce to the form

$$\int\frac{F(x)\,y^{\alpha}\,dx}{(x-a)^{l-1}\,my^{m-1}};$$

and the process may evidently be continued till we are reduced to the form

$$\int\frac{F(x)\,y^{\alpha}\,dx}{my^{m-1}}\,.$$

In this we may reduce the degree of F by the method already given and find again only integrals of the first type.

All the integrals connected with the curve $f=0$ are now reduced to the two types

$$\left.\begin{array}{lll}\text{I.} & \displaystyle\int\frac{x^{\beta}\,y^{\alpha}\,dx}{my^{m-1}}\,, & \begin{array}{l}\beta\leqq mn-2\\ \alpha\leqq m-1\end{array}\\[2ex] \text{II.} & \displaystyle\int\frac{F(x)\,y^{\alpha}\,dx}{(x-a)\,my^{m-1}}\,. & \begin{array}{l}R(a)\neq 0\\ \alpha\leqq m-1\end{array}\end{array}\right\}\qquad(21)$$

V.

Integrals of the First Kind. Their Number.

Consider the type I and put it in the form

$$\int \frac{x^\beta (R(x))^{\frac{a}{m}} dx}{m (R(x))^{\frac{m-1}{m}}}.$$

When x becomes very great this becomes comparable to

$$\int x^{\beta + an - n(m-1)} dx,$$

which remains finite for x very great so long as $\beta + an - n(m-1) < -1$. This inequality for β positive is possible for all values of $a < m - 1$. In order therefore to find the number of integrals of the first type which remain finite for x very great, we have the formula

$$\sum_{a=0}^{a=m-2} [mn - an - n - 1] = \frac{(m-1)(mn-2)}{2}.$$

The hyperelliptic analogy would lead us to call these $\frac{(m-1)(mn-2)}{2}$ integrals, "integrals of the first kind," but the analogy fails at this point. In the hyperelliptic case $m = 2$, $R_i(x)$ is therefore reducible by bi-rational transformation to the form

$$y^2 - R_{2n}(x) = 0,$$

where $R_{2n}(x)$ has no multiple factors. The curve has therefore no multiple points except the $2(n-1)$-ple point at $x = z = 0$. At this the integrals remain finite, and therefore the hyperelliptic integral which remains finite for x very great remains finite throughout the entire plane.

In our case, if $x - a$ is a multiple factor of $R_{mn}(x)$, the expression $\frac{x^\beta y^a}{my^{m-1}}$ is, in general, infinite of an order $\overline{\gtrless} 1$ at the point $x = a$; and the corresponding integral is therefore not of the first kind. If, however, $R_{mn}(x)$ has no repeated factors, we have $\frac{(m-1)(mn-2)}{2}$ integrals of the first kind, and this is equal to the genus of the corresponding curve $f = 0$.

If a is a root of $R_{mn}(x) = 0$ of order k, we take the general integral of the first type

$$\int \frac{Q(x) y^a dx}{my^{m-1}}; \qquad (21)$$

and subtract from it

$$d\left(\frac{Cy^{\alpha+1}}{(x-a)^{k-1}}\right)$$

$$=\frac{C(\alpha+1)\,y^{\alpha}\,(x-a)\dfrac{dy}{dx}dx-C(k-1)\,y^{\alpha+1}dx}{(x-a)^{k}}$$

$$=\frac{y^{\alpha}\,[C(\alpha+1)\,(x-a)\,R'-Cm\,(k-1)\,R]\,dx}{(x-a)^{k}my^{m-1}},$$

which, if we define G_1 and G_2 as before, is equal to

$$\frac{y^{\alpha}\,[C(\alpha+1)\,G_2(x)-Cm\,(k-1)\,G_1(x)\,]\,dx}{my^{m-1}}.$$

If now C be defined by the relation

$$Q\,(a)-C[(\alpha+1)\,G_2(a)+m\,(k-1)\,G_1(a)]=0,$$

our integral reduces to an integrated part and to an integral of the form

$$\int\frac{(x-a)\,Q_1(x)\,y^{\alpha}dx}{my^{m-1}},$$

where $Q_1(x)$ is of the degree $mn-3$. This reduction may evidently be continued till we have the form

