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ALMAR NÆSS

(Videnskapsselskapets Skrifter. I. Mat.-naturv. Klasse. 1922. No. 13)

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A. W. BRØGGERS BOKTRYKKERI A/S

CONTENTS.

§ 1. Introduction.

The object of this paper is to develop some of the chief properties of a special determinant polyadic, deriving — by the definition given in § 4 (a) — from any number of independent vectors, and which we shall call their space complement. From the definition will be seen that the vector product of ordinary vector analysis is nothing but a special space complement. It is further our object to show that the equations expressing characteristic properties of the space complement, from a formal point of view can be regarded as generalized vector product formulæ, and thus formally the space complement may be considered to be a kind of a generalized vector product.

As will be known, by the vector product of two vectors is in modern tensor analysis usually understood the skew symmetric tensor which is determined by the same two vectors. This tensor is of the second order in any space. But as in S_3 only three of its six components are independent quantities, there may in this case be associated with it a vector whose components are those three quantities taken in a definite order. But this tensor, which in S_3 is different from, but representable by, the vector product of classical vector analysis, can hardly from a formal point of view be characterized as a generalization of the latter. In fact, it only means an old name on a new and different quantity. It is, of course, in this connection of perfect indifference whether or not this new quantity (the tensor) is a more suitable or convenient representation of those physical phenomena which formerly were represented by the vector product.

Notwithstanding that the language and conceptions of vector analysis are always used in the sequel, it may equally well be regarded as dealing with (an extended) algebra, the unit vectors playing the rôle of positional symbols, and their Gibbsian indeterminate products — to which any polyadic can be reduced — only being new positional symbols. A few of our theorems concern properties of matrices only, as for example § 12 (a), quite independent of vector analysis notations and conceptions.

Rather often reference is given to the writer's paper on triadics, where a few of the theorems are worked out for the three-dimensional case.

§ 2. Preliminaries.

Firstly we lay down a few definitions:

In an ordinary n-dimensional space S_n be given a fixed set of rectangular (i. e. mutually perpendicular) axes $o\, x_1, x_2 \ldots . x_n$ defining a coordinate system. To any given set of n real numbers

$$x_1, x_2, \ldots . x_n$$

corresponds a point in this space. Further let

$$\mathfrak{e}_1, \mathfrak{e}_2, \ldots . \mathfrak{e}_n$$

designate a ***normal system of unit vectors*** in this coordinate system, i. e.: n vectors of length one, originating from any point in S_n and parallel to the coordinate axes respectively, i. e. each of them is at right angles to the other $(n-1)$. These vectors, therefore, determine the coordinate system.

Any scalar function v of n variables $x_1, x_2, \ldots . x_n$ determines for each set of the variables a scalar quantity. Hence: to each point in S_n is thus made to correspond a scalar; v defines a scalar field.

The $\mathfrak{e}$'s are n linearly independent vectors. Any other vector in S_n is expressible by them. This contains our axiom of dimensions. A vector is then a quantity of the form

$$(1) \qquad \mathfrak{v} = \mathfrak{e}_1 v_1 + \mathfrak{e}_2 v_2 + \ldots . + \mathfrak{e}_n v_n = \mathfrak{e}_i v_i.$$

Summation with respect to a subscript appearing twice is always understood. The v's are called the *components* of the vector $\mathfrak{v}$. Supposing the v's are functions of the variables $x_1, x_2, \ldots . x_n$. With each point $(x_1, x_2 \ldots . x_n)$ in S_n is then associated a set of the v's, that is a vector. The point is called its origin. An expression as (1) thus determines in each point a vector. $\mathfrak{v}$ is a *vector function of position in space*, defining a *vector field*, but is in what follows nevertheless usually spoken of as a vector.

If $\mathfrak{v} = \mathfrak{e}_i v_i$ and $\mathfrak{v}' = \mathfrak{e}_i v_i'$ then the scalar quantity $v_i v_i'$ is called the scalar product of $\mathfrak{v}$ and $\mathfrak{v}'$ and denoted by $\mathfrak{v} \cdot \mathfrak{v}'$. If $\mathfrak{v} \cdot \mathfrak{v}'$ vanishes, the two vectors are said to be perpendicular on one another. $\mathfrak{v} \cdot \mathfrak{v}$ is the square of the length of $\mathfrak{v}$. The fundamental properties of the unit vectors can thus be written:

$$(2) \qquad \mathfrak{e}_i \cdot \mathfrak{e}_j = \delta_{ij}$$

where δ_{ij} is a symbol equal to *one* for $i = j$ and equal to *zero* for $i \neq j$.

The definition of GIBBS's indeterminate product of vectors (dyads, triads and in general polyads) can evidently be extended to S_n without further explanation, as there is nothing in the mathematical nature of those conceptions which limits them to three-space only. This is simply a consequence of the fact that a dyad (and a dyadic) is expressible as (BÔCHER says: identical with) a square matrix. Here may briefly be mentioned:

Given a linear transformation, — the matrix of which is a_{ij}, i and $j = 1, 2 \ldots n$ — which transforms a vector $\mathfrak{v} = \mathfrak{e}_i v_i$ into a vector $\mathfrak{v}' = \mathfrak{e}_i v_i'$ as follows;

$$
\begin{aligned}
v_1' &= a_{11} v_1 + a_{12} v_2 + \ldots + a_{1n} v_n = a_{1i} v_i \\
v_2' &= a_{21} v_1 + a_{22} v_2 + \ldots + a_{2n} v_n = a_{2i} v_i \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
v_n' &= a_{n1} v_1 + a_{n2} v_2 + \ldots + a_{nn} v_n = a_{ni} v_i
\end{aligned} \tag{3}
$$

If we here introduce n vectors $\mathfrak{a}_1, \mathfrak{a}_2 \ldots \mathfrak{a}_n$ defined by

$$\mathfrak{a}_k = \mathfrak{e}_1 a_{k1} + \mathfrak{e}_2 a_{k2} + \ldots + \mathfrak{e}_n a_{kn} = \mathfrak{e}_i a_{ki}$$

we see that (3) can simply be written:

$$
\begin{aligned}
v_1' &= \mathfrak{a}_1 \cdot \mathfrak{v} \\
v_2' &= \mathfrak{a}_2 \cdot \mathfrak{v} \\
&\ldots\ldots \\
v_n' &= \mathfrak{a}_n \cdot \mathfrak{v}
\end{aligned} \tag{4}
$$

(5) or, briefly: $v_k' = \mathfrak{a}_k \cdot \mathfrak{v}$

and accordingly:

$$\mathfrak{v}' = \mathfrak{e}_1 \mathfrak{a}_1 \cdot \mathfrak{v} + \mathfrak{e}_2 \mathfrak{a}_2 \cdot \mathfrak{v} + \ldots + \mathfrak{e}_n \mathfrak{a}_n \cdot \mathfrak{v} = \mathfrak{e}_i \mathfrak{a}_i \cdot \mathfrak{v}. \tag{6}$$

The expression:

$$\mathfrak{e}_i \mathfrak{a}_i = \mathfrak{e}_1 \mathfrak{a}_1 + \mathfrak{e}_2 \mathfrak{a}_2 + \ldots + \mathfrak{e}_n \mathfrak{a}_n = A \tag{7}$$

is a dyadic in S_n. It is completely defined by the vector system $\mathfrak{a}_i$. The vectors $\mathfrak{e}_i$ and $\mathfrak{a}_i$ in $\mathfrak{e}_i \mathfrak{a}_i$ are said to be multiplied *indeterminately* with one another. A dyad is the indeterminate product of any two vectors, its corresponding matrix is of rank one, a dyadic is a sum of dyads. It is frequently in literature called a tensor of the second order.

Thus $A \cdot \mathfrak{v}$ is nothing but a linear transformation, and the matrix of A or of the vector system $\mathfrak{a}_i$ is simply the matrix of the transformation.

We also call to memory that a dyadic can be resolved into a sum of elementary dyads, i. e. indeterminate products of two unit vectors multiplied by a scalar factor. This is obtained by putting

$$\mathfrak{a}_i = \mathfrak{e}_j a_{ij}$$

and expanding according to the distributive law of multiplication.

Therefore:

$$A = \mathfrak{e}_i \mathfrak{e}_j a_{ij} \qquad i, j = 1, 2 \ldots n \tag{8}$$

As it is immaterial to which vector the scalar factor is applied, this evidently may be written:

(9) $$A = \mathfrak{e}_i a_{i_1} \mathfrak{e}_1 + \mathfrak{e}_i a_{i_2} \mathfrak{e}_2 + \ldots + \mathfrak{e}_i a_{in} \mathfrak{e}_n$$

Let us here introduce a vector system $\varkappa_i$ defined by

(10) $$\varkappa_i = \mathfrak{e}_j a_{ji}$$

Then we have

(11) $$A = \varkappa_i \mathfrak{e}_i.$$

The system $\varkappa_i$ is said to be *conjugate* to the system $\mathfrak{a}_i$. Two conjugate systems of vectors are determined by the rows and columns of the same square matrix.* The dyadic A_c, the conjugate to A, is then the following:

(12) $$A_c = \mathfrak{e}_i \varkappa_i.$$

In an analogous way the definition of triadics, tetradics . . . . polyadics is extended to S_n. A triadic, or tensor of the third order, is any sum of the form:

(13) $$\mathfrak{e}_i \mathfrak{e}_j \mathfrak{e}_k a_{ijk}, \qquad i, j, k = 1, 2. \ldots n$$

or any quantity, which can be broken up into terms of this kind, and thus wholly determined by a cubic matrix a_{ijk}. If we instead of $\mathfrak{e}_i \mathfrak{e}_j \mathfrak{e}_k$ have the indeterminate product of p unit vectors multiplied by a scalar, *i. e.*:

(14) $$\mathfrak{e}_{i_1} \mathfrak{e}_{i_2} \ldots \mathfrak{e}_{i_p} a_{i_1 i_2 \ldots i_p}$$

we get an elementary polyad of the p^{th} order, and any sum of such quantities is called a polyadic (or tensor) of the p^{th} order. As above, the n^p scalars $a_{i_1 i_2 \ldots i_p}$ suffice for the determination of the polyadic, which is called complete when these n^p scalars are independent of one another.

The special dyadic which transforms any vector into itself is called the *idemfactor* (Einheitsdyade) and denoted by I. It is always reducible to the form

(15) $$I = \mathfrak{e}_i \mathfrak{e}_i \qquad \text{(sum for } i\text{)}$$

which follows immediately from the fact that the corresponding matrix of transformation in this case must be the unit matrix.

The scalar (dot) product of two dyadics, which is frequently used in the following, is defined in S_n exactly in the same way as in S_3.** It may be expanded, according to the distributive law of multiplication, into a sum

* Concerning the properties of conjugate vector systems in three-space, see Zur Theorie der Triaden von ALMAR NÆSS (Kristiania 1921).

** See GIBBS-WILSON: Vector Analysis, p. 276.

of products of dyads, this sum being, of course, independent of the particular form in which the dyadics are written. Let the dyadics be for example:

(16) $$A = \mathfrak{e}_i \mathfrak{a}_i \text{ and } B = \mathfrak{e}_i \mathfrak{b}_i.$$

Hence the product, which is also a dyadic, may be written

(17) $$A \cdot B = \mathfrak{e}_i \mathfrak{a}_i \cdot B$$

and the vector system defining this new dyadic (*i. e.* the i^{th} vector of the system, i running from 1 to n) is:

(18) $$\mathfrak{a}_i \cdot B = \mathfrak{a}_i \cdot \mathfrak{e}_j \mathfrak{b}_j = a_{ij} \mathfrak{b}_j$$

or, $\mathfrak{b}_j$ being equal to $\mathfrak{e}_k b_{jk}$:

(18[1]) $$\mathfrak{a}_i \cdot B = \mathfrak{e}_k a_{ij} b_{jk}, \quad \text{sum for } j \text{ and } k$$

Let us by $\varkappa_i^b$ denote the vector system which is conjugate to the $\mathfrak{b}$'s (i. e. a system such that its i^{th} vector has its components in the i^{th} column of the matrix b_{ij} defining the dyadic B). That is:

(19) $$\varkappa_i^b = \mathfrak{e}_j b_{ji}.$$

Therefore:

(20) $$\mathfrak{a}_i \cdot B = \mathfrak{e}_k a_{ij} b_{jk} = \mathfrak{e}_k \mathfrak{a}_i \cdot \varkappa_k^b = \mathfrak{a}_i \cdot \varkappa_k^b \mathfrak{e}_k$$

a result which is obtained directly by observing that:

(21) $$B = \mathfrak{e}_k \mathfrak{b}_k = \varkappa_k^b \mathfrak{e}_k$$

and, accordingly:

(22) $$\mathfrak{a}_i \cdot B = \mathfrak{a}_i \cdot (\varkappa_k^b \mathfrak{e}_k) = (\mathfrak{a}_i \cdot \varkappa_k^b) \mathfrak{e}_k = \mathfrak{e}_k \mathfrak{a}_i \cdot \varkappa_k^b$$

This only means that if c_{ij} is the matrix of the dyadic $A \cdot B$, then

(23) $$c_{ij} = \mathfrak{a}_i \cdot \varkappa_j^b$$

As, for any vector $\mathfrak{v}$, $A \cdot B \cdot \mathfrak{v} = A \cdot (B \cdot \mathfrak{v})$ is the resulting vector when B and A acting in succession upon the vector $\mathfrak{v}$, this simply contains the multiplication law of two matrices, which, hence, is compatible with the law of (scalar) multiplication of two dyadics.

§ 3. Remarks concerning the vector product and the reciprocal vector system.

As is well known, the vector product of two vectors $\mathfrak{a}$ and $\mathfrak{b}$, denoted by $\mathfrak{a} \times \mathfrak{b}$, in three-space is a vector whose components are the two rowed determinants which can be formed from the matrix of the components of the factors, i. e. from the matrix:

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix}$$

thus giving as the components of the product the three quantities

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

which also is written:

(1) $$\mathfrak{a} \times \mathfrak{b} = \begin{vmatrix} \mathfrak{i} & \mathfrak{j} & \mathfrak{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_3 & b_3 \end{vmatrix}$$

$\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$, being the unit vectors of S_3.

If we in this way shall obtain a vector, it is, of course, necessary that the number of determinants which can be picked out of the matrix, is equal to the number of dimensions of the space concerned. Since this only is the case when $n = 3$, the operation of forming the vector product from two given vectors has been considered to be unique for S_3, without any possibility of generalizing to S_n. But, of course, it is not obviously given beforehand, that such a generalized product — giving in S_3 the Gibbsian vector product as a particular case — necessarily shall be a vector, nor that it shall be derived from *two* given vectors. On the contrary, we will show by an example that we even in elementary vector analysis may meet with quantities, deriving from another number of vectors than two, which with respect to fundamental properties must be considered to be analogous to the vector product.

Let in three-space a system of three vectors be given: $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$. To this system there corresponds one, and only one, definite system of vectors, say $\mathfrak{a}^*$, $\mathfrak{b}^*$, $\mathfrak{c}^*$, called the *reciprocal* to the first, such that

(2) $$\mathfrak{a}\mathfrak{a}^* + \mathfrak{b}\mathfrak{b}^* + \mathfrak{c}\mathfrak{c}^* = I = \mathfrak{a}^*\mathfrak{a} + \mathfrak{b}^*\mathfrak{b} + \mathfrak{c}^*\mathfrak{c}.$$

The starred system is easily determined by elementary matrix operations. Let $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$ be the unit vectors in S_3, and

(3) $$\Psi = \mathfrak{i}\mathfrak{a} + \mathfrak{j}\mathfrak{b} + \mathfrak{k}\mathfrak{c}.$$

(4) $$\Psi^* = \mathfrak{i}\mathfrak{a}^* + \mathfrak{j}\mathfrak{b}^* + \mathfrak{k}\mathfrak{c}^*.$$

Then:

(5) $$\mathfrak{a}^*\mathfrak{a} + \mathfrak{b}^*\mathfrak{b} + \mathfrak{c}^*\mathfrak{c} = \Psi^*_c \cdot \Psi.$$

And since this shall be equal to the idemfactor, the matrix of Ψ^*_c must be the inverse of that of Ψ, and the matrix of Ψ^*, accordingly, the conjugate to the inverse of that of Ψ. Then we get from this immediately:

(6) $$\mathfrak{a}^* = \frac{1}{|\Psi|}\mathfrak{b}\times\mathfrak{c};\quad \mathfrak{b}^* = \frac{1}{|\Psi|}\mathfrak{c}\times\mathfrak{a};\quad \mathfrak{c}^* = \frac{1}{|\Psi|}\mathfrak{a}\times\mathfrak{b}$$

where $|\Psi|$ designates the determinant of the matrix of Ψ. Each vector in the reciprocal system is thus determined as a *vector product* of two vectors.

We will carry out the analogous operations in two-space (unit vectors being $\mathfrak{i}$ and $\mathfrak{j}$). Assuming given two vectors $\mathfrak{a}$ and $\mathfrak{b}$ in S_2, we determine two others $\mathfrak{a}^*$ and $\mathfrak{b}^*$ such that

(7) $$\mathfrak{a}^*\mathfrak{a} + \mathfrak{b}^*\mathfrak{b} = \mathfrak{i}\,\mathfrak{i} + \mathfrak{j}\,\mathfrak{j}.$$

As we have

$$\mathfrak{a} = \mathfrak{i}\,a_1 + \mathfrak{j}\,a_2 \text{ and } \mathfrak{b} = \mathfrak{i}\,b_1 + \mathfrak{j}\,b_2$$

and by putting:

$$d = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

we easily get:

(8) $$\mathfrak{a}^* = \frac{1}{d}\begin{vmatrix} \mathfrak{i} & \mathfrak{j} \\ b_1 & b_2 \end{vmatrix};\quad \mathfrak{b}^* = -\frac{1}{d}\begin{vmatrix} \mathfrak{i} & \mathfrak{j} \\ a_1 & a_2 \end{vmatrix}$$

where the two vectors $\begin{vmatrix} \mathfrak{i} & \mathfrak{j} \\ b_1 & b_2 \end{vmatrix}$, etc. must be considered to be quite analogous to $\mathfrak{b}\times\mathfrak{c}$, etc. above. I. e.: each of the corresponding vectors in the two-dimensional case derives only from *one* of the primary vectors, by an operation given by (8).

If therefore a generalization of the vector product also shall cover this operation as a particular case, it is readily understood that the generalization cannot exactly be limited to a quantity deriving from *two* vectors only. On the other hand, we cannot very well characterize e. g. $\begin{vmatrix} \mathfrak{i} & \mathfrak{j} \\ b_1 & b_2 \end{vmatrix}$, which is completely determined by $\mathfrak{b}$ alone, as a "product" of $\mathfrak{b}$. It seems merely to be accidental that the number of vectors in the analogous quantity in S_3, viz.: $\begin{vmatrix} \mathfrak{i} & \mathfrak{j} & \mathfrak{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, is two, and it may be questioned whether the term "product" is a proper name for the quantity also in this case. As a matter of fact, the idea that the vector product cannot naturally be characterized as a *product* of its two vectors is not new. It has been set forth for example by E. W. Hyde.

§ 4. The Space Complement.

Our view point in the following is to consider the vector product as being a particular case of a (somewhat special) polyadic that can be derived from any number ($\leqq n$) of independent vectors in S_n by means of the following

Definition: (a) *In an n-dimensional space let there be given p linearly independent vectors* $\mathfrak{a}_1 = \mathfrak{e}_i a_{1i}$, $\mathfrak{a}_2 = \mathfrak{e}_i a_{2i}$, ... $\mathfrak{a}_p = \mathfrak{e}_i a_{pi}$, (sum for i from 1 to n), $\mathfrak{e}_1$, $\mathfrak{e}_2$, ... $\mathfrak{e}_n$ *being an orthogonal system of unit vectors. By the space complement of those p vectors we understand a determinant whose last p rows are formed from the components of the* $\mathfrak{a}$'s *and whose first n—p rows are the unit vectors,* i. e.:

$$
(\mathrm{a}^1) \qquad \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ \dots & \dots & \dots & \dots \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots & a_{pn} \end{vmatrix}
$$

As the vectors in these $n-p$ rows are, of course, to be multiplied *indeterminately* in the developed determinant, we see that the space complement of p vectors is a polyadic (tensor) of the $(n-p)^{\text{th}}$ order. The simplest and for our purpose most convenient way of expressing it as a sum of (elementary) polyadics of the same order is by expanding it according to the $(n-p)$-rowed determinants of the first $n-p$ rows.

What we in the following will try to show is that, by deriving the fundamental properties of this space complement we arrive at equations which can be regarded as generalized vector product equations of S_3, and from which, therefore, we get the formulæ of the Gibbsian cross product as special results.

We see that the space complement is a vector if and only if the number of vectors is $n-1$, and that this vector then is perpendicular to each of the primary ones, i. e.: it is perpendicular to the hyperplane containing the $(n-1)$ vectors from which it is derived. For the components of the space complement are in this case the cofactors of the elements (i. e. the unit vectors) of the first row. Hence the scalar product of the vector $\mathfrak{a}_i$ and the space complement by definition is:

$$
(\mathrm{b}) \qquad \begin{vmatrix} a_{i1} & a_{i2} & \dots & a_{in} \\ a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots & a_{pn} \end{vmatrix} \qquad p = n - 1.
$$

which vanishes identically, two rows being equal.

If $n = 3$ (i. e.: $p = n - 1 = 2$) we get the ordinary vector product of two vectors. The space complement is a scalar if $p = n$, viz. equal to the determinant of the n vectors.

For brevity we will denote the space complement of $\mathfrak{a}_1, \mathfrak{a}_2 \ldots . \mathfrak{a}_p$ by

(c) $$\langle^p \, \mathfrak{a}_1 \, \mathfrak{a}_2 \ldots \mathfrak{a}_p \qquad \text{or:} \quad \mathfrak{a}_1 \, \mathfrak{a}_2 \ldots \mathfrak{a}_p \, {}^p\rangle$$

Hence the operation sign $\langle^p$ or ${}^p\rangle$ indicates that p vectors written to the right, or respectively to the left, shall be combined into their space complement. If we are going to derive the complement of $s + t$ vectors, s to the left and t to the right, we write ${}^s\times^t$, e. g.:

(1) $$\mathfrak{a}_1 \, \mathfrak{a}_2 \, \mathfrak{a}_3 \, {}^2\times^3 \, \mathfrak{b}_1 \, \mathfrak{b}_2 \, \mathfrak{b}_3 \, \mathfrak{b}_4 \, \mathfrak{b}_5 = \mathfrak{a}_1 \begin{vmatrix} \mathfrak{e}_1 & \ldots\ldots & \mathfrak{e}_n \\ \ldots & \ldots\ldots & \ldots \\ \mathfrak{e}_1 & \ldots\ldots & \mathfrak{e}_n \\ a_{21} & \ldots\ldots & a_{2n} \\ a_{31} & \ldots\ldots & a_{3n} \\ b_{11} & \ldots\ldots & b_{1n} \\ b_{21} & \ldots\ldots & b_{2n} \\ b_{31} & \ldots\ldots & b_{3n} \end{vmatrix} \mathfrak{b}_4 \, \mathfrak{b}_5$$

evidently a polyadic of order $1 + (n - 5) + 2 = n - 2$. If s and t both are equal to one, we write $\times$. Thus the space complement of two vectors $\mathfrak{a}$ and $\mathfrak{b}$ may be written:

(2) $$\mathfrak{a} \times \mathfrak{b} = \langle^2 \, \mathfrak{a} \, \mathfrak{b} = \mathfrak{a} \, \mathfrak{b} \, {}^2\rangle$$

which in S_3 coincides with the ordinary vector product of $\mathfrak{a}$ and $\mathfrak{b}$.

§ 5. Invariance with regard to orthogonal transformations of coordinates.

First we will show that the space complement of any number (say p) of vectors is independent of the particular (orthogonal) coordinate system which we may choose:

Let

$$\mathfrak{e}'_1, \mathfrak{e}'_2, \ldots \mathfrak{e}'_n$$

be a system of orthogonal unit vectors, defining a new coordinate system, defined by:

(1) $$\mathfrak{e}'_1 = \mathfrak{e}_i \, \varepsilon_{1i}; \quad \mathfrak{e}'_2 = \mathfrak{e}_i \, \varepsilon_{2i}; \ldots . \mathfrak{e}'_n = \mathfrak{e}_i \, \varepsilon_{ni}$$

where consequently

(2) $$\varepsilon_{j1}^2 + \varepsilon_{j2}^2 + \ldots . + \varepsilon_{jn}^2 = 1 \quad \text{for all } j\text{'s}$$

(3) and $$\varepsilon_{i1} \, \varepsilon_{j1} + \varepsilon_{i2} \, \varepsilon_{j2} + \ldots . + \varepsilon_{in} \, \varepsilon_{jn} = 0 \quad i \neq j$$

Further, let the components of the vectors $\mathfrak{a}_1, \mathfrak{a}_2 \ldots . \mathfrak{a}_n$ with respect to this new coordinate system be primed, such that for any j

(4) $$\mathfrak{a}_j = \mathfrak{e}'_i \, a'_{ji}$$

We then get by intuition that

$$(5) \qquad a'_{ji} = \mathfrak{a}_j \cdot \mathfrak{e}'_i = a_{jk}\,\varepsilon_{ik}$$

which also, more exactly, can be found in the following wellknown way:

$$(6) \qquad \mathfrak{a}_j = a'_{ji}\,\mathfrak{e}'_i = a'_{ji}\,\varepsilon_{ik}\,\mathfrak{e}_k$$

But as we also have

$$\mathfrak{a}_j = a_{jk}\,\mathfrak{e}_k$$

we get

$$a'_{ji}\,\varepsilon_{ik} = a_{jk}$$

which involves the following n^2 equations:

$$(7) \qquad \begin{aligned} a'_{ji}\,\varepsilon_{i1} &= a_{j1} \\ a'_{ji}\,\varepsilon_{i2} &= a_{j2} \\ &\cdots\cdots \\ a'_{ji}\,\varepsilon_{in} &= a_{jn} \end{aligned}$$

If we by $\bar{\varepsilon}_{ij}$ denote the cofactor of the element ε_{ij} in the determinant of the ε's, these equations (7) give:

$$(8) \qquad a'_{ji} = \frac{a_{jk}\,\bar{\varepsilon}_{ik}}{|\varepsilon_{ij}|}$$

But as the ε's form an orthogonal matrix, we have:

$$|\varepsilon_{ij}| = 1 \text{ and } \bar{\varepsilon}_{ik} = \varepsilon_{ik}$$

Therefore:

$$(9) \qquad a'_{ji} = a_{jk}\,\varepsilon_{ik}$$

We now will form the space complement of the vectors $\mathfrak{a}_1, \mathfrak{a}_2 \ldots \mathfrak{a}_p$ with respect to the new (primed) coordinate system. By definition it clearly is:

$$(10) \qquad \begin{vmatrix} \mathfrak{e}'_1 & \mathfrak{e}'_2 & \cdots\cdots & \mathfrak{e}'_n \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \mathfrak{e}'_1 & \mathfrak{e}'_2 & \cdots\cdots & \mathfrak{e}'_{1n} \\ a'_{11} & a'_{12} & \cdots\cdots & a'_{1n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a'_{p1} & a'_{p2} & \cdots\cdots & a'_{pn} \end{vmatrix} = \begin{vmatrix} \mathfrak{e}_i\,\varepsilon_{1i} & \mathfrak{e}_i\,\varepsilon_{2i} & \cdots\cdots & \mathfrak{e}_i\,\varepsilon_{ni} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \mathfrak{e}_i\,\varepsilon_{1i} & \mathfrak{e}_i\,\varepsilon_{2i} & \cdots\cdots & \mathfrak{e}_i\,\varepsilon_{ni} \\ a_{1i}\,\varepsilon_{1i} & a_{1i}\,\varepsilon_{2i} & \cdots\cdots & a_{1i}\,\varepsilon_{ni} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{pi}\,\varepsilon_{1i} & a_{pi}\,\varepsilon_{2i} & \cdots\cdots & a_{pi}\,\varepsilon_{ni} \end{vmatrix}$$

$$
(10)\qquad = \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ \dots & & & \dots \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ a_{11} & a_{12} & \dots & a_{1n} \\ \dots & & & \dots \\ a_{p1} & a_{p2} & \dots & a_{pn} \end{vmatrix} \begin{vmatrix} \varepsilon_{11} & \varepsilon_{21} & \dots & \varepsilon_{n1} \\ \varepsilon_{12} & \varepsilon_{22} & \dots & \varepsilon_{n2} \\ \dots & & & \dots \\ \dots & & & \dots \\ \varepsilon_{1n} & \varepsilon_{2n} & \dots & \varepsilon_{nn} \end{vmatrix}
= \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ \dots & & & \dots \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ a_{11} & a_{12} & \dots & a_{1n} \\ \dots & & & \dots \\ a^{p}{}_{1} & a_{p2} & \dots & a_{pn} \end{vmatrix}
$$

which shows that the space complement of any p vectors is invariant with regard to any orthogonal transformation of coordinates (invariant under the group of orthogonal transformations).

Now let us assume that the p vectors $\mathfrak{a}_1 \dots \mathfrak{a}_p$ all are expressible by the same p unit vectors, i. e.: the p-space containing $\mathfrak{a}_1 \dots \mathfrak{a}_p$ also contains p of the unit vectors,and we may assume without loss of generality that those are the first p vectors $\mathfrak{e}_1, \mathfrak{e}_2 \dots \mathfrak{e}_p$. Then all the components a_{ij} vanish for $j > p$ and we evidently get:

$$
(11)\qquad \langle^p \mathfrak{a}_1 \dots \mathfrak{a}_p = (-1)^{(n+1)p} \begin{vmatrix} \mathfrak{e}_{p+1} & \dots & \mathfrak{e}_n \\ \dots & & \dots \\ \mathfrak{e}_{p+1} & \dots & \mathfrak{e}_n \end{vmatrix} \begin{vmatrix} a_{11} & \dots & a_{1p} \\ \dots & & \dots \\ a_{p1} & \dots & a_{pp} \end{vmatrix}
$$

i. e.: the space complement is expressed by the other unit vectors (and a scalar). This proposition is general. In other words:

(a). *The space complement of any p independent vectors is expressible by vectors lying in the $(n-p)$-space which is absolutely perpendicular to the p-space containing the p primary vectors.*

In order to show this it is sufficient to transform the p vectors into a new rectangular coordinate system and to choose the first p unit vectors of this system such that they are contained in the p-space on the p given vectors in question, which is always possible. This done the problem is reduced to the case mentioned above (under (11)), and our proposition is proved.

It follows directly from the definition § 4 (a) that:

(b). *The space complement of any permutation of a given set of vectors is equal to the space complement of the given set with the same or opposite sign according as the permutation can be obtained from the given set by means of an even or odd number of transpositions.*

§ 6. The space complement regarded as a function of the indeterminate product of its vectors.

By the elementary law for addition of determinants, we get:

$$(1)\quad (\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \ldots) \times \mathfrak{v} = \mathfrak{a} \times \mathfrak{v} + \mathfrak{b} \times \mathfrak{v} + \mathfrak{c} \times \mathfrak{v} + \ldots .$$

The combination of vectors in the space complement is thus evidently in this case distributive, which — according to Gibbs's general view of multiplication — might justify the consideration of the space complement as a kind of product of the two vectors of which it is formed.

Clearly it is immaterial whether $\mathfrak{v}$ in (1) is post- or pre-factor.

As we have not yet defined what we understand by the space complement of a complete polyadic (i. e.: a sum of polyads) we cannot rightaway extend (1) to the case when we instead of $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} \ldots$ etc. have a sum of polyads. In order to obtain a meaning to (1) also in this case, we proceed as follows:

The space complement:

$$\langle^p \mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_p \rangle$$

can be considered as a function of the polyad of the p^{th} order

$$\mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_p$$

i. e.: as a function of the indeterminate product of the same p vectors. This is in accordance with the fact that the scalar and vector product of ordinary vector analysis are considered to be special functions of the corresponding dyad.

Firstly it is then necessary to show: (a) *The space complement of the vectors of a polyad is independent of the particular form in which the polyad is expressed.*

It is sufficient to prove that if the polyad $\mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_p$[1] is reduced into a sum of elementary polyads, and if we derive the space complement of each of these and sum, this sum is equal to the space complement of the primary polyad.

Let us expand the space complement (i. e. the determinant) according to the $(n-p)$-rowed determinants of the first $n-p$ rows. Let $k_1, k_2 \ldots k_p$ denote any set of p numbers picked out of the set $1, 2, \ldots n$, such that:

$$k_1 < k_2 < \ldots < k_p$$

[1] $p \overline{\overline{<}} n$ and the $\mathfrak{a}$'s are independent vectors; if not, the theorem is true, but trivial.

Then:

$$(2)\quad \langle_p\, \mathfrak{a}_1\,\mathfrak{a}_2 \ldots \mathfrak{a}_p = \begin{vmatrix} \mathfrak{e}_1 & \ldots & \mathfrak{e}_{k_1} & \ldots & \mathfrak{e}_{k_2} & \ldots & \mathfrak{e}_{k_p} & \ldots & \mathfrak{e}_n \\ \ldots & & & & & & & & \\ \mathfrak{e}_1 & \ldots & \mathfrak{e}_{k_1} & \ldots & \mathfrak{e}_{k_2} & \ldots & \mathfrak{e}_{k_p} & \ldots & \mathfrak{e}_n \\ a_{11} & \ldots & a_{1k_1} & \ldots & a_{1k_2} & \ldots & a_{1k_p} & \ldots & a_{1n} \\ \ldots & & & & & & & & \\ a_{p1} & \ldots & a_{pk_1} & \ldots & a_{pk_2} & \ldots & a_{pk_p} & \ldots & a_{pn} \end{vmatrix}$$

$$= \Sigma(-1)^{(n-p+1)+\ldots+n+\sum_1^n k_i} \begin{vmatrix} \mathfrak{e}_1 & \ldots & {}_{\prime}\mathfrak{e}'_{k_i} & \ldots & \mathfrak{e}_n \\ \ldots & & & & \\ \ldots & & & & \\ \mathfrak{e}_1 & \ldots & & \ldots & \mathfrak{e}_n \end{vmatrix} \begin{vmatrix} a_{1k_1} & a_{1k_2} & \ldots & a_{1k_p} \\ \ldots & & & \\ \ldots & & & \\ a_{pk_1} & a_{pk_2} & \ldots & a_{pk_p} \end{vmatrix}$$

The ${}_{\prime}\mathfrak{e}'_{k_i}$ indicates that the first determinant is formed from the rest of the unit vectors after $\mathfrak{e}_{k_1}\,\mathfrak{e}_{k_2} \ldots \mathfrak{e}_{k_p}$ have been stricken out. The sum is understood to be taken for all possible sets of the k's. On the other hand, we can express the indeterminate product $\mathfrak{a}_1\,\mathfrak{a}_2 \ldots \mathfrak{a}_p$ as a sum of elementary polyads by putting $\mathfrak{a}_i = \mathfrak{e}_j a_{ij}$ (sum for j from 1 to n, $i = 1, 2 \ldots p$) and multiplying according to the distributive law:

$$(3)\quad \mathfrak{a}_1\,\mathfrak{a}_2 \ldots \mathfrak{a}_p = \Sigma\, \mathfrak{e}_{j_1}\,\mathfrak{e}_{j_2} \ldots \mathfrak{e}_{j_p}\, a_{1j_1}\, a_{2j_2} \ldots a_{pj_p}$$

$j_1 j_2 \ldots j_p$ here denotes any set of p integers in *any* order picked out of the numbers $1, 2 \ldots n$, and the sum is to be taken for all possible sets of the j's.

Now let $k_1, k_2 \ldots k_p$ as before be a set of p integers picked out of $1, 2 \ldots n$ such that $k_1 < k_2 < \ldots k_p$. Then we have:

$$(4)\quad \langle_p\, \mathfrak{e}_{k_1}\,\mathfrak{e}_{k_2} \ldots \mathfrak{e}_{k_p} = \begin{vmatrix} \mathfrak{e}_1 & \ldots & \mathfrak{e}_{k_1} & \ldots & \mathfrak{e}_{k_p} & \ldots & \mathfrak{e}_n \\ \ldots & & & & & & \\ \mathfrak{e}_1 & \ldots & & & & \ldots & \mathfrak{e}_n \\ 0 & \ldots & 1 & \ldots & 0 & \ldots & 0 \\ \ldots & & & & & & \\ 0 & \ldots & 0 & \ldots & 1 & \ldots & 0 \end{vmatrix}$$

If we expand this according to the determinants of the first $n-p$ rows, we notice that all but one of the plain complements of these $(n-p)$-rowed determinants vanish, the non-vanishing plain complement having the value *one* (each element in its principal diagonal is one, all the others zero).

Thus we get:

$$(5)\quad \langle_p\, \mathfrak{e}_{k_1}\,\mathfrak{e}_{k_2} \ldots \mathfrak{e}_{k_p} = (-1)^{(n-p+1)+\ldots n+\sum_1^n k_i} \begin{vmatrix} \mathfrak{e}_1 & \ldots & {}_{\prime}\mathfrak{e}'_{k_i} & \ldots & \mathfrak{e}_n \\ \ldots & & & & \\ \mathfrak{e}_1 & \ldots & & \ldots & \mathfrak{e}_n \end{vmatrix}$$

Let us further consider the set $k_1, k_2 \ldots . k_p$ with all its possible permutations; let $j_{k1}, j_{k2} \ldots . j_{k_p}$ be any such permutation. We then first observe that

$$\langle^p\, \mathfrak{e}_{j_{k1}}\, \mathfrak{e}_{j_{k2}} \ldots . \mathfrak{e}_{j_{k_p}} = \pm \langle^p\, \mathfrak{e}_{k1}\, \mathfrak{e}_{k2} \ldots . \mathfrak{e}_{k_p}$$

where + or — is to be chosen according as the set $j_{k1}\, j_{k2} \ldots . j_{k_p}$ is an even or odd permutation of the k's (s. § 5 (b)).

Let us now consider those $p!$ terms in (3) which are of the form:

$$\mathfrak{e}_{j_{k1}}\, \mathfrak{e}_{j_{k2}} \ldots . \mathfrak{e}_{j_{k_p}}\, a_{1\, j_{k1}}\, a_{2\, j_{k2}} \ldots . a_{p\, j_{k_p}}$$

i. e.: all those $p!$ terms which contain the same unit vectors, viz.

$$\mathfrak{e}_{k1}\, \mathfrak{e}_{k2} \ldots . \mathfrak{e}_{k_p},$$

in all possible order. We will take the space complement of each of those $p!$ terms and then sum. By what is said above, we get:

$$\Sigma \langle^p\, \mathfrak{e}_{j_{k1}}\, \mathfrak{e}_{j_{k2}} \ldots \mathfrak{e}_{j_{k_p}}\, a_{1 j_{k1}}\, a_{2 j_{k2}} \ldots a_{p j_{k_p}} = \langle^p\, \mathfrak{e}_{k1}\, \mathfrak{e}_{k2} \ldots \mathfrak{e}_{k_p} \Sigma \pm a_{1 j_{k1}}\, a_{2 j_{k2}} \ldots a_{p j_{k_p}}$$

$$= \langle^p\, \mathfrak{e}_{k1}\, \mathfrak{e}_{k2} \ldots \mathfrak{e}_{k_p} \begin{vmatrix} a_{1\,k1}\, a_{1\,k2} \ldots a_{1\,k_p} \\ \ldots\ldots\ldots \\ a_{p\,k1}\, a_{p\,k2} \ldots a_{p\,k_p} \end{vmatrix} \tag{6}$$

$$= (-1)^{(n-p+1)+\ldots:n+\sum_1^n k_i} \begin{vmatrix} \mathfrak{e}_1 \ldots \mathfrak{e}'_{k_i} \ldots \mathfrak{e}_n \\ \ldots\ldots\ldots \\ \mathfrak{e}_1 \ldots\ldots\ldots \mathfrak{e}_n \end{vmatrix} \begin{vmatrix} a_{1\,k1}\, a_{1\,k2} \ldots a_{1\,k_p} \\ \ldots\ldots\ldots \\ a_{p\,k1}\, a_{p\,k2} \ldots a_{p\,k_p} \end{vmatrix}$$

Therefore: The sum of the space complements of all terms in (3) is equal to the sum of all possible terms of this kind, i. e.: the sum for all possible sets of the k's, $k_1 < k_2 \ldots . k_p$. And, by (2), this shows that the sum is equal to $\langle^p\, \mathfrak{a}_1\, \mathfrak{a}_2 \ldots . \mathfrak{a}_p$.

Now let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two different forms of the same polyad of the p^{th} order (i. e.: two equivalent polyads, $\mathbf{P}_1 = \mathbf{P}_2$) and thus giving, when expressed by elementary polyads, the same form $\mathbf{P}_e$. Then

(7) $\langle^p\, \mathbf{P}_1 = \langle^p\, \mathbf{P}_e$ and (7[1]) $\langle^p\, \mathbf{P}_2 = \langle^p\, \mathbf{P}_e$

accordingly:

(8) $\langle^p\, \mathbf{P}_1 = \langle^p\, \mathbf{P}_2$

That is: the space complement of the vectors of a polyad (we will say, shorter: of a polyad) is independent of the particular form in which the latter is expressed, which is the desired result.

This can always be applied to any sum of elementary polyads which can be summed up to a single polyad, but, strictly speaking, not to a sum of such polyads in general. But what we have found above very naturally leads to an extension of our definition, such that we by the space complement of any sum of elementary polyads understand the sum of the space complements of each polyad. Once this extension established, it follows immediately that it must hold good for sums of all kinds of polyads, as they always can be reduced to elementary ones. That is: we can lay down the

Definition (b). *By the space complement of a polyadic is understood the sum of the space complements of each of its polyads.*

Or:

(9)
$$\langle^p (\mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_p + \mathfrak{b}_1 \mathfrak{b}_2 \ldots \mathfrak{b}_p + \ldots) \\ = \langle^p \mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_p + \langle^p \mathfrak{b}_1 \mathfrak{b}_2 \ldots \mathfrak{b}_p + \ldots$$

Accordingly we get from any equation between polyadics a new equation by inserting the sign $\langle^p$ (or $\times$) in the same way in each of its terms on both sides of the equation.

And from this follows that the operation of forming the space complement obeys the distributive law because the indeterminate multiplication does. Since we e. g. have:

(10)
$$\mathfrak{v} (\mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_p + \mathfrak{b}_1 \mathfrak{b}_2 \ldots \mathfrak{b}_p + \ldots) \\ = \mathfrak{v} \mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_p + \mathfrak{v} \mathfrak{b}_1 \mathfrak{b}_2 \ldots \mathfrak{b}_p + \ldots$$

we know that those two equal polyadics (of order $n + 1$) must also have equal space complements, i. e.:

(11)
$$\mathfrak{v} \times^s (\mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_p + \mathfrak{b}_1 \mathfrak{b}_2 \ldots \mathfrak{b}_p + \ldots) \\ = \mathfrak{v} \times^s \mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_p + \mathfrak{v} \times^s \mathfrak{b}_1 \mathfrak{b}_2 \ldots \mathfrak{b}_p +$$

where $s \gtreqless p$.

§ 7. The space complement of a determinant of the form:

$$\begin{vmatrix} \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \\ \ldots & \ldots & \ldots \\ \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \end{vmatrix}$$

Each row here consists of the same p independent vectors ($p \leqq n$). The multiplication being indeterminate (or general) the determinant is a polyadic of the p^{th} order.

If we expand this determinant we get $p!$ terms (polyads). One of them is the principal diagonal $\mathfrak{a}_1 \mathfrak{a}_2 \ldots \mathfrak{a}_p$, all the others are permutations of this term. And, by what is said above, we get the desired space complement by taking the space complement of each of these terms and summing.

Now is:

$$(1) \qquad \langle_p \, \mathfrak{a}_1 \, \mathfrak{a}_2 \, \ldots \, \mathfrak{a}_p = \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots & \mathfrak{e}_n \\ \ldots & & & \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots & \mathfrak{e}_n \\ a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & & & \\ a_{p1} & a_{p2} & \ldots & a_{pn} \end{vmatrix}$$

And, by § 5 (b), the space complement of each of the *even* permutations of $\mathfrak{a}_1 \, \mathfrak{a}_2 \ldots \mathfrak{a}_p$ is equal to $\langle_p \, \mathfrak{a}_1 \, \mathfrak{a}_2 \ldots \mathfrak{a}_p$, but of any *odd* permutation equal to the same quantity taken negatively. But the odd permutations have, in the developed determinant, minus sign, which reverses the sign. I. e.: the space complements of each term of the determinant in question are — the sign of the term taken into account — all equal to the space complement of the principal diagonal.

Thus we get:

$$(2) \qquad \langle_p \begin{vmatrix} \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \\ \ldots & & \\ \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \end{vmatrix} = p! \, \langle_p \, \mathfrak{a}_1 \ldots \mathfrak{a}_p$$

We get a similar result if we expand the space complement of a polyadic of the form (order being $p + 1$):

$$(3) \qquad \mathfrak{v} \begin{vmatrix} \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \\ \ldots & & \\ \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \end{vmatrix} \qquad p < n$$

That is:

$$(4) \qquad \langle_{p+1} \, \mathfrak{v} \begin{vmatrix} \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \\ \ldots & & \\ \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \end{vmatrix} = p! \, \langle_{p+1} \, \mathfrak{v} \, \mathfrak{a}_1 \ldots \mathfrak{a}_p$$

which we also can write:

$$(5) \qquad \mathfrak{v} \times_p \begin{vmatrix} \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \\ \ldots & & \\ \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \end{vmatrix} = p! \, \mathfrak{v} \times_p \mathfrak{a}_1 \ldots \mathfrak{a}_p.$$

We readily see that this quantity vanishes if $\mathfrak{v}$ is equal to one of the $\mathfrak{a}$'s or, in general, linearly dependent on the $\mathfrak{a}$'s.

§ 8. A generalization of the expansion for the vector triple product.

An equation which in ordinary vector analysis is of importance on account of its frequent occurrence, is that of the vector triple product. In quaternion notation it is written:

(1) $$V\mathfrak{a}(V\mathfrak{b}\,\mathfrak{c}) = \mathfrak{c}\,S\,\mathfrak{a}\,\mathfrak{b} - \mathfrak{b}\,S\,\mathfrak{c}\,\mathfrak{a}$$ *

which equation GIBBS writes

(2) $$\mathfrak{a}\times(\mathfrak{b}\times\mathfrak{c}) = -\mathfrak{a}\cdot\{\mathfrak{b}\,\mathfrak{c} - \mathfrak{c}\,\mathfrak{b}\}.$$

It may be found more convenient, in this and similar equations, to write such dyadics (and also triadics etc.) in determinant form, as thereby greater symmetry is obtained:

(3) $$\mathfrak{a}\times(\mathfrak{b}\times\mathfrak{c}) = -\mathfrak{a}\cdot\begin{vmatrix}\mathfrak{b} & \mathfrak{c}\\ \mathfrak{b} & \mathfrak{c}\end{vmatrix}$$ **

In this form the equation can be generalized to n-space. It must only be kept in mind that $\mathfrak{b}\times\mathfrak{c}$ in S_n is not a vector, but a polyadic of order $n-2$. The vector $\mathfrak{a}$ and this polyadic then combine to form the final space complement of (3). We can then prove that in any space S_n the following equation is valid:

(4) $$\mathfrak{a}\times^{n-2}(\mathfrak{b}\times\mathfrak{c}) = -(n-2)!\,\mathfrak{a}\cdot\begin{vmatrix}\mathfrak{b} & \mathfrak{c}\\ \mathfrak{b} & \mathfrak{c}\end{vmatrix}$$

But this equation can be still more generalized. We are going to show that it holds good, not only for the triple product, i. e.: when we have to derive the space complement of two vectors $\mathfrak{b}$ and $\mathfrak{c}$ and then combine this with $\mathfrak{a}$, but also in the case when we instead of $\mathfrak{b}$ and $\mathfrak{c}$ have *any set of p independent vectors*: $\mathfrak{a}_1, \mathfrak{a}_2 \ldots\ldots \mathfrak{a}_p, (p < n)$. (If the vectors are dependent the theorem is true, but trivial.) Hence, the equation which we will consider to be the generalization of the expansion for the vector triple product, and which we now are going to prove, is:

(5) $$\mathfrak{v}\times^{n-p}(\langle^p\,\mathfrak{a}_1\,\mathfrak{a}_2\ldots\ldots\mathfrak{a}_p) = -(-1)^{np}(n-p)!\,\mathfrak{v}\cdot\begin{vmatrix}\mathfrak{a}_1 & \ldots\ldots & \mathfrak{a}_p\\ \ldots & \ldots & \ldots\\ \mathfrak{a}_1 & \ldots\ldots & \mathfrak{a}_p\end{vmatrix}$$

n being the number of dimensions of the space considered. We can tell at a glance that it gives (4) as well as (3) as special cases.

In order to prove (5) we first expand $\langle^p\,\mathfrak{a}_1\,\mathfrak{a}_2\ldots\ldots\mathfrak{a}_p$. By definition we have:

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* As will be known, $V\mathfrak{a}\,\mathfrak{b} = -\mathfrak{a}\times\mathfrak{b}$, $S\,\mathfrak{a}\,\mathfrak{b} = -\mathfrak{a}\cdot\mathfrak{b}$.

** See: Zur Theorie der Triaden von ALMAR NÆSS (24), p. 108.

$$(6)\qquad \langle^{p}\, \mathfrak{a}_1\, \mathfrak{a}_2 \, .\, .\, .\, .\, \mathfrak{a}_p = \left| \begin{array}{ccccccccc} \mathfrak{e}_1 & \ldots & \mathfrak{e}_{k_1} & \ldots & \mathfrak{e}_{k_2} & \ldots & \mathfrak{e}_{k_p} & \ldots & \mathfrak{e}_n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathfrak{e}_1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \mathfrak{e}_n \\ a_{11} & \ldots & a_{1\,k_1} & \ldots & a_{1\,k_2} & \ldots & a_{1\,k_p} & \ldots & a_{1\,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{p1} & \ldots & a_{p\,k_1} & \ldots & a_{p\,k_2} & \ldots & a_{p\,k_p} & \ldots & a_{p\,n} \end{array} \right|$$

We will expand this determinant according to the p-rowed determinants of the last p rows. Let p columns be determined by $k_1, k_1 \, .\, .\, .\, .\, k_p$, such that $k_1 < k_2 < \, .\, .\, .\, . < k_p$. The plain complement of the p-rowed determinant in question containing these columns is obtained by striking out from the set of unit vectors all the $\mathfrak{e}_{k_i}$ and forming the $(n-p)$-rowed determinant of the rest. With regard to the sign of the *algebraic* complement we observe that the sum of the indices of the last p rows in (6) is

$$\frac{(n - p + 1 + n)p}{2} = np - \frac{p(p-1)}{2}$$

According to Laplace's theorem, the space complement of $\langle^{p}\, \mathfrak{a}_1\, \mathfrak{a}_2 \, .\, .\, .\, .\, \mathfrak{a}_p$ is equal to the sum of all $\binom{n}{p}$ terms of the form

$$(A)\qquad (-1)^{np - \frac{p(p-1)}{2} + \sum_1^n k_i} \left| \begin{array}{ccccc} \mathfrak{e}_1 & \ldots & \not{\mathfrak{e}}_{k_i} & \ldots & \mathfrak{e}_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathfrak{e}_1 & \ldots & \ldots & \ldots & \mathfrak{e}_n \end{array} \right| \quad \left| \begin{array}{ccc} a_{1\,k_1} & \ldots & a_{1\,k_p} \\ \ldots & \ldots & \ldots \\ a_{p\,k_1} & \ldots & a_{p\,k_p} \end{array} \right|$$

each being a (special) polyadic of the $(n-p)^{\text{th}}$ order.

In order to obtain the left member of the equation (5) we take the indeterminate product of $\mathfrak{v}$ by each of these terms (A) and then deriving the space complement of each of the polyadics, obtained in this way, of order $n - p + 1$. But *each* of these polyadics can be expanded into a sum of n others by putting $\mathfrak{v} = \mathfrak{e}_j v_j$ and then multiplying distributively. Neglecting the scalar factor we thus all together get $\binom{n}{p} n$ terms of the following form:

$$(B)\qquad \mathfrak{e}_j \left| \begin{array}{ccccc} \mathfrak{e}_1 & \ldots & \not{\mathfrak{e}}_{k_i} & \ldots & \mathfrak{e}_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathfrak{e}_1 & \ldots & \ldots & \ldots & \mathfrak{e}_n \end{array} \right|$$

and our final task is to derive the space complement of each of these, i. e.:

$$(C)\qquad \mathfrak{e}_j \times^{n-p} \left| \begin{array}{ccccc} \mathfrak{e}_1 & \ldots & \not{\mathfrak{e}}_{k_i} & \ldots & \mathfrak{e}_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathfrak{e}_1 & \ldots & \ldots & \ldots & \mathfrak{e}_n \end{array} \right|$$

then multiply by the corresponding scalar and sum.

But, by § 7 (5), all the space complements (C) vanish where $\mathfrak{e}_j$ is equal to one of the unit vectors in the determinant. Hence it is sufficient to take into account those terms only where $\mathfrak{e}_j$ is equal to one of the vectors $\mathfrak{e}_{k1}, \mathfrak{e}_{k2} \dots \mathfrak{e}_{k_p}$, which are stricken out when forming the determinant. For each set of the k's we thus get only p terms of the form (C).

Let us consider a *fixed* set of the k's and form all the space complements with regard to this set. The first one will be:

$$\mathfrak{e}_{k1} \times^{n-p} \begin{vmatrix} \mathfrak{e}_1 \dots \mathfrak{e}'_{k_i} \dots \mathfrak{e}_n \\ \dots\dots\dots \\ \mathfrak{e}_1 \dots\dots\dots \mathfrak{e}_n \end{vmatrix} = (n-p)!\, \mathfrak{e}_{k1} \times^{n-p} (\mathfrak{e}_1 \mathfrak{e}_2 \dots \mathfrak{e}'_{k_i} \dots \mathfrak{e}_n)$$

$$(7) \qquad = (n-p)! \begin{vmatrix} \mathfrak{e}_1 \dots \mathfrak{e}_{k1} \dots \mathfrak{e}_{k_i} \dots \mathfrak{e}_n \\ \dots\dots\dots\dots\dots\dots \\ \mathfrak{e}_1 \dots\dots\dots\dots\dots\dots \mathfrak{e}_n \\ 0 \dots 1 \dots 0 \dots 0 \\ 1 \dots 0 \dots 0 \dots 0 \\ 0\,1 \dots 0 \dots 0 \dots 0 \\ \dots\dots\dots\dots\dots\dots \\ 0 \dots 0 \dots 0 \dots 1 \end{vmatrix}$$

Each of the last $n-p+1$ rows, being components of a unit vector, consists of 1 and $n-1$ zeros. Of all the determinants which can be formed from these rows there is therefore only one which is different from zero. The sum of the indices of the columns of this nonvanishing determinant is

$$\frac{(n+1)n}{2} - \sum_{2}^{p} k_i$$

and the sum of indices of the rows is

$$\frac{(p+n)(n-p+1)}{2} = \frac{n^2-p^2+n+p}{2}$$

Expanding (7) after Laplace according to these determinants of the last $n-p+1$ rows, we thus get only one term, the following:

$$\text{(D)} \qquad (-1)^{\frac{n^2-p^2+n+p}{2} + \frac{(n+1)n}{2} - \sum_2^p k_i} (n-p)! \begin{vmatrix} \mathfrak{e}_{k2} \dots \mathfrak{e}_{k_p} \\ \dots\dots \\ \mathfrak{e}_{k2} \dots \mathfrak{e}_{k_p} \end{vmatrix} \begin{vmatrix} 0 \dots 1 \dots 0 \\ 1\,0 \dots\dots 0 \\ \dots\dots\dots \\ 0 \dots 1\,0 \dots 0 \\ \dots 0\,1 \dots \\ \dots\dots\dots \\ 0 \dots\dots 1 \end{vmatrix}$$

We must especially notice that the columns stricken out of the last $n-p+1$ rows to form the second determinant of (D) (the last factor of the term) are $k_2, k_3 \ldots . k_p$, which are *all to the right* of the column k_1. Hence the element 1 in the first row also in this determinant belongs to the column k_1. The value of the determinant therefore is

(8)
$$(-1)^{k_1+1} \begin{vmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{vmatrix} = (-1)^{k_1+1}$$

Hence the final sign of the expression (D) is:

(9)
$$(-1)^{\frac{n^2-p^2+n+p+n^2+n}{2} - \sum_2^p k_i + 1 + k_1}$$
$$= (-1)^{n^2+n-\frac{p^2-p}{2} - \sum_1^p k_i + 1 + 2k_1} = (-1)^{-\frac{p(p-1)}{2} - \sum_1^p k_i + 1}$$

since $n^2 + n + 2k_1$ is an *even* number, and therefore cancel out. Then we get:

(10)
$$\mathfrak{e}_{k_1} \times^{n-p} \begin{vmatrix} \mathfrak{e}_1 \ldots \mathfrak{e}'_{k_i} \ldots \mathfrak{e}_n \\ \ldots\ldots\ldots \\ \mathfrak{e}_1 \ldots\ldots \mathfrak{e}_n \end{vmatrix} = (-1)^{-\frac{p(p-1)}{2}+1-\sum_1^p k_i} (n-p)! \begin{vmatrix} \mathfrak{e}_{k_2} \ldots \mathfrak{e}_{k_p} \\ \ldots\ldots \\ \mathfrak{e}_{k_2} \ldots \mathfrak{e}_{k_p} \end{vmatrix}$$

The scalar factor belonging to this term is:

(E)
$$(-1)^{np-\frac{p(p-1)}{2}+\sum_1^p k_i} v_{k_1} \begin{vmatrix} a_{1\,k_1} \ldots\ldots a_{1\,k_p} \\ \ldots\ldots\ldots \\ a_{p\,k_1} \ldots\ldots a_{p\,k_p} \end{vmatrix}$$

where we can write $\mathfrak{v} \cdot \mathfrak{e}_{k_1}$ instead of v_{k_1}. Multiplying by this scalar, the term (10) takes the following form:

(F)
$$(-1)^{np+1} (n-p)! \, \mathfrak{v} \cdot \mathfrak{e}_{k_1} \begin{vmatrix} \mathfrak{e}_{k_2} \ldots\ldots \mathfrak{e}_{k_p} \\ \ldots\ldots \\ \mathfrak{e}_{k_2} \ldots\ldots \mathfrak{e}_{k_p} \end{vmatrix} \begin{vmatrix} a_{1\,k_1} \ldots a_{1\,k_p} \\ \ldots\ldots\ldots \\ a_{p\,k_1} \ldots a_{p\,k_p} \end{vmatrix}$$

When we now form the space complement:

(G)
$$\mathfrak{e}_{k_2} \times^{n-p} \begin{vmatrix} \mathfrak{e}_1 \ldots \mathfrak{e}'_{k_i} \ldots \mathfrak{e}_n \\ \ldots\ldots\ldots \\ \mathfrak{e}_1 \ldots\ldots \mathfrak{e}_n \end{vmatrix}$$

we, of course, get an expression which can be obtained from (D) by interchanging k_1 with k_2, only it must be noticed that the single „*one*" in the first row (of the last determinant of (D)) belongs to the column $k_2 - 1$, because, in forming this determinant from the last $n - p + 1$ rows of (7), we have also stricken out the k_1th column, which is to the left of k_2. Thus the value of the „one-determinant" in this case (compare (8)) is

$$(-1)^{(k_2-1)+1} = -(-1)^{k_2+1}$$

and, accordingly, for (G) we get an expression completely analogous to (10) *with change of sign.* If we put k_3 instead of k_2 we have change of sign once more (*two* columns to the left of k_3 stricken out) and so on. If we then sum all those p space complements of the type which we get from that fixed set of the k's which we have considered, we arrive at the following expression:

$$(-1)^{np+1}(n-p)!\,\mathfrak{v}\cdot\left\{\mathfrak{e}_{k_1}\begin{vmatrix}\mathfrak{e}_{k_2}\,\mathfrak{e}_{k_3}\ldots\ldots\mathfrak{e}_{k_p}\\ \cdots\cdots\cdots\\ \mathfrak{e}_{k_2}\,\mathfrak{e}_{k_3}\ldots\ldots\mathfrak{e}_{k_p}\end{vmatrix} - \mathfrak{e}_{k_2}\begin{vmatrix}\mathfrak{e}_{k_1}\,\mathfrak{e}_{k_3}\ldots\ldots\mathfrak{e}_{k_p}\\ \cdots\cdots\cdots\\ \mathfrak{e}_{k_1}\,\mathfrak{e}_{k_3}\ldots\ldots\mathfrak{e}_{k_p}\end{vmatrix}\right.$$

$$(11) \qquad \left. + \mathfrak{e}_{k_3}\begin{vmatrix}\mathfrak{e}_{k_1}\,\mathfrak{e}_{k_2}\,\mathfrak{e}_{k_4}\ldots\mathfrak{e}_{k_p}\\ \cdots\cdots\cdots\\ \mathfrak{e}_{k_1}\,\mathfrak{e}_{k_2}\,\mathfrak{e}_{k_4}\ldots\mathfrak{e}_{k_p}\end{vmatrix} - \text{etc.}\right\}\begin{vmatrix}a_{1k_1}\ldots\ldots a_{1k_p}\\ \cdots\cdots\cdots\\ a_{pk_1}\ldots\ldots a_{pk_p}\end{vmatrix}$$

$$= (-1)^{np+1}(n-p)!\,\mathfrak{v}\cdot\begin{vmatrix}\mathfrak{e}_{k_1}\,\mathfrak{e}_{k_2}\ldots\ldots\mathfrak{e}_{k_p}\\ \cdots\cdots\cdots\\ \mathfrak{e}_{k_1}\,\mathfrak{e}_{k_2}\ldots\ldots\mathfrak{e}_{k_p}\end{vmatrix}\begin{vmatrix}a_{1k_1}\,a_{1k_2}\ldots a_{1k_p}\\ \cdots\cdots\cdots\\ a_{pk_1}\,a_{pk_2}\ldots a_{pk_p}\end{vmatrix}$$

Therefore:

$$12) \quad \mathfrak{v}\times^{n-p}(\langle^p\,\mathfrak{a}_1\,\mathfrak{a}_2\ldots\mathfrak{a}_p) = -(-1)^{np}(n-p)!\,\mathfrak{v}\cdot\Sigma\begin{vmatrix}\mathfrak{e}_{k_1}\ldots\mathfrak{e}_{k_p}\\ \cdots\cdots\\ \mathfrak{e}_{k_1}\ldots\mathfrak{e}_{k_p}\end{vmatrix}\begin{vmatrix}a_{1k_1}\ldots a_{1k_p}\\ \cdots\cdots\\ a_{pk_1}\ldots a_{pk_p}\end{vmatrix}$$

where the sum is to be taken for all possible sets of the k's.

We now only have to show that:

$$(13) \qquad \begin{vmatrix}\mathfrak{a}_1\ldots\ldots\mathfrak{a}_p\\ \cdots\cdots\\ \mathfrak{a}_1\ldots\ldots\mathfrak{a}_p\end{vmatrix} = \Sigma\begin{vmatrix}\mathfrak{e}_{k_1}\ldots\ldots\mathfrak{e}_{k_p}\\ \cdots\cdots\\ \mathfrak{e}_{k_1}\ldots\ldots\mathfrak{e}_{k_p}\end{vmatrix}\begin{vmatrix}a_{1k_1}\ldots a_{1k_p}\\ \cdots\cdots\\ a_{pk_1}\ldots a_{pk_p}\end{vmatrix}$$

This formula follows from elementary properties of vector determinants, and is well known in literature. For the sake of completeness we shall also give this last step of the proof.

We put $\mathfrak{a}_i = \mathfrak{e}_j a_{ij}$, and inserting this in (13) (left side) we get for the principal diagonal:

$$\text{(H)} \qquad \Sigma\,\mathfrak{e}_j\,a_{1j}\,\Sigma\,\mathfrak{e}_j\,a_{2j}\ldots\ldots\Sigma\,\mathfrak{e}_j\,a_{pj} \qquad j = 1, 2\ldots n$$

while the other terms in the expansion are all possible permutations of this one. We carry out the multiplication. Let one term thus obtained from the principal diagonal be:

$$\text{(I)} \qquad \mathfrak{e}_{\alpha_1}\,\mathfrak{e}_{\alpha_2}\ldots\ldots\mathfrak{e}_{\alpha_p}\,a_{1\alpha_1}\,a_{2\alpha_2}\ldots\ldots a_{p\alpha_p}$$

but to this there is a corresponding one in each of the permutations of the diagonal; i. e.: a term consisting of the same vectors and scalars in another order, and, therefore, the multiplication of scalars being commutative, the product of the scalars is equal in each of them. Moreover, each term having the sign of its permutation, it has the sign + or — according as it is an even or odd permutation of the first one (those terms cancel out where two of the α's are equal). Hence the sum of all these terms is

$$\text{(K)} \qquad \begin{vmatrix} \mathfrak{e}_{\alpha_1} \mathfrak{e}_{\alpha_2} \dots \dots \mathfrak{e}_{\alpha_p} \\ \dots\dots\dots\dots \\ \mathfrak{e}_{\alpha_1} \mathfrak{e}_{\alpha_2} \dots \dots \mathfrak{e}_{\alpha_p} \end{vmatrix} a_{1\,\alpha_1}\, a_{2\,\alpha_2} \dots \dots a_{p\,\alpha_p}.$$

And the expanded determinant (13) is then reduced to the sum of all terms of this form, the α's being any p numbers in *any* order of $1, 2 \dots . n$. But then there will be $p!$ of these terms which contain the same set of unit vectors, but with the *columns* of the determinant in different order. In one of them the indices will occur in order of magnitude, say $k_1 < k_2 < \dots . < k_p$, the term accordingly:

$$\text{(L)} \qquad \begin{vmatrix} \mathfrak{e}_{k1} \mathfrak{e}_{k2} \dots \dots \mathfrak{e}_{k_p} \\ \dots\dots\dots\dots \\ \mathfrak{e}_{k1} \mathfrak{e}_{k2} \dots \dots \mathfrak{e}_{k_p} \end{vmatrix} a_{1\,k_1}\, a_{2\,k_2} \dots \dots a_{p\,k_p}$$

And as the columns of a vector determinant can be interchanged as in an ordinary one, all the others can be written:

$$\text{(M)} \qquad \pm \begin{vmatrix} \mathfrak{e}_{k1} \mathfrak{e}_{k2} \dots \dots \mathfrak{e}_{k_p} \\ \dots\dots\dots\dots \\ \mathfrak{e}_{k1} \mathfrak{e}_{k2} \dots \dots \mathfrak{e}_{k_p} \end{vmatrix} a_{1\,\beta_1}\, a_{2\,\beta_2} \dots \dots a_{p\,\beta_p}$$

where the β's stand for all permutations of the k's, the sign being determined as usual. Hence the sum of these $p!$ terms is:

$$\text{(N)} \qquad \begin{vmatrix} \mathfrak{e}_{k1} \dots \dots \mathfrak{e}_{k_p} \\ \dots\dots\dots\dots \\ \mathfrak{e}_{k1} \dots \dots \mathfrak{e}_{k_p} \end{vmatrix} \begin{vmatrix} a_{1\,k1} \dots a_{1\,k_p} \\ \dots\dots\dots\dots \\ a_{p\,k1} \dots a_{p\,k_p} \end{vmatrix}$$

And, consequently, the vector determinant in (13) is equal to the sum of the $\binom{n}{p}$ terms of this form, from which follows the desired result. Formula (5) is thereby proved.

If we in (5) put $p = 2$ we get the more special formula (4), which also can be written:

$$\text{(14)} \qquad \mathfrak{a} \times^{n-2} (\mathfrak{b} \times \mathfrak{c}) = -(n-2)!\, \mathfrak{a} \cdot \{\mathfrak{b}\,\mathfrak{c} - (\mathfrak{b}\,\mathfrak{c})_c\}.$$

§ 9. Expressions of the form $\sum_{i=1}^{n} \mathfrak{f}'_i \times \mathfrak{f}_i$ and $\sum_{i=1}^{n} \mathfrak{e}_i \times \mathfrak{f}_i$. The symmetric differences of a matrix.

Given in S_n two systems of n vectors:

$$\text{(A)} \qquad \begin{matrix} \mathfrak{f}'_1, \mathfrak{f}'_2, \dots \mathfrak{f}'_n \\ \mathfrak{f}_1, \mathfrak{f}_2, \dots \mathfrak{f}_n \end{matrix}$$

Let the two conjugate systems of these be denoted by $\varkappa'_i$ and $\varkappa_i$ respectively.

We will find an expression for the quantity $\Sigma \mathfrak{f}'_i \times \mathfrak{f}_i$, evidently a polyadic of order $n-2$. It is a vector in three-space, the $\times$ then denoting the ordinary vector product, and we know that this vector is expressible in the form[1] ($\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ being the unit vectors of S_3):

$$\text{(B)} \qquad \begin{vmatrix} \mathfrak{i} & \mathfrak{j} & \mathfrak{k} \\ \varkappa'_1 \cdot & \varkappa'_2 \cdot & \varkappa'_3 \cdot \\ \varkappa_1 & \varkappa_2 & \varkappa_3 \end{vmatrix}$$

where the scalar product is to be taken of each two corresponding vectors of the last two rows, i. e.: the dot is here written after the vector where it is to be used in the developed determinant.

We are going to show that we in S_n arrive at an analogous expression. According to our definition we get the sum of n determinants:

$$\text{(1)} \qquad \sum_i \mathfrak{f}'_i \times \mathfrak{f}_i = \sum_{i=1}^{n} \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ f'_{i1} & f'_{i2} & \dots & f'_{in} \\ f_{i1} & f_{i2} & \dots & f_{in} \end{vmatrix}$$

the last two rows being the components of $\mathfrak{f}'_i$ and $\mathfrak{f}_i$ respectively.

We develop each of these n determinants in terms of the $(n-2)$-rowed determinants of the first $n-2$ rows of unit vectors. We get, j and l being any two columns, $j < l$:

$$\text{(2)} \quad \sum_i \mathfrak{f}'_i \times \mathfrak{f}_i = \sum_{(jl)} \left\{ -(-1)^{j+l} \begin{vmatrix} \mathfrak{e}_1 \dots \not{\mathfrak{e}}_j \dots \not{\mathfrak{e}}_l \dots \mathfrak{e}_n \\ \dots\dots\dots\dots \\ \mathfrak{e}_1 \dots\dots\dots\dots \mathfrak{e}_n \end{vmatrix} \sum_{i=1}^{n} \begin{vmatrix} f'_{ij} & f'_{il} \\ f_{ij} & f_{il} \end{vmatrix} \right\}$$

the sum $\sum_{(jl)}$ being taken for all the $\binom{n}{2}$ possible sets of $(j\,l)$. As before, $\not{\mathfrak{e}}_j$ and $\not{\mathfrak{e}}_l$ indicate that $\mathfrak{e}_j$ and $\mathfrak{e}_l$ are stricken out.

[1] See: Zur Theorie der Triaden von ALMAR NÆSS. (5) and (6), p. 16.

But evidently is:

$$\sum_i f'_{ij} f_{il} = \varkappa'_j \cdot \varkappa_l \tag{3}$$

$$\sum_i f'_{il} f_{ij} = \varkappa'_l \cdot \varkappa_j \tag{4}$$

and, accordingly:

$$\sum_i \begin{vmatrix} f'_{ij} & f'_{il} \\ f_{ij} & f_{il} \end{vmatrix} = \begin{vmatrix} \varkappa'_j \cdot & \varkappa'_l \cdot \\ \varkappa_j & \varkappa_l \end{vmatrix} \tag{5}$$

Therefore, we can write:

$$\sum_i \mathfrak{f}'_i \times \mathfrak{f}_i = \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots & \mathfrak{e}_n \\ \ldots & \ldots & \ldots & \ldots \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots & \mathfrak{e}_n \\ \varkappa'_1 \cdot & \varkappa'_2 \cdot & \ldots & \varkappa'_n \cdot \\ \varkappa_1 & \varkappa_2 & \ldots & \varkappa_n \end{vmatrix} \tag{6}$$

One special case of this formula is of particular interest.

We know from three-space, that the *vector* of a dyadic (GIBBS) is obtained by insertion of the cross between each pair of its vectors. The dyadic be* $\Psi = \mathfrak{i}\,\mathfrak{a} + \mathfrak{j}\,\mathfrak{b} + \mathfrak{k}\,\mathfrak{c}$. Then Ψ_v (GIBBS writes $\Psi_\times$) $= \mathfrak{i} \times \mathfrak{a} + \mathfrak{j} \times \mathfrak{b} + \mathfrak{k} \times \mathfrak{c}$. We also know that the components of this vector are the so-called *symmetric differences* of the matrix of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$,**. They play a rôle in the theory of triadics in S_3***. In any square matrix there are in general $\frac{n(n-1)}{2}$ pairs of elements such that the elements of each pair are symmetric with respect to the principal diagonal of the matrix. We thus can form $\frac{n(n-1)}{2}$ differences („the symmetric differences") by subtracting one of these two elements (a definite one) from the other. The number of symmetric differences is equal to n if and only if $n = 3$. Of the matrix of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ they are†

$$b_3 - c_2, \; c_1 - a_3, \; a_2 - b_1.$$

We observe that the minuend is chosen in a definite way, alternately in the upper and lower half of the matrix††.

* In order to be able to tell at a glance, whether we are speaking of three-space or n-space, we will in the following (usually) denote a dyadic in S_3 by $\Psi = \mathfrak{i}\,\mathfrak{a} + \mathfrak{j}\,\mathfrak{b} + \mathfrak{k}\,\mathfrak{c}$, in S_n by Φ.

** Zur Theorie der Triaden von ALMAR NÆSS, § 4.

*** loc. cit. § 33 & § 45.

† loc. cit. § 4 (1) or p. 71.

†† loc. cit. p. 70, footnote.

Let a dyadic in S_n be defined as $\Phi = \mathfrak{e}_i \mathfrak{f}_i$ (sum for i as usual from 1 to n). Then the quantity which is analogous to Ψ_v of S_8, must be:

(7) $$\Phi_v = \mathfrak{e}_i \times \mathfrak{f}_i$$

i. e.: a polyadic of order $n-2$, the *space complement* of Φ. We obtain a formula for Φ_v by putting $\mathfrak{f}'_i = \mathfrak{e}_i$ in (6). Thus we get:

(8) $$\Phi_v = \mathfrak{e}_i \times \mathfrak{f}_i = \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ \dots & & & \dots \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ \mathfrak{e}_1 \cdot & \mathfrak{e}_2 \cdot & \dots & \mathfrak{e}_n \cdot \\ \varkappa_1 & \varkappa_2 & \dots & \varkappa_n \end{vmatrix}$$

But as:

(9) $$\mathfrak{e}_j \cdot \varkappa_l = f_{jl} = \mathfrak{e}_l \cdot \mathfrak{f}_j$$

we can write:

(10) $$\Phi_v = \mathfrak{e}_i \times \mathfrak{f}_i = - \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ \dots & & & \dots \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ \mathfrak{e}_1 \cdot & \mathfrak{e}_2 \cdot & \dots & \mathfrak{e}_n \cdot \\ \mathfrak{f}_1 & \mathfrak{f}_2 & \dots & \mathfrak{f}_n \end{vmatrix}$$

Here we have for any j and l ($j < l$):

(11) $$\begin{vmatrix} \mathfrak{e}_j \cdot & \mathfrak{e}_l \cdot \\ \mathfrak{f}_j & \mathfrak{f}_l \end{vmatrix} = \mathfrak{e}_j \cdot \mathfrak{f}_l - \mathfrak{e}_l \cdot \mathfrak{f}_j = -(f_{jl} - f_{lj})$$

If we develop (10) in terms of determinants of this kind, the sign of (11) will be $(-1)^{n-1+n+j+l} = -(-1)^{j+l}$. Let us by E_{jl} denote the $(n-2)$-rowed determinant defined by the unit vectors after erasing $\mathfrak{e}_j$ and $\mathfrak{e}_l$ ($j < l$), i. e.:

(12) $$E_{jl} = \begin{vmatrix} \mathfrak{e}_1 & \dots & \not{\mathfrak{e}}_j & \dots & \not{\mathfrak{e}}_l & \dots & \mathfrak{e}_n \\ \dots & & & & & & \dots \\ \mathfrak{e}_1 & \dots & & & & \dots & \mathfrak{e}_n \end{vmatrix}$$

i. e.: The E's are defined by the equation:

(13) $$\Sigma - (-1)^{j+l} E_{jl} \begin{vmatrix} \mathfrak{e}_j & \mathfrak{e}_l \\ \mathfrak{e}_j & \mathfrak{e}_l \end{vmatrix} = \begin{vmatrix} \mathfrak{e}_1 & \dots & \mathfrak{e}_n \\ \dots & & \dots \\ \mathfrak{e}_1 & \dots & \mathfrak{e}_n \end{vmatrix}$$

Moreover, we put:

(14) $$-(-1)^{j+l}(f_{jl} - f_{lj}) = d_{jl}.$$

Then:

$$\Phi_v = \sum_{(j\,l)} E_{j\,l}\, d_{j\,l} \tag{15}$$

Summing for all possible sets of $(j\,l)$. As j and l are not quite independent of one another $(j < l)$, it is convenient to consider $j\,l$ in this and similar summations as a single index running from 1 to $\binom{n}{2} = \frac{n(n-1)}{2}$.

The $\frac{n(n-1)}{2}$ quantities: $d_{j\,l} = -(-1)^{j+l}(f_{j\,l} - f_{l\,j})$, $(j < l)$, we will call the *symmetric differences of the matrix:*

$$\begin{Vmatrix} f_{11}\, f_{12} \cdot \cdot \cdot \cdot \cdot \cdot f_{1n} \\ f_{21}\, f_{22} \cdot \cdot \cdot \cdot \cdot \cdot f_{2n} \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ f_{n1}\, f_{n2} \cdot \cdot \cdot \cdot \cdot \cdot f_{nn} \end{Vmatrix}$$

whereby they are defined also for the n-dimensional case. We now can easily see which of the two quantities $f_{j\,l}$ and $f_{l\,j}$ (the sign taken into account) that is to be subtracted from the other, as we have:

(a) $d_{j\,l} = f_{j\,l} - f_{l\,j}$ if $j + l$ is an *odd* number; (minuend in the upper half of the matrix),

and

(b) $d_{j\,l} = f_{l\,j} - f_{j\,l}$ if $j + l$ is an *even* number; (minuend in the lower half of the matrix).

We readily see that this gives the well-known formula for Ψ_v in S_3. For in the case $n = 3$ we have:

$$E_{23} = \mathfrak{e}_1 = \mathfrak{i};\; E_{13} = \mathfrak{e}_2 = \mathfrak{j};\; E_{12} = \mathfrak{e}_3 = \mathfrak{k}. \tag{16}$$

But the formula (15) also holds good in two-space. For if we have $\Phi = \mathfrak{e}_1\, \mathfrak{f}_1 + \mathfrak{e}_2\, \mathfrak{f}_2$, Φ_v must by definition in this case be a scalar:

$$\Phi_v = \begin{vmatrix} 1 & 0 \\ f_{11} & f_{12} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ f_{21} & f_{22} \end{vmatrix} = f_{12} - f_{21}. \tag{17}$$

But according to (13) we must have $E_{12} = 1$*, and the formula (15) gives the same as (17), viz.:

$$\Phi_v = 1\, d_{12} = -(-1)^{1+2}(f_{12} - f_{21}) = f_{12} - f_{21}. \tag{18}$$

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* „The complement of the n-rowed minor (the determinant itself) is 1". BÔCHER, M., Introduction to Higher Algebra, p. 23.

As we, by § 6 (a), have:

(19) $$\mathfrak{e}_i \times \mathfrak{f}_i = \varkappa_i \times \mathfrak{e}_i,$$

we obviously get:

(20) $$\mathfrak{e}_i \times \mathfrak{f}_i = -\mathfrak{e}_i \times \varkappa_i$$

an equation which is well-known for the three-dimensional case*.

A few other properties of Φ_v, which are completely analogous to well-known vector product properties in S_3, shall also be mentioned.

The equation § 8 (14) is valid if we instead of $\mathfrak{b} \times \mathfrak{c}$ put a sum of such expressions. From this we deduce:

(21) $$\mathfrak{v} \times^{n-2} \Phi_v = -(n-2)!\, \mathfrak{v} \cdot (\Phi - \Phi_c)$$

analogous to the equation in S_3:

(22) $$\mathfrak{v} \times \Psi_v = -\mathfrak{v} \cdot (\Psi - \Psi_c).$$

If we put $\mathfrak{v} = \mathfrak{e}_i$ in (21) we get the n equations:

(23) $$\mathfrak{e}_i \times^{n-2} \Phi_v = -(n-2)!\, (\mathfrak{f}_i - \varkappa_i)$$

corresponding to the following three in S_3:**

(24) $$\begin{aligned} \mathfrak{i} \times \Psi_v &= -(\mathfrak{a} - \varkappa_1) \\ \mathfrak{j} \times \Psi_v &= -(\mathfrak{b} - \varkappa_2) \\ \mathfrak{k} \times \Psi_v &= -(\mathfrak{c} - \varkappa_3) \end{aligned}$$

$\varkappa_1$, $\varkappa_2$, $\varkappa_3$ denoting here, of course, the conjugate system to $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$.

§ 10. The reciprocal system and the „Ergänzungen" of a given set of vectors.

Let the reciprocal system, say $\mathfrak{f}_i^*$, to a given system $\mathfrak{f}_i$ be defined (as in S_3) by the equation

(1) $$\mathfrak{f}_i^* \mathfrak{f}_i = \mathfrak{e}_i \mathfrak{e}_i = \mathfrak{f}_i \mathfrak{f}_i^*.$$

It is here convenient to introduce, as we have done in S_3, the „Ergänzungssystem" of a primary system.† If the latter be $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ (in S_3), the „Ergänzungen" are: $\mathfrak{w}_1 = \mathfrak{b} \times \mathfrak{c}$, $\mathfrak{w}_2 = \mathfrak{c} \times \mathfrak{a} = -\mathfrak{a} \times \mathfrak{c}$; $\mathfrak{w}_3 = \mathfrak{a} \times \mathfrak{b}$; $\mathfrak{w}_1$ is the Ergänzung of $\mathfrak{a}$, $\mathfrak{w}_2$ that of $\mathfrak{b}$, etc. The reciprocal system of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ is, as mentioned § 3 (6), obtained from the „Ergänzungssystem" by division by the determinant of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$.

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* Almar Næss: loc. cit. § 4 (4).

** loc. cit. § 4 (2).

† loc. cit. § 13.

The Ergänzungssystem has a few properties which may be worth noting.† Here we shall only mention that the Ergänzungssystems of two conjugate vector systems are conjugate. This follows from:††

$$(\Psi \overset{\times}{\times} \Psi)_c = \Psi_c \overset{\times}{\times} \Psi_c$$

where $\overset{\times}{\times}$ denotes the (GIBBSIAN) double cross product.

From our point of view, the Ergänzung of a vector of a system oı n vectors in S_n must be the *space complement of all the others,* taken alternately with positive or negative sign. We will give the definition the following form:

(a) *The i^{th} Ergänzungsvector of a given vector system $\mathfrak{f}_i$ is obtained by striking out the i^{th} row in the determinant of the $\mathfrak{f}$'s and replacing it by the unit vectors.*

If the i^{th} Ergänzungsvector is denoted by $\mathfrak{w}_i$, we get:

$$(2) \qquad \mathfrak{w}_i = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{i-1\,1} & f_{i-1\,2} & \cdots & f_{i-1\,n} \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \cdots & \mathfrak{e}_n \\ f_{i+1\,1} & f_{i+1\,2} & \cdots & f_{i+1\,n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

$$= \mathfrak{e}_1 F_{i1} + \mathfrak{e}_2 F_{i2} + \ldots \ldots \mathfrak{e}_n F_{in} = \mathfrak{e}_j F_{ij}$$

where F_{ij} is the cofactor of f_{ij}. We thus see that the matrix of the Ergänzungssystem is the matrix of the cofactors, i. e. conjugate to the adjoint of the matrix of the $\mathfrak{f}$'s.

Now (2) evidently can be written:

$$(3) \qquad \mathfrak{w}_i = (-1)^{i-1} \langle_{n-1}\, \mathfrak{f}_1 \ldots \mathfrak{f}_{i-1}\, \mathfrak{f}_{i+1} \ldots \mathfrak{f}_n.$$

It is now easily shown that the reciprocal system of the $\mathfrak{f}$'s is determined by the n equations:

$$(4) \qquad \mathfrak{f}_i^* = \frac{1}{|f|} \mathfrak{w}_i$$

analogous to what we have found in S_3. $|f|$ is the determinant of the $\mathfrak{f}$'s.

Let us put:

$$(5) \qquad \Phi = \mathfrak{e}_i \mathfrak{f}_i; \;\; \Phi^* = \mathfrak{e}_i \mathfrak{f}_i^*; \;\; \Phi_c^* = \mathfrak{f}_i^* \mathfrak{e}_i.$$

Moreover:

$$(6) \qquad \mathfrak{f}_i^* \mathfrak{f}_i = (\mathfrak{f}_i^* \mathfrak{e}_i) \cdot (\mathfrak{e}_j \mathfrak{f}_j) = \Phi_c^* \cdot \Phi.$$

† loc. cit. § 13, § 37, § 46.

†† loc. cit, § 12 (5) and § 13 (1).

If now $\mathfrak{v}$ be any vector, and $\mathfrak{v}' = \Phi \cdot \mathfrak{v}$, then:

(7) $$(\Phi_c^* \cdot \Phi) \cdot \mathfrak{v} = \Phi_c^* \cdot (\Phi \cdot \mathfrak{v}) = \Phi_c^* \cdot \mathfrak{v}'.$$

But $\mathfrak{f}_i^* \mathfrak{f}_i$ is equal to the idemfactor if, and only if, $\Phi_c^* \cdot \mathfrak{v}' = \mathfrak{v}$. I. e. the transformation Φ_c^* must be the *inverse* of Φ,† and its matrix accordingly the inverse of the matrix of Φ, Hence the matrix of Φ^*, being the conjugate of that of Φ_c^*, consequently is:

(A) $$\left\| \begin{matrix} \frac{F_{11}}{|f|} & \cdots\cdots & \frac{F_{1n}}{|f|} \\ \cdots & \cdots & \cdots \\ \frac{F_{n1}}{|f|} & \cdots\cdots & \frac{F_{nn}}{|f|} \end{matrix} \right\|$$

whereby the validity of (4) is shown.

From (a) follows immediately that the Ergänzungssystem of $\varkappa_i$ is conjugate to $\mathfrak{w}_i$ (where $\varkappa_i$ is the conjugate system of the $\mathfrak{f}$'s).

We also have as in S_3:

(8) $$\mathfrak{f}_1 \cdot \mathfrak{w}_i = \mathfrak{f}_2 \cdot \mathfrak{w}_2 = \ldots\ldots = \mathfrak{f}_n \cdot \mathfrak{w}_n = |f|.$$

The dyadic determined by the $\mathfrak{w}$'s, the *Ergänzungsdyadic*, is in S_3 given by the following determinant, the primary system being $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$:††

(9) $$\frac{1}{2} \begin{vmatrix} \mathfrak{i} & \mathfrak{j} & \mathfrak{k} \\ \mathfrak{a}\times & \mathfrak{b}\times & \mathfrak{c}\times \\ \mathfrak{a} & \mathfrak{b} & \mathfrak{c} \end{vmatrix} = \frac{1}{2}\left\{ \mathfrak{i}\,(\mathfrak{b} \times \mathfrak{c} - \mathfrak{c} \times \mathfrak{b}) - \text{etc.} \ldots \right\}$$

As we see deriving from a (somewhat special) determinant-triadic by taking — as the crosses indicate — the vector product of the two last vectors in each of its triads.

In the analogous way we can derive the Ergänzungsdyadic $\Omega = \mathfrak{e}_i \mathfrak{w}_i$ in S_n by means of the space complement. It is readily shown that:

(10) $$\Omega = \frac{1}{(n-1)!} \begin{vmatrix} \mathfrak{e}_1\, \mathfrak{e}_2 \ldots\ldots\ldots \mathfrak{e}_n \\ \mathfrak{f}_1\, \mathfrak{f}_2 \ldots\ldots\ldots \mathfrak{f}_n \\ \ldots\ldots\ldots\ldots \\ \mathfrak{f}_1\, \mathfrak{f}_2 \ldots\ldots\ldots \mathfrak{f}_n \end{vmatrix}_{n-1}\rangle$$

where the space complement is to be derived of the last $n-1$ vectors in each of the polyads of the polyadic, represented by the determinant.

(10) can also be written:

† Usually in literature denoted by Φ^{-1}.

†† Almar Næss, loc. cit. § 13 (1) and (2), and § 12 (4).

(11) $$\Omega = \frac{1}{(n-1)!}\sum_i (-1)^{i+1} \mathfrak{e}_i \langle_{n-1} \begin{vmatrix} \mathfrak{f}_1 \dots \mathfrak{f}_{i-1}\, \mathfrak{f}_{i+1} \dots \mathfrak{f}_n \\ \dots\dots\dots\dots \\ \mathfrak{f}_1 \dots \mathfrak{f}_{i-1}\, \mathfrak{f}_{i+1} \dots \mathfrak{f}_n \end{vmatrix}$$

And as, by § 7 (2), the determinant in this expression is equal to $(n-1)! \langle_{n-1} \mathfrak{f}_1 \dots \mathfrak{f}_{i-1}\, \mathfrak{f}_{i+1} \dots \mathfrak{f}_n$, it follows immediately from (3) that the second member of the equation (11) is equal to $\mathfrak{e}_i \mathfrak{w}_i$, q. e. d.

§ 11. The space complement of the Ergänzungsdyadic.

As is known, the „*vector*" of the Ergänzungsdyadic in S_3 can be written: †

(1) $$\Omega_v = (b_3 - c_2)\,\mathfrak{a} + (c_1 - a_3)\,\mathfrak{b} + (a_2 - b_1)\,\mathfrak{c} = - \begin{vmatrix} \mathfrak{i}\cdot & \mathfrak{j}\cdot & \mathfrak{k}\cdot \\ \mathfrak{a} & \mathfrak{b} & \mathfrak{c} \\ \mathfrak{a} & \mathfrak{b} & \mathfrak{c} \end{vmatrix}$$

The analogous equation holds in S_n. We put:

(2) $$\Omega_v = \mathfrak{e}_i \times \mathfrak{w}_i.$$

By § 8 (5) we get, noticing that here $p = n-1$, and therefore
$$(-1)^{np}(n-p)! = (-1)^{n(n-1)}(n-(n-1))! = 1:$$

$$\Omega_v = \mathfrak{e}_i \times \mathfrak{w}_i = \sum_i \mathfrak{e}_i \times \{(-1)^{i+1} \langle_{n-1} \mathfrak{f}_1 \dots \mathfrak{f}_{i-1}\, \mathfrak{f}_{i+1} \dots \mathfrak{f}_n\}$$

(3) $$= \sum_i (-1)^{i+1} \mathfrak{e}_i \times (\langle_{n-1} \mathfrak{f}_1 \dots \mathfrak{f}_{i-1}\, \mathfrak{f}_{i+1} \dots \mathfrak{f}_n)$$

$$= -\sum_i (-1)^{i+1} \mathfrak{e}_i \cdot \begin{vmatrix} \mathfrak{f}_1 \dots \mathfrak{f}_{i-1}\, \mathfrak{f}_{i+1} \dots \mathfrak{f}_n \\ \dots\dots\dots\dots \\ \mathfrak{f}_1 \dots \mathfrak{f}_{i-1}\, \mathfrak{f}_{i+1} \dots \mathfrak{f}_n \end{vmatrix}$$

$$= - \begin{vmatrix} \mathfrak{e}_1\cdot & \mathfrak{e}_2\cdot & \dots & \mathfrak{e}_n\cdot \\ \mathfrak{f}_1 & \mathfrak{f}_2 & \dots & \mathfrak{f}_n \\ \dots & \dots & \dots & \dots \\ \mathfrak{f}_1 & \mathfrak{f}_2 & \dots & \mathfrak{f}_n \end{vmatrix}$$

We notice that the two-rowed determinants of the first two rows are all scalars of the form:

(4) $$\begin{vmatrix} \mathfrak{e}_j\cdot & \mathfrak{e}_l\cdot \\ \mathfrak{f}_j & \mathfrak{f}_l \end{vmatrix} = f_{lj} - f_{jl} = -(f_{jl} - f_{lj}). \quad j < l$$

Thus we can write:

(5) $$\Omega_v = -\sum_{\substack{(j\,l)\\ j<l}} (-1)^{j+l} (f_{jl} - f_{lj}) \begin{vmatrix} \mathfrak{f}_1 \dots ,\mathfrak{f}_j \dots ,\mathfrak{f}_l \dots \mathfrak{f}_n \\ \dots\dots\dots\dots \\ \mathfrak{f}_1 \dots ,\mathfrak{f}_j \dots ,\mathfrak{f}_l \dots \mathfrak{f}_n \end{vmatrix}$$

† loc. cit. § 13 (7).

Here we write, as in § 9 (14): $-(-1)^{j+l}(f_{jl}-f_{lj})=d_{jl}$. Further, let the $(n-2)$-rowed vector-determinant, formed from the $\mathfrak{f}$'s after $\mathfrak{f}_j$ and $\mathfrak{f}_l$ have been stricken out, be denoted by F_{jl}, i. e.: F_{jl} is formed from the $\mathfrak{f}$'s similarly as E_{jl} of § 9 (12) and (13) is formed from the $\mathfrak{e}$'s. Then we can write the expression for Ω_v in the very simple way:

(6) $$\Omega_v=\sum_{(jl)} F_{jl}\,d_{jl}$$

the sum extending to all the $\binom{n}{2}$ possible combinations of $j, l (j<l)$. We may think of these two sets as arranged in some definite order, for example:

(A) $$\begin{matrix} F_{12} & F_{13} & F_{23} & \dots\dots & F_{n-1\,n} \\ d_{12} & d_{13} & d_{23} & \dots\dots & d_{n-1\,n} \end{matrix}$$

and then regard the sum as the GRASSMANN „inneres Produkt“ of these two ordered sets. Thus we realize that j, l here plays the rôle of a single, not a double, index running from 1 to $\binom{n}{2}=\frac{n(n-1)}{2}$.

Let us consider a definite determinant F_{jl}. We will expand it as § 8 (13). Let us in the $(n-2)$-rowed matrix of the vectors of F_{jl}, viz.:

(B) $$\left\|\begin{matrix} f_{11} & f_{12} & \dots\dots & f_{1n} \\ \dots & \dots & \dots\dots & \dots \\ f_{n1} & f_{n2} & \dots\dots & f_{nn} \end{matrix}\right\| \qquad (n-2 \text{ rows}),$$

strike out the r^{th} and t^{th} columns $(r<t)$ and denote the determinant thus obtained by $\mathfrak{F}_{jl,rt}$. Thus we see that $\mathfrak{F}_{jl,rt}$ is *the second minor* of *the determinant* of *the* $\mathfrak{f}$'s, obtained by striking out its j^{th} and l^{th} rows and its r^{th} and t^{th} columns.

Further, let us put, as we have done § 9 (12):

(7) $$\left|\begin{matrix} \mathfrak{e}_1 & \dots & \not{\mathfrak{e}}_r & \dots & \not{\mathfrak{e}}_t & \dots & \mathfrak{e}_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathfrak{e}_1 & \dots & \dots & \dots & \dots & \dots & \mathfrak{e}_n \end{matrix}\right| = E_{rt}.$$

Thus we have, by § 8 (13):

(8) $$F_{jl}=\sum_{(rt)}\mathfrak{F}_{jl,rt}\,E_{rt}$$

and, accordingly:

(9) $$\Omega_v=\sum_{(rt)}E_{rt}\sum_{(jl)}\mathfrak{F}_{jl,rt}\,d_{jl}.$$

§ 12. A theorem of the symmetric differences of a matrix.

We readily see that the expression (9) for Ω_v in the preceding § (11) is simply a transformation of the form:

(1) $$E_\alpha \mathfrak{F}_{\beta\alpha} d_\beta$$

to sum, as usual, for α and β which here as above must be thought of as indices running from 1 to $\frac{n(n-1)}{2}$. The elements of the matrix of this transformation, i. e. of the matrix $\mathfrak{F}_{\beta\alpha}$, are the minors of the second order of the matrix of the $\mathfrak{f}$'s. In full $\mathfrak{F}_{\beta\alpha}$ can be written:

(A) $$\left\| \begin{matrix} \mathfrak{F}_{12,12} & \mathfrak{F}_{12,13} & \cdots & \mathfrak{F}_{12,n-1n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathfrak{F}_{n-1n,12} & \mathfrak{F}_{n-1n,13} & \cdots & \mathfrak{F}_{n-1n,n-1n} \end{matrix} \right\|$$

i. e.: (conjugate to) the adjoint of F of the second class. It may be denoted by $[F_2]$. (F stands for the primary matrix.)

But is should be emphasized that the matrix of the transformation $\mathfrak{F}_{\beta\alpha} d_\beta$, where we have to sum for the first index, is the *conjugate* (transposed) of this matrix (A), that is, the matrix of the transformation $\mathfrak{F}_{\beta\alpha} d_\beta$ is $([F]_2)_c = [F_c]_2$.

The two transformations $[F]_2$ and $[F_c]_2$ are, of course, different just as F and F_c are. But we can prove that in this case, where the transformed quantities are the d's, it does not make any difference, because there is one particular set of $\binom{n}{2}$ quantities with that property that the two matrices $[F]_2$ and $[F_c]_2$ effect the same transformation on it. This particular set is the symmetric differences of the matrix. This theorem, which we now are going to prove, can be expressed in the following form:

(a) *The two matrices which can be formed from the second minors of a primary matrix and from the second minors of the conjugate of this, transform the symmetric differences of the primary matrix into the same set of quantities.*

In order to prove this, we must show that the following equation between the two transformations in question:

(2) $$\mathfrak{F}_{\beta\alpha} d_\beta = \mathfrak{F}_{\alpha\beta} d_\beta$$

holds good for any α, i. e. for any combination of two rows and columns respectively.

We can without loss of generality assume that a stands for the first and the second rows, or respectively columns. Then more explicitely we write the equation which we have to prove, thus:

$$\sum_{(r\,t)} \mathfrak{F}_{12,\,r\,t}\, d_{r\,t} = \sum_{(j\,l)} \mathfrak{F}_{j\,l,\,12}\, d_{j\,l}. \tag{3}$$

The symmetric differences can be expressed as the scalar values of all the two-rowed determinants — taken with the sign $-(-1)^{j+l}$, j and l being the two columns represented in the determinant — of the following matrix:

$$\left\| \begin{matrix} \mathfrak{f}_1 & \mathfrak{f}_2 & \cdots\cdots & \mathfrak{f}_n \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \cdots\cdots & \mathfrak{e}_n \end{matrix} \right\| \tag{B}$$

i. e.: we have to take the scalar product of each two vectors to be multiplied. But we also notice that the symmetric differences in the same way can be formed from the matrix.

$$\left\| \begin{matrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \cdots\cdots & \mathfrak{e}_n \\ \varkappa_1 & \varkappa_2 & \cdots\cdots & \varkappa_n \end{matrix} \right\| \tag{C}$$

$\varkappa_i$ being the conjugate system of the $\mathfrak{f}$'s.

Now all the quantities $\mathfrak{F}_{j\,l,\,12}$ are all the $(n-2)$-rowed determinants of the matrix:

$$\left\| \begin{matrix} f_{13} & f_{14} & \cdots\cdots & f_{1n} \\ f_{23} & f_{24} & \cdots\cdots & f_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ f_{n3} & f_{n4} & \cdots\cdots & f_{nn} \end{matrix} \right\| \tag{D}$$

obtained from the matrix of the $\mathfrak{f}$'s by striking out the columns 1 and 2. And in order to form $\mathfrak{F}_{j\,l,\,12}\, d_{j\,l}$ we have to multiply each $d_{j\,l}$ by the corresponding one of these determinants and add up all the products. But then we see that this sum is simply got as a determinant, obtained from (D) by replacing the two missing columns by the matrix (B), whose two-rowed determinants — as said above — exactly give the quantities $d_{j\,l}$ as their scalar values. Changing rows and columns in this determinant we thus obviously have:

$$\sum_{(j\,l)} \mathfrak{F}_{j\,l,\,12}\, d_{j\,l} = \left| \begin{matrix} \mathfrak{f}_1\cdot & \mathfrak{f}_2\cdot & \cdots\cdots & \mathfrak{f}_n\cdot \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \cdots\cdots & \mathfrak{e}_n \\ f_{13} & f_{23} & \cdots\cdots & f_{n3} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ f_{1n} & f_{2n} & \cdots\cdots & f_{nn} \end{matrix} \right| \tag{4}$$

The validity of this equation is also readily shown by expanding its second member in terms of the two-rowed determinants of the first two rows.

We here put:

$$\text{(5)} \qquad \mathfrak{f}_1 = \sum_i f_{1\,i}\,\mathfrak{e}_i, \; \mathfrak{f}_2 = \sum_i f_{2\,i}\,\mathfrak{e}_i, \; \ldots \; \mathfrak{f}_n = \sum_i f_{n\,i}\,\mathfrak{e}_i$$

and inserting this in (4) we get:

$$\text{(6)} \qquad \sum_{(j\,l)} \mathfrak{G}_{j\,l,\,1\,2}\, d_{j\,l} = \begin{vmatrix} \sum_i f_{1\,i}\,\mathfrak{e}_i\cdot & \sum_i f_{2\,i}\,\mathfrak{e}_i\cdot & \ldots & \sum_i f_{n\,i}\,\mathfrak{e}_i\cdot \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots & \mathfrak{e}_n \\ f_{1\,3} & f_{2\,3} & \ldots & f_{n\,3} \\ \ldots & \ldots & \ldots & \ldots \\ f_{1\,n} & f_{2\,n} & \ldots & f_{n\,n} \end{vmatrix}$$

But, according to an elementary theorem of determinants, this simply means that (6) can be expressed as a sum of all the n determinants of the following type:

$$\text{(7)} \qquad \begin{vmatrix} f_{1\,i}\,\mathfrak{e}_i\cdot & f_{2\,i}\,\mathfrak{e}_i\cdot & \ldots & f_{n\,i}\,\mathfrak{e}_i\cdot \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots & \mathfrak{e}_n \\ f_{1\,3} & f_{2\,3} & \ldots & f_{n\,3} \\ \ldots & \ldots & \ldots & \ldots \\ f_{1\,n} & f_{2\,n} & \ldots & f_{n\,n} \end{vmatrix} = \mathfrak{e}_i\cdot \begin{vmatrix} f_{1\,i} & f_{2\,i} & \ldots & f_{n\,i} \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots & \mathfrak{e}_n \\ f_{1\,3} & f_{2\,3} & \ldots & f_{n\,3} \\ \ldots & \ldots & \ldots & \ldots \\ f_{1\,n} & f_{2\,n} & \ldots & f_{n\,n} \end{vmatrix}$$

where especially the subscript i in this case does not indicate a summation in ordinary sence; it only means that i can be any one of the numbers 1, 2, 3 . . . n. And the „dotted" vector $\mathfrak{e}_i\cdot$ is, of course, to be applied to the „nearest" vectors, i. e. to those in the second row.

But we now readily see, that by putting $i \gtreqless 3$ we get determinants in which two rows of scalars are equal, i. e.: vanishing determinants. Thus we have:

$$\text{(8)} \quad \sum_{(j\,l)} \mathfrak{G}_{j\,l,\,1\,2}\, d_{j\,l} = \mathfrak{e}_1\cdot \begin{vmatrix} f_{1\,1} & f_{2\,1} & \ldots & f_{n\,1} \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots & \mathfrak{e}_n \\ f_{1\,3} & f_{2\,3} & \ldots & f_{n\,3} \\ \ldots & \ldots & \ldots & \ldots \\ f_{1\,n} & f_{2\,n} & \ldots & f_{n\,n} \end{vmatrix} + \mathfrak{e}_2\cdot \begin{vmatrix} f_{1\,2} & f_{2\,2} & \ldots & f_{n\,2} \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots & \mathfrak{e}_n \\ f_{1\,3} & f_{2\,3} & \ldots & f_{n\,3} \\ \ldots & \ldots & \ldots & \ldots \\ f_{1\,n} & f_{2\,n} & \ldots & f_{n\,n} \end{vmatrix}$$

Each of these two determinants is a vector, whose components are the cofactors of the elements in the second row. If we now expand in terms of these elements (i. e.: in terms of the unit vectors) and then multiply distributively by $\mathfrak{e}_1\cdot$ and $\mathfrak{e}_2\cdot$ respectively, all the scalar products vanish except one in each determinant, as $\mathfrak{e}_i \cdot \mathfrak{e}_j = 0$ $(i \neq j)$ and $= 1$ $(i = j)$. Therefore:

$$\text{(9)} \quad \sum_{(j\,l)} \mathfrak{G}_{j\,l,\,1\,2}\, d_{j\,l} = - \begin{vmatrix} f_{2\,1} & f_{3\,1} & \ldots & f_{n\,1} \\ f_{2\,3} & f_{3\,3} & \ldots & f_{n\,3} \\ \ldots & \ldots & \ldots & \ldots \\ f_{2\,n} & f_{3\,n} & \ldots & f_{n\,n} \end{vmatrix} + \begin{vmatrix} f_{1\,2} & f_{3\,2} & \ldots & f_{n\,2} \\ f_{1\,3} & f_{3\,3} & \ldots & f_{n\,3} \\ \ldots & \ldots & \ldots & \ldots \\ f_{1\,n} & f_{3\,n} & \ldots & f_{n\,n} \end{vmatrix}$$

In order to get an expression for the sum $\sum\limits_{(rt)} \mathfrak{F}_{12,\,rt}\, d_{rt}$, we can proceed in a completely analogous way. We get:

$$(10)\qquad \sum_{(rt)} \mathfrak{F}_{12,\,rt}\, d_{rt} = \begin{vmatrix} \mathfrak{e}_1\cdot & \mathfrak{e}_2\cdot & \dots & \mathfrak{e}_n\cdot \\ \varkappa_1 & \varkappa_2 & \dots & \varkappa_n \\ f_{31} & f_{32} & \dots & f_{3n} \\ \dots & & & \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix}$$

readily seen by expanding according to the two-rowed determinants of the first two rows (i. e. according to the quantities d_{rt}), because we now shall combine d_{rt} with determinants of that matrix which is obtained by striking out the first two rows of the matrix of the f's.

We here put:

$$(11)\qquad \varkappa_1 = \sum_i f_{i1}\,\mathfrak{e}_i,\ \varkappa_2 = \sum_i f_{i2}\,\mathfrak{e}_i,\ \dots\dots\ \varkappa_n = \sum_i f_{in}\,\mathfrak{e}_i$$

and inserting this in (10) we get:

$$(12)\qquad \sum_{(rt)} \mathfrak{F}_{12,\,rt}\, d_{rt} = \begin{vmatrix} \mathfrak{e}_1\cdot & \mathfrak{e}_2\cdot & \dots & \mathfrak{e}_n\cdot \\ \sum\limits_i f_{i1}\,\mathfrak{e}_i & \sum\limits_i f_{i2}\,\mathfrak{e}_i & \dots & \sum\limits_i f_{in}\,\mathfrak{e}_i \\ f_{31} & f_{32} & \dots & f_{3n} \\ \dots & & & \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix}$$

and this determinant can be reduced to the sum of the n determinants of the following form $(i = 1, 2, \dots . n)$:

$$(13)\qquad \begin{vmatrix} \mathfrak{e}_1\cdot & \mathfrak{e}_2\cdot & \dots & \mathfrak{e}_n\cdot \\ f_{i1}\,\mathfrak{e}_i & f_{i2}\,\mathfrak{e}_i & \dots & f_{in}\,\mathfrak{e}_i \\ f_{31} & f_{32} & \dots & f_{3n} \\ \dots & & & \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix} = \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ f_{i1} & f_{i2} & \dots & f_{in} \\ f_{31} & f_{32} & \dots & f_{3n} \\ \dots & & & \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix} \cdot \mathfrak{e}_i$$

But if we here put $i \gtreqless 3$, we get vanishing determinants. Therefore we have:

$$(14)\qquad \sum_{(rt)} \mathfrak{F}_{12,\,rt}\, d_{rt} = \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ f_{11} & f_{12} & \dots & f_{1n} \\ f_{31} & f_{32} & \dots & f_{3n} \\ \dots & & & \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix} \cdot \mathfrak{e}_1 + \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \dots & \mathfrak{e}_n \\ f_{21} & f_{22} & \dots & f_{2n} \\ f_{31} & f_{32} & \dots & f_{3n} \\ \dots & & & \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix} \cdot \mathfrak{e}_2$$

or:

(14′) $$\sum_{(rt)} \mathfrak{F}_{12,\,rt}\, d_{rt} = \begin{vmatrix} f_{12} f_{13} & \cdot\cdot\cdot\cdot\cdot & f_{1n} \\ f_{32} f_{33} & \cdot\cdot\cdot\cdot\cdot & f_{3n} \\ \cdot\cdot\cdot & \cdot\cdot\cdot\cdot\cdot & \cdot\cdot\cdot \\ f_{n2} f_{n3} & \cdot\cdot\cdot\cdot\cdot & f_{nn} \end{vmatrix} - \begin{vmatrix} f_{21} f_{23} & \cdot\cdot\cdot\cdot\cdot & f_{2n} \\ f_{31} f_{33} & \cdot\cdot\cdot\cdot\cdot & f_{3n} \\ \cdot\cdot\cdot & \cdot\cdot\cdot\cdot\cdot & \cdot\cdot\cdot \\ f_{n1} f_{n3} & \cdot\cdot\cdot\cdot\cdot & f_{nn} \end{vmatrix}$$

$$= \sum_{(jl)} \mathfrak{F}_{jl,\,12}\, d_{jl}, \quad \text{q. e. d.}$$

As obviously the method in this proof is entirely general, this result holds good when we instead of 1, 2 have any other two possible numbers, and our theorem is hereby proved.

If $n = 3$, the matrix of the second minors is simply the primary matrix.† Let the latter be that of the transformation (dyadic) Ψ; then we know that the symmetric differences in this case are the components of the vector Ψ_v. For the three-dimensional case our theorem thus takes the particular form

(15) $$\Psi \cdot \Psi_v = \Psi_c \cdot \Psi_v$$

which simply means that each vector of the triple:

$$\mathfrak{a} - \varkappa_1, \ \mathfrak{b} - \varkappa_2, \ \mathfrak{c} - \varkappa_3$$

is perpendicular to Ψ_v ($\varkappa_i$ conjugate to $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$), a proposition previously stated.††

A still more particular case of the same theorem is that if $\mathfrak{v}$ is any vector in S_3, with the three components P, Q, R, then each of the three vectors

$$\frac{\partial \mathfrak{v}}{\partial x} - \nabla P, \ \ \frac{\partial \mathfrak{v}}{\partial y} - \nabla Q, \ \ \frac{\partial \mathfrak{v}}{\partial z} - \nabla R$$

is perpendicular to the curl of $\mathfrak{v}$.†††

§ 13. Application to Cramer's Rule.

Let a system of n equations of the first degree be given, the unknowns being $x_1, x_2, \ldots . x_n$.

(1) $$\begin{aligned} f_{11} x_1 + f_{12} x_2 + \ldots . + f_{1n} x_n &= v_1 \\ f_{21} x_1 + f_{22} x_2 + \ldots . + f_{2n} x_n &= v_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots &\cdots \\ f_{n1} x_1 + f_{n2} x_2 + \ldots . + f_{nn} x_n &= v_n \end{aligned}$$

† though, according to (A), with the elements in another order. This is, however, of no consequence as the order of d_{jl} is altered correspondingly.

†† Zur Theorie der Triaden von ALMAR NÆSS, § 4 (2) and (a), and § 8 (1).

††† loc. cit. § 4 (e).

Let us here by $\mathfrak{x}$ denote an unknown vector, $\mathfrak{x} = \mathfrak{e}_i x_i$, and by $\mathfrak{v}$ the known vector $\mathfrak{v} = \mathfrak{e}_i v_i$. Putting, moreover, $\mathfrak{f}_j = f_{ji} \mathfrak{e}_i$, then (1) can be written:

$$\begin{aligned} \mathfrak{f}_1 \cdot \mathfrak{x} &= v_1 \\ \mathfrak{f}_2 \cdot \mathfrak{x} &= v_2 \\ &\cdots\cdots \\ &\cdots\cdots \\ \mathfrak{f}_n \cdot \mathfrak{x} &= v_n \end{aligned} \tag{2}$$

Multiplying these equations by $\mathfrak{e}_1, \mathfrak{e}_2 \ldots . \mathfrak{e}_n$ respectively, and adding, we get

$$\mathfrak{e}_1 \mathfrak{f}_1 \cdot \mathfrak{x} + \mathfrak{e}_2 \mathfrak{f}_2 \cdot \mathfrak{x} + \ldots . + \mathfrak{e}_n \mathfrak{f}_n \cdot \mathfrak{x} = \mathfrak{v} \tag{3}$$

or

$$\Phi \cdot \mathfrak{x} = \mathfrak{v}. \tag{4}$$

To solve the equations (1) then simply means to find that unknown vector $\mathfrak{x}$ which by the known dyadic Φ is transformed into the known vector $\mathfrak{v}$. We know that the equations (1) are always solvable if the $\mathfrak{f}$'s are not all contained in a subspace, S_p, of S_n. For in this case $\Phi \cdot \mathfrak{x}$ will also be lying in a p-space, viz. the p-space which contains the conjugate vectors to the $\mathfrak{f}$'s, and which in general is different from S_p.

Now (1) is solved by multiplying (4) by Φ_c^*, Φ^* being the dyadic determined by *the reciprocal system of the* $\mathfrak{f}$'s. From (4) then we get:

$$\Phi_c^* \cdot \Phi \cdot \mathfrak{x} = \Phi_c^* \cdot \mathfrak{v} \tag{5}$$

which reduces to

$$\mathfrak{x} = \Phi_c^* \cdot \mathfrak{v}. \tag{6}$$

This single equation involves CRAMER's formulæ. Let $\mathfrak{w}_i^{\varkappa}$ be the Ergänzungssystem of the $\varkappa$'s, i. e. the conjugate system to the Ergänzungen of the $\mathfrak{f}$'s (see § 10). Then:

$$\Phi_c^* = \mathfrak{e}_i \frac{\mathfrak{w}_i^{\varkappa}}{|f|} \tag{7}$$

and (6) may be written:

$$\mathfrak{e}_i x_i = \mathfrak{e}_i \frac{\mathfrak{w}_i^{\varkappa} \cdot \mathfrak{v}}{|f|} = \mathfrak{e}_i \frac{\mathfrak{v} \cdot \mathfrak{w}_i^{\varkappa}}{|f|} \tag{8}$$

The components here being equal each to each, we get:

$$x_i = \frac{\mathfrak{v} \cdot \mathfrak{w}_i^{\varkappa}}{|f|} \tag{9}$$

which are CRAMER's formulæ. We notice that the space complement in this very compact formula serve to determine the unknowns exactly in the analogous way as the vector product does in the particular case that we have three equations with three unknowns.†

Written in full, (9) becomes:

$$x_i = \frac{\mathfrak{v}}{|f|} \cdot \begin{vmatrix} f_{11} & f_{21} & \cdots & f_{n1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{1\,i-1} & f_{2\,i-1} & \cdots & f_{n\,i-1} \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \cdots & \mathfrak{e}_n \\ f_{1\,i+1} & f_{2\,i+1} & \cdots & f_{n\,i+1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} \end{vmatrix} = \frac{1}{|f|} \begin{vmatrix} f_{11} & f_{21} & \cdots & f_{n1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{1\,i-1} & f_{2\,i-1} & \cdots & f_{n\,i-1} \\ v_1 & v_2 & \cdots & v_n \\ f_{1\,i+1} & f_{2\,i+1} & \cdots & f_{n\,i+1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} \end{vmatrix} \tag{10}$$

$$= \frac{1}{|f|} \begin{vmatrix} f_{11} & \cdots & f_{1\,i-1} & v_1 & f_{1\,i+1} & \cdots & f_{1n} \\ f_{21} & \cdots & f_{2\,i-1} & v_2 & f_{2\,i+1} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{n1} & \cdots & f_{n\,i-1} & v_n & f_{n\,i+1} & \cdots & f_{nn} \end{vmatrix}$$

which is the usual form.

Another related application shall also be mentioned:

Let there be given the two systems of independent vectors $\mathfrak{f}_i$ and $\mathfrak{f}_i'$. We will find the dyadic X which transforms the vectors $\mathfrak{f}_i$ into the vectors $\mathfrak{f}_i'$ respectively. X is hereby completely determined by the n equations:

$$X \cdot \mathfrak{f}_i = \mathfrak{f}_i'. \tag{11}$$

Let $\Phi' = \mathfrak{e}_i \mathfrak{f}'_i = \varkappa_i' \mathfrak{e}_i$, else the notations given above. From (11) we then get:

$$X \cdot \mathfrak{f}_i \mathfrak{e}_i = \mathfrak{f}_i' \mathfrak{e}_i \tag{12}$$

(13) or: $$X \cdot \Phi_c = \Phi_c'$$

Multiplying by Φ^* we get:

$$X \cdot \Phi_c \cdot \Phi^* = \Phi_c' \cdot \Phi^* \tag{14}$$

(15) or: $$X = \Phi_c' \cdot \Phi^*$$

whereby X is determined.

Let us put:

$$X = \mathfrak{e}_i \mathfrak{x}_i = \mathfrak{e}_i \mathfrak{e}_j x_{ij}$$

(16) then, by (15) $$\mathfrak{e}_i \mathfrak{x}_i = \mathfrak{e}_i \varkappa_i' \cdot \Phi^*$$

(17) or: $$x_{ij} \mathfrak{e}_j = \mathfrak{x}_i = \varkappa_i' \cdot \Phi^* = \varkappa_i' \cdot \frac{\mathfrak{w}_j^*}{|f|} \mathfrak{e}_j$$

† C. Runge: Vektoranalysis (des dreidimensionalen Raumes), (Leipzig 1919) § 12.

That is, the matrix of X is determined by:

(18) $$x_{ij} = \frac{1}{|f|} \varkappa_i' \cdot \mathfrak{w}_j^{\varkappa}$$

By the definition of the j^{th} Ergänzungsvector, § 10 (a), this means: $|f|\, x_{ij}$ is obtained from the determinant

$$\begin{vmatrix} f_{11}\, f_{21} \cdot \cdot \cdot \cdot \cdot \cdot \cdot f_{n1} \\ f_{12}\, f_{22} \cdot \cdot \cdot \cdot \cdot \cdot \cdot f_{n2} \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ f_{1n}\, f_{2n} \cdot \cdot \cdot \cdot \cdot \cdot f_{nn} \end{vmatrix}$$

by interchanging the j^{th} row with the quantities $f'_{1i}, f'_{2i}, \ldots . f'_{ni}$. Or, in other words:

$|f|\, x_{ij}$ is obtained from the determinant of the $\mathfrak{f}$'s by interchanging its j^{th} column with the i^{th} column of the determinant of the $\mathfrak{f}'$'s.

§ 14. Miscellaneous Formulæ.

The space complement of a set of p unit vectors must, by § 5 (a), be expressible by the other $n-p$ unit vectors. Let the set be $\mathfrak{e}_{k_1}, \mathfrak{e}_{k_2} \ldots . \mathfrak{e}_{k_p}$: We will assume that they are arranged in order of magnitude, i. e.. $k_1 < k_2 < \ldots . < k_p$.

$$\langle^p\, \mathfrak{e}_{k_1} \mathfrak{e}_{k_2} \ldots . \mathfrak{e}_{k_p} = \begin{vmatrix} \mathfrak{e}_1 \ldots . \mathfrak{e}_{k_1} \ldots . \mathfrak{e}_{k_2} \ldots . \mathfrak{e}_{k_p} \ldots . \mathfrak{e}_n \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \mathfrak{e}_1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \mathfrak{e}_n \\ 0 \ldots . 1 \ldots \ldots \ldots \ldots \ldots . 0 \\ 0 \ldots \ldots \ldots . 1 \ldots \ldots \ldots . 0 \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ 0 \ldots \ldots \ldots \ldots \ldots . 1 \ldots . 0 \end{vmatrix}$$

(1) $$= (-1)^{np - \frac{p(p-1)}{2} + \sum_1^n k_i} \begin{vmatrix} \mathfrak{e}_1 \ldots . \not{\mathfrak{e}}_{k_i} \ldots . \mathfrak{e}_n \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \mathfrak{e}_1 \ldots \ldots \ldots . \mathfrak{e}_n \end{vmatrix}$$

where, as before, $\mathfrak{e}_1 \ldots . \not{\mathfrak{e}}_{k_i} \ldots . \mathfrak{e}_n$ denotes the set of the $n-p$ unit vectors which are left after erasing the $\mathfrak{e}_{k_i}$'s. This formula taken into account, we can write § 8 (6) and (A):

(2) $$\langle^p\, \mathfrak{a}_1 \mathfrak{a}_2 \ldots . \mathfrak{a}_p = \Sigma \langle^p\, \mathfrak{e}_{k_1} \mathfrak{e}_{k_2} \ldots . \mathfrak{e}_{k_p} \begin{vmatrix} a_{1k_1} \ldots . a_{1k_p} \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ a_{pk_1} \ldots . a_{pk_p} \end{vmatrix}$$

But, of course, from this does not follow that the equation holds if we remove the operation sign $\langle_p$. i. e.: the two polyad(ic)s whose space complements form the left and right member of this equation, are not equal.

From (1) again follows that, for example ($I = \mathfrak{e}_i \mathfrak{e}_i$ and $\langle$ stands for $\langle_1$):

$$\langle I = \begin{vmatrix} \mathfrak{e}_1 \, \mathfrak{e}_2 & \ldots\ldots\ldots & \mathfrak{e}_n \\ \mathfrak{e}_1 \, \mathfrak{e}_2 & \ldots\ldots\ldots & \mathfrak{e}_n \\ \ldots & \ldots\ldots\ldots & \ldots \\ \mathfrak{e}_1 \, \mathfrak{e}_2 & \ldots\ldots\ldots & \mathfrak{e}_n \end{vmatrix} \tag{3}$$

while $I \rangle$ is equal to the same determinant times $(-1)^{n-1}$.

We also have:

$$\mathfrak{e}_i \times \mathfrak{a} = (-1)^n (-1)^{i+1} \begin{vmatrix} \mathfrak{e}_1 \ldots\ldots \mathfrak{e}_{i-1} \; \mathfrak{e}_{i+1} \ldots\ldots \mathfrak{e}_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathfrak{e}_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathfrak{e}_n \\ a_1 \ldots\ldots a_{i-1} \; a_{i+1} \ldots\ldots a_n \end{vmatrix} \tag{4}$$

from which we get:

$$I \times \mathfrak{a} = (-1)^n \langle \mathfrak{a}. \tag{5}$$

In the same way we can prove the more general formula:

$$I \times_p \mathfrak{a}_1 \mathfrak{a}_2 \ldots\ldots \mathfrak{a}_p = -(-1)^{n-p} \langle_p \mathfrak{a}_1 \mathfrak{a}_2 \ldots\ldots \mathfrak{a}_p. \tag{6}$$

We will expand

$$\Phi \times \Phi'$$

where Φ and Φ' are two dyadics, $\Phi = \mathfrak{e}_i \mathfrak{f}_i = \varkappa_i \mathfrak{e}_i$, $\Phi' = \mathfrak{e}_i \mathfrak{f}'_i$. $\Phi \times \Phi'$ is always a polyadic of order n. In three-space it is called the vector product of two dyadics. We get:

$$\Phi \times \Phi' = \varkappa_i \mathfrak{e}_i \times \mathfrak{e}_j \mathfrak{f}'_j, \quad i, j = 1, 2 \ldots n. \tag{7}$$

Here we only consider $i \neq j$ as $\mathfrak{e}_i \times \mathfrak{e}_i = 0$.

Firstly we assume that $i < j$. Then we have:

$$\mathfrak{e}_i \times \mathfrak{e}_j = \begin{vmatrix} \mathfrak{e}_1 \ldots \mathfrak{e}_i \ldots \mathfrak{e}_j \ldots \mathfrak{e}_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \mathfrak{e}_1 \ldots\ldots\ldots\ldots\ldots \mathfrak{e}_n \\ 0 \ldots 1 \ldots 0 \ldots 0 \\ 0 \ldots 0 \ldots 1 \ldots 0 \end{vmatrix} = -(-1)^{i+j} \begin{vmatrix} \mathfrak{e}_1 \ldots \acute{\mathfrak{e}}_i \ldots \acute{\mathfrak{e}}_j \ldots \mathfrak{e}_n \\ \ldots\ldots\ldots\ldots\ldots \\ \mathfrak{e}_1 \ldots\ldots\ldots\ldots \mathfrak{e}_n \end{vmatrix} \tag{8}$$

$$= -(-1)^{i+j} E_{ij}.$$

If $i > j$ we get:

$$\mathfrak{e}_i \times \mathfrak{e}_j = (-1)^{i+j} E_{ij}. \tag{9}$$

Therefore:

(10) $$\Phi \times \Phi' = \Sigma\Sigma \pm \varkappa_i E_{ij} \mathfrak{f}_j'$$

$$= -\sum_i \varkappa_i \{\sum_j (-1)^{i+j} [\underset{i<j}{E_{ij}} \mathfrak{f}_j' - \underset{i>j}{E_{ij}} \mathfrak{f}'_j]\}$$

$$= \sum_i (-1)^{1+i} \varkappa_i \{\sum_j (-1)^j [\underset{i<j}{E_{ij}} \mathfrak{f}_j' - \underset{i>j}{E_{ij}} \mathfrak{f}'_j]\}.$$

The sum in the brackets is equal to the following determinant of order $(n-1)$, multiplied by $(-1)^n$:

(A) $$\begin{vmatrix} \mathfrak{e}_1 \dots \mathfrak{e}_{i-1} \; \mathfrak{e}_{i+1} \dots \mathfrak{e}_n \\ \dots\dots\dots\dots \\ \mathfrak{e}_1 \dots\dots\dots\dots \mathfrak{e}_n \\ \mathfrak{f}'_1 \dots \mathfrak{f}'_{i-1} \; \mathfrak{f}'_{i+1} \dots \mathfrak{f}'_n \end{vmatrix}$$

For all the $(n-2)$-rowed determinants of the first $n-2$ rows are of the form E_{ij}, where $j = 1, 2 \dots i-1, i+1 \dots n$. The plain complement of E_{ij} is $\mathfrak{f}_j'$. It must be noticed that $\mathfrak{f}_j'$ stands in the j^{th} column of this determinant if $i > j$, but in the $(j-1)^{\text{th}}$ column if $i < j$. Hence the algebraic complement of E_{ij} in the first case is:

(11) $$(-1)^{n-1+j} \mathfrak{f}_j' = -(-1)^n (-1)^j \mathfrak{f}_j', \text{ for } i > j$$

but $$= (-1)^n (-1)^j \mathfrak{f}_j', \text{ for } i < j$$

But then (10) readily shows that:

(12) $$\Phi \times \Phi' = (-1)^n \begin{vmatrix} \varkappa_1 \; \varkappa_2 \dots\dots \varkappa_n \\ \mathfrak{e}_1 \; \mathfrak{e}_2 \dots\dots \mathfrak{e}_n \\ \dots\dots\dots\dots \\ \mathfrak{e}_1 \; \mathfrak{e}_2 \dots\dots \mathfrak{e}_n \\ \mathfrak{f}'_1 \; \mathfrak{f}'_2 \dots\dots \mathfrak{f}'_n \end{vmatrix}$$

which gives the formula for the vector product of two dyadics in three-space† as a particular case.

By comparing (12) with § 4 (a^1) we observe that (12), as in S_8, holds good also if Φ and Φ' are vectors, i. e.: if $\varkappa_i$ and $\mathfrak{f}_i'$ are scalars.

† Almar Næss, loc. cit., § 37 (7).

§ 15. The skew-symmetric dyadic (tensor) of two vectors expressed as a space complement.

From two given vectors $\mathfrak{a}$ and $\mathfrak{b}$ we can derive a skew-symmetric tensor defined by the following scalars:

$$(1) \qquad c_{ij} = a_i b_j - a_j b_i$$

involving $\frac{n(n-1)}{2}$ independent scalars, as $c_{ii} = 0$ and $c_{ij} = -c_{ji}$. This tensor (by some authors called the vector product of $\mathfrak{a}$ and $\mathfrak{b}$†) is in vector analysis notations:

$$(2) \qquad \mathfrak{e}_i \mathfrak{e}_j c_{ij} = \mathfrak{a}\,\mathfrak{b} - \mathfrak{b}\,\mathfrak{a} = \begin{vmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{a} & \mathfrak{b} \end{vmatrix}$$

the multiplication of the vectors being indeterminate.

This tensor (dyadic) and the space complement of $\mathfrak{a}$ and $\mathfrak{b}$ are very closely related to one another, as either of them in a simple way can be derived from the other. We will here show that the tensor c_{ij} can be obtained as the space complement of $\mathfrak{a} \times \mathfrak{b}$ times a scalar.

By definition we get as an expression for $\mathfrak{a} \times \mathfrak{b}$ the sum of all possible terms (when $i < j$) of the following form:

$$(3) \qquad \mathfrak{a} \times \mathfrak{b} = \sum_{(i<j)} -(-1)^{i+j} E_{ij} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$$

So we take the space complement of this. We get by § 7 (2) and § 14 (1), putting $p = 2, \sum_1^p k_i = i + j$:

$$\langle_{n-2} E_{ij} = (n-2)!\ \langle_{n-2}\, \mathfrak{e}_1 \ldots \mathfrak{e}'_i \ldots \mathfrak{e}'_j \ldots \mathfrak{e}_n$$

$$= (n-2)!\ (-1)^{i+j-1} \begin{vmatrix} \mathfrak{e}_i & \mathfrak{e}_j \\ \mathfrak{e}_i & \mathfrak{e}_j \end{vmatrix}$$

† Hermann Weyl, Raum, Zeit, Materie, p. 40.

Hence:

(5) $$\langle_{n-2}\, \mathfrak{a} \times \mathfrak{b} = (n-2)!\, \Sigma \begin{vmatrix} \mathfrak{e}_i & \mathfrak{e}_j \\ \mathfrak{e}_i & \mathfrak{e}_j \end{vmatrix} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$$

$$= (n-2)!\ \Sigma\, [\mathfrak{e}_i\, \mathfrak{e}_j\, (a_i\, b_j - a_j\, b_i) + \mathfrak{e}_j\, \mathfrak{e}_i\, (a_j\, b_i - a_i\, b_j)]$$

with the restriction that $i < j$. But this sum is, of course, equal to the sum of all terms of the form:

(A) $$(n-2)!\ \mathfrak{e}_i\, \mathfrak{e}_j\, c_{ij}$$

i and $j = 1, 2 \ldots . n$

or:

(6) $$\langle_{n-2}\, \mathfrak{a} \times \mathfrak{b} = (n-2)!\ \{\mathfrak{a}\, \mathfrak{b} - \mathfrak{b}\, \mathfrak{a}\} = (n-2)! \begin{vmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{a} & \mathfrak{b} \end{vmatrix}$$

which also follows from (5) directly, by § 8 (13).

From this we easily get the space complement of Φ_v, viz.:

(7) $$\langle_{n-2}\Phi_v = (n-2)!\ (\Phi - \Phi_c)$$

The equation (6) can be considered as a particular case of a formula for the space complement of the space complement of a set of any number of vectors less than n (say p).

We found § 8 (A) that:

(8) $$\langle_p\, \mathfrak{a}_1 \ldots \mathfrak{a}_p = \Sigma(-1)^{np - \frac{p(p-1)}{2} + \Sigma k_i} \begin{vmatrix} \mathfrak{e}_1 \ldots, \mathfrak{e}'_{k_i} \ldots \mathfrak{e}_n \\ \ldots\ldots\ldots \\ \mathfrak{e}_1 \ldots\ldots\ldots \mathfrak{e}_n \end{vmatrix} \begin{vmatrix} a_{1k_1} & \ldots & a_{1k_p} \\ \ldots & & \ldots \\ a_{pk_1} & \ldots & a_{pk_p} \end{vmatrix}$$

Now we will derive the space complement of this quantity. We notice that:

(9) $$\langle_{n-p} \begin{vmatrix} \mathfrak{e}_1 \ldots, \mathfrak{e}'_{k_i} \ldots \mathfrak{e}_n \\ \ldots\ldots\ldots \\ \mathfrak{e}_1 \ldots\ldots\ldots \mathfrak{e}_n \end{vmatrix} = (n-p)!\ \langle_{n-p}\, \mathfrak{e}_1 \ldots, \mathfrak{e}'_{k_i} \ldots \mathfrak{e}_n$$

This can be developed according to § 14 (1) by putting $n-p$ instead of p and, accordingly, $\frac{n(n+1)}{2} - \sum_1^p k_i$ instead of $\sum_1^p k_i$ whereby the final sign of (9) will be $(-1)^{-\frac{p(p+1)}{2} - \sum_1^p k_i}$. Noticing that:

$$(-1)^{-p^2} = (-1)^{-p} = (-1)^p$$

we get from (8):

(10) $$\langle_{n-p}(\langle_p\, \mathfrak{a}_1 \ldots \mathfrak{a}_p) = (n-p)!\ \Sigma(-1)^{np+p} \begin{vmatrix} \mathfrak{e}_{k_1} & \ldots & \mathfrak{e}_{k_p} \\ \ldots & & \ldots \\ \mathfrak{e}_{k_1} & \ldots & \mathfrak{e}_{k_p} \end{vmatrix} \begin{vmatrix} a_{1k_1} & \ldots & a_{1k_p} \\ \ldots & & \ldots \\ a_{pk_1} & \ldots & a_{pk_p} \end{vmatrix}$$

which by § 8 (13) gives:

(11) $$\langle_{n-p}(\langle_p\, \mathfrak{a}_1 \ldots . \mathfrak{a}_p) = (n-p)!\ (-1)^{(n+1)p} \begin{vmatrix} \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \\ \ldots & & \ldots \\ \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \end{vmatrix}$$

The equation § 8 (5) can also be obtained from this by the following theorem:

(a) *The space complement of any number of vectors (say p) is equal to the scalar product of the first vector by the space complement of the others, taken with the sign* $(-1)^{n-p}$.

Let P_r be a polyad of order r. Then the theorem says:

(12) $$\mathfrak{v} \times^r P_r = (-1)^{n-r-1}\, \mathfrak{v} \cdot \langle^r P_r.$$

It is easily proved. Let $P_r = \mathfrak{b}_1 \ldots \mathfrak{b}_r$ (it is readily seen that the proof is valid also in the case that P_r is a sum of such polyads). Then:

(13) $$\mathfrak{v} \times^r P_r = \begin{vmatrix} \mathfrak{e}_1 & \ldots & \mathfrak{e}_n \\ \ldots & \ldots & \ldots \\ \mathfrak{e}_1 & \ldots & \mathfrak{e}_n \\ v_1 & \ldots & v_n \\ b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{r1} & \ldots & b_{rn} \end{vmatrix} = (-1)^{n-r-1} \begin{vmatrix} v_1 & \ldots & v_n \\ \mathfrak{e}_1 & \ldots & \mathfrak{e}_n \\ \ldots & \ldots & \ldots \\ \mathfrak{e}_1 & \ldots & \mathfrak{e}_n \\ b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{r1} & \ldots & b_{rn} \end{vmatrix}$$

$$= (-1)^{n-r-1} \begin{vmatrix} \mathfrak{v}\cdot\mathfrak{e}_1 & \ldots & \mathfrak{v}\cdot\mathfrak{e}_n \\ \mathfrak{e}_1 & \ldots & \mathfrak{e}_n \\ \ldots & \ldots & \ldots \\ \mathfrak{e}_1 & \ldots & \mathfrak{e}_n \\ b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{r1} & \ldots & b_{rn} \end{vmatrix} = (-1)^{n-r-1}\, \mathfrak{v} \cdot \langle^r P_r$$

As $\langle^p \mathfrak{a}_1 \ldots \mathfrak{a}_p$ is of order $n-p$, we can put: $P_r = \langle^p \mathfrak{a}_i \ldots \mathfrak{a}_p = P_{n-p}$, and inserting this in (12), we get immediately from (11):

(14) $$\mathfrak{v} \times^{n-p} (\langle^p \mathfrak{a}_1 \ldots \mathfrak{a}_p) = -(-1)^{np} (n-p)!\, \mathfrak{v} \cdot \begin{vmatrix} \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \\ \ldots & \ldots & \ldots \\ \mathfrak{a}_1 & \ldots & \mathfrak{a}_p \end{vmatrix}$$

If we in (12) put $P_r = \mathfrak{a}$, we get

(15) $$\mathfrak{v} \times \mathfrak{a} = (-1)^n\, \mathfrak{v} \cdot \langle \mathfrak{a}$$

which by § 14 (5) can be written:

(16) $$\mathfrak{v} \times \mathfrak{a} = \mathfrak{v} \cdot (I \times \mathfrak{a}).$$

The well-known equation of the same form in S_3† is thus valid unaltered in S_n. In S_2 the equation is self-evident. $\mathfrak{v} \times \mathfrak{a}$ then simply means the

† Gibbs, Scientific Papers II, p. 59.

area of the parallelogram on $\mathfrak{v}$ and $\mathfrak{a}$, and $I \times \mathfrak{a}$ is the vector $\mathfrak{a}$ turned one right angle in negative direction, that is in the direction from $\mathfrak{a}$ to $\mathfrak{v}$ if $\mathfrak{v} \times \mathfrak{a}$ is a positive scaler. Then (16) only says that two opposite sides of a parallelogram are equal in length.

§ 16. Remarks concerning the divergence and the curl.

By the Nabla vector ∇ we understand the symbolic vector differentiator $\mathfrak{e}_i \frac{\partial}{\partial x_i}$. Hence:

$$\nabla \mathfrak{a} = \mathfrak{e}_i \frac{\partial \mathfrak{a}}{\partial x_i}. \tag{1}$$

In the three-dimensional vector analysis the scalar and vector of this dyadic is called the divergence and curl of $\mathfrak{a}$ respectively.

As the first of these conceptions only depends upon the definition of the scalar product of two vectors — which is valid in any space — we put also in S_n:

$$\operatorname{div} \mathfrak{a} = \nabla \cdot \mathfrak{a} = \mathfrak{e}_i \cdot \frac{\partial \mathfrak{a}}{\partial x_i} = \frac{\partial a_i}{\partial x_i} \tag{2}$$

As in S_3, we will apply this equation also to the case when we instead of $\mathfrak{a}$ have in general a polyad(ic), whereby the divergence of any polyad(ic) is defined. Particularly we notice that if a polyadic is written as a determinant whose first row consists of the unit vectors, the divergence of it is obtained by interchanging the first row with the operators $\frac{\partial}{\partial x_i}$.

The generalisation of the curl to S_n is not so obvious. We here want to emphasize that by the term curl we only understand the (special) vector function, such as it is defined in classical vector analysis, not the physical phenomena (the rotation) which this vector may represent. And it is outside our province to consider whether or not there may be a more suitable mathematical representation for those phenomena (e. g. a skew-symmetric tensor of the second order).† But from this point of view, the curl is nothing but a certain vector product (i. e.: a sum of such ones), and a way of extending the latter to S_n once defined or adopted, necessarily leads to a corresponding generalization of the curl.

Hence, the quantity which we here will consider to be the generalized „vector“ of the dyadic $\nabla \mathfrak{a}$, is the following:

$$\nabla \times \mathfrak{a} = \mathfrak{e}_i \times \frac{\partial \mathfrak{a}}{\partial x_i}, \tag{3}$$

† Weyl, H.: loc. cit. p. 54.

the cross as before denoting the space complement of two vectors. From this equation we get the ordinary curl of a vector as a particular case (viz. $n = 3$), and we will also call (3) the curl of $\mathfrak{a}$.

We will derive a few properties of this quantity:

It is a tensor (polyadic) of order $n - 2$, thus a vector only in S_3. From § 9 (10) we immediately get:

$$(4) \qquad \nabla \times \mathfrak{a} = - \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots\ldots\ldots & \mathfrak{e}_n \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots\ldots\ldots & \mathfrak{e}_n \\ \mathfrak{e}_1 \cdot & \mathfrak{e}_2 \cdot & \ldots\ldots\ldots & \mathfrak{e}_n \cdot \\ \dfrac{\partial \mathfrak{a}}{\partial x_1} & \dfrac{\partial \mathfrak{a}}{\partial x_2} & \ldots\ldots\ldots & \dfrac{\partial \mathfrak{a}}{\partial x_n} \end{vmatrix}$$

But as:

$$(5) \qquad \begin{vmatrix} \mathfrak{e}_i \cdot & \mathfrak{e}_j \cdot \\ \dfrac{\partial \mathfrak{a}}{\partial x_i} & \dfrac{\partial \mathfrak{a}}{\partial x_j} \end{vmatrix} = \frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} = - \begin{vmatrix} \dfrac{\partial}{\partial x_i} & \dfrac{\partial}{\partial x_j} \\ a_i & a_j \end{vmatrix}$$

(4) evidently can be written:

$$(6) \qquad \nabla \times \mathfrak{a} = \begin{vmatrix} \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots\ldots\ldots & \mathfrak{e}_n \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \ldots\ldots\ldots & \mathfrak{e}_n \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \ldots\ldots\ldots & \dfrac{\partial}{\partial x_n} \\ a_1 & a_2 & \ldots\ldots\ldots & a_n \end{vmatrix}$$

of which the well-known formula in three-space:

$$(7) \qquad \text{curl } \mathfrak{a} = \begin{vmatrix} \mathfrak{i} & \mathfrak{j} & \mathfrak{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

is a particular case.

When — as in (6) — one or more rows of a determinant consist of operators, it is always understood that these are to be applied to the quantities in all of the following rows, i. e.: to the determinants formed from the matrix of the following rows.

According to what is said above, we get:

$$(8) \qquad \nabla \cdot (\nabla \times \mathfrak{a}) = \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots\cdots & \frac{\partial}{\partial x_n} \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \cdots\cdots & \mathfrak{e}_n \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \mathfrak{e}_1 & \mathfrak{e}_2 & \cdots\cdots & \mathfrak{e}_n \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots\cdots & \frac{\partial}{\partial x_n} \\ a_1 & a_2 & \cdots\cdots & a_n \end{vmatrix}$$

which vanishes identically. Hence the curl of $\mathfrak{a}$, defined as we have done above, satisfies the characteristic equation

$$(9) \qquad \text{div curl } \mathfrak{a} \equiv 0.$$

We also find:

(a) *The divergence of the space complement of a vector is equal to the curl of the same vector times* $(-1)^n$.

For remembering that the $\mathfrak{e}$'s are constant vectors, we get

$$(10) \; \nabla \cdot (\langle \mathfrak{a}) = \begin{vmatrix} \frac{\partial}{\partial x_1} & \cdots\cdots & \frac{\partial}{\partial x_n} \\ \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ \cdots & \cdots\cdots & \cdots \\ \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ a_1 & \cdots\cdots & a_n \end{vmatrix} = (-1)^{n-2} \begin{vmatrix} \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ \cdots & \cdots\cdots & \cdots \\ \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ \frac{\partial}{\partial x_1} & \cdots\cdots & \frac{\partial}{\partial x_n} \\ a_1 & \cdots\cdots & a_n \end{vmatrix}$$

from which the proposition follows. This may be written:

$$(11) \qquad \nabla \cdot \langle \mathfrak{a} = (-1)^n \, \nabla \times \mathfrak{a}$$

and in this form it can be regarded as a particular case of § 15 (15), $\mathfrak{v}$ being interchanged with Nabla, and $r = 1$, i. e.: $P_r = \mathfrak{a}$.

Also in (9) (or (8)) Nabla plays the rôle of an ordinary vector, as $\mathfrak{v} \cdot (\mathfrak{v} \times \mathfrak{a})$ vanishes identically too.

By § 14 (5), (11) can be written:

$$(12) \qquad \nabla \cdot (I \times \mathfrak{a}) = \nabla \times \mathfrak{a}$$

which is only a special case of § 15 (16). This equation is well-known in S_3.†

By § 9 (14) (15) and remembering that in this special case:

$$f_{ij} = \frac{\partial a_j}{\partial x_i},$$

† Zur Theorie der Triaden von ALMAR NÆSS, p. 121.

the curl of $\mathfrak{a}$ can also be written:

$$(13) \qquad \nabla \times \mathfrak{a} = -(-1)^{i+j} E_{ij} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \right)$$

the sum to be taken for all possible sets of i, j, when $i < j$.

And exactly in the same way as in § 15 we here prove that the space complement of the curl is equal to $(n-2)!$ times the dyadic whose matrix is:

$$\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}$$

which dyadic sometimes is called the curl of $\mathfrak{a}$. Thus we get:

$$\langle_{n-2} \nabla \times \mathfrak{a} = (n-2)! \, \{\nabla \mathfrak{a} - (\nabla \mathfrak{a})_c\} = (n-2)! \, \{\nabla \mathfrak{a} - \mathfrak{a} \nabla\}.$$

The formula for the divergence of the vector product in S_3 is a particular case of the following equation:

$$(15) \qquad \operatorname{div} \mathfrak{a}_1 \times \mathfrak{a}_2 = -(-1)^n \begin{vmatrix} \operatorname{curl} \mathfrak{a}_1 \cdot & \operatorname{curl} \mathfrak{a}_2 \cdot \\ \mathfrak{a}_1 & \mathfrak{a}_2 \end{vmatrix}$$

We have:

$$(16) \quad \operatorname{div} \mathfrak{a}_1 \times \mathfrak{a}_2 = \begin{vmatrix} \frac{\partial}{\partial x_1} & \cdots\cdots & \frac{\partial}{\partial x_n} \\ \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ \cdots & \cdots\cdots & \cdots \\ \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ a_{11} & \cdots\cdots & a_{1n} \\ a_{21} & \cdots\cdots & a_{2n} \end{vmatrix} = (-1)^{n-3} \begin{vmatrix} \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ \cdots & \cdots\cdots & \cdots \\ \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ \frac{\partial}{\partial x_1} & \cdots\cdots & \frac{\partial}{\partial x_n} \\ a_{11} & \cdots\cdots & a_{1n} \\ a_{21} & \cdots\cdots & a_{2n} \end{vmatrix}$$

But this last determinant is obviously equal to the sum of two determinants obtained by applying the operators $\frac{\partial}{\partial x_i}$ to the rows a_{1i} and a_{2i} respectively. The first of these clearly is:

$$(17) \quad \begin{vmatrix} \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ \cdots & \cdots\cdots & \cdots \\ \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ a_{21} & \cdots\cdots & a_{2n} \\ \frac{\partial}{\partial x_1} & \cdots\cdots & \frac{\partial}{\partial x_n} \\ a_{11} & \cdots\cdots & a_{1n} \end{vmatrix} = \begin{vmatrix} \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ \cdots & \cdots\cdots & \cdots \\ \mathfrak{e}_1 & \cdots\cdots & \mathfrak{e}_n \\ \mathfrak{e}_1 \cdot \mathfrak{a}_2 & \cdots\cdots & \mathfrak{e}_n \cdot \mathfrak{a}_2 \\ \frac{\partial}{\partial x_1} & \cdots\cdots & \frac{\partial}{\partial x_n} \\ a_{11} & \cdots\cdots & a_{1n} \end{vmatrix} = \operatorname{curl} \mathfrak{a}_1 \cdot \mathfrak{a}_2$$

and the second:

$$(18) \qquad -\begin{vmatrix} \mathfrak{e}_1 & \cdots & \mathfrak{e}_n \\ \cdots & \cdots & \cdots \\ \mathfrak{e}_1 & \cdots & \mathfrak{e}_n \\ a_{11} & \cdots & a_{1n} \\ \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \\ a_{21} & \cdots & a_{2n} \end{vmatrix} = -\operatorname{curl} \mathfrak{a}_2 \cdot \mathfrak{a}_1$$

whereby our theorem (15) is proved.

Let α be any fixed integer of the set $1, 2, \ldots n$. Then applying § 8 (4) we get:

$$(19) \qquad \mathfrak{v} \times^{n-2} \left(\mathfrak{e}_\alpha \times \frac{\partial \mathfrak{a}}{\partial x_\alpha}\right) = -(n-2)!\, \mathfrak{v} \cdot \left\{\mathfrak{e}_\alpha \frac{\partial \mathfrak{a}}{\partial x_\alpha} - \frac{\partial \mathfrak{a}}{\partial x_\alpha} \mathfrak{e}_\alpha\right\}$$

and by summing all the expressions of this form we get:

$$(20) \qquad \mathfrak{v} \times^{n-2} (\nabla \times \mathfrak{a}) = -(n-2)!\, \mathfrak{v} \cdot \{\nabla \mathfrak{a} - \mathfrak{a} \nabla\}$$

and from this, putting $\mathfrak{v} = \mathfrak{e}_i$:

$$\mathfrak{e}_i \times^{n-2} (\nabla \times \mathfrak{a}) = -(n-2)! \left\{\frac{\partial \mathfrak{a}}{\partial x_i} - \nabla a_i\right\}$$

which can be regarded as a particular case of § 9 (23).

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On a special polyadic of order n-p which can be derived from any p independent vectors in an n-dimensional space and which can be regarded as a generalization of the vector product