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On Symmetric Cournot-Nash Equilibrium Distributions in a Finite-Action, Atomless Game

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by

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Abstract. We show that in a finite-action, atomless game, every Cournot-Nash equilibrium distribution can be "symmetrized." This yields an elementary proof of a result of Mas-Colell.

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I. Introduction

In [4], Mas-Colell showed the existence of a Cournot-Nash equilibrium distribution (CNED) as a consequence of the Fan-Glicksberg theorem. Mas-Colell also showed the existence of a symmetric CNED in finite-action, atomless games as a consequence of the Kakutani fixed point theorem and results in the theory of integration of correspondences. These results consist, in particular, of Lyapunov's theorem on the range of a vector measure, Aumann's measurable selection theorem, as well as his theorem on the upper hemicontinuity of the integral of a correspondence with upper-hemicontinuous values; on all of this [1] is a standard reference.

In this note, we show that in a finite-action, atomless game every CNED can be "symmetrized" to yield a symmetric CNED. This allows us to deduce Mas-Colell's result on the existence of a symmetric CNED from his first result on the existence of a CNED. The proof of our result is elementary in the sense that it uses only Lyapunov's theorem on the convexity of the range of a scalar measure.

Section 2 recalls the model and presents the results. Section 3 gives the basic idea of the proof and Section 4 is devoted to the formalities of the proof. Section 5 concludes with a remark.

2. The Model and Results

We recall for the reader's convenience the basic definitions from [4]. Let A be a compact, metric space of actions, $\mathcal{M}$ the set of Borel probability measures on A endowed with the weak * topology and $\mathcal{U}_A$ is the space of continuous from $A \times \mathcal{M}$ into R and endowed with the supremum-norm topology. A game is a Borel probability measure on $\mathcal{U}_A$.

A Borel probability measure τ is said to be a <u>Cournot-Nash equilibrium distribution (CNED)</u> of the game μ if the marginal of τ on $\mathcal{U}_A$, $\tau_{\mathcal{U}}$, is μ and $\tau(B_\tau) = 1$ where $B_\tau = \{(a,u) \in A \times \mathcal{U}_A : u(a,\tau_A) \geq u(a,\tau_A)$ for all $a \in A\}$ and τ_A denotes the marginal of τ on A. τ is said to be a <u>symmetric Cournot-Nash equilibrium distribution</u> if τ is a CNED and there exists a measurable function h: $\mathcal{U}_A \to A$ such that τ (graph h) = 1. We shall say that every CNED τ can be <u>symmetrized</u> if there exists a symmetric CNED τ^s such that $B_{\tau^s} = B_\tau$.

We can now state

<u>Theorem</u>. <u>Every Cournot-Nash equilibrium distribution of a game μ with action set A can be symmetrized if μ is atomless and A is finite</u>.

This yields as a corollary

<u>Corollary (Mas-Colell)</u>: <u>A symmetric Cournot-Nash equilibrium distribution exists for a game μ with action set A whenever μ is atomless and A is finite</u>.

The Corollary is an easy consequence of our theorem and Theorem 1 of [4].

3. <u>Heuristics of the Proof</u>

We illustrate the basic idea of the proof of our theorem by considering an action set with two elements. The reader may wish to keep Figure 1 in mind as we go through the argument.

Let τ be the CNED of a game μ with action set $\{a_1, a_2\}$. Let the set B_τ of all pay-offs and corresponding pay-off maximizing actions be

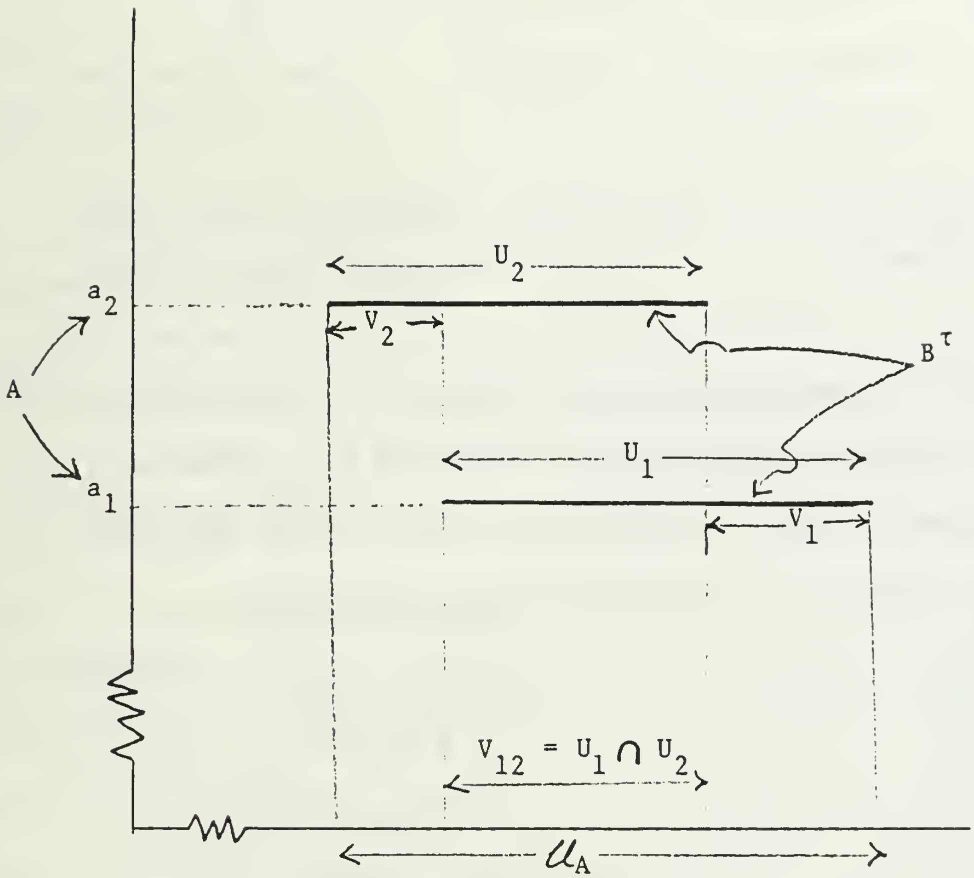

Figure 1

denoted by the set $(a_1 \times U_1) \cup (a_2 \times U_2)$. Unlike Figure 1, U_1 and U_2 need not necessarily be connected sets. Suppose, again unlike Figure 1, that $U_1 \cap U_2 = \phi$. Since $U_1 \cup U_2 = \mathcal{U}_A$, τ can be shown to be symmetric CNED simply by letting $h(u) = a_i$ for all $u \in U_i$, for all $i = 1, 2$. Certainly h is measurable and τ (graph h) = 1. Thus, in the case $U_1 \cap U_2 = \phi$, there is nothing to prove.

Suppose $U_1 \cap U_2 \neq \phi$. The basic idea in this case is to "disjointify" U_1 and U_2, i.e., to construct measurable subsets $U_i^* \subset U_i$ for all $i = 1, 2$, such that $U_1^* \cap U_2^* = \phi$. Since μ is atomless, this can be done in a number of ways but the important consideration is to do this in such a way that the marginal of τ on A, τ_A, does not change. Since B_τ depends only on τ_A, this ensures that B_τ does not change. We now briefly spell out the mechanics of such a procedure.

Let $V_i = U_i - U_j$, $i = 1, 2$, $j \neq i$, and $V_{12} = U_1 \cap U_2$. Find measurable subsets V_{12}^1, V_{12}^2 of V_{12} such that $V_{12}^1 \cap V_{12}^2 = \phi$, $V_{12}^1 \cup V_{12}^2 = V_{12}$ and $\mu(V_{12}^i) = \tau(a_i \times V_{12})$, $i = 1, 2$. Since $\sum_{i=1}^{2} \tau(a_i \times V_{12}) = \tau(\bigcup_{i=1}^{2} (a_i \times V_{12})) = \tau(\{a_1, a_2\} \times V_{12}) = \tau_A(V_{12}) = \mu_A(V_{12})$, Lyapunov's theorem on the range of an atomless scalar measure guarantees that V_{12}^1 and V_{12}^2 can be found. Now let $U_i^* = V_i \cup V_{12}^i$, $i = 1, 2$. These are the sets that work by letting $h: \mathcal{U}_A \to A$ be a function such that $h(u) = a_i$ for all $u \in U_i^*$, for all $i = 1, 2$. Now let $\tau^s(B) = \mu\{u \in \mathcal{U}_A : (h(u), u) \in B\}$ for any measurable subset B of $A \times \mathcal{U}_A$. τ^s is the symmetric CNED. The only point which needs to be checked is that $\tau_A^s = \tau_A$. But $\tau_A(\{a_i\}) = \tau(a_i \times \mathcal{U}_A) = \tau(a_i \times U_i) = \tau(a_i \times V_i) + \tau(a_i \times V_{12}) = \mu(V_i) + \mu(V_{12}^i) = \mu(V_i \cup V_{12}^i) = \mu(U_i^*) = \mu\{u \in \mathcal{U}_A : (h(u), u) \in (a_i \times \mathcal{U}_A)\} = \tau_A^s(\{a_i\})$.

4. Proof of the Theorem

We begin with an elementary lemma.

Lemma 1. *Let A_i (i = 1, ..., k) and B be arbitrary sets. Then* $\bigcup_{i=1}^{k} (A_i \times B) = ((\bigcup_{i=1}^{k} A_i)\times B)$.

Proof: Straightforward. ||

Our next lemma is a simple consequence of Lyapunov's theorem on the range of a scalar measure.

Lemma 2. *Let $(S, \mathcal{S}, \mu)$ be an atomless measure space. If $V \in \mathcal{S}$, $\mu(V) = \sum_{i=1}^{n} \lambda_i$ with $\lambda_i \geq 0$ for all i, there exist for all i = 1, ..., n, $V^i \in \mathcal{S}$ such that $V^i \cap V^j = \phi$ $(i \neq j)$, $\bigcup_{i=1}^{n} V_i = V$ and $\mu(V^i) = \lambda_i$.*

Proof: We shall prove the lemma by induction. The lemma is trivially true for n = 1. Assume it to be true for n = k and let $V \in \mathcal{S}$ with $\mu(V) = \sum_{i=1}^{k+1} \lambda_i$, $\lambda_i \geq 0$ for all i = 1, ..., k+1. If $\lambda_i = 0$ for any i, we are reduced to the case of n = k and the proof is completed by letting $V_i = \phi$ for that i. Thus, suppose $\lambda_i > 0$ for all i. Let $\lambda(1) = \sum_{i=1}^{k} \lambda_i / \sum_{i=1}^{k+1} \lambda_i$ and $\lambda(2) = 1 - \lambda(1)$. By Lyapunov's theorem [1, p. 45], we can find $V^{k+1} \in \mathcal{S}$ such that $\mu(V^{k+1}) = \lambda_{k+1}$. Since $(V-V^{k+1}) \in \mathcal{S}$, and $\mu(V-V^{k+1}) = \sum_{i=1}^{k} \lambda_i$, we use the induction hypothesis to complete the proof. ||

Before we present the proof of Theorem, we develop some notation. Let I denote the set {1, 2, ..., n} and P(I) the set of subsets of I, including the empty set. For any $\pi \in P(I)$, let π^c denote the complement of π in I. Let $P^m(I) = \{\pi \in P(I): m \in \pi\}$. We shall use the convention that a union over the empty set is the empty set. We also use the same notation for a point and a set consisting solely of that point.

Proof of Theorem

Let τ be the Cournot-Nash equilibrium distribution of the game μ. Let $U_i = \mathrm{proj}_{\mathcal{U}_A}(B_\tau \cap (a_i \times \mathcal{U}_A))$ for all $i \in I$

(1) $\bigcup_{i \in I} U_i = \mathcal{U}_A$

Certainly $U_i \subset \mathcal{U}_A$ for $i \in I$. On the other hand, let $u \in \mathcal{U}_A$. Certainly there exists $k \in I$ such that $u(a_k, \tau) \geq u(a_i, \tau)$. Then $(a_k, u) \in B_\tau$ and hence $u \in U_k$.

(2) $B_\tau = \bigcup_{i \in I} (a_i \times U_i)$

Certainly $(a_i \times U_i) \subset B_\tau$ for all $i \in I$. Now any element x of B_τ can be written as (a_i, u) for some $i \in I$ and some $u \in \mathcal{U}_A$. Hence $u \in U_i$ and $x \in (a_i \times U_i)$.

(3) $\tau(a_i \times U_i) = \tau(a_i \times \mathcal{U}_A)$

Since $(a_i \times U_i) \subset (a_i \times \mathcal{U}_A)$, certainly $\tau(a_i \times U_i) \leq \tau(a_i \times \mathcal{U}_A)$. Suppose there exists $i \in I$ such that strict inequality holds for that i. Then $1 = \tau(B_\tau) = \tau(\bigcup_{i \in I} (a_i \times \mathcal{U}_A)) = \tau(A \times \mathcal{U}_A)$, a contradiction to the fact that τ is a probability measure.

For any $\pi \in P(I)$, let $V_\pi = (\bigcap_{i \in \pi} U_i) - (\bigcup_{i \in \pi^c} U_i)$.

(4) (a) $\bigcup_{\pi \varepsilon P(I)} V_\pi = \mathcal{U}_A$, (b) $V_\pi \cap V_\sigma = \phi(\pi, \sigma \varepsilon P(I), \pi \neq \sigma)$, (c) $\bigcup_{\pi \varepsilon P^i(I)} V_\pi = U_i$.

For (a), pick $u \varepsilon \mathcal{U}_A$. Let $\sigma = \{i \ \varepsilon\ I: u \ \varepsilon\ U_i\}$. By (1), $\sigma \neq \phi$. Then $u \ \varepsilon\ V_\sigma$. On the other hand, $u \ \varepsilon \bigcup_{\pi \varepsilon P(I)} V_\pi$ implies that there exists $\sigma \ \varepsilon\ P(I)$, $\sigma \neq \phi$ such that $u \ \varepsilon\ V_\sigma$. Hence $u \ \varepsilon\ U_i$ for all $i \ \varepsilon\ \sigma$ and hence, by (1), $u \ \varepsilon\ \mathcal{U}_A$. For (b), suppose there exists π, σ in $P(I)$ such that $\pi \neq \sigma$ and $V_\pi \cap V_\sigma \neq \phi$. Since V_π and V_σ are nonempty, π and σ are nonempty. Then there exists $i \ \varepsilon\ \pi$, $i \not\varepsilon\ \sigma$. Now $u \ \varepsilon\ V_\pi \cap V_\sigma$ implies $u \ \varepsilon\ U_i$. Since $i \ \varepsilon\ \sigma^c$, $u \not\varepsilon\ V_\sigma$ which is a contradiction. For (c), pick $u \ \varepsilon \bigcup_{\pi \varepsilon P^i(I)} V_\pi$. Then there exists $\pi \ \varepsilon\ P^i(I)$ such that $u \ \varepsilon\ V_\pi$. Since $i \ \varepsilon\ \pi$, $u \ \varepsilon\ U_i$. On the other hand, for any $u \ \varepsilon\ U_i$, let $\sigma = \{j \ \varepsilon\ I, u \ \varepsilon\ U_j\}$ and $\pi = \{i\} \cup \sigma$. Certainly $u \ \varepsilon\ V_\pi$ and $\pi \ \varepsilon\ P^i(I)$.

(5) For any $\pi \ \varepsilon\ P(I)$, $\exists$ measurable $V_\pi^i (i \varepsilon I)$, $V_\pi^i \cap V_\pi^j = \phi (i \neq j)$, $\bigcup_{i \varepsilon \pi} V_\pi^i = V_\pi$

$$\text{and } \mu(V_\pi^i) = \tau(a_i \times V_\pi)$$

Observe that $\mu(V_\pi) = \tau_{\mathcal{U}_A}(V_\pi) = \tau(A \times V_\pi) = \tau((\bigcup_{i \varepsilon I} a_i) \times V_\pi)$ which, by Lemma 1, equals $\tau((\bigcup_{i \varepsilon I} a_i \times V_\pi)) = \sum_{i \varepsilon I} \tau(a_i \times V_\pi)$. We can now apply Lemma 2 to complete the proof of (5).

Now let $U_i^* = \bigcup_{\pi \varepsilon P^i(I)} V_\pi^i$.

(6) For all $i \ \varepsilon\ I$, (a) $U_i^* \subset U_i$, (b) $U_i^* \cap U_j^* = \phi \ (i \neq j)$, (c) $\bigcup_{i \varepsilon I} U_i^* = \mathcal{U}_A$

To see (a), pick $u \ \varepsilon\ U_i^*$. Then there exists $\pi \ \varepsilon\ P^i(I)$ such that $u \ \varepsilon\ V_\pi^i$. This implies $u \ \varepsilon\ V_\pi$. Since $i \ \varepsilon\ \pi$, $u \ \varepsilon\ U_i$. (b) follows from the fact that for $i \neq j$, $V_\pi^i \cap V_\pi^j = \phi$ on the one hand, and from

$V_\pi \cap V_\sigma = \phi$ for $\pi \neq \sigma$ on the other. For (c), note that

$$\bigcup_{i\epsilon I} U_i^* = \bigcup_{i\epsilon I} \bigsqcup_{\pi\epsilon P^i(I)} V_\pi^i = \bigcup_{i\epsilon I} \bigsqcup_{\pi\epsilon P(I)} V_\pi^i = \bigcup_{\pi\epsilon P(I)} \bigcup_{i\epsilon I} V_\pi^i = \bigcup_{\pi\epsilon P(I)} V_\pi =$$

$\mathcal{U}_A$, the last step from (4a).

(7) $\mu(U_i^*) = \tau(a_i \times U_i)$ for all $i \in I$.

The left hand side equals $\mu(\bigsqcup_{\pi\epsilon P^i(I)} (V_\pi^i))$. Since $V_\pi^i \subset V_\pi$ by (5), and $V_\pi \cap V_\sigma = \phi$ for $\pi \neq \sigma$ by (4c), this equals $\Sigma_{\pi\epsilon P^i(I)} \mu(V_\pi^i)$. By (5), this equals $\Sigma_{\pi\epsilon P^i(I)} \tau(a_i \times V_\pi)$ which equals $\tau(\bigsqcup_{\pi\epsilon P^i(I)} (a_i \times V_\pi))$. By Lemma 1, this can be written as $\tau(a_i \times \bigsqcup_{\pi\epsilon P^i(I)} V_\pi)$ and hence by (4b) as $\tau(a_i \times U_i)$.

We are now ready to construct our symmetric Cournot-Nash equilibrium distribution. Let $h: \mathcal{U}_A \to A$ be such that $h(u) = a_i$ for all $u \in U_i^*$, for all $i \in I$. Since V_π^i are measurable, U_i^* are measurable. Moreover, from (vi), h is a well-defined function. Now let τ^s be a measure on $A \times \mathcal{U}_A$ such that for any measurable B, $\tau^s(B) = \mu\{u \in \mathcal{U}_A: (h(u),u) \in B\}$. Given measurability of h and the identity map, τ^s is well-defined. Also

$$\tau^s(\text{graph } h) = \mu\{u \in \mathcal{U}_A: (h(u),u) \in (\text{graph } h)\} = \mu\{u \in \mathcal{U}_A\} = 1.$$

All that remains to be shown is that τ^s is a Cournot-Nash equilibrium distribution. Towards this end, we first show that $\tau^s_{\mathcal{U}_A} = \mu$. Pick any measurable subset W of $\mathcal{U}_A$. Then $\tau^s_{\mathcal{U}_A}(W) = \tau^s(A \times W) = \mu\{u \in \mathcal{U}_A: (h(u),u) \in A \times W\} = \mu\{u \in (\mathcal{U}_A \cap W)\} = \mu(W)$.

Next, we show $\tau^s_A = \tau_A$. Pick any measurable subset of A. If this set is empty, there is nothing to be shown. Hence, let this set be

$\bigcup_{i\in\pi} a_i$ for some $\pi \in P(I)$. Now $\tau_A^s(\bigcup_{i\in\pi} a_i) = \tau^s(\bigcup_{i\in\pi} a_i \times \mathcal{U}_A) =$

$\mu\{u \in \mathcal{U}_A: (h(u),u) \in ((\bigcup_{i\in\pi} a_i) \times \mathcal{U}_A)\} = \mu\{u \in \mathcal{U}_A: h(u) = a_i,$

$i \in \pi\} = \mu(\bigcup_{i\in\pi} h^{-1}(a_i)) = \sum_{i\in\pi} \mu(U_i^*)$. Now

$$\begin{aligned}
\sum_{i\in\pi} (U_i^*) &= \sum_{i\in\pi} \tau(a_i \times U_i) && \text{(by (7))} \\
&= \sum_{i\in\pi} \tau(a_i \times \mathcal{U}_A) && \text{(by (2))} \\
&= \tau(\bigcup_{i\in\pi} (a_i \times \mathcal{U}_A)) \\
&= \tau((\bigcup_{i\in\pi} a_i) \times \mathcal{U}_A) && \text{(by Lemma 1)} \\
&= \tau_A(\bigcup_{i\in\pi} a_i).
\end{aligned}$$

We are done.

Since $\tau_A^s = \tau_A$ and since B_τ depends only on τ_A, $B_{\tau^s} = B_\tau$. Thus to show $\tau^s(B_{\tau^s}) = 1$. But by the definition of h, graph $h \subset B_\tau$. Since $\tau^s(\text{graph } h) = 1$, $\tau^s(B_{\tau^s}) = \tau^s(B_\tau) = 1$. The proof of the theorem is complete. ||

5. Concluding Remark

In [2, 3], the authors present an alternative formulation of Mas-Colell's result in games where pay-offs are represented by preference relations or by functions which are upper-semicontinuous in actions. We remark that the theorem proved here applies to that generalized set-up.

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