$$\int\frac{(x-a)^{k-1}Q_{k-1}(x)\,y^{\alpha}dx}{my^{m-1}}.$$

Denote now by the symbol $[z]$, where z is a real number, integer or fractional, the greatest integer contained in z. We note, first, that

$$k-1\,\overline{\gtrless}\left[\frac{k}{m}(m-1-\alpha)\right],$$

and therefore, second, that all our integrals of the first type reduce to a set of integrals of the form

$$\int\frac{(x-a)^{\left[\frac{k}{m}(m-1-\alpha)\right]}Q\,(x)\,dx}{my^{m-1-\alpha}},$$

which are finite when $x=a$. Separate this into its terms, put $m-1-\alpha=\delta$, and we have the integrals

$$\int\frac{(x-a)^{\left[\frac{k\delta}{m}\right]}x^{\beta}dx}{my^{\delta}}.\qquad \beta\,\overline{\gtrless}\,mn-2-\left[\frac{k\delta}{m}\right]$$

$$0<\delta\,\overline{\gtrless}\,m-1$$

In order that these may remain finite for x very great we must have

$$n\delta - \left[\frac{k\delta}{m}\right] - \beta > 1,$$

or

$$\beta < n\delta - 1 - \left[\frac{k\delta}{m}\right].$$

The number of integers satisfying this condition is

$$\sum_{\delta=1}^{\delta=m-1}(n\delta-1)-\left[\frac{k\delta}{m}\right]=\frac{(mn-2)(m-1)}{2}-\sum_{\delta=1}^{\delta=m-1}\left[\frac{k\delta}{m}\right].$$

When k is prime to m a known theorem gives us

$$\sum_{\delta=1}^{\delta=m-1}\left[\frac{k\delta}{m}\right]=\frac{(k-1)(m-1)}{2}, \tag{22}$$

a quantity which we have already found as the equivalence of a k-ple point on $f=0$, when k is prime to m.

If k is not prime to m, put $k=l\rho$ and $m=\lambda\rho$ and an immediate extension of the above-mentioned theorem gives us

$$\sum_{\delta=1}^{\delta=m-1}\left[\frac{k\delta}{m}\right]=\frac{(k-1)(m-1)+\rho-1}{2}, \tag{23}$$

which is the equivalence of the k-ple point when k is not prime to m.

If similar reductions are made for all the multiple factors of $R(x)$, the number of integrals which remain finite throughout the plane, *i. e.* the number of integrals of the first kind, will evidently be

$$\frac{(mn-2)(m-1)}{2}-\sum E_i=p, \tag{24}$$

where $\sum E_i$ is the sum of the double points and cusps equivalent to those multiple points of $f=0$ which result from repeated factors of $R(x)$. The number of integrals of the first kind is therefore in all cases equal to the genus of the curve.

We have now formed p integrals of the first kind, and they are evidently linearly independent. But if we ask for the most general form of an integral of the first kind, and whether there may or may not be more than p of them, we must turn our attention to the more general theory of Abelian Functions and follow the analogy which it presents. We know that the most general integral connected with the curve $f=0$ is of the form

$$\int \frac{A_1 y^{m-1} + A_2 y^{m-2} + \ldots + A_m}{(x-a_1)^{l_1}(x-a_2)^{l_2}\ldots(x-a_\nu)^{l_\nu} m y^{m-1}}\,dx. \qquad (25)$$

What are the conditions under which this will always remain finite?

Consider the point $x=a_i$. The integral is evidently infinite at this point unless the numerator vanish also. Two cases arise corresponding to the two conditions $R(a_i) \neq 0$, and $R(a_i)=0$.

When $R(a_i)\neq 0$, y has for $x=a_i$ m values all different and all different from zero. In order then that the integral remain finite when $x=a_i$ it is necessary and sufficient that the numerator, a polynomial in y of degree $m-1$, vanish for m different values of y. Its coefficients must therefore all vanish, *i. e.* the A's must all have a factor $x-a_i$; and the integral reduces to the form

$$\int \frac{A_1 y^{m-1} + \ldots + A_m}{(x-a_1)^{l_1}\ldots(x-a_i)^{l_i-1}\ldots(x-a_\nu)^{l_\nu} m y^{m-1}}\,dx.$$

A repetition of the argument will evidently remove from the denominator all the factors $(x-a_i)$, where $R(a_i)\neq 0$.

