THEORY AND SOLUTION

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TO

GEORGE BIRKBECK, M.D., F.G.S.

&c. &c. &c.

MY DEAR SIR,

I BEG to inscribe this little volume to you: but I am anxious that you should receive it less as a memento of the personal obligations which your numerous acts of kindness have laid me under, than as an humble testimony of admiration of your splendid and varied attainments in every department of Science and Philosophy; and of the untiring zeal with which you devote them to the advancement of the best interests of mankind.

I am,

My dear sir,

Your obedient and obliged servant,

J. R. YOUNG.

Belfast College;
Aug. 25, 1835.

PREFACE.

THE discoveries which have recently been made in the general theory and solution of Algebraical Equations, are of sufficient importance to render a treatise on the subject embodying those discoveries, in an elementary form, peculiarly acceptable to the British student.

There has long appeared to me to be great want of a compendious work on the theory and solution of Equations in the English language; and the successful researches of *Horner*, *Budan*, *Fourier*, and *Sturm*, have now so entirely changed the state of this department of analysis, as to render such a publication almost indispensable.

The improvements of Mr. Horner, which have so greatly contributed to perfect the numerical solution of equations of all orders, were first published in the Philosophical Transactions of the Royal Society, in the year 1819; but strange as it may appear, have as yet excited but little attention from British mathematicians.

This apparent neglect of a valuable discovery may possibly be owing to the very limited circulation of the work in which the discovery appears, or it may perhaps be attributable to the somewhat uninviting mode of investigation which Mr. Horner has adopted. It cannot arise from indifference on the part of English mathematicians, in reference to so celebrated a problem as the general solution of numerical equations, seeing that the recent publication of the French mathematician, *Fourier*, where this solution is attempted by a process analogous to, although far less continuous and compact than, that of Horner, has received from English mathematicians the most unmeasured praise.

Much of this praise I cannot help considering as greatly misapplied; for Fourier has occupied a considerable portion of his work in establishing a rule for determining the number and situation of the real roots of an equation, which rule was discovered and satisfactorily demonstrated by *Budan*, and presented to the Institute of France, twenty years before the publication of Fourier's work! This fact is the more important, because the rule in question will be considered by most persons as the principal deduction of consequence to be found in Fourier's book. In giving an account of the researches of Fourier, in the following pages, I have certainly associated his name with that of Budan, in the investigation of this rule, because the *Analyse des Equations Determinées*, of Fourier, professes to be an original performance, and may have been composed, independently of the Mémoire of Budan. This treatise of Fourier is a posthumous publication, put forth under the superintendence of *M. Navier*, of the Paris Academy of Sciences; and it is very remarkable that in the long advertisement which he has prefixed to the work, bearing date July, 1831, he has not once mentioned the Mémoire of *Budan*, but has taken great pains to show that the work of Fourier is entirely original.* That he has not succeeded with his own countrymen, appears from the fact that the latest publications on the subject refer to this rule as the "Théorême de Budan."† There is no doubt, however, that Fourier, in the work adverted to, has extended his enquiries beyond the point where those of Budan terminated, since he has furnished criteria for the detection of the number and situation of the imaginary roots of an equation; but these criteria are, in many cases, of very difficult application, and are not always free from perplexing circumstances. Happily, however, this (undoubtedly the most original part of Fourier's work,) is now entirely superseded by a very remarkable theorem of M. Sturm, published in the *Mémoires presentés par des Savans Etrangers à l'Académie Royale des Sciences*, for

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* The academicians appointed to examine the Mémoire of Budan, were *Lagrange* and *Legendre*, who make no mention in their report of any similar theorem by *Fourier*, although *Navier* labours to show that Fourier had demonstrated it long before. The report of the commissioners on Budan's paper closes as follows:—Nous croyons que le théorème trouvé par M. Budan mérite l'attention de la classe, comme étant une extension de la règle de Descartes, et que son Mémoire peut etre imprimé dans le Recueil des Mémoires présentés, accompagné du présent rapport,

Signé—LAGRANGE; LEGENDRE, *rapporteur*."

See the *Nouvelle Méthode* of Budan, 2d edit. 1822.

† See *Bourdon's Algèbre*, 1831, pa. 720; and that of *Lefébure de Fourcy*, pp. 447, 1835.

1835,* and by means of which the important problem of the complete separation and enumeration of the real and imaginary roots of an equation is now satisfactorily solved. This valuable discovery I have developed and applied at great length in the VIIth Chapter, and have thus, I trust, rendered an acceptable service to the English student.

With regard to the numerical process of Mr. Horner, and which, in conjunction with the theorem of Sturm, renders the theory and solution of numerical equations complete, it may be here remarked, that I have done little else than fully to explain and copiously to illustrate Mr. Horner's views: one or two modifications of trifling moment I have ventured to suggest, and I have also detached the subject of cubic equations for a separate and distinct investigation, and have thence deduced a new and easy rule for finding the cube root of a number; but whether these innovations be for the better or the worse, must be left with the reader to determine.†

The names of Mr. Atkinson and Mr. Holdred have often been associated with that of Mr. Horner, in connexion with this discovery of the numerical solution of equations. Mr. Horner's paper was published in 1819, and Mr. Holdred's Tract in 1820. Mr. Holdred's work contains two modes of

* This important paper will, I am happy to find, be shortly published in an English translation, by my accomplished friend, *W. H. Spiller, Esq.* of Highgate.

† It should be remarked that the chapter on cubic equations was first published in the author's "Treatise on Algebra," 8vo. edition, 1823.

solving a numerical equation, one a tedious method, discovered, according to the Preface, forty years before publication; and the other, printed as a supplement to the former, discovered, Mr. Holdred says, after the work had been announced for publication. Between this latter method and the one published by Mr. Horner, there is a remarkable resemblance. The name of this latter gentleman is not, however, mentioned in Mr. Holdred's Tract, nor is there any hint given as to the precise period when the alleged discovery was made, although from a remark in the last page of the Preface, it would seem that the publication of the work was delayed "some weeks," in consequence. This, however, is matter of but little moment as respects Mr. Horner's claims, for as this gentleman's paper had been six months before the public when that of Mr. Holdred first appeared, there can be no reasonable doubt as to whom the honor of the discovery belongs.

Mr. Atkinson's method of solution is identical with that of Mr. Holdred's first method,—so completely identical that both papers appear like the work of one person.

Mr. Atkinson, in the Preface to his "New Method of Extracting the Roots of Equations," published in 1831, at Newcastle, brings forward convincing proof that his method was publicly read at the Literary and Philosophical Society of Newcastle-upon-Tyne in August, 1809; and there seems to be no reason for doubting that this method was equally the independent discovery of Mr. Holdred and Mr. Atkinson; as, however, it is entirely superseded by that of Mr. Horner, there would be little interest created by either establishing or controverting this opinion.

The present publication is intended to embrace all that is important in the researches to which I have just alluded, and thus to supply the English student with a treatise on the subject of numerical equations, adapted to the present improved state of that important branch of analysis. I shall rejoice if I be found to have succeeded in rendering these researches intelligible to the young mathematician; or if this little work shall be the means of facilitating their introduction into the mathematical courses of instruction prescribed in our public seminaries of education.

J. R. YOUNG.

Belfast College;
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Page 21, line 19, *for* all the signs, *read* the alternate signs.
.. 37, .. 5 from bottom, *for* $f(x)$, *read* $f(-x)$; and *for* negative, *read* positive.
.. 47, .. 8, *for* in which is, *read* independent of.
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ELEMENTS

OF THE

THEORY OF ALGEBRAICAL EQUATIONS,

&c.

INTRODUCTION.

(*Article* 1.) THE object of almost every mathematical enquiry, of a practical nature, is the determination of numerical values for unknown quantities, by the help of given relations between them and others which are known. The algebraical expressions of these relations, when announced in their most convenient form, give rise to *equations*; and it is the evolution of the unknown quantities from these which forms the chief business of Algebra. The desire to effect this evolution in every possible instance, and thus to render the science of Algebra complete, has prompted the laborious exertions of the ablest analysts; and, although their efforts have not fully accomplished their wishes, yet there has resulted from them a very full and satisfactory theory of the subject of equations, in the practical solution of which improvements have recently been made, so comprehensive, that any considerable extension of them is scarcely to be expected, nor indeed likely to be required. In the present volume we shall endeavour to give, in moderate compass, a perspicuous view of the elements of this theory, and a clear exposition of the numerical process of solution; and, for the better accommodation of the student, we shall divide the subject into two parts; the first, comprehending all the more useful particulars, and the second, discussing certain details which have lately been entered into with success, and which are necessary to the satisfactory completion of the subject. To the second part will also be consigned several collateral enquiries which are usually expected to have a place in every treatise on the theory of equations.

It may be proper to mention here that in representing the different classes of equations, involving one unknown quantity, we shall usually employ the following notation, viz.

A simple equation,

$$Ax + N = 0.$$

A quadratic equation,

$$A_2 x^2 + Ax + N = 0.$$

A cubic equation,

$$A_3 x^3 + A_2 x^2 + Ax + N = 0.$$

A biquadratic equation,

$$A_4 x^4 + A_3 x^3 + A_2 x^2 + Ax + N = 0.$$

And, in general, an equation of the *n*th degree will be written,

$$A_n x^n \ldots . + A_3 x^3 + A_2 x^2 + Ax + N = 0;$$

in which the absolute term N, and the coefficients A, A_2, A_3, &c. represent real numbers, either positive or negative, integral or fractional. The polynomial on the left of the sign of equality we shall frequently call the *first side* or the *first member* of the equation.

(2.) It is common in algebraical enquiries, involving frequent reference to complicated expressions, to designate those expressions by some more brief and commodious form; and to facilitate this abridgment, a new word, the word *function*, has been introduced into algebra, and represented symbolically by the initial letter f, or F or ϕ, or f', or f_1, &c.

Thus any expression involving x, as, for instance, the left-hand member of either of the foregoing equations, is called, in brief, a *function of* x, and represented by one or other of the forms

$$f(x),\quad F(x),\quad \phi(x),\quad \psi(x),\quad f'(x),\quad F'(x),\quad f_1(x),\ \&c.$$

when, however, one of these forms is fixed upon to represent any algebraical expression, it is plain that, in order to avoid confusion, we must adhere to that form of representation throughout the enquiry; and must not employ the same form to characterize other expressions, or other functions.

If, for example, we agree to represent the foregoing general equation of the nth degree by $f(x) = 0$, we are not afterwards at liberty to represent any other different function, occurring in the same enquiry, by the characteristic f, any more than we are at liberty to denote two different magnitudes by one and the same algebraical character. We see, therefore, that while the term *function* has the most extended signification, comprehending all algebraical combinations possible, yet, by varying the form of the initial letter, or characteristic, which stands for the word, the various forms of functions may all be represented in the proposed notation by distinctive symbols.

The expression $f(x) = 0$, which we have just employed to denote, in short, the general equation of the nth degree, includes in it, of course, all the particular equations written above, as n may be any positive and integral exponent whatever. The symbol $f'(x)$ or $f_1(x)$, &c. denotes, as already remarked, a function of the same quantity, x, although different from the function $f(x)$; yet, as the preceding forms are derived from this last, by simply supplying an accent, or subscribed numeral, they are the forms usually employed to express functions derived from, or dependent on, a primitive function $f(x)$. For example, if the function $ax^6 + bx^5$ be represented by $f(x)$, and we have occasion to exhibit the successive quotients which arise from dividing this primitive function by x repeatedly, it would be convenient to use the following notation:

$$f(x) = ax^6 + bx^5$$
$$f_1(x) = ax^5 + bx^4$$
$$f_2(x) = ax^4 + bx^3$$
$$f_3(x) = ax^3 + bx^2$$
$$\&c. \qquad \&c.$$

where $f(x)$ is the primitive, and the others the derived functions, each being derived from the preceding, by a repetition of a known process, viz. the process of division by x. Again, suppose we had to deduce from the function, $3x^4 + 5x^3 - 2x^2 + 7x - 12$, a series of others in succession, by the following uniform process, viz. each term in the derived function is to be deduced from the corresponding term in the preceding function by multiplying that term by the exponent of x in it, and then diminishing the exponent by unity; the several functions would be as follows:

primitive function,	$f(x) = 3x^4 + 5x^3 - 2x^2 + 7x - 12$
1st derived function,	$f_1(x) = 12x^3 + 15x^2 - 4x + 7$
2nd derived function,	$f_2(x) = 36x^2 + 30x - 4$
3rd derived function,	$f_3(x) = 72x + 30$
4th derived function,	$f_4(x) = 72$

This last expression, 72, not containing x, cannot in strictness be regarded as a function of that quantity; its symbolical representation, however, $f_4(x)$, carrying the subscribed numeral 4, informs us that it has arisen from four repetitions of some uniform process to a primitive function, $f(x)$.

If in any function we change the quantity of which it is a function for any other, preserving however the form of the function unaltered, then we must introduce a like change in the abridged representation, merely altering the letter inclosed in the parenthesis, without changing the characteristic outside: thus, if $f(x)$ denote, as in the last example, then $f(y)$, $f(a)$, &c. will be the respective representatives of

$$3y^4 + 5y^3 - 2y^2 + 7y - 12, \quad 3a^4 + 5a^3 - 2a^2 + 7a - 12, \text{ \&c.}$$

(3.) The expression, *root of an equation*, is applied to every quantity which, when substituted for the unknown, x, in it, actually reduces the first member to zero, thus satisfying the condition implied in the equation; so that if there exist p quantities, which substituted for x in the polynomial $f(x)$, reduce it to zero, then the equation $f(x) = 0$ is said to have p roots.

The existence of at least one such root, real or imaginary, is necessarily assumed in the bare announcement of the condition

$$A_n x^n \; . \; . \; . \; . + A_3 x^3 + A_2 x^2 + Ax + N = 0;$$

and, therefore, in discussing the theory of existing equations, it seems unnecessary to *prove* that every such equation must have a root. It would, however, be too much to affirm without demonstration that whatever arbitrary values we assume for N, A, A_2, A_3, &c. the equation will always subsist, or, in other words, that a value exists for x, which will cause the polynomial $f(x)$, in every circumstance, to become zero, and the proposition has accordingly been submitted to rigorous investigation. But the demonstrations of its truth which have been given are not sufficiently simple for this place, we have, however, given *Cauchy's* proof in note A, at the end of the volume.

PART I.

CHAPTER I.

ON THE FUNDAMENTAL PROPERTIES OF EQUATIONS IN GENERAL.

PROPOSITION I.

(4.) If any two quantities be severally raised to the same power, the difference of those powers will always be divisible by the difference of the original quantities.

Let x and y represent any two quantities, then it is to be proved that $x^n - y^n$ is divisible by $x - y$.

In order to this, put z for $x - y$; then $x = z + y$, and, therefore,

$$x^n - y^n = (z + y)^n - y^n.$$

But, by developing $(z + y)^n$ by the binomial theorem, we have

$$(z + y)^n - y^n =$$

$$z^n + nz^{n-1}y + \frac{n(n-1)}{2}z^{n-2}y^2 + \ldots\ldots nzy^{n-1} + y^n - y^n =$$

$$z(z^{n-1} + nz^{n-2}y + \ldots\ldots ny^{n-1});$$

which is evidently divisible by the factor z. Hence, $x^n - y^n$ is divisible by $x - y$. If the division of $x^n - y^n$ by $x - y$ be actually performed, the quotient will be

$$x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + \ldots\ldots xy^{n-2} + y^{n-1};$$

which, when $x = y$, becomes nx^{n-1}, because the quotient will then consist of n terms each equal to x^{n-1}.

PROPOSITION II.

(5.) If a is a root of any equation

$$N + Ax + A_2 x^2 + A_3 x^3 + \dots A_n x^n = 0;$$

the first member of this equation is divisible by $x - a$, and conversely, if the first member is divisible by a factor of the form $x - a$, then is a a root of the equation.

First let a be a root of the equation; then, if a be substituted for x in the proposed, the polynomial will be reduced to zero; that is, we shall have

$$N + Aa + A_2 a^2 + A_3 a^3 + \dots A_n a^n = 0;$$

and, consequently, by transposition, we have for N the expression

$$N = - Aa - A_2 a^2 - A_3 a^3 - \dots - A_n a^n;$$

so that the proposed equation is the same as

$$\left.\begin{matrix} - Aa - A_2 a^2 - A_3 a^3 - \dots - A_n a^n \\ + Ax + A_2 x^2 + A_3 x^3 + \dots + A_n x^n \end{matrix}\right\} = 0;$$

or, as

$$A(x - a) + A_2 (x^2 - a^2) + A_3 (x^3 - a^3) + \dots A_n (x^n - a^n) = 0,$$

the several terms of which are, by the preceding proposition, divisible by $x - a$. Hence, when a is a root of the equation, the first member is divisible by $x - a$, so that $f(x)$ is of the form $(x - a)f_1(x)$.

Again, let the first member of the equation be divisible by $x - a$, giving a quotient which we shall represent by $f(x)$, then the polynomial will be of the form

$$f(x) = (x - a)f_1(x);$$

and, as this becomes zero, for $x = a$, it follows that a is a root of the equation $f(x) = 0$.

PROPOSITION III.

(6.) Every equation containing but one unknown quantity has as many roots as there are units in the number denoting its degree, that is, an equation of the nth degree has n roots.

Let an equation of the nth degree, which, for simplicity, we suppose to be freed from the coefficient of x^n by division, be represented by

$$f(x)=0,$$

then, since this equation has a root a, $f(x)$ is, by last proposition, of the form $(x-a) f_1(x)$, consequently,

$$(x-a) f_1(x)=0;$$

or, dividing by $x-a$,

$$f_1(x)=0,$$

and, since this equation has a root, a_2, $f_1(x)$ is, by last proposition, of the form $(x-a_2) f_2(x)$, consequently,

$$(x-a_2) f_2(x)=0;$$

or, dividing by $(x-a_2)$,

$$f_2(x)=0.$$

In like manner, this equation having a root a_3, it may be written

$$(x-a_3) f_3(x)=0,$$

whence, dividing by $(x-a_3)$,

$$f_3(x)=0,$$

and so on. These successive divisions being continued $n-1$ times, the proposed polynomial will at length be reduced to the single factor, or quotient, $x-a_n$, which is incapable of further decomposition. Hence, multiplying the several divisors and the final quotient together, we have, for the original polynomial, the form

$$f(x)=(x-a)(x-a_2)(x-a_3)\ .\ .\ .\ .\ (x-a_n),$$

which becomes zero for any one of the n conditions

$$x = a, \quad x = a_2, \quad x = a_3, \quad \ldots \quad x = a_n,$$

and, consequently, the equation has the n roots

$$a, \quad a_2, \quad a_3 \quad \ldots \quad a_n.$$

(7.) It must be observed, that all that this proposition proves is, that the left hand member of the equation of the nth degree, represented by $f(x) = 0$, may be continually depressed by division, and finally exhausted after n operations. The divisors are not necessarily different; any number, or indeed all of them, may be equal: so that when it is said that an equation of n dimensions has n roots, all that is meant is, that the polynomial is decomposable into n simple factors, equal or unequal, each containing a root, determined by equating that factor to zero. Actually to effect this decomposition in every case, and thus to exhibit the roots, is the most difficult, as well as the most important, problem in algebra. It will be considered in a future chapter. The inverse problem, however, viz. to determine the equation when the roots are given, is very readily performed, because, knowing the roots, we know the component factors of the polynomial. Thus, if it be required to form an equation whose roots shall be

$$a, \quad a_2, \quad a_3 \quad \ldots \quad a_n,$$

we shall merely have to multiply together the factors

$$x - a, \quad x - a_2, \quad x - a_3 \quad \ldots \quad x - a_n,$$

and to equate the result, $f(x)$, to zero. As a particular example in numbers, let it be required to form an equation whose roots shall be 2, -3, and 4. The polynomial, which constitutes the first member, will be the product of the factors

$$x - 2, \ x + 3, \ x - 4;$$

hence the equation having the proposed roots will be

$$x^3 - 3x^2 - 10x + 24 = 0.$$

PROPOSITION IV.

(8.) No equation can have a greater number of roots than there are units in the number denoting its degree; that is, an equation of the nth degree can have no more than n roots.

It is proved, in last proposition, that an equation of the nth degree has *as many* as n roots, or that the polynomial has n binomial divisors:

$$x-a, \quad x-a_2, \quad x-a_3 \quad \ldots \quad x-a_n.$$

It cannot, *after* division by all these, admit of another division by a factor, either equal or unequal, to any of them, because, by last proposition, these completely exhaust the polynomial. Nor can it *before* division by these admit of division by a binomial, $x-\alpha$, different from either; for if this were possible, calling the quotient $f_1(x)$, we should have

$$f(x)=(x-\alpha)\,f_1(x),$$

that is,

$$(x-a)\,(x-a_2)\,(x-a_3)\,\ldots\,(x-a_n)=(x-\alpha)f_1(x).$$

Now, for the value $x=\alpha$, the second member vanishes, because $\alpha-\alpha$ is zero, whereas the first member does not vanish, because α is not equal to any of the quantities

$$a, \quad a_2, \quad a_3 \quad \ldots \quad a_n;$$

hence the supposed division is impossible. But if α were a root, the division might be effected (prop. ii.), therefore α is not a root of the equation; hence the number of roots of an equation always equals, but never exceeds, the number denoting the degree of the equation: and this amounts to the same as saying that every equation is compounded (by multiplication) of as many simple equations

$$x-a=0, \quad x-a_2=0, \quad \&c.,$$

as there are units in the number denoting its degree, and no more.

PROPOSITION V.

(9.) Having one of the roots of an equation, given to determine the equation containing the remaining roots.

Let the original equation be

$$A_4 x^4 + A_3 x^3 + A_2 x^2 + Ax + N = 0 \quad . \; . \; . \; . \quad (1),$$

and the given root $x = a$. Then, dividing by the simple equation $x - a = 0$, containing this root, we have a result of the form

$$A'_3 x^3 + A'_2 x^2 + A' x + N' = 0 \quad . \; . \; . \; . \quad (2),$$

which equation must contain the other roots, since the first member is made up of all the factors which compose the original polynomial, with the exception of that involving the root a. Hence, if the first member of (2) be multiplied by the factor $x - a$, the first member of (1) will be reproduced. The result of this multiplication is

$$A'_3 x^4 + (A'_2 - a A'_3) x^3 + (A' - a A'_2) x^2 + (N' - a A') x - a N',$$

and, as this must be identical with the first member of (1), the coefficients of the like powers of x must be identical; that is, we must have the conditions

$$A'_3 = A_4$$

$$A'_2 - a A'_3 = A_3 \quad \therefore \quad A'_2 = A_3 + a A_4$$

$$A' - a A'_2 = A_2 \quad \therefore \quad A' = A_2 + a A'_2$$

$$N' - a A' = A \quad \therefore \quad N' = A + a A'.$$

(10.) It thus appears that the coefficients in the reduced equation (2), are successively obtained, by uniform steps, from the coefficients in the original equation (1). Hence this rule for finding the coefficients of the depressed equation, from knowing one root of the primitive, viz.

The coefficient of the first term in the depressed equation is always the same as that of the first term in the original; and the coefficient of the pth term in the depressed equation is always equal to the root times (viz. a times) the coefficient of the preceding term, plus the coefficient of the pth term in the proposed.

1. As an example, let one root of the equation

$$x^3 - 3x^2 - 10x + 24 = 0,$$

viz. $x = 2$, be given to determine the depressed equation involving the other roots.

Coefficients of the proposed,	1	-3	-10
		2	-2
The equation required . . .	x^2	$-x$	$-12 = 0$

2. Again, let one root of the equation

$$15x^5 - 19x^4 + 6x^3 + 15x^2 - 19x + 6 = 0,$$

viz. $x = \cdot 6$, be given to determine the biquadratic equation containing the other four roots.

Coefficients of the proposed,	15	-19	$+6$	$+15$	-19
		9	-6	$+0$	$+9$
Equation required . . .	$15x^4$	$-10x^3$	$+0x^2$	$+15x$	$-10 = 0$

This process obviously furnishes a very short method of dividing a polynomial of the form

$$A_n x^n + \ldots . A_2 x^2 + Ax + N,$$

by a binomial of the form $x \pm a$.*

* In the examples in the text, the polynomials are divisible by the binomial without remainder, a being a root; but in other cases it will be necessary to insert N in the upper row, and to continue the work up to it, in order to get the remainder.

The following additional exercises in this method are left for the student to perform.

3. One root of the cubic equation,

$$x^3 - 7x^2 + 36 = 0,$$

is found to be 3; required the other two roots?

Ans. 6 and -2.

4. One root of the cubic equation,

$$x^3 + x^2 - 16x + 20 = 0,$$

is -5; required the other two roots?

Ans. 2 and 2.

5. Two roots of the biquadratic equation,

$$x^4 - 3x^3 - 14x^2 + 48x - 32 = 0,$$

are 1 and 2; required the remaining roots?

Ans. 4 and -4.

6. The equation

$$9x^6 + 30x^5 + 22x^4 + 10x^3 + 17x^2 - 20x + 4 = 0$$

has two roots equal to -2; required the biquadratic containing the other four roots?

Ans. $9x^4 - 6x^3 + 10x^2 - 6x + 1 = 0$.

PROPOSITION VI.

(11.) To determine the forms of the functions which the coefficients in the general equation

$$x^n + A_{n-1}x^{n-1} + \dots A_3x^3 + A_2x^2 + Ax + N = 0,$$

are of the roots

$$a, \ a_2, \ a_3, \ a_4 \ \dots \ a_n.$$

The polynomial which forms the first member of the proposed equation being equal to the product

$$(x-a)\,(x-a_2)\,\ldots\,(x-a_n),$$

we shall arrive at it by performing the actual multiplication here indicated, and the result of this process will necessarily exhibit the formation of the several coefficients. This formation will be sufficiently evident after two or three steps of the multiplication:

$$\begin{array}{l}
x - a \\
x - a_2 \\
\hline
\end{array}$$

$$\begin{array}{r|l}
x^2 - a & x + a\,a_2 \\
- a_2 &
\end{array}$$

$$\begin{array}{l}
x - a_3 \\
\hline
\end{array}$$

$$\begin{array}{r|r|l}
x^3 - a & x^2 + a\,a_2 & x - a\,a_2\,a_3 \\
- a_2 & + a\,a_3 & \\
- a_3 & + a_2\,a_3 &
\end{array}$$

$$\begin{array}{l}
x - a_4 \\
\hline
\end{array}$$

$$\begin{array}{r|r|r|l}
x^4 - a & x^3 + a\,a_2 & x^2 - a\,a_2\,a_3 & x + a\,a_2\,a_3\,a_4 \\
- a_2 & + a\,a_3 & - a\,a_2\,a_4 & \\
- a_3 & + a_2\,a_3 & - a\,a_3\,a_4 & \\
- a_4 & + a\,a_4 & - a_2\,a_3\,a_4 & \\
 & + a_2\,a_4 & & \\
 & + a_3\,a_4 & &
\end{array}$$

hence, by continuing this process, we have, for the coefficients of the proposed equation, the values

$$A_{n-1} = -a - a_2 - a_3 - \ldots\ldots - a_n$$
$$A_{n-2} = aa_2 + aa_3 + a_2 a_3 + \ldots\ldots + a_{n-1} a_n$$
$$A_{n-3} = -aa_2 a_3 - aa_2 a_4 - \ldots\ldots - a_{n-2} a_{n-1} a_n$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$N = aa_2 a_3 a_4 \ldots\ldots a_n (-1)^n.$$

We infer, therefore, that in any equation in which the first term, or highest power of x, has the coefficient unity, the coefficient of the second term is equal to the sum of the roots with their signs changed; the coefficient of the third term is equal to the sum of the products of every two roots with their signs changed; the coefficient of the fourth term is equal to the sum of the products of every three roots with their signs changed; and so on: and the last term is equal to the product of all the roots with their signs changed.

Cor. 1. It follows from this, that, if the coefficient of the second term in any equation be 0, that is, if the term be absent, the sum of the positive roots must be equal to the sum of the negative roots.

Cor. 2. Every root of an equation is a divisor of the last term. It appears, moreover, that if one root of an equation be changed, every one of the coefficients will be changed.

Cor. 3. If the roots of an equation be all positive, it is plain, from the foregoing composition of the coefficients, that the terms will be alternately positive and negative; and if the roots be all negative, the terms will be all positive.

PROPOSITION VII.

(12.) If the signs of the alternate terms in an equation be changed, the signs of all the roots will be changed.*

Let

$$x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + A_{n-3} x^{n-3} + \&c. = 0 \ldots\ldots (1)$$

---

* The equation is understood to be complete; if any term be absent it must be replaced by a cipher.

be any equation, and a a root; then, if a be substituted for x in the first member, the result will be zero; and if we change the alternate signs, writing the equation thus,

$$x^n - A_{n-1}x^{n-1} + A_{n-2}x^{n-2} - A_{n-3}x^{n-3} + \&c. = 0 \quad . \; . \; . \; . \quad (2),$$

and instead of a substitute $-a$ for x, the result, should n be even, will be the very same as before, and consequently zero; but if n be odd, then the result will differ from what it was before only in this, viz. that all the signs merely of the polynomial will be changed, so that as it was zero before, it must be zero still. Hence, for every root a in (1), there is an equal root with contrary sign, $-a$, in (2).

It is obvious that if the signs of *all* the terms are changed, the roots remain unaltered, because whatever values of x cause the polynomial to become zero in one case, make it zero in the other also.

PROPOSITION VIII.

(13.) If all the coefficients in an equation, whose leading term has the coefficient unity, be whole numbers, that equation cannot have a fractional root.

Let, if possible, the equation

$$x^n + A_{n-1}x^{n-1} + \; . \; . \; . \; . \; A_3x^3 + A_2x^2 + Ax + N = 0,$$

whose coefficients are integral, have a fractional root, and let the fraction in its lowest terms be $\frac{r}{s}$; then

$$\left(\frac{r}{s}\right)^n + A_{n-1}\left(\frac{r}{s}\right)^{n-1} + \; . \; . \; . \; . \; A_3\left(\frac{r}{s}\right)^3 + A_2\left(\frac{r}{s}\right)^2 + A\frac{r}{s} + N = 0,$$

or, multiplying by s^{n-1},

$$\frac{r^{n-1}}{s} + A_{n-1}r^{n-1} + \; . \; . \; . \; . \; A_3r^3s^{n-4} + A_2r^2s^{n-3} + Ars^{n-2} + Ns^{n-1} = 0.$$

Now in this polynomial every term after the first is integral; hence, transposing these to the other side, the first term must be integral.

But, $\frac{r}{s}$ being in its lowest terms, r and s have no common factor; and it is obvious that there can be no simple factors in r^{n-1}, which are not also in r; hence r^{n-1} and s can have no common factor, consequently $\frac{r^{n-1}}{s}$ is also a fraction in its lowest terms, and yet it is integral, which is absurd; therefore the proposed equation cannot have a fractional root. Hence, when the coefficients are whole numbers, the root of the equation must either be a whole number, or else an interminable decimal.

PROPOSITION IX.

(14.) Every equation has an even number of impossible roots, or else none at all.

For let the equation contain one impossible root, as $a + \sqrt{-\beta^2}$, then, substituting this expression for x in the first member of the equation

$$N + Ax + A_2 x^2 + \dots A_n x^n = 0,$$

it becomes

$$N + A(a + \sqrt{-\beta^2}) + A_2(a + \sqrt{-\beta^2})^2 + \dots A_n(a + \sqrt{-\beta^2})^n = 0.$$

Now it is obvious, that if the several terms in the first member be developed, we shall have a series of monomials, of which all those will be imaginary which involve odd powers of $\sqrt{-\beta^2}$, and the others will be real. Hence, this first member consisting of a real and imaginary part, the imaginary factor being always $\sqrt{-\beta^2}$, may be written in short, thus:

$$P + \phi\sqrt{-\beta^2} = 0 \dots (1),$$

$$\therefore P = -\phi\sqrt{-\beta^2},$$

an equation which exists only under the conditions

$$P = 0, \quad \phi = 0 \dots (2),$$

otherwise we should have the absurdity of a real quantity equalling an imaginary one.

If, instead of $a + \sqrt{-\beta^2}$, we had substituted $a - \sqrt{-\beta^2}$, in the proposed, the only difference in the result would have been that the terms involving the odd powers of $\sqrt{-\beta^2}$ would have had contrary signs, because the developments of $(p+q)^m$ and of $(p-q)^m$ differ only in the signs of the terms involving odd powers of q. Hence, consistently with our former notation, the result of this new substitution for x would have been

$$P - \phi\sqrt{-\beta^2},$$

which result, as well as the former (1), is zero, seeing that the former involves the conditions (2). The first member of the proposed is therefore divisible by the two simple factors

$$x - a - \sqrt{-\beta^2}$$

$$x - a + \sqrt{-\beta^2},$$

or by their product

$$x^2 - 2ax + a^2 + \beta^2;$$

and the resulting polynomial put equal to zero, will be the depressed equation containing the remaining roots of the proposed equation. Now as the original polynomial involves, by hypothesis, no imaginary term, the quotient of the division to which we have adverted can involve no imaginary term; hence, if the depressed equation contain one imaginary root, $b + \sqrt{-\gamma^2}$, it follows, from the foregoing investigation, that it must also contain another imaginary root, $b - \sqrt{-\gamma^2}$. Consequently, imaginary roots always enter into equations in pairs of the form

$$a + \sqrt{-\beta^2}$$

$$a - \sqrt{-\beta^2}.$$

(15.) It is obvious that the quadratic factor

$$x^2 - 2ax + a^2 + \beta^2,$$

must always give a positive result, whatever real value we put for x in it; for $2ax$ can never exceed $a^2 + x^2$, since, if $a = x$, the two expressions are but equal; and, if $a = x \pm p$, the first expression is $2x^2 \pm 2px$, and the second $2x^2 \pm 2px + p^2$, exceeding the former by p^2.

By a similar mode of reasoning to that employed in the proposition, it may be proved that roots of the form $a + \sqrt{\gamma}$ enter equations, whose terms are rational, in pairs.

Pairs of roots entering equations under the forms here noticed, are frequently called *conjugate roots*.

Cor. 1. An equation of an even degree may have all its roots impossible; but, if they are not all impossible, two of them at least must be possible.

Cor. 2. Since the product of every pair of conjugate impossible roots is of the form $a^2 + \beta^2$, β^2 being positive, it follows that when all the roots are impossible, their product is essentially positive, and hence, in such a case, the absolute term N in the equation must be positive, (prop. VI.)

Cor. 3. Hence, every equation of an odd degree has at least one real root of a contrary sign to that of the last term; and every equation of an even degree, whose last term is negative, has at least two real roots with contrary signs.

NOTE. It must not be concealed from the student that our supposition of $a + \sqrt{-\beta^2}$, as the form which the impossible root of an equation must take, is an assumption. The proof that every impossible root necessarily comes under this form cannot be conveniently given in this place, but it will be found in the second part.

PROPOSITION X.

(16.) No equation can have a greater number of positive roots than there are changes of sign from + to —, and from — to +, in the terms of its first member, nor can it have a greater number of negative roots than of permanencies, or successive repetitions of the same sign.

To demonstrate this remarkable proposition, it will be necessary merely to show that, if any polynomial, whatever be the signs of its terms, be multiplied by a factor $x - a$, corresponding to a *positive* root, the resulting polynomial will present at least one more *change of sign* than the original; and that if it be multiplied by $x + a$, corresponding to a *negative* root, the result will exhibit at least one more *permanence of sign* than the original.

Suppose the signs of the proposed polynomial to succeed each other in any order, as

$$+ - - + - + + + - - +,$$

then the multiplication of the polynomial, by $x - a$, will give rise to two rows of terms, which, added vertically, furnish the product. The first row will, obviously, present the very same series of signs as the original, and the second, arising from the multiplication by the negative term $- a$, will present the same series of signs as we should get by changing every one of the signs of the first row. In fact, the two rows of signs would be

$$\begin{array}{l} + - - + - + + + - - + \\ \quad - + + - + - - - + + - \\ \hline \end{array}$$

and signs of prod. $+ - \pm + - + \pm \pm - \pm + -.$

We have written the ambiguous sign $\pm$ in the product when the addition of unlike signs in the partial products occurs, and it is very plain that these ambiguities will, in this and in every other arrangement, be just as numerous as the permanencies in the proposed; thus, in the present arrangement, the proposed furnishes four permanencies, viz. $- -$, $+ +$, $+ +$, $- -$, and there are, accordingly, in the product four ambiguities, the other signs remaining the same as in the proposed, with the exception of the final sign, which is superadded, and which is always contrary to the final sign in the proposed.

It is an easy matter, therefore, when the signs of the terms of any polynomial are given to write down immediately the signs in the product of that polynomial, by $x - a$, as far, at least, as these signs are determinable without knowing the values of the quantities employed; for we shall merely have to change every permanency in the proposed into a sign of ambiguity, and to superadd the final sign changed. For instance, if the proposed arrangement were

$$+ - + + - - - + - + + +,$$

the signs of the product would be

$$+ - + \pm - \pm \pm + - + \pm \pm -.$$

Again, if the signs in the proposed were in the order

$$+ + + - + - + - - -,$$

the signs in the product would be in the order

$$+ \pm \pm - + - + - \pm \pm +.$$

As therefore in passing from the multiplicand to the product, it is the *permanencies* only of the former that can suffer any change, it is impossible that the *variations* can ever be diminished, however they may be increased. Consequently the most unfavorable supposition for our purpose is, that the permanencies (omitting the superadded sign,) remain the same in number, and, in this case, if the proposed terminate with a variation, the superadded sign in the product will introduce another variation; but if it terminate with a permanency, then the corresponding ambiguity in the result will obviously, substitute for it what sign we will, form a variation, either with the preceding or with the superadded sign. It follows, therefore, that no equation can have a greater number of positive roots than variations of sign.

