Faculty Working Papers

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1. Introduction

Consider a finite horizon economy with outside money which has no intrinsic (i.e., fiat money). At a self-fulfilling (or rational) expectation equilibrium, not only the price of money must be zero but also money has no exchange value. For, at the end of the horizon all the traders attempt to spend away money unless price of money is zero and rational expectation makes price of money to be zero over the horizon.

If we consider an infinite horizon economy, the story is different. In any period, a trader with positive cash balance may find some other trader who is willing to accept money. In this case, an equilibrium with positive price of money may exist.

In [7], Samuelson showed that, in a simple balanced growth generation overlapping economy, there are indeed cases where traders are willing to accept money and equilibrium exists with positive price of money. Moreover, it was shown that, to achieve a Pareto efficient allocation through a competitive market, the creation of money is necessary in these cases.

Cass and Yaari [3] argued that this pecuniary feature arises because of a lack of financial intermediary. In a generation overlapping economy not all traders can trade among each other. Nor there is any central market as in a usual finite economy. Therefore, some type of financial intermediation like money is necessary to establish Pareto efficiency of a competitive equilibrium. Then, one might ask when this type of financial intermediation is called for.

In terms of Gale [4], a simple economy of the type of Samuelson possesses two distinct possibilities of stationary competitive equilibrium

(golden-rule or no-trade). Moreover, the optimal (or Pareto efficient) equilibrium requires the existence of fiat money if the economy satisfies a condition which Gale called the Samuelson case. Hayashi [5] showed, in more general but still restricted class of economies, that money is necessary for Pareto efficiency of an equilibrium when initial endowment is not Pareto efficient.

However, most of the models above are of very simple nature, i.e., each generation live for two period and/or there is only one good except money, in addition to the property that the economy is either stationary or of balanced growth nature. If these restrictions are removed, a simple result like in [3] or [5] no longer holds. It is worthwhile to mention here that the question of existence of a monetary equilibrium in non-stationary economy was investigated by Okuno [6]; it was shown that, under some very restrictive assumptions, there is an equilibrium where money has a positive exchange value.

We consider the following model. In each period, a finite number of consumers are born. All the consumers live the same, but finite, number of periods. The number of goods is (finite) constant over time. Those, and only those, consumers who live on the first day of the economy may have positive amount of fiat money. Therefore, the total amount of fiat money is constant over the time. Transactions are costless even for future markets. However, transactions with consumers who are either not born or have died are prohibited. In other words, consumers can issue I.O.U. (inside money) but they must clear their obligations by the time of their death. We treat these I.O.U. and fiat money separately. We

call equilibrium monetary if fiat money has a positive value and barter otherwise.

In view of Cass and Yaari's work the following questions arise; (1) If there is a system of financial intermediaries, notably that of the "social contrivance of money", are competitive equilibria which utilize these intermediaries necessarily Pareto efficient? Namely, in our set-up, are monetary equilibria always Pareto efficient? (2) If the answer is no, what makes equilibria Pareto inefficient and in what situations equilibria are Pareto efficient? This paper answers these questions.

In Section 2, we present a model which is slightly different from an economy described above. (The equivalence of the economy above and our model is shown in the Appendix). In Section 3, we prove the existence of a competitive equilibrium, although it can be monetary or barter, Pareto efficient or inefficient. In Section 4, the question (1) of Pareto efficiency of competitive equilibria, especially in regard to the role of money, is answered with the help of a few examples. In Section 5, sufficient conditions for efficient equilibrium are discussed by using theories on efficiency prices (especially, by Benveniste [2]). In Section 6, we discuss relationship between our result and that by Cass-Yaari [3], present an open question accompained by an interesting example.

2. Description of the Economy

There are an infinite number of time periods ($t = 1, 2, \ldots$) and in each period there are m identical goods. All the goods are completely perishable. There is also (outside) money which has no intrinsic value but it is the economy's only store of value. We assume, for simplicity, that all consumers (except those who were born before period 1) live for the same number of periods, $N + 1$ ($N \geq 1$). Consumers are denoted by $i = 1, 2, \ldots$. Let $\theta_i \in \{1, 2, \ldots\}$ be the period in which i is born. Hence i-th consumer is born in θ_i and dies in $\theta_i + N$. We also use the notation $L_i = \{\theta_i, \theta_i + 1, \ldots \theta_i + N\}$, a set of periods in which i lives. The set of individuals who are born in the same period t is called t-th generation and denoted by G_t, i.e., $G_t = \{i | \theta_i = t\}$. We assume that for all $t = 1, 2, \ldots$, G_t is a non-empty and finite set.

Let $E = \overset{\infty}{\underset{t=1}{X}} E^{m_t}$ where $m_t = m$ for all t. Let E_+ be the non-negative cone of E. We assume that the ordering of all the consumers in this economy is made according to their age, i.e., for any i, j, $i \geq j$ if and only if $\theta_i \geq \theta_j$. We denote by x_i the following type of sequence in E; $x_i = \{x_i(1), x_i(2), \ldots\} \in E$ where $x_i(t) \in E^m$ for all t and $x_i(t) = 0$ for all $t \notin L_i = \{\theta_i, \ldots, \theta_i + N\}$. We denote by $\underline{x}_i$ the projection of x_i on the coordinates of L_i. Namely, $\underline{x}_i = \{x_i(\theta_i), x_i(\theta_i + 1), \ldots, x_i(\theta_i + N)\} \in E^{m(N+1)}$.

Each consumer i has a vector of initial endowments in each period t in L_i, $w_i(t)$. A sequence of initial endowments is denoted by $w_i \in E_+$ where $w_i(t) = 0$ for all $t \notin L_i$. The <u>consumption possibility set</u> for i, X_i, is assumed as

$$X_i = \{c_i \in E_+ | c_i(t) = 0 \text{ for all } t \notin L_i\} .$$

We denote by $c = (c_i)_{i=1}^{\infty} = (c_1, c_2, \ldots)$ the allocation of consumptions in the economy. The set of feasible consumption allocations is defined as

$$X = \{c = (c_i)_{i=1}^{\infty} \,|\, c_i \in X_i \text{ for all } i \text{ and } \sum_{i=1}^{\infty} c_i(t) \leq \sum_{i=1}^{\infty} w_i(t) \text{ for all } t\} ,$$

Each consumer i has the preference ordering, $\succ_i$, defined over $E_+^{m(N+1)}$, i.e., over the lifetime bundles of consumption. We assume that for each i his preference ordering is represented by a utility function $u_i : E_+^{m(N+1)} \longrightarrow E^1$. Finally, we assume that consumers in the first period may have a positive amount of initial money holding, $\bar{m}_i \in E_+^1$. For generality, we denote by $\bar{m}_i$ an amount of initial money holding for i, but we assume $\bar{m}_i \geqq 0$ for all i, $\bar{m}_i = 0$ for all i with $\theta_i > 1$ and $\bar{m} = \sum_{i \in G_1} \bar{m}_i > 0$.

The following assumptions on initial endowments and the utility functions will be made.

Assumption I: For all i $w_i(t) > 0$ for all $t \in L_i$, $\sum_{i=1}^{\infty} w_i(t) >> 0$ and $\|\sum_{i=1}^{\infty} w_i(t)\| \leq M$ for all t, for some constant M.

Assumption II: For all i, u_i is continuous, increasing and strictly quasi-concave and $u_i(w_i) = 0$.

Let $c^* = (c_i^*)_{i=1}^{\infty}$ in X. c^* is Pareto efficient if (1) $c_i^* \succsim_i w_i$ for all i and (2) there is no $c = (c_i)_{i=1}^{\infty}$ in X such that $c_i \succsim_i c_i^*$ for all i and $c_j \succ_j c_j^*$ for some j. Note that our definition of Pareto efficiency requires individual rationality as well.

Let X^* be the set of all Pareto efficient allocations in X .

A sequence of prices of goods will be denoted by $\underset{\sim}{p} = \{p(1), p(2), \ldots\} \in E_+$. Given prices $\underset{\sim}{p}$ and a consumption allocation $\underset{\sim}{c}$ in X , we have the price of money, $p^m(\underset{\sim}{p}, \underset{\sim}{c})$. The fact that the price of money can be taken constant over time is shown in the Appendix.

Let $U = \{\underset{\sim}{u} = (\alpha_1, \alpha_2, \ldots) | \alpha_i = u_i(\underset{\sim}{c}_i)$ for all i for some $\underset{\sim}{c} \in X\}$ and let $U^* = \{\underset{\sim}{u} = (\alpha_1, \alpha_2, \ldots) \in U |$ there is $\underset{\sim}{c} \in X^*$ such that $u_i(\underset{\sim}{c}_i) = \alpha_i$ for all $i\}$.