When a_i is a root of $R(x)=0$ of order k, we write

$$R(x)=(x-a_i)^k G(x),$$

and put for y its value

$$(R(x))^{\frac{1}{m}}=(x-a_i)^{\frac{k}{m}}(G(x))^{\frac{1}{m}}.$$

The integral now becomes

$$\int \frac{A_1(x-a_i)^{\frac{k(m-1)}{m}}(G(x))^{\frac{m-1}{m}} + A_2(x-a_i)^{\frac{k(m-2)}{m}}(G(x))^{\frac{m-2}{m}} + \ldots}{(x-a_1)^{l_1}\ldots(x-a_i)^{l_i+k\left(\frac{m-1}{m}\right)}\ldots(x-a_\nu)^{l_\nu}(G(x))^{\frac{m-1}{m}}}\,dx. \qquad (26)$$

In order that this remain finite for $x=a_i$ it must reduce to the form

$$\int \frac{\varphi(x)\,dx}{(x-a_i)^k},$$

where $\varphi(x)$ is finite for $x=a_i$, and k is less than unity. The numerator must therefore contain $(x-a_i)^{l_i+\left[\frac{k(m-1)}{m}\right]}$ as a factor. For this it is necessary and sufficient that we have

$$\left.\begin{aligned} A_1 &= (x-a_i)^{l_i} B_1\,, \\ A_2 &= (x-a_i)^{l_i+\left[\frac{k}{m}\right]} B_2\,, \\ &\vdots \\ A_m &= (x-a_i)^{l_i+\left[\frac{k(m-1)}{m}\right]} B_m\,. \end{aligned}\right\} \tag{27}$$

The repetition of this process will remove from the denominator of (25) all the factors $(x-a_i)$, where $R(a_i)=0$; and (25) accordingly takes the form

$$\int \frac{A_1 y^{m-1} + A_2 y^{m-2} + \ldots + A_m}{m y^{m-1}}\,dx\,. \tag{28}$$

This must still be examined for x very great and for those values of x which satisfy $my^{m-1}=0$. For x very great we see at once that, if (28) is to remain finite, we must have

$$\left.\begin{aligned} &A_1 = 0\,, \\ &A_2 \text{ of degree } n-2 \\ &\vdots \\ &A_m \text{ of degree } (m-1)n-2\,. \end{aligned}\right\} \tag{29}$$

The points which satisfy $my^{m-1}=0$ are the roots of $R(x)=0$. If a_i be a simple root of $R(x)=0$, the integral reduces to the form

$$\int \frac{\varphi(x)\,dx}{(x-a_i)^{\frac{m-r}{m}} G(x)}\,, \qquad \begin{matrix} R(x) = (x-a_i)\,G(x) \\ 0 < r < m \end{matrix}$$

and is therefore finite. If all the roots of $R(x)$ are simple, the most general integral of the first kind will be of the form (28), the degrees of the A's will be given by (29), and the number of arbitrary constants included in the integral is therefore

$$\sum_{i=2}^{i=m}((i-1)n-1) = \frac{(mn-2)(n-1)}{2}.$$

If a_i is a k-fold root of $R(x)=0$, we have only to put $l_i=0$ in the discussion above in order to show that the number of constants is reduced by $\sum\limits_{\delta=1}^{\delta=m-1}\left[\frac{k\delta}{m}\right]$, which we know to be equal to the number of double points and cusps equivalent to the multiple point a_i on the curve $f=0$. The number of arbitrary constants in the most general integral of the first kind is therefore

$$\frac{(mn-2)(m-1)}{2} - \Sigma E_i = p\,.$$

In the more general theory, based on the curve $\varphi(xy) = 0$ of degree m in x and y, the general integral of the first kind is found to be of the form

