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by

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Abstract. We present variations on a theme of Mas-Colell and report results on the existence of Cournot-Nash equilibrium distributions in which individual action sets depend on the distribution of actions and the payoffs are represented by relations that are not necessarily complete, or transitive.

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1. Introduction

In [29], Mas-Colell showed the existence of a Cournot-Nash equilibrium distribution for games with a compact metric space of actions and with continuous payoffs. Mas-Colell's result was generalized in [22] to situations where the space of actions is not necessarily metric, and hence not necessarily separable, and where the payoffs need only be upper semicontinuous in the individual actions.

The economics of this problem go back to Cournot [7]. A player is characterized by his action set A and his payoff function $u(\cdot)$ which depends not only on the action taken but also on the probability distribution of the actions taken by everybody else. In equilibrium, each player, in choosing his best action, leads to the same distribution on the (common) action space on which his choice is conditioned. Thus, the problem of finding a Cournot-Nash equilibrium distribution is a natural fixed point problem though the original formulation of Cournot predated the discovery of fixed point theorems.

Stripped of their economic motivation, the results constitute an interesting application of the Ky Fan, Glicksberg fixed point theorem [11,13] to a problem that is without an explicit linear structure and, as such, essentially topological. The data of the problem consists of a compact Hausdorff space A; the space $\mathcal{M}(A)$ of Radon probability measures endowed with the topology of weak * convergence (as in [34] or [37]); the space $\mathcal{U}_A$ of continuous real valued functions on $A \times \mathcal{M}(A)$ endowed with the sup-norm topology (as in [29]); and a Radon probability measure μ on $\mathcal{U}_A$. The problem is to find a Radon probability measure τ on $A \times \mathcal{U}_A$ such that the marginal of τ on $\mathcal{U}_A$, $\tau_{\mathcal{U}}$, equals μ and such that τ gives full measure to the set $\{(a,u) \in (A \times \mathcal{U}_A): u(a,\tau_A) \geq u(x,\tau_A)$ for all $x \in A\}$, a set which, it bears emphasis, is conditioned on τ_A and hence on τ.

In this paper, we present a reformulation of this basic problem by

(i) dispensing with payoff functions and focussing instead on preference relations;

(ii) allowing strategy or action sets to depend on the distribution of actions.

In terms of (i), we consider two classes of preference relations, one in which preferences are reflexive, transitive and continuous, though not necessarily complete, and the other in which transitivity is replaced by a suitable convexity assumption. In equilibrium theory, Schmeidler [32] was the first to work with incomplete but transitive preferences while incomplete and non-transitive preferences date to Mas-Colell [28]; also see [13] and [35]. It should be noted that our second setting departs from [29] and [22] and explicitly brings in a linear topological structure on the action space A. The motivation for extension (ii) goes back at least to Debreu's 1952 paper [8].

Important as these extensions are from an economic point of view, they also add to the technical diversity of our problem. Under our reformulation, the question we pose involves the interplay between the compact Hausdorff space A, the space $\mathcal{M}(A)$, the hyperspace $\mathcal{H}(A \times A)$ of closed subsets of $A \times A$ endowed with a "suitable" topology, the space of continuous functions $\mathcal{C}(A)$ from $\mathcal{M}(A)$ to a subspace of $\mathcal{H}(A \times A)$ endowed with a suitable topology, and finally a Radon measure on (A). Our search for a suitable topology on a particular subspace of the hyperspace $\mathcal{H}(A \times A)$ leads us to "neighboring" economic agents as formalized by Kannai [19], Debreu [9], Hildenbrand [16,17], Grodal [15], Chichilnisky [5] and Back [2]. Moreover, the setup of our problem leads us to formalize "neighboring" economic agents in the presence of externalities.

The plan of the paper is as follows. Section 2 is devoted to the formalization of neighboring economic agents in the presence of externalities. In this section, we motivate our formalization by relating it to previous work. Section 3 is devoted to mathematical preliminaries. This section enables us to develop the necessary terminology and notation and presents the mathematical results on which our formulations depend. Section 4 presents the model and results and Section 5 collects separately two results on the existence of maximal elements in compact sets. These are essential ingredients in the proofs which follow in Section 6. References and proofs for the technical results in Section 3 are presented in Section 7.

2. Neighboring Economic Agents in the Presence of Externalities

In a pioneering paper circulated in 1964, Kannai [19] gave a formalization of the intuitive idea that one economic agent is "similar" or "close" to another. Since an economic agent is characterized by his preferences and endowments, this formalization amounted, in particular, to endowing the space of preference relations with a topology. Kannai considered a setting in which each agent's preference relation $\succsim$ is defined on the non-negative orthant of n-dimensional Euclidean space and that there is enough structure on $\succsim$

that it can be represented by a utility function $u(\succsim,\cdot)$ on R^n_+. He could then define "closeness" of two preference relations $\succsim$, $\succsim'$ by the metric d where

$$d(\succsim,\succsim') = \underset{x \in R^n_+}{\text{Max}} \frac{|u(\succsim,x)-u(\succsim', x)|}{1 + |x|^2}.$$

An alternative way of viewing Kannai's work is to say that he endows the space of monotonic preference relations $\mathcal{R}$ with the weakest Hausdorff topology making the set $\{(x,y,\succsim) \in R^n_t \times R^n_t \times \mathcal{R}: \ x \succsim y\}$ closed.

In a subsequent contribution, Debreu [9] exploited the observation that continuous preference relations can be injectively mapped into the space of closed subsets of the product space and hence, in the context of a metric space of commodities, can be endowed with the topology induced by the Hausdorff metric. Debreu also observed that a uniformity, rather than a metric, would suffice for his results.