<u>Definition</u>: $(\underset{\sim}{p}^*, \underset{\sim}{c}^*)$ is a competitive equilibrium if $\underset{\sim}{p}^* \in E_+$, $\underset{\sim}{p}^* > 0$ $\underset{\sim}{c}^* = (\underset{\sim}{c}_i^*)_{i=1}^{\infty} \in X$, $p^m(\underset{\sim}{p}^*, \underset{\sim}{c}^*) \geq 0$, and for all i

(a) $\underset{\sim}{p}^* \cdot \underset{\sim}{c}_i^* \leq \underset{\sim}{p}^* \cdot \underset{\sim}{w}_i + p^m(\underset{\sim}{p}^*, \underset{\sim}{c}^*) \cdot \bar{m}_i$

(b) $\underset{\sim}{c}_i^*$ maximizes u_i over $\{\underset{\sim}{c}_i \in X_i | \underset{\sim}{p}^* \cdot \underset{\sim}{c}_i \leq \underset{\sim}{p}^* \cdot \underset{\sim}{w}_i + p^m(\underset{\sim}{p}^*, \underset{\sim}{c}^*) \cdot \bar{m}_i\}$.

<u>Definition</u>: $(\underset{\sim}{p}^*, \underset{\sim}{u}^*, \underset{\sim}{c}^*)$ where $\underset{\sim}{u}^* = (u_1^*, u_2^*, \ldots)$ with $u_i^* \in E^1$ is a <u>compensated equilibrium</u> if $\underset{\sim}{p}^* > 0$, $p^m(\underset{\sim}{p}^*, \underset{\sim}{c}^*) \geq 0$, $\underset{\sim}{c}^* \in X$, and for all i .

(a') $\underset{\sim}{c}_i^*$ minimizes $\underset{\sim}{p}^* \cdot \underset{\sim}{c}_i$ over $\{\underset{\sim}{c}_i \in X_i | u_i(\underset{\sim}{c}_i) \geq u_i^*\}$

(b') $\underset{\sim}{p}^* \cdot \underset{\sim}{c}_i^* = \underset{\sim}{p}^* \cdot \underset{\sim}{w}_i + p^m(\underset{\sim}{p}^*, \underset{\sim}{c}^*) \cdot \bar{m}_i$.

Given a competitive equilibrium, $(\underset{\sim}{p}^*, \underset{\sim}{c}^*)$, we call it a <u>monetary equilibrium</u> if $p^m(\underset{\sim}{p}^*, \underset{\sim}{c}^*) > 0$ and a <u>barter equilibrium</u> if $p^m(\underset{\sim}{p}^*, \underset{\sim}{c}^*) = 0$.

3. Existence of a Competitive Equilibrium

In this section, we shall prove the existence of a competitive equilibrium in this economy. For this purpose, we shall introduce a few definitions and an assumption.

For each time period $T \geq N + 1$ let F_T denote the set of of all consumers who will die on or before T, i.e., $F_T = \{i \mid \theta_i + N \leq T\}$. Let X^T be the projection of X on F_T, namely

$$X^T = \{\underset{\sim}{c}^T = (\underset{\sim}{c}_i)_{i \in F_T} \mid \sum_{i \in F_T} c_i(t) \leq \sum_{i=1}^{\infty} w_i(t) \quad \text{for all} \quad t \leq T\} .$$

Note that the aggregate consumption for F_T may exceed their aggregate endowment. Let U^T be the projection of U on F_T, or

$$U^T = \{\underset{\sim}{u}^T = (\alpha_i)_{i \in F_T} \mid \alpha_i = u_i(\underset{\sim}{c}_i) \quad \text{for all} \quad i \in F_T \quad \text{for some } \underset{\sim}{c}^T \text{ in } X^T\}$$

Next, the sets of (individually rational and) Pareto efficient allocations in X^T, and the corresponding utility allocations are

$$X^{T*} = \{\underset{\sim}{c}^T = (\underset{\sim}{c}_i)_{i \in F_T} \text{ in } X^T \mid \underset{\sim}{c}_i \succsim_i \underset{\sim}{w}_i \quad \text{for all} \quad i \in F_T \quad \text{and}$$

$$\text{there is no } \hat{\underset{\sim}{c}}^T \in X^T \text{ s.t. } \hat{\underset{\sim}{c}}_i \succsim_i \underset{\sim}{c}_i \quad \text{for all} \quad i \in F_T$$

$$\text{and } \hat{\underset{\sim}{c}}_j \succ_j \underset{\sim}{c}_j \quad \text{for some} \quad j \in F_T\}$$

$$U^{T*} = \{\underset{\sim}{u}^T = (\alpha_i)_{i \in F_T} \text{ in } U^T \mid \text{there exists } \underset{\sim}{c}^T \in X^{T*} \text{ such that}$$

$$\alpha_i = u_i(\underset{\sim}{c}_i) \quad \text{for all} \quad i \in F_T\}$$

Let us introduce now the following definitions.

<u>Definition</u>: $\underset{\sim}{c}^T = (\underset{\sim}{c}_i)_{i \in F_T}$ in X^T is <u>T-efficient</u> if $\underset{\sim}{c}_i \succsim_i \underset{\sim}{w}_i$ for

all $i \in F_T$ and there is no $\hat{c}^T = (\hat{c}_i)_{i \in F_T}$, $\sum_{i \in F_T} \hat{c}_i \leq \sum_{i \in F_T} c_i$, $\hat{c}_i \succsim_i c_i$ for all i in F_T and for some $j \in F_T$, $\hat{c}_j \succ_j c_j$.

$c^* = (c_i^*)_{i=1}^{\infty}$ in X is short-run efficient if for all T, $(c_i^*)_{i \in F_T}$ is T-efficient. Obviously, a short-run efficient allocation is not necessarily Pareto efficient.

Definition: Given $c^T = (c_i)_{i \in F_T}$ in X^T, let j and j' be any two commodities, t and $t' \leq T$ be any two periods. We say that timed commodities (t, j) and (t', j') are resource-related (with respect to c^T), if there exists i in F_T, such that $c_{ij}(t) > 0$ and $c_{ij'}(t') > 0$. We say that (t, j) and (t', j') are indirectly-recource-related (w. r. t. c^T) if there is a finite sequence $\{(t_k, j_k)\}_{k=0}^{\ell}$ such that $(t_o, j_o) = (t, j)$, $(t_\ell, j_\ell) = (t', j')$ and (t_{k-1}, j_{k-1}) is resource-related to (t_k, j_k) for $k = 1, \ldots, \ell$.

Assumption III: Any (t, j) and (t', j') $(1 \leq j, j' \leq m, 1 \leq t, t' \leq T)$ are indirectly resource related with respect to any T-efficient c^T, for any T.

We shall use Assumption III to prove existence of competitive equilibrium. However, we show first that this assumption implies another important property of intergenerational substitution of consumption.

Definition: Let $c^{T*} \in X^T$ be T-efficient ($c^* \in X$ be short-run efficient, respectively) and $\varepsilon > 0$. Period 1 is said to be ε-resource related to period $t \leq T$ with respect to c^{T*} (c^*, resp.) if there exists $\hat{c}^T = (\hat{c}_i)_{i \in F_T}$ such that

$$\text{(a)} \quad \sum_{i \in F_T} \hat{c}_i(1) = \sum_{i \in F_T} c_i^*(1) + (1, 1, \ldots, 1),$$

(b) $\sum_{i \in F_T} \hat{c}_i(t) = \sum_{i \in F_T} c_i^*(t) - \varepsilon(1, 1, \ldots, 1)$

(c) $\sum_{i \in F_T} \hat{c}_i(\tau) = \sum_{i \in F_T} c_i^*(\tau)$ for all $\tau \leq T$, $\tau \neq 1, t$,

(d) $\hat{c}_i \in X_i$ and $\hat{c}_i \succ_i c_i^*$ for all $i \in F_T$.

Lemma 1: For each T , there exist positive scalers $\{\varepsilon_1^T, \varepsilon_2^T, \ldots \varepsilon_T^T\}$ such that for all T-efficient allocations, $c^T \in X^{T*}$, period 1 is ε_t^T-resource related to period t .