$$\int \frac{Q(xy)\,dx}{\varphi'_y},$$

where Q is of degree $m-3$ in both x and y, and has the k-ple points of $\varphi = 0$ as $(k-1)$-ple points. Q has therefore $\frac{(m-1)(m-2)}{2}$ coefficients which are subject to $\Sigma a_i \frac{i(i-1)}{2}$ conditions ($a_i =$ the number of i-ple points of $\varphi = 0$). Consequently, if there is no reduction in the number of these conditions, there will be p and only p integrals of the first kind linearly independent. But in order to show that there are only p we must show that no such reduction takes place. In the case we have been considering we find not only the number of the conditions but the conditions themselves. We thus at once see that there is no reduction in their number, and affirm that there are p and only p linearly independent integrals of the first kind.

VI.

Periods of Integrals of the First Kind. Their Form, and Reductions in their Number.

If we ask after the number and character of the periods of these integrals of the first kind, we can proceed in two ways. We can, in the first place, form the Rieman surface connected with the curve $f = 0$. It will be m sheeted and of genus p. We know that by bending and stretching it can be deformed into a sphere* with p handles,† and therefore that the integral of a uniform function has on it $2p$ periods.

In general these $2p$ periods are distinct, *i. e.* there exists between them no linear homogeneous relation with integer coefficients. The proof of this last statement need not be given here; as it is the same as in the case of the general Abelian Functions; except that, in the consideration of the inequality

$$\int_K X dY > 0,‡$$

*Jordan, Liouville's Journal, Series 2, t. XI (1866).

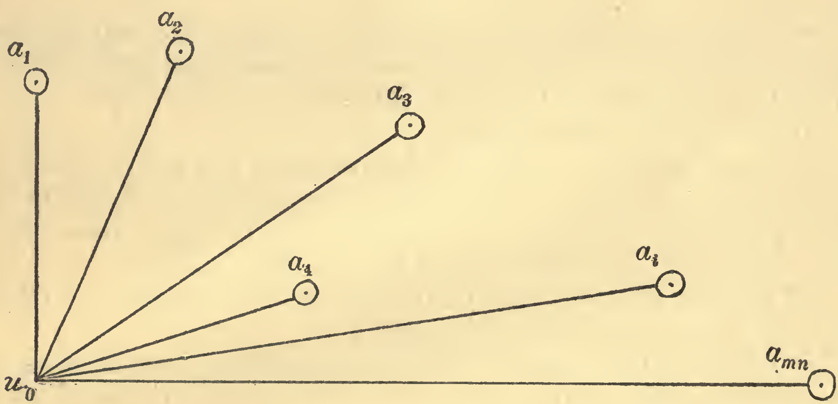
†Klein. On Rieman's Theory of Functions. Section 8 of Miss Hardcastle's translation.

‡Picard, Traité d'Analyse, Tome 2, pp. 405–409. $X + iY$ is an integral of the first kind. K is the contour made up of the p cuts through the holes, the p cuts around the holes, and the $p-1$ cuts joining these into a continuous contour.

we note that, in the region of the branch points, $\frac{dY}{ds}$ is no longer of the order $\frac{1}{\sqrt{r}}$, but of the order $\frac{1}{\sqrt[m]{r}}$ in the case of a simple branch point; and of the order $\frac{1}{\sqrt[m]{r^i}} (i < m)$ in the case of a multiple branch point. (r is the radius of the circle of integration about the branch point.)

We might proceed to form a system of normal integrals and to discuss the relations existing among their periods; but the discussion is so strictly analogous to that usually given for the general Abelian Functions that we prefer to turn to the system of integrals already formed, and consider the form, and certain reductions in the number, of their periods.

We assume, first, that all the factors of $R(x) = 0$ are distinct. Take upon the plane an arbitrary point u_o, and draw from it loops to all the mn branch points of f (*i. e.* the roots of $R(x) = 0$). Denote the loop around a_i by a_i^1 when it is described in the positive direction, and by a_i^{-1} when it is described in the negative direction.