This identification of a preference relation with a closed subset of the product of the underlying commodity space prompts one to consider any topology on the space of closed subsets of a topological space, the hyperspace of that topological space, as potentially relevant to the problem of defining "neighboring" economic agents. The relative merit of one topology as opposed to another would then depend on the economic problem that was being studied. Accordingly, in [16], Hildenbrand worked with the topology of closed convergence. He was motivated, in part, by the fact that the Hausdorff metric topology proposed by Debreu is not separable whereas the topology of closed convergence is not only separable but also metrizable. Moreover, on the space of monotonic preferences, this topology coincides with that proposed by Kannai. Hildenbrand allowed consumption sets to differ among agents but stayed within the context of n-dimensional Euclidean space. On all of this, see [17, Section 1.2] for details. Hildenbrand's work was extended by Grodal [15], and more recently by Back [2], but both remain in R^n.

Chichilnisky [5], on the other hand, was motivated in part by a search for a topology that was sensitive to the Borel measure of the "lower contour set" of a preference relation. Accordingly, she formalized the idea of "neighboring" economic agents through the use of the order topology on the space of closed subsets of the product of a connected, normal space. She showed, in particular, that her formalization led to a topology on the space of agents that was strictly finer than that provided both by the Hausdorff metric as well as the closed convergence topologies in the setting of a compact metric space of commodities.

None of this literature allows for "externalities" in consumption. Put another way, it does not allow consumption sets and preference relations to respond to changes in the "actions" of other economic agents. Such dependence is, of course, an essential aspect of the Cournot-Nash problem as can be seen in the original formulations of Nash [30] and Debreu [8]. Nash and Debreu, however, considered games with a finite set of players and, in such a setting, allowed each player's strategy set and payoff function to depend continuously on the action of each and every other player. In the context of games with a continuum of players, the situation is much less straightforward. We turn to this.

One can discern four distinct formulations of "externalities" in games with a continuum of players. Two of these go back to Schmeidler [33] in the context of a measure of space of players. Under the first formulation, Schmeidler allows each player's payoff function to depend on the actions taken by almost all players as embodied in the equivalence class of measurable functions from the space of players to a common strategy set in R^n. Schmeidler's formulation has been the object of extensive work; see, for example, [20], [21], [25], [38] and their references. Under the second formulation, Schmeidler specializes to a situation under which each player's payoff is made to depend on the "average response" of the other players. This average response is formalized as an integral of the "actions" or "plays" of the other players. This formulation has also been extended to situations when the payoffs are represented by preference relations over strategy sets that are subsets of a Banach space, see [20] and the references therein.

The third formulation is due to Mas-Colell [29]. Unlike Schmeidler's first formulation, Mas-Colell also limits himself to dependence on the "average response" but formalizes this response as a distribution, rather than an integral, of the actions of the other players. Thus, if A denotes the common strategy set and $\mathcal{M}(A)$, the space of probability measures on A, the payoff functions in Mas-Colell are continuous real-valued functions on $A \times \mathcal{M}(A)$.

In [22], the payoff functions of Mas-Colell are viewed in a different way and this gives us our fourth formulation. Under this formulation, a continuous real-valued function on $A \times \mathcal{M}(A)$ is viewed as a family of continuous real-valued functions on A and parameterized by elements of $\mathcal{M}(A)$, i.e., as a function from $\mathcal{M}(A)$ into R^A. We shall use this alternative viewpoint to present the formulation of externalities and dependence that is studied in this paper. This can now be used to generalize the work of Kannai, Debreu, Hildenbrand and Chichilnisky to situations with externalities.

An economic agent or a player is now characterizated as a continuous function from $\mathcal{M}(A)$ to the space of preference relations on A, $\mathcal{F}(A)$. Thus, for any f ε $\mathcal{L}(A)$ and any τ ε $\mathcal{M}(A)$, f (τ) gives a strategy set and a preference relation defined on that strategy set. Two economic agents or players, f and g, are considered to be "close" if f and g are "close." Our formulation thus proceeds in two distinct steps. In the first place, using the topology on A, we endow the space $\mathcal{F}(A)$ with a topology. This could be any of the topologies discussed in the first part of this section. Next, and dependent on the topology on $\mathcal{F}(A)$, we endow the space of continuous functions from $\mathcal{M}(A)$ to $\mathcal{F}(A)$ with a suitable topology. This typically involves the compact-open topology and another topology that we introduce and that seems more natural for our problem. However, a discussion of these necessarily involves a more formal exposition.

3. Mathematical Preliminaries

Let A be a compact Hausdorff space and $\mathcal{H}(A)$ denote the space of closed subsets of A. Note that the empty set ϕ is an element of $\mathcal{H}(A)$. We shall say that P ε $\mathcal{H}$ (A x A) is reflexive if (a,a) ε P for all a ε A and that P is complementedly transitive if (a,b) ε P^c, (b,c) ε P^c implies (a,c) ε P^c where P^c represents the complement of P in A x A and a,b,c are arbitrary elements in A. Any $(x,y) \in P^c$ can be read as "x is preferred to y" and alternatively denoted $x > y$. Analogously, any $(x,y) \in P$ can be read as "y is preferred or indifferent to x" and alternatively denoted $y \geq x$. For any $B \in \mathcal{H}(A), B \neq \phi$, let

$$\mathcal{F}_B(A) = \{P \in \mathcal{H}(B \times B) : P \text{ is reflexive and complementedly transitive}\},$$

$$(A) = \bigcup_{\substack{B \in \mathcal{H}(A) \\ B \neq \phi}} \mathcal{F}_B(A).$$

$\mathcal{F}(A)$ represents the space of preference relations and we shall view it as a subspace of $\mathcal{H}(A \times A)$ endowed with one of two possible topologies.

The first topology we consider is the topology of closed convergence which was first applied in equilibrium theory by Hildenbrand [16] and proposed by Mertens (see [17, page 108]). This topology has as its sub-base sets of the following form

$$\{F \in \mathcal{H}(A \times A) : F \cap K = \phi\} \text{ and } \{F \in \mathcal{H}(A \times A) : F \cap G \neq \phi\}$$

where K and G are respectively compact and open subsets of A x A. For more details into this topology, the reader is referred to [17,26] and their references. We shall need the following preliminary result.

