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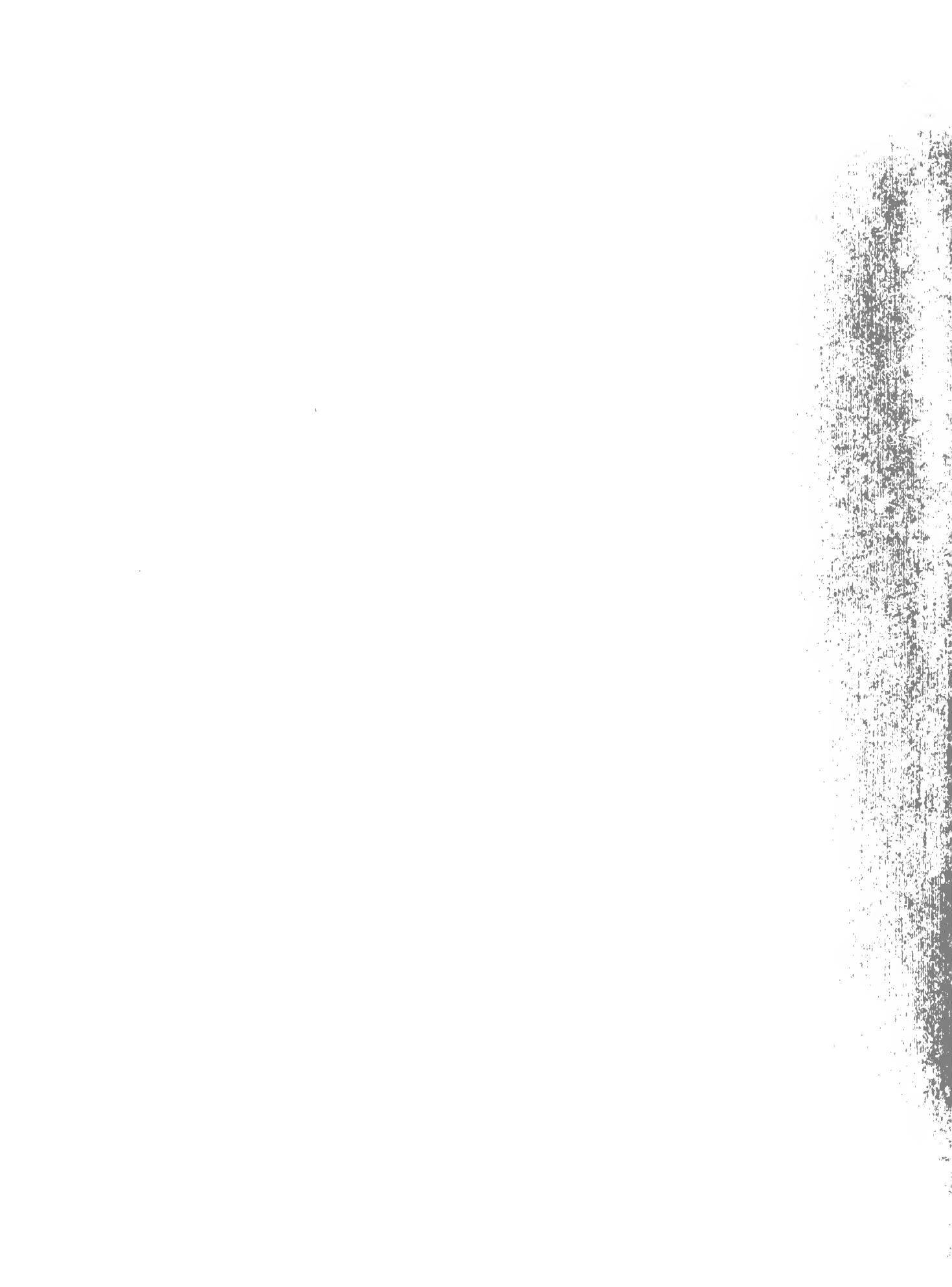
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On Sufficient Conditions for the Sum of Two  
Weak \* Closed Convex Sets to be Weak \* Closed

*M. Ali Khan*  
*Rajiv Vohra*



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College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

October, 1985

On Sufficient Conditions for the Sum of Two  
Weak \* Closed Convex Sets to be Weak \* Closed

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On Sufficient Conditions for the Sum of Two Weak \* Closed  
Convex Sets to be Weak \* Closed

by

M. Ali Khan and Rajiv Vohra<sup>†</sup>

September 1985

Abstract. We give a sufficient condition for the sum of two weak \* closed convex subsets of a strictly hypercomplete locally convex Hausdorff topological vector space to be weak \* closed. Our sufficient condition relates to the mutual position of the two sets and our result may be seen as a simple consequence of the Alaoglu-Bourbaki Theorem.