Take now any one of our integrals of the first kind

$$U_\delta = \int \frac{\varphi(x)\, dx}{my^\delta}.$$

The effect on U_δ of a small circle around a_i is to multiply it by $e^{\frac{-2\pi i\delta}{m}} \equiv \lambda^{-\delta}$. The value of U along the straight line from u_o to the small circle about a_i will be denoted by A_i.

The inverse function y will have as periods the values of U_δ along any contours which return y to its original value. Among such contours we choose the following,

$$a_1^{m-1} a_2,\ a_1^{m-1} a_3,\ a_1^{m-1} a_4,\ \ldots\ a_1^{m-1} a_{mn};$$

and denote the corresponding periods by

$$\omega_2, \omega_3, \ldots . \omega_{mn}.$$

Consider the first of these. We have evidently

$$\omega_2 = A_1(1-\lambda^{(m-1)\delta}) + A_1(\lambda^{(m-1)\delta} - \lambda^{(m-2)\delta}) + \ldots . + A_1(\lambda^2 - \lambda) + A_2(\lambda - 1).$$

If now we multiply this by λ^δ, we have a new period $\lambda^\delta \omega_2$ to which corresponds the contour $a_1^{m-2} a_2 a_1$. In the same way we have $\lambda^{2\delta}\omega_2$ corresponding to $a_1^{m-3} a_2 a_1^2$, etc. Treating the other ω's in the same way, we have the following table of periods associated with U_δ,

$$\left.\begin{array}{lllll} \omega_2, & \lambda^\delta \omega_2, & \lambda^{2\delta}\omega_2, & \ldots . & \lambda^{(m-1)\delta}\omega_2, \\ \omega_3, & \lambda^\delta \omega_3, & \lambda^{2\delta}\omega_3, & \ldots . & \lambda^{(m-1)\delta}\omega_3, \\ \omega_4, & \lambda^\delta \omega_4, & \lambda^{2\delta}\omega_4, & \ldots . & \lambda^{(m-1)\delta}\omega_4, \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \omega_{mn}, & \lambda^\delta \omega_{mn}, & \lambda^{2\delta}\omega_{mn}, & \ldots & \lambda^{(m-1)\delta}\omega_{mn}. \end{array}\right\} \qquad (30)$$

We say farther that this table of periods is complete, *i. e.* that all possible periods associated with U_δ can be expressed as linear homogeneous functions, with integer coefficients, of the periods (30).

In order to show this we note:

1° The most general period of y corresponds to a contour made up of $km+l$ loops described in the same direction, and l loops described in the opposite direction, where k and l may have any positive integer values including zero. (The same loop described r times is regarded as r loops.)

2° Between any two loops of any period we may introduce the nugatory contour $a_i^k a_i^{-k}$, corresponding to the period 0.

3° Given a general period ω_x and its corresponding contour, there exists a period $\lambda^{k\delta}\omega_x$, whose contour consists of the same loops arranged in the same cyclic order as in the contour belonging to ω_x, but beginning at an arbitrary point in the cycle. (This is merely a generalization of what has already been done in the case of $\omega_2, \lambda^\delta\omega_2, \ldots . \lambda^{(m-1)\delta}\omega_2$.)

4° The new periods, $-\omega_i + \omega_k$, $i \neq k$, corresponding to the new contours $a_i^{-1}a_k$; together with the period ω_2, which corresponds to both the contours $a_1^{m-1}a_2$ and $a_1^{-1}a_2$, enable us to replace the last loop of any contour by any other loop.

1°, 2°, 3°, and 4° being granted, let it be required to form from the periods (30) an arbitrary period corresponding to a contour, denoted by D, consisting of $km+l$ positive and l negative loops. (The argument will be the same for $km + l$ negative and l positive loops.) To do this repeat the contour corresponding to ω_2 k times; and from this, by the introduction of the proper nugatory contours $\alpha_i^k \alpha_i^{-k}$, we get a new contour Δ which has the same number of positive and negative loops as D and in the same order. By successive cyclic permutation of the loops of Δ; and the addition, after each permutation, of the proper one of the periods deduced in 4°; Δ becomes identical with D; and the corresponding period is given as a linear homogeneous function, with integer coefficients, of the periods (30). The system (30) is therefore complete.