Theorem 3.1. *If A is a compact Hausdorff space, then $\mathcal{F}(A)$ is a compact Hausdorff space in the topology of closed convergence.*

The second topology that we consider is the order topology first applied in equilibrium theory by Chichilnisky [5]. Consider the subset $\mathcal{H}_o(A \times A)$ of $\mathcal{H}(A \times A)$ given by

$$\mathcal{H}_o(A \times A) = \{ B \in \mathcal{H}(A \times A) : B = \text{cl Int}(B)\}$$

where cl and Int denote closure and interior respectively. The space $\mathcal{H}_o(A \times A)$ can be endowed with an order structure $\bar{>}$ under which for any X,Y in $\mathcal{H}_o(A \times A)$

$$X \bar{>} Y \text{ if and only if } Y \subset \text{Int}(X).$$

Following [5, especially footnote 10], $\mathcal{H}_o(A \times A)$ can be endowed with a topology, the order topology, which has its sub-base sets of the following form:

$$\{X \in \mathcal{H}_o(A \times A) : X \bar{>} Y\} \text{ and } \{X \in \mathcal{H}_o(A \times A) : Y \bar{>} X\}$$

for any Y in $\mathcal{H}_o(A \times A)$. Let $\mathcal{F}_o(A) = \mathcal{H}_o(A \times A) \cap \mathcal{F}(A)$.

We can state

Theorem 3.2. *If* A *is a compact Hausdorff space, then* $\mathcal{H}_o(A \times A)$ *endowed with the order topology is a completely regular space and hence the relative topology on* $\mathcal{F}_o(A)$ *is completely regular*.

In [5], it is shown that the order topology is finer than the topology of closed convergence in the case when A is compact metric. Our next result offers a generalization of this to a non-metric setting.

Theorem 3.3. *If* A *is a compact Hausdorff space, then the order topology on* $\mathcal{H}_o(A \times A)$ *is finer than the topology of closed convergence on* $\mathcal{H}_o(A \times A)$.

Next, we consider the space of Radon probability measures defined on $\mathcal{B}(A)$, the Borel σ-algebra on A. Denote this space by $\mathcal{M}(A)$ and endow it with the relative weak * (or narrow) topology (see [34] or [37] for details). We can now state the following preliminary result.

Theorem 3.4. *If* A *is a compact Hausdorff space*, $\mathcal{M}(A)$ *is a compact Hausdorff space in the (relative) narrow topology*.

Our final set of preliminary concepts concerns the space of continuous functions from $\mathcal{M}(A)$ into the space of preference relations. When the space of preference relations $\mathcal{F}(A)$ is endowed with the topology of closed convergence, we shall denote the space of continuous functions from $\mathcal{M}(A)$ into $\mathcal{F}(A)$ by $\mathcal{C}(A)$. When the space of preference relations is chosen from $\mathcal{H}_o(A \times A)$ and endowed with the order topology, we shall use the notation $\mathcal{C}_o(A)$ for the space of continuous functions from $\mathcal{M}(A)$ into $\mathcal{F}_o(A)$. We shall endow the spaces $\mathcal{C}(A)$ and $\mathcal{C}_o(A)$ with one of two possible topologies.

The first topology that we consider on $\mathcal{C}(A)$ is the graph topology. This is the relativization of the topology of closed convergence on $\mathcal{M}(A) \times \mathcal{F}(A)$ to the space consisting of the graphs of the elements in $\mathcal{C}(A)$. Accordingly, we shall say that a net $\{f^\nu\}$ chosen from $\mathcal{C}(A)$ converges to an element f in $\mathcal{C}(A)$ if and only if the graph of f^ν, Grf^ν, converges in the topology of closed convergence to Grf., Since f^ν and f are continuous functions and their range is a Hausdorff space, their graphs are closed subsets of $\mathcal{M}(A) \times \mathcal{F}(A)$. Moreover, given Theorems 3.1 and 3.4, $\mathcal{M}(A) \times \mathcal{F}(A)$ is compact and hence the topology is well defined (see [23]). Indeed, one can say more.

Theorem 3.5. *If* A *is a compact Hausdorff space, then* $\mathcal{C}(A)$ *endowed with the graph topology is Hausdorff and completely regular*.

The second topology that we consider on $\mathcal{C}(A)$ is the compact-open topology. This has as its sub-base sets of the following form:

$$\{f \in \mathcal{C}(A) : f(K) \subset U\} \equiv (K,U), \text{ K compact, U open.}$$

The relationship between the topology of closed convergence and the compact-open topology is given by the following result.

<u>Theorem 3.6.</u> <u>If</u> A <u>is a compact Hausdorff space, then the graph topology and the compact-open topology are identical on</u> $\mathcal{C}(A)$.

Next, we turn to $\mathcal{C}_o(A)$. The first point to be noted is that $\mathcal{F}_o(A)$ is not necessarily compact but, as established in Theorem 3.2, only completely regular. Nevertheless, we can state a result on which the importance of the compact-open topology is partly based.

<u>Theorem 3.7.</u> <u>If</u> A <u>is a compact Hausdorff space, then</u> $\mathcal{C}_o(A)$, <u>endowed with the compact-open topology is a completely regular Hausdorff space</u>.

The question remains as to whether the graph topology has any relevance for $\mathcal{C}_o(A)$. A little reflection leads to one to an affirmative, and somewhat surprising, answer. Since $\mathcal{F}_o(A)$ is completely regular, it can be embedded in a compact Hausdorff space $\mathcal{K}(A)$; see, for example, [10, p. 243, paragraph 2]. Furthermore, a continuous function from $\mathcal{M}(A)$ into $\mathcal{F}_o(A)$ has a closed (indeed compact) graph in $\mathcal{M}(A) \times \mathcal{K}(A)$. This observation allows us to define the graph topology on $\mathcal{C}_o(A)$ and to state a generalization of Theorem 3.6.