<sup>†</sup>This research was supported, in part, by a NSF grant to Brown University.



1.

The question as to when the sum of two closed sets is closed arises naturally in applied mathematics and is of interest in its own right. For subsets of an infinite dimensional space, answers to this question have been given by Choquet [1], Ky Fan [3] and Dieudonné [2]. Ky Fan, in particular, has shown that the sum of two weakly closed convex subsets of a locally convex topological vector space is weakly closed under a condition that takes account of the mutual position of the two sets. In this note, we offer another sufficient condition of this genre for subsets in the dual of a locally convex topological vector space. We also give simple examples which bring out the difference between our sufficient condition and those of the others. It is worth mentioning that our result is directly inspired by a problem in mathematical economics, see [4].

2.

Let  $(E, \tau)$  be a real, Hausdorff locally convex vector space and  $E'$  its topological dual, i.e., the space of all  $\tau$ -continuous linear functions on  $E$ .  $\sigma(E', E)$  denotes the weak \* topology on  $E'$ . Following [5, Definition 12-3-7], we say that  $E$  is strictly hypercomplete if for any  $S \subset E'$ ,  $S \cap U^\circ$  is weak \* compact implies  $S$  is weak \* closed, where  $U^\circ$  is the polar of any neighborhood of zero in  $E$ .

We can now state

Theorem. Let  $X$  and  $Y$  be two  $\sigma(E', E)$ -closed, convex subsets of  $E'$  such that  $X \cap (-Y + U^\circ)$  is  $\tau$ -equicontinuous for any neighborhood  $U$  of zero in  $E$ . If  $E$  is strictly hypercomplete, then  $X + Y$  is  $\sigma(E', E)$ -closed.

Proof. Let  $W = X+Y$  and for any neighborhood  $U$  of zero in  $E$ , let  $W_U = (X+Y) \cap U^\circ$ . Given Alaoglu-Bourbaki theorem, the strict hypercompleteness of  $E$  and the convexity of  $X$  and  $Y$ , we need only show that  $W_U$  is  $\sigma(E',E)$ -closed. Towards this end, pick any net  $\{w^\nu\}$  from  $W_U$  and such that the  $\sigma(E',E)$  limit of  $\{w^\nu\}$  is  $w$ . For each  $w^\nu$ , there exist  $x^\nu \in X$ ,  $y^\nu \in Y$  such that  $x^\nu + y^\nu = w^\nu$ . Under the hypothesis of the theorem, this implies that  $\{x^\nu\}$  lie in a  $\tau$ -equicontinuous set. By a second application of Alaoglu-Bourbaki, we can assert the existence of a subnet  $\{x^{\nu\rho}\}$  which tends in  $\sigma(E',E)$  to a limit, say  $x$ . Since  $X$  is  $\sigma(E',E)$ -closed,  $x \in X$ . Furthermore, since  $y^{\nu\rho} = w^{\nu\rho} - x^{\nu\rho}$ ,  $y^{\nu\rho}$  also tends in  $\sigma(E',E)$  to a limit, say  $y$ . Since  $Y$  is  $\sigma(E',E)$  - closed,  $y \in Y$ . Certainly  $y = w - x$  which implies that  $w \in W_U$  and the proof is finished. □

Corollary. The theorem is true if  $E$  is assumed, in addition, to be quasibarrelled and the condition  $X \cap (-Y+U^\circ)$  is  $\tau$ -equicontinuous is weakened to the requirement that it be  $\beta(E',E)$ -bounded, where  $\beta$  denotes the strong topology.

Proof. Given the definition of quasibarrelled [5; Definition 10-1-7 and Theorem 10-1-11], an obvious modification in the proof of the theorem yields the corollary. □

3.

We now present six examples to relate our result to previous work. Our first two examples are taken from Ky Fan.

Example 1. In the Euclidean plane  $\mathbb{R}^2$ , let  $X = \{(a_1, a_2) \in \mathbb{R}^2: a_1 > 0, a_1 a_2 \geq 1\}$  and  $Y = \{(a_1, a_2) \in \mathbb{R}^2: -\infty < a_1 < \infty, a_2 = 0\}$ . In the context

of our theorem  $X+Y$  cannot be shown to be closed since  $X \cap (Y+S)$  or  $Y \cap (-X+S)$  is not a compact set, where  $S$  denotes the closed unit ball in  $\mathbb{R}^2$ . Note that Ky Fan's reason for  $X+Y$  not being closed is that the interior of  $X^\circ$  does not intersect  $Y^\circ$ .

Example 2. Let  $X$  be as in Example 1 but  $Y = \{(a_1, a_2) \in \mathbb{R}^2: a_1 + a_2 = 0\}$ .  $X+Y$  is closed since  $X \cap (Y+kS)$  is compact for any positive real number  $k$ . This example also satisfies the sufficiency conditions of the results of Ky Fan and Dieudonné but not for that of Choquet.

Example 3. Let  $X = \{(a_1, a_2) \in \mathbb{R}^2: -\infty < a_1 < \infty, a_2 = 0\}$  and  $Y = \{(a_1, a_2) \in \mathbb{R}^2: -\infty < a_2 < \infty, a_1 = 0\}$ . Since  $X^\circ$  and  $Y^\circ$  have empty interiors in  $\mathbb{R}^2$ , we cannot apply Ky Fan's Theorem 2. Since  $X$  and  $Y$  are not subsets of a convex set that contains no straight line, we cannot apply Choquet's result either. However  $(X+Y)$  is closed as a consequence of our theorem, since  $X \cap (Y+S)$  is compact. Note that this example also satisfies the sufficiency conditions of Dieudonné's result.

Example 4. In the space  $\ell_\infty$ , let  $X$  and  $Y$  be the positive cone  $\{a \in \ell_\infty: a_i \geq 0\}$ .  $X^\circ = Y^\circ = \{a \in \ell_1: a_i \leq 0\}$ . Since the positive cone in  $\ell_1$  has an empty norm interior and the norm topology is identical to the Mackey topology  $\tau(\ell_1, \ell_\infty)$ , we cannot appeal to Ky Fan's theorem to assert the closedness of  $X+Y$ . Dieudonné's theorem does not apply since  $X$  and  $Y$  are not locally compact. Since  $\ell_\infty$  is not weakly complete, neither are  $X$  and  $Y$  and thus Choquet's result does not apply. However,  $X+Y$  is closed as a consequence of our theorem since  $X \cap (-Y+U^\circ)$  is  $\sigma(\ell^\infty, \ell_1)$  compact for any zero neighborhood  $U$  in  $\ell_1$ .

Next, we present an example that does not satisfy any of the sufficient conditions required by the results in [1], [2] or [3] but does satisfy those required for our theorem.

Example 5. In the Hilbert space  $\ell_2$ , let  $X = \{a \in \ell_2: a_i = 0, i \text{ even}; -\infty < a_i < \infty, i \text{ odd}\}$  and  $Y = \{a \in \ell_2: a_i = 0, i \text{ odd}; -\infty < a_i < \infty, i \text{ even}\}$ . Certainly  $X$  and  $Y$  are not subsets of a convex set that contains no straight line as required in [1] and  $X, Y$  are not locally compact in the weak topology,  $\sigma(\ell_2, \ell_2)$ , as required in [2]. Finally since  $X^\circ = Y$  and  $Y^\circ = X$ , the Mackey topology is identical to the norm topology, and the norm interior of  $X$  and  $Y$  is empty, the sufficiency condition for the result in [3] is violated. However  $X+Y$  is  $\sigma(\ell_2, \ell_2)$ -closed since  $X$  and  $Y$  are  $\sigma(\ell_2, \ell_2)$ -closed and  $Y \cap (-X+kU)$  is  $\sigma(\ell_2, \ell_2)$ -compact for any real positive number  $k$  and  $U$  the unit ball in  $\ell_2$ . This implies that  $Y \cap (-X+kU)$  is norm equicontinuous in  $\ell_2$ .

Finally, we present an example which does not satisfy any of the sufficient conditions required by the results in [1], [2], [3] or by those in our theorem but the sets are nevertheless closed.

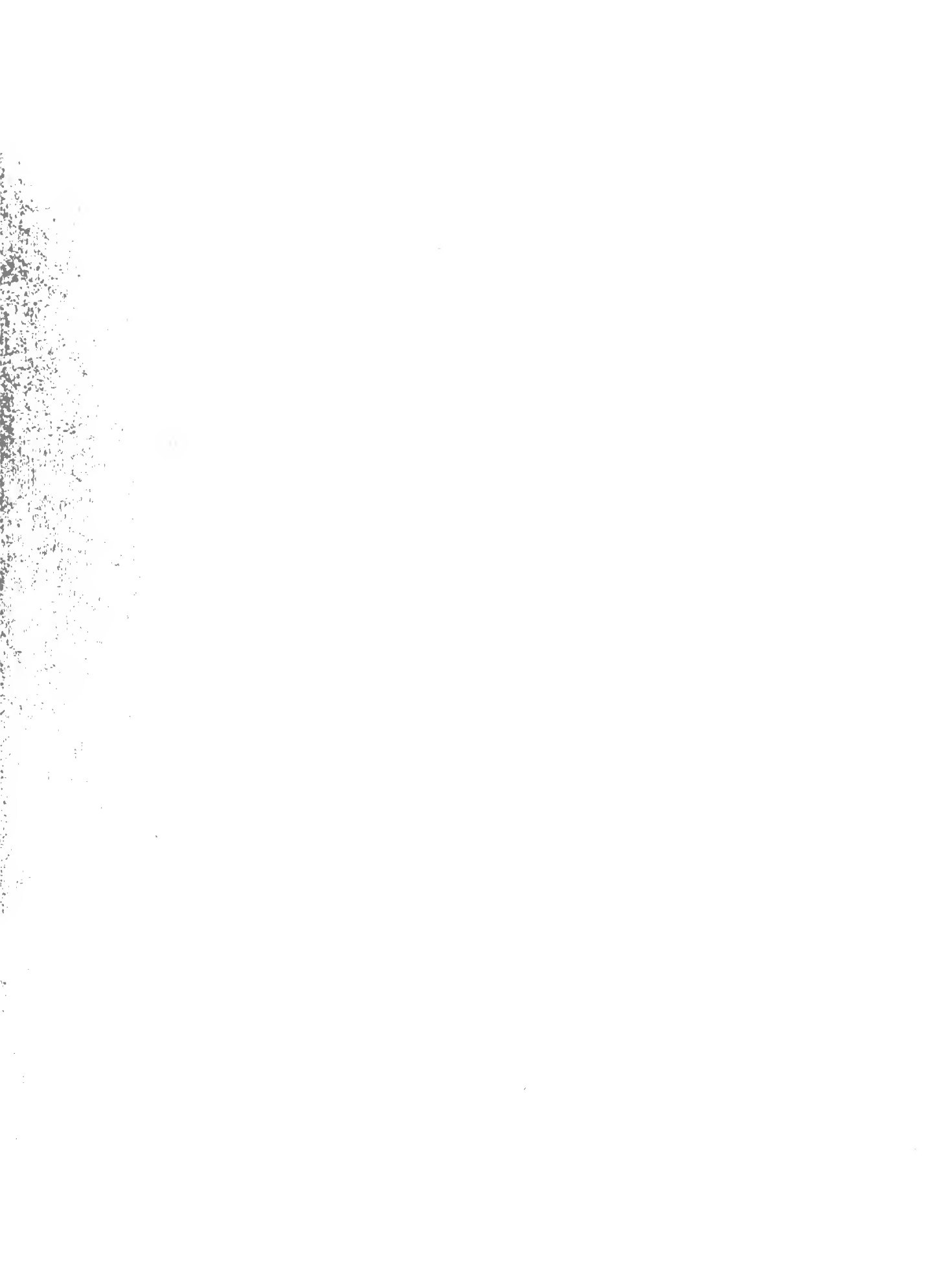
Example 6. Let  $X = \{(a_1, a_2) \in \mathbb{R}^2: a_1 \geq 0, a_2 \geq 0\}$  and  $Y = \{(a_1, a_2) \in \mathbb{R}^2: -\infty < a_1 < \infty, a_2 \geq 0\}$ . Certainly,  $Y$  is not a subset of a set that contains no straight line as required in [1]. Also, the intersection of the asymptotic cones of  $X$  and  $Y$  is not  $\{0\}$  as required in [2]. The interior of  $X^\circ$  does not intersect  $Y^\circ$  as required in [3]. Finally, it is clear that  $X \cap (-Y+S)$  is not relatively compact. Nevertheless a simple computation yields that  $X+Y$  is closed.

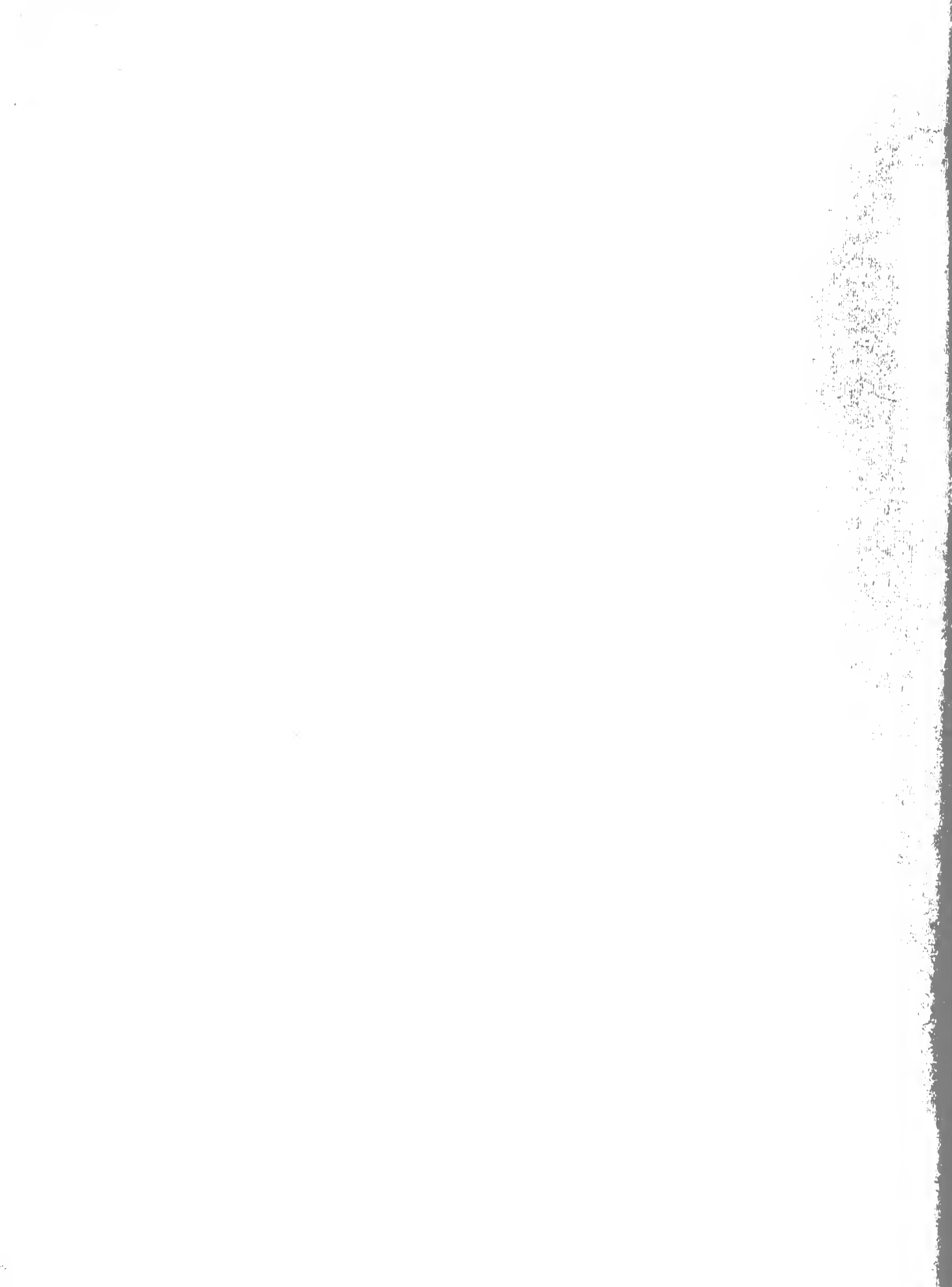
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