On the Representation of a Function by a Trigonometric Series.

DISSERTATION

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PREFACE.

In this thesis I have considered the representation, in trigonometric series, of a function of a real variable only.

Since this subject has already been so fully treated I could hardly expect to obtain many new results. Accordingly it has been my special aim, after carefully examining the work of others, to investigate independently certain phases of the question, and thereby to obtain if possible a simpler development of the subject. To some extent my efforts have been rewarded, since, in several cases, I have arrived at results by methods considerably shorter than those which others have employed.

The first two sections of this paper are introductory. Section 1 is purely historical, giving a brief account of the origin of the question and an outline of the principal work done upon it up to the present time.* In section 2 is presented the theorem of du Bois-Reymond which gives directly the form of the coefficients in the trigonometric series and proves for all cases the uniqueness of the development.

In the next two sections I have shown that the convergence of this development of an integrable function $f(x)$ to $f(x_0)$, for $x = x_0$, depends only upon the behavior of the function in the vicinity of x_0. This is followed in sections 5 and 6 by a proof that the series thus converges to $f(x_0)$:

1°. When $f(x)$, in the neighborhood of x_0, is finite, possesses only a finite number of discontinuities and only a finite number of maxima and minima.

2°. When it satisfies the very general condition

$$\lim_{\epsilon=0}\int_0^\epsilon \frac{|f(x_0 \pm 2\gamma) - f(x_0 \pm 0)|}{\gamma}\, d\gamma = 0.$$

This condition I have compared with other conditions and have shown its application to an important class of functions in modern analysis, viz., to functions having an infinite number of maxima and minima, or an infinite number of discontinuities in a finite region. In section 7 I have given an illustration of a function of this kind whose trigonometric development diverges for one

*For a more complete historical treatment see the papers of Riemann (*Math. Werke*, p. 213; found also in *Bulletin des Sci. Math.*, Vol. V, 1873, p. 20), Sachse (*Bulletin des Sci. Math.*, 1880, p. 43), and Gibson (*Proceedings of the Edinburgh Math. Society*, Vol. XI, p. 137.)

value of the variable. Finally, in the last section, I have discussed the nature of the convergence of the series, treating especially the question of uniform convergence.

I desire here to express my acknowledgment to Professor Craig, who suggested to me the subject of this thesis and who has had general supervision over its preparation, and also to Professor Franklin, who, by his advice and suggestions, has afforded me valuable assistance on several points.

BALTIMORE, *April* 3, 1894.

CONTENTS.

ON THE REPRESENTATION OF A FUNCTION BY A TRIGONOMETRIC SERIES.

1. The question as to the possibility of representing an arbitrary function of a real variable by means of a trigonometric series, first suggested nearly a century and a half ago, has received considerable attention. Much has been written upon it, for it is important, not only in pure analysis, but also in the practical applications of mathematics to problems in physics and astronomy.

The question arose from a comparison of two different forms of solution of the partial differential equation for a vibrating chord, viz:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}. \qquad (1)$$

Integrals of this equation were obtained in the form of a general solution containing an arbitrary function and in the form of a trigonometric series. The general solution, subjected to the condition that the extremities of the chord are at rest, becomes

$$y = f(x + ct) - f(ct - x),$$

where f is an arbitrary function having for a period the double length of the chord. This general solution was first obtained by d'Alembert,* who, however, supposed that f was of such a nature that it could always be represented by a continuous curve, this being the idea conveyed to mathematicians of that time by the expression "arbitrary function." But Euler, who in the following year took up the question,† recognized the fact that f could be perfectly arbitrary and said that d'Alembert had imposed an unnecessary restriction.‡

In the same article, as a special case, Euler gave, for the first time, a solution in the form of a trigonometric series.§ He showed that if the initial position was expressed by

$$y = \alpha \sin\frac{\pi x}{a} + \beta \sin\frac{2\pi x}{a} + \gamma \sin\frac{3\pi x}{a} + \dots,$$

* *Mémoires de l'Acad. de Berlin*, 1747, p. 214.

† *Ibid.*, 1748, p. 69.

‡ *Ibid.*, 1748, p. 70. (See also *Mémoires*, 1753, §§ III, V, IX, pp. 197–200.) In this way the broader conception of an arbitrary function which we have at the present time was first introduced.

§ *Mémoires de l'Acad. de Berlin*, 1748, p. 85.

where a is the length of the chord and $\alpha, \beta, \gamma, \ldots$ are any constants, the form of the chord after a time t would be given by the equation

$$y = \alpha \sin \frac{\pi x}{a} \cos \frac{\pi \nu}{a} + \beta \sin \frac{2\pi x}{a} \cos \frac{2\pi \nu}{a} + \gamma \sin \frac{3\pi x}{a} \cos \frac{3\pi \nu}{a} + \ldots, \qquad (2)$$

where ν is a constant multiplied by t. (It is readily seen that if $\nu = ct$, (2) is a solution of the differential equation (1).) This solution Bernoulli, who next took up the subject, claimed was perfectly general,* and later Lagrange, from a comparison of the two different forms of solution, was led to believe† that a given analytical function could always be represented by a trigonometric series.

Not much progress was made in the discussion of this question until the time of Fourier, who in 1807 made to the French Academy of Sciences the announcement:

Every arbitrary function of a variable, whether simple or composed of any number of different parts defined by different laws, can be represented by a trigonometric series.

The well known method by which Fourier obtained the coefficients a_n, b_n of the trigonometric development

$$f(x) = \sum_{n=0}^{n=\infty} (a_n \sin nx + b_n \cos nx)$$

was by multiplying this equation by $\sin nx$ and $\cos nx$ respectively and integrating from $-\pi$ to π.‡ In this way is obtained readily

$$b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha)\, d\alpha\,; \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) \sin n\alpha \, d\alpha\,; \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) \cos n\alpha \, d\alpha. \qquad (3)$$

This method, however, is not due to Fourier. His merit lies principally in the fact that he was the first to recognize the possibility of representing a completely arbitrary function by a trigonometric series.§

Fourier did not prove generally that the series obtained converged to the value of the function. He showed, however, in several examples that this was so, and he considered that in the actual application of the development to any particular case the proof of the convergence would be easy.

The first important contribution to the theory of trigonometric series after Fourier's work upon this subject, was Dirichlet's memoir, published in Crelle's

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* *Mémoires de l'Acad. de Berlin*, 1753, p. 157, § XIII.

† *Miscellanea Taurinensia*, Vol. III, *Pars Math.*, p. 221, Art. XXV.

‡ Fourier first obtained the coefficients by integrating from 0 to π (*Theory of Heat*, Arts. 219–221). Afterwards he showed that they might be obtained by integrating from $-\pi$ to π (Art. 231).

§ See in regard to this Arnold Sachse's memoir, *Bulletin des Sci. Math.*, 1880, p. 47.

Journal in 1829.* In this memoir he demonstrated rigorously for the first time the possibility of representing by trigonometric series functions which, in the interval under consideration, fulfil the three conditions, known as Dirichlet's conditions, of being finite, possessing only a finite number of points of discontinuity and only a finite number of maxima and minima. His demonstration was based upon the proof that the following two equations

$$\lim_{k=\infty}\int_0^h \frac{f(\beta)\sin k\beta\, d\beta}{\sin\beta} = \frac{\pi}{2}f(0),\ \lim_{k=\infty}\int_g^h \frac{f(\beta)\sin k\beta\, d\beta}{\sin\beta} = 0\quad \left[0<g<h\leqq\frac{\pi}{2}\right] \tag{4}$$

are satisfied whenever f fulfils the above conditions. His demonstration showed that whenever f satisfies these two equations the trigonometric development is applicable. A few years later (1837) he showed† that the function might become infinite for isolated points between zero and h provided that the integral

$$\int_0^\beta f(\beta)\, d\beta = F(\beta)$$

remain finite and continuous as β varies from zero to h.

The next important memoir published on this subject was that of Lipschitz which appeared in Crelle's Journal in 1864.‡ In this paper he made an important extension to the class of functions which satisfy equations (4), and to which therefore Fourier's development can be applied. He presented a theorem§ which was, in effect, as follows:

If the function $f(\beta)$ is of such a nature that

$$\begin{array}{ll} |f(\beta)| < A & \left[\begin{array}{l} g\leqq\beta\leqq h \\ 0\leqq g<h\leqq\frac{\pi}{2}\end{array}\right], \\ \lim\limits_{\delta=0}\left[f(g+\delta)-f(g)\right] = 0 & \\ \lim\limits_{\delta=0}\dfrac{|f(\beta+\delta)-f(\beta)|}{\delta^a} < B & [g<\beta<h], \end{array} \tag{5}$$

where A, B, a denote finite positive constants, then

$$\lim_{k=\infty}\int_g^h f(\beta)\frac{\sin k\beta}{\sin\beta}\, d\beta$$

equals $\frac{\pi}{2}f(0)$ or zero according as $g \lesseqgtr 0$.

The theorem thus stated is not quite accurate, however, since simple continuity of the function at the lower limit of the integral is not sufficient, as is shown by the illustration given on page 21 of this paper. In fact, Lipschitz's

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* Vol. IV, p. 157; *Werke*, Vol. I, p. 117.
† Crelle's Journal, XVII, p. 54; *Werke*, Vol. I, p. 305.
‡ Vol. LXIII, p. 296.
§ P. 301.

demonstration requires that the difference $f(0+\delta)-f(0)$ when δ tends to zero approach zero with a certain degree of rapidity,* a point which he apparently overlooked. This defect, however, is removed at once by making the condition (5) include the limits.† The class of functions satisfying this condition includes many functions having an infinite number of maxima and minima which would be excluded by Dirichlet's conditions.

In 1867 Riemann's well known work upon trigonometric series was published.‡ After an historical sketch he considered the question: What must be the properties of a function which is supposed to be already represented by Fourier's series? From this point of view Riemann arrived at the important conclusion§ that the convergence of the series for any particular value of the variable depends only upon the behavior of the function in the vicinity of that value.

Attention was now directed more particularly to the nature of the convergence of Fourier's series, and to the question as to whether the development was unique. Heine first demonstrated in 1870‖ that a function satisfying Dirichlet's conditions possesses a development uniformly convergent in each interval comprised within the interval $(-\pi, \pi)$ and containing no point of discontinuity for the function. He also showed that there can be only one such development by showing that a *uniformly* convergent development representing zero except for a finite number of points cannot exist, that each coefficient must be identically zero.

Shortly afterwards Cantor proved** the more general theorem that *any* trigonometric series of the form

$$\sum_{n=0}^{n=\infty}(c_n \sin nx + d_n \cos nx), \tag{6}$$

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* See the first two of his inequalities on the top of page 307. For his proof it is necessary that $\lim \frac{\lambda+\mu}{2}\log 2m = 0$.

† It can then be easily shown that $\lim \frac{\lambda+\mu}{2}\log 2m = 0$. For $\frac{\lambda+\mu}{2} < \frac{1}{2} B\left(\frac{2m\pi}{k}\right)^{\alpha}$ and, if we write $\sqrt{k} = m + \sigma$ $[0 \leqq \sigma \leqq 1]$, we have

$$\lim\left(\frac{m}{k}\right)^{\alpha}\log m = \lim \frac{\log m}{\left(\frac{m^2+2m\sigma+\sigma^2}{m}\right)^{\alpha}} = \lim_{m=\infty}\frac{\log m}{m^{\alpha}} = 0.$$

‡ *Abhandlungen der Gesellschaft der Wissenschaft zu Göttingen*, 1867, *Math. Classe*, p. 87. It is also published in Riemann's *Werke*, p. 213, and *Bulletin des Sci. Math.*, 1873, Vol. V, p. 20. Although not published until 1867, it had been written for some time, having been presented by Riemann in 1854 for admission to the Faculty of Philosophy at the University of Göttingen.

§ *Werke*, p. 239; *Bulletin des Sci. Math.*, 1873, Vol. V, p. 82.

‖ Crelle's Journal, LXXI, p. 353, §§ 7, 8, 9.

** Crelle's Journal, LXXII, p. 139; LXXIII, p. 294.

convergent and representing zero, except for a finite number of values of x, cannot exist. This theorem Cantor soon extended* still further to the case where the series represents zero except for values of x corresponding to the points of a system of points P of the ν^{th} species† comprised within the interval. Finally in 1875 du Bois-Reymond‡ completed this part of the theory very satisfactorily by showing that *whenever* a function f can be represented by a series of the form (6), the coefficients must always have the definite form (3), thus showing that the development is unique.

Some other interesting results, which I will briefly mention, have been obtained more recently in other directions. In the *Comptes rendus* for 1881, p. 228, M. Camille Jordan has shown that functions having limited oscillation satisfy Dirichlet's equations (4) and hence are developable in Fourier's series. He also showed by an example that the class of functions possessing the property of limited oscillation includes some functions having an infinite number of discontinuities scattered all along over a finite interval.

In the same volume of the *Comptes rendus*,§ du Bois-Reymond gives an account of the results of his researches upon integrals similar to Dirichlet's, but more general. He obtained as a sufficient condition that equations similar to (4) be satisfied,

$$\lim_{\epsilon=0} \int_0^\epsilon \frac{da}{a} \bmod [f(a)-f(0)] = 0.$$

Du Bois-Reymond has also investigated functions of the form

$$\frac{\cos \psi(x)}{\rho(x)},$$

where $\rho(x)$ and $\psi(x)$ tend to infinity as x tends to zero, $\rho(x)$ being always positive. He has shown that there are functions of this kind which cannot be represented by Fourier's series at the point $x=0$.

Hölder, Kronecker, Weierstrass and some others have also written upon the subject, but it is not necessary to dwell upon their work here. Kronecker‖

* *Mathematische Annalen*, V, p. 123; *Acta Mathematica*, II, p. 336.

† If P comprises an infinite number of points, then there must be one or more points, called point limits, in the neighborhood of which there are an infinite number of the points of P. The aggregate of these point-limits is called the first derived system of P and is denoted by P'. The first derived system of P' is called the second derived system of P and is denoted by P'', and so on. If $P^{(\nu)}$ is the last derived system which P admits, *i. e.*, if $P^{(\nu)}$ comprises only a finite number of points, P is said to be of the ν^{th} species.

‡ *Beweis dass die Coefficienten*, etc., *Abhandl. d. k. bayer Akad. d. W.*, 1875, Vol. XII, pp. 117–167.

§ Pages 915, 962; see also his memoir, *Untersuchungen über die Convergenz und Divergenz der Fourierschen Darstellungsformeln*, *Abhandl. d. k. bayer Akad.*, Vol. XII, p. 1.

‖ *Sitzungsberichte der Ak. der W. zu Berlin*, 1885, p. 641.

obtained in a number of different forms, conditions that equations (4) be satisfied, and in the course of his paper showed that his forms included many of the conditions which had already been given by others.

2. In considering the development of a function $f(x)$ in trigonometric series going according to the sines and cosines of increasing multiples of x, it is natural to consider first what form, if $f(x)$ is thus developable, the coefficients of the sines and cosines will take. With this question naturally arises another, viz: Can the coefficients take more than one form? *i. e.* Is the development unique? As these questions are answered very satisfactorily by an important theorem published by du Bois-Reymond in 1875,* I will give that theorem here.

In whatever manner a function $f(x)$ can be developed in the series

$$f(x) = \sum_{n=0}^{n=\infty} (a_n \sin nx + b_n \cos nx), \tag{7}$$

holding within the interval $(-\pi, \pi)$, whose coefficients a_n, b_n become infinitely small† *with $\frac{1}{n}$, the coefficients have always the form*

$$b_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\, d\alpha; \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha) \sin n\alpha\, d\alpha; \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha) \cos n\alpha\, d\alpha,$$

provided that the integrals have sense.‡

The proof of this theorem is as follows:

Form the function

$$F(x) = b_0 \frac{x^2}{2} - \sum_{n=1}^{n=\infty} \frac{a_n \sin nx + b_n \cos nx}{n^2} \tag{8}$$

derived from (7) by two successive integrations. The function $F(x)$ possesses the following properties:§

1°. It is uniformly convergent for every value of x.

2°. $\lim\limits_{\epsilon=0} \dfrac{F(x+\epsilon) + F(x-\epsilon) - 2F(x)}{\epsilon^2} = f(x)$ except for values of x which make the series (7) divergent or discontinuous.

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* *Abhandl. der k. bayer Akad. d. W.*, Vol. XII, 1875, pp. 117–167, *Beweis dass die Coefficienten*, etc. As I have not had access to du Bois-Reymond's memoir, I follow here Sachse's presentation of the proof (*Bulletin des Sci. Math.*, 1880, p. 104). Since this is rather condensed, I have expanded it a little in a few places. In particular, I have given a proof for the case where the function becomes infinite, du Bois-Reymond's treatment of which Sachse has omitted.

† This will be the case if they will have a form like that given in the theorem, as is shown in the footnote on page 8.

‡ This requires of course that $f(x)$ be integrable in the interval $(-\pi, \pi)$.

§ Shown in Riemann's *Math. Werke*, pp. 231–34: *Bull. des Sci. Math.*, Vol. V, 1873, pp. 41–45; Picard, *Traité d'Analyse*, Vol. I, pp. 240–44.

3°. $\lim\limits_{\epsilon=0} \dfrac{F(x+\varepsilon)+F(x-\varepsilon)-2F(x)}{\varepsilon}=0$ for every value of x.

Let us consider first a function $f(x)$ which is finite and continuous and does not have an infinite number of maxima and minima. We must have necessarily for every value of x between $-\pi$ and $+\pi$,

$$\frac{d^2}{dx^2}\int_{-\pi}^{x} da \int_{-\pi}^{a} f(\beta)\, d\beta = f(x). \tag{9}$$

Form the expression

$$\Phi(x) = F(x) - \int_{-\pi}^{x} da \int_{-\pi}^{a} f(\beta)\, d\beta.$$

From the property 2° and from (9) it follows that

$$\lim_{\epsilon=0} \frac{\Phi(x+\varepsilon)-2\Phi(x)+\Phi(x-\varepsilon)}{\varepsilon^2} = \lim_{\epsilon=0} \frac{\Delta^2\Phi(x)}{\varepsilon^2} = 0,$$

and therefore

$$\Phi(x) = c_0 + c_1 x.$$

Hence

$$F(x) = \int_{-\pi}^{x} da \int_{-\pi}^{a} f(\beta)\, d\beta + c_0 + c_1 x. \tag{10}$$

If now, according to the method employed by Fourier, we multiply (8) by sin nx and cos nx successively, and integrate from $-\pi$ to $+\pi$, we will get

$$\left.\begin{aligned} &\int_{-\pi}^{\pi} F(a)\, da = \frac{\pi^3}{3} b_0, \quad \int_{-\pi}^{\pi} F(a) \sin na\, da = -\frac{a_n}{n^2}\pi, \\ &\int_{-\pi}^{\pi} F(a) \cos na\, da = -\frac{b_n \pi}{n^2} + (-1)^n \frac{2\pi b_0}{n^2}.^* \end{aligned}\right\} \tag{11}$$

In these equations substitute for $F(a)$ its value given in (10) and integrate by parts.

$$\begin{aligned} \int_{-\pi}^{\pi} F(x)\, dx &= \int_{-\pi}^{\pi} \left\{ \int_{-\pi}^{x} da \int_{-\pi}^{a} f(\beta)\, d\beta + c_0 + c_1 x \right\} dx\,; \quad \int_{-\pi}^{\pi} dx \int_{-\pi}^{x} da \int_{-\pi}^{a} f(\beta)\, d\beta \\ &= x \int_{-\pi}^{x} da \int_{-\pi}^{a} f(\beta)\, d\beta \Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} x dx \int_{-\pi}^{x} f(\beta)\, d\beta \\ &= \pi a \int_{-\pi}^{a} f(\beta)\, d\beta \Big]_{-\pi}^{\pi} - \pi \int_{-\pi}^{\pi} a f(a)\, da - \frac{x^2}{2} \int_{-\pi}^{x} f(\beta)\, d\beta \Big]_{-\pi}^{\pi} + \frac{1}{2} \int_{-\pi}^{\pi} x^2 f(x)\, dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} (\pi - a)^2 f(a)\, da\,; \quad \int_{-\pi}^{\pi} (c_0 + c_1 x)\, dx = 2\pi c_0. \end{aligned}$$

Hence

$$\frac{1}{2} \int_{-\pi}^{\pi} (\pi - a)^2 f(a)\, da + 2\pi c_0 = \frac{\pi^3}{3} b_0. \tag{12}$$

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* Integrating by parts, it is readily seen that

$$\int_{-\pi}^{\pi} \frac{x^2}{2} \cos nx\, dx = (-1)^n \frac{2\pi}{n^2}, \quad \int_{-\pi}^{\pi} \frac{x^2}{2} \sin nx\, dx = 0.$$

$$\int_{-\pi}^{\pi} F(a)\sin na\,da = \int_{-\pi}^{\pi}\sin na\,da\left\{\int_{-\pi}^{a}da\int_{-\pi}^{a}f(\beta)\,d\beta + c_0 + c_1 a\right\}$$
$$= \left\{\frac{1}{n}\cos na\left[\int_{-\pi}^{a}da\int_{-\pi}^{a}f(\beta)\,d\beta + c_0 + c_1 a\right]\right\}_{-\pi}^{\pi}$$
$$+\frac{1}{n}\int_{-\pi}^{\pi}\cos na\left[\int_{-\pi}^{a}f(\beta)\,d\beta + c_1\right]da$$
$$= \frac{(-1)^{n+1}}{n}\int_{-\pi}^{\pi}da\int_{-\pi}^{a}f(\beta)\,d\beta + \frac{(-1)^{n+1}2\pi c_1}{n} - \frac{1}{n^2}\int_{-\pi}^{\pi}f(a)\sin na\,da$$
$$= \frac{(-1)^{n+1}}{n}\,a\int_{-\pi}^{a}f(\beta)\,d\beta\Big]_{-\pi}^{\pi} + \frac{(-1)^{n}}{n}\int_{-\pi}^{\pi}af(a)\,da$$
$$+\frac{(-1)^{n+1}2\pi c_1}{n} - \frac{1}{n^2}\int_{-\pi}^{\pi}f(a)\sin na\,da.$$

Therefore

$$\frac{(-1)^{n+1}}{n}\left[\pi\int_{-\pi}^{\pi}f(a)\,da - \int_{-\pi}^{\pi}af(a)\,da + 2\pi c_1\right] - \frac{1}{n^2}\int_{-\pi}^{\pi}f(a)\sin na\,da = -\frac{a_n\pi}{n^2}. \quad (13)$$

$$\int_{-\pi}^{\pi}F(a)\cos na\,da = \int_{-\pi}^{\pi}\cos na\,da\left\{\int_{-\pi}^{a}da\int_{-\pi}^{a}f(\beta)\,d\beta + c_0 + c_1 a\right\}$$
$$= \left\{\frac{1}{n}\sin na\left[\int_{-\pi}^{a}da\int_{-\pi}^{a}f(\beta)\,d\beta + c_0 + c_1 a\right]\right\}_{-\pi}^{\pi}$$
$$-\frac{1}{n}\int_{-\pi}^{\pi}\sin na\left[\int_{-\pi}^{a}f(\beta)\,d\beta + c_1\right]da$$
$$= \left\{\frac{1}{n^2}\cos na\left[\int_{-\pi}^{a}f(\beta)\,d\beta + c_1\right]\right\}_{-\pi}^{\pi} - \frac{1}{n^2}\int_{-\pi}^{\pi}f(a)\cos na\,da.$$

Hence

$$\frac{(-1)^n}{n^2}\int_{-\pi}^{\pi}f(a)\,da - \frac{1}{n^2}\int_{-\pi}^{\pi}f(a)\cos na\,da = -\frac{b_n\pi}{n^2} + (-1)^n\frac{2\pi b_0}{n^2}. \quad (14)$$

From the three equations (12), (13), (14) we can readily obtain the values of c_0, c_1, b_0, a_n, b_n, by remembering that these equations are true for all integer values of n greater than zero, and that a_n, b_n and the integrals

$$\int_{-\pi}^{\pi}f(a)\sin na\,da,\quad \int_{-\pi}^{\pi}f(a)\cos na\,da$$

tend to zero with $\dfrac{1}{n}$.* Thus from equation (14) by making n tend to infinity

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* That these integrals tend to zero whenever $f(x)$ is integrable is proved by the same method as the theorem given on page 14. In the integral $\int_{-\pi}^{\pi}f(a)\sin na\,da$, $f(a)$ takes the place of the function $\dfrac{\phi(\gamma)}{\sin\gamma}$ in the proof of that theorem. For the integral $\int_{-\pi}^{\pi}f(a)\cos na\,da$ it is only necessary in that proof to make the subdivision of the interval $(-\pi, \pi)$ in such a way that each small interval equals $\dfrac{\pi}{k}$ and the points of subdivision are integer multiples of $\dfrac{\pi}{k}$.

we see first that

$$b_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a)\,da,$$

and that then we obtain the value of b_n for any value of n. The results which we obtain from the three equations are

$$c_0 = \frac{1}{4\pi}\int_{-\pi}^{\pi} f(a)\left[\frac{\pi^2}{3} - (\pi - a)^2\right] da, \quad c_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a)(\pi - a)\,da,$$

$$b_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a)\,da, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(a)\sin na\,da, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(a)\cos na\,da.$$

If now, however, supposing that $f(x)$ is finite, we subject it to the single additional condition of being integrable, the above proof that $\Phi(x) = c_0 + c_1 x$ will not hold. We must make another investigation for this case. $f(x)$ will not have necessarily at each point a determinate limit, but its value will lie between a superior and an inferior limit. Let us designate the half-sum of these two limits by $S(x)$, the half-difference by $D(x)$. We can evidently give to the function the form $S(x) + jD(x)$, where j denotes a real number lying between -1 and $+1$. Let us now, according to the method employed by Riemann, find an expression for $\lim\limits_{\varepsilon=0} \dfrac{\Delta^2 F(x)}{\varepsilon^2}$. Put

$$\sum_{n=0}^{n=\lambda}(a_n \sin nx + b_n \cos nx) = S(x) + \alpha_\lambda.$$

Taking δ an arbitrary small quantity, we can find a value of m such that $|\alpha_m| < D(x) + \delta$. We can now write*

$$\frac{\Delta^2 F(x)}{\varepsilon^2} = f(x) + \sum_{n=1}^{n=\infty} \alpha_n \left\{\left[\frac{\sin (n-1)\frac{\varepsilon}{2}}{(n-1)\frac{\varepsilon}{2}}\right]^2 - \left[\frac{\sin n\frac{\varepsilon}{2}}{n\frac{\varepsilon}{2}}\right]^2\right\}.$$

Taking ε sufficiently small to have $m\dfrac{\varepsilon}{2} < \pi$, let us divide this series into three parts. In the first let n increase from 1 to m; in the second from $m+1$ to s, the greatest integer in $\dfrac{\pi}{\frac{1}{2}\varepsilon}$, and in the third from $s+1$ to infinity. The first part will tend to zero as ε diminishes. The second part will be less in absolute value than

$$\left[\left(\frac{\sin m\frac{1}{2}\varepsilon}{m\frac{1}{2}\varepsilon}\right)^2 - \left(\frac{\sin s\frac{1}{2}\varepsilon}{s\frac{1}{2}\varepsilon}\right)^2\right](D(x) + \delta).$$

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* Riemann, *Math. Werke*, p. 223; *Bul. des Sci. Math.*, 1873, p. 43; Picard, *Traité d'Analyse*, Vol. I, p. 241.

The third part will be less in absolute value than

$$\left[\frac{1}{(\pi-\frac{1}{2}\varepsilon)^2}+\frac{1}{\pi-\frac{1}{2}\varepsilon}\right](D(x)+\delta).$$

Now passing to the limit we see that the first of these expressions will tend to $D(x)+\delta$, and the second to $\left(\frac{1}{\pi^2}+\frac{1}{\pi}\right)(D(x)+\delta)$. Hence

$$\lim_{\epsilon=0}\frac{\Delta^2 F(x)}{\varepsilon^2}=S(x)+j\left(1+\frac{1}{\pi^2}+\frac{1}{\pi}\right)[D(x)+\delta] \qquad [-1\overline{\gtrless} j\leqq 1].$$

Putting

$$F_1(x)=\int_{-\pi}^{x}d\alpha\int_{-\pi}^{\alpha}f(\beta)\,d\beta,$$

we have that

$$\lim_{\epsilon=0}\frac{\Delta^2 F_1(x)}{\varepsilon^2}=S(x)+j_1D(x) \qquad [-1\overline{\gtrless} j_1\leqq 1].$$

But $\Phi(x)=F(x)-F_1(x)$, and hence, neglecting the arbitrary small quantity δ, the modulus of the greatest value which $\lim\limits_{\epsilon=0}\dfrac{\Delta^2\Phi(x)}{\varepsilon^2}$ can take is

$$\left(2+\frac{1}{\pi^2}+\frac{1}{\pi}\right)D(x).$$

We can now, by making use of the condition that $f(x)$ is integrable, show that $\Phi(x)$ is a linear function of x. Let us divide the interval $(x, x+a)$ into smaller intervals limited by the points x, $x+\delta_1$, $x+\delta_1+\delta_2, \ldots, x+\delta_1+\ldots+\delta_{n-1}$, $x+a$ and form the sum

$$\delta_1 D(x+\rho_1\delta_1)+\delta_2 D(x+\delta_1+\rho_2\delta_2)+\ldots+\delta_n D(x+\delta_1+\delta_2+\ldots+\rho_n\delta_n), \tag{15}$$

where the quantities ρ are positive fractions. This sum, in virtue of the condition of integrability, must tend to zero with δ. Hence when the δ's become infinitely small and ε tends to zero, the limit of

$$\frac{\Delta^2[\delta_1\Phi(x+\rho_1\delta_1)+\delta_2\Phi(x+\delta_1+\rho_2\delta_2)+\ldots+\delta_n\Phi(x+\delta_1+\ldots+\rho_n\delta_n)]}{\varepsilon^2},$$

which in absolute value cannot exceed the sum (15) multiplied by $\left(2+\frac{1}{\pi^2}+\frac{1}{\pi}\right)$, must be zero. That is, we must have

$$\lim_{\epsilon=0}\frac{\Delta^2}{\varepsilon^2}\int_0^a\Phi(x+\alpha)\,d\alpha=0.$$

It follows therefore that

$$\int_0^a\Phi(x+\alpha)\,d\alpha=c_0+c_1x.$$

But since $F(x)$ and $F_1(x)$ are continuous functions of x,* $\Phi(x), = F(x) - F_1(x)$ is continuous also. Hence, employing the theorem of means, we can write

$$\int_0^a \Phi(x+\alpha)\,d\alpha = a\,\Phi(x+\alpha_1) \qquad [0 < \alpha_1 < a].$$

Consequently

$$\Phi(x+\alpha_1) = \frac{c_0}{a} + \frac{c_1}{a}x.$$

Now let a tend to zero. The first member will tend to a definite limit, and therefore the second member will also, *i. e.*, $\frac{c_0}{a}$ and $\frac{c_1}{a}$ will tend to fixed quantities c_0' and c_1'. We have then

$$\Phi(x) = c_0' + c_1' x,$$

and the demonstration can be completed as before.

Suppose now that $f(x)$ becomes infinite in the interval $(-\pi, \pi)$, but still satisfies the condition of integrability. Over each portion of the interval containing no infinite point for $f(x)$ $\Phi(x)$ is a linear function of x by the above proof. Hence since $\Phi(x)$ is continuous in the interval $(-\pi, \pi)$, the curve $y = \Phi(x)$, as x varies from $-\pi$ to π, if it does not represent a straight line, at least represents a continuous broken line, the corners corresponding to values of x which make $f(x)$ infinite. We will show that the line is straight.

Writing

$$\int_{-\pi}^{\alpha} f(\beta)\,d\beta = \Psi(\alpha),$$

it is easy to show that $\Psi(\alpha)$ is continuous in the interval $(-\pi, \pi)$. For

$$\Psi(\alpha+\delta) - \Psi(\alpha) = \int_{\alpha}^{\alpha+\delta} f(\beta)\,d\beta$$

can evidently be made less than any assigned small quantity ε by choosing δ sufficiently small since, if $f(x)$ becomes infinite for $x = x_0$, the condition of integrability requires that

$$\lim_{\epsilon=0}\int_{x_0-\epsilon}^{x_0} f(\beta)\,d\beta = 0, \quad \lim_{\epsilon=0}\int_{x_0}^{x_0+\epsilon} f(\beta)\,d\beta = 0.$$

Employing the theorem of means, and remembering that $\Psi(\alpha)$ is continuous, we get at once

$$\lim_{\epsilon=0}\frac{\Delta^2 F_1(x)}{\varepsilon} = \lim_{\epsilon=0}\frac{\Delta^2}{\varepsilon}\int_{-\pi}^{x}\Psi(\alpha)\,d\alpha = \lim_{\epsilon=0}\frac{1}{\varepsilon}\left[\int_x^{x+\epsilon}\Psi(\alpha)\,d\alpha - \int_{x-\epsilon}^{x}\Psi(\alpha)\,d\alpha\right] = 0.$$

---

* $F_1(x)$ is continuous whenever f is integrable, since F_1 is the integral from $-\pi$ to x of a finite function.

But since by property 3°, p. 7, $\lim\limits_{\epsilon=0} \frac{\Delta^2 F(x)}{\varepsilon} = 0$, we must have

$$\lim_{\epsilon=0} \frac{\Phi(x+\varepsilon)+\Phi(x-\varepsilon)-2\Phi(x)}{\varepsilon}=0.$$

That is,

$$\lim_{\epsilon=0} \frac{\Phi(x+\varepsilon)-\Phi(x)}{\varepsilon}=\lim_{\epsilon=0} \frac{\Phi(x)-\Phi(x-\varepsilon)}{\varepsilon},$$

which tells us that the directions of the two straight lines meeting at any vertex are the same. Hence, for values of x in the interval $(-\pi, \pi)$, $y=\Phi(x)$ represents a single straight line, and we can write

$$\Phi(x)=c+c'x$$

and complete the demonstration as in the first case.

The theorem just proved shows that whenever $f(x)$ is integrable and can be represented by a development of the form (7), the coefficients a_n and b_n in this development must always be of the form originally given by Fourier. But the integrals giving these coefficients are perfectly determinate quantities. It follows therefore that a given function can be developed in a trigonometric series of the nature mentioned in only one way, or, in other words, that the development (7) must be unique.

We have considered above a development available in the interval $(-\pi, \pi)$. If, however, we had desired to obtain the form of the coefficients in a development holding for some other interval of 2π, say $[m\pi, (m+2)\pi]$ where m is an integer, the method of obtaining them would have been precisely the same except that instead of integrating over the interval $(-\pi, \pi)$, as on page 7, we would have integrated over the interval $[m\pi, (m+2)\pi]$. If we should do this, getting first equations similar to (11), and then after partial integration equations similar to (12), (13) and (14), we would find that while the expressions for c_0 and c_1 would be different, b_0, a_n and b_n would be the same as before (p. 9), except that the field of integration would be $[m\pi, (m+2)\pi]$.*

3. We have seen that if a function $f(x)$ is capable of being represented by a trigonometric series of the form (7), this development must be

$$f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(a)\,da + \frac{1}{\pi}\left\{\begin{aligned} &\sin x \int_{-\pi}^{\pi} f(a)\sin a\,da + \sin 2x \int_{-\pi}^{\pi} f(a)\sin 2a\,da + \ldots \\ &+\cos x \int_{-\pi}^{\pi} f(a)\cos a\,da + \cos 2x \int_{-\pi}^{\pi} f(a)\cos 2a\,da + \ldots \end{aligned}\right\}. \qquad (16)$$

* Evidently also a development of $f(x)$, available in the neighborhood of any point a, is at once obtained by developing $f(a+x)$, regarded as a new function of x, in the ordinary manner. The two developments, holding for the interval $[m\pi, (m+2)\pi]$, obtained by the two methods are easily shown to be identical.

We need then to consider under what conditions this series will converge to the value of the function $f(x)$. The sum of the first n terms, S_n say, gives us

$$S_n = \frac{1}{\pi}\int_{-\pi}^{\pi}[\tfrac{1}{2} + \cos(a - x) + \cos 2(a - x) + \ldots + \cos n(a - x)]f(a)\,da$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(a)\sin(2n+1)\frac{a-x}{2}\,da}{2\sin\frac{a-x}{2}}.$$

Let us find to what limit S_n tends when n increases indefinitely. By making a change of variable S_n can be written

$$S_n = \frac{1}{\pi}\int_{-\frac{1}{2}(\pi+x)}^{-\frac{1}{2}(\pi-x)}\frac{f(x+2\gamma)\sin(2n+1)\gamma\,d\gamma}{\sin\gamma}$$

$$= \frac{1}{\pi}\int_{0}^{\frac{1}{2}(\pi-x)}\frac{f(x+2\gamma)\sin(2n+1)\gamma\,d\gamma}{\sin\gamma}$$

$$+ \frac{1}{\pi}\int_{0}^{\frac{1}{2}(\pi+x)}\frac{f(x-2\gamma)\sin(2n+1)\gamma\,d\gamma}{\sin\gamma}, \tag{17}$$

where $-\pi < x < \pi$. This form of expressing S_n leads one very naturally to consider the limit for $k = \infty$ of the following integral, known as Dirichlet's integral,

$$\int_0^h \frac{\varphi(\gamma)\sin k\gamma}{\sin\gamma}\,d\gamma \qquad [0 < h < \pi],$$

where $\varphi(\gamma)$ has been put for $f(x \pm 2\gamma)$. We will consider under what circumstances

$$\lim_{k=\infty}\int_0^h \frac{\varphi(\gamma)\sin k\gamma\,d\gamma}{\sin\gamma} = \lim_{\epsilon=0}\frac{\pi}{2}\,\varphi(\varepsilon). \tag{18}$$

For if (18) holds for all values of h lying between 0 and π, then when n increases indefinitely, S_n will tend to the limit

$$\lim_{\epsilon=0}\tfrac{1}{2}[f(x+2\varepsilon) + f(x-2\varepsilon)],$$

i. e., to $f(x)$ unless f has a point of discontinuity at the point x, and in this case to a mean between the two values taken by f at the point.

It is only necessary, however, to consider $0 < h \leq \frac{\pi}{2}$, for if (18) holds for such values of h, it will hold also when $\frac{\pi}{2} < h < \pi$. In order to see this, suppose (18) is true when $0 < h \leq \frac{\pi}{2}$. This will evidently require that

$$\lim_{k=\infty}\int_g^h \frac{\varphi(\gamma)\sin k\gamma\,d\gamma}{\sin\gamma} = 0 \qquad \left[0 < g < h \leq \frac{\pi}{2}\right]. \tag{19}$$

If now $\frac{\pi}{2} < h < \pi$, then

$$\int_0^h \frac{\varphi(\gamma)\sin k\gamma}{\sin\gamma}\,d\gamma = \int_0^{\frac{\pi}{2}} \frac{\varphi(\gamma)\sin k\gamma}{\sin\gamma}\,d\gamma + \int_{\frac{\pi}{2}}^h \frac{\varphi(\gamma)\sin k\gamma}{\sin\gamma}\,d\gamma.$$

In the case where k is an odd integer, $k = 2n+1$, the second integral in the right-hand member of this last equation will be zero, for, on changing γ into $\pi - \gamma$, it will become

$$\int_{\pi-h}^{\frac{\pi}{2}} \frac{\varphi(\pi-\gamma)\sin k\gamma\, d\gamma}{\sin\gamma},$$

which is zero by (19), provided that, over the interval, $\varphi(\pi-\gamma)$ fulfil the conditions required of φ by equation (19).

4. We have shown in the last section that the question of the convergence of the trigonometric series (16) to the value $f(x)$ reduces to the consideration of the circumstances under which

$$\lim_{k=\infty}\int_0^h \frac{\varphi(\gamma)\sin k\gamma d\gamma}{\sin\gamma} = \lim_{\epsilon=0}\frac{\pi}{2}\varphi(\varepsilon) \qquad \left[0 < h \leqq \frac{\pi}{2}\right]. \tag{20}$$

We will now show that it is only necessary to consider what conditions φ must fulfil in order that (20) be satisfied when h is a quantity greater than zero, but as small as we please. This can be seen at once from the following theorem:

Whenever $\varphi(\gamma)$ is integrable over the interval (ε, h) and $\varphi(\varepsilon)$ is finite,

$$\lim_{k=\infty}\int_\epsilon^h \frac{\varphi(\gamma)\sin k\gamma\, d\gamma}{\sin\gamma} = 0 \qquad \left[0 < \varepsilon < h \leqq \frac{\pi}{2}\right].$$

To show this let us divide the interval (ε, h) into n parts $\delta_1, \delta_2, \ldots \delta_n$, making the points of division odd integer multiples of $\frac{\pi}{2k}$, distant from each other by an amount equal to $\frac{\pi}{k}$. Let us also take ε itself an odd integer multiple of $\frac{\pi}{2k}$, so that we shall have

$$\delta_1 = \delta_2 = \delta_3 = \ldots = \delta_{n-1} = \frac{\pi}{k} \geqq \delta_n.$$

We will now seek the limit of

$$\int_\epsilon^h \frac{\varphi(\gamma)\sin k\gamma\, d\gamma}{\sin\gamma} \tag{21}$$

as k tends to infinity. Let γ_p denote the value of γ, in the interval δ_p, for which $\varphi(\gamma)$ approaches nearest to zero, and in each interval δ_p let

$$\frac{\varphi(\gamma)}{\sin\gamma} = \frac{\varphi(\gamma_p)}{\sin\gamma_p} + J\left[\frac{\varphi(\gamma)}{\sin\gamma}\right].$$

We can evidently express the integral (21) as the following sum of integrals,

$$\sum_{p=1}^{p=n-1}\left[\frac{\varphi(\gamma_p)}{\sin\gamma_p}\int_{\lambda_p\frac{\pi}{2k}}^{\lambda_p+1\frac{\pi}{2k}}\sin k\gamma\,d\gamma\right]+\int_{\epsilon}^{\lambda_n\frac{\pi}{2k}}\varDelta\left[\frac{\varphi(\gamma)}{\sin\gamma}\right]\sin k\gamma\,d\gamma+\int_{\lambda_n\frac{\pi}{2k}}^{h}\frac{\varphi(\gamma)\sin k\gamma\,d\gamma}{\sin\gamma},\quad(22)$$

where $\lambda_1\dfrac{\pi}{2k}=\varepsilon$, $\lambda_{i+1}=\lambda_i+2$, $\lambda_n\dfrac{\pi}{2k}=h-\delta_n$. But since, in virtue of the condition of integrability,

$$\sum_{p=1}^{p=n-1}\left[\frac{\varphi(\gamma_p)}{\sin\gamma_p}\int_{\lambda_p\frac{\pi}{2k}}^{\lambda_p+1\frac{\pi}{2k}}d\gamma\right]=\sum_{p=1}^{p=n-1}\left[\frac{\varphi(\gamma_p)\,\delta_p}{\sin\gamma_p}\right]=\text{constant},$$

the first term of (22) is zero because

$$\int_{\lambda_p\frac{\pi}{2k}}^{\lambda_p+1\frac{\pi}{2k}}\sin k\gamma\,d\gamma=-\frac{\cos k\gamma}{k}\Bigg]_{\lambda_p\frac{\pi}{2k}}^{\lambda_p+1\frac{\pi}{2k}}=0.$$

Now let k increase indefinitely. The first term of (22) will continue to be zero. As for the second term we have always

$$\left|\int_{\epsilon}^{\lambda_n\frac{\pi}{2k}}\varDelta\left[\frac{\varphi(\gamma)}{\sin\gamma}\right]\sin k\gamma\,d\gamma\right|<\int_{\epsilon}^{\lambda_n\frac{\pi}{2k}}\left|\varDelta\left[\frac{\varphi(\gamma)}{\sin\gamma}\right]\sin k\gamma\right|d\gamma<\int_{\epsilon}^{\lambda_n\frac{\pi}{2k}}\left|\varDelta\left[\frac{\varphi(\gamma)}{\sin\gamma}\right]\right|d\gamma,$$

which tends to zero from the condition of integrability. Finally, for the last term of (22), since $\dfrac{\sin k\gamma}{\sin\gamma}$ varies always in the same sense from $h-\delta_n$ to h, we have, from Bonnet's theorem, that

$$\int_{h-\delta_n}^{h}\frac{\varphi(\gamma)\sin k\gamma}{\sin\gamma}\,d\gamma$$

is not greater in absolute value than one of the two quantities

$$\frac{1}{\sin(h-\delta_n)}\int_{h-\delta_n}^{\xi}\varphi(\gamma)\,d\gamma,\quad\frac{1}{\sin(h-\delta_n)}\int_{\xi}^{h}\varphi(\gamma)\,d\gamma\qquad[h-\delta_n<\xi<h].$$

But each of these approaches zero as k increases indefinitely, even though $\varphi(\gamma)$ becomes infinite, because if γ_0 is a point at which $\varphi(\gamma)$ becomes infinite, the condition of integrability requires that

$$\lim_{\alpha=0}\int_{\gamma_0-\alpha}^{\gamma_0}\varphi(\gamma)\,d\gamma=0,\quad\lim_{\alpha=0}\int_{\gamma_0}^{\gamma_0+\alpha}\varphi(\gamma)\,d\gamma=0.$$

Hence the sum (22) tends to zero as k becomes infinite, and we have

$$\lim_{k=\infty}\int_{\epsilon}^{h}\frac{\varphi(\gamma)\sin k\gamma\,d\gamma}{\sin\gamma}=0.$$

This conclusion is very important, for (see equation (17), p. 13) it shows that if a function $f(x)$ is to be developed in Fourier's series, it only needs to satisfy the condition of integrability except for a region as small as we please on each side of the point at which the development is to be made, or in other words, that the convergence of the series for any particular value of x depends only upon the behavior of the function in the vicinity of that value.

A particular case of the above theorem is when $\varphi(\gamma)$ is a constant, say unity. The theorem will then become

$$\lim_{k=\infty}\int_{\epsilon}^{h}\frac{\sin k\gamma\, d\gamma}{\sin\gamma}=0 \qquad \left[0<\epsilon<h\leq\frac{\pi}{2}\right]. \tag{23}$$

This can also be shown very readily by partial integration.*

5. The theorem proved in the preceding section reduces our problem to the consideration of the conditions which $\varphi(\gamma)$ must satisfy in order that

$$\lim_{k=\infty}\int_{0}^{\epsilon}\frac{\varphi(\gamma)\sin k\gamma\, d\gamma}{\sin\gamma}=\lim_{\sigma=0}\frac{\pi}{2}\varphi(\sigma)=\frac{\pi}{2}\varphi(+0),\ \text{say}, \tag{24}$$

where ϵ denotes a quantity as small as we please but greater than zero.

Now, k being an odd integer,

$$\int_{0}^{\frac{\pi}{2}}\frac{\sin k\gamma}{\sin\gamma}\, d\gamma=\frac{\pi}{2},\dagger$$

and hence from (23) we must have

$$\lim_{k=\infty}\int_{0}^{\epsilon}\frac{\sin k\gamma}{\sin\gamma}\, d\gamma=\frac{\pi}{2}.$$

Accordingly we may write

$$\lim_{k=\infty}\int_{0}^{\epsilon}\frac{\varphi(\gamma)\sin k\gamma\, d\gamma}{\sin\gamma}=\lim_{k=\infty}\int_{0}^{\epsilon}\frac{[\varphi(\gamma)-\varphi(+0)]\sin k\gamma\, d\gamma}{\sin\gamma}+\frac{\pi}{2}\varphi(+0).$$

Equation (24) will therefore be satisfied if

$$\lim_{k=\infty}\int_{0}^{\epsilon}\frac{[\varphi(\gamma)-\varphi(+0)]\sin k\gamma\, d\gamma}{\sin\gamma}=0. \tag{25}$$

---

* $$\int_{\epsilon}^{h}\frac{\sin k\gamma}{\sin\gamma}\, d\gamma=-\frac{1}{k}\frac{\cos k\gamma}{\sin\gamma}\bigg]_{\epsilon}^{h}-\frac{1}{k}\int_{\epsilon}^{h}\frac{\cos k\gamma\cos\gamma\, d\gamma}{\sin^2\gamma},$$

which tends to zero with $\frac{1}{k}$.

† This follows at once from the equation

$$\frac{\sin(2n+1)x}{\sin x}=1+2\cos 2x+2\cos 4x+\ldots+2\cos 2nx.$$

Let us first suppose that φ is a function possessing the property of limited oscillation or limited variation, as Jordan has defined it,* in the interval $(0, \varepsilon)$. It can then be written in the form

$$\varphi(\gamma) = \varphi(+0) + P_\gamma - N_\gamma,$$

where P_γ represents the sum of the positive oscillations of φ from zero to γ, and N_γ represents the sum of the negative oscillations. In this case the left hand member of (25) will take the form

$$\lim_{k=\infty} \int_0^\varepsilon \frac{(P_\gamma - N_\gamma)\sin k\gamma}{\sin\gamma}\,d\gamma.$$

But from Bonnet's theorem, since P_γ is a positive function, never decreasing,

$$\int_0^\varepsilon \frac{P_\gamma \sin k\gamma\,d\gamma}{\sin\gamma} = P_\varepsilon \int_\xi^\varepsilon \frac{\sin k\gamma}{\sin\gamma}\,d\gamma \qquad [0 < \xi < \varepsilon].$$

Suppose now that we divide the interval $\left(0, \frac{\pi}{2}\right)$ into the partial intervals

$$\left(0, \frac{\pi}{k}\right), \left(\frac{\pi}{k}, \frac{2\pi}{k}\right), \left(\frac{2\pi}{k}, \frac{3\pi}{k}\right), \ldots \left(\frac{r\pi}{k}, \frac{\pi}{2}\right),$$

r being the largest number of times that $\frac{\pi}{k}$ is contained in $\frac{\pi}{2}$. We can write

$$\frac{\pi}{2} = \int_0^{\frac{\pi}{2}} \frac{\sin k\gamma}{\sin\gamma}\,d\gamma = \rho_0 - \rho_1 + \rho_2 - \ldots (-1)^r \rho_r,$$

where

$$\rho_{n-1} = \int_{\left(\frac{n-1}{k}\right)\pi}^{\frac{n\pi}{k}} \frac{\sin k\gamma\,d\gamma}{\sin\gamma}, \quad \rho_{n-1} > \rho_n > 0, \quad \rho_n < \frac{2}{k}\,\frac{1}{\sin\frac{n\pi}{k}} \qquad [n = 1, 2, 3, \ldots r].$$

This gives at once

$$\rho_1 < \frac{2}{\pi}\,\frac{\frac{\pi}{k}}{\sin\frac{\pi}{k}} < \frac{2}{\pi}, \quad \rho_0 < \frac{\pi}{2} + \frac{2}{\pi}, \tag{26}$$

and hence *whatever* value ξ takes in the interval $\left(0, \frac{\pi}{2}\right)$,

$$0 < \int_0^\xi \frac{\sin k\gamma}{\sin\gamma}\,d\gamma < \frac{\pi}{2} + \frac{2}{\pi}. \tag{27}$$

But since

$$\lim_{k=\infty} \int_0^\varepsilon \frac{\sin k\gamma\,d\gamma}{\sin\gamma} = \frac{\pi}{2},$$

we have

$$\lim_{k=\infty} \left| \int_\xi^\varepsilon \frac{\sin k\gamma\,d\gamma}{\sin\gamma} \right| < \frac{\pi}{2}.$$

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* See *Comptes rendus*, 1881, p. 228; also Jordan's *Cours d'Analyse*, Vol. II, p. 216.

It follows therefore that

$$\lim_{k=\infty}\left|\int_0^\epsilon \frac{P_\gamma \sin k\gamma}{\sin\gamma}\,d\gamma\right| < \frac{\pi}{2} P_\epsilon .$$

Similarly

$$\lim_{k=\infty}\left|\int_0^\epsilon \frac{N_\gamma \sin k\gamma}{\sin\gamma}\,d\gamma\right| < \frac{\pi}{2} N_\epsilon .$$

Hence

$$\lim_{k=\infty}\left|\int_0^\epsilon \frac{(P_\gamma - N_\gamma)\sin k\gamma\,d\gamma}{\sin\gamma}\right| < \pi M,$$

where M is the largest of the quantities P_ϵ and N_ϵ. It follows therefore that (24) will be satisfied provided that P_ϵ and N_ϵ tend to zero with ε. This will evidently be the case under the following circumstances:

1°. If the function is continuous and has only a finite number of maxima and minima.

2°. If the function has a finite number of discontinuities,* but is finite, and has only a finite number of maxima and minima.

These conditions are precisely those of Dirichlet, and hence we see that for functions satisfying Dirichlet's conditions we must have

$$\lim_{k=\infty}\int_0^\epsilon \frac{\varphi(\gamma)\sin k\gamma\,d\gamma}{\sin\gamma} = \frac{\pi}{2}\varphi(+0).$$

6. If the function φ is continuous or has a finite number of discontinuities, but has an infinite number of maxima and minima, it may still be true, when φ possesses the property of limited oscillation in the interval $(0, \varepsilon)$, that P_ϵ and N_ϵ tend to zero with ε. But for the general case of a function with an infinite number of maxima and minima (with no discontinuities or a finite number of them), and also for the case of an infinite number of discontinuities, a different investigation is necessary. We can, however, derive a very general condition which it is sufficient for functions of this kind to fulfil in order that the equation just written be satisfied.

In equation (25) we can replace $\sin\gamma$ by γ since their ratio is very near unity when ε is very small. Now

$$\left|\int_0^\epsilon \frac{[\varphi(\gamma)-\varphi(+0)]\sin k\gamma\,d\gamma}{\gamma}\right| < \int_0^\epsilon \left|\frac{\varphi(\gamma)-\varphi(+0)}{\gamma}\sin k\gamma\right| d\gamma$$

$$< \int_0^\epsilon \left|\frac{\varphi(\gamma)-\varphi(+0)}{\gamma}\right| d\gamma .$$

* For if the function has a finite number of discontinuities, we can take ε so small that in the interval $(0, \varepsilon)$ the function will be continuous.

But ε can be taken as small as we please, and hence a sufficient condition that

$$\lim_{k=\infty}\int_0^h \frac{\varphi(\gamma)\sin k\gamma}{\sin\gamma}\,d\gamma=\frac{\pi}{2}\,\varphi(+0) \qquad \left[0<h\leqq\frac{\pi}{2}\right]$$

is

$$\lim_{\epsilon=0}\int_0^\epsilon \frac{|\,\varphi(\gamma)-\varphi(+0)\,|}{\gamma}\,d\gamma=0.* \tag{28}$$

Let us compare this condition with one or two other conditions which have been given. Lipschitz's condition was

$$\lim_{\delta=0}\frac{\varphi(\gamma+\delta)-\varphi(\gamma)}{\delta^a}<B, \tag{29}$$

where B and a denote any positive finite quantities. His condition is less general than (28). For if (29) is satisfied we can write

$$\lim_{\epsilon=0}\int_0^\epsilon \frac{|\,\varphi(\gamma)-\varphi(+0)\,|}{\gamma}\,d\gamma<\lim_{\epsilon=0}\int_0^\epsilon\frac{B\gamma^a\,d\gamma}{\gamma}.$$

But

$$B\int_0^\epsilon \gamma^{a-1}\,d\gamma=B\left.\frac{\gamma^a}{a}\right]_0^\epsilon$$

which tends to zero with ε. Hence (28) is satisfied in this case. But on the other hand (28) can be satisfied without Lipschitz's condition being fulfilled. Thus if φ was of such a nature that

$$\lim_{\delta=0}\left(\log\delta\right)^2\left[\varphi(\gamma+\delta)-\varphi(\gamma)\right]=K,$$

where K is a finite constant, Lipschitz's condition would not be fulfilled, since

$$\lim_{\delta=0}\frac{K}{\delta^a(\log\delta)^2}=\infty.$$

The condition (28), however, would be satisfied in this case, for we can write

$$\lim_{\epsilon=0}\int_0^\epsilon \frac{|\,\varphi(\gamma)-\varphi(+0)\,|}{\gamma}\,d\gamma=\lim_{\epsilon=0}K\int_0^\epsilon\frac{d\gamma}{\gamma(\log\gamma)^2}=\lim_{\epsilon=0}\left[-\frac{K}{\log\gamma}\right]_0^\epsilon=0.$$

Another condition which has been given is, denoting by D the difference $\varphi(\gamma+\delta)-\varphi(\gamma)$,

$$\lim_{\delta=0}D\log\delta=0. \tag{30}$$

The two conditions (28) and (30) differ from each other in comprehensiveness very little, but the latter is slightly more general. Thus both can easily be shown to be satisfied by a function for which

$$\lim_{\delta=0}D\,(\log\delta)^a=K \qquad [a>1],$$

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* See page 5. This condition, for integrals similar to Dirichlet's but more general, was obtained by du Bois-Reymond in another way. See his article, *Comptes rendus*, 1881, p. 915.

where K is a finite constant. But both cease to be satisfied for $a \overline{<} 1$. If, however, we consider a function for which

$$\lim_{\delta=0} D \log \delta \, [\log(-\log \delta)]^{a} = K \qquad [0 < a \overline{<} 1],$$

the condition (30) will be satisfied while (28) will not. For

$$\lim_{\delta=0} D \log \delta = \lim_{\delta=0} \frac{K}{[\log(-\log \delta)]^{a}} = 0.$$

But

$$\lim_{\epsilon=0} \int_0^{\epsilon} \frac{d\gamma}{\gamma \log \gamma \, [\log(-\log \gamma)]^{a}} = \lim_{\epsilon=0} \int_0^{\epsilon} \frac{d\,[\log(-\log \gamma)]}{[\log(-\log \gamma)]^{a}}$$
$$= \lim_{\epsilon=0} \Big[\frac{1}{1-a} [\log(-\log \gamma)]^{1-a} \Big]_0^{\epsilon} = \infty.$$

Let us now apply the condition (28) to one or two functions having an infinite number of maxima and minima in a finite region. The function $f(x) \equiv x \sin \frac{1}{x}$ becomes zero for $x = 0$, having at that point an infinite number of maxima and minima. For this function the left-hand member of (28) would become, since for $x = 0$ φ becomes $f(\pm 2\gamma) = f(2\gamma)$,

$$\lim_{\epsilon=0} \int_0^{\epsilon} 2 \sin \frac{1}{2\gamma} \, d\gamma.$$

But

$$\left| \int_0^{\epsilon} \sin \frac{1}{2\gamma} \, d\gamma \right| < \int_0^{\epsilon} \left| \sin \frac{1}{2\gamma} \right| d\gamma < \epsilon.$$

Hence, since the condition (28) is satisfied, $f(x)$ is capable of being represented by Fourier's series at the point zero as well as elsewhere.

Again consider the function

$$f(x) \equiv \frac{1}{\left(\log \left| \frac{x}{\lambda \pi} \right| \right)^{a}} \sin \frac{1}{x} \qquad [\lambda > 1].$$

For this case we have to consider

$$\lim_{\epsilon=0} \int_0^{\epsilon} \frac{\sin \frac{1}{2\gamma}}{\left(\log \frac{2\gamma}{\lambda\pi}\right)^{a}} \frac{d\gamma}{\gamma}.$$

Now

$$\left| \int_0^{\epsilon} \frac{\sin \frac{1}{2\gamma}}{\left(\log \frac{2\gamma}{\lambda\pi}\right)^{a}} \frac{d\gamma}{\gamma} \right| < \int_0^{\epsilon} \left| \frac{\sin \frac{1}{2\gamma}}{\left(\log \frac{2\gamma}{\lambda\pi}\right)^{a} \gamma} \right| d\gamma < \left| \frac{\left(\log \frac{2\gamma}{\lambda\pi}\right)^{1-a}}{1-a} \right| \Bigg]_0^{\epsilon},$$

which tends to zero with ε if $a > 1$. Hence $f(x)$ is developable in Fourier's series at the point zero.

7. Although the condition

$$\lim_{\epsilon=0}\int_0^\epsilon \frac{|\varphi(x)-\varphi(+0)|}{x}dx$$

imposes very little restriction upon the function φ, yet, as has already been mentioned,* du Bois-Reymond has shown that there are continuous functions which cannot be represented by Fourier's series at every point. The following illustration given by Schwartz† is a special case of functions of this kind discussed by du Bois-Reymond.‡

Recalling the value of the sum of the first n terms of Fourier's series,

$$S_n = \frac{1}{\pi}\int_0^{\frac{1}{2}(\pi-x)}\frac{f(x+2\gamma)\sin k\gamma}{\sin\gamma}d\gamma + \frac{1}{\pi}\int_0^{\frac{1}{2}(\pi+x)}\frac{f(x-2\gamma)\sin k\gamma}{\sin\gamma}d\gamma \quad [k=2n+1],$$

and the theorem given on page 14, it is evident that we only need to find an integrable function f such that for some value of x,

$$\lim_{k=\infty}\int_0^\epsilon \frac{f(x+2\gamma)\sin k\gamma\, d\gamma}{\gamma} = \infty,$$

where ε is a small quantity greater than zero, and of such a nature over the interval $(x-\varepsilon, x)$ that the second integral of S_n is finite.

Divide the interval $\left(\frac{\pi}{2}, 0\right)$ into intervals becoming smaller and smaller, as follows:

$$\left[\frac{\pi}{2}, \frac{\pi}{(1)}\right], \left[\frac{\pi}{(1)}, \frac{\pi}{(2)}\right], \ldots \left[\frac{\pi}{(\lambda-1)}, \frac{\pi}{(\lambda)}\right], \ldots \left[\frac{\pi}{(\mu-1)}, \frac{\pi}{(\mu)}\right], \left[\frac{\pi}{(\mu)}, 0\right], \tag{31}$$

where $(\lambda)=1.3.5\ldots[2\lambda+1]$, and consider a function which in the λ^{th} interval is defined by the formula

$$f(\beta) = c_\lambda \sin(\lambda)\frac{\beta}{2},$$

where the constants $c_1, c_2, \ldots c_{\mu+1}$ are positive and decrease indefinitely to zero as μ becomes infinite. When $\mu=\infty$ this function, evidently continuous, tends to zero with β, presenting an infinite number of maxima and minima in

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* Page 5.

† *Bulletin des Sci. Math.*, 1880, p. 109. I give here the function given by Schwartz, but prove the divergence of the series differently.

‡ *Untersuchungen über die Convergenz und Divergenz der Fourierschen Darstellungsformeln. Abhandl. d. k. bayer Akad. d. W.*, II Cl., XII Bd., II Abth., §§ 35-37.

the region of the point zero. We will assume that for negative values of β, $f(\beta)$ is of such a nature that it fulfils the condition (28). This is allowable since f is an arbitrary function.

In order to show that the Fourier series, which for all other values of β represents the function, diverges for $\beta = 0$, we need to examine the integral

$$\int_0^\epsilon \frac{f(2\gamma)\sin k\gamma}{\gamma}\,d\gamma\,.$$

We will show that this integral, with proper choice of the c's, becomes infinite with k. To do this we will give to k only values of the form (μ), where μ increases indefinitely.* Let us divide the integral into partial integrals corresponding to the intervals of (31), beginning with the right, and let λ_1 be the greatest integer for which $\frac{\pi}{(\lambda_1)} \overline{\gtrless} \varepsilon$. Putting

$$J = \int_0^{\frac{\pi}{(\lambda_1)}} \frac{f(2\gamma)\sin(\mu)\gamma\,d\gamma}{\gamma}\,,$$

we have

$$J = c_{\mu+1}\int_0^{\frac{\pi}{(\mu)}} \frac{\sin(\mu+1)\gamma\sin(\mu)\gamma\,d\gamma}{\gamma} + \sum_{\mu}^{\lambda_1+1} J_\lambda\,,$$

where

$$J_\lambda = c_\lambda \int_{\frac{\pi}{(\lambda)}}^{\frac{\pi}{(\lambda-1)}} \frac{\sin(\lambda)\gamma\sin(\mu)\gamma\,d\gamma}{\gamma}\,.$$

Over the interval $\left[0\,,\frac{\pi}{(\mu)}\right]$ $\sin(\mu)\gamma$ is always positive, and hence applying the theorem of means, we have

$$|J_{\mu+1}| = \left| c_{\mu+1}\int_0^{\frac{\pi}{(\mu)}} \frac{\sin(\mu+1)\gamma\sin(\mu)\gamma\,d\gamma}{\gamma}\right| < c_{\mu+1}\int_0^{\frac{\pi}{(\mu)}} \frac{\sin(\mu)\gamma\,d\gamma}{\gamma}\,.$$

But this last integral by equation (27), p. 17, is less than $\frac{\pi}{2} + \frac{2}{\pi}$, and since $\lim\limits_{\mu=\infty} c_{\mu+1} = 0$, we must have

$$\lim_{\mu=\infty} J_{\mu+1} = 0\,.$$

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*This of course is allowable, because increasing k from (μ) to $(\mu+1)$ amounts simply to adding a large number of the terms of the series at one time instead of taking only one term additional. Thus putting λ equal to the number of terms of the series taken when k is (μ), and λ' the number taken when k is $(\mu+1)$, we have

$$\lambda = \frac{1.3.5.7\ldots.[2\mu+1]-1}{2}\,,\qquad \lambda' = \frac{1.3.5\ldots.[2\mu+1]\,[2\mu+3]-1}{2}\,.$$

Hence
$$\lambda' = \lambda + [\mu+1]\,(\mu)\,.$$

To the integral J_λ add and subtract

$$\tfrac{1}{2} c_\lambda \int_{\frac{\pi}{(\lambda)}}^{\frac{\pi}{(\lambda-1)}} \frac{\cos(\lambda)\gamma \cos(\mu)\gamma}{\gamma} d\gamma .$$

We get readily

$$J_\lambda = \tfrac{1}{2} c_\lambda \left[\int_{\frac{\pi}{(\lambda)}}^{\frac{\pi}{(\lambda-1)}} \frac{\cos[(\mu)\gamma - (\lambda)\gamma]\, d\gamma}{\gamma} - \int_{\frac{\pi}{(\lambda)}}^{\frac{\pi}{(\lambda-1)}} \frac{\cos[(\mu)\gamma + (\lambda)\gamma]\, d\gamma}{\gamma} \right].$$

Integrating by parts we get

$$J_\lambda = \tfrac{1}{2} c_\lambda \left[\frac{1}{(\mu)-(\lambda)} \int_{\frac{\pi}{(\lambda)}}^{\frac{\pi}{(\lambda-1)}} \frac{\sin[(\mu)\gamma - (\lambda)\gamma]\, d\gamma}{\gamma^2} \right.$$

$$\left. - \frac{1}{(\mu)+(\lambda)} \int_{\frac{\pi}{(\lambda)}}^{\frac{\pi}{(\lambda-1)}} \frac{\sin[(\mu)\gamma + (\lambda)\gamma]\, d\gamma}{\gamma^2} \right]. \tag{32}$$

In particular,

$$J_\mu = \tfrac{1}{2} c_\mu \int_{\frac{\pi}{(\mu)}}^{\frac{\pi}{(\mu-1)}} \frac{d\gamma}{\gamma} - \frac{\tfrac{1}{2} c_\mu}{2(\mu)} \int_{\frac{\pi}{(\mu)}}^{\frac{\pi}{(\mu-1)}} \frac{\sin 2(\mu)\gamma}{\gamma^2} d\gamma .$$

Employing the theorem of means, the second integral of J_μ is seen to be less than

$$\frac{\tfrac{1}{2} c_\mu}{2(\mu)} \left[-\frac{1}{\gamma} \right]_{\frac{\pi}{(\mu)}}^{\frac{\pi}{(\mu-1)}} = \frac{c_\mu}{4\pi(\mu)} \Big[(\mu) - (\mu - 1) \Big] = \frac{c_\mu}{4\pi} \left[1 - \frac{1}{2\mu + 1} \right],$$

which tends to zero when μ increases indefinitely, since $\lim\limits_{\mu=\infty} c_\mu = 0$. Consequently

$$\lim_{\mu=\infty} J_\mu = \lim_{\mu=\infty} \tfrac{1}{2} c_\mu \log[2\mu + 1].$$

Applying the theorem of means to (32) we have

$$| J_\lambda | < \tfrac{1}{2} c_\lambda \left| \left[\frac{1}{(\mu)-(\lambda)} + \frac{1}{(\mu)+(\lambda)} \right] \frac{1}{\gamma} \right\}_{\frac{\pi}{(\lambda)}}^{\frac{\pi}{(\lambda-1)}} \Bigg|$$

$$< \frac{1}{2\pi} c_\lambda \left[\frac{1}{(\mu)-(\lambda)} + \frac{1}{(\mu)+(\lambda)} \right] (\lambda) < \frac{c_\lambda}{\frac{(\mu)}{(\lambda)} - \frac{(\lambda)}{(\mu)}} .$$

Hence,

$$\left| \sum_{\mu-1}^{\lambda_1+1} J_\lambda \right| < \sum_{\mu-1}^{\lambda_1+1} \frac{c_\lambda}{\frac{(\mu)}{(\lambda)} - \frac{(\lambda)}{(\mu)}} .$$

But this series evidently has zero for its limit, as is seen at once by writing it in the form

$$\frac{c_{\mu-1}}{2\mu+1-\frac{1}{2\mu+1}}+\frac{c_{\mu-2}}{[2\mu+1][2\mu-1]-\frac{1}{[2\mu+1][2\mu-1]}}+\dots$$

$$+\frac{c_{\lambda_1+1}}{\frac{(\mu)}{(\lambda_1+1)}-\frac{(\lambda_1+1)}{(\mu)}}.$$

But

$$J=J_{\mu+1}+J_\mu+\sum_{\mu-1}^{\lambda_1+1}J_\lambda$$

and therefore

$$\lim_{\mu=\infty} J=\lim_{\mu=\infty}\tfrac{1}{2}c_\mu \log[2\mu+1].$$

Now since the c's are unrestricted, except that they are positive and $\lim\limits_{\mu=\infty} c_\mu=0$, it is possible to choose them in such a way that the above product shall become infinitely great with μ. This will be the case for example if we take

$$c_\mu=\frac{1}{\sqrt{\log[2\mu+1]}}.$$

It follows, therefore, that with such a choice of the c's, the Fourier series, which represents the function $f(\beta)$ for values of β different from zero, diverges for $\beta=0$.

8. Let us determine as far as possible the nature of the convergence of the trigonometric series

$$\sum_{n=0}^{n=\infty}(a_n\sin nx+b_n\cos nx)$$

which represents the function $f(x)$, say, in the interval $(-\pi,\pi)$. Now

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(a)\sin na\,da=-\frac{1}{\pi}\left[\frac{f(a)\cos na}{n}\right]_{-\pi}^{\pi}+\frac{1}{n\pi}\int_{-\pi}^{\pi}f'(a)\cos na\,da$$

$$=\frac{(-1)^{n+1}}{n\pi}[f(\pi)-f(-\pi)]-\frac{1}{n^2\pi}\int_{-\pi}^{\pi}f''(a)\sin na\,da,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(a)\cos na\,da=-\frac{1}{n\pi}\int_{-\pi}^{\pi}f'(a)\sin na\,da$$

$$=\frac{(-1)^n}{n^2\pi}[f'(\pi)-f'(-\pi)]+\frac{1}{n^2\pi}\int_{-\pi}^{\pi}f''(a)\cos na\,da.$$

This shows that, if the first and second derivatives of f are finite, the n^{th} term is of the order $\frac{1}{n}$, except when $f(-\pi)=f(\pi)$, and hence that in general the series is only semi-convergent. But if $f(-\pi)=f(\pi)$, or if the development contains only terms of the form $b_n \cos nx$, the series is absolutely convergent.

The question also arises: Is the series uniformly convergent? Manifestly, it cannot be so in any interval containing a point of discontinuity. Let us then consider the question for any interval (a, b), comprised within $(-\pi, \pi)$, containing no point of discontinuity for the function, and over which the function has not an infinite number of maxima and minima. We need to show that we can take n so large that the sum of the first n terms of the series

$$S_n = \frac{1}{\pi}\int_0^{\frac{1}{2}(\pi-x)} \frac{f(x+2\gamma)\sin k\gamma\, d\gamma}{\sin\gamma} + \frac{1}{\pi}\int_0^{\frac{1}{2}(\pi+x)} \frac{f(x-2\gamma)\sin k\gamma\, d\gamma}{\sin\gamma} \qquad [k=2n+1]$$

shall, for any value of x within the interval (a, b), differ from $f(x)$ by a quantity whose modulus is less than α, where α is an arbitrary small quantity chosen in advance. Let us put

$$A = \frac{1}{\pi}\int_0^{\epsilon} \frac{f(x+2\gamma)\sin k\gamma}{\sin\gamma}\, d\gamma + \frac{1}{\pi}\int_0^{\epsilon} \frac{f(x-2\gamma)\sin k\gamma}{\sin\gamma}\, d\gamma.$$

$f(x)$ under the conditions mentioned above evidently possesses the property of limited oscillation in the interval (a, b), and hence we can write

$$f(x+2\gamma) = f(x) + P'_\gamma - N'_\gamma, \quad f(x-2\gamma) = f(x) + P''_\gamma - N''_\gamma,$$

as long as $x+2\gamma$ and $x-2\gamma$ do not pass outside the interval (a, b). Or we can write

$$f(x+2\gamma) + f(x-2\gamma) = 2f(x) + P - N_\gamma.$$

Consequently we have

$$A = \frac{1}{\pi}\int_0^{\epsilon} [2f(x) + P_\gamma - N_\gamma] \frac{\sin k\gamma}{\sin\gamma}\, d\gamma.$$

Since P_γ and N_γ are positive increasing functions, we get, employing Bonnet's theorem,

$$A = \frac{2}{\pi} f(x)\int_0^{\epsilon} \frac{\sin k\gamma}{\sin\gamma}\, d\gamma + \frac{P_\epsilon}{\pi}\int_\xi^{\epsilon} \frac{\sin k\gamma}{\sin\gamma}\, d\gamma - \frac{N_\epsilon}{\pi}\int_{\xi'}^{\epsilon} \frac{\sin k\gamma}{\sin\gamma}\, d\gamma \quad \begin{bmatrix} 0 < \xi < \varepsilon \\ 0 < \xi' < \varepsilon \end{bmatrix}.$$

But from (26) and (27), page 17, we must have

$$-\frac{2}{\pi} < \int_\xi^{\epsilon} \frac{\sin k\gamma}{\sin\gamma}\, d\gamma < \frac{\pi}{2} + \frac{2}{\pi}, \quad -\frac{2}{\pi} < \int_{\xi'}^{\epsilon} \frac{\sin k\gamma}{\sin\gamma}\, d\gamma < \frac{\pi}{2} + \frac{2}{\pi}.$$

Hence A will differ from

$$\frac{2}{\pi} f(x)\int_0^{\epsilon} \frac{\sin k\gamma}{\sin\gamma}\, d\gamma = f(x) - \frac{2}{\pi} f(x)\int_{\epsilon}^{\frac{\pi}{2}} \frac{\sin k\gamma}{\sin\gamma}\, d\gamma$$

by a quantity which is less in absolute value than

$$\frac{M}{\pi}\left(\frac{\pi}{2} + \frac{4}{\pi}\right) < M,$$

where M is the largest of the quantities P_ϵ and N_ϵ. But since $f(x)$ satisfies Dirichlet's conditions, we can choose ε sufficiently small to have $M < \frac{1}{4}a$. Having thus chosen ε, it follows, from the theorem proved in §4, that we can choose $n = \frac{k-1}{2}$ so large that

$$\frac{2}{\pi} f(x) \int_\epsilon^{\frac{\pi}{2}} \frac{\sin k\gamma}{\sin \gamma}\, d\gamma, \quad \frac{1}{\pi}\int_\epsilon^{\frac{1}{2}(\pi - x)} \frac{f(x+2\gamma)\sin k\gamma\, d\gamma}{\sin\gamma},$$
$$\frac{1}{\pi}\int_\epsilon^{\frac{1}{2}(\pi + x)} \frac{f(x-2\gamma)\sin k\gamma\, d\gamma}{\sin\gamma} \tag{33}$$

shall, for any value of x in the interval (a, b), each be less in absolute value than $\frac{1}{4}a$, and consequently we will have S_n differing from $f(x)$ by a quantity whose modulus is less than a.

Suppose now that $f(x)$ has an infinite number of maxima and minima in the interval (a, b), but satisfies the condition

$$\lim_{\epsilon=0} \int_0^\epsilon \left| \frac{f(x \pm 2\gamma) - f(x)}{\gamma} \right| d\gamma = 0. \tag{28}$$

We can write A, defined as above, in the form

$$\frac{2}{\pi} f(x) \int_0^\epsilon \frac{\sin k\gamma}{\sin\gamma}\, d\gamma + \frac{1}{\pi}\int_0^\epsilon \frac{[f(x+2\gamma) - f(x)]\sin k\gamma}{\sin\gamma}\, d\gamma$$
$$- \frac{1}{\pi}\int_0^\epsilon \frac{[f(x) - f(x-2\gamma)]\sin k\gamma}{\sin\gamma}\, d\gamma.$$

Since the condition (28) is satisfied, it is easily seen, by employing the process used at the beginning of §6, that we can choose ε so small that the last two terms in this expression for A shall each be less than $\frac{1}{5}a$ in absolute value. Having thus chosen ε, we can, as above, now choose n sufficiently large to make the modulus of each of the expressions in (33) less than $\frac{1}{5}a$, and hence we will have S_n differing from $f(x)$ by a quantity whose modulus is less than a.

We have thus shown that the Fourier series representing a function $f(x)$ is uniformly convergent in each interval (a, b), comprised within the interval $(-\pi, \pi)$ and containing no point of discontinuity for the function:

1°. When $f(x)$ does not possess an infinite number of maxima and minima in this interval.

2°. When it fulfils the condition

$$\lim_{\epsilon=0} \int_0^\epsilon \frac{|f(x \pm 2\gamma) - f(x)|}{\gamma}\, d\gamma = 0.$$

Biographical Sketch.

Edward Payson Manning was born in Antwerp, N. Y., March 21st, 1865. In 1866 his parents removed to Raynham, Mass., where he lived until the fall of 1882, when he entered the High School of Providence, R. I.

In 1885 he entered Brown University, from which he received the degree of Bachelor of Arts in 1889.

After spending one year in private teaching in Baltimore, he entered the Johns Hopkins University as a candidate for the degree of Doctor of Philosophy, selecting Mathematics as his principal subject, with Physics and Astronomy as subordinate subjects. During the last three years he has held successively the positions of University Scholar, Fellow and Fellow by Courtesy, this past year serving also as an assistant in the Mathematical Department.