There are, however, certain reductions among the periods (30). We know that the value of U_8 along a contour composed of all the loops is zero. But this contour, by a process entirely analogous to that used in the case of D and Δ, may be reduced to the contour corresponding to ω_2 repeated $n-1$ times and followed by ω_i. Moreover, the periods used in this reduction are none of them derived from ω_i. We may therefore express ω_i in terms of the other periods, and strike out from the table 30 the row of periods

$$\omega_i, \; \lambda^{\delta}\omega_i, \; \lambda^{2\delta}\omega_i, \; \ldots . \; \lambda^{(m-1)\delta}\omega_i .$$

The remaining $mn-2$ periods in the first column of (30) are in general distinct, since each of them contains an A that does not appear in any of the others.

If δ is prime to m, the table (30), by virtue of the relation $\lambda^m = 1$, and by a permutation of the columns, takes the form

$$\left.\begin{array}{lllll}
\omega_2, & \lambda\omega_2, & \lambda^2\omega_2, & \ldots . & \lambda^{m-1}\omega_2, \\
\omega_3, & \lambda\omega_3, & \lambda^2\omega_3, & \ldots . & \lambda^{m-1}\omega_3, \\
\cdot & & & & \\
\cdot & & & & \\
\cdot & & & & \\
\omega_{mn-1}, & \lambda\omega_{mn-1}, & \lambda^2\omega_{mn-1}, & \ldots . & \lambda^{m-1}\omega_{mn-1} .
\end{array}\right\} \qquad (31)$$

(We have chosen the last row as the one to be dropped.) If m is odd, we have between the periods of any row the one relation

$$\omega_i \sum_{k=0}^{k=m-1} \lambda^k = 0 .$$

We may therefore strike out any column (say the last). The remaining periods are in general distinct; and the integral has, under the hypotheses made above as to δ and m, the maximum number of independent periods, *i. e.*

$$(mn-2)(m-1) = 2p .$$

If m is even (δ still prime to m) we have the relation

$$\lambda^{\frac{m}{2}} = -1. \quad \therefore \lambda^{\frac{m}{2}+i} = -\lambda^i.$$

When $\frac{m}{2}$ is odd, we can express all of the even powers of λ in terms of the odd powers; and the table of periods (31) may be replaced by the table

$$\left.\begin{array}{llll} \lambda\omega_2, & \lambda^3\omega_2, & \ldots . \lambda^{m-1}\omega_2, \\ \lambda\omega_3, & \lambda^3\omega_3, & \ldots . \lambda^{m-1}\omega_3, \\ \cdot & & \\ \quad\cdot & & \\ \qquad\cdot & & \\ \lambda\omega_{mn-1}, & \lambda^3\omega_{mn-1}, & \ldots . \lambda^{m-1}\omega_{mn-1}. \end{array}\right\} \qquad (32)$$

But λ is now a primitive $\frac{m}{2}$th root of unity, and therefore

$$\lambda + \lambda^3 + \lambda^5 + \ldots . + \lambda^{m-1} = \sum_{k=0}^{k=\frac{m}{2}-1} \lambda^{2k+1} = 0.$$

We may therefore drop the last column of (32), and the integral has now only $(mn-2)\left(\frac{m}{2}-1\right)$ periods, which are in general distinct. (We note that this is twice the genus of the curve $y^{\frac{m}{2}} - R_{\left(\frac{m}{2}\right)2n}(x) = 0$, all the factors of R being distinct.)

If, on the other hand, $\frac{m}{2}$ is even, we can drop half the columns of (31); but we know of no relation connecting those which remain. The integral has in this case $(mn-2)\frac{m}{2}$ periods which are in general distinct.

