

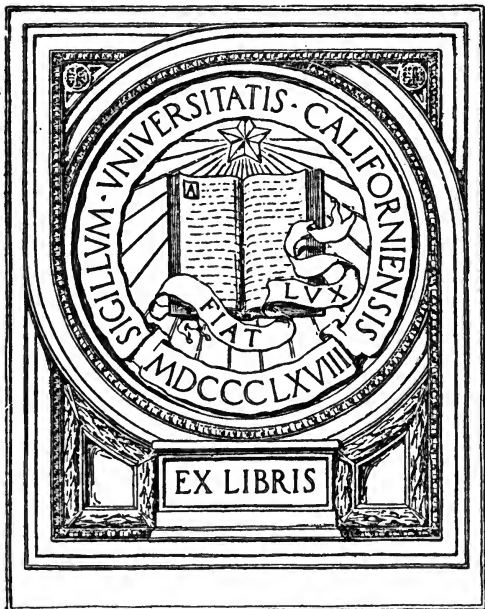
UC-NRLF



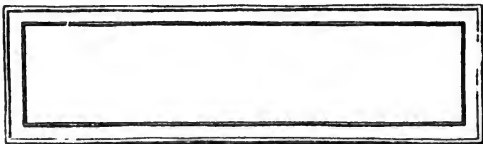
\$B 278 797

YB 17296

IN MEMORIAM
FLORIAN CAJORI



EX LIBRIS

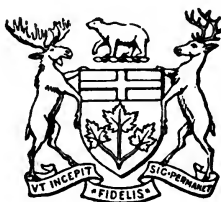


Florian Cajori,

THE ONTARIO

HIGH SCHOOL GEOMETRY

THEORETICAL



BY

A. H. McDOUGALL, B.A.

PRINCIPAL OTTAWA COLLEGIATE INSTITUTE

Authorized by the Minister of Education for Ontario

TORONTO

THE COPP, CLARK COMPANY, LIMITED

Copyright, Canada, 1910, by THE COPP, CLARK COMPANY, LIMITED,
Toronto, Ontario.

FIRST EDITION, 1910.

REPRINTED, 1911, 1912, 1913, 1914.

QA453
M3

PREFACE

The Ontario High School Geometry is intended to cover the course in Theoretical Geometry, begun in the Lower School and completed in the Middle School, as defined in the Programme of Studies for High Schools and Collegiate Institutes of the Province of Ontario.

In deference to the wish of the teachers of mathematics of the Province, this Geometry is divided into Books with numbered propositions.

While the theoretical course is complete in itself, it is assumed that its study has been preceded by the usual course in drawing and measurement. A considerable number of practical problems are given in the exercises. These should be worked out carefully, and, in fact, all diagrams should be accurately and neatly made.

The book contains an abundant supply of carefully selected and graded exercises. Those given in sets throughout the Books will be found suitable for the work of average classes, and just about sufficient in number to fix the subject-matter of the propositions in the minds of the pupils. All the problems contained in the miscellaneous collections at the ends of the Books could be worked through by a few of the best pupils only, and should be used also by the teachers as a store from which to draw suitable material for review purposes from time to time.

While the requirements of class-work have been constantly kept in mind in the choice of proofs, it should not be assumed that other proofs, just as good, cannot in many cases be given.

Students should be constantly encouraged to work out methods of their own, and to keep records of the best in their note books.

Symmetry has been used to an unusual extent in giving a more concise form to the proofs of constructions.

The treatment of parallels, in accord with the method of many of the best English text-books, is based on Playfair's Axiom.

Tangents are treated both by the method of limits and as lines which meet the circle in only one point.

Areas of triangles and parallelograms are compared with rectangles, thereby not only giving a simple method of treatment, but also promoting facility in numerical computations.

Similarly, the treatment of proportion is correlated with the algebraic knowledge of the pupil.

OTTAWA, June, 1910.

SYMBOLS AND ABBREVIATIONS

The following symbols and abbreviations are used :—

Fig.	Figure.
Const.	Construction.
Hyp.	Hypothesis.
Cor.	Corollary.
<i>e.g.</i>	<i>exempli gratia</i> , for example.
<i>i.e.</i>	<i>id est</i> , that is.
p.	page.
∴	because, since.
∴	therefore.
rt.	right.
st.	straight.
∠, ∠s, ∠d	angle, angles, angled.
△, △s	triangle, triangles.
, s	parallel, parallels.
gm, gms	parallelogram, parallelograms.
sq., sqs.	square, squares.
AB ²	the square on AB.
rect.	rectangle.
AB.CD	the rectangle contained by AB and CD.
AB : CD, or	$\frac{AB}{CD}$ the ratio of AB to CD.
+	plus, together with.
-	minus, diminished by.
⊥	is perpendicular to, a perpendicular.
=	is equal to, equals.
>	is greater than.
<	is less than.
≡	is congruent to, congruent.
	is similar to, similar.

Digitized by the Internet Archive
in 2007 with funding from
Microsoft Corporation

CONTENTS

BOOK I	PAGE
PRELIMINARY DEFINITIONS AND EXPLANATIONS . . .	1
GEOMETRICAL REASONING	3
ANGLES AND TRIANGLES	8
FIRST CASE OF THE CONGRUENCE OF TRIANGLES . .	16
SECOND CASE OF THE CONGRUENCE OF TRIANGLES .	22
CONSTRUCTIONS	25
PARALLEL STRAIGHT LINES	35
TRIANGLES	45
THIRD CASE OF THE CONGRUENCE OF TRIANGLES . .	54
THE AMBIGUOUS CASE IN THE COMPARISON OF TRIANGLES	56
INEQUALITIES	59
PARALLELOGRAMS	66
CONSTRUCTION	74
LOCI	75
MISCELLANEOUS EXERCISES	83
 BOOK II	
AREAS OF PARALLELOGRAMS AND TRIANGLES . . .	93
CONSTRUCTIONS	110
AREAS OF SQUARES	116
MISCELLANEOUS EXERCISES	134

	PAGE
BOOK III	
THE CIRCLE	141
CONSTRUCTIONS	143
ANGLES IN A CIRCLE	152
TANGENTS AND CHORDS	169
CONSTRUCTION	173
ANGLE BETWEEN CHORD AND TANGENT	177
CONSTRUCTIONS	182
CONTACT OF CIRCLES	196
MISCELLANEOUS EXERCISES	200
BOOK IV	
RATIO AND PROPORTION	213
CONSTRUCTIONS	226
BISECTOR THEOREMS	230
SIMILAR TRIANGLES	236
GEOMETRIC MEANS	244
RECTANGLES	247
PROOF OF PYTHAGOREAN THEOREM BY PROPORTION	249
CHORDS AND TANGENTS	252
MISCELLANEOUS EXERCISES	258
BOOK V	
AREAS OF SIMILAR FIGURES	271
CONSTRUCTIONS	274
AREAS OF SIMILAR POLYGONS	278
CONSTRUCTIONS	280
ARCS AND ANGLES	284
ANALYSIS OF A PROBLEM—COMMON TANGENTS	285
MISCELLANEOUS EXERCISES	289
INDEX	299

THEORETICAL GEOMETRY

BOOK I

PRELIMINARY DEFINITIONS AND EXPLANATIONS

1. A **point** is that which has position but no size.

The position of a point on the blackboard, or on paper, is represented by a mark. This mark has some small size and therefore only roughly represents the idea of a point.

2. A **line** is that which has length but neither breadth nor thickness.

Again, the mark that we use to represent a line has breadth and some small thickness, and consequently, only roughly represents the idea.

The intersection of two lines is a point.

3. Lines may be either straight or curved.

The following property distinguishes straight lines from curved lines and may be used as the definition of a straight line:—

Two straight lines cannot have any two points of one coincide with two points of the other without the lines coinciding altogether.

This is sometimes stated as follows:—Joining two points there is always one and only one straight line.

It follows from this definition that **two straight lines cannot enclose a space.**

Can the circumferences of two equal circles coincide in two points without coinciding altogether?

4. A **surface** is that which has length and breadth but no thickness.

A sheet of tissue paper has length and breadth and very little thickness. It thus roughly represents the idea of a surface. In fact the sheet of paper has two well-defined surfaces separated by the substance of the paper.

The boundary between two parts of space is a surface.

5. Surfaces may be either plane or curved.

The following property distinguishes plane surfaces from curved surfaces and may be used as the definition of a plane surface:—

The straight line joining any two points on a plane surface lies wholly on that surface.

Give examples of curved surfaces on which straight lines may be drawn in certain directions. Notice the force of the word “any” in the definition above.

6. A **solid** is that which has length, breadth and thickness.

7. Any combination of points, lines, surfaces and solids is called a **figure**.

8. **Geometry** is the science which investigates the properties of figures and the relations of figures to one another.

9. In **Plane Geometry** the figure, or figures, considered in each proposition are confined to one plane, while **Solid Geometry** treats of figures the parts of which are not all in the same plane.

Plane Geometry is also called Geometry of Two Dimensions (length and breadth), and Solid Geometry is called Geometry of Three Dimensions (length, breadth and thickness).

GEOMETRICAL REASONING

10. Two general methods of investigating the properties or relations of figures may be distinguished as the Practical Method and the Theoretical Method.

Some properties may be tested by measurement, paper-folding, etc., while in the same or other cases it may be shown that the property follows as a necessary result from others that are already known to be true.

The Theoretical Method, has certain advantages over the Practical method. Measurements, etc., are never exact, and in many cases cannot be made directly; but in the Theoretical Method, starting from certain simple statements, called **axioms**, the truth of which is self-evident, or, it may be in some cases, assumed, the consequent statements follow with absolute certainty.

The Practical Method is also known as the Inductive Method of Reasoning, and the Theoretical Method as the Deductive Method.

11. Figures may be compared by making a tracing of one of them and fitting the tracing on the other. In many cases the process may be made a mental operation and the comparison made with absolute certainty by means of the following axiom:—

A figure may be, actually or mentally, transferred from one position to another without change of form or size.

When two figures are shown to be exactly equal in all respects by supposing one to be made to fit exactly on the other, the proof is said to be by the **method of superposition**.

Figures which exactly fill the same space are said to **coincide** with each other.

12. In general a **proposition** is that which is stated or affirmed for discussion.

In mathematics a **proposition** is a statement of either a truth to be demonstrated or of an operation to be performed. It is called a **theorem** when it is something to be proved, and a **problem** when it is a construction to be made.

Example of Theorem:—If two straight lines cut each other, the vertically opposite angles are equal.

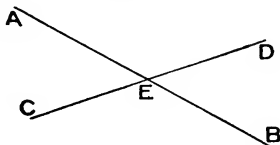
Example of Problem:—It is required to bisect a given straight line.

13. Theorems are commonly stated in two ways:—First, the **General Enunciation**, in which the property is stated as true for all figures of a class, but without naming any particular figure, as in the first example given in § 12; second, the **Particular Enunciation**, in which the theorem is stated to be true of the particular figure in a certain diagram.

Similarly general and particular enunciations are commonly given for problems.

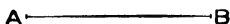
Examples of Particular Enunciation:—

1. Let **AB** and **CD** be two st. lines cutting at **E**.



It is required to show that $\angle AEC = \angle BED$, and that $\angle AED = \angle BEC$.

2. Let **AB** be a given st. line.



It is required to bisect **AB**.

14. In general, the enunciation of a theorem consists of two parts: the hypothesis and the conclusion.

The **hypothesis** is the formal statement of the conditions that are supposed to exist, *e.g.*, in the first example of § 12, "If two straight lines cut each other."

The **conclusion** is that which is asserted to follow necessarily from the hypothesis, *e.g.*, "the vertically opposite angles are equal to each other."

Commonly, the hypothesis of a theorem is stated first, introduced by the word "if," and the two parts hypothesis and conclusion are separated by a comma. Sometimes, however, the two parts are not so formally

distinguished, *e.g.*, in the proposition:—The angles at the base of an isosceles triangle are equal to each other. In order to show the two parts, this statement may be changed as follows:—If a triangle has two sides equal to each other, the angles opposite these equal sides (or angles at the base) are equal to each other.

15. The demonstration of a theorem depends either on definitions and axioms, or on other theorems that have been previously shown to be true.

The following are some of the axioms commonly used in geometrical reasoning:—

1. Things that are equal to the same thing are equal to each other.

If $A = B$, $B = C$, $C = D$, $D = E$ and $E = F$, what about A and F ?

2. If equals be added to equals the sums are equal.

A	—————	C	—————
B	—————	D	—————

Thus if A , B , C , D be four st. lines such that $A = B$ and $C = D$, then the sum of A and $C =$ the sum of B and D .

Exercise:—Mark four successive points A , B , C , D on a st. line such that $AB = CD$. Show that $AC = BD$.

3. If equals be taken from equals the remainders are equal.

Give example.

Exercise:—Mark four successive points **A**, **B**, **C**, **D** on a st. line such that $AC = BD$. Show that $AB = CD$.

4. If equals be added to unequals the sums are unequal, the greater sum being obtained from the greater unequal.

Give example. Show also, by example, that if unequals be added to unequals the sums may be either equal or unequal.

5. If equals be taken from unequals the remainders are unequal, the greater remainder being obtained from the greater unequal.

6. Doubles of the same thing, or of equal things, are equal to each other.

7. Halves of the same thing, or of equal things, are equal to each other.

8. The whole is greater than its part, and equal to the sum of all its parts.

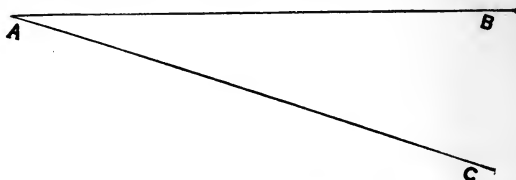
Give examples.

9. Magnitudes that coincide with each other, are equal to each other.

These simple propositions, and others that are also plainly true, may be freely used in proving theorems.

ANGLES AND TRIANGLES

16. **Definitions.**—When two straight lines are drawn from a point they are said to form an **angle**.



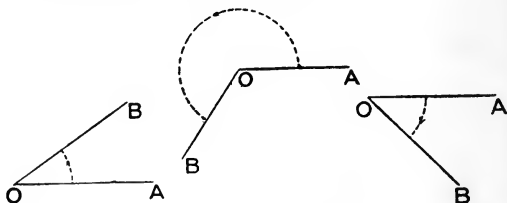
The point from which the two lines are drawn is called the **vertex** of the angle.

The two lines are called the **arms** of the angle.

The angle in the figure may be called the angle **BAC**, or the angle **CAB**. The letter at the vertex must be the middle one in reading the angle.

The single letter at the vertex is sometimes used to denote the angle when there can be no doubt as to which angle is meant.

17. Suppose a straight line **OB** to be fixed, like a rigid rod on a pivot at the point **O**, and be free to rotate in the plane of the paper.



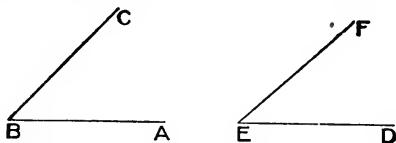
If the line **OB** start from any position **OA**, it may rotate in either of two directions—that in which the hands of a clock rotate, or in the opposite.

When **OB** starts from **OA** and stops at any position an angle is formed with **O** for its vertex and **OA** and **OB** for its arms.

18. An angle is said to be positive or negative according to the direction in which the line that traces out the angle is supposed to have rotated. The direction contrary to that in which the hands of a clock rotate is commonly taken as positive.

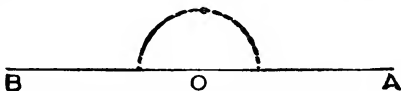
19. The magnitude of an angle depends altogether on the amount of rotation, and is quite independent of the lengths of its arms.

20. If we wish to compare two angles **ABC** and **DEF** we may suppose the angle **ABC** to be placed on



the angle **DEF** so that **B** falls on **E** and **BA** along **ED**. The position of **BC** with respect to **EF** will then show which of the angles is the greater and by how much it is greater than the other.

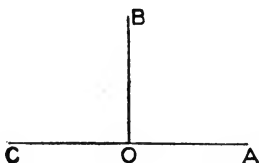
21. Definition.—When a revolving line **OB** has made half of a complete revolution from the initial position **OA** the angle formed is a **straight angle**.



The arms of a straight angle are thus in the same straight line and extend in opposite directions from

the vertex. At the point O , in the diagram, there are two straight angles on opposite sides of the straight line AOB , the two straight angles making up the complete revolution.

22. Definition.—If a straight line, starting from OA , rotates in succession through two equal angles AOB ,



BOC , the sum of which is a straight angle, each of these angles is called a **right angle**.

A right angle is thus one-half of a straight angle, or one-quarter of a complete revolution.

Each arm of a right angle is said to be **perpendicular** to the other arm.

What is a *vertical* line? a *horizontal* line?

An angle which is less than a right angle is called an **acute angle**.

An angle which is greater than a right angle is called an **obtuse angle**.

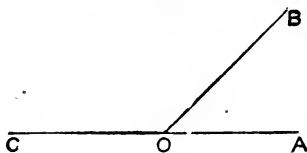
23. If a right \angle be divided into ninety equal parts, each of these parts is called a degree.

$$\text{Thus 1 rt. } \angle \quad = 90^\circ,$$

$$1 \text{ st. } \angle \quad = 180^\circ$$

$$1 \text{ revolution} = 360^\circ.$$

24. Let a st. line starting from OA revolve through two successive \angle s



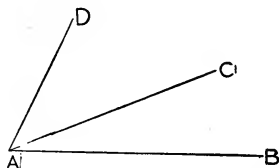
AOB , BOC such that OC is in the same st. line with OA , but in the opposite direction from the point O , and consequently AOC is a st. \angle .

$$\therefore \angle AOB + \angle BOC = \text{the st. } \angle AOC,$$

$$\therefore \angle AOB + \angle BOC = 2 \text{ rt. } \angle \text{s.}$$

Thus the angles which one straight line makes with another on the same side of that other are together equal to two right angles.

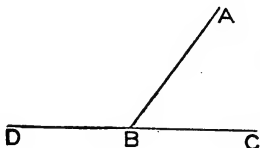
25. **Definition.**—When two angles have the same vertex and a common arm, and the remaining arms on opposite sides of the common arm, they are said to be **adjacent angles**.



Thus BAC and CAD are adjacent angles having the same vertex A and the common arm AC .

But angles BAD and CAD , with the same vertex and the common arm AD are not adjacent angles.

26. Let the adjacent \angle s ABC , ABD be together equal to two rt. \angle s.



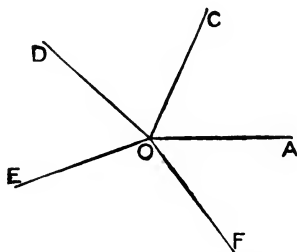
$\angle ABD + \angle ABC = \text{two rt. } \angle\text{s} = \text{a st. } \angle.$

That is, $\angle DBC$ is a st. \angle ,

and \therefore line DBC is a st. line.

Thus, if two adjacent angles are together equal to two right angles, the exterior arms of the angles are in the same straight line.

27. Let a st. line OB , starting from the position OA , and rotating in the positive direction, trace out the successive \angle s: AOC , COD , DOE , EOF , FOA .

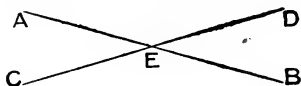


The sum of the successive \angle s is a complete revolution, and therefore equal to four rt. \angle s.

Thus, if any number of straight lines meet at a point, the sum of the successive angles is four right angles.

THEOREM 1

Each of the angles formed by two intersecting straight lines is equal to the vertically opposite angle.



Hypothesis.—The two st. lines AB, CD cut each other at E.

To prove that (1) $\angle AEC = \angle BED$,
(2) $\angle AED = \angle BEC$.

Proof.— \because CED is a st. line,
 $\angle AEC + \angle AED =$ two rt. \angle s.
 \because AEB is a st. line,
 $\angle AED + \angle DEB =$ two rt. \angle s.
 $\therefore \angle AEC + \angle AED = \angle AED + \angle DEB$.

From each of these equals take away the common $\angle AED$ and the remainders must be equal to each other.

$$\therefore \angle AEC = \angle DEB.$$

In the same manner it may be shown that $\angle AED = \angle BEC$.

28. **Definitions.**—When two angles are such that their sum is two right angles, they are said to be **supplementary** angles, or each angle is said to be the **supplement** of the other.

If two \angle s are equal, what about their supplementary \angle s?

When two angles are such that their sum is one right angle, they are said to be **complementary** angles, or each angle is said to be the **complement** of the other.

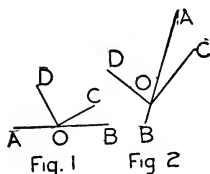
29.—Exercises

1. If one of the four \angle s made by two intersecting st. lines be 17° , find the number of degrees in each of the other three.

2. Two st. lines ABD , CBE cut at B , and $\angle ABC$ is a rt. \angle . Prove that the other \angle s at B are also rt. \angle s.

3. If in the figure of Theorem 1 the $\angle AEC = \frac{2}{3} \angle AED$, find the number of degrees in each \angle of the figure.

4.



$\angle DOC$ is a rt. \angle , and through the vertex O a st. line AOB is drawn.

Prove that:—

In Fig. 1, $\angle BOC + \angle AOD =$ a rt. \angle .

In Fig. 2, $\angle BOC - \angle AOD =$ a rt. \angle .

5. In the diagram,

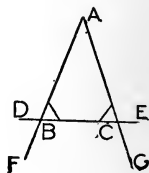
$$\angle ABC = \angle ACB.$$

Prove that

$$(1) \angle ABD = \angle ACE,$$

$$(2) \angle FBC = \angle GCB,$$

$$(3) \angle DBF = \angle ECG.$$

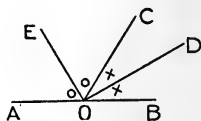


6. In the diagram,

AOB is a st. line,

$$\angle COD = \angle DOB \text{ and}$$

$$\angle AOE = \angle EOC.$$

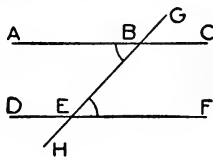


Prove that $\angle EOD$ is a rt. \angle , and that $\angle AOE$ is the complement of $\angle BOD$.

7. E is a point between A and B in the st. line AB ; DE , FE are drawn on opposite sides of AB and such that $\angle DEA = \angle FEB$. Show that DEF is a st. line.

8. Four st. lines, OA, OB, OC, OD , are drawn in succession from the point O , and are such that $\angle AOB = \angle COD$ and $\angle BOC = \angle DOA$. Show that AOC is a st. line, and also that BOD is a st. line.

9. In the diagram, $ABC, DEF, GBEH$ are st. lines and $\angle ABE = \angle BEF$.



Prove that

- (1) $\angle CBE = \angle BED$,
- (2) $\angle GBC = \angle DEH$,
- (3) $\angle ABG = \angle BED$,
- (4) $\angle s$ CBE, BEF are supplementary,
- (5) $\angle s$ ABE, BED are supplementary.

30. **Definitions.**—A figure formed by straight lines is called a **rectilinear figure**.

The figure formed by three straight lines which intersect one another is called a **triangle**.

The three points of intersection are called the **vertices** of the triangle.

The lines between the vertices of the triangle are called the **sides** of the triangle.

31. Figures that are equal in all respects, so that one may be made to fit the other exactly, are said to be **congruent**.

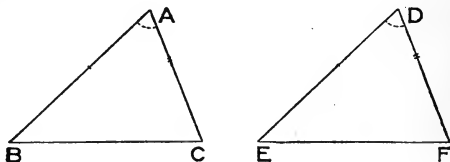
The sign \equiv is used to denote the congruence of figures.



FIRST CASE OF THE CONGRUENCE OF TRIANGLES

THEOREM 2

If two triangles have two sides and the contained angle of one respectively equal to two sides and the contained angle of the other, the two triangles are congruent.



Hypothesis.— $\triangle ABC$ and $\triangle DEF$ are two \triangle s having $AB = DE$, $AC = DF$ and $\angle A = \angle D$.

To prove that (1) $BC = EF$,
 (2) $\angle B = \angle E$,
 (3) $\angle C = \angle F$,
 (4) area of $\triangle ABC =$ area of $\triangle DEF$;
 and, hence, $\triangle ABC \equiv \triangle DEF$.

Proof.—Let $\triangle ABC$ be applied to $\triangle DEF$ so that vertex A falls on vertex D and AB falls along DE .

$\therefore AB = DE$,

\therefore vertex B must fall on vertex E .

$\therefore \angle A = \angle D$,

$\therefore AC$ must fall along DF ,

and \therefore , as $AC = DF$,

the vertex C must fall on the vertex F .

$\therefore \triangle ABC$ coincides with $\triangle DEF$.

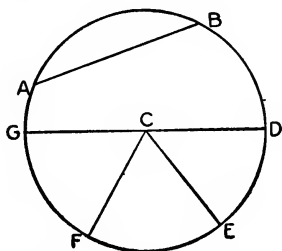
and $\therefore \triangle ABC \equiv \triangle DEF$.

32. **Definitions.**—A closed figure formed by four straight lines is called a **quadrilateral**.

In a quadrilateral a straight line joining two opposite vertices is called a **diagonal**.

A quadrilateral having its four sides equal to each other is called a **rhombus**.

A **circle** is a figure consisting of one closed curved line, called the **circumference**, and is such that all straight lines drawn from a certain point within the figure, called the **centre**, to the circumference are equal to each other.



In a circle a st. line drawn from the centre to the circumference is called a **radius**. (Plural—radii.)

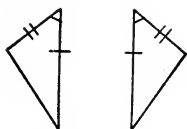
A st. line, as AB , joining two points in the circumference is called a **chord**.

If a chord passes through the centre, as GD , it is called a **diameter**.

A part of the circumference, as the curved line FED , is called an **arc**.

A line drawn from a point in one arm of an angle to a point in the other arm is said to **subtend** the angle. In the diagram the arc FE subtends the $\angle FCE$; or in any \triangle each side subtends the opposite \angle .

33.—Exercises



1. Prove Theorem 2 when one \triangle has to be supposed to be turned over before it can be made to coincide with the other.

2. The $\angle B$ of a $\triangle ABC$ is a rt. \angle , and CB is produced to D making $BD = BC$. Prove $AD = AC$.

3. A, B, C are three points in a st. line such that $AB = BC$. DB is $\perp AC$. Show that any point in DB , produced in either direction, is equidistant from A and C .

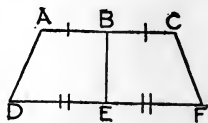
4. Two st. lines AOB, COD cut one another at O , so that $OA = OB$ and $OC = OD$; join AD and BC , and prove $\triangle s AOD, BOC$ congruent.

5. Prove that all chords of a circle which subtend equal angles at the centre are equal to each other.

6. If with the same centre O , two circles be drawn, and st. lines ODB, OEC be drawn to meet the circumferences in D, E, B, C ; prove that $BE = DC$.

7. $ABCD$ is a quadrilateral having the opposite sides AB, CD equal and $\angle B = \angle C$. Show that $AC = BD$.

8. In the diagram, ABC and DEF are both $\perp BE$. Also $AB = BC$ and $DE = EF$. Prove that $AD = CF$.



9. Two st. lines AOB, COD cut one another at rt. $\angle s$ at O . AO is cut off $= OB$, and $CO = OD$. Prove that the quadrilateral $ACBD$ is a rhombus.

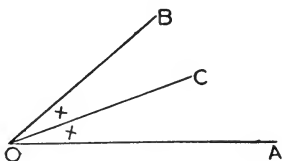
10. Two quadrilaterals $ABCD, EFGH$ have $AB = EF, BC = FG, CD = GH, \angle B = \angle F, \angle C = \angle G$. Prove that they are congruent.

34. **Definitions.**—A triangle having its sides all equal to each other is called an **equilateral** triangle.

A triangle having two sides equal to each other is called an **isosceles** triangle.

A triangle having no two of its sides equal to each other is called a **scalene** triangle.

35.



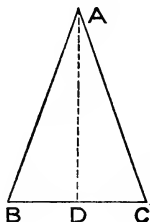
If a straight line revolve in the positive direction about the point **O** from the position **OA** to the position **OB**, it must pass through some position **OC** such that $\angle AOC = \angle COB$.

A straight line which divides an angle into two equal angles is called the **bisector** of the angle.

When a construction is represented in a diagram, although it has not previously been proved that it can be made, it is called a **hypothetical construction**. Thus **OC** has been drawn to represent the bisector of $\angle AOB$.

THEOREM 3

The angles at the base of an isosceles triangle are equal to each other.



Hypothesis.— ABC is an isosceles \triangle having $AB = AC$.

To prove that $\angle B = \angle C$.

Hypothetical Construction.—Draw the st. line AD to represent the bisector of $\angle BAC$.

Proof.—In the two \triangle s ADB, ADC ,

$$\begin{cases} AB = AC, & (\text{Hyp.}) \\ AD \text{ is common,} \\ \angle BAD = \angle CAD, & (\text{Const.}) \end{cases}$$

$\therefore \triangle ADB \equiv \triangle ADC,$ (I—2, page 16.)

$\therefore \angle B = \angle C.$

36. The two \triangle s ADB, ADC , in the diagram of Theorem 3, are congruent, and if the isosceles \triangle be folded along the bisector of the vertical \angle as crease, the parts on one side of the bisector will exactly fit the corresponding parts on the other side.

Definition.—When a figure can be folded along a line so that the part on one side exactly fits the part on the other side, the figure is said to be **symmetrical** with respect to that line.

The line along which the figure is folded is called an **axis of symmetry** of the figure.

Hence the bisector of the vertical \angle of an isosceles \triangle is an axis of symmetry of the \triangle .

It follows from the above definition of a symmetrical figure that—

If a figure is symmetrical with respect to a st. line, for every point on one side of this axis of symmetry there is a corresponding point on the other side.

Show by folding, in the diagram of Theorem 3, that if $\angle B = \angle C$, the side $AB =$ the side AC .

37.—Exercises

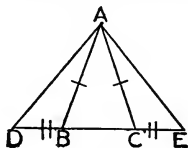
1. An equilateral \triangle is equiangular.

2. ABC is an equilateral \triangle , and points D, E, F , are taken in BC, CA, AB respectively, such that $BD = CE = AF$. Show that DEF is an equilateral \triangle .

3. Show that the exterior \angle s at the base of an isosceles \triangle are equal to each other.

4. The opposite \angle s of a rhombus are equal to each other.

5. ABC is an isosceles \triangle having $AB = AC$, and the base BC produced to D and E such that $BD = CE$. Prove that ADE is an isosceles \triangle .



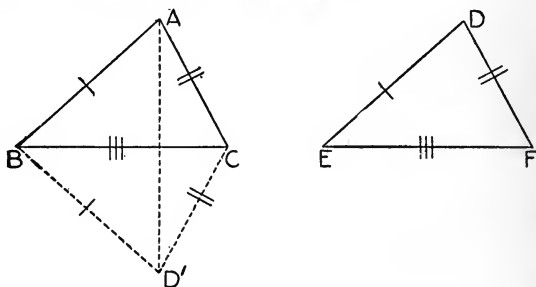
6. AC, AD are two st. lines on opposite sides of AB . Prove that if the bisectors of \angle s BAC, BAD are at rt. \angle s, AC, AD must be in the same st. line.

7. If a figure be symmetrical with respect to a st. line, the st. line joining any two corresponding points cuts the axis at rt. \angle s.

SECOND CASE OF THE CONGRUENCE OF TRIANGLES

THEOREM 4

If two triangles have the three sides of one respectively equal to the three sides of the other, the two triangles are congruent.



Hypothesis.— $\triangle ABC$, $\triangle DEF$ are two \triangle s having $AB = DE$, $AC = DF$ and $BC = EF$.

To prove that $\triangle ABC \equiv \triangle DEF$.

Proof.—Let $\triangle DEF$ be applied to $\triangle ABC$ so that the vertex E falls on the vertex B and EF falls along BC .

Then $\because EF = BC$, the vertex F falls on C . Let D take the position D' on the side of BC remote from A .

Join AD' .

$$\begin{aligned} \therefore BA &= BD', \\ \therefore \angle BAD' &= \angle BD'A. \quad (\text{I—3, p. 20.}) \end{aligned}$$

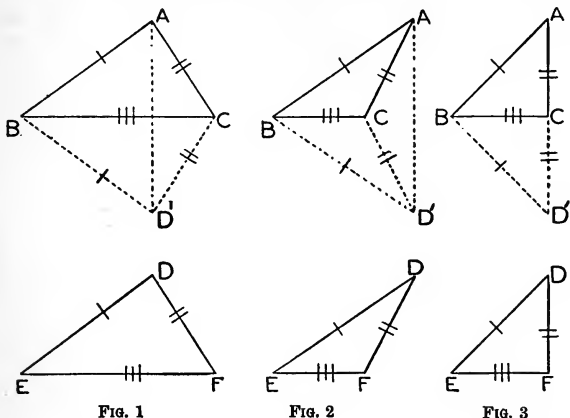
Similarly $\angle CAD' = \angle CD'A$.

$$\begin{aligned} \therefore \angle BAD' + \angle CAD' &= \angle BD'A + \angle CD'A, \\ \text{i.e., } \angle BAC &= \angle BD'C. \end{aligned}$$

Then in \triangle s BAC , $BD'C$ $\left\{ \begin{array}{l} BA = BD', \\ CA = CD', \\ \angle BAC = \angle BD'C, \end{array} \right.$

$$\begin{aligned} \therefore \triangle ABC &\equiv \triangle BD'C; \quad (\text{I—2, p. 16.}) \\ \text{i.e., } \triangle ABC &\equiv \triangle DEF. \end{aligned}$$

Note.—In the proof of this theorem three cases may occur :— AD' may cut BC as in Fig. 1, or not cut BC as in Fig. 2, or pass through one end of BC as in Fig. 3.



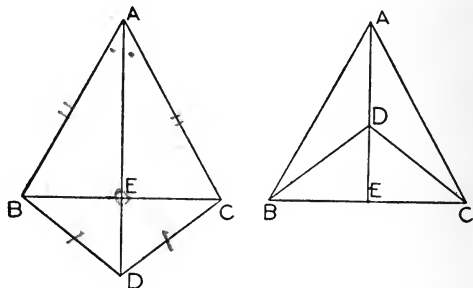
The proof given above is that of the first case. The pupil should work out the proofs of the other two cases.

38.—Exercises

1. If the opposite sides of a quadrilateral be equal, the opposite \angle s are equal.
2. A diagonal of a rhombus bisects each of the \angle s through which it passes, and consequently, the diagonal is an axis of symmetry in the rhombus.
3. If in a quadrilateral $ABCD$ the sides AB, CD be equal and $\angle ABC = \angle BCD$, prove that $\angle CDA = \angle DAB$.
4. Show that equal chords in a circle subtend equal \angle s at the centre.
5. Prove that the diagonals of a rhombus bisect each other at rt. \angle s.

THEOREM 5

If two isosceles triangles are on the same base, the straight line joining their vertices is an axis of symmetry of the figure; and the ends of the base are corresponding points.



Hypothesis.— ABC , DBC are two isosceles \triangle s on the same base BC .

To prove that AD is an axis of symmetry of the figure.

Proof.— AD , or AD produced, cuts BC at E .

In \triangle s ABD , ACD , $\left\{ \begin{array}{l} AB = AC \\ BD = CD, \\ AD \text{ is common,} \end{array} \right.$

$\therefore \triangle BAD \equiv \triangle CAD.$ (I—4, p. 22.)

and $\therefore \angle BAD = \angle CAD.$

In \triangle s BAE , CAE , $\left\{ \begin{array}{l} BA = CA, \\ AE \text{ is common,} \\ \angle BAE = \angle CAE, \end{array} \right.$

$\therefore \triangle BAE \equiv \triangle CAE.$ (I—2, p. 16.)

Similarly, $\triangle BDE \equiv \triangle CDE.$

Hence, each part of the figure on one side of AD is congruent to the corresponding part on the other side, and if the figure be folded on AD , as crease, the corresponding parts will coincide.

$\therefore AD$ is an axis of symmetry of the figure; and B, C are corresponding points.

39.—Exercises

1. If two circles cut at two points, the st. line which joins their centres bisects at rt. \angle s the st. line joining the points of section.

2. A, B, C are three points each of which is equidistant from two fixed points P, Q . Show that A, B, C are in a st. line which bisects the st. line joining P, Q and cuts it at rt. \angle s.

CONSTRUCTIONS

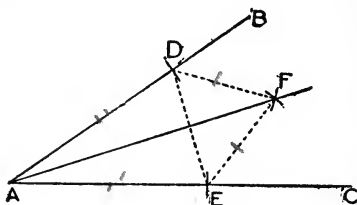
40. In Theoretical Geometry the use of instruments in making constructions is generally restricted to an ungraduated straight edge and a pair of compasses. With these instruments we can:—

1. Draw a st. line from one point to another.
2. Produce a st. line.
3. Describe a circle with any point as its centre and radius equal to any given st. line.
4. Cut off from one st. line a part equal to another st. line.

NOTE.—*All constructions should be accurately and neatly drawn by the pupil, and, by means of theorems already proved, the correctness of the method of construction should be shown.*

PROBLEM 1

To bisect a given angle.



Let BAC be the given \angle .

Construction.—With the compasses cut off equal distances AD and AE from the arms of the \angle .

With centre D describe an arc.

With centre E and the same radius describe another arc cutting the first at F .

Join AF .

Then AF is the bisector of $\angle BAC$.

Proof.—Join DF , EF , DE .

ADE , FDE are isosceles \triangle s on the same base DE ,

$\therefore AF$ is an axis of symmetry of the figure, (I—5, p. 24.)

$\therefore AF$ bisects $\angle BAC$.

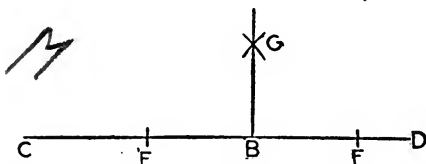
NOTE.—The equal radii for the arcs with centres D and E must be taken long enough for the arcs to intersect.

41.—Exercises

1. Divide a given \angle into four equal parts.
2. Prove that the bisectors of a pair of vertically opposite \angle s are in the same st. line.
3. Bisect a st. \angle .

PROBLEM 2

To draw a perpendicular to a given straight line from a given point in the line.



Let CD be the given st. line and B the given point.

Construction.—Bisect the st. $\angle CBD$ by the st. line BG .

Proof.—Then each of the \angle s CBG , DBG is half of a st. \angle and \therefore each is a rt. \angle .

$\therefore BG$ is $\perp CD$.

42.—Exercises

Using ruler and compasses only, construct \angle s of (1), 45° ; (2), $22\frac{1}{2}^\circ$; (3), 135° ; (4), $67\frac{1}{2}^\circ$; (5), 225° .

43. **Definitions.**—If one angle of a triangle be a right angle, the triangle is called a **right-angled triangle**.

In a right-angled triangle the side opposite the right angle is called the **hypotenuse**.

If one angle of a triangle be an obtuse angle, the triangle is called an **obtuse-angled triangle**.

If all three angles of a triangle be acute angles, the triangle is called an **acute-angled triangle**.

The **altitude** of a triangle is the length of the perpendicular from any vertex to the opposite side.

44.—**Exercises**

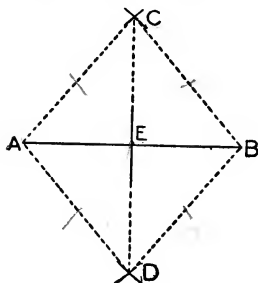
1. Construct a rt.- \angle d \triangle having one of the arms of the rt. \angle three times the other.
2. Construct a rt.- \angle d \triangle having the hypotenuse three times one of the arms of the rt. \angle .
3. Given the length of the hypotenuse and of one of the sides of a rt.- \angle d \triangle , construct the \triangle .
4. Construct a rhombus having each of its diagonals equal to twice a given st. line.
5. Construct a rhombus having one diagonal twice and the other four times a given st. line.
6. Construct an isosceles \triangle having given its altitude and the length of one of the equal sides.
7. Construct an isosceles rt.- \angle d \triangle .

45. **Definitions.**—Sometimes when a proposition has been proved the truth of another proposition follows as an immediate consequence of the former; such a proposition is called a **corollary**.

A straight line which bisects a line of given length at right angles is called the **right bisector** of the line.

PROBLEM 3

To bisect a given straight line.



Let AB be the given st. line.

Construction.—With centre A and any radius that is plainly greater than half of AB , draw two arcs, one on each side of AB .

With centre B and the same radius draw two arcs cutting the first two at C and D .

Join CD , cutting AB at E .

E is the middle point of AB .

Proof.—Join CA , AD , DB , BC .

CAB , DAB are isosceles \triangle s on the same base AB ,

$\therefore CD$ is an axis of symmetry of the figure; and A , B are corresponding points. (I—5, p. 24.)

$\therefore AE = EB$.

Corollary.—From the above proof it follows that the \angle s at E are rt. \angle s, and hence, CD is the right bisector of AB .

46. **Definition.**—The straight line drawn from a vertex of a triangle to the middle point of the opposite side is called a **median** of the triangle.

47.—Exercises

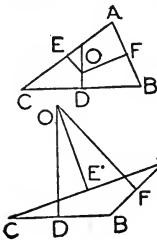
1. Divide a given st. line into four equal parts.

✓ 2. In an isosceles \triangle prove that the bisector of the vertical \angle is a median of the \triangle .

3. In an equilateral \triangle prove that the bisectors of the \angle s are medians of the \triangle .

✓ 4. Show that any point in the right bisector of a given st. line is equidistant from the ends of the given line.

✓ 5. In any \triangle the point of intersection of the right bisectors of any two sides is equidistant from the three vertices.



✓ 6. The right bisectors of the three sides of a \triangle pass through one point.

The right bisectors of AB, BC meet at O. Bisect AC at E. Join EO. Prove $OE \perp AC$.

7. Describe a circle through the three vertices of a \triangle .

8. Describe a circle to pass through three given points that are not in the same st. line.

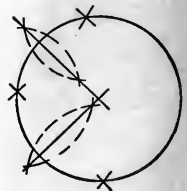
9. Show how any number of circles may be drawn through two given points.

What line contains the centres of all these circles?

10. In a given st. line find a point that is equally distant from two given points.

11. On a given base describe an isosceles \triangle so that the sum of the two equal sides may equal a given st. line.

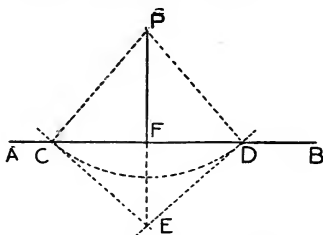
In what case is this impossible?



12. Construct a rhombus having its diagonals equal to two given st. lines.
- ✓ 13. In $\triangle ABC$ find in CA , produced if necessary, a point D so that $DC = DB$.
- ✓ 14. In $\triangle s$ ABC , DEF , $AB = DE$, $AC = DF$ and the medians drawn from B and E are equal to each other. Prove that $\triangle ABC \equiv \triangle DEF$.

PROBLEM 4

To draw a perpendicular to a given straight line from a given point without the line.



Let P be the given point and AB the given st. line.

Construction.—Describe an arc with centre P to cut AB at C and D .

With centres C and D , and equal radii, describe two arcs cutting at E .

Join PE , cutting AB at F .

PF is the required perpendicular.

Proof.—Join PC , CE , ED , DP .

\therefore PCD , ECD are isosceles $\triangle s$ on the same base CD ,

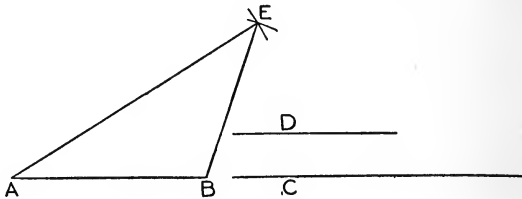
\therefore PE is an axis of symmetry of the figure; and

C , D are corresponding points. (I—5, p. 24.)

\therefore $\angle s$ at F are rt. $\angle s$, and PF is $\perp AB$.

PROBLEM 5

To construct a triangle with sides of given length.



Let AB , C and D be the given lengths.

Construction.—With centre A and radius C describe an arc.

With centre B and radius D describe an arc cutting the first arc at E .

Join EA , EB .

AEB is the required \triangle .

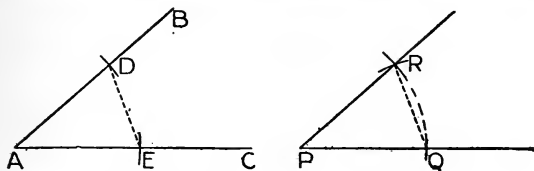
QUESTION—*In what case would the above construction fail?*

48.—Exercises

1. On a given st. line describe an equilateral \triangle .
2. On a given base describe an isosceles \triangle having each of the equal sides double the base.
3. Construct a rhombus having given a diagonal and the length of one of the equal sides.

PROBLEM 6

To construct an angle equal to a given angle.



Let BAC be the given \angle .

Construction.—From AC , AB cut off equal parts AE , AD .

Draw a line and mark a point P in it.

Cut off $\text{PQ} = \text{AE}$.

With centre P and radius PQ describe an arc.

With centre Q and radius DE describe an arc cutting the arc with centre P at R .

Join RP .

RPQ is the required \angle .

Proof.—Join DE , RQ .

In \triangle s PRQ , ADE . $\left\{ \begin{array}{l} \text{PQ} = \text{AE}, \\ \text{PR} = \text{AD}, \\ \text{RQ} = \text{DE}, \end{array} \right.$

$\therefore \angle \text{RPQ} = \angle \text{BAC}$. (I—4, p. 22.)

49.—Exercises

1. Construct a rhombus having given one of its \angle s and the length of one of its equal sides.
 2. Construct a quadrilateral equal in all respects to a given quadrilateral.
 3. On a given st. line BC construct a \triangle having the \angle s B, C equal to two given acute \angle s.
 4. Construct an \angle equal to the complement of a given acute \angle .
 5. Construct an \angle equal to the supplement of a given \angle .
 6. On a given base describe an isosceles \triangle having its altitude equal to a given st. line.
 7. In the side BC of a $\triangle ABC$ find a point E , such that AE is half the sum of AB and AC .
 8. The \triangle formed by joining the middle points of the three sides of an isosceles \triangle is isosceles.
 9. AB is a given st. line and C is a given point without the line. Find the point D so that C and D may be symmetrical with respect to AB .
 10. C, D are given points, (1) on opposite sides, (2) on the same side of a given st. line AB . Find a point P in AB so that CP, DP make equal \angle s with AB .
 11. The right bisectors of the two sides AB, AC of $\triangle ABC$ meet at D , and E is the middle point of BC . Show that $DE \perp BC$.
-

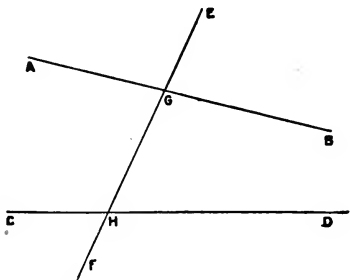
PARALLEL STRAIGHT LINES

50. **Definitions.**—Two straight lines *in the same plane* which do not meet when produced for any finite distance in either direction are said to be **parallel** to each other.

A straight line which cuts two, or more, other straight lines is called a **transversal**.

A quadrilateral that has both pairs of opposite sides parallel to each other is called a **parallelogram**.

Draw a st. line EF cutting two other st. lines AB and CD at G and H .



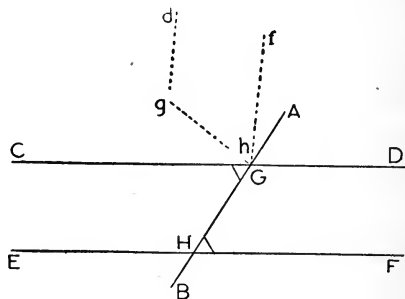
Eight \angle s are thus formed, four of which, AGH , BGH , CHG , DHG , being between AB and CD , are called **interior** \angle s. The other four are called **exterior** \angle s.

The interior \angle s AGH and GHD , on opposite sides of the transversal, are called **alternate** \angle s. Thus also, BGH and GHC are alternate \angle s.

Name four pairs of equal angles in the diagram.

THEOREM 6

If a transversal meeting two straight lines makes the alternate angles equal to each other, the two straight lines are parallel.



Hypothesis.—The transversal **AB** meeting **CD** and **EF** makes $\angle \text{CGH} =$ the alternate $\angle \text{GHF}$.

To prove that **CD** \parallel **EF**.

Proof.—Detach the part **DGHF** from the figure and mark it *dghf*.

Slide *dghf*, from its original position, along the transversal until *h* comes to the point **G**.

Then, rotate *dghf*, in either direction, through a st. \angle about the point **G**.

When the rotation is complete *h g* coincides with **GH**.

$$\text{And, } \because \angle fhg = \angle \text{CGH},$$

$$\therefore hf \text{ coincides with } \text{GC}.$$

$$\text{Also, } \because \angle dgh = \angle \text{GHE},$$

$$\therefore gd \text{ coincides with } \text{HE}.$$

If it be possible let **CD** and **EF** when produced meet towards **D** and **F**.

Then $h f$ and $g d$ must meet towards f and d ,

\therefore **GC** and **HE** must meet towards **C** and **E**.

Hence, **CD** and **EF** when produced must meet in two points.

This is impossible by the definition of a st. line.

\therefore **CD** and **EF** do not meet towards **D** and **F**, and hence cannot meet towards **C** and **E**.

\therefore **CD** \parallel **EF**.

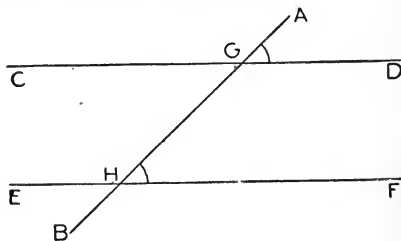
*NOTE.—If this proof is not at once clear to the pupil he should make a drawing of the diagram, cut out the part $d g h f$, and turning it about, fit it to **E H G C**.*

51.—Exercises

1. Lines which are \perp to the same st. line are \parallel to each other.
2. If both pairs of opposite sides of a quadrilateral are equal to each other, the quadrilateral is a \parallel gm.
3. A rhombus is a \parallel gm.
4. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a \parallel gm.
5. No two st. lines drawn from two vertices of a \triangle , and terminated in the opposite sides, can bisect each other.

THEOREM 7

If a transversal meeting two straight lines makes (1) an exterior angle equal to the interior and opposite angle on the same side of the transversal, or, (2) the two interior angles on the same side of the transversal supplementary, in either case the two straight lines are parallel.



(1) *Hypothesis*.— AB meeting CD , EF makes $\angle AGD = \angle GHF$.

To prove.— $CD \parallel EF$.

Proof.— $\angle CGH = \angle AGD$, (I—1, p. 13.)
 but $\angle AGD = \angle GHF$, (*Hyp.*)
 $\therefore \angle CGH = \angle GHF$.
 $\therefore CD \parallel EF$. (I—6, p. 36.)

(2) *Hypothesis*.— AB meeting CD , EF makes $\angle DGH + \angle GHF = \text{two rt. } \angle\text{s}$.

To prove.— $CD \parallel EF$.

Proof.— $\angle CGH + \angle DGH = \text{two rt. } \angle\text{s}$,
 but $\angle DGH + \angle GHF = \text{two rt. } \angle\text{s}$, (*Hyp.*)
 $\therefore \angle CGH + \angle DGH = \angle DGH + \angle GHF$.

From each take the common $\angle DGH$, and $\angle CGH = \angle GHF$,

$\therefore CD \parallel EF$ (I—6, p. 36.)

52. The following statement of a fundamental property of parallel straight lines is called **Playfair's axiom** :—

Through any point one, and only one, straight line can be drawn parallel to a given straight line.