Proof: For each $c^T \in X^{T*}$, by Assumption III and monotonicity of u_i , 1 is ε-resource related to t for some $\varepsilon > 0$. Let $\varepsilon_t(c^T)$ be the maximum of such $\varepsilon > 0$ (one can take maximum, for ε is defined over a compact set as is easily seen by the definition). For any T-efficient $c^{T*} \in X^{T*}$, and for any δ, $0 < \delta < \varepsilon_\tau(c^{T*})$ by Assumption III, there exists $\hat{c}^T$ satisfying condition (a), (c) and

(b') $\sum_{i \in F_T} \hat{c}_i(t) = \sum_{i \in F_T} c_i^*(t) - \delta(1, 1, \ldots, 1)$

(d') $\hat{c}_i \in X_i$ and $\hat{c}_i \succ_i c_i^*$ for all $i \in F_T$.

Since u_i is continuous, there is a (relatively) open neighborhood of c^{T*} in X^T , $N(c^{T*}, \delta) \subset X^T$, such that for all $c^T \in N(c^{T*}, \delta)$, peirod 1 is δ-resource related to period t . Therefore $\varepsilon_t(c^T)$ is continuous. Since by assumption III, the set of all T-efficient c^T in X is compact, there is $\varepsilon_t^T = \min\{\varepsilon_t(c^T) | c^T \in X^T$ is T-efficient$\} > 0$.

Given Assumptions I-III, we can prove

Theorem 1: There exists a competitive equilibrium.

Let us prove first the following Lemma.

<u>Lemma 2</u>: <u>For each</u> T, $T \geq N+1$, <u>there exists</u> $\underset{\sim}{c}^{T*} = (\underset{\sim}{c}_i^*)_{i \in F_T}$ <u>in</u> X^{T*}, $\underset{\sim}{p}^{T*} \in E_+$ <u>with</u> $p^{T*}(t) = 0$ <u>for all</u> $t > T$, <u>and</u> $p^{mT*} > 0$ <u>such that</u> for all $i \in F_T$

$$\text{(a)}\quad \underset{\sim}{p}^{T*}\cdot \underset{\sim}{c}_i^* \leq \underset{\sim}{p}^{T*}\cdot \underset{\sim}{w}_i + p^{mT*}\cdot \bar{m}_i$$

$$\text{(b)}\quad \underset{\sim}{c}_i^* \text{ maximizes } u_i \text{ over } \{c_i \in X^i \mid \underset{\sim}{p}^{T*}\cdot \underset{\sim}{c}_i \leq p^{T*}\cdot w_i + p^{mT*}\cdot \bar{m}_i\}$$

<u>Proof</u>: The proof is essentially similar to Arrow-Hahn's [1, ch. 5]; However, due to our special structure and existence of money, some parts are substantially different.

Given T, $T \geq N+1$, define

$S^T = \{\underset{\sim}{p}^T \in E_+ \mid \|p^T(1)\| = 1$, $\|p^T(t)\| = \frac{1}{\varepsilon_t^T}$ for all $2 \leq t \leq T$ and $p^T(t) = 0$ for all $t > T\}$, where the sequence of positive scalars $\{\varepsilon_t^T\}_{t=1}^T$ is given by Lemma 1. Define a mapping $p^m : S^T \longrightarrow E_+^1$ by

$$p^m(\underset{\sim}{p}^T) = \frac{1}{\bar{m}} \sum_{i \in F_T} \sum_{t=T-N}^{T} p^T(t)\, w_i(t)$$

In other words, $p^m(\underset{\sim}{p}^T)\bar{m}$, the value of the money, is made so that it is exactly equal to the value of endowments in X^T, which does not properly belong to the consumers in F_T. Given prices $\underset{\sim}{p}^T \in S^T$ and allocation $c^T \in X^T$, we define budget surplus for all i in F^T,

$$s_i(\underset{\sim}{p}^T, \underset{\sim}{c}^T) = p^T\cdot(\underset{\sim}{w}_i - \underset{\sim}{c}_i) + p^m(p^T)\bar{m}_i .$$

Let $\Sigma^T = \{\underset{\sim}{v}^T = (v_i)_{i \in F_T} \mid v_i \in E_+^1$ and $\sum_{i \in F_T} v_i = 1\}$. Define $\psi^T : \Sigma^T \longrightarrow U^{T*}$ in the following way: For $\underset{\sim}{v}^T \in \Sigma^T$, define

$\psi^T(\underset{\sim}{v}^T) = \max \{\lambda > 0 \mid \lambda \underset{\sim}{v}^T \in U^T\} \cdot \underset{\sim}{v}^T$. ψ^T is well defined because (1) $\underset{\sim}{u}^T \notin U^{T*}$ if $u_i = 0$ for all $i \in F_T$, since $\sum_{i=1}^{\infty} w_i(t) - \sum_{i \in F_t} w_i(t) > 0$ for all t $T - N + 1 \leqq t \leqq T$ (note the definition of X^T) and (2) any Pareto efficient allocation $c^T \in X^{T*}$ is also weakly Pareto efficient because of Assumption III (i.e., if $\hat{\underset{\sim}{c}}^T$ Pareto-dominates $\underset{\sim}{c}^T$ then $u_i(\hat{\hat{c}}_i) > u_i(\underset{\sim}{c}_i)$ for all $i \in F_T$ for some $\hat{\hat{\underset{\sim}{c}}}$ in X^T). ψ^T is a continuous function.

For each $\underset{\sim}{u}^T = (\alpha^i)_{i \in F_T}$ in U^{T*}, define corresponding consumptions $\underset{\sim}{c}^T$ in X^{T*} by,

$$W^T(\underset{\sim}{u}^T) = \{\underset{\sim}{c}^T \in X^{T*} \mid u_i(\underset{\sim}{c}_i) = \alpha_i \quad \text{for all} \quad i \in F_T\} .$$

W^T is single-valued and continuous by Assumption II.

For each $\underset{\sim}{c}^T$ in X^{T*}, define

$$B^T(\underset{\sim}{c}^T) = \{(x(1), \ldots, x(T)) \in E^{mT} \mid \text{there exists}$$

$$\hat{\underset{\sim}{c}}^T = (\hat{\underset{\sim}{c}}_i)_{i \in F_T} \quad \text{such that} \quad \hat{\underset{\sim}{c}}_i \succsim_i \underset{\sim}{c}_i$$

$$\text{for all} \quad i \in F_T \quad \text{and}$$

$$\sum_{i \in F_T} \hat{c}_i(t) \leqq \sum_{i \in F_T} c_i(t) + x(t) \text{ for all } t \leq T\}$$

$B^T(\underset{\sim}{c}^T)$ is strictly covex and contains a non-empty interior. Furhtermore, $(0, \ldots, 0) \in \text{Int } B^T(\underset{\sim}{c}^T)$. Define $P^T : X^{T*} \longrightarrow S^T$ as follows:

$$P^T(\underset{\sim}{c}^T) = \{\underset{\sim}{p} \in S^T \mid \sum_{t=1}^{T} p(t)x(t) \geq 0 \quad \text{for all } (x(1), \ldots, x(T)) \text{ in } B^T(\underset{\sim}{c}^T)\}$$

By the well-known separation theorem, there exists a non-trivial prices $\{p(1), \ldots, p(T)\} \in E_+^{mT}$ satisfying the condition of $P^T(c^T)$ except that $p^T \in S^T$. By the definition of ε-resource relatedness and Lemma 1, $\{x(1), \ldots, x(T)\} \supseteq B^T(c^T)$ when $x(1) = (1, \ldots, 1)$, $x(t) = -\varepsilon_t^T \cdot (1, \ldots, 1)$ and $x(\tau) = 0$ for all $\tau \geq T$, $\tau \neq 1, t$. Therefore $\|p(1)\| - \varepsilon_t^T \|p(t)\| \geq 0$ or $\|p(t)\| \leq \|p(1)\| / \varepsilon_t^T$. Hence $p^T \in S^T \cap P^T(c^T)$ is non-empty, convex, compact-valued, and upper semi-continuous.

Finally, define $V^T : S^T \times X^T \longrightarrow \Sigma^T$,

$$V^T(p^T, c^T) = \{v^T \in \Sigma^T \mid v_i = 0 \text{ if } s_i(p^T, c^T) < 0\} .$$

$V^T(p^T, c^T) \neq \phi$, convex, compact valued, and upper-semi-continuous. Now, consider the following correspondence.

$$P^T \circ W^T \circ \psi^T \times V^T \times W^T \circ \psi^T : S^T \times \Sigma^T \times X^T \longrightarrow S^T \times \Sigma^T \times X^T$$

This correspondence satisfies all the conditions of Kakutani's fixed point theorem.

Let (p^{T*}, v^{T*}, c^{T*}) be the fixed point. By our definition, $c^{T*} \in X^{T*}$. Let us show that when $u^{T*} = \psi^T(v^{T*})$, (p^{T*}, u^{T*}, c^{T*}) satisfies (a') and (b') of the definition of compensated equilibrium for all $i \in F_T$. Assume $s_i(p^{T*}, c^{T*}) < 0$ for some i. By definition of V^T, $v_i^* = 0$ and $u_i(c_i^*) = 0$. Since $u_i(w_i) = 0$ we come to $0 > s_i(p^{T*}, c_i^*) \geq p^{T*}(w_i - c_i^*)$ which contradicts the fact that $p^{T*} \in P^T(c^{T*})$. Therefore $s_i(p^{T*}, c^{T*}) \geq 0$ for all i. By the definition of $p^m(p^{T*})$ and noting that $\sum_{i \in F_T} \bar{m}_i = \bar{m}$, we see that

$$\sum_{i \in F_T} s_i(p^{T*}, c^{T*}) = \sum_{i \in F_T} \{p^{T*}(w_i - c_i^*)\} - p^m(p^{T*}) \cdot \bar{m} = 0$$

Hence $s_i(\underset{\sim}{p}^{T*}, \underset{\sim}{c}^{T*}) = 0$ for all i in F_T.

Finally, because of Assumption III and monotonicity of u_i's, conditions for compensated equilibrium are equivalent to those of competitive equilibrium (i.e., $\underset{\sim}{p}^{T*} \cdot \underset{\sim}{c}_i^* > 0$ for all $i \in F_T$, since $p^{T*}(t) \gg 0$ for all $t \leq T$).

<u>Proof of Theorem 1</u>: Let $(\underset{\sim}{c}^{T*}, \underset{\sim}{p}^{T*}, p^{mT*})$ be the fixed point we obtained in Lemma 1. Let $\underset{\sim}{u}^{T*}$ be the corresponding utility values, which is in U^*. Let $(\bar{p}^T, \bar{u}^T, \bar{c}^T, \bar{p}^{mT})_{T=1}^{\infty}$ satisfy; $\bar{p}^T = p^{T*}$; $\bar{u}^T \in U$ such that $\operatorname{Proj}_{F_T} \bar{u}^T = u^{T*}$; $\bar{c}^T \in X$ such that $\operatorname{Proj}_{F_T} \bar{c}^T = c^{T*}$; $\bar{p}^{mT} = p^{mT*}$.

Then there is a subsequence of $\{\bar{p}^T(1), \bar{u}^T, \bar{c}^T\}_{T=1}^{\infty}$ which converges coordinate-wise to $(p^*(1), \underset{\sim}{u}^*, c^*)$ such that $\| p^*(1) \| = 1$, $\underset{\sim}{u}^* \in U$, and $\underset{\sim}{c}^* \in X$.

To show that $\underset{\sim}{c}^*$ is short run efficient, notice that by Assumption III it can be proved, in a straightforward manner, that a limit of k-efficient allocations (for some finite k) is a k-efficient allocation. This follows from the fact that if $\underset{\sim}{c}^k = (\underset{\sim}{c}_i^k)_{i \in F_k}$ is Pareto-dominated, then there exists some k-efficient $(\hat{c}_i^k)_{i \in F_k}$ s.t. $u_i(\hat{c}_i^k) > u_i(\underset{\sim}{c}_i^k)$ for all i in F_k.

Given any T_o for all $T \geq T_o$, $(\underset{\sim}{c}_i^{T*})_{i \in F_{T_o}}$ is T_o-efficient (otherwise, for some $(\tilde{c}_i^T)_{i \in F_{T_o}}$ with $\sum_{i \in F_{T_o}} \tilde{c}_i^{T} \leq \sum_{i \in F_{T_o}} \underset{\sim}{c}_i^{T*}$ we have $u_i(\tilde{c}_i^T) > u_i(\underset{\sim}{c}_i^{T*})$ for all i in F_{T_o}. This implies $\sum_{i \in F_{T_o}} \underset{\sim}{p}^{T*}(\tilde{c}_i^T - \underset{\sim}{c}_i^{*T}) > 0$ which is impossible.) This implies that $(\underset{\sim}{c}_i^*)_{i \in F_{T_o}}$ is T_0-efficient for any T_o, hence $\underset{\sim}{c}^*$ is short-run efficient.

Since $\underset{\sim}{c}^*$ is short-run efficient, by Assumption III, there exists a sequence of positive scalars $\{\varepsilon_1, \varepsilon_2, \ldots, \}$ such that 1 is ε_t - resource related to t for all t with respect to $\underset{\sim}{c}^*$. Therefore, for given t , there exist T_o and $(\hat{c}_i)_{i \in F_{T_o}}$ which satisfy the conditions of δ-resource relatedness with $\delta(0 < \delta < \varepsilon_t)$ and with $\hat{c}_i \sim_i c_i^*$ for all $i \in F_{T_o}$ (by Assumption III again). Because of the coordinatewise convergence, period 1 must be δ-resource related to t for all $\underset{\sim}{c}^{T*}$ for T sufficiently large. Therefore $\{p^{T*}(t)\}_{T=1}^{\infty}$ must be bounded and we can find a converging subsequence. Finally, for such a subsequence, $\{p^{mT*}\}$ is bounded, for $\sum_{i \in G_1} p^{mT*} \cdot \bar{m}_i = p^{mT*} \cdot \bar{m} \leq \sum_{i \in G_1} p^{T*} \cdot c_i^* \leq \sum_{i=1}^{\infty} \sum_{t=1}^{N+1} p^{T*}(t) w_i(t)$, which is unformly bounded. Let $(\underset{\sim}{p}^*, p^{m*})$ be the limit of such a converging subsequence. Then $p^{m*} \geq 0$, $\underset{\sim}{p}^* > 0$ since $\|p^*(1)\| = 1$, and $s_i(\underset{\sim}{p}^*, \underset{\sim}{c}^*) = 0$ for all i . It can be shown easily that $(\underset{\sim}{p}^*, \underset{\sim}{u}^*, \underset{\sim}{c}^*, p^{m*})$ is a compensated equilibrium, which, in turn, is a competitive equilibrium by Assumption III,

Q.E.D.

4. Efficiency of an Equilibrium

In their paper [3], Cass and Yaari, in effect, claim as follows. In an economy with overlapping generations, there is no "Walras market", for some consumers in different generations may not be allowed to trade each other. To achieve an efficient equilibrium in a limited market structure, some device like a financial intermediary is called for. Therefore, there may exist an equilibrium with money actively traded.

Example 1

Consider the following economy, which has the same structure as our model. Let m (the number of goods) be one. Let each consumer live for two periods. We index them as $0,1,2,\ldots$. $\theta_i = i$ for all $i \geqq 1$. For $i = 0$, he lives only at $t = 1$. Their initial endowment is as in the Table 1 while we assume $\bar{m}_0 = 1$ and $\bar{m}_i = 0$ for all $i \geqq 1$.

i \ t	1	2	3	...
0	1			
1	3	1		
2		3	1	
⋮			⋱	⋱

Table 1

Finally, let utility $u_i = \ell n\ c_i(\theta_i) + \ell n\ c_i(\theta_i + 1)$ for all $i \geqq 1$ and $u_0 = \ell n\ c_0(1)$.

There are two equilibria, one is barter and the other monetary. Barter equilibrium is such that consumption is exactly the same as initial

allocation and prices satisfy $p(t) = 3^{t-1}p(1)$ for all t. Monetary equilibrium is such that consumption is as in Table 2, prices satisfy $p(t) = p(1) = p^m$ for all t. The Barter equilibrium is Pareto inefficient, for it is dominated by the monetary one and the monetary equilibrium is Pareto efficient. Therefore, the example not only seems to confirm Cass-Yarri contention but also indicate importance of distinguishing a Pareto efficient equilibrium from an inefficient one.

i \ t	1	2	3	...
0	2			
1	2	2		
2		2	2	
⋱				

Table 2

Example 2

However, it is not too difficult to construct an inefficient monetary equilibrium. For example, let us keep essentially the same structure of the first example but change their initial endowment slightly as in Table 3 with $\varepsilon > 0$. Then the barter equilibrium (of consumption and prices) in the first example becomes a monetary equilibrium in the new situation with $p^m = \varepsilon p(1)$. Since the aggregate endowments are the same as before, our monetary equilibrium is Pareto inefficient.

$i \backslash t$	1	2	3	4	...
0	$1-\varepsilon$				
1	$3+\varepsilon$	$1-3^{-1}\varepsilon$			
2		$3+3^{-1}\varepsilon$	$1-3^{-2}\varepsilon$		
3			$3+3^{-2}\varepsilon$	$1-3^{-3}\varepsilon$	
				$\ddots$	$\ddots$

Table 3

As a matter of fact, a monetary equilibrium and a barter equilibrium are not too different in some cases. Suppose we obtain a barter equilibrium with equilibrium prices $\tilde{p}$ which satisfy $\liminf_{t\to\infty} \| p(t) \| > 0$. Then, in general, we can redistribute initial endowments in each period from a young generation to a old generation, and make the original barter equilibrium a monetary equilibrium in a new economy. Moreover, we can make this redistribution as small as possible, as is the case of ε in our example. Namely, any barter equilibrium with $\liminf_{t\to\infty} \| p(t) \| > 0$ is considered as the limit of a sequence of monetary equilibria where each monetary equilibrium belongs to a different economy but these economies converge to the economy in question.

More disturbingly, if at any equilibrium, $\limsup_{t\to\infty} \| p(t) \| = 0$, then the equilibrium can never be monetary because the value of money

$p^{m}\cdot\bar{m}$ must sooner or later exceed the value of economy's total initial endowments. But as we show in Theorem 2 below, such a (batter) equilibrium is always Pareto efficient.

Some readers may observe that the Example 2 is of very absurd nature. If p is unbounded, one unit of good in period 1 has the same value as a very small units of goods in a distant future. Hence one can make a sequence of transfers of goods, of the same value, from young generations to older generations in each period, and eventually make this transfer disappear in the limit, thus creating a Pareto-dominating allocation. A seemingly natural conclusion, in view of strict-quasi-concavity of the utilities (Assumption II), seems to be the following one. If $\limsup_{t\to\infty} \| p(t) \| < \infty$, then a (monetary) equilibrium is Pareto efficient. Unfortunately, this contention is, again, incorrect as the following example shows.

<u>Example 3</u>

Let the generational structure be the same as in Example 1. Let the initial endowments be as in the Table 4. Namely, $\omega_0(1) = 1 - \varepsilon\cdot 3^{-q}$, $\omega_1(1) = 3 + \varepsilon\cdot 3^{-q}$,

$$\omega_i(\theta_i) = 3 + \varepsilon\cdot 3^{-q+\frac{1}{(i-1)^2}} \quad \text{for all} \quad i \geqq 2 ,$$

$$\omega_i(\theta_i+1) = 1 - \varepsilon\cdot 3^{-q+\frac{1}{i^2}} \quad \text{for all} \quad i \geqq 1 ,$$

where ε is an arbitrarily small positive number and $q = \pi^2/6$. Let utility functions be

i \ t	1	2	3	
0	$1-\varepsilon\cdot 3^{-q}$			
1	$3+\varepsilon\cdot 3^{-q}$	$1-\varepsilon\cdot 3^{-q+1}$		
2		$3+\varepsilon\cdot 3^{-q+1}$	$1-\varepsilon\cdot 3^{-q+\frac{1}{4}}$	
3			$3+\varepsilon\cdot 3^{-q+\frac{1}{4}}$	$1-\varepsilon\cdot 3^{-q+\frac{1}{9}}$
			$\ddots$	$\ddots$

Table 4

$$u_i\big(c(\theta_i),c(\theta_i+1)\big) = \left[c(\theta_i)^{\frac{i^2-1}{i^2}} + c(\theta_i+1)^{\frac{i^2-1}{i^2}}\right]^{\frac{i^2}{i^2-1}} \quad \text{for all} \quad i \geqq 2 ,$$

$$u_1\big(c(1),c(2)\big) = \ln c(1) + \ln c(2) \quad \text{and} \quad u_0\big(c(1)\big) = \ln c(1) .$$

Then there is a monetary equilibrium with consumption allocation as in Table 1 with prices

$$p(t) = p(1)\cdot \prod_{i=1}^{t-1} 3^{-\frac{1}{i^2}} = 3^{-\sum_{i=1}^{t-1} 1/i^2}\cdot p(1) \quad \text{for all} \quad t \geqq 2 .$$

To see this, observe that for any $i \geqq 1$, marginal rate of substitution of $c(\theta_i)$ for $c(\theta_i+1)$ is $[c(\theta_i)/c(\theta_i+1)]^{1/i^2}$. Therefore, at the equilibrium, $MRS = 3^{1/i^2}$. Moreover, since $\sum_{i=1}^{\infty} \frac{1}{i^2} = \pi^2/6 = q$, $\lim_{t\to\infty} p(t)$ is finite. However, again, this equilibrium allocation is Pareto

dominated by the allocation in Table 2. Because for any $i \geq 1$, the equilibrium consumption $\{c^*(S_i), c^*(S_i+1)\} = (3,1)$ is indifferent to $(1,3)$. Since consumption in Table 2 is a convex combination of these two, strict quasi-concavity of utility functions asserts that Table 2 consumption dominates the equilibrium allocation.

5. Characterization of Efficient Equilibria

In this section, we shall ask the following question; under what conditions there is a (monetary or barter) Pareto efficient equilibrum? Let us start with a relatively simple case.

Theorem 2: Let $(\underset{\sim}{p}^*, \underset{\sim}{c}^*, p^{m*})$ be a competitive equilibrium. If $\limsup_{t \to \infty} \|p^*(t)\| = 0$, then $\underset{\sim}{c}^*$ is an efficient equilibrium.

Proof: Suppose that c^* is not efficient. Then there exists an allocation $\hat{\underset{\sim}{c}} \in X$ which dominates $\underset{\sim}{c}^*$, i.e., $\hat{c}_i \succsim_i c_i^*$ for all i and for some j, $\hat{c}_j \succ_j c_j^*$. Therefore, $\underset{\sim}{p}^* \cdot \hat{c}_i \geq \underset{\sim}{p}^* \cdot c_i^*$ for all i and $\underset{\sim}{p}^* \cdot \hat{c}_j > \underset{\sim}{p}^* \cdot c_j^*$. Let $T_0 = \tau_j + N$, the last period in which j lives. Then for any $T \geq T_0$, F_T contains j and

$$0 < \sum_{i \in F_T} \underset{\sim}{p}^* \cdot (\hat{c}_j - c_i^*)$$

$$= \sum_{i \in F_T} \sum_{t=1}^{T-N+1} p^*(t)[\hat{c}_i(t) - c_i^*(t)] + \sum_{i \in F_T} \sum_{t=T-N+2}^{T} p^*(t)[\hat{c}_i(t) - c_i^*(t)]$$

$$\leq \sum_{i \in F_T} \sum_{t=T-N+2}^{T} p^*(t)[\hat{c}_i(t) - c_i^*(t)] .$$

However the right hand side goes to zero as $T \longrightarrow \infty$, for $\sum_{i \in F_T} [\hat{c}_i(t) - c_i^*(t)]$ is uniformly bounded by Assumption I, a contradiction.

Q.E.D.

If $\limsup_{t \to \infty} \|p(t)\| > 0$, we need some conditons for Pareto efficiency. The following approach makes direct use of the result by Benveniste [2] on efficiency prices. In the proof of Theorem 1, we show that a competitive equilibrium is always short-run efficient.

Let $H_T = \{i \mid T \leq \theta_i \leq T + N - 1\}$, i.e., the set of all the consumers who live in both periods $T + N - 1$ and $T + N$. Given a short-run efficient allocation $\underset{\sim}{c} \in X$, define

$$D^T(\underset{\sim}{c}) = \{(\bar{X}_1^T;\ \bar{X}_2^T) = (\bar{x}^T(T),\ \ldots,\ \bar{x}^T(T+N-1);\ \bar{x}^T(T+N),\ \ldots,$$

$$\bar{x}^T(T+2N-1)) \in E^{2mN} \mid \text{There exists } (\hat{\underset{\sim}{c}}^i)_{i \in H_T} \text{ such that}$$

$$\hat{\underline{c}}_i \succsim_i \underline{c}_i \text{ for all } i \in H_T \text{ and}$$

$$\sum_{i \in H_T} \hat{c}_i(t) = \sum_{i \in H_T} c_i(t) + \bar{x}^T(t) \text{ for all } t,$$

$$T \leq t \leq T + 2N - 1\}$$

Since $\underset{\sim}{c}$ is short-run efficient, we can define a social (aggregate) indifference curve at $\underset{\sim}{c}$. $D^T(\underset{\sim}{c})$ is (the translate of) upper contour set for this indifference curve. In the examples in the previous section where there is only one good, there is only one consumer in each generation and each consumer lives only for two periods, $D^T(\underset{\sim}{c})$ is exactly equal to upper contour set for the consumption. In Example 3, we showed that if the indifference curve loses "strict" curvature over the time, we might find a Pareto inefficient equilibrium. Therefore, we shall define the "coefficient of strictness" of the aggregate indifference curve similar to Benveniste [2].

Let $(p^*, \underset{\sim}{c}^*, p^{m^*})$ be a competitive equilibrium, then $\underset{\sim}{c}^*$ is short-run efficient as was shown in the proof of Theorem 1.

Define

$$Q^T(\underset{\sim}{c}^*, \underset{\sim}{p}^*) = \{(\bar{X}_1^T, \bar{X}_2^T) \quad D^T(\underset{\sim}{c}^*) \mid \sum_{t=T}^{T+N-1} p^*(t)\cdot \bar{x}^T(t) < 0\}$$

The following definition of μ-strictness guarantees that if $\underset{\sim}{c}^*$ is short-run efficient with support p^*, then given T we can find a tangent "paraboloid" at $(\underset{\sim}{c}_i^*)_i \quad H_T$ containing $D^T(\underset{\sim}{c}^*)$.

Definition: Given a competitive equilibrium $(\underset{\sim}{p}^*, \underset{\sim}{c}^*, p^{m^*})$, $\underset{\sim}{c}^*$ is μ_T-strict at T if for all $(\bar{X}_1^T, \bar{X}_2^T)$ in $Q^T(\underset{\sim}{c}^*, p^*)$, we have

$$\sum_{t=T+N}^{T+2N-1} p^*(t)\cdot \bar{x}^T(t) \geq -\sum_{t=T}^{T+N-1} p^*(t)\cdot \bar{x}^T(t) + \frac{\mu_T}{\sum_{t=T}^{T+N-1} p^*(t)\cdot w(t)}\left\{\sum_{t=T}^{T+N-1} p^*(t)\cdot \bar{x}^T(t)\right\}^2 .$$

where $w(t) = \sum_{i=1}^{\infty} w_i(t)$, i.e., the aggregate endowment of the economy at t. Denote by $\bar{\mu}_T = \sup\{\mu_T \mid \underset{\sim}{c}^* \text{ is } \mu_T\text{-strict at } T\}$.

Remark: Our definition of μ_T-strictness generalizes that of Benveniste. The reader is referred to [2] for detailed explanation of this condition.

Given $\underset{\sim}{c}^*$ in X, $\{\varepsilon_t\}_{t=1}^{\infty}$ are coefficients of resource relatedness if for all t $\varepsilon_t = \sup\{\varepsilon > 0 \mid \text{period 1 is } \varepsilon\text{-resource related to period } t \text{ (w.r.t. } \underset{\sim}{c}^*)\}$.

Theorem 3 A competitive equilibrium $(p^*, \underset{\sim}{c}^*, p^{m^*})$ is Pareto efficient if, at $\underset{\sim}{c}^*$,

(a) the coefficients of resource relatedness $(\varepsilon_t)_{t=1}^{\infty}$ satisfy $\sum_{t=1}^{\infty} \varepsilon_t = \infty$, and

(b) <u>there exists</u> $\bar{\mu} > 0$ <u>such that for all</u> $T > 0$ $\bar{\mu}_T \geq \bar{\mu}$, <u>where</u> $\underset{\sim}{c}^*$ <u>is</u> $\bar{\mu}_T$<u>-strict at</u> T .

<u>Remark</u>: Examples 1 and 2 can be considered as counter examples for the theorem without (a) and example 3 as a counter example for the theorem without (b). By assuming uniform smoothness in the analogous manner (see [2]), we can prove that condition (a) is sufficient for Pareto efficiency of a competitive equilibrium.

<u>Proof</u>: Because of (a), an argument similar to the one in the proof of Lemma 1 shows that $\sum_{t=1}^{T} 1/\|p(t)\| \longrightarrow \infty$ as $T \longrightarrow \infty$.

Suppose there exists $\hat{\underset{\sim}{c}} \in X$ such that $\hat{\underset{\sim}{c}}_i \succsim_i \underset{\sim}{c}_i^*$ for all i and $\hat{\underset{\sim}{c}}_j \succ_j c_j^*$ for some j . Define a sequence $t_k = (k - 1)N + 1$ for $k \geq k_o$ where k_o is such that $j \in H_{t_{k_o}}$. Define $\bar{p}^{t_k} = \{p^*(t)\}_{t=t_k}^{t_k+N-1}$ and $(\bar{X}_1^{t_k}, \bar{X}_2^{t_k})$ such that $\bar{X}^{t_k}(t) = \sum_{i \in H_{t_k}} \{\hat{c}_i(t) - c_i(t)\}$ for all t with $t_k \leq t \leq t_k + 2N - 1$. Define $\delta_1^k = \bar{p}^{t_k} \cdot \bar{X}_1^{t_k}$ for all $k \geq 1$ and $\delta_2^k = \bar{p}^{t_k} \cdot \bar{X}_2^{t_{k-1}}$, for all $k \geq 2$. Then

$$\sum_{i \in H_{t_k}} p^*[\hat{\underset{\sim}{c}}_i - \underset{\sim}{c}_i^*] = \bar{p}^{t_k} \cdot \bar{X}_1^{t_k} + \bar{p}^{t_{k+1}} \cdot \bar{X}_2^{t_k} = \delta_1^k + \delta_2^{k+1} \geq 0$$

for all k with strict inequality with $k = k_o$. Since $\underset{\sim}{c}^*$ is an equilibrium allocation and $\hat{\underset{\sim}{c}} \in X$, $\sum_{i=1}^{\infty} \sum_{t=t_k}^{t_k+N-1} p^*(t)[\hat{c}_i(t) - c_i^*(t)]$ $= \bar{p}^{t_k}[\bar{X}_1^{t_k} + \bar{X}_2^{t_{k-1}}] = \delta_1^k + \delta_2^k \leq 0$ for all $k \geq 2$ and

$$\sum_{i=1}^{\infty} \sum_{t=1}^{N-1} p^*(t)[c_i(t) - c_i^*(t)] = \bar{p}^{t_1} \cdot \bar{X}^{t_1} = \delta_1^1 \leq 0 \, .$$

Therefore, δ_2^k (δ_1^k, respectively) is a non-decreasing (non-increasing) sequence of non-negative (non-positive) numbers and $\delta_2^k > 0$ ($\delta_1^k < 0$) for all $k \geq k_o + 1$.

Hence, $(\bar{X}_1^{t_k}, \bar{X}_2^{t_k}) \in Q^{t_k}(c^*, p^*)$ for all $k \geq k_o + 1$ and using (b), there exists $\bar{\mu}_k \geq \bar{\mu} > 0$ such that

$$\delta_2^{k+1} \geq -\delta_1^k + \frac{\bar{\mu}_k}{\eta_k}(\delta_1^k)^2 \geq -\delta_1^k + \frac{\bar{\mu}}{\eta_k}(\delta_1^k)^2 > 0 \quad \text{for all} \quad k \geq k_o + 1 \qquad (1)$$

where $\eta_k = \sum_{t=t_k}^{t_k+N-1} p^*(t) \cdot w(t) > 0$. Since $-\delta_1^k \geq \delta_2^k > 0$ for all $k \geq k_o + 1$, (1) is rewritten as

$$\delta_2^{k+1} \geq \delta_2^k + \frac{\bar{\mu}}{\eta_k}(\delta_2^k)^2 > 0 \quad \text{for all} \quad k \geq k_o + 1 \qquad (2)$$

Define $V_k = \delta_2^k/\eta_k$. For all $k \geq k_o + 1$, $V_k > 0$, and since $\bar{x}^{t_k}(t) \leq w(t)$ for all t and k , $V_k \leq 1$. Then, (2) becomes

$$\eta_{k+1} V_{k+1} \geq \eta_k V_k + \bar{\mu} \eta_k (V_k)^2 > 0 \quad \text{for all} \quad k \geq k_o + 1 \qquad (3)$$

Taking the inverse of the both sides,

$$\frac{1}{\eta_{k+1} V_{k+1}} \leq \frac{1}{\eta_k V_k} - \frac{\bar{\mu}}{1 + \bar{\mu}} \frac{1}{\eta_k}$$

or

$$\frac{1}{\eta_k} \leq \frac{1+\bar{\mu}}{\bar{\mu}} \left[\frac{1}{\eta_k V_k} - \frac{1}{\eta_{k+1} V_{k+1}} \right] \text{ for all } k \geq k_o + 1 \tag{4}$$

Adding (4) from $k = k_o + 1$ to $K - 1$ and cancelling terms,

$$\sum_{k=k_o+1}^{K-1} \frac{1}{\eta_k} \leq \frac{1+\bar{\mu}}{\bar{\mu}} \left[\frac{1}{\eta_{k_o+1} V_{k_o+1}} - \frac{1}{\eta_K V_K} \right] \leq \frac{1+\bar{\mu}}{\bar{\mu}} \left[\frac{1}{\eta_{k_o+1} V_{k_o+1}} \right] < \infty ,$$

and $\sum_{k=k_o+1}^{\infty} \frac{1}{\eta_k} < \infty$.

However,

$$\infty > \sum_{k=k_o+1} \frac{1}{\eta_k} = \sum_{k=k_o+1}^{\infty} \sum_{t=t_k}^{t_k+N-1} \frac{1}{p^*(t)\cdot w(t)} \geq \sum_{k=k_o+1}^{\infty} \sum_{t=t_k}^{t_k+N-1} 1/\|p^*(t)\|\cdot\|w(t)\| .$$

Since $w(t)$ is uniformly bounded by Assumption I, it implies

$\sum_{t=t_{k_o}+1}^{\infty} 1/\|p^*(t)\| < \infty$, which is a contradiction.

Q.E.D.

6. Conclusions and Remarks

In an economy with a generational structure, one would not expect to have the existence of Walras market. Hence, unless there is some "double coincidence of wants" between generations, an equilibrium may be Pareto inefficient in a sequence of markets generated by a generational structure. This is precisely the reason why Cass-Yaari looked into a "closed-loop" model, which is a finite analogue of a stationary generational economy. It seems that they conclude that if money is introduced to the market, a central clearing house, which was lacking without money, was effectively restored because money plays a role of intermediation. As we saw in Section 4, it is incorrect if one extends his attention beyond a stationary economy. It is true, in some cases, that the introduction of money makes a new equilibrium Pareto efficient. But not all monetary equilibrium are Pareto efficient, namely financial intermediation <u>does not always</u> remedy the lack of Walras markets.

As we observed in Section 5, the question of Pareto efficiency of a competitive (barter or monetary) equilibrium is closely related to the question of efficiency prices. The problems seem to be inherent to the infinity of horizon more than anything else.

However, it should be noted that our findings do not refute the claim by Cass-Yaari altogether, for we have no example where there is neither monetary nor barter Pareto-efficient equilibrium. We believe that the following conjecture is true: if an economy cannot find a Pareto efficient equilibrium without intermediation (money), there is a Pareto efficient equilibrium in the same economy with money. To this end, one

must show the existence of a Pareto efficient (barter or monetary) equilibrium. One might hope that a direct application of Arrow-Hahn [1] approach may do just this. Unfortunately, there is some technical obstacle, that is, the utility frontier U^* is not closed in the topology of convergence in coordinates (the most natural topology in this economy).

So, it is an open question if the non-existence of a Pareto efficient barter equilibrium is a sufficient condition for the existence of a Pareto efficient monetary equilibrium. Let us conclude this paper by presenting yet another example, which shows that it is not a necessary condition. Namely, an example which contains both Pareto efficient barter and Pareto efficient monetary equilibria.

There are two goods in the economy in addition to money. In each period, consumers $2t$ and $2t + 1$ are born. We refer the first consumer as an "even" consumer and the second as an "odd" consumer. All the even consumers and all the odd consumers have the same characteristics, respectively, except the date of birth. Namely for an even consumer, $2t$, initial endowment is $\left(w_i(t), w_i(t+1)\right) = (\frac{71}{2}, \frac{1}{2}; \frac{71}{2}, \frac{1}{2})$, $i = 2t$, his utility function

$$u_i = \ln\{\min[c_{i1}(t), c_{i2}(t)]\} + \ln\{\min[c_{i1}(t+1), c_{i2}(t+1)]\}, \; i = 2t .$$

An odd consumer $2t + 1$ has initial endowment

$$\left(w_i(t), w_i(t+1)\right) = (4, 2; 0, 12), \; i = 2t + 1 ,$$

$$u_i = \{4[c_{i1}(t)]^{-1} + [c_{i2}(t)]^{-1}\}^{-\frac{1}{2}} + \{[c_{i1}(t+1)]^{-1} + [c_{i2}(t+1)]^{-1}\}^{-\frac{1}{2}}, \; i = 2t+1 .$$

Consumers 0 and 1 live only in period 1, with

$$w_0(1) = (\frac{71}{2}, \frac{1}{2}) \text{ and } w_1(1) = (0, 12)$$

$$u_0 = \ln \{\min [c_{01}(1), 11c_{02}(1)]\}$$

$$u_1 = \ln \{[c_{11}(1)]^{-1} + [c_{12}(1)]^{-1}\}^{-\frac{1}{2}} .$$

Finally, let $\bar{m}_i = 0$ for all $i \neq 1$ and $\bar{m}_1 = \frac{12}{5}$. The initial endowment allocation is summed up in Table 5. Then there exist at leat two distinct equilibria, one monetary and the other barter.

i \ t	1	2	3	. . .
0	$(\frac{71}{2}, \frac{1}{2})$			
1	(0, 12)			
2	$(\frac{71}{2}, \frac{1}{2})$	$(\frac{71}{2}, \frac{1}{2})$		
3	(4, 2)	(0, 12)		
4		$(\frac{71}{2}, \frac{1}{2})$	$(\frac{71}{2}, \frac{1}{2})$	
5		(4, 2)	(0, 12)	
.		.	.	
.		.	.	
.		.	.	

Table 5

A monetary equilibrium consumption allocation is shown in

Table 6 while prices are $p(t) = (1, 1)$ for all t and $p^m = 1$. A barter equilibrium consumption allocation is shown in Table 7 while prices are $p(t) = (1, 4)$ for all t and $p^m = 0$. Since both allocations satisfy conditions for Theorem 3, both are Pareto efficient.

i \ t	1	2	3	. . .
0	(33, 3)			
1	$(\frac{36}{5}, \frac{36}{5})$			
2	(33, 3)	(33, 3)		
3	$(\frac{9}{5}, \frac{9}{5})$	$(\frac{36}{5}, \frac{36}{5})$		
4		(33, 3)	(33, 3)	
5		$(\frac{9}{5}, \frac{9}{5})$	$(\frac{36}{5}, \frac{36}{5})$	
		⋱	⋱	

Table 6

i \ t	1	2	3	. . .
0	$(\frac{55}{2}, \frac{5}{2})$			
1	(16, 8)			
2	$(\frac{55}{2}, \frac{5}{2})$	$(\frac{55}{2}, \frac{5}{2})$		
3	(4, 2)	(16, 8)		
4		$(\frac{55}{2}, \frac{5}{2})$	$(\frac{55}{2}, \frac{5}{2})$	
5		(4, 2)	(16, 8)	
		⋱	⋱	

Table 7

APPENDIX

We consider the following economy. As before there are m goods and a finite number of consumers in each period t. Let H_t be a set of individuals living at t, i.e., $H_t = \{i; t - N \leq \theta_i \leq t\}$. Each consumer i is characterized by θ_i so that he lives from period θ_i up to period $\theta_i + N$. All the transactions are costless but we assume that transactions can be made only among living generations for the periods they live. Hence in each period t, there are m spot markets for each commodity, m futures markets each for delivery date $\tau = t+1, \ldots, t+N$ and a bond market each for maturization date $\tau = t+1, \ldots, t+N$. Let $x_\tau^i(t)$ be a vector of net purchases that consumer i makes in period t to be delivered at $\tau = t, t+1, \ldots t+N$. Each $x_\tau^i(t) \in E^m$ and we write $\underline{x}_t^i = \{x_t^i(t), x_{t+1}^i(t), \ldots x_{t+N}^i(t)\}$ which is the list of spot and future transactions i made at time t. $\underline{x}_t^i$ is in $E^{m(N+1)}$.

A unit of bond is a contract to deliver one unit of money at the maturization date. Let $x_\tau^{ib}(t) \in E^1$ be a net purchase of bond in period t for delivery at $\tau, \tau > t$ and $\underline{x}_t^{ib} = \{x_{t+1}^{ib}(t), \ldots, x_{t+N}^{ib}(t)\}$ is the bond transactions at t. Let us use infinite sequences of vectors in $E^{m(N+1)}$. x_t^i is defined as $x_t^i = \{0, 0, \ldots, x_t^i(t), \ldots, x_{t+N}^i(t), 0, \ldots\}$ and x_t^{ib} is is defined as $x_t^{ib} = \{0, 0, \ldots, x_{t+1}^{ib}(t), \ldots, x_{t+N}^{ib}(t), 0, \ldots\}$. Let $m_t^i \in E^1$ be the cash balance of consumer i at the end of period t and $m^i = (0, 0, \ldots, m_{\theta_i}^i, \ldots, m_{\theta_i+N}^i, 0, \ldots)$. Let us write the commodity price in period t as $p_t = \{0, 0, \ldots, 0, p_t(t), \ldots p_{t+N}(t), 0, \ldots\}$ where $p_\tau(t) \in E_+^m$ is a vector of prices at t in the future market for delivery at τ, $t \leq \tau \leq t+N$. Also bond price at period t.
$p_t^b = \{0, \ldots, 0, p_{t+1}^b(t), \ldots p_{t+N}^b(t), 0, \ldots\}$ where $p_\tau^b(t) \in E_+^1$ is the

price of bond in t which matures at τ, $\tau > t$. Finally $p_t^m = E_+^1$ is the price of money in t.

A consumption plan for consumer i, $(\underset{\sim}{c}^i, (\underset{\sim}{x}_t^i)_{t=\theta_i}^{\theta_i+N}, (\underset{\sim}{x}_t^{ib})_{t=\theta_i}^{\theta_i+N}, \underset{\sim}{m}^i)$ is *budget feasible* if

a) $\underset{\sim}{c}^i = \underset{\sim}{w}^i + \sum_t \underset{\sim}{x}_t^i \geqq 0$

b) $p_t^m m_{t+1}^i = p_t^m m_t^i - \underset{\sim}{p}_t \underset{\sim}{x}_t^i - \underset{\sim}{p}_t^b \underset{\sim}{x}_t^{ib} + p_t^m \sum_{\tau < t} x_t^{ib}(\tau) \geqq 0$

where $m_{\theta_i - 1}^i = \bar{m}^i$.

A consumption plan for consumer i is *optimal* if it is budget feasible and it is a u^i maximizer (over the set of all budget feasible consumption plan of i).

An equilibrium is a pair of prices and feasible consumtion plans (i.e., budget feasible for each i), namely

$((\underset{\sim}{p}_t, \underset{\sim}{p}_t^b, p_t^m)_{t=1}^{\infty}, (\underset{\sim}{c}^i, (\underset{\sim}{x}_t^i)_{t=\theta_i}^{\theta_i+N}, (\underset{\sim}{x}_t^{ib})_{t=\theta_i}^{\theta_i+N}, m^i)_{i=1}^{\infty})$ such that

a) The consumption plan of each consumer i is optimal,

b) $\sum_i \underset{\sim}{x}_t^i = 0$ for all t,

) $\sum_i \underset{\sim}{x}_t^{ib} = 0$ for all t,

d) $\sum_i m_t^i = \bar{m}$ for all t ($\bar{m}$ is the fixed amount of money).

Proposition: *The equilibrium of the economy described in the Appendix is equivalent to the equilibrium of the economy in the text.*

Proof: $\longrightarrow$: An equilibrium, in the economy described in this Appendix requires either $p_t^m = 0$ for all t or $p_t^m > 0$ for all t. For if

$p_t^m = 0$ and $p_{t-1}^m > 0$ for some t, it would contradict to the monotonicity of u^i and the fact that $\sum_i m_{t-1}^i = \bar{m}^1$. Also if $p_{t-1}^m = 0$ and $p_t^m > 0$ the above conditions are violated. Secondly, the set of feasible plans is homogeneous of degree zero (for p^t, p_t^b and p_τ^m for any t). Therefore we can normalize prices in each period. We can do it by taking $p_t^m = p^m \in E_+^1$ for all t. Next, we shall see that $p_\tau(t) = p_\tau(\tau)$ and $p_\tau^b(t) = p^m$ for all τ, $t \leq \tau$.

$p_\tau(t) \leq p_\tau(\tau)$ and $p_\tau^b(t) \leq p^m$ are obvious, for otherwise consumer i with $\theta_i \leq t \leq \tau \leq \theta_i+N$ can sell the good (bond) at t and obtain the arbitrage profit. Also $p_\tau(t) \geq p_\tau(t+1)$ and $p_\tau^b(t) \geq p_\tau^b(t+1)$ and $p_\tau^b(t-1) \geq p^m$ for all t otherwise by the equilibrium conditions, there exists i with $\theta_i \leq t \leq t+1 \leq \theta_i+N$ and $m_t^i > 0$ who can make arbitrage profit by buying the good (or bond). Proceeding by induction, $p_\tau(t) = p_\tau(\tau)$ and $p_\tau^b(t) = p^m$ for all t and τ.

Given the equilibrium in the Appendix define $\hat{x}_t^i = \sum_{\tau<t} x_t^i(\tau)$ for all i, the consumption plan c with $c = (c^i)_{i=1}^{\infty}$. Then this consumption allocation with normalized prices and p^m are a competitive equilibrium in the economy described in the text.

$\longleftarrow$: Given the equilibrium in the text (p, c), let us find an equilibrium of the economy in the Appendix. Define $\hat{p}_\tau(t) = \hat{p}_\tau$, $\hat{p}_\tau^b(t) = p^m$, $p_t^m = p^m$ for all t and τ. Let $\hat{c}_t^i = c_t^i$ for all i and t.

Assume $\hat{p}^m > 0$, for the case $\hat{p}^m = 0$ can be treated analogously. By the monotonicity of u^i, $p_t \gg 0$ for all t. Hence for any arbitrary index k for good, $p_{kt} > 0$ for all t. Fix the index k. Let $\hat{x}_\tau^{ib}(t) = 0$ for i, τ, t, and let $\hat{x}_{j\tau}^i(t) = 0$ for all i and t, and all $\tau \geq t+1$ if $j \neq k$ and all $\tau \geq t+1$ if $j = k$.

Let $\hat{x}^i_1(1) = x^i_1$ for all i. Let $P_1 \subset H_1$ be the set of households with $\hat{p}_1(1)\hat{x}^i_1(1) - \hat{p}^m_1 \bar{m}^i = \delta^i_1 > 0$ and $N_1 \subset H_1$ be those with $\hat{p}_1(1)\hat{x}^i_1(1) - \hat{p}^m_1 \bar{m}^i = \delta^i_1 \leq 0$. Define $\alpha_1 = \sum_{i \in N_1} \delta^i_1 / \sum_{i \in P_1} \delta^i_1$. α_1 is well defined for $\hat{p}^m \bar{m} = \hat{p}^m \sum_i \bar{m}^i > 0$ by assumption. For each $i \in N_1$, define $\hat{x}^i_{2k}(1) = \delta^i_1 / p_k$ and for each $i \in P_1$, let $\hat{x}^i_{2k}(1) = \alpha_1 \delta^i_1 / p_k$. By construction, $\hat{p}_1 \hat{x}^i_1 - \hat{p}^m_1 \bar{m}^i \leq 0$ and $\sum_i \hat{x}^i_1 = 0$. Define $\hat{m}^i_1 = (\hat{p}^m_1 \bar{m}^i - \hat{p}_1 \hat{x}^i_1)/\hat{p}^m_1$.

Given $\hat{m}^i_t$, $\hat{x}^i_t$ for all i and $t \leq T - 1$, define $\hat{x}^i_T(T) = x^i_T + \hat{x}^i_T(T-1)$. Define P_T as the set of households with $\hat{p}_T(T)\hat{x}^i_T(T) - \hat{p}^m_T \hat{m}^i_{T-1} = \delta^i_T < 0$ and N_T as the set of households with $\hat{p}_T(T)\hat{x}^i_T(T) - \hat{p}^m_T \hat{m}^i_{T-1} = \delta^i_T \geq 0$. Proceeding exactly the same way as before, we can find $\hat{x}^i_T$ and $\hat{m}^i_T$ which satisfies both budget condition and market clearing condition.

By induction, we can construct $\hat{x}^i_t$ and $\hat{m}^i_t$ for all i and t. This is an equilibrium in the Appendix.

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