If δ is not prime to m, put $\delta = s\mu$ and $m = r\mu$, where s is prime to r. We have then the relation $\lambda^{r\delta} = 1$, and from this the relation

$$\sum_{k=0}^{k=m-1} \lambda^{k\delta} = \mu \sum_{k=0}^{k=r-1} \lambda^{k\delta}.$$

The table of periods (30) accordingly takes the form

$$\left.\begin{array}{lllll} \omega_2, & \lambda^{\delta}\omega_2, & \lambda^{2\delta}\omega_2, & \ldots . \lambda^{(r-1)\delta}\omega_2, \\ \omega_3, & \lambda^{\delta}\omega_3, & \lambda^{2\delta}\omega_3, & \ldots . \lambda^{(r-1)\delta}\omega_3, \\ \cdot & & & \\ \quad\cdot & & & \\ \qquad\cdot & & & \\ \omega_{mn-1}, & \lambda^{\delta}\omega_{mn-1}, & \lambda^{2\delta}\omega_{mn-1}, & \ldots . \lambda^{(r-1)\delta}\omega_{mn-1}. \end{array}\right\} \qquad (33)$$

But $\lambda^\delta = e^{\frac{2\pi i\delta}{m}} = e^{\frac{2\pi i s}{r}} \equiv \beta^s$, where β is a primitive rth root of unity. Since r is prime to s and $\beta^r = 1$, (33) may, by use of this value of λ^δ and proper permutation of the columns, be put in the form

$$\left.\begin{array}{lllll} \omega_2, & \beta\omega_2, & \beta^2\omega_2, & \ldots & \beta^{r-1}\omega_2, \\ \omega_3, & \beta\omega_3, & \beta^2\omega_3, & \ldots & \beta^{r-1}\omega_3, \\ \vdots & & & & \\ \omega_{mn-1}, & \beta\omega_{mn-1}, & \beta^2\omega_{mn-1}, & \ldots & \beta^{r-1}\omega_{mn-1}. \end{array}\right\} \tag{34}$$

When r is odd, $\sum_{k=0}^{k=r-1}\beta^k = 0$, we strike out the last column, and have in general $(mn-2)(r-1)$ independent periods. (We note that this is twice the genus of the curve $y^2 - R_{r\mu n}(x) = 0$, all the factors of $R(x)$ being distinct.)

If r is even, a discussion exactly analogous to that made in the case when m was even and δ prime to m shows that we have in this case $(mn-2)\left(\frac{r}{2}-1\right)$ or $(mn-2)\frac{r}{2}$ independent periods, according as $\frac{r}{2}$ is odd or even.

We may tabulate all these results as follows:

$$\begin{array}{c} \delta \text{ prime to } m \\ \text{and} \end{array} \left\{\begin{array}{llll} m \text{ odd, the no. of periods is} & & & (mn-2)(m-1). \\ \frac{m}{2} \text{ odd,} & \text{"} & \text{"} & (mn-2)\left(\frac{m}{2}-1\right). \\ \frac{m}{2} \text{ even,} & \text{"} & \text{"} & (mn-2)\left(\frac{m}{2}\right). \end{array}\right.$$

$$\begin{array}{c} \delta = s\mu.\ m = r\mu. \\ s \text{ prime to } r \\ \text{and} \end{array} \left\{\begin{array}{llll} r \text{ odd,} & \text{"} & \text{"} & (mn-2)(r-1). \\ \frac{r}{2} \text{ odd,} & \text{"} & \text{"} & (mn-2)\left(\frac{r}{2}-1\right). \\ \frac{r}{2} \text{ even,} & \text{"} & \text{"} & (mn-2)\left(\frac{r}{2}\right). \end{array}\right.$$

The meaning of a portion of these reductions is very evident. When $\delta = s\mu$ and $m = r\mu$, the integral

$$\int\frac{x^\beta dx}{my^\delta} = \int\frac{x^\beta dx}{m(R_{mn}(x))^{\frac{\delta}{m}}} = \int\frac{x^\beta dx}{m(R_{mn}(x))^{\frac{s}{r}}},$$

which last is an integral connected with the curve $y^r - R_{mn}(x) = 0$. I am able at present to offer no satisfactory explanations of the other reductions. We have said that the periods, after the above reductions have been made, are in general distinct. It is evident that the only farther reductions which can arise, so long as the factors of $R(x)$ are distinct, must come from relations among the ω's themselves. It may be possible to so choose the roots of $R(x)=0$ that some at least of the integrals connected with the curve shall have less than $mn-2$ distinct ω's. For example, the integral of the first kind,