From this axiom it follows that:—

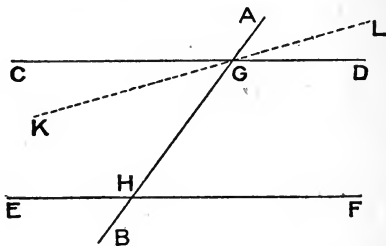
No two intersecting straight lines can be parallel to the same straight line.

∴ straight lines which are parallel to the same straight line are not intersecting lines, *i.e.* :—

Straight lines which are parallel to the same straight line are parallel to each other.

THEOREM 8

If a transversal cuts two parallel straight lines, the alternate angles are equal to each other.



Hypothesis.—The transversal **AB** cuts the \parallel st. lines **CD**, **EF** at **G**, **H**.

To prove that $\angle \text{CGH} = \angle \text{GHF}$.

Proof.—If $\angle \text{CGH}$ be not equal to $\angle \text{GHF}$, make the $\angle \text{KGH} = \angle \text{GHF}$, and produce **KG** to **L**.

Then \because **AB** cuts **KL** and **EF**, making $\angle \text{KGH} =$ the alternate $\angle \text{GHF}$.

\therefore **KL** is \parallel to **EF**. (I—6, p. 36.)

But **CD** is, by hypothesis, \parallel to **EF**.

That is, two intersecting st. lines, **KL** and **CD**, are both \parallel **EF**, which is impossible.

$\therefore \angle \text{CGH} = \angle \text{GHF}$.

53. Consider the method of proof used in Theorem 8.

To prove that $\angle \text{CGH} = \angle \text{GHF}$ we began by assuming that these \angle s are not equal, and then showed that something absurd or contrary to the hypothesis must follow, and concluded that $\angle \text{CGH} = \angle \text{GHF}$.

This method of proof, in which we begin by assuming that the conclusion is not true, is called the **indirect method of demonstration**.

54. Compare Theorems 6 and 8.

In both cases a transversal cuts two straight lines.

In Theorem 6 the hypothesis is that the alternate angles are equal, and the conclusion is that the lines are parallel.

In Theorem 8 the hypothesis is that the lines are parallel, and the conclusion is that the alternate angles are equal.

Thus in these propositions the hypothesis of each is the conclusion of the other.

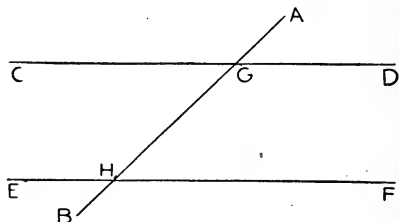
When two propositions are such that the hypothesis of each is the conclusion of the other, they are said to be **converse propositions**; or each is said to be the converse of the other.

The converse of a true proposition may, or may not, be true. The converse propositions in Theorems 6 and 8 are both true; but consider the true proposition:— All rt. \angle s are equal to each other; and its converse:— All equal \angle s are rt. \angle s. The last is easily seen to be untrue. Consequently proof must in general be given for each of a pair of converse propositions.

When a proposition is known to be true and we wish to prove the converse we commonly use the indirect method.

THEOREM 9

If a transversal cuts two parallel straight lines, it makes (1) an exterior angle equal to the interior and opposite angle on the same side of the transversal, and (2) the interior angles on the same side of the transversal supplementary.



Hypothesis.— AB cuts the \parallel st. lines CD , EF .

To prove that (1) $\angle AGD = \angle AHF$.

(2) $\angle DGH + \angle GHF =$ two rt. \angle s.

Proof.—(1) $\because CD \parallel EF$,

$\therefore \angle GHF = \angle CGH$. (I—8, p. 40.)

but $\angle CGH = \angle AGD$, (I—1, p. 13.)

$\therefore \angle AGD = \angle GHF$.

(2) $\because \angle GHF = \angle CGH$,

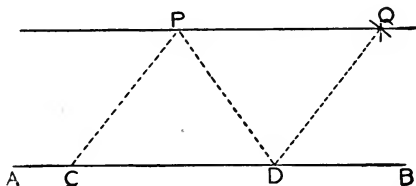
$\therefore \angle GHF + \angle DGH = \angle CGH + \angle DGH$;

but $\angle CGH + \angle DGH =$ a st. \angle

$\therefore \angle$ s GHF , DGH are supplementary.

PROBLEM 7

Through a given point to draw a straight line parallel to a given straight line.



Let P be the given point and AB the given st. line.

Construction.—Take two points C, D , in AB .

With centre P and radius CD describe an arc.

With centre D and radius CP describe an arc cutting the first at Q .

Join PQ .

Then $PQ \parallel AB$.

Proof.—Join PC, DQ, PD .

$$\text{In } \triangle s \text{ } PCD, DQP, \begin{cases} PC = DQ, \\ CD = QP, \\ PD \text{ is common,} \end{cases}$$

$$\therefore \angle CDP = \angle DPQ. \quad (\text{I—4, p. 22.})$$

$$\therefore PQ \parallel AB. \quad (\text{I—6, p. 36.})$$

55.—**Exercises**

¶ 1. If a st. line be \perp to one of two \parallel st. lines, it is also \perp to the other.

¶ 2. Prove, by using a transversal, that st. lines which are \parallel to the same st. line are \parallel to each other.

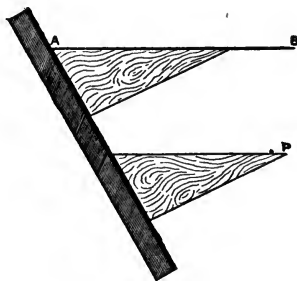
3. Any st. line \parallel to the base of an isosceles \triangle makes equal \angle s with the sides, or the sides produced.

4. Construct a \triangle having two of its \angle s respectively equal to two given \angle s, and the length of the \perp from the vertex of the third \angle to the opposite side equal to a given st. line.

5. Construct a rt.- \angle d \triangle having given one side and the opposite \angle .

6. If one \angle of a \parallel gm be a rt. \angle , the other three \angle s are also rt. \angle s.

7. Give a proof for the following method of drawing a line through $P \parallel AB$:—



Place the set-square with the hypotenuse along the st. line AB .

Place a ruler against another side of the set-square as in the diagram.

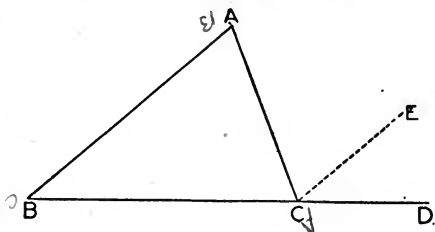
Hold the ruler firmly in position and slide the set-square along it until the hypotenuse comes to the point P .

A line drawn through P along the set-square is $\parallel AB$.

TRIANGLES

THEOREM 10

The exterior angle, made by producing one side of a triangle, equals the sum of the two interior and opposite angles; and the three interior angles are together equal to two right angles.



Hypothesis.— ABC is a \triangle having BC produced to D .

To prove that (1) $\angle ACD = \angle A + \angle B$.

(2) $\angle A + \angle B + \angle ACB =$ two rt. \angle s.

Construction.—Through C draw $CE \parallel AB$.

Proof.—

$\therefore CE \parallel AB$,

and AC is a transversal,

$\therefore \angle ACE = \angle A$.

(I—8, p. 40.)

$\therefore BD$ is a transversal,

$\therefore \angle ECD = \angle B$.

(I—9, p. 42.)

$\therefore \angle ACE + \angle ECD = \angle A + \angle B$.

i.e., $\angle ACD = \angle A + \angle B$.

Hence, $\angle A + \angle B + \angle ACB = \angle ACD + \angle ACB$.

But $\angle ACD + \angle ACB =$ two rt. \angle s,

$\therefore \angle A + \angle B + \angle ACB =$ two rt. \angle s.

Cor.—The exterior angle of a triangle is greater than either of the interior and opposite angles.

56.—Exercises

×1. Prove Theorem 10 by means of a st. line drawn through the vertex \parallel the base.

2. If two \triangle s have two \angle s of one respectively equal to two \angle s of the other, the third \angle of one is equal to the third \angle of the other.

×3. The sum of the \angle s of a quadrilateral is equal to four rt. \angle s.

4. The sum of the \angle s of a pentagon is six rt. \angle s.

5. Each \angle of an equilateral \triangle is an \angle of 60° .

6. Find a point **B** in a given st. line **CD** such that, if **AB** be drawn to **B** from a given point **A**, the \angle **ABC** will equal a given \angle .

7. Show that the bisectors of the two acute \angle s of a rt.- \angle d \triangle contain an \angle of 135° .

✦ 8. If both pairs of opposite \angle s of a quadrilateral are equal, the quadrilateral is a \parallel gm.

9. **C** is the middle point of the st. line **AB**. **CD** is drawn in any direction and equal to **CA** or **CB**. Prove that **ADB** is a rt. \angle .

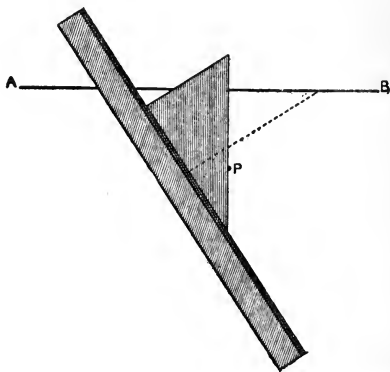
×10. On **AB**, **AC**, sides of a \triangle **ABC**, equilateral \triangle s **ABD**, **ACE** are described externally. Show that **DC** = **BE**.

×11. **AB** is any chord of a circle of which the centre is **O**. **AB** is produced to **C** so that **BC** = **BO**. **CO** is joined, cutting the circle at **D** and is produced to cut it again at **E**. Show that \angle **AOE** = three times \angle **BCD**.

×12. If the exterior \angle s at **B** and **C** of a \triangle **ABC** be bisected and the bisectors be produced to meet at **D**, the \angle **BDC** equals half the sum of \angle s **ABC**, **ACB**.

13. Show that a \triangle must have at least two acute \angle s.
14. In an acute- \angle d \triangle show that the \perp from a vertex to the opposite side cannot fall outside of the \triangle .
15. In an obtuse- \angle d \triangle show that the \perp from the vertex of the obtuse \angle on the opposite side falls within the \triangle , but that the \perp from the vertex of either acute \angle on the opposite side falls outside of the \triangle .
16. In a rt.- \angle d \triangle where do the \perp s from the vertices on the opposite sides fall?
17. Only one \perp can be drawn from a given point to a given st. line.
18. Not more than two st. lines each equal to the same given st. line can be drawn from a given point to a given st. line.
19. D is a point taken within the $\triangle ABC$. Join DB , DC ; and show, by producing BD to meet AC , that $\angle BDC > \angle BAC$.
20. With compasses and ruler only, construct the following \angle s: -30° , 15° , 120° , 105° , 75° , $67\frac{1}{2}^\circ$, 150° , 195° , 210° , 240° , 255° , 285° , -30° , -75° , -135° .
21. If a transversal cut two st. lines so as to make the interior \angle s on one side of the transversal together less than two rt. \angle s, the two lines when produced shall meet on that side of the transversal.
22. The bisector of the exterior vertical \angle of an isosceles \triangle is \perp to the base.
23. Give a proof for the following method of drawing a line through $P \perp AB$:—

First place the set-square in the position shown by the dotted line, with its hypotenuse along AB .



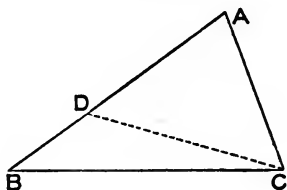
Place a ruler along one of the sides of the set-square and hold it firmly in that position.

Rotate the set-square through its right \angle , thus bringing the other side against the ruler, and slide the set-square along the ruler to the position shown by the shaded \triangle .

A line drawn through P , along the hypotenuse of the set-square, is perpendicular to AB .

THEOREM 11

If one side of a triangle is greater than another side, the angle opposite the greater side is greater than the angle opposite the less side.



Hypothesis.— $\triangle ABC$ is a \triangle having $AB > AC$.

To prove that $\angle ACB > \angle ABC$.

Construction.—From AB cut off $AD = AC$. Join DC .

Proof.—In $\triangle ADC$,

$$\because AD = AC,$$

$$\therefore \angle ADC = \angle ACD. \quad (\text{I—3, p. 20.})$$

But $\angle ACB > \angle ACD$,

$$\therefore \angle ACB > \angle ADC.$$

In $\triangle BDC$,

$\because BD$ is produced to A ,

\therefore exterior $\angle ADC >$ interior and opposite

$$\angle DBC. \quad (\text{I—10, Cor., p. 45.})$$

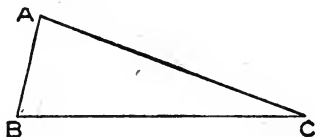
But $\angle ACB > \angle ADC$;

much more \therefore is $\angle ACB > \angle ABC$.

THEOREM 12

(Converse of Theorem 11)

If one angle of a triangle is greater than another angle of the same triangle, the side opposite the greater angle is greater than the side opposite the less.



Hypothesis.—In $\triangle ABC$ $\angle B > \angle C$.

To show that $AC > AB$.

Proof.—If AC be not $> AB$,

then either $AC = AB$,

or $AC < AB$.

If $AC = AB$,

then $\angle B = \angle C$.

(I—3, p. 20.)

But this is not so, $\therefore AC$ is not $= AB$.

If $AC < AB$,

then $\angle B < \angle C$.

(I—11, p. 49.)

But this also is not so, $\therefore AC$ is not $< AB$.

Hence $\therefore AC$ is neither $=$ nor $< AB$,

$\therefore AC > AB$.

57.—**Exercises**

1. The perpendicular is the shortest st. line that can be drawn from a given point to a given straight line.



The length of the \perp from a given point to a given st. line is called the distance of the point from the line.

2. $ABCD$ is a quadrilateral, of which AD is the longest side, and BC the shortest. Show that $\angle B > \angle D$, and that $\angle C > \angle A$.

3. The hypotenuse of a rt.- \angle \triangle is greater than either of the other two sides.

4. A st. line drawn from the vertex of an isosceles \triangle to any point in the base is less than either of the equal sides.

5. A st. line drawn from the vertex of an isosceles \triangle to any point in the base produced is greater than either of the equal sides.

6. If one side of a \triangle be less than another, the \angle opposite the less side is acute.

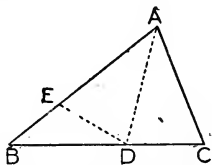
7. If D be any point in the side BC of a $\triangle ABC$, the greater of the sides AB, AC , is greater than AD .

8. AB is drawn from $A \perp CD$. E, F are two points in CD on the same side of B , and such that $BE < BF$. Show that $AE < AF$. Prove the same proposition when E, F are on opposite sides of B .

9. ABC is a \triangle having $AB > AC$. The bisector of $\angle A$ meets BC at D . Show that $BD > DC$. Give a general statement of this proposition.

10. ABC is a \triangle having $AB > AC$. If the bisectors of $\angle s B, C$ meet at D , show that $BD > DC$.

11. Prove Theorem 11 from the following construction: Bisect $\angle A$ by AD which meets BC at D ; from AB cut off $AE = AC$, and join ED .



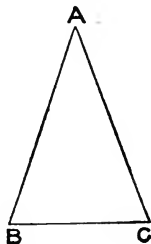
12. The $\angle s$ at the ends of the greatest side of a \triangle are acute.

13. If $AB > AD$ in the $\parallel gm ABCD$, $\angle ADB > \angle BDC$.

THEOREM 13

(Converse of Theorem 3)

If two angles of a triangle are equal to each other, the sides opposite these equal angles are equal to each other.



Hypothesis.—In $\triangle ABC$ $\angle B = \angle C$.

To prove that $AB = AC$.

Proof.— If AB is not $= AC$,
let $AB > AC$.

Then $\angle C > \angle B$. (I—11, p. 49.)

But this is not so.

$\therefore AB$ is not $> AC$.

Similarly it may be shown that

AB is not $< AC$.

$\therefore AB = AC$.

58.—Exercises

1. An equiangular \triangle is equilateral.
2. BD , CD bisect the \angle s ABC , ACB at the base of an isosceles $\triangle ABC$. Show that $\triangle DBC$ is isosceles.
3. ABC is a \triangle having AB , AC produced to D , E respectively. The exterior \angle s DBC , ECB are bisected by

BF, CF, which meet at F. Show that, if $FB = FC$, the $\triangle ABC$ is isosceles.

4. On the same side of AB the two \triangle s ACB, ADB have $AC = BD$, $AD = BC$, and AD, BC meet at E. Show that $AE = BE$.

5. On a given base construct a \triangle having one of the \angle s at the base equal to a given \angle , and the sum of the sides equal to a given st. line.

6. On a given base construct a \triangle having one of the \angle s at the base equal to a given \angle and the difference of the sides equal to a given st. line.

7. If the bisector of an exterior \angle of a \triangle be \parallel to the opposite side, the \triangle is isosceles.



8. Through a point on the bisector of an \angle a line is drawn \parallel to one of the arms. Prove that the \triangle thus formed is isosceles.

9. A st. line drawn \perp to BC, the base of an isosceles $\triangle ABC$, cuts AB at X and CA produced at Y. Show that AXY is an isosceles \triangle .

10. ACB is a rt.- \angle d \triangle having the rt. \angle at C. Through X, the middle point of AC, XY is drawn \parallel CB cutting AB at Y. Show that Y is the middle point of AB.

11. The middle point of the hypotenuse of a rt.- \angle d \triangle is equidistant from the three vertices.

12. The st. line joining the middle points of two sides of a \triangle is \parallel to the third side.

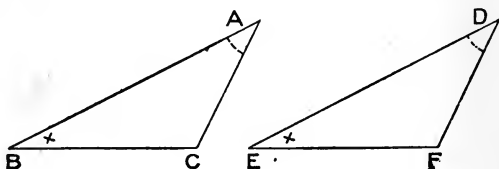
13. Construct a rt.- \angle d \triangle , having the hypotenuse equal to one given st. line, and the sum of the other two sides equal to another given st. line.

14. If one \angle of a \triangle equals the sum of the other two, show that the \triangle is a rt.- \angle d \triangle .

THIRD CASE OF THE CONGRUENCE OF TRIANGLES

THEOREM 14

If two triangles have two angles and a side of one respectively equal to two angles and the corresponding side of the other, the triangles are congruent.



Hypothesis.— $\triangle ABC$, $\triangle DEF$ are two \triangle s having $\angle A = \angle D$, $\angle B = \angle E$, and $BC = EF$.

To prove that $\triangle ABC \equiv \triangle DEF$

Proof.— $\therefore \angle A = \angle D$,

and $\angle B = \angle E$,

$\therefore \angle A + \angle B = \angle D + \angle E$.

But $\angle A + \angle B + \angle C = \angle D + \angle E + \angle F$. (I—10, p. 45.)

$\therefore \angle C = \angle F$.

Apply $\triangle ABC$ to $\triangle DEF$ so that BC coincides with the equal side EF .

$\therefore \angle B = \angle E$,

$\therefore BA$ falls along ED , and A is on the line ED .

$\therefore \angle C = \angle F$,

$\therefore CA$ falls along FD , and A is on the line FD .

But D is the only point common to ED and FD ,

$\therefore A$ falls on D .

$\therefore \triangle ABC$ coincides with $\triangle DEF$,

and $\therefore \triangle ABC \equiv \triangle DEF$.

59.—Exercises

- ✕ 1. If the bisector of an \angle of a \triangle be \perp to the opposite side, the \triangle is isosceles.
- ✕ 2. Any point in the bisector of an \angle is equidistant from the arms of the \angle .
3. In the base of a \triangle find a point that is equidistant from the two sides.
4. In a given st. line find a point that is equidistant from two other given st. lines.
- ✕ 5. Within a \triangle find a point that is equally distant from the three sides of the \triangle .
6. Without a \triangle find three points each of which is equally distant from the three st. lines that form the \triangle .
- ✕ 7. The ends of the base of an isosceles \triangle are equidistant from the opposite sides.
8. Two rt.- \angle d \triangle s are congruent, if the hypotenuse and an acute \angle of one are respectively equal to the hypotenuse and an acute \angle of the other.
9. Construct a \triangle with a side and two \angle s respectively equal to a given st. line and two given \angle s.
10. The \perp from the vertex of an isosceles \triangle to the base, bisects the base and the vertical \angle .
11. Prove I—13 by drawing the bisector of the vertical \angle , and using I—14.
12. $\triangle ABC \cong \triangle DEF$ and AX, DY are \perp to BC, EF respectively. Prove that $AX = DY$.
13. $\triangle ABC \cong \triangle DEF$ and AM, DN bisect \angle s A, D and meet BC, EF at M, N respectively. Prove that $AM = DN$.
14. If the diagonal AC of a quadrilateral $ABCD$ bisects the \angle s at A and C , AC is an axis of symmetry of $ABCD$.
15. The middle point of the base of an isosceles \triangle is equidistant from the equal sides.

THE AMBIGUOUS CASE IN THE COMPARISON OF
TRIANGLES

THEOREM 15

If two triangles have two sides of one respectively equal to two sides of the other and have the angles opposite one pair of equal sides equal to each other, the angles opposite the other pair of equal sides are either equal or supplementary.

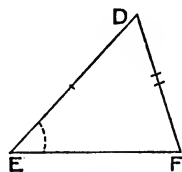
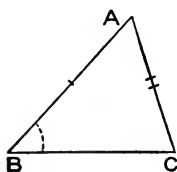


FIG. 1

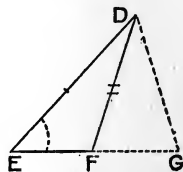


FIG. 2

Hypothesis.— ABC , DEF are two \triangle s having $AB = DE$, $AC = DF$ and $\angle B = \angle E$.

To prove that either $\angle C = \angle F$,
or $\angle C + \angle F = \text{two rt. } \angle\text{s}$.

Proof.—**Case I.** Suppose $\angle A = \angle D$. (Fig. 1.)

Then in the two \triangle s ABC , DEF ,

$$\therefore \angle A = \angle D,$$

$$\text{and } \angle B = \angle E,$$

$$\therefore \angle A + \angle B = \angle D + \angle E.$$

But $\angle A + \angle B + \angle C = \angle D + \angle E + \angle F$. (I—10, p. 45.)

$$\therefore \angle C = \angle F.$$

Case II. Suppose $\angle A \text{ not } = \angle D$. (Fig. 2.)

Make $\angle EDG = \angle BAC$, and produce its arm to meet EF , produced if necessary, at G .

$$\text{In } \triangle\text{s } ABC, DEG, \left\{ \begin{array}{l} \angle A = \angle EDG, \\ \angle B = \angle E, \\ AB = DE, \end{array} \right. \quad (I-14, \text{ p. } 54.)$$

$$\text{and } AC = DG. \left. \vphantom{\begin{array}{l} \angle A = \angle EDG, \\ \angle B = \angle E, \\ AB = DE, \end{array}} \right\} \quad (I-14, \text{ p. } 54.)$$

$$\text{But } DF = AC, \quad (\text{Hyp.})$$

$$\therefore DF = DG.$$

$$\therefore \angle DFG = \angle G. \quad (I-3, \text{ p. } 20.)$$

$$\text{But } \angle C = \angle G.$$

$$\therefore \angle C = \angle DFG.$$

$$\angle DFG + \angle DFE = \text{two rt. } \angle\text{s,}$$

$$\therefore \angle C + \angle DFE = \text{two rt. } \angle\text{s.}$$

NOTE.—There are six parts in a triangle, viz., three sides and three angles, and in the cases in which the congruence of two triangles has been established three parts of one triangle, one at least a side, have been given respectively equal to the corresponding parts of the other.

The following general cases occur:—

1. Two sides and the contained angle. The triangles are congruent—Theorem 2.

2. Three sides. The triangles are congruent—Theorem 4.

3. Two angles and a side. The triangles are congruent—Theorem 14.

4. Two sides and an angle opposite one of them. In this case the triangles are congruent if the angle is opposite the greater of the two sides—§ 60, Ex. 3, but

if the angle is opposite the less of the two sides, they are not necessarily congruent—Theorem 15.

5. Three angles. The triangles are not necessarily congruent—§ 60, Ex. 7.

60.—Exercises

1. If two rt.- \angle Δ s have the hypotenuse and a side of one respectively equal to the hypotenuse and a side of the other, the Δ s are congruent.

2. If the bisector of the vertical \angle of a Δ also bisects the base, the Δ is isosceles.

3. If two Δ s have two sides of one respectively equal to two sides of the other and the \angle s opposite the greater pair of equal sides equal to each other, the Δ s are congruent.

4. Construct a Δ having given two sides and the \angle opposite one of them.

When will there be: (a) no solution, (b) two solutions, (c) only one solution?

5. If two \angle s of a Δ be bisected and the bisectors be produced to meet, the line joining the point of intersection to the vertex of the third \angle bisects that third \angle . Hence.—**The bisectors of the three \angle s of a Δ pass through one point.**

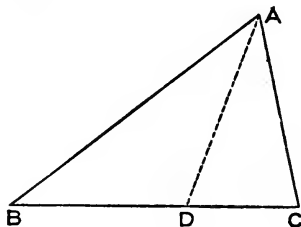
6. If two exterior \angle s of a Δ be bisected and the bisectors be produced to meet, the line joining the point of intersection of the bisectors to the vertex of the third \angle of the Δ bisects that third \angle .

7. Draw diagrams to show that if the three \angle s of one Δ are respectively equal to the three \angle s of another Δ , the two Δ s are not necessarily congruent.

INEQUALITIES

THEOREM 16

Any two sides of a triangle are together greater than the third side.



Hypothesis.— ABC is a \triangle .

To prove that $AB + AC > BC$.

Construction.—Bisect $\angle A$ and let the bisector meet BC at D .

Proof.— $\angle ADC$ is an exterior \angle of $\triangle ABD$,

$$\therefore \angle ADC > \angle BAD. \text{ (I—10, Cor., p. 45.)}$$

But $\angle BAD = \angle DAC$.

$$\therefore \angle ADC > \angle DAC.$$

$$\therefore AC > DC. \text{ (I—12, p. 50.)}$$

Similarly it may be shown that

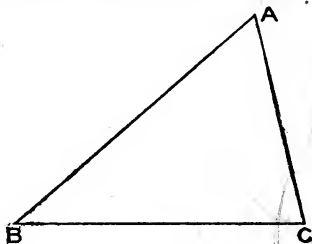
$$AB > BD.$$

$$\therefore AB + AC > BD + DC,$$

$$\text{i.e., } AB + AC > BC.$$

In the same manner it may be shown that $AB + BC > AC$ and that $AC + CB > AB$.

Cor.—The difference between any two sides of a triangle is less than the third side.



ABC is a \triangle .

It is required to show that $AB - AC < BC$.

$$AB < AC + BC. \quad (\text{I-16, p. 59.})$$

From each of these unequals take AC ,

$$\text{and } AB - AC < BC.$$

In the same manner it may be shown that $AB - BC < AC$ and that $BC - AC < AB$.

61.—Exercises

1. Show that the sum of any three sides of a quadrilateral is greater than the fourth side.
2. The sum of the four sides of a quadrilateral is greater than the sum of its diagonals.
3. The sum of the diagonals of a quadrilateral is greater than the sum of either pair of opposite sides.
4. The sum of the st. lines joining any point, except the intersection of the diagonals, to the four vertices of a quadrilateral, is greater than the sum of the diagonals.
5. If any point within a \triangle be joined to the ends of a side of the \triangle , the sum of the joining lines is less than the sum of the other two sides of the \triangle .

6. If any point within a \triangle be joined to the three vertices of the \triangle , the sum of the three joining lines is less than the perimeter of the \triangle , but greater than half the perimeter.

7. The sum of any two sides of a \triangle is greater than twice the median drawn to the third side.

8. The median of a \triangle divides the vertical \angle into parts, of which the greater is adjacent to the less side.

9. The perimeter of a \triangle is greater than the sum of the three medians.

10. **A** and **B** are two fixed points, and **CD** is a fixed st. line. Find the point **P** in **CD**, such that **PA** + **PB** is the least possible ;

(a) When **A** and **B** are on opposite sides of **CD** ;

(b) When **A** and **B** are on the same side of **CD**.

11. **A** and **B** are two fixed points, and **CD** is a fixed st. line. Find the point **P** in **CD**, such that the difference between **PA** and **PB** is the least possible ;

(a) When **A** and **B** are on the same side of **CD** ;

(b) When **A** and **B** are on opposite sides of **CD**.

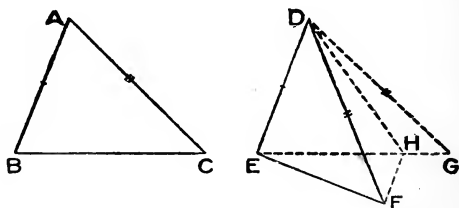
12. Prove Theorem 16 by producing **BA** to **E**, making **AE** = **AC**, and joining **EC**.

13. Prove that the shortest line which can be drawn with its ends on the circumferences of two concentric circles, will, when produced, pass through the centre.

14. Prove the Corollary under Theorem 16, (a) by cutting off from **AB** a part **AD** = **AC** and joining **DC** ; (b) by producing **AC** to **E** making **AE** = **AB** and joining **BE**.

THEOREM 17

If two triangles have two sides of one respectively equal to two sides of the other but the contained angle in one greater than the contained angle in the other, the base of the triangle which has the greater angle is greater than the base of the other.



Hypothesis.— ABC, DEF are two \triangle s having $AB = DE$, $AC = DF$ and $\angle BAC > \angle EDF$.

To show that $BC > EF$.

Construction.—Make $\angle EDG = \angle BAC$ and cut off $DG = AC$, or DF . Join EG . Bisect $\angle FDG$ and let the bisector meet EG at H . Join FH .

Proof.—

In \triangle s ABC, DEG , $\left\{ \begin{array}{l} AB = DE, \\ AC = DG, \\ \angle A = \angle EDG, \end{array} \right.$
 $\therefore BC = EG.$ (I—2, p. 16.)

In \triangle s FDH, GDH , $\left\{ \begin{array}{l} DF = DG, \\ DH \text{ is common,} \\ \angle FDH = \angle GDH, \end{array} \right.$
 $\therefore FH = HG.$

In $\triangle EHF$, $EH + HF > EF.$ (I—16, p. 59.)

But $HF = HG$,

$\therefore EH + HG > EF$.

i.e., $EG > EF$.

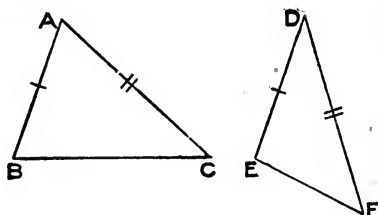
But $BC = EG$,

$\therefore BC > EF$.

THEOREM 18

(Converse of Theorem 17)

If two triangles have two sides of one respectively equal to two sides of the other but the base of one greater than the base of the other, the triangle which has the greater base has the greater vertical angle.



Hypothesis.— $\triangle ABC$, $\triangle DEF$ are two \triangle s having $AB = DE$, $AC = DF$ and $BC > EF$.

To prove that $\angle A > \angle D$.

Proof.— If $\angle A$ is not $> \angle D$,

either $\angle A = \angle D$,

or $\angle A < \angle D$.

(1) If $\angle A = \angle D$.

In \triangle s ABC , DEF , $\begin{cases} AB = DE, \\ AC = DF, \\ \angle A = \angle D, \end{cases}$
 $\therefore BC = EF.$ (I—2, p. 16.)

But this is not so.

$\therefore \angle A$ is not $= \angle D$.

(2) If $\angle A < \angle D$.

$$\text{In } \triangle s \text{ } ABC, DEF, \begin{cases} AB = DE, \\ AC = DF, \\ \angle A < \angle D, \end{cases} \\ \therefore BC < EF. \quad (\text{I—17, p. 62.})$$

But this is not so.

$$\therefore \angle A \text{ is not } < \angle D.$$

Then since $\angle A$ is neither = nor $< \angle D$,


$$\therefore \angle A > \angle D.$$

62.—Exercises

/ 1. $ABCD$ is a quadrilateral having $AB = CD$ and $\angle BAD > \angle ADC$. Show that $\angle BCD > \angle ABC$.

2. In $\triangle ABC$, $AB > AC$ and D is the middle point of BC . If any point P in the median AD be joined to B and C , $BP > CP$.

If AD be produced to any point Q show that $BQ < QC$.

/ 3. D is a point in the side AB of the $\triangle ABC$. AC is produced to E making $CE = BD$. BE and CD are joined. Show that $BE > CD$. 

4. If two chords of a circle be unequal the greater subtends the greater angle at the centre.

5. Two circles have a common centre at O . A, B are two points on the inner circumference and C, D two on the outer. $\angle AOC > \angle BOD$. Show that $AC > BD$.

/ 6. CD bisects AB at rt. $\angle s$. A point E is taken not in CD . Prove that EA, EB are unequal.

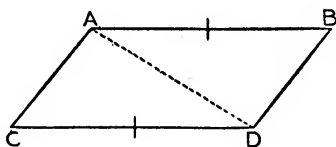
/ 7. In $\triangle ABC$, $AB > AC$. Equal distances BD, CE are cut off from BA, CA respectively. Prove $BE > CD$.

/ 8. In $\triangle ABC$, $AB > AC$. AB, AC are produced to D, E making $BD = CE$. Prove $CD > BE$.

PARALLELOGRAMS

THEOREM 19

Straight lines which join the ends of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Hypothesis.— AB, CD are = and \parallel .

To prove that (1) $AC = BD$,

(2) $AC \parallel BD$.

Construction.—Join AD .

Proof.—

$\therefore AB \parallel CD$,

and AD is a transversal,

$\therefore \angle BAD = \angle CDA$. (I—8, p. 40.)

In $\triangle s$ BAD, CDA , $\left\{ \begin{array}{l} BA = CD, \\ AD \text{ is common,} \\ \angle BAD = \angle CDA, \\ \therefore BD = AC, \end{array} \right\}$ (I—2, p. 16.)

and $\angle BDA = \angle CAD$,

\therefore transversal AD

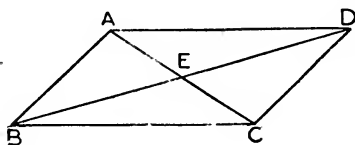
makes $\angle BDA = \angle CAD$,

$\therefore BD \parallel AC$. (I—6, p. 36.)

THEOREM 20

In any parallelogram :

- (1) The opposite sides are equal ;
- (2) The opposite angles are equal ;
- (3) The diagonal bisects the area ;
- (4) The diagonals bisect each other.



Hypothesis.—ABCD is a ||gm, AC, BD its diagonals.

To prove that (1) $AD = BC$ and $AB = CD$.

(2) $\angle BAD = \angle BCD$ and $\angle ABC = \angle ADC$.

(3) $\triangle ABC = \triangle ACD$.

(4) $AE = EC$ and $BE = ED$.

Proof.— \because AC cuts || lines AD, BC,

$\therefore \angle DAC = \angle ACB$. (I—8, p. 40.)

\because AC cuts || lines DC, AB,

$\therefore \angle DCA = \angle CAB$.

In \triangle s ACD, ACB, $\left\{ \begin{array}{l} \angle DAC = \angle ACB, \\ \angle DCA = \angle CAB, \\ AC \text{ is common,} \end{array} \right.$

\therefore (1) $AD = BC$, and $CD = AB$,

(2) also $\angle ADC = \angle ABC$,

(3) and $\triangle ADC = \triangle ABC$.

(I—14, p. 54.)

Similarly it may be shown that $\angle BAD = \angle BCD$.

In \triangle s AED, BEC, $\left\{ \begin{array}{l} AD = BC, \\ \angle DAE = \angle BCE, \\ \angle ADE = \angle CBE, \end{array} \right.$

(4) $\therefore AE = EC,$
and $DE = EB.$

(I—14, p. 54.)

63. **Definitions.**—A parallelogram of which the angles are right angles is called a **rectangle**.

A rectangle of which all the sides are equal to each other is called a **square**.

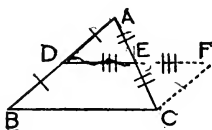
A figure bounded by more than four straight lines is called a **polygon**.

The name polygon is sometimes used for a figure having any number of sides.

A polygon in which all the sides are equal to each other and all the angles are equal to each other is called a **regular polygon**.

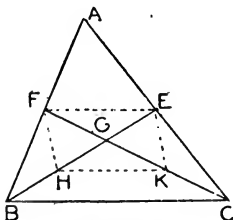
64.—Exercises

1. The diagonals of a rectangle are equal to each other.
2. If the diagonals of a \parallel gm are equal to each other, the \parallel gm is a rectangle.
3. A rectangle has two axes of symmetry.
4. A square has four axes of symmetry.



5. The st. line joining the middle points of the sides of a \triangle is \parallel the base, and equal to half of it.

NOTE.—D, E are the middle points of AB, AC. Produce DE to F making $EF = DE$. Join FC.



6. Of two medians of a \triangle each cuts the other at the point of trisection remote from the vertex.

NOTE.—Medians BE, CF cut at G. Bisect BG, CG at H, K. Join FH, HK, KE, EF.

7. The medians of a \triangle pass through one point.

Definition.—The point where the medians of a \triangle intersect is called the **centroid** of the \triangle .

× 8. A st. line drawn through the middle point of one side of a \triangle , \parallel to a second side, bisects the third side.

9. In any \parallel gm the diagonal which joins the vertices of the obtuse \angle s is shorter than the other diagonal.

× 10. If two sides of a quadrilateral be \parallel , and the other two be equal to each other but not \parallel , the diagonals of the quadrilateral are equal.

11. Through a given point draw a st. line, such that the part of it intercepted between two given \parallel st. lines is equal to a given st. line.

Show that, in general, two such lines can be drawn.

12. Through a given point draw a st. line that shall be equidistant from two other given points.

Show that, in general, two such lines can be drawn.

× 13. Draw a st. line \parallel to a given st. line, and such that the part of it intercepted between two given intersecting lines is equal to a given st. line.

× 14. $\angle BAC$ is a given \angle , and P is a given point. Draw a st. line terminated in the st. lines AB , AC and bisected at P .

× 15. Construct a \triangle having given the middle points of the three sides.

16. If the diagonals of a \parallel gm cut each other at rt. \angle s, the \parallel gm is a rhombus.

× 17. Every st. line drawn through the intersection of the diagonals of a \parallel gm, and terminated by a pair of opposite sides, is bisected, and bisects the \parallel gm.

18. Bisect a given \parallel gm by a st. line drawn through a given point.

19. Divide a given \triangle into four congruent \triangle s.

× 20. The bisectors of two opposite \angle s of a \parallel gm are \parallel to each other.

× 21. In the quadrilateral $ABCD$, $AB \parallel CD$ and $AD = BC$. Prove that (1) $\angle C = \angle D$; (2) if E, F are the middle points of AB, CD respectively, $EF \perp AB$.

22. On a given st. line construct a square.

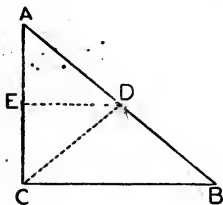
23. Construct a square having its diagonal equal to a given st. line.

× 24. ABC is a \triangle and DE a st. line. Draw a st. line $= DE, \parallel BC$ and terminated in AB, AC , or in these lines produced.

× 25. Inscribe a rhombus in a given \parallel gm, such that one vertex of the rhombus is at a given point in a side of the \parallel gm.

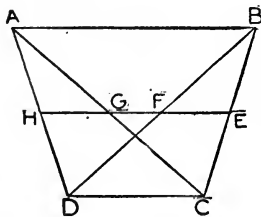
26. ABC is an isosceles \triangle in which $AB = AC$. From P , any point in BC , PX, PY are drawn $\perp AB, AC$ respectively and BM is $\perp AC$. Prove that $PX + PY = BM$.

If P is taken on CB produced, prove that $PY - PX = BM$.



27. The middle point of the hypotenuse of a rt.- \angle d \triangle is equidistant from the three vertices.

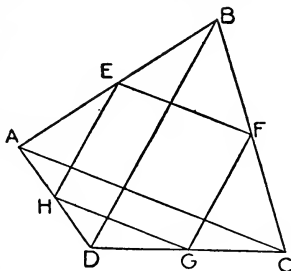
NOTE.—Through D , the middle point of the hypotenuse AB , draw $DE \parallel BC$. Join DC .



28. $ABCD$ is a quadrilateral in which $AB \parallel CD$. E, F, G, H are the middle points of BC, BD, AC, AD . Prove that: (1) the st. line through $E \parallel AB$, or DC , passes through F, G and H ; (2) $HE =$ half the sum of

AB and DC ; (3) $GF =$ half the difference of AB and DC .

29. E, F, G, H are the middle points of the sides AB, BC, CD, DA of the quadrilateral ABCD. Prove that EFGH is a \parallel gm. Show also that: (1) the perimeter of EFGH = $AC + BD$; (2) if $AC = BD$, EFGH is a rhombus; (3) if $AC \perp BD$, EFGH is a rectangle; (4) if $AC =$ and $\perp BD$, EFGH is a square.



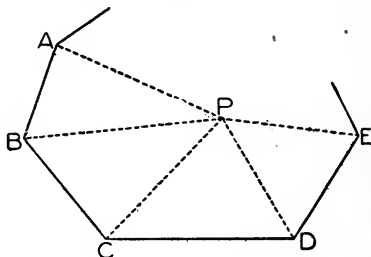
30. The middle points of a pair of opposite sides of a quadrilateral and the middle points of the diagonals are the vertices of a \parallel gm.

31. The st. lines joining the middle points of the opposite sides of a quadrilateral and the st. line joining the middle points of the diagonals are concurrent.



THEOREM 21

The sum of the interior angles of a polygon of n sides is $(2n - 4)$ right angles.



Hypothesis.— $ABCDE$, etc., is a closed polygon of n sides.

To prove that the sum of the interior angles is $(2n - 4)$ rt. \angle s.

Construction.—Take any point P within the polygon and join P to the vertices.

Proof.—The polygon is divided into n \triangle s PAB , PBC , PCD , etc.

The sum of the interior \angle s of each \triangle is two rt. \angle s. (I—10, p. 45.)

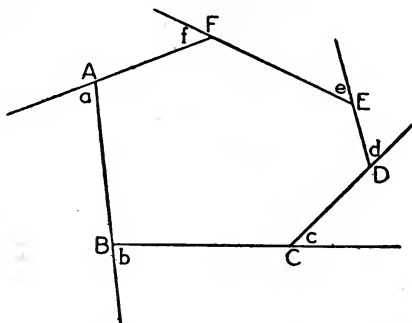
\therefore the sum of the \angle s of the n \triangle s is $2n$ rt. \angle s.

But the \angle s of the n \triangle s make up the interior \angle s of the polygon together with the \angle s about the point P .

And the sum of the \angle s about P equals 4 rt. \angle s.

\therefore the sum of the interior \angle s of the polygon = $(2n - 4)$ rt. \angle s.

Cor.—If the sides of a polygon are produced in order, the sum of the exterior angles thus formed is four right angles.



If the polygon has n sides, the sum of all the st. \angle s at the vertices = $2n$ rt. \angle s.

But, the sum of the interior \angle s = $(2n - 4)$ rt. \angle s. (I—21, p. 72.) *

\therefore , subtracting, $\angle a + \angle b + \text{etc.} = 4$ rt. \angle s.

65.—Exercises

1. Find the number of degrees in an exterior \angle of an equiangular polygon of twelve sides.

Hence, find the number of degrees in each interior \angle .

2. Find the number of degrees in each \angle of (a) an equiangular pentagon; (b) an equiangular hexagon; (c) an equiangular octagon; (d) an equiangular decagon.

3. Each \angle of an equiangular polygon contains 162° . Find the number of sides.

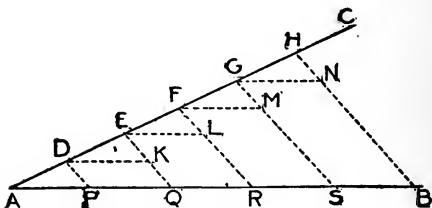
4. Each \angle of an equiangular polygon contains 170° . Find the number of sides.

5. Show that the space around a point may be exactly filled in by six equilateral \triangle s, four squares, or three equiangular hexagons. Draw the diagram in each case.

CONSTRUCTION

PROBLEM 8

To divide a straight line into any number of equal parts.



Let AB be the given st. line.

To divide AB into five equal parts.

Construction.—From A draw a st. line AC .

From AC cut off five equal parts AD, DE, EF, FG, GH .

Join HB .

Through D, E, F, G draw lines $\parallel HB$ cutting AB at P, Q, R, S .

AB is divided into five equal parts at P, Q, R, S .

Proof.—Through D, E, F, G draw $DK, EL, FM, GN \parallel AB$.

$\therefore AE$ cuts the parallels AP, DK ,

$\therefore \angle EDK = \angle DAP$.

(I—9, p. 42.)

$\therefore AE$ cuts the parallels DP, EQ ,

$\therefore \angle ADP = \angle DEQ$.

In $\triangle s ADP, DEK$, $\left\{ \begin{array}{l} \angle DAP = \angle EDK, \\ \angle ADP = \angle DEK, \\ AD = DE, \end{array} \right.$

$\therefore AP = DK$

(I—14, p. 54.)

But $PQ = DK$.

(I—20, p. 67.)

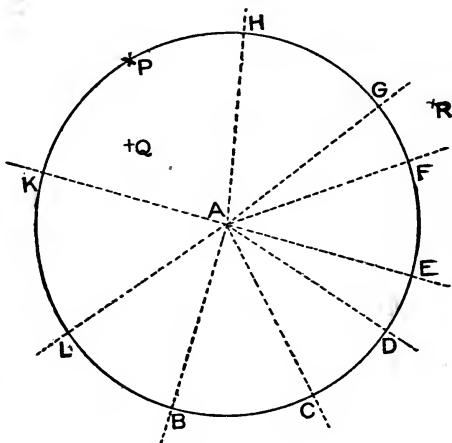
$\therefore PQ = AP$.

Similarly it may be shown that each of QR , RS , $SB = AP$.

By this method a st. line may be divided into any number of equal parts.

LOCI

66. **Example I.**— A is a point and from A straight lines are drawn in different directions in the same plane.



On each line a distance of one inch is measured from A and the resulting points are B, C, D , etc.

Is there any one line that contains all of the points in the plane that are at a distance of one inch from A ?

To answer this question describe a circle with centre A and radius one inch. The circumference of this circle is a line that passes through all the points.

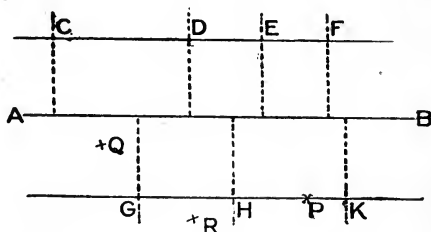
Mark any other point P on the circumference. What is the distance of P from A ? From the definition of a circle the answer to this question is one inch.

If any point Q be taken within the circle, its distance from A is less than one inch, and if any point R be taken without the circle, its distance from A is greater than one inch.

Thus every point in the circumference satisfies the condition of being just one inch from A , and no point, in the plane, that is not on the circumference does satisfy this condition.

This circumference is called the **locus** of all points in the plane that are at a distance of one inch from A .

Example 2:— AB is a straight line of indefinite length, to which any number of perpendiculars are drawn.



On each of these perpendiculars a distance of one centimetre is measured from AB , and the resulting points are C, D, E , etc.

Are there any lines that contain all of the points, such as C, D , etc., that are at a distance of one centimetre from AB ?

Draw two straight lines parallel to **AB**, each at a distance of one centimetre from **AB**, and one or other of these lines will pass through each of the points.

Any point **P** in **CF**, or in **GK**, is at a distance of one centimetre from **AB**; any point **Q** in the space between **CF** and **GK** is less than one centimetre from **AB**, and any point **R** in the plane and neither between **CF** and **GK** nor in one of these lines is more than one centimetre from **AB**.

Thus every point in **CF** and **GK** satisfies the condition of being just one centimetre from **AB**, and no point outside of these lines and in the plane does satisfy this condition.

The two lines **GF**, **GK** make up the locus of all points in the plane that are at a distance of one centimetre from **AB**.

Definition.—When a figure consisting of a line or lines contains all the points that satisfy a given condition, and no others, this figure is called the locus of these points.

67. In place of speaking of the “locus of the points which satisfy a given condition,” the alternative expression “locus of the point which satisfies a given condition” may be used.

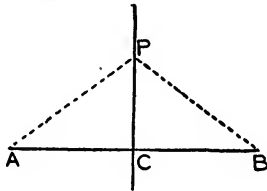
Suppose a point to move in a plane so that it traces out a continuous line, but its distance from a fixed point **A** in the plane is always one inch; then it must move on the circumference of the circle of centre **A** and radius one inch, and the locus of the point in its different positions is that circumference.

The following definition of a locus may thus be given as an alternative to that in § 66.

Definition.—If a point moves on a line, or on lines, so that it constantly satisfies a given condition, the figure consisting of the line, or lines, is the locus of the point.

THEOREM 22

The locus of a point which is equidistant from two given points is the right bisector of the straight line joining the two given points.



Hypothesis.—P is a point equidistant from A and B.

To prove that P is on the right bisector of AB.

Construction.—Bisect AB at C.

Join PC, PA, PB.

Proof.—

In \triangle s PAC, PBC, $\left\{ \begin{array}{l} PA = PB, \\ AC = CB, \\ PC \text{ is common,} \end{array} \right.$

$\therefore \triangle PAC \equiv \triangle PCB,$ (I—4, p. 22.)

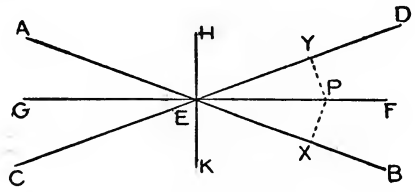
$\therefore \angle PCA = \angle PCB,$

and \therefore P is on the right bisector of AB.

Wrong - J.J.W.
 LOCI May 8/79

THEOREM 23 Do same as Th. 2

The locus of a point which is equidistant from two given intersecting straight lines is the pair of straight lines which bisect the angles between the two straight lines.



Hypothesis. — **AB, CD** are two st. lines cutting at **E**; **GF, HK** are the bisectors of \angle s made by **AB, CD**.

To prove that the locus of a point equidistant from **AB, and CD** consists of **GF and HK**.

Construction. — Take any point **P** in **GF**. Draw **PX \perp AB, PY \perp CD**.

Proof. —

In \triangle s **PEX, PEY**, $\begin{cases} \angle PEX = \angle PEY, \\ \angle PXE = \angle PYE, \\ PE \text{ is common,} \end{cases}$
 $\therefore PX = PY.$ (I—14, p. 54.)

\therefore every point in **GF** is equidistant from **AB and CD**.

Similarly it may be shown that every point in **HK** is equidistant from **AB and CD**.

\therefore the locus of points equidistant from **AB, CD** consists of **GF and HK**.

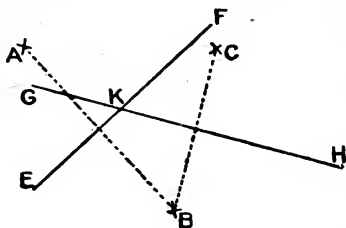
There may be other points as

68. Problem:--To find the point that is equally distant from three given points, that are not in the same straight line.

Let A, B, C be the three given points.

It is required to find a point equally distant from A, B and C .

Draw EF the locus of all points that are equally distant from A and B . (I—22, p. 78.)



Draw GH the locus of all points that are equally distant from B and C .

Let EF and GH meet at K .

Then K is the required point.

K is on EF , $\therefore KA = KB$.

K is on GH , $\therefore KB = KC$.

Consequently K is equally distant from A, B and C .

69.—Exercises

1. Find the locus of the centres of all circles that pass through two given points.

2. Describe a circle to pass through two given points and have its centre in a given st. line.

3. Describe a circle to pass through two given points and have its radius equal to a given st. line. Show that

generally two such circles may be described. When will there be only one? and when none?

4. Find the locus of a point which is equidistant from two given \parallel st. lines.

5. In a given st. line find two points each of which is equally distant from two given intersecting st. lines.

When will there be only one solution?

6. Find the locus of the vertices of all Δ s on a given base which have the medians drawn to the base equal to a given st. line.

7. Find the locus of the vertices of all Δ s on a given base which have one side equal to a given st. line.

8. Construct a Δ having given the base, the median drawn to the base, and the length of one side.

9. Find the locus of the vertices of all Δ s on a given base which have a given altitude.

10. Construct a Δ having given the base, the median drawn to the base, and the altitude.

11. Construct a Δ having given the base, the altitude and one side.

12. Find the locus of a point such that the sum of its distances from two given intersecting st. lines is equal to a given st. line.

13. Find the locus of a point such that the difference of its distances from two given intersecting st. lines is equal to a given st. line.

14. Find the locus of the vertices of all Δ s on a given base which have the median drawn from one end of the base equal to a given st. line.

15. Show that, if the ends of a st. line of constant length slide along two st. lines at rt. \angle s to each other, the locus of its middle point is a circle.

16. **AB** is a st. line and **C** is a point at a distance of 2 cm. from **AB**. Find a point which is 1 cm. from **AB** and 4 cm. from **C**. How many such points can be found?

17. Two st. lines, **AB**, **CD**, intersect each other at an \angle of 45° . Find all the points that are 3 cm. from **AB** and 2 cm. from **CD**.

18. **ABC** is a scalene \triangle . Find a point equidistant from **AB** and **AC**, and also equidistant from **B** and **C**.

19. Find a point equidistant from the three vertices of a given \triangle .

20. Find four points each of which is equidistant from the three sides of a \triangle .

NOTE.—*Produce each side in both directions.*

21. Find the locus of a point at which two equal segments of a st. line subtend equal \angle s.

22. Find the locus of the centre of a circle which shall pass through a given point and have its radius equal to a given st. line.

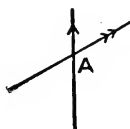
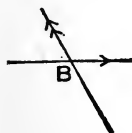
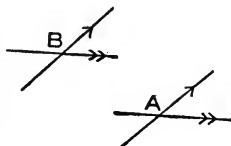
23. A st. line of constant length remains always \parallel to itself, while one of its extremities describes the circumference of a fixed circle. Find the locus of the other extremity.

24. The locus of the middle points of all st. lines drawn from a fixed point to the circumference of a fixed circle is a circle.

Miscellaneous Exercises

1. If a st. line be terminated by two \parallel s, all st. lines drawn through its middle point and terminated by the same \parallel s are bisected at that point.

2. If two lines intersecting at A be respectively \parallel to two lines intersecting at B, each \angle at A is either equal to or supplementary to each \angle at B.



3. If two lines intersecting at A be respectively \perp to two lines intersecting at B, each \angle at A is either equal to or supplementary to each \angle at B.

4. If from any point in the bisector of an \angle st. lines be drawn \parallel to the arms of the \angle and terminated by the arms, these st. lines are equal to each other.

5. In the base of a \triangle find a point such that the st. lines drawn from that point \parallel to the sides of the \triangle and terminated by the sides are equal to each other.

6. One \angle of an isosceles \triangle is half each of the others. Calculate the \angle s.

7. If the \perp from the vertex of a \triangle to the base falls within the \triangle , the segment of the base adjacent to the greater side of the \triangle is the greater.

8. If a star-shaped figure be formed by producing the alternate sides of a polygon of n sides, the sum of the \angle s at the points of the star is $(2n - 8)$ rt. \angle s.

9. In a quadrilateral ABCD, $\angle A = \angle B$ and $\angle C = \angle D$. Prove that $AD = BC$.

10. The bisectors of the \angle s of a \parallel gm form a rectangle, the diagonals of which are \parallel to the sides of the original \parallel gm ; and equal to the difference between them.

11. From **A, B** the ends of a st. line \perp s **AC, BD** are drawn to any st. line. **E** is the middle point of **AB**. Show that **EC = ED**.

X12. If through a point within a \triangle three st. lines be drawn from the vertices to the opposite sides, the sum of these st. lines is greater than half the perimeter of the \triangle .

13. **A, D** are the centres of two circles, and **AB, DE** are two \parallel radii. **EB** cuts the circumferences again at **C, F**. Show that **AC \parallel DF**.

X14. The bisectors of the interior \angle s of a quadrilateral form a quadrilateral of which the opposite \angle s are supplementary.

15. In a given square inscribe an equilateral \triangle having one vertex at a vertex of the square.

16. Through two given points draw two st. lines, forming an equilateral \triangle with a given st. line.

17. Draw an isosceles \triangle having its base in a given st. line, its altitude equal to a given st. line, and its equal sides passing through two given points.

18. If a \perp be drawn from one end of the base of an isosceles \triangle to the opposite side, the \angle between the \perp and the base = half the vertical \angle of the \triangle .

19. If any point **P** in **AD** the bisector of the \angle **A** of \triangle **ABC** be joined to **B** and **C**, the difference between **PB** and **PC** is less than the difference between **AB** and **AC**.

X20. If any point **P** in the bisector of the exterior \angle at **A** in the \triangle **ABC** be joined to **B** and **C**, **PB + PC > AB + AC**.

21. $\triangle BAC$ is a rt. \angle and D is any point. DE is drawn \perp AB and produced to F , making $EF = DE$. DG is drawn \perp AC and produced to H , making $GH = DG$. Show that F, A, H are in the same st. line.

X 22. Construct a \triangle having its perimeter equal to a given st. line and its \angle s respectively equal to the \angle s of a given \triangle .

X 23. In any quadrilateral, the sum of the exterior \angle s at one pair of opposite vertices = the sum of the interior \angle s at the other vertices.

24. If the arms of one \angle be respectively \parallel to the arms of another \angle , the bisectors of the \angle s are either \parallel or \perp .

25. In a given \triangle inscribe a \parallel gm the diagonals of which intersect at a given point.

26. Show that the \perp s from the centre of a circle to two equal chords are equal to each other.

27. Construct a quadrilateral having its sides equal to four given st. lines and one \angle equal to a given \angle .

X 28. The bisector of $\angle A$ of $\triangle ABC$ meets BC at D and BC is produced to E . Show that $\angle ABC + \angle ACE =$ twice $\angle ADC$.

29. The bisectors of \angle s A and B of $\triangle ABC$ intersect at D . Show that $\angle ADB = 90^\circ +$ half of $\angle C$.

X 30. The sides AB, AC of a $\triangle ABC$ are bisected at D, E ; and BE, CD are produced to F, G , so that $EF = BE$ and $DG = CD$. Show that F, A, G are in the same st. line, and that $FA = AG$.

31. $\triangle ABC$ is an isosceles \triangle , having $AB = AC$. AE, AD are equal parts cut off from AB, AC respectively. BD, CE cut at F . Show that FBC and FDE are isosceles \triangle s.

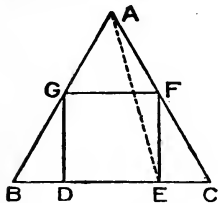
32. In a $\triangle ABC$, the bisector of $\angle A$ and the right bisector of BC meet at D . DE, DF are drawn $\perp AB, AC$ respectively. Show that the point D is not within the \triangle , that $AE = AF$ and that $BE = CF$.

X 33. $ABCD$ is a quadrilateral having $\angle B = \angle C$ and $AB < CD$. Prove that $\angle A > \angle D$.

34. Through a given point draw a st. line cutting two intersecting st. lines and forming an isosceles \triangle with them.

Show that two such lines can be drawn through the given point.

35. If ACB be a st. line and ACD, BCD two adjacent \angle s, any \parallel to AB will meet the bisectors of these \angle s in points equally distant from where it meets CD .



36. Inscribe a square in a given equilateral \triangle .

NOTE.—Draw a sketch as in the diagram given here. Join AE .

What is the number of degrees in $\angle CAE$?

37. ABC is a \triangle , AX is $\perp BC$, and AD bisects $\angle BAC$. Show that $\angle XAD$ equals half the difference of \angle s B and C .

38. Construct a \parallel gm having its diagonals and a side respectively equal to three given st. lines.

39. Find a point in each of two \parallel st. lines such that the two points are equally distant from a given point and the st. line joining them subtends a rt. \angle at the given point.

40. P, Q are two given points on the same side of a given st. line BC . Find the position of a point A in BC such that $\angle PAB = \angle QAC$.

NOTE.—If P, Q are two points on a billiard table and BC the side of the table, a ball starting from P and reflected from BC at A would pass through Q .

41. Find the path of a billiard ball which, starting from a given point, is reflected from the four sides of the table and passes through another given point.

42. $\angle BAC$ is a given \angle and D, E are two given st. lines. Find a point P such that its distances from AB, AC equal D, E respectively.

43. Find in a side of a \triangle a point such that the sum of the two st. lines drawn from the point \parallel to the other sides and terminated by them is equal to a given st. line.

44. AEB, CED are two st. lines, and each of the quadrilaterals $CEAF, BEDG$ is a rhombus. Prove that FEG is a st. line.

45. F is a point within the $\triangle ABC$ such that $\angle FBC = \angle FCB$. BF, CF produced meet AC, AB at D, E respectively. Prove that if $\angle AFD = \angle AFE$, $\triangle ABC$ is isosceles.

46. D is a point in the base BC of an equilateral $\triangle ABC$. E is the middle point of AD . Prove that $EC > ED$.

47. ABC is a \triangle of which $\angle BAC$ is obtuse, O a point within it; BO, CO meet AC, AB at D, E respectively. Prove that $BD + CE > BE + ED + DC$.

48. D, E, F are points in the sides BC, CA, AB of an equilateral \triangle and are such that $BD = CE = AF$. If AD, BE, CF do not all pass through one point, they form an equilateral \triangle .

49. The bisector of $\angle A$ of $\triangle ABC$ meets BC at D . DE, DF drawn $\parallel AB, AC$ respectively meet AC, AB at E, F . Prove that $AEDF$ is a rhombus.

50. Through each angular point of a \triangle a st. line is drawn \parallel the opposite side: prove that the \triangle formed by these three st. lines is equiangular to the given \triangle .

51. AD , BE , CF respectively bisect the interior $\angle A$ and the exterior \angle s at B and C of the $\triangle ABC$. Show that no two of the lines AD , BE , CF can be \parallel .

52. DE is \parallel to the base AB of the isosceles $\triangle CAB$ and cuts CA , CB , or those sides produced, at D , E respectively. AE , BD cut at F . Prove that DEF is an isosceles \triangle .

53. Through A , B the extremities of a diameter of a circle \parallel chords AC , BD are drawn. Prove that $AC = BD$; and that CD is a diameter of the circle.

54. The median drawn from the vertex of a \triangle is $>$, $=$ or $<$ half the base according as the vertical \angle is acute, right or obtuse.

55. ABC is a \triangle , obtuse- \angle d at C ; st. lines are drawn bisecting CA , CB at rt. \angle s, cutting AB in D , E respectively. Prove that $\angle DCE$ is equal to twice the excess of $\angle ACB$ over a rt. \angle .

56. With one extremity C of the base BC of an isosceles $\triangle ABC$ as centre, and radius CB , a circle is described cutting AB , AC at D , E respectively. Prove that $DE \parallel$ to the bisector of $\angle B$.

57. In $\triangle ABC$ side BC is produced to D . Prove that the \angle between the bisectors of \angle s ABC , $ACD =$ half the $\angle A$.

58. Through the vertices of $\triangle ABC$, st. lines falling within the \triangle are drawn making equal \angle s BAL , CBM , ACN ; if these lines intersect in D , E , F , prove $\triangle DEF$ equiangular to $\triangle ABC$.

59. If the \angle between two adjacent sides of a \parallel gm be increased, while their lengths do not alter, the diagonal through the point of intersection will decrease.

60. A, B, C are three given points. Find a point equidistant from A, B and such that its distance from C equals a given st. line. When is the problem impossible?

61. Through a fixed point draw a st. line which shall make with a given st. line adjacent \angle s the difference of which = a given \angle .

62. Construct a \triangle having given one \angle and the lengths of the \perp s from the vertices of the other \angle s on the opposite sides.

63. Construct an isosceles \triangle having given the vertical \angle and the altitude.

64. Construct an isosceles \triangle having given the perimeter and altitude.

65. Prove that the quadrilateral formed by joining the extremities of two diameters of a circle is a rectangle.

66. In a given \parallel gm inscribe a rhombus, such that one diagonal passes through a given point.

67. St. lines are drawn from a given point to a given st. line. Find the locus of the middle points of the st. lines.

68. St. lines are drawn from a given point to the circumference of a given circle. Find the locus of the middle points of the st. lines.

69. The sum of the \perp s from any point within an equilateral \triangle to the three sides is equal to the altitude of the \triangle .

70. Draw a square which has the sum of a side and a diagonal equal to 3 inches.

71. Draw a square in which the difference between a diagonal and a side is 1 inch.

72. Draw a rectangle having one side 2 inches in length, and subtending an \angle of 40° at the point of intersection of the diagonals.

(Use a protractor in Exercises 72 to 86.)

73. Draw a \parallel gm with diagonals 2 inches and 4 inches and their \angle of intersection 50° .

74. Draw a \parallel gm with diagonals 4 inches and 7 inches and one side 5 inches.

75. Draw a \parallel gm with side 3 inches, diagonal $2\frac{1}{2}$ inches and \angle 35° . Show that there are two solutions.

76. Draw a \parallel gm with side $2\frac{3}{8}$ inches, \angle 70° and diagonal opposite \angle of 70° equal to 4 inches.

77. Draw a rectangle having the perimeter 8 inches and an \angle between the diagonals 80° .

78. Draw a rectangle having the difference of two sides 1 inch and an \angle between the diagonals 50° .

79. Draw a rectangle which has the perimeter 9 inches and a diagonal $3\frac{1}{2}$ inches.

80. Draw an \angle of 55° . Find within the \angle a point which is 1 inch from one arm and 2 inches from the other.

81. Construct a \triangle in which side $a = 7$ cm., $b + c = 10.6$ cm. and $\angle A = 78^\circ$.

82. Construct a \triangle with perimeter 4 inches and \angle s 70° and 50° .

83. AB , CD are two \parallel st. lines; P , Q two fixed points. Find a point equidistant from AB , CD and also equidistant from P and Q . When is this impossible?

84. Through two given points on the same side of a given st. line draw two st. lines so as to form with the given st. line an equilateral \triangle .

85. Construct a rhombus with one diagonal 2 inches and the opposite \angle 100° .

86. Construct a \triangle in which $a = 8$ cm., $b - c = 2$ cm., $\angle C = 50^\circ$.

87. Squares $ABGE$, $ACHF$ are described externally on two sides of a $\triangle ABC$. Prove that the median AD of the \triangle is $\perp EF$ and equal to half of EF .

NOTE.—Rotate $\triangle ABC$ through a rt. \angle making AC coincide with AF .

88. Prove also in Ex. 87 that EC is \perp and $= BF$.

89. Trisect a rt. \angle .

90. From any point in the base of an isosceles \triangle st. lines are drawn \parallel to the equal sides and terminated by them. Prove that the sum of these lines = one of the equal sides.

91. ABC is a st. line such that $AB = BC$. \perp s are drawn from A, B, C to another st. line EF . Prove that the \perp from B = half the sum of the \perp s from A and C , unless EF passes between A and C , and then the \perp from B = half the difference of the \perp s from A and C .

92. AD is the bisector of $\angle A$ of $\triangle ABC$, and M the middle point of BC . BE and CF are $\perp AD$. Prove that $ME = MF$.

93. E, F are the middle points of AD, BC respectively in the \parallel gm $ABCD$. Prove that BE, DF trisect AC .

94. Find a point P in the side AC of a $\triangle ABC$ so that AP may be equal to the \perp from P to BC .

95. If the st. line AB be bisected at C and produced to D , prove that CD is half the sum of AD , BD .

96. In $\triangle ABC$ side $AC >$ side AB ; $AX \perp BC$ and AD is a median. Prove that (1) $\angle CAX > \angle BAX$; (2) $\angle CAD < \angle DAB$; (3) the bisector of $\angle BAC$ falls between AX and AD .

97. The median of a $\triangle ABC$ drawn from A is not less than the bisector of $\angle A$.

98. In a quadrilateral $ABCD$, $AB = DC$ and $\angle B = \angle C$. Prove that $AD \parallel BC$.

99. If two medians of a \triangle are equal, the \triangle is isosceles.

NOTE.—Use *Ex. 6*, § 64.

100. If both pairs of opposite \angle s of a quadrilateral are equal, the quadrilateral is a \parallel gm.

101. Find the point on the base of a \triangle such that the difference of the \perp s from it to the sides is equal to a given st. line.

102. Find the point on the base of a \triangle such that the sum of the \perp s from it to the sides is equal to a given st. line.

103. Show that the three exterior \angle s at A , C , E , in the hexagon $ABCDEF$, are together less than the three interior \angle s at B , D , F by two rt. \angle s.

BOOK II

AREAS OF PARALLELOGRAMS AND TRIANGLES

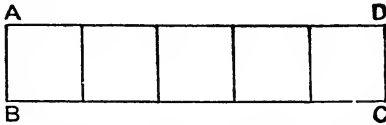
70. A square unit of area is a square, each side of which is equal to a unit of length.

Examples:—A square inch is a square each side of which is one inch; a square centimetre is a square each side of which is one centimetre.

The acre is an exceptional case.

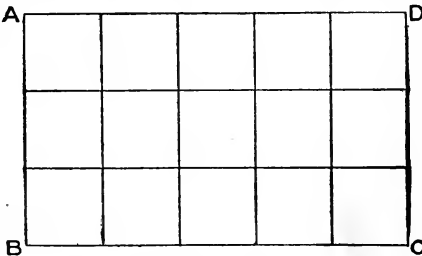
71. A numerical measure of any area is the number of times the area contains some unit of area.

ABCD is a rectangle one centimetre wide and five centimetres long.



This rectangle is a strip divided into five square centimetres, and consequently the numerical measure of its area in square centimetres is 5.

72. **ABCD** is a rectangle 3 cm. wide and 5 cm. long.



This rectangle is divided into 5 strips of 3 sq. cm. each, or into 3 strips of 5 sq. cm. each, and consequently

the measure of the area in square centimetres is 5×3 sq. cm., or 3×5 sq. cm.

Similarly, if the length of a rectangle is 2·34 inches and its breadth ·56 of an inch, the one-hundredth of an inch may be taken as the unit and the rectangle can be divided into 234 strips each containing 56 square one-hundredths of an inch. The measure of the area then is 234×56 of these small squares, ten thousand (100×100) of which make one square inch.

This method of expressing the area of a rectangle may be carried to any degree of approximation, so that in all cases the numerical measure of its area is equal to the product of its length by its breadth.

In a rectangle any side may be called the base, and then either of the adjacent sides is the altitude.

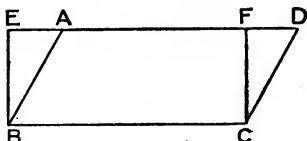
A rectangle, as **ABCD**, is commonly represented by the symbol **AB · BC**, where **AB** and **BC** may be taken to represent the number of units in the length and the breadth respectively.

Or, if a be the measure of the base of a rectangle and b the measure of its altitude, the area is ab .

In the case of a square, the base is equal to the altitude, and if the measure of each be a , the area is a^2 .

THEOREM 1

The area of a parallelogram is equal to that of a rectangle on the same base and of the same altitude.



Hypothesis.— $ABCD$ is a \parallel gm and $EBCF$ a rectangle on the same base BC and of the same altitude EB .

To prove that the area of the \parallel gm $ABCD$ = the area of rect. $EBCF$.

Proof.— \because ED cuts the \parallel s AB, DC ,
 $\therefore \angle EAB = \angle FDC$. (I—9, p. 42.)
 \because $ABCD$ is a \parallel gm,
 $\therefore AB = CD$. (I—20, p. 67.)

In \triangle s EAB, FDC , $\begin{cases} \angle EAB = \angle FDC, \\ \angle AEB = \angle DFC, \\ AB = DC, \end{cases}$
 $\therefore \triangle AEB = \triangle FDC$, (I—14, p. 54.)

Figure $EBCD - \triangle EAB = \parallel$ gm $ABCD$,

Figure $EBCD - \triangle FDC = \text{rect. } EBCF$;

and as equal parts have been taken from the same area, the remainders are equal.

$\therefore \parallel$ gm $ABCD = \text{rect. } EBCF$.

Cor.—If a be the measure of the base of a \parallel gm and b the measure of its altitude, the area, being the same as that of a rect. of the same base and altitude, = ab .

73.—Practical Exercises

1. Draw a \parallel gm having two adjacent sides 6.4 cm. and 7.3 cm. and the contained \angle 30° . Find its area.

2. Draw a \parallel gm having the two diagonals 4.8 cm. and 6.8 cm. and an \angle between the diagonals 75° . Find its area.

3. The area of a \parallel gm is 50 sq. cm., one side is 10 cm. and one \angle is 60° . Construct the \parallel gm, and measure the other side.

4. Draw a rectangle of base 7 cm. and height 4 cm. On the same base construct a \parallel gm having the same area as the rectangle and two of its sides each 65 mm. Measure one of the smaller \angle s of the \parallel gm.

5. Make a \parallel gm having sides 10 and 7 cm. and one \angle 60° . Make a rhombus equal in area to the \parallel gm and having each side 10 cm. Measure the shorter diagonal of the rhombus.

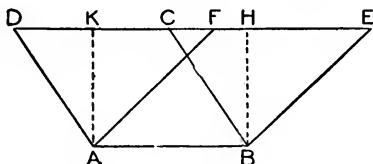
6. Make a rectangle 8 cm. by 5 cm. Construct a \parallel gm equal in area to the rectangle and having two sides 7 cm. and 8 cm. Construct a rhombus equal in area to the \parallel gm and having each side 7 cm. Measure the shorter diagonal of the rhombus.

7. Make a rhombus having each side 8 cm. and its area 50 sq. cm. Measure the shorter diagonal.

ANSWERS:—1. 23.4 sq. cm. nearly. 2. 15.8 sq. cm. nearly.
3. 57.7 mm. nearly. 4. 38° nearly. 5. 64 mm. nearly.
6. 64 mm. nearly. 7. 69 mm. nearly.

THEOREM 2

Parallelograms on the same base and between the same parallels are equal in area.



Hypothesis.— $ABCD$, $ABEF$ are \parallel gms on the same base AB and between the same \parallel s AB , DE .

To prove that \parallel gm $ABCD = \parallel$ gm $ABEF$.

Construction.—Draw AK , BH each \perp to both AB and DE .

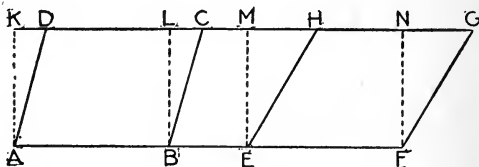
Proof.— $\because \parallel$ gm $ABCD = \text{rect. } ABHK$, (II—1, p. 95.)

and \parallel gm $ABEF = \text{rect. } ABHK$,

$\therefore \parallel$ gm $ABCD = \parallel$ gm $ABEF$.

THEOREM 3

Parallelograms on equal bases and between the same parallels are equal in area.



Hypothesis.— $ABCD$, $EFGH$ are \parallel gms on the equal bases AB , EF and between the same \parallel s AF , DG .

To prove that \parallel gm $ABCD = \parallel$ gm $EFGH$.

Construction.—Draw AK , BL , EM , FN each \perp to both AF , DG .

Proof.— $\because AB = EF$,

and $AK = EM$,

(I—20, p. 67.)

\therefore rect. $KB =$ rect MF .

But \parallel gm $ABCD =$ rect. KB ,

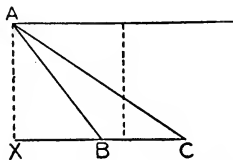
(II—1, p. 95.)

and \parallel gm $EFGH =$ rect. MF ,

$\therefore \parallel$ gm $ABCD = \parallel$ gm $EFGH$.

74. Draw an acute- \angle d $\triangle ABC$. Draw the \perp from **A** to **BC**. Draw through **A**, a st. line \parallel **BC**. Show that the \perp distance between these \parallel lines at any place = the altitude of $\triangle ABC$.

Draw an obtuse- \angle d $\triangle ABC$, having the obtuse \angle at **B**. Draw the altitude **AX**. Show that it falls without the \triangle . Draw through **A**, a st. line \parallel **BC**. Show that the distance between these \parallel lines at any place = the altitude of the \triangle .



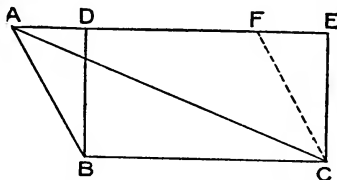
Taking **C** as the vertex and **AB** as the base, draw the altitude.

If a \triangle is between two \parallel s, having its base in one of the \parallel s and its vertex in the other, its altitude is the distance between the \parallel s.



THEOREM 4

The area of a triangle is half that of the rectangle on the same base and of the same altitude as the triangle.



Hypothesis.— ABC is a \triangle and $DBCE$ a rectangle on the same base and of the same altitude BD .

To prove that area of $\triangle ABC =$ half that of rect. $DBCE$.

Construction.—Through C draw $CF \parallel BA$.

Proof.— $\because AC$ is a diagonal of $\parallel\text{gm } ABCF$,

$$\therefore \triangle ABC = \text{half of } \parallel\text{gm } ABCF. \text{ (I—20, p. 67.)}$$

$$\text{But } \parallel\text{gm } ABCF = \text{rect. } DBCE, \quad \text{(II.—1, p. 95.)}$$

$$\therefore \triangle ABC = \text{half of rect. } DBCE.$$

Cor.—If a be the measure of the base of a \triangle and b the measure of its altitude, the measure of its area is $\frac{1}{2}ab$.

75.—Practical Exercises

1. Draw a rt.- \angle \triangle having the sides that contain the right \angle 56 mm. and 72 mm. Find the area of the \triangle .
2. Make a $\triangle ABC$, having $b = 6$ cm., $c = 8$ cm., and $\angle A = 72^\circ$. Find its area.
3. Draw a \triangle having its sides 73 mm., 57 mm. and 48 mm. Find its area.

4. Find the area of the \triangle : $a = 10$ cm., $\angle B = 42^\circ$, $\angle C = 58^\circ$.

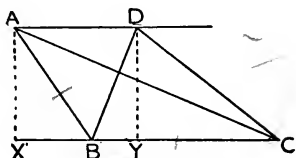
5. The sides of a triangular field are 36 chains, 25 chains and 29 chains. Draw a diagram and find the number of acres in the field. (Scale: 1 mm. to the chain.)

6. Two sides of a triangular field are 41 and 38 chains and the contained \angle is 70° . Find its area in acres.

ANSWERS:—1. 20.16 sq. cm.; 2. 23 sq. cm. nearly; 3. 13.7 sq. cm. nearly; 4. 28.8 sq. cm.; 5. 36 ac.; 6. 73 ac. nearly.

THEOREM 5

Triangles on the same base and between the same parallels are equal in area.



Hypothesis.— ABC , DBC are \triangle s on the same base BC and between the same \parallel s AD , BC .

To prove that $\triangle ABC = \triangle DBC$.

Construction.—Draw AX , $DY \perp BC$.

Proof.— $\triangle ABC = \frac{1}{2}$ rect. $AX \cdot BC$. (II—4, p. 100.)

$\triangle DBC = \frac{1}{2}$ rect. $DY \cdot BC$.

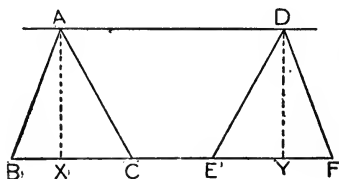
But, $\because AX = DY$,

\therefore rect. $AX \cdot BC =$ rect. $DY \cdot BC$.

and $\therefore \triangle ABC = \triangle DBC$.

THEOREM 6

Triangles on equal bases and between the same parallels are equal in area.



Hypothesis.— $\triangle ABC$, $\triangle DEF$ are \triangle s on equal bases BC , EF and between the same \parallel s AD , BF .

To prove that $\triangle ABC = \triangle DEF$.

Construction.—Draw AX , $DY \perp BF$.

Proof.— $\triangle ABC = \frac{1}{2}$ rect. $AX \cdot BC$. (II.—4, p. 100.)

$\triangle DEF = \frac{1}{2}$ rect. $DY \cdot EF$.

But, $\because BC = EF$,

and $AX = DY$, (I.—20, p. 67.)

\therefore rect. $AX \cdot BC =$ rect. $DY \cdot EF$.

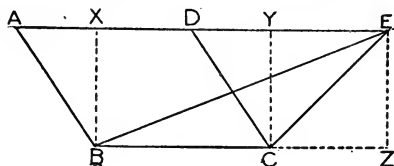
Hence, $\triangle ABC = \triangle DEF$.

Cor. 1.—Triangles on equal bases and of the same altitude are equal in area.

Cor. 2.—A median bisects the area of the triangle.

THEOREM 7

If a parallelogram and a triangle are on the same base and between the same parallels, the parallelogram is double the triangle.



Hypothesis.— $ABCD$ is a \parallel gm and EBC a \triangle on the same base BC and between the same \parallel s AE, BC .

To prove that \parallel gm $ABCD =$ twice $\triangle EBC$.

Construction.—Draw $BX, CY, EZ \perp BC$ and AE .

Proof.— \parallel gm $ABCD =$ rect. $BX \cdot BC$. (II—1, p. 95.)

$\triangle EBC = \frac{1}{2}$ rect. $EZ \cdot BC$. (II—4, p. 100.)

But, $\because BX = EZ$ (I—20, p. 67.)

\therefore rect. $BX \cdot BC =$ rect. $EZ \cdot BC$.

And $\therefore \parallel$ gm $ABCD =$ twice $\triangle EBC$.

76.—Exercises

✕ 1. \triangle s ABC, DEF are between the same \parallel s AD and $BCEF$, and $BC > EF$. Prove that $\triangle ABC > \triangle DEF$.

2. On the same base with a \parallel gm construct a rectangle equal in area to the \parallel gm.

3. On the same base with a given \parallel gm, construct a \parallel gm equal in area to the given \parallel gm, and having one of its sides equal to a given st. line.

✕ 4. Construct a rect. equal in area to a given \parallel gm, and having one of its sides equal to a given st. line.

5. Make a \parallel gm with sides 5 cm. and 3 cm., and contained $\angle 125^\circ$. Construct an equivalent rect. having one side 1.5 cm.

6. On the same base as a given \triangle construct a rect. equal in area to the \triangle .

7. Construct a rect. equal in area to a given \triangle , and having one of its sides equal to a given st. line.

✕ 8. On the same base with a \parallel gm construct a rhombus equal in area to the \parallel gm.

✕ 9. Construct a rhombus equal in area to a given \parallel gm, and having each of its sides equal to a given st. line.

10. On the same base with a given \triangle , construct a rt.- \angle d \triangle equal in area to the given \triangle .

✕ 11. On the same base with a given \triangle , construct an isosceles \triangle equal in area to the given \triangle .

✕ 12. If, in the \parallel gm $ABCD$, P be any point between AB , CD produced indefinitely, the sum of the \triangle s PAB , PCD equals half the \parallel gm; and if P be any point not between AB , CD , the difference of the \triangle s PAB , PCD equals half the \parallel gm.

✕ 13. AB and ECD are two \parallel st. lines; BF , DF are drawn \parallel AD , AE respectively; prove that \triangle s ABC , DEF are equal to each other.

14. On the same base with a given \triangle , construct a \triangle equal in area to the given \triangle , and having its vertex in a given st. line.

/ 15. If two \triangle s have two sides of one respectively equal to two sides of the other and the contained \angle s supplementary, the \triangle s are equal in area.

/ 16. $ABCD$ is a \parallel gm, and P is a point in the diagonal AC . Prove that $\triangle PAB = \triangle PAD$.

10
17. P is a point within a \parallel gm ABCD. Prove that $\triangle PAC$ equals the difference between \triangle s PAB, PAD.

X 18. In $\triangle ABC$, BC and CA are produced to P and Q respectively, such that CP = one-half of BC, and AQ = one-half of CA. Show that $\triangle QCP =$ three-fourths of $\triangle ABC$.

X 19. The medians BE, CD of the $\triangle ABC$ intersect at F. Show that $\triangle BFC =$ quadrilateral ADFE.

X 20. On the sides AB, BC of a \triangle the \parallel gms ABDE, CDFG are described external to the \triangle : ED and GF meet at H and BH is joined. On AC the \parallel gm CAKL is described with CL and AK \parallel and = HB. Prove \parallel gm AL = \parallel gm AD + \parallel gm CF.

X 21. Two \triangle s are equal in area and between the same \parallel s. Prove that they are on equal bases.

X 22. Of all \triangle s on a given base and between the same \parallel s, the isosceles \triangle has the least perimeter.

23. ABCD is a \parallel gm, and E is a point such that AE, CE are respectively \perp and \parallel to BD. Show that BE = CD.

X 24. The side AB of \parallel gm ABCD is produced to E and DE cuts BC at F. AF and CE are joined. Prove that $\triangle AFE = \triangle CBE$.

25. In the quadrilateral ABCD, AB \parallel CD. If AB = a, CD = b and the distance between AB and CD = h, show that the area of ABCD = $\frac{1}{2} h (a + b)$.

26. Two sides AB, AC of a \triangle are given in length, find the $\angle A$ for which the area of the \triangle will be greatest.

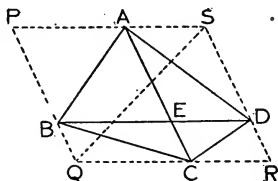
27. The medians AD, BE of $\triangle ABC$ intersect at G, and CG is joined. Prove that the three lines AG, BG, CG trisect the area of the \triangle .

28. Bisect the area of a \triangle by a st. line drawn through a vertex.

29. Trisect the area of a \triangle by two st. lines drawn through a vertex.

30. Bisect the area of a \triangle by a st. line drawn through a given point in one of the sides.

31. Trisect the area of a \triangle by two st. lines drawn through a given point in one of the sides.



32. The area of any quadrilateral **ABCD** is equal to that of a \triangle having two sides and their included \angle respectively equal to the diagonals of the quadrilateral and their included \angle .

NOTE.—Draw **PS** and **QR** \parallel **BD**, **PQ** and **SR** \parallel **AC**. Join **SQ**.

33. Prove that in a rhombus the distance between one pair of opposite sides equals the distance between the other pair.

34. \parallel gms are described on the same base and between the same \parallel s. Find the locus of the intersection of their diagonals.

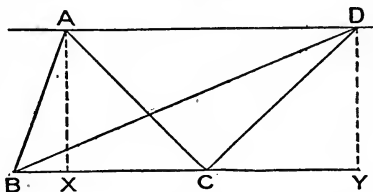
35. Prove that the area of a rhombus is half the product of the lengths of its diagonals.

36. **ABCD** is a quadrilateral in which **AB** \parallel **CD**, **E** is the middle point of **AD**. Prove that \triangle **BEC** = $\frac{1}{2}$ quadrilateral **ABCD**.

37. Divide a given \triangle into seven equal parts.

THEOREM 8

If two equal triangles are on the same side of a common base, the straight line joining their vertices is parallel to the common base.



Hypothesis.— $\triangle ABC$, $\triangle DBC$ are two equal \triangle s on the same side of the common base BC .

To prove that $AD \parallel BC$.

Construction.—Draw AX and $DY \perp BC$.

Proof.— $\triangle ABC = \frac{1}{2}$ rect. $BC \cdot AX$. (II—4, p. 100.)

$\triangle DBC = \frac{1}{2}$ rect. $BC \cdot DY$;

but $\triangle ABC = \triangle DBC$,

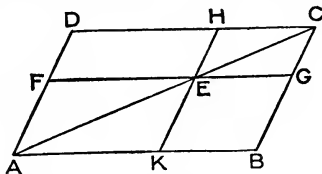
$\therefore \frac{1}{2}$ rect. $BC \cdot AX = \frac{1}{2}$ rect. $BC \cdot DY$

and hence $AX = DY$,

that is, AX and DY are both $=$ and \parallel to each other

$\therefore AD \parallel XY$. (I—19, p. 66.)

77. If, through any point E , in the diagonal AC of a parallelogram BD , two straight lines FEG , HEK be drawn parallel respectively to the sides DC , DA of the parallelogram,

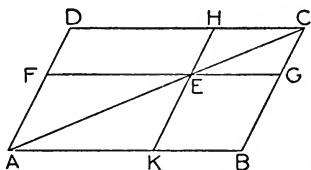


the \parallel gms FK and HG are said to be parallelograms

about the diagonal **AC**, and the \parallel gms **DE**, **EB** are called the complements of the \parallel gms **FK**, **HG**, which are about the diagonal.

THEOREM 9

The complements of the parallelograms about the diagonal of any parallelogram are equal to each other.



Hypothesis.—**FK** and **HG** are \parallel gms about the diagonal **AC** of the \parallel gm **ABCD**.

To prove that the complements **DE**, **EB** are equal to each other.

Proof.— \because **AE** is a diagonal of \parallel gm **FK**,
 $\therefore \triangle AFE = \triangle AKE$. (I—20, p. 67.)

Similarly $\triangle HEC = \triangle EGC$.

$\therefore \triangle AFE + \triangle HEC = \triangle AKE + \triangle EGC$.

But, \because **AC** is a diagonal of \parallel gm **ABCD**

$\therefore \triangle ADC = \triangle ABC$.

$\therefore \triangle ADC - (\triangle AFE + \triangle HEC)$

$= \triangle ABC - (\triangle AKE + \triangle EGC)$.

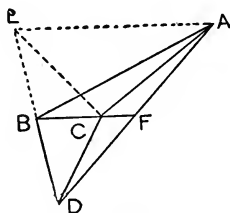
$\therefore \parallel$ gm **DE** = \parallel gm **EB**.

78.—Exercises

1. If two equal \triangle s be on equal segments of the same st. line and on the same side of the line, the st. line joining their vertices is \parallel to the line containing their bases.

2. Through P, a point within the \parallel gm ABCD, EPF is drawn \parallel AB and GPH is drawn \parallel AD. If \parallel gm AP = \parallel gm PC, show that P is on the diagonal BD. (Converse of Theorem 9.)

3. Two equal \triangle s ABC, DBC are on opposite sides of the same base. Prove that AD is bisected by BC, or BC produced.



NOTE.—Produce DB making BE = DB. Join EA, EC.

Give another proof of this proposition using \perp s from A and D to BC and II—4, p. 100.

4. The median drawn to the base of a \triangle bisects all st. lines drawn \parallel to the base and terminated by the sides, or the sides produced.

5. P is a point within a \triangle ABC and is such that \triangle PAB + \triangle PBC is constant. Prove that the locus of P is a st. line \parallel AC.

6. \parallel gms about the diagonal of a square are squares.

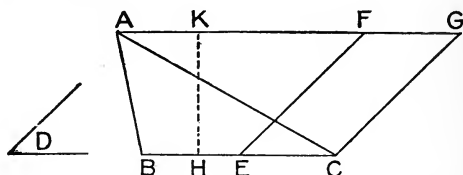
7. D, E, F are respectively the middle points of the sides BC, CA, AB in the \triangle ABC. Prove \triangle BEF = \triangle CEF and hence that EF \parallel BC.

8. In the diagram of II—9, show that FK \parallel HG.

CONSTRUCTIONS

PROBLEM 1

To construct a parallelogram equal in area to a given triangle and having one of its angles equal to a given angle.



Let ABC be the given \triangle and D the given \angle .

It is required to construct a \parallel gm equal in area to $\triangle ABC$ and having one \angle equal to $\angle D$.

Construction.—Through A draw $AFG \parallel BC$. Bisect BC at E . At E make $\angle CEF = \angle D$. Through C draw $CG \parallel EF$.

FC is the required \parallel gm.

Proof.—Draw any line $HK \perp$ to the two \parallel st. lines.

HK is the common altitude of the \parallel gm FC and the $\triangle ABC$.

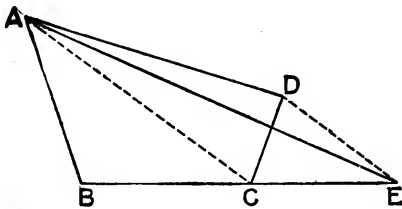
$$\parallel\text{gm } FC = \text{rect. } EC \cdot HK. \quad (\text{II—1, p. 95.})$$

$$= \frac{1}{2} \text{ rect. } BC \cdot HK, \because EC = \frac{1}{2} BC,$$

$$= \triangle ABC. \quad (\text{II—4, p. 100.})$$

PROBLEM 2

To construct a triangle equal in area to a given quadrilateral.



Let $ABCD$ be the given quadrilateral.

It is required to construct a \triangle equal in area to $ABCD$.

Construction.—Join AC . Through D draw $DE \parallel AC$ and meeting BC produced at E . Join AE .

$$\triangle ABE = \text{quadrilateral } ABCD.$$

Proof.— $\because DE \parallel AC$,

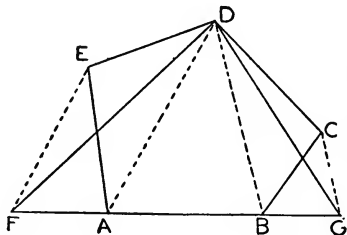
$$\therefore \triangle EAC = \triangle DAC. \quad (\text{II—5, p. 101.})$$

To each of these equals add $\triangle ABC$.

Then $\triangle ABE = \text{quadrilateral } ABCD$.

PROBLEM 3

To construct a triangle equal in area to a given rectilinear figure.



Let the pentagon $ABCDE$ be the given rectilinear figure.

Construction.—Join AD , BD . Through E , draw $EF \parallel AD$ and meeting BA at F . Through C draw $CG \parallel BD$ and meeting AB at G .

Join DF , DG .

$$\triangle DFG = \text{figure } ABCDE.$$

Proof.—

$$\begin{aligned} &\because EF \parallel AD, \\ \therefore \triangle DFA &= \triangle DEA. \quad (\text{II—5, p. 101.}) \end{aligned}$$

$$\because CG \parallel DB,$$

$$\therefore \triangle DGB = \triangle DCB.$$

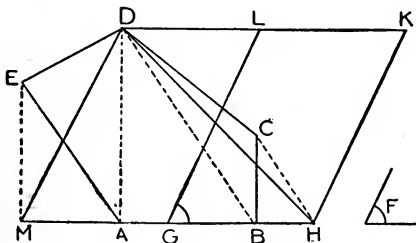
$$\begin{aligned} \therefore \triangle DFA + \triangle DAB + \triangle DBG \\ &= \triangle DEA + \triangle DAB + \triangle DCB; \end{aligned}$$

$$\text{i.e., } \triangle DFG = \text{figure } ABCDE.$$

By this method a \triangle may be constructed equal in area to a given rectilinear figure of any number of sides; *e.g.*, for a figure of seven sides, an equivalent figure of five sides may be constructed, and then, as in the construction just given, a \triangle may be constructed equal to the figure of five sides.

PROBLEM 4

To describe a parallelogram equal to a given rectilinear figure, and having an angle equal to a given angle.



Let $ABCDE$ be the given rectilinear figure and F the given \angle .

It is required to construct a $\parallel\text{gm} = ABCDE$, and having an $\angle = \angle F$.

Construction.—Make $\triangle DMH$ equal in area to figure $ABCDE$. (II—Prob. 3, p. 112.)

Make $\parallel\text{gm} LGHK = \triangle DMH$, and having $\angle LGH = \angle F$. (II—Prob. 1, p. 110.)

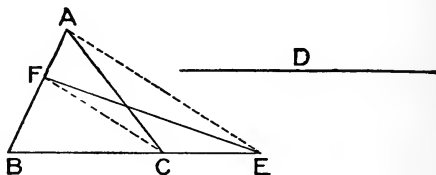
Then $\parallel\text{gm} LGHK = \text{figure } ABCDE$, and has $\angle LGH = \angle F$.

79.—Exercises

1. Construct a rect. equal in area to a given \triangle .
2. Construct a rect. equal in area to a given quadrilateral.
3. Construct a quadrilateral equal in area to a given hexagon.
4. On one side of a given \triangle construct a rhombus equal in area to the given \triangle .
5. Construct a \triangle equal in area to a given $\parallel\text{gm}$, and having one of its \angle s = a given \angle .

PROBLEM 5

To construct a triangle equal in area to a given triangle and having one of its sides equal to a given straight line.



Let ABC be the given \triangle and D the given st. line.

It is required to make a $\triangle = \triangle ABC$ and having one side = D .

Construction.—From BC , produced if necessary, cut off $BE = D$. Join AE . Through C draw $CF \parallel EA$ and meeting BA , or BA produced at F . Join FE .

FBE is the required \triangle .

Proof.— $\because FC \parallel AE,$

$$\therefore \triangle FCE = \triangle AFC. \quad (\text{II—5, p. 101.})$$

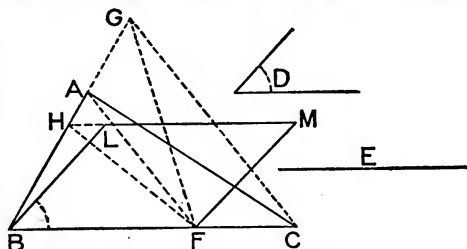
$$\therefore \triangle FBC + \triangle FCE = \triangle FBC + \triangle AFC,$$

$$\text{i.e., } \triangle FBE = \triangle ABC,$$

and side BE was made = D .

PROBLEM 6

On a straight line of given length to make a parallelogram equal in area to a given triangle and having an angle equal to a given angle.



Let ABC be the given \triangle , E the given st. line and D the given \angle .

It is required to make a \parallel gm equal in area to $\triangle ABC$, having one side equal in length to E , and one \angle equal to D .

Construction.—From BC , produced if necessary, cut off $BF = E$. Join AF . Through C draw $CG \parallel FA$ meeting BA , or BA produced, at G . Join GF . Bisect BG at H . Through H draw $HM \parallel BC$. At B make $\angle CBL = \angle D$. Through F draw $FM \parallel BL$.

$LBFM$ is the required \parallel gm.

Proof.—Join HF .

\triangle s GAF , AFC are on the same base AF and have the same altitude, \therefore they are equal. (II—5, p. 101.)

To each of these equal \triangle s add the $\triangle ABF$, and

$$\triangle GBF = \triangle ABC.$$

$$\triangle GBF = \text{twice } \triangle HBF, \quad (\text{II—6, Cor. 2, p. 102.})$$

$$= \parallel\text{gm } LBFM, \quad (\text{II—7, p. 103.})$$

$$\therefore \parallel\text{gm } LBFM = \triangle ABC.$$

Also $\angle LBF = \angle D$ and side $BF = E$.

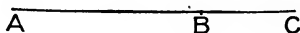
AREAS OF SQUARES

80.—A rectangle is said to be contained by two st. lines when its length is equal to one of the st. lines, and its breadth is equal to the other.

The symbol AB^2 should be read:—"the square on AB ," and not " AB squared."

THEOREM 10

The square on the sum of two straight lines equals the sum of the squares on the two straight lines increased by twice the rectangle contained by the straight lines.



Hypothesis.— AB , BC are the two st. lines placed in the same st. line so that AC is their sum.

To prove that

$$AC^2 = AB^2 + BC^2 + 2 \cdot AB \cdot BC.$$

Algebraic Proof

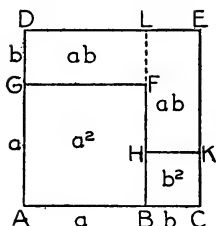
Proof.—Let a , b represent the number of units of length in AB , BC respectively.

Area of the square on AC

$$= (a + b)^2$$

$$= a^2 + b^2 + 2ab$$

= area of square on AB + area of square on BC + twice the area of the rectangle contained by AB , and BC .

Geometric Proof

Construction.—On AC , AB , BC draw squares $ACED$, $ABFG$, $BCKH$. Produce BF to meet DE at L .

Proof.—

$$GD = AD - AG = AC - AB = BC, \text{ and } GF = AB.$$

$$\therefore GL = \text{rect. } AB \cdot BC.$$

$$KE = CE - CK = AC - BC = AB, \text{ and } HK = BC.$$

$$\therefore HE = \text{rect. } AB \cdot BC.$$

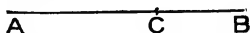
$$AC^2 = AE$$

$$= AF + BK + GL + HE$$

$$= AB^2 + BC^2 + 2 AB \cdot BC.$$

THEOREM 11

The square on the difference of two straight lines equals the sum of the squares on the two straight lines diminished by twice the rectangle contained by the straight lines.



Hypothesis.—**AB**, **BC** are two st. lines, of which **AB** is the greater, placed in the same st. line, and so that **AC** is their difference.

To prove that

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC.$$

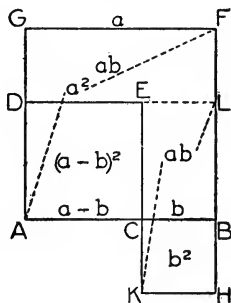
Algebraic Proof

Proof.—Let a , b represent the number of units of length in **AB**, **BC** respectively.

$$\begin{aligned} \text{Area of square on } AC &= (a - b)^2 \\ &= a^2 + b^2 - 2ab. \end{aligned}$$

= the sum of the squares on **AB** and **BC** diminished by twice the area of the rectangle contained by **AB** and **BC**.

Geometric Proof



Construction.—On **AC**, **AB**, **BC** draw the squares **ACED**, **ABFG**, **BCKH**. Produce **DE** to meet **BF** at **L**.

Proof.— $DG = AG - AD = AB - AC = BC$, and $DL = AB$.

$$\therefore DF = \text{rect. } AB \cdot BC.$$

$KE = KC + CE = BC + AC = AB$, and $KH = BC$.

$$\therefore KL = \text{rect. } AB \cdot BC.$$

$$AC^2 = AE$$

$$= AF + KB - (DF + KL)$$

$$= AB^2 + BC^2 - 2 AB \cdot BC.$$

THEOREM 12

The difference of the squares on two straight lines equals the rectangle of which the length is the sum of the straight lines and the breadth is the difference of the straight lines.

A ————— B —————

A, B are two st. lines, of which $A > B$.

To prove that the square on A diminished by the square on B = the rect. contained by A + B and A - B.

Proof.—Let a, b represent the number of units in A and B respectively.

The difference of the squares on A and B

$$= a^2 - b^2$$

$$= (a + b)(a - b)$$

$$= \text{the area of the rectangle}$$

contained by A + B and A - B.

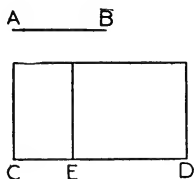
81.—Exercises

1. Draw a diagram illustrative of Theorem 12.
2. The square on the sum of three st. lines equals the sum of the squares on the three st. lines increased by twice the sum of the rectangles contained by each pair of the st. lines.

Illustrate by diagram.

3. The sum of the squares on two unequal st. lines $>$ twice the rectangle contained by the two st. lines.
4. The sum of the squares on three unequal st. lines $>$ the sum of the rectangles contained by each pair of the st. lines.
5. Construct a rectangle equal to the difference of two given squares.

6. If there be two st. lines **AB** and **CD**, and **CD** be divided at **E** into any two parts, the rect. **AB.CD** = rect. **AB.CE** + rect. **AB.ED**.



Let **AB** = p units of length

CE = q " " "

ED = r " " "

Area of **AB.CD** = $p(q+r)$

" " **AB.CE** = pq

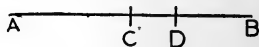
" " **AB.ED** = pr .

But $p(q+r) = pq + pr$.

\therefore **AB.CD** = **AB.CE** + **AB.ED**.

7. Give a diagram illustrating the identity $(a + b)(c + d) = ac + ad + bc + bd$, taking a, b, c, d to be respectively the number of units in four st. lines.

8. **C** is the middle point of a st. line **AB**, and **D** is any other point in the line. Prove:

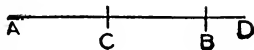


(1) $AD \cdot DB = AC^2 - CD^2$;

(2) $AD^2 + DB^2 = 2 AC^2 + 2 CD^2$.

(Let $AC = CB = p$, $CD = q$).

9. C is the middle point of a st. line AB , and D is any point in AB produced. Prove:



(1) $AD \cdot DB = CD^2 - AC^2$;

(2) $AD^2 + DB^2 = 2 AC^2 + 2 CD^2$.

10. Draw diagrams to illustrate the four results in exercises 8 and 9.

11. Draw a diagram illustrating the identity $(a + b)^2 - (a - b)^2 = 4 ab$.

12. If A, B, C, D be four points in order in a st. line, $AB \cdot CD + AD \cdot BC = AC \cdot BD$.

Illustrate by a diagram.

13. AB is a st. line in which C is any point. Prove that $AB^2 = AB \cdot AC + AB \cdot CB$.

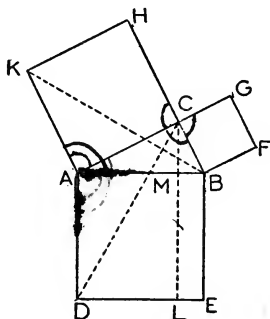
14. Construct a \triangle having two sides and the median drawn to one of these sides equal to three given st. lines.

15. Construct a \triangle having two sides and the median drawn to the third side equal to three given st. lines.

16. In a given $\parallel gm$ inscribe a rhombus having one vertex at a given point in a side of the $\parallel gm$.

THEOREM 13

The square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.



Hypothesis.— ABC is a \triangle in which $\angle ACB$ is a rt. \angle , and AE , BG , CK are squares on AB , BC and CA .

To prove that $AB^2 = AC^2 + BC^2$.

Construction.—Through C draw $CL \parallel AD$.

Join KB . CD .

Proof.— $\because \angle$ s HCA , ACB , BCG are rt. \angle s,

$\therefore \angle$ s HCB , ACG are st. \angle s.

and $\therefore HCB$, ACG are st. lines.

$\angle BAD = \angle KAC$,

to each add $\angle CAB$,

then $\angle CAD = \angle KAB$.

In \triangle s CAD , KAB , $\left\{ \begin{array}{l} CA = KA \\ AD = AB \\ \angle CAD = \angle KAB \end{array} \right.$

$\therefore \triangle CAD = \triangle KAB$

(I—2, p. 16.)

\therefore rect. $ADLM$ and $\triangle CAD$ are on the same base AD and between the same \parallel s CL, AD ,

\therefore rect. $AL =$ twice $\triangle CAD$. (II—7, p. 103.)

Similarly, sq. $HA =$ twice $\triangle KAB$.

\therefore rect. $AL =$ sq. HA .

In the same manner, by joining CE and AF , it may be shown that

rect. $BL =$ sq. BG .

\therefore rect. $AL +$ rect. $BL =$ sq. $HA +$ sq. BG ,

i.e., $AB^2 = AC^2 + BC^2$.

82. Many proofs have been given for this important theorem. Pythagoras (570 to 500 B.C.) is said by tradition to have been the first to prove it, and from that it is commonly called the Theorem of Pythagoras, or the Pythagorean Theorem. The proof given above is attributed to Euclid (about 300 B.C.). An alternative proof is given in Book IV.

83.—Exercises

1. Draw two st. lines 5 cm. and 6 cm. in length. Describe squares on both, and make a square equal in area to the two squares. Measure the side of this last square and check your result by calculation.

2. Draw three squares having sides 1 in., 2 in. and $2\frac{1}{2}$ in. Make one square equal to the sum of the three. Check by calculation.

3. Draw two squares having sides $1\frac{1}{2}$ in. and $2\frac{1}{2}$ in. Make a third square equal to the difference of the first two. Check by calculation.

4. Draw two squares having sides 9 cm. and 6 cm. Make a third square equal to the difference of the first two. Check your result by calculation.

5. Draw any square and one of its diagonals. Draw a square on the diagonal and show that it is double the first square.

6. Draw a square having each side 4 cm. Draw a second square double the first. Measure a side, and check by calculation.

7. Draw a square having one side 45 mm. Draw a second square three times the first. Measure its side, and check by calculation.

8. Draw three lines in the ratio 1:2:3. Draw squares on the lines, and divide the two larger so as to show that the squares are in the ratio 1:4:9.

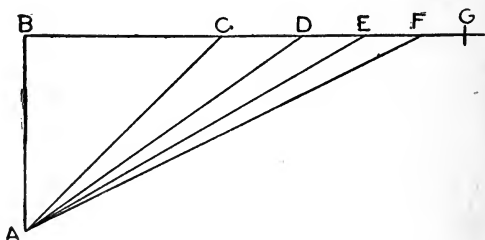
9. Draw a st. line $\sqrt{2}$ in. in length.

10. Draw a st. line $\sqrt{3}$ in. in length.

11. Draw a st. line $\sqrt{5}$ in. in length.

12. Draw any rt.- \angle d \triangle . Describe equilateral \triangle s on the three sides. Find the areas of the \triangle s and compare that on the hypotenuse with the sum of those on the other two sides.

13.



AB is one inch in length, $\angle B$ a rt. \angle , BC is one inch
 BD is cut off = AC, BE = AD, BF = AE, BG = AF, etc.
 Show that $BD = \sqrt{2}$ in., $BE = \sqrt{3}$ in., $BF = \sqrt{4} = 2$
 in., $BG = \sqrt{5}$ in., etc.

14. Construct a square equal to half a given square.

15. If a \perp be drawn from the vertex of a \triangle to the base, the difference of the squares on the segments of the base = the difference of the squares on the other two sides.

Hence, prove that the altitudes of a \triangle pass through one point.

16. A is a given st. line. Find another st. line B , such that the difference of the square on A and B may be equal to the difference of two given squares.

✓17. If the diagonals of a quadrilateral cut at rt. \angle s, the sum of the squares on one pair of opposite sides equals the sum of the squares on the other pair.

18. The sum of the squares on the diagonals of a rhombus equals the sum of the squares on the four sides.

✓19. Five times the square on the hypotenuse of a rt.- \angle \triangle equals four times the sum of the squares on the medians drawn to the other two sides.

20. In an isosceles rt.- \angle \triangle the sides have the ratios $1:1:\sqrt{2}$.

21. If the angles of a \triangle be 90° , 30° , 60° , the sides have the ratios $2:1:\sqrt{3}$.

✓22. Divide a st. line into two parts such that the sum of the squares on the parts equals the square on another given st. line. When is this impossible?

✓23. In the st. line AB produced find a point C such that the sum of the squares on AC , BC equals the square on a given st. line.

✓24. Divide a given st. line into two parts such that the square on one part is double the square on the other part.

✓25. $ABCD$ is a rect., and P is any point. Show that $PA^2 + PC^2 = PB^2 + PD^2$.

✓26. ABC is a \triangle rt.- \angle d at A . E is a point on AC and F is a point on AB . Show that $BE^2 + CF^2 = EF^2 + BC^2$.

27. If two rt.- \angle d \triangle s have the hypotenuse and a side of one respectively equal to the hypotenuse and a side of the other, the \triangle s are congruent.

28. The square on the side opposite an acute \angle of a \triangle is less than the sum of the squares on the other two sides.

29. The square on the side opposite an obtuse \angle of a \triangle is greater than the sum of the squares on the other two sides.

30. Construct a square that contains 20 square inches.

31. In the diagram of II—13, show that **KB**, **CD** cut at rt. \angle s.

32. In the diagram of II—13, if **KD** be joined, show that \triangle **KAD** = \triangle **ABC**.

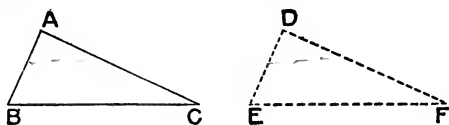
33. In the diagram of II—13, the distance of **E** from **AC** = **AC** + **CB**.

34. **ABC** is an isosceles rt.- \angle d \triangle in which **C** is the rt. \angle . **CB** is produced to **D** making **BD** = **CB**. \perp s to **AB**, **BD** at **A**, **D** respectively meet at **E**. Prove that **AE** = 2 **AB**.

THEOREM 14

(Converse of Theorem 13)

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by these sides is a right angle.



Hypothesis.— ABC is a \triangle in which $BC^2 = AB^2 + AC^2$.

To prove that $\angle A$ is a rt. \angle .

Construction.—Make a rt. $\angle D$ and cut off $DE = AB$, $DF = AC$.

Join EF .

$$BC^2 = AB^2 + AC^2 \quad (\text{Hyp.})$$

$$= DE^2 + DF^2$$

$$= EF^2 \quad (\because D \text{ is a rt. } \angle). \quad (\text{II—13, p. 122.})$$

$$\therefore BC = EF.$$

$$\text{In } \triangle\text{s } ABC, DEF, \begin{cases} AB = DE, \\ AC = DF, \\ BC = EF, \end{cases}$$

$$\therefore \angle A = \angle D. \quad (\text{I—4, p. 22.})$$

$$\therefore \angle A \text{ is a rt. } \angle.$$

84.—Exercises

1. The sides of a \triangle are 3 in., 4 in. and 5 in. Prove that it is a rt.- \angle d \triangle .

2. The sides of a \triangle are 13 mm., 84 mm. and 85 mm. Prove that it is a rt.- \angle d \triangle .

3. In the quadrilateral $ABCD$, $AB^2 + CD^2 = BC^2 + AD^2$. Prove that the diagonals AC , BD cut at rt. \angle s.

4. If the sq. on one side of a \triangle be less than the sum of the squares on the other two sides, the \angle contained by these sides is an acute \angle . (Converse of § 83, Ex. 28.)

5. State and prove a converse of § 83, Ex. 29.

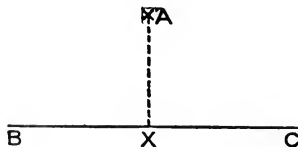
6. Using a tape-measure, or a knotted cord, and Ex. 1, draw a st. line at rt. \angle s to a given st. line.

7. Show that, if the sides of a \triangle are represented by $m^2 + n^2$, $m^2 - n^2$, $2mn$, where m and n are any numbers, the \triangle is rt.- \angle d.

Use this result to find numbers representing the sides of a rt.- \angle d \triangle .

85. **Definition.**—If a perpendicular be drawn from a given point to a given straight line, the foot of the perpendicular is said to be the **projection of the point on the line.**

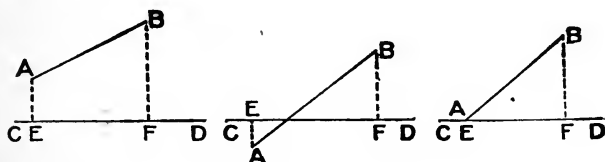
From the point A the \perp AX is drawn to the line BC .



The point X is the projection of the point A on the st. line BC .

86. **Definition.**—If from the ends of a given straight line perpendiculars be drawn to another given straight line, the segment intercepted on the second straight line is called the **projection of the first straight line on the second straight line.**

AB is a st. line of fixed length and **CD** another st. line. **AE, BF** are drawn \perp **CD**.



EF is the projection of **AB** on **CD**.

87.—Exercises

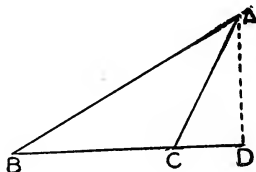
1. Show that a st. line of fixed length is never less than its projection on another st. line. In what case are they equal? In what case is the projection of one st. line on another st. line just a point?

2. **ABC** is a \triangle having $a = 36$ mm., $b = 40$ mm. and $c = 45$ mm. Draw the \triangle and measure the projection of **AB** on **BC**. (*Ans.* 23.9 mm. nearly.)

3. **ABC** is a \triangle having $a = 5$ cm., $b = 7$ cm., $c = 10$ cm. Draw the \triangle and measure the projection of **AB** on **BC**. (*Ans.* 76 mm.)

THEOREM 15

In an obtuse-angled triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the sides that contain the obtuse angle increased by twice the rectangle contained by either of these sides and the projection on that side of the other.



Hypothesis.— $\triangle ABC$ is a \triangle in which $\angle C$ is obtuse, and CD is the projection of CA on CB .

To prove that $AB^2 = AC^2 + BC^2 + 2 BC \cdot CD$.

Proof.— \because $\triangle ADB$ is a rt. \angle ,

$$\therefore AB^2 = BD^2 + AD^2. \quad (\text{II—13, p. 122.})$$

$$\because BD = BC + CD,$$

$$\therefore BD^2 = BC^2 + CD^2 + 2 BC \cdot CD. \quad (\text{II—10, p. 116.})$$

$$\therefore AB^2 = BC^2 + CD^2 + 2 BC \cdot CD + AD^2.$$

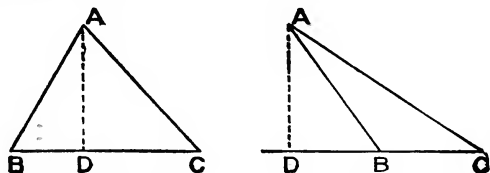
But $\because \triangle ADC$ is a rt. \angle ,

$$\therefore CD^2 + AD^2 = AC^2.$$

$$\therefore AB^2 = AC^2 + BC^2 + 2 BC \cdot CD.$$

THEOREM 16

In any triangle, the square on the side opposite an acute angle is equal to the sum of the squares on the sides which contain the acute angle diminished by twice the rectangle contained by either of these sides and the projection on that side of the other.



Hypothesis.— ABC is a \triangle in which $\angle C$ is acute, and CD is the projection of CA on CB .

To prove that $AB^2 = AC^2 + BC^2 - 2 BC \cdot CD$.

Proof.— $\because ADB$ is a rt. \angle ,

$$\therefore AB^2 = BD^2 + AD^2. \quad (\text{II—13, p. 122.})$$

$\because BD$ is the difference between BC and CD ,

$$\therefore BD^2 = CD^2 + BC^2 - 2 BC \cdot CD. \quad (\text{II—11, p. 118.})$$

$$\therefore AB^2 = CD^2 + BC^2 - 2 BC \cdot CD + AD^2.$$

But, $\because ADC$ is a rt. \angle ,

$$\therefore CD^2 + AD^2 = AC^2.$$

$$\therefore AB^2 = AC^2 + BC^2 - 2 BC \cdot CD.$$

88.—Exercises

1. ABC is a \triangle having C an \angle of 60° . Show that sq. on $AB = \text{sq. on } BC + \text{sq. on } AC - \text{rect. } BC \cdot AC$.

2. ABC is a \triangle having C an \angle of 120° . Show that sq. on $AB = \text{sq. on } BC + \text{sq. on } AC + \text{rect. } BC \cdot AC$.

3. $\triangle ABC$ is a \triangle , CD the projection of CA on CB , and CE the projection of CB on CA . Show that $\text{rect. } BC \cdot CD = \text{rect. } AC \cdot CE$.

4. In any \triangle the sum of the squares on two sides equals twice the square on half the base together with twice the square on the median drawn to the base.

NOTE.—Draw a \perp from the vertex to the base, and use II—15 and II—16.

5. In any quadrilateral the sum of the squares on the four sides exceeds the sum of the squares on the diagonals by four times the square on the st. line joining the middle points of the diagonals.

What does this proposition become when the quadrilateral is a $\parallel\text{gm}$?

6. $\triangle ABC$ is a \triangle having $a = 47$ mm., $b = 62$ mm., and $c = 84$ mm. D, E, F are the middle points of BC, CA, AB respectively. Calculate the lengths of AD, BE and CF . Test your results by drawing and measurement.

7. The squares on the diagonals of a quadrilateral are together double the sum of the squares on the st. lines joining the middle points of opposite sides.

8. If the medians of a \triangle intersect at G ,

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

9. C is the middle point of a st. line AB . P is any point on the circumference of a circle of which C is the centre. Show that $PA^2 + PB^2$ is constant.

10. Two circles have the same centre. Prove that the sum of the squares of the distances from any point on the circumference of either circle to the ends of the diameter of the other is constant.

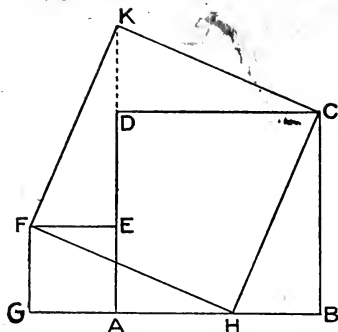
11. The square on the base of an isosceles \triangle is equal to twice the rect. contained by either of the equal sides and the projection on it of the base.

12. Prove II—13 from the following construction:—

Draw two squares, $ABCD$, $AEGF$, having AD , AE in the same st. line.

Cut off GH and EK each $= AB$.

Join FH , HC , CK , KF .



13. If two sides of a \triangle be unequal, the median drawn to the shorter side is greater than the median drawn to the longer side.

14. If, from any point P within $\triangle ABC$, \perp s PX , PY , PZ be drawn to BC , CA , AB respectively,

$$BX^2 + CY^2 + AZ^2 = CX^2 + AY^2 + BZ^2.$$

15. D , E , F are the middle points of BC , CA , AB respectively in $\triangle ABC$. Prove that

$$3(AB^2 + BC^2 + CA^2) = 4(AD^2 + BE^2 + CF^2).$$

16. G is the centroid of $\triangle ABC$, and P is any point. Show that

$$PA^2 + PB^2 + PC^2 = AG^2 + BG^2 + CG^2 + 3PG^2.$$

17. Find the point P in the plane of the $\triangle ABC$ such that the sum of the squares on PA , PB , PC may be the least possible.

18. Check the results in Exs. 2 and 3, §87, by calculation.

19. If, in II—15, the obtuse \angle becomes greater and greater and finally becomes a st. \angle , what does the theorem become?

20. If, in the diagram of II—16, the $\angle C$ becomes more and more acute and finally the point A comes down to the line BC , what does the theorem become?

Miscellaneous Exercises

1. If a quadrilateral be bisected by each of its diagonals, it is a \parallel gm.

2. If any point P in the diagonal AC of the \parallel gm $ABCD$ be joined to B and D , the \parallel gm is divided into two pairs of equal \triangle s.

3. The diagonals of a \parallel gm divide the \parallel gm into four equal parts.

4. If two sides of a quadrilateral are \parallel to each other, the st. line joining their middle points bisects the area of the quadrilateral.

5. If two sides of a quadrilateral are \parallel to each other, the st. line joining their middle points passes through the intersection of the diagonals.

6. If P is any point in the side AB of \parallel gm $ABCD$, and PC , PD are joined,

$$\triangle PAD + \triangle PBC = \triangle PDC.$$

7. Prove that the following method of bisecting a quadrilateral by a st. line drawn through one of its vertices is correct:—Let $ABCD$ be the quadrilateral. Join AC , BD . Bisect BD at E . Through E draw $EF \parallel AC$ and meeting BC , or CD , at F . Join AF . AF bisects the quadrilateral.

NOTE.—Join AE , and EC .

8. If the diagonals of \parallel gm $ABCD$ cut at O , and P is any point within the $\triangle AOB$, $\triangle CPD = \triangle APB + \triangle APC + \triangle BPD$.

NOTE.—Join PO .

9. ABC is an isosceles \triangle having $AB = AC$, and D is a point in the base BC , or BC produced. Prove that the difference between the squares on AD and $AC = \text{rect. } BD \cdot DC$.

10. P, Q, R, S are respectively the middle points of the sides AB, BC, CD, DA in the quadrilateral $ABCD$. Prove that $AB^2 + CD^2 + 2PR^2 = CB^2 + DA^2 + 2QS^2$.

11. $BY \perp AC$ and $CZ \perp AB$ in $\triangle ABC$. Prove that $BC^2 = \text{rect. } AB.BZ + \text{rect. } AC.CY$.

12. L, M, N are three given points, and PQ a given st. line. Construct a rhombus $ABCD$, having its angular points A, C lying on the line PQ , and its three sides AB, BC, CD (produced if necessary) passing through L, M, N respectively.

13. Through D the middle point of the side BC of $\triangle ABC$ a st. line XDY is drawn cutting AB at X and AC produced through C at Y . Prove $\triangle AXY > \triangle ABC$.

14. From the vertex A of $\triangle ABC$ draw a st. line terminated in BC and equal to the average of AB and AC .

15. AB and CD are two equal st. lines that are not in the same st. line. Find a point P such that $\triangle PAB \equiv \triangle PCD$.

Show that, in general, two such points may be found.

16. EF drawn \parallel to the diagonal AC of $\parallel\text{gm } ABCD$ meets AD, DC , or those sides produced, in E, F respectively. Prove that $\triangle ABE = \triangle BCF$.

17. Construct a rect. equal to a given square and such that one side equals a given st. line.

18. Find a point in one of two given intersecting st. lines such that the perpendiculars drawn from it to both the given lines may cut off from the other a segment of given length.

19. In the diagram of II—9, if BD, BE and DE be drawn, $\parallel\text{gm } FK - \parallel\text{gm } HG = 2 \triangle EBD$.

20. ABC is an isosceles \triangle in which C is a rt. \angle , and the bisector of $\angle A$ meets BC at D . Prove that $CD = AB - AC$.

21. Place a st. line of given length between two given st. lines so as to be \parallel a given st. line.

22. Describe a $\triangle =$ a given \parallel gm and such that its base = a given st. line, and one \angle at the base = a given \angle .

23. Construct a \parallel gm equal and equiangular to a given \parallel gm, and such that one side is equal to a given st. line.

24. Construct a \parallel gm equal and equiangular to a given \parallel gm, and such that its altitude is equal to a given st. line.

25. $ABCD$ is a quadrilateral. On BC as base construct a \parallel gm equal in area to $ABCD$, and having one side along BA .

26. Squares $ABDE$, $ACFG$ have a common $\angle A$, and A, B, C are in the same st. line. AH is drawn $\perp BG$ and produced to cut CE at K . Prove that $EK = KC$.

27. Make a rhombus $ABCD$ in which $\angle A = 100^\circ$. A circle described with centre A and radius AB cuts BC, CD at E, F respectively. Prove that AEF is an equilateral \triangle .

28. A st. line AB is bisected at C and divided into two unequal parts at D . Prove that $AD^2 + DB^2 = 2AD \cdot DB + 4CD^2$.

29. $ABCD$ is a quadrilateral in which $AB \parallel CD$. Prove that

$$AC^2 + BD^2 = AD^2 + BC^2 + 2AB \cdot CD.$$

30. Trisect a given \parallel gm by st. lines drawn through one of its angular points.

31. The base BC of the $\triangle ABC$ is trisected at D, E . Prove that

$$AB^2 + AC^2 = AD^2 + AE^2 + 4DE^2.$$

32. ACB, ADB are two rt.- \angle d \triangle s on the same side of the same hypotenuse AB , and AX, BY are $\perp CD$ produced. Prove that

$$XC^2 + CY^2 = XD^2 + DY^2.$$

33. ABC is an isosceles \triangle , and XY is $\parallel BC$ and terminated in AB, AC . Prove

$$BY^2 = CY^2 + BC \cdot XY.$$

34. Any rect. = half the rect. contained by the diagonals of the squares on two of its adjacent sides.

35. $ABCD$ is a $\parallel gm$ in which $BD = AB$. Prove that $BD^2 + 2 BC^2 = AC^2$.

36. A rect. $BDEC$ is described on the side BC of a $\triangle ABC$. Prove that

$$AB^2 + AE^2 = AC^2 + AD^2.$$

37. BE, CD are squares described externally on the sides AB, AC of a $\triangle ABC$. Prove that

$$BC^2 + ED^2 = 2(AB^2 + AC^2).$$

NOTE.—Draw $EX, CY \perp DA, AB$ respectively, and rotate $\triangle ABC$ to the position in which AB coincides with AE .

38. ABC is a \triangle in which $AX \perp BC$, and D is the middle point of BC . Prove that the difference of the squares on $AB, AC = 2 BC \cdot DX$.

39. BC is the greatest and AB the least side in $\triangle ABC$. D, E, F are the middle points of BC, CA, AB respectively; and X, Y, Z are the feet of the \perp s from A, B, C to the opposite sides. Prove that $CA \cdot EY = AB \cdot FZ + BC \cdot DX$.

40. $ABCD$ is a rect. in which E is any point in BC and F is any point in CD . Prove that $ABCD = 2 \triangle AEF + BE \cdot DF$.

41. A and B are two fixed points. Find the position of a point P such that $PA^2 + PB^2$ may be the least possible.

42. From a given point A draw three st. lines AB, AC, AD respectively equal to three given st. lines, and such that B, C, D are in the same st. line and $BC = CD$.

43. Find the locus of a point such that the sum of the squares on its distances from two given points is constant.

44. Find the locus of a point such that the difference of the squares on its distances from two given points is constant.

45. $ABCD$ is a \parallel gm, P any point in BC , and Q any point in AP . Prove that $\triangle BQC = \triangle PQD$.

46. $ABCD$ is a quadrilateral having $AB \parallel CD$, and $AB + CD = BC$. Prove that the bisectors of \angle 's B and C intersect on AD .

47. ABC is a \triangle in which $\angle A$ is a rt. \angle , and $AB > AC$. Squares $BCDE$, $CAHF$, $ABGK$ are described outwardly to the \triangle . Prove that

$$DG^2 - EF^2 = 3(AB^2 - AC^2).$$

48. In the hypotenuse AB of a rt.- \angle d $\triangle ACB$, points D and E are taken such that $AD = AC$ and $BE = BC$. Prove that

$$DE^2 = 2 BD \cdot AE.$$

49. A st. line is 8 cm. in length. Divide it into two parts such that the difference of the squares on the parts = 5 sq. cm.

50. A and B are two given points and CD is a given st. line. Find a point P in CD such that the difference of the squares on PA and PB may be equal to a given rectangle.

51. AD is a median of the acute- \angle d $\triangle ABC$; $DX \perp AB$, $DY \perp AC$. Prove that

$$BA \cdot AX + CA \cdot AY = 2 AD^2.$$

52. Find a point P within a given quadrilateral $KLMN$ such that $\triangle PLM = \triangle PMN = \triangle PNK$.

53. ABC is an isosceles \triangle in which $AB = AC$. $AP \parallel BC$. Prove that the difference between PB^2 and PC^2 equals $2 AP \cdot BC$.

54. If the sum of the squares on the diagonals of a quadrilateral be equal to the sum of the squares on the sides, the quadrilateral is a \parallel gm.

55. D is a point in the side BC of a $\triangle ABC$ such that $AB^2 + AC^2 = 2AD^2 + 2BD^2$. $AX \perp BC$. Prove that either $BD = DC$, or $2DX = BC$.

56. $ABCD$ is a \parallel gm, and P is a point such that $PA^2 + PC^2 = PB^2 + PD^2$. Prove that $ABDC$ is a rectangle.

57. A, B, C, D are four fixed points. Find the locus of a point P such that $PA^2 + PB^2 + PC^2 + PD^2$ is constant.

58. A, B, C, D are four fixed points. Find the locus of a point P such that $PA^2 + PB^2 = PC^2 + PD^2$.

59. D and E are taken in the base BC of $\triangle ABC$ so that $BD = EC$. Through D, E st. lines are drawn $\parallel AB$ and AC forming two \parallel gms with AD, AE as diagonals. Prove the \parallel gms equal in area.

60. A st. line EF drawn \parallel to the diagonal AC of a \parallel gm $ABCD$ meets AB in E and BC in F . Prove that BD bisects the quadrilateral $DEBF$.

61. ABC is an isosceles rt.- \angle d \triangle in which $AB = AC$. E is taken in AB and D in AC produced such that $EB = CD$. Prove that $\triangle EAD < \triangle ABC$.

62. L and M are respectively the middle points of the diagonals BD and AC of a quadrilateral $ABCD$. ML is produced to meet AD at E . Prove that $\triangle EBC =$ half the quadrilateral.

63. DE is $\parallel BC$ the base of $\triangle ABC$, and meets AB, AC at D, E respectively. DE is produced to F making $DF = BC$. Prove that $\triangle AEF = \triangle BDE$.

64. Construct the minimum \triangle which has a fixed vertical \angle , and its base passing through a fixed point situated between the arms of the \angle .

65. BE, BD are the bisectors of the interior and exterior \angle s at B in the $\triangle ABC$. $AE \perp BE$ and $CD \perp BD$. AE and CD intersect at F . Prove that $\text{rect. } BEFD = \triangle ABC$.

66. $ABCD$ is a square. St. lines drawn through A and D make with BC produced in both directions the $\triangle EFG$. $EX \perp FG$. Prove that $BC(EX + FG) = 2 \triangle EFG$.

67. The $\triangle ABC$ is rt.- \angle d at C , and the bisectors of \angle s A and B meet at E . $ED \perp AB$. Prove that $\text{rect. } AD \cdot DB = \triangle ABC$.

68. Calculate the area of an equilateral \triangle of which the side is 2 inches.

69. If the side of an equilateral \triangle is a inches, show that its area is $\frac{a^2\sqrt{3}}{4}$ sq. in.

70. Calculate the side of an equilateral \triangle of which the area is 10 sq. cm.

71. Construct a \triangle having two sides 4 cm. and 4.5 cm., and the area 7 sq. cm.

Show that there are two solutions.

72. M is a point in the side QR of $\triangle PQR$ such that $QM = 2 MR$. Prove that $PQ^2 + 2 PR^2 = 3 PM^2 + 6 MR^2$.

73. The rectangle contained by the two segments of a st. line is a maximum when the st. line is bisected. (Use Ex. 8 (1), §81.)

74. The sum of the squares on the two segments of a st. line is a minimum when the st. line is bisected. (Use Ex. 8 (2), §81.)

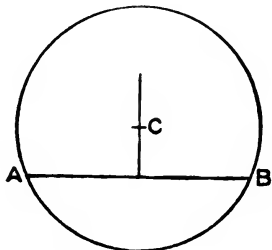
BOOK III

THE CIRCLE

89. A definition of a circle was given in § 32, and from the explanation given in § 66 we may take the following alternative definition of it:—

A **circle** is the locus of the points that lie at a fixed distance from a fixed point.

90. As the centre of a circle is a point equally distant from the two ends of any chord of the circle, the three following statements follow at once from I—22, p. 78:—



(a) The straight line drawn from the centre of a circle perpendicular to a chord bisects the chord.

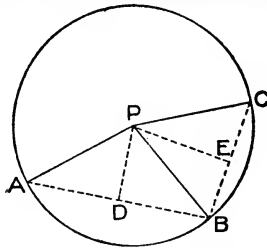
(b) The straight line drawn from the centre of a circle to the middle point of a chord is perpendicular to the chord.

(c) The right bisector of a chord of a circle passes through the centre of the circle.

As an exercise the pupil should give independent proofs of theorems (a), (b) and (c).

THEOREM I

If from a point within a circle more than two equal straight lines are drawn to the circumference, that point is the centre.



Hypothesis.— P is a point within the circle ABC such that $PA = PB = PC$.

To prove that P is the centre of the circle.

Construction.—Join AB , BC , and from P draw $PD \perp AB$ and $PE \perp BC$.

Proof.— $\because PA = PB$,

$\therefore P$ is in the right bisector of AB . (I—22, p. 78.)

And $\therefore PD$ produced is the locus of the centres of all circles through A and B .

\therefore the centre of the circle ABC is somewhere in PD .

In the same manner it may be shown that the centre of the circle ABC is somewhere in PE .

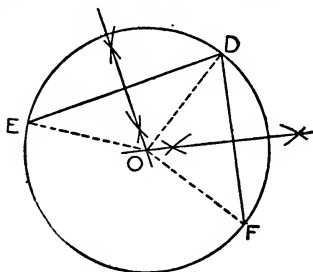
But P is the only point common to PD and PE .

$\therefore P$ is the centre of circle ABC .

CONSTRUCTIONS

PROBLEM 1

To find the centre of a given circle.



Let DEF be the given circle.

Construction.—From any point D on the circumference draw two chords DE , DF .

Draw the right bisectors of DE , DF meeting at O .

O is the centre of circle DEF .

Join OF , OE , OD .

Proof.— $\because O$ is on the right bisector of DE ,

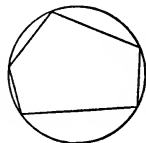
$$\therefore OE = OD. \quad (\text{I—22, p. 78.})$$

Similarly $OD = OF$.

$$\therefore OE = OD = OF,$$

$$\therefore O \text{ is the centre of the circle. } (\text{III—1, p. 142.})$$

91. **Definitions.**—If a circle passes through all the vertices of a rectilineal figure, it is said to be **circumscribed** about the figure.



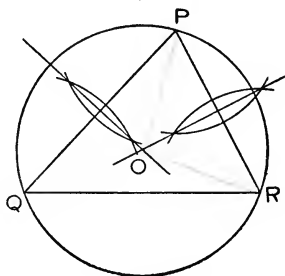
Four points so situated that a circle may be described to pass through all of them are said to be **conyclic**.

If the four vertices of a quadrilateral are on the circumference of the same circle, it is said to be a **cyclic quadrilateral**.

The centre of a circle circumscribed about a triangle is called the **circumcentre** of the triangle.

PROBLEM 2

To circumscribe a circle about a given triangle.



Let PQR be the given \triangle .

Construction.—Draw the right bisectors of PQ , PR meeting at O .

$\therefore O$ is on the right bisector of PQ .

$\therefore OP = OQ$.

(I—22, p. 78.)

Similarly $OP = OR$.

$\therefore OP = OQ = OR$,

And a circle described with centre O and radius OP will pass through Q and R , and be circumscribed about the \triangle .

92.—Exercises

1. Through a given point within a circle draw a chord that is bisected at the given point.

2. Complete a circle of which an arc only is given.

3. Circumscribe a circle about a given square.

4. Circumscribe a circle about a given rectangle.

5. Describe a circle with a given centre to cut a given circle at the ends of a diameter.

6. The locus of the middle points of a system of \parallel chords in a circle is a diameter of the circle.

7. If two circles cut each other, the st. line joining their centres bisects their common chord at rt. \angle s.

8. If each of two equal st. lines has one extremity on one of two concentric circles, and the other extremity on the other circle, the st. lines subtend equal \angle s at the common centres.

9. A st. line cuts the outer of two concentric circles at **E**, **F**; and the inner at **G**, **H**. Prove that $EG = FH$.

10. A st. line cannot cut a circle at more than two points.

11. Two chords of a circle cannot bisect each other unless both are diameters.

12. A circle cannot be circumscribed about a \parallel gm unless the \parallel gm is a rectangle.

13. A st. line which joins the middle points of two \parallel chords in a circle is \perp to the chords.

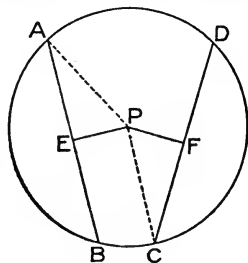
14. If two circles cut each other, a st. line through a point of intersection, \parallel to the line of centres and terminated in the circumferences, is double the line joining the centres.

15. If two circles cut each other, any two \parallel st. lines through the points of intersection, and terminated by the circumferences, are equal to each other.

16. If two circles cut each other, any two st. lines through one of the points of intersection, making equal \angle s with the line of centres and terminated by the circumferences, are equal to each other.

THEOREM 2

Chords that are equally distant from the centre of a circle are equal to each other.



Hypothesis.— ABC is a circle of which P is the centre and AB, CD are two chords such that the \perp s PE, PF from P to AB, CD respectively are equal to each other.

To prove that $AB = CD$.

Construction.—Join AP, CP .

Proof.—Rotate $\triangle PFC$ about point P making PF fall on PE .

$$\therefore PF = PE,$$

\therefore point F falls on point E .

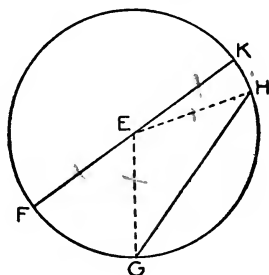
$$\therefore \angle PFC = \angle PEA,$$

$\therefore FC$ falls along EA .

hence, \therefore PC is a radius and
 \therefore C remains on the circumference,
 C must fall on A .
 \therefore FC coincides with EA ,
 and \therefore $FC = EA$,
 But $CD = 2 CF$,
 and $AB = 2 AE$,
 \therefore $CD = AB$.

THEOREM 3

In a circle any chord which does not pass through the centre is less than a diameter.



Hypothesis.—In the circle FGH , GH is a chord which does not pass through the centre and FK is a diameter. E is the centre.

To prove that $GH < FK$.

Construction.—Join EG , EH .

Proof.— \therefore $GE = EF$ and $EH = EK$,

\therefore $GE + EH = FK$.

\therefore GEH is a \triangle ,

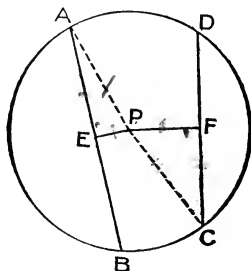
\therefore $GH < GE + EH$.

(I—16, p. 59.)

And \therefore $GH < FK$.

THEOREM 4

Of two chords in a circle the one which is nearer to the centre is greater than the one which is more remote from the centre.



Hypothesis.— P is the centre of a circle ABC , and AB , CD are two chords such that PE , the distance of AB from the centre, is less than PF , the distance of CD from the centre.

To prove that $AB > CD$.

Construction.—Join PA , PC .

Proof.—

$\therefore PEA$ is a rt. \angle ,

$\therefore AE^2 + EP^2 = AP^2$. (II—13, p. 122.)

Similarly $CF^2 + FP^2 = CP^2$.

But $\therefore AP = CP$,

$\therefore AP^2 = CP^2$.

And $\therefore AE^2 + EP^2 = CF^2 + FP^2$.

$\therefore EP < PF$,

$\therefore EP^2 < PF^2$.

And $\therefore AE^2 > CF^2$,

$\therefore AE > CF$.

But $AB = 2 AE$,

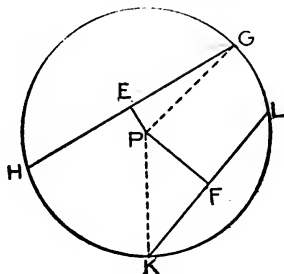
and $CD = 2 CF$,

$\therefore AB > CD$.

THEOREM 5

(Converse of Theorem 4)

If two chords of a circle are unequal, the greater is nearer to the centre than the less.



Hypothesis.—Chord $GH >$ chord KL , and PE, PF are respectively perpendiculars from the centre P to GH, KL .

To prove that $PE < PF$.

Construction.—Join PG, PK .

Proof.— $\because PEG$ is a rt. \angle ,

$$GE^2 + EP^2 = GP^2. \quad (\text{II—13, p. 122.})$$

Similarly $PF^2 + FK^2 = PK^2$.

$$\therefore GE^2 + EP^2 = PF^2 + FK^2.$$

But $\because GH = 2 GE$ and $KL = 2 KF$,

And also $GH > KL$,

$$\therefore GE > KF.$$

$$\therefore GE^2 > KF^2.$$

Hence, $EP^2 < PF^2$.

And $\therefore EP < PF$.

93.—Exercises

1. If two chords of a circle are equal to each other, they are equally distant from the centre. (Converse of Theorem 2.)

2. A chord 6 cm. in length is placed in a circle of radius 4 cm. Calculate the distance of the chord from the centre.

3. A chord a inches long is placed in a circle of radius b inches. Find an algebraic expression for the distance of the chord from the centre.

4. In a circle of radius 5 cm. a chord is placed at a distance of 3 cm. from the centre. Calculate the length of the chord.

5. Through a given point within a circle draw the shortest chord.

6. In a circle of radius 4 cm., a point P is taken at the distance 3 cm. from the centre. Calculate the length of the shortest chord through P .

7. The length of a chord 2 cm. from the centre of a circle is 5.5 cm. Find the length of a chord 3 cm. from the centre. Verify your result by measurement.

8. In a circle of radius 5 cm., two \parallel chords of lengths 8 cm. and 6 cm. are placed. Find the distance between the chords. Show that there are two solutions.

9. ACB is a diameter, and C the centre of a circle. D is any point on AB , or on AB produced, and P is any point on the circumference except A and B . Show that DP is intermediate in magnitude between DA and DB .

10. O is the centre of a circle, and P is any point. If two st. lines be drawn through P , cutting the circle, and

making equal \angle s with PO , the chords intercepted on these lines by the circumference are equal to each other.

11. O is the centre of a circle, and P is any point. On two lines drawn through P chords AB , CD are intercepted by the circumference. If the \angle made by AB with $PO >$ \angle made by CD with PO , the chord $AB <$ chord CD .

12. From any point in a circle which is not the centre equal st. lines can be drawn to the circumference only in pairs.

13. Find the locus of the middle points of chords of a fixed length in a circle.

14 K and L are two fixed points. Find a point P on a given circle such that $KP^2 + LP^2$ may be the least possible.

15. Chords equally distant from the centre of a circle subtend equal \angle s at the centre.

16. The nearer to the centre of two chords of a circle subtends the greater \angle at the centre.

ANSWERS :—2, 26.5 mm. nearly ; 4, 8 cm. ; 6, 5.3 c.m. nearly ; 7, 32 mm. nearly ; 8, 1 cm. or 7 cm.

ANGLES IN A CIRCLE

THEOREM 6

The angle which an arc of a circle subtends at the centre is double the angle which it subtends at any point on the remaining part of the circumference.

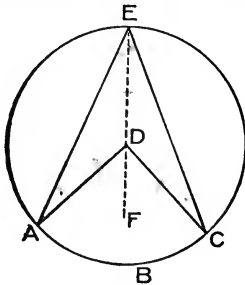


FIG. 1

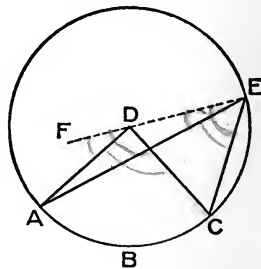


FIG. 2

Hypothesis.—**ABC** is an arc of a circle, **D** the centre, and **E** any point on the remaining part of the circumference.

To prove that $\angle ADC = 2 \angle AEC$.

Construction. — Join **ED** and produce **ED** to any point **F**.

Proof.—

In both figures:—

$$\text{In } \triangle DAE, \because DA = DE$$

$$\therefore \angle DAE = \angle DEA \quad (\text{I—3, p. 20}).$$

\because **ADF** is an exterior \angle of $\triangle ADE$,

$$\begin{aligned} \therefore \angle ADF &= \angle DAE + \angle DEA & (\text{I—10, p. 45.}) \\ &= 2 \angle DEA. \end{aligned}$$

Similarly $\angle CDF = 2 \angle DEC$.

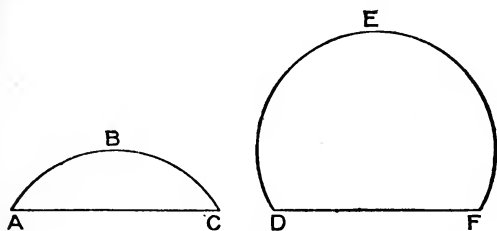
In Fig. 1:—

$$\begin{aligned} \angle ADF &= 2 \angle DEA \\ \angle CDF &= 2 \angle DEC, \\ \text{adding, } \angle ADC &= 2(\angle DEA + \angle DEC) \\ &= 2 \angle AEC. \end{aligned}$$

In Fig. 2:—

$$\begin{aligned} \angle CDF &= 2 \angle DEC, \\ \angle ADF &= 2 \angle DEA, \\ \text{subtracting, } \angle ADC &= 2(\angle DEC - \angle DEA). \\ &= 2 \angle AEC. \end{aligned}$$

94. **Definitions.**—The figure bounded by an arc of a circle and the chord which joins the ends of the arc is called a **segment of a circle**.



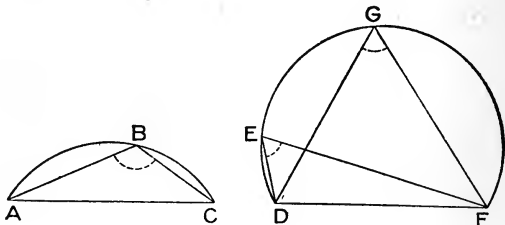
ABC, DEF are segments of circles.

A semi-circle is a particular case of a segment.

An arc is called a **major arc** or a **minor arc** according as it is greater or less than half the circumference.

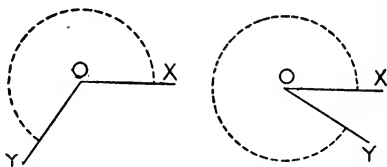
A segment is called a **major segment** or a **minor segment** according as the arc of the segment is a major or a minor arc.

95. **Definitions.**—If the ends of a chord of a segment are joined to any point on the arc of the segment, the angle between the joining lines is called an **angle in the segment**.



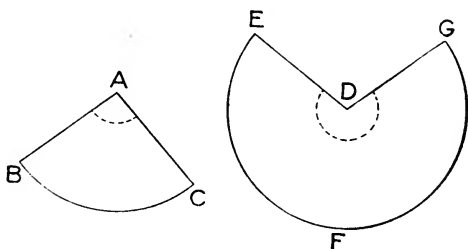
ABC is an \angle in the segment **ABC**, and **DEF** is an \angle in the segment **DEF**. **DGF** is also an \angle in the segment **DEF**.

96. **Definitions.**—An angle which is greater than two right angles but less than four right angles is called a **reflex angle**.



A straight line starting from the position **OX** and rotating in the direction opposite to that of the hands of a clock to the position **OY**, in either diagram, traces out the reflex angle **XOY**.

The figure bounded by two radii of a circle, and either of the arcs intercepted by the radii is called a **sector of the circle**.



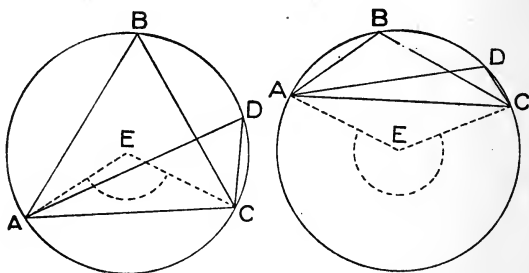
ABC, DEFG are sectors of circles.

BAC is the \angle of the sector **ABC**, and the reflex \angle **EDG** is the \angle of the sector **DEFG**.



THEOREM 7

Angles in the same segment of a circle are equal to each other.



Hypothesis.— $\angle ABC$, $\angle ADC$ are two \angle s in the same segment $ABDC$.

To prove that $\angle ABC = \angle ADC$.

Construction.—Find E the centre of the circle. Join AE , EC .

Proof.—The $\angle AEC$ at the centre and the \angle s ABC and ADC at the circumference are subtended by the same arc,

$$\begin{aligned} \therefore \angle ABC &= \frac{1}{2} \angle AEC, & (\text{III—6, p. 152.}) \\ \text{and } \angle ADC &= \frac{1}{2} \angle AEC, \\ \therefore \angle ABC &= \angle ADC. \end{aligned}$$

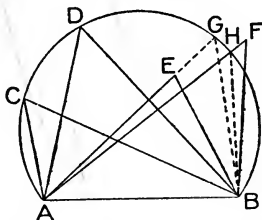
Alternative statement of the preceding theorem:—

The angle in a given segment is constant in magnitude for all positions of the vertex of the angle on the arc of the segment.

THEOREM 8

(Converse of Theorem 7)

The locus of all points on one side of a straight line at which the straight line subtends equal angles is the arc of a segment of which the straight line is the chord.



Hypothesis.—**AB** is a st. line, and **C** one of the points. Circumscribe a circle about the $\triangle ACB$.

To prove that arc **ACB** is the locus of all points on the same side of **AB** at which **AB** subtends \angle s equal to $\angle ACB$.

Construction.—Take any other point **D** on arc **ACB**, **E** any point within the segment, and **F** any point without the segment.

Join **AD**, **DB**, **AE**, **EB**, **AF**, **FB**.

Proof.—Then $\angle ADB = \angle ACB$. (III—7, p. 156.)

Produce **AE** to meet arc **ACB** at **G**. Join **BG**.

$\therefore \angle AEB$ is an exterior \angle of $\triangle EGB$,

$$\therefore \angle AEB > \angle AGB; \quad (\text{I—10, Cor., p. 45.})$$

$$\text{but } \angle AGB = \angle ACB, \quad (\text{III—7, p. 156.})$$

$$\therefore \angle AEB > \angle ACB;$$

In a similar manner it may be shown that

$$\angle AFB < \angle ACB;$$

and consequently arc **ACB** is the locus.

The vertices of equal angles subtended by the same straight line are on the same arc.

97. **Definition:**—If the three angles of one triangle are respectively equal to the three angles of another triangle, the triangles are said to be **similar**. |||

98. There are two conditions implied when figures are said to be similar: not only are the angles of one respectively equal to the angles of the other, but a certain relationship must exist between the lengths of the sides of the two figures. For triangles, it will be shown in Book IV that, if one of these conditions is given, the other is also true. For figures of more than three sides this is not the case, and a definition including both conditions must be given. (See § 131.)

The symbol ||| may be used for the word similar, or for “is similar to.”

99.—Exercises

- ✓ 1. Prove Theorem 6 when the arc is half the circumference.
2. Construct a circular arc on a chord of 3 inches and having the apex 3 inches from the chord. Calculate the radius of the circle.
3. If the chord of an arc is a inches, and the distance of its apex from the chord b inches, show that the radius of the circle is $\frac{a^2 + 4b^2}{8b}$.
- √ 4. Two chords **AOB**, **COD**, intersect at a point **O** within the circle. Show that **AOC**, **BOD** are similar Δ s. **BOC**, **AOD** are also similar Δ s. Read the segments that contain the equal \angle s.
- √ 5. **ABC** is a Δ inscribed in a circle, and the bisector of $\angle A$ meets the circumference again at **D**. Show that the st. line drawn from **D** \perp **BC** is a diameter.

✓ 6. A circle is divided into two segments by a chord equal to the radius. Show that the \angle in the major segment is 30° and that in the minor segment is 150° .

7. The locus of the vertices of the rt. \angle s of all rt.- \angle d \triangle s on the same hypotenuse is a circle.

8. Prove Theorem 6 when the arc is greater than half the circumference.

✓ 9. PQR is a \triangle inscribed in a circle. The bisector of $\angle P$ cuts QR at D and meets the circle at E. Prove that $\triangle PQD \parallel \triangle PER$.

✓ 10. DPQ and EPQ are two fixed circles, and D, P and E are in the same st. line. The bisector of $\angle DQE$ meets DE at F. Show that the locus of F is an arc of a circle.

✓ 11. If the diagonals of a quadrilateral inscribed in a circle cut at rt. \angle s, the \perp from their intersection on any side bisects the opposite side.

12. If the diagonals of a quadrilateral inscribed in a circle cut at rt. \angle s, the distance of the centre of the circle from any side is half the opposite side.

13. If the diagonals of a quadrilateral inscribed in a circle cut each other at rt. \angle s, the \angle s which a pair of opposite sides of the quadrilateral subtend at the centre of the circle are supplementary.

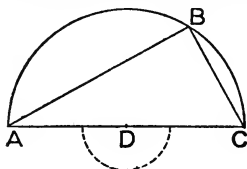
14. XYZ, XYV are two equal circles, the centre of each being on the circumference of the other. ZXV is a st. line. Prove that YZV is an equilateral \triangle .

15. EFGH is a quadrilateral inscribed in a circle and $EF = GH$. Prove that $EG = FH$.

16. ABCD is a quadrilateral inscribed in a circle; the diagonals AC, BD cut at E; F the centre of the circle is within the quadrilateral. Prove that $\angle AFB + \angle CFD = 2 \angle AEB$.

THEOREM 9

The angle in a semi-circle is a right angle.



Hypothesis.— $\angle ABC$ is an \angle in the semi-circle ABC , of which D is the centre.

To prove that $\angle ABC$ is a rt. \angle .

Proof.—The $\angle ABC$ at the circumference, and the st. $\angle ADC$ at the centre, would each subtend the same arc, if the circle were complete.

$$\begin{aligned} \therefore \angle ABC &= \frac{1}{2} \angle ADC. && \text{(III—6, p. 152.)} \\ &= \text{a rt. } \angle. \end{aligned}$$

THEOREM 10

(a) The angle in a major segment of a circle is acute.

(b) The angle in a minor segment of a circle is obtuse.

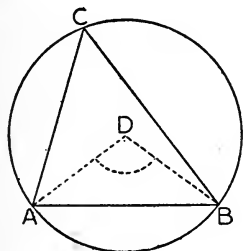


Fig 1

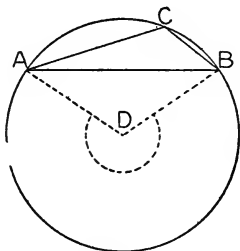


Fig.2

(a) *Hypothesis.*— $\angle ACB$ is an \angle in a major segment of a circle. (Fig. 1.)

To prove that $\angle ACB$ *is acute.*

Construction.—Join A and B to the centre D .

Proof.— $\angle ACB$ at the circumference and $\angle ADB$ at the centre stand on the same arc,

$$\therefore \angle ACB = \frac{1}{2} \angle ADB. \quad (\text{III—6, p. 152.})$$

But $\angle ADB$ is $<$ a st. \angle .

$$\therefore \angle ACB \text{ is acute.}$$

(b) *Hypothesis.*— $\angle ACB$ is an \angle in a minor segment of a circle. (Fig. 2.)

To prove that $\angle ACB$ *is obtuse.*

Construction.—Join A and B to the centre D .

Proof.—

$$\angle ACB = \frac{1}{2} \text{ the reflex } \angle ADB. \quad (\text{III—6, p. 152.})$$

$$\therefore \angle ACB \text{ is obtuse.}$$

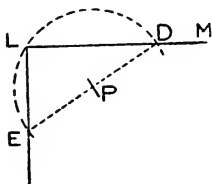
100.—Exercises

1. A circle described on the hypotenuse of a rt.- \angle \triangle as diameter passes through the vertex of the rt. \angle . (Converse of III-9).

2. Circles described on two sides of a \triangle as diameters, intersect on the third side, or the third side produced.

Where is the point of intersection when the circles are described on the equal sides of an isosceles \triangle ?

3. LM is a st. line and L a point from which it is required to draw a \perp to LM .



Construction.—With a convenient point P as centre describe a circle to pass through L and cut LM at D . Join DP , and produce DP to cut the circle at E . Join LE .

Prove $LE \perp LM$.

4. EF , EG are diameters of two circles FEH , GEH respectively. Show that FHG is a st. line.

5. ST is a diameter of the circle SVT . A circle is described with centre S and radius ST . Show that any chord of this latter circle drawn from T is bisected by the circle SVT .

6. Chords of a given circle are drawn through a given point. Find the locus of the middle points of the chords when the given point is (a) on the circumference, (b) within the circle, (c) without the circle.

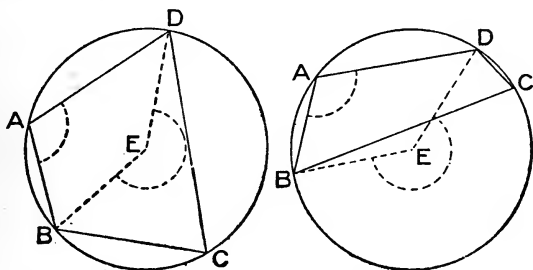
7. F is any point on the arc of a semi-circle of which DE is a diameter. The bisectors of \angle s FED , FDE meet at P . Find the locus of P .

8. F is a point on the arc of a semi-circle of which DE is a diameter. $FG \perp DE$. Show that the \triangle s FDG , FEG , FDE are similar.

9. $PQRS$ is a st. line and circles described on PR , QS as diameters cut at E . Prove that $\angle PEQ = \angle RES$.

THEOREM 11

If a quadrilateral is inscribed in a circle, its opposite angles are supplementary.



Hypothesis.— $ABCD$ is a quadrilateral inscribed in a circle.

To prove that $\angle A + \angle C = 2 \text{ rt. } \angle$ s.

Construction.—Find the centre E . Join BE , ED .

Proof.— $\angle BED$ at the centre and $\angle C$ at the circumference are subtended by the same arc BAD .

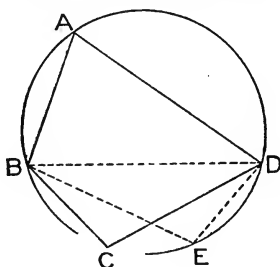
$$\therefore \angle C = \frac{1}{2} \angle BED. \quad (\text{III—6, p. 152.})$$

Similarly $\angle A = \frac{1}{2} \text{ reflex } \angle BED$.

Hence $\angle A + \angle C = \frac{1}{2}$ the sum of the two \angle s BED at the centre = $\frac{1}{2}$ of $4 \text{ rt. } \angle$ s
 $= 2 \text{ rt. } \angle$ s.

THEOREM 12

If the opposite angles of a quadrilateral are supplementary, its vertices are concyclic.



Hypothesis.— $ABCD$ is a quadrilateral in which $\angle A + \angle C = 2 \text{ rt. } \angle s.$

To prove that A, B, C, D are on the circumference of a circle.

Construction.—Draw a circle through the three points $A, B, D.$ On this circumference and on the side of BD remote from A take a point $E.$ Join $BE, ED.$

Proof.— $\because ABED$ is a quadrilateral inscribed in a circle,

$$\therefore \angle A + \angle E = 2 \text{ rt. } \angle s; \quad (\text{III—11, p. 163.})$$

$$\text{but } \angle A + \angle C = 2 \text{ rt. } \angle s. \quad (\text{Hyp.})$$

$$\therefore \angle A + \angle E = \angle A + \angle C,$$

$$\text{and } \therefore \angle E = \angle C.$$

Consequently, as C, E are on the same side of $BD,$ the circle $BADE$ passes through $C.$ (III—8, p. 157.)

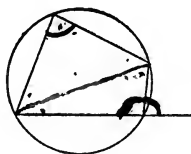
101.—Exercises

1. If one side of an inscribed quadrilateral be produced, the exterior \angle thus formed at one vertex equals the interior \angle at the opposite vertex of the quadrilateral.

State and prove the converse.

2. From a point O without a circle two st. lines OAB , OCD are drawn cutting the circumference at A , B , C , D .

Show that \triangle s OBC , OAD are similar, and that \triangle s OAC , OBD are similar.



3. If a \parallel gm be inscribed in a circle, the \parallel gm is a rect.

4. A , D , C , E , B are five successive points on the circumference of a circle; and A , B are fixed. Show that the sum of the \angle s ADC , CEB is the same for all positions of D , C , E .

5. A circle is circumscribed about an equilateral \triangle . Show that the \angle in each segment outside the \triangle is an \angle of 120° .

6. A scalene \triangle is inscribed in a circle. Show that the sum of the \angle s in the three segments outside the \triangle is 360° .

7. A quadrilateral is inscribed in a circle. Show that the sum of the \angle s in the four segments outside the quadrilateral is 540° .

8. P is a point on the diagonal KM of the \parallel gm $KLMN$. Circles are described about PKN and PLM . Show that LN passes through the other point of intersection of the circles.

✕ 9. A circle drawn through the middle points of the sides of a \triangle passes through the feet of the \perp s from the vertices to the opposite sides.

✓ 10. If the opposite sides of a quadrilateral inscribed in a circle be produced to meet at L and M , and about the \triangle s

*Term
Prob*

so formed outside the quadrilateral circles be described intersecting again at N , then L, M, N are in the same st. line.

✓ 11. In a $\triangle DEF$, $DX \perp EF$ and $EY \perp DF$. Prove that $\angle XYF = \angle DEF$.

✓ 12. $PQRS, PQTV$ are circles and SPV, RQT are st. lines. Prove that $SR \parallel VT$.

✓ 13. The st. lines that bisect any \angle of a quadrilateral inscribed in a circle and the opposite exterior \angle meet on the circumference.

14. XYZ is a \triangle ; $YD \perp ZX$, and $DE \perp XY$; $ZF \perp XY$ and $FG \perp ZX$. Show that $EG \parallel YZ$.

15. EGD, FGD are two circles with centres H, K respectively. EGF is a st. line. EH, FK meet at P . Show that H, K, D, P are concyclic.

16. KL, MN are two \parallel chords in a circle; KE, NF two \perp chords in the same circle. Show that $LF \perp ME$.

17. The bisectors of the \angle s formed by producing the opposite sides of a quadrilateral inscribed in a circle are \perp to each other.

18. HKM, LKM are two circles, and HKL is a st. line. HM, LM cut the circles again at E, F respectively, and HF cuts LE at G . Show that a circle may be circumscribed about $MEGF$.

19. $PQRS$ is a quadrilateral and the bisectors of the \angle s $P, Q; Q, R; R, S; S, P$ meet at four points. Show that a circle may be circumscribed about the quadrilateral thus formed.

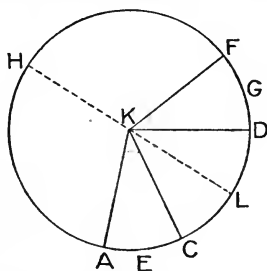
20. EF is the diameter of a semi-circle and G, H any two points on its arc. EH, FG cut at K and EG, FH cut at L . Show that $KL \perp EF$.

21. DE is the diameter, O the centre and P any point on the arc of a semi-circle. $PM \perp DE$. Show that the bisector of $\angle MPO$ passes through a fixed point.

22. PQR is a \triangle and PDQ , PFQ are two circles cutting PR at D, F and QR at E, G . Prove that $DE \parallel FG$.

THEOREM 13

If two angles at the centre of a circle are equal to each other, they are subtended by equal arcs.



Hypothesis.— $\angle AKC, \angle DKF$ are equal \angle s at the centre K of the circle ACD .

To prove that arc AEC equals arc DGF .

Construction.—Draw the diameter HKL bisecting $\angle CKD$.

Proof.—Suppose the circle to be folded along the diameter HKL , and the semi-circle HFL will coincide throughout with the semi-circle HAL .

$$\therefore \angle LKD = \angle LKC,$$

$$\therefore KD \text{ falls along } KC;$$

$$\text{and } \therefore D \text{ falls on } C.$$

$$\therefore \angle DKF = \angle CKA,$$

$$\therefore KF \text{ falls along } KA;$$

$$\text{and } \therefore F \text{ falls on } A.$$

$$\therefore \text{the arc } DGF \text{ coincides with the arc } CEA.$$

$$\therefore \text{arc } DGF = \text{arc } CEA.$$

102. — Exercises

1. If two arcs of a circle be equal to each other, they subtend equal \angle s at the centre. (Prove either by indirect demonstration, or by the construction and method used in III—13.)

2. If two \angle s at the circumference of a circle be equal to each other, they are subtended by equal arcs.

3. If two arcs of a circle be equal to each other, they subtend equal \angle s at the circumference.

4. In equal circles equal \angle s at the centres (or circumferences) stand on equal arcs.

5. In equal circles equal arcs subtend equal \angle s at the centres (or circumferences).

6. If two arcs of a circle (or of equal circles) be equal, they are cut off by equal chords.

7. If two chords of a circle be equal to each other, the major and minor arcs cut off by one are respectively equal to the major and minor arcs cut off by the other.

8. If two sectors of a circle have equal \angle s at the centre, the sectors are congruent.

9. Bisect a given arc of a circle.

10. Parallel chords of a circle intercept equal arcs.

Show also that the converse is true.

11. If two equal circles cut one another, any st. line drawn through one of the points of intersection will meet the circles again at two points which are equally distant from the other point of intersection.

12. The bisectors of the opposite \angle s of a quadrilateral inscribed in a circle meet the circumference at the ends of a diameter.

13. If two \angle s at the centre of a circle be supplementary, the sum of the arcs on which they stand is equal to half the circumference.

14. If any number of \angle s be in a segment, their bisectors all pass through one point.

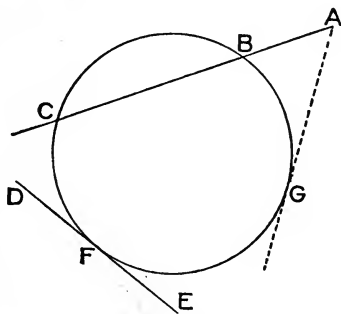
TANGENTS AND CHORDS

103. **Definitions.**—Any straight line which cuts a circle is called a **secant**.

A straight line which, however far it may be produced, has one point on the circumference of a circle, and all other points without the circle is called a **tangent** to the circle.

A tangent is said to **touch** the circle.

The common point of a tangent and circle, that is, the point where the tangent touches the circle, is called the **point of contact**.



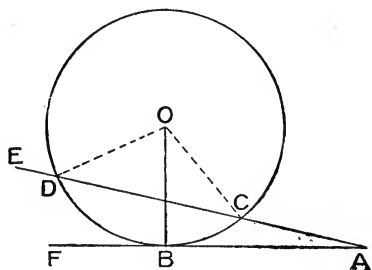
$\triangle ABC$ is a secant drawn to the circle BCF from the point A .

$\triangle DFE$ is a tangent to the circle BCF , touching the circle at the point of contact F .

If the secant ABC rotate about the point A until the two points B, C where it cuts the circle coincide at G , the secant becomes a tangent having G for the point of contact.

THEOREM 14

The radius drawn to the point of contact of a tangent is perpendicular to the tangent.



Hypothesis.—**ABF** is a tangent to the circle **CBD** at the point **B**, **O** is the centre and **OB** the radius drawn to the point of contact.

To prove that **OB** is \perp **AF**.

Construction.—From any point **A**, except **B**, in **AF** draw a secant **AE** cutting the circle in **C** and **D**. Join **OC**, **OD**.

Proof.— \because $OD = OC$,
 $\therefore \angle ODC = \angle OCD$. (I—3, p. 20.)

But, st. $\angle EDC =$ st. $\angle DCA$,

$\therefore \angle ODE = \angle OCA$.

Rotate **AE** about **A** until it coincides with **AF**. As **AE** rotates about **A** the \angle s **ODE**, **OCA** are continually equal to each other and finally $\angle ODE$ becomes $\angle OBF$ and $\angle OCA$ becomes $\angle OBA$.

$\therefore \angle OBF = \angle OBA$.

and $\therefore OB \perp AF$.

Cor. 1.—Only one tangent can be drawn at any point on the circumference of a circle.

\therefore only one st. line can be \perp to the radius at that point.

Hence, also:—The straight line drawn perpendicular to a radius at the point where it meets the circumference is a tangent.

Cor. 2.—The perpendicular to a tangent at its point of contact passes through the centre of the circle.

\therefore only one st. line can be \perp to the tangent at that point.

Cor. 3.—The perpendicular from the centre on a tangent passes through the point of contact.

\therefore only one \perp can be drawn from a given external point to a given st. line.

104.—Exercises

1. Draw a tangent to a given circle from a given point on the circumference.
2. Describe a circle with its centre on a given st. line DE to pass through a given point P in DE and touch another given st. line DF .
3. Find the locus of the centres of all circles that touch a given st. line at a given point.
4. Describe a circle to pass through a given point and touch a given st. line at a given point.
5. Tangents at the ends of a diameter are \parallel .

6. C is any point on the tangent of which A is the point of contact. The st. line from C to the centre O cuts the circumference at B . AD is $\perp OC$. Show that BA bisects the $\angle DAC$.

7. Find the locus of the centres of all circles which touch two given \parallel st. lines.

✓8. Draw a circle to touch two given \parallel st. lines and pass through a given point between the \parallel s. Show that two such circles may be drawn.

✓9. To a given circle draw two tangents, each of which is \parallel to a given st. line.

10. To a given circle draw two tangents, each of which is \perp to a given st. line.

11. Give an alternative proof for III—14 by supposing the radius OB drawn to the point of contact of the tangent ABF not \perp to AF and drawing $OG \perp AF$.

12. Two tangents to a circle meet each other. Prove that they are equal to each other.

13. EF is a diameter of a circle and EG is a chord. EH is a chord bisecting the $\angle FEG$. Prove that the tangent at H is $\perp EG$.

14. Draw a circle to touch a given st. line at a given point and have its centre on another given st. line.

15. Draw a tangent to a given circle making a given \angle with a given st. line.

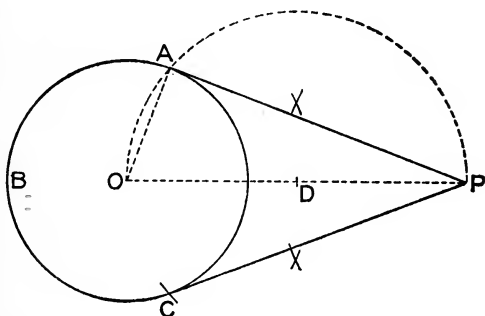
Show that, in general, four such tangents may be drawn.



CONSTRUCTION

PROBLEM 3

To draw a tangent to a given circle from a given point without the circle.



Let ABC be the given circle, and P the given point.

It is required to draw a tangent from P to the circle ABC .

Join P to the centre O . Bisect OP at D . With centre D and radius DO , describe a circle cutting the circle ABC at A and C . Join PA , PC .

Either PA or PC is a tangent to the given circle.

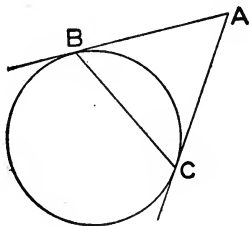
Join OA .

OAP is an \angle in a semi-circle, and is \therefore a rt. \angle . (III—9, p. 160.)

$\therefore PA$ is a tangent. (III—14, Cor. 1, p. 171.)

In the same manner it may be shown that PC is a tangent.

105. **Definition.**—The straight line joining the points of contact of two tangents to a circle is called the **chord of contact** of the tangents.



BC is the chord of contact of the tangent **AB**, **AC**.



106.—Exercises

1. Draw a circle of radius 4 cm. Take a point 9 cm. from the centre of the circle. From this point draw two tangents to the circle. Measure the length of each tangent and check your result by calculation.

2. Draw a circle of radius 5 cm. Mark a point 7 cm. from the centre. From this point draw two tangents to the circle and measure the \angle between the tangents. (91° nearly.)

3. Draw a circle with a radius of 3 cm. Mark any point **A** on the circumference, and from this point draw a tangent **AB** 4 cm. long. Measure the distance of **B** from the centre and check your result.

4. Draw a circle with 43 mm. radius. Draw any st. line through the centre, and find a point, in this line, from which the tangent to the circle will be 5 cm. in

length. Measure the distance of the point from the centre and check your result.

5. Mark two points **A** and **B** 7 cm. apart. Draw two st. lines from **A** such that the length of the perpendicular from **B** to either of them is 4 cm.

6. Draw a circle of radius 6 cm. Mark a point **P** 4 cm. from the centre. Draw a chord through **P** such that the perpendicular from the centre to the chord is 3 cm. in length. Measure the length of the chord and check your result by calculation.

7. Draw a circle of radius 36 mm. Mark any point **P** without the circle. Draw a st. line from **P** such that the chord cut off on it by the circle is 4 cm. in length.

8. Draw a circle of radius 47 mm. Mark a point **P** 4 cm. from the centre. Draw two chords through **P**, each of which is 65 mm. in length.

9. If from a point without a circle two tangents be drawn, the st. line drawn from this point to the centre bisects the chord of contact and cuts it at rt. \angle s.

✓10. If a quadrilateral be circumscribed about a circle, the sum of one pair of opposite sides equals the sum of the other pair.

✱11. Through a given point draw a st. line, such that the chord intercepted on the line by a given circle is equal to a given st. line.

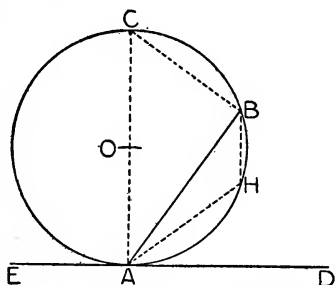
✓12. If a \parallel gm be circumscribed about a circle, the \parallel gm is a rhombus.

13. If two tangents to a circle be \parallel , their chord of contact is a diameter.

- ✓ 14. If two \parallel tangents to a circle be cut by a third tangent to the circle at A, B ; show that AB subtends a rt. \angle at the centre.
- ✓ 15. If a quadrilateral be circumscribed about a circle, the \angle s subtended at the centre by a pair of opposite sides are supplementary.
16. To a given circle draw two tangents containing an \angle equal to a given \angle .
17. Find the locus of the points from which tangents drawn to a given circle are equal to a given st. line.
- ✓ 18. Find a point P in a given st. line, such that the tangent from P to a given circle is of given length. What is the condition that this is possible?
- ✓ 19. E is a point outside a circle the centre of which is D . In DE produced find a point F , such that the length of the tangent from F may be twice that of the tangent from E .
20. Two tangents, LM, LN are drawn to a circle; P is any point on the circumference outside the $\triangle LMN$. Prove that $\angle LMP + \angle LNP$ is constant.
21. Find the \angle between the tangents to a circle from a point whose distance from the centre is equal to a diameter.
22. Show that all equal chords of a given circle touch a fixed concentric circle.
23. From a given point without a circle draw a st. line such that the part intercepted by the circle subtends a rt. \angle at the centre.
-

THEOREM 15

If at one end of a chord of a circle a tangent is drawn, each angle between the chord and the tangent is equal to the angle in the segment on the other side of the chord.



Hypothesis.—**AB** is a chord and **EAD** a tangent to the circle **ABC**.

To prove that $\angle \text{DAB} = \angle \text{ACB}$ and that $\angle \text{EAB} = \angle \text{AHB}$.

Construction.—From **A** draw the diameter **AOC**. Join **BC**. Join any point **H** in the arc **AHB** to **A** and **B**.

Proof.— \because **ABC** is an \angle in a semi-circle,

$$\therefore \text{ABC is a rt. } \angle. \quad (\text{III—9, p. 160.})$$

$$\therefore \angle \text{BAC} + \angle \text{BCA} = \text{a rt. } \angle \quad (\text{I—10, p. 45.})$$

$$= \angle \text{CAD. (III—14, p. 170.)}$$

Take away the common $\angle \text{BAC}$,

$$\therefore \angle \text{BAD} = \angle \text{ACB},$$

$$= \angle \text{ in the segment } \text{ACB}.$$

\because **AHBC** is an inscribed quadrilateral,

$$\therefore \angle \text{H} + \angle \text{C} = \text{a st. } \angle \quad (\text{III—11, p. 163.})$$

$$= \text{st. } \angle \text{ DAE}.$$

But $\angle \text{C} = \angle \text{BAD}$.

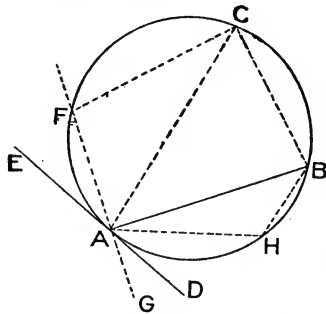
$$\therefore \angle \text{BAE} = \angle \text{H}$$

$$= \angle \text{ in the segment } \text{AHB}.$$

THEOREM 15

(Alternative Proof)

If at one end of a chord of a circle a tangent is drawn, each angle between the chord and the tangent is equal to the angle in the segment on the other side of the chord.



Hypothesis.—**AB** is a chord and **EAD** a tangent to the circle **ABC**.

To prove that $\angle DAB = \angle ACB$, and that $\angle EAB = \angle AHB$.

Construction.—In arc **AFC** take any point **F**. Join **CF**, and draw the line **FAG**.

Proof.— \because **AFCB** is an inscribed quadrilateral,

$\therefore \angle FCB$ is supplementary to $\angle FAB$,
(III—11, p. 163.)

But, $\angle BAG$ is supplementary to $\angle FAB$.

$\therefore \angle BAG = \angle FCB$.

These \angle s are equal however near **F** is to **A**.

Let **F** move along the circumference towards **A** and finally coincide with **A**.

The line **FAG** rotates about the point **A** and finally coincides with **EAD**. The \angle **GAB** becomes \angle **DAB** and \angle **FCB** becomes \angle **ACB**.

$$\therefore \angle \text{DAB} = \angle \text{ACB}.$$

\because \angle **EAB** is supplementary to \angle **DAB**,
and, \angle **AHB** is supplementary to \angle **ACB**.

(III—11, p. 163.)

$$\therefore \angle \text{EAB} = \angle \text{AHB}.$$

107.—Exercises

1. **AB** is a chord of a circle and **AC** is a diameter. **AD** is \perp to the tangent at **B**. Show that **AB** bisects the \angle **DAC**.

2. Two circles intersect at **A** and **B**. Any point **P** on the circumference of one circle is joined to **A** and **B** and the joining lines are produced to meet the circumference of the other circle at **C**, **D**. Show that **CD** is \parallel to the tangent at **P**.

3. **LMN** is a \triangle . Show how to draw the tangent at **L** to the circumscribed circle, without finding the centre of this circle.

4. If either of the \angle s which a st. line, drawn through one end of a chord of a circle, makes with the chord is equal to the \angle in the segment on the other side of the chord, the st. line is a tangent. (*Converse of III—15.*)

5. The tangent at a point **P** on a circle meets the chord **MN** produced through **N**, at **Q**. Prove \angle **Q** = \angle **PNM** - \angle **PMN**.

6. A tangent drawn \parallel to a chord of a circle bisects the arc cut off by the chord.

7. FGE , HKE are two circles, and FEH , GEK two st. lines. Prove that FG , KH meet at an \angle which = the \angle between the tangents to the circles at E .

8. G is the middle point of an arc EGF of a circle. Show that G is equidistant from the chord EF and the tangent at E .

9. A st. line EF is trisected in G , H , and an equilateral $\triangle PGH$ is described on GH . Show that the circle FGP touches EP .

10. D , E , F are respectively the points of contact of the sides MN , NL , LM of a \triangle circumscribed about a circle. DG , EH are respectively $\perp EF$, DF . Prove $GH \parallel LM$.

11. The tangent at L to the circumscribed circle of $\triangle LMN$ meets MN produced at D , and the internal and external bisectors of the $\angle MLN$ meet MN at E , F respectively. Prove that D is the middle point of EF .

12. GEF , HEF are two circles and GEH is a st. line. The tangents at G , H meet at K . Show that K , G , F , H are concyclic.

13. Points P , Q are taken on two st. lines LM , LN so that $LP + LQ =$ a given st. line. Prove that the circle PLQ passes through a second fixed point.

14. E , F , G , H are the points of contact of the sides XY , YZ , ZV , VX of a quadrilateral circumscribed about a circle. If X , Y , Z , V are concyclic, show that $EG \perp FH$.

15. $XYZV$ is a quadrilateral inscribed in a circle, and XZ , YV cut at E . Prove that the tangent at E to the circle XEY is $\parallel ZV$.

16. F is the point of contact of a tangent EF to the circle FGH . GK drawn $\parallel EF$ meets FH , or FH produced,

at **K**. Show that the circle through **G**, **K**, **H** touches **FG** at **G**.

17. If from an external point **P** a tangent **PT** and a secant **PMN** be drawn to a circle, the \triangle s **PTM**, **PNT** are similar.

18. Use III—15 to prove that the tangents drawn to a circle from an external point are equal.

19. From an external point **T** a tangent **TR** and a secant **TQP** through the centre are drawn to a circle. Prove that $\angle T + 2 \angle TRQ = \text{a rt. } \angle$.

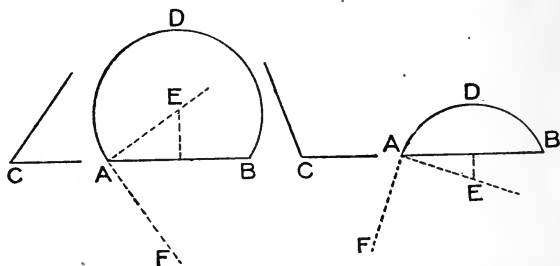
20. The tangents **OT**, **OS** from a fixed point **O** to a given circle contain an \angle of x degrees. A third tangent is drawn to the circle at any point on the minor arc **TS**. Show that the portion of this tangent intercepted by **OT** and **OS** subtends an \angle of $(90 - \frac{x}{2})$ degrees at the centre.

Show that if the moving point be taken on the major arc **TS**, the \angle at the centre will be $(90 + \frac{x}{2})$ degrees.

CONSTRUCTIONS

PROBLEM 4

On a given straight line to construct a segment containing an angle equal to a given angle.



Let AB be the given st. line, and C the given \angle .

Construction.—Make $\angle BAF = \angle C$.

Draw $AE \perp AF$.

Draw the right bisector of AB and produce it to cut AE at E .

$\therefore E$ is in the right bisector of AB , it is equidistant from A and B . (I—22, p. 78.)

With centre E and radius EA describe the arc ADB .
 ADB is the required arc.

Proof.— $\therefore AF$ is $\perp AE$,

$\therefore AF$ is a tangent to the circle ADB .

(III—14, Cor. 1, p. 171.)

$\therefore AB$ is a chord drawn from the point of contact of the tangent AF ,

$\therefore \angle$ in segment $ADB = \angle FAB$. (III—15, p. 177.)

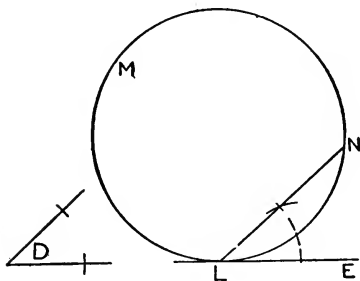
But, $\angle FAB = \angle C$,

(*Const.*)

$\therefore \angle$ in segment $ADB = \angle C$.

PROBLEM 5

From a given circle to cut off a segment containing an angle equal to a given angle.



Let LMN be the given circle, and D the given \angle .

Construction.—Draw a tangent LE to the given circle.

At L make the $\angle ELN = \angle D$.

LMN is the required segment.

Proof.— \because LE is a tangent, and LN a chord,

$$\therefore \angle \text{ in segment } LMN = \angle NLE.$$

(III—15, p. 177.)

$$\text{But, } \angle NLE = \angle D.$$

(*Const.*)

$$\therefore \angle \text{ in segment } LMN = \angle D.$$

108.—**Exercises**

1. On st. lines each 4 cm. in length, describe segments containing \angle s of (a) 45° , (b) 150° , (c) 72° , (d) 116° . (*Use the protractor for (c) and (d).*)

2. On a given base construct an isosceles \triangle with a given vertical \angle .

3. Divide a circle into two segments such that the \angle in one segment is (a) twice, (b) three times, (c) five times, (d) seven times the \angle in the other segment.

4. Construct two \triangle s ABC_1 , ABC_2 on the same base $AB = 4$ cm., having $\angle AC_1B = \angle AC_2B = 50^\circ$, and $AC_1 = AC_2 = 5$ cm.

Prove that $\angle ABC_1 + \angle ABC_2 = 2$ rt. \angle s.

5. Construct a $\triangle LMN$ having $LM = 5$ cm., $\angle N = 110^\circ$, and the median from $N = 2$ cm.

Measure the greatest and least values the median from N could have, with $LM = 5$ cm., and $\angle N = 110^\circ$.

6. Construct a \triangle having its base 5 cm., its vertical $\angle 70^\circ$, and its altitude 3 cm.

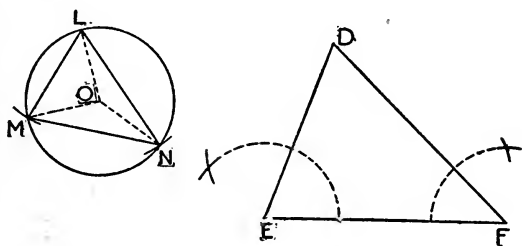
7. Construct a $\triangle XYZ$, having $XY = 4$ cm., $\angle Z = 40^\circ$, and $XZ + ZY = 10$ cm.

8. Construct a $\triangle XYZ$, having $XY = 6$ cm., $\angle Z = 50^\circ$ and $XZ - ZY = 4$ cm.

9. Through a given point draw a st. line to cut off from a given circle a segment containing an \angle equal to a given \angle .

PROBLEM 6

In a given circle to inscribe a triangle similar to a given triangle.



Let LMN be the given circle, and DEF the given \triangle .

Construction.—Draw a radius OL of the circle.

Make $\angle LON = 2 \angle E$, and $\angle LOM = 2 \angle F$.

Join LM , MN , NL .

LMN is the required \triangle .

Join OM , ON .

Proof.— $\because \angle LON$ at the centre and $\angle LMN$ at the circumference stand on the same arc.

$$\therefore \angle LON = 2 \angle LMN, \text{ (III—6, p. 152.)}$$

$$\text{But } \angle LON = 2 \angle E, \quad (\text{Const.})$$

$$\therefore \angle LMN = \angle E.$$

$$\text{Similarly } \angle LNM = \angle F.$$

$$\therefore \angle LMN = \angle E,$$

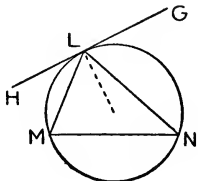
$$\text{and } \angle LNM = \angle F,$$

$$\therefore \angle MLN = \angle D. \quad (\text{I—10, p. 45.})$$

$$\text{and } \therefore \triangle LMN \parallel \triangle DEF.$$

109.—Exercises

1. Prove the following construction for inscribing a \triangle similar to a given $\triangle DEF$ in the circle LMN .

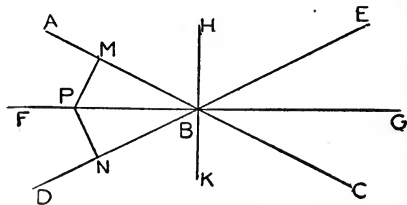


Draw a tangent HLG . Make $\angle GLN = \angle E$, and $\angle HLM = \angle F$. Join MN .

2. Inscribe an equilateral \triangle in a given circle.
3. Inscribe a square in a given circle.
4. Inscribe a regular pentagon in a given circle. (Use protractor).
5. Inscribe a regular hexagon in a given circle. (Without protractor).
6. Inscribe a regular octagon in a given circle.
7. Two \triangle s LMN , DEF , each similar to a given $\triangle GHK$, are inscribed in a given circle. Prove $\triangle LMN \equiv \triangle DEF$.
8. In a given circle inscribe a \triangle having its sides \parallel to the sides of a given \triangle .

PROBLEM 7

To find the locus of the centres of circles touching two given intersecting straight lines.



Let ABC , DBE be the two st. lines.

Construction.—Draw the bisectors **FBG**, **HBK** of the \angle s made by **AC** and **DE**.

These bisectors make up the required locus.

Proof.—Take a point **P** in either **FG** or **HK**, and draw **PM** \perp **AC**, **PN** \perp **DE**.

$$\text{In } \triangle\text{s } \text{PMB, PNB, } \begin{cases} \angle \text{PBM} = \angle \text{PBN,} \\ \angle \text{PMB} = \angle \text{PNB,} \\ \text{and } \text{PB} \text{ is common,} \end{cases}$$

$$\therefore \text{PM} = \text{PN.} \quad (\text{I—14, p. 54.})$$

Hence, a circle described with centre **P** and radius **PM** will pass through **N**.

$\therefore \angle$ s at **M**, **N** are rt. \angle s,

\therefore **AC**, **DE** are tangents to the circle.

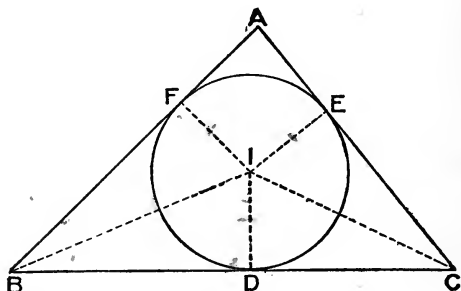
(III—14, Cor. 1, p. 171.)

110. Definitions.—When a circle is within a triangle, and the three sides of the triangle are tangents to the circle, the circle is said to be **inscribed in the triangle**, and is called the **inscribed circle of the triangle**.

When a circle lies without a triangle, and touches one side and the other two sides produced, the circle is called an **escribed circle of the triangle**.

PROBLEM 8

To inscribe a circle in a given triangle.



Let ABC be the given \triangle .

Bisect \angle s B and C and produce the bisectors to meet at I .

Draw $ID, IE, IF, \perp BC, CA, AB$ respectively.

$$\text{In } \triangle\text{s } BID, BIF, \begin{cases} \angle IBD = \angle IBF, \\ \angle IDB = \angle IFB, \\ IB \text{ is common,} \end{cases}$$

$$\therefore ID = IF. \quad (\text{I—14, p. 54.})$$

Similarly, $ID = IE$.

\therefore a circle described with centre I and radius ID will pass through E and F .

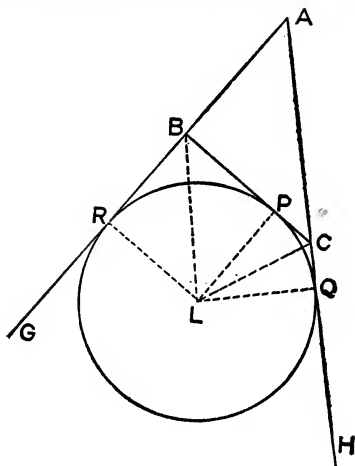
And \because the \angle s at D, E and F are rt. \angle s,

\therefore the circle will touch BC, CA and AB .

(III—14, Cor. 1, p. 171.)

PROBLEM 9

To draw an escribed circle of a given triangle.



Let ABC be a given \triangle having AB, AC produced to G, H .

It is required to describe a circle touching the side BC and the two sides AB, AC produced.

Bisect \angle s GBC, HCB and let the bisectors meet at L . Draw \perp s LP, LQ, LR to BC, CH, BG respectively.

$$\text{In } \triangle\text{s } LBP, LBR, \begin{cases} \angle PBL = \angle RBL, \\ \angle LPB = \angle LRB, \\ LB \text{ is common,} \end{cases}$$

$$\therefore LP = LR. \quad (\text{I—14, p. 54.})$$

Similarly $LP = LQ$.

\therefore a circle described with centre L and radius LP will pass through R and Q .

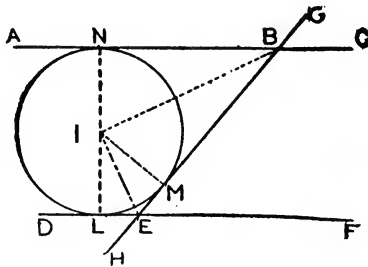
\therefore the \angle s at P, Q and R are rt. \angle s,

\therefore the circle will touch BC , and CA and AB produced. (III—14, Cor. 1, p. 171.)

PROBLEM 10

To describe a circle to touch three given straight lines.

(a) If two of the lines are \parallel to each other, and the third cuts them, two circles may be drawn to touch the three lines.



Let ABC , DEF and $GBEH$ be the three lines of which $AC \parallel DF$.

Bisect \angle s ABE , BED , and produce the bisectors to meet at I .

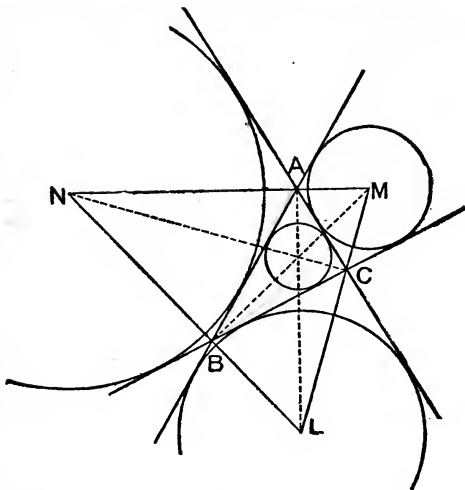
Draw IL , IM , $IN \perp DE$, EB , BA respectively.

As in problems 8 and 9 it may be shown that a circle described with centre I and radius IL will touch DE , EB and BA .

Similarly, a circle may be described on the other side of BE to touch the three given st. lines.

(b) If the lines intersect each other forming a Δ , four circles may be drawn to touch the three lines.

Let $\triangle ABC$ be the \triangle formed by the lines.



Draw the inscribed circle and the three escribed circles of $\triangle ABC$.

These four circles touch the three given st. lines.

111.—Exercises

1. Make an $\angle YXZ = 45^\circ$. Find a point P such that its distance from XY is 3 cm., and its distance from XZ is 4 cm.
2. Make an $\angle YXZ = 60^\circ$. Find a point P such that its distance from XY is 4 cm., and its distance from XZ is 5 cm.
3. The bisectors of the \angle s of a \triangle are concurrent.
4. The bisectors of the exterior \angle s at two vertices of a \triangle and the bisector of the interior \angle at the third vertex are concurrent.
5. If a, b, c represent the numerical measures of the sides BC, CA, AB respectively of $\triangle ABC$, and $s = \frac{1}{2}(a + b + c)$,

(a) $AF = s - a$, $BD = s - b$, $CE = s - c$, when **D**, **E** and **F** are the points of contact of **BC**, **CA**, **AB** with the inscribed circle. (Diagram of Problem 8.)

(b) $AR = s$, $BP = s - c$, $CP = s - b$, where **R** and **P** are the points of contact of **AB** produced and of **BC** with an escribed circle. (Diagram of Problem 9.)

(c) If r be the radius of the inscribed circle, $rs =$ the area of $\triangle ABC$.

(d) If r_1 be the radius of the escribed circle touching **BC**, $r_1(s - a) =$ the area of $\triangle ABC$.

6. If the base and vertical \angle of a \triangle be given, find the locus of the inscribed centre.

7. If the base and vertical \angle of a \triangle be given, find the loci of the escribed centres.

8. **L**, **M**, **N** are the centres of the escribed circles of $\triangle PQR$. Show that the sides of $\triangle LMN$ pass through the vertices of $\triangle PQR$.

9. If the centres of the escribed circles be joined, and the points of contact of the inscribed circle with the sides be joined, the \triangle s thus formed are similar.

10. Construct a \triangle having given the base, the vertical \angle and the radius of the inscribed circle.

11. Describe a circle cutting off three equal chords of given length from the sides of a given \triangle .

12. An escribed circle of $\triangle ABC$ touches **BC** at **D** and also touches **AB** and **AC** produced. The inscribed circle touches **BC** at **E**. Show that **DE** equals the difference of **AB** and **AC**.

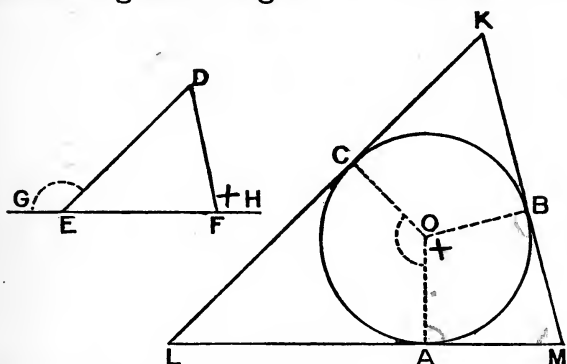
13. Circumscribe a square about a given circle.

14. Inscribe a circle in a given square.

15. Circumscribe a circle about a given square.

PROBLEM 11

About a given circle to circumscribe a triangle similar to a given triangle.



Let ABC be the given circle and DEF the given \triangle .

Construction.—Produce EF to G and H .

Draw any radius OA of the circle, and at O make $\angle AOB = \angle DFH$, and $\angle AOC = \angle DEG$; and produce the arms to cut the circle at B, C .

At A, B, C draw tangents to the circle meeting at K, L and M .

KLM is the required \triangle .

Proof.— $\because \angle s$ MAO and MBO in the quadrilateral $MBOA$ are rt. $\angle s$,

$$\begin{aligned} \therefore \angle M + \angle AOB &= 2 \text{ rt. } \angle s. \\ &= \angle DFE + \angle DFH. \end{aligned}$$

But, $\angle AOB = \angle DFH$,

$$\therefore \angle M = \angle DFE.$$

Similarly, $\angle L = \angle DEF$.

$$\therefore \angle L + \angle M = \angle DEF + \angle DFE,$$

and $\therefore \angle K = \angle EDF$. (I—10, p. 45.)

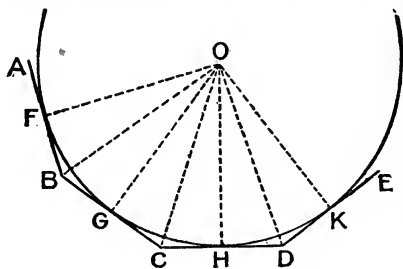
$$\therefore \triangle KLM \parallel \triangle DEF.$$

112.—Exercises

1. About a given circle circumscribe an equilateral \triangle .
2. If two similar \triangle s be circumscribed about the same circle, the \triangle s are congruent.
3. Describe a \triangle LMN similar to a given \triangle and such that a given circle is touched by MN and by LM and LN produced.

PROBLEM 12

To inscribe a circle in a given regular polygon.



Let AB, BC, CD, DE be four consecutive sides of a given regular polygon.

It is required to inscribe a circle in the polygon.

Bisect \angle s BCD, CDE and produce the bisectors to meet at O. Join OB. From O draw \perp s OF, OG, OH, OK to AB, BC, CD, DE respectively.

$$\text{In } \triangle\text{s } OCB, OCD, \begin{cases} BC = CD, \\ CO \text{ is common,} \\ \angle OCB = \angle OCD, \end{cases}$$

$$\therefore \angle OBC = \angle ODC.$$

(I—2, p. 16.)

$$\text{But } \angle ODC = \frac{1}{2} \angle CDE \text{ and } \angle ABC = \angle CDE,$$

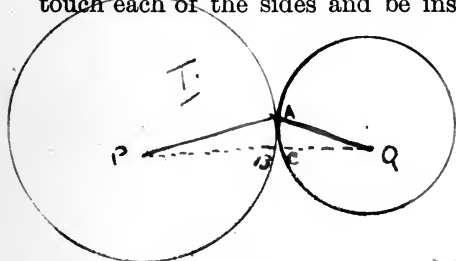
$$\therefore \angle OBC = \frac{1}{2} \angle ABC.$$

In the same manner it may be shown that if O be joined to all the vertices of the polygon the joining lines will bisect the \angle s at the vertices.

$$\text{In } \triangle\text{s } OCG, OCH, \begin{cases} \angle OCG = \angle OCH, \\ \angle OGC = \angle OHC, \\ OC \text{ is common,} \end{cases}$$

$$\therefore OG = OH. \quad (\text{I-14, p. 54.})$$

In the same manner it may be shown that the \perp s from O to all of the sides are equal to each other, and as the \angle s at $F, G, H,$ etc., are rt. \angle s, a circle described with O as centre and OF as radius will touch each of the sides and be inscribed in the polygon.



Theorem 16

I. $P + Q$ are centres of two circles touching at A

Prove PQ passes through A

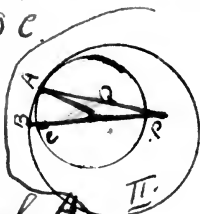
Suppose PQ does not pass through A but cuts the circles at B, C . Join PA, QA .

$$\therefore PA = PB + QA = QC$$

$$\begin{aligned} PA + AQ &< PQ \\ PA - AQ &> PQ \end{aligned}$$

Which is impossible.

$\therefore PQ$ passes through A



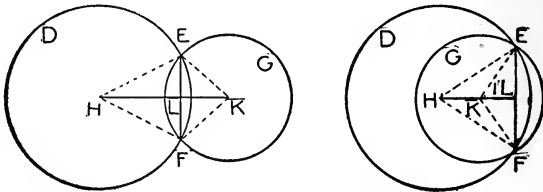
I.
II.
K.

CONTACT OF CIRCLES

113. **Definition.**—If two circles meet each other at one and only one point, they are said to **touch** each other at that point.

THEOREM 16

If two circles touch each other, the straight line joining their centres passes through the point of contact.

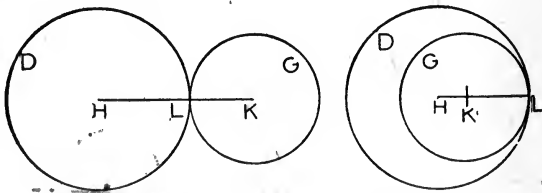


Let two circles DEF, GEF, of which the centres are H, K respectively, cut each other at E, F.

Join HE, HF, KE, KF.

\because HEF, KEF are isosceles Δ s on the same base EF,
 \therefore HK is an axis of symmetry of the quadrilateral HEKF and E, F are corresponding points. (I—5, p. 24.)

\therefore HK bisects EF.

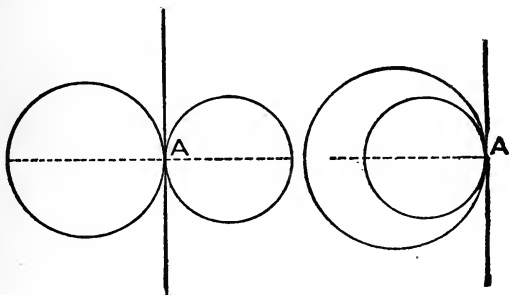


Let the circle GEF move so that the points E, F approach each other and finally coincide.

$\therefore L$ is the middle point of EF ,

$\therefore L$ coincides with E and F , the circles touch at L , and the st. line HK passes through the point of contact L .

Cor. 1.—The straight line drawn from the point of contact perpendicular to the line of centres is a common tangent to the two circles.



Definition.—If two circles which touch each other are on opposite sides of the common tangent at their point of contact, and consequently each circle outside the other, they are said to touch **externally**; if they are on the same side of the common tangent, and consequently one within the other, they are said to touch **internally**.

Cor. 2.—If two circles touch externally, the distance between their centres is equal to the sum of their radii; and conversely.

Cor. 3.—If two circles touch internally, the distance between their centres is equal to the difference of their radii; and conversely.

114.—**Exercises**

1. If the st. line joining the centres of two circles pass through a point common to the two circumferences, the circles touch each other at that point.

2. Find the locus of the centres of all circles which touch a given circle at a given point.

✓ 3. Draw three circles with radii 23, 32 and 43 mm. each of which touches the other two externally.

✓ 4. Draw a circle of radius 9 cm., and within it draw two circles of radii 3 cm. and 4 cm., to touch each other externally, and each of which touches the first circle internally.

✓ 5. Draw a circle of radius 85 mm., and within it draw two circles of radii 25 mm. and 35 mm., to touch each other externally, and each of which touches the first circle internally.

6. Draw a $\triangle ABC$ with sides 5, 12 and 13 cm. Draw three circles, with centres **A**, **B** and **C** respectively, each of which touches the other two externally.

* 7. Construct the $\triangle ABC$, having $a = 5$ cm., $b = 4$ cm., and $c = 3$ cm. Draw three circles with centres **A**, **B** and **C** respectively, such that the circles with centres **B** and **C** touch externally, and each touches the circle with centre **A** internally.

8. Mark two points **P** and **Q** 10 cm. apart. With centres **P** and **Q**, and radii 4 cm. and 3 cm., describe two circles. Draw a circle of radius 5 cm. which touches each of the first two circles externally. Find the distance of the centre from **PQ**.

• 9. Describe a circle to pass through a given point, and touch a given circle at a given point.

• 10. If two circles touch each other, any st. line drawn through the point of contact will cut off segments that contain equal \angle s.

✓ 11. Two circles ACO , BDO touch, and through O , st. lines AOB , COD are drawn. Show that $AC \parallel BD$.

✓ 12. If two \parallel diameters be drawn in two circles which touch one another, the point of contact and an extremity of each diameter are in the same st. line.

13. Describe a circle which shall touch a given circle, have its centre in a given st. line, and pass through a given point in the st. line.

14. Describe three circles having their centres at three given points and touching each other in pairs. Show that there are four solutions.

15. Two circles touch a given st. line at two given points, and also touch each other; find the locus of their point of contact.

16. If through the point of contact of two touching circles a st. line be drawn cutting the circles again at two points, the radii drawn to these points are \parallel .

• 17. In a given semi-circle inscribe a circle having its radius equal to a given st. line.

• 18. Inscribe a circle in a given sector.

19. A circle of 2.5 cm. radius has its centre at a distance of 5 cm. from a given st. line. Describe four circles each of 4 cm. radius to touch both the circle and the st. line.

20. If DE be drawn \parallel to the base GH of a $\triangle FGH$ to meet FG , FH at D , E respectively, the circles described about the \triangle s FGH , FDE touch each other at F .

21. Two circles with centres P , Q touch externally and a third circle is drawn, with centre R , which both the first circles touch internally. Prove that the perimeter of $\triangle PQR$ = the diameter of the circle with centre R .

Miscellaneous Exercises

1. If two chords of a circle intersect at rt. \angle s, the sum of the squares on their segments is equal to the square on the diameter.

2. Find a point in the circumference of a given circle, the sum of the squares on whose distances from two given points may be a maximum or minimum.

3. AOB , COD are chords cutting at a point O within the circle. Show that $\angle BOC$ equals an \angle at the circumference, subtended by an arc which is equal to the sum of the arcs subtending \angle s BOC , AOD .

4. Two chords AB , CD intersect at a point O without a circle. Show that $\angle AOC$ equals an \angle at the circumference subtended by an arc which is equal to the difference of the two arcs BD , AC intercepted between OBA and ODC .

5. Two circles touch externally at E , and are cut by a st. line at A , B , C , D . Show that $\angle AED$ is supplementary to $\angle BEC$.

6. If at a point of intersection of two circles the tangents drawn to the circles be at rt. \angle s, the st. line joining the points where these tangents meet the circles again, passes through the other point of intersection of the circles.

7. Find a point within a given \triangle at which the three sides subtend equal \angle s. When is the solution possible?

8. Through one of the points of intersection of two given circles draw the greatest possible st. line terminated in the two circumferences.

9. Through one of the points of intersection of two given circles draw a st. line terminated in the two circumferences and equal to a given st. line.

10. Describe a circle of given radius to touch two given circles.

11. DEF is a st. line cutting BC , CA , AB , the sides of $\triangle ABC$, at D , E , F respectively. Show that the circles circumscribed about the \triangle s AEF , BFD , CDE , ABC , all pass through one point.

12. Two circles touch each other at A and BAC is drawn terminated in the circumferences at B , C . Show that the tangents at B , C are \parallel .

13. D , E , F are any points on the sides BC , CA , AB of $\triangle ABC$. Show that the circles circumscribed about the \triangle s AFE , BDF , CED pass through a common point.

14. Two arcs stand on a common chord AB . P is any point on one arc and PA , PB cut the other arc at C , D . Show that the length of CD is constant.

15. ACB is an \angle in a segment. The tangent at A is \parallel to the bisector of $\angle ACB$ and meets BC produced at D . Show that $AD = AB$.

16. Describe a circle of given radius to touch two given intersecting st. lines.

17. In the $\triangle ABC$, the bisector of $\angle A$ meets BC at D . O is the centre of a circle which touches AB at A and passes through D . Prove that $OD \perp AC$.

18. The st. line BC of given length moves so that B and C are respectively on two given fixed st. lines AX and AY . Prove that the circumcentre of $\triangle ABC$ lies on the circumference of a circle with centre A .

19. ABC is an isosceles \triangle in which $AB = AC$. D is any point in BC . Show that the centre of the circle ABD is the same distance from AB that the centre of the circle ACD is from AC .

20. **E, F, G, H** are the points of contact of the sides of a quadrilateral **ABCD** circumscribed about a circle. Prove that the difference of two opposite \angle s of **ABCD** = twice the difference of two adjacent \angle s of **EFGH**.

21. **ABC** is a \triangle in which **AX, BY** are \perp **BC, CA** respectively. Prove that the tangent at **X** to the circle **CXY** passes through the middle point of **AB**; and the tangent at **C** to the same circle \parallel **AB**.

22. The inscribed circle of \triangle **ABC** touches **BC** at **D**. Prove that the circles inscribed in \triangle s **BAD, CAD** touch each other.

23. **O** is the circumcentre of the \triangle **ABC**, and **AO, BO, CO** produced meet the circumference in **D, E, F**. Prove \triangle **DEF** \equiv \triangle **ABC**.

24. **ABC** is a rt.- \angle d \triangle , **A** being the rt. \angle . Prove that **BC** = the difference between the radius of the inscribed circle and the radius of the circle which touches **BC** and the other two sides produced.

25. Describe two circles to touch two given circles, the point of contact with one of these given circles being given.

26. Circles through two fixed points **A** and **B** intersect fixed st. lines, which terminate at **A** and are equally inclined to **AB** on opposite sides of it, in the points **L, M**. Prove that **AL + AM** is constant.

27. **AB** is a diameter and **CD** a chord of a given circle. **AX** and **BY** are both \perp **CD**. Prove that **CX = DY**.

28. Through a fixed point **A** on a circle any chord **AB** is drawn and produced to **C** making **BC = AB**. Find the locus of **C**.

29. Construct a \triangle having given the base, the vertical \angle , and the length of the median drawn from one end of the base.

30. If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, a circle may be inscribed in the quadrilateral.

31. Construct a \triangle having given the vertical \angle , the base, and the point where the bisector of the vertical \angle cuts the base.

32. From the ends of a diameter BC of a circle, \parallel chords BE , CF are drawn, meeting the circle again in E and F . Prove that EF is a diameter.

33. $ACFB$ and $ADEB$ are fixed circles; CAD , CBE and DBF are st. lines. Prove that CF and DE meet at a constant \angle .

34. A , B , C , D are four points in order on the circumference of a circle, and the arc $AB =$ the arc CD . If AC and BD cut at E , the chord which bisects \angle s AEB , CED is itself bisected at E .

35. AB , AC are tangents at B , C to a circle, and D is the middle point of the minor arc BC . Prove that D is the centre of the inscribed circle of the $\triangle ABC$.

36. Construct an equilateral \triangle whose side is of given length so that its vertices may be on the sides of a given equilateral \triangle .

37. D , E , F are the points of contact of the sides BC , CA , AB of a $\triangle ABC$ with its inscribed circle. FK is $\perp DE$, and EH is $\perp FD$. Prove $HK \parallel BC$.

38. Tangents are drawn from a given point to a system of concentric circles. Find the locus of their points of contact.

39. From a given point A without a given circle draw a secant ABC such that $AB = BC$.

40. EF is a fixed chord of a given circle, P any point on its circumference. $EM \perp FP$ and $FN \perp EP$. Find the locus of the middle point of MN .

41. K is the middle point of a chord PQ in a circle of which O is the centre. LKM is a chord. Tangents at L, M meet PQ produced at G, H respectively. Prove $\triangle OGL \equiv \triangle OHM$.

42. LM is the diameter of the semi-circle LNM in which arc $LN >$ arc NM , and $ND \perp LM$. A circle inscribed in the figure bounded by ND, DM and the arc NM touches DM at E . Show that $LE = LN$; and hence give a construction for inscribing the circle.

43. GK is a diameter and O the centre of a circle. A tangent $KD = KO$. From O a $\perp OE$ is drawn to GD . KE is joined and produced to meet the circumference in F . Prove that $FE = FG$.

44. LPM and $LQRM$ are two given segments on the same chord LM . If P moves on the arc LPM such that LQP and MRP are st. lines, the length of QR is constant.

45. $EFP, EFRS$ are two circles and PFR, PES are st. lines. O is the centre of the circle EFP . Prove that $PO \perp RS$.

46. E, F are fixed points on the circles EPD, FQD , and PDQ is a variable st. line. PE, QF intersect at R . Find the locus of R .

47. The circle $PEGF$ passes through the centre G of the circle QEF , and P, E, Q are in a st. line. Prove that $PQ = PF$.

48. Through two points on a diameter equally distant from the centre of a circle, \parallel chords are drawn, show that

these chords are the opposite sides of a rectangle inscribed in the circle.

49. If through the points of intersection of two circles any two \parallel st. lines be drawn and the ends joined towards the same parts, the figure so formed is a \parallel gm.

50. Any two \parallel tangents are drawn, one to each of two given circles; a st. line is drawn through the points of contact, show that the tangents to the circles at the other points of intersection are also \parallel .

51. The hypotenuse of a rt.- \angle d \triangle is fixed and the other two sides are moveable, find the locus of the point of intersection of the bisectors of the acute \angle s of the \triangle .

52. From the middle point L of the arc MLN of a circle two chords are drawn cutting the chord MN and the circumference. Show that the four points of intersection are concyclic.

53. If from one end of a diameter of a circle, two st. lines be drawn to the tangent at the other end of the diameter, the four points of intersection—with the circle, and with the tangent—are concyclic.

54. ABC is a diameter of a circle, B being the centre. AD is a chord, and $BE \perp$ to AC cutting the chord at E . Show that $BCDE$ is a cyclic quadrilateral; and that the circles described about ABE and the quadrilateral $BCDE$, are equal.

55. Two circles intersect at A and B . From A two chords AC and AE are drawn one in each circle making equal \angle s with AB , st. lines CBD and EBF are drawn to cut the circles at D and F , prove C, F, D, E concyclic; also prove \triangle s FCA and DEA similar.

56. ABC is a \triangle and any circle is drawn passing through B , and cutting BC at D and AB at F ; another circle is

drawn passing through **C** and **D** and intersecting the former circle at **E** and **AC** at **G**. Prove **A, F, E, G** are concyclic.

57. If two equal circles intersect, the four tangents at the points of intersection form a rhombus.

58. If two equal circles cut, and at **G**, one of the points of intersection, chords be drawn in each circle, to touch the other circle, these chords are equal.

59. Two equal circles, centres **O** and **P**, touch externally at **S**, **SQ** and **SR** are drawn \perp to each other cutting the circumferences at **Q** and **R** respectively. Show that **O, P, Q** and **R** are the vertices of a \parallel gm.

60. **AB, CD, and EF** are \parallel chords in a circle, prove that the Δ s **ACE** and **BDF** are congruent; also **ACF** and **BDE**; also **ADF** and **BCE**.

61. On the circumference of a circle are two fixed points which are joined to a moveable point either inside or outside the circle. If these lines intercept a constant arc, find the locus of the point.

62. **KL** is any chord of a circle and **H** the middle point of one of the arcs, any st. line **HED** cuts **KL** at **E** and the circumference at **D**. Show that **HL** is a tangent to the circle about **LED**, and **HK** a tangent to that about **KED**.

63. Two circles intersect at **E** and **F**. From any point **P** on the circumference of one of them st. lines **PE** and **PF** are drawn to meet the circumference of the other at **Q** and **R**, show that the length of the straight line **QR** is constant. [Take **P** both on the major arc and on the minor arc.]

64. **HKL** is a Δ having $\angle H$ acute; on **KL** as diameter a circle, centre **O**, is described cutting **HK** at **D** and **HL** at **E**. Show that $\angle ODE = \angle H$.

65. P is a point external to two concentric circles whose centre is O , PQ is a tangent to the outer circle and PR and PS are tangents to the inner circle. Show that $\angle RQS$ is bisected by QO .

66. If the extremities of two \parallel diameters in two circles be joined by a st. line which cuts the circles, the tangents at the points of intersection are \parallel . Show that this is true for the four cases that arise.

67. $KLMN$ is a \parallel gm, through L and N two \parallel st. lines are drawn cutting MN at F and KL at E , show that the circles described about the \triangle s KNE and LMF are equal.

68. $EFGH$ is a quadrilateral having $EF \parallel HG$ and $EH = FG$. From E a st. line EK is drawn $\parallel FG$ meeting HG at K . Show that circles described about the \triangle s EHG , EKG are equal.

69. From any point P on the circumference of a circle PD , PE and PF are perpendiculars to a chord QR , and to the tangents QT and RT . Show that the \triangle s PED and PFD are similar.

70. A quadrilateral having two \parallel sides is described about a circle. Show that the st. line drawn through the centre \parallel to the \parallel sides and terminated by the nonparallel sides is one quarter of the perimeter of the quadrilateral.

71. CD is a diameter of a circle centre O ; chords CF and DG intersect within the circle at E . Show that OF is a tangent to the circle passing through F , G and E .

72. EF is a chord of a circle and EP a tangent; a st. line $PG \parallel EF$ meets the circle at G ; prove that the \triangle s EFG and EPG are similar.

73. The diagonals of a quadrilateral are \perp ; show that the st. lines joining the feet of the perpendiculars from

the intersection of the diagonals on the sides form a cyclic quadrilateral.

74. Two chords of a circle intersect at rt. \angle s and tangents are drawn to the circle from the extremities of the chords; show that the resulting quadrilateral is cyclic.

75. A quadrilateral is described about a circle and its vertices are joined to the centre cutting the circumference in four points. Show that the diagonals of the quadrilateral formed by joining these four points are \perp .

76. DEF is a \triangle inscribed in a circle whose centre is O . On EF any arc of a circle is described and ED , FD , or these lines produced, meet the arc at P , Q . Show that OD , or OD produced, cuts PQ at rt. \angle s.

77. $PQRS$ is a \parallel gm and the diagonals intersect at E . Show that the circles described about PES and QER touch each other; and likewise those about PEQ and RES .

78. Two equal circles intersect at E and F ; with centre E and radius EF a circle is described cutting the circles at G and H . Show that FG and FH are tangents to the equal circles.

79. If from any point on the circumference of a circle perpendiculars be drawn to two fixed diameters, the line joining their feet is of constant length.

80. From the extremities of the diameter of a circle perpendiculars are drawn to any chord. Show that the centre is equally distant from the feet of the perpendiculars.

81. EF and GH are \parallel chords in a circle, F and H being towards the same parts; a point K is taken on the circumference such that GF bisects $\angle HGK$. Prove $GK = EF$.

82. Two circles intersect at D and E , and KEL and PEQ are two chords terminated by the circumferences. Show that the \triangle s DKP and DLQ are similar.

83. If from two points outside a circle, equally distant from the centre and situated on a diameter produced, tangents be drawn to the circle, the resulting quadrilateral is a rhombus.

84. If the arcs cut off by the sides of a quadrilateral inscribed in a circle be bisected and the opposite points be joined, these two lines shall be \perp . (*Note.*—*Use Ex. 3.*)

85. PQ is a fixed st. line and PM , QN are any two \parallel st. lines, M and N being towards the same parts. The \angle s MPQ and NQP are bisected by PR and QR . Find the locus of R .

86. If the \angle s of a \triangle inscribed in a circle be bisected by lines which meet the circumference, and a new \triangle be formed by joining these points on the circumference, its sides shall be \perp to the bisectors.

87. If two circles touch each other internally, and a st. line be drawn \parallel to the tangent at the point of contact, the two intercepts between the circumferences subtend equal \angle s at the point of contact.

88. ABC is a \triangle inscribed in a circle and BA is produced to E ; D is any point in AE ; circles are described through B, C, D and through B, C, E ; $CFDG$ cuts the circles ABC, EBC in F and G . Prove that \triangle s ADF and DEG are similar.

89. Draw a tangent to a circle which shall bisect a given \parallel gm which is outside the circle.

90. In a given circle draw a chord of fixed length which shall be bisected by a given chord.

91. In a given circle draw a chord which shall pass through a given point and be bisected by a given chord. How many such chords can be drawn?

92. Describe a circle with given radius to touch a given st. line and have its centre in another given st. line.

93. Describe a circle of given radius to pass through a given point and touch a given st. line.

94. Describe a circle to touch a given circle at a given point and a given st. line.

95. In a given st. line find a point such that the st. lines joining it to two given points may be (a) \perp s, (b) make a given \angle with each other.

96. Describe a circle of given radius to touch a given circle and a given st. line.

97. Describe a circle to touch a given circle and a given st. line at a given point.

98. Inscribe in a given circle a \triangle one of whose sides shall be equal to a given st. line, and such that the other two may pass through two given points respectively.

99. Place a chord PQ in a circle so that it will pass through a given point O within the circle, and such that the difference between OP and OQ may be equal to a given st. line.

100. Find two points on the circumference of a given circle which shall be concyclic with two given points P and Q outside the circle.

101. Describe a square ($EFGH$) having given the point F and two points P and Q in the sides FE and EH respectively.

102. Describe a square ($EFGH$) having given the point G and two points P and Q in the sides FE and EH respectively.

103. Describe a square so that its sides shall pass respectively through four given points.

104. If three circles touch externally at P, Q, R and PQ and PR meet the circumference of QR at D and E , then DE is a diameter, and is \parallel to the line joining the centres of the other two circles.

105. Two equal circles intersect so that the tangents at one of the points of intersection are \perp s. Show that the square on the diameter is twice the square on the common chord.

106. LMN is a rt.- \angle d \triangle , L being the rt. \angle , and LD is \perp to MN . Show that LM is a tangent to the circle LDN .

107. PQ is a tangent to a circle and PRS a secant passing through the centre, QN is \perp to PS . Show that QR bisects $\angle PQN$.

108. LMN is a \triangle inscribed in a circle whose centre is O . Show that the radius OL makes the same \angle with LM that the \perp from L to MN makes with LN .

109. If two chords of a circle be \perp , the sum of one pair of opposite intercepted arcs is equal to the sum of the other pair.

110. On the sides of a quadrilateral as diameters circles are described. Show that the common chords of every adjacent pair of circles is \parallel to the common chord of the remaining pair.

111. Two equal circles are so situated that the distance between their nearest points is less than the diameter of either circle. Show how to draw a st. line cutting them so as to be trisected by the circumferences.

112. LMN is a \triangle and D, E, F are the middle points of MN, NL and LM respectively; if LP is the perpendicular on MN , show that D, P, E, F are concyclic.

113. QR is a fixed chord of a circle and P a moveable point on the circumference. Find the locus of the intersection of the diagonals of the \parallel gm having PQ and QR for adjacent sides.

114. If a quadrilateral having two \parallel sides is inscribed in a circle, show that the four perpendiculars from the middle point of an arc cut off by one of the \parallel sides, to the two diagonals and to the nonparallel sides, are equal.

115. $ABCD$ and $A'B'C'D'$ are any rectangles inscribed in two concentric circles respectively. P is on the circumference of the former circle and P' on the latter. Prove $PA'^2 + PB'^2 + PC'^2 + PD'^2 = P'A^2 + P'B^2 + P'C^2 + P'D^2$.

116. A point Y is taken in a radius of a circle whose centre is O ; on OY as base an isosceles $\triangle XOY$ is described having X on the circumference; XO and XY are produced to meet the circumference at D and Z respectively, and E is the point between D and Z where the perpendicular from O to OY cuts the circle. Show that the arc DE is one-third of arc EZ .

BOOK IV

RATIO AND PROPORTION

115. **Definitions.**—The **ratio** of one magnitude to another *of the same kind* is the number of times that the first contains the second; or it is the part, or fraction, that the first magnitude is of the second.

Thus the ratio of one magnitude to another is the same as the measure of the first when the second is taken as the unit.

If a st. line is 5 cm. in length, the ratio of its length to the length of one centimetre is 5, that is, the st. line is to one centimetre as 5 is to 1.

If two st. lines **A**, **B** are respectively 8 inches and 3 inches in length, then the ratio of **A** to **B** is 8 to 3.

The ratio of one magnitude **A** to another **B** is written either $\frac{A}{B}$, or **A** : **B**.

When the form $\frac{A}{B}$ is used, the upper magnitude is called the **numerator**, and the lower the **denominator**; and when the form **A** : **B** is used, the first magnitude is called the **antecedent**, and the second the **consequent**. The two magnitudes are called the **terms** of the ratio.

116. **Definitions.**—**Proportion** is the equality of ratios, *i.e.*, when two ratios are equal to each other, the four magnitudes are said to be in proportion.

The equality of the ratios of **K** to **L** and of **M** to **N** may be written in any one of the three forms:—

$\frac{K}{L} = \frac{M}{N}$, **K** : **L** = **M** : **N** or **K** : **L** :: **M** : **N**; and is read

“**K** is to **L** as **M** is to **N**.”

The four magnitudes in a proportion are called **proportionals**.

The first and last are called the **extremes**, and the second and third are called the **means**.

The first two magnitudes of a proportion must be of the same kind, and the last two must be of the same kind; but the first two need not be of the same kind as the last two. Thus in the proportion $\frac{D}{E} = \frac{F}{H}$, **D** and **E** may be lengths of lines, while **F** and **H** are areas.

117. **Definitions.**—Three magnitudes are said to be in **continued proportion**, or in **geometric progression**, when the ratio of the first to the second equals the ratio of the second to the third.

Three magnitudes **L**, **M**, **N**, of the same kind, are in continued proportion, if $\frac{L}{M} = \frac{M}{N}$.

e. g.:— **L** = 4 cm., **M** = 6 cm., **N** = 9 cm.

The second magnitude of a continued proportion is called the **mean proportional**, or **geometric mean**, of the other two.

118. Two magnitudes of the same kind are **commensurable** when each contains some common measure an integral number of times.

Two magnitudes of the same kind are **incommensurable** when there is no common measure, however small, contained in each of them an integral number of times.

The diagonal and side of a square are incommensurable; the ratio of the diagonal to the side being $\sqrt{2} : 1$.

The side of an equilateral triangle and the perpendicular from a vertex to the opposite side are incommensurable; the ratio of a side to the perpendicular being $2 : \sqrt{3}$.

$\sqrt{2} = 1.414$ nearly, and $\sqrt{3} = 1.732$ nearly, but while these roots may be calculated to any required degree of accuracy they cannot be exactly found. Thus there is no straight line however short that is contained an integral number of times in both the diagonal and side of a square; or in both the side and altitude of an equilateral triangle.

The treatment of incommensurable magnitudes is too difficult for an elementary text-book, but as in algebra, the relations that are obtained in geometry for commensurable magnitudes hold good also for incommensurable magnitudes.

119. The following simple algebraic theorems are used in geometry:—

$$1. \text{ If } \frac{a}{b} = \frac{c}{d}, \quad ad = bc.$$

If four numbers be in proportion, the product of the extremes is equal to the product of the means.

$$2. \text{ If } \frac{a}{b} = \frac{c}{d}, \quad \frac{a}{c} = \frac{b}{d}$$

If four numbers be in proportion, the first is to the third as the second is to the fourth.

When a proportion is changed in this way the second proportion is said to be formed from the first by **alternation**.

In order that a given proportion may be changed by alternation, the four magnitudes must be of the same kind.

e. g.:— $\frac{2 \text{ ft.}}{5 \text{ ft.}} = \frac{4 \text{ ft.}}{10 \text{ ft.}}$ and, by alternation, $\frac{2 \text{ ft.}}{4 \text{ ft.}} = \frac{5 \text{ ft.}}{10 \text{ ft.}}$; but from the proportion $\frac{\text{st. line D}}{\text{st. line E}} = \frac{\text{area F}}{\text{area G}}$ another proportion cannot be inferred by alternation.

$$3. \text{ If } \frac{a}{b} = \frac{c}{d}, \quad \frac{b}{a} = \frac{d}{c}.$$

If four numbers be in proportion, the second is to the first as the fourth is to the third.

When a proportion is changed in this way the second proportion is said to be formed from the first by **inversion**.

$$4. \text{ If } \frac{a}{b} = \frac{c}{d}, \quad \frac{a+b}{b} = \frac{c+d}{d}.$$

If four numbers be in proportion, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.

$$5. \text{ If } \frac{a}{b} = \frac{c}{d}, \quad \frac{a-b}{b} = \frac{c-d}{d}.$$

If four numbers be in proportion, the difference of the first and second is to the second as the difference of the third and fourth is to the fourth.

$$6. \text{ If } \frac{a}{b} = \frac{c}{d}, \quad \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

If four numbers be in proportion, the sum of the first and second terms is to the difference of the first and second terms as the sum of the third and fourth terms is to the difference of the third and fourth terms.

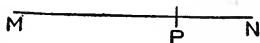
7. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \text{etc.}$, then each of the equal fractions $= \frac{a + c + e + \text{etc.}}{b + d + f + \text{etc.}}$

If any number of ratios, the terms of which are all magnitudes of the same kind, be equal to each other, the sum of the numerators divided by the sum of the denominators equals each of the given ratios.

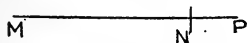
8. If $ad = bc$, $\frac{a}{b} = \frac{c}{d}$, and $\frac{a}{c} = \frac{b}{d}$.

If the product of two numbers be equal to the product of two other numbers, one factor of the first product is to a factor of the second product as the remaining factor of the second is to the remaining factor of the first.

120. If a given straight line MN be divided internally at a point P , the internal segments PM , PN are the distances from P to the ends of the given straight line.



Similarly, if a point P be taken in a given straight line MN produced, the distances from P to the ends of the given straight line, PM ,



PN , are called the external segments of the straight

line, or the given straight line is said to be divided externally at the point P.



121. There is only one point where a straight line MN is divided internally into segments MP, PN that have a given ratio $\frac{a}{b}$.

For, if possible, let it be divided internally at P and Q such that $\frac{MP}{PN}$ and $\frac{MQ}{QN}$ each equals $\frac{a}{b}$.

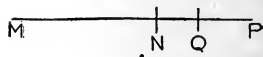
$$\begin{aligned} \text{Then} \quad & \frac{MP}{PN} = \frac{MQ}{QN} \\ \therefore \quad & \frac{MP + PN}{PN} = \frac{MQ + QN}{QN} \quad (4, \text{§ 119.}) \end{aligned}$$

$$\text{i.e.,} \quad \frac{MN}{PN} = \frac{MN}{QN}$$

$$\therefore \quad PN = QN.$$

and \therefore Q coincides with P.

Similarly, there is only one point where a straight line MN is divided externally into segments MP, PN that have a given ratio $\frac{a}{b}$.



For, if possible, let it be divided externally at P and Q such that $\frac{MP}{PN}$ and $\frac{MQ}{QN}$ each equals $\frac{a}{b}$.

$$\begin{aligned} \text{Then} \quad & \frac{MP}{PN} = \frac{MQ}{QN} \\ \therefore \quad & \frac{MP - PN}{PN} = \frac{MQ - QN}{QN} \quad (5, \text{§ 119.}) \end{aligned}$$

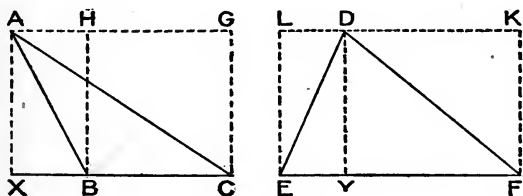
$$\text{i.e.,} \quad \frac{MN}{PN} = \frac{MN}{QN}$$

$$\therefore \quad PN = QN.$$

and \therefore Q coincides with P.

THEOREM 1

Triangles of the same altitude are to each other as their bases.



Hypothesis.—In \triangle s ABC , DEF ; $AX \perp BC$, $DY \perp EF$ and $AX = DY$.

To prove that $\frac{\triangle ABC}{\triangle DEF} = \frac{BC}{EF}$.

Construction.—On BC and EF construct the rectangles HC and LF , having $HB = AX$ and $LE = DY$.

Proof.—Let BC and EF contain a and b units of length respectively, and AX or DY contain c units.

$$\triangle ABC = \frac{1}{2} HB \cdot BC = \frac{1}{2} ca. \quad (\text{II—4, p. 100.})$$

$$\triangle DEF = \frac{1}{2} LE \cdot EF = \frac{1}{2} cb.$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} ca}{\frac{1}{2} cb} = \frac{a}{b} = \frac{BC}{EF}.$$

122.—Exercises

1. \triangle s on equal bases are to each other as their altitudes.
2. If two \triangle s are to each other as their bases, their altitudes must be equal.
3. \parallel gms of equal altitudes are to each other as their bases.
4. Construct a \triangle equal to $\frac{5}{4}$ of a given \triangle .
5. Construct a \parallel gm equal to $\frac{5}{2}$ of a given \parallel gm.

6. $\triangle ABC$, $\triangle DEF$ are two \triangle s having $AB = DE$ and $\angle B = \angle E$. Show that $\triangle ABC : \triangle DEF = BC : EF$.

7. The rectangle contained by two st. lines is a mean proportional between the squares on the lines.

8. If two equal \triangle s be on opposite sides of the same base, the st. line joining their vertices is bisected by the common base, or the base produced.

9. The sum of the \perp s from any point in the base of an isosceles \triangle to the two equal sides equals the \perp from either end of the base to the opposite side.

10. The difference of the \perp s from any point in the base produced of an isosceles \triangle to the equal sides equals the \perp from either end of the base to the opposite side.

11. The sum of the \perp s from any point within an equilateral \triangle to the three sides equals the \perp from any vertex to the opposite side.

12. If st. lines AO , BO , CO are drawn from the vertices of a $\triangle ABC$ to any point O and AO , produced if necessary, cuts BC at D ,

$$\frac{\triangle AOB}{\triangle AOC} = \frac{BD}{DC}.$$

13. In any $\triangle ABC$, F is the middle point of AB , E is the middle point of AC , and BE , CF intersect at O . Show that AO produced bisects BC ; that is, the medians of a \triangle are concurrent.

14. ABC is a \triangle and O is any point. AO , BO , CO , produced if necessary cut BC , CA , AB at D , E , F respectively, a_1 , a_2 , b_1 , b_2 , c_1 , c_2 , are respectively the numerical measures of BD , DC , CE , EA , AF , FB . Show that $a_1 b_1 c_1 = a_2 b_2 c_2$. (This is known as Ceva's Theorem.)

15. The four \triangle s into which a quadrilateral is divided by its diagonals are proportional.

16. $\triangle DEF$ is a \triangle ; G is a point in DE such that $DG = 3GE$, and H is a point in DF such that $FH = 3HD$. Show that $\triangle FGH = 9 \triangle EGH$.

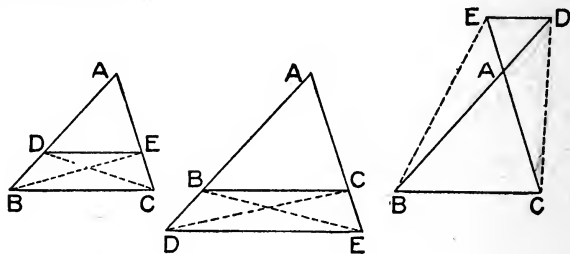
17. St. lines DG , EH , FK drawn from the vertices of $\triangle DEF$ to meet the opposite sides at G , H , K pass through a common point O . Prove that $\frac{DO}{DG} + \frac{EO}{EH} + \frac{FO}{FK} = 2$.

18. In $\triangle DEF$, G is taken in side EF such that $EG = 2GF$, and H is taken in side FD such that $FH = 2HD$. DG and EH intersect at O . Prove that $\frac{\triangle DOH}{\triangle DEF} = \frac{1}{21}$.

Imp

THEOREM 2

A straight line drawn parallel to the base of a triangle cuts the sides, or the sides produced, proportionally.



Hypothesis.—In $\triangle ABC$, $DE \parallel BC$.

To prove that $\frac{BD}{DA} = \frac{CE}{EA}$.

Construction.—Join BE and DC .

Proof.— $\because DE \parallel BC$,
 $\therefore \triangle BDE = \triangle CDE$ (II—5, p. 101.)
 $\therefore \frac{\triangle BDE}{\triangle ADE} = \frac{\triangle CDE}{\triangle ADE}$.

$\because \triangle s BDE, ADE$ have the same altitude, viz., the \perp from E to AB ,

$$\therefore \frac{\triangle BDE}{\triangle ADE} = \frac{BD}{DA}. \quad (\text{IV—1, p. 219.})$$

In the same way,

$$\frac{\triangle CDE}{\triangle ADE} = \frac{CE}{EA}.$$

$$\therefore \frac{BD}{DA} = \frac{CE}{EA}.$$

N.B.—By placing **D** on **AB** and **E** on **AC** in all three figures the proof applies to all.

Cor.—In the first figure,

$$\therefore \frac{BD}{DA} = \frac{CE}{EA}, \therefore \frac{BD + DA}{DA} = \frac{CE + EA}{EA} \quad \text{by addition.}$$

$$\therefore \frac{AB}{AD} = \frac{AC}{AE}.$$

$$\therefore \frac{AD}{AB} = \frac{AE}{AC} \quad \text{by inverting.}$$

Again,

$$\therefore \frac{BD}{DA} = \frac{CE}{EA}, \therefore \frac{DA}{BD} = \frac{EA}{CE} \quad \text{by inverting.}$$

$$\therefore \frac{DA + BD}{BD} = \frac{EA + CE}{CE} \quad \text{by addition.}$$

$$\therefore \frac{AB}{BD} = \frac{AC}{CE}.$$

$$\therefore \frac{BD}{AB} = \frac{CE}{AC} \quad \text{by inverting.}$$

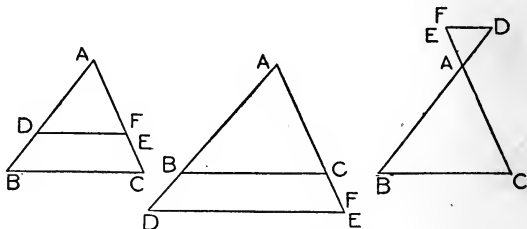
Similar proofs may be given for the second and third figures.

Thus we see that where a line is parallel to the base of a triangle we may form a proportion by taking the whole side or either of the segments, in any order, for the terms of the first ratio, provided we take the corresponding parts of the other side to form the terms of the other ratio in the proportion.

THEOREM 3

(Converse of Theorem 2)

If two sides of a triangle, or two sides produced, be divided proportionally, the straight line joining the points of section is parallel to the base.



Hypothesis.—In $\triangle ABC$, $\frac{BD}{DA} = \frac{CE}{EA}$.

To prove that $DE \parallel BC$.

Construction.—Draw $DF \parallel BC$, to cut AC at F .

Proof.—

$$\because DF \parallel BC,$$

$$\therefore \frac{BD}{DA} = \frac{CF}{FA}. \quad (\text{IV—2, p. 222.})$$

But $\frac{BD}{DA} = \frac{CE}{EA}$.

$$\therefore \frac{CE}{EA} = \frac{CF}{FA},$$

And $\therefore E$ coincides with F .

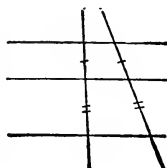
$$\therefore DE \parallel BC.$$

123.—**Exercises**

1. The st. line drawn through the middle point of one side of a \triangle , and \parallel to a second side bisects the third side.
2. The st. line joining the middle points of two sides of a \triangle is \parallel to the third side.
3. If two sides of a quadrilateral be \parallel , any st. line drawn \parallel to the \parallel sides and cutting the other sides, will cut these other sides proportionally.

4. $ABCD$ is a quadrilateral having $AB \parallel DC$. P, Q are points in AD, BC respectively such that $AP : PD = BQ : QC$. Show that $PQ \parallel AB$ or DC .

5. If two st. lines are cut by a series of \parallel st. lines, the intercepts on one are proportional to the corresponding intercepts on the other.



6. D, E are points in AB, AC , the sides of $\triangle ABC$, such that $DE \parallel BC$; BE, CD meet at F . Show that $\triangle ADF = \triangle AEF$.

Show also that AF bisects DE and BC .

7. Through D , any point in the side BC of $\triangle ABC$, DE, DF are drawn $\parallel AB, AC$ respectively and meeting AC, AB at E, F . Show that $\triangle AEF$ is a mean proportional between $\triangle s$ FBD, EDC .

8. ACB, ADB are two $\triangle s$ on the same base AB . E is any point in AB . $EF \parallel AC$ and meets BC at F . $EG \parallel AD$ and meets BD at G . Prove $FG \parallel CD$.

9. D is a point in the side AB of $\triangle ABC$; DE is drawn $\parallel BC$ and meets AC at E ; EF is drawn $\parallel AB$ and meets BC at F . Show that $AD : DB = BF : FC$.

10. From a given point M in the side DE of $\triangle DEF$, draw a st. line to meet DF produced at N so that MN is bisected by EF .

11. $PQRS$ is a $\parallel gm$, and from the diagonal PR equal lengths PK, RL are cut off. SK, SL when produced meet PQ, RQ respectively at E, F . Prove $EF \parallel PR$.

12. DEF is a \triangle in which K, M are points in the side DE and L, N are points in the side DF such that KL and MN are both $\parallel EF$. Find the locus of the intersection of KN and LM .

13. O any point within a quadrilateral $PQRS$ is joined to the four vertices and in OP any point X is taken. XY

is drawn $\parallel PQ$ to meet OQ at Y ; YZ is drawn $\parallel QR$ to meet OR at Z ; and ZV is drawn $\parallel RS$ to meet OS at V . Prove that $XV \parallel PS$.

14. O is a fixed point and P moves along a fixed st. line. Q is a point in OP , or in OP produced in either direction, such that $OQ : QP$ is constant. Find the locus of Q .

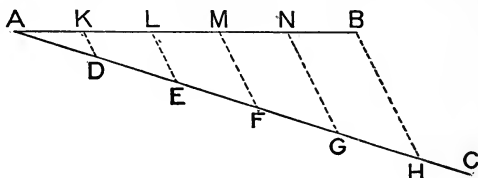
15. L is any point in the side DE of a $\triangle DEF$. From L a line drawn $\parallel EF$ meets DF at M . From F a line drawn $\parallel ME$ meets DE produced at N . Prove that $DL : DE = DE : DN$.

16. If from the vertex of a \triangle perpendiculars are drawn to the bisectors of the exterior \angle s at the base, the line joining the feet of the perpendiculars is \parallel the base.

PROBLEM 1

To divide a given straight line into any number of equal parts.

(Alternative proof for I Prob. 8)



Let AB be the given straight line.

At A draw AC making any angle with AB and from AC cut off in succession the required number of equal parts. AD, DE, EF, FG, GH .

Join HB and through D, E, F, G draw lines $\parallel BH$ cutting AB at K, L, M, N .

Then $AK = KL = LM = MN = NB$.

In $\triangle AEL$, $DK \parallel EL$,
 $\therefore \frac{AD}{DE} = \frac{AK}{KL}$. (IV—2, p. 222.)

But $AD = DE$, $\therefore AK = KL$.

In $\triangle AFM$, $EL \parallel FM$,
 $\therefore \frac{AE}{EF} = \frac{AL}{LM}$.

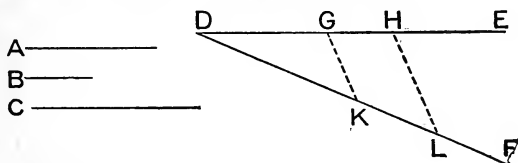
But $AE = 2 EF$, $\therefore AL = 2 LM$

$\therefore LM = AK$ or KL .

In the same way it may be proved that $AK = KL = LM = MN = NB$.

PROBLEM 2

To find a fourth proportional to three given straight lines taken in a given order.



Let A , B , C be the three given st. lines.

From a point D draw two st. lines DE , DF .

Cut off $DG = A$, $GH = B$, $DK = C$.

Join GK . Through H draw $HL \parallel GK$ meeting DF in L .

Then KL is the required fourth proportional.

In $\triangle DHL$, $GK \parallel HL$.

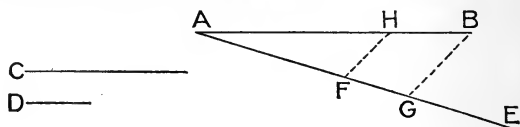
$$\therefore \frac{DG}{GH} = \frac{DK}{KL} \quad (\text{IV—2, p. 222.})$$

$$\text{i.e., } \frac{A}{B} = \frac{C}{KL}$$

$\therefore KL$ is the required fourth proportional.

PROBLEM 3

To divide a given straight line in a given ratio.



Let AB be the given st. line, and $\frac{C}{D}$ the given ratio.

Draw AE making any \angle with AB .

On AE cut off $AF = C$, $FG = D$.

Join BG , and through F draw $FH \parallel GB$.

In $\triangle ABG$, $\because FH \parallel GB$,

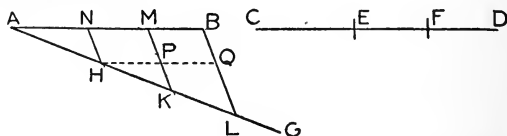
$$\therefore \frac{AH}{HB} = \frac{AF}{FG}. \quad (\text{IV--2, p.222.})$$

But $AF = C$, and $FG = D$,

$$\therefore \frac{AH}{HB} = \frac{C}{D}.$$

PROBLEM 4

To divide a given straight line similarly to a given divided line.



Let AB be the given st. line, and CD the given line divided at E and F .

At A draw AG making any angle with AB .

From AG cut off $AH = CE$, $HK = EF$, $KL = FD$. Join BL . Through H, K draw HN, KM both $\parallel BL$.

Then **AB** is divided at **N** and **M** similarly to **CD**.
Through **H** draw **HPQ** \parallel **AB**.

Proof.—In \triangle **AMK**, **NH** \parallel **MK**,

$$\therefore \frac{AN}{NM} = \frac{AH}{HK}. \quad (\text{IV—2, p. 222.})$$

In \triangle **HQL**, **PK** \parallel **QL**,

$$\therefore \frac{HP}{PQ} = \frac{HK}{KL}.$$

But **HP** = **NM** and **PQ** = **MB**,

$$\therefore \frac{NM}{MB} = \frac{HK}{KL}. \quad (\text{I—20, p. 67.})$$

$$\therefore \frac{AN}{NM} = \frac{CE}{EF} \text{ and } \frac{NM}{MB} = \frac{EF}{FD}.$$

Both these relations are contained in

$$\frac{AN}{CE} = \frac{NM}{EF} = \frac{MB}{FD}.$$

124.—Exercises

1. Divide the area of a given \triangle into parts that are in the ratio of two given st. lines.

2. Divide the area of a \parallel gm into parts that are in the ratio of two given st. lines.

3. Find a third proportional to two given st. lines. Show how two third proportionals, one greater than either of the given st. lines and the other less than either, may be found.

4. Divide a given st. line externally so that the ratio of the segments may equal the ratio of two given st. lines.

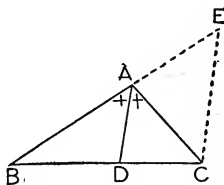
5. **BAC** is a given \angle and **P** is a given point. Through **P** draw a st. line **DPE** cutting **AB** at **D** and **AC** at **E** such that **DP** : **PE** equals the ratio of two given st. lines.

6. Divide a given st. line in the ratio 2 : 3 : 5.
7. Construct a \triangle having its sides in the ratio 2 : 3 : 4, and its perimeter equal to a given st. line.
8. From a given point P outside the $\angle XOY$ draw a line meeting OX at Q and OY at R so that $PQ : QR =$ a given ratio.

BISECTOR THEOREMS

THEOREM 4

If the vertical angle of a triangle is bisected by a straight line which cuts the base, the segments of the base are proportional to the other sides of the triangle.



Hypothesis.—In $\triangle ABC$, AD bisects $\angle BAC$.

To prove $\frac{BD}{DC} = \frac{BA}{AC}$.

Construction.—Through C draw $CE \parallel AD$ to meet BA produced at E .

Proof.— $AD \parallel EC$, $\therefore \angle BAD = \angle AEC$, (I—9, p. 42.)

and $\angle DAC = \angle ACE$. (I—8, p. 40.)

But $\angle BAD = \angle DAC$, by hypothesis,

$\therefore \angle AEC = \angle ACE$.

$\therefore AC = AE$. (I—13, p. 52.)

In $\triangle EBC$, $AD \parallel EC$,

$$\therefore \frac{BD}{DC} = \frac{BA}{AE}. \quad (\text{IV—2, p. 222.})$$

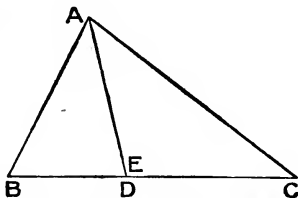
But $AE = AC$,

$$\therefore \frac{BD}{DC} = \frac{BA}{AC}.$$

THEOREM 5

(*Converse of Theorem 4*)

If the base of a triangle is divided internally into segments that are proportional to the other sides of the triangle, the straight line which joins the point of section to the vertex bisects the vertical angle.



Hypothesis.— In $\triangle ABC$, $\frac{BD}{DC} = \frac{BA}{AC}$.

To prove that AD bisects $\angle BAC$.

Construction.—Bisect $\angle BAC$ and let the bisector cut BC at E .

Proof.— $\therefore AE$ bisects $\angle BAC$

$$\therefore \frac{BE}{EC} = \frac{BA}{AC}. \quad (\text{IV—4, p. 230.})$$

But, by hypothesis, $\frac{BD}{DC} = \frac{BA}{AC}$.

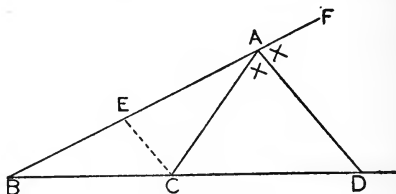
$$\therefore \frac{BE}{EC} = \frac{BD}{DC}.$$

$\therefore E$ and D coincide.

$\therefore AD$ bisects $\angle BAC$.

THEOREM 6

The bisector of the exterior vertical angle of a triangle divides the base externally into segments that are proportional to the sides of the triangle.



Hypothesis.—In $\triangle ABC$, BA is produced to F .

$\angle FAC$ is bisected by AD which cuts BC produced at D .

To prove
$$\frac{BD}{CD} = \frac{BA}{AC}.$$

Construction.—Through C draw $CE \parallel AD$ to meet AB at E .

Proof.— $\because EC \parallel AD, \therefore \angle FAD = \angle AEC.$ (I—9, p. 42.)

and $\angle DAC = \angle ACE.$ (I—8, p. 40.)

But, by hypothesis, $\angle FAD = \angle DAC.$

$$\therefore \angle AEC = \angle ACE.$$

$$\therefore AC = AE. \quad (\text{I—13, p. 52.})$$

In $\triangle BAD$, $EC \parallel AD$,

$$\therefore \frac{BA}{AE} = \frac{BD}{DC}. \quad (\text{IV—2, Cor., p. 223.})$$

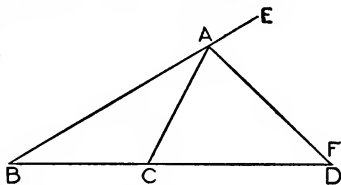
But $AE = AC.$

$$\therefore \frac{BA}{AC} = \frac{BD}{CD}.$$

THEOREM 7

(Converse of Theorem 6)

If the base of a triangle is divided externally so that the segments of the base are proportional to the other sides of the triangle, the straight line which joins the point of section to the vertex bisects the exterior vertical angle.



Hypothesis.—In $\triangle ABC$, $\frac{BD}{CD} = \frac{BA}{AC}$, and BA is produced to E .

To prove that AD bisects $\angle CAE$.

Construction.—Bisect $\angle EAC$ by AF .

Proof.— $\because AF$ bisects exterior $\angle EAC$,

$$\therefore \frac{BF}{CF} = \frac{BA}{AC}. \quad (\text{IV—6, p. 232.})$$

But, by hypothesis, $\frac{BD}{CD} = \frac{BA}{AC}$.

$$\therefore \frac{BF}{CF} = \frac{BD}{CD}.$$

$\therefore D$ and F coincide

$\therefore AD$ bisects $\angle EAC$.

125.—**Exercises**

1. The sides of a \triangle are 4 cm., 5 cm., 6 cm. Calculate the lengths of the segments of each side made by the bisector of the opposite \angle .

2. AD bisects $\angle A$ of $\triangle ABC$ and meets BC at D . Find BD and CD in terms of a , b , and c .

3. In $\triangle ABC$, $a = 7$, $b = 5$, $c = 3$. The bisectors of the exterior \angle s at **A**, **B**, **C** meet **BC**, **CA**, **AB** respectively at **D**, **E**, **F**. Calculate **BD**, **AE** and **AF**.

4. In $\triangle ABC$, the bisector of the exterior \angle at **A** meets **BC** produced at **D**. Find **BD** and **CD** in terms of a , b and c .

5. If a st. line bisects both the vertical \angle and the base of a \triangle , the \triangle is isosceles.

6. The bisectors of the \angle s of a \triangle are concurrent. (Use IV—4 and 5.)

7. **AD** is a median of $\triangle ABC$; \angle s **ADB**, **ADC** are bisected by **DE**, **DF** meeting **AB**, **AC** at **E**, **F** respectively. Prove **EF** \parallel **BC**.

8. The bisectors of \angle s **A**, **B**, **C** in $\triangle ABC$ meet **BC**, **CA**, **AB** at **D**, **E**, **F** respectively. Show that **AF**.**BD**.**CE** = **FB**.**DC**.**EA**.

9. If the bisectors of \angle s **A**, **C** in the quadrilateral **ABCD** meet in the diagonal **BD**, the bisectors of \angle s **B**, **D** meet in the diagonal **AC**.

10. If the bisectors of \angle s **ABC**, **ADC** in the quadrilateral **ABCD** meet at a point in **AC**, the bisectors of the exterior \angle s at **B** and **D** meet in **AC** produced.

11. If **O** is the centre of the inscribed circle of $\triangle DEF$ and **DO** produced meets **EF** at **G**, prove that **DO** : **OG** = **ED** + **DF** : **EF**.

12. **PQ** is a chord of a circle \perp to a diameter **MN** and **D** is any point in **PQ**. The st. lines **MD**, **ND** meet the circle at **E**, **F** respectively. Prove that any two adjacent sides of the quadrilateral **PEQF** are in the same ratio as the other two.

13. The bisector of the vertical \angle of a \triangle and the bisectors of the exterior \angle s at the base are concurrent.

14. One circle touches another internally at M . A chord PQ of the outer circle touches the inner circle at T .

Prove that $\frac{PT}{TQ} = \frac{PM}{MQ}$.

15. LMN is a \triangle in which $LM = 3 LN$. The bisector of $\angle L$ meets MN in D , and $MX \perp LD$. Prove that $LD = DX$.

16. The $\angle A$ of $\triangle ABC$ is bisected by AD , which cuts the base at D , and O is the middle point of BC . Show that OD is to OB as the difference of AB and AC is to their sum.

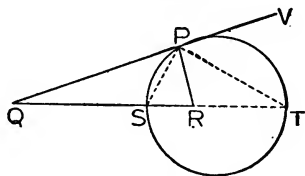
17. The bisectors of the interior and exterior \angle s at the vertex of a \triangle divide the base internally and externally in the same ratio.

18. A point P moves so that the ratio of its distances from two fixed points Q, R is constant. Prove that the locus of P is a circle. (The Circle of Apollonius.)

Divide QR internally at S and externally at T so that

$$\frac{QS}{SR} = \frac{QT}{TR} = \frac{PQ}{PR}.$$

Join PS, PT ; and produce QP to V .



$$\therefore \frac{QS}{SR} = \frac{PQ}{PR} \therefore \angle QPS = \angle SPR.$$

$$\therefore \frac{QT}{TR} = \frac{PQ}{PR} \therefore \angle RPT = \angle TPV.$$

$$\begin{aligned} \therefore \angle SPT &= \angle QPS + \angle TPV \\ &= \frac{1}{2} \text{ st. } \angle QPV \\ &= \text{a rt. } \angle ; \end{aligned}$$

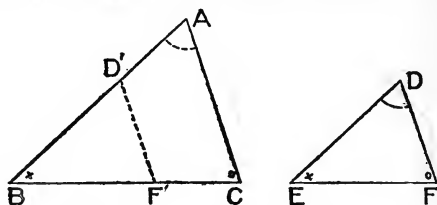
and, hence, a circle described on ST as diameter passes through P .

19. If L, M, N be three points in a st. line, and P a point at which LM and MN subtend equal \angle s, the locus of P is a circle.

SIMILAR TRIANGLES

THEOREM 8

If the angles of one triangle are respectively equal to the angles of another, the corresponding sides of the triangles are proportional.



Hypothesis.—In \triangle s ABC , DEF ; $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

To prove $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

Proof.—Apply $\triangle DEF$ to $\triangle ABC$ so that $\angle E$ coincides with $\angle B$; the $\triangle DEF$ taking the position $D'BF'$.

$\therefore \angle BD'F' = \angle A$, $\therefore D'F' \parallel AC$. (I—7, p. 38.)

$\therefore \frac{AB}{D'B} = \frac{CB}{F'B}$ (IV—2, Cor., p. 223.)

$\therefore \frac{AB}{DE} = \frac{BC}{EF}$.

In the same way, by applying the \triangle s so that \angle s C and F coincide, it may be proved that $\frac{BC}{EF} = \frac{CA}{FD}$.

$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

Note.— $\therefore \frac{AB}{DE} = \frac{BC}{EF}$, $\therefore \frac{AB}{BC} = \frac{DE}{EF}$,

and in the same way $\frac{BC}{CA} = \frac{EF}{FD}$ and $\frac{CA}{AB} = \frac{FD}{DE}$.

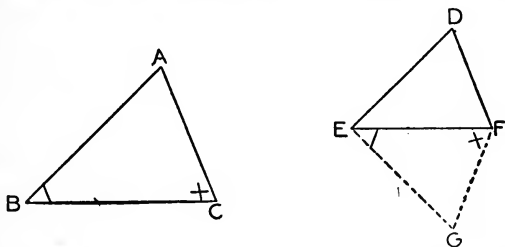
\therefore If two triangles are similar, the corresponding sides about the equal angles are proportional.

Imp.

THEOREM 9

(Converse of Theorem 8)

If the sides of one triangle are proportional to the sides of another, the triangles are similar, the equal angles being opposite corresponding sides.



Hypothesis.—In $\triangle s$ ABC , DEF ; $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

To prove $\angle A = \angle D$, $\angle B = \angle DEF$, $\angle C = \angle DFE$.

Construction.—Make $\angle FEG = \angle B$, $\angle EFG = \angle C$.

Proof.—In $\triangle s$ ABC , GEF $\left\{ \begin{array}{l} \angle A = \angle G, \\ \angle B = \angle GEF, \\ \angle C = \angle EFG. \end{array} \right.$

$\therefore \triangle ABC \equiv \triangle GEF$.

$\therefore \frac{AB}{GE} = \frac{BC}{EF}$. (IV—8, p. 236.)

But, by hypothesis, $\frac{AB}{DE} = \frac{BC}{EF}$.

$\therefore \frac{AB}{GE} = \frac{AB}{DE}$, $\therefore GE = DE$.

Similarly it may be proved that $GF = DF$.

In $\triangle s$ DEF , GEF $\left\{ \begin{array}{l} DE = GE, \\ EF \text{ is common,} \\ FD = FG. \end{array} \right.$

$\therefore \triangle DEF \equiv \triangle GEF$. (I—4, p. 22.)

$\therefore \angle DEF = \angle GEF = \angle B$, $\angle DFE = \angle GFE = \angle C$.

\therefore remaining $\angle D =$ remaining $\angle A$.

126.—Exercises

1. The st. line joining the middle points of the sides of a \triangle is \parallel to the base, and equal to half of it.

2. If two sides of a quadrilateral be \parallel , the diagonals cut each other proportionally.

3. In the $\triangle ABC$ the medians BE , CF cut at G . Show that $BG =$ twice GE , and $CG =$ twice GF .

4. Using the theorem in Ex. 3, devise a method of trisecting a st. line.

5. If three st. lines meet at a point, they intercept on any \parallel st. lines portions which are proportional to one another.

6. In similar \triangle s \perp s from corresponding vertices to the opposite sides are in the same ratio as the corresponding sides.

7. In similar \triangle s the bisectors of two corresponding \angle s, terminated by the opposite sides, are in the same ratio as the corresponding sides.

~~8.~~ $ABCD$ is a \parallel gm, and a line through A cuts BD at E , BC at F and meets DC produced at G . Show that $AE : EF = AG : AF$.

~~9.~~ If two \parallel st. lines AB , CD be divided at E , F respectively so that $AE : EB = CF : FD$, then AC , BD and EF are concurrent.

10. The median drawn to a side of a \triangle bisects all st. lines \parallel to that side and terminated by the other two sides, or those sides produced.

~~11.~~ $ABCD$ is a \parallel gm. AD is bisected at E and BC at F . Show that AF and CE trisect the diagonal BD .

~~12.~~ If the st. lines OAB , OCD , OEF be similarly divided, the \triangle s ACE , BDF are similar.

13. If the corresponding sides of two similar \triangle s be \parallel , the st. lines joining the corresponding vertices are concurrent.

14. $\triangle LMN \sim \triangle PQR$, $\angle L = \angle P$ and $\angle M = \angle Q$. $LM = 7$ cm., $MN = 5$ cm., $LN = 9$ cm., $QR = 4$ cm. Find PQ and PR .

15. In $\triangle DEF$, $DE = 13$ cm., $EF = 5$ cm. and $DF = 12$ cm. The \triangle is folded so that the point D falls on the point E . Find the length of the crease.

16. LMN is a \triangle and X is any point in MN . Prove that the radii of the circles circumscribing LMX , LNX are proportional to LM , LN .

17. St. lines POQ , ROS are drawn so that $PO = 2 OQ$ and $RO = 2 OS$. RQ and PS are produced to meet at T . Prove that $PS = ST$ and $RQ = QT$.

18. FDE , GDE are two circles and FDG is a st. line. FE , GE are drawn. Prove that FE is to GE as diameter of circle FDE is to diameter of GDE .

19. P is any point on either arm of an $\angle XOY$, and $PN \perp$ to the other arm. Show that $\frac{PN}{OP}$ has the same value for all positions of P .

Show also that $\frac{ON}{OP}$ has the same value for all positions of P ; and that $\frac{PN}{ON}$ has the same value for all positions of P .

(NOTE.—The ratio $\frac{PN}{OP}$ is called the sine of the $\angle XOY$, $\frac{ON}{OP}$ is the cosine of that \angle , and $\frac{PN}{ON}$ is the tangent of the same \angle .)

20. PQRS is a quadrilateral inscribed in a circle. The diagonals PR, QS cut at X. Prove that $\frac{PQ}{SR} = \frac{XP}{XS}$.

21. OX, OY, OZ are three fixed st. lines, and P is any point in OZ. From P, PL is drawn \perp OX and PM \perp OY. Prove that the ratio PL : PM is constant.

22. In the quadrilateral DEFG the side DE \parallel GF and the diagonals DF, EG cut at H. Through H the line LHM is drawn \parallel DE and meeting EF, DG at L, M respectively. Prove HL = HM.

23. KLMN is a quadrilateral in which KL \parallel NM. Prove that the line joining the middle points of KL and MN passes through the intersection of the diagonals KM, LN.

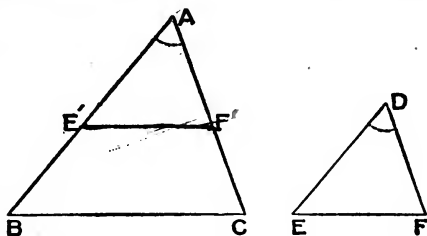
24. DEF is a \triangle and G is any point in EF. The bisector of \angle DGF meets DF in H. EH cuts DG at K. FK meets DE at L. Prove that LG bisects \angle DGE.

25. DG and DH bisect the interior and exterior \angle s at D of a \triangle DEF, and meet EF at G and H; and O is the middle point of EF. Show that OE is a mean proportional between OG and OH.

26. DG bisects \angle D of \triangle DEF and meets EF at G. GK bisects \angle DGE and meets DE at K. GH bisects \angle DGF and meets DF at H. Prove that \triangle EKH : \triangle FKH = ED : DF.

THEOREM 10

If two triangles have one angle of one equal to one angle of the other and the sides about these angles proportional, the triangles are similar, the equal angles being opposite corresponding sides.



Hypothesis.—In $\triangle s$ ABC , DEF , $\angle A = \angle D$

$$\text{and } \frac{AB}{DE} = \frac{AC}{DF}.$$

To prove $\triangle ABC \parallel \triangle DEF$.

Proof.—Apply the $\triangle s$ so that $\angle D$ coincides with $\angle A$ and $\triangle DEF$ takes the position $AE'F'$.

$$\therefore \frac{AB}{DE} = \frac{AC}{DF},$$

$$\therefore \frac{AB}{AE'} = \frac{AC}{AF'}, \therefore E'F' \parallel BC. \text{ (IV-3, p. 224.)}$$

$$\therefore \angle B = \angle AE'F', \angle C = \angle AF'E', \text{ (I-9, p. 42.)}$$

$$\therefore \triangle ABC \parallel \triangle AE'F'.$$

But $\triangle AE'F'$ is the triangle DEF in its new position,

$$\therefore \triangle ABC \parallel \triangle DEF.$$

The equal $\angle s$ B , E are respectively opposite the corresponding sides AC , DF , also the equal $\angle s$ C , F are respectively opposite the corresponding sides AB , DE .

THEOREM 11

If two triangles have two sides of one proportional to two sides of the other, and the angles opposite one pair of corresponding sides in the proportion equal, the angles opposite the other pair of corresponding sides in the proportion are either equal or supplementary.

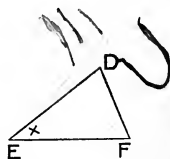
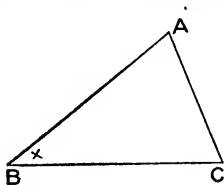


FIG. 1

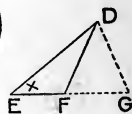


FIG. 2

Hypothesis.—In $\triangle s$ ABC , DEF , $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle B = \angle E$.

To prove either $\angle C = \angle F$ or $\angle C + \angle DFE = 2 \text{ rt. } \angle s$.

Proof.—(1) If $\angle A = \angle D$. (Fig. 1.)

$\therefore \angle A = \angle D$, and $\angle B = \angle E$, $\therefore \angle C = \angle F$.

In this case $\triangle ABC \parallel \triangle DEF$.

(2) If $\angle A$ is not equal to $\angle D$. (Fig. 2.)

At D make $\angle EDG = \angle A$ and produce DG to meet EF , produced if necessary, at G .

In $\triangle s$ ABC , DEG $\left\{ \begin{array}{l} \angle A = \angle EDG, \\ \angle B = \angle E, \\ \therefore \angle C = \angle G. \end{array} \right.$

$\therefore \triangle ABC \parallel \triangle DEG$.

$\therefore \frac{AB}{DE} = \frac{AC}{DG}$. (IV—8, p. 236.)

But, by hypothesis, $\frac{AB}{DE} = \frac{AC}{DF}$.

$$\therefore \frac{AC}{DG} = \frac{AC}{DF}, \therefore DG = DF.$$

In $\triangle DFG$, $\therefore DF = DG$, $\therefore \angle DGF = \angle DFG$.

But $\angle DGF = \angle C$, $\therefore \angle DFG = \angle C$.

$$\angle DFE + \angle DFG = 2 \text{ rt. } \angle s,$$

$$\therefore \angle DFE + \angle C = 2 \text{ rt. } \angle s.$$

127.—Exercises

1. Show that certain propositions of Book I are respectively particular cases of Theorems 9, 10 and 11 of Book IV.

2. In similar \triangle s medians drawn from corresponding vertices are proportional to the corresponding sides.

3. In a $\triangle ABC$, AD is drawn $\perp BC$. If $BD : DA = DA : DC$, prove that $\angle BAC$ is a rt. \angle .

4. If the diagonals of a quadrilateral divide each other proportionally, one pair of sides are \parallel .

5. A point D is taken within a $\triangle LMN$ and joined to L and M . A $\triangle EMN$ is described on the other side of MN from $\triangle LMN$ having $\angle EMN = \angle DML$, and $\angle ENM = \angle DLM$. Prove that $\triangle DME \parallel \triangle LMN$.

6. M, N are fixed points on the circumference of a given circle, and P is any other point on the circumference. MP is produced to Q so that $PQ : PN$ is a fixed ratio. Find the locus of Q .

7. EOD, GOF are two st. lines such that $GO : DO = EO : FO$. Prove that E, F, D, G are concyclic.

8. OEF, OGD are two st. lines such that $OE : OG = OD : OF$. Prove that E, F, G, D are concyclic.

9. DEF is a \triangle , and $FX \perp DE$. Prove that, if $DF : FX = DE : EF$, $\angle XFE = \angle D$.

10. Similar isosceles \triangle s DEF , DEG are described on opposite sides of DE such that $DF = DE$ and $GD = GE$. H is any point in DF and K is taken in GD such that $GK : GD = DH : DF$. Prove $\triangle KHE \parallel \triangle GDE$.

11. LMN is a \triangle , and D is any point in LM produced. E is taken in NM such that $NE : EM = LD : DM$. Prove that DE produced bisects LN .

12. O is the centre and OD a radius of a circle. E is any point in OD , and F is taken in OD produced such that OF is a third proportional to OE , OD . P is any point on the circumference. Prove $\angle FPD = \angle DPE$.

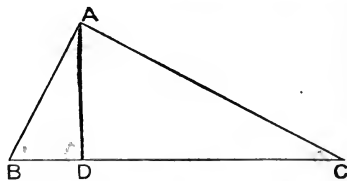
13. The bisectors of the interior and exterior \angle s at L in the $\triangle LMN$ meet MN and MN produced at D , E respectively. FNG drawn $\parallel LM$ meets LE at F and LD produced at G . Prove $FN = NG$.

14. If one pair of \angle s of two \triangle s be equal and another pair of \angle s be supplementary, the ratios of the sides opposite to these pairs of \angle s are equal to each other.

GEOMETRIC MEANS

THEOREM 12

The perpendicular from the right angle to the hypotenuse in a right-angled triangle divides the triangle into two triangles which are similar to each other and to the original triangle.



Hypothesis.—In $\triangle ABC$, $\angle BAC$ is a rt. \angle and $AD \perp BC$.

To prove $\triangle ABD \parallel \triangle CAD \parallel \triangle CBA$.

Proof.—

In \triangle s ABD, CBA $\left\{ \begin{array}{l} \angle B \text{ is common.} \\ \angle BDA = \angle BAC, \text{ both rt. } \angle \text{s.} \\ \therefore \angle BAD = \angle BCA. \end{array} \right.$

$$\therefore \triangle ABD \parallel \triangle CBA$$

Similarly $\triangle ADC \parallel \triangle CBA$.

$$\therefore \triangle ABD \parallel \triangle CAD \parallel \triangle CBA.$$

Cor. 1.— $\therefore \triangle ABD \parallel \triangle CAD,$

$$\therefore \frac{BD}{AD} = \frac{AD}{CD}.$$

$\therefore AD$ is the mean proportional between BD and DC .

Cor. 2.—Because $\triangle ABD \parallel \triangle CBA$

$$\therefore \frac{BD}{AB} = \frac{AB}{BC}.$$

$\therefore AB$ is the mean proportional between BD and BC .

Similarly—

AC is the mean proportional between DC and CB .

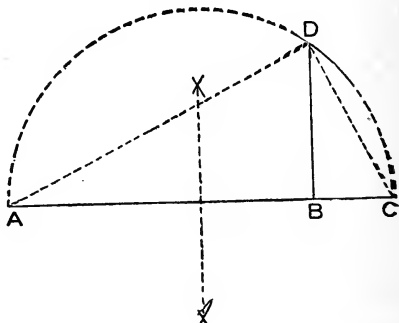
Cor. 3.—Because $\triangle CBA \parallel \triangle CAD,$

$$\therefore \frac{CB}{BA} = \frac{CA}{AD}.$$

i.e., the hypotenuse is to one side as the other side is to the perpendicular.

PROBLEM 5

To find the mean proportional between two given straight lines.



From a st. line cut off AB , BC respectively equal to the two given st. lines.

It is required to find the mean proportional to AB , BC .

On AC as diameter describe a semi-circle ADC . From B draw $BD \perp AC$ and meeting the arc ADC at D .

BD is the required mean proportional.

Join AD , DC .

Proof.— \because ADC is a semi-circle,
 $\therefore \angle ADC$ is a rt. \angle . (III—9, p. 160.)

In $\triangle ADC$, $\angle ADC$ is a rt. \angle ,
 and $DB \perp AC$.

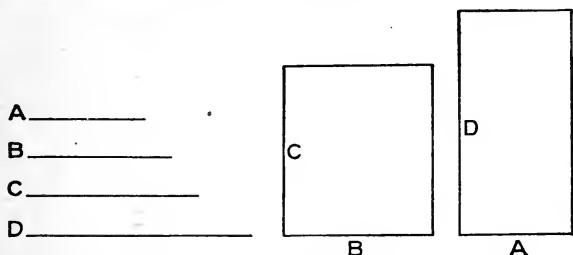
$\therefore \frac{AB}{BD} = \frac{DB}{BC}$. (IV—12, Cor. 1, p. 245.)

$\therefore BD$ is the mean proportional between AB and BC .

RECTANGLES

THEOREM 13

If four straight lines are proportionals, the rectangle contained by the means is equal to the rectangle contained by the extremes.



Hypothesis.—A, B, C, D are four st. lines such that

$$\frac{A}{B} = \frac{C}{D}.$$

To prove that rect. B.C = rect. A.D.

Let a, b, c, d be the numerical measures of A, B, C, D respectively.

$$\text{Then } \frac{a}{b} = \frac{c}{d}.$$

$$\therefore bc = ad.$$

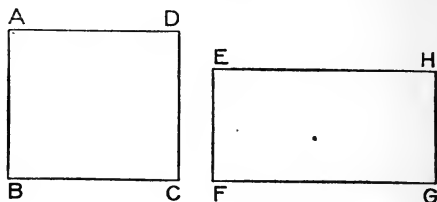
But bc is the numerical measure of B.C and ad is the numerical measure of A.D,

$$\therefore \text{rect. B.C} = \text{rect. A.D.}$$

THEOREM 14

(Converse of Theorem 13)

If two rectangles are equal to each other, the length of one is to the length of the other as the breadth of the second is to the breadth of the first.



Hypothesis.—Rect. $ABCD =$ rect. $EFGH$.

To prove $\frac{BC}{FG} = \frac{EF}{AB}$.

Proof.—Let a, b, c, d be the numerical measures of BC, BA, FG, EF respectively.

Then since the rectangles are equal,

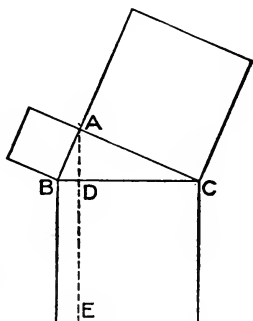
$$ab = cd.$$

$$\therefore \frac{a}{c} = \frac{d}{b},$$

$$\therefore \frac{BC}{FG} = \frac{EF}{AB}.$$

Alternative proof of the Pythagorean Theorem.
(II—13, p. 122.)

The square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides.



Hypothesis.— $\triangle BAC$ is a \triangle having $\angle BAC$ a rt. \angle , and having squares described on the three sides.

To prove that $BC^2 = BA^2 + AC^2$.

Construction.—Draw $AD \perp BC$.

Proof.— $\because \triangle BAC$ is a rt.- \angle d \triangle with $AD \perp$ the hypotenuse BC ,

$$\therefore \frac{BC}{BA} = \frac{BA}{BD}. \quad (\text{IV—12, Cor. 2, p. 245.})$$

$$\therefore BA^2 = BC \cdot BD. \quad (\text{IV—13, p. 247.})$$

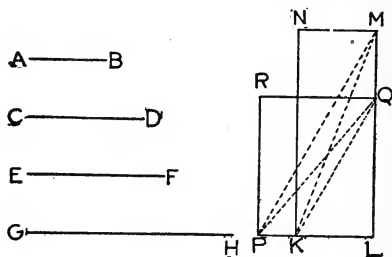
Similarly $CA^2 = BC \cdot CD$.

$$\begin{aligned} \therefore BA^2 + CA^2 &= BC \cdot BD + BC \cdot CD \\ &= BC (BD + CD) \\ &= BC \cdot BC \\ &= BC^2 \end{aligned}$$

$$\text{i.e., } BC^2 = BA^2 + CA^2.$$

128.—Exercises

1. Give a general enunciation of IV—12, Cor. 1.
2. Give a general enunciation of IV—12, Cor. 2.
3. Give an alternative proof of IV—13, using the construction indicated in the following diagram:—



$\frac{AB}{CD} = \frac{EF}{GH}$. In the rectangles NL, RL, KL = AB, LM = GH, PL = CD and LQ = EF.

Using a similar construction give also an alternative proof of IV—14.

4. In any two equal \triangle s ABC, DEF, if AG, DH be \perp s to BC, EF respectively, $AG : DH = EF : BC$.

5. In any \triangle the \perp s from the vertices to the opposite sides are inversely as the sides.

6. In the diagram of IV—12, show that rect. AD.BC = rect. BA.AC. Give a general statement of this theorem.

7. ABC, DEF are two equal \triangle s having also $\angle B = \angle E$. Show that $\frac{BC}{EF} = \frac{DE}{AB}$.

8. ABCD, EFGH are two equal \parallel gms having also $\angle B = \angle F$. Show that $\frac{BC}{FG} = \frac{FE}{BA}$.

9. ABCD is a given rect. and EF a given st. line. It is required to make a rect. equal in area to ABCD and having one of its sides equal to EF.

10. Make a rect. equal in area to a given \triangle and having one of its sides equal to a given st. line.

11. Show how to construct a rect. equal in area to a given polygon and having one of its sides equal to a given st. line.

12. If from any point on the circumference of a circle a \perp be drawn to a diameter, the square on the \perp equals the rect. contained by the segments of the diameter.

13. Construct a square equal to a given rect.

14. Construct a square equal to a given \parallel gm.

15. Construct a square equal to a given \triangle .

16. Draw a square having its area 12 sq. inches.

17. Divide a given st. line into two parts such that the rect. contained by the parts is equal to the square on another given st. line.

18. If a st. line be divided into two parts, the rect. contained by the parts is greatest when the line is bisected.

19. **AB** and **C** are two given st. lines. Find a point **D** in **AB** produced such that rect. **AD.DB** = sq. on **C**.

20. Construct a rect. equal in area to a given square and having its perimeter equal to a given st. line.

When will the solution be impossible?

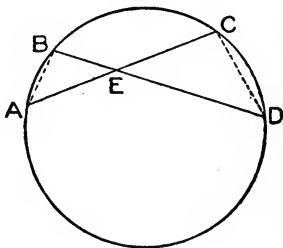
21. Show how to construct a square equal in area to a given polygon.

22. In the corresponding sides **BC**, **EF** of the similar \triangle s **ABC**, **DEF** the points **G**, **H** are taken such that **BG : GC** = **EH : HF**. Prove **AG : DH** = **BC : EF**.

CHORDS AND TANGENTS

THEOREM 15

If two chords intersect within a circle, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.



Hypothesis.—In the circle **ABC**, the chords **AC**, **BD** intersect at **E**.

To prove that $\text{rect. AE.EC} = \text{rect. BE.ED}$.

Construction.—Join **AB**, **CD**.

Proof.— \because \angle s **ABD**, **ACD** are in the same segment,

$$\therefore \angle \text{ABD} = \angle \text{ACD}. \quad (\text{III—7, p. 156.})$$

Similarly, $\angle \text{BAC} = \angle \text{BDC}$.

And $\angle \text{AEB} = \angle \text{CED}$. (I—1, p. 13.)

$$\therefore \triangle \text{AEB} \parallel \triangle \text{DCE}.$$

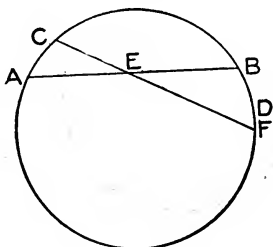
$$\therefore \frac{\text{AE}}{\text{ED}} = \frac{\text{BE}}{\text{EC}}. \quad (\text{IV—8, p. 236.})$$

$$\therefore \text{rect. AE.EC} = \text{rect. BE.ED}. \quad (\text{IV—13, p. 247.})$$

THEOREM 16

(Converse of IV—15)

If two straight lines cut each other so that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other, the four extremities of the two straight lines are concyclic.



Hypothesis—The st. lines **AB**, **CD** cut at **E** so that $\text{rect. } \mathbf{AE.EB} = \text{rect. } \mathbf{CE.ED}$.

To prove that **A**, **C**, **B**, **D** are concyclic.

Construction.—Describe a circle through **A**, **C**, **B**, and let it cut **ED**, produced if necessary, at **F**.

Proof.— \because **AB**, **CF** are chords of a circle,
 $\therefore \mathbf{AE.EB} = \mathbf{CE.EF}$. (IV—15, p. 252.)

But, $\mathbf{AE.EB} = \mathbf{CE.ED}$. (*Hyp.*)

$$\therefore \mathbf{CE.EF} = \mathbf{CE.ED}.$$

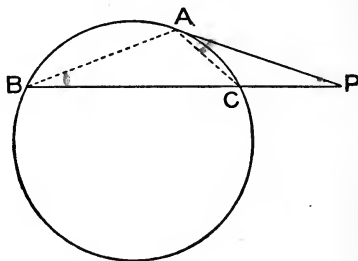
And $\therefore \mathbf{EF} = \mathbf{ED}$.

\therefore **F** coincides with **D**,

and the points **A**, **C**, **B**, **D** are concyclic.

THEOREM 17

If from a point without a circle, a secant and a tangent are drawn, the square on the tangent is equal to the rectangle contained by the secant, and the part of it without the circle.



Hypothesis.—PA is a tangent and PCB a secant to the circle ABC.

To prove that $PA^2 = PB \cdot PC$.

Construction.—Join AB, AC.

Proof.— \because AP is a tangent, and AC is a chord from the same point A,

$$\therefore \angle PAC = \angle ABC. \quad (\text{III—15, p. 177.})$$

$$\text{In } \triangle\text{s PAB, PCA, } \left\{ \begin{array}{l} \angle P \text{ is common,} \\ \angle PBA = \angle PAC, \\ \text{and } \therefore, \angle PAB = \angle PCA, \end{array} \right.$$

$$\therefore \triangle PAB \parallel \triangle PCA.$$

$$\therefore \frac{PB}{PA} = \frac{PA}{PC}. \quad (\text{IV—8, p. 236.})$$

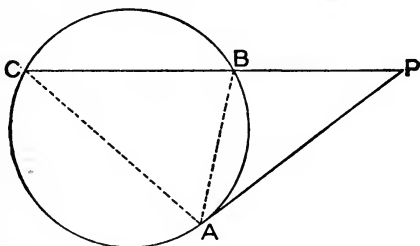
$$\therefore PA^2 = PB \cdot PC. \quad (\text{IV—13, p. 247.})$$

Imp

THEOREM 18

(Converse of IV—17)

If from a point without a circle two straight lines are drawn, one of which is a secant and the other meets the circle so that the square on the line which meets the circle is equal to the rectangle contained by the secant and the part of it without the circle, the line which meets the circle is a tangent.



Hypothesis.—PA and PBC are drawn to the circle ABC so that $PA^2 = PB \cdot PC$.

To prove that PA is a tangent.

Construction.—Join AB, AC.

Proof.—In \triangle s PAB, PAC, $\angle P$ is common,

$$\text{and } \therefore PA^2 = PB \cdot PC,$$

$$\therefore \frac{PA}{PB} = \frac{PC}{PA}. \quad (\text{IV—14, p. 248.})$$

$$\therefore \triangle PAB \parallel \triangle PCA. \quad (\text{IV—10, p. 241.})$$

$$\therefore \angle PAB = \angle PCA.$$

$$\therefore PA \text{ coincides with the tangent at A.} \quad (\text{III—15, p. 177.})$$

i.e., PA is a tangent to the circle.

NOTE.—Prove this proposition with the following construction:—Draw a tangent from P, and join the point of contact and the points A, P to the centre.

129.—Exercises

1. PAB , PCD are two secants drawn from a point P without a circle. Show that $\text{rect. } PA.PB = \text{rect. } PC.PD$.

From this exercise deduce a proof for IV—17.

2. If in two st. lines PB , PD points A , C respectively be taken such that $\text{rect. } PA.PB = \text{rect. } PC.PD$, the four points A , B , C , D are concyclic.

3. If two circles intersect, their common chord bisects their common tangents.

4. If two circles intersect, the tangents drawn to them from any point in their common chord produced are equal to each other.

5. Through P any point in the common chord, or the common chord produced, of two intersecting circles two lines are drawn cutting one circle at A , B and the other at C , D . Show that A , B , C , D are concyclic.

6. Through a point P within a circle, any chord APB is drawn. If O be the centre, show that $\text{rect. } AP.PB = OA^2 - OP^2$.

7. From any point P without a circle any secant PAB is drawn. If O be the centre, show that $\text{rect. } PA.PB = OP^2 - OA^2$.

8. From a given point as centre describe a circle cutting a given st. line in two points, so that the rectangle contained by their distances from a given point in the st. line may be equal to a given square.

9. Describe a circle to pass through two given points and touch a given st. line.

10. If three circles be drawn so that each intersects the other two, the common chords of each pair meet at a point.

11. Find a point **D**, in the side **BC** of $\triangle ABC$, such that the sq. on **AD** = rect. **BD.DC**. When is the solution possible?

12. Use IV—17 to find a mean proportional to two given st. lines.

13. **P** is a point at a distance of 7 cm. from the centre of a circle. **PDE** is a secant such that **PD** = 5 cm. and **DE** = 3 cm. Find the length of the radius of the circle.

14. In a circle of radius 4 cm. a chord **DE** is drawn 7 cm. in length. **F** is a point in **DE** such that **DF** = 5 cm. Find the distance of **F** from the centre of the circle.

15. **DEF** is an isosceles \triangle in which **ED** = **EF**. A circle, which passes through **D** and touches **EF** at its middle point cuts **DE** at **H**. Prove that **DH** = 3 **HE**.

16. In a circle two chords **DE**, **FG** cut at **H**. Prove that
 $(FH - HG)^2 - (DH - HE)^2 = FG^2 - DE^2$.

17. **LND**, **MNE** are two chords intersecting inside a circle and **LM** is a diameter. Prove that

$$LN.LD + MN.ME = LM^2.$$

18. **DEF**, **HGF** are two circles and **DFG** is a fixed st. line. Show how to draw a st. line **EFH** such that **EF.FH** = **DF.FG**.

19. **P** is a point in the diameter **DE** of a circle, and **PT** is the \perp on the tangent at a point **Q**. Prove that **PT.DE** = **DP.PE** + **PQ**².

20. **P**, **Q**, **R**, **S** are four points in order in the same st. line. Find a point **O** in this st. line such that **OP.OR** = **OQ.OS**.

21. The tangent at **P** to a circle, whose centre is **O** meets two \parallel tangents in **Q**, **R**. Prove that **PQ.PR** = **OP**².

Miscellaneous Exercises

1. EFGH is a \parallel gm, P a point in EF such that $EP:PF = m:n$. What fraction is $\triangle EPH$ of the \parallel gm?

2. EFGH is a \parallel gm, P is a point in the diagonal FH such that $FP:PH = 2:5$. What fraction of the \parallel gm is $\triangle EFP$? If $FP:PH = m:n$ find the fraction.

3. EFGH is a \parallel gm, P is a point in the diagonal FH produced such that $FP:PH = 9:5$. What fraction of the \parallel gm is the $\triangle PEH$?

4. KLMN is a \parallel gm. Any st. line EKG is drawn cutting the sides ML and MN produced at E and G. Show that half the \parallel gm is a mean proportional between \triangle s EKL and NKG.

5. The $\triangle PQR$ has PQ and QR divided at D and E such that $PD:DQ = QE:ER = 1:3$. PE and RD intersect at O. Find the ratios of the \triangle s PDO:OPR:OER:PQR.

6. D and E are points in PQ and PR sides of the $\triangle PQR$ such that $QD:DP = PE:ER = m:n$. Compare the areas of the \triangle s QDE and DER.

7. Either of the complements of the \parallel gms about the diagonal of a \parallel gm is a mean proportional between the two \parallel gms about the diagonal.

8. LMN is an isosceles \triangle having $LM = LN$, LD is perpendicular to MN, P is a point in LN such that $LP:LM = 1:3$. Prove that MP bisects LD.

9. Through E one of the vertices of a rectangle EFGH any st. line is drawn, and HP and FQ are \perp s to PEQ. Prove $PE \cdot EQ = HP \cdot FQ$.

10. DEF is a \triangle , P and Q are points in DE and DF, and $DP:PE = 3:5$ and $DQ:QF = 7:8$. In what ratio is PQ cut by the median DG?

11. $DEFG$ is a \parallel gm, and EF is produced to K so that $FK = EF$; DK cuts EG at P . Show that $GP = \frac{1}{3} EG$.

12. The diagonals of the \parallel gm $EFGH$ intersect at O ; if E be joined to the middle point P of OH , and EP and FG meet at K , find $GK:EH$.

13. DEF is a right-angled \triangle , E being the right angle. G is taken in DE produced such that $DG:GF = DF:EF$. Prove that $\angle DFG$ is right.

14. If the perpendicular to the base of a \triangle from the vertex be a mean proportional to the segments of the base, the triangle is right angled.

15. DGH is any \triangle , and from K the middle point of GH a line is drawn cutting DH at E and GD produced at F . Prove $GF:FD = HE:ED$. Prove the converse also.

16. AD and AE are the interior and exterior bisectors of the vertical angle of $\triangle ABC$ meeting the base at D and E . Through C , FCG is drawn \parallel to AB meeting AD and AE at F and G . Prove that $FC = CG$.

17. HKL is an isosceles \triangle , having $HK = HL$; KL is produced to D and DEF is drawn cutting HL at E , and HK at F . Prove $DE:DF = EL:KF$.

18. DP and DQ are perpendiculars to the bisectors of the interior angles E and F of any $\triangle DEF$. Prove $PQ \parallel EF$.

19. PX and QY are perpendiculars from P and Q to XY ; PY and QX intersect at R , and RZ is perpendicular to XY . Prove $\angle PZX = \angle QZY$.

20. ABC is any \triangle , and AD is taken along AC such that $AC:AB = AB:AD$; also CF is taken along AC such that $AC:CB = CB:CF$. Prove $BF = BD$.

21. The perpendicular KD to the hypotenuse HL of a right-angled $\triangle KHL$ is produced to E such that $KD:DH = DH:DE$. Prove $HE \parallel KL$.

22. DEF is a \triangle inscribed in a circle, and P and Q are taken in DE and DF such that $DP:PE = DQ:QF$. Show that the circle described about D, P, Q touches the given circle at D .

23. D is a point in LM a side of $\triangle LMN$, DE is \parallel to MN and $EF \parallel$ to LM , meeting the sides at E and F . Prove $LD:DM = MF:FN$.

24. A variable line through a fixed point O meets two \parallel st. lines at P and Q . Prove $OP:OQ$ a constant ratio.

25. If the nonparallel sides of a trapezium are cut in the same ratio by a st. line, show that this line is \parallel to the \parallel sides.

26. $ABCDE$ is a polygon, O a point within it. If X, Y, Z, P, Q are points in OA, OB, OC, OD, OE such that $OX:OA = OY:OB = \text{etc.}$, show that the sides of $XYZPQ$ are \parallel to those of $ABCDE$.

27. DE is a st. line, F any point in it; find a point P in DE produced such that $PD:PE = DF:FE$.

28. St. lines PD, PE, PF and PG are such that each of the \angle s DPE, EPF, FPG is equal to half a right angle. $DEFG$ cuts them such that $PD = PG$. Prove that $DG:FG = FG:EF$.

29. GH is a chord of a circle, K and D points on the two arcs respectively; KH and KD are joined and GD meets KH produced at E ; $EF \parallel$ to GH meets KD produced. Show that EF is equal to the tangent from F .

30. DEF, DEG are two circles, the centre P of DEG being on the circumference of DEF . A st. line $PHGF$ cuts the common chord at H . Prove that $PH:PG = PG:PF$.

31. EF is the diameter of a circle. PQ is a chord \perp to EF , a chord QXR cuts EF at X , and PR, EF produced meet at Y . Show that $EX:EY = FX:FY$.

32. O is a fixed point and P a variable point on the circumference of a circle; PO is produced to Q such that $OQ:OP = m:n$. Find the locus of Q .

33. LMN is a \triangle inscribed in a circle, $\angle L$ is bisected by LED cutting MN at E and the arc at D . Prove $\triangle LEN$ and $\triangle LMD$ similar.

34. The $\angle D$ of the $\triangle DEF$ is bisected by DP cutting EF in P ; QPR is \perp to DP meeting DE and DF at Q and R ; RS is \parallel to EF meeting DE at S . Prove $SE = EQ$.

35. AOB , COD and EOF are any three st. lines; ACE is \parallel to FDB . Prove $AC:CE = BD:DF$. State and prove a converse to this theorem.

36. Two circles DEF and DEG intersect; a tangent DF is drawn to DEG , and EG to DEF . Show that DE is a mean proportional between FE and DG .

37. $EFGH$ is a quadrilateral, the diagonals EG and FH meet at Q . Prove $\triangle EFH:\triangle FGH = EQ:QG$.

38. $EFGH$ is a quadrilateral of which the sides EH and FG produced meet at P . Prove $\triangle EFG:\triangle FGH = EP:PH$.

39. G is the middle point of the st. line MN , PE a st. line \parallel to MN . Any st. line $EFGH$ cuts PN at F and PM produced at H . Prove $EF:FG = EH:HG$.

40. ABC is a \triangle having $\angle B = \angle C =$ twice $\angle A$, BD bisects the $\angle B$ meeting AC at D . Prove $AC:AD = AD:DC$; also prove $\triangle ABC:\triangle ABD = \triangle ABD:\triangle BDC$.

41. $EFGH$ is a cyclic quadrilateral, EG and FH intersect at O , and OP and OQ are \perp s to EH and FG . Show that $OP:OQ = EH:FG$.

42. EF is the diameter of a circle and P and Q any points on the circumference on opposite sides of EF ; QR is \perp to EF meeting EP at S . Prove $\triangle ESQ \parallel \triangle EQP$.

43. ABC is a \triangle inscribed in a circle, centre O , $AD \perp$ to BC , AOE a diameter. Prove $\triangle s$ ADC and ABE similar: and $AD \cdot AE = AB \cdot AC$.

44. EFG is a \triangle inscribed in a circle, $ED \parallel$ to the tangent at G meets the base at D . Prove that $FG : FE = EG : ED$.

45. Find the ratio of the segments of the hypotenuse of a right- \angle d \triangle made by a perpendicular on it from the vertex, if the ratio of the sides be (1) $1 : 2$; (2) $m : n$.

46. PQ is the diameter of a circle; a tangent is drawn from a point R on the circumference, PS and QT are \perp to the tangent. Prove $\triangle s$ PRQ , RPS and RTQ similar; also show that $\triangle PRQ$ is half of $PSTQ$.

47. PQ and PR are tangents to a circle, PST is a secant meeting the circle at S and T . Prove $QT : QS = RT : RS$.

48. Two circles intersect at E and F ; from P , any point on one of them, chords PE , PF are drawn, EF and DG meet at Q and PQ cuts the circle PEF at R . Prove R, F, G, Q concyclic; also that PQ^2 is equal to the sum of the squares on the tangents to the circle $EFGD$ from P and Q .

49. PBR is a st. line, and similar segments of circles, PAB and BAR , are described on PB and BR and on the same side of PR . PAC and RAD are drawn to meet the circles at C and D . Prove $PD : RC = PB : BR$.

NOTE.—*Segments of circles are said to be similar when they contain equal angles.*

50. PMQ is the diameter of a circle PRQ , PX and QY are \parallel tangents, XRY is any other tangent, PY and XQ meet at O . Show that RO is \parallel to PX ; that RO produced to M is \perp to the diameter; and that $MO = OR$.

51. $ABCD$ is a rectangle, a st. line $APQR$ is drawn cutting BC at P , the circle circumscribing the rectangle at Q and DC produced at R , and such that AC bisects $\angle DAR$. Prove $DC : CR = PQ : PA$.

52. PQRS is a square. A st. line PFED cuts QS at F, SR at E and QR produced at D. Prove FR a tangent to the circle described about DER; also that $EF:PF = PF:FD$.

53. FGHK is a cyclic quadrilateral, the $\angle GFE$ is made equal to $\angle HFK$ and E is in GK. Prove $\triangle s$ FEK and FGH similar.

54. PA and PB are tangents to a circle, centre O, AB meets PO in R; PCD is any secant, OS is \perp to PD, and AB and OS produced meet at Q. Prove (1) P, R, S, Q concyclic; (2) $PO \cdot OR = OA^2$; (3) QD and QC are tangents to the given circle.

55. DEF is a \triangle and P and Q are points in ED and FE such that $EP:PD = FQ:QE$, and PQ meets DF produced at R. Prove $RF:RD = PE^2:PD^2$. (*Through F draw a st. line \parallel to DE to meet PR.*)

56. If a square is inscribed in a rt.- $\angle d$ \triangle having one side on the hypotenuse, show that the three segments of the base are in continued proportion.

57. FGH is a \triangle and $\angle G$ and $\angle H$ are bisected by st. lines which cut the opposite sides at D and E; if DE is \parallel to GH, then $FG = FH$.

58. From P, the middle point of an arc of a circle cut off by a chord QR, any chord PDE is drawn cutting QR at D. Show that $PQ^2 = PD \cdot PE$.

59. Draw a st. line through a given point so that the perpendiculars on it from two other given points may be (1) equal, (2) one twice the other, (3) three times the other, (4) in a given ratio.

60. LMN is an isosceles \triangle , the base MN is produced both ways, in NM produced any point P is taken, and in MN produced NQ is taken a third proportional to PM and LM. Prove $\triangle s$ PLQ and PLM similar.

61. $EDOF$ is the diameter of a circle, centre O . PE and PG are tangents to the circle; GD is \perp to EF . Prove $GD : DE = OE : EP$.

62. DEF is a \triangle inscribed in a circle, centre O . The diameter \perp to EF cuts DE at P and FD produced at Q . Prove \triangle s EPO and FOQ similar; and hence $OE^2 = OP \cdot OQ$.

63. ABC is a \triangle inscribed in a circle. The exterior \perp at A is bisected by a st. line cutting BC produced at D and the circumference at E . Prove $BA \cdot AC = EA \cdot AD$.

64. $EFGH$ is a cyclic quadrilateral, P a point on the circumference, PQ, PR, PS, PT are \perp to EF, FG, GH, HE respectively. Prove \triangle s PTQ and PSR similar; and $PT \cdot PR = PS \cdot PQ$.

65. Any three \parallel chords AB, CD, EF are drawn in a circle, AC and BD meet EF produced at Q and R , P is a point in the arc EF , and PA and PD meet EF at M and N . Prove \triangle s AQM and NDR similar; hence show that, for all positions of P , $QM \cdot NR$ is constant.

66. Two tangents TMP and TNQ are drawn to a circle, centre O ; and the st. line POQ is \perp to TO . MN is any other tangent to the circle. Prove \triangle s MPO and NQO similar.

67. DH is a median of the $\triangle DEF$, PQ is \parallel to EF cutting DE at P and DF at Q . Show that PF and EQ intersect on DH .

68. LNM is a \triangle inscribed in a semicircle, diameter LM . NM is greater than NL . On opposite sides of LN the $\angle LNP$ is made equal to $\angle LNQ$, P and Q lying along LM . Prove $PL : LQ = PM : QM$.

69. $EFGH$ is a \parallel gm, and RS is drawn \parallel to HF meeting EH and EF at R and S . Show that RG and SG cut off equal segments of the diagonal FH . Prove a converse of this.

70. ABC is a \triangle and AB, AC are produced to D, E so that $BD = CE$; DE and BC produced meet at F . Show that $AD : AE = FC : FB$.

71. Two circles, centres O, P intersect, the centre O being on the circumference of the other circle. GDE touches the circle with centre O at G and cuts the other at D, E , and EPF is a diameter. Prove $\triangle OGD \parallel \triangle OEF$; and hence, that $OD.OE$ is constant for all positions of the tangent.

72. Two circles touch externally at P ; EF a chord of one circle touches the other at D . Prove $PE : PF = ED : DF$.

73. EOF is the diameter of a circle, with centre O , DP any chord cutting the diameter; $OSQR \perp$ to DP meets DP at S , DE at Q , and PE at R . Prove $\triangle s$ EDF and RSP similar; also $OQ.OR = OD^2$.

74. Divide an arc of a circle into two parts so that the chords which cut them off shall have a given ratio to each other.

75. LMN is a \triangle , and $XY \parallel MN$ meets LM at X and LN at Y ; MN is produced to D so that $ND = XY$, and $XP \parallel$ to LD meets MN at P . Prove $MN : ND = ND : NP$.

76. Two circles intersect and a st. line $CDOEF$ cuts the circumferences at C, D, E, F and the common chord at O . Show that $CD : DO = EF : OE$

77. $DX \perp EF$ and $EY \perp DF$ in $\triangle DEF$. The lines DX, EY cut at O . Prove that $EX : XO = DX : XF$.

78. From a point P without a circle two secants PKL, PMN are drawn to meet the circle in K, L, M, N . The bisector of $\angle KPM$ meets the chord KM at E and the chord LN at F . Prove that $LF : FN = ME : EK$

79. QR is a chord \parallel to the tangent at P to a circle. A chord PD cuts QR at E . Prove that PQ is a mean proportional between PE and PD .

80. DEF , DEG are two fixed circles and FEG is a st. line. Show that the ratio $FD : DG$ is constant for all positions of the st. line FEG .

81. DEF is a st. line, and EG , FH are any two \parallel st. lines on the same side of DEF such that $EG : FH = DE : DF$. Prove that D , G , H are in a st. line.

82. From a given point on the circumference of a circle draw two chords which are in a given ratio and contain a given \angle .

83. DEF is a \triangle and on DE , DF two \triangle s DLE , DFM are described externally such that $\angle FDM = \angle EDL$ and $\angle DFM = \angle DLE$. Prove $\triangle DLF \parallel \triangle DEM$.

84. $DEFG$ is a \parallel gm and P is any point in the diagonal EG . The st. line KPL meets DE at K and FG at L , and MPN meets EF at M and GD at N . Prove $KM \parallel NL$.

85. $ABCD$ is a \parallel gm and PQ is a st. line $\parallel AB$. The st. lines PA , QB meet at R and PD , QC meet at S . Prove $RS \parallel AD$.

86. If the three sides of one \triangle are respectively \perp to the three sides of another \triangle , the two \triangle s are similar.

87. Find a point whose \perp distances from the three sides of a \triangle are in the ratio $1 : 2 : 3$.

88. Squares are described each with one side on one given st. line and one vertex on another given st. line. Find the locus of the vertices which are on neither.

89. If the sides of a rt.- \angle d \triangle are in the ratio $3 : 2$, prove that the \perp from the vertex of the rt. \angle to the hypotenuse divides it in the ratio $9 : 4$.

90. HK is a diameter of a circle and L is any point on the circumference. A st. line $\perp HK$ meets HK at D , HL at E , KL at G , and the circumference at F . Show that $DF^2 = DE \cdot DG$.

91. The st. line joining a fixed point to any point on the circumference of a given circle is divided in a given ratio at P . Prove that the locus of P is a circle.

92. $DEFG$ is a quadrilateral and P, Q, R, S are points on DE, EF, FG, GD such that $DP : DE = FQ : FE = FR : FG = DS : DG$. Prove that $PQRS$ is a \parallel gm.

93. $DEFG$ is a \parallel gm, and a line is drawn from E cutting DF in P , DG in Q and FG produced in R . Prove that $PQ : PR = DP^2 : PF^2$; and that $PQ \cdot PR = EP^2$.

94. If $\triangle DEF : \triangle GHK = DE \cdot EF : GH \cdot HK$, prove that $\angle s$ E, H are either equal or supplementary.

95. From a point P without a circle draw a secant PQR , such that QR is a mean proportional between PQ and PR .

96. Through a point of intersection of two circles draw a line such that the chords intercepted by the circles are in a given ratio.

97. If two $\triangle s$ are on equal bases and between the same $\parallel s$, the intercepts made by the sides of the $\triangle s$ on any st. line \parallel to the base are equal.

98. The radius of a fixed circle is 38 mm., and a chord LM of the circle is divided at P such that $LP \cdot PM = 225$ sq. mm. Construct the locus of P .

99. If the tangents from a given point to any number of intersecting circles are all equal, all the common chords of the circles pass through that point.

100. Circles are described passing through two fixed points; find the locus of a point from which the tangents to all the circles are equal.

101. DEF is a \triangle having $\angle E$ a rt. \angle . A circle is described with centre D and radius DE ; from F a secant is drawn cutting the circle at G, H ; and EX is drawn $\perp DF$. Show that D, X, G, H are concyclic.

102. GD is a chord drawn \parallel to the diameter LM of a circle. LG , LD cut the tangent at M at E , F respectively. Prove that $LG \cdot GE + LD \cdot DF = LM^2$.

103. LM is a diameter of a circle, and on the tangent at L equal distances LP , PQ are cut off. MP , MQ cut the circumference at R , S respectively. Prove that $LR : RS = LM : MS$.

104. GH drawn in the $\triangle DEF$ meets DE in G and DF in H . From D any line DLK is drawn cutting GH in L and EF in K . From L the st. lines LM , LN are drawn $\parallel KH$, KG and meeting DH , DG at M , N respectively. Prove $\triangle LMN \parallel \triangle KHG$.

105. In a given \triangle inscribe an equilateral \triangle so as to have one side \parallel to a side of the given \triangle .

106. In a given $\triangle DEF$ draw a st. line $PQ \parallel ED$ meeting EF in P and DF in Q , so that PQ is a mean proportional between EP and PF .

107. Two circles intersect at E , F , and DEG is the st. line $\perp EF$ and terminated in the circumferences. HEK is any other st. line through E terminated in the circumferences. HF , DF , KF , GF are drawn. Prove, by similar \triangle s, that $DG > HK$.

108. In $\triangle ABC$ the bisectors of $\angle A$ and of the exterior \angle at A meet the st. line BC at D and E . Show that $DE = \frac{2abc}{c^2 - b^2}$.

109. If two circles intercept equal chords PQ , RS on any st. line, the tangents PT , RT to the circles at P , R are to one another as the diameters of the circles.

110. DEF is a \triangle having $DF > DE$. From DF a part DG is cut off equal to DE , and GH is drawn $\parallel DE$ to meet

EF at **H**. From **GF** a part **GK** is cut off equal to **GH**, and **KL** is drawn \parallel **GH** to meet **EF** at **L**; etc. Prove that **DE**, **GH**, **KL**, etc., are in continued proportion.

111. A circle **P** touches a circle **Q** internally, and also touches two \parallel chords of **Q**. Prove that the \perp from the centre of **P** on the diameter of **Q** which bisects the chords is a mean proportional between the two extremes of the three segments into which the diameter is divided by the chords.

112. **PX** is the \perp from a point **P** on the circumference of a circle to a chord **QR**, and **QY**, **RZ** are \perp s to the tangent at **P**. Prove that $PX^2 = QY.RZ$.

113. Prove, by using 112, that if \perp s are drawn to the sides and diagonals of a cyclic quadrilateral from a point on the circumference of the circumscribed circle, the rectangle contained by the \perp s on the diagonals is equal to the rectangle contained by the \perp s on either pair of opposite sides.

114. The projections of two \parallel st. lines on a given st. line are proportional to the st. lines.

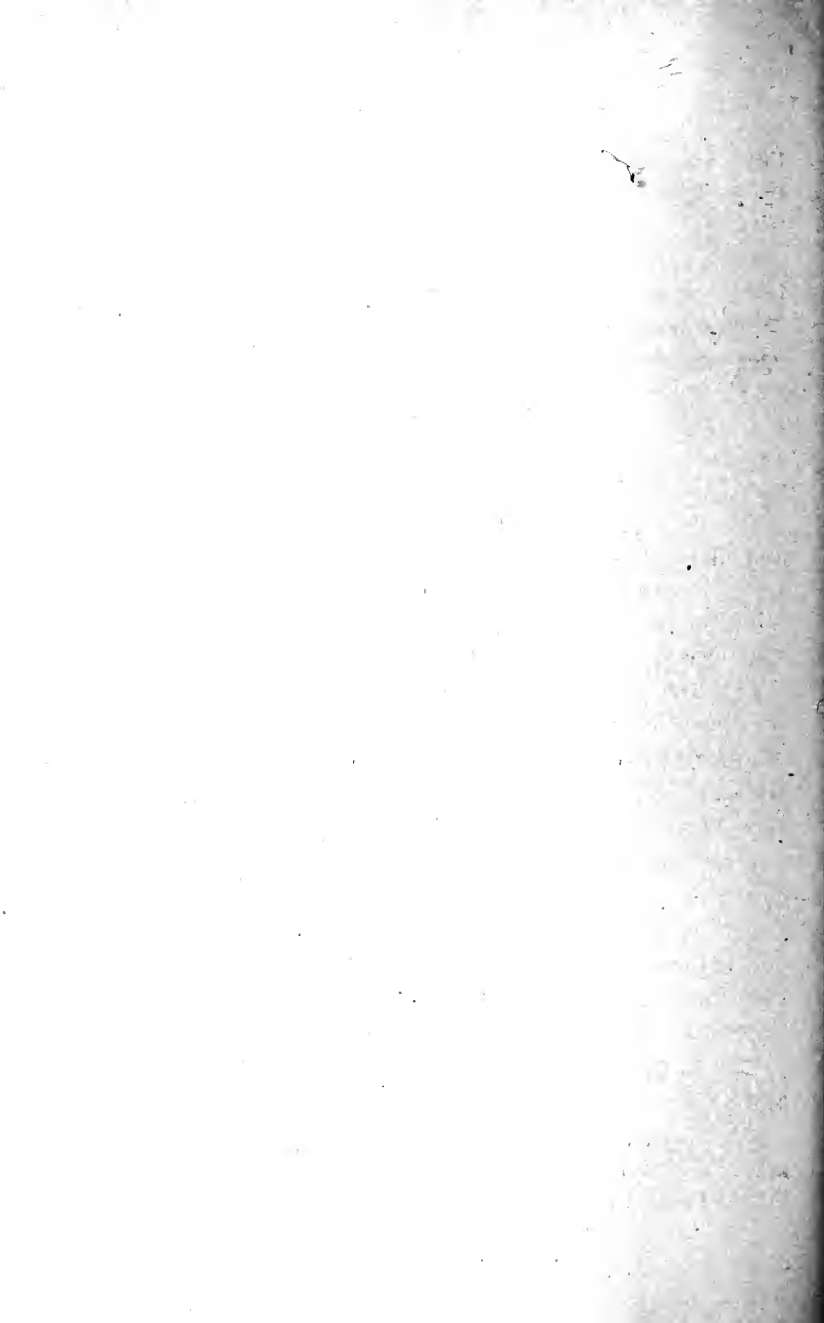
115. **DEFG** is a square, and **P** is a point in **GF** such that $DP = FP + FE$. Prove that the st. line from **D** to the middle point of **EF** bisects $\angle PDE$.

116. **DEF**, **GEF** are Δ s on opposite sides of **EF**, and **DG** cuts **EF** at **H**. Prove that $\Delta DEF : \Delta GEF = DH : HG$.

117. From the intersection of the diagonals of a cyclic quadrilateral \perp s are drawn to a pair of opposite sides: prove that these \perp s are in the same ratio as the sides to which they are drawn.

118. **P**, **Q**, **R**, **S** are points in a st. line, $PX \parallel QY$, $RX \parallel SY$, and **XY** meets **PS** at **O**. Prove that $OP.OS = OQ.OR$.

119. From a point **T** without a circle tangents **TP**, **TQ** and a secant **TRS** are drawn. Prove that in the quadrilateral **PRQS** the rect. **PR.QS** = the rect. **RQ.SP**.

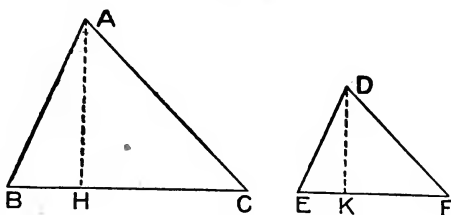


BOOK V

AREAS OF SIMILAR FIGURES

THEOREM I

The areas of similar triangles are proportional to the squares on corresponding sides.



Hypothesis.— ABC , DEF are similar \triangle s of which BC , EF are corresponding sides.

To prove that $\frac{\triangle ABC}{\triangle DEF} = \frac{BC^2}{EF^2}$

Construction.—Draw $AH \perp BC$ and $DK \perp EF$.

Proof.—

$$\because \triangle AHC \parallel \triangle DKF,$$

$$\text{And } \triangle ABC \parallel \triangle DEF,$$

$$\therefore \frac{AH}{DK} = \frac{AC}{DF} = \frac{BC}{EF}. \quad (\text{IV—8, p. 236.})$$

$$\triangle ABC = \frac{1}{2} AH \cdot BC, \quad (\text{II—4, p. 100.})$$

$$\triangle DEF = \frac{1}{2} DK \cdot EF,$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} AH \cdot BC}{\frac{1}{2} DK \cdot EF}$$

$$= \frac{AH \cdot BC}{DK \cdot EF}$$

$$= \frac{BC \cdot BC}{EF \cdot EF}$$

$$= \frac{BC^2}{EF^2}.$$

130.—Exercises

1. Two similar \triangle s have corresponding sides in the ratio of 3 to 5. What is the ratio of their areas?

2. The ratio of the areas of two similar \triangle s equals the ratio of 64 to 169. What is the ratio of their corresponding sides?

3. Draw a \triangle having sides 4 cm, 5 cm., 6 cm. Make a second \triangle having its area four times that of the first, and divide it into parts each equal and similar to the first \triangle .

4. Show that the areas of similar \triangle s are as:—

(a) the squares on corresponding altitudes;

(b) the squares on corresponding medians;

(c) the squares on the bisectors of corresponding \angle s.

5. $\triangle ABC$, $\triangle DEF$ are two similar \triangle s such that area of $\triangle DEF$ is twice that of $\triangle ABC$. What is the ratio of corresponding sides?

Draw $\triangle ABC$ having sides 5 cm., 6 cm., 7 cm., and make $\triangle DEF$ similar to $\triangle ABC$, and of double the area.

✓ 6. If $\triangle ABC$, $\triangle DEF$ be similar \triangle s of which BC , EF are corresponding sides, and the st. line G be such that $BC : EF = EF : G$, then $\triangle ABC : \triangle DEF = BC : G$; that is:—

If three st. lines be in continued proportion, the first is to the third as any \triangle on the first is to the similar \triangle similarly described on the second.

NOTE.—*Similar \triangle s are said to be similarly described on corresponding sides.*

7. $\triangle ABC$ is a \triangle and G is a st. line. Describe a $\triangle DEF$ similar to $\triangle ABC$ and such that $\triangle ABC : \triangle DEF = BC : G$.

Describe another $\triangle HKL$ similar to $\triangle ABC$ and such that $\triangle ABC : \triangle HKL = AB : G$.

8. Bisect a given \triangle by a st. line drawn \parallel to one of its sides.

9. From a given \triangle cut off a part equal to one-third of its area by a st. line drawn \parallel to one of its sides.

10. Trisect a given \triangle by st. lines drawn \parallel to one of its sides.

11. Show that the equilateral \triangle described on the hypotenuse of a rt.- \angle d \triangle equals the sum of the equilateral \triangle s on the two sides.

12. In $\triangle DEF$, $DX \perp EF$ and $EY \perp FD$. Prove that $\triangle FXY : \triangle DEF = FX^2 : FD^2$.

13. In the acute- \angle d $\triangle DEF$, $DX \perp EF$, $EY \perp FD$, $FZ \perp DE$, $YG \perp EF$ and $ZH \perp EF$. Prove that XY and XZ divide the $\triangle DEF$ into three parts that are proportional to FG , GH and HE .

14. LMN is an equilateral \triangle . The st. lines RLQ , PMR , QNP are respectively \perp LM , MN , NL . Find the ratio of $\triangle PQR$ to $\triangle LMN$.

15. A point O is taken in the diameter PQ produced of a circle. OT is a tangent, and the tangent at P cuts OT at N . If D is the centre of the circle, prove that $\triangle OPN : \triangle OTD = OP : OQ$.

16. H is a point on the circumference of a circle of which FG is a diameter, and O is the centre. $HD \perp FG$, and tangents at F and H meet at E . Prove that $\triangle FEH : \triangle OHG = FD : DG$.

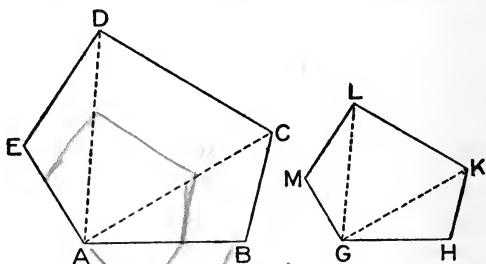
17. DEF , LMN are two \triangle s in which $\angle E = \angle M$. Prove that $\triangle DEF : \triangle LMN = DE \cdot EF : LM \cdot MN$.

18. Similar \triangle s are to one another as the squares on the radii of their circumscribing circles.

131. **Definition.**—If two polygons of the same number of sides have the angles of one taken in order around the figure respectively equal to the angles of the other in order, and have also the corresponding sides in proportion, the polygons are said to be **similar polygons**.

PROBLEM 1

To describe a polygon similar to a given polygon, and with the corresponding sides in a given ratio.



Let $ABCDE$ be the given polygon, and GH a st. line taken such that AB is to GH in the given ratio.

It is required to describe on GH a polygon similar to $ABCDE$ and such that AB and GH are corresponding sides.

Join AC , AD .

Make $\angle H = \angle B$, $\angle HGK = \angle BAC$ and produce the arms to meet at K . Make $\angle KGL = \angle CAD$, $\angle GKL = \angle ACD$, and produce the arms to meet at L . Make $\angle LGM = \angle DAE$, $\angle GLM = \angle ADE$ and produce the arms to meet at M .

$GHKLM$ is the required polygon.

$$\angle H = \angle B, \angle HGK = \angle BAC, \therefore \angle HKG = \angle BCA.$$

Similarly $\angle GLK = \angle ADC$, and $\angle M = \angle E$.

Hence $\angle HKL = \angle BCD$, $\angle KLM = \angle CDE$ and $\angle HGM = \angle BAE$.

\therefore polygon $GHKLM$ has its \angle s equal respectively to the \angle s of polygon $ABCDE$.

From the similar \triangle s GHK , ABC , $\frac{GH}{AB} = \frac{HK}{BC} = \frac{KG}{CA}$;

and from the similar \triangle s GKL , ACD , $\frac{KG}{CA} = \frac{KL}{CD}$;

$$\therefore \frac{GH}{AB} = \frac{HK}{BC} = \frac{KL}{CD}.$$

In the same manner it may be shown that each of these ratios equals $\frac{LM}{DE}$ and \therefore equals $\frac{MG}{EA}$.

Hence the corresponding sides of the two polygons are proportional; \therefore polygon $GHKLM$ is similar to polygon $ABCDE$; and the two polygons have their corresponding sides in the given ratio.

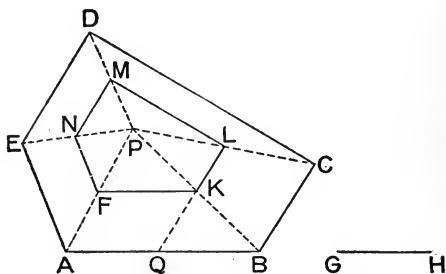
132.—Exercises

1. Draw diagrams to show that two quadrilaterals may have the sides of one respectively proportional to the sides of the other, but the \angle s of one not equal to the corresponding \angle s of the other.

2. Draw diagrams to show that two quadrilaterals may have the \angle s of one respectively equal to the \angle s of the other, but the corresponding sides not in the same proportion.

3. $KLMN$ is a polygon. Construct a polygon similar to $KLMN$, and having each side one-third of the corresponding side of $KLMN$.

4. $ABCDE$ is a given polygon and GH a given st. line. Cut off $AQ = GH$. Take any point P within $ABCDE$.



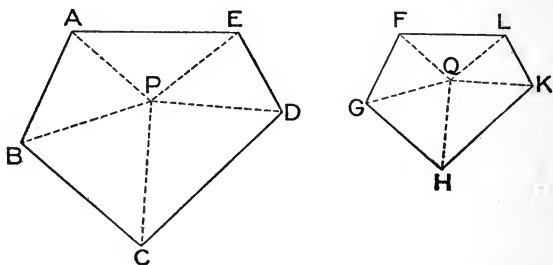
Join P to A, B, C, D, E . Draw $QK \parallel AP$, $KF \parallel AB$, $FN \parallel AE$, $NM \parallel ED$, $KL \parallel BC$. Join LM .

Show that $FKLMN$ is similar to $ABCDE$.

5. Twice as many polygons may be described on a given st. line GH , each similar to a given polygon, as the given polygon has sides.

PROBLEM 2

To divide similar polygons into similar triangles.



Let $ABCDE, FGHKL$ be similar polygons of which AB and FG are corresponding sides.

It is required to divide $ABCDE$, and $FGHKL$ into similar Δ s.

Take any point P within the polygon $ABCDE$. Join PA , PB , PC , PD , PE .

Make $\angle GFQ = \angle BAP$ and $\angle FGQ = \angle ABP$, and let the arms of these \angle s meet at Q .

Join QH , QK , QL .

$\angle PAB = \angle QFG$ and $\angle PBA = \angle QGF$; $\therefore \angle FQG = \angle APB$, and consequently Δ s ABP , FGQ are similar;

$$\therefore \frac{QG}{PB} = \frac{FG}{AB}.$$

But, by definition of similar polygons,

$$\frac{FG}{AB} = \frac{GH}{BC}.$$

$$\therefore \frac{QG}{PB} = \frac{GH}{BC}.$$

Also $\angle FGH = \angle ABC$ and $\angle FGQ = \angle ABP$;

$$\therefore \angle QGH = \angle PBC.$$

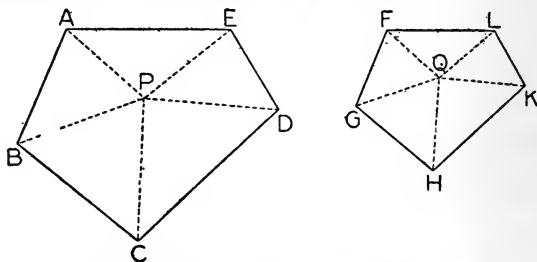
Then in Δ s QGH , PBC , $\frac{QG}{PB} = \frac{GH}{BC}$, and $\angle QGH = \angle PBC$.

\therefore these Δ s are similar. (IV—10, p. 241.)

In the same manner it may be shown that the remaining pairs of corresponding Δ s are similar.

THEOREM 2

The areas of similar polygons are proportional to the squares on corresponding sides.



Using the diagram and construction of Problem 2.

It is required to show that $\frac{\text{polygon FGHLK}}{\text{polygon ABCDE}} = \frac{FG^2}{AB^2}$.

$\therefore \triangle s$ **FGQ**, **ABP** are similar,

$$\therefore \frac{\triangle FGQ}{\triangle ABP} = \frac{GQ^2}{BP^2}. \quad (\text{V—1, p. 271.})$$

Similarly $\frac{\triangle QGH}{\triangle PBC} = \frac{GQ^2}{BP^2}$.

$$\therefore \frac{\triangle QGF}{\triangle PAB} = \frac{\triangle QGH}{\triangle PBC} = (\text{in the same manner})$$

$$\frac{\triangle QHK}{\triangle PCD} = \frac{\triangle QKL}{\triangle PDE} = \frac{\triangle QLF}{\triangle PEA}.$$

But, if any number of fractions be equal to each other, the sum of their numerators divided by the sum of their denominators equals each of the fractions.

Now the sum of the numerators of the equal fractions is the polygon FGHLK, and the sum of the denominators is the polygon ABCDE;

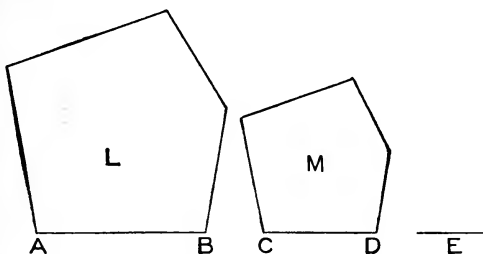
$$\therefore \frac{\text{polygon FGHLK}}{\text{polygon ABCDE}} = \frac{\triangle QFG}{\triangle PAB}.$$

$$\text{But } \frac{\triangle QFG}{\triangle PAB} = \frac{FG^2}{AB^2}.$$

$$\therefore \frac{\text{polygon } FGHKL}{\text{polygon } ABCDE} = \frac{FG^2}{AB^2}.$$

THEOREM 3

If three straight lines are in continued proportion, the first is to the third as any polygon on the first is to the similar and similarly described polygon on the second.



Hypothesis.—AB, CD, E are three st. lines such that $AB : CD = CD : E$, and L, M, similar polygons having AB, CD corresponding sides.

To prove that polygon L : polygon M = AB : E.

$$\text{Proof.}— \frac{\text{Polygon L}}{\text{Polygon M}} = \frac{AB^2}{CD^2} \quad (\text{V—2, p. 278.})$$

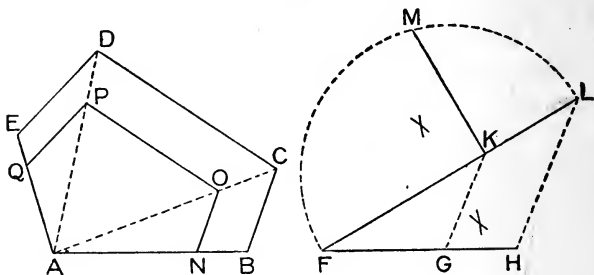
$$= \frac{AB}{CD} \cdot \frac{AB}{CD}$$

$$= \frac{AB}{CD} \cdot \frac{CD}{E} \quad (\text{Hyp.})$$

$$= \frac{AB}{E}.$$

PROBLEM 3

To make a polygon similar to a given polygon and such that their areas are in a given ratio.



Let $ABCDE$ be the given polygon and FG , GH two given st. lines.

It is required to make a polygon similar to $ABCDE$, and such that its area is to that of $ABCDE$ as GH is to FG .

Construction.—Find KL a fourth proportional to FG , GH , AB . (IV—Prob. 2, p. 227.)

Find KM a mean proportional to FK , KL . (IV—Prob. 5, p. 246.)

Cut off $AN = KM$, and on AN construct a polygon $ANOPQ$ similar to $ABCDE$.

Proof.—

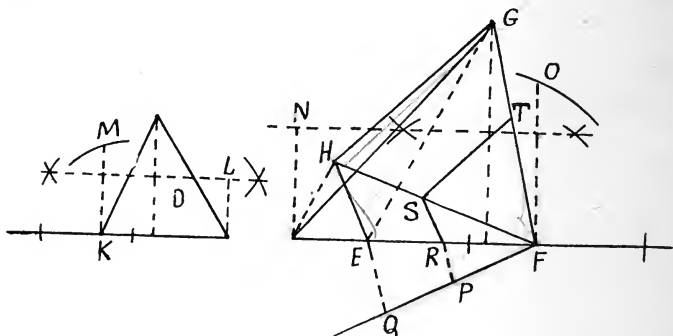
$$\therefore \frac{AB}{AN} = \frac{AN}{KL},$$

$$\begin{aligned} \therefore \frac{\text{polygon } ABCDE}{\text{polygon } ANOPQ} &= \frac{AB}{KL} && (\text{V—3, p. 279.}) \\ &= \frac{FG}{GH}. \end{aligned}$$

$$\text{And } \therefore \frac{\text{polygon } ANOPQ}{\text{polygon } ABCDE} = \frac{GH}{FG}.$$

PROBLEM 4

To make a figure equal to one given rectilinear figure and similar to another.



Let D and $EFGH$ be the given figures.

It is required to make a figure similar to $EFGH$ and equal to D .

Construction.—Construct the rect. $KL = D$, and the rect. $FN = EFGH$.

Make KM the side of a square which is equal to KL , and FO the side of a square which is equal to FN ; so that, $KM^2 = D$ and $FO^2 = EFGH$.

From F draw a st. line FQ and from it cut off $FP = KM$ and $FQ = FO$.

Join QE , and draw $PR \parallel QE$ cutting EF at R .

On RF describe $RFTS$ similar to $EFGH$.

$RFTS$ is the required figure.

Proof.—

$$\therefore RFTS \parallel EFGH,$$

$$\therefore \frac{RFTS}{EFGH} = \frac{RF^2}{EF^2} \quad (\text{V—1, p. 271.})$$

$$= \frac{PF^2}{QF^2} \quad (\text{IV—2, p. 222.})$$

$$= \frac{KM^2}{FO^2} = \frac{D}{EFGH}.$$

$$\therefore \frac{RFTS}{EFGH} = \frac{D}{EFGH};$$

and $\therefore RFTS = D$;

also $RFTS$ was made similar to $EFGH$.

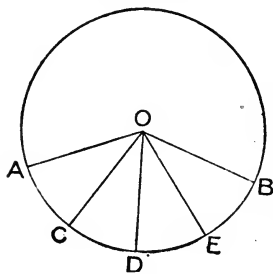
133.—Exercises

1. On a plan of which the scale is 1 inch to 2 feet, a room is represented by 30 sq. in. Find the area of the room.
2. On a map of which the scale is 4 inches to the mile, a farm is represented by 10 sq. in. Find the number of acres in the farm.
3. Construct an equilateral \triangle equal in area to a given square.
4. Construct a square equal in area to a given \triangle .
5. Construct a rectangle similar to a given rectangle and equal in area to a given square.
6. Construct a square the area of which is 15 sq. in.
7. Bisect a given \triangle by a st. line drawn \perp to one side.



ARCS AND ANGLES

134.—Suppose an angle AOB at the centre of a circle to be divided into a number of equal parts AOC , COD , DOE , EOB .

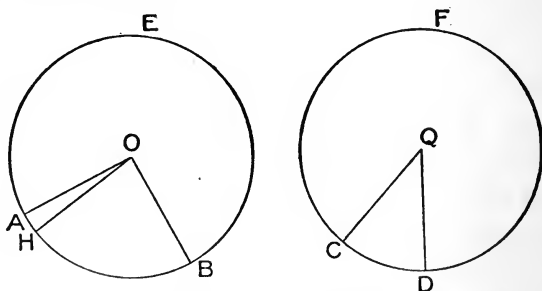


Then, by III—13, p. 167, the arcs AC , CD , DE , EB are equal to each other, and whatever multiple the angle AOB is of the angle AOC , the arc AB is the same multiple of the arc AC .

Thus, if an angle at the centre of a circle be divided into degrees and contain a of them, the arc subtending the angle will contain the arc subtending one degree a times.

THEOREM 4

In equal circles, angles, whether at the centres or circumferences, are proportional to the arcs on which they stand.



Hypothesis.—In the equal circles AEB , CFD , the \angle s AOB , CQD at the centres stand respectively on the arcs AB , CD .

To prove that $\frac{\angle AOB}{\angle CQD} = \frac{\text{arc AB}}{\text{arc CD}}$.

Proof.—Let the \angle s AOB , CQD be commensurable having $\angle AOH$ a common measure. Suppose $\angle AOB$ contains $\angle AOH$ a times, and $\angle CQD$ contains $\angle AOH$ b times.

Then arc AB contains arc AH a times, and arc CD contains arc AH b times.

$$\therefore \frac{\angle AOB}{\angle CQD} = \frac{a \times \angle AOH}{b \times \angle AOH} = \frac{a}{b}.$$

And $\frac{\text{arc AB}}{\text{arc CD}} = \frac{a \times \text{arc AH}}{b \times \text{arc AH}} = \frac{a}{b}.$

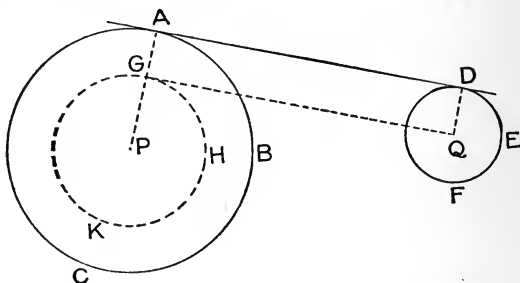
$$\therefore \frac{\angle AOB}{\angle CQD} = \frac{\text{arc AB}}{\text{arc CD}}.$$

Again, since the \angle s at the circumferences are respectively half the \angle s at the centres, on the same arcs, the \angle s at the circumferences are also in the ratio of the arcs on which they stand.

ANALYSIS OF A PROBLEM—COMMON TANGENTS OF CIRCLES

135. A common method of discovering the solution of a problem begins with the drawing of the given figure or figures. The required part is then sketched in, and a careful examination is made to determine the connection between the given parts and the required result. Properties of the figure are noted, and lines are drawn that may help in finding the solution. This method of attack is known as the **Analysis of the Problem**. Its use is illustrated in the following sections.

136. Problem.—To draw the direct common tangents to two given circles.



Let ABC , DEF be two circles, with centres P , Q .

It is required to draw a direct common tangent to the circles ABC , DEF .

Suppose AD to be a direct common tangent touching the circles at A , D .

Join PA , QD .

PA , QD are both $\perp AD$, and $\therefore PA \parallel QD$.

Cut off $AG = DQ$. Join QG .

AG is both $=$ and $\parallel QD$, $\therefore AQ$ is a \parallel gm, and as $\angle GAD$ is a rt. \angle , AQ is a rect.

Draw a circle with centre P and radius PG .

PGQ is a rt. \angle , $\therefore QG$ is a tangent to the circle GHK and this tangent is drawn from the given point Q . The radius PG of the circle GHK is the difference of the radii of the given circles.

Using the construction suggested by the above analysis the pupil should make the direct drawing and prove that it is correct.

Show that two direct common tangents may be drawn.

138.—Exercises

1. Draw diagrams to show that the number of common tangents to two circles may be 4, 3, 2, 1 or 0.

2. Draw a st. line to cut two given circles so that the chords intercepted on the line may be equal respectively to two given st. lines.

3. P , Q are the centres of two circles. A common tangent (either direct or transversal) meets the line of centres at R . Show that the ratio $PR : QR$ equals the ratio of the radii of the circles.

4. The transverse common tangents and the line of centres of two circles are concurrent.

5. The direct common tangents and the line of centres of two circles are concurrent.

6. P , Q are the centres of two circles and PA , QB any two \parallel radii drawn in the same direction from P , Q . Show that AB produced and the direct common tangents meet the line of centres at the same point.

7. P , Q are the centres of two circles and PA , QB any two \parallel radii drawn in opposite directions from P , Q . Show that AB and the transverse common tangents meet the line of centres at the same point.

8. Draw the direct common tangents to two equal circles.

Miscellaneous Exercises

1. Draw four circles each of radius $1\frac{3}{4}$ inches, touching a fixed circle of radius 1 inch and also touching a st. line $1\frac{1}{2}$ inches distant from the centre of the circle.

2. DE, FG are \parallel chords of the circle $DEGF$. Prove that $DE \cdot FG = DG^2 - DF^2$.

3. If two circles touch externally at A and are touched at B, C by a st. line, the st. line BC subtends a rt. \angle at A .

4. Of all Δ s of given base and vertical \angle , the isosceles Δ has the greatest area.

5. ABC is an equilateral Δ inscribed in a circle, P is any point on the circumference. Of the three st. lines PA, PB, PC , shew that one equals the sum of the other two.

6. Construct a rt.- \angle d Δ , given the radius of the inscribed circle and an acute \angle of the Δ .

7. The diagonals AC, BD of a cyclic quadrilateral $ABCD$ cut at E . Show that the tangent at E to the circle circumscribed about ΔABE is \parallel to CD .

8. A, B, C are three points on a circle. The bisector of $\angle ABC$ meets the circle again at D . DE is drawn \parallel to AB and meets the circle again at E . Show that $DE = BC$.

9. The side of an equilateral Δ circumscribed about a circle is double the side of the equilateral Δ inscribed in the same circle.

10. AB is the diameter of a circle and CD a chord. EF is the projection of AB on CD . Show that $CE = DF$.

11. Construct an isosceles Δ , given the base and the radius of the inscribed circle.

12. Two circles touch externally. Find the locus of the points from which tangents drawn to the circles are equal to each other.

13. Two circles, centres C, D , intersect at A, B . PAQ is a st. line cutting the circles at P, Q . PC, QD intersect at R . Find the locus of R .

14. Two circles touch internally at A ; BC , a chord of the outer circle, touches the inner circle at D . Show that AD bisects $\angle BAC$.

15. P is a given point on the circumference of a circle, of which AB is a given chord. Through P draw a chord PQ that is bisected by AB .

16. On a given base construct a \triangle having given the vertical \angle and the ratio of the two sides.

17. AB is a given st. line and P, Q are two points such that $AP : PB = AQ : QB$. Show that the bisectors of $\angle s$ APB, AQB cut AB at the same point.

18. AB is a given st. line and P, Q are two points such that $AP : PB = AQ : QB$. Show that the bisectors of the exterior $\angle s$ at P, Q of the $\triangle s$ APB, AQB meet AB produced at the same point.

19. AB is a given st. line and P is a point which moves so that the ratio $AP : PB$ is constant. The bisectors of the interior and exterior $\angle s$ at P of the $\triangle APB$, meet AB and AB produced at C, D respectively. Show that the locus of P is a circle on CD as diameter.

20. AB is a st. line 2 inches in length. P is a point such that AP is twice BP . Construct the locus of P .

21. Two circles touch externally, and A, B are the points of contact of a common tangent. Show that AB is a mean proportional between their diameters.

22. If on equal chords segments of circles be described containing equal \angle s, the circles are equal.

23. Construct a quadrilateral such that the bisectors of the opposite \angle s meet on the diagonals.

24. Draw a circle to pass through a given point and touch two given st. lines.

25. Draw a circle to touch a given circle and two given st. lines.

26. Draw a circle to pass through two given points and touch a given circle.

27. Construct a rt.- \angle d \triangle given the hypotenuse and the radius of the inscribed circle.

28. In $\triangle ABC$ the inscribed circle touches **AB**, **AC** at **D**, **E** respectively. The line joining **A** to the centre cuts the circle at **F**. Show that **F** is the centre of the inscribed circle of $\triangle ADE$.

29. The inscribed circle of the rt.- \angle d $\triangle ABC$ touches the hypotenuse **BC** at **D**. Show that $\text{rect. } BD \cdot DC = \triangle ABC$.

30. If on the sides of any \triangle equilateral \triangle s be described outwardly, the centres of the circumscribed circles of the three equilateral \triangle s are the vertices of an equilateral \triangle .

31. Describe three circles to touch each other externally and a given circle internally.

32. Show that two circles can be described with the middle point of the hypotenuse of a rt.- \angle d \triangle as centre to touch the two circles described on the two sides as diameters.

33. A st. line **AB** of fixed length moves so as to be constantly \parallel to a given st. line and **A** to be on the circumference of a given circle. Show that the locus of **B** is an equal circle.

34. Construct an isosceles \triangle equal in area to a given \triangle and having the vertical \angle equal to one of the \angle s of the given \triangle .

35. If two chords AB, AC , drawn from a point A in the circumference of the circle ABC , be produced to meet the tangent at the other extremity of the diameter through A in D, E respectively, then the $\triangle AED$ is similar to $\triangle ABC$.

36. If a st. line be divided into two parts, the sq. on the st. line equals the sum of the rectangles contained by the st. line and the two parts.

37. $ABCD$ is a quadrilateral inscribed in a circle. AB, DC meet at E and BC, AD meet at F . Show that the sq. on EF equals the sum of the sqs. on the tangents drawn from E, F to the circle.

38. The st. line AB is divided at C so that $AC = 3CB$. Circles are described on AC, CB as diameters and a common tangent meets AB produced at D . Show that BD equals the radius of the smaller circle.

39. DE is a diameter of a circle and A is any point on the circumference. The tangent at A meets the tangents at D, E at B, C respectively. BE, CD meet at F . Show that AF is \parallel to BD .

40. TA, TB are tangents to a circle of which C is the centre. AD is $\perp BC$. Show that $TB : BC = BD : DA$.

41. $ABCD$ is a quadrilateral inscribed in a circle. BA, CD produced meet at P , and AD, BC produced meet at Q . Show that $PC : PB = QA : QB$.

42. Divide a given arc of a circle into two parts, so that the chords of these parts shall be to each other in the ratio of two given st. lines.

43. Describe a circle to pass through a given point and touch a given st. line and a given circle.

44. LMN is a rt.- \angle d \triangle with L the rt. \angle . On the three sides equilateral \triangle s LEM , MFN , NDL are described outwardly. LG is \perp MN . Prove that $\triangle FGM = \triangle LEM$ and $\triangle FGN = \triangle NDL$.

45. L is the rt. \angle of a rt.- \angle d $\triangle LMN$ in which $LN = 2 LM$. Also $LX \perp MN$. Prove that $LX = \frac{2}{3} MN$.

46. A st. line meets two intersecting circles in P and Q , R and S and their common chord in O . Prove that OP , OQ , OR , OS , taken in a certain order, are proportionals.

47. LMN is a semi-circle of which O is the centre, and $OM \perp LN$. A chord LDE cuts OM at D . Prove that LM is a tangent to the circle MDE .

48. The bisector of $\angle F$ of $\triangle FGH$ meets the base GH in E and the circumcircle in D . Prove that $DG^2 = DE \cdot DF$.

49. POQ , ROS are two st. lines such that $PO : OQ = 3 : 4$ and $RO : OS = 2 : 5$. Compare areas of \triangle s POR , QOS ; and also areas of \triangle s POS , QOR .

50. Trisect a given square by st. lines drawn \parallel to one of its diagonals.

51. Construct a \triangle having its base 8 cm., the other sides in the ratio of 3 to 2, and the vertical $\angle = 75^\circ$.

52. In two similar \triangle s, the parts lying within the \triangle of the right bisectors of corresponding sides have the same ratio as the corresponding sides of the \triangle .

53. KMN , LMN are \triangle s on the same base and between the same \parallel s. KN , LM cut at E . A line through E , \parallel MN , meets KM in F and LN in G . Prove that $FE = EG$.

54. Construct a \triangle having given the vertical \angle , the ratio of the sides containing that \angle , and the altitude drawn to the base.

55. From a point P without a circle two secants PFG , PED are drawn, and PQ drawn $\parallel FD$ meets GE produced at Q . Prove that PQ is a mean proportional between QE , QG .

56. LD bisects $\angle L$ of $\triangle LMN$ and meets MN at D . From D the line $DE \parallel LM$ meets LN at E , and $DF \parallel LN$ meets LM at F . Prove that $FM : EN = LM^2 : LN^2$.

57. LMN is a $\triangle \angle$ rt.-d at L . $LD \perp MN$ and meets a line drawn from $M \perp LM$ at E . Prove that $\triangle LMD$ is a mean proportional between \triangle s LDN , MDE .

58. Two circles touch externally at D and PQ is a common tangent. PD and QD produced meet the circumferences at L , M respectively. Show that PM and QL are diameters of the circles.

59. The common tangent to two circles which intersect subtends supplementary \angle s at the points of intersection.

60. Two circles intersect at Q and R , and ST is a common tangent. Show that the circles described about \triangle s STR , STQ are equal.

61. A st. line DEF is drawn from D the extremity of a diameter of a circle cutting the circumference at E and a fixed st. line \perp to the diameter at F . Show that the rect. $DE \cdot DF$ is constant for all positions of DEF .

62. A chord LM of a circle is produced to E such that ME is one-third of LM ; a tangent EP is drawn to the circle and produced to D such that $PD = EP$. Prove that $\triangle ELD$ is isosceles.

63. Draw a st. line to touch one circle and to cut another, the chord cut off being equal to a given st. line.

64. Two equal circles are placed so that the transverse common tangent is equal to the radius. Show that the tangent from the centre of one circle to the other equals the diameter of each circle.

65. Construct a \triangle having its medians respectively equal to three given st. lines.

66. Construct a \triangle given one side and the lengths of the medians drawn from the ends of that side.

67. Construct a \triangle given one side, the median drawn to the middle point of that side, and a median drawn from one end of that side.

68. Construct a \triangle having $\angle A = 20^\circ$, $\angle C = 90^\circ$, and $c - a = 4$ cm.

69. Construct a \triangle having $\angle C = 90^\circ$, $b = 6$ cm., and $c - a = 3.5$ cm.

70. Construct a \triangle having $a = 7$ cm., $c - b = 3$ cm., and $\angle C - \angle B = 28^\circ$.

71. If a st. line be drawn in any direction from one vertex of a \parallel gm, the \perp to it from the opposite vertex equals the sum or difference of the \perp s to it from the two remaining vertices.

72. PQ is a chord of a circle \perp to the diameter LM , and E is any point in LM . If PE , QE meet the circumference in S , R respectively, show that $PS = QR$; and that $RS \perp LM$.

73. P is any point in a diameter LM of a circle, and QR is a chord $\parallel LM$. Prove that $PQ^2 + PR^2 = PL^2 + PM^2$.

74. On the hypotenuse EF of the rt.- \angle d $\triangle DEF$ a $\triangle GEF$ is described outwardly having $\angle GEF = \angle DEF$ and $\angle GFE$ a rt. \angle . Prove that $\triangle GFE : \triangle DEF = GE : ED$.

75. Two quadrilaterals whose diagonals intersect at equal \angle s are to one another in the ratio of the rectangles contained by the diagonals.

76. P is any point in the side LM of a $\triangle LMN$. The st. line $MQ, \parallel PN$, meets LN produced at Q ; and X, Y are points in LM, LQ respectively, such that $LX^2 = LP.LM$ and $LY^2 = LN.LQ$. Prove that $\triangle LXY = \triangle LMN$.

77. EFP, EFQ are circles and PFQ is a st. line. ER is a diameter of circle EFP and ES a diameter of EFQ . Prove $\triangle EPR : \triangle EQS$ as the squares on the radii of the circles.

78. If P is the point of intersection of an external common tangent PQR to two circles with the line of centres, prove that $PQ : PR$ as the radii of the circles. Also, if $PCDEF$ is a secant, prove that $PC : PE = PD : PF$

79. A point E is taken within a quadrilateral $FGHK$ such that $\angle EFK = \angle GFH$ and $\angle EKF = \angle GHF$. GE is joined. Prove $\triangle FEG \parallel \triangle FHK$.

80. Through a given point within a circle, draw a chord that is divided at the point in a given ratio

81. From P , a point on the circumference of a circle, tangents PE, PF are drawn to an inner concentric circle. $GEFH$ is a chord, and PE meets the circumference at Q . Prove \triangle s PGF, PEH, GEQ similar; also show that $GQ^2 : GP^2 = GE : GF$.

82. L is the vertex of an isosceles $\triangle LMN$ inscribed in a circle, LRS is a st. line which cuts the base in R and meets the circle in S . Prove that $SL.RL = LM^2$.

83. PQR is a rt.- \angle d \triangle with P the rt. \angle . $PD \perp QR$; $DM \perp PQ$ and $DN \perp PR$. Prove that $\angle QMR = \angle QNR$.

84. DEF is an isosceles \triangle with $\angle D = 120^\circ$. Show that if EF be trisected at G and H , the $\triangle DGH$ is equilateral.

85. AS and AT, BP and BQ are tangents from two points A and B to a circle. C, D, E, F are the middle points of AS, AT, BP, BQ respectively. Prove that CD, EF, produced if necessary, meet on the right bisector of AB. (Let O be the centre of the circle; L and M the points where OA, OB cut the chords of contact. Prove A, L, M, B concyclic, etc.)

86. If from the middle point of an arc two st. lines be drawn cutting the chord of the arc and the circumference, the four points of intersection are concyclic.

87. If a st. line be divided at two given points, find a third point in the line, such that its distances from the ends of the line may be proportional to its distances from the two given points.

88. Prove geometrically that the arithmetic mean between two given st. lines is greater than the geometric mean between the two st. lines.

89. A square is inscribed in a rt.-angled triangle, one side of the square coinciding with the hypotenuse: prove that the area of the square is equal to the rectangle contained by the extreme segments of the hypotenuse.

90. Any regular polygon inscribed in a circle is the geometric mean between the inscribed and circumscribed regular polygons of half the number of sides.

91. The diagonal and the diagonals of the complements of the parallelograms about the diagonal of a parallelogram are concurrent.

92. Develop the formula for the area of a \triangle , $\sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a + b + c$ and a, b, c are the sides.

Solution of 92. In $\triangle ABC$, draw $AX \perp BC$, and let $AX = h$, $BX = x$. Then $CX = a - x$.

Area of $\triangle ABC = \frac{1}{2} a h$.

$$h^2 = b^2 - (a - x)^2 = c^2 - x^2,$$

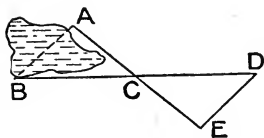
$$\therefore x = \frac{a^2 - b^2 + c^2}{2a}.$$

$$h^2 = c^2 - \frac{(a^2 - b^2 + c^2)^2}{4a^2}.$$

$$\begin{aligned} 4a^2 h^2 &= 4a^2 c^2 - (a^2 - b^2 + c^2)^2 \\ &= (2ac + a^2 - b^2 + c^2)(2ac - a^2 + b^2 - c^2) \\ &= \{(a + c)^2 - b^2\} \{b^2 - (a - c)^2\} \\ &= (a + b + c)(a - b + c)(a + b - c)(b - a + c) \\ &= 2s(2s - 2b)(2s - 2c)(2s - 2a). \end{aligned}$$

$$\therefore \frac{1}{4} a^2 h^2 = s(s - a)(s - b)(s - c),$$

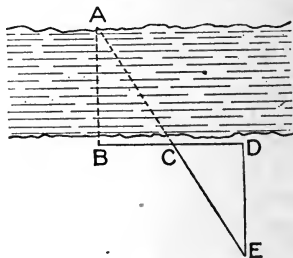
$$\text{And } \frac{1}{2} ah = \sqrt{s(s - a)(s - b)(s - c)}.$$



93. Show from the diagram how the distance between two points, A, B at opposite sides of a pond may be found by measurements on land.

94. Show from the diagram how the breadth of a river may be found by measurements made on one side of it.

95. Given a st. line AB, construct a continuation of it CD, AB and CD being separated by an obstacle.



96. AB, CD are two lines which would meet off the paper. Draw a st. line which would pass through the point of intersection of AB, CD, and bisect the \angle between them.

INDEX TO DEFINITIONS

	PAGE
Acute angle :—An \angle which is $<$ a rt. \angle	10
Acute-angled triangle :—A \triangle which has three acute \angle s.....	27
Adjacent angles :—Two \angle s which have the same vertex, an arm common, and the remaining arms on opposite sides of the common arm.....	11
Altitude of a triangle :—The length of the \perp from any vertex of the \triangle to the opposite side.....	27
Angle :—The amount of rotation made by a st. line when it revolves about a fixed point in itself from one position to another.....	8
Antecedent :—The first term in a ratio.....	213
Arc of a circle :—A part of the circumference.....	17
Axiom :—A statement that is self-evident, or assumed.....	3
Axis of symmetry :—The line about which a symmetrical figure can be folded so that the parts on one side will exactly fit the corresponding parts on the other side.....	21
Centroid :—The point where the medians of a \triangle intersect one another.....	69
Chord of a circle :—The st. line joining two points on the circumference.....	17
Chord of contact :—The st. line joining the points of contact of two tangents.....	174
Circle :—The locus of the points that are at a fixed distance from a fixed point..... 17, 141 (The name circle is also used for the area inclosed by the circumference).	
Circumcentre :—The centre of the circumscribed circle of a \triangle	144
Circumference :—Same as circle.....	17
Circumscribed circle :—A circle which passes through all the vertices of a rectilineal figure.....	143
Coincide :—Magnitudes which fill exactly the same space are said to coincide with each other.....	4
Commensurable magnitudes :—Magnitudes which have a common measure.....	214
Complementary angles :—Two \angle s of which the sum is one rt. \angle	13
Concyclic points :—Points through which a circle may be described.....	143
Congruent figures :—Figures equal in all respects, so that one may be made to fit the other exactly.....	15
Consequent :—The second term of a ratio.....	213
Converse propositions :—Two propositions of which the hypothesis of each is the conclusion of the other.....	41

	PAGE
Cyclic quadrilateral :—A quadrilateral of which all the vertices are on the circumference of the same circle.....	144
Diagonal :—A st. line joining opposite vertices of a rectilinear figure.....	17
Diameter of a circle :—A chord passing through the centre.....	17
Equilateral triangle :—A \triangle having three equal sides.....	19
Escribed circle of a triangle :—A circle which is touched by one side of a \triangle and by the other two sides produced.....	187
External segments of a line :—The distances from the two ends of a line to a point taken in the line produced.....	217
Figure :—Any combination of points, lines, surfaces and solids... 2	2
Geometric mean :—The second of three terms that are in continued proportion.....	214
Hypotenuse :—The side opposite the rt. \angle in a rt.- \angle d \triangle	27
Hypothesis :—The part of a proposition in which the conditions that are supposed to exist are stated.....	5
Incommensurable magnitudes :—Magnitudes that have no common measure however small.....	214
Indirect method of demonstration :—Method of proof that begins by assuming that the proposition to be proved is not true....	41
Inscribed Circle :—A circle that is within a closed, rectilinear figure, and which is touched by each side of the figure....	187
Internal segments of a line :—The distances from the ends of a line to a point in the line.....	217
Isosceles triangle :—A \triangle having two equal sides.....	19
Line :—That which has length, but neither breadth nor thickness..	1
Locus :—A figure consisting of a line (or lines) which contains all the points that satisfy a given condition, and no others.....	77
(See also page 78).	
Major arc of a circle :—An arc that is greater than half the circumference.....	153
Major segment of a circle :—A segment of which the arc is a major arc.....	153
Means of a proportion :—The second and third of four magnitudes that are in proportion.....	214
Mean proportional :—The second of three magnitudes that are in continued proportion.....	214
Median :—A st. line drawn from any vertex of a \triangle to the middle point of the opposite side.....	30
Minor arc of a circle :—An arc that is less than half the circumference.....	153

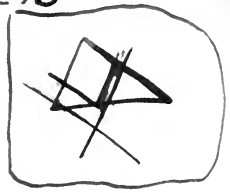
	PAGE
Minor segment of a circle :—A segment of which the arc is a minor arc	153
Obtuse angle :—An \angle which is $>$ a rt. \angle	10
Obtuse-angled triangle :—A \triangle one \angle of which is an obtuse \angle	27
Parallel straight lines :—St. lines in the same plane which do not meet when produced for any finite distance in either direction.	35
Parallelogram :—A quadrilateral that has both pairs of opposite sides \parallel to each other	35
Perpendicular :—Each arm of a rt. \angle is said to be \perp to the other arm.	10
Plane surface :—A surface such that if any two points on it be joined by a st. line the joining line lies wholly on the surface.	2
Point :—That which has position but no size	1
Point of contact :—The common point of a tangent and circle.	169
Polygon :—A figure bounded by more than four st. lines. Also sometimes used for a figure bounded by any number of st. lines.	68
Problem :—The statement of a construction to be made.	4
Projection of the point on the line :—The foot of the \perp from the point to the line	128
Projection of the line on the line :—The intercept on the second line between the projections of the two ends of the first line on the second	129
Proportion :—The equality of ratios.	213
Proposition :—That which is stated or affirmed for discussion	4
Quadrilateral :—A closed figure formed by four st. lines.	17
Radius :—A st. line drawn from the centre of a circle to the circumference	17
Ratio :—The measure of one magnitude when another magnitude of the same kind is taken as the unit.	213
Rectangle :—A \parallel gm of which the \angle s are rt. \angle s.	68
Rectilinear figure :—A figure formed by st. lines	15
Reflex angle :—An \angle which is $>$ two rt. \angle s, but $<$ four rt. \angle s.	154
Regular polygon :—A polygon in which all the sides are equal to each other, and all the \angle s are equal to each other.	68
Rhombus :—A quadrilateral having its four sides equal to each other	17
Right angle :—An \angle which is half of a st. \angle	10
Right-angled triangle :—A \triangle one \angle of which is a rt. \angle	27
Right bisector :—A st. line which bisects a st. line of given length at rt. \angle s.	28
Scalene triangle :—A \triangle having no two of its sides equal to each other	19

	PAGE
Secant of a circle :—A st. line drawn from a point without to cut a circle.....	169
Sector of a circle :—A figure bounded by two radii of a circle and either of the arcs intercepted by these radii	155
Segment of a circle :—A figure bounded by an arc of a circle and the chord which joins the ends of the arc.....	153
Similar polygons :—Two polygons of the same number of sides which have the \angle s of one taken in order around the figure respectively equal to the \angle s of the other in order, and have also the corresponding sides in proportion	274
Similar segments of circles :—Segments which contain equal \angle s.	263
Similar triangles :—Two \triangle s which have the three \angle s of one respectively equal to the three \angle s of the other	158
Solid :—That which has length, breadth and thickness.....	2
Square :—A rectangle of which all the sides are equal to each other.	68
Straight angle :—Half of a complete revolution made by a st. line revolving about a point in itself.....	9
Straight lines :—Lines which cannot have any two points of one coincide with two points of the other without the lines coinciding altogether.....	1
Subtend :—A line drawn from a point in one arm of an \angle to a point in the other arm subtends the \angle	17
Supplementary angles :—Two \angle s of which the sum is two rt. \angle s.	13
Surface :—That which has length and breadth but no thickness...	2
Symmetrical figure :—A figure which can be folded along a st. line so that the parts on one side exactly fit the corresponding parts on the other side.....	20
Tangent to a circle :—A st. line which, however far it may be produced, has one point on the circumference of a circle and all other points without the circle.....	169
Theorem :—The statement of a truth to be proved.....	4
Touch :—A tangent is said to touch a circle.....	169
Two circles which meet each other at one point and only one point are said to touch each other.....	196
Transversal :—A st. line which cuts two, or more, other st. lines..	35
Triangle :—A figure formed by three st. lines which intersect one another.....	15
Vertex of an angle :—The point from which the two arms of the \angle are drawn.....	8





YB 17296



≡ congruence 13

∠ angle

∠ ∠

||

||

△

∠

∠

||

M306170

QA453

M3

8

THE UNIVERSITY OF CALIFORNIA LIBRARY