To demonstrate the second part of the proposition it will suffice to remark that, if we change all the signs in an equation, we change the roots from positive to negative, and vice versâ, (prop. VII.) The equation thus changed would have its permanencies replaced by variations, and its variations by permanencies, and, since by the foregoing, the changed equation cannot have a greater number of positive roots than variations, the proposed cannot have a greater number of negative roots than permanencies.

(17.) This proposition constitutes the *rule of Descartes*, and serves to point out limits which the number of the positive and negative roots of an equation can never exceed. It does not, however, furnish us with the means of ascertaining how many real roots, of either kind, any proposed equation may involve; nor indeed does it enable us to affirm that even one positive or negative root actually exists in any equation; it merely shows that *if* real roots exist, those which are positive, or those which are negative cannot exceed a certain number; they may, however, fall greatly short of this number, and, indeed, be all imaginary. But the rule is not without its use, even in detecting imaginary roots, as it sometimes discovers discrepancies incompatable with the existence of real roots, in those equations which are incomplete, or have terms wanting. For example, suppose we wished to ascertain the nature of the roots of the cubic equation

$$x^3 + Ax + N = 0,$$

in which A and N are positive. Putting the equation in a complete form, it is

$$x^3 \pm 0x^2 + Ax + N = 0.$$

Now, when we take the second term, $+$, there are no variations, so that there can be no positive roots; but, when we take the same term, $-$, there is only one permanence, so that there cannot be more than one negative root; these conclusions would be contradictory if the roots were real, we therefore infer that the proposed has a pair of imaginary roots.

If the equation had been

$$x^3 - Ax + N = 0,$$

we could not have pronounced any thing respecting the nature of the roots from the application of the *rule of signs;* for, supplying the absent term, we have

$$x^3 \pm 0x^2 - Ax + N = 0;$$

which presents one permanence and two variations, whichever sign we give to the second term; so that all we can affirm is, that *if* the roots are real, two must be positive and one negative. Two roots, however, *may* be imaginary, in which case the third will be negative, because the sign of N is positive, (prop. IX. cor. 3.)

(18.) Unfailing criteria for the detection of imaginary roots will be given in the seventh chapter; but we may remark here, that when we know beforehand that an equation contains none but real roots, then the rule of Descartes will enable us to ascertain exactly the number of each kind, as may be readily proved as follows:

Let n be the degree of the equation, p the number of permanencies, and v the number of variations, then $n = p + v$. Let also p' be the number of negative roots, and v' the number of positive roots, then

$n = p' + v'$, whence

$$p + v = p' + v'.$$

Now it is proved above that p' cannot be greater than p, nor can v' be greater than v; hence, necessarily,

$$p = p' \text{ and } v = v';$$

consequently, when the roots are all real, the number of positive roots will be equal to the number of variations, and the number of negative roots equal to the number of permanencies.*

CHAPTER II.

ON THE TRANSFORMATION OF EQUATIONS.

(19.) Algebraical equations do not always present themselves in the most convenient forms for solution, and hence the expediency of being provided with the means of changing them from one form to another. Depriving the leading term of its coefficient, by division or otherwise, is the most simple change of this kind, and is a desirable preparative to the usual methods of solution, as it simplifies the form without affecting the roots of the equation. In most transformations, however, the roots themselves become also changed, but still bear such known and simple relations to those of the original equation, as to render the determination of these latter from them an easy operation. Generally indeed, to change the roots into others bearing given relations to them,

* The rule of Descartes has recently received considerable extension, both from *Fourier* and *Budan*, in their researches into the theory of equations. These will be more particularly adverted to in the fourth chapter.

is the direct object of the transformation; so that this part of the subject, in its full extent, involves the solution of the following comprehensive problem, viz. To transform an equation into another such that the roots of the latter shall be any given functions of those of the former; but we have no occasion to enter upon the investigation of so general a problem here, our attention at present being confined to those transformations which are useful or necessary in the actual solution of equations, and which may be comprised in the four propositions following:

PROPOSITION I.

(20.) To transform an equation into another whose roots shall differ, either in excess or defect, from the roots of the original by any given quantity.

Let us suppose that the original equation is

$$A_4 x^4 + A_3 x^3 + A_2 x^2 + Ax + N = 0 \;.\;.\;. \quad (1),$$

and that we wish to transform it into another whose roots shall be the same in number, but shall differ from them in magnitude each by the quantity r; then the relation between the x in the original equation and the x' in the transformed, will be

$$x = x' + r,$$

in which r will be plus or minus, according as the new roots, or values of x' are to differ from the original roots, or values of x, in defect or in excess. Substituting, therefore, this value of x in the original, we shall obviously have the transformed equation, which will be of the form

$$A_4 x'^4 + A'_3 x'^3 + A'_2 x'^2 + A' x' + N' = 0 \;.\;.\;.\;. \; (2).$$

Now, if instead of x', we put its value $x - r$ in this equation, we shall have

$$A_4 (x-r)^4 + A'_3 (x-r)^3 + A'_2 (x-r)^2 + A' (x-r) + N' = 0 \,.\, (3);$$

an equation which, when reduced to a series of monomials by actually developing the terms, must be identical with the original, for, in fact, we have now returned from (2) to (1), by restoring to x' its value $x - r$. Hence we have the identity

$$A_4(x-r)^4 + A'_3(x-r)^3 + A'_2(x-r)^2 + A'(x-r) + N' =$$
$$A_4x^4 + A_3x^3 + A_2x^2 + Ax + N;$$

which plainly shows that if we divide the first member by $x - r$, the remainder must be N'; but, the two members being identical, the division of either by $x - r$ must give the same remainder, and the same quotient. The division, therefore, of the second member, that is of the original polynomial, by $x - r$, gives, for remainder N', and for quotient,

$$A_4(x-r)^3 + A'_3(x-r)^2 + A'_2(x-r) + A'.$$

Also, dividing this by $x - r$, we have for remainder A', and for quotient,

$$A_4(x-r)^2 + A'_3(x-r) + A'_2.$$

Again, dividing this by $x - r$, the remainder becomes A'_2, and quotient,

$$A_4(x-r) + A'_3.$$

And lastly, dividing this by $x - r$, we have, for the final remainder, A'_3; and, for the final quotient, A_4; and in this manner may the coefficients in the transformed equation (2) be severally determined.

Now we have exhibited at (10) a very easy way of performing the division of a polynomial, $f(x)$, by such a divisor as $x - r$; and, by employing that method in the present problem, the required transformation may always be rapidly effected, as the following examples will show.

1. Transform the equation

$$x^4 + 5x^3 + 4x^2 + 3x - 105 = 0,$$

into another, whose roots shall be less than those of the proposed, by 2. Here the constant divisor is $x - 2$, and the process directed by the above investigation, and conducted according to the plan at (10), will be as follows:

$$
\begin{array}{llllll}
A_4 & A_3 & A_2 & A & N & \\
1 + & 5 + & 4 + & 3 - & 105 & (2 = r \\
 & 2 + & 14 + & 36 + & 78 & \\
\hline
1 + & 7 + & 18 + & 39 - & 27 & \therefore N' = -27 \\
 & 2 + & 18 + & 72 & & \\
\hline
1 + & 9 + & 36 + & 111 & \therefore A' = 111 & \\
 & 2 + & 22 & & & \\
\hline
1 + & 11 + & 58 & \therefore A'_2 = 58 & & \\
 & 2 & & & & \\
\hline
1 + & 13 & \therefore A'_3 = 13. & & &
\end{array}
$$

Hence the transformed equation is

$$x'^4 + 13x'^3 + 58x'^2 + 111x' - 27 = 0.$$

(21.) After what has been done in Proposition V. p. 11, it is presumed that the student will require no verbal explanation of the foregoing process. It will no doubt be sufficient to remark that, calling the numbers below the black lines *results*, each result is formed by adding r times the result immediately before it to the result immediately above it. We may observe, however, that the operation would be somewhat abbreviated by omitting the repetition of the first coefficient in the commencement of each row of results, by suppressing the plus signs, and by reserving the determinations of A'_3, A'_2, A', and N', till we come to the last result, thus:

1	5	4	3	— 105 (2
	2	14	36	78
	7	18	39	
	2	18	72	
	9	36		
	2	22		
	11			
	2			

$$x'^4 + 13x'^3 + 58x'^2 + 111x' - 27 = 0.$$

By suppressing also the several addends, and performing the addition operations mentally, we should, of course, abridge the space occupied by the process, very considerably. The whole would then, in fact, be reduced to this, viz.

1	5	4	3	— 105 (2
	7	18	39	— 27
	9	36	111	
	11	58		
	13			

Other means might be easily contrived for shortening the apparent work, but we would recommend to the student the exhibition of the entire process rather than incur the risk of error by suppressing any of the steps. When r is 1, then indeed, as there is no effective multiplication, the process naturally takes the form here given, as in the following example.

2. It is required to transform the equation

$$2x^4 - 13x^2 + 10x - 19 = 0,$$

into another, whose roots shall be less than the roots of this equat by 1.

$$\begin{array}{llll} 2 & 0-13 & 10-19\ (1 & \\ & 2-11- & 1-20 & \therefore\ N'=-20 \\ & 4-\ \ 7- & 8 & \therefore\ A'=-8 \\ & 6-\ \ 1 & \therefore\ A'_2=-1 & \\ & 8 & \therefore\ A'_3=8. & \end{array}$$

Hence, the transformed equation is

$$2x'^4+8x'^3-x'^2-8x'-20=0.$$

3. It is required to transform the preceding equation into anot whose roots are less by 3.

$$\begin{array}{rrrrr} 2 & 0 & -13 & 10 & -19 \\ & 6 & 18 & 15 & 75 \\ \hline & 6 & 5 & 25 & \\ & 6 & 36 & 123 & \\ \hline & 12 & 41 & & \\ & 6 & 54 & & \\ \hline & 18 & & & \\ & 6 & & & \end{array}$$

trans. equa. $2x'^4+24x'^3+95x'^2+148x'+56=0.$

4. It is required to transform the equation

$$6x^3-3x^2+4x-1=0,$$

into another, whose roots shall exceed the roots of this by 3.

Here the multiplier will be -3.

6	− 3	4	− 1	(− 3
	− 18	63	− 201	
	− 21	67		
	− 18	117		
	− 39			
	− 18			

trans. equation, $6x'^3 - 57x'^2 + 184x' - 202 = 0$.

5. Transform the equation

$$x^3 - 7x + 7 = 0,$$

into one whose roots shall be less than the roots of this by 2.

The transformed equation is $x^3 + 6x^2 + 5x + 1 = 0$.

6. Transform the equation

$$19x^4 - 22x^3 - 35x^2 - 16x - 2 = 0,$$

into another, in which the roots shall be diminished by 3.
The transformed equation is $19x^4 + 206x^3 + 793x^2 + 1232x + 580 = 0$.

7. Transform the equation

$$3x^4 - 13x^3 + 7x^2 - 8x - 9 = 0,$$

into another, whose roots shall each be smaller than those of the proposed by $\frac{1}{3}$.

The transformed equation is $3x^4 - 9x^3 - 4x^2 - \frac{65}{9}x - \frac{102}{9} = 0$.

PROPOSITION II.

(22.) To transform an equation into another whose roots shall be the reciprocals of those of the former.

In the proposed equation

$$N + Ax + A_2 x^2 + A_3 x^3 + \dots A_n x^n = 0,$$

substitute $\frac{1}{y}$ for x, then the values of $\frac{1}{y}$ will be the same as those of x, and, consequently, the values of y will be the reciprocals of those of x; that is, the roots of the equation

$$N + \frac{A}{y} + \frac{A_2}{y^2} + \frac{A_3}{y^3} + \dots \frac{A_n}{y^n} = 0,$$

or, rather of

$$Ny^n + Ay^{n-1} + A_2 y^{n-2} + A_3 y^{n-3} + \dots A_n = 0,$$

will be the reciprocals of the roots of the proposed equation. Hence the transformed equation is deduced from the original, simply by reversing the order of the coefficients; as many terms, therefore, as are absent in the original equation, so many and no more will be absent in the transformed.

Cor. 1. Hence we may transform an equation into another, whose roots shall be less or greater than the reciprocals of those of the proposed, by applying the process employed in last proposition to the coefficients of the given equation, written in reverse order. For example, let it be required to transform the equation

$$x^4 - 12x^2 + 12x - 3 = 0,$$

into another, whose roots shall be equal to the reciprocals of those of the given equation, diminished by 1.

− 3	12	− 12	0	1 (1
	9	− 3	− 3	− 2
	6	− 3	0	
	3	6		
	0.			

Hence, the transformed equation is

$$-3y^4 \pm 0y^3 + 6y^2 \pm 0y - 2 = 0;$$

or rather

$$3y^4 - 6y^2 + 2 = 0.$$

Cor. 2. If the coefficients of the proposed equation be the same when taken in reverse order, as when taken in direct order, it is obvious, from the foregoing investigation, that the reciprocals of the roots will furnish the same series of quantities as the roots themselves, seeing that the equation which involves the reciprocal roots will be the same as the original equation; the roots of the original equation must, therefore, under such circumstances, be of the form

$$a, \frac{1}{a}; \ a_2, \frac{1}{a_2}; \ a_3, \frac{1}{a_3}; \ \&c.$$

of which the reciprocals produce the same series of quantities.

If the equation be of an odd degree, and the coefficients taken in reverse order, be in magnitude the same as when taken in direct order, but in signs all different, then also will the roots of the transformed equation be identical with those of the original equation; for, by changing all the signs of the transformed equation, which of course produces no change in the roots, the equations will become the same as the original, and must, therefore, have the same roots. The same thing evidently has place in equations of an even degree, under like circumstances, provided only the middle term be absent.

Equations whose coefficients exhibit this law, and whose roots are, in consequence, of the above form, are called *recurring equations*, or *reciprocal equations*.

Cor. 3. In a recurring equation of an odd degree, one root will always be $+ 1$ or $- 1$, according as the sign of the last term is $-$ or $+$, for, as the roots of the transformed are always the same as those of the original in recurring equations, and yet at the same time the roots of the transformed are the reciprocals of those of the original, one of the odd number of roots must be $+ 1$, or $- 1$; moreover, as the remaining roots consist of pairs, having the same sign, the last term of the equation, which is the product of all the roots with their signs changed, must take the opposite sign to the root unity.

PROPOSITION III.

(23.) To transform an equation into another, whose roots shall be given multiples or submultiples of those of the proposed equation.

Let the given equation be freed by division from the coefficient of the first term;* then, in the resulting equation, the coefficient of the second term will be the sum of the roots with contrary signs; the next coefficient, the sum of the products, two and two; the next, the sum of the products, three and three, signs being changed, and so on (prop. vi. p. 15): hence, for the roots to be m times as great, we must multiply the second term by m, the third by m^2, the fourth by m^3, and so on. If, for example, it be required to transform the equation

$$2x^3 - 5x^2 + 7x - 12 = 0,$$

into another, whose roots are three times as great, we shall merely have to multiply the second term by 3, the third by 9, and the fourth by 27; the transformed will therefore be

$$2x^3 - 15x^2 + 63x - 324 = 0.$$

Cor. 1. If in an equation the coefficients of the second, third, fourth, &c. terms be divisible by m, m^2, m^3, &c., respectively, the roots will have the common measure m.

Cor. 2. By this transformation the coefficient of the first term of an equation may be removed without introducing fractions; for, if m be the coefficient of the first term, and we transform the equation into another, whose roots are m times those of the former, we shall introduce the factor m into all the terms; dividing by it will therefore free the first term, and introduce no fractions. The transformed equation

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* It is necessary to say freed by *division*, in order that the roots may be preserved unaltered. The present proposition furnishes other means of removing the first coefficient, but not without changing the roots.

will therefore be obtained by expunging the coefficient of the first term, preserving the second term, multiplying the third by m, the fourth by m^2, &c. and the roots of the transformed will be m times those of the original. Thus, taking the equation

$$3x^3 - 5x + 2 = 0,$$

which, completed, is

$$3x^3 + 0x^2 - 5x + 2 = 0;$$

we have for the transformed, whose roots are three times as great, the equation

$$x^3 + 0x^2 - 15x + 18 = 0,$$

or, rather

$$x^3 - 15x + 18 = 0.$$

Fractions may be removed from an equation by transforming the equation into another, whose roots are those of the former, multiplied by the product of the denominators of the fractions. For example, the equation $x^3 + \frac{1}{2} x^2 - \frac{1}{3} x + 2 = 0$, will be transformed into $x^3 + 3x^2 - 12x + 432 = 0$, by multiplying the terms, commencing at the second, by the successive powers of 6; and, if the roots of the former equation be a, a_2, a_3, those of the latter will be $6a$, $6a_2$, $6a_3$.

PROPOSITION IV.

(24.) To transform an equation into another, in which any proposed term shall be absent.

If the transformed equation is to be deprived of its *second* term, which is the term generally required to be removed, the transformation may be effected by the process in Problem I. p. 24, as it will be merely required to diminish the roots by such a quantity, r, as will cause the second coefficient in the resulting equation to vanish. Now, in the process of diminishing the roots, it is seen that r is added to the second term n times, so that for the result of these additions to be zero, r must be minus the nth part of the second coefficient in the proposed equation. To illustrate this, let it be required to remove the second term from the equation

$$x^4 - 12x^3 + 17x^2 - 9x + 7 = 0.$$

Here $r = \frac{12}{4} = 3$, and the operation is as follows:

$$\begin{array}{ccccc}
1 & -12 & 17 & -9 & 7\ (3 \\
 & 3 & -27 & -30 & -117 \\
\hline
 & -9 & -10 & -39 & \\
 & 3 & -18 & -84 & \\
\hline
 & -6 & -28 & & \\
 & 3 & -9 & & \\
\hline
 & -3 & & & \\
 & 3 & & & \\
\hline
x^4 & +\ 0x^3 & -\ 37x^2 & -\ 123x & -\ 110 = 0.
\end{array}$$

hence the transformed equation is

$$x^4 - 37x^2 - 123x - 110 = 0,$$

the roots of which are those of the proposed diminished by 3.

(25.) But in order to determine the value of r, necessary to cause any other coefficient to vanish, let us actually substitute $x' + r$ for x, in the general equation

$$x^n + A_{n-1}\, x^{n-1} + \ldots\ Ax + N = 0,$$

and develope the several powers by the binomial theorem, arranging the result according to the decreasing powers of x'; we shall thus have

$$\begin{array}{l|l|l|l}
x'^n + nr & x'^{n-1} + \frac{n(n-1)}{2} r^2 & x'^{n-2} + \dots\dots + r^n & = 0 \\
+ A_{n-1} & + (n-1) A_{n-1} r & + A_{n-1} r^{n-1} & \\
& + A_{n-2} & + A_{n-2} r^{n-2} & \\
& & \dots\dots & \\
& & \dots\dots & \\
& & + Ar & \\
& & + N &
\end{array}$$

In order that the second term of this transformed equation may vanish, we must have the condition

$$nr + A_{n-1} = 0 \therefore r = -\frac{A_{n-1}}{n},$$

as before determined.

That the third term may vanish, we must have the condition

$$\frac{n(n-1)}{2} r^2 + (n-1) A_{n-1} r + A_{n-2} = 0;$$

which, being a quadratic equation, will furnish two values for r, each of which will cause the third term in the transformed equation to vanish.

The determination of values for r, that will cause the fourth term to vanish, will require the solution of an equation of the third degree, and, to remove the last term N, would require the solution of the following equation of the nth degree in r, viz. the equation

$$r^n + A_{n-1} r^{n-1} + \dots\dots Ar + N = 0;$$

which is no other than the proposed, x being replaced by r, so that the removal of the last term requires a preparatory process, equivalent to solving the original equation. But the removal of any term, other than the second, is an operation of little or no use in the solution of equations.

By removing the second term from a quadratic equation, we shall be immediately conducted to the well-known formula for its solution. Thus, the equation being

$$x^2 + Ax + N = 0,$$

the transformed in $x' + r$, will be

$$\left.\begin{array}{l|l} x'^2 + 2r & x' + r^2 \\ \quad + A & \quad + Ar \\ & \quad + N \end{array}\right\} = 0;$$

and, that its second term may vanish, we must have

$$2r + A = 0 \therefore r = -\tfrac{1}{2} A,$$

which condition reduces the transformed to

$$x'^2 - \tfrac{1}{4} A^2 + N = 0$$

$$\therefore x' = \pm \sqrt{\tfrac{1}{4} A^2 - N}$$

$$\therefore x = x' + r = -\tfrac{1}{2} A \pm \sqrt{\tfrac{1}{4} A^2 - N};$$

which is the common formula for the solution of a quadratic equation.

CHAPTER III.

ON THE LIMITS OF THE ROOTS OF EQUATIONS.

(26.) Limits to a root of an equation are any two numbers between which that root lies. The extreme values 0 and $\frac{1}{0}$ are obviously limits to every positive root in any equation, and the values 0 and $-\frac{1}{0}$ are limits to every negative root. But in order to evolve the numerical value of the root in the higher equations, we must be prepared with much narrower limits than these, so narrow indeed that the inferior limit may furnish the first figure of the root itself. To speak of numerical limits to imaginary roots would of course be absurd, as an imaginary quantity can have no intermediate value between two real ones; in the following propositions, therefore, it will be remembered that the roots spoken of are always the real roots.

A superior limit to the positive roots of an equation, $f(x)=0$, is any positive number which is greater than the greatest root of the equation, and consequently its character is that when it, or any number greater than it, is substituted for x in the polynomial $f(x)$, the result will always be too great, that is, always positive; and an inferior limit to the negative roots is any negative number which (abstracting from the sign,) numerically exceeds the greatest negative root, and consequently its character is that when it, or any greater number with the same sign, is substituted for x in $f(x)$, the result is always negative.

PROPOSITION I.

(27.) In any equation whose second term is negative, and all the other terms positive, the coefficient of the second term taken positively, is a superior limit to the positive roots.

Let the equation be

$$x^n - A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \ldots . Ax + N = 0,$$

then, since

$$x^n - A_{n-1} x^{n-1} = (x - A_{n-1}) x^{n-1},$$

the equation may be written thus, viz.

$$(x - A_{n-1}) x^{n-1} + A_{n-2} x^{n-2} + \ldots . Ax + N = 0;$$

in which, if A_{n-1} be substituted for x, the first term will vanish, and, all the other terms being positive, the result of the substitution must be positive. If any other quantity, greater than A_{n-1}, be substituted for x, then the first term, as well as all the others, will be positive; hence, A_{n-1} is a superior limit to the positive roots.

PROPOSITION II.

(28.) In any equation the greatest negative coefficient, increased by unity, is a superior limit to the positive roots.

Let A_m be the greatest negative coefficient in the equation, and call it $-q$, and suppose all the coefficients to be made negative, and equal to $-q$; then the equation will be changed into

$$x^n - qx^{n-1} - qx^{n-2} - \ldots . - qx - q = 0,$$

and it is clear that whatever substitutions for x give a positive result, here the same must give a positive result in the original.

Writing our changed equation in the form

$$x^n - q(x^{n-1} + x^{n-2} + \ldots . + x + 1) = 0,$$

and, summing the geometrical series within the brackets, we have (Alg. page 92,)

$$x^n - q\left\{\frac{x^n - 1}{x - 1}\right\} = 0.$$

In the first member of this, substitute $q+1$ for x, and the result will be

$$(q+1)^n - q\left\{\frac{(q+1)^n - 1}{q}\right\} = (q+1)^n - (q+1)^n + 1 = 1,$$

a positive quantity. If we substitute for x, a quantity greater than $q+1$, as s, then

$$s^n - q\left\{\frac{s^n - 1}{s - 1}\right\} > 1,$$

for

$$s^n - (s^n - 1) = 1,$$

and, by the supposition,

$$s^n - 1 > q\left\{\frac{s^n - 1}{s - 1}\right\},$$

inasmuch as $q < s - 1$.

Hence $q+1$ is such a quantity, that when it, or any number greater than it, is substituted for x in the equation, the result will always be positive, and therefore $q+1$ is a superior limit to the positive roots.

PROPOSITION III.

(29.) In any equation of the nth degree, if x^{n-s} be the power involved in the first negative term, and $-P$ be the greatest negative coefficient, then will $P^{\frac{1}{s}} + 1$ be a superior limit to the positive roots.

Let us take the most unfavorable case, viz. that in which all the terms, from the one involving x^{n-s}, are negative, and affected with the coefficient P, and let us endeavour to satisfy the condition

$$x^n > Px^{n-s} + Px^{n-s-1} + \ldots + Px + P \ldots (1).$$

Suppose we first try $x^s = P$, or $x = P^{\frac{1}{s}}$. The first member of the inequality will become $P^{\frac{n}{s}}$, and the second member, when put under the more convenient form

$$\frac{Px^n}{x^s} + \frac{Px^{n-1}}{x^s} + \dots \qquad \dots (2),$$

will become

$$P^{\frac{n}{s}} + P^{\frac{n-1}{s}} + \dots$$

which exceeds $P^{\frac{n}{s}}$, therefore $x = P^{\frac{1}{s}}$ does not satisfy the inequality. Again, make $x = P^{\frac{1}{s}} + 1 = P' + 1$, then the first member of (1) is $(P' + 1)^n$, and the second member, which, when summed, is

$$P\frac{x^{n-s+1} - 1}{x - 1},$$

becomes

$$P\frac{(P' + 1)^{n-s+1} - 1}{P' + 1 - 1};$$

or, because $P = P'^s$,

$$P'^{s-1}\{(P' + 1)^{n-s+1} - 1\},$$

which expression is the same as

$$\{\frac{P'}{P' + 1}\}^{s-1}(P' + 1)^n - P'^{s-1},$$

which is evidently less than $(P' + 1)^n$.

It is plain that when once the inequality (1) is satisfied for any value of x, it will be satisfied for any higher value; for the second member is x^n times a series of fractions in x, which consequently decrease as x increases. Hence, $P^{\frac{1}{s}} + 1$ exceeds the greatest root of the equation.

Let the following equations be proposed, to determine a superior limit to the positive roots in each.

1. $x^4 - 5x^3 + 37x^2 - 3x + 39 = 0,$

$$\therefore P^{\frac{1}{s}} + 1 = 5 + 1 = 6.$$

2. $x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0,$

$$\therefore \; P^{\frac{1}{r}} + 1 = 49^{\frac{1}{2}} + 1 = 8.$$

3. $x^4 + 11x^2 - 25x - 67 = 0,$

$$\therefore \; P^{\frac{1}{r}} + 1 = 67^{\frac{1}{3}} + 1 = 6.$$

4. $3x^3 - 2x^2 - 11x + 4 = 0,$

$$\therefore \; P^{\frac{1}{r}} + 1 = \frac{11}{3} + 1 = 5.$$

PROPOSITION IV.

(30.) If, in an equation, $-P$ be the greatest negative coefficient, and if, among those positive terms which precede the first negative term, the greatest coefficient S be taken, then will $\frac{P}{S} + 1$ be a superior limit to the positive roots of the equation.

The most unfavorable case will be that in which all the terms that follow the first negative term are also negative, and their coefficients equal to P. Under these circumstances, the equation may be written

$$-P(1 + x + x^2 + \ldots . x^m) + Sx^{m+1} + Tx^{m+2} + Ux^{m+3} + \ldots . = 0 \ldots (1).$$

Now the negative portion of this polynomial will be

$$-P\frac{x^{m+1} - 1}{x - 1},$$

which, by substituting $\frac{P}{S} + 1$ for x, becomes

$$-S\left\{\frac{P}{S} + 1\right\}^{m+1} + S.$$

Also the positive portion, by a like substitution, becomes

$$S\left\{\frac{P}{S} + 1\right\}^{m+1} + T\left\{\frac{P}{S} + 1\right\}^{m+2} + U\left\{\frac{P}{S} + 1\right\}^{m+3} + \ldots .$$

and, as the first term alone exceeds the former portion, it is plain that the aggregate of both portions must be positive. If the coefficient S belonged to a term more advanced, it is obvious that the excess of the positive portion above the negative would be increased. It is easy to see that, when any value of x is found that will cause the positive part of (1) to exceed the negative, every higher value of x will have a similar effect; for, if we divide both portions by x^{m+1}, the first will consist of a series of fractions in x, and will consequently diminish as x increases; while the second part will continually increase with x. Hence $\frac{P}{S}+1$ is a superior limit to the positive roots of the equation.

Applying this method of finding a limit to the examples in the preceding proposition, we have, for the limit in the first example,

$$\frac{P}{S}+1=\frac{5}{1}+1=6.$$

$$\text{2d,}\quad \frac{P}{S}+1=\frac{49}{7}+1=8.$$

$$\text{3d,}\quad \frac{P}{S}+1=\frac{67}{11}+1=8.$$

$$\text{4th,}\quad \frac{P}{S}+1=\frac{11}{3}+1=5.$$

The limits given by this method are, in these examples, the same as those before determined, with the exception of that in the 3d example, to which the former method is applied with more success. In the following example, however, this latter method of finding a near superior limit has greatly the advantage:

$$x^4+16x^3-2x^2-12x+6=0,$$

$$\therefore \frac{P}{S}+1=\frac{12}{16}+1=2.$$

By the former method the limit would be

$$P^{\frac{1}{r}}+1=12^{\frac{1}{2}}+1=5.$$

PROPOSITION V.

(31.) If the real roots of an equation, ranged in the order of their magnitudes, be

$$a \quad a_2 \quad a_3 \quad a_4 \ldots\ldots$$

a being the greatest, a_2 the next in magnitude, &c.; and if a number b, greater than a, be substituted for x, the result will be positive; if a number b_2, in magnitude between a and a_2, be substituted for x, the result will be negative; if a number b_3, between a_3 and a_4, be substituted, the result will be positive, and so on.

The first member of the proposed equation is the product of the simple factors

$$(x-a)\,(x-a_2)\,(x-a_3)\,(x-a_4)\ldots.$$

multiplied by the quadratic factors involving the imaginary roots. Omitting these latter for the present, let us examine the effect of our proposed substitutions upon the product of the real factors. Putting then b for x in these factors, we have

$$(b-a)\,(b-a_2)\,(b-a_3)\,(b-a_4) = \text{a positive number,}$$

because all the factors are positive.

Putting b_2 for x, we have

$$(b_2-a)\,(b_2-a_2)\,(b_2-a_3)\,(b_2-a_4) = \text{a negative number,}$$

because the first factor is negative, and all the others positive.

Putting b_3 for x, we have

$$(b_3-a)\,(b_3-a_2)\,(b_3-a_3)\,(b_3-a_4) = \text{a positive number,}$$

because the first two factors are positive, and the others negative, &c. Now the quadratic factors which we have omitted, always give a positive result for every real value of x (*Cor.* 2, p. 19); hence the introduction of these factors would cause no change in the foregoing results.

Cor. 1. Hence, if two numbers be successively substituted for x in any equation, and give results affected with *different* signs, then there lie between those numbers, one, three, five, or some *odd* number of roots.

Cor. 2. And if two numbers, when substituted successively for x, give results affected with the *same* sign, then there lie between those numbers, two, four, six, or some even number of roots, or else none at all.

Cor. 3. If any two consecutive numbers in the arithmetical scale, 1, 2, 3, &c.; or —1, —2, —3, &c.; or ·1, ·2, ·3, &c.; or 10, 20, 30, &c. &c. be separately substituted for x, and give results affected with different signs, then one root, at least, must lie between those numbers, and therefore that which has the smallest numerical value will be the first figure of the root; but if the results have the same sign, whatever substitution be made, then, unless all the roots are imaginary, which cannot however be if the result of any substitution is negative (p. 19), an even number of real roots must lie between those two numbers, the substitution of which furnish results nearest to 0, and consequently the least of these two numbers must be the first figure of each of the roots that lie between them.

1. Suppose, for example, it were required to find the first figure in one of the roots of the equation

$$x^3 + 1{\cdot}5x^2 + {\cdot}3x - 46 = 0.$$

It is here obvious that x must be less than 4, for, if it were so great as 4, the first term alone would be 64, therefore the result would be positive, as also for all values greater than 4. Let us then try 3, and there is found to result a negative quantity, viz. — 4·6; hence, one root must lie between 3 and 4, and consequently its first figure is 3.

The result of our substitution 3 will be most rapidly obtained by writing the coefficients in a row, and proceeding one step in the process for diminishing the roots by 3; thus:

1	1·5	·3	—46
	3	13·5	41·4
	4·5	13·8	— 4·6

where it is obvious that, on the supposition of $x = 3$,

$$4{\cdot}5 = x + 1{\cdot}5, \quad 13{\cdot}8 = (x + 1{\cdot}5)\,x + {\cdot}3 = x^2 + 1{\cdot}5x + {\cdot}3$$

$$-4{\cdot}6 = (x^2 + 1{\cdot}5x + {\cdot}3)\,x - 46 = x^3 + 1{\cdot}5x + {\cdot}3x - 46.$$

2. Let it be proposed to find the first figure of a root of the equation

$$x^4 + 3x^3 + 2x^2 + 6x - 148 = 0.$$

The first term shows that the root must be below 4, let us therefore try 3:

1	3	2	6	− 148
	3	18	60	198
	6	20	66	+ 50

This is also too great; trying then 2, we have

1	3	2	6	− 148
	2	10	24	60
	5	12	30	− 88

as we have now a change sign, we conclude that 2 is the first figure of a root.

3. Find the first figure of one of the roots of the equation

$$x^3 - 17x^2 + 54x - 350 = 0.$$

It is obvious that 20 exceeds the greatest root; let us try 10:

1	− 17	54	− 350
	10	− 70	− 160
	− 7	− 16	− 510

This result being negative, and the result of $x = 20$ positive, we infer that the first figure of the root is 1 in the ten's place.

PROPOSITION VI.

(32.) Having an equation given, to determine another, of an immediately inferior degree, such that the real roots of the former may be limits to those of the latter.

Let the proposed equation be

$$f(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots\ldots A_2 x^2 + Ax + N = 0 \quad \ldots\ldots (1),$$

and its roots taken in order

$$a, \quad a_2, \quad a_3 \ldots\ldots a_n,$$

then it is required to determine an equation of the $n - 1$th degree whose real roots shall severally range themselves between those above.

It is evident that if the roots of the given equation be all diminished by r, the roots of the transformed will be

$$a - r, \quad a_2 - r, \quad a_3 - r, \ldots\ldots a_n - r;$$

and, therefore, that the coefficient of the last term but one, in the transformed equation, will be (11)

$$\left.\begin{array}{l} A' = (r-a)\ (r-a_2)\ (r-a_3) \ldots\ldots \\ \quad + (r-a)\ (r-a_2)\ (r-a_4) \ldots\ldots \\ \quad + (r-a)\ (r-a_3)\ (r-a_4) \ldots\ldots \\ \quad \vdots \\ \quad + (r-a_2)\ (r-a_3)\ (r-a_4) \ldots\ldots \\ \quad + \&c. \qquad \&c. \end{array}\right\} \ldots\ldots (2).$$

But the coefficient of the last term but one is obtained from the original coefficients, by the process described in (20), and which is as follows, viz.

A_n	A_{n-1}	A_{n-2}	$A_{n-3} \quad \ldots\ldots$	..
	$A_n r + A_{n-1}$	$A_n r^2 + A_{n-1} r + A_{n-2}$	$A_n r^3 + A_{n-1} r^2 + A_{n-2} r + A_{n-3}$	..
	$2A_n r + A_{n-1}$	$3A_n r^2 + 2A_{n-1} r + A_{n-2}$	$4A_n r^3 + 3A_{n-1} r^2 + 2A_{n-2} r + A_{n-3}$	..

and, as the last but one, A′, is the nth, this term will obviously be

$$A' = nA_n r^{n-1} + (n-1)A_{n-1} r^{n-2} + (n-2)A_{n-2} r^{n-3} + \ldots . 2A_2 r + A \ldots . (3);$$

which, equated to zero, will furnish the equation, whose real roots are between those of the proposed. For, if in A′, as given by (2), we put a for r, each group of factors must vanish, except one, because there is only one group in which $(r - a)$ is absent, and this will become

$$(a - a_2)\ (a - a_3)\ (a - a_4) \quad \text{\&c. positive.}$$

If we put a_2 for r, all will vanish except that in which is $(r - a_2)$, and this will become

$$(a_2 - a)\ (a_2 - a_3)\ (a_2 - a_4) \quad \text{\&c. negative.}$$

In like manner, putting a_3 for r, we shall have

$$(a_3 - a)\ (a_3 - a_2)\ (a_3 - a_4) \quad \text{\&c. positive};$$

and so on. But when a series of quantities a, a_2, a_3, &c. substituted for the unknown in any equation, give results alternately, positive, and negative, every pair of results comprehends a real root of the equation. Hence, representing the real roots of the equation

$$n A_n r^{n-1} + (n - 1) A_{n-1} r^{n-2} + (n - 2) A_{n-2} r^{n-3} + \ldots . 2A_2 r + A = 0 \ldots . (4),$$

by

$$b, \quad b_2, \quad b_3, \quad \text{\&c.}$$

their situation, with respect to the real roots a, a_2, a_3, &c., will be thus marked

$$\begin{array}{ccccccccc} a, & & a_2, & & a_3, & & a_4 & & \ldots . \\ & b, & & b_2, & & b_3 & & & \ldots . \end{array}$$

and, consequently, if the real roots b, b_2, b_3 . . ., be found, we may express the limits, between which the real roots of the original equation will be situated, by

$$\begin{array}{ccccccccc} \infty & & b & & b_2 & & b_3 & & \ldots . - \infty \\ & a & & a_2 & & a_3 & & a_4 \ldots . & \end{array}$$

(33.) The equation (3), whose roots thus furnish limits to those of the proposed, is called the *limiting equation* to the proposed, and is usually written, for uniformity sake, with x instead of r; thus:

$$f'(x) = nA_n x^{n-1} + (n-1) A_{n-1} x^{n-2} + (n-2) A_{n-2} x^{n-3} + \\ \ldots . \; 2A_2 x + A = 0 \; \ldots . \; (5),$$

an equation which may be immediately written down from inspecting the proposed; for the kth being any term in the limiting equation, it is obtained from the kth in the proposed, by multiplying this latter by the exponent of x in it, and diminishing the exponent by unity. Thus $A_{n-1} x^{n-1}$ being the second term in the proposed, the second term in the limiting equation will be $(n-1) A_{n-1} x^{n-2}$, &c.

Suppose, for example, the proposed equation is

$$2x^4 - 7x^3 + 4x^2 + 2x - 12 = 0,$$

then the limiting equation is

$$8x^3 - 21x^2 + 8x + 2 = 0.$$

In like manner, the limiting equation to this is

$$24x^2 - 42x + 8 = 0,$$

and, finally, the limiting equation to this last is the simple equation

$$48x - 42 = 0;$$

hence the value of x here is between the two values in the preceding equation, and these two are limits between the roots of the preceding cubic, and, finally, the roots of the cubic are limits between those of the proposed equation.

(34.) The expression (3), determined above for the coefficient A′ of the last term but one of the transformed equation, would of course represent the coefficient of the *second* term, if the terms of the transformed were reversed, or arranged according to the ascending powers of x'. We might, therefore, have readily obtained it, as also the coefficients of the succeeding powers of x', by substituting $x' + r$ for x in the original equation, and actually developing the terms according to the *ascending* powers of x, instead of according to the *descending* powers, as at page 35. As this development of $f(r+x')$, according to the ascending powers of x', may be expressed in a formula of great elegance; and, as it is frequently referred to by modern writers on the theory of equations, we shall here insert it, although, for the purposes of the present proposition, the coefficient of the second term of the development, that is, the expression A′ before determined, is all that we require. But we shall have occasion to refer to the entire development in the closing Chapter of this FIRST PART, when we come to explain the recent researches of *M. Sturm*, on the nature of the roots of equations.

Substituting, then, $r + x'$ for x in (1), and developing by the binomial theorem, we have

$$\begin{array}{ll|l|l|l}
A_n r^n & + nA_n r^{n-1} & x' + \frac{n(n-1)}{2} A_n r^{n-2} & x'^2 + \frac{n(n-1)(n-2)}{2\,.\,3} A_n r^{n-3} & x'^3 + \ldots A_n x'^n = 0. \\
+ A_{n-1} r^{n-1} & + (n-1) A_{n-1} r^{n-2} & + \frac{(n-1)(n-2)}{2} A_{n-1} r^{n-3} & + \frac{(n-1)(n-2)(n-3)}{2\,.\,3} A_{n-1} r^{n-4} & \\
+ A_{n-2} r^{n-2} & + (n-2) A_{n-2} r^{n-3} & + \frac{(n-2)(n-3)}{2} A_{n-2} r^{n-4} & + \frac{(n-2)(n-3)(n-4)}{2\,.\,3} A_{n-2} r^{n-5} & \\
+ A_{n-3} r^{n-3} & + (n-3) A_{n-3} r^{n-4} & + \frac{(n-3)(n-4)}{2} A_{n-3} r^{n-5} & + \frac{(n-3)(n-4)(n-5)}{2\,.\,3} A_{n-3} r^{n-6} & \\
\vdots & \vdots & \vdots & \vdots & \\
+ A_2 r^2 & + 2A_2 r & + A_2 & A_3 & \\
+ Ar & + A & & & \\
+ N & & & &
\end{array}$$

The coefficient of x' is the expression (3) before found, and may be written down immediately from the preceding term, or coefficient of x'^0, as explained in Art. (33) above, and agreeably to the notation there used it will be represented by $f'(r)$. In like manner, by inspecting the coefficient of x'^2, we see that it also may be formed, according to the same law, from the preceding, only here we must divide the entire result by 2; this coefficient therefore will, in the same notation, be $\frac{f''(r)}{2}$. In the coefficient of x'^3 the same law of formation is still preserved, only here the division is by 2·3; hence the representation of this coefficient is $\frac{f'''(r)}{2\cdot 3}$, and, from the known form of the binomial development, it is obvious that every coefficient is derivable from the preceding, by the same uniform process, the only variation being that each derived function of r has one divisor more than the preceding; that additional divisor being the number immediately following in the arithmetical scale. Hence, then, we may conclude that the development of $f(r + x')$, that is, of $f(x)$ in (1), when x is changed into $r + x'$, is

$$f(r+x') = f(r) + f'(r)x' + \frac{f''(r)}{2}x'^2 + \frac{f'''(r)}{2\cdot 3}x'^3 + \ldots \ldots x'^n;$$

where the function in each numerator is derived from the preceding one, as explained in (33). Thus, if

$$f(r) = 2r^4 - r^3 + 3r^2 + 6r - 28$$
$$\text{then } f'(r) = 8r^3 - 3r^2 + 6r + 6$$
$$f''(r) = 24r^2 - 6r + 6$$
$$f'''(r) = 48r - 6$$
$$f''''(r) = 48$$
$$f'''''(r) = 0.$$

(35.) From the reasoning in the proposition, we derive the following consequences, viz.

Since

$$f(x) = (x - a)(x - a_2)(x - a_3)(x - a_4) \ldots .$$

and

$$f'(x) = (x-a)(x-a_2)(x-a_3)\ldots + (x-a)(x-a_3)(x-a_4)\ldots + \&c.$$

it follows that

$$\frac{f'(x)}{f(x)} = \ldots \frac{1}{x-a_4} + \frac{1}{x-a_3} + \frac{1}{x-a_2} + \frac{1}{x-a} \ldots (1).$$

In like manner, for any other equation $F(x) = 0$, we have

$$\frac{F'(x)}{F(x)} = \ldots \frac{1}{x-b_4} + \frac{1}{x-b_3} + \frac{1}{x-b_2} + \frac{1}{x-b} \ldots (2).$$

Suppose the two equations

$$f(x) = 0, \qquad F(x) = 0,$$

have a root in common, viz. $a = b$, then, dividing (1) by (2), we have

$$\frac{f'(x)}{F'(x)} \cdot \frac{F(x)}{f(x)} = \frac{\ldots \frac{1}{x-a_4} + \frac{1}{x-a_3} + \frac{1}{x-a_2} + \frac{1}{x-a}}{\ldots \frac{1}{x-b_4} + \frac{1}{x-b_3} + \frac{1}{x-b_2} + \frac{1}{x-a}}.$$

Hence, multiplying numerator and denominator of the second member by $x - a$, and then substituting for x, its value $x = a$, we have

$$\frac{f'(a)}{F'(a)} \cdot \frac{F(a)}{f(a)} = 1$$

$$\therefore \frac{f'(a)}{F'(a)} = \frac{f(a)}{F(a)};$$

from which we learn, that if any two equations have a common root a, and their limiting equations be taken, the ratio of the original polynomials, when a is put for x, will be equal to the ratio of the limiting polynomials when a is put for x.

This property furnishes us with a ready method of determining the value of a fraction, such as $\frac{f(x)}{F(x)}$, when both numerator and denominator vanish for a particular value of x, as, for instance, for

$x = a$. For we shall merely have to replace the polynomials in numerator and denominator by their limiting polynomials, and then make the substitution of a for x. If, however, the terms of the new fraction should also vanish for this value of x, we must treat it as we did the original, and so on, till we arrive at a fraction of which the terms do not vanish for the proposed value of x. The advantage of this method, over that hitherto given in books on Algebra, will be readily seen by comparing the solutions to the first two of the following examples with those given by Dr. Wood in his Algebra.

Examples in Vanishing Fractions.

1. Required the value of $\frac{x^2 - a^2}{x - a}$, when $x = a$?

$$\text{Here } \frac{f'(a)}{F'(a)} = \frac{2a}{1} = 2a, \text{ the required value.}$$

2. Required the value of

$$\frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2},*$$

when $x = 1$?

$\frac{f'(x)}{F'(x)} = \frac{n(n+1)x^n - n(n+1)x^{n-1}}{2(1-x)}$. This still becomes $\frac{0}{0}$ for $x=1$, $\therefore$

$$\frac{f''(x)}{F''(x)} = \frac{n^2(n+1)x^{n-1} - n(n+1)(n-1)x^{n-2}}{-2}$$

$$\therefore \frac{f''(1)}{F''(1)} = \frac{n(n+1)}{2},$$

the value sought.

---

* This is the expression for the sum of n terms of the series

$$1 + 2x + 3x^2 + 4x^3 + \&c.$$

3. Required the value of

$$\frac{1-x^n}{1-x},$$

when $x = 1$?

$$\frac{f'(1)}{F'(1)} = \frac{-n}{-1} = n.$$

4. Required the value of

$$\frac{b(a-\sqrt{ax})}{a-x},$$

for $x = a$?

We may here put $\sqrt{x} = y$, and thus change the fraction into

$$\frac{b(a-a^{\frac{1}{2}}y)}{a-y^2}$$

$$\frac{f'(y)}{F'(y)} = \frac{-ba^{\frac{1}{2}}}{-2y} \therefore \frac{f'(a^{\frac{1}{2}})}{F'(a^{\frac{1}{2}})} = \frac{b}{2} \text{ the value required.}$$

5. Required the value of

$$\frac{f(y)}{F(y)} = \frac{(a+x)^{\frac{m}{n}} - (a+y)^{\frac{m}{n}}}{x-y},$$

when $x = y$? (see Algebra, second edition, page 179.)

Put $a + y = z^n$, then the fraction is changed into

$$\frac{(a+x)^{\frac{m}{n}} - z^m}{x - z^n + a}$$

$$\therefore \frac{f'(z)}{F'(z)} = \frac{-mz^{m-1}}{-nz^{n-1}} = \frac{m}{n} \cdot \frac{z^m}{z^n} = \frac{m}{n} \cdot \frac{(a+y)^{\frac{m}{n}}}{a+y};$$

and therefore the value, when $x = y$, is

$$\frac{m}{n} \cdot \frac{(a+x)^{\frac{m}{n}}}{a+x}$$

(36.) Fractions of the above kind are of very common occurrence in analytical enquiries, and the determination of their values, in all circumstances, may be regarded as the ultimate object of the Differential Calculus, when received under its simplest aspect. Some persons, however, object to the conclusions derived from the theory of vanishing fractions, and maintain that quantities, or magnitudes, which have vanished, can no longer continue the objects of computation. This affirmation, when taken without any qualifying circumstances, is undoubtedly true. We can, of course, have no further control over what has been annihilated; nor are such the nonentities with which the theory of vanishing fractions, and the differential calculus, have to do. In mathematical investigations a quantity is understood to vanish when it no longer exists in its former state, but passes into an inferior state from one or more of its dimensions, or one or more of its factors, becoming zero. In this way a solid vanishes, when, by the continual diminution and final exhaustion of its thickness, it passes into another state, viz. that of a superficies. As a *solid*, it has doubtless disappeared; and although it would have been absurd to have compared it, in its original state, to a surface, yet the comparison may be made now with entire consistency. It is this *loss of dimension*, and, consequently, *change of state* in the terms of a fraction, that is attempted to be expressed by the symbol $\frac{0}{0}$, and nothing more. When quantities, or algebraical functions, are connected together by the signs of addition or subtraction, it is an admitted principle that the component parts must be all of the same dimensions, that is, the expression must be *homogeneous*. If, by introducing particular values for the variables, one of these functions should lose a dimension, by a factor vanishing, to retain it in the expression would be absurd; for its value, as one of the component magnitudes, has become nothing; it has changed its state, and, as a part of the original expression, is destroyed; still, as its other dimensions remain, it is not annihilated, but may legitimately enter into comparison with other quantities.*

---

* It appears to the author that if this idea, of rejecting what are called vanishing terms from an algebraical expression, not because they

Equal Roots.

(37.) The foregoing proposition also readily leads to a method of freeing an equation from all repetitions of the same root, whenever such occur; as also of ascertaining whether an equation has equal roots or not. For, as in the limiting equation $f'(x) = 0$, the polynomial $f'(x)$ consists of the sum of the products arising from multiplying together every $n - 1$ of the factors of $f(x)$, each group of factors in $f'(x)$ will differ from $f(x)$ only by the absence of a single factor. Hence, if there be two equal factors in $f(x)$, that is, if $f(x) = 0$ have two equal roots, one of these factors must occur in each of the groups which compose $f'(x)$, so that $f(x)$ and $f'(x)$ have this factor for a common measure. If there be three equal roots in $f(x) = 0$, then will $f(x)$ and $f'(x)$ have for a common measure the quadratic factor involving two of them, and generally if $f(x) = 0$ have p roots equal to a, then will $(x - a)^{p-1}$ be a common measure of $f(x)$ and $f'(x)$, since in none of the component parts of $f'(x)$ can more than one of the p equal factors be absent.

Again, if besides the p factors equal to $(x - a)$, there also enter q factors equal to $(x - b)$ in the composition of $f(x)$, then, besides the former common measure, the polynomials $f(x), f'(x)$, will also have the common measure $(x - b)^{q-1}$, and, generally, if the equation $f(x) = 0$ have p roots equal to a, q roots equal to b, r roots equal to c, &c. then the greatest common measure of the polynomials $f(x), f'(x)$, will be

$$(x - a)^{p-1} (x - b)^{q-1} (x - c)^{r-1} \dots$$

In order, therefore, to discover whether or not an equation $f(x) = 0$ has equal roots, we have only to ascertain whether or not $f(x)$ and $f'(x)$ have a common measure $\phi(x)$; if they have, the division of $f(x)$ by $\phi(x)$ will give a polynomial involving the roots of the proposed equation without any repetition; it is indeed practicable to deduce a polyno-

---

are annihilated, but because, by having lost one or more dimensions, they have become heterogeneous, were kept in view and acted upon, the *Infinitesimal Calculus* of *Leibnitz* might be freed from the objections to which its different orders of infinitesimals or zeros expose it.

mial which shall involve only those roots which enter singly into the proposed, as we shall shortly show in general terms; at present, we shall apply the method to one or two particular examples.

1. It is required to determine whether the equation

$$f(x) = 2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0,$$

has equal roots?

$$f'(x) = 8x^3 - 36x^2 + 38x - 6,$$

the greatest common measure $\phi(x)$ of the polynomials $f(x), f'(x)$, is $x - 3$; hence the equation has *two* roots each equal to 3. Dividing, therefore, $f(x)$ by $(x - 3)^2$, we have $2x^2 + 1$; hence the other roots are involved in the equation

$$2x^2 + 1 = 0 \therefore x = \pm \tfrac{1}{2} \sqrt{-2},$$

that is, the four roots of the proposed equation are

$$3, \quad 3, \quad \tfrac{1}{2} \sqrt{-2}, \quad -\tfrac{1}{2} \sqrt{-2}.$$

2. It is required to determine whether the equation

$$f(x) = x^7 + 5x^6 + 6x^5 - 6x^4 - 15x^3 - 3x^2 + 8x + 4 = 0,$$

has equal roots?

$$f'(x) = 7x^6 + 30x^5 + 30x^4 - 24x^3 - 45x^2 - 6x + 8,$$

$$\phi(x) = x^4 + 3x^3 + x^2 - 3x - 2.$$

The equation has therefore equal roots involved in the equation $\phi(x) = 0$. As in this last equation the roots all occur once less often than in the original, these roots will be all different if those of the original enter only in pairs; but, if any enter in *threes*, or in a greater number, the equation $\phi(x) = 0$ will also contain equal roots. Let us ascertain this in the present case:

$$\phi'(x) = 4x^3 + 9x^2 + 2x - 3,$$

the common measure of $\phi(x)$, $\phi'(x)$, is $x + 1$; hence the equation

$\phi(x) = 0$ has two roots equal to -1, and, consequently, the equation $f(x) = 0$ must have three roots equal to -1.

By division,

$$\frac{\phi(x)}{(x+1)^2} = x^2 + x - 2,$$

and from

$$x^2 + x - 2 = 0$$

we get

$$x = 1, \quad x = -2;$$

hence

$$\phi(x) = (x+1)^2 (x-1)(x+2),$$

and consequently,

$$f(x) = (x+1)^3 (x-1)^2 (x+2)^2,$$

that is, the roots of the proposed are

$$-1, -1, -1, 1, 1, -2, -2.$$

If the equal roots in the proposed had all entered in pairs, $\phi(x)$, $\phi'(x)$ would have had no common measure, and the determination of the equal roots would have required the solution of the equation $\phi(x) = 0$, which contains each of those roots once. And in general the solution of the proposed equation, when equal roots enter, may always be reduced to the solution of a series of others of inferior degrees, of which the first contains only the unequal roots of the proposed, the second each one of the double roots, the third each of the triple roots, &c. This may be proved as follows:

Let X represent the product of the factors which enter *singly*.

X^2_2 . . the product of all the *pairs*.

X^3_3 . . the product of all the *threes*.

X^4_4 . . the product of all the *fours*.

&c.

so that

$$f(x) = X X^2_2 X^3_3 X^4_4 \ldots .$$

then the greatest common divisor, $\phi(x)$, of $f(x)$ and $f'(x)$, will b

$$\phi(x) = X_2 X^2_3 X^3_4 X^4_5 \dots$$

Again, calling the greatest common measure of $\phi(x)$, and its d function $\phi'(x)$, $\phi_1(x)$, we have

$$\phi_1(x) = X_3 X^2_4 X^3_5 \dots$$

In like manner, calling the greatest common measure of $\phi_1(x)$ its derived function $\phi'_1(x)$, $\phi_2(x)$, and continuing the operatio have

$$\phi_2(x) = X_4 X^2_5 \dots$$

$$\phi_3(x) = X_5 \dots$$

&c. &c.

Hence, by division,

$$F(x) = \frac{f(x)}{\phi(x)} = X X_2 X_3 X_4 X_5 \dots$$

$$F_1(x) = \frac{\phi(x)}{\phi_1(x)} = X_2 X_3 X_4 X_5 \dots$$

$$F_2(x) = \frac{\phi_1(x)}{\phi_2(x)} = X_3 X_4 X_5 \dots$$

$$F_3(x) = \frac{\phi_2(x)}{\phi_3(x)} = X_4 X_5 \dots$$

&c. &c.

and, consequently, the determination of the roots of the proposed tion is reduced to the solution of the following series of equation

$$\frac{F(x)}{F_1(x)} = X = 0,$$

$$\frac{F_1(x)}{F_2(x)} = X_2 = 0,$$

$$\frac{F_2(x)}{F_3(x)} = X_3 = 0,$$

&c. &c.

The first of these equations involves the single roots only, the second each one of the double roots, the third each one of the triple roots, &c.

From inspecting the formation of the coefficient of x, in an equation (11), it is plain that when equal roots exist, this coefficient and the absolute term N will have a common factor; when, therefore, such is not the case, we may immediately conclude that the equation has no equal roots. A common factor, however, may evidently exist, although the roots are all unequal.

CHAPTER IV.

ON THE METHOD OF NEWTON FOR FINDING A SUPERIOR LIMIT TO THE POSITIVE ROOTS OF AN EQUATION, AND ON THE SEPARATION OF THE REAL ROOTS.

(38.) To find a number greater than the greatest root of an equation, Newton proposed to transform the equation into another whose roots should be less than those of the former by an undetermined quantity r, and then to determine r by trial, so as to cause all the coefficients in the transformed equation to become positive. Such a value of r would obviously exceed the greatest positive root of the proposed equation; for the real roots of the transformed, which are those of the original diminished by r, would all be negative, since, on account of all the terms being positive, the substitution of even 0, as well as any positive number for x', would be positive. As an example of Newton's method, let us take the equation

$$x^3 - 5x^2 + 7x - 1 = 0;$$

then, substituting $x' + r$ for x, the transformed is

$$
\begin{array}{llll}
0 = x'^3 + 3r & x'^2 + 3r^2 & x' + r^3 \\
\quad\;\; -5 & \;\;\, -10r & \;\;\, -5r^2 \\
 & \;\;\, +7 & \;\;\, +7r \\
 & & \;\;\, -1
\end{array}
$$

Now, after a few trials, we find that 3 is the smallest value for r which causes the several compound coefficients to become positive; therefore 3 exceeds the greatest positive root of the equation.

(39.) We should arrive at the same result, by diminishing the roots of the proposed successively by unity, according to the process in (31), and stopping as soon as the transformed coefficients became all positive; thus:

$$
\begin{array}{rrrrl}
1 & -5 & 7 & -1 & (1 \\
 & -4 & 3 & 2 & \\
 & -3 & 0 & & \\
 & -2 & & & \\
\hline
1 & -2 & \pm 0 & +2 & (1 \\
 & -1 & -1 & 1 & \\
 & 0 & -1 & & \\
 & 1 & & & \\
\hline
1 & +1 & -1 & +1 & (1 \\
 & 2 & 1 & 2 & \\
 & 3 & 4 & & \\
 & 4 & & & \\
\hline
1 & +4 & +4 & +2 &
\end{array}
$$

It may be observed of this method, that it not only furnishes a superior limit to the greatest positive root in every case, but when the roots are all real the limit thus determined is *immediately* above the true value of the greatest root; that is, the preceding number in the arithmetical scale is the first figure of that root. This is obvious, for the coefficients of the transformed become all positive as soon as all the roots become negative, and not before (18).

Even without knowing whether the roots are all real, we can pronounce the limit thus found to be the immediately superior limit, if the last coefficient in the immediately preceding set be negative; so that, in this case, we shall also know the first figure of the greatest root. This will appear plain, from considering that the last coefficient in any set (which is in fact the absolute number,) is the result of the corresponding polynomial for *root* = 0; and the last coefficient in the succeeding set is the result of the same polynomial for *root* = 1: and, as these results, in the case supposed, have opposite signs, one root at least must have been passed over, and that the greatest, as the final coefficients are all positive.

The same process, as we go on, supplies a like indication of every passage we make over a real root, or over an odd number of roots; every such indication being a change of sign in the last terms of two consecutive transformations. In the example above, the very first transformation presents a change of sign in the last term; we infer, therefore, that a root of the equation lies between 0 and 1.

If, however, the last term vanish in any transformed, the circumstance will prove that our last diminution has exhausted one of the roots, for one root of the transformed will then be zero, this being the value which it is obvious will always satisfy every equation whose final or absolute term is zero. Should not only the last, but also the last but one, vanish, we may, in like manner, conclude that two roots have been exhausted, and, if p last terms vanish, p roots will have been exhausted; so that the equation proposed will have p roots, each equal to the number of transformations: in seeking, therefore, the superior limit by the foregoing process, we shall always detect in our progress every positive integral root of the equation.

Again, if any intermediate term vanish from one of the transformed equations, the circumstance may lead to the detection of imaginary roots of the equation; for, if on each side of the vanishing term the contiguous terms have *like* signs, the rule of Descartes will show that the roots cannot be all real; such an occurrence will be, therefore, a sure indication of the existence of at least one pair of imaginary roots. The following example will illustrate these remarks.

2. Let the equation be

$$x^3-3x^2+4x-2=0,$$

and diminish the roots by unity:

1	−3	4	−2 (1
	−2	2	0
	−1	1	
	0		

At the close of the first step, we immediately infer that $x=1$ is a root of the equation. The other two roots are involved in the equation

$$x^2 \pm 0x + 1 = 0;$$

and, as 0 occurs between the two like signs +, we infer that both roots are imaginary.

In seeking the superior limit, therefore, by the process recommended, we may *sometimes* detect the existence of imaginary roots, although they do not *always* furnish the above indication of their presence.*

3. Again, let us take the example

$$x^4-4x^3+10x^2-12x+9=0,$$

which, as it has no permanencies, cannot have any negative roots. Diminishing the roots by 1, we get the transformed coefficients

$$1 \quad 0 \quad 4 \quad 0 \quad 4.$$

This transformation detects the existence of two pair of imaginary roots; we need not, therefore, proceed to another transformation, but conclude immediately that all the roots of the proposed are imaginary.

The foregoing advantages, with some others which might be mentioned, are considerable; and are peculiar to this method of applying Newton's rule to the discovery of limits.

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* Unequivocal tests for detecting the existence of imaginary roots, will be furnished in the Seventh Chapter.

It may be enquired, however, here, Is it, under all circumstances, possible to obtain, by successive diminutions of the roots, a transformed equation involving *only positive* coefficients? To this it may be replied, that, whenever we diminish the roots by a number exceeding the greatest positive root, the result of the real simple factors in the polynomial is necessarily positive in every term; and it continues so for every further diminution. Now, if there be any imaginary factors, the continual diminutions of which we speak must at length annihilate the real parts of these imaginaries, or render them positive, in which case every quadratic factor into which the several pairs of imaginaries enter will have all its coefficients essentially positive, and therefore those of the transformed polynomial will be all positive.

(40.) But the same conclusion may be otherwise established as follows: it is evident, in diminishing the roots of an equation by 1, 2, 3, &c., that the second coefficient in any transformed is always equal to the second coefficient in the preceding equation, plus a certain number of times the first; so that, should there be a variation of sign between the first two terms of the proposed, we may, by continuing the transformations, change this variation into a permanency; whilst, on the contrary, if there be a permanency between the first two terms of the original, no transformation can change them into a variation. A permanency of sign may, therefore, in all cases be established between the first two terms of a transformed equation.

Again, since the third coefficient in any transformed is always equal to the third in the preceding transformation, plus a certain number of times the second, plus a certain number of times the first, it is plain that a variation between the second and third terms of a transformed, whose first and second terms do not vary, must be eventually converted into a permanency; whilst, on the contrary, if the first three terms had originally a permanency of sign, no subsequent transformation could introduce among them a variation.

By similar reasoning, we prove that, having obtained a permanency for the first three terms, we shall arrive, by continuing the transformations, at a permanency between the third and fourth, and so on, till we shall necessarily be led at length to a transformed equation exhibiting only permanencies of sign. Of course this *necessary* increase of permanencies in the leading terms of the successive transformed polynomials, will not prevent an *accidental* increase of them among other terms to the right, and these will facilitate the close of the process.

Let us take for a fourth example the equation given at p. 45:

1	3	2	6	— 148
	4	6	12	— 136.

The — 136 in this step is indication sufficient that 1 is not the limit. Diminishing then by 2, we find, for the final term, — 88; hence 2 is not the limit: but, by diminishing by 3, the numbers in the first step are

6   20   66   60,

which being all positive, the succeeding numbers must be positive; so that, without continuing the process, we infer that 2 is the first figure of the greatest positive root of the equation. We might, in like manner, have stopped the work at the second step of the third transformation, in the former example, and have inferred the value of the limit.

(41.) Hitherto we have considered only the positive roots of the equation; but this might seem sufficient for our purpose, because, by changing the signs of the alternate terms of an equation, the negative roots become changed into positive, and, after this change, the superior limit to the positive roots would, when taken with the negative sign, be the inferior limit to the negative roots.

There is, however, no absolute necessity to effect this change in the signs of the terms of an equation. For it is plain, after the foregoing reasoning, that, if instead of diminishing we increase the roots of the proposed by 1, 2, 3, &c., we shall ultimately obtain a *variation* between the first and second term, then a variation between the second and third, then between the third and fourth, and so on; so that we shall finally arrive at a set of transformed coefficients, presenting only *variations* of sign, and the number of transformations required to lead to this will express the number, which, taken negatively, is the inferior limit of the negative roots; that is, a larger negative number than any of them. Whenever, in the progress of these transformations, we pass over a single, or indeed over any odd number of negative roots, a change of sign in the last coefficient will always give notice of the circumstance; and, when we have entirely exhausted a negative root by these continual additions of unity, the reduction to zero of the same coefficient will apprize us of the fact.

(42.) From what has now been said of the progressive tendency of the successive transformations to terminate, when the roots are diminished, in a series of permanencies, and when they are increased in a series of variations, we may conclude that,

1. If p, q be any positive numbers, of which p is less than q, and if the roots of an equation be diminished first by p and then by q, the coefficients of the first transformed equation, that is, of the equation in $(x - p)$, cannot have fewer variations than the coefficients of the second transformed, that is, of the equation in $(x - q)$.

2. If the roots be *increased* first by p, and then by q, the coefficients of the *second* transformed equation, or that in $(x + q)$, cannot have fewer variations than the coefficients of the transformed in $(x + p)$.

Hence, under no circumstances can the number of variations, furnished by any transformed equation in $(x \pm r)$, be diminished by further *increasing* the root x.

(43.) We are now prepared to demonstrate the following theorem, which is that to which allusion was made at page 23, and which may be regarded as an extension of the rule of Descartes:

Let p *and* q *be any two numbers, with signs like or unlike, but such that* q *with its sign is greater than* p *with its sign; then, if an equation in* x *has* m *real roots comprised between* p *and* q, *the transformed equation in* (x — p) *has at least* m *variations more than the transformed in* (x — q).

Suppose first, that but one real root lies between p and q; then (31) the last terms of the transformations in $(x - p)$ and $(x - q)$ must have contrary signs, which requires that these transformations have not the same number of variations; but, by what is shown above, the first cannot have *fewer* variations than the second: it must necessarily, therefore, have at least one variation more.

Again: let there be m real roots comprised between p and q, and let us suppose them to be all unequal, and represented in the order of their increasing magnitude by

$$a_1, \; a_2, \; a_3, \; a_4 \; \ldots \; a_m.$$

Let, moreover, the numbers

$$b, \quad b_2, \quad b_3, \quad b_4 \ldots\ldots b_{m-1}$$

be respectively comprised between a and a_2; between a_2 and a_3; between a_3 and a_4, &c.; so that we may have the continued inequality

$$p < a < b < a_2 < b_2 < a_3 \ldots\ldots < a_{m-1} < b_{m-1} < a_m < q;$$

it will then follow, that if we form successively the equations in $(x-p)$, in $(x-b)$, in $(x-b_2)$, in $(x-b_3)$ .... $(x-b_{m-1})$ up to that in $(x-q)$, each of these equations will have at least one variation more than the following one. Hence the equation in $(x-p)$ must have at least m variations more than the equation in $(x-q)$, which was to be proved. When the roots are all real, it is obvious that the number of variations which disappear in the successive transformations, is precisely equal to the number of roots comprised between p and q.

It will have been remarked, that in the foregoing examination we have supposed that the real roots between p and q are unequal; this we have done because, previously to seeking the nature and situation of the roots, the first member of an equation may always be disencumbered of its multiple factors (37).

(44.) The substance of what has now been proved amounts to this, viz.

1. If two transformed equations, the one in $(x-p)$, and the other in $(x-q)$, both exhibit the same number of variations, there is no root comprised between p and q.

2. If there be a variation between the last term in one, and the last term in the other, an odd number of roots must be comprehended between p and q, and there cannot be an odd number without this variation.

3. If the only change in the two series of signs, furnished by the transformed equations, is that arising from a variation between the last term in one, and the last term in the other, then one root, and only one, lies between p and q, otherwise there would be more roots than variations lost between the two transformations.

It is very obvious, that the loss of a *single* variation, in passing from one transformed to another, can never take place, except a change occur in the *final* term; so that when the sign of the final term remains the same, if any changes are lost, two, or some even number, must be lost. Moreover, as in passing over *one* root the final term changes sign, and, as this passage is attended with the loss of *one* variation, it follows that the immediately preceding term preserves its sign during the passage.

4. If the number of variations lost be two, the equation *may* have two real roots between p and q; but it may happen also that there are none in this interval. It is certain that the equation cannot have more than two roots in the interval p, q, otherwise the series would have lost more than two variations.

5. If the variations lost amount to any even number m, the equation *may* have m roots between the proposed limits; but if the number of real roots be not m, then the true number can differ from m only by an even number k, and the additional loss of variations will be attributable to k imaginary roots in the proposed equation, as we are now about to show.

(45.) If, between the two transformed equations which we are considering, we could interpose all the intermediate transformations which would arise from passing continuously from p to q, we should readily detect the cause of this loss of an even number of variations between the extreme transformations; for, as no quantity can proceed continuously from + to —, or from — to +, without first passing through zero, we should necessarily arrive, in the course of our intermediate transformations, at one or more containing vanishing terms. The corresponding terms, in the immediately preceding transformation, would make known the *signs* with which the consecutive ones vanished; or the corresponding terms, in the immediately subsequent transformation, would also make known the proper signs in which the same terms would vanish, in returning from the latter transformation to the former. Now should it happen that when the signs of the zeros, determined in the former way, or by means of the antecedent transformation, cause the terms among which these zeros occur to have more variations than when the signs are determined by the subsequent transformation, it is plain that this loss of variations will never be replaced in the following transformations, but will go to augment the loss arising from passing over roots between p and q, (art. 42.) But a loss of variation, any where within the extreme terms of any transformed equation, implies the change of *two* variations into two permanencies (page 67); hence an *even number* of variations are thus lost, and yet the real roots of the transformed, involving the zeros, remain the same. It follows, therefore, by the rule of Descartes, that this equation (and consequently the proposed,) has that even number of imaginary roots.

If the signs of the zeros in the transformation in question present no ambiguity, whether determined from the antecedent or from the subsequent equation, then the several transformed equations must all exhibit the same number of variations, till we arrive at a root, when the last term will vanish, and in the next transformed reappear with a changed sign. This will continue till all the real roots between the proposed limits are passed over, when there will have disappeared as many variations as roots between p and q. Hence the additional variations, which may have disappeared, can have done so from no other cause than that above stated, and these additional disappearances therefore mark the number of imaginary roots.

We have noticed before (page 62) the importance of attending to the signs of the terms contiguous to any simple vanishing term in a transformed equation, and have shown that when the contiguous terms have like signs we may infer the existence of a pair of imaginary roots; a conclusion which harmonizes with that just deduced, and which is, in fact, included in it, as the case referred to is contained in the more general one here considered. When, however, but one term vanishes, the signs are very readily supplied, the zero being always of one sign, +, or —, when the term is deduced from the antecedent contiguous equation, and of the opposite sign, —, or +, when it is deduced from the subsequent contiguous equation. But when several terms vanish, we must actually write down the two series of signs which the contiguous equations referred to exhibit, and which, as before remarked, may equally replace the intermediate series, in order to discover the indications of imaginary roots. This supposes, of course, that we know what the contiguous series of signs are; and that we may in all cases find them with great ease, will be seen from the following considerations.

(46.) Let us suppose that in the course of any transformations we have arrived at an equation or at a series of coefficients containing zeros, and that we want to determine the series of signs due to the immediately succeeding transformation. Represent the indefinitely small quantity by which the roots of the transformed at which we have arrived must be diminished, in order to furnish the next transformation, by δ; then, from what has been said about the influence of the signs in one transformation upon those of the next (40), it will be seen that, on account of the minuteness of δ, the sign of any term to be deduced must always be the same as that of the corresponding term above it; for, by making the multiplier δ smaller and smaller, we may render every product by it as small as we please; so that the final addend, which, added to any term in the proposed series, is to produce the desired term in the new one, may always be made smaller than the term to which it is added, when that term is of any magnitude at all, and therefore the new term will have the same sign as the corresponding term preceding; when, however, this corresponding term is zero, then the sign of the result will obviously be the same as that immediately before this zero. For example, if the series in which the zeros occur be

$$+\ 0\ 0\ 0\ 0\ -\ 0\ 0\ 0\ -\ +\ 0\ +\ 0\ 0\ 0\ 0\ 0\ -,$$

the immediately subsequent series must be

$$\therefore\ +\ +\ +\ +\ +\ -\ -\ -\ -\ -\ +\ +\ +\ +\ +\ +\ +\ +\ -.$$

To form the immediately preceding series from the proposed, and thus to go back a step, requires that we regard our minute factor δ as negative; and as multiplying by $-\delta$ has the effect of changing the sign of every addend, which we must always remember is numerically less than the term to which it is to be added, on account of the minuteness of the multiplier which forms it, the antecedent series will be

$$+\ -\ +\ -\ +\ -\ +\ -\ +\ -\ +\ -\ +\ -\ +\ -\ +\ -\ -.$$

The order, therefore, and the signs, of the three consecutive transformations are as follow:

$$
\begin{array}{ccccccccccccccccccc}
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - & - \\
+ & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & 0 & 0 & 0 & - \\
+ & + & + & + & + & - & - & - & - & - & + & + & + & + & + & + & + & + & -;
\end{array}
$$

in which the lower series has fourteen changes of sign fewer that the upper series, showing that, in the insensibly minute transit from the first to the third, fourteen variations have been lost, and yet no real root passed over: hence the equation from which such results have been deduced contains fourteen imaginary roots, besides whatever others may manifest themselves in transforming between other intervals.

(47.) The foregoing considerations lead to this *rule of the double sign*, viz.

To obtain the upper series, repeat the signs in the middle series, commencing at the left hand, till we come to a zero, over which write the contrary sign to that last inserted, so that every sign exhibited in the middle series is to have the same sign above it in the upper series, and every zero is to have above it a sign contrary to that previously written in the upper series.

To obtain the lower series, put under every zero the same sign as that last inserted instead of the contrary sign; in other respects proceed as in the former precept.

It is plain that, although when but one zero occurs, the upper and lower series *may* preserve the same variations, yet, when two or more consecutive zeros occur, this will be impossible; so that when any transformation has two or more consecutive vanishing terms we may be sure of the existence of imaginary roots. The rule will make known how many are indicated.

(48.) From what has now been said we gather the following directions for determining the nature and situation of the roots of an equation.

1. From the given equation deduce a series of transformed equations, by means of the multipliers

$$\ldots\ldots -1000,\ \ -100,\ \ -10,\ \ -1,\ 0$$
$$1,\ 10,\ 100,\ 1000\ \ldots\ldots$$

taken in order, commencing sufficiently low down in the scale to cause the terms in the first transformed equation to have variations only. If our first transformed exhibit any permanencies we are not to reject the step, but to ascend from it, through the preceding transformations, till we arrive at a series of variations. This is to be regarded as the first series. The last series, or that which terminates the process in the other direction, is to present only permanencies. The interval between the first *transforming multiplier* and the last, will comprise all the real roots of the equation, and will also conceal the indications of the imaginary roots.

2. When zeros occur in any of the transformations, the signs of the terms are to be ascertained by the *rule of the double sign.*

3. Those partial intervals, from step to step, during which no loss of variation occurs, are to be rejected, as no roots can lie in this region of the entire interval.

4. Those partial intervals, wherein *one* variation is lost, embrace one real root of the equation, and only one.

5. Those partial intervals, in which any *odd* number of variations is lost, comprehend at least one real root; and *may* inclose as many real roots as there are variations lost; at all events, either so many real roots will be comprehended, or else so many indications of imaginary roots.

6. Those partial intervals, in which any even number of variations disappear, *may* comprehend as many real roots. They either actually do this, or else they comprehend indications of so many imaginary roots.

(49.) The last two of these statements point to certain regions of doubt, occurring within the entire interval, which limits the range of the system of roots. To remove this doubt, and to evolve the information respecting the roots, which really lies concealed in these regions, would, agreeably to the foregoing theory, require us to pass continuously over the space, without allowing the minutest interval to escape examination. This tedious scrutiny may, however, be dispensed with, and the desired information obtained by the application to the doubtful intervals of a certain criterion, by means of which, the indications of the real and of the imaginary roots are much more readily detected. The investigation and subsequent application of this criterion would doubtless complete this part of our subject, viz. the separation of the real roots, and the enumeration of the imaginary roots of an equation. Yet we do not propose to enter upon the enquiry; not merely because the investigation is somewhat lengthy, but because, in the application of the criterion, the particulars sought do not, in general, immediately offer themselves, and are in some cases tediously slow in appearing. The subject, however, is fully developed by *M. Fourier*,* in his posthumous work on the Theory of Equations, in which work all our conclusions above, respecting the separation of the roots, are deduced, although by trains of reasoning very different from those which we have employed for the purpose, and to which we have been chiefly guided by the views of *Budan*† on the same subject.

The complete separation of the imaginary roots from the real, and the determination of the number of each, may be effected far more readily by help of a theorem, unknown to Fourier and Budan, recently discovered by *M. Sturm*,‡ a young foreign mathematician of great promise. This theorem we shall investigate and apply in the seventh chapter, as we are anxious not to detain the student longer than absolutely necessary from the numerical solution of equations, the great object which the entire theory is intended to subserve, and which the talents of an English mathematician, *Mr. W. G. Horner*,§ of Bath, have brought to the highest degree of perfection.

We shall now show the application of the foregoing principles to one or two examples.

1. Let there be proposed the equation

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* *Analyse des Equations Déterminées*, par M. Fourier, 1831.

† *Nouvelle Méthode pour la Résolution des Equations Numériques*, par F. D. Budan, 1807. See also the Note at the end of the *Algèbre* of *Bourdon*, 1831.

‡ *Mémoires présentés, par des Savans Etrangers*, tom. vi.

§ *Philosophical Transactions*, 1819.

$$x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 = 0.$$

To determine the intervals, between which the roots are to be found.

In order to this we must deduce a series of transformed equations by means of the multipliers . . . . — 10, — 1, 0, 1, 10 . . . ., which we shall call *factors of transformation* or *transforming factors*, and we shall thus have the series of signs,

$$\begin{array}{rl} (-10) \ldots & + - + - + - \\ (-1) \ldots & + - + - + + \\ (0) \ldots & + - - + - - \\ (1) \ldots & + + - + + - \\ (10) \ldots & + + + + + + \end{array}$$

As the first factor of transformation gives only variations, — 10 is the inferior limit to all the negative roots, and, as the factor of transformation 10 gives only permanencies, 10 is the superior limit to the positive roots. Hence the roots all lie between — 10 and 10, and within these limits lie concealed the indications of the imaginary roots.

By comparing the two series given by the factors — 10 and — 1, we conclude, from the change of sign in the final term of the latter, which is the only change that has taken place, that one root exists between — 10 and — 1.

The series given by the factors — 1 and 0, intimate the existence of one root between these limits, for the final signs are contrary, and only one variation is lost.

The series given by the factors 0 and 1 show that no root exists between these limits, nor yet any indications of imaginary roots, for no variations are lost.

The series given by the factors 1 and 10 show, by the change in the final sign, that one root at least exists between these limits; there may be three, because three variations are lost; at all events, the interval 1, 10, is the only interval in which indications of imaginary roots can occur, it would, however, be tedious to seek for these indications by trying intermediate factors of transformation, and we have already promised a more convenient method of proceeding, to be given hereafter. (See Chapter VII. of this First Part.)

2. Let the equation

$$x^4-4x^3-3x+23=0,$$

be proposed.

The transforming factors 0, 1, 10, give

$$\begin{array}{l}(0) \ldots\ldots + - 0 - + \\ (1) \ldots\ldots + 0 - - + \\ (10) \ldots\ldots + + + + +\end{array}$$

Hence, applying the rule of the double sign, we have

$$\begin{array}{ll}(0)\left\{\begin{array}{l}(<0) \ldots\ldots + - + - + \\ (>0) \ldots\ldots + - - - +\end{array}\right. \\ (1)\left\{\begin{array}{l}(<1) \ldots\ldots + - - - + \\ (>1) \ldots\ldots + + - - +\end{array}\right. \\ (10) \qquad \ldots\ldots + + + + +\end{array}$$

The first of these series give four variations, and the second two, this loss of two variations indicates the existence of one pair of imaginary roots.

The third and fourth series exhibit the same number of variations; hence the zero, produced by the transforming factor (1), does not arise from imaginary roots.

Let us now examine the series (0) and (1), for which purpose we must compare the signs of (> 0) and (< 1), and we thus find that no root is comprised in the interval 0, 1, because there is no loss of variation.

For the interval 1, 10, we must examine the series (> 1) and (10), which we find to indicate the existence of two roots, because two variations are lost, but whether they are real or not cannot as yet be ascertained; this, however, is the only doubtful interval.

3. Let the proposed equation be

$$x^5+x^4+x^2-25x-36=0.$$

The transforming factors

$$-10, \quad -1, \quad 0, \quad 1, \quad 10,$$

give the following series of results:

$$\begin{array}{l} (-10) \ldots\ldots + - + - + - \\ (-1) \ldots\ldots + - + - - - \\ (0) \ldots\ldots + + \ 0 \ + - - \\ (1) \ldots\ldots + + + + - - \\ (10) \ldots\ldots + + + + + +. \end{array}$$

Applying the rule of the double sign, we have

$$\begin{array}{l} (-1) \ldots\ldots + - + - - - \\ (0) \begin{cases} (<0) \ldots\ldots + + - + - - \\ (>0) \ldots\ldots + + + + - - \end{cases} \\ (1) \qquad \ldots\ldots + + + + - -. \end{array}$$

Comparing now these results, we find that all the real roots exist in the interval between — 10 and + 10.

That two of these roots *may* lie between — 10 and — 1, because, in passing over this interval, two variations have disappeared; the interval may, however, contain indications of two imaginary roots.

That a pair of imaginary roots are indicated by (0), because the signs of (<0) and (>0) differ by two variations.

That no root exists between — 1 and 0, because the series (— 1) and (<0) have the same number of variations.

That no root exists between 0 and 1, because the series (>0) and (1) have the same number of variations.

That one real root exists between 1 and 10, because one variation has disappeared.

The only doubtful interval here is that between — 10 and — 1.

We shall give but one more example of the determination of the intervals of the roots.

4. Let the proposed equation be

$$x^7 - 2x^5 - 3x^3 + 4x^2 - 5x + 6 = 0.$$

The transforming factors

$$-10, \quad -1, \quad 0, \quad 1, \quad 10,$$

give the following results:

(− 10) . . . . + − + − + − + −
(− 1) . . . . + − + − + + − +
(0) . . . . + 0 − 0 − + − +
(1) . . . . + + + + + − − +
(10) . . . . + + + + + + + +

And, applying the rule of the double sign, we have

(− 1) . . . . + − + − + + − +
(0) { (< 0) . . . . + − − + − + − +
(0) { (> 0) . . . . + + − − − + − +
(1) . . . . + + + + + − − +

We deduce, therefore, the following particulars:

There is one root between the limits − 10 and − 1, and only one.

The series (0) shows the existence of two imaginary roots in the equation, because the series (< 0) and (> 0) differ by two variations.

There is no real root between − 1 and 0.

There *may* be two real roots between 0 and 1, as two variations disappear between (> 0) and (1); but if there are not two real roots in this doubtful interval, there exists within it an indication of two imaginary roots.

There *may* also be two more real roots between 1 and 10.

The only intervals, therefore, in which we ought to seek for roots are those between − 10 and − 1, between 0 and 1, and between 1 and 10; and we know also that the equation has two imaginary roots at least.

(50.) It may not be improper to remark here that when the equation proposed for examination has any of its terms wanting, as in the three last examples, we may always, by applying to it the rule of the double sign at once, determine the least number of imaginary roots that the equation can possibly have. Thus, in the last example, the signs of the proposed are

$$+\,0-0-+-+;$$

instead of which, the rule of the double sign gives the two series,

$$+--+-+-+$$
$$++---+-+;$$

which, because they differ by two variations, establish the existence of at least two imaginary roots in the equation.

(51.) In equations of the form

$$x^n + N = 0,$$

this method makes known immediately the number of imaginary roots. For example, suppose the equation is

$$x^6 - 1 = 0;$$

which gives the series

$$+\;0\;\;0\;\;0\;\;0\;\;0\;-,$$

and, by the rule of the double sign,

$$+-+-+--$$
$$++++++-,$$

in which the upper series has five variations, and the lower but one. Hence, there are four imaginary roots in the equation, which is, obviously the entire number; the two real roots being $+1, -1$.

From a mere inspection of this upper and lower series, it is obvious that, in all cases, when $m+1$ zeros intervene in an equation between *unlike* signs, there must exist at least m imaginary roots; and when $m+1$ zeros intervene between *like* signs, there must exist at least $m+2$ imaginary roots.*

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* These particulars are not overlooked by *Fourier* in his *Analyse des Equations*, before referred to; but, to establish them, he seems to

think it necessary first to go through the entire system of derived or limiting equations obtained by successive differentiation; not reflecting that, even in perfect accordance with his own views of the subject, the proposed equation may be regarded as a *derivee* from a preceding one; and the rule of signs applied immediately, without the aid of any subsequently derived functions, (see Analyse des Equations, page 109.) We may take this opportunity of remarking further that, in the actual determination of the successive series of signs . . . . (— 10), (— 10), (0), (1), (10) . . . ., in the examples in the text, we have always considered the transformations to be performed agreeably to the process explained in (21). This is very different from the manner in which Fourier directs the several series to be determined, and is much more easy and expeditious. Obvious as this shorter method may appear, after the simple proposition at (20) is established, yet, as far as known to the author, it was first practically adopted by *Mr. Horner*, it being the process by which he continually diminishes the roots of an equation in his "New method of solving Numerical Equations of all Orders," published in the Philosophical Transactions for 1819. It is true that *Budan*, in his "Nouvelle Méthode pour la Résolution des Equations Numériques," published in 1807, has used a similar method of transformation when the roots are to be diminished or increased by *unity* only; but his work contains no intimation that the same process is universally applicable. In fact, his mode of arriving at a transformed equation in $(x - u)$, is to proceed, step by step, through all the preceding transformations in $(x - 1)$, $(x - 2)$, &c. unless u is 10, 20, 30, ... 100, 200 . . . . &c. in which case he recommends a preliminary transformation of the proposed equation, obtained by substituting for x, $10x'$, $20x'$, &c. and shows that the diminution of the roots of this by first one unit, then another, and so on, is equivalent to diminishing the roots of the proposed by first one *ten*, then another, and so on. (See Nouvelle Méthode, &c. pp. 16, 20.)

CHAPTER V.

ON THE SOLUTION OF CUBIC EQUATIONS.

(52.) In the foregoing chapter we have explained at length the method of separating the roots of an equation, that is, of determining the intervals within which they must be sought. Some of these intervals we have, however, called *doubtful intervals*, because of the uncertainty as to whether they comprehend real roots or only indications of imaginary ones. Hence, before applying any process of numerical solution, it would seem more methodical first to ascertain the real character of those intervals, in order that in the business of solution we may not waste our efforts in attempting to evolve real roots from where none but imaginary ones exists. We have already remarked that this important inquiry has been successfully entered upon by *Fourier*, and more satisfactorily still by *Sturm*; and although it must be acknowledged that this is the proper place for such an inquiry, yet, rather than detain the student longer from the praxis, we have thought it expedient to defer it till the close of this First Part; and therefore in exemplifying the numerical process, we shall always take care, throughout this and the following chapter, so to frame our examples that the intervals shall not *all* be doubtful. For cubic equations, indeed, any such caution is altogether unnecessary, because, as every equation of an odd degree has at least one real root, whose situation in the numerical scale is always discoverable by the method in last chapter, there will be at least one interval free from doubt, to which our process of evolution may be effectively applied; and when one root of a cubic equation is determined, the equation may be depressed to a quadratic by an easy operation (10), and thence the nature of the other two roots at once ascertained. As far, therefore, as the requisitions of the present chapter are concerned, the materials furnished by the preceding one are abundantly sufficient. But our principal reasons for devoting a distinct chapter to this class of equations are, first, because we shall thus the better prepare the student for the general method of solution to be afterwards given; and, secondly, because the investigation of the rule for cubics may be rendered independent of the principles which, in the general method, we shall find it necessary to refer to.

(53.) Let

$$x^3 + A_2 x^2 + Ax = N \quad \ldots \ldots (\text{I}),$$

be any cubic equation, N, for convenience sake, being transposed to the right hand side; and let r represent the first figure or highest denomination of the root, determined by the method at page 44, then it is plain that r is such a number that, if N be divided by

$$r^2 + A_2 r + A \quad \ldots \ldots (1),$$

the quotient will be r, or rather r will be the first figure of that quotient. Let the remaining part of the root be called x', and let it consist of the several figures r', r'', r''', &c. then, since $x = r + x'$, or, which is the same thing, $x' + r = x$, we have

$$\begin{array}{rcl} Ax' + Ar & = & Ax \\ A_2 x'^2 + 2A_2 rx' + A_2 r^2 & = & A_2 x^2 \\ x'^3 + 3r x'^2 + 3r^2 x' + r^3 & = & x^3 \\ \hline x'^3 + A'_2 x'^2 + A'x' + R & = & N; \end{array}$$

the terms in the last line being the sums of those under which they are respectively placed, A'_2, A', R, and N, being known numbers.

Now, by transposing R, and putting N′ for N — R, we have

$$x'^3 + A'_2 x'^2 + A' x' = N' \quad \ldots \ldots (\text{II}),$$

an equation similar to the first; and, since r' is the first figure of the root x' of this equation, r' must obviously be such that, if N′ be divided by

$$r'^2 + A'_2 r' + A' \quad \ldots \ldots (2),$$

the quotient, or at least the first figure of it, must be r'. If therefore we knew this divisor, we could immediately determine r', the second figure in the root x; but it is impossible that we can know the entire divisor without knowing r', seeing that r' enters into its formation; a part of this divisor, however, we do know, viz. the term A', since this portion of it is entirely independent of r', and if we examine into the formation of A', and reflect that r', being a place lower in the arithmetical scale than r, must in all cases be less than r, we shall readily perceive that, in most instances, A' must form a very large portion of the entire divisor; and, being known, may therefore be advantageously employed as a search-divisor, or trial-divisor for finding r'; the accuracy of this estimated value of r' may then be readily tested by completing the divisor with it, and trying whether or not our estimation of r' is correct. In this method of anticipating the new figure r' of the root x, the student will at once recognize an analogy to the arithmetical process of extracting the square root, where each succeeding figure of the quotient is suggested by a trial divisor, formed from the preceding figures, and verified by the true divisor which the new figure is employed to complete.

Supposing then r' to be thus found, and let the remainder of the root x' be called x''; then $x'' + r' = x'$, whence

$$\begin{array}{rcl} A'x'' + A'r' & = & A'x' \\ A'_2 x''^2 + 2A'_2 r' x'' + A'_2 r'^2 & = & A'_2 x'^2 \\ x''^3 + 3r' x''^2 + 3r'^2 x'' + r'^3 & = & x'^3 \\ \hline x''^3 + A''_2 x''^2 + A'' x'' + R' & = & N'; \end{array}$$

therefore, by transposing R′, and putting N″ for N′ — R′, we have

$$x''^3 + A''_2 x''^2 + A'' x'' = N'' \quad . \; . \; . \; . \; (\text{III});$$

an equation also similar to the first, and in which A''_2, A'', and N'', are known numbers; and as r'' is the first figure of the root x'' of this equation, r'' must be such that, if N'' be divided by

$$r''^2 + A''_2 r'' + A'' \quad . \; . \; . \; . \; (3),$$

the quotient must be r''. Now r'' will be suggested by the trial divisor A'', which is known, and its correctness afterwards tested by the true divisor (3), as in the former step. It is plain that, by carrying on this process, the entire root, whose first figure is r, may be evolved figure by figure, and that this evolution may be rapidly effected, provided we can devise ready methods for the formation of the successive divisors (2), (3), (4), &c. which, as we have just seen, are always discoverable by help of the trial divisors, A′, A″, A‴, &c.

The first divisor (1) is formed at once by means of the first figure r of the root found by trial; but, to render the derivation of the divisors a uniform operation throughout, we shall consider (1) to be obtained by adding r to A_2, multiplying the sum by r, and then adding A to the result: thus

$$\begin{array}{l} A \\ r^2 + A_2 r \\ \hline r^2 + A_2 r + A \quad \text{the first divisor.} \end{array}$$

To determine now the trial divisor A′, which is to assist us in discovering the next figure r' of the root, we have merely to place r^2 underneath the first divisor, and to add it, and the two expressions immediately above it, into one sum, thus:

$$\begin{array}{l} A \\ r^2 + A_2 r \\ \hline r^2 + A_2 r + A = \text{first divisor} \\ r^2 \\ \hline 3r^2 + 2A_2 r + A = A'. \end{array}$$

With this trial divisor A′ the next figure r' may be ascertained, and then the second divisor (2) completed. In order to this, we must first get A'_2, thus:

$$3r + A_2 = A'_2;$$

and then form the true divisor, as also the next trial divisor, as before, viz.

$$\begin{array}{l} A' \\ r'^2 + \ A'_2 r' \\ \hline r'^2 + \ A'_2 r' + A' = \text{second divisor} \\ r'^2 \\ \hline 3r'^2 + 2A'_2 r' + A' = A''. \end{array}$$

We have now the trial divisor A″ for the determination of the next figure r'', which, when found, we have as before,

$$3r' + A'_2 = 3(r + r') + A_2 = A''_2;$$

and, consequently, by continuing the process, we have

$$\begin{array}{l} A''_2 \\ r''^2 + \ A''_2 r'' \\ \hline r''^2 + \ A''_2 r'' + A'' = \text{third divisor} \\ r''^2 \\ \hline 3r''^2 + 2A''_2 r'' + A'' = A'''; \end{array}$$

a new trial divisor, and so on. Hence the several figures of the root may be evolved by a uniform operation; and, to render the arithmetical process as concise as possible, the work may be arranged agreeably to the following rule.

(54.) Put down A, the coefficient of x, and a little to the right place the absolute number, which is to be considered as a dividend, the figures of the root forming the quotient.

Place the first figure of the root, found by trial, in the quotient, above which write the coefficient of x^2, observing that its units' place be over the units' place of the quotient.

Multiply the value of the quotient figure, taking in those above, by that value; add the product to A, and the sum is the first divisor.

Write the square of the quotient figure, just found under the first divisor, add it to the two sums immediately above, and the result will be the trial divisor for finding the next figure.

Find now the next figure of the root, and to its value, including those above it, prefix three times the preceding, taking in the value of the figure above it, multiply the result by the last found figure, add the product to the trial divisor, and we shall have the true divisor; and in the same manner are the succeeding divisors to be obtained.

EXAMPLES.

1. Extract the root of the equation $x^3 + 8x^2 + 6x = 75·9$.

Here we find the first figure of the root to be 2, therefore the operation will be as follows:

		$8 = A_2$	
	$6 = A$	75·9(2·4257 the root.	$r = 2$
$A_2 + r$ =	10...20	52	
	26	$23·9 = N'$	$r' = ·4$
	4	22·304	
	$50 = A'$	$1·596 = N''$	$r'' = ·02$
$A'_2 + r'$ =	14·4... 5·76	1·239688	
	55·76	$·356312 = N'''$	$r''' = ·005$
	·16	·311827625	
	$61·68 = A''$	$44484375 = N''''$	$r'''' = ·0007$
$A''_2 + r''$ =	15·22... ·3044	43716797593	
	61·9844	$767577407 = N'''''$	
	4	&c.	
	$62·2892 = A'''$		
$A'''_2 + r'''$ =	15·265... 76325		
	62·365525		
	25		
	$62·441875 = A''''$		
$A''''_2 + r''''$ =	15·2757.. 1069299		
	62·45256799		
	&c.		

In the foregoing operation, which has been performed exactly according to the rule, it will be perceived, that after the first decimal place in the root has been found, more decimals have been used in the succeeding parts of the work than were absolutely necessary for the extent to which the root has been carried; for if, after having got so many as three places of decimals in the last column of the work, we had desisted admitting any more, and had rejected all the other places to the right, we should still have had the root equally correct to four places of decimals. Now, in order that the number of decimals in the last column may not exceed three, it is obvious that the divisor corresponding to the first decimal in the root must contain but two decimals; the divisor corresponding to the next decimal of the root must contain but one; and that corresponding to the next succeeding root figure must not contain any; and for every subsequent decimal in the root the right-hand digit in the corresponding divisor must be struck off.* In the same manner, after one decimal in the root is obtained, the numbers in the first column are to be diminished, in order that the decimals in the second column may not exceed the necessary number: it therefore follows, that the operation of annexing each new figure of the root to thrice the preceding, and also that of placing the square of each new figure under the preceding divisor, become quite unnecessary after the first decimal in the root is found. Hence the work of the preceding example may be rendered more concise, and will stand as follows:

---

* It must be observed, however, that although a figure is thus cut off each time, yet, in the multiplication, the product that would have arisen from this figure is to be ascertained; and although nothing is to be put down, yet, what would have been carried, is still to be carried for the increase of the next figure; and, indeed, if the figure that would have been put down be 5, or upwards, then a *unit more* is to be carried to the next figure, exactly the same as in *contracted multiplication.* Thus, although in the operation in the text the figure 8 is struck off from the divisor 6198, yet, since the product of the 8 by 2 is 16, 2 is carried for the increase of the next figure.

		8
	6	75·9 (2·4257 the root
10	20	52
	26	23·9
	4	22·304
	50	1·596
14·4	5·76	1·240
	55·76	·356
	·16	312
	61·68	44
15·2	·30	43
	61·98	1
	62·3	
	1	
	62·4	

Here the last two figures of the root are obtained by common division, the figures in the first column having all been struck off.

By comparing the former arithmetical process, p. 84, with the several steps of the general investigation, we immediately see that the equations marked (I), (II), (III), have, in our particular example, the forms

$$x^3 + 8x^2 + 6x = 75{\cdot}9 \quad \ldots . \ (\text{I}),$$

$$x'^3 + 14x'^2 + 50x' = 23{\cdot}9 \quad \ldots . \ (\text{II}),$$

$$x''^3 + 15{\cdot}2x''^2 + 61{\cdot}68x'' = 1{\cdot}596 \quad \ldots . \ (\text{III}).$$

The roots of (II) are those of (I) diminished by $r = 2$; the roots of (III) are those of (I) diminished by $r + r' = 2{\cdot}4$. In like manner, taking the next transformed equation, the roots of

$$x^3 + 15{\cdot}26x^2 + 62{\cdot}2892x = {\cdot}356312$$

are those of (I) diminished by $r + r' + r'' = 2{\cdot}42$. The roots of

$$x^3 + 15{\cdot}275x^2 + 62{\cdot}441875x = {\cdot}044484375$$

are those of (I) diminished by $r + r' + r'' + r''' = 2{\cdot}425$; and, finally, by extending the first two columns of the work one step further, we have, for the last transformation, the equation

$$x^3 + 15{\cdot}2771x^2 + 62{\cdot}46326147x = {\cdot}000767677407,$$

whose roots are those of (1) diminished by $r + r' + r'' + r''' + r'''' =$ 2·4257; and, consequently, this is the very same equation that we should get by diminishing at once the roots of (1) by 2·4257, according to the method explained in (21). As 2·4257 is equal to one of the roots, as far at least as the decimals have been carried, its defect from the complete root is regarded as nothing; so that, as one of the roots is thus exhausted, the other two diminished by 2·4257 must be given by the equation

$$x^2 + 15{\cdot}2771x + 62{\cdot}46326147 = 0;$$

the operation, therefore, while it supplies us with one root, furnishes also the depressed equation for determining the others. As it is easy to see that the absolute number is greater than one fourth the square of the coefficient of x, we know that the roots of the quadratic are imaginary (Algebra, page 100).

2. Extract the root of the equation

$$x^3 + 5x^2 + 7x = 47.$$

Here the first figure of the root is 2, and the operation is as follows:

		5
	7	47 (2·1238 the root
7	14	42
	21	5
	4	4·011
	39	·989
11·1 . . .	1·11	·829
	40·11	·160
	1	·125
	41·23	35
11·3	·23	33
	41·46	2
	41·7	

3. Extract the root of the equation

$$x^3 + 2x^2 + 3x = 13089030.$$

Here the highest denomination of the root is 200.

		2
	3	13089030 (235 the root
202	40400	80806
	40403	500843
	4	419289
	120803	815540
632	1896	815540
	139763	
	9	
	159623	
697	3485	
	163108	

By carrying the first two columns one step further, we have, for depressed equation containing the other roots diminished by 235, quadratic

$$x^2 + 707x + 16618 = 0,$$

the roots of which are real.

4. Extract the root of the equation

$$x^3 + 24x = 68.$$

Here the first figure of the root is found to be 2.

	24	68 (2·3158 the root
	4	56
	28	12
	4	11·367
	36	·633
6·3	1·89	·399
	37·89	·234
	9	·200
	39·87	34
6·9	7	32
	39·94	2
	400	

5. Extract the root of the equation

$$x^3 + x^2 = 500,$$

to about eight places of decimals.

In this example the first figure of the root is 7.

		1
	0	500 (7·61727975 the root
8	56	392
	56	108
	49	104·736
	161	3·264
22·6	13·56	1·887181
	174·56	1·376819
	36	1·323862
	188·48	52957
23·81	2381	37859
	188·7181	15098
	1	13251
	188·9563	1847
23·84	1669	1704
| | |	189·1232	143
	——|	133
	189·290	10
	5	9
	189·295	1
	| | | | |	

6. Extract the root of the equation

$$x^3 - 17x^2 + 54x = 350,$$

to about 10 places of figures.

Here the first denomination of the root is 10.

		−17
	54	350 (14·95406861 the roo
−7	−70	−160
	−16	510
	100	328
	14	182
17	68	170·379
	82	11·621
	16	10·740875
	166	·880125
25·9	23·31	·865276
	189·31	14849
	·81	12986
	213·43	1863
27·75	1·3875	1731
	214·8175	132
	25	130
	216·2075	2
27·85	·1114	2
	216·3189	
	216·430	

7. Extract the root of the equation

$$x^3 + 24{\cdot}84x^2 - 67{\cdot}613x = 3721{\cdot}2758,$$

:o about 10 places of figures.

Here the highest denomination of the root is 10.

		24·84
	−67·613	3761·2758 (11·19733377 the root
34·84	348·4	2807·87
	280·787	953·405
	100	785·027
	729·187	168·3788
55·84	55·84	84·7661
	785·027	83·6127
	1	77·283513
	841·867	6·329187
57·94	5·794	6·050544
	847·661	·278643
	1	·259437
	853·465	29206
58·23	5·2407	25944
	858·7057	3262
	81	2594
	863·9545	668
58·41	·4089	605
|||	864·3634	63
	|	60
	864·772	3
	17	
	864·789	
	|	
	864·81	
	||||	

8. Required a positive root of the equation

$$x^3 - 7x = -7.$$

In this equation, whatever term in the series 0, 1, 2, 3, &c. is substituted for x, the result is always greater than -7, therefore, two roots must lie between that pair of numbers in the above series, the substitutions of which produce results the nearest to -7; these numbers are found to be 1 and 2; therefore 1 must be the first figure of each of the positive roots.

```
                  — 7              — 7 (1·356895 the root
                    1              — 6
                  ———              ———
                  — 6              — 1
                    1              — ·903
                  ———              ———
                  — 4              — ·097
3·3 . . . .       ·99              — 86625
                  ———              ———
                  — 3·01           — 10375
                       9           — 9049
                  ———              ———
                  — 1·93           — 1326
3·95 . . . .      ·1975            — 1185
                  ———              ———
                  — 1·7325         — 141
                         25        — 133
                  ———              ———
                  — 1·5325         — 8
4·05 . . . .             243       — 7
 | |              ———              ———
                  — 1·5082         — 1
                  ——— |
                  — 1·484
                           3
                  ———
                  — 1·481
                      | | |
```

As no change of sign occurs in the last column, we infer that no root has been passed over in the process, and that, therefore, the second figure of the other root must exceed ·3.

It is plain that the roots which seem to lie between 1 and 2 are not imaginary, because, if for 3 in the second figure we had put 4, a change of sign would have taken place, thus showing that a real root lies between 1·3 and 1·4.

The approximation to the other positive root is as follows:

	−7	−7 (1·6920214 the root
	1	−6
	−6	−1
	1	−1·104
	−4	·104*
3·6	2·16	·100809
	−1·84	3191
	36	3156868
	·68	34112
4·89	·4401	31774
	1·1201	2338
	81	1589
	1·5683	749
5·072	10144	635
	1·578444	114
	4	111
	1·588592	3
	101	3
	1·588693	

As the coefficient of the second term, with its sign changed, is always equal to the sum of the roots, and, as in this example, the coefficient of the second term is 0, it follows that the remaining root of

* The change of sign here shows that a root lies between 1 and 1·6, and this is the root determined in the preceding page.

the proposed equation is

$$-1{\cdot}356895 - 1{\cdot}692021 = -3{\cdot}048916.$$

9. Extract the root of the equation $x^3 - 6x^2 + 18x = 22$.

Ans. $x = 2{\cdot}3274$.

10. Extract the root of the equation $x^3 + 4x^2 + 2x = 2328$.

Ans. $x = 12$.

11. Extract the root of the equation $x^3 + 8x = 34648584$.

Ans. $x = 326$.

12. Extract the root of the equation $x^3 + 2{\cdot}5x^2 + 2x = {\cdot}5$.

Ans. $x = {\cdot}1974$.

13. Extract the root of the equation $x^3 + 4{\cdot}73x = 1{\cdot}746$.

Ans. $x = {\cdot}3594$.

14. Extract the root of the equation $x^3 + 9x^2 + 4x = 80$.

Ans. $x = 2{\cdot}4721$.

15. Extract the root of the equation $x^3 - 12x = -8$.

Ans. $x = {\cdot}6945928$.

16. Extract the root of the equation $x^3 + x^2 + x = 100$.

Ans. $x = 4{\cdot}26442997$.

17. Extract the root of the equation $x^3 + 10x^2 + 5x = 2600$.

Ans. $x = 11{\cdot}00679934$.

18. Extract the root of the equation $x^3 - 2x - 5 = 0$.

Ans. $x = 2{\cdot}09455148$.

19. Extract the root of the equation $x^3 + 2x^2 - 23x = 70$.

Ans. $x = 5{\cdot}13457873$.

20. Extract the root of the equation $x^3 - 7035x^2 + 15262754x = 10000730880$.

Ans. $x = 2345$.

21. It is required to determine the three roots of the equation $x^3 - 21x + 21 = 0$.

$$\text{Ans. } x = \begin{cases} 1{\cdot}05608970 \\ 3{\cdot}96233441 \\ -5{\cdot}01842411. \end{cases}$$

ON THE EXTRACTION OF THE CUBE ROOT.

(55.) From the preceding method of extracting the roots of cubic equations, may be derived a new method of extracting the cube root of numbers, which will be much more easy and concise than the method usually given.

Suppose, for example, it were required to extract the cube root of the number 12326391, or which is the same thing, to extract the root of the cubic equation

$$x^3 = 12326391.$$

By proceeding according to the method in article 52, the process will be as follows:

	0	12326391 (231 the root
	4	8
	4	4326
	4	4167
	12	159391
63	189	159391
	1389	
	9	
	1587	
691	691	
	159391	

From inspecting the above operation, it will be obvious that some of the work is superfluous; thus, the first trial divisor, 12, might have been easily found at once, by multiplying the square of the root figure 2, by 3; also, since the numbers that are placed under the trial divisors to be added thereto, always have two figures to the right, when the addition is performed they are written down again; but this repetition would be avoided if these two numbers were placed *at first* a line lower down, and only the other figures placed *immediately under* the trial divisor, but then, in afterwards adding the square of the new figure, these two figures must be repeated *twice* in the addition, so that we have the following

New Method to extract the Cube Root of any given Number.

(56.) Divide the given number into periods of three figures each, as in the common method, and find the nearest cube to the first period, subtract it therefrom, and put the root in the quotient; then thrice the square of this root will be the trial divisor for finding the next figure.

Draw a line a little below the trial divisor, multiply the new figure, with thrice the preceding prefixed, by the new figure, and place the first two figures of the product *below* this line, to the right of the trial divisor, and the others *above* the line; add them to the trial divisor, and the sum will be the true divisor.

Under this divisor write the square of the last root figure, which add to the two sums above, repeating the two final figures of the divisor, and the result is the next trial divisor; the true divisor is found as before, &c.

NOTE. After the first or second decimal place in the root is found, the square of the root figure used in forming the trial divisor may be omitted, and also those two figures that would fall below the line in forming the true divisor, as the value of these figures will be too small for their omission to affect the truth of the result. But, if the number of decimals in the root is required to be very great, these omissions must not be made till after the third or fourth decimal in the root is found.*

* At whatever divisor these contractions take place, as many more decimals of the root will be obtained as there are figures in this divisor, *minus* one, although the last decimal thus obtained, if the root has been extended to fourteen or fifteen places, is not always to be depended upon.

The preceding example, by this rule, will stand thus:

	12	12326391 (231
	1	8
63	——	——
	1389	4326
	9	4167
	——	——
	1587	159391
	6	159391
691	——	——
	159391	——
	——	

2. Extract the cube root of the number 673373097125.

Here the nearest root of the first period, 673, is 8; hence the operation is as follows:

	192	673373097125 (8765
	17	512
247	——	——
	20929	161373
	49	146503
	——	——
	22707	14870097
	156	13718376
2616	——	——
	2286396	1151721125
	36	1151721125
	——	——
	2302128	——
	1314	
26285	——	
	230344225	
	——	

3. Extract the cube root of 3 to three places of decimals.

	3	3 (1·442
	1	1
3·4		
	4·36	2
	·16	1·744
	5·88	·256
	·17	·242
4·2		
	605	14
		12
	62	
		2

4. Extract the cube root of 3 to six places of decimals.

	3	3 (1·442249
	1	1
3·4		
	4·36	2
	·16	1·744
	5·88	·256
	·16	·241984
4·24		
	6·0496	14016
	16	12459
	6·2208	1557
	86	1248
4·32		
	6·2294	309
		250
	6·238	
	1	59
		56
	6·239	
		3
	624	

5. Extract the cube root of 9 to about fourteen or fifteen places of :imals.

```
                       12                    9 (2·080083823051904*
                                             8
 6·08 . . . .   ───────                      ─
                12·4864                      1
                     64                       ·998912
                ───────                      ───────
                12·9792                        1088
                      4                        1038375936512
 6·24008 . . . . ─────────────                 ─────────────
                12·9796992064                    49624063488
                           64                    38940651420
                ─────────────                    ───────────
                12·9801984192                    10683412068
                       187207                    10384192682
 6·24024 . . . . ─────────────                   ───────────
   | | |        12·9802171399                      299219386
                ─────────────|                     259604919
                12·980235861                       ─────────
                        4992                        39614467
                ────────────                        38940738
                12·980240853                        ────────
                ────────────|                         673729
                12·98024585                           649012
                         12                           ──────
                ───────────                            24717
                12·98024597                         ·· 12980
                ───────────|                           ─────
                129802461                              11737
                  |||||||                              11682
                                                       ─────
                                                          55
                                                          52
                                                          ──
                                                           3
```

* Another figure might have been obtained, viz. 2, but, on account the extent to which the root has been carried, this figure could not e been depended on as true; all the fifteen places, however, that have found, are true to the last figure.

6. Extract the cube root of 6 to about six places of decimals.
Ans. 1·817120.

7. Extract the cube root of 18609625.
Ans. 265.

8. Extract the cube root of 5 to six places of decimals.
Ans. 1·709975.

9. Extract the cube root of 469640998917.
Ans. 7773.

10. Extract the cube root of 2.
Ans. 1·25992105.

11. Extract the cube root of 1·25992105.
Ans. 1·08005974.

12. Extract the cube root of 6692234354139671875.
Ans. 1884475.

CHAPTER VI.

ON THE SOLUTION OF EQUATIONS OF THE HIGHER ORDERS.

(57.) The general method of solving numerical equations of all orders, which we now propose to explain, is the discovery of W. G. Horner, esq., of Bath, and was first published in 1819, in the Philosophical Transactions of the Royal Society of London. It is a process of remarkable simplicity and accuracy, consisting merely of a series of easy transformations, conducted according to the directions in Arts. (20), (21,) and blended in a continuous course of recurring operations, by which the figures of the root are evolved one by one.

When the first figure of a root of the equation

$$A_n x^n + \ldots . A_3 x^3 + A_2 x^2 + Ax + N = 0$$

is determined, we have seen, by the examples in (21), how easy it is to obtain the transformed equation

$$A'_n x'^n + \ldots . A'_3 x'^3 + A'_2 x'^2 + A' x' + N' = 0,$$

involving the remaining portion of the root; and, as this portion forms one of the entire roots of the transformed, if the first figure of it be found, we shall have the second figure in the original root, and, by a repetition of the process of transformation, we shall get a new equation, involving the following figures of the root. The evolution of any root would, therefore, be effected, by finding the first figure by trial, or otherwise, and diminishing the roots by it; then finding the first figure of the transformed, and diminishing the roots by it, and so on till the proposed root be entirely evolved, or determined to any required number of decimals.

It is evident that after the determination of the first figure, and thence of the first transformed equation, we shall not be left to conjecture the value of the following figure; for, as in the case of cubics in the last chapter, we may regard N′, when transposed to the right, as a dividend; and, if the true first figure of the root x' be r', we shall have so to determine r' that, when the dividend is divided by

$$A'_n r'^{n-1} + \ldots . A'_3 r'^2 + A'_2 r' + A',$$

the quotient may be r'; and we are evidently assisted in this determination of r' by A′, the known portion of the true divisor. The influence of this *trial divisor* will indeed be readily foreseen, after what has been done in the preceding chapter.

When, by means of the trial divisor, the new figure r' of the root is ascertained, and the divisor completed, we may proceed to the next transformation by diminishing the roots of the last transformed equation by r'; we shall thus have an equation of the form

$$A''_n x''^n + \ldots . A''_3 x''^3 + A''_2 x'' = N'';$$

the first figure r'' in the root of which must be such that, when N″ is divided by

$$A''_n r''^{n-1} + \ldots . A''_3 r'' + A''_2,$$

the quotient must be r'', and, for discovering r'', we have the trial divisor A''_2, which is previously known.

It is plain, therefore, that the determination of the several root figures, r, r', r'', &c. in succession, is effected by a continuous and uniform arithmetical process; the several trial divisors A$'$, A''_2, &c., all presenting themselves as they are wanted, in passing from one transformation to another.* This process may be described in words as follows:

GENERAL RULE.

Arrange the coefficients of the given equation in a row, commencing with that of the first term.

Add the product of the first root figure, found by trial, and the first coefficient to the second coefficient; the product of the sum and same figure to the third coefficient, and so on to the last coefficient, A; and the last sum will be the divisor.

Repeat this process with the first coefficient, and these sums, and the number under the last sum, will be the trial divisor, A$'$, for the next figure.

Perform a similar process with the first coefficient and these second sums, stopping under the n — 1th coefficient, A_2.

Perform again a similar process with the same first coefficient and these last sums, stopping here under the preceding, or n — 2th, coefficient, A_3; and so on till the last sum falls under the second coefficient.

---

* Notwithstanding the great value of the trial divisors, yet, when the roots of an equation are not all real, the figures suggested by these divisors are sometimes liable to suspicion, which suspicion would not be completely removed, although the suggested figures be submitted to all the tests hitherto proposed to determine their character. Even when the roots are all real, and differ but little from each other, perplexities would occur in our search for the leading figure of the root of each transformed equation, were it not for the theorem of *Budan*, (page 65 and 67.) This theorem, however, always enables us to detect, by means of the variations of sign in any transformed, whether or not roots have been passed over, and thus to retrace any faulty step. But it is to the theorem of Sturm, discussed in the next Chapter, that we are indebted for the complete removal of every doubt and perplexity in the numerical solution of equations.

Find now, from the trial divisor, the next figure of the root, and proceed with the last set of sums, and this new figure, exactly the same as with the original coefficients and the first figure, in finding the preceding divisor, and the next divisor will be obtained; and in a similar manner are the other divisors to be determined.

NOTE. After the first or second decimal of the root is obtained, the work of each column may be contracted as in cubic equations.*

(58.) We shall now exhibit several examples of the arithmetical process conducted agreeably to the above rule, which will be seen to involve only repetitions of the operation at page 27.

Quadratic equations, whose coefficients are high numbers, are solved much more expeditiously by this rule than by that given in the Algebra, as the trouble is little more than that of extracting the square root: we shall commence with a few of these.

EXAMPLES.

1. Given the equation $x^2 - 700\,x = 59829$ to find the value of x.

The first denomination in the root is soon seen to be 700; hence the process is as follows:

1	-700	59829 (777, the root.
	700	00000
true divisor	000	$59829 = N'$
	700	53900
trial divisor	700	$5929 = N''$
	70	5929
true divisor	770	——
	70	
trial divisor	840	
	7	
true divisor	847	

* In biquadratic, and higher equations, these contractions may always be made after the first decimal in the root is found, unless the root be required to an unusual number of places.

2. Given the equation $x^2 + x = 60$, to find the value of x.

Here the first figure of the root is 7.

1	
7	
8	60 (7·26208734816, the root.
7	56
15·2	$4 = n'$
·2	3·04
15·46	$·96 = n''$
6	·9276
15·522	324
2	31044
1|5|·5,2|4,0|8	1356
	12419264
	1140736
	1086686
	54050
	46572
	7478
	6210
	1268
	1242
	26
	16
	10
	9
	1

3. Given the equation $x^3 + 1728\frac{1}{3}x = 123578$, to find the value of x.

Here the value of x is between 60 and 70; hence the first figure of e root is 6 in the tens' place.

1728·3̇	123578	(68·7652857, the root.
60	10730	
1788·3̇	16278	
60	14850·6̇	
1848·3̇	1427·3̇	
8	1305·523̇	
1856·3̇	121·81	
8	111·9476	
1864·3̇	9·8624	
7	9·3293	
1865·03̇	5331	
7	3732	
1865·73̇	1599	
6	1493	
1865·793̇	106	
6	93	
1865·853̇	13	
5	13	
1|8|6|5·|8|5|8	—	

4. Given $x^2 + 1{\cdot}41421356x = 1{\cdot}73205081$, to find the value

```
1·41421356     1·73205081   (·78689818, the r
  7            1·47994949
-----------    ----------
2·1142135|6     ·25210132
  7             ·23153708
-----------    ----------
2·8142135        2056424
  8              1788128
-----------    ----------
2·894213|5        268296
  8               238961
-----------    ----------
2·974213           29335
  6                26890
-----------    ----------
2·98021|3           2445
  6                 2390
-----------    ----------
2·98621               55
   8                  30
-----------    ----------
2·9870|1              25
   8                  24
-----------    ----------
2·9|8|7|8              1
```

5. Extract the root of the biquadratic equation $x^4 - 3x^2 +$
= 10000 to three places of decimals.

Here the first figure of the root is 10.

```
1     0     — 3     75     10000(9·886, the root
      9      81     702     6993
     --     ---     ---     ----
      9      78     777     3007
      9     162    2160     2678
     --     ---    ----     ----
     18     240    2937      329
      9     243     410      306
     --     ---    ----     ----
     27     483    334|7      23
      9      29     43        23
     --     ---    ----     ----
     3|6    51|2   378        —
              3     4
            ---    ----
            5|4   3|8|2
```

6. Extract the root of the preceding equation to about six places of decimals.

<table>
<tr><td>1</td><td>0</td><td>— 3</td><td>76</td><td>10000 (9·8860027, the root.*</td></tr>
<tr><td></td><td>9</td><td>81</td><td>702</td><td>6993</td></tr>
<tr><td></td><td>9</td><td>78</td><td>777</td><td>3007</td></tr>
<tr><td></td><td>9</td><td>162</td><td>2160</td><td>2677·6616</td></tr>
<tr><td></td><td>18</td><td>240</td><td>2937</td><td>329·4384</td></tr>
<tr><td></td><td>9</td><td>243</td><td>409·952</td><td>306·1662</td></tr>
<tr><td></td><td>27</td><td>483</td><td>3346·952</td><td>23·2722</td></tr>
<tr><td></td><td>9</td><td>29·44</td><td>434·016</td><td>23·2616</td></tr>
<tr><td></td><td>36·8</td><td>512·44</td><td>3780·968</td><td>106</td></tr>
<tr><td></td><td>8</td><td>30·08</td><td>46·110</td><td>78</td></tr>
<tr><td></td><td>37·6</td><td>542·52</td><td>3827·07|8</td><td>28</td></tr>
<tr><td></td><td>·8</td><td>30·72</td><td>46·36</td><td>27</td></tr>
<tr><td></td><td>38·4</td><td>573·24</td><td>3873·44</td><td>1</td></tr>
<tr><td></td><td>·8</td><td>3·14</td><td>3·50</td><td></td></tr>
<tr><td></td><td>3|9·|2</td><td>576·3|8</td><td>3876·9|4</td><td></td></tr>
<tr><td></td><td></td><td>3·1</td><td>3·5</td><td></td></tr>
<tr><td></td><td></td><td>579·|5</td><td>3|8|80·4</td><td></td></tr>
<tr><td></td><td></td><td>3</td><td></td><td></td></tr>
<tr><td></td><td></td><td>5|8|3</td><td></td><td></td></tr>
</table>

* Thus, we have found the root to seven places of decimals, by bringing down only one period of decimals; if another period had been brought down, we should have found the root to be 9·88600270094, true to twelve places of figures.

7. Extract the root of the biquadratic equation

$$x^4 + 3x^3 + 2x^2 + 6x = 148{\cdot}6.$$

Here the first figure of the root is found to be 2.

1	3	2	6	148·6(2·734400, the ro
	2	10	24	60
	5	12	30	88·6
	2	14	52	82·9731
	7	26	82	5·6269
	2	18	36·533	4·8977
	9	44	118·533	·7292
	2	8·19	42·609	·6627
	11·7	52·19	161·142	665
	7	8·68	2·114	664
	13·4	60·87	163·256	1
	7	9·17	2·13	
	13·1	70·04	165·39	
	7	41	28	
	13·8	70·45	165·67	
		4	2	
		70·9	165·9	

8. Extract the root of the equation

$$x^5 + 4x^4 - 3x^3 + 10x^2 - 2x = 962.$$

Here the first figure of the root is found to be 3.

1	4	—3	10	—2	962 (3·35484874
	3	21	57	201	597
	7	19	67	199	365
	3	30	147	642	299·14833
	10	49	214	841	65·85167
	3	39	264	156·1611	59·87805
	13	88	478	997·1611	5·97362
	3	48	42·537	169·4514	4·92583
	16	136	520·537	1166·6125	1·04779
	3	5·79	44·301	30·9484	·98760
	19·3	141·79	564·838	1197·5609	6019
	3	5·88	46·092	31·3531	4940
	19·6	147·67	610·930	1228·914	1079
	3	5·97	8·037	2·544	988
	19·9	153·64	618·967	1231·458	91
	3	6·06	8·091	2·541	86
	20·2	159·70	627·06	1234·00	5
	3	1·03	8·21	·50	5
	20·5	160·73	635·3	1234·50	
	||	1 |	·6	·5|	
		161·7	635·9	1235·0	
		1 |	|||	||||	
		163			
		||			

9. **Extract the positive root of the equation**

$$x^5 + 6x^4 - 10x^3 - 112x^2 - 207x = 110.$$

Here the first figure of the root is 4.

<table>
<tr><td>1</td><td>6</td><td>−10</td><td>−112</td><td>−207</td><td>110(4·464101</td></tr>
<tr><td></td><td>4</td><td>40</td><td>120</td><td>32</td><td>−700</td></tr>
<tr><td></td><td>10</td><td>30</td><td>8</td><td>−175</td><td>810</td></tr>
<tr><td></td><td>4</td><td>56</td><td>344</td><td>1408</td><td>667·05984</td></tr>
<tr><td></td><td>14</td><td>86</td><td>352</td><td>1233</td><td>142·94016</td></tr>
<tr><td></td><td>4</td><td>72</td><td>632</td><td>434·6496</td><td>133·46395</td></tr>
<tr><td></td><td>18</td><td>158</td><td>984</td><td>1667·6496</td><td>9·47621</td></tr>
<tr><td></td><td>4</td><td>88</td><td>102·624</td><td>477·4144</td><td>9·24089</td></tr>
<tr><td></td><td>22</td><td>246</td><td>1086·624</td><td>2145·0640</td><td>·23532</td></tr>
<tr><td></td><td>4</td><td>10·56</td><td>106·912</td><td>79·3352</td><td>·23158</td></tr>
<tr><td></td><td>26·4</td><td>256·56</td><td>1193·536</td><td>2224·399|2</td><td>374</td></tr>
<tr><td></td><td>·4</td><td>10·72</td><td>111·264</td><td>80·389</td><td>232</td></tr>
<tr><td></td><td>26·8</td><td>267·28</td><td>1304·800</td><td>2304·788</td><td>142</td></tr>
<tr><td></td><td>·4</td><td>10·88</td><td>17·453</td><td>5·434</td><td>139</td></tr>
<tr><td></td><td>27·2</td><td>278·16</td><td>1322·25|3</td><td>2310·22|2</td><td>3</td></tr>
<tr><td></td><td>·4</td><td>11·04</td><td>17·56</td><td>5·44</td><td>2</td></tr>
<tr><td></td><td>27·6</td><td>289·20</td><td>1339·8|1</td><td>2315·66</td><td>1</td></tr>
<tr><td></td><td>·4</td><td>1·68</td><td>17·6</td><td>14</td><td></td></tr>
<tr><td></td><td>2|8·|0</td><td>290·8|8</td><td>1357·4</td><td>2315·8|0</td><td></td></tr>
<tr><td></td><td></td><td>1·7</td><td>1·2</td><td>1</td><td></td></tr>
<tr><td></td><td></td><td>292·|6</td><td>1358·|6</td><td>2|3|1|5·9</td><td></td></tr>
<tr><td></td><td></td><td>1</td><td>1</td><td></td><td></td></tr>
<tr><td></td><td></td><td>|2|9|4</td><td>1|3|6|0</td><td></td><td></td></tr>
</table>

(59.) In each of the preceding examples every precept of the general rule has been strictly obeyed, and even the most obvious arithmetical abridgments have been purposely disregarded, in order that the entire process of solution by the new method might appear without the slightest disguise. It would, however, be unjust towards *Mr. Horner*, the accomplished author of this method, were we to conclude this part of our subject without adverting to some of the expedients which he recommends for reducing the numerical space and labour. The following example embodies the principal of these abridgments, slightly modified, to conform the process to that which we have given in the preceding Chapter for cubic equations.

10. Required a root of the equation

$$x^4 + 3x^3 + 2x^2 + 6x = 148\cdot6,$$

which has been solved by the unabridged process at page 108.

			3
	2	6	148·6 (2·734400
5	10	24	60
	12	30	88·6
	14	28	82·9731
	18	82	5·6269
	44	36·533	4·8977
11·7	8·19	118·533	·7292
	52·19	6·076	·6626
	8·68	161·142	666
	9·17	2·114	662
	70·04	163·256	4
13·8	41	1	
	70·45	165·38	
	4	28	
	70·9	165·66	

In this process the first column on the left is formed, like that in the operation for cubics, the multiplier being the index of the degree, in this case 4. In the second column each addend is formed from the immediately preceding one, by adding to it the square of the last root figure. The addends in the third column are as usual formed by multiplying those of the preceding column by the last root figure; and every *trial* divisor is the sum of the *three* numbers above it, as in cubic equations.

Precisely in this way may the first and second columns, as also the trial divisors, be always formed, whatever be the degree of the equation; but when there are intermediate columns of work between the second and that which supplies the trial divisors, as must always happen when the equation is above the fourth degree, the addends in these must each be formed, in the abridged method, by multiplying the corresponding addend in the preceding column by the last root figure, and, *at the same time*, taking in the addend immediately above; so that every addend in these intermediate columns helps to form the one immediately under it, by being incorporated with the product, which, in the unabridged process, is carried from column to column. The following example, taken with but little alteration from Mr. Horner's paper in the Philosophical Transactions, will sufficiently illustrate our meaning.*

* In the following operation we have marked the several *true* and *trial* divisors by the letters T, *t*, respectively.

11. **Required a root of the equation**

$$x^5 + 12x^4 + 59x^3 + 150x^2 + 201x = 207.$$

	59	150	201	12
12·6	7·56	39·936	113·9616	207 (·638605803327
	66·56	189·936	T = 314·9616	18·897696
	7·92	44·688	26·8128	18·02304
	8·28	49·656		13·9304119743
	8·64		t = 455·7360	
		284·280	8·61106581	4·0926280257
	91·40	2·755527		3·8031030022
15·03	·4509		T = 464·34706581	
		287·035527	8307243	·2895260235
	91·8509	2·769081		·2867504754
	·4518	·2782662	t = 473·04120405	
	·4527		2·34667123	27745481
	4536	292·587270	T = 475·38787528	23904795
		746634		
	93·2090		·0059807	3840686
	1202	293·333904		3824781
		74759	t = 477·7405272	
	93·3292		·1769318	15905
				14343
	120	74855	T = 477·9174590	
	120			1562
	120	294·8300	34	1434
	94·	564	t = 478·094425	128
			1475	96
		294·8864	T = 478·095900	32
		56		33
			t = 478·0974	
		56	2	
		295	T = 478·0976	

Hence the root is ·638605803327.

The only objection that could be brought against this mode of arranging the numerical process, is that in the third column of the work, such arrangement requires us to perform the operations of multiplication and addition simultaneously. But, in the case of a biquadratic equation, no such objection can apply, and, consequently, the foregoing arrangement of the work must be preferred to that at page 108, on the score of practical facility. We would, therefore, recommend the student to rework examples 5 and 6 by this shorter method, and to employ it in the equations of the fourth degree following.

(60.) We have observed above, that in the foregoing example, from Mr. Horner's paper, we have slightly modified the process, and it ought to be mentioned that, in so doing, we have, in fact, increased the length of the operation. This has arisen from our having *actually exhibited* the trial divisor derivable at every step from the last root figure.

By dispensing, however, with this, and merely writing under the true divisor, the addend which is due to the formation of the next trial divisor, we may, without actually performing the addition, readily foresee what the leading figures in that divisor would be, and thence discover the new figure of the root. In the foregoing example, where the root is entirely decimal, no inconvenience can arise from this suppression of the trial divisors, even from the commencement of the operation, on account of the smallness of the addends in the divisor column. But, where the addends are of considerable influence, we think it preferable always to exhibit the trial divisors. The work of the last example stands in Mr. Horner's paper as follows:

12	59	150	2 0 1	907·00000 (6388605803321
6	75 6	3993 6	1 1 3 9 6 1 6	188·97696
126	665 6	18993 6	3 1 4 9 6 1 6	18·0230400000
24	79 2	4468 8	1 4 0 7 7 4 4	13·9304 19743
3	82 8	4965 6	8 6 1 1 0 6 5 8 1	4·0926280257
1503	86 4	275 55 27	4 6 4 3 4 7 0 6 5 8 1	3·8031030021
12	4 509	28703 55 27	8 6 9 4 1 3 8 2 4	·2895250236
15	918 509	276 90 81	2 3 4 6 6 7 1 2 1 6	·2867504754
	4 518	278 26 62	4 7 5 3 8 7 8 7 5 2 6 6	27745482
	4 527	74 66 32	2 3 5 2 6 5 1 9 5 2	23904795
	4 536	29333 39 02	1 7 6 9 3 1 8 4	3840687
	1 20	74 75 92	4 7 7 9 1 7 4 5 9 0 6	3824781
	933 29	74 85 50	1 7 6 9 6 5 6 8	15906
	3 60	5 64	1 4 7 5	14343
	94	2 9488 64	4 7 8 0 9 5 9 0 0	1563
		11 28	1 4 7 5	1434
		2 95	2 3	129
			4 7 8 0 9 7 6 1	96
			2 3	33
				33

(61.) The plan of avoiding the double operation of multiplica and addition in the divisor column, as practised in the two exam before given, might, we think, have been introduced with advantag this solution.

We shall give another example, conducted upon the same pla the preceding, from Mr. Horner's paper, and shall afterwards ext the work with the divisor column modified, as proposed.

12. Required a root of the equation

$$x^3 + 3{\cdot}9x^2 - 1{\cdot}93x = -{\cdot}097.$$

which is the transformed equation obtained at page 92, after hav diminished the roots of the equation $x^3 - 7x = -7$ by 1·3.

3 9	— 1 9 3	— 97 (·056895867
5	1 9 7 5	— 86625
3 9 5	— 1 7 3 4 5	— 10375000
1 0 6	2 0 0 0	— 9048984
	2 4 3 3 6	
4 0 5 6		— 1326016
1 2	— 1 5 0 8 1 6 4	— 1184430
	2 4 3 7 2	
· · 40 6 8		— 14586
	3 2 5 4 4	
		— 132923
	— 1 4 8 0 5 3 7 6	
		— 8663
	3 2 5 5 0	
		— 7383
	3 6 6	
		— 1280
	— 1 4 7 7 9 1 7	
		— 1181
	3 6 6	
	2	— 99
	— 1 4 7 6 5 3	— 89
		— 10
		— 10

The solution, modified as explained above, will be as follows:

		3·9
	— 1·93	·097 (·056695867
3·95	1975	86625
	— 17325	10375
	25	9048984
4·056	24336	1326016
	— 1508164	1184430
	36	141586
4·068	3254	132923
	— 1480538	8663
	1	7383
	365	1280
	— 147692	1181
	2	99
	14765	88
		11
		10
		1

13. As a last illustration of this mode of working, we shall take the example at page 84, in which the process of extraction will be as follows:

		8
	6	75·9 (2·4257
10	20	52
	26	23·9
	4	22·304
14·4	5·76	1·596
	55·76	1·240
	16	356
15·2	30	312
	61·98	44
	8	43
	62·4	1

Examples for Practice.

14. Extract the root of the equation

$$x^4 - 12x^2 + 12x = 3.$$

Ans. $x = 2{\cdot}858083$.

15. Extract the root of the equation

$$x^3 - 17x^2 + 54x = 350$$

by the general rule.

Ans. $x = 14{\cdot}95406861$.

16. Required a root of the equation

$$x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321.$$

Ans. $x = 8{\cdot}4144547$.

17. Find the value of x in the equation

$$7x^2 + 2x = 36,$$

by the general rule.

Ans. $x = 2{\cdot}129424817$.

18. Find the value of x in the equation

$$x^4 + 4x^3 - 4x^2 - 11x + 4 = 0.$$

Ans. $x = 1{\cdot}63691356$.

19. Find the value of x in the equation

$$x^5 - 5x^3 + 5x - 1 = 0.$$

Ans. $x = {\cdot}20905693$.

20. Find the value of x in the equation

$$x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x = 654321.$$

Ans. $x = 8{\cdot}95697957$.

(62.) After having obtained one of the real roots of an equation by the foregoing process, we may employ the final transformation, as directed at p. 87, to obtain the other roots; or, by the application of the precepts in (10), we may determine the depressed equation involving the remaining roots. It will be found, however, that in most cases the readiest way will be to evolve the other roots directly from the proposed equation, by commencing a-new with every first figure. This, however, would require that we were possessed of à priori means of ascertaining the exact number and place in the arithmetical scale of the real roots of every equation. But, till lately, it was not in the power of algebra to furnish these means, except in a few instances, and the computist was in consequence often involved in doubt and perplexity, from being unable to separate the imaginary roots from the real. This separation had indeed long ago been shown to be theoretically possible, by the celebrated *Lagrange*, but the numerical labour, which the process involved, rendered it practically useless, except in the simplest and least doubtful cases; it is now, however, entirely superseded by the method of *Sturm*, the exposition of which will form the subject of the next chapter.

For further researches on the subject of the present Chapter, the student is referred to Mr. Horner's series of papers, in vol. 5 of Leybourn's Repository.

On the Determination of the Integral Roots by the Method of Divisors.

(63.) It was demonstrated at (13) that no equation in which the coefficients of the first term is unity, and those of the other terms integers, can have a fractional root; so that the roots of every such equation can comprise only whole numbers, and interminable decimals. These latter we have shown above how to approximate to as closely as we please; and, although the same method will furnish us, figure by figure, with every integral root also, yet it is worth while to explain here a distinct process for the discovery and determination of every such root. The method we advert to was proposed by *Newton*, and is called the *method of divisors*.

Let

$$N + Ax + A_2x^2 + A_3x^3 + A_4x^4 + \dots x^n = 0$$

be an equation of the *n*th degree, in which the coefficients are all whole numbers; and let a be an integral root of it, then we must have

$$N + Aa + A_2 a^2 + A_3 a^3 + A_4 a^4 + \dots a^n = 0$$

$$\therefore \frac{N}{a} = - A - A_2 a - A_3 a^2 - A_4 a^3 - \dots - a^{n-1},$$

from which we infer that $\frac{N}{a}$ must be a whole number; hence every integral root must always be a divisor of the last term N. Call the quotient of this division Q, then, by transposing $-$ A, and dividing by a, the last equation will become

$$\frac{Q + A}{a} = - A_2 - A_3 a - A_4 a^2 - \dots - a^{n-2};$$

consequently, $\frac{Q + A}{a}$ is also a whole number, which, calling Q_2, and transposing $-A_2$, we have, after division by a,

$$\frac{Q_2 + A_2}{a} = - A_3 - A_4 a - \dots - a^{n-3};$$

hence $\frac{Q_2 + A_2}{a}$, or Q_3, is also a whole number; and, continuing this process, we shall obviously have the quotients

$$Q, \quad Q_2, \quad Q_3, \quad Q_4 \dots Q_n$$

all whole numbers, and the last, Q_n, will be -1.

(64.) We infer, therefore, that for a to be an integral root of an equation, the last term must be divisible by it, and so must the sum of the quotient and next coefficient; and throughout, the sum of each coefficient and preceding quotient must be divisible by a, the final quotient being always -1.

Hence, after having determined all the divisors of the absolute term in an equation, we must submit all those of them which are between the limits $-L$ and $+L'$ of the roots, found by the rules in Chapter III., to the foregoing tests, and retain only those divisors which satisfy them all.

(65.) When, however, one divisor is found to succeed, we need not, in order to test the others, return to the original coefficients, since, as it is easy to show, the quotients Q, Q_2, Q_3, &c., are no other than the coefficients of the depressed equation with their signs changed, or, which is the same thing, the coefficients in the quotient of $N + Ax + A_2 x^2 \dots x^n$ by $a - x$; for, by actually performing the division, and recollecting that

$$N = Qa, \quad Q + A = Q_2 a, \quad Q_2 + A_2 = Q_3 a, \quad \&c. \dots (1),$$

we have

$$
\begin{array}{l}
a - x)\ N + Ax \ + A_2 x^2 + A_3 x^3 \dots (Q + Q_2 x + Q_3 x^2 \dots \\
\quad\quad N - Qx \\
\hline
\quad\quad\quad Q_2 ax + A_2 x^2 \\
\quad\quad\quad Q_2 ax - Q_2 x^2 \\
\hline
\quad\quad\quad\quad Q_3 ax^2 + A_3 x^3 \\
\quad\quad\quad\quad Q_3 ax^2 - Q_3 x^3 \\
\hline
\quad\quad\quad\quad\quad Q_4 ax^3 + A_4 x^4 \\
\quad\quad\quad\quad\quad \&c.
\end{array}
$$

It follows, therefore, that a being a root of the proposed equation, the equation

$$Q + Q_2 x + Q_3 x^2 \dots - x^{n-1} = 0 \dots (2)$$

will be the depressed equation involving the remaining roots, for changing the signs of all the terms does not change the roots. Hence the other integral roots of the original equation will also be roots of this; so that, for the discovery of them, we may employ this depressed equation instead of the proposed. If we multiply every term of the depressed equation by a, keeping in mind the conditions (1) above, it will become

$$N + (Q + A) x + (Q_2 + A_2) x^2 \dots - ax^{n-1} = 0 \dots (3),$$

the roots of which are, of course, the same as those of (2); so that, for the discovery of another integral root, we may, if we please, use the form (3) instead of (2), in which case the final quotient must be $-a$.

As an example, let us take the equation

$$x^5+5x^4+x^3-16x^2-20x-16=0.$$

The divisors of 16 are

$$\pm 16, \quad \pm 8, \quad \pm 4, \quad \pm 2, \quad \pm 1.$$

A superior limit to the positive roots is, by (29),

$$1+\sqrt[3]{16} \text{ or } 4;$$

and, by substituting $-x$ for x in the proposed, or, which is the same thing, by changing the signs of the alternate terms, the equation will be

$$x^5-5x^4+x^3+16x^2-20x+16=0,$$

and a superior limit to its positive roots is, by (29), 21; but it is easy to see at a glance that 5 must also exceed the greatest positive root, therefore -5 is a limit to the negative roots of the proposed. Hence the divisors not within the limits -5, 4, that is, the divisors

$$\pm 16, \quad \pm 8, \quad +4,$$

must be rejected; we have, therefore, to try only the divisors ± 2 and -4:

$+2)$	-16	-20	-16	$+1$	$+5$	$+1$
		-8	-14	-15	-7	-1
$-2)$		-28	-30	-14	-2	0
		8	10	10	2	
$-4)$		-20	-20	-4	0	
		4	4	4		
		-16	-16	0		

Hence $+2$, -2, and -4, are integral roots, and the depressed equation is

$$-16x^2-16x-16=0,$$

or rather

$$x^2 + x + 1 = 0,$$

the roots of which are imaginary, (*Algebra*, art. 87).

We have not applied the method to the divisors $+1$ and -1, because it is easy to ascertain whether or not these are roots of the equation, and to depress the equation accordingly by (10). In fact the method of art. (10) will equally serve for the discovery of all the suitable divisors, and is perhaps on the whole but little inferior in facility to that above. We should indeed, by the method of (10), have in all cases to arrive at the final term of the transformation, before we could affirm that the number under trial was a root or not; whereas, in the method here explained, there is a chance of detecting the unsuitable divisors at every division, as the quotient may be fractional. It is scarcely necessary to observe that, when such quotients occur, the work is to be erased, and a new divisor tried; thus: suppose it were required to find whether the equation

$$x^3 - 37x + 72 = 0$$

has any integral positive roots. We readily see that 5 is a superior limit to the positive roots; so that the only divisors of 72 to be tried are 2, 3, and 4. Trying 2, we have

$$\begin{array}{rrrr} 2)\ 72 & -37 & 0 & +1 \\ & 36 & & \\ \hline & -\ 1 & & \end{array}$$

the divisor 2 must be rejected, as the next quotient would be fractional. Trying 3, we have

$$\begin{array}{rrrr} 3)\ 72 & -37 & 0 & +1 \\ & 24 & & \\ \hline & -13 & & \end{array}$$

the divisor 3 is also unsuitable, as this gives Q_2 fractional. Lastly, trying 4, we have

4) 72	— 37	0	+ 1
	18		
	— 19		

which must be rejected for a like reason, so that there are no positive integral roots.

When the divisors of the last term between the limits $-$ L and $+$ L′ are very numerous, the trials may become tiresome; but it is easy to devise a contrivance for diminishing the number of superfluous divisors thus:

(66.) We have seen (65) that

$$\frac{N + Ax + A_2x^2 + A_3x^3 + \dots}{a - x} = Q + Q_2x + Q_3x^2 + \dots$$

the second member being an integer for every integral value of x, because the coefficients are all integral; the simplest integral values of x are $+1$ and -1; hence the first member shows that when $+1$ is put for x, in the original polynomial $f(x)$, no divisor a can be admissible which does not render $\frac{f(1)}{a-1}$ an integer; and, putting -1 for x, we see that no divisor can be admissible which does not render $\frac{f(-1)}{a+1}$ an integer. The divisors between the limits may, therefore, be advantageously submitted to these tests before those at (64) are applied to them. We know from (34) that $f(1)$ will be the last term of the transformed equation in $(x+1)$, and $f(-1)$ will be the last term of the transformed equation in $(x-1)$; hence the best mode of proceeding will be, to effect one step of each transformation by (20), and to divide the final term in the first by each divisor *minus* 1, and the final term in the second by the same, *plus* 1; and then to employ only those divisors which furnish integral quotients. Should the final term in either transformation be 0, it will be a proof that the divisor unity is a root, and then we must employ the depressed equation for the other roots; the coefficients of this depressed equation will have been written down in proceeding to the final term, as at (10).

(67.) Let the equation

$$x^3 - 5x^2 - 18x + 72 = 0$$

be proposed.

The limit of the positive roots is $18 + 1 = 19$; and, changing the alternate signs, we have (30)

$$-\left(\frac{72}{5} + 1\right) = -15\tfrac{2}{5}$$

for the limit of the negative roots; hence the only divisors of 72 which can comprise the integral roots, are

$$\begin{array}{cccccccc} 2, & 3, & 4, & 6, & 8, & 9, & 12, & 18, \\ -2, & -3, & -4, & -6, & -8, & -9, & -12. & \end{array}$$

Let us, therefore, now proceed to determine $f(1)$ and $f(-1)$,

$$\begin{array}{cccl} 1 & -5 & -18 & +72\ (1, -1 \\ & -4 & -22 & 50 = f(1) \\ & -6 & -12 & 84 = f(-1). \end{array}$$

Now those among the foregoing divisors, which, diminished by 1, divide 50, and which, increased by 1, divide 84, are

$$2,\ 3,\ -4,\ 6;$$

and, by trying these in succession, we find 2 to fail, but for 3, -4, and 6, we have

$$\begin{array}{lrrr} 3)\ 72 & -18 & -5 & +1 \\ & 24 & 2 & -1 \\ \hline -4) & 6 & -3 & 0 \\ & -18 & 3 & \\ \hline 6) & -12 & 0 & \\ & 12 & & \\ \hline & 0 & & \end{array}$$

hence the roots are all integral, and are 3, -4, and 6.

Newton's Method of approximating to the Incommensurable Roots of an Equation.

(68.) The method proposed by Newton for approximating to the incommensurable roots which may still exist in an equation, after the integral roots have been removed by the method of divisors, requires like all other approximative methods that we know the intervals in which the roots are situated. It requires, moreover, that before commencing the approximation to any root, we render the interval so narrow, that the extreme limits of it may not differ by more than $\frac{1}{10}$, in which case, either limit must be within the fraction $\frac{1}{10}$ of the value of the root. Call the initial value, thus obtained, x', and its difference from the true root δ, then, in the proposed equation $f(x) = 0$, we have

$$x = x' + \delta;$$

and, consequently, (34)

$$f(x) = f(x' + \delta) = f(x') + f'(x')\,\delta + \frac{f''(x')}{2}\,\delta^2 + \&c. = 0;$$

and, since δ is less than $\frac{1}{10}$, δ^2 must be less than $\frac{1}{100}$, δ^3 less than $\frac{1}{1000}$, &c.; hence, rejecting the terms into which these diminishing factors enter, we have, for a first approximation to the value of the correction δ, the expression

$$\delta = -\frac{f(x')}{f'(x')};$$

which will give the value true to two places of decimals, adding, therefore, this approximate correction to x', we obtain a nearer value, x'', to the root, the error δ' being below $\frac{1}{100}$.

For a second approximation, put

$$x = x'' + \delta',$$

then, proceeding as before, we have

$$\delta' = -\frac{f(x'')}{f'(x'')},$$

which will usually give the value of the correction, as far as four places of decimals, and this correction applied to x'' will give the more correct value x''' for x, being the true value, as far as about four decimals, and, by repeating the operation, we shall get a new value, true to about eight decimals, and so on.

The following is the example chosen by Newton to illustrate his method, viz.

$$x^3 - 2x - 5 = 0.$$

The root of this equation lies between 2 and 3; to narrow these limits, diminish the roots of the transformed in $x - 3$, by ·5, and we shall find no change of sign in the final term; hence the root is between 2 and 2·5. Diminish the roots of this transformed by ·4, and still the final sign is preserved; hence the root is between 2 and 2·1, so that the first two figures of it must be 2·0, that is,

$$x = 2{\cdot}0 + \delta;$$

also,

$$\delta = -\frac{f(2{\cdot}0)}{f'(2\ 0)} = -\frac{-1}{10} = {\cdot}1$$

$$\therefore\ x = 2{\cdot}1 + \delta'$$

$$\delta' = -\frac{f(2{\cdot}1)}{f'(2{\cdot}1)} = -\frac{{\cdot}061}{11{\cdot}23} = -{\cdot}0054$$

$$\therefore\ x = 2{\cdot}0946$$

$$\delta'' = -\frac{f(2{\cdot}0946)}{f'(2{\cdot}0946)} = \frac{{\cdot}0005417}{11{\cdot}16196} = {\cdot}00004853.$$

$$\therefore\ x = 2{\cdot}09455147.$$

In this particular example the approximation is very rapid; this arises from the circumstance that, in the expressions for δ', δ'', &c. the numerators are very small when compared with the denominators, such, however, will not be the case, when the root, to which we are approaching, differs but little from another root; because, as the roots approach to equality, the expression $f'(x)$, when the value of one of these roots is put for x, approaches to zero (37); and hence the denominators of the foregoing fractions will be very small, as well as the numerators. In such a case, too, the terms rejected in the values of δ', δ'', &c. might exceed in magnitude those preserved, and thus no approximation to the true corrections would be obtained. These imperfections in Newton's process render its application unsafe, when the root sought differs by only a small decimal from any of the other real roots, unless, indeed, at each approximation, we test the value obtained, by actually substituting it in the proposed equation.

As an illustration, let the equation,

$$x^3 - 7x + 7 = 0,$$

be proposed.

After a few trials, a root is found to lie between 1·3 and 1·4, and to be nearer to 1·4 than to 1·3. Let us assume then

$$x = 1{\cdot}4 + \delta,$$

then we have

$$\delta = -\frac{f(1{\cdot}4)}{f'(1{\cdot}4)} = -\frac{{\cdot}056}{1{\cdot}12} = -{\cdot}05$$

$$\therefore\ x = 1{\cdot}35 + \delta'.$$

To verify this approximation, let 1·35 and 1·36 be separately put for x in the proposed equation, the results are

$$\text{for } x = 1{\cdot}35,\quad f(x) = +{\cdot}010375$$

$$\text{for } x = 1{\cdot}36,\quad f(x) = -{\cdot}004544$$

which, being of contrary signs, shows that our approximation is correct.

For a second approximation, we have

$$\delta'' = -\frac{f(1{\cdot}35)}{f'(1{\cdot}35)} = \frac{{\cdot}010375}{1{\cdot}5325} = {\cdot}0068$$

$$\therefore\ x = 1{\cdot}3568.$$

To verify this approximation, let 1·3568 and 1·3569 be substituted for x, in the proposed, the results will be

$$\text{for } x = 1{\cdot}3568,\quad f(x) = +\,{\cdot}000141586432$$

$$\text{for } x = 1{\cdot}3569,\quad f(x) = -\,{\cdot}000006100991;$$

which, being of contrary signs, proves the correctness of our approximation: hence the root is between 1·3568 and 1·3569, the former number is, therefore, the true value, as far as four places of decimals.

It will not escape the observation of the student, that the process for the determination of the successive values of $f(x')$, $f(x'')$, $f'(x')$, $f'(x'')$, &c. as also the operations for verifying the several approximations, may all be conducted with great advantage, agreeably to the method of transformation, uniformly employed throughout this volume.

Solution of Recurring Equations.

(69.) It has been shown at (22) that every equation of an even degree, of the form

$$x^{2n} + Ax^{2n-1} + A_2 x^{2n-2} + A_3 x^{2n-3} + \ldots + A_3 x^3 + A_2 x^2 + Ax + 1 = 0,$$

in which the coefficients of any two terms, equally distant from the extremes, are alike both in magnitude and sign, has one half of the entire system of roots, the reciprocals of the other half; that is, if n of the roots be

$$a,\ a_2,\ a_3,\ \ldots\ a_n,$$

then the other n roots will be

$$\frac{1}{a},\ \frac{1}{a_2},\ \frac{1}{a_3},\ \ldots\ \frac{1}{a_n};$$

and, moreover, that even when the equidistant coefficients are like only in magnitude, and unlike in sign, the same relations will exist, provided only the middle term of the equation be absent.

It has also been shown, that if the equation is of an odd degree, then, whether the equal and equidistant coefficients have like signs or not, the same relations among the roots will have place, and that one root will always be $+ 1$ or $- 1$, according as the sign of the last term is $-$ or $+$; so that a recurring equation of an odd degree may always be depressed to a recurring equation of a degree lower.

On account of these peculiar properties of recurring equations, they may always be reduced to others of inferior degrees; in fact, every such equation of an odd degree may, as we have just remarked, be at once reduced to the next inferior even degree; and this, as we shall now prove, may be further reduced to an equation of half the dimensions.

(70.) Suppose the exponent $2n$, in the general equation above, to be successively 2, 4, 6, &c. then dividing every term by x^n, we have the several equations

$$x + \frac{1}{x} + A = 0, \text{ which may be written } z + A = 0$$

$$(x^2 + \frac{1}{x^2}) + A(x + \frac{1}{x}) + A_2 = 0 \quad \ldots \quad z^2 - 2 + Az + A_2 = 0$$

$$(x^3 + \frac{1}{x^3}) + A(x^2 + \frac{1}{x^2}) + A_2(x + \frac{1}{x}) + A_3 = 0 \quad \therefore$$

$$(z^3 - 3z) + A(z^2 - 2) + A_2 z + A_3 = 0$$

&c. &c.

These several equations in z are of a lower degree, by one half, than those from which they have been deduced; and, if in either of these the value of z be found, x will be obtained by the solution of a quadratic, from the condition

$$x + \frac{1}{x} = z.$$

It is worthy of remark, that the depressed equations in z are formed according to a certain law, easily discovered from the general relation,

$$(x^n + \frac{1}{x^n})(x + \frac{1}{x}) = x^{n+1} + \frac{1}{x^{n+1}} + x^{n-1} + \frac{1}{x^{n-1}};$$

which, by replacing $x + \frac{1}{x}$ by z, gives

$$x^{n+1} + \frac{1}{x^{n+1}} = (x^n + \frac{1}{x^n})z - (x^{n-1} + \frac{1}{x^{n-1}});$$

a formula from which the expression $x^{n+1} + \frac{1}{x^{n+1}}$ is obtained in terms of the two preceding expressions; hence we have

$$x^2 + \frac{1}{x^2} = (x + \frac{1}{x})z - (x^0 + \frac{1}{x^0}) = z^2 - 2$$

$$x^3 + \frac{1}{x^3} = (x^2 + \frac{1}{x^2})z - (x + \frac{1}{x}) = z^3 - 3z$$

$$x^4 + \frac{1}{x^4} = (x^3 + \frac{1}{x^3})z - (x^2 + \frac{1}{x^2}) = z^4 - 4z^2 + 2$$

$$x^5 + \frac{1}{x^5} = (x^4 + \frac{1}{x^4})z - (x^3 + \frac{1}{x^3}) = z^5 - 5z^3 + 5z$$

&c. &c. &c.

the expressions in z, obviously forming a recurring series, of which the scale of relation is $(-1, z)$, (*Algebra*, art. 179).

(71.) Let now the recurring equation,

$$4x^6 - 24x^5 + 57x^4 - 73x^3 + 57x^2 - 24x + 4 = 0,$$

be proposed for solution; or, which is the same thing, the equation

$$4(x^3 + \frac{1}{x^3}) - 24(x^2 + \frac{1}{x^2}) + 57(x + \frac{1}{x}) - 73 = 0,$$

which, by putting

$$x+\frac{1}{x}=z, \quad \text{or } x^2-zx=-1,$$

and, taking account of the foregoing expressions, becomes

$$4z^3-24z^2+45z-25=0;$$

an equation of a degree, lower by one half than the proposed.

One root of this equation we find to be 1; thus

4	— 24	45	— 25 (1	
	4	— 20	25	
	—	—	—	
	— 20	25	0	

and, for the depressed equation, containing the other roots, we have

$$4z^2-20z+25=0;$$

of which the first member is a perfect square, because the square of half the middle term is equal to the product of the extremes; its root is evidently $2z-5$; hence z has two values equal to $\frac{5}{2}$, and, therefore, the six values of x are given by the three quadratic equations,

$$x^2-x=-1, \quad x^2-\frac{5}{2}x=-1, \quad x^2-\frac{5}{2}x=-1;$$

the roots of the proposed equation are, therefore,

$$\frac{1\pm\sqrt{-3}}{2}, \quad \frac{1}{2}, \quad 2, \quad \frac{1}{2}, \quad 2.$$

That the first pair of roots, viz.

$$\frac{1+\sqrt{-3}}{2} \text{ and } \frac{1-\sqrt{-3}}{2}$$

are the reciprocals of each other, will be readily seen by multiplying the terms of the latter by

$$1+\sqrt{-3}.$$

Again, let the equation

$$x^5-11x^4+17x^3+17x^2-11x+1=0,$$

be proposed for solution.

Then, as this equation has necessarily the root $x=-1$, we immediately get the depressed biquadratic,

$$x^4-12x^3+29x^2-12x+1=0,$$

or, dividing by x^2, and bringing the equidistant terms together,

$$(x^2+\frac{1}{x^2})-12(x+\frac{1}{x})+29=0,$$

which, by means of the assumed relation,

$$x+\frac{1}{x}=z, \quad \text{or } x^2-zx=-1,$$

becomes

$$z^2-12z+27=0.$$

By solving this quadratic, we have, for z, the values 9 and 3; and, consequently, the values of x in the preceding biquadratic equation are involved in the two quadratics following, viz.

$$x^2-9x=-1, \quad \text{and} \quad x^2-3x=-1;$$

these values are, consequently,

$$\frac{9}{2}\pm\frac{1}{2}\sqrt{77}, \quad \frac{3}{2}\pm\frac{1}{2}\sqrt{5};$$

hence the five roots of the proposed equation are

$$-1, \quad \frac{9+\sqrt{77}}{2}, \quad \frac{9-\sqrt{77}}{2}, \quad \frac{3+\sqrt{5}}{2}, \quad \frac{3-\sqrt{5}}{2};$$

or, if the terms of the second of these fractions be multiplied by $9 - \sqrt{77}$, and those of the last fraction by $3 + \sqrt{5}$, the four last roots will assume the following form, viz.

$$\frac{9+\sqrt{77}}{2}, \frac{2}{9+\sqrt{77}}; \frac{3+\sqrt{5}}{2}, \frac{2}{3+\sqrt{5}};$$

each being accompanied by its reciprocal.

(72.) It has been observed above, that an equation of an even degree is recurring only when the equidistant coefficients are like in sign as well as magnitude; if, however, the signs are unlike, the equation may be reduced to a recurring one, by dividing its first member by $x - 1$; for it is plain that a root of the equation

$$x^{2n} + Ax^{2n-1} + A_2x^{2n-2} + \ldots - A_2x^2 - Ax - 1 = 0$$

is 1, since the substitution of this for x renders the first member zero; this first member is, therefore, divisible by $x - 1$, and the resulting quotient must evidently be the same as that which we should get by dividing

$$1 + Ax + A_2x^2 + \ldots - A_2x^{2n-2} - A_3x^{2n-1} - x^{2n}$$

by $1 - x$, because this dividend and divisor are no other than the former with changed signs; the terms, however, of the latter quotient would be those of the former, reversed.

The *coefficients* of the first quotient would, it is plain, be all obtained by dividing

$$1 + A + A_2 + \ldots - A_2 - A - 1$$

by $1 - 1$; and the coefficients of the second quotient would be obtained by dividing

$$1 + A + A_2 + \ldots - A_2 - A - 1$$

by $1 - 1$; the same series of coefficients are, therefore, produced in both cases; but this latter series is the same as the former, taken in reverse order, therefore the coefficients in the quotient, arising from dividing the proposed polynomial by $x - 1$, furnish the same series, whether taken in the direct or in the reverse order. The depressed equation, therefore, resulting from the elimination of the root 1, is a recurring equation of an odd degree, whose equidistant terms are equal in magnitude and sign. This depressed equation has, therefore, the root -1, and, consequently, equations of the kind, here considered, have always two roots equal to $+1$, and -1, which may be eliminated, and the resulting equation lowered to one of half its degree.

Binomial equations, or those of the form

$$x^m \pm 1 = 0,$$

evidently belong to the class considered in the last two articles; we shall devote a Chapter to their examination in the SECOND PART.

CHAPTER VII.

ON THE THEOREM OF STURM.

(73.) It is obvious, from what has been delivered in the preceding Chapters, that nothing is wanted to render the numerical solution of equations complete, but the discovery of some unfailing method whereby we may always ascertain the nature of those doubtful intervals which frequently occur within the limits of the real roots of an equation. This important object has, at length, been attained, and the accurate determination of the number and situation of all the real roots of any equation is a problem which is now completely resolvable, by means of a theorem discovered by M. Sturm, and published in the *Memoires présentés par des Savans Etrangers*, for 1835; and which gained for its author the mathematical prize of the French Academy of Sciences for 1834. This theorem we shall now explain.

Let

$$X = 0$$

be any equation whose coefficients are real, and whose roots are unequal, and let X_1 be the polynomial, derived from X, agreeably to the process in (34). Let the operation of finding the greatest common measure of X and X_1 be performed, and, in the several remainders which successively arise in the course of the process, change all the signs from + to —, and from — to +, and call the remainders thus modified, X_2, X_3, X_4 .... Putting also the several quotients equal to Q_1, Q_2, Q_3 ...., we shall obviously have these equations, viz.

$$\begin{aligned} X &= X_1\, Q - X_2 \\ X_1 &= X_2\, Q_2 - X_3 \\ X_2 &= X_3\, Q_3 - X_4 \\ &\dots\dots\dots \\ &\dots\dots\dots \\ X_{m-2} &= X_{m-1}\, Q_{m-1} - X_m\,. \end{aligned}$$

The final remainder, X_m, is necessarily independent of x, and different from zero, since, by hypothesis, the equation has no equal roots (37). Suppose now, that in the several functions,

$$X, \ X_1, \ X_2, \ X_3 \ldots . X_m,$$

two numbers, p, q, such that $p > q$ be successively substituted for x, these substitutions will furnish two series of signs, and it is the object of Sturm's theorem to prove that

The difference between the number of variations of the first series, and that of the second, expresses exactly the number of real roots of the proposed equation, which are comprised between p *and* q; whence results the following rules for determining the entire number of real roots of an equation.

1. Apply to the two polynomials, X, X_1, the process for finding the greatest common measure, modifying every remainder, or new divisor, by changing the signs; we shall thus have the series of functions,

$$X, \ X_1, \ X_2, \ X_3, \ldots . X_m;$$

which are of continually decreasing dimensions in x, X_m being independent of x.*

2. Substitute in this series, $-\infty$, and $+\infty$, successively, for x, noting the signs of the results.

3. Count the number of variations in each row of signs; the difference of these numbers expresses the total number of real roots in the equation.

Having now stated the nature and object of Sturm's theorem, we shall proceed to establish the principles from which it is deduced.

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* In finding the common measure, each division is of course to be continued till the remainder, by losing the requisite dimensions renders the continuance impossible. (See the operation at page 149, in the Note.)

(74.) Let r be a real root of the equation

$$x = f(x) = 0.$$

Put $r + x'$ for x, in the function $f(x)$, and we shall have a result of the form,

$$f(r) + f_1(r)x' + \frac{f_2(r)}{2}x'^2 + \frac{f_3(r)}{2 \,.\, 3}x'^3 + \ldots . x'^n,$$

in which $f_1(r)$, $f_2(r)$, &c. are derived from $f(r)$ by the process explained in (34).

As, by hypothesis, r is a root of $f(x) = 0$, we must have $f(r) = 0$; so that the foregoing development is

$$x' \{f_1(r) + \frac{f_2(r)}{2}x' + \frac{f_3(r)}{2 \,.\, 3}x'^2 + \ldots . x'^{n-1}\} \ldots . (1),$$

and it may be easily proved that a value may be found for x' sufficiently small to render the first term within the brackets greater than the sum of all that follow, which value will therefore be such as to cause the sign of the entire quantity within the brackets to be the same as the sign of the first term $f_1(r)$, this term being, by hypothesis, different from zero, because there are no equal roots in the proposed equation (37).

In order to prove this, let us suppose the most unfavorable case, viz. that in which all the terms after the first have the same sign, and let the greatest of these terms be called K; then the series within the brackets, omitting the first term, cannot exceed

$$Kx' \{1 + x' + x'^2 + \ldots . x'^{n-2}\} =$$

$$Kx' \{\frac{1 - x'^{n-1}}{1 - x'}\} \ldots . (2),$$

and it is obvious that when x' is less than unity, this quantity is less than $\frac{Kx'}{1 - x'}$, because $1 - x'^{n-1}$ must be less than unity. If, therefore, we wish to discover such a value for x' as will make (2) less than any proposed quantity, $f_1(r)$, we have only to assume

$$\frac{Kx'}{1-x'} =, \text{ or } < f_1(r) \ldots . (3),$$

which leads to

$$x' =, \text{ or } < \frac{f_1(r)}{f_1(r)+K} \ldots . (4);$$

and every value of x' which satisfies this condition, necessarily satisfies the inequality

$$Kx'\{1 + x' + x'^2 + \ldots . x'^{n-2}\} < f_1(r),$$

and, *à fortiori*, the inequality

$$\{\frac{f_2(r)}{2}x' + \frac{f_3(r)}{2.3}x'^2 + \ldots . x'^{n-1}\} < f_1(r).$$

It is obvious, from (4), that when any value of x' fulfils this last condition, every smaller value of x' must fulfil it also.

The next principle to be proved is, that if in the functions

$$X, \ X_1, \ X_2 \ldots .$$

we put a for x, it can never happen that two consecutive functions vanish at once.

Let

$$X_{p-1}, \ X_p, \ X_{p+1},$$

be any three consecutive functions; then (p. 139,)

$$X_{p-1} = X_p Q_p - X_{p+1},$$

and, if it were possible that there could exist together the conditions

$$X_{p-1} = 0, \ \ X_p = 0,$$

it would necessarily follow that

$$X_{p+1} = 0;$$

and, as moreover

$$X_p = X_{p+1} Q_{p+1} - X_{p+2},$$

it would further follow that

$$X_{p+2} = 0,$$

and so on. We should thus have finally the condition

$$X_m = 0,$$

that is to say, the last remainder is zero, which is impossible, because, as there are no equal roots, X and X_1 cannot have a common measure.

This immediately leads to the third principle, viz.

If one of the functions, as X_p, becomes zero for any particular value of x, the two functions X_{p-1}, X_{p+1}, between which it is placed, have necessarily contrary signs for the same value of x. This is evident from the relation

$$X_{p-1} = X_p Q_p - X_{p+1}.$$

These principles being admitted, let us now represent by k any quantity, positive or negative, which may be nearer to $-\frac{1}{0}$ than any of the real roots of the equations

$$X = 0, \quad X_1 = 0, \quad X_2 = 0 \; . \; . \; . \; . \; X_{m-1} = 0;$$

and let k be conceived to increase continuously towards $+\frac{1}{0}$, and that all the successive values are substituted for x in the functions

$$X, \quad X_1, \quad X_2 \; . \; . \; . \; . \; X_m,$$

the last of which, X_m, being independent of x, will of course remain unaffected by these substitutions; and, with respect to the others, we know that the *signs* of the results which they give will be continually reproduced in the same order, so long as k does not reach a value sufficiently great to render one of the functions zero.

Suppose, however, that such a value is attained, and let it be a; then the substitution of this value for x will either cause one or more of the functions

$$X_1, \quad X_2, \quad X_3 \; . \; . \; . \; . \; X_{m-1},$$

to become zero without rendering X zero, or else the substitution will render X zero, and may besides cause one or more of the other functions to vanish. Here are then two cases, and we shall now prove that in *the first case* no variation can be lost in the passage of x through the three consecutive states $a - \delta,\ a,\ a + \delta$; that in *the second case* one variation will disappear, and only one, in passing from the state $x = a - \delta$ through $x = a$ to the immediately succeeding state $x = a + \delta$.

(75.) Let us examine the first case, viz. that in which one of the intermediate functions, as X_p, becomes zero for $x = a$, for which value X does not vanish.

As for the same value $x = a$, X_{p-1} and X_{p+1} give results with contrary signs (p. 140), it follows that the consecutive functions

$$X_{p-1},\quad X_p,\quad X_{p+1},$$

must furnish one or other of these combinations of signs, viz.

$$\begin{array}{ccc} + & 0 & - \\ - & 0 & + \end{array};$$

so that, whether 0 be regarded as + or —, there is always one variation and one permanence; but whatever be the signs given by X_{p-1} and X_{p+1}, they have been preserved unaltered through all the passages of x, from $x = k$ up to $x = a$, as, by hypothesis, no root of $X_{p-1} = 0$, or of $X_{p+1} = 0$, has been passed over in this interval; nor will these signs change in passing to the immediately succeeding state $x = a + \delta$, because, however near a may be to a root of one of these equations, yet δ may be made so small as to render it impossible that a root can be comprised between a and $a + \delta$.

We may, therefore, conclude that the three functions above, which for $x = a$ furnish one variation and one permanence, give equally a variation and a permanence for all values of x comprised between $x = k$ and $x = a + \delta$. No variation, therefore, is either lost or gained in the series $X, X_1, X_2 \ldots.$, in passing through the state $x = a$, however many of these functions may vanish in the passage.

(76.) Let us now consider the second case, or that in which X or $f(x)$ becomes zero for $x = a$.

Substitute in X and X_1, that is, in $f(x)$ and $f_1(x)$, the value $a + \delta$ for x, and we shall have (34)

$$f(a + \delta) = f(a) + f_1(a)\,\delta + \frac{f_2(a)}{2}\,\delta^2 + \frac{f_3(a)}{2\,.\,3}\,\delta^3 + \&c.$$

$$f_1(a+\delta)=f_1(a)+f_2(a)\,\delta+\frac{f_3(a)}{2}\,\delta^2+\frac{f_4(a)}{2\,.\,3}\,\delta^3+\&c.$$

But, by hypothesis,

$$f(a)=0,$$

$$\therefore f(a+\delta)=f_1(a)\,\delta+\frac{f_2(a)}{2}\,\delta^2+\frac{f_3(a)}{2\,.\,3}\,\delta^3+\&c.$$

Hence (74), taking δ sufficiently small, $f(a+\delta)$ and $f_1(a+\delta)$ have the same signs as $f_1(a)\,\delta$ and $f_1(a)$; and these have like signs when δ is positive, and unlike signs when δ is negative. Consequently, when δ is negative, $f(a+\delta)$ and $f_1(a+\delta)$ have contrary signs, and when δ is positive they have the same signs; so that in the passage from $x=a-\delta$ to $x=a+\delta$, a variation is changed into a permanence. No other loss of variation is due to this passage, because although other functions should vanish in the transition, yet, as we have seen above, their vanishing does not affect the number of variations.

It hence appears, that whatever be the previous state of the series

$$X, \quad X_1, \quad X_2 \ldots .$$

with respect to signs, immediately before the passage of a root, one variation, and only one, will be lost in consequence of that passage.

Now it is plain that this loss cannot be recovered in the interval between the passage of one root and of that next following; because, as in this interval X does not vanish, the variations throughout remain in number the same, as we have already proved. Yet, from the foregoing deductions, it clearly follows that immediately before the passage of the second root there must be a variation between the signs of the first two functions; we must conclude, therefore, that this change of a permanency into a variation cannot add to the total number of changes; hence the variations immediately before the passage of the second root, are precisely the same in number as immediately after the passage of the first. When the second root passes, a variation is necessarily lost, but only one; so that, immediately after the passage, the variations are in number fewer by two than at first, and thus the passage of every successive root is attended with the loss of one additional variation, and one only.

We may, therefore, now conclude, that the number of variations lost during the increase of x from $x = p$, to $x = q$, is exactly equal to the number of real roots which are comprised between p and q; and thus the theorem at (63) is fully established.

From the foregoing investigation we gather the following useful particulars, viz.

(77.) 1. In order to ascertain the total number of real roots in any equation, we shall not be required by this theorem first to determine close limits, $-L$ and $+L'$; it will obviously be sufficient to substitute in the series of functions X, X_1, X_2, &c. the extreme values $-\infty$ and $+\infty$, between which all the real roots are necessarily comprehended; and the difference between the variations furnished by these substitutions, will be equal in number to the number of real roots in the equation. Having thus ascertained how many real roots there are in the equation, we may determine their nearest extreme limits by substituting the successive numbers of the series

$$0, \quad -1, \quad -2, \quad -3, \&c. \ldots . (1),$$

till we have as many variations as were given by the substitution of $-\infty$; after which we may substitute, in like manner, the numbers of the series

$$0, \quad 1, \quad 2, \quad 3, \&c. \ldots . (2),$$

till we arrive at as many variations as were before given by $+\infty$; the numbers at which we stop will be the extreme limits, and, moreover, the intermediate numbers will mark out the situations of the roots themselves, as the difference between the variations given by one number, and those given by any other, will always express the number of real roots which lie between the numbers substituted. The extreme limits thus obtained will obviously be the nearest integral limits possible.

2. It must here be observed that $-\infty$ and $+\infty$ need be substituted only in the terms containing the highest power of x in each function, because this term must, for $x = \pm\infty$, be numerically greater than all the other terms in the function together, so that the sign of this first term will determine the sign of the whole.

It is, moreover, obvious that when all the roots are real, the functions must be $n + 1$ in number; more numerous than this they cannot be, because they are of continually descending dimensions, and, from x^n to x^0 inclusively, comprehends but $n+1$ grades at most; nor can the number of functions be fewer than $n+1$, in the case supposed, for else there would not be n variations to lose, and, therefore, not n real roots. These same functions, too, must have the leading terms all of one sign, in order that the substitutions in them, of $-\infty$ and $+\infty$ for x, may in the one case give all variations, and, in the other, all permanences. When, therefore, the functions X, X_1, X_2, &c. are $n+1$ in number, and have the first term in each, uniformly $+$, or uniformly $-$, we may conclude that the roots are all real; when, however, such conditions have not place, then imaginary roots exist; of which, the exact number may be determined, as above directed.

3. But in all cases where there are so many as $n+1$ functions, however their leading signs may vary, the determination of the number of real and of imaginary roots, may still be effected by a rule easily deducible from, but more simple than, the general one just established; and it is of consequence to notice this simplification of the general theorem, because the functions of which we speak usually amount in number to $n+1$, inasmuch as in seeking the greatest common measure of X and X_1, each divisor is usually of a degree immediately below that of the preceding divisor. Now in every such case, the number of imaginary roots in the equation $X=0$ may be readily discovered, by the simple inspection of the signs of the leading terms of the $n+1$ functions; in fact

The equation $X=0$ has as many pairs of imaginary roots as there are variations in the series of signs of the leading terms of the functions

$$X_1,\ X_2,\ X_3 \ldots . X_n,$$

these being supposed to diminish in degree regularly by unity.

This is proved by Sturm thus:

It follows from the hypothesis which has just been admitted, that every two consecutive functions X_{p-1}, X_p, are the one of an even degree, and the other of an odd degree. Hence, if these two functions have the same sign for $x=+\infty$, they must have contrary signs for $x=-\infty$, and *vice versâ*, if they have contrary signs for $x=+\infty$, they must have the same sign for $x=-\infty$; so that if we write one below the other, the two series of signs of the functions

$$X, \ X_1, \ X_2 \dots . X_n,$$

for $x = -\infty$, and for $x = +\infty$, each variation in either of these two series will correspond to a permanence in the other series; therefore the number of permanencies for $x = -\infty$ is equal to the number of variations for $x = +\infty$.

Let i be the number of variations for $x = +\infty$, and which may be zero. These variations are entirely due to the signs of the leading terms in the n functions

$$X_1, \ X_2, \ X_3 \dots . X_n,$$

because the leading term of X and the leading term of X_1 are necessarily positive.

Now we have just seen that the series of signs for $x = -\infty$ must furnish i permanencies; it must contain then $n - i$ variations, since the functions $X, X_1 \dots . X_n$ are $n + 1$ in number; and that in a series of $n + 1$ signs, the number of variations and permanencies combined amount to the sum n.

But, by the general theorem, the number of real roots of the equation $X = 0$, all comprised between $-\infty$ and $+\infty$, must equal the excess of the number, $n - i$, of variations due to $x = -\infty$, above the number, i, of variations due to $x = +\infty$. The equation $X = 0$ has, therefore, $n - 2i$ real roots, and consequently $2i$ imaginary roots: these we know enter in pairs of the form $a \pm b\sqrt{-1}$; hence the number of these pairs is i.

4. If, in substituting the two numbers, p and q, in the functions, in order to ascertain how many roots lie between them, we find that any intermediate function vanishes, we may pass over the zero in estimating the number of variations; for, as it was shown that in such a case the contiguous functions are always of contrary signs, the intervening one, whether taken + or —, will cause the three to furnish but one variation; so that the number of variations will not be affected by its omission.

When the first function, X, vanishes, we may also omit the zero in estimating the variations; for the vanishing of X shows that the number substituted is a root, and that a variation has just been lost by the change of sign of X, (76) the remaining variations, therefore, are all that are concerned with the other roots.

5. If, after having obtained the series of functions, we find that one of them, as X_r, is of such a nature as always to preserve the same sign, whatever number between p and q be substituted for x in it, then, in order to ascertain the number of roots between p and q, we may reject all the functions beyond X_r, and confine our substitutions to the series

$$X, \ X_1, \ X_2, \dots X_r;$$

for, so long as X_r preserves the same sign, and, consequently, does not pass through zero, no alteration can take place in the number of variations furnished by it, and the following functions, which is proved precisely as for X in (76). Hence, whatever changes take place, occur in the functions, as far as X_r only. From this result the following consequences, viz.

6. If, in the course of the operations, by which X_1, X_2, X_3, &c. are determined, we ascertain that a certain function, X_r, can have only imaginary roots, then, as the result of every substitution in it must be positive, (15) we need not extend the process to the other functions, X_{r+1}, X_{r+2}, &c.

7. As, therefore, in the case just supposed, the number of real roots in the equation is determinable from an examination of the $r + 1$, first functions only, viz. the functions

$$X, \ X_1, \ X_2 \dots X_r;$$

we may, obviously, apply to these all the remarks which have hitherto been made in reference to the entire series; we may affirm, for instance, that when these functions regularly diminish in degree, by unity, and have all the same leading sign, that the equation has r real roots, and no more; and further, that when the leading signs are not all the same, but present i variations, the number of real roots will be only $r - 2i$. Hence we may extend Sturm's second rule, as follows:

(78) *When the series of functions*

$$X_1,\ X_2,\ X_3 \ldots . X_r,$$

in which X_r *is either the final quotient, or else such that the roots of* $X_r = 0$ *are imaginary, regularly descend in degree, by unity, and present* i *variations in their leading signs, there are exactly* r — 2i *real roots in the equation* X = 0. *If* i = 0, *that is, if there are no variations, the equation has* r *real roots, but no more.*

We shall now proceed to a few applications of Sturm's theorem.

1. Let it be required to determine the number and situation of the real roots of the equation

$$8x^3 - 6x - 1 = 0.$$

By taking the derived function, we have

$$24x^2 - 6,$$

for the first divisor; or, since this has a factor 6, not common to the dividend X, we may suppress it* (Alg. 27), and, instead, take

$$X_1 = 4x^2 - 1.$$

Performing the division of X by this, we have the remainder, $-4x - 1$, consequently, changing signs agreeably to the rule,

$$X_2 = 4x + 1.$$

Performing the division of X_1 by this, we have the remainder, — 3, therefore, changing the sign,

$$X_3 = 3.$$

Hence, to find the number of real roots, we must substitute, first, $-\infty$, and then $+\infty$, in the first term of each of the expressions,

$$8x^3 - 6x - 1,\quad 4x^2 - 1,\quad 4x + 1,\quad 3,$$

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* We must always take care that the factors suppressed or introduced be *positive*, in order that the signs of the several remainders may suffer no change.

and we shall thus have the two series of signs following, viz.

$-\ +\ -\ +$ *three variations,*

$+\ +\ +\ +$ *no variation.*

Therefore the three roots are all real.

Or, since the functions are $n+1$ in number, and all their leading terms positive, we may, at once, infer the reality of all the roots, from the precept 2, page 143.

To determine the situation of the roots, we have

for $x=-1$, the signs $-\ +\ -\ +$ *three variations*

$x=\ \ 0$ . . . . . $-\ -\ +\ +$ *one variation*

$x=+1$ . . . . . $+\ +\ +\ +$ *no variation:*

hence there are two roots between 0 and -1, and one between 0 and $+1$.

2. Required the number and situation of the real roots of the equation

$$x^3-7x+7=0.$$

In this example we have

$$X=x^3-7x+7,\quad X_1=3x^2-7,\quad X_2=2x-3,\quad X_3=+.$$

The substitution of $-\infty$ for x, in the first term of each expression, gives three variations, and the substitution of $+\infty$ gives no variation, therefore all the roots are real; an inference which we might have immediately reached by a mere inspection of the signs of the leading terms of the functions, (page 144.)

To determine the situation of the roots, we have

for $x=-4$ the signs $-\ +\ -\ +$

$x=-3$ . . . . . $+\ +\ -\ +$

$x=-2$ . . . . . $+\ +\ -\ +$

$x=\ \ 0$ . . . . . $+\ -\ -\ +$

$x=\ \ 1$ . . . . . $+\ -\ -\ +$

$x=\ \ 2$ . . . . . $+\ +\ +\ +$.

Hence there is one root between -3 and -4, and two between 1 and 2.

3. Required the number and situation of the real roots of the equation

$$x^3 - 5x^2 + 8x - 1 = 0.$$

In this example the functions are

$$X = x^3 - 5x^2 + 8x - 1,\ X_1 = 3x^2 - 10x + 8,\ X_2 = 2x - 31,\ X_3 = -,^*$$

in the first of each of which, if $-\infty$ and $+\infty$ be successively substituted, we have, first, two variations, and then one; so that there is but one real root which is positive, because the last term of the equation is negative (page 19).

As in this case, too, the functions are $n + 1$ in number, and, as the signs of the leading terms present one variation, we might have inferred, at once, from the rule at page 147, that the equation has one pair of imaginary roots.

Putting 0, 1, &c. for x, we have

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* The operation for determining these functions is as follows:

$$3x^2 - 10x + 8)\ 3x^3 - 15x^2 + 24x - 3\ (x - \tfrac{5}{3}$$
$$3x^3 - 10x^2 + 8x$$
$$- 5x^2 + 16x - 3$$
$$- 5x^2 + \tfrac{50}{3}x - \tfrac{40}{3}$$

changing signs . . . . $\tfrac{2}{3}x - \tfrac{31}{3}$ or $2x - 31)\ 3x^2 - 10x + 8\ (\tfrac{3}{2}x$

$$3x^2 - \tfrac{93}{2}x$$
$$\tfrac{73}{2}x + 8.$$

It will be unnecessary to seek the final remainder, as we plainly see that it will be positive. In fact, the sign of the final remainder due to the binomial, may be always readily foreseen by mentally applying the process at (10).

for $x=0$ the signs $- + - -$

for $x=1$ . . . . . $+ + - -$;

therefore, as a variation is lost in the second line, we conclude that the real root is between 0 and 1.

4. Required the number and situation of the real roots of the equation

$$x^4-2x^3-7x^2+10x+10=0.$$

Here the functions are

$$X = x^4 - 2x^3 - 7x^2 + 10x + 10$$
$$X_1 = 2x^3 - 3x^2 - 7x + 5$$
$$X_2 = 17x^2 - 23x - 45$$
$$X_3 = 152x - 305$$
$$X_4 = +;$$

and the substitutions of $-\infty$ and $+\infty$ give the series

$+ - + - +$

$+ + + + +$;

hence all the roots are real, as we also know from the signs of the leading terms being all $+$.

To determine their situations, we have

for $x=-3$ the signs $+ - + - +$

$x=-2$ . . . . . $- + + - +$

$x=-1$ . . . . . $- + + - +$

$x=\ \ 0$ . . . . . $+ + - - +$

$x=\ \ 1$ . . . . . $+ - - - +$

$x=\ \ 2$ . . . . . $+ - - - +$

$x=\ \ 3$ . . . . . $+ + + + +$.

The variations show that one negative root is situated between -2

and — 3, another between 0 and — 1, and two positive roots between 2 and 3.

5. Required the number and situation of the real roots of the equation

$$2x^4 - 13x^2 + 10x - 19 = 0.$$

The three first functions are

$$X = 2x^4 - 13x^2 + 10x - 19$$
$$X_1 = 4x^3 - 13x + 5$$
$$X_2 = 13x^2 - 15x + 38.$$

It is useless to continue the process, for we have now arrived at a polynomial, X_2, such that the roots of $X_2 = 0$ are imaginary. This we know, because, as it is easy to see, $13 \times 38 > \frac{(15)^2}{4}$ (Alg. p. 100.) Hence X_2 must preserve the same sign, whatever number be substituted in it for x, and, therefore, the subsequent functions will, for every substitution, furnish the same number of variations (page 146).

The values $x = -\infty$, and $x = \infty$, give the following series of signs, viz.

$$+ \quad - \quad +$$
$$+ \quad + \quad +;$$

so that the equation has *two* real roots, and no more, therefore two of the roots are imaginary; and this fact might have been at once ascertained by the theorem at (78).

The real roots are of contrary signs, because the last term of the proposed equation is negative (15).

In order to determine the situation of each real root, we need not employ any but the first function X, for, as there is only one positive root, the two positive numbers, which cause X to change its sign, must comprehend them. For the numbers 2 and 3, we have contrary results, therefore the positive root lies between these numbers. Also for — 3, and — 4, we have contrary results, therefore the negative root is between these numbers.

(79.) The preceding illustrations of Sturm's theorem are fully sufficient to exemplify its importance in the numerical solution of equations, and to show that it will always infallibly direct us to the leading figure of every real root in any proposed equation. While the theorem thus furnishes us with a sure basis, on which to found our operations for evolving any real root of an equation, precluding the possibility of our being misled by the imaginary values, it also enables us to proceed with certainty in every step of our approximation in those hitherto perplexing cases where two or some even number of roots are nearly equal to each other, and, to meet the difficulties of which, was the principal object to which *Lagrange* devoted his celebrated *Traité de la Résolution des Equations numériques.*

The success of this illustrious mathematician extended itself no farther than to prove that the difficulties in question might be completely removed by help of certain rules, which he himself supplied for the purpose, but, of which, the practical application was so excessively laborious as to render them nearly useless in the actual solution of an equation above the fourth degree.* The method of Lagrange effects the object in view, by the aid of an auxiliary equation, technically called the *Equation of the Squares of the Differences*, the formation of which involves the dispiriting labour to which we have alluded. Till the discovery, however, of the theorem at (73), it was necessary to go through this labour before we could rest, with confidence,

* Newton was the first to remark that the complete separation of the real roots would be effected by the determination of a number less than the least of the differences of the roots; and this determination is accomplished by the *Equation of the Squares of the Differences*, as was remarked by *Waring* before the publication of Lagrange's Mémoire. This is admitted by Lagrange, who says "Mais je ne connaissais pas cet Ouvrage de *Waring (Miscellanea Analytica*, 1762,) lorsque je composai mon premier Mémoire sur la resolution des équations numériques; d'ailleurs cette remarque n'étant présentée dans l'Ouvrage de *Waring*, que d'une manière isolée serait peut-être restée long-temps stérile sans les recherches dont elle était accompagnée dans ce Mémoire." *Résolution des Equa. Numériques*, page 110.

upon any approximative process when the roots of the equation were nearly equal to each other.*

(80.) We shall now extract from Sturm's *Mémoire* an example of the application of his theorem to an equation, having two roots nearly equal to each other.

6. Let it be required to approximate to the positive roots of the equation

$$x^3 + 11x^2 - 102x + 181 = 0 \; . \; . \; . \; . \; (1).$$

First, in order to ascertain the number and situation of the roots, we form the functions

$$X \;= x^3 + 11x^2 - 102x + 181$$
$$X_1 = 3x^2 + 22x - 102$$
$$X_2 = 854x - 2751$$
$$X_3 = + \,;$$

from which, as all the leading signs are $+$, we infer that all the roots are real (page 144).

To determine the intervals of the positive roots, we make the substitutions

$x = 0$, which gives $+ - - +$ *two variations*
$x = 1$ . . . $+ - - +$
$x = 2$ . . . $+ - - +$
$x = 3$ . . . $+ - - +$ *two variations*
$x = 4$ . . . $+ + + +$ *no variation.*

Hence the equation has two positive roots, both comprised between 3 and 4, so that the first figure, common to both, is 3. Therefore, by our method of approximation (Chap. V.) the first step of the process will be as follows:

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* The method of forming the equation of the series of the differences is given in Chapter III., Part II.

		11
	— 102	— 181 (3
14 . . .	42	— 180
	— 60	— 1
	9	
	— 9	
20 . . .		

and the resulting transformed equation, whose roots are those of the original, diminished by 3, is

$$x'^3 + 20x'^2 - 9x' + 1 = 0 \quad . \; . \; . \; . \; (2).$$

The first figure of the root of this, or the second figure in the quotient above, appears to be ·2, because, of all numbers occupying the place of the second figure, we find this to be the one which produces a result nearest to — 1. Still we cannot affirm that the number which produces a result nearest to the absolute number, or which, when the terms are all arranged on one side, produces a result the nearest to zero, is necessarily the first figure of the root, unless the next figure in the scale produces a change of sign, which is not the case here. To test the figure ·2, therefore, we transform all the other functions, as well as the first X, by diminishing the value of x in each, by 3, as above, and we find these results, viz.

$$X' = x'^3 + 20x'^2 - 9x' + 1$$
$$X'_1 = 3x'^2 + 40x' - 9^*$$
$$X'_2 = 854x' - 189$$
$$X_3 = +;$$

which, for $x = {\cdot}2$, gives the series + — — + *two variations*

and for $x = {\cdot}3$ . . . . + + + + *no variation* ;

* This transformation is always the limiting polynomial, derived from the preceding. There is, therefore, in cubic equations, only one

so that two roots of (2) are comprised between ·2 and ·3, and thus ·2 is the correct second figure of both roots of (1).

If the substitutions, ·2, ·3, had not given series of signs, differing by two variations, we should have concluded that the root figure, ·2, was incorrect, and should have continued to substitute, in the transformed functions, the successive values, 0, ·1, ·2, ·3, ·4 . . . . 1, till such a difference of variations had been obtained, and should have taken the least of the two numbers, to which the change was due, for the true second figure. We may now, therefore, complete the second step of our solution, and shall thus have

		11
	— 102	— 181 (3·2
14 . . .	42	— 180
	— 60	— 1
	9	— ·992
	— 9	— ·008
20·2 . . .	4·04	
	— 4·96	
	4	
	— ·88	
20·6 . . .		

and we may now proceed, without hesitation, to any extent, with our approximation; for the trial divisor indicates ·01 as the third figure of the root, and, trying ·02, we find a change of sign in the absolute number, a sure indication that the two roots separate at the second figure, and become distinct. The roots, as far as three decimals, are 3·213 and 3·229.

It ought to be remarked here, that a reference to Sturm's theorem,

---

extra transformation to effect in any doubtful step, viz. the simple transformation in X'_2. But, in every case, the transformations will be effected with very little extra trouble.

in order to test the accuracy of the second root figure, ·2, was not absolutely necessary, because the transformation due to this figure presents the same series of signs as the preceding transformation, from which circumstance we know, by the theorem of *Budan* (43), that no root is comprised between 3 and 3·2; and, as the substitution of ·3 for ·2 would cause the loss of two variations in the resulting transformation, we know from the same theorem that, as the roots are real, two must be comprised between 3·2 and 3·3, (see page 66). When, however, imaginary roots enter an equation, it would be unsafe to conclude that the loss of an even number of variations in passing from one transformation to another arises from the presence of an even number of real roots in the interval between the transformations, although such may actually be the case, (see remark 5, page 67). But the theorem of Sturm removes all doubt of this kind.

7. As a last application of Sturm's theorem, let it be required to ascertain the conditions necessary, in order that the equation

$$x^3 + Ax + N = 0,$$

may have all its roots real.

The several functions are as follow:

$$X = x^3 + Ax + N$$
$$X_1 = 3x^2 + A$$
$$X_2 = -2Ax - 3N$$
$$X_3 = -4A^3 - 27N^2.$$

Now in order that the proposed equation may have all its roots real, it is necessary, and it is sufficient, that the first terms of these functions may, for x *negative*, give only variations, and for x *positive* only permanencies. When x is negative, the first term of X has the sign —, and that of X_1 the sign +; hence, $-2Ax$ must have the sign —, and the constant X_3 the sign +; these conclusions involve the conditions

$$A < 0, \quad 4A^3 + 27N^2 < 0,$$

which conditions are also sufficient to render the leading terms all permanent in sign for x *positive*, as is at once seen. Such then are the relations which must exist among the coefficients in order that all the roots of the proposed may be real. The first of these relations, however, viz. $A < 0$, is necessarily comprised in the second, for the term $27N^2$ being essentially positive, the second condition necessarily requires that A be negative; the only relation absolutely necessary is, therefore,

$$4A^3 + 27N^2 < 0;$$

and whenever this has place in the cubic equation all its roots must be real; but when this condition is not fulfilled, two roots must be imaginary.

Application of the Theorem to the Case of Equal Roots.

(81.) That nothing may be wanted to complete the present inquiry, we shall, in conclusion, briefly show how the theorem at (73) may be applied when the equation, $X = 0$, contains equal roots.

In order to this, let us consider X_1 to be, not the function derived from X, as heretofore, but any polynomial, subject to the conditions of having no factor in common with X, and of taking a sign contrary to that of X, for values, $a - \delta$, very little below the value of a root a of the equation $X = 0$; and let us employ this polynomial to calculate X_2, X_3 . . . . instead of the derived polynomial. The theorem of Sturm will equally apply to the series X, X_1, X_2, X_3 . . . . thus obtained. For, if we follow the details of the demonstration, we shall see that the new intermediate functions possess the same properties as those formerly examined; and it is, moreover, plain that in passing from a value of x, a little below a root of $X = 0$, to a value a little above, the function X must change its sign; while X_1, which has no factor in common with X, must preserve its sign unchanged.

This being admitted, let us assume

$$X = (x - a)^p (x - b)^q (x - c)(x - d) \ . \ . \ . \ .;$$

then, by performing upon the function X, and its derivee, X_1, the operation already directed, we shall, because of the equal roots, be length conducted to a divisor, X_r, which will be the greatest comm measure of X and X_1 (37), and, by actually performing the divisic we shall have

$$V = \frac{X}{X_r} = (x-a)(x-b)(x-c)(x-d) \ldots\ldots$$

$$V_1 = \frac{X_1}{X_r} = p(x-b)(x-c)(x-d) \ldots\ldots +$$

$$q(x-a)(x-c)(x-d) \ldots\ldots +$$

$$(x-a)(x-b)(x-d) \ldots\ldots + \text{\&c.}$$

Let $x - a$ be one of the real factors of X, and, for brevity, put

$$K = (x-b)(x-c)(x-d) \ldots\ldots$$

$$K_1 = q(x-c)(x-d) \ldots\ldots + (x-b)(x-d) \ldots\ldots + \text{\&c}$$

we shall thus have

$$V = (x-a) K$$

$$V_1 = pK + (x-a) K_1.$$

Now we may give to x a value so little below the value of a, that the factor $(x - a)$ may be small enough to render V_1 of the same sign as K, which sign is evidently contrary to the sign of V.

Hence, from the remark above, in applying the theorem of Sturm to the equation $V = 0$, we may employ the quotient V_1 instead of the function derived from V.

Now it is easy to see that the functions V_2, V_3 . . . ., which we deduce from V, V_1, according to the rule already prescribed, are no other than the quotients of X_2, X_3 . . . . X_r by X_r; therefore the two series

$$X, \; X_1, \; X_2 \ldots\ldots X_r$$

$$V, \; V_1, \; V_2 \ldots\ldots V_r,$$

differ the one from the other, only by a common factor; and, whatever be the sign of this factor, it is evident that the series must present the same variations when we substitute in them a value for x which does not cause X to vanish; hence, in applying the theorem to the first series, we may determine how many roots of the equation are comprised between two numbers, α and β, neither of which causes X to become zero, and, consequently, how many real roots there are in the equation $X = 0$, abstracting from the degree of multiplicity of the roots.* We may, therefore, conclude that

The number of real roots in the equation $X = 0$, *comprised between* α *and* β, *is equal to the excess of the number of variations in the signs of the functions*

$$X, \ X_1, \ X_2 \ldots\ldots,$$

for $x = \alpha$, *above the number of variations for* $x = \beta$, *abstraction being made of the multiplicity of the roots.*

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* The preceding examination of the case of equal roots, which is somewhat shorter than that proposed by Sturm, is from the *Algébre* of *M. Lefebure de Fourcy*, 1835.

END OF PART I.

PART II.

CHAPTER I.

ON CONTINUED FRACTIONS.

(82.) Let α represent either a fractional or an irrational number; and let a be the greatest integer below the value of α, and which, if α be less than 1, will of course be 0. Then, since $\alpha - a$ is less than 1, it follows that $\frac{1}{\alpha - a}$ must be greater than 1. Put

$$\frac{1}{\alpha - a} = \beta \therefore \alpha = a + \frac{1}{\beta},$$

and let b be the integer which in value is immediately below β; then $\beta - b$ is less than 1, and consequently $\frac{1}{\beta - b}$ must be greater than 1. Put

$$\frac{1}{\beta - b} = \gamma \therefore \beta = b + \frac{1}{\gamma},$$

and let c be the greatest integer below the value of γ; then will $\gamma - c$ be less than 1, and therefore $\frac{1}{\gamma - c}$ greater than 1. Put

$$\frac{1}{\gamma - c} = \delta \therefore \gamma = c + \frac{1}{\delta}.$$

Continuing this process, we obviously have, by substituting in the foregoing expression for α the values of β, γ, &c. in succession, the following development of the value α, viz.

$$\alpha = a + \cfrac{1}{\beta}$$

$$= a + \cfrac{1}{b + \cfrac{1}{\gamma}}$$

$$= a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \&c.}}}$$

which development is called a *continued fraction*.

If either of the quantities β, γ, δ, &c. is a whole number, the development must of course terminate at that number, and this will necessarily be the case if α be rational, or a finite fraction; but if α be irrational, then the fraction representing its development must be interminable. This is readily admissible; it is, however, an unavoidable conclusion from what follows.

(83.) If α be a rational fraction $\frac{A}{B}$, we may very easily arrive at its equivalent continued fraction. For the first term a will be the quotient of A by B, and, calling the remainder C, we shall have

$$\frac{A}{B} - a = \frac{C}{B} \therefore \beta = \frac{B}{C}.$$

In like manner, the division of B by C gives b; and, putting D for the remainder, we have

$$\frac{B}{C} - b = \frac{D}{C} \therefore \gamma = \frac{C}{D}.$$

Similarly the division of C by D gives c, and so on.

Hence a, b, c, &c. are no other than the quotients which successively arise in the process of finding the common measure of the terms of the proposed fraction $\frac{A}{B}$; thus:

B) A (a
Ba
——
C) B (b
Cb
——
D) C (c
Dc
——
E
&c.

It is easy to see that when α is a rational fraction, the expression deduced for it in the preceding article is readily derivable from this operation of the common measure; indeed the form of the continued fraction, as deduced from this process, will have greater generality than that given in last article. For without restricting the foregoing quotients to be integral and positive, we shall evidently have, in every case,

$$\alpha = \frac{A}{B} = a + \frac{C}{B} = a + \frac{1}{\beta}$$

$$\beta = \frac{B}{C} = b + \frac{D}{C} = b + \frac{1}{\gamma}$$

$$\gamma = \frac{C}{D} = c + \frac{E}{D} = c + \frac{1}{\delta}$$

&c. &c. &c.

so that

$$\frac{A}{B} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \&c.}}}$$

in which a, b, c, &c. are quotients, positive or negative, integral or fractional, derived by the foregoing operation. In most applications of continued fractions, integral and positive quotients only are employed; but it is useful to show that these restrictions are not essential to the form of the development, which is preserved, whatever be the character of the quotients. This is a truth that we shall have occasion to avail ourselves of at the close of the Chapter; at present, however, we shall require only positive and integral quotients.

(84.) As a particular application, let the proposed fraction be $\frac{1103}{887}$; then, by applying the process for the common measure, the several quotients furnish the following development, viz.

$$\frac{1103}{887} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{4},$$

and if the fraction be $\frac{1171}{9743}$, we have the following equivalent development, viz.

$$\frac{1171}{9743} = \frac{1}{8} + \frac{1}{3} + \frac{1}{8} + \frac{1}{6} + \frac{1}{1} + \frac{1}{1} + \frac{1}{3}.$$

Since the process of seeking the greatest common divisor of two numbers always terminates, it follows that every rational fraction may be expressed in a finite continued fraction.

(85.) We might obviously, by reduction, collect into one the successive portions

$$\frac{1}{a}, \quad \frac{1}{a} + \frac{1}{b}, \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \text{ \&c.}$$

of a continued fraction, by putting for a, in the first, $a + \frac{1}{b}$; then $b + \frac{1}{c}$ for b, and so on; we should thus have the results

$$\frac{1}{a}, \quad \frac{b}{ab + 1}, \quad \frac{bc + 1}{a\,(ab + 1) + C}, \text{ \&c.}$$

so that every finite continued fraction may be reduced to an ordinary finite fraction; hence an incommensurable quantity cannot be expressed by a terminate continued fraction.

The partial sums which we have just obtained are called *converging fractions*, for, as we shall presently demonstrate, they approach nearer and nearer to the whole value of the continued fraction.

For the sake of simplicity, let us represent the series of converging fractions by

$$\frac{A}{A'},\ \frac{B}{B'},\ \frac{C}{C'},\ \&c.,$$

then we shall always be able to recognize the particular fraction represented, by observing that the capitals A, B, C, &c. correspond to the quotients a, b, c, &c. last introduced; so that $\frac{B}{B'}$ will represent

$$\cfrac{1}{a + \cfrac{1}{b}},$$

$\frac{C}{C'}$ will represent

$$\cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c}}},$$

and so on. This notation being agreed upon, we may readily demonstrate the following proposition, viz.

In any three consecutive converging fractions

$$\frac{P}{P'},\ \frac{Q}{Q'},\ \frac{R}{R'},$$

we shall always have the property

$$\frac{R}{R'} = \frac{Qr + P}{Q'r + P'},$$

r being, as observed above, the quotient last introduced into the value of $\frac{R}{R'}$.

As to the three first converging fractions, viz.

$$\frac{1}{a}, \quad \frac{b}{ab+1}, \quad \frac{bc+1}{a(ab+1)+c},$$

or

$$\frac{A}{A'}, \quad \frac{B}{B'}, \quad \frac{C}{C'},$$

it is plain that the properly announced has place; for we immediately recognize the relation

$$\frac{C}{C'} = \frac{Bc+A}{B'c+A'}.$$

If then we can show from this, that the succeeding fraction must have the same property, similar reasoning would apply to the next following fraction, and so on throughout the whole. We have only then, in order to establish the proposition, to prove that from the condition

$$\frac{C}{C'} = \frac{Bc+A}{B'c+A'} \text{ we must have } \frac{D}{D'} = \frac{Cd+B}{C'd+B'}.$$

The expression for $\frac{D}{D'}$ differs from the expression for $\frac{C}{C'}$ only by having $c + \frac{1}{d}$ in place of c; so that, by changing in $\frac{C}{C'}$, c into $c + \frac{1}{d}$, we must have $\frac{D}{D'}$; therefore

$$\frac{D}{D'} = \frac{B(c+\frac{1}{d})+A}{B'(c+\frac{1}{d})+A'} = \frac{(Bc+A)d+B}{(B'c+A')d+B'}$$

$$= \frac{Cd+B}{C'd+B'};$$

hence, generally,

$$\frac{R}{R'} = \frac{Qr+P}{Q'r+P'},$$

which shows that both numerators and denominators go on continually increasing. By means of this property we may form the series of converging fractions with great facility, when only the first two are given; and we may thence arrive at the entire sum of the series when it terminates, and thus obtain the value of the original fraction.

For example, let it be required to determine the fraction of which the development is

$$1+\cfrac{1}{4+\cfrac{1}{9+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4}}}}}}.$$

Here the first two converging fractions are $\frac{1}{1}$, $\frac{5}{4}$, from which we deduce the third by multiplying the two terms of the second each by 9, and adding in the corresponding terms of the first fraction; from the third we get the fourth, using the next quotient 2 as the multiplier, and adding in the corresponding terms of the second fraction, and so on, as follows:

$$\begin{array}{ccccccc} & & 9 & 2 & 1 & 1 & 4 \\ \frac{1}{1}, & \frac{5}{4}, & \frac{46}{37}, & \frac{97}{78}, & \frac{143}{115}, & \frac{240}{193}, & \frac{1103}{887}. \end{array}$$

(86.) We can now show the propriety of calling these results *converging fractions*, by proving that they continually approach nearer and nearer to the value of the continued fraction.

That these fractions are alternately less and greater than the developed form may be readily seen, without the aid of the above property; for, calling the entire value x, we have the first, $\frac{a}{1}$, less than x, because the positive quantity $\frac{1}{b+\&c.}$ is neglected. The second $a+\frac{1}{b}$ is greater than x, for the denominator is less than it ought to be, by the positive quantity $\frac{1}{c + \&c.,}$ yet, if we take in $\frac{1}{c}$, that denominator will be increased too much, because $\frac{1}{c}$ is greater than $\frac{1}{c + \&c.}$; so that $a + \frac{1}{b + \frac{1}{c}}$ is less than x, and so on. But to prove the proposition announced in a general manner, we shall employ the equation

$$\frac{R}{R'} = \frac{Qr + P}{Q'r + P'}$$

before established, either member of which will necessarily express the value of the entire fraction x, if we substitute in it $r + \frac{1}{s + \&c.}$ for r. The quantity $r + \frac{1}{s + \&c.}$ is always greater than unity, because r is not less than unity. Calling it y, we have

$$x = \frac{Qy + P}{Q'y + P'};$$

and, consequently, by subtracting first $\frac{P}{P'}$ and then $\frac{Q}{Q'}$ from each side, we have the equations

$$x - \frac{P}{P'} = \frac{(QP' - Q'P)y}{(Q'y + P')P'}, \quad x - \frac{Q}{Q'} = \frac{Q'P - QP'}{(Q'y + P')Q'},$$

which show that the differences $x - \frac{P}{P'}$, $x - \frac{Q}{Q'}$, have contrary signs; so that if x be greater than $\frac{P}{P'}$, it will be less than $\frac{Q}{Q'}$, and vice versâ; and, as x is greater than the first converging fraction $\frac{a}{1}$, (or $\frac{0}{1}$ if a is 0), it follows that, throughout the series of converging fractions, the 1st, 3d, 5th, 7th, &c. of them are each below the true value; and the 2d, 4th, 6th, 8th, &c. above the true value.

As to the relative values of the differences $x - \frac{P}{P'}$, $x - \frac{Q}{Q'}$, it is plain that the latter is less than the former, because y is greater than 1, and Q' greater than P', since the denominators increase as the fractions advance (85). It follows, therefore, that the converging fractions approach continually nearer and nearer to the true value of the continued fraction, and, therefore, this value may be approximated to as closely as we please when the first two converging fractions are found. It follows, moreover, that the odd terms of the series of converging fractions form an increasing series of values, approximating to the truth, and that the even terms form a decreasing series of approximating values.

(87.) Let us now inquire what is the limit to the error we commit, in taking any one of these converging fractions for the complete value.

In the first place, it is clear that this error cannot be so great as the difference between the fraction taken and that which immediately follows it, because the true value lies between these two. Now the *numerator* of the difference between two consecutive fractions is obtained by multiplying the terms crosswise, and subtracting; the denominator is obtained by multiplying together those of the given fractions. Let, then, $\frac{P}{P'}$, $\frac{Q}{Q'}$, be any two consecutive fractions, and we shall have, for the numerator of their difference, the expression

$$PQ' - P'Q;$$

and, for the numerator of the difference between $\frac{Q}{Q'}$, $\frac{R}{R'}$, or, which is the same thing, between $\frac{Q}{Q'}$, $\frac{Qr + P}{Qr' + P'}$, we shall have the expression

$$QQ'r + P'Q - QQ'r - PQ' = P'Q - PQ';$$

the very same as the former difference, only with contrary sign. Hence, throughout the series, if the difference between each fraction and the next following be taken, the numerators of the results will always be the same in magnitude, but will have alternate signs. To determine the actual value of the numerators, we have, therefore, only to ascertain it in one instance. Let us then take the two leading fractions, which are $\frac{1}{a}$, $\frac{b}{ab+1}$, and we have

$$(ab+1)-ab=1;$$

hence the numerators in question are always unity, so that the error we commit in taking the converging fraction, $\frac{Q}{Q'}$, for the true value, is always less than $\frac{1}{Q'R'}$. This leads to a valuable property of these fractions, which is that between any two consecutive terms; it will be impossible to insert a fraction of intermediate value, whose denominator shall not be greater than those of the proposed fractions, for it is obvious that no fraction can be smaller than $\frac{1}{Q'R'}$, unless its denominator be greater. Hence, the series of converging fractions not only approximate continually to the value of x, but they present themselves in the most simple forms possible, so that it would be impracticable to substitute for any one of them another, more approximative, that would not be more complex.

These converging fractions are, therefore, highly useful for the purpose of enabling us to express, in small numbers, a near value of a ratio of which the terms may be too large to be easily managed in computation. For instance, the ratio of the diameter of a circle, to its circumference, is known to be very nearly as 100000 to 314159, and to get a series of other ratios more simply expressed, and continually approximating to this, we proceed as follows:

```
100000) 314159 (3
        300000
        ------
         14159) 100000 (7
                 99113
                 -----
                   887) 14159 (15
                        887
                        ----
                        5289
                        4435
                        ----
                         854) 887 (1
                              854
                              ---
                               33) 854 (25
                                   66
                                   ---
                                   194
                                   165
                                   ---
                                    29) 33 (1
                                        29
                                        --
                                         4) 29 (7
                                            28
                                            --
                                             1) 4 (4
                                                4
                                                -
```

∴ the quots. are 3 7 15 1 25 1 7 4

and conv. frac. $\frac{1}{3}, \frac{7}{22}, \frac{106}{333}, \frac{113}{355}, \frac{2931}{9208}, \frac{3044}{9563}, \frac{24239}{76149}, \frac{100000}{314159}.$

The second of these ratios, viz. 7 to 22, is that which was first given by Archimedes, and is sufficiently near the truth for many practical purposes; the ratio, 113 to 355, is that of Metius, and is a still nearer approximation. The ratio of Archimedes differs from the truth, by a quantity less than $\frac{1}{22 \times 333}$, and the ratio of Metius differs from the truth, by a quantity less than $\frac{1}{355 \times 9208}$, as appears from the foregoing expression, for the limit of the error.

(88.) We may easily obtain a limit to the error, that shall be independent of the denominator of the fraction, which follows that at which we stop, although such a limit will not be so small as that just deduced. For, since the denominators increase, we must have

$$R' > Q' \therefore Q'R' > Q'^2 \therefore \frac{1}{Q'R'} < \frac{1}{Q'^2};$$

hence the error committed by taking the converging fraction, $\frac{Q}{Q'}$, for the value of x, must be less than $\frac{1}{Q'^2}$.

From this expression, for the limit of error, we can always determine a converging fraction, which shall approach as near to the true value as we please, or which shall differ from that value by less than any assigned quantity Δ; for, in order that $\frac{Q}{Q'}$ may be the fraction, it will be sufficient that $\frac{1}{Q'^2}$ do not exceed Δ, or, that Q' be not less than $\sqrt{\frac{1}{\Delta}}$.

The property established in (87), that $P'Q - PQ' = 1$, will also furnish an expression for the inferior limit of the error, as well as for the superior limit; for, in consequence of this property, we have (86)

$$x - \frac{P}{P'} = \frac{y}{(Q'y + P')P'};$$

and, therefore, dividing the numerator, and only *part* of the denominator by y, we have

$$x - \frac{P}{P'} > \frac{1}{(Q' + P')P'}.$$

Hence, taking either of the converging fractions, $\frac{P}{P'}$, for the true value x, we have the following limits to the error, viz.

$$x - \frac{P}{P'} < \frac{1}{P'Q'} < \frac{1}{P'^2}$$

$$x - \frac{P}{P'} > \frac{1}{(Q'+P')P'} > \frac{1}{2P'Q'}.$$

(89.) In the examples hitherto given of the development of α, in the form of a continued fraction, α has been considered to be a rational fraction, and the several quantities a, b, c, &c. have been obtained by means of the operation to find the greatest common measure of the terms of the proposed fraction. But, when α is an irrational quantity, it is obvious that we must determine a, b, c, &c. by some other means. Let us here recall the principles with which we set out at the commencement of the Chapter, and, from which, without any restriction as to the rationality of α, we arrived at the expressions

$$\alpha = a + \frac{1}{\beta} = a + \frac{1}{b + \frac{1}{\gamma}} = a + \frac{1}{b + \frac{1}{c + \frac{1}{\delta}}};$$

and so on; and let us follow the successive steps there pointed out, in order to effect the reduction of $\sqrt{19}$ into a continued fraction; that is, let $\alpha = \sqrt{19}$.

Now the greatest integer in $\sqrt{19}$ is 4, $\therefore a = 4$, and, consequently, $\frac{1}{\sqrt{19}-4} = \beta$; in order to perceive more readily the greatest integer in this, multiply the numerator and denominator by $\sqrt{19}+4$, then $\frac{\sqrt{19}+4}{3} = \beta$, in which the greatest integer is obviously 2; hence $\frac{\sqrt{19}+4}{3} - 2 = \frac{\sqrt{19}-2}{3} \therefore \frac{3}{\sqrt{19}-2} = \gamma$, and, by proceeding in this way, we have

$$\alpha = \sqrt{19} = 4 + \frac{\sqrt{19}-4}{1} \quad \ldots\ldots\ldots\ldots \quad \therefore a = 4$$

$$\beta = \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3} = 2 + \frac{\sqrt{19}-2}{3} \quad \therefore b = 2$$

$$\gamma = \frac{3}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{5} = 1 + \frac{\sqrt{19}-3}{5} \quad \therefore c = 1$$

$$\delta = \frac{5}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{2} = 3 + \frac{\sqrt{19}-3}{2} \quad \therefore d = 3$$

$$\epsilon = \frac{2}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{5} = 1 + \frac{\sqrt{19}-2}{5} \quad \therefore e = 1$$

$$\xi = \frac{5}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{3} = 2 + \frac{\sqrt{19}-4}{3} \quad \therefore f = 2$$

$$\eta = \frac{3}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{1} = 8 + \sqrt{19}-4 \quad \therefore g = 8$$

$$\theta = \frac{1}{\sqrt{19}-4}.$$

As we have now arrived at the same expression as that which we have already had for β, it is plain that the series b, c, &c. must recur; and that the continued fraction, as far as one period, will be

$$\sqrt{19} = 4 + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{2} + \frac{1}{8} + \text{\&c.}$$

and the series of converging fractions, which may be carried to any extent, now that we have got a, b, c, for a complete period, will be

2	1	3	1	2	8	2	&c.
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{4}{11}$	$\frac{5}{14}$	$\frac{14}{39}$	$\frac{117}{326}$	$\frac{248}{691}$	&c.

$$\therefore \sqrt{19} = 4 + \frac{248}{691} \text{ nearly};$$

which does not differ from the truth by so much as $\frac{1}{(691)^2}$.

It is not only in the particular example which we have here chosen that the continued fraction is periodical, for it is the property of all quadratic surds to give rise to these recurring fractions; but, for the proof of it, we must refer the student to note B, at the end of the volume, or to the *Théorie des Nombres* of *Legendre*, page 43.

Application of continued Fractions to the Summation of Infinite Series.

(90.) In our Treatise on Algebra, page 220, we promised to furnish, in the present volume, a direct and easy method of summing every infinite series of which the generating function is rational. The method to which we alluded, is one of the many deductions from the doctrine of continued fractions, and may, therefore, without impropriety, be given in this place.

1. Let the sum of the infinite series

$$1 - 3x + 5x^2 - 7x^3 + 9x^4 - 11x^5 + 13x^6 - \&c.$$

be required.

Regarding this series as the numerator of a fraction, whose denominator is unity, and, dividing the denominator by the numerator, we obtain for quotient 1 and for remainder

$$3x - 5x^2 + 7x^3 - 9x^4 + 11x^5 - \&c.$$

dividing the former divisor, that is, the original series, by this remainder, we have, for quotient $\frac{1}{3x}$, and for remainder,

$$-\frac{4}{3}x+\frac{8}{3}x^2-\frac{12}{3}x^3+\frac{16}{3}x^4-\frac{20}{3}x^5+\&c.$$

dividing the last divisor by this, we obtain for quotient $-\frac{9}{4}$, and for remainder,

$$x^2-2x^3+3x^4-4x^5+5x^6-6x^7+\&c.$$

and, lastly, dividing the preceding divisor by this, we get, for quotient $-\frac{4}{3x}$, and for remainder, zero. Hence the proposed series may be replaced by the continued fraction,

$$\frac{1}{1}+\cfrac{1}{\cfrac{1}{3x}+\cfrac{1}{-\cfrac{9}{4}+\cfrac{1}{-\cfrac{4}{3x}}}}$$

We may get rid of the fractional denominators, one by one, in the usual way, thus: omitting the leading term, multiply numerator and denominator of the remaining fraction, by $3x$, then omitting the second term, multiply numerator and denominator of the remaining fraction by 4, and, finally, omitting the preceding terms, multiply numerator and denominator of the remaining fraction by $3x$, and the continued fraction will then be

$$\frac{1}{1}+\cfrac{3x}{1+\cfrac{12x}{-9+\cfrac{12x}{-4}}};$$

or rather

$$\frac{1}{1} + \frac{3x}{1} - \frac{4x}{3} + \frac{x}{1}.$$

This may be easily reduced to an ordinary fraction, by collecting the several terms, commencing at the last; and we thus find, for the sum of the proposed series, the expression

$$\frac{1-x}{(1+x)^2}.$$

2. As a second example, let the series

$$1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \&c.$$

be proposed.

By proceeding as in the former example, we find the following series of quotients and remainders, viz.

Quotients.	*Remainders.*
1	$2x - 3x^2 + 4x^3 - 5x^4 + 6x^5 - \&c.$
$\frac{1}{2x}$	$-\frac{x}{2} + \frac{2x^2}{2} - \frac{3x^3}{2} + \frac{4x^4}{2} - \frac{5x^5}{2} + \&c.$
-4	$x^3 - 2x^3 + 3x^4 - 4x^5 + \&c.$
$-\frac{1}{2x}$	0

Hence the equivalent continued fraction is

$$\frac{1}{1} + \cfrac{1}{\frac{1}{2x} + \cfrac{1}{-4 + \cfrac{1}{-\frac{1}{2x}}}};$$

or rather,

$$\frac{1}{1} + \frac{2x}{1} - \frac{x}{2} + \frac{x}{1};$$

which, reduced to an ordinary fraction, is

$$\frac{1}{(1+x)^2};$$

of which the proposed series is the development.

3. Let the series be

$$1 + 5x + 9x^2 + 13x^3 + 17x^4 + 21x^5 + \&c.$$

Then proceeding, as in the last example, we have the following table of quotients and remainders, viz.

Quotients.	*Remainders.*
1	$-5x - 9x^2 - 13x^3 - 17x^4 - 21x^5 - \&c.$
$-\frac{1}{5x}$	$\frac{16x}{5} + \frac{32x^2}{5} + \frac{48x^3}{5} + \frac{64x^4}{5} + \frac{80x^5}{5} + \&c.$
$-\frac{25}{16}$	$x^2 + 2x^3 + 3x^4 + 4x^5 + \&c.$
$\frac{16}{5x}$	0

hence the equivalent continued fraction is

$$\frac{1}{1} + \cfrac{1}{-\frac{1}{5x} + \cfrac{1}{-\frac{25}{16} + \cfrac{1}{\frac{16}{5x}}}},$$

or rather

$$\frac{1}{1} - \cfrac{5x}{1 + \cfrac{16x}{5 - \cfrac{x}{1}}};$$

which, reduced to a common fraction, is

$$\frac{1+3x}{(1-x)^2};$$

the development of which is the proposed series.

By treating, in a similar way, the series

$$4x + 15x^2 + 40x^3 + 85x^4 + 156x^5 + \&c.$$

we find its generating rational fraction to be

$$\frac{x(1+x^2)(4-x)}{(1-x)^4}.$$

These examples are sufficient to show that the foregoing process, founded on the determination of the greatest common divisor, between unity and the proposed series, furnishes a direct and simple method of summing every infinite series of which the generating function is rational. We are indebted for it to a paper by *M. Le Barbier*, published in the *Annales de Mathématiques*, for March, 1831.

Application of Continued Fractions to the Solution of Equations.

The method of approximating to the incommensurable roots of an equation, by continued fractions, is due to Lagrange. A single example or two will suffice to illustrate it.

1. Let the equation

$$x^3 - 2x - 5 = 0,$$

be proposed. It is soon seen that 2 is the first figure of the real root, the other two are imaginary, because $4(-2)^3 + 27 \times 5^2 > 0$ (page 156). Substitute, then, $2 + \frac{1}{x'}$ for x, and we have the following transformed equation, in which the root x' must necessarily exceed unity:

$$x'^3 - 10x'^2 - 6x' - 1 = 0.$$

Of course we effect this transformation, not by the actual substitution of $2 + \frac{1}{x'}$ for x, in the proposed equation, as Lagrange did, but by operating as in (22), thus:

1	0	−2	−5 (2
	2	4	4
	2	2	1
	2	8	
	4	10	
	2		
	6		

and, consequently (22), the transformed equation is

$$x'^3 - 10x'^2 - 6x' - 1 = 0.$$

The first figure of the root of this equation, found by trial, is 10; putting, therefore,

$$x' = 10 + \frac{1}{x''},$$

we have, for a new transformed, the equation

$$61x''^3 - 94x''^2 - 20x'' - 1 = 0;$$

the first figure in the root of which is 1. Put, therefore,

$$x'' = 1 + \frac{1}{x'''},$$

and effect a third transformation, which will be

$$54x'''^3 + 25x'''^2 - 89x''' - 61 = 0;$$

in which the first figure of the root is 1. Continue this process, and we shall have, for the leading figures of the root of the original equation, the expression

$$x = 2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \&c.}}}}$$

which furnishes the converging fractions following,

$$\begin{array}{cccccccccc} & 1 & 1 & 2 & 1 & 3 & 1 & 1 & 12 & \\ 2, & \frac{21}{10}, & \frac{23}{11}, & \frac{44}{21}, & \frac{111}{53}, & \frac{155}{74}, & \frac{576}{275}, & \frac{731}{349}, & \frac{1307}{624}, & \frac{16415}{7837}, \text{ \&c.} \end{array}$$

and these are alternately below and above the true value of the root. The fraction $\frac{16415}{7837}$ is greater than the true value, but the error being less than $\frac{1}{(7837)^2}$, by (88), that is, less than $\cdot 0000000163$, it follows that the approximation, $\frac{16415}{7837}$, will be true, as far as the seventh decimal. The root is, therefore, 2·0945514, true to seven places.

In each of the transformed equations, which occur in the foregoing operation, the root is necessarily greater than unity, and but one real root exists in each, so that, in searching for the first figure, we are to limit our trials to the numbers 0, 1, 2, 3, ... 10, 11, &c.

When the equation has several real roots, they may all be separately evolved, as above, provided we know their number and situation. This knowledge the application of Sturm's theorem will always supply, and the method of Lagrange may thus be perfected. The practical tediousness, however, of the numerical process, renders it greatly inferior to the method of Mr. Horner, already explained.

We shall give one more example of Lagrange's method, as applied to the equation

$$x^3 - 7x + 7 = 0,$$

which we have already solved, by continuous approximation, at p. 92.

By Sturm's theorem we find that two roots of this equation are comprised between 1 and 2, and one between -3 and -4, (see p. 148). In order to approximate to the positive roots, put, as in last example,

$$x = 1 + \frac{1}{x'};$$

and we shall have the transformed equation

$$x'^3 - 4x'^2 + 3x' + 1 = 0;$$

which, because 1 is the first figure of *two* values of x, shall necessarily have two roots greater than unity. The situation of these roots are found, by Sturm's theorem, or that of Budan,* to be the one between 1 and 2, and the other between 2 and 3. Hence, in order to approximate to the first of these, put

$$x' = 1 + \frac{1}{x''},$$

and we shall have the transformed equation

$$x''^3 - 2x''^2 - x'' + 1 = 0;$$

to which there will be no necessity to apply the theorem of Sturm, because we know that it has one root, and only one, greater than unity, so that two consecutive numbers in the series 0, 1, 2 . . . . 10, must, when substituted for x, give results with contrary signs. These numbers are 2 and 3.

To approximate to the other positive root, we must put

$$x' = 2 + \frac{1}{x_{\prime\prime}},$$

which will furnish the transformed equation

$$x_{\prime\prime}^3 + x_{\prime\prime}^2 - 2x_{\prime\prime} - 1 = 0;$$

which has one, and only one, root greater than unity, and, therefore, its situation may be easily found by trial to be between 1 and 2. We have, therefore, now to make the substitutions

$$x'' = 2 + \frac{1}{x'''}$$

$$x_{\prime\prime} = 1 + \frac{1}{x_{\prime\prime\prime}},$$

and we thus have the new equations

$$x'''^3 - 3x'''^2 - 4x''' - 1 = 0$$

$$x_{\prime\prime\prime}^3 - 3x_{\prime\prime\prime}^2 - 4x_{\prime\prime\prime} - 1 = 0,$$

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* Budan's theorem (p. 65,) is applicable here, because all the roots are known to be real.

each of which has one, and only one, root greater than unity; hence the first figure of each is readily found to be 4, and the next transformation is consequently

$$x''''^3 - 20x''''^2 - 9x'''' - 1 = 0,$$

the first figure of the root of which is 2; the next transformed equation is

$$197x'''''^3 - 568x'''''^2 - 695x''''' - 181 = 0,$$

the first figure of the root of which is 3; and, by continuing these transformations, we have, for the values of x sought, the following developments, viz.

$$x = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{20} + \frac{1}{2} + \frac{1}{3} +, \text{ \&c.}$$

$$x = 1 + \frac{1}{2} + \frac{1}{1} + \frac{1}{4} + \frac{1}{20} + \frac{1}{2} + \frac{1}{3} +, \text{ \&c.}$$

The converging fractions deduced from these are

$$\frac{1}{1},\ \frac{2}{1},\ \overset{2}{\frac{5}{3}},\ \overset{4}{\frac{22}{13}},\ \overset{20}{\frac{445}{263}},\ \overset{2}{\frac{912}{539}},\ \overset{3}{\frac{3181}{1880}},\ \text{\&c.}$$

$$\frac{1}{1},\ \frac{3}{2},\ \overset{1}{\frac{4}{3}},\ \overset{4}{\frac{19}{14}},\ \overset{20}{\frac{384}{283}},\ \overset{2}{\frac{787}{580}},\ \overset{3}{\frac{2745}{2023}},\ \text{\&c.}$$

Hence, for near values of x, we have

$$x = \frac{3181}{1880},\quad x = \frac{2745}{2023};$$

or, in decimals,

$$x = 1\cdot6920213,\quad x = 1\cdot3568957,$$

which are true as far as six decimals, and are but a unit below the truth in the seventh place.

CHAPTER II.

ON BINOMIAL EQUATIONS.

(91.) Binomial Equations are those which consist of merely two terms; the one being some power of the unknown quantity, and the other the absolute number. The general form of such equations is, therefore,

$$y^n \pm a^n = 0,$$

in which a^n represents a known quantity. By substituting ax for y, the form becomes

$$a^n x^n \pm a^n = 0;$$

or, more simply,

$$x^n \pm 1 = 0,$$

to which form we shall always suppose the equation to be reduced.

(92.) The following particulars respecting these equations, result from the simplest considerations.

1. If n be even, the equation $x^n - 1 = 0$, or $x^n = 1$, has two real roots, viz. $+1$ and -1, and no more. That it has these two roots is plain, for an even power of ± 1 is always $+1$; and that it has no other real root is equally obvious, because no other number can, by its involution, produce 1. Hence the binomial $x^n - 1$ is divisible by $(x+1)(x-1) = x^2 - 1$. By actually performing the division, we have the equation

$$x^{n-2} + x^{n-4} + x^{n-6} + \ldots\ldots + x^4 + x^2 + 1 = 0,$$

a recurring equation in which all the $n-2$ roots must be imaginary. For example, the equation

$$x^6 - 1 = 0,$$

divided by $x^2 - 1$, gives

$$x^4 + x^2 + 1 = 0,$$

whence

$$x = \pm \sqrt{\frac{-1 \pm \sqrt{-3}}{2}};$$

so that the six roots of the proposed equation are

$$\begin{array}{cc} +1 & -1 \\ +\sqrt{\frac{-1+\sqrt{-3}}{2}}, & -\sqrt{\frac{-1+\sqrt{-3}}{2}} \\ +\sqrt{\frac{-1-\sqrt{-3}}{2}}, & -\sqrt{\frac{-1-\sqrt{-3}}{2}}. \end{array}$$

2. If n be odd, the equation $x^n - 1 = 0$ has only one real root, viz. $+1$; for $+1$ is the only real number of which the odd powers are $+1$; hence $x - 1$ is the only real simple factor of x^{n-1}; dividing by this, we have the recurring equation

$$x^{n-1} + x^{n-2} + x^{n-3} + \ldots + x^2 + x + 1 = 0,$$

of which all the $n - 1$ roots are imaginary.

For example, the equation

$$x^3 - 1 = 0,$$

divided by $x - 1$, gives

$$x^2 + x + 1 = 0,$$

whence

$$x = \frac{-1 \pm \sqrt{-3}}{2};$$

so that the three roots of the proposed equation are

$$1, \quad \frac{-1+\sqrt{-3}}{2}, \quad \frac{-1-\sqrt{-3}}{2}.$$

3. If n be even, the equation $x^n + 1 = 0$, or $x^n = -1$, has no real root, for $\sqrt[n]{-1}$ is then impossible; so that all the roots of this equation are imaginary. For example, the four roots of the equation

$$x^4 + 1 = 0,$$

as determined by the rules for recurring equations (71), are

$$\frac{-1+\sqrt{-1}}{\sqrt{2}},\quad \frac{-1-\sqrt{-1}}{\sqrt{2}},\quad \frac{1+\sqrt{-1}}{\sqrt{2}},\quad \frac{1-\sqrt{-1}}{\sqrt{2}}.$$

4. If n be odd, the equation $x^n+1=0$, or $x^n=-1$, has one real root, viz. -1, and no more, because this is the only real number of which an odd power is -1; hence, if the equation $x^3+1=0$ be proposed, the first member being divisible by $x+1$, we have the equation

$$x^2-x+1=0;$$

so that the three roots of the proposed equation are

$$-1,\quad \frac{1+\sqrt{-3}}{2},\quad \frac{1-\sqrt{-3}}{2}.$$

5. The roots of the equation

$$x^n \pm 1 = 0$$

are all unequal; for the limiting polynomial nx^{n-1} having no divisor in common with $x^n \pm 1$, the proposed cannot have equal roots (37).

PROPOSITION I.

(93.) If a be one of the imaginary roots of the equation $x^n-1=0$, then any power of a will be also a root.

For, since a is one root of the equation $x^n-1=0$, therefore $a^n=1$, and consequently,

$$a^{2n}=1,\ a^{3n}=1,\ a^{4n}=1,\ \&c., \text{ also } a^{-n}=1,\ a^{-2n}=1,\ a^{-3n}=1,\ \&c.$$

the values

$$a,\ a^2,\ a^3\ \ldots.,\ a^{-1},\ a^{-2},\ a^{-3}\ \ldots.,$$

thus satisfying, the conditions of the equation are roots of it.

Cor. 1. It hence appears that the roots of the equation $x^n-1=0$ may be represented under an infinite variety of forms, each term in the following series being a root, viz.

$$\ldots\ldots a^{-3},\ a^{-2},\ a^{-1},\ 1,\ a,\ a^2,\ a^3,\ \ldots\ldots a^{n-1},\ a^n,\ a^{n+1},\ \ldots\ldots a^{2n},\ a^{2n+1},\ \ldots\ldots$$

in which series, however, there cannot be more than n quantities essentially different, otherwise the equation would have more than n roots.

PROPOSITION II.

(94.) If a be one of the imaginary roots of the equation $x^n+1=0$, then any odd power of a will be also a root.

For, since a is one root of the equation $x^n=-1$, therefore $a^n=-1$; and, since every odd power, whether positive or negative, of -1 is also -1, therefore

$$a^{3n}=-1,\quad a^{5n}=-1,\quad a^{7n}=-1,\quad \text{\&c., also}\quad a^{-3n}=-1,$$
$$a^{-5n}=-1,\quad a^{-7n}=-1,\quad \text{\&c.};$$

so that the quantities

$$a,\ a^3,\ a^5\ \ldots\ldots,\ a^{-1},\ a^{-3},\ a^{-5}\ \ldots\ldots,$$

are roots of the equation. These roots, therefore, assume an infinite variety of forms, although there cannot be more than n different values.

PROPOSITION III.

(95.) When n is a prime number, the roots of the equation $x^n-1=0$ are all exhibited in the series

$$1,\ a,\ a^2,\ a^3\ \ldots\ldots a^{n-1},$$
$$\text{or}\ a^n,\ a^{n+1},\ a^{n+2},\ a^{n+3}\ \ldots\ldots a^{2n-1},$$
$$\text{or}\ a^{2n},\ a^{2n+1},\ a^{2n+2},\ a^{2n+3}\ \ldots\ldots a^{3n-1}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\text{or}\ 1,\ a^{-1},\ a^{-2},\ a^{-3}\ \ldots\ldots a^{-(n-1)}$$
$$\text{or}\ a^{-n},\ a^{-n-1},\ a^{-n-2},\ a^{-n-3}\ \ldots\ldots a^{-n-(n-1)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It has been shown in Prop. I., that every one of the foregoing quantities is a root of the equation if a is a root; if, therefore, no two of the n quantities in each series are the same under a different form, each series will exhibit all the n roots of the equation.

Now, if we suppose any two of the roots in either series to be equal, as for instance a^p and a^t, in which $t > p$, then, by dividing the equation $a^t = a^p$ by a^p, we have $a^{t-p} = 1$; and that this equation is impossible, may be proved as follows:

Because n is a prime number, and $t - p$ necessarily less than n, therefore the numbers n and $t - p$ are prime to each other, and consequently two whole numbers, x' and y', may always be found such that

$$(t - p)\, x' = ny' + 1;^*$$

and, as $a^{t-p} = 1$, therefore $a^{(t-p)x'} = 1$, and consequently $a^{ny'+1} = 1$, that is, $a^{ny'} \,.\, a = 1$; but $a^{y'}$ is a root (Prop. I.); hence $a^{ny'} = 1$, therefore, from the last equation, $a = 1$, which is impossible, because, by hypothesis, a is imaginary. Hence, each of the series announced above, comprises the n roots of the equation under different forms.

PROPOSITION IV.

(96.) When p and q have no common measure, then the equations $x^p - 1 = 0$ and $x^q - 1 = 0$ have no common root except unity.

If possible, let a be a root common to both equations, and different from unity, then we have $a^p = 1$ and $a^q = 1$; and, since p and q are prime to each other, two whole numbers, x' and y', may always be found such that $px' = qy' + 1$ (Algebra, p. 241). Consequently $a^{px'} = a^{qy'+1} = a^{qy'} \,.\, a$. But $a^{y'}$ is a root of each equation; hence $a^{qy'} = 1$, therefore $a = 1$, which is contrary to the hypothesis. Hence the root common to the two proposed equations can be no other than unity.

Cor. When the equations $x^n - 1 = 0$, $x^m - 1 = 0$, have an imaginary root in common, the exponents m, n, must have a common measure.

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* See the Algebra, second edition, page 241.

PROPOSITION V.

(97.) **When n is a composite number, formed of the factors p, q, r, &c., then the roots of the equations $x^p - 1 = 0$, $x^q - 1 = 0$, $x^r - 1 = 0$, &c., must all of them be roots of the equation $x^n - 1 = 0$.**

This is obvious from Prop. I., Part I.; for the two quantities x^n, and 1, may be regarded as the result of the two quantities x^p, and 1, raised to the qr, &c. power, or as the result of x^q and 1 raised to the pr, &c. power, &c.

PROPOSITION VI.

(98.) **When n is the product of two prime numbers, p and q, the roots of the equation $x^n - 1 = 0$ will be expressed by the n products arising from multiplying every root of the equation $x^p - 1 = 0$, by every root of the equation.**

Let the roots of the equation $x^p - 1 = 0$ be

$$1 \quad \alpha, \quad \alpha^2, \quad \alpha^3 \ldots \alpha^{p-1},$$

and those of the equation $x^q - 1 = 0$,

$$1 \quad \beta, \quad \beta^2, \quad \beta^3 \ldots \beta^{q-1}.$$

These two series of roots are all different, with the exception of the common root unity (Prop. IV.), and are, therefore, so many different roots of the equation $x^n - 1 = 0$, (Prop. V.) The pq products, also arising from multiplying the one series by the other, will be so many roots of the proposed equation. For, let $\alpha^h \beta^k$ represent any one of these products, then, since α^h and β^k are roots of $x^n - 1 = 0$, we have $\alpha^{hn} = 1$ and $\beta^{kn} = 1$, and consequently, $(\alpha^h \beta^k)^n = 1$, or $(\alpha^h \beta^k)^n - 1 = 0$; hence $\alpha^h \beta^k$ must be a root of $x^n - 1 = 0$. The products are all different, for, if possible, let

$$\alpha^h \beta^k = \alpha^g \beta^m$$

$$\therefore \alpha^{h-g} = \beta^{m-k}.$$

Now, whether $h - g$ and $m - k$ be positive or negative numbers, the expression α^{k-r} is, necessarily, a root of $x^p - 1 = 0$, and the expression β^{m-k}, a root of $x^q - 1 = 0$; and as these roots are, by Prop. IV., essentially different, except when they are both unity, it follows that the equation deduced from our hypothesis cannot exist; that hypothesis, therefore, is not true, so that no two products can be equal to each other. As, therefore, the products are pq in number, all different, and all satisfy the equation $x^n - 1 = 0$, they must express the pq roots of that equation.

In the foregoing reasoning, it is, of course, presumed that the component factors, p, q, are unequal. If they are equal, then the roots of the equation, $x^n - 1 = 0$, will not all be comprised in the aforesaid products.

As an example of the application of this proposition, let it be required to determine the six roots of the equation, $x^6 - 1 = 0$.

As 6 is composed of the two prime numbers, 2 and 3, we have first to find the roots of

$$x^2 - 1 = 0, \quad \text{and} \quad x^3 - 1 = 0.$$

The roots of $x^2 - 1 = 0$ are 1 and -1. The roots of $x^3 - 1 = 0$ are (p. 184),

$$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}.$$

Consequently, the six roots sought are the six products, arising from multiplying these three roots by 1, -1, and are, therefore,

$$1, \quad -1, \quad \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}, \frac{1+\sqrt{-3}}{2}.$$

PROPOSITION VII.

(99.) To determine the roots of the equation $x^n - 1 = 0$, when n is the square of a prime number p.

Put $x^p = z$, then $x^p - z = 0$, and $z^p - 1 = 0$, and let the roots of this last equation be $1, \beta, \beta^2, \beta^3 \ldots \beta^{p-1}$, then, by substitution,

$$x^p - z = \begin{cases} x^p - 1 = 0, \\ x^p - \beta = 0, \\ x^p - \beta^2 = 0, \\ x^p - \beta^3 = 0, \\ \&c. \quad \&c. \end{cases}$$

hence the pp values of x, in these p equations, will evidently be all different, and will be the roots of the equation $x^{pp} - 1 = 0$.

To determine these roots, it will be sufficient to advert to Art. (23), which proves that the roots of $x^p - \beta = 0$ are equal to the roots of $x^p - 1 = 0$ multiplied by $\sqrt[p]{\beta}$; and, in a similar manner, the roots of $x^p - \beta^2 = 0$ are equal to the roots of $x^p - 1 = 0$, multiplied by $\sqrt[p]{\beta^2}$, &c.; therefore we immediately conclude that the roots of

$$\left.\begin{array}{l} x^p - 1 = 0 \text{ are } 1, \beta, \beta^2, \beta^3 \ldots\ldots\ldots \beta^{p-1} \\ x^p - \beta = 0 \ldots\ldots \sqrt[p]{\beta}, \beta\sqrt[p]{\beta}, \beta^2\sqrt[p]{\beta} \ldots\ldots \beta^{p-1}\sqrt[p]{\beta} \\ x^p - \beta^2 = 0 \ldots\ldots \sqrt[p]{\beta^2}, \beta\sqrt[p]{\beta^2}, \beta^2\sqrt[p]{\beta^2} \ldots \beta^{p-1}\sqrt[p]{\beta^2} \end{array}\right\} = \text{the } n \text{ roots of } x^n - 1 = 0,$$

&c. &c. &c.

For example, let it be required to find the roots of $x^9 - 1 = 0$. The roots of $x^3 - 1 = 0$ are

$$1, \quad \frac{-1 + \sqrt{-3}}{2}, \quad \frac{-1 - \sqrt{-3}}{2};$$

hence the roots of $x^9 - 1 = 0$ are

$$1, \quad \frac{-1 + \sqrt{-3}}{2}, \quad \frac{-1 - \sqrt{-3}}{2}$$

$$\sqrt[3]{\frac{-1 + \sqrt{-3}}{2}}, \quad \frac{-1 + \sqrt{-3}}{2} \sqrt[3]{\frac{-1 + \sqrt{-3}}{2}},$$

$$\frac{-1 - \sqrt{-3}}{2}, \sqrt[3]{\frac{-1 + \sqrt{-3}}{2}} \sqrt[3]{\frac{-1 - \sqrt{-3}}{2}},$$

$$\frac{-1 + \sqrt{-3}}{2} \sqrt[3]{\frac{-1 - \sqrt{-3}}{2}}, \quad \frac{-1 - \sqrt{-3}}{2} \sqrt[3]{\frac{-1 - \sqrt{-3}}{2}}$$

The preceding propositions fully establish the truth, that every root has as many values as there are units in its index; for, as there are n different quantities which satisfy the equation $x^n = 1$, it follows that $\sqrt[n]{1}$ has n different values; and it is plain that if each of these values be multiplied by the common arithmetical value of $\sqrt[n]{a}$, the n products will all be different, and such that, if each be raised to the nth power, the result will always be a; hence the products of which we speak are so many different values of $\sqrt[n]{a}$. The determination, therefore, of the n roots of $\sqrt[n]{a}$ requires that we are able to extract the nth arithmetical root of a, and to exhibit all the imaginary roots of $\sqrt[n]{1}$. The foregoing propositions have been devoted chiefly to an examination of the properties and relations of these roots, and not to the actual exhibition of their values, although, as in the proposition above, one or two examples of the solution have been given to illustrate the reasoning. To obtain the imaginary roots, however, in their simplest form, that is, in the form $a + b\sqrt{-1}$, and for all values of the exponent, requires the aid of a theorem, borrowed from the science of Trigonometry.

(100.) The theorem to which we refer, is the well known formula of *De Moivre* given in most books on Analytical Trigonometry, viz. (see Trigonometry, page 52,)

$$(\cos. a \pm \sin. a \,.\, \sqrt{-1})^n = \cos. na \pm \sin. na \,.\, \sqrt{-1};$$

which, if the arc $2k\pi$, (π being a semicircumference,) be substituted for na, becomes

$$(\cos. \frac{2k\pi}{n} \pm \sin. \frac{2k\pi}{n} \,.\, \sqrt{-1})^n = \cos. 2k\pi \pm \sin. 2k\pi \,.\, \sqrt{-1};$$

that is, since

$$\cos. 2k\pi = 1, \quad \text{and } \sin. 2k\pi = 0,$$

$$(\cos. \frac{2k\pi}{n} \pm \sin. \frac{2k\pi}{n} \,.\, \sqrt{-1})^n = 1;$$

so that the expression

$$\cos. \frac{2k\pi}{n} \pm \sin. \frac{2k\pi}{n} \,.\, \sqrt{-1},$$

comprehends in it all the n roots of unity, or all the particular values of x, which satisfy the equation $x^n - 1 = 0$.

Although, in this general expression, the value of k is quite arbitrary, yet, assume it what we will, the expression can never furnish more than n different values. These different values will arise from the several substitutions of

$$0, \quad 1, \quad 2, \quad 3 \;.\;.\;.\;.$$

up to the number $\frac{n-1}{2}$, inclusively, if n is odd, and up to $\frac{n}{2}$, if n is even; and for substitutions beyond these limits the preceding results will recur. To prove this, let us actually substitute as proposed; we shall thus have the following series of results, viz.

$$\text{for } k = 0 \;.\;.\;.\;.\; x = \cos. 0 \pm \sin. 0 \,.\, \sqrt{-1} = 1$$

$$k = 1 \;.\;.\;.\;.\; x = \cos. \frac{2\pi}{n} \pm \sin. \frac{2\pi}{n} \,.\, \sqrt{-1}$$

$$k = 2 \;.\;.\;.\;.\; x = \cos. \frac{4\pi}{n} \pm \sin. \frac{4\pi}{n} \,.\, \sqrt{-1}$$

$$k = 3 \;.\;.\;.\;.\; x = \cos. \frac{6\pi}{n} \pm \sin. \frac{6\pi}{n} \,.\, \sqrt{-1}$$

$$\vdots \qquad\qquad \vdots$$

$$k = \frac{n-1}{2} \;.\;.\;.\; x = \cos. \frac{(n-1)\pi}{n} \pm \sin. \frac{(n-1)\pi}{n} \,.\, \sqrt{-1}$$

Each of these expressions, except the first, involves two distinct values, so that, omitting the value given by $k = 0$, there are $n - 1$ values, and, consequently, altogether, there are n values, and that they are all different, is plain, because the arcs

$$0, \; \frac{2\pi}{n}, \; \frac{4\pi}{n}, \; \frac{6\pi}{n}, \;.\;.\;.\;.\; \frac{(n-1)\pi}{n},$$

being all different, and less than π, have all different cosines. The arcs which would arise from continuing the substitutions, are

$$\frac{(n+1)\pi}{n}, \quad \frac{(n+3)\pi}{n}, \quad \frac{(n+5)\pi}{n}, \;\&c.$$

or, which are the same,

$$2\pi - \frac{(n-1)\pi}{n}, \quad 2\pi - \frac{(n-3)\pi}{n}, \quad 2\pi - \frac{(n-5)\pi}{n}, \text{ \&c.,}$$

and the sines and cosines of these are respectively the same as the sines and cosines of the arcs

$$\frac{(n-1)\pi}{n}, \quad \frac{(n-3)\pi}{n}, \quad \frac{(n-5)\pi}{n}, \text{ \&c.,}$$

which have already occurred.*

If n is an even number, the final substitution for k must be $\frac{n}{2}$ instead of $\frac{n-1}{2}$, as above; and therefore the final pair of conjugate values for x will be

$$x = \cos.\pi \pm \sin.\pi . \sqrt{-1} = -1,$$

which values of x differ from all the other values because in them no arc occurs so great as π.

The arcs which would arise from continuing the substitutions beyond $k = \frac{n}{2}$ are

$$\frac{(n+2)\pi}{n}, \quad \frac{(n+4)\pi}{n}, \quad \frac{(n+6)\pi}{n}, \text{ \&c.;}$$

or, which are the same,

$$2\pi - \frac{(n-2)\pi}{n}, \quad 2\pi - \frac{(n-4)\pi}{n}, \quad 2\pi - \frac{(n-6)\pi}{n}, \text{ \&c.,}$$

and the sines and cosines of these are respectively the same as the

* The *signs* of the sines will, however, be different, but the only effect of this difference is evidently to furnish each pair of conjugate roots in an inverse order.

sines and cosines of the arcs

$$\frac{(n-2)\pi}{n}, \quad \frac{(n-4)\pi}{n}, \quad \frac{(n-6)\pi}{n}, \text{ \&c.,}$$

which have already occurred.

It is easy to see that in every pair of conjugate roots, each is the reciprocal of the other. In fact whatever be k,

$$(\cos.\frac{2k\pi}{n} + \sin.\frac{2k\pi}{n}.\sqrt{-1})(\cos.\frac{2k\pi}{n} - \sin.\frac{2k\pi}{n}.\sqrt{-1}) =$$
$$\cos.^2\frac{2k\pi}{n} + \sin.^2\frac{2k\pi}{n} = 1,$$

which shows that the two factors in the first member are of the form $a, \frac{1}{a}$.

We have proved (93) that every power of an imaginary root of the binomial equation is also a root; but unless n be a prime number, we could not infer that all the roots would ever be produced by involving any one of them. Such would not indeed be the case. There is always, however, one among the imaginary roots of which the involution will furnish all the others; it is the first imaginary root, or that due to the substitution $k = 1$, in the foregoing series of values; for, by De Moivre's formula, the powers of this produce all the others, thus:

$$(\cos.\frac{2\pi}{n} + \sin.\frac{2\pi}{n}.\sqrt{-1})^2 = \cos.\frac{4\pi}{n} + \sin.\frac{4\pi}{n}.\sqrt{-1}$$

$$(\cos.\frac{2\pi}{n} + \sin.\frac{2\pi}{n}.\sqrt{-1})^3 = \cos.\frac{6\pi}{n} + \sin.\frac{6\pi}{n}.\sqrt{-1}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$(\cos.\frac{2\pi}{n} + \sin.\frac{2\pi}{n}.\sqrt{-1})^{\frac{n-1}{2}} = \cos.\frac{n-1}{n}\pi + \sin.\frac{n-1}{n}\pi.\sqrt{-1}.$$

These powers of the first imaginary root, which we may call a, thus furnish one half of the entire number of imaginary roots, and their reciprocals of these being the other half, all of them are determined from the first; the imaginary roots are, therefore,

$$a,\quad a^2,\quad a^3\ \ldots\ldots\ a^{\frac{n-1}{2}}$$

$$\frac{1}{a},\quad \frac{1}{a^2},\quad \frac{1}{a^3}\ \ldots\ldots\ \frac{1}{a^{\frac{n-1}{2}}}.$$

When n is even, the last power will be

$$(\cos.\frac{2\pi}{n}+\sin.\frac{2\pi}{n}.\sqrt{-1})^{\frac{n}{2}}=\cos.\pi+\sin.\pi.\sqrt{-1};$$

and the imaginary roots are, therefore,

$$a,\quad a^2,\quad a^3\ \ldots\ldots\ a^{\frac{n}{2}}$$

$$\frac{1}{a},\quad \frac{1}{a^2},\quad \frac{1}{a^3}\ \ldots\ldots\ \frac{1}{a^{\frac{n}{2}}}.$$

(101.) By the same general formula (page 191), we are enabled to determine all the roots of the equation

$$x^n+1=0,$$

for, since

$$\cos.(2k+1)\pi=-1,\quad\text{and }\sin.(2k+1)\pi=0,$$

that formula gives

$$(\cos.\frac{2k+1}{n}\pi\pm\sin.\frac{2k+1}{n}\pi.\sqrt{-1})^n=$$

$$\cos.(2k+1)\pi\pm\sin.(2k+1)\pi.\sqrt{-1}=-1;$$

hence the n values of x are all comprised in the general expression

$$x=\cos.\frac{2k+1}{n}\pi\pm\sin.\frac{2k+1}{n}\pi.\sqrt{-1};$$

which, by putting for k the values 0, 1, 2, 3, &c. in succession, furnishes the following series of separate values, viz.

$$\text{for } k=0 \;\ldots\; x=\cos.\frac{\pi}{n} \pm \sin.\frac{\pi}{n}.\sqrt{-1}$$

$$k=1 \;\ldots\; x=\cos.\frac{3\pi}{n} \pm \sin.\frac{3\pi}{n}.\sqrt{-1}$$

$$k=2 \;\ldots\; x=\cos.\frac{5\pi}{n} \pm \sin.\frac{5\pi}{n}.\sqrt{-1}$$

$$\vdots$$

$$k=\frac{n-1}{2} \;\ldots\; x=\cos.\pi \pm \sin.\pi.\sqrt{-1}=-1;$$

or, when n is even,

$$k=\frac{n-2}{2} \;\ldots\; x=\cos.(\pi-\frac{\pi}{n}) \pm \sin.(\pi-\frac{\pi}{n}.\sqrt{-1}).$$

Now that the foregoing system of n roots are all different is obvious, since

$$\frac{\pi}{n},\ \frac{3\pi}{n},\ \frac{5\pi}{n}\ \ldots\ \frac{n\pi}{n},\ \text{or}\ \pi-\frac{\pi}{n},$$

are all different arcs, of which the greatest does not exceed a semicircumference. If the preceding series be extended, it will be easy to prove, after what has been done at page 192, that the values formerly obtained will recur.

As in the former case of the general problem, so here, each root may be derived from the first pair of the series: thus designing the first root, $\cos.\frac{\pi}{n} \pm \sin.\frac{\pi}{n}.\sqrt{-1}$, by a or $\frac{1}{a}$, according as the upper or lower sign is taken we evidently have, for the preceding series, the following equivalent expressions, viz.

$$\left.\begin{array}{l} a,\ a^3,\ a^5\ \ldots\ a^n \\ \frac{1}{a},\ \frac{1}{a^3},\ \frac{1}{a^5}\ \ldots\ \frac{1}{a^n} \end{array}\right\}\text{when } n \text{ is odd}$$

and

$$\left.\begin{matrix} a, & a^3, & a^5 & \dots & a^{n-1} \\ \frac{1}{a}, & \frac{1}{a^3}, & \frac{1}{a^5} & \dots & \frac{1}{a^{n-1}} \end{matrix}\right\} \text{when } n \text{ is even}$$

For further researches on the theory of binomial equations, the student may consult *Lagrange's Traité de la Résolution des Equations numériques*, NOTE 14; and *Legendre's Théorie des Nombres*, PART V.; as also *Sir James Ivory's* article on EQUATIONS, in the Supplement to the Ençyclopædia Britannica.

CHAPTER III.

ON THE SOLUTION OF TWO EQUATIONS, CONTAINING TWO UNKNOWN QUANTITIES.

(102.) Two equations, each containing two unknown quantities, x, y, together with known numbers, may be thus expressed, viz.

$$F(x, y)=0, \quad f(x,y)=0 \dots (1);$$

and their solution consists in determining the system of values for x and y, which simultaneously satisfy both equations.

In order that y may have a value β, which will equally belong to both equations, it is obviously necessary, and it is sufficient, that there exist a value for x, competent to satisfy the two equations

$$F(x,\beta)=0, \quad f(x,\beta)=0;$$

that is, these two equations in x must have a common root, and, therefore, the polynomials $F(x, \beta)$, $f(x, \beta)$, must have a common factor, or admit of a common measure in x. In order, therefore, to ascertain whether any proposed value, β, for y is consistent with the conditions (1), we should have to perform the operation for the common measure upon the functions $F(x, \beta)$, $f(x, \beta)$. If a common measure, which is a function of x, be found to exist, the proposed value for y is admissible, and the common measure, equated to zero, will be an equation whose roots will be the corresponding values of x; but if no such common measure exist, we must then reject the assumed value of y, as being incompatable with the conditions (1).

To assume different values for one of the unknowns, and, in this way, to try their eligibility, would, in many cases, require an endless series of operations. The most direct and obvious mode of proceeding, in order to obtain values for y, which must necessarily cause the functions in x to have a common measure, would seem to be this, viz. to arrange the terms of each polynomial according to the powers of x, and to operate upon them, for the common measure, till we arrive at a remainder independent of x, and then to equate this remainder in y to zero. For the values of y, which satisfy this equation, are all such as to cause the remainder to vanish.

It must be remembered, however, that, in the operation of finding the greatest common measure of two algebraical expressions, we have frequent occasion to suppress certain factors, and to introduce others, and, before we could affirm with confidence that the values of y, which cause the remainder to vanish, necessarily fulfil the proposed conditions, we must examine whether or not this remainder is affected by the factors, which may have been rejected or introduced. If, however, the process for the common measure, in any particular case, requires neither the suppression nor the introduction of a factor, we may then safely infer that the final remainder, or that which is independent of x, will, when equated to zero, furnish all the values of y consistent with the proposed conditions; because, if each value thus determined were to be put for y, in the original polynomials, and the common measure in each case found, we should, obviously, arrive at the very same series of collateral expressions for x.

For example, suppose the equations

$$x^3 + 3yx^2 + (3y^2 - y + 1)\,x + y^3 - y^2 + 2y = 0$$
$$x^2 + 2yx + y^2 - y = 0,$$

were proposed for solution.

As the polynomials are already arranged according to the decreasing powers of x, we may at once commence the operation for the common measure, which is as follows:

$$x^2+2yx+y^2-y]x^3+3yx^2+(3y^2-y+1)x+y^3-y^2+2y\,[x+y$$
$$x^3+2yx^2+(\ y^2-y)x$$
$$\overline{\qquad yx^2+(2y^2+1)x+y^3-y^2+2y}$$
$$yx^2+2y^2x+y^3-y^2$$
$$\overline{\qquad x+2y]x^2+2yx+y^2-y\,[x}$$
$$x^2+2yx$$
$$\overline{\qquad y^2-y.}$$

Having now got a remainder, independent of x, we have, for the determination of all those values of y, which cause the proposed polynomials to have a common measure, the equation

$$y^2-y=0\ \therefore\ y=0,\quad y=1;$$

and the values of x, corresponding to these, are, of course, those furnished by equating the common measure to zero; they are, therefore,

$$x=0,\quad x=-2.$$

It is plain that $y=0$, and $y=1$, are the only values which, when substituted in the proposed expression, will cause the preceding operation to terminate; if other values for y existed, the final remainder above would necessarily contain them.

If a like process be performed with the two equations

$$x^4+2yx^3+(2y^2+1)x^2+(y^3+9y^2+y-81)x+y^2=0$$
$$x^3+2yx^2+2y^2x+y^3+9y^2-81=0,$$

we should find, without suppressing or introducing any factor, the expression $9y^2-81$, for the remainder in y, and the expression x^2+yx+y^2, for the corresponding divisor; hence the final equations for determining x and y, are

$$y^2 - 9 = 0, \quad x^2 + yx + y^2 = 0,$$

the solution of which will furnish the values which satisfy the proposed equations.

But let us examine the consequences of introducing or suppressing factors in the course of the process for finding the common measure, or of arriving at a remainder independent of x.

There are three distinct cases to consider, viz.

1. The value attributed to y may reduce to zero neither of the factors which have been introduced or suppressed.

2. It may reduce to zero one of the factors which have been introduced.

3. The value may be such as to reduce to zero one of the factors which have been suppressed.

(103.) 1. Suppose that the value attributed to y does not render any of the factors introduced or suppressed zero. If we substitute this value in the two polynomials, and perform the operation with the resulting functions of x, we shall obtain for remainder the same value that was before furnished by the substitution of y in Y, or else a value equal to the result of this substitution multiplied or divided by a numerical factor. For every algebraic factor introduced or suppressed is, by the substitution of the proposed value for y, reduced to a number, because, by hypothesis, none of them are rendered zero; these factors, therefore, affect only the numerical factors of the several remainders which arise in the course of the operation. Hence, in order that the value of y may satisfy the proposed equations, it is necessary and sufficient that it satisfies the equation $Y = 0$.

(104.) 2. Let the value attributed to y destroy one of the factors introduced into a dividend to render the division possible; the dividend thus modified will, for that particular value of y, become zero; so that, in order to carry on the division, we have introduced a factor that causes a dividend to vanish, which is of course not allowable, for, with such a dividend, the process would always terminate, whether there was a common measure or not; we cannot, therefore, affirm that the value of y, which causes one of the factors that have been introduced to vanish, satisfies the proposed equations, although it may fulfil the condition $Y = 0$.

(105.) 3. Lastly, let the value attributed to y destroy one of the factors which have been suppressed, and yet not satisfy the condition $Y = 0$; then, such a value of y causes the process to terminate at that remainder in which the factor has been suppressed, because, when the assumed value is put for y in the polynomials, this remainder becomes zero; hence the preceding divisor is a common measure of those polynomials, and thus a common measure may exist for values of y which do not satisfy the condition $Y = 0$. It must be remarked, however, that if in any part of the operation which precedes the suppression of the vanishing factor, a factor has been introduced which also vanishes for the same value of y, the above conclusion would not necessarily follow.

(106.) From the foregoing considerations we see, that to obtain the values of y which belong to the proposed equation, we must equate to zero the remainder, which is independent of x, as also each of the factors in y which have been suppressed in the course of the operation, and resolve each equation separately; secondly, that among the values thus obtained, there may be found some which are extraneous, and which must therefore be rejected as not being consistent with the proposed conditions. If no factor has been suppressed in the course of the operation, the equation $Y = 0$ alonè will furnish all the suitable values of y, and may also contain values not admissible, provided factors have been introduced; but when no factor has been either introduced or suppressed, then the values of y in the equation $Y = 0$ all belong to the proposed equations.

Having thus examined the influence of the factors introduced or suppressed in the course of the operