FOUNDATION AND TECHNIC OF ARITHMETIC

HALSTED

On the Foundation and Technic of Arithmetic

By

George Bruce Halsted

A. B. and A. M., Princeton; Ph. D., Johns Hopkins; F. R. A. S.

Chicago
The Open Court Publishing Company
1912

CONTENTS.

CHAPTER PAGE

Introduction 1

I. The Prehuman Contributions to Arithmetic 3
The natural individual, 3.—The artificial individual, 4.—Primary number, 5.—Our base ten, 7.

II. The Genesis of Number 8

III. Counting and Numerals 10
Correlation, 10.—To count, 11.—The primitive standard sets, 11.—The abacus, 12.—The word-numeral system, 12.—Periodicity, 13.—A partitioned unit, 14.—Number without counting, 14.—Decimal word-numerals, 14.—Invariance of cardinal, 15.

IV. Genesis of our Number Notation 17
Positional counting, 17.—The abacus, 17.—Recorded symbols, 18.—The Hindu numerals, 19.—The zero, 20.—Our present notation, 22.

V. The Two Direct Operations, Addition and Multiplication. 26
Notation, 26.—The symbol =, 26.—Inequality, 27.—Parentheses, 28.—Expressions, 28.—Substitution, 29.—Addition, 29.—Formulas, 32.—Ordinal addition, 33.—Properties of addition, 33.—Multiplication, 35.

VI. The Two Inverse Operations, Subtraction and Division.. 39
Inversion, 39.—Subtraction, 39.—Division, 41.

VII. Technic 44
Addition, 44.—Subtraction, 44.—Multiplication, 45.—Verify multiplication, 46.—Shorter forms, 47.—Division, 47.—Verify division, 48.

VIII. Decimals 49
Decimals, 49.—Product, 52.—Quotient, 53.

IX. Fractions 55
Generalizations of number, 55.—Principle of permanence, 56.—Fractions, 56.—Fractions ordered, 59.—Division of fractions, 61.—Multiplication of fractions, 61.

X. Relation of Decimals to Fractions 63
1st. Decimals into fractions, 63.—2d. Fractions into decimals, 63.—Base, 65.—Change of base, 67.

CHAPTER PAGE

XI. Measurement 68
Why count? 68.—The measure device, 70.—Counting prior to measuring, 70.—New assumptions, 72.

XII. Mensuration 75
Geometry, 75.—Length of a sect. 76.—Length of the circle, 76.—Area, 77.—Volume, 78.

XIII. Order 81
Depiction, 82.—Infinite, 82.—Sense, 82.—Analysis of order, 83.—Ordered set, 84.—Finite ordinal types, 85.—Number series, type of order, 85.—Well-ordered sets, 86.

XIV. Ordinal Number 88
Ordinal number, 88.—Children's counting, 88.—Uses of ordinals, 89.—Nominal number, 92.

XV. The Psychology of Reading a Number 94

XVI. Arithmetic as Formal Calculus 101

XVII. On the Presentation of Arithmetic 110
1st Grade: Previous blunders, 110. — Begin with ordinals, 110. —Cardinal from ordinal, 111. — Ordinal counting, 111. — Cardinal counting, 112.—Number precedes measure, 112.—Cardinal number, 113.—How to begin, 113.—Ordinal games, 114.—The call, 114.—Ordinal operations, 116.—The simplest cardinal, 116.—Triplets and quartets, 117.—The "how many" idea, 117.—Symbols, 117.—Cardinal counting, 117.—Recognition of the cardinal, 117.—Cardinal addition, 118.—Summary, 118.—2d Grade: Measurement, 119.—The decimal, 120.—Carrying, 121.—Subtraction, 121.—Fractions, 122. — Multiplication, 122. — Division, 123. — Summary, 124. — 3d Grade, 125.—4th Grade, 127.—5th Grade, 127.—6th Grade, 128.—7th Grade, 129.

Index 131

INTRODUCTION.

In the French Revolution, when called before the tribunal and asked what useful thing he could do to deserve life, Lagrange answered: "I will teach arithmetic."

Almost invariably now arithmetic is taught by those whose knowledge of mathematics is most meager. No wonder it and the children suffer. In this day of the arithmetization of mathematics and later its logicization, are the beauty, the elegance of arithmetical procedures to remain still unexplained? Is the singular, the lonely precision of this science and art to remain unheralded, unexpounded?

In arithmetic a child may taste the joy of the genius, the joy of creative activity.

Arithmetic is for man an integrant part of his world construction. Thus do his fellows make their world, and so must he. Now this is not by passive apprehension of something presenting itself, but by permeating vitalization spreading life and its substance through what the ignorant teacher would present as the dead mechanism of mechanical computation.

More than in any other science, there has been in mathematics an outburst of most unexpected, most deep-reaching progress. Its results, if made available for the teacher, will revivify this first, most precious of educational organisms; the more so since mathematics is seen to possess of all things the most essential, most fundamental objective reality.

CHAPTER I.

THE PREHUMAN CONTRIBUTIONS TO ARITHMETIC.

Properly to understand or to teach arithmetic, one should have a glimpse of its origin, foundation, meaning, aim.

Arithmetic is the science of number, but for the ordinary school-teacher it is to be chiefly the doctrine of primary natural number, the decimal and later the fraction, and the art of reckoning with them.

Numbers are of human make, creations of man's mind; but they are first created upon and influenced by a basis which comes from the prehuman.

The natural individual.

Before our ancestors were men, they represented to themselves, as do some animals now, the world as consisting of or containing individuals, definite objects of thought, things. They exercised an individuating creative power. In now understanding by *thing* a definite object of thought, conceived as individual, we are using a method of world presentation which served animals before there were any men to serve.

The child's consciousness certainly begins with a sense-blur into which specification is only gradually introduced. At what stage of animal development the vague and fluctuating fusion, which was the world, begins to be broken up into persistently separate entities would be an interesting comparative biologico-psychologic investigation. However that might turn out, yet things, separate objective things, are a gift to man from the prehuman. Yet simple multiplicity of objects present to perception or even to consciousness does not give number. The duck does not count its young. The crow, wise old bird, has no real counting power to help its cunning. The animals' senses may be keener than ours, yet they never give number.

A babe sees nothing numeric. Even an older child may attend to diverse objects with no suggestion from them of number. Sense-perception may be said to have to do with natural individuals, but never, unaided by other mind-act, does it give number.

The artificial individual.

To the animal habit of postulating entities as separate must be added, before cardinal number comes, the human unification of certain of them into one whole, one totality, one assemblage or group or set, one discrete aggregate or artificial individual man-made.

This artificial whole, this discrete aggregate it is to which cardinal number pertains. Thus number rests upon a prehuman basis, yet is not number itself prehuman. Cardinal number involves more than the animal or natural individuals or things. It comes only with a human creation, the creation of artificial individuals, discrete aggregates taken each as an individual, an individual of human make, fleeting perhaps as our thought, transient, yet the necessary substratum for cardinal number. Unification is necessary. The mind must make of the distinct things a whole, a totality. Else no cardinal number.

Now to an educated man a number concept is suggested when a specific simple aggregate of objects is attended to. Not so to any animal, though just the same individual objects be recognized and attended to. The animal has the unity of the natural object or individual, but that unity is not enough. There is needed the new, the artificial, the man-made individuality of the total aggregate. To this artificial individual it is that the cardinal number pertains. There is thus a unity, man-made, of the aggregate of natural individuals, of the set of constituent units. To this unity made of units cardinal number belongs.

Going for quite different articles, or to accomplish entirely different things, may we not help and check memory by fixing in our mind that we are to get *three* things, or that we are to do three things? How man-made, arbitrary, and artificial, this conjoining of acts most diverse into a fleeting unified whole!

Each finger of the left hand is different. A dog might be taught to recognize each as a separate and distinct individual. Only a man can make of all at once an individual which, conceived as a whole, is yet multiple, multiplex, a manifold, fivefold, a five of fingers, a product of rational creation beyond the dog.

Primary number.

A primary cardinal number is a character or attribute of an artificial unit made of natural units. It needs this single individuality and this multiplicity of individuals. The fingered hand has fiveness only if taken as an individual made of individuals.

Number is a quality of a construct. If three things are completely amalgamated, emulsified, like the components of bronze or the ingredients of a cake, there remains no threeness. If some things are in no way taken together the number concept is still inapplicable, we do not see them as a trio.

The animally originated primitive individuals, however complete in their distinctness, have no numeric suggestion. The creative synthesis of a manifold must precede the conscious perception of its numeric quality. The set must be conceived as a whole before discriminated as a dozen. It is only to man-made conceptual unities that the numeric quality pertains. This "number of natural individuals" in an artificial individual is called its *cardinal number* or *cardinal*. The cardinal n of a set s is the class of all sets similar to s.

Primary number would seem in some sense a normal creation of man's mind. No primitive language has ever been investigated without therein finding records of the number idea, unmistakable though perhaps slight, limited, meager, it may be not going beyond our baby stage, one, two, many.

There is a baby stage when no *many* is specialized but *two*. One, two, many, then baby waits how long before that many called *three* is specialized? Numeric *one* as cardinal only comes into existence in contrast with *many*. It involves a distinction between the class whose only member is x, and the thing x itself. The Stoic Chrysippos (282-209 B. C.) spoke of the "aggregate or assemblage one." Number comes when we make a vague *many* specific.

The world-mind rose from the animal to the human when it grouped, aggregated, made wholes of, made artificial individuals of the distinct individual objects previously created by the animal mind. We may see babies recapitulating the race in this.

The number of a particular totality represents the particular multiplicity of its individual elements and nothing more. So far as represented in a number, each natural individual loses everything but its distinctness; all are alike, indistinguishably equivalent. The idea of unity is doubly involved in number, which applies to a unity of a plurality of units. The units are arithmetically identical; not so the complex unities man-made out of collections of the units. To these pertain the differing cardinal numbers.

Our base ten.

In our developed number systems certain *manys* take on a peculiar prominence, are of basal character. Of these ten has now permanently the upper hand.

What is the origin of this preeminence?

Its origin is prehuman. Our system is decimal, not because ten is scientifically, arithmetically a good base, a superior number, but solely because our prehuman ancestors gave us five fingers on each of two hands.

CHAPTER II.

THE GENESIS OF NUMBER.

Cardinals.

In nature, distinct things are made and perceived as individual. Each distinct thing is a whole by itself, a qualitative whole. The individual thing is the only whole or distinct object in nature. But the human mind takes individuals together and makes of them a single whole of a new kind, and names it. Thus we have made the concept a flock, a herd, a bevy, a covey, a genus, a species, a bunch, a gang, a host, a class, a family, a group, an array, a crowd, a party, an assemblage, an aggregate, a manifold, a throw, a set, etc. These are artificial units, discrete magnitudes; the unity is wholly in the concept, not in nature; it is artificial. We constitute of certain things an artificial individual when we distinguish them collectively from the rest of the world, making out of subsidiary individuals a single thing, a system, of which each component is recognizable as distinct from all others. From the contemplation of the natural individual or element in relation to the artificial individual, the group, spring the related ideas "many" and "one." We must have numeric many before we can have cardinal one. A natural qualitative unit thought of in contrast to a "many" as *not-many* gives the idea "one" as cardinal. An aggregate may contain only a single element. Thus we have a set containing an element with which every element is identical. So we get "one." A unity, a "many" composed of a "one" and another "one" is characterized as *two.*

The unity, the "many" composed of "one" and the special many "two" is characterized as *three.*

Among the primitive ideas of cardinal number, the idea of "two" is the first to be formed definitely. There are ever present doublets, things which can be grasped in pairs. This two is the very simplest many, the simplest recognized form of plurality. It is incalculably simpler than three, as witness whole savage tribes whose spoken number system is "one, two, many"; as witness the mind-wasting primitive stupidities of the dual number in Greek grammar.

The special many, a one made of three, a trinity, a trio, triplets, here is an advance. When to the grasp of the pair, the dominance over the trio is added, when the three is created, then after-progress is rapid.

With a couple of pairs goes four; with a couple of threes, six. A hand represents five coming in between four and six. A pair of hands says ten. A pair of tens is twenty, a score. A pair of fours is eight. A trio of threes is nine. A pair of sixes or a trio of fours is twelve, a dozen.

Arithmetic flowers like a rocket. That seven is left out, is missed, makes it the sacred, the mystic number of superstition. To numbers, however complicated their genesis, is finally ascribed a certain objective reality. In our mind the number concepts finally become simple things, objectively real.

CHAPTER III.

COUNTING AND NUMERALS.

Correlation. The ability of mind to relate things to things, to correlate, to represent something by something else, to make or perceive a correspondence between things or thought creations is fundamental, essential, necessary.

The operation of establishing such a correspondence between two sets that every thing or element of each set is mated with, paired with, just one particular thing or element of the other, is called establishing a one-to-one correspondence between the sets. Two sets which can be so mated are said to be *equivalent* as regards plurality, or to have the same *potency*. Two sets equivalent to the same are equivalent to each other, their elements correlated to the same element being thereby mated. Two sets between which a one-to-one relation exists have the same cardinal number and are said to be cardinally similar.

A set's cardinal number is what is common to the set and every equivalent set. Thus a set's cardinal is independent of every characteristic or quality of any element beyond its distinctness. To find the cardinal of a set, we count the set.

Counting is the establishing of a one-to-one correspondence of aggregates, one of which belongs to a well-known series of aggregates. If a group of things have this correspondence with this standard group, then those properties of this standard group which are carried over by the correspondence will belong to the new group. They are properties of the group's cardinal number.

To count.

To count an aggregate, an artificial individual, is to identify it as to numeric quality with a familiar assemblage by setting up a one-to-one correspondence between the elements of the two groups. Thus counting consists in assigning to each natural individual of an aggregate one distinct individual in a familiar set, originally a group of fingers, now usually a set of words or marks. So counting is essentially the numeric identification, by setting up a one-to-one correspondence, of an unfamiliar with a familiar group. Thus it ascertains, it fixes the nature of the less familiar through the preceding knowledge of the more familiar.

The primitive standard sets.

Primitively the known groups were the groups of fingers. The fingers gave the first set of standard groups and formed the original apparatus for counting, and served for the symbolic transmission of the concepts, the number ideas generated. More than that, this finger counting gave the names of the numbers, the numeric words so helpful in the further development of numeric creation. The name of a number, when referring to an artificial unit, as of sheep, denoted that a certain group of fingers would touch successively the natural units in the discrete magnitude indicated, or a certain finger would stand as a symbol for the numerical characteristic of that group of natural units.

Our word "five" is cognate with the Latin *quinque,* Greek *pente,* Sanskrit *pancha,* Persian *pendji*; now in Persian *penjeh* or *pentcha* means an outspread hand.

In Eskimo "hand me" is *tamuche*; "shake hands" is *tallalue*; "bracelet" is *talegowruk*; "five" is *talema.*

In the language of the Tamanocs of the Orinoco, five means "whole hand"; six is "one of the other hand"; and so up to ten or "both hands."

Philology confirms that the original counting series or outfit was the series of sets of fingers, and this primitive method preceded the formation of numeral words. The use of visible signs to represent numbers and aid reckoning is not only older than writing, but older than the development of numerical language. In very many languages the counting words come directly and recognizably from the finger procedure.

But of the fingers there are only a few distinct aggregates, only ten. Developing man needs more, needs to enlarge and extend his standards.

The abacus.

The Chinese, even at the present day, extend the series of primary groups, the finger-groups, by substituting groups of counters movably strung on rods fixed in an oblong frame. With this *abacus,* which they call *shwanpan,* reckoning board, and the Japanese call *soroban,* they count and perform their arithmetical calculations.

The word-numeral system.

In many languages there are not even words for the first ten groups. Higher races have not only named these groups, but have extended indefinitely this system of names. They no longer count directly with their fingers, but use a series of names, so that the operation of counting an assemblage of things consists in assigning to each of them one of these numeral words, the words being always taken in order, and none skipped, each word being thus capable of representing not merely the individual with which it is associated, but the entire named group of which this individual is the last named.

In making this series of word-numerals, there is evidently need for a system of periodic repetition. The prehuman fixes five, ten, or twenty as the number after which repetition begins. Of these, ten has become predominant. Thus come our word-numerals, each applicable to just one of a counted set and to the aggregate ending with this one. This dekadic word-system makes easy, with a simple, a light numerational equipment, the perfectly definite expression of any number, however advanced.

Periodicity.

So for us to count is to assign the numerals one, two, three, etc., successively and in order, to all the individual objects of a collection, one to each. The collection is said to be given in number, the number of things in it, by the cardinal number signified by the numeral assigned to the last natural unit or component of the collection in the operation of counting it. Numerals are also called numbers. The numeral and a word specifying the kind of objects counted make what is called a concrete number. In distinction from this, a number is called an abstract number.

When children are to count, the things should be sufficiently distinct to be clearly and easily recognizable as individual, yet not so disparate as to hinder the human power to make from them an artificial individual. The objects should not be such as to individually distract the attention from the assemblage of them.

With little children use a binary system. Build with twos. Then go on, as did the Romans, to a quinary-binary system, which suits counting on the fingers.

In counting, an artificial individual may take the place of a natural individual. Children enjoy counting by fives. Inversely, a unit may be thought of as an artificial individual, composed of subsidiary individuals, as a dollar of 100 cents.

A partitioned unit.

An interesting exercise is the instantaneous recognition of the cardinal, the particular numeric quality of the collection, its specification without counting. But this power to picture all the separate individuals and to recognize the specific given picture is very limited. If it be attempted to facilitate this recognition by arrangement, the recognition may easily become that of form instead of number. It is then simply recognizing a shape which we know should have just so many elements. Every teacher should remember when using blocks in developing the number-concept that only if very few can their number be perceived without the help of counting or addition. If 4 blocks lie close their number may be perceived immediately, but seven are dealt with as two groups. It is believed that the limit, even for adults and under favorable conditions, is about 4. We know that even IIII was replaced by IV. Try the children to see if their primitive number perception, that of II, has grown, and how far.

Number without counting.

In the making of numeral words it is necessary to fix upon one after which repetition is to begin. Otherwise there would be no end to the number of different words required. We have noted that the prehuman has narrowed the choice, by the fiveness of the extremities of mammalian limbs, to five, ten or twenty. The majority of races, especially the higher, in prehistoric time chose ten, the number of our fingers. Then was developed a system to express

Decimal word-numerals.

by a few number-names a vast series of numbers. If we interpret eleven as "one and ten" and twelve as "two and ten," *teen* as "and ten," *ty* as "tens," then English, until it took "million," ("great thousand," Latin *mille,* a thousand,) bodily from the French and Italian, used only a dozen words in naming numbers, in making a series of word-numerals with fixed order.

The systematic formation of numerical words is called *numeration.*

The cardinal number of any finite set of things is the same in whatever order we count them.

Invariance of cardinal. This is so fundamental a theorem of arithmetic, it may be well to make its realization more intuitive.

That the number of any finite group of distinct things is independent of the order in which they are taken, that beginning with the little finger of the left hand and going from left to right, a group of distinct things comes ultimately to the same finger in whatever order they are counted, follows simply from the hypothesis that they are distinct things. If a group of distinct things comes to, say, five when counted in a certain order, it will come to five when counted in any other order.

For a general proof of this, take as objects the letters in the word "triangle," and assign to each a finger, beginning with the little finger of the left hand and ending with the middle finger of the right hand. Each of these fingers has its own letter, and the group of fingers thus exactly adequate is always necessary and sufficient for counting this group of letters in this order.

That the same fingers are exactly adequate to touch this same group of letters in any other order, say the alphabetical, follows because, being distinct, any pair attached to two of my fingers in a certain order can also be attached to the same two fingers in the other order.

In the new order I want *a* to be first. Now the letters *t* and *a* are by hypothesis distinct. I can therefore interchange the fingers to which they are assigned, so that each finger goes to the object previously touched by the other, without using any new fingers or setting free any previously employed. The same is true of *r* and *e*, of *i* and *g*, etc.

As I go to each one, I can substitute by this process the new one which is wanted in its stead in such a way that the required new order shall hold good behind me, and since the group is finite, I can go on in this way until I come to the end, without changing the group of fingers used in counting, that is without altering the cardinal number, in this case 8.

The group of fingers exactly adequate to touch a group of objects in any one definite order is thus exactly adequate for every order. But when touching in one definite order each finger has its own particular object and each object its own particular finger, so that the group of fingers exactly adequate for one peculiar order is always necessary and sufficient for that one order. But we have shown it then exactly adequate for every order; therefore it is necessary and sufficient for every order.

CHAPTER IV.

GENESIS OF OUR NUMBER NOTATION.

Positional counting.

The systematic decimal system in accordance with which, even in the times of our prehistoric ancestors, a few number names were used to build all numeral words, is paralleled by the procedure, even at the present day, of those Africans who in counting use a row of men as follows: The first begins with the little finger of the left hand, and indicates, by raising it and pointing or touching, the assignment of this finger as representative of a certain individual from the group to be counted; his next finger he assigns to another individual; and so on until all his fingers are raised. And now the second man raises the little finger of his left hand as representative of this whole ten, and the first man, thus relieved, closes his fingers and begins over again. When this has been repeated ten times, the second man has all his fingers up, and is then relieved by one finger of the third man, which finger therefore represents a hundred; and so on to a finger of the fourth man, which represents a thousand, and to a finger of the fifth man, which represents a myriad (ten thousand).

The abacus.

An advance on this actual use of fingers with a positional value depending only on the man's place in the row, is seen in the widely occurring *abacus,* a rough instance of which is just a row of grooves in which pebbles can slide. With most races, as with the Egyptians, Greeks, Japanese, the grooves or columns are vertical, like a row of men. The counters in the right-most column correspond to the fingers of the man who actually touches or checks off the individuals counted; it is the units column.

But in the abacus a simplification occurs. One finger of the second man is raised to picture the whole ten fingers of the first man, so that he may lower them and begin again to use them in representing individuals. Thus there are two designations for ten, either all the fingers of the first man or one finger of the second man. The abacus omits the first of these equivalents, and so each column contains only nine counters.

For purposes of counting, a group of objects can be represented by a graphic picture so simple that it can be produced whenever wanted by just making a mark for each distinct object. Thus the marks I, II, III, IIII, picture the simplest groups with a permanence beyond gesture or word; and for many important purposes, one of these stroke-diagrams, though composed of individuals all alike, is an absolutely perfect picture, as accurate as the latest photograph, of any group of real things no matter how unlike.

Recorded symbols.

The ancient Egyptians denoted all numbers under ten by the corresponding number of strokes; but with ten a new symbol was introduced. The Romans regularly used strokes for numbers under five, using V for five. The ancient Greeks and Romans both however indicated numbers by simple strokes as high as ten. The Aztecs carried this system as high as twenty, but they used a small circle in place of the straight stroke. I have seen the same thing done in Japan.

Each stroke of such a picture-group may be called a unit. Each group of such units will correspond always to the same group of fingers, to the same numeral word.

Though to this primitive graphic system of number-pictures there is no limit, yet it soon becomes cumbrous.

The Hindu numerals.

Abbreviations naturally arise. Those the world now uses, the Hindu numerals, have been traced back to inscriptions in India probably dating from the early part of the second century B. C.

The oldest inscription using them positionally with local value and developed form is of 595 A. D. The Egyptians had no positional notation for number, though they had a hieroglyph for nothing, which they substituted for one side when applying their formula for a quadrilateral to a triangle. The Babylonians had a sign of this kind, not used in calculation, consisting of two angular marks, one above the other. About A. D. 130, Ptolemy in Alexandria used, in his Almagest, the Babylonian sexagesimal fractions, and designated voids by the first letter of the word *οὐδέν*, nothing. This letter was not used as a zero.

M. F. Nau gives in French translation in *Journal asiatique,* Vol. 16 (10th series), 1910, pp. 225-227, a quotation from Severus Sebokt, of Quennesra, on the Euphrates, near Diarbekr, written in 662 A. D., more than two centuries before the earliest known appearance of the numerals in Europe:

"I refrain from speaking of the science of the Hindus, who are not Syrians, of their subtile discoveries in this science of astronomy—more ingenious than those of the Greeks and even of the Babylonians—and of their facile method of calculating and computing, which surpasses words. I mean that made with nine symbols."

But probably a long time was yet to pass before the creation of the most useful symbol in the world, the naught, the zero, not merely a sign for nothing, but a mark for the absence of quantity, the cipher, whose first known use in ring form in a document is in 738 A. D.*

The zero.

This little ellipse, picture for airy nothing, is an indispensable corner-stone of modern civilization. It is an Ariel lending magic powers of computation, promoting our kindergarten babies at once to an equality with Cæsar, Plato or Paul in matters arithmetical.

The user of an abacus might instead rule columns on paper and write in them the number of pebbles or counters. But zero, 0, shows an empty column and so at once relieves us of the need of ruling the columns, or using the abacus. Modern arithmetic comes from ancient counting on the columns of the abacus, immeasurably improved by the creation of a symbol for an empty column.

The importance of the creation of the zero mark can never be exaggerated. This giving to airy nothing not merely a local habitation and a name, a picture, a symbol, but helpful power, is characteristic of the Hindu race whence it sprang. It is like coining the Nirvana into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power. From the second half of the eighth century Hindu writings were current at Bagdad. After that the Arabs knew positional notation. They called the zero *çifr*. The Arab word, a substantive use of the adjective *çifr* ("empty"), was simply a translation of the Sanskrit name *śūnya,*

* E. C. Bayley, 1882. Doubted by G. F. Hill, 1910, who substituted an inscription of 876 A. D.

literally "empty." It gave birth to the low-Latin *zephirum* or *zefirum* (used by Leonard of Pisa, 1202), whence the Italian form *zefiro,* contracted to *zefro,* and (1307) *zeuero,* then *zero,* whose introduction in print goes back to the 15th century (1491).

In the oldest known French treatise on algorithm (author unknown, of the thirteenth century) we read, "iusca le darraine ki est appellee *cifre* 0." In the thirteenth century in Latin the word *cifra* for "naught" is met in Jordan Nemorarius and in Sacrabosco who wrote at Paris about 1240.

In MS. Egerton 2622, one of the earliest arithmetics in our language, on leaf 137*b,* we read:

"Nil cifra significat sed dat signare sequenti.

"Expone this verse. A cifre tokens noyt, bot he makes the figure to betoken that comes aftur hym more than he schuld & he were away, as thus 10. here the figure of one tokens ten. it may happe aftur a cifre schuld come a nothur cifre, as thus 200."

Maximus Planudes (1330) uses *tziphra.* Euler used (1783) in Latin the word *cyphra.* We still say "cipher" or "cypher." In German *Ziffer* has taken a more general meaning, as has the equivalent French word *chiffre,* the most important numeral coming to mean any. The oldest coin positionally dated is of 1458.

Zero, originally the sign of a blank or nil or vacant column, may be looked upon as indicating that a class is void, containing no object whatever, that it is the null class. Thus it is one of the answers to the question, "How many?", and so is a cardinal. It is also given a place in the ordinal series of natural numbers, and is chief in the series of algebraic numbers. Only in the sixteenth century does naught appear as common symbol for all differences in which minuend and subtrahend are equal, and thus show itself as ready for its second great application, to standardize algebraic forms.

By the first meaning of cipher, "empty," we have $20 =$ twain ten, but $2+0=2$. Hankel, 1867, calls *modulus of an operation* that which combined by the operation with something leaves this unchanged. So to-day we use nine digits and have no digit corresponding to the Roman X, for X is all the fingers of the first man, while we, like the abacus, use 10, which is one finger of the second man. Thus the ten, hundred, thousand are only expressed by the position of the number which multiplies them.

In the written numeral IIII, we still see in the symbol the units of which the fourfold unit four is composed. Later abbreviation veils the constituent units, but their independence and all-alike-ness remain fundamental, giving to cardinal number its independence of the order in which the things are enumerated.

Our present notation.

The use of the digits (Latin, *digitus,* a "finger"), the substitution of a single symbol for each of the first nine picture-groups, and that splendid creation of the Hindus, the zero, 0, naught, cipher, made possible our present notation for number. This still has a base, ten, in which the sins of our fathers, the mammals, are visited on their children. Its perfection is in its use of position with digits and zero, a positional notation for number, which the decimal point (or unital point) empowers to run down below the units, giving the indispensable *decimals.*

This positional notation for number consists in the very refined artifice of representing every number as a sum of terms expressed by a row of digits each standing for a product of two factors, one factor the *intrinsic,* the face factor, indicated by the digit itself, the other factor, the *local,* the place factor, indicated by the place of this digit in the row, the local factor being a power of the base, for units' place, or column b^0 or one, for the next place to the left b^1 or b (the base), to the right b^{-1} or $1/b$, etc. The summation of these binary products is indicated by the juxtaposition in the row of the digits representing them by their form and their place in the row with reference to units' place.

Calculus, (Latin, "a pebble"), ciphering, which thus by the aid of zero attains an ease and facility which would have astounded the antique world, consists in combining given numbers according to fixed laws to find certain resulting numbers.

Teaching is to enable the ordinary child to do what the genius has done untaught.

A Hindu genius created the zero. The common, even the stupid, child is now to be taught to understand and use this wonderful creation just as it is taught to use the telephone. So the teacher incites, provokes the self-activity of the child's mind and guides it and confirms it, stopping this kaleidoscope at a certain turn, when the evershifting picture is near enough for life to the picture in the teacher's mind.

Without theory, no practice, yet need not the theory be conscious. There is a logic of it, yet the child need not necessarily know, had perhaps better not know, that logic. The teacher should know, the child practise.

It is striking to realize the centuries that passed after the present system of number-naming, numeration, had been developed, before it had analogous, adequate symbolization, adequate written notation.

As compared with their number-names, how bungling the Greek and Roman numerals, how arithmetically helpless the men of classic antiquity for lack of just one written symbol, the Hindu naught, giving us a written system which, except for its base ten, seems to be final and for all time, a world sign-language more perspicuous and compendious than any word-language. That prehuman parasite, the ten, is fixed on us like an Old Man of the Sea, else we could take the easily superior base twelve. The number of digit figures required is one less than the base; since 10 represents the base, whatever it be.

In each case the prebasal figures, by help of the zero, always express as written in succession to left or to right of the units place (fixed by the unital point) multiples of ascending and descending powers of the base. But while the two and six of twelve are like the two and five of ten, yet twelve has three and four besides as divisors, as submultiples, for which tremendous advantage ten offers no equivalent whatsoever. The prehuman imposition of ten as base, disbarring twelve, is thus a permanent clog on human arithmetic.

The mere numerals, 1, 2, 3,.... or the numeral words, "one," "two," "three,".... are signs for what are called "natural numbers," or positive integers. Integer with us shall always mean positive integer. If pure numbers, integers, have an intrinsic order, so do these, their symbols.

The unending series, 1, 2, 3, 4, 5,.... or one, two, three, four, five,.. ..is called the "natural scale," or the scale of the natural numbers, or the number series. Each symbol in it, besides its ordinal, positional significance in the sequence of symbols, is used also to indicate the cardinal number of the symbols in the piece of the scale it ends, and so of any group correlated to that piece.

Thus the ordinal system is the original from which the cardinal system is derived.

In the primary ordinal system the symbols refer to the individual objects, while in the derived cardinal system these same symbols refer to the successively larger sets whose names are determined as the name of the last individual counted ordinally.

CHAPTER V.

THE TWO DIRECT OPERATIONS, ADDITION AND MULTIPLICATION.

Notation.

The symbolic representation of numbers and ways of combining numbers comes under the head of what is called *notation.*

The natural numbers, as shown in the primitive numeral pictures, I, II, III, IIII, begin with a single unit, and, cardinally considered, are changed to the next always by taking another single unit.

The symbol =.

A number, an integer, is said to be *equal* to, or the same as, a number otherwise expressed, when their units being counted come to the same finger, the same numeral word. The symbol =, read *equals,* is called the sign of equality, and takes the part of verb in this symbolic language. It was invented by an Englishman, Robert Recorde, replacing in his algebra, *The Whetstone of Witte,** the sign z used for equality in his arithmetic, *The Grounde of Artes,* 1540. Equality is a relation reflexive, symmetric, invertible. Equality is a mutual relation of its two members. If $x=y$, then $y=x$. Equality is a transitive relation. If $x=y$ and $y=z$, then $x=z$. A symbolic sentence using this verb is called an equality.

Ordinally, $x=y$ means that x and y denote the same

* London (no date, preface 1557).

number in the natural scale. Formally, $x=y$ means that either can at will be substituted for the other anywhere.

Inequality. When the process of counting the units of one number simultaneously one-to-one with units of a second number ends because no unit of the second number remains uncounted, but the units of the first number are not all counted, then the first number is said to contain more units than the second number, and the second number is said to contain less units than the first.

If a number contains more units than a second, it is called *greater* than this second, which is called the *lesser*. By successively incorporating single units with the lesser of two primitive numbers we can make the greater.

Thomas Harriot* (1560-1621), tutor to Sir Walter Raleigh and one of "the three magi of the Earl of Northumberland," devised the symbol $>$, published 1631, read "is greater than," and called the sign of inequality. Inequality is a sensed relation. Turned thus $<$ its symbol is read "is less than." Inequality in the same sense is transitive. If $x>y$ and $y>z$, then $x>z$.

Since the result of counting is independent of the order of the individuals counted, therefore of two unequal natural numbers the one once found greater is always the greater. Without knowing the number n, we can write "either $n>5$, or $n=5$, or $n<5$." Any number which succeeds another in the natural scale is greater than this other. Ordinally, $x<y$ means that x precedes y in the scale.

* Harriot was sent to America by Raleigh in the year 1585. He made the first survey of Virginia and North Carolina, the maps of these being subsequently presented to Queen Elizabeth. He started the standardizing of algebraic forms and the theory of functions by writing every equation as a function equal to zero.

When by any definite process we select one or more elements of any aggregate A, these form another aggregate B, called a *part* of A. If any element of A remains unselected, B is called a *proper part* of A. It is possible for an aggregate to be equivalent to a proper part of itself; the aggregate is then called infinite. For example: for every number there is an even number; again, for every point on a foot there is a point on an inch.

When we can get a third number from two given numbers by a definite operation, the two given numbers joined by the sign for the operation and enclosed in parentheses may be taken to mean the result of that combination. The result can now be again combined with another given number, and so we may get combinations of several numbers though each operation is performed only with two.

Parentheses.

Parentheses indicate that neither of the two numbers enclosed, but only the number produced by their combination, is related to anything outside the parentheses.

Parentheses (first used by the Flemish geometer Albert Girard in 1629) may without ambiguity be omitted:

First, When of two operations of like rank the preceding (going from left to right) is to be first carried out;

Second, When of two operations of unlike rank the higher is the first to be carried out.

The representation of one number by others with symbols of combination and operation is called an expression. By enclosing it in parentheses, any expression however complex in any way representing a number, may be operated upon as if it were a single symbol of that number. If an expression already involving parentheses is enclosed in parentheses,

Expressions.

each pair, to distinguish it, can be made different in size or shape. The three most usual forms are the parenthesis (, the bracket [, and the brace {. In translating the expression into English, (should be called first parenthesis, and) second parenthesis; [first bracket,] second bracket; { first brace, } second brace.

Substitution. No change of resulting value is made in any expression by substituting for any number its equal however expressed. From this it follows that two numbers each equal to a third are equal to one another. This process, putting one expression for another, substitution, is a primitive yet most important proceeding. A single symbol may be substituted for any expression whatever.

Permutation consists in a simultaneous carrying out of mutual substitution, interchange. Thus *a* and *b* in an expression, as *abc*, are permuted when they are interchanged, giving *bac*. More than two symbols are permuted when each is replaced by one of the others, as in *abc* giving *bca* or *cab*.

Addition. Suppose we have two natural numbers written in their primitive form, as III and IIII; if we write all these units in one row we indicate another natural number; and the process of getting from two numbers the number belonging to the group formed by putting together their groups to make a single group is called *addition*. This operation of incorporating other units into the preceding diagram is indicated by a symbol first met in print in the arithmetic by John Widman, (Leipsic, 1489), a little Greek cross, +, read plus.

If one artificial individual be combined with another to give a new artificial individual in which each unit of the components appears retaining its natural independence and natural individuality, while the artificial individuality of the two components vanishes, the number of the new artificial individual is called the *sum* of the numbers of the two components, and is said to be obtained by *adding* these two numbers (the terms or summands). The first of two summands may be called the augment; the second, the increment. The sum of two numbers, two terms, is the numeric attribute of the total system constituted of two partial systems to which the two terms respectively pertain.

In the child as in the savage, the number idea is not dissociated from the group it characterizes. But education should help on the stage where the number exists as an independent concept, say the number five with its own characteristics, its own life. Therefore we have number-science, pure arithmetic. So though it might perhaps be argued that there is only one number 5, yet we may properly speak of combining 5 with 5 so as to retain the units unaffected while the fiveness vanishes in the compound, the sum, 10.

Addition is a taking together of the units of two numbers to constitute the units of a third, their sum. This may be obtained by a repetition of the operation of forming a new number from an old by taking with it one more unit; thus $3+2=3+1+1$.

If given numbers are written as groups of units, e. g. (*exempli gratia*), $2=1+1$, $3=1+1+1$, the result of adding is obtained by writing together these rows of units, e. g., $2+3=(1+1)+(1+1+1)=1+1+1+1+1=5$.

Since cardinal number is independent of the order of counting, therefore in any natural number expressed in its primitive form, as IIII, the permutation of any pair of units produces neither apparent nor real change.

The units of numeration are completely interchangeable. Therefore we may say adding numbers is finding one number which contains in itself as many units as the given numbers taken together.

In defining addition, we need make no mention of the order in which the given numbers are taken to make the sum. A sum is independent of the order of its parts or terms. This is an immediate consequence of the theorem of the invariance of the number of a set. For a change in the order of the parts added is only a change in the order of the units, which change is without influence when all are counted together.

To write in symbols, in the universal language of mathematics, that addition is an operation unaffected by permutation of the order of the parts added, though applied to any numbers whatsoever, we cannot use numerals, since numerals are always absolutely definite, particular. If, following Vieta's book of 1591, we use letters as general symbols to denote numbers left otherwise indefinite, we may write a to represent the first number not only in the sum $2+3$, but in the sum $4+1$ and in the sum of any two numbers. Taking b for a second number, the symbolic sentence $a+b=b+a$ is a statement about all numbers whatsoever. It says, addition is a *commutative* operation.

The words *commutative* and *distributive* were used for the first time by F. J. Servois in 1813.

The previous grouping of the parts added has no effect upon the sum. Brackets occurring in an indicated sum may be omitted as not affecting the result. The general statement or formula $(a+b)+c=a+(b+c)$ says, addition is an *associative* operation, an operation having associative freedom.

Rowan Hamilton in 1844 first explicitly stated and named the associative law. For addition it follows from the theorem of the invariance of the number of a group.

Formulas. Equalities having to do only with the very nature of the operations involved, and not at all with the particular numbers used are called *formulas*.

A formula is characterized by the fact that for any letter in it any number whatsoever may be substituted without destroying the equality or restricting the values of any other letter. In a formula a letter as symbol for any number may be replaced not only by any digital number, but also by any other symbol for a number whether simple or compound, in the last case bracketed. Thus $a+b=b+a$ gives $(a+c)+b=b+(a+c)$. So from a formula we can get an indefinite number of formulas and special numerical equations.

Each side or member of a formula expresses a method of reckoning a number, and the formula says that both reckonings produce the same result. A formula translated from symbols into words gives a rule. As equality is a mutual relation always invertible, a formula will usually give two rules, since its second member may be read first.

By definition, from the inequality $a > b$ we know that a could be obtained by adding units to b. Calling this unknown group of units n, we have $a=b+n$.

Inversely, if $a=b+n$ then $a > b$, that is a sum of finite natural numbers is always greater than one of its parts. A sum increases if either of its parts increases.

Addition may also be defined and its properties established from the ordinal view-point.

Ordinal addition.

Start from the natural scale. To add 1 to the number x is to replace x by the next following ordinal. So if we know x, we know $x+1$.

When we have defined adding some particular number a to x, when we have defined the operation $x+a$, the operation $x+(a+1)$ shall be defined by the formula

$$(1) \ldots\ldots x+(a+1)=(x+a)+1.$$

We shall know then what $x+(a+1)$ is when we know what $x+a$ is, and as we have, to start with, defined what $x+1$ is, we thus have successively and "by recurrence" the operations $x+2$, $x+3$, etc.

The sum $a+b$ is thus defined ordinally as the bth term after the ath.

It serves to represent conventionally a new number univocally deduced by a definite given procedure from the numbers summed or added together.

Associativity: $a+(b+c)=(a+b)+c$.

Properties of addition.

This theorem is by definition true for $c=1$, since, by formula (1), $a+(b+1)=(a+b)+1$. Now supposing the theorem true for $c=y$, it will be true for $c=y+1$. For supposing

$$(a+b)+y=a+(b+y),$$

it follows that

$$(2) \ldots\ldots [(a+b)+y]+1=[a+(b+y)]+1,$$

which is only adding one to the same number, to equal numbers.

Now by definition (1) the first member of this equation (2)

$$[(a+b)+y]+1=(a+b)+(y+1) \ldots\ldots (3),$$

as we recognize that it should be, since y is the number preceding $y+1$.

But by the same formula (1), read backward, the second member of equation (2)

$$[a+(b+y)]+1=a+[(b+y)+1] \ldots . (4),$$

as we see it should be, since $b+y$ is the number preceding $b+y+1$. But again by (1), the second member of (4),

$$a+[(b+y)+1]=a+[b+(y+1)] \ldots . (5).$$

Therefore [by (5), (4) and (3)], (2) may be written,

$$a+[b+(y+1)]=(a+b)+(y+1).$$

Hence the theorem is true for $c=y+1$.

Being true for $c=1$, we thus see successively that so it is for $c=2$, for $c=3$, etc.

This is a proof by *mathematical induction* or demonstration by recurrence, a procedure first explicitly used, although without a general enunciation, by Maurolycus in his work, *Arithmeticorum libri duo* (Venice, 1575).

Commutativity: 1° .... $a+1=1+a$.

This theorem is identically true for $a=1$.

Now we can verify that if it is true for $a=y$ it will be true for $a=y+1$; for then

$$(y+1)+1=(1+y)+1=1+(y+1)$$

by associativity. But it is true for $a=1$, therefore it will be true for $a=2$, for $a=3$, etc.

2° .... $a+b=b+a$.

This has just been demonstrated for $b=1$; it can be verified that if it is true for $b=x$, it will be true for $b=x+1$. For, if true for $b=x$, then we have by hypothesis $a+x=x+a$; whence, by formula (1), by 1° and associativity, $a+(x+1)=(a+x)+1=(x+a)+1=x+(a+1)$ $=x+(1+a)=(x+1)+a$.

The proposition is therefore established by recurrence.

Sums in which all the parts are equal frequently occur. Such additions are often laborious and liable to error.

Multiplication. But such a sum is *determined* if we know one of the equal parts and the number of parts. The operation of combining these two numbers to get the result is called *multiplication*; the result is then called the *product*. The part repeated is called the *multiplicand*, and the number which indicates how often it occurs is called the *multiplier*. Multiplicand and multiplier are each *factors* of the product. Such a product is a *multiple* of each of its factors. In forming such a product, the multiplicand is taken once as summand for each unit in the multiplier. More generally, *a product is the number related to the multiplicand as the multiplier to the unit.*

Following Wm. Oughtred (1631), we use the sign × to denote multiplication, writing it before the multiplier but after the multiplicand. Thus 1×10, read one multiplied by ten, or simply one by ten, stands for the product of the multiplication of 1 by 10, which by definition equals 10. The multiplication sign may be omitted when the product cannot reasonably be confounded with anything else, thus $1a$ means $1\times a$, read one by a, which by definition equals a.

From our definition also $a\times1$, that is a multiplied by 1, must equal a.

Commutativity. Multiplication of a number by a number is commutative.

Multiplier and multiplicand may be interchanged without altering the product.

1 1 1 1 1
1 1 1 1 1
1 1 1 1 1

For if we have a rectangular array of a rows each containing b units, it is also b columns each containing a units.

Therefore $b \times a = a \times b$.

Taking apposition to mean successive multiplication, for example, $abcde = \{[(ab)c]d\}e$, calling the numbers involved *factors*, and the result their *product*, we may prove that commutative freedom extends to any or all factors in any product.

For changing the order of a pair of factors which are next one another does not alter the product. $abcd = acbd$.

a a a a a
a a a a a
a a a a a

For c rows of a's, each row containing b of them, is b columns of a's, each containing c of them. So c groups of ab units comes to the same number as b groups of ac units.

This reasoning holds no matter how many factors come before or after the interchanged pair. For example

$$abcdefg = abc\ ed\ fg,$$

since in this case the product abc simply takes the place which the number a had before. And e rows with d times abc in each row come to the same number as d columns with e times abc in each column. It remains only to multiply this number successively by whatever factors stand to the right of the interchanged pair.

It follows therefore that no matter how many numbers are multiplied together, we may interchange the places of any two of them which are adjacent without altering the product. But by repeated interchanges of adjacent pairs we may produce any alteration we choose in the order of the factors.

This extends the commutative law of freedom to all the factors in any product.

Associativity. To show with equal generality that multiplication is associative, we have only to prove that in any product any group of the successive factors may be replaced by their product.

$$abcdefgh = abc(def)gh.$$

By the commutative law we may arrange the factors so that this group comes first. Thus $abcdefgh = def\ abc\ gh.$

But now the product of this group is made in carrying out the multiplication according to definition. Therefore

$$abcdefgh = def\ abc\ gh = (def)\ abc\ gh.$$

Considering this bracketed product now as a single factor of the whole product, it can, by the commutative law, be brought into any position among the other factors, for example, back into the old place; so $abcdefgh = def\ abc\ gh = (def)\ abc\ gh = abc\ (def)\ gh.$

Distributivity. Multiplication combines with addition according to what is called the *distributive* law.

Instead of multiplying a sum and a number we may multiply each part of the sum with the number and add these partial products.

$$a(b+c) = (b+c)a = ab + ac.$$

$$4\times 5 = 4(2+3) = (2+3)4 = 2\times 4 + 3\times 4 = 5\times 4.$$

Four by five equals five by four, and four rows of $(2+3)$ units may be counted as four rows of two units together with 4 rows of 3 units.

As the sum of two numbers is a number, we may substitute $(a+b)$ for b in the formula $(b+c)d=bd+cd$, which thus gives

$$[(a+b)+c]d=(a+b)d+cd=ad+bd+cd.$$

So the distributive law extends to the sum of however many numbers or terms.

Since $a(b+c)>ab$ and $(a+b)b>ab$, therefore a product changes if either of its factors changes. A product increases if either of its factors increases.

Notwithstanding the historical origin of addition from counting and of multiplication from the addition of equal terms, it is now advantageous to consider multiplication, not as repeated addition, but as a separate operation, only connected with addition by the distributive law, an operation for finding from two elements, x, y, an element univocally determined, xy, called "the product, x by y," which by commutativity equals x times y.

CHAPTER VI.

THE TWO INVERSE OPERATIONS, SUBTRACTION AND DIVISION.

Inversion.

In the preceding direct operations, in addition and multiplication, the simplest problem is, from two given numbers to make a third.

If a and b are the given numbers, and x the unknown number resulting, then

$$x=a+b, \text{ or } x=a\times b,$$

according to the operation.

An *inverse* of such a problem is where the result of a direct operation is given and one of the components, to find the other component. The operation by which such a problem is solved is called an inverse operation.

Since by the commutative law we are free to interchange the two parts or terms of a given sum, as also the two factors of a given product, therefore here the inverse operation does not depend upon which of the two components is also given, but only upon the direct operation by which they were combined.

Subtraction.

Suppose we are given a sum which we designate by s, and one part of it, say, p, to find the corresponding other part, which, yet unknown, we represent by x. Since the sum of the numbers p and x is what $p+x$ expresses, we have the equality $x+p=s$.

But this equation differs in kind from the literal equalities heretofore used. It is not a formula, for any digital number substituted for one of these letters restricts the simultaneous values permissible for the others. Such an equality is called a conditional equality or a *synthetic* equation, or simply an *equation.*

The inverse problem for addition now consists just in this,—to solve the synthetic equation

$$b+x=a,$$

when a and b are given; in other words, to find a definite number which placed as value for x will satisfy the equation, that is which added to b will give a, and thus *verify* the equation. The number found, which satisfies the equation is called a *root* of the equation.

If the operation by which from a given sum a and a given part of it b we find a value for the corresponding other part x is called *from a subtracting b,* then, using the minus sign ($-$) to denote subtraction, we may write the result $a-b$, read a minus b.

We may get this result, remembering that a number is a sum of units, by pairing off every unit in b with a unit in a, and then counting the unpaired units. This gives a number which added to b makes a.

The expression or result $a-b$ is called a *difference.*

The term preceded by the minus sign is called the *subtrahend*; the other the *minuend.*

Thus $(a-b)+b=a-b+b=a$; also

$$b+(a-b)=b+a-b=a.$$

Ordinally, to subtract y from x is to find the number occupying the yth place before x.

Postulating the "rule of signs," that $a-(b-c)=a-b+c$, subtraction is associative and commutative.

Division. The term division has two distinct meanings in elementary mathematics. There are two operations called division: 1°, Remainder division; 2°, Multiplication's inverse.

1°, Given two numbers, $a > b$, a the *dividend,* and b the *divisor,* the aim of *remainder division* may be considered the putting of a under the form $bq+r$, where $r < b$, and b not 0. We call q the *quotient,* and r the *remainder.* Both are integral. There is a definite probability that r will not be 0.

The remainder division of a by b answers the two questions: 1°, What multiple of b if subtracted from a gives a difference or remainder less than b? 2°, What is this remainder?

Remainder division will regroup a given set, the dividend, into smaller sets each with the same cardinal as a given set, the divisor, and a remaining set whose cardinal is less than that of the divisor.

The number of the equivalent subsets is here the quotient. There is no implication that the original units are equal in size. So it would be a blunder to call this process measuring.

Again remainder division will regroup a given set, the dividend, into equivalent subsets and a less remainder, when the number of subsets, the divisor, is given. The cardinal of each subset is here the quotient. This has sometimes been called partitive division. But these two applications of remainder division are not two kinds of division, and should not be emphasized. In arithmetical division, dividend and divisor are two given numbers fixing a third, the quotient. So the division of 15 by 4 tells how often 15 eggs contain 4 eggs and equally well how many dollars in each of the 4 equivalent pieces of 15 dollars.

When r is 0, then a is a *multiple* of b, and a is *exactly divisible* by b.

The case $b=0$ is excluded. In this excluded case the problem would be impossible if a were not 0. But if $a=0$ and $b=0$, every number, q, would satisfy the equality $a=bq$. So this case must be excluded to make the operation of division unequivocal, that is, in order that the problem of division shall have always one and only one solution. A second solution q', r' would give $a=bq+r$ $=bq'+r'$, $b(q-q')=r'-r$. But $r'-r<b$, while $b(q-q')$ not $<b$.

2°, Division may also be regarded as the inverse of multiplication. Its aim is then considered to be the finding of a number q (quotient) which mulitplied by b (the divisor) gives a (the dividend). Here division is the process of finding one of two factors when their product and the other factor are given.

The result q is represented by a/b. If $a=0$, then $q=0$. This definition of division gives the equality

$$(a/b)b=a.$$

Remember $b\neq 0$, that is, b not equal to 0.

In particular $a/1=a$.

Postulating the rule $a/(b/c)=a/b\times c$, division is associative, commutative, and distributive.

$$(a+b)/c=(a/c)+(b/c)\text{ ; but}$$
$$a/(b+c)\neq(a/b)+(a/c).$$

In general 1° $(a+b)/m=a/m+b/m.$

2° $(a-b)/m=a/m-b/m.$

3° $a(b/c)=ab/c$.
4° $a/(bc)=(a/b)/c$.
5° $a/(b/c)=(a/b)c$.
6° $a/b=am/bm$.
7° $a/b=(a/m)/(b/m)$.

The Arabs, as early as 1000 A. D., used the *solidus,* or slant-sect / and also the horizontal sect, as in $\frac{2}{3}$ or ⅔, to denote the quotient of the first or upper number by the other.

The symbol ÷ is not found until about 1630. It may have been suggested by the use of the horizontal sect in $\frac{n}{d}$. Turned on end ·|· I use it for *symmetrical,* as in ·|· △ for isosceles triangle.

CHAPTER VII.

TECHNIC.

Addition.

In adding a column of digits, consider two numbers together, but only *think* their sum.

3	23
8	48
5	35
9	59
2	62
7	87
4	74
5	95
43	3

Now in adding up this column only think 9, 16, 18, 27, 32, 43, stressing forty, and writing down the three while thinking it.

The stress on the forty is to hold the four in mind for use in the next column to the left. Such a number is said to be *carried*. Begin adding up the next column to the left by thinking 13.

To check the work, add the column downward, since mere repetition of work tends to repeat the mistake also.

Subtraction.

Look at the question of subtracting as asking what number added to the subtrahend gives the minuend. Always work subtraction by adding. Thus subtract 1978 from 3139 as follows: Think 8 and one make 9; 7 and six make 13, carry 1; 10 and one make 11, carry 1; 2 and one make 3. Write down the spelled digits just while thinking them.

3139
1978
1161

Explain "carrying" by the principle that the difference between two numbers remains the same though they be given equal increments.

9254	Again think, 5 and nine make 14, carry 1;
8365	7 and eight make 15, carry 1; 4 and eight make
889	12, carry 1; 9 equals 9.

In working the examples we have added *downwards*, so check by adding *upwards* the difference (the answer) to the subtrahend; think (for 9 and 5) 14, (for 9 and 6) 15, (for 9 and 3) 12, (for 1 and 8) 9.

It is preferable for several reasons to perform numerical operations from the left. An operation thus corresponds more closely with the process it represents. Again this way fixes the attention at once upon the greater, more important parts of the quantities concerned, permitting immediate approximations, and so giving speed in dealing with life realities, thus increasing practical efficiency.

Though the immediate conception of a large multiple of a small number, perhaps because of our mastery of the number series, is simpler than that of a small multiple of a large number, yet operatively, as a multiplication, the latter gives the easier process. Hence choose the smaller as your multiplier. To find thrice 2104 it would be best to apply the distributive law from the left, giving 3(2000+100+4). This is the way of the lightning calculator. Meantime, as a concession, we teach the backward application of the law, (2000+100+4)3.

Multiplication.

Set down the multiplier precisely in column under the multiplicand, units under units. Begin by multiplying the units figure of the multiplicand by the leftmost figure of the multiplier, writing under this leftmost figure the first figure thus obtained.

```
   35427
    1324
--------
 35427
 106281
   70854
  141708
--------
 46905348
```

Then use the successive figures in order. The figure set down from multiplying the units always comes precisely under its multiplier.

The advantage of this method is that it gives the most important partial product first, and in abridged or approximate work one or two of the leftmost figures may be all that are wanted.

Rule: If of two figures multiplied one is in units column, the figure set down stands under the other.

Check by casting out nines.

Verify multiplication.

Proceed as follows: Add the single figures of the *multiplicand,* but always diminish the partial sums by dropping nine. The remainder is identical with the remainder found much more laboriously by dividing by nine. Thus 35427 gives 3, since 7 and 2 give nine as also 4 and 5. Find just so the remainder of the *multiplier.* Here 1324 gives 1. If our work is correct, the remainder, or *excess,* of the product of these two remainders equals the remainder, or excess, for our product. Here 46905348 gives 3.

The complete proof of this method of verification lies simply in the fact that the remainder left when any number is divided by nine is the same as that left when the sum of its digits is divided by nine. For $10-1=9$, $100-1=99$, $1000-1=999$, etc. Hence if from any number be taken its units, also a unit for each of its tens, a unit for each of its hundreds, a unit for each of its thousands, etc., the remainder is a multiple of nine. But the part taken away is the sum of the number's digits.

(*a*) When the multiplier contains only two digits, shorten the work by adding in the results of the multiplication by the second digit to that already obtained. Here, after multiplying by 3, think four*teen*; 16, 17 eigh*teen*; 10, 11, seven*teen*; 18, 19, *twenty*-six; ten; three. Write down the unaccented part of these spelled numbers while thinking it.

Shorter forms.

```
  9587
    32
------
 28761
------
306784
```

(*b*) If in a multiplier of only two digits either is unity, write only the answer.

```
  9867
    15
------
148005
```

Here think *thirty*-five; 30, 33, *forty*; 40, 44, *fifty*; 45, 50, *fifty*-eight; fourteen.

```
  7968
    41
------
326688
```

Here think eight; 32, *thirty*-eight; 24, 27, *thirty*-six; 36, 39, *forty*-six; 28, thirty-two.

(*c*) When in a three-place multiplier taking away either end-digit leaves a multiple of it, shorten by adding to the digit's partial product the proper multiple of it.

```
  1234
   568
------
  9872
 69104
------
700912
```

After multiplying by 8, multiply this partial product by 7 (tens).

```
   4213
    864
-------
 33704
  269632
-------
3640032
```

After multiplying by the 8, (hundreds), multiply this partial product by 8. This gives units.

(Divisor an integer):

Division.

Write the first figure of the quotient precisely over the last figure of the first partial dividend. Use no bar to separate them.

Omit the partial products, the multiples of the divisor, writing down the differences while doing the multiplication.

```
      318
 19) 6054
      3
      16
       12
```

Nineteen into 60 thrice. Three nines are 27, and three makes 30. Carry 3. Three ones are 3; say 6.

Nineteen into 35 once. One nine is 9, and six makes 15. Carry 1. One 1 is 1; say 2, and one makes 3.

Nineteen into 164 eight times. Eight nines are 72, and two makes 74. Carry 7. Eight ones are 8; say 15, and one makes 16.

```
       27
358) 9762
     260
       96
```

Here, using the 2, think 16 and naught, 1'6. Carry 1. 10, say 11 and six, 1'7. Carry 1. 6, say 7 and two, 9. Thus we get the new partial dividend 2602, which gives in our quotient 7. Using this 7, think 56 and six, 6'2. Carry 6. 35, say 41 and nine, 5'0. Carry 5. 21, say 26. Thus we get our remainder 96.

This method gives at once the true value of each partial quotient. Moreover its using the partial products instead of setting them down, actually diminishes error, besides being easier and quicker and more compact.

Verify division. The excess of the product of excesses of divisor and quotient increased by excess of remainder equals excess of dividend.

In our example the excess from the quotient is 0. So the excess from the dividend, 6, equals that from the remainder.

CHAPTER VIII.

DECIMALS.

Decimals.

A decimal is a number whose expression in our positional notation contains digits to the right of units column. A decimal is a basal subunital; a number containing subunits which are multiples of minus powers of the base.

It is the characteristic of our positional notation for number that shifting a digit one place to the left multiplies it by the base of the system. The zero enables us to indicate such shifting. Thus since our base is ten, 1 shifted one place to the left, 10, becomes ten; two shifted two places to the left, 200, becomes two hundred.

Inversely, shifting a digit one place to the right, divides it by the base of the system. Thus 3 in the thousands place, 3000, shifted one place to the right becomes 300.

We now create that this shifting to the right may go on beyond the units' place with no change of meaning or effect.

In order to write this, we use a device, a notation to mark or point out the units place, a point immediately to its right called the decimal point or unital point. Our present decimal notation, a development of that of Simon Stevinus of Bruges, 1585, was not generally used before the eighteenth century, although the decimal point appears first in 1617 on page 21 of Napier's *Rabdologiae*. Thus

4 shifted one place to the right becomes 0.4 and of course means a number which multiplied by the base gives 4. Such numbers have been called decimals. Their theory is independent of the base, which might be say 12 or 2, in which case the word decimals would be a distinct misnomer.

The perfection of our system is in its subtle use of a base-number, not in that number ten. Our system is a miraculous instrument for easy reckoning, not because it is decimal, but because the digit figures, by aid of a potential zero, always express, in their orderly position, to left or right of a point, multiples of ascending and descending powers of one basal number. Thus $9(10)^4 + 0(10)^3 + 8(10)^2 + 7(10)^1 + 6(10)^0 + 5(10)^{-1} + 4(10)^{-2} + 3(10)^{-3} = 90876.543$.

If however the base be ten, then shifting a digit one place to the left multiplies it by ten. But this is accomplished for every digit in the number simply by shifting the point one place to the right. Thus .05 is tenfold .005. If our unit is a dollar, $1, then the first place to the right will be dimes. Thus $0.6 means six dimes. The next place to the right of dimes means cents. Thus $.07 means seven cents. The next place to the right of cents means mills. Thus $.008 means eight mills.

Ten mills make a cent. Ten cents make a dime. Ten dimes make a dollar.

In general we name these basal subunitals so as to indicate by symmetry their place with reference to the units' column. As the first column to the left of units is tens, so the first column to the right of units is called tenths. As the second column to the left of the units' column is called hundreds, so the second column to the right of the units' column is called hundredths. As the third column to the left of the units' column is called thousands, so the third column to the right of the units' column is called thousandths.

But these names need not be used in reading a subunital. Thus 0.987 may be read: Point, nine, eight, seven. So mathematicians read it, and all educated scientists.

One-tenth is a number ten of which are together equal to a unit. "Point, one," says this.

If an integer be read by merely pronouncing in succession the names of its digits, as in reading 7689 as seven, six, eight, nine, we do not know the rank and so all the value of any figure read until after all have been read.

Hence the advantage of reading 7689 seven thousand six hundred and eighty-nine. But in reading the decimal .7689 as "point, seven, six, eight, nine" we know every thing about each figure as it is read, which on the contrary we do not know if it be read seven thousand six hundred and eighty-nine ten-thousandths.

Morever such a habit of reading decimals detracts from our confident certainty of understanding integers step by step as read. There may be coming at the end a wretched subunital designation like this "ten-thousandths" to metamorphose everything read.

So always read decimals by pronouncing the word *point* and the names of the separate single digits.

Read 700.008 seven hundred, point, naught, naught, eight. Read .708 point, seven, naught, eight.

This wholly obviates the imaginary difficulty of the hysterical country school ma'am (unmarried), whose hypothetical man she supposed could not properly inflect his voice, and so could not by tone indicate the difference marked by punctuation, between "seven hundred, and eight thousandths," and "seven-hundred-and-eight thousandths." To relieve her wooden man, her femininity suggested the crime of suppressing the "and" in all such good English phrases as The Thousand and One Nights.

$$9(10)^3+8(10)^2+7(10)^1+6(10)^0+5(10)^{-1}+ \\ 4(10)^{-2}+3(10)^{-3}+2(10)^{-4}=9876.5432.$$

To add decimals, write the terms so that the decimal points fall precisely under one another, in a vertical column. Then proceed just as with integers, the point in the sum falling under those of the terms.

Just so it is with subtraction.

Product. In multiplying decimals remember we are dealing simply with a symmetrical completion, extension of positional notation to the *right* from units' place. Realize the perfect balance resting on the units' column. 4321.234.

A shift of the decimal point changes the rank of each of the digits. So to multiply or divide by any power of ten is accomplished by a simple shift of the point.

Thus 98.76×10 is 987.6. Just so 98.76/10 is 9.876, and is identical with 98.76×0.1. Twice this is $98.76\times0.1\times2$ or $98.76\times0.2=19.752$.

So to multiply by a decimal is to multiply by an integer and shift the point.

Hence the rule, useful for check, that the number of decimal places in the product is the sum of the places in the factors. There is no need for thinking of tenths as fractions to realize that two-tenths of a number is twice one-tenth of it.

In multiplying decimals, write the multiplier so that its point comes precisely under the point in the multiplicand, and in vertical column with these put the point in each partial product. The figure obtained from multiplying the *units* figure of the multiplicand must come precisely under the figure by which we are multiplying.

```
   1293.015
    132.02
 ----------
 129301.5
  38790.45
   2586.030
     25.86030
 ----------
 170703.8403
```

Here, beginning to multiply by the 1, think five while writing it two places to the left of the figure multiplied because the 1 is two places to the left of the units' column. Proceed to multiply by the 3, thinking fif*teen*; 3, four; naught; nine; *twenty*-seven; etc.

Rule: Multiplying shifts as many places right or left as the multiplier is from the units' column.

```
 41.27
   .03
 ------
 1.2381
```

Here think *twenty*-one while writing the 1 two places more to the right than the 7 because the 3 is two places to the right of the units' column.

Quotient. In division of decimals place the decimal point of the quotient precisely over the decimal point of the dividend and, when the divisor is an integer, the first figure of the quotient over the last figure of the first partial dividend.

```
       638.
 .021)13.4
         8
        17
         2
```

Rule: The first figure of the quotient stands as many places to the left of the last figure of the first partial dividend as there are decimal places in the divisor.

Here the quotient 638 is an integer.

The sign + at the end of a number means there is a remainder, or that the number to which it is attached

```
       6+
2.1).0134
        8
```

falls short of completely, exactly expressing all it represents, though increasing the last figure by unity would overpass exactitude and so should be followed by the sign – (minus).

Thus $\pi = 3.14+$ and

$\pi = 3.1416-$

This is historically the first meaning of the signs + and –, which arose from the marks chalked on chests of goods in German warehouses, to denote excess or defect from some standard weight.

When there is a remainder we may get additional places in the quotient by annexing ciphers to the dividend and continuing the division.

```
     63
.21)13.4
      8
      17
```

The phrase "true to 2 (or 3, etc.) places of decimals" means that a closer approximation can not be written without using more places.

Thus as a value for π, 3.14 is true or "correct" to two places of decimals, since $\pi = 3.14159+$; while 3.1416 is true to four places.

As an approximation to 1.235 we may say either 1.23 or 1.24 is true to two places of decimals.

CHAPTER IX.

FRACTIONS.

Generality is the essence of modern mathematics. The creative extension of its previously attained system marks the growth of its powers as our greatest instrument for that ordering and simplification of our universe, that transforming of chaos into cosmos, which is the vocation of science. Such an extension of the original integral numbers we see in decimals. But just here we have one of the sharp rebuttals found everywhere in mathematics to the pedagogic principle that education should recapitulate the path of the race. Decimals, roughly two centuries old, should be taught before fractions, probably more than five thousand years old. The romantic treatise we still possess, entitled "Directions for obtaining the Knowledge of all Dark Things," written by the scribe Ahmes about 1700 B. C., and founded on an older work believed by the Egyptologist Birch to date back as far as 3400 B. C., contains, to solve the problem of representing any fraction as a sum of fractions each with numerator one, a table of solutions for all fractions with numerator 2 and all denominators from 3 to 99; e. g., $2/99 = 1/66 + 1/198$.

Generalizations of number.

Expansions of the number-idea are guided by one criterion, that there be no break in the applicability of the old formal conventions of procedure. They are motived by the desire to obviate exceptions. Thus after decimals and fractions or *rationals,* mathematicians created *reals,* and signed numbers, and complex numbers.

For the new numbers hold the old laws.

Principle of permanence.

1st. Every number combination which gives no already existing number, is to be given such an interpretation that the combination can be handled according to the same rules as the previously existing numbers.

2d. Such combination is to be defined as a number, thus enlarging the number idea.

3d. Then the usual laws (freedoms) are to be proved to hold for it.

4th. Equal, greater, less are to be defined in the enlarged domain.

This was first given by Hankel as generalization of a principle given by G. Peacock, British Association, III, London, 1834, p. 195. *Symbolic Algebra,* Cambridge, 1830, p. 105; 2d ed., 1845, p. 59.

Fractions.

If unity in pure number be considered as indivisible, fractions may be introduced by conventions. Take two integers in a given order and regard them as forming a couple with sense; create that this ordered couple shall be a number of a new kind, and define the equality, addition, and multiplication of such numbers by the conventions,

$$a/b = c/d \text{ if } ad = bc;$$
$$a/b + c/d = (ad + bc)/bd;$$
$$(a/b)(c/d) = (ac)/(bd).$$

The preceding number is called the *numerator* of the fraction; the succeeding number, the *denominator.*

Fractions have application only to objects capable of partition into equal portions equal in number to the denominator. No fraction is applicable to a person.

In accordance with the principle of permanence, we create that the compound symbol of the form a/b, two natural numbers separated by the slant, shall designate a number. Either the symbol or the number may be called a *fraction*. The slant is to stand for the division of a by b, of the preceding by the succeeding number, where this is possible. When a is exactly divisible by b, that is, without remainder, the fraction designates a natural number. Always notationally a fraction represents an unperformed operation, a division, and any approximate result of the performance of this division is an approximate value of the fraction; but the number represented by the fraction is always exact, precise, definite, perfect.

When a is a multiple of b, and a' of b', the equality $ab'=a'b$ is the necessary and sufficient condition for the symbols a/b, a'/b' to represent the same number. By this same condition we define the equality of the new numbers, the fractions.

A fraction is *irreducible* when its numerator and denominator contain no common factor other than 1.

To compare two fractions, reduce them to a common denominator, then that which has the greater numerator is called the greater.

A *proper* fraction is a fraction with numerator less than denominator. It is less than 1.

Subtraction is given by the equality $a/b-a'/b'=(ab'-a'b')/bb'$.

The *multiplication* of fractions is covered by the statement: A product is the number related to the multiplicand as the multiplier is to unity.

$$(a/b)\ (a'/b')=aa'/bb'.$$

Thus $(5/7)\times(2/3)$ means trisect, take one of these three parts, then double, giving 10/21.

So $(a/b)\times(b/a)=1$. Two numbers whose product is unity are called *reciprocal.*

Extending the meaning of "times" so that 2/3 times thrice equals twice, and n/d times d times equals n times, we have $(x/z)z=x$. Hence 5/7 times Q is a quantity such that 7 times it gives 5Q. Therefore it equals 5 times a quantity seven of which make Q, that is five-sevenths of Q.

So 2/3 times Q is 2/3 of Q.

Division is taken as the inverse of multiplication, hence $(c/d)/(a/b)$ means to find a number whose product with (a/b) is (c/d). Such is $(c/d)(b/a)$.

So $(c/d)/(a/b)=(c/d)(b/a)=bc/ad$.

1°. This last expression may be considered simply a more compact form of the first, obtained by reducing to a common denominator and cancelling this denominator. This compact form can be obtained by a procedure sometimes called the rule for division by a fraction: *Invert the divisor and multiply.*

2°. If we interchange numerator and denominator of a fraction we get its *inverse* or *reciprocal.* So the inverse of a is $1/a$.

$(a/b)(b/a)=1$.

Now $(x/y)/(a/b)$ means to find a number which multiplied by a/b gives x/y, and so the answer is $(x/y)(b/a)$. Hence: *To divide by a fraction, multiply by its reciprocal.*

3°. Again to find $(a/b)/(c/d)$, note that c/d is contained in 1 d/c times, and hence in a/b it is contained $(a/b)(d/c)$ times.

A *reduced* fraction is one whose numerator and denominator contain no common factor.

Fractions ordered.

The fractions arranged according to size are an ordered set, but not well ordered; for no fraction has a determinate next greater fraction, since between any two numbers, however near in size, lie always innumerable others.

But all reduced fractions can be arranged in a well-ordered set arranged according to groups in which the sum of numerator and denominator is the same:

1/1, 1/2, 2/1, 1/3, 3/1, 1/4, 2/3, 3/2, 4/1, 1/5, 5/1, 1/6, 2/5, 3/4, 4/3, 5/2, 6/1,....

Thus they make a simply infinite series equivalent to the number series.

Proper fractions can be arranged by denominators:

$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \ldots .$

To turn a fraction a/b into a decimal $c/10^k$ must give $a10^k = bc$, where c is a whole number. Since a/b is in reduced form, therefore a and b have no common factor. So 10^k must be exactly divisible by b. Thus only fractions with denominator of the form $2^n 5^m$ can be turned exactly into decimals.

Fractions may be thought of as like decimals in being also subunitals. The unit operated with in a fraction, the fractional unit, is a subunit, and the *denominator* is to tell us just what subunit, just what certain part of the whole or original or primal unit is taken as this subunit; while the *numerator* is the number of these subunits. The denominator tells the *scale* of the subunit, its relation to the primary integral unit. Thus 3/10 is a three of subunits ten of which make a unit. Thus, like an integer, a fraction is a unity of units (or one unit), but these are subunits. Different subunits may be very simply related, as are 1/2, 1/4, 1/8.

To add 3/4 and 1/2 we first make their subunits the same by bisecting the subunit of 1/2, which thus becomes 2/4. Then 3/4 and 2/4 may be counted together to give 5/4.

Fractions having the same subunit are added by adding their numerators, the same denominator being retained since the subunit is unchanged. The like is true of subtraction.

To add unlike fractions change to one same subunit. The technical expression for this is "reduce to a common denominator."

Since we already know that to be counted together the things must be taken as indistinguishably equivalent, the procedure of changing to one same subunit is crystal clear.

To change a half to twelfths is simply to split up the one-half, the first subunit, into subunits twelve of which make the whole or original unit.

Thus, operatively, to express a fraction in terms of some other subunit, the procedure is simply to multiply (or divide) numerator and denominator by the same number.

Thus $1/2 = (1 \times 6)/(2 \times 6) = 6/12$.

So $6/12 = (6/3)/(12/3) = 2/4$.

This principle in the form: "The value of a fraction is unaltered by dividing both numerator and denominator by the same number," is freely applied in what is technically called "reducing fractions to their lowest terms."

It should be applied just as freely and directly in the form: "The value of a fraction is unaltered by multiplying both numerator and denominator by the same number." Thus the complex fraction (2+2/3)/(3+2/9), multiplying both terms by 9, gives at once 24/29. Again (3 feet 5 inches)/(2 feet 7 inches), multiplying both terms by 12, gives 41/31.

$$\begin{array}{r} 13\frac{1}{4} \\ 7\frac{3}{4} \\ \hline 5\frac{2}{4} \end{array}$$

To subtract 7+3/4 from 13+1/4, that is to evaluate $13\frac{1}{4}-7\frac{3}{4}$, think 3/4 and two-fourths make 5/4, carry 1; 8 and five make thirteen.

The 1 in $1/n$ is the suounit, the n specifying what particular subunit. In division of a fraction by an in-

Division of fractions.

teger we meet the same limitation which theoretically led to the creation of fractions; namely 2/5 is no more divisible by three than any other two. But here we can easily transform our fraction into an equivalent divisible by 3. Just trisect the subunit. Thus 2/5 becomes 6/15, which is divisible by 3 giving 2/15.

Such result is always at once attained simply by multiplying the given denominator by the given integral divisor. Hence the rule: To divide a fraction by an integer, multiply its denominator by the integer.

Our multiplication is to be associative, so when the multiplier is increased any whole number of times, the

Multiplication of fractions.

product will be increased the same number of times. For instance, thrice 5 is 15. Doubling the multiplier, twice thrice 5 is 30, which is double the former product 15. So for fractions, as 4/7 and 9/10, the product is such that when the multiplier 4/7 is increased 7 times, so is the product. Now 7 times 4/7 is 4. Thus 4 times the fraction 9/10 will be 7 times the required product. But 4 times 9/10 is 36/10, and the seventh part of this is 36/70. Let this then be our product of 4/7 and 9/10. We reach thus for the product of two fractions the rule: Multiply the numerators together for the new numerator, and the denominators for the new denominator.

CHAPTER X.

RELATION OF DECIMALS TO FRACTIONS.

6.214
$3\frac{1}{3}$
18.642
$2.071\frac{1}{3}$
$20.713\frac{1}{3}$

Fractions may be freely combined with decimals. Thus $1/24 = .04\frac{1}{6}$.

1 meter = 39.37 inches = 3 feet $3\frac{3}{8}$ inches.

In finding the product of a decimal and a fraction use the fraction as multiplier.

1st. Decimals into fractions.

By our positional notation, 0.1 means one subunit such that ten of them make the unit. But just this same thing is meant by 1/10. Therefore any decimal may be instantly written as a fraction; e. g., $0.234 = 2/10 + 3/100 + 4/1000 = 234/1000$.

First Method.

2d. Fractions into decimals.

Any fraction equals the quotient of its numerator divided by its denominator. Consider the fraction, then, simply as indicating an example in division of decimals, and proceed to find the quotient.

Thus for 1/2 we have: .5
2)1.0 So 1/2=0.5.

For 3/4 we have .75
4)3.00 So 3/4=0.75.

For 7/8 we have: .875
8)7.000 So 7/8=0.875.

Second Method.

Apply the principle: The value of a fraction is unaltered by multiplying both numerator and denominator by the same number.

Thus $7/8 = 7/(2\times2\times2)$

$$= (7\times5\times5\times5)/(2\times5\times2\times5\times2\times5)$$
$$= 875/1000 = 0.875.$$

Considering the application of this second method to 1/3, we see there is no multiplier which will convert 3 into a power of 10, since 10 contains no factors but 2 and 5. Ten does not contain 3 as a factor, so we cannot convert 1/3 into an ordinary decimal. We cannot, as an example in division of decimals, divide 1 by 3 *without remainder.* But we can freely apply remainder-division, at any length. Thus

```
   .333
3)1.
   .1
   .01
   .001
```

The procedure is recurrent, and if continued the 3 would simply recur.

```
   .142857
7)1.
   .3
     2
      6
       4
        5
         1
```

In division by n, not more than $n-1$ different remainders can occur. But as soon as a preceding dividend thus recurs, the procedure begins to repeat itself. Here then this division by 7 must begin to repeat, and the figures in the quotient must begin to recur.

If the recurring cycle begins at once, immediately after the decimal point, the decimal is called a pure recurring decimal. As notation for a pure recurring decimal, we write the recurring period, the repetend, dotting its first and last figures thus

$$1/11 = \cdot\dot{0}\dot{9};\ 1/9 = \cdot\dot{1}.$$

Every fraction is a product of a decimal by a pure recurring decimal. Thus

$$1/6 = (1/2)(1/3) = 0\cdot 5 \times \cdot\dot{3}.$$

To convert recurring decimals into fractions:

$$\begin{array}{l} \cdot\dot{1}\dot{2} \times 100 = 12\cdot\dot{1}\dot{2} \\ \underline{\cdot\dot{1}\dot{2} \times 1 = \cdot\dot{1}\dot{2}} \\ \cdot\dot{1}\dot{2} \times 99 = 12 \\ \cdot\dot{1}\dot{2} = 12/99 = 4/33 \end{array} \qquad \text{Therefore subtracting,}$$

Rule: Any pure recurring decimal equals the fraction with the repeating period for a numerator, and that many nines for denominator.

Base.

The base of a number system is the number which indicates how many units are to be taken together into a composite unit, to be named, and then to be used in the count instead of the units composing it, this first composite unit to be counted until, upon reaching as many of them as units in the base, this set of composite units is taken together to make a complex unit, to be named, and in turn to be used in the count, and enumerated until again the basal number of these complex units be reached, which manifold is again to be made a new unit, named, etc.

Thus twenty-five, twain ten + five, uses ten as base. Using twelve as base, it would be two dozen and one. Using twenty, it would be a score and five. In positional notation for number, a digit in the units' place means so many units, but in the first place to the left of units' place it means so many times the base, while in the first place to the right of the units' place it means so many subunits each of which multiplied by the base gives the unit. And so on, for the second, etc., place to the left of the units' column, and for the second, etc., place to the right of the units' column.

It is the systematic use of a base in connection with the significant use of position, which constitutes the formal perfection of our Hindu notation for number. The actual base itself, ten, is a concession to our fingers.

The complete formula for a number in the Hindu positional notation is

$$db^n \ldots + db^4 + db^3 + db^2 + db^1 + db^0 + db^{-1} + db^{-2} + \ldots db^{-n}$$

where juxtaposition of the d (digit) and b (base) means multiplication. This we condense to $d \ldots ddd.ddd \ldots d$, where the omitted b-factor is indicated by the position of the d with reference to the units column, fixed by the unital point written to its right in the ordered row. Juxtaposition here means addition. If no base be specified, ten is understood.

Compare these subunital expressions for the fundamental fractions, to base ten, to base twelve, to base two.

DECIMALLY. [IN THE DENARY SCALE.]	DUODECIMALLY. [DUODENARY SCALE.]	DUALLY. [DYADIC SCALE.]
$1/2 = 0 \cdot 5$	$1/2 = 0 \cdot 6$	$1/2 = 1/10 = 0 \cdot 1$
$1/3 = \cdot \dot{3}$	$1/3 = 0 \cdot 4$	$1/3 = 1/11 = \cdot \dot{0}\dot{1}$
$2/3 = \cdot \dot{6}$	$2/3 = \cdot 8$	$1/4 = 1/100 = \cdot 01$
$1/4 = 0 \cdot 25$	$1/4 = 0 \cdot 3$	$1/6 = 1/110 = \cdot 0\dot{0}\dot{1}$
$3/4 = \cdot 75$	$3/4 = \cdot 9$	$1/8 = 1/1000 = \cdot 001$
$1/5 = \cdot 2$	$1/5 = \cdot 249+$	$1/9 = 1/1001 = \cdot \dot{0}0011\dot{1}$
$1/6 = 0 \cdot 1\dot{6}$	$1/6 = 0 \cdot 2$	
$1/8 = 0 \cdot 125$	$1/8 = 0 \cdot 16$	
$3/8 = \cdot 375$	$3/8 = \cdot 46$	
$1/9 = \cdot \dot{1}$	$1/9 = 0 \cdot 14$	

Change of base.

To express a given number to a new base, divide it and the successive quotients by the new base until a quotient is reached less than the new base; this quotient and the successive remainders will be digits.

Express 1594 to base twelve.

11 – 0   Using x for ten and ӿ for eleven, the answer is ӿ0x.
12)132 – 10
12)1594

Express ӿxӿ (base twelve) to base ten.

1 – 7   Answer 1715.
x)15 – 1
x)123 – 5
x)ӿxӿ

Express 98 to base two.

1 – 1   Answer 1100010.
2)3 – 0
2)6 – 0
2)12 – 0
2)24 – 1
2)49 – 0
2)98

Express 1111 (base two) to base ten.

1 – 101   Answer 15.
1010)1111

CHAPTER XI.

MEASUREMENT.

Says Dr. E. W. Hobson: "It is a very significant fact that the operation of counting, in connection with which numbers, integral and fractional, have their origin, is the one and only absolutely exact operation of a mathematical character which we are able to undertake upon the objects which we perceive. On the other hand, all operations of the nature of measurement which we can perform in connection with the objects of perception contain an essential element of inexactness. The theory of exact measurement in the domain of the ideal objects of abstract geometry is not immediately derivable from intuition."

Arithmetic is a fundamental engine for our creative construction of the world in the interests of our dominance over it. The world so conceived bends to our will and purpose most completely. No rival construct now exists. There is no rival way of looking at the world's discrete constituents. One of the most far-reaching achievements of constructive human thinking is the arithmetization of that world handed down to us by the thinking of our animal predecessors.

Why count?

In regard to an aggregate of things, why do we care to inquire "how many"? Why do we count an assemblage of things? Why not be satisfied to look upon it as an animal would? How does the cardinal number of it help?

First of all it serves the various uses of identification. Then the inexhaustible wealth of properties individual and conjoined of exact science is through number assimilated and attached to the studied set, and its numeric potential revealed. Mathematical knowledge is made applicable and its transmission possible.

Thus the number is basal for effective domination of the world social as well as natural.

Number arises from a creative act whose aim and purpose is to differentiate and dominate more perfectly than do animals the perceived material, primarily when perceived as made of individuals. Not merely must the material be made of individuals, but primarily it must be made of individuals in a way amenable to treatment of this particular kind by our finite powers. Powers which suffice to make specific a clutch of eggs, say a dozen, may be transcended by the stars in the sky.

Number is the outcome of an aggressive operation of mind in making and distinguishing certain multiplex objects, certain manifolds. We substitute for the things of nature the things born of man's mind and more obedient, more docile. They, responsive to our needs, give us the result we are after, while economizing our output of effort, our life. The number series, the ordered denumerable discrete infinity is the prolific source of arithmetic progress. Who attempts to visualize 90 as a group of objects? It is nine tens. Then the fingers tell us what it is, no graphic group visualization. First comes the creation of artificial individuals having numeric quality. The cardinal number of a group is a selective representation of it which takes or pictures only one quality of the group but takes that all at once. This selective picture process only applies primarily to those particular artificial wholes which may be called discrete aggregates. But these are of inestimable importance for human life.

The measure device.

The overwhelming advantages of the number picture led, after centuries, to a human invention as clearly a device of man for himself as the telephone. This was a device for making a primitive individual thinkable as a recognizable and recoverable artificial individual of the kind having the numeric quality, having a number picture. This is the recondite device called measurement.

Measurement is an artifice for making a primitive individual conceivable as an artificial individual of the group kind with previously known elements, conventionally fixed elements, and so having a significant number-picture by which knowledge of it may be transmitted, to any one knowing the conventionally chosen standard unit, in terms of this previously known standard unit and an ascertained number.

From the number and the standard unit for measure the measured thing can be approximately reproduced and so known and recovered. No knowledge of the thing measured must be requisite for knowledge of the standard unit for the measurement. This standard unit of measure must have been familiar from previous direct perception. So the picturing of an individual as three-thirds of itself is not measurement.

All measurement is essentially inexact. No exact measurement is ever possible.

Counting prior to measuring.

Counting is essentially prior to measuring. The savage, making the first faltering steps, furnished number, an indispensable prerequisite for measurement, long ages before measurement was ever thought of. The primitive function of number was to serve the purposes of identification. Counting, consisting in associating with each primitive individual in an artificial individual a distinct primitive individual in a familiar artificial individual, is thus itself essentially the identification, by a one-to-one correspondence, of an unfamiliar with a familiar thing. Thus primitive counting decides which of the familiar groups of fingers is to have its numeric quality attached to the group counted. To attempt to found the notion of number upon measurement is a complete blunder. No measurement can be made exact, while number is perfectly exact.

Counting implies first a known ordinal series or a known series of groups; secondly an unfamiliar group; thirdly the identification of the unfamiliar group by its one-to-one correspondence with a familiar group of the known series. Absolutely no idea of measurement, of standard unit of measure, of value is necessarily involved or indeed ordinarily used in counting. We count when we wish to find out whether the same group of horses has been driven back at night that was taken out in the morning. Here counting is a process of identification, not connected fundamentally with any idea of a standard measurement-unit-of-reference, or any idea of some value to be ascertained. We may say with perfect certainty that there is no implicit presence of the measurement idea in primitive number. The number system is not in any way based upon geometric congruence or measurement of any sort or kind.

The numerical measurement of an extensive quantity consists in approximately making of it, by use of a well-known extensive quantity used as a standard unit, a collection of approximately equal, quantitatively equal, quantities, and then counting these approximately equal quantities. The single extensive quantity is said to be numerically measured in terms of the convened standard quantitative extensive unit. Any continuous magnitude is measured by discreting it into a standardized set and a negligible residue, and counting the standard units in this set.

For measurement, assumptions are necessary which are not needed for counting or number. Spatial measurement depends upon the assumption that there is available a standard body which may be transferred from place to place without undergoing any other change. Therein lies not only an assumption about the nature of space but also about the nature of space-occupying bodies. Kindred assumptions are necessary for the measuring of time and of mass.

New assumptions.

Now in reality none of these assumptions requisite for measurement are exactly fulfilled. How fortunate then that number involves no measurement idea!

But still other assumptions are made in measurement. After this device for making counting apply to something all in one piece has marked off the parts which are to be assumed as each equal to the standard, their order is unessential to their cardinal number. But it is also assumed that such pieces may be marked out beginning anywhere, then again anywhere in what remains, without affecting the final remainder or the whole count. Moreover measurement, even the very simplest, must face at once incommensurability. Whatever you take as standard for length, neither it nor any part of it is exactly contained in the diagonal of the square on it. This is proven. But the great probabilities are that your standard is not exactly contained in anything you may wish to measure. There is a remainder large or small, perceptible or imperceptible. Measurement then can only be a way of pretending that a thing is a discrete aggregate of parts equal to the standard, or an aliquot part of it. We must neglect the remainder. If we do it unconsciously, so much the worse for us.

No way has been discovered of describing an object exactly by counting and words and a standard. Any man can count exactly. No man can measure exactly.

Arithmetic applies to our representation of the world, to the constructed phenomena the mind has created to help, to explain, its own perceptions. This representation of things lends itself to the application of arithmetic. Arithmetic is a most powerful instrument for that ordering and simplification of perception which is fundamental for dominance over so-called nature.

Measurement may be analyzed into three primary procedures: 1°. The conventional acceptance or determination of a standard object, the unit of measure. 2°. The breaking up of the object to be measured into pieces each congruent to the standard object. 3°. The counting of these pieces.

The standard unit for any particular sort of magnitude might have been any magnitude of the same kind. Race, locality, convenience, chance, have contributed to establish and maintain diverse units for magnitudes of the same kind, some wholly bad, stupid, indefensible, like the acre (160 times 30¼ square yards).

A magnitude is often measured indirectly, perhaps by substituting for it and its standard unit two other magnitudes know to have the same quantuplicity relation; thus an angle may be measured indirectly by using two arcs; the thermometer serves for the indirect measurement of a temperature by use of two volumes; a mass is usually measured indirectly by use of two weights at the same station.

CHAPTER XII.

MENSURATION.

Never forget that no exact measurement is ever possible, that no theorem of arithmetic, algebra, or geometry could ever be proved by measurement, that measure could never have been the basis or foundation or origin of number.

But the approximate measurements of life are important, and the best current arithmetics give great space to mensuration.

Geometry is an ideal construct.

Geometry.

Of course the point and the straight are to be assumed as elements, without definition. They are equally immeasurable, the straight in Euclidean geometry being infinite. What we first measure and the standard with which we measure it are both *sects*. A sect is a piece of a straight between two points, the end points of the sect. The sides of a triangle are sects.

A *ray* is one of the parts into which a straight is divided by a point on it.

An *angle* is the figure consisting of two coinitial rays. Their common origin is its vertex. The rays are its sides.

When two straights cross so that the four angles made are congruent, each is called a *right angle*.

One ninetieth of a right angle is a *degree* (1°).

A *circle* is a line on a plane, equidistant from a point of the plane (the center). A sect from center to circle is its radius.

An *arc* is a piece of a circle. If less than a semicircle it is a minor arc.

One quarter of a circle is a *quadrant.*

One ninetieth of a quadrant is called a *degree of arc.*

A sect joining the end points of an arc is its *chord.*

A straight with one, and only one, point in common with the circle is a *tangent.*

Length of a sect.

To measure a sect is to find the number L (its length) when the sect is conceived as $Lu+r$, where u is the standard sect and r a sect less than u. In science, u is the centimeter.

Thus the length, L, of the diagonal of a square centimeter, true to three places of decimals, is 1.414.

Since there are different standard sects in use, it is customary to name u with the L. Here 1.414 cm.

Knowing the length of a sect, from our knowledge of the number and the standard sect it multiplies we get knowledge of the measured sect, and can always approximately construct it.

Length of the circle.

We assume that with every arc is connected one, and only one, sect not less than the chord, and if the arc be minor, not greater than the sum of the sects on the tangents from the extremities of the arc to their intersection, and such that if the arc be cut into two arcs, this sect is the sum of their sects. The length of this sect we call the *length of the arc.*

If r be the length of its radius, the length of the semicircle is πr.

Archimedes expressed π approximately as $3+1/7$.

True to two places of decimals, $\pi=3.14$ or 3.1416 true to four places.

The approximation $\pi = 3 + 1/7$ is true to three significant figures. But since $\pi = 3.1416 = 3 + 1/7 - 1/800$, a second approximation, true to five significant figures, can be obtained by a correction of the first.

Again $\pi = 3.1416 = (3 + 1/7)(1 - .0004)$, which gives the advantage that in a product of factors including π, the value $3 + 1/7$ can be used and the product corrected by subtracting four ten-thousandths of itself.

The circle with the standard sect for radius is called the *unit circle.* The length of the arc of unit circle intercepted by an angle with vertex at center is called the *size* of the angle.

The angle whose size is 1, the length of the standard sect, is called a *radian.*

A radian intercepts on any circle an arc whose length is the length of that circle's radius.

The number of radians in an angle at the center intercepting an arc of length L on circle of radius length r, is L/r. $180° = \pi\rho$.

An arc with the radii to its endpoints is called a *sector.*

Area.

The area of a *triangle* is half the product of the length of either of its sides (the base) by the length of the corresponding altitude, the perpendicular upon the straight of that side from the opposite vertex.

A figure which can be cut into triangles is a *polygon,* whose area is the sum of theirs. Its *perimeter* is the sum of its sides.

Area of Circle. In area, an inscribed regular polygon (one whose sides are equal chords) of $2n$ sides equals a triangle with altitude the circle's radius r and base the perimeter of an inscribed regular polygon of n sides.

A circumscribed regular polygon (one with sides on tangents) of *n* sides equals a triangle with altitude *r* and base the polygon's perimeter.

There is one, and only one, triangle intermediate between the series of inscribed regular polygons and the series of circumscribed regular polygons, namely that with altitude *r* and base equal in length to the circle. This triangle's area, $rc/2=r^2\pi$, is the *area of the circle,* $r^2\pi$.

From analogous considerations, the *area of a sector* is the product of the length of its arc by the length of half its radius.

Volume.

A *tetrahedron* is the figure constituted by four non-coplanar points, their sects and triangles.

The four points are called its *summits,* the six sects its *edges*, the four triangles its *faces.*

Every summit is said to be *opposite* to the face made by the other three; every edge opposite to that made by the two remaining summits.

A *polyhedron* is the figure formed by *n* plane polygons such that each side is common to two. The polygons are called its faces; their sects its edges; their vertices its summits.

One-third the product of the area of a face by the length of the perpendicular to it from the opposite vertex is the *volume of the tetrahedron.*

The *volume of a polyhedron* is the sum of the volumes of any set of tetrahedra into which it is cut.

A *prismatoid* is a polyhedron with no summits other than the vertices of two parallel faces.

The altitude of a prismatoid is the perpendicular from top to base.

A number of different prismatoids thus have the same base, top, and altitude.

If both base and top of a prismatoid are sects, it is a tetrahedron.

A *section* or *cross-cut* of a prismatoid is the polygon determined by a plane perpendicular to the altitude.

To find the *volume of any prismatoid*. Rule: Multiply one-fourth its altitude by the sum of the base and three times the cut, at two-thirds the altitude from the base.

Halsted's Formula: $V=(a/4)(B+3C)$.

All the solids of ordinary mensuration, and very many others heretofore treated only by the higher mathematics, are nothing but prismatoids or covered by Halsted's Formula.

A *pyramid* is a prismatoid with a point as top. Hence its volume is $aB/3$.

A circular *cone* is a pyramid with circular base.

A *prism* is a prismatoid with all lateral faces parallelograms.

Hence the volume of any prism $=aB$.

A circular *cylinder* is a prism with circular base.

A *right* prism is one whose lateral edges are perpendicular to its base.

A *parallelopiped* is a prism whose base and top are parallelograms.

A *cuboid* is a parallelopiped whose six faces are rectangles.

A *cube* is a cuboid whose six faces are squares.

Hence the volume of any cuboid is the product of its length, breadth and thickness.

The cube whose edge is the standard sect has for volume 1.

Therefore the volume of any polyhedron tells how oft it contains the cube on the standard sect, called the unit cube.

Such units, like the unit square, though traditional, are unnecessary.

A *sphere* is a surface equidistant from a point (the center).

A sect from the center to sphere is its radius.

A *spherical segment* is the piece of a sphere between two parallel planes.

If a sphere be tangent to the parallel planes containing opposite edges of a tetrahedron, and sections made in the sphere and tetrahedron by one plane parallel to these are of equal area, so are sections made by any parallel plane. Hence the volume of a sphere is given by Halsted's Formula.

$$V=(a/4)(B+3C)=(3/4)aC.$$

But $a=2r$ and $C=(2/3)r\pi(4/3)r$.

So Vol. sphere $=(4/3)\pi r^3$.

Hence also the volume of a spherical segment is given by Halsted's Formula.

Area of sphere $=4\pi r^2$.

The area of a sphere is quadruple the area of its great circle.

As examples of solids which might now be introduced into elementary arithmetic, since they are covered by Halsted's Formula, may be mentioned: oblate spheroid, prolate spheroid, ellipsoid, paraboloid of revolution, hyperboloid of revolution, elliptic hyperboloid, and their segments or frustums made by planes perpendicular to their axes, all solids uniformly twisted, like the squarethreaded screw, etc.

CHAPTER XIII.

ORDER.

In the counting of a primitive group, any element is considered equivalent to any other. But in the use even of the primitive counting apparatus, the fingers, appeared another and extraordinarily important character, order.

If always when any two elements a, b of a set are taken, a definite criterion fixes one or other of two alternative relations, symbolized by a generalized use of $>$ and $<$, such that if $a < b$ then $b > a$, while if $a > b$ then $b < a$, and such that if $a < b$ and $b < c$, then $a < c$, we say the criterion arranges the set in order. So arranged, it becomes an ordered set.

The savage in counting systematically begins his count with the little finger of the left hand, thence proceeding toward the thumb, which is fifth in the count. When number-words or number-symbols come to serve as extended counting apparatus, order is a salient characteristic. Each is associated with a definite next succeeding number. The set possesses intrinsic order.

By one-to-one adjunction of these numerals the individuals of a collection are given a factitious order, the familiar order of the number-set.

When the order is emphasized the number-names are modified, becoming first, second, third, fourth, etc., and are called ordinal numbers or ordinals, but this designation is now applied also to the ordinary forms, one, two, three, etc., when order is made their fundamental characteristic.

Depiction.

If we can so correlate each element of the set A with a definite element of the set B that two different elements of A are never correlated with the same element of B, the element of A is considered as depicted or pictured or imaged by the correlated element of B, its picture or image.

Such a correlation we call a *depiction* of the set A upon the set B. The elements of A are called the *originals.*

An assemblage contained entirely in another is called a component of the latter.

A *proper component* or *proper part* of an assemblage is an aggregate made by omitting some element of the assemblage.

Infinite.

An assemblage is called *infinite* if it can be depicted upon some proper part of itself, or distinctly imaged, element for element, by a constituent portion, a proper component of itself. Otherwise it is *finite.*

Stand between two mirrors and face one of them. Your image in the one faced will be repeated by the other. If this replica could be separately reflected in the first, this reflection imaged by itself in the second, this image pictured as distinct in the first, this in turn depicted in the second, and so on forever, this set would be infinite, for it is depicted upon the proper part of it made by omitting you. It is *ordered.* You may be called 1, your image 2, its image 3, and so on.

Sense.

A relation has what mathematicians call *sense,* if, when A has it to B, then B has to A a relation different, but only in being correlatively opposite. Thus "greater than" is a sensed relation. "Greater than" and "less than" are different relations, but differ only in sense.

Any number of numbers, all individually given, form a finite set. If numbers be potentially given through a given operand and a given operation, law, of successive eduction, they are still said to form a set. If the law educes the numbers one by one in definite succession, they have an *order,* taking on the order inherent in time or in logical or causal succession.

A set in order is a *series.*

Analysis of order.

Intrinsic order depends fundamentally upon relations having sense, and, for three terms, upon a relation and its opposite in sense attaching to a given term.

The unsymmetrical sensed relation which determines the fixed order of sequence may be thought of as a logic-relation, that an element shall involve a logically sequential element creatively or as representative. An individual or element 1 has its shadow 2, which in turn has its shadow 3, and so on.

Linear order is established by an unsymmetrical relation for one sense of which we may use the word "precede," for the opposite sense "follow."

The ordering relation may be envisaged as an operation, a transformation, which performed upon a preceding gives the one next succeeding it; turns 1 into 2, and 2 into 3, and so on.

If we have applicable to a given individual an operation which turns it into a new individual to which in turn the operation is applicable with like result, and so on without cease, we have a recurrent operation which recreates the condition for its ongoing. If in such a set we have one and only one term not so created from any other, a first term, and if every term is different from all others, we have a commencing but unending ordered series. The number series, 1, 2, 3, and so on, may be thought of as the outcome of a recurrent operation, that of the ever repeated adjunction of one more unit. It is a system such that for every element of it there is always one and only one next following. This successor may be thought of as the depiction of its predecessor. Every element is different from all others. Every element is imaged. There is an element which though imaged is itself no image.

Thus the series is depicted without diminution upon a proper part of itself; is infinite, and by constitution endless. It has a first element, but no element following all others, no "last" element.

Any set which can be brought into one-to-one correspondence with some or all of the natural numbers is said to be *countable,* and if not finite, is called *countably infinite.*

Ordered set.

An order of a set is constituted by a relation between the elements of the set. The same set may have at the same time many different orders. The particular order is defined by the particular serial or arranging relation.

A set of elements is said to be in simple order if it has two characteristics:

1°. Every two distinct arbitrarily selected elements, A and B, are always connected by the same unsymmetrical relation, in which relation we know what rôle one plays, so that always one, and only one, say A, comes before B, is source of B, precedes B, is less than B; while B comes after A, is derived from A, follows A, is greater than A.

2°. Of three elements ABC, if A precedes B, and B precedes C, then A precedes C.

So an arranging relation implies diversity of the elements, is transitive, and connects any two different elements related by it to a third. Thus the moments of time between twelve and one o'clock, and the points on the sect AB as passed in going from A to B are simply ordered sets.

Two ordered sets A, B are called *similar* when a one-to-one correspondence can be established between their elements such that if $a < a'$ in A then their correlates $b < b'$. Similar ordered sets are said to have the same *order-type.*

Finite ordinal types.

An arranged finite set of, say, n elements can be brought into one-to-one correspondence with the first n integers.

Such an ordered set has a first and a last element; so has each ordered component.

Inversely an ordered set with a first and last element, whose every component has a first and a last element, is finite. For let a_1 be the first element. The remaining elements form an ordered component; let a_2 be the first of these elements. In the same way determine a_3. We must thus reach the last, else were there an ordered component without last element, contrary to hypothesis. These then are the characteristics of the finite ordinal types.

Number series, type of order.

Any set equivalent to the natural number series (the natural scale) is called *countably infinite.*

The characteristic property of a countably infinite set, when arranged in countable order, is that we know of any element a whether, or no, it corresponds to a smaller integer than does the element b. Should a and b correspond to the same integer they would be identical. Thus when arranged in countable order, the order of any countably infinite set is that of the natural numbers. The defining characteristics of this ordered set are that it, as well as each of its ordered components, has a first element, and that every element, except the first, has another immediately preceding it; while each element has one next following, and consequently, there is no last element.

Any simply ordered set between any pair of whose elements there is always another element is said to be in *close order.*

Well-ordered sets.

A simply-ordered set is said to be "well-ordered" if the set itself, as well as every one of its components, has a first element.

In a well-ordered set its elements so follow one another according to a given law that every element is immediately followed by a completely determined element, if by any. As typical of well-ordered sets we may take first the finite sets of the ordinal numbers: 1st; 1st, 2d; 1st, 2d, 3d; and so on.

As typical of the first transfinite well-ordered set we may take the set of all the ordinal numbers, the ascending order of the natural numbers.

The thousandth even number is immediately followed by the number 2001.

But if a point B is taken on a sect AC, there is no next consecutive point to B determinable.

The way in which an iterative operation develops from an individual operand not only infinity but endless variety unthought of and so waiting to be thought of, lights up the fact that mathematics though deductive is not troubled with the syllogism's tautology but offers ever green fields and pastures new. Thus in the number series is the series of even numbers, in this the set of even even numbers, 4, 8, 12, 16, 20, etc., each a system in which every element of every preceding system of this series of systems can have its own uniquely determined picture, the first term depicting any first term, the second any second, etc.

CHAPTER XIV.

ORDINAL NUMBER.

Ordinal number.

Numbers are ordinal as individuals in a well-ordered set or series, and used ordinally when taken to give to any one object its position in an arrangement and thus to individually identify and place it in a series.

The ordinal process has also as outcome knowledge of the cardinal; when we have in order ticketed the ninth, we have ticketed nine. Thus the last ordinal used tells the result of the count, being given a cardinal meaning to denote the particular plurality of the set now ticketed.

Children's counting.

The assignment of order to a collection and ascertainment of place in the series made by this putting in order is shown by that use of *count* which occurs in children's games, in their *counting out* or counting to fix who shall be *it*. This counting is the use of a set of words not ever investigated as to multiplicity, but characterized by order. Such is the actually-used set: ana, mana, mona, mike; bahsa, lona, bona, strike; hare, ware, frounce, nack; halico, baliko, we, wo, wy, wak. Applied to an assemblage, it gives order to the assemblage until exhausted, and the last one of the ordered but unnumbered group is *out* or else *it*. How many individual words the ordering group contains is never once thought of. There is successive enumeration without simultaneous apprehension.

Every element has an ordinal significance. No element has any cardinal significance.

E nee, me nee, my nee, mo;
Crack ah, fee nee, fy nee, fo;
Amo neu ger, po po tu ger;
Rick stick, jan jo.

Such a group but indefinitely extensible, having a first but no last term, is the ordinal number series.

Uses of ordinals.

But in our ordinary system of numeral words, with fixed and rote-learned order, each word is used to convey also an exact notion of the multiplicity of individuals in the group whose tagging has used up that and all preceding numerals. Thus each one characterizes a specific group, and so has a cardinal content.

Yet it is upon the ordered system itself that we chiefly rely to get a working hold of the number when beyond the point where we try to have any complete appreciation, as simultaneous, of the collection of natural units involved. Thus it is to the ordinal system that we look for succor and aid in getting grasp and understanding particularly of numbers too great for their component individual units to be at once and together separately picturable. Thus the ideas we get of large numbers come not from any attempt to realize the multiplicity of the discrete manifold, but rather from place in the number-set.

Number in its genesis is independent of quantity, and number-science consists chiefly, perhaps essentially, in relations of one number in the number series to another and to the series.

That a concept is dependent for its existence upon a word or language-symbol is a blunder. The savage has number-concepts beyond words. On the other hand, the modern child gets the words of the ordinal series before the cardinal concepts we attach to them. If a little child says, "Yes, I can count a hundred," it simply means it can repeat the series of number-names in order. Its slips would be skips or repetitions. The ordinal idea has been formed. It is used by the child who recognizes its errors in this ordinal counting. The ordinal idea has been made, has been embodied perhaps in rythmical movements. The child's rudimentary counting set is a sing-song ditty. The number series when learned is perhaps chanted. Just so there is a pleasurable swing in the count by fives.

The use of the terms of the number series as instruments for individual identification appears in the primitive child's game. Before making or using number, children delight in making series. Succession is one of the earliest made thoughts.

We think in substituted symbols. It is folly to attempt to hold back the child in this substitution. The abstractest number becomes a thing, an objective reality.

Number has not originated in comparison of quantity nor in quantity at all. Number and quantity are wholly independent categories, and the application of number to quantity, as it occurs in measurement, has no deeper motive than one of convenience.

It has often been stressed, that children knowing the number-names, if asked to count objects, pay out the series far faster than the objects; the names far outstrip the things they should mate.

The so-called passion of children for counting is a delight in ordinal tagging, in ordinal depiction with names, with no attempt to carry the luggage of cardinals.

The "which one" is often more primitive and more important than the "how many." The hour of the day is an ordinal in an ordered set. Its interest for us is wholly ordinal. It identifies one element in an ordered set. The strike of the clock is a word. The striking clock has a vocabulary of 12 words. These words are distinguished by the cardinal number of their syllables. But even when recognized by the cardinal number of syllables in its clock-spoken name, the hour is in essence an ordinal.

So the number series as a word-song may well in our children precede any application to objects. Objects are easily over-estimated by those who have never come to the higher consciousness that objects are mind-made, that every perception must partake of the subjective.

Children often apply the number-names to natural individuals as animals might, that is without making any artificial or man-made individual, and so without any cardinal number. Each name depicts a natural individual, but not as component of a unity composed of units. What passes for knowledge of number among animals is only recognition of an individual or an individual form.

Serial depiction under the form of tallying or beats or strokes may precede all thought of cardinal number. Nine out of ten children learn number names merely as words, not from objects or groups.

The typical case is given of the girl who could "count" 100 long before she could recognize a group of seven objects.

The names of the natural numbers are an unending child's ditty, primarily ordinal, but a ditty to whose terms cardinal meanings have also been attached. Ordinally the number name "one" is simply the initial term of this series; any number name is simply a term of this series. The ordinal property it designates is the positional property of an element in a well-ordered set, the place of an individual in a series.

The natural scale is the standard for civilized counting. Its symbols in sequence are mated with the elements of an aggregate and the last symbol used gives the outcome of the count, tells the cardinal number of the counted aggregate. The cardinal, *n*, of a set is that attribute by which when the set's elements are coupled with ordinals, the ordinal *n* and all ordinals preceding *n* are used.

The very first step in the teaching of arithmetic should be the child's chanting of the number names in order. Then the first application should be ordinal. Use the numbers as specific tags, conveying at first only order and individual identification. Afterward connect with each group, as *its* name, the last numeral it uses, which thus takes on a cardinal significance.

Nominal number.

Modern civilization has brought out a use of numbers neither ordinal nor cardinal. It is their employment as mere proper nouns. His number is the convict's name. This use of what may be called *nominal* number has reached its highest social development in the telephone. Since the size of the number and its place in the number series are here alike irrelevant, the whole stress falls upon its recognition as a unique name made by the juxtaposition in linear order of ten simple symbols, the nine digits and the zero. And these symbols must be orally conveyed to a girl whose vocabulary is so meager it does not contain the word triple. So 333 is read three, three, three. But the profoundest development is that zero has dropped everything but its adventitious Italian ending o, and so evolved a new name, oh! Thus *The Saturday Evening Post,* Aug. 5, 1911, p. 11, has "Six-oh-nine-two Nassau"; for the telephone rejoices also in a family name. Thus "The Thousand and One Nights," as a telephone name reads: One-oh-oh-one Nights. But surely in these proper names the family should come first as in Magyar, Bolyai John, and we should have Greeley oh-oh-oh. Since both intrinsic and local value have vanished, there are 111 more nominal than cardinal numbers before 100. Among nominal numbers, the additional class corresponding to no new cardinals may be called roundheads, e. g., 00, 01, 02, etc., 000, 030, 099.

CHAPTER XV.

THE PSYCHOLOGY OF READING A NUMBER.

Our marvelous positional notation for number is built of three elements, digit, base, column. The base it is which interprets the column. With base ten, 100 means a ten of tens. With base two, 100 means two twos. With base twelve, 100 means a dozen dozen.

The Romans had a base, or rather two bases, but neither digits nor columns. Their V is a trace of the more primitive base five, seen also in the Greek πεμπάζω, to finger fit by fives, to count. This, combining with the more final base ten, X, explains their having a separate symbol, L, for fifty, and D for five hundred.

Their ten of tens has its unitary symbol, C, and their ten of hundreds is M, a thousand.

Each basal number is a new unit, an atom, a monad, a neomon, squeezing into an individual the components, making thus one ball to be further played with.

Our present basal number-word, hundred, is properly a collective noun, a hundred, literally a tenth count or tale; for its *red* is the root in German *Rede,* talk, our *rate,* reckon, and its *hund* is the Old English word, cognate with Latin *centum,* Greek ἑκατόν, to be found in Bosworth's *Anglo-Saxon Dictionary,* but seldom used after A. D. 1200.

The *Century Dictionary,* to which I may be forgiven for being attached, says *hund* is from the root of ten, and this leads it far, into the postulating of an assumed type *kanta* which it gives as a reduced form of an equally hypothetical *dakanta* for an assumed original *dakan-dakan-ta*, "ten-ten-th," from assumed *dakan*, on the analogy of the Gothic *taihun-taihund, taihun-tēhund,* a hundred, of which it regards *hund* as an abbreviation or reduced form. The same original elements, it says, without the suffix *d* = *th*, appear in Old High German *zehanzo* = Anglo-Saxon *teón-tig, ten-ty* = *ten-ten.*

The element *hund* occurring in the Anglo-Saxon *hund-seofontig,* seventy, etc., *hund-endlefontig,* eleventy, *hund-twelftig,* twelfty, it gives as representing "ten" or "tenth," and these words as developed by cumulation (*hund* and *tig* being ultimately from the same root, that of "ten") from the theoretically assumed *hund-seofon,* "tenth seven," etc. Murray is not well persuaded of all this, and says there is no satisfactory explanation of the use of *hund* in these Anglo-Saxon words.

However that may be, just as, in Latin, *de-cem* gives *centum,* so *t-enth* gives *hund,* in each case the dental, or better, lingual, dropping away. Moreover, with us this *enth* or *hund,* with Saxon dogged persistence, reappears in *thous-and,* as shown by the Icelandic *thúsund, thús-hund, thúshundrad,* though Latin here takes a new start with *mille,* the Sanskrit root *mil,* to unite, to combine, seen also in *miles,* a soldier, and *militia.* Perhaps our prefix *thous,* Icelandic *thús,* is Teutonic *thu,* Aryan *tu,* to swell, seen in *tumor.*

So our "a hundred" is an abbreviation for the phrase "a tenth reckoning [of decads]."

This is consonant with the fact that in Old Norse the word *hundrath,* "hundred," "tenthtale," originally meant 120; it was a tenthtale not of tens but of dozens, the rival base twelve, against which the bestial base ten, an Old-Man-of-the-Sea saddled upon us by our pre-human simian ancestors, has been continuously fighting down to this very day. And even in modern English, remnants of this older usage remain. The *Glasgow Herald* of September 13, 1886, says: "A mease [of herring]....is five hundreds of 120 each."

Chambers Cyclopaedia says: "Deal boards are six score to the hundred."

This hundred was legal for balks, deals, eggs, spars, stone, etc.

Peacock, in the *Encyclopaedia Metropolitana,* I, 381, says: "The technical meaning attached by merchants to the word 'hundred' associated with certain objects, was six score—a usage which is commemorated in the popular distich or Old Saw:

> "Five score of men, money and pins,
> Six score of all other things."

Just so the Norwegians and Icelanders have two sorts of thousand, the lesser and the greater, the lesser $=10\times 100$, but the greater $=12\times 100$; and this latter is called *tolfraed,* twelfth-red, a word the exact analogue of our hund-red, tenth-reckoning.*

All this abundantly proves that our hundred is very far from being a simple numeral adjective, like e. g., seventy; so that while we properly say seventy-five, to say a hundred-five is a hideous blunder.

Hundred is strictly not an adjective at all, but a collective noun; it is always preceded by a definitive, usually an article or a numeral, and if followed by a numeral, this must invariably be preceded by the word "and."

A following noun is, historically, a genitive partitive,

* Hickes, *Institutiones Grammaticae,* p. 43.

in Old English a genitive plural, later a plural preceded by "of." Thus 1663, Gerbier, *Counsel,* "About one hundred of Leagues." Hale (1668): "These many hundred of years." Cowper (1782) *Loss of Royal George*: "Eight hundred of the brave." To-day: "A hundred of my friends," "A hundred of bricks," "Some hundreds of men were present." [Murray].

Even if there be an ellipsis of "of" before the noun, the word hundred retains its substantival character so far as to be always preceded by "a" or some adjective. Compare "dozen," which has precisely parallel constructions, e. g., "a dozen of eggs." Hooke (1665): "A hundred and twenty-five thousand times bigger." Murray's *Dictionary* (1901) gives as model modern English: "Mod. The hundred and one odd chances." Again it says: "c. The cardinal form *hundred* is also used as an ordinal when followed by other numbers, the last of which alone takes the ordinal form: e. g., 'the hundred-and-first,' 'the hundred-and-twentieth,' 'the six-hundred-and-fortieth part of a square mile.'" Goold Brown, *The Grammar of English Grammars*: "Four hundred and fiftieth."

All this furnishes complete explanation and warranty of the "and" which must always separate "hundred" from a following numeral. It marks a complete change of construction: "a hundred of leagues and three leagues"; "a hundred and three leagues." This fine English usage is unbroken throughout the centuries. Thus, Byrhtferth's *Handboc* (about 1050): "twa hundred & tyn"; Cursor MS. 8886 (before 1300): "O quens had he [Solomon] hundrets seuen." *Myrr. our Ladye* (1450-1530) 309:

"Twyes syxe tymes ten, that ys to a hundereth and twenty."

Oliver Wendell Holmes, "The Deacon's Masterpiece":

"Seventeen hundred and fifty-five.
Georgius Secundus was then alive,—
Snuffy old drone from the German hive."

The *London Times* of February 20, 1885: "The hundred and one forms of small craft used by the Chinese to gain an honest livelihood."

The new *Encyclopaedia Britannica,* 11th Edition, 1911, Vol. 2, p. 523: "Thus we speak of one thousand eight hundred and seventy-six, and represent it by MDCCCLXXVI or 1876." Again, p. 526: "A set of written symbols is sometimes read in more than one way. Thus 1820 might be read as *one thousand eight hundred and twenty* if it represented a number of men, but it would be read as *eighteen hundred and twenty* if it represented a year of the Christian era."

Though all the numerals up to a hundred belong in common to all the Indo-European languages, the word thousand is found only in the Teutonic and Slavonic languages, and maybe the Slavs borrowed the word in prehistoric times from the Teutons.

Very naturally thousand is construed precisely like hundred: "Land on him like a thousand of brick"; "The Thousand and One Nights."

And just so it is with that marvelous makeshift *million,* "big thousand," Old French (1359) augmentative (Latin *mille,* a thousand +*-one* augmentative suffix).

Says Langland in Piers Plowman (1362) A, III, 255:

"Coveyte not his goodes
For Milions of Moneye."

And the divine Shakespeare [Henry V, Prol.], anticipating the telephonic oh for naught:

"Or may we cram
Within this wooden O the very casques
That did affright the air at Agincourt?
O, pardon! since a crooked figure may
Attest, in little space, a million!
And let us, ciphers to this great accompt,
On your imaginary forces work."

"Thus, we say six million three hundred and twenty thousand four hundred and thirty-six,"* which does not at all militate against our reading 0033 to the telephone girl as "oh, oh, three, three." The word which specifies the local value of the digit is best omitted when this local value is unimportant or is otherwise determined. The date 1911 read "nineteen eleven." The approximation $\pi = 3.14159265+$ read "*pi* equals three, point, one, four, one, five, nine, two, six, five, plus." Here, as in all decimals, the "point" fixes the local factor for every subsequent digit.

The country schoolmaster's use of "and" solely to indicate the decimal point is not merely bad form and stupid; it is criminal. It introduces a completely unnecessary ambiguity, doubt, anxiety into the understanding even of oral whole numbers, since he may end with a wretched fractional, such as hundredths, a retroactive dampener over all that has preceded it.

When that most spectacular of Frenchmen, who, like so many great Frenchmen, was an Italian, witness Mazarin, Lagrange, Cassini, etc, etc.,—when the comparatively unlettered Corsican, Napoleon, sat upon his white horse at a German jubilee while an official opened at him an address of felicitation, the great Captain began to be puzzled at the silent strained attention of those listeners who were supposed to understand German speech. He

* Whitney, *Essentials of English Grammar*, p. 94.

whispered to his aide, "Why do they not applaud?" "Sire," was the answer, "on attend le verbe." Just so when the country schoolmaster reads a number, one awaits the fractional!

Thus though we may now read Room 203 as "room two-oh-three" or as "room two, naught, three," or as "room two hundred and three," reading it "room two hundred three" remains an abominable *gaucherie,* a nauseating blunder.

CHAPTER XVI.

ARITHMETIC AS FORMAL CALCULUS.

The propositions of arithmetic, as the body of doctrine concerning numbers and certain operations by which numbers may be combined, are all deducible from a few assumptions.

In a formal calculus we suppose ourselves to know nothing of the elements (represented by letters) or their rules of combination (conventions by which two elements give a third) (represented by symbols) except our assumptions, which themselves are empty frames or forms.

If a specific meaning be read into the letters and symbols, a true proposition may result, or a false.

The logical deductions made from such empty frames must needs be formal, but this is of advantage in keeping the logic pure and unaffected by additional *unconscious* assumptions which might vitiate it.

We propose to treat a system, a Formal Calculus, which has arithmetic as a special interpretation.

Whatever has the properties laid down in the assumptions will of necessity have also the properties therefrom deducible.

We shall set up therefore a Formal Calculus of which Rational Arithmetic shall be merely a true special case.

This chapter is essentially a contribution from Dr. R. L. Moore, of the University of Pennsylvania.

Our elements are denoted by small italics, *a, b, c* ....

x, y, z, and may for convenience be called "positive integers," for which "integers" is only an abbreviation.

Equality is denoted by $=$, inequality by $\neq$; the equality of two elements meaning that either is everywhere replaceable by the other. Our two rules of combination are symbolized by $+$ and $\times$, and may for convenience be called addition and multiplication.

Assumptions.

I. 1. If a and b are given elements ($a=b$ or $a\neq b$), then $a+b$ (order considered) is a univocally determined element called "the sum, a plus b."

I. 2. Commutativity: $a+b=b+a$.

I. 3. Associativity: $a+(b+c)=(a+b)+c$.

I. 4. If $a\neq b$, then there is not more than one element, z, such that $a+z=b$.

II. 1. If a and b are given elements ($a=b$ or $a\neq b$), then $a\times b$ (written also simply ab) (order considered) is a univocally determined element called "the product, a by b."

II. 2. Commutativity: $a\times b=b\times a$.

II. 3. Associativity: $a\times(b\times c)=(a\times b)\times c$.

II. 4. If $a\times x=a\times y$, then $x=y$.

III. Distributivity: $a\times(b+c)=(a\times b)+(a\times c)$.

Theorem I. If $m=n\times x$, then $mn'=m'n$ is a necessary and sufficient condition that $m'=n'x$.

Proof: Firstly if $m=nx$* and $m'=n'x$, then $m(n'x)=mm'$.

Hence, by II 3, $(mn')x=(nx)m'$;
by II 2, $(mn')x=m'(nx)$;
by II 3, $(mn')x=(m'n)x$;

* The sign $\times$ between two letters will hereafter often be omitted and understood.

by II 2, $x(mn')=x(m'n)$;

by II 4, $mn'=m'n$.

Conversely, if $m=nx$ and $mn'=m'n$, then $(m'n)m=(mn')(nx)$.

Hence, by II 2, II 3, $(mn)m'=(mn)(n'x)$;

by II 4, $m'=n'x$

Definition 1: If $m=nx$, then and only then $x=m/n$.

If for a certain pair of integers, m and n, there is no integer x such that $m=nx$, and thus no integer equal to m/n, then if one wishes that in this case also there should be something which is equal to m/n, that m/n should enter our Formal Calculus as a new kind of element, he may choose something other than an integer which it would be convenient to *call* m/n. He is at liberty in this case to call anything (except an integer) m/n. Such a definition could never possibly contradict our assumptions or previous definitions since according to hypothesis there is no x such that, in sense of previous Definition 1, $m/n=x$. Now what shall we in this case call "m/n"? It is desired to establish a Formal Calculus which shall contain ordinary arithmetic. It is desired then that m/n, in this case also, and operations in which it is to figure, should be such that certain laws may be obeyed. One thing which is desirable then is that (as in the case when m/n is an integer) m/n and m'/n' here also shall mean the same thing only in case $mn'=m'n$. What sort of definition for m/n would satisfy this condition?

Evidently the following does:

Definition 2: If there is no x such that $m=nx$, then m/n means the set of all sensed pairs (p,q) such that $mq=pn$.

For: *Theorem* 2: If $mq=pn$, then $mn'=m'n$ is a necessary and sufficient condition that $m'q=pn'$.

Proof: If $mn'=m'n$ and $mq=pn$ then $(mq)(m'n)=(pn)(mn')$. Hence, by II 2 and II 3, $(mn)(m'q)=(mn)(pn')$.

From Definition 2 and Theorem 2 it is seen that if there is no x such that $m=nx$, then $mn'=m'n$ is a necessary and sufficient condition that $m/n=m'/n'$; this is easily seen except perhaps for an obstacle which one may indicate thus: Suppose it should happen that, even though $mn'=m'n$ and there is no x such that $m=nx$, still there is an x such that $m'=n'x$ and thus m'/n' may not be the set which m/n is according to Definition 2. But if $mn'=m'n$, and there is no x such that $m=nx$ then, by Theorem 1, there can be no x such that $m'=n'x$.

From this result and Theorem 1 we have finally:

Theorem 3: In any case $mn'=m'n$ is a necessary and sufficient condition that $m/n=m'/n'$.

Theorem 4: If $m=nx$ and $m'=n'x'$ then 1° $m/n+m'/n'=(mn'+nm')/nn'$ and 2° $m/n\times m'/n'=mm'/nn'$.

Proof: 1°. $mn'+nm'=(nx)n'+n(n'x')$.

Hence, by II 2, II 3, and III.

$$mn'+nm'=nn'(x+x').$$

Hence, by Definition 1

$$x+x'=(mn'+nm')/nn'.$$

2°. $mm'/nn'=(nx)(n'x')/nn'$.

Hence, by II 2, II 3,

$$mm'/nn'=(nn')(xx')/nn'.$$

Hence, by Theorem 3,

$$(mm')(nn')=[(nn')(xx')](nn').$$

Hence, by II 2 and II 4,

$$mm'=(nn')(xx').$$

Hence, by Definition 1,

$$mm'/nn' = xx'.$$

Definition 3: If for any particular integers m, n, m', n', either there is no x such that $m = nx$ or there is no x' such that $m' = n'x'$, then 1°. $m/n + m'/n'$ means $(mn' + nm')/nn'$ and 2°. $m/n \times m'/n'$ means mm'/nn'.

In order that there should be no contradiction here, in order that this may not be defining one thing as being the same as two different things, it is necessary that this following theorem should be true:

Theorem 4: If $m/n = a/b$, and $m'/n' = a'/b'$, then 1°. $(mn' + nm')/nn' = (ab' + ba')/bb'$; and 2°. $mm'/nn' = aa'/bb'$.

Proof: 1°: By hypothesis and Theorem 3, $mb = an$ and $m'b' = a'n'$. Hence,

$$(mb)(b'n') = (an)(b'n').$$
$$(m'b')(bn) = (a'n')(bn).$$

Hence, $(mb)(b'n') + (m'b')(bn) = (an)(b'n') + (a'n')(bn)$.

Hence, by II 2, II 3, III, $(mn' + nm')bb' = nn'(ab' + ba')$.

Hence, by Theorem 3, $(mn' + nm')/nn' = (ab' + ba')/bb'$.

2°. From $mb = an$ and $m'b' = a'n'$ it follows that

$$(mb)(m'b') = (an)(a'n').$$

Hence, by II 2 and II 3, $(mm')(bb') = (aa')(nn')$.

Hence, by Theorem 3, $mm'/nn' = aa'/bb'$.

Theorem 5: In any case, $m/n + m'/n' = (mn' + nm')/nn'$, and $(m/n \times m'/n') = mm'/nn'$.

Proof: See Theorem 4 and Definition 3.

Definition 4: If m and n are any two integers (the same or different) then m/n is called a *positive fraction,* for which *fraction* is only an abbreviation. Conversely, every fraction is m/n, where m and n are integers (the same or different).

Theorem 6: Every integer is a fraction.

Proof: If m is an integer, then, by Definition 1, $mm/m = m$. Hence, by II 2 and Definition 4, m is a fraction.

Capital letters are used here to designate fractions only.

Theorem 7: If A, B, C are fractions, then the following statements are true:

F 1. $A+B$ and $A \times B$ are fractions.

F 2. $A+B=B+A$ and $A \times B = B \times A$.

F 3. $(A+B)+C=A+(B+C)$, and $(AB)C= A(BC)$.

F 4. There is not more than one fraction, D, such that $A+D=B$, and there is not more than one fraction E such that $A \times E = B$.

F 5. There is a fraction F such that $A \times F = B$.

F 6. There is a fraction G such that, if H is any fraction whatsoever, then, $GH=H$.

F 7. $A(B+C)=AB+AC$.

Proof of F 1: See Theorem 5, Definition 4, I 1 and II 1.

Proof of F 2:

a. By Theorem 5, $m/n+m'/n'=(mn'+nm')/nn'$.

Hence, by I 2 and II 2, $m/n+m'/n'=(m'n+n'm)/nn'$.

Hence, by Theorem 5, $m/n+m'/n'=m'/n'+m/n$.

b. By Theorem 5, $(m/n\times m'/n')=mm'/nn'$.

Hence, by II 2 and Theorem 5, $(m/n\times m'/n')=(m'/n'\times m/n)$.

Proof of F 3:

a. By Theorem 5, $m/n+(m'/n'+m''/n'')$
$=m/n+(m'n''+n'm'')/n'n''$
$=[m(n'n'')+n(m'n''+n'm'')]/n(n'n'')$, which by III,
$=[m(n'n'')+n(m'n'')+n(n'm'')]/n(n'n'')$, which by II 3,
$=[(mn')n''+(nm')n''+(nn')m'']/(nn')n''$, which by II 2 and III,
$=[(mn'+nm')n''+(nn')m'']/(nn')n''$,
$=(mn'+nm')/nn'+m''/n''=(m/n+m'/n')+m''/n''$.

b. By Theorem 5 and II, 3,
$m/n\times(m'/n'\times m''/m'')=m/n\times(m'm''/n'n'')$
$=m(m'm'')/n(h'n'')=(mm')m''/(nn')n''$
$=(m/n\times m'/n')\times m''/n''$.

Proof of F 4:

a. If $a/b+x/y=c/d$ and $a/b+x'/y'=c/d$, then, by Theorem 5, $(ay+bx)/by=(ay'+bx')/by'$.

Hence, by Theorem 3, II 2 and II 3, $(ay)(by')+(bb)(xy')=(ay)(by')+(bb)(x'y)$.

Hence, by I 4, $(bb)(xy')=(bb)(x'y)$.

Hence, by II 4, $xy'=x'y$.

Hence, by Theorem 3, $x/y=x'/y'$.

b. If $(a/b \times x/y) = c/d$ and $(a/b \times x'/y') = c/d$, then by Theorem 5, $ax/by = ax'/by'$.

Hence, by Theorem 3, II 3 and II 2, $(ab)(xy') = (ab)(x'y)$.

Hence, by II 4, $xy' = x'y$.

Hence, by Theorem 3, $x/y = x'/y'$.

Proof of F 5:

A$=m/n$ and B$=m'/n'$, then A$\times(m'n/n'm) = m/n \times m'n/n'm$, which, by II 2 and II 3, $= (mn)m'/(mn)n'$.

But, by II 2 and II 3, $[(mn)m']n' = m'[(mn)n']$.

Hence, by Theorem 3, $(mn)m'/(mn)n' = m'/n'$.

Thus A$\times(m'n/n'm) =$ B.

Proof of F 6:

If H$=m/n$, and k is any integer, then by Theorem 5, $(k/k)H = km/kn$.

But, by II 2 and II 3, $(km)n = m(kn)$.

Hence, $km/kn = m/n$.

Hence, for any H whatever, (k/k)H$=$H.

Proof of F 7:

By Theorem 5, $m/n \times (m'/n' + m''/n'') = m/n \times [(m'n'' + n'm'')/n'n''] = m \times (m'n'' + n'm'')/[n \times (n'n'')]$, which by Theorem 3, II 2 and II 3, $= [(mm')(nn'') + (nn')(mm'')]/(nn')(nn'')$, which by Theorem 5, $= (m/n \times m'/n') + (m/n \times m''/n'')$.

Definition:

$m/n > m'/n'$ means there exist x, y such that $m/n = m'/n' + x/y$. $m/n < m'/n'$ means $m'/n' > m/n$.

Assumption IV: A necessary and sufficient condition that integer a should be different from integer b is the existence of an integer x such that either $a+x=b$ or $b+x=a$.

If this assumption IV is added to the others, then the following additional statements may be added in Theorem 7:

F 8. Either $A < B$, $A = B$, or $A > B$. But no two of these three statements are simultaneously true.

F 9.* If $A > B$, and $B > C$, then $A > C$.

F 10. If $A > B$, then $A + C > B + C$, and $AC > BC$.

* F 9 and F 10 may be proved without use of IV.

CHAPTER XVII.

ON THE PRESENTATION OF ARITHMETIC.

FIRST GRADE.

All schools heretofore have commenced the study of number by asking and considering the answers to the questions, "how many?" "how much?" "how far?" "how long?" They have thus begun with the cardinal, and with it alone have continued. Thus all teaching of the beginnings of arithmetic has unconsciously overlooked and missed the more fundamental and prerequisite question, "which one?", and so remained unconscious of, and blind to the infinitely precious and in fact indispensable succor and aid of order, of the ordinal.

Previous blunders.

Had study of the child been fructified by foreknowledge of the modern higher mathematics, it could not have overlooked in the spontaneous creative activities of the child, the prominence and absolutely basal character of the ordinal, non-cardinal ideas, the serial, arranging and identifying ideas, historically and developmentally preceding and prerequisite for the very apparatus subsequently used for the ascertainment of the "how many."

Begin with ordinals.

In the counting of a primitive group, any element is considered equivalent to any other. But in the use even of the primitive counting apparatus, the fingers, appeared another and extraordinarily important character, order.

The savage, in counting, systematically begins his count with the little finger of the left hand, thence proceeding toward the thumb, which is fifth in the count. When number-words come to serve as extended counting apparatus, order is not only a salient but an absolutely essential and indispensable characteristic of the apparatus. The number series, 1, 2, 3, and so on, is a system such that for every element of it there is always one and only one next following.

Numbers are ordinal as individuals in a well-ordered set or series, and used ordinally when taken to give to any one object its position in an arrangement and thus individually to identify and place it.

Cardinal from ordinal.

The ordinal process has also as outcome knowledge of the cardinal. When we have in order ticketed the ninth, we have ticketed nine. Thus the last ordinal used tells the result of the count. But this very ordering process precedes all cardinal ideas, as is shown by that use of *count* which occurs in the spontaneous games of little children, in their *counting* out or counting to fix who shall be *it*.

Ordinal counting.

This counting is characterized by order pure and simple. There is successive designation with no attempt at simultaneous apprehension, simply the assignment of order to a collection and the ascertainment of place in the series made by this putting in order. Our instrument for this is the number series, and it is upon the order in the system that we ourselves rely to get a working hold of the individual number, especially when beyond the point where we can have any complete appreciation of the simultaneous multiplicity of the units involved in the corresponding cardinal.

It is fortunate then, and natural, that the modern child, despite the blindness of its teachers hitherto, gets the words of the ordinal series before it gets the cardinal concepts we attach to them.

The ordinal coherence of the number series and its independence of cardinal concepts is shown by the child. Each name depicts a natural individual, not the so-far group of natural individuals, not a new kind of unity composed of units.

Our apparatus for the ascertainment of cardinal number involves, is based upon and uses order, ordinal number.

Cardinal counting.

The child should be counting up to a hundred before it can recognize a group of seven objects. When the symbols of the number series, the natural scale, are mated in sequence with the elements of an aggregate, the last symbol used is also taken as designation of the particular whole set so far used, and this identification of the unknown set with a known set it is which gives the cardinal property or quality of the hitherto unknown set. In this sense we say the last symbol used gives the outcome of the count, tells the cardinal number of the counted aggregate.

First of all then let the teacher put out of her mind the blunder, pedagogic as well as scientific, that number

Number precedes measure.

was in any way dependent upon measurement for origin. Number was created and used for individual ordering and identification and for group identification centuries before any measurement. There are tribes now using number that never have used measurement. All natural children use number long before measurement can even be explained to them. Measurement is a recondite device. Number is enormously more simple and primitive. Its uses in identification both of individuals and groups are vastly important and quite independent of measurement. They long precede any thought of measurement.

The number concepts are wholly apart from measurement, from length, from size, from the late-coming conventional standards for measurement, from the yard, the mile, the grain, the liter or any other standard for measurement. Valuation is a false associate for primitive number. Number implies no exact size image. Cardinal number is a quality of a group. Two eyes and an ear-ache is a less dangerous trio than three yards, lest the teacher make the mistake of supposing number in any way dependent upon measurement.

Cardinal number.

It is the acme of stupidity to attempt to found the number concepts upon "how much"; for example, "my desk is greater in length than in width."

Begin by letting the child sing the number names as far as it enjoys the singing. Follow this up by exercises in designating or tagging objects with these number-names as identifying tags. Paper horses may be used, named one, two, three, etc. Paper automobiles may be named, as the real ones are tagged, one, two, three, etc. Objects so tagged may be jumbled up and then arranged in the order of their names. Then differing objects, say the various differing animals in animal crackers, may be named, each with a number. Then the qualities of No. 2 may be contrasted with those of No. 4.

How to begin.

The children may each be given a number as name. The teacher and the children may invent games using the ordinal properties, carefully avoiding as yet any "how many."

Ordinal games.

(1) Thus an instructive ordinal game is using a set of ordinals to *count out* the class. Choose a set of ordinals, say the first nine. Distribute them in order and let the child to whom the nine comes be *out*. Then begin again with the remaining children and again distribute the ordinals in order, dropping as *out* the child upon whom the nine now falls. When there are only eight children remaining, the count will more than go around, and the child tagged with one will also be tagged with nine and so be *out,* etc.

(2) Give each child the same set of disarranged numbers. See who can arrange quickest.

(3) One, two;
Buckle my shoe.
Three, four;
Open the door.
Five, six;
Pick up sticks.
Seven, eight;
Lay them straight.
Nine, ten;
A big, fat hen.
Eleven, twelve;
Dig and delve.

(4) One, two, three, four, five;
I caught a bird alive.
Six, seven, eight, nine, ten;
I let it go again.

The call.

(5) One, two;
Glad to see you.
Three, four;

Open the door.
Five, six;
My dog does tricks.
Seven, eight;
Walk to the gate.
Nine, ten;
Please come again.

(6) Mix up nine blocks numbered from one to nine. Let the child draw them out of the heap and put down each in its relative place when drawn until all are arranged in their proper order.

(7) Hang about the neck of each of nine children a numbered tag. Let the children arrange themselves in order in line. Bend the line into a closed curve. Call out one number. The child so numbered goes within the enclosure. The others march about him. At a signal he calls a number. The child so designated takes his stand within the encircling line, and the caller finds his proper place in the line.

(8) Give a number to each animal in a Noah's ark. This so far is only a nominal number, a name for a natural individual. Then introduce the ordinal by letting the child arrange the numbered animals in accordance with their number-names. Animal crackers may be substituted for a Noah's ark.

(9) Have colored strips of paper numbered consecutively in correspondence with the colors in the primary rain-bow. Let the children arrange them in order to make a rain-bow.

(10) Shuffle a pack of numbered cards. Give the pack to the child to arrange in the order of the numbers.

(11) Let the aisles in the school-room be numbered streets, and the broad cross passage-ways numbered avenues, and each desk a numbered house. Let the children write and address notes giving the house address, and let a messenger-child carry and deliver the letters.

Addition:

Ordinal operations. In the ordered row of children ask: Which is the third after the second? Answer: the fifth.

Subtraction:

Which is the third before the fifth? Answer: the second.

Multiplication:

Which is the third second? Answer: the sixth.

Which is the second third? Answer: the sixth.

When the child is thoroughly familiar with the ordered names as applied to natural individuals, we are ready for their first application to artificial individuals of the group kind, and first the application of the ordinal two to a pair. Make couples, partners, pairs, mates, and call each pair two.

The simplest cardinal.

The cardinal two, the simplest cardinal, is that property of a set whereby it can be mated, one to one, with a child's thumbs, or it is the class of such sets.

The idea of a cardinal, belonging as it does to a set of things as a whole, is a comparatively late concept. It must follow the concept of a whole composed of parts, constituents permanently distinguishable. Later comes the attribution of the geometirc quality of relative size, big and little, to numbers.

For the next step make trios. The cardinal three is the class of all triplets, or that quality of a set whereby it can be mated, one to one, with a child's eyes and nose, also with the ordinal set one, two, three; the last of which is used as a tag or name for the group, the trio.

Triplets and quartets.

Quartets are groups mateable, individual to individual, with the fingers of the left hand, or the words one, two, three, four; the last of which is to be used as a name for every such set; and so on. There may follow in rich variety the construction, the identification, the tagging, of small groups. This is at last the "how many" idea. Let it first be the natural and useful question of simple identification of groups, recognition of like or unlike cardinal.

The "how many" idea.

Herein lies abundant opportunity for constructive work. Give the child the first five ordinals. Let him then construct groups whose name shall be five, consequently whose "how many" shall be five, the cardinal.

Explain how simple groups were used as symbols for the numeric quality of all like groups. Thus, II, III, IIII, are symbols for their own cardinal quality two, three, four. Then may come the Hindu symbols 2, 3, 4, primarily as ordinals, then secondarily as cardinals. Now is the time for cardinal counting, counting as group-identification, using first the ten different groups of fingers as known groups with one of which the unknown group is to be identified by setting up a one-to-one correspondence between the individuals of the unknown group and the individuals of a finger group. Then we go to cardinal counting using the first dozen groups of ordinal words as known groups. All in good time, a test that the idea of

Symbols.

Cardinal counting.

Recognition of the cardinal.

the cardinal has taken root and germinated, is practice in the instantaneous recognition of the cardinal of a small group suddenly exhibited, then veiled. The question "how many" is to be answered without conscious counting. Then larger groups may be used recognizable by use of symmetry in the arrangement or grouping, as on playing cards.

Cardinal addition.

Then we may begin to train for the instantaneous recognition from two components of the cardinal of their compound, for example, the thinking of seven upon seeing three and four. Here we should stop to train until every pair from one plus one up to nine plus nine arouses the image of its sum instantly and automatically. Coins, cents, nickels, dimes, dollars, are admirably adapted at this stage as anchors for the ideas created, while at the same time bringing home to the child the precious aid of number (anterior to any measurement) in the child's social relations, in the interest growing out of and attaching to the very life of the child itself. Games of buying, and perhaps actual buying, with the consequent paying and change-making, are here in place.

Constructive processes familiarize and endear to the child the ideal numeric creations.

Summary (First Grade).

Ordinal arithmetic, then cardinal.

A. Ordinal counting. Utilize the spontaneously child-created ordinal systems. Also rhymes and jingles.

B. The number symbols, 1, 2, 3, 4, 5, 6 etc. as ordinals.

C. Ordinal applications, identification, arrangement, factitious order.

D. Ordinal tagging.

E. Ordinal games.

F. Group-making, group distinction, group familiarization.

G. Group identification, cardinal counting.

H. Cardinal applications.

I. Cardinal games.

J. The number symbols as cardinals.

K. Positional notation for number.

L. Addition tables.

M. Coins and their applications and games.

N. Exercises in making conscious the number-needs of the child's own life, individual and social.

O. Problems oral and motor; ordinal; to be solved by ordinal identification and arrangement. Cardinal; to be solved by cardinal identification, by addition, by correlation.

SECOND GRADE.

The number work of the second grade, as in all grades, is to be related as closely as may be to the actually existing interests and immediate needs of the child.

Measurement.

Do not bend for a moment to the false and exploded idea that number was originated or created by measurement, a palpable absurdity, since we must already be able to count before we can measure, and since the preexistent counting is absolutely exact while no measuring ever can be exact. But now that the child has the prerequisite number-equipment, we may envisage measurement.

The "muchness" of a quantity is not determined by the "how many" parts in it, unless these be all of a fixed, a preestablished size. Hence in addition to, and outside of the number-ideas, the child must now be confronted with the new and difficult idea of definite conventional standards, the so-called units-for-measure or units of measure, the inch, the quart, the pound, the second, the degree. To measure is to break the thing up into pieces each equal to one of these standards, or a like standard, and to count the pieces. The child must combine his old knowledge of the number obtained with his new knowledge of the standard now used.

Measurement then can only come after much practice in counting. Finally begin measuring by measuring a length. Show that nothing would be gained here by actually breaking off the pieces, as we do in measuring milk. We need only see where they could be broken off. Now we are ready for the consideration of the actual problems presented to the child by its own occupations. It may be called upon to use so much rope or board or food. The outcome of the measurement is a graphic description in known terms, a number and a unit; and now inversely a metric description should evoke a graphic image, a picture.

Since mensuration is combined with arithmetic, there may be training to familiarize the various units and their subunits, yard, foot, inch, gallon, quart, pint, hour, minute, second, pound, ounce, gram, etc.

The decimal.

Now should be given a thorough-going presentation of our positional notation for number, and as the necessary extension of it, the decimal. Decimals are made up of the subunits inevitably designated by the extension of our positional notation to the right of the units' column.

As the self-interpreting extension of this positional notation for number to the right of the units' column, we have decimals. We need no new elements, nothing but the already mastered digits, base, column. The decimal is not a fraction; it has no denominator. Decimals are significant figures to the right of the units' column; to indicate units' column, we henceforth use the decimal point. One thousand (1000) means ten of such units as stand in the adjacent column to the right; and one of these, one hundred (100), means ten of such as stand in the next column; and one of these, ten (10), means ten of our primal units, such as stand in our units' column; and one of these, One (1), means ten of such as stand in the next column to the right, that is in the first column to the right of our units' column; and one of these, one-tenth, .1, has the same relation to one in the next column. We have an excellent available illustration in our coins. Taking the dollar as the primal unit, one-tenth, .1, is one dime or ten cents; .01 is one cent, or ten mills. These columns are to be named so that units' column be axis of symmetry; twenty (20) gives tens; so 0.2 gives tenths; three hundred (300) gives hundreds; so 0.03 gives hundredths; then 4000 gives thousands; so 0.004 gives thousandths.

As no new elements come with decimals, nothing but our old digits, base, column, so no new principle is involved in their addition, subtraction, multiplication and division. The child who has the equipment for interpreting 23 has that for interpreting 3.14159265. Our explanation of positional notation contains the explanation of "carrying" in addition. Whenever the digit X is reached in any column, it is carried, appearing as one in the next column to the left. So we have this word already available when we reach subtraction, which is always to be

Carrying.

Subtraction.

worked by addition. Look upon *difference* as the number which if added to the subtrahend gives the minuend. Thus to subtract,

$$\begin{array}{r} 9004 \\ 5126 \\ \hline 3878 \end{array}$$

Think six and *eight* make fourteen; carry 1; three and *seven* make ten; carry 1; two and *eight* make ten; carry 1; six and *three* make nine. We carry one to balance a one put in to facilitate our procedure. Thus in subtracting,

$$\begin{array}{r} 8\tfrac{1}{5} \\ 6\tfrac{2}{5} \\ \hline 1\tfrac{4}{5} \end{array}$$

say two-fifths and *four-fifths* make six-fifths; carry 1; seven and *one* make eight.

Fractions.

A fraction is an ordered number-pair where the second number, the denominator, tells what sort of units are represented by the first number, the numerator. Thus 2/3 means two of such units (subunits) that three of them make the primal unit.

Multiplication.

When we come to multiplication, the idea of column is to dominate. The fundamental admonition is: *Always keep your columns.* Always begin to multiply with left-most figure of the multiplier. Thus we get the most important partial product first. *Rule: The figure put down stands as many places to the right or left of the digit multiplied as the multiplier is from units' column.*

$$\begin{array}{l} 21.354 \\ 200.003 \\ \hline 4270.8 \\ 64062 \\ \hline 4270.864062 \end{array}$$

Another form of the rule is: Multiplying shifts as many places right or left as the multiplier is from units' column. Note as an important special case of our rule: *If of two* *figures multiplied one is in units' column, the figure put down stands under the other.*

Division. There are two interpretations of division, namely Remainder Division and Multiplication's Inverse. Remainder division may be taught before the multiplication of fractions. It is to find how many times one number, the divisor, is contained in another, the dividend; and what then remains. For example, if eggs are four cents apiece, how many can be bought for three nickels? Answer three Or in counting with a compound unit, the divisor, how many times is it taken before overstepping the dividend?

Historically it was in connection with measurement that fractions had their origin. By way of review and advance combined, we may now introduce subtraction, of course never to be worked by anything but addition, the "making change" method.

Again multiplication may now be introduced, with the tables for doubling, tripling, quadrupling. Here may be given the symbols, +, −, ×, /, ÷, =.

Pairs of numbers may now be exhibited for the child to give their difference; then pairs of numbers, the second number a 1, 2, 3, 4, or 5, for the child to give the product. For games we have dominoes, bean matching, and the like. Use the savage device of a row of men for counting, to make easy our positional notation for number. Thus familiarize digits of different orders.

Sticks and stick-bundles can be correlated with cents, nickels, dollars, halves, quarters. If the sticks be marked off in tenths, decimals may be illustrated.

Thus numbers of two and three orders are familiarized, as also the shifting of the decimal point. 9876 mills are 987.6 cents, or 98.76 dimes, or 9.876 dollars. Decimals and fractions are made simple by the idea of a principal unit and subunits.

Give each child a cheap foot rule; here inches are subunits. Actual familiarity with standards for measure is essential, the more so as these are no part of pure arithmetic or number, but only extraneous components of a device for the application of number, namely measurement.

Practice in simple multiplication, envisaged first as condensed addition, may go up through doubling, tripling, quadrupling, quintupling. Multiplication by ten is equivalent to shifting the decimal point to the right. Quintupling is shifting the point and halving. Measurements for the application of number knowledge to the attainment of ends desired by the child are in place, but the so-called "formal work" and "mechanical drill" may give more joy and interest to the child than any measurement.

From *Teachers College Record* we quote: "Upon being given their choice one morning between going to the new gymnasium and remaining in the room to learn a new multiplication table, all but three of a class of thirty chose the mental gymnastics. This is cited to show that much of the so-called 'formal work,' 'systematic mechanical drill,' which sounds so formidable to an outsider, may bring much delight to one of our eight year old children, and that the mechanism of number may be secured with no sacrifice of interest."

Summary (Second Grade)

A. The extra-arithmetical idea of a standard for measurement.

B. The usual standards for measure.

C. Explanation of "to measure."

D. Knowledge obtained by measuring is a combine of number-knowledge and knowledge of the standard.

E. Length, area, volume, capacity, weight, temperature; with their standards, foot, square, cube, quart, pound, degree.

F. Metric description evoking visual image.

G. Positional notation for number.

H. Decimals. Basal subunits. Significant figures to right of units column.

I. Fractions. Any subunits.

J. Subtraction. Difference.

K. Multiplication.

L. Symbols.

M. Games.

O. Change of unit. Shifting the decimal point.

P. Problems; written work.

Q. Multiplication tables through quintupling.

THIRD GRADE.

We are more than ever to aim at helping the development of the child in mental power, accuracy, and precision, mind-mastery, ability to direct and fix the attention, and withal to a distinct growth in technically arithmetical equipment for efficiency and life.

There very often seems here to bloom out spontaneously in the child a love for what has sometimes been called the abstract formal part of arithmetic. It is seen to give delight. The play-joy, which is perhaps a greater ingredient in pure science than has been suspected, now shows forth to illumine the work, and beautify the seemingly mechanical.

Review Work.

A. Counting with a compound unit, by 2's, by 3's, by 4's, by 5's, by 10's. Beginning with zero or any number.

B. Addition with "carrying."

C. Subtraction, with "carrying." (Never use any but the addition method.)

Advance Work.

D. Multiplication. Complete the tables through 8 constructively. Explain the nine, ten, and eleven tables, so that they need not be memorized. For example, to "nine-times" a digit, write the preceding digit and adjoin what it lacks of being nine: e. g., $9 \times 8 = 72$. Connect the eights with the fours. Written multiplication; begin always with the left-most figure of the multiplier.

E. Division. Two kinds, but teach first Remainder Division. First utilize the multiplication work. Teach to divide by one digit, then by two. Contrast remainder division and multiplication's inverse.

F. Decimals. The point in addition and subtraction. Shifting the point in multiplication and division.

G. Fractions. 1/2, 1/4, 1/8, 1/3, 1/6; change the subunit. Addition; subtraction.

H. Measurement. Square measure.

When objects are used, it should be remembered that after they have once served their purpose they only hamper children and teacher. But buying, selling, making change may often be used. Let the children, where possible, make their own problems. Groups of objects may be used to introduce division. Let a child realize what he is working to accomplish.

FOURTH GRADE.

A. A review of addition, subtraction, and multiplication; but a very extensive presentation and mastery of remainder division.

B. Verifications. Verify addition and subtraction by the commutative principle. Verify multiplication and division by the simplest method of casting out nines.

C. Multiplication's inverse. No remainder. Fraction in quotient.

D. Invention of problems.

E. Tests of accuracy and speed.

F. Measurements. Include decimals and fractions in the problems apart and together. Cubic.

G. Plotting on squared paper. Graphic representation.

H. Illustrations of the life-value of facility and accuracy in the four operations.

I. Divisibility. Factors. Multiples.

K. Emphasize the form of arrangement of written work.

FIFTH GRADE.

A. Decimals. The identity of decimal notation with the ordinary positional notation used throughout the first four grades.

B. Reading of all decimals in the new method.

C. Addition and subtraction shown to involve nothing new.

D. Illustrations from our money.

E. Multiplication of decimals; (all multiplications begin with the left-most figure of the multiplier). Shifting the point.

F. Division of decimals. Shifting the point.

G. Problems.

H. Checking results.

I. Percentage. Applications to discount, commission, simple interest. The one hundred months method for interest.

1. Find a percent of a number, (given number and rate).
2. Find what percent one number is of another.
3. To find a number from a given percent of it.

J. Geometric forms. Denominate numbers.

K. Prime numbers. Prime factors.

L. Business problems.

SIXTH GRADE.

Fractions.

Meaning of fractions. Gain by the notation.

A. Reduction, that is, change of the subunit.

B. Addition and subtraction. Meaning of these operations for fractions.

C. Least common multiple. Common denominator. Simplest form.

D. Extension of the idea of multiplication.

E. Multiplication by a fraction.

F. "Of" not multiplication symbol, yet $\frac{2}{3} \times Q$ or $\frac{2}{3}$ times Q equals $\frac{2}{3}$ of Q.

G. Cancellation.

H. Division by a fraction.

I. The so-called business fractions and their percent equivalents.

J. Expression of decimals as fractions and fractions as decimals. Show by squared paper and diagrams the identity of different expressions for the same fraction.

K. Scale drawing.

SEVENTH GRADE.

Review.

A. Symbols: Row of savages. Zero. Decimal point. Fractional notation. Parentheses. Units added counted together are thereby taken as equivalent. Illustrations. Adding with time limit.

B. Business forms and operations. Banks. Interest. Deposit slips. Checks. Drafts. Notes. Discount. Stocks. Bonds. Coupons.

C. Meaning of per cent and percentage. Decimals and fractions in percentage.

D. Percentage equivalents of 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 1/6, 1/8, 3/8, 5/8, 7/8, when considered as operators; and *vice versa.* Percentage equivalents of decimals when considered as operators.

E. Problems on percentage. Commission, taxes tariff, insurance.

F. Longitude and time.

G. *Hundred Months Method.*

Interest for one hundred months at twelve percent equals principal. Interest for one month at twelve percent equals .01 of principal. Interest for a number of months, an aliquot part of one hundred, is just that part of the principal. Interest for 3 days is .001 of the principal.

Thus to get interest at twelve percent for eight months, shift point two places to left in principal and multiply by eight.

Interest at 8, 6, 4, 3, 2 % is 2/3, 1/2, 1/3, 1/4, 1/6 of that at 12.

H. Mensuration; rectangle; parallelogram; trapezoid; regular polygon; circle; prismatoid; Halsted's Formula: $V=(a/4)(B+3C)$; prism; cylinder; pyramid; cone; sphere.

I. Evolution; use of tables. Logarithms. Negative and positive numbers. The equation. The unknown. The variable. The constant. The parameter. Coordinates. The graph. The function.

J. General review.

INDEX.

abacus 12, 17
addition 29, 33, 44, 60, 116
Ahmes 55
angle 75
Archimedes 76
area 77
arithmetic 68, 73, 101
artificial 4
associative 32
assumptions 72, 102

base 7, 24, 65
Bayley 20
begin 113
binary 13
Birch 55
blunders 110
Bosworth 94
Britannica 98
Brown 97
Byrhtferth 97

calculus 22, 101
cardinal 5, 111, 113
cardinals 8
carrying 121
Cassini 99
Century 94
Chambers 96
child 4, 110
Chrysippos 6
cipher 20
columns 122
commutative 31
correlation 10
count 11, 68, 111
countable 84
counting 10, 71, 88, 117
Cowper 97
cross-cut 79
cuboid 79
Cursor 97

decimal 14, 51, 120
decimals 22, 49, 63
degree 75, 76
difference 40, 122
digits 22
distributive 31
division 41, 47, 58, 61, 123
dozen 97

Egerton 21
equivalent 10
Eskimo 12

fingers 11
five 11
formulas 32
fraction 106
fractions 56, 63, 121

games 113
geometry 75
Gerbier 97
Girard 28
grades 110, 118

Hale 97
Halsted's 79, 80, 130
Hamilton 32
Hankel 22, 56
Harriot 27
Hickes 96
Hill 20
Hindu 19
Holmes 97
Hooke 97
hundred 94, 129
hundredths 50

individual 3
induction 34
inequality 27
infinite 82
integer 24
interest 129
intrinsic 23
invariance 15
inverse 39

Lagrange 1, 99
Langland 98
length 76
Leonard 21
local 23

Maurolycus 34
Mazarin 99
measurement 68, 73, 119
million 98
modulus 22
Moore 101
multiple 42
multiplication 35, 38, 45, 61
Murray 95, 97

Napier 49
Napoleon 99
natural 24
Nau 19
Nemorarius 21
nines 46
nine-times 126
nominal 92
notation 26
number 5, 14, 69, 88, 92
numbers 3
numeral 12
numeration 15, 23

one 6
order 81, 83
ordered 59
ordinal 25, 33, 88, 111
ordinals 81, 89, 110
Oughtred 35

parentheses 28
part 28
partitioned 14
Peacock 56, 96
periodicity 13
permanence 56
Planudes 21
play-joy 125
plus 29, 53
point 51
position 17
positional 22
prehuman 3
presentation 110
prism 79
prismatoid 78
product 35, 52
Ptolemy 19

quartets 117
quotient 41, 53

radian 77
Raleigh 27
ray 75
read 51, 94
reciprocal 58
Recorde 26
recur 64
remainder 41
roundheads 93

Sacrabosco 21
scale 59
schoolmaster's 99
Sebokt 19
sect 75
sense 82
Servois 31
seven 9
Shakespeare 98
shift 50
solidus 43
sphere 80
standard 11, 73, 124
Stevinus 49
straight 75
substitution 29
subtraction 39, 44, 57, 116
sum 30
summits 78
symbol 26
symbols 18
symmetrical 43

teaching 23
technic 44
telephone 92
ten 7
tenths 50

terms 30
thousand 96, 98
thousandths 51
three 9, 116
twenty 22
two, 6, 9, 116

unification 4
unit, 8, 73

verify 46, 48
Vieta 31
volume 78, 80

well-ordered 86
Whitney 99
Widman 29

zero 20

A Brief History of Mathematics.

By the late DR. KARL FINK, Tübingen, Germany. Translated by Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Professor of Mathematics in Teachers' College, Columbia University, New York City. With biographical notes and full index. Second revised edition. Pages, xii, 333. Cloth, $1.50 net. (5s. 6d. net.)

"Dr. Fink's work is the most systematic attempt yet made to present a compendious history of mathematics."—*The Outlook.*

"This book is the best that has appeared in English. It should find a place in the library of every teacher of mathematics."
—*The Inland Educator.*

Lectures on Elementary Mathematics.

By JOSEPH LOUIS LAGRANGE. With portrait and biography of Lagrange. Translated from the French by T. J. McCormack. Pages, 172. Cloth, $1.00 net. (4s. 6d. net.)

"Historical and methodological remarks abound, and are so woven together with the mathematical material proper, and the whole is so vivified by the clear and almost chatty style of the author as to give the lectures a charm for the readers not often to be found in mathematical works."—*Bulletin American Mathematical Society.*

A Scrapbook of Elementary Mathematics.

By WM. F. WHITE, State Normal School, New Paltz, N. Y. Cloth. Pages, 248. $1.00 net. (5s. net.)
A collection of Accounts, Essays, Recreations and Notes, selected for their conspicuous interest from the domain of mathematics, and calculated to reveal that domain as a world in which invention and imagination are prodigiously enabled, and in which the practice of generalization is carried to extents undreamed of by the ordinary thinker, who has at his command only the resources of ordinary language. A few of the seventy sections of this attractive book have the following suggestive titles: Familiar Tricks, Algebraic Fallacies, Geometric Puzzles, Linkages, A Few Surprising Facts, Labyrinths, The Nature of Mathematical Reasoning, Alice in the Wonderland of Mathematics. The book is supplied with Bibliographic Notes, Bibliographic Index and a copious General Index.

"The book is interesting, valuable and suggestive. It is a book that really fills a long-felt want. It is a book that should be in the library of every high school and on the desk of every teacher of mathematics."
—*The Educator-Journal.*

On the Foundation and Technic of Arithmetic.

By GEORGE BRUCE HALSTED. Cloth, $1.50
Pages, 140.

A practical presentation of arithmetic for the use of teachers. There has been in mathematics an outburst of unexpected deep reaching progress and properly to understand or to teach arithmetic, one should have a glimpse of its origin, foundation, meaning and aim.

Non-Euclidean Geometry, a Critical and Historical Study of its Development.

By ROBERTO BONOLA. With an Introduction by FEDERIGO ENRIQUES. Translated by H. S. CARSLAW. Cloth, $2.00. Pages, 268. Illustrated.

A clear exposition of the principles of elementary geometry especially of that hypothesis on which rests Euclid's theory of parallels, and of the long discussion to which that theory was subjected; and of the final discovery of the logical possibility of the different Non-Euclidean Geometries.

In Preparation: Bibliography of 100 selected books on the History and Philosophy of Mathematics.

Price, $1.00.

Geometric Exercises in Paper-Folding.

By T. SUNDARA ROW. Edited and revised by W. W. BEMAN and D. E. SMITH. With half-tone engravings from photographs of actual exercises, and a package of papers for folding. Pages, x, 148. Price, cloth, $1.00 net. (4s. 6d. net.)

"The book is simply a revelation in paper folding. All sorts of things are done with the paper squares, and a large number of geometric figures are constructed and explained in the simplest way."
—*Teachers' Institute.*

Magic Squares and Cubes.

By W. S. ANDREWS. With chapters by PAUL CARUS, L. S. FRIERSON and C. A. BROWNE, JR., and Introduction by PAUL CARUS. Price, $1.50 net. (7s. 6d. net.)

The first two chapters consist of a general discussion of the general qualities and characteristics of odd and even magic squares and cubes, and notes on their construction. The third describes the squares of Benjamin Franklin and their characteristics, while Dr. Carus adds a further analysis of these squares. The fourth chapter contains "Reflections on Magic Squares" by Dr. Carus, in which he brings out the intrinsic harmony and symmetry which exists in the laws governing the construction of these apparently magical groups of numbers. Mr. Frierson's "Mathematical Study of Magic Squares," which forms the fifth chapter, states the laws in algebraic formulas. Mr. Browne contributes a chapter on "Magic Squares and Pythagorean Numbers," in which he shows the importance laid by the ancients on strange and mystical combinations of figures. The book closes with three chapters of generalizations in which Mr. Andrews discusses "Some Curious Magic Squares and Combinations," "Notes on Various Constructive Plans by Which Magic Squares May Be Classified," and "The Mathematical Value of Magic Squares."

"The examples are numerous; the laws and rules, some of them original, for making squares are well worked out. The volume is attractive in appearance, and what is of the greatest importance in such a work, the proof-reading has been careful."—*The Nation.*

The Foundations of Mathematics.

A Contribution to The Philosophy of Geometry. BY DR. PAUL CARUS. 140 pages. Cloth. Gilt top. 75 cents net. (3s. 6d. net.)

The Open Court Publishing Co.

623-633 Wabash Avenue **Chicago**