$$\int\frac{x^{n-1}dx}{y^2},$$

connected with the curve

$$y^m-(x^n-a_1^n)(x^n-a_2^n)\ldots.(x^n-a_m^n)=0$$

reduces to the integral

$$\int\frac{dx}{y^2},$$

connected with the curve

$$y^m-(x-a_1^n)(x-a_2^n)\ldots.(x-a_m^n)=0,$$

when we take x^n as our new variable; and the new integral has only $m-2$ ω's. A similar case is the reduction of the integrals connected with the sextic

$$y^2-(x^2-a_1^2)(x^2-a_2^2)(x^2-a_3^2)=0$$

to elliptic integrals.*

It is evident, however, that no farther reductions can take place among the periods derived from any one ω, so long as the factors of $R(x)$ are distinct. Therefore, while in the case of the hyperelliptic integrals we can only say that there are at least two periods, we are able in the present case to say that the integral

$$\int\frac{x^\beta dx}{my^\delta}\qquad \begin{aligned}\delta&=s\mu\\ m&=r\mu\end{aligned}$$

has at least $(mn-2)\left(\frac{r}{2}-1\right)$ distinct periods.

We have limited ourselves so far to the case where all the factors of $R(x)$ are distinct. The case where some of them are the same presents no insuperable difficulties, but introduces a great deal of complexity. We shall limit ourselves to a simple case.

Suppose k $(k<m)$ of the factors of $R(x)$ to be of the form $(x-a_j)$, the others remaining distinct. We shall have in this case $mn-2-k$ ω's corre-

* Picard, Traité d'Analyse, Tome I, p. 217.

sponding to contours of the form $a_1^{m-1}a_i$, and one, ω_j, corresponding to a contour of the form $a_j^{m-1}a_1^k$. If k is prime to m, the point a_j will accordingly give rise to $m-1$ periods and we have in all $(mn-k-1)(m-1)=2p$ periods.

If k is not prime to m, put $k=l\rho$ and $m=\lambda\rho$, and by the argument used when δ was not prime to m we see that the point a_j gives rise to $\lambda-1$ periods, and we have in all $(mn-k-2)(m-1)+\lambda-1$.

The number of periods is in this case therefore $(\rho-1)(\lambda-1)$ less than the maximum $2p$. The number of periods in both these cases will of course be subject to reduction when m is even or when δ is not prime to m.

We have been speaking so far of curves reduced to the standard form

$$y^m - R_{mn}(x) = 0,$$

but similar relations exist among the periods associated with the curve,

$$y^m - R_s(x) = 0,$$

where R is rational and entire and s is any integer. The difference in the discussion will arise from the fact that the point infinity is now a branch point where all the values of y permute in one or more cycles. We will have then a similar complete system of periods ω_i, and their multiples by mth roots of unity.

In particular, if we make $m=3$ and $s=4$, we have for the three integrals connected with the curve $y^3=x(x-a)(x-b)(x-t)$, the periods given by Picard,*

$$\begin{array}{ll} \omega_1', \omega_1'', \omega_1''', & \lambda\omega_1', \lambda\omega_1'', \lambda\omega_1''', \\ \omega', \omega_2'', \omega_2''', & \lambda^2\omega_2', \lambda^2\omega_2'', \lambda^2\omega_2''', \\ \omega_3', \omega_3'', \omega_3''', & \lambda\omega_3', \lambda\omega_3'', \lambda\omega_3'''. \end{array}$$

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* Comptes Rendus, Tome 93, p. 835.

Biographical Sketch.

The author, William H. Maltbie, was born in Toledo, Ohio, August 26, 1867. He was under the care of private instructors and in various elementary and high schools until 1885, at which time he entered the Ohio Wesleyan University at Delaware, Ohio, from which institution he received the degree of A. B. in 1890, and of A. M. in 1892. In 1890 he was appointed Professor of Mathematics in Hedding College at Abingdon, Ill. In 1891 he entered the Johns Hopkins University as a candidate for the degree of Doctor of Philosophy, selecting Mathematics as his principal subject, with Astronomy as first and Physics as second subordinate. In January, 1893, he was appointed University Scholar, and in June, 1894, Fellow in Mathematics.