<u>Theorem 3.8.</u> <u>If</u> A <u>is a compact Hausdorff space, then the compact-open topology on</u> $\mathcal{C}_o(A)$ <u>is identical to the relativization on</u> $\mathcal{C}_o(A)$ <u>of the topology of closed convergence on</u> $\mathcal{M}(A) \times \mathcal{K}(A)$, <u>where</u> $\mathcal{K}(A)$ <u>is a compact Hausdorff space in which</u> $\mathcal{F}_o(A)$ <u>is imbedded</u>.

We shall refer to this relative topology on $\mathcal{L}_o(A)$ as the induced graph topology on $\mathcal{L}_o(A)$.

4. The Model and Results

We now have all the technical machinery we need to develop the principal results of this paper. We state a preliminary definition with the remark that a Radon probability measure on a topological space is to be understood as being defined on the Borel σ-algebra generated by the topology on that space.

Definition 4.1. A game on $\mathcal{L}(A)$ is a Radon probability measure on $\mathcal{L}(A)$, i.e., an element of $\mathcal{M}(\mathcal{L}(A))$.

It should be noted that Definition 4.1 is not specific as to the topology on $\mathcal{L}(A)$. The same is true for the definition to follow. It is only in the statement of our results that we shall specify the topologies on $\mathcal{L}(A)$.

For any $P \in \mathcal{F}_B(A)$, let

$$M(P) = \{a \in B : \forall\, b \in B,\ (b,a) \notin P^c\}.$$

$M(P)$ thus denotes the set of maximal elements in B for the preference relation P defined on B x B. We can now state our reformulation of a Cournot-Nash equilibrium distribution.

Definition 4.2. A Borel probability measure τ on $A \times \mathcal{L}(A)$ is a Cournot-Nash equilibrium distribution of a game μ on $\mathcal{L}(A)$ if

(i) the marginal distribution of τ on $\mathcal{L}(A)$, $\tau_{\mathcal{L}(A)}$, equals μ,

(ii) $\tau(B_\tau) = 1$ where $B_\tau = \{(a,f) \in A \times \mathcal{L}(A) : a \in M(f(\tau_A))\}$.

We can now present

<u>Theorem 4.1.</u> <u>If</u> A <u>is a compact Hausdorff space, there exists a Cournot-Nash equilibrium distribution of a game on</u> $\mathcal{C}(A)$ <u>if</u> $\mathcal{C}(A)$ <u>is endowed with the graph topology or equivalently with the compact-open topology</u>.

Next, we consider the space $\mathcal{C}_o(A)$. We can define a game on $\mathcal{C}_o(A)$ as well as the Cournot-Nash equilibrium distributions of such a game by substituting $\mathcal{C}_o(A)$ for $\mathcal{C}(A)$ in Definitions 4.1 and 4.2.

We can now state

<u>Theorem 4.2.</u> <u>If</u> A <u>is a compact Hausdorff space, there exists a Cournot-Nash equilibrium distribution of a game on</u> $\mathcal{C}_o(A)$ <u>if</u> $\mathcal{C}_o(A)$ <u>is endowed with the induced graph topology or equivalently with the compact-open topology</u>.

For our next result, we assume that A is a convex, compact subset of a topological vector space. We shall say that $P \in \mathcal{H}(A \times A)$ is <u>irreflexively convex</u> if for any $a \in A$, $a \notin \text{con}\,\{b : (b,a) \in P^c\}$ where con W denotes the convex hull of a set W. For any $B \in \mathcal{H}(A)$ with B convex, let

$$\mathcal{F}_B^{co}(A) = \{P \in \mathcal{H}(B \times B) : P \text{ is irreflexively convex}\}$$

$$\mathcal{F}^{co}(A) = \bigcup_{\substack{B \in \mathcal{H}(A) \\ B \text{ convex}}} \mathcal{F}_B^{co}(B).$$

In keeping with our earlier notation, we shall let $\mathcal{C}^{co}(A)$ denote the space of continuous functions from $\mathcal{M}(A)$ into $\mathcal{F}^{co}(A)$ when the latter is endowed with the topology of closed convergence. $\mathcal{C}_o^{co}(A)$ denotes the space of continuous functions from $\mathcal{M}(A)$ into a subset of

$\mathcal{F}^{co}(A)$ obtained by considering sets in $\mathcal{H}_o(A \times A)$ and with that subset endowed with the order topology.

We can now state analogues of Theorems 4.1 and 4.2 to a setup where the payoffs are not necessarily generated by transitive relations.

Theorem 4.3. *If* A *is a compact convex subset of Hausdorff topological vector* space, there exists a *Cournot-Nash equilibrium distribution of a game on* $\mathcal{L}^{co}(A)$ *or on* $\mathcal{L}_o^{co}(A)$ *if these spaces are endowed with their induced graph topologies or equivalently with their compact-open topologies*.

Note that we have no result on the existence of Cournot-Nash equilibria in games on $\mathcal{L}^{co}(A)$ when the latter is endowed with the graph topology. The reason for this lies in the absence of the compactness property of $\mathcal{F}^{co}(A)$ when the latter is endowed with the topology of closed convergence; see Grodal [15] for a counter example.

5. On the Existence of Maximal Elements in Compact Sets

In this section we present two results on the existence of maximal elements with respect to P^c in a compact set. These results constitute essential ingredients in the proofs of Theorems 4.1 and 4.2 but since they may have independent interest, we have collected them in a separate section. It is worth stating, however, that the results themselves are essentially known in the mathematical economics literature and we detail this in the accompanying remarks below.

We can now present

Theorem 5.1. *For any* $P \in \mathcal{F}_B(A)$, $M(P) \neq \phi$.

Corrolary 5.1. Theorem 5.1 is valid for any $P \in \mathcal{F}_B(A) \cap \mathcal{H}_o(A \times A)$.

Next, we present analogous results for the convex non-transitive case.

Theorem 5.2. For any $P \in \mathcal{F}_B^{co}(A)$, $M(P) \neq \phi$.

Corrolary 5.2. Theorem 5.2 is valid for any $P \in \mathcal{F}_B^{co}(A) \cap \mathcal{H}_o(A \times A)$.

Proof of Theorem 5.1.

Suppose $M(P) = \phi$ and let $0_x = \{y \in B : (x,y) \in P^c\}$ for any $x \in B$. For any $a \in B$, there exists $b \in B$ such that $a \in 0_b$. If not, then for all $b \in B$, $a \notin 0_b$. This means that $(b,a) \notin P^c$, i.e., $a \in M(P)^b$, a contradiction to the emptiness of $M(P)$. Also note that for any $b \in B$, 0_b, being a projection of an open set, is also open. Hence $\{0_b\}_{b \in B}$ is an open cover of B. Since B is compact, it has a finite subcover. Pick a finite subcover with the smallest number of elements and let this number be k. Hence there exists $b_i \in B$, $i = 1, \ldots, k$ such that $B \subset \bigcup_{i=1}^{k} 0_{b_i}$.

Next we show that $k = 1$. Suppose not, i.e., k is an integer greater than one. Certainly $b_k \notin 0_{b_k}$. Then there exists $i < k$ such that $b_k \in 0_{b_i}$. Now pick any $y \in 0_{b_k}$. Then by complemented transitivity of P, $y \in 0_{b_i}$. Hence $0_{b_k} \subset 0_{b_i}$ and we have contradicted the fact that we had a smallest subcover. Hence $B \subset 0_{b_1}$. But this implies that $b_1 \in 0_{b_1}$, a final contradiction completes the proof. ||

Remark 1. Theorem 5.1 was first stated and proved by Schmeidler [32, Lemma 2] for the case when X is a subset of R^n. On comparison, the reader will find that our proof is slightly different from his.

Proof of Theorem 5.2.

For any $a \in B$, let $T(a) = \text{con}\,\{b \in B : (b,a) \in P^c\}$. Suppose there exists $a \in B$ such that $T(a) = \phi$. Then for all $b \notin B$, $(b,a) \in P^c$, i.e., $a \in M(P)$ and the proof is finished. Thus, assume that $T(a) \neq \phi$ for all $a \in B$.

For any $a \in B$, $T(a)$ is a convex set by assumption. Now for any $b \in B$, consider $T^{-1}(b) = \{a \in B : b \in T(a)\} = \text{con}\,\{a \in B : (b,a) \in P^c\}$. This is precisely the convex hull of the set O_b defined in the proof of Theorem 5.1 and it is open in B; see for example, Lemma 5.1 in [40]. (If O_b is empty, the claim is a trivial one).

Thus all the conditions of Browder's fixed point theorem [4] are satisfied and there exists $a^* \in B$ such that $a^* \in T(a^*)$, i.e., $a^* \in \text{con}$ $\{b: (b,a^*) \in P^c\}$, an impossibility. Hence we have contradicted our assumption that $T(a) \neq \phi$ for all $a \in B$. The proof is finished. ||

Remark 2. Theorem 5.2 goes back to Sonnenschein [36] for the case when $X = R^n$; also see Anderson [1, Theorem 1]. The most up-to-date reference is Yannelis-Prabhakar [39].

6. Proofs of the Principal Results

Proof of Theorem 4.1.

We prove the theorem for the case when $\mathcal{C}(A)$ is endowed with the compact-open topology. The proof for the case when $\mathcal{C}(A)$ is endowed with the graph topology then follows from Theorem 3.6.

As in Mas-Colell [29], the proof is an application of the Ky Fan, Glicksberg fixed point theorem [11,14]; (also see, for example, Berge

[3, p. 251]). We show in a series of claims that all the conditions for the applicability of this theorem are satisfied. Let

$$\mathcal{J} = \{\tau \in \mathcal{M}(\mathcal{L}(A) \times A) : \tau_{\mathcal{L}(A)} = \mu\}.$$

<u>Claim 1</u>: $\mathcal{J}$ <u>is nonempty</u>.

Since Dirac measures are in $\mathcal{M}(A)$, certainly we are guaranteed that $\mathcal{M}(A) \neq \phi$. Since $\mu \in \mathcal{M}(\mathcal{L}(A))$, we can now appeal to Theorem 17 in Schwartz [34, p. 63] to assert the existence of a unique Radon measure λ on $A \times \mathcal{L}(A)$ such that

$$(^{*}) \qquad \lambda(B \times C) = \nu(B)\mu(C) \text{ for all } B \in \mathcal{B}(A),\ C \in \mathcal{B}(\mathcal{L}(A)).$$

Note that Schwartz states $(^{*})$ in his theorem in terms of the <u>essential outer measure</u> but since we are dealing with probability measures, the distinction can be neglected. The reason for this is that we have defined a Radon measure in terms of Definition R_3 of Schwartz [34, p. 13] and from the proof of $R_3 => R_1$ (Schwartz [34, p. 13], we see that the measure of a set with finite measure equals the essential outer measure of that set.

From $(^{*})$ we obtain

$$\lambda_{\mathcal{L}(A)}(B) = \lambda(A \times B) = \nu(A)\mu(B) = \mu(B) \text{ for all } B \in \mathcal{B}(\mathcal{L}(A)).$$

Since $\lambda(A \times \mathcal{L}(A)) = \nu(A)\mu(\mathcal{L}(A)) = 1$ and since, by definition, measures are non-negative, $\lambda \in \mathcal{J}$ and the proof of the claim is complete.

Claim 2. $\mathcal{J}$ is convex.

Pick μ^1, μ^2 from $\mathcal{J}$ and λ a real number such that $0 < \lambda < 1$. Then it is routinely checked that the marginal of $\lambda\mu^1 + (1-\lambda)\mu^2$ on $\mathcal{C}(A)$ is μ and that $\lambda\mu^1 + (1-\lambda)\mu^2 \in \mathcal{M}(\mathcal{C}(A) \times A)$.

Claim 3. $\mathcal{J}$ is compact.

We show first that $\mathcal{J}$ is tight (equally interiorly regular or equally tight in the terminology of Schwartz [34, p. 379, Definition 4]). Towards this end, pick any $\delta > 0$. Since $\mu \in M(\mathcal{C}(A))$, there exists a compact set $K \subset \mathcal{C}(A)$ such that $\mu(K) > 1 - \delta$. Since the marginal of γ on $\mathcal{C}(A)$ for any $\gamma \in \mathcal{J}$ is μ, certainly

$$\gamma(A \times K) = \gamma_{\mathcal{C}(A)}(K) = \mu(K) > 1-\delta.$$

Hence we can appeal to Theorem 3 in [34, p. 379] (also see Topsoe [37; Theorem 9.1]) to assert that $\mathcal{J}$ is relatively compact.

All that remains to be shown is that $\mathcal{J}$ is closed in $\mathcal{M}(A \times \mathcal{C}(A))$. To show this, pick a net $\{\gamma^\alpha\}$ from $\mathcal{J}$ such that $\gamma^\alpha \to \gamma$. For any closed set F in $\mathcal{C}(A)$, we know by the Portmanteau theorem (Theorem 8.1 in [37]) that $\limsup \gamma^\alpha(A \times F) \leq \gamma(A \times F)$. This can be rewritten as

$$\limsup \gamma^\alpha_{\mathcal{C}(A)}(F) \leq \gamma_{\mathcal{C}(A)}(F).$$

Since $\gamma^\alpha(A \times \mathcal{C}(A)) = 1$ for all α, and $\gamma(A \times \mathcal{C}(A)) = 1$, we conclude by a second application of the Portmanteau theorem that $\gamma^\alpha_{\mathcal{C}(A)} \to \gamma_{\mathcal{C}(A)}$. Since $\gamma^\alpha_{\mathcal{C}(A)} = \mu$ for all α, $\gamma_{\mathcal{C}(A)} = \mu$ and the proof is complete.

<u>Remark 3</u>. The proof of Claim 3 offered here is different from that in Khan [22] which relied on a lemma of Hoffman-Jorgenson [18], rather than on the relative compactness of a tight subset of a space.

<u>Claim 4</u>. $B : \mathcal{T} \to 2^{A \times \mathcal{L}(A)}$ <u>has a closed graph</u>.

Let $\tau^\nu \to \tau$, $(a^\nu, f^\nu) \to (a,f)$, and $(a^\nu, f^\nu) \in B_{\tau^\nu}$. We have to show that $(a,f) \in B_\tau$.

As in the proof of Claim 3, we can show that $\tau_A^\nu \to \tau_A$. Furthermore, since $f^\nu \to f$ in the compact-open topology, and since $\mathcal{M}(A)$ is compact Hausdorff by Theorem 3.2, we can appeal to the fact that the evaluation map is continuous; see, for example, Dugundji [10, Theorem XII.2.4, p. 260]. Hence $f^\nu(\tau_A^\nu) \to f(\tau_A)$.

Since $(a^\nu, f^\nu) \in B_{\tau^\nu}$, $a^\nu \in M(f^\nu(\tau_A^\nu))$ and hence $(a^\nu, a^\nu) \in f^\nu(\tau_A^\nu)$. Since $f^\nu(\tau_A^\nu) \to f(\tau_A)$ and $(a^\nu, a^\nu) \to (a,a)$, certainly $a \in D$ where $D \in \mathcal{H}(A)$ is such that $f(\tau_A) \in \mathcal{F}_D(A)$.

Now suppose $a \notin M(f(\tau_A))$. Then there exists $b \in D$ such that $(b,a) \in (f(\tau_A))^c$. Since D is compact, Hausdorff, certainly $D \times D$ is regular. Hence there exists a neighborhood U of (b,a) and an open set $V \supset f(\tau_A)$ such that $U \cap V = \phi$. Relying on the definition of the topology of closed convergence this means that there exist $D^\nu \in \mathcal{H}(A)$ such that $f^\nu(\tau_A^\nu) \in \mathcal{F}_{D^\nu}(A)$ and $b^\nu \in D^\nu$ and $\bar{\nu}$ such $(b^\nu, a^\nu) \in (f^\nu(\tau_A^\nu))^c$ for all $\nu > \bar{\nu}$. But this is a contradiction to the fact that $(a^\nu, f^\nu) \in B_{\tau^\nu}$ for all ν.

<u>Claim 5</u>. <u>For any</u> $\tau \in \mathcal{T}$, B_τ <u>is a closed set in</u> $\mathcal{L}(A) \times A$.

This is a simple consequence of Claim 4.

Next, we consider the map $Q : \mathcal{T} \to 2^{\mathcal{T}}$ such that

$$Q(\tau) = \{\rho \in \mathcal{J} : \rho(B_\tau) = 1\}.$$

Claim 6. For any $\tau \in \mathcal{J}$, $Q(\tau) \neq \phi$.

The proof of this claim relies on the fact that the Dirac point measures on a topological measure space are dense in the space of probability measures on that space. Specifically, Theorem 11.1 in Topsoe [37, p. 48] assures us of a net $\{\mu^\nu\}$ converging to μ such that each μ^ν has a finite support. Pick a particular μ^ν and assume that its support consists of k elements, $f_1, f_2, \ldots, f_k$. From Theorem 5.1, we know that $M(f_i(\tau_A)) \neq \phi$. Let $a_i \in M(f_i(\tau_A))$. Certainly $(a_i, f_i) \in A \times \mathcal{L}(A)$. Now for any $W \in \mathcal{B}(A) \times \mathcal{B}(\mathcal{L}(A))$ let

$$\begin{aligned} \delta^{\nu i}(W) &= 1 \text{ if } (a_i, f_i) \in W \\ &= 0 \text{ otherwise.} \end{aligned}$$

It is easy to see that $\delta^{\nu i} \in \mathcal{M}(A \times \mathcal{L}(A))$. Since $\{(a_i, f_i)\}$ is a compact set in $A \times \mathcal{L}(A)$, for any $W \in \mathcal{B}(A) \times \mathcal{B}(\mathcal{L}(A))$ and for any $\varepsilon > 0$, certainly $\delta^{\nu i}(\{a, f_i\}) > \delta^{\nu i}(W) - \varepsilon$. Now define, for any $W \in \mathcal{B}(A) \times \mathcal{B}(\mathcal{L}(A))$,

$$\delta^\nu(W) = \sum_{i=1}^{k} \delta^{\nu i}(W)\mu^\nu(f_i).$$

Again, it is clear that $\delta^\nu \in \mathcal{M}(A \times \mathcal{L}(A))$.

We now show that the marginal of δ^ν on $\mathcal{L}(A)$ is μ^ν. Pick any $F \in \mathcal{B}(\mathcal{L}(A))$. Then

$$\delta^\nu_{\mathcal{L}(A)}(F) = \delta^\nu(A \times F) = \sum_{i=1}^{k} \delta^{\nu i}(A \times F)\mu^\nu(f_i)$$

$$= \sum_{i \varepsilon I} \mu^{\nu}(f_i) \text{ where } I = \{i \ \varepsilon \ (1...k) : f_i \ \varepsilon \ F\}$$

$$= \mu^{\nu}(F).$$

We can also show that $\delta^{\nu}(B_{\tau}) = 1$. To see this, simply note that $\bigcup_{i=1}^{k} \{a_i, f_i\}$ is a compact set and this is contained in B_{τ}.

Next consider the marginal of δ^{ν} on A. Since $\mathcal{M}(A)$ is compact by virtue of [38, p. 76] (also [34, p. 379]), there exists a subnet δ^{ρ} such that δ^{ρ}_{A} converges to a limit α in $\mathcal{M}(A)$.

Given that $\mathcal{L}(A)$ is completely regular by virtue of Theorem 3.5, and that A, being compact Hausdorff, is also completely regular, we can appeal to Dugundji [10; Theorem VII. 7.2, p. 154] to assert that A x (A) is completely regular. We can now apply Lemma 5.1 in Hoffman-Jorgenson [18] to assert the existence of a measure $\delta^{*} \ \varepsilon \ \mathcal{M}(A \times \mathcal{L}(A))$ such that $\delta^{*}_{A} = \alpha$ and $\delta^{*}_{\mathcal{L}(A)} = \mu$ and such that $\delta^{\rho} \to \delta^{*}$.

Since B_{τ} is a closed set by virtue of Claim 4, we can appeal to the Portmanteau theorem in [37] to assert that $\delta^{*}(B_{\tau}) \geq \limsup \delta^{\rho}(B_{\tau}) = 1$. Since $\delta^{*}(A \times \mathcal{L}(A)) = \mu(\mathcal{L}(A)) = 1$, $\delta^{*}(B_{\tau}) = 1$. Hence $\delta^{*} \ \varepsilon$ $Q(\tau)$ and the proof of the claim is complete.

<u>Claim 8</u>. <u>Q is an upper hemicontinuous correspondence</u>.

Since $\mathcal{J}$ is compact, it suffices to show that Q has a closed graph (see Berge [3, p. 112, Corollary]). Towards this end, let $\tau^{\nu} \to \tau$, $\rho^{\nu} \ \varepsilon \ Q(\tau^{\nu})$ and $\rho^{\nu} \to \rho$. We have to show that $\rho(B_{\tau}) = 1$. Since ρ is a Radon measure, it suffices to show that $\rho(K) = 0$ for all compact $K \subset B^{c}_{\tau}$. By Claim 4, we know that $\limsup B_{\tau^{\nu}} \subset B_{\tau}$. Hence $K \subset (\limsup B_{\tau^{\nu}})^{c}$. From Klein-Thompson [26; Proposition 3.2.11], we obtain

$$K \subset \Big(\bigcap_{\alpha} \overline{\bigcup_{\gamma \geq \alpha} B_{\tau^{\gamma}}}\Big)^{c} = \bigcup_{\alpha} \Big(\overline{\bigcup_{\gamma \geq \alpha} B_{\tau^{\gamma}}}\Big)^{c}$$

where $\overline{A}$ denotes the closure of A. Since K is compact, every open cover has a finite subcover. Hence there exists α' such that

$$K \subset \Big(\overline{\bigcup_{\gamma \geq \alpha'} B_{\tau^{\gamma}}}\Big)^{c}.$$

By the monotonicity property of a measure

$$\rho(K) \leq \rho\Big(\Big(\overline{\bigcup_{\gamma \geq \alpha'} B_{\tau^{\gamma}}}\Big)^{c}$$

and by the Portmanteau theorem [38] we obtain

$$\rho(K) \leq \liminf \rho^{\nu}(K) \leq \liminf \rho^{\nu}\Big(\Big(\overline{\bigcup_{\gamma \geq \alpha'} B_{\tau^{\gamma}}}\Big)^{c}\Big).$$

Now for any $\nu > \alpha'$

$$B_{\tau^{\nu}} \subset \overline{\bigcup_{\gamma \geq \alpha'} B_{\tau^{\gamma}}} \text{ which implies } B^{c}_{\tau^{\nu}} \supset \Big(\overline{\bigcup_{\gamma \geq \alpha'} B_{\tau^{\gamma}}}\Big)^{c}.$$

Hence, by the monotonicity property of a measure,

$$\rho^{\nu}\Big(\Big(\overline{\bigcup_{\gamma \geq \alpha'} B_{\tau^{\gamma}}}\Big)^{c}\Big) \leq \rho^{\nu}(B^{c}_{\tau^{\nu}}) = 0.$$

The proof of the claim in finished.

Remark 4. The proof presented here is a simplification of the one presented for an analogous claim in [22].

We can now apply the Ky Fan, Glicksberg fixed point theorem (see, for example, Berge [3, p. 251]) to the map Q to complete the proof of the theorem.

Remark 5. We have provided the proof of Theorem 4.1 with $\mathcal{L}(A)$ endowed with the compact-open topology. In particular, we used in the proof of Claim 4, the fact that the evaluation map $\mathcal{M}(A) \times \mathcal{L}(A) \to \mathcal{F}(A)$ is continuous. If, instead, we had considered the case where $\mathcal{L}(A)$ is endowed with the graph topology, a direct proof is available. As in the proof of Claim 3, let $\tau_A^\nu \to \tau_A$ and $f^\nu \to f$ in the graph topology on $\mathcal{L}(A)$. We shall show $f^\nu(\tau_A^\nu) \to f(\tau_A)$.

Since the net $(\tau_A^\nu, f^\nu(\tau_A^\nu))$ lies in a compact set, we can find a subnet, also indexed by ν, such that $(\tau_A^\nu, f^\nu(\tau_A^\nu))$ converges to (τ_A, b) where $b \in \mathcal{F}(A)$. Since $(\tau_A^\nu, f^\nu(\tau_A^\nu)) \in Gr(f^\nu)$, and since $Gr(f^\nu) \to Gr(f)$, we obtain that $(\tau_A, b) \in Gr(f)$. But f is a function and hence $b = f(\tau_A)$.

Proof of Theorem 4.2. The proof is identical to that of the proof of Theorem 4.1 except for the fact that we appeal to Theorem 3.8 to show the equivalence between the compact-open topology and the induced graph topology on $\mathcal{L}_o(A)$. ‖

Proof of Theorem 4.3. Again, we follow the proofs of Theorems 4.1 and 4.2 but with Theorem 5.2 substituted for Theorem 5.1 in the proof of Claim 6. ‖

7. Proofs of Results in Section 3

We begin with a

<u>Proof of Theorem 3.1</u>. It is well known that the space of closed subsets of a locally compact space endowed with the topology of closed convergence is compact; see, for example, Problem 3 in [17]. Hence, we have to show that $\mathcal{F}(A)$ is a closed subset of $\mathcal{H}(A \times A)$. Towards this end, pick a net $\{P_\nu\}$ from $\mathcal{F}(A)$ such that $P_\nu \to P$. We have to show $P \in \mathcal{F}(A)$. Let $B_\nu \in \mathcal{H}(A)$ such that $P_\nu \in \mathcal{F}_{B_\nu}(A)$. Since $\mathcal{H}(A)$ is compact, Hausdorff, there exists a subnet, also indexed by ν, such that $B_\nu \to B$. We shall show that $P \in \mathcal{F}_B(A)$.

We first show that P is an element of $\mathcal{H}(B \times B)$ and is reflexive. Since A is compact, certainly $B \neq \phi$. Pick any $(a,b) \in P$. There exists $(a^\nu, b^\nu) \in P_\nu$ for each ν such that $(a^\nu, b^\nu) \to (a,b)$. Since $a^\nu, b^\nu \in B_\nu$, (a,b) BxB, and hence $P \subset B \times B$. Pick any $b \in B$. By the definition of closed convergence, we can construct a subnet $\{B_\sigma\}$ of $\{B_\nu\}$ and $b_\sigma \in B_\sigma$ such that $b_\sigma \to b$. Since $P_\sigma \in \mathcal{F}_{B_\sigma}(A)$, certainly $(b_\sigma, b_\sigma) \in P_\sigma$. Since $P_\sigma \to P$, again by the definition of closed convergence, $(b,b) \in P$.

Next, we show that P is complementedly transitive. In what follows, we shall take complements of P and P_ν in BxB and $B_\nu \times B_\nu$ respectively. Suppose $(a,b) \in P^c$, $(b,c) \in P^c$ but $(a,c) \notin P^c$. Since $(a,c) \in P$ and $P_\nu \to P$, there exists $(a^\nu, c^\nu) \in P_\nu$ such that $(a^\nu, c^\nu) \to (a,c)$. Since $B_\nu \to B$, there exists $b^\nu \in B_\nu$ such that $b^\nu \to b$. Hence $(a^\nu, b^\nu) \to (a,b)$ and $(b^\nu, c^\nu) \to (b,c)$. By [12, Proposition 3.1], there exists $\bar{\nu}$ such that $(a^\nu, b^\nu) \notin P_\nu$ and $(b^\nu, c^\nu) \notin P_\nu$ for all $\nu > \bar{\nu}$. This implies $(a^\nu, c^\nu) \notin P_\nu$ or that $(a^\nu, c^\nu) \in P_\nu$, a contradiction.

The proof of the theorem is complete. ||

<u>Proof of Theorem 3.2</u>. See the proof of Theorem 2 in Khan-Sun [24]. ||

<u>Proof of Theorem 3.3</u>. See the proof of Theorem 1 in Khan-Sun [24]. ||

Proof of Theorem 3.4. See, for example, the Notes to Section 11 in Topsoe [37, p. 76]. Also Schwartz [34, p. 379]. ||

Proof of Theorem 3.5. Since $\mathcal{M}(A)$ is compact Hausdorff by virtue of Theorem 3.4 and since $\mathcal{F}(A)$ endowed with the topology of closed convergence is compact by virtue of Theorem 3.1, $\mathcal{C}(A)$ endowed with the graph topology is a subspace of a compact Hausdorff space. Hence it is normal and therefore completely regular and Hausdorff. Since a subspace of a completely regular Hausdorff space is completely regular and Hausdorff (see, for example, [10, VII.7.2(1)], we are done. ||

Proof of Theorem 3.6. Since A is compact Hausdorff, $\mathcal{M}(A)$ is compact Hausdorff by Theorem 3.4. Furthermore, $\mathcal{F}(A)$, endowed with the topology of closed convergence, is compact Hausdorff by Theorem 3.1. We can now apply Corollary 1 in Khan-Sun [23]. ||

Proof of Theorem 3.7. Since the range of an element in $\mathcal{C}_o(A)$ is a completely regular Hausdorff space by virtue of Theorem 3.3, the claim is a direct consequence of Nagata [31, Theorem F, p. 272]. ||

Proof of Theorem 3.8. See the proof of Corollary 2 in Khan-Sun [23]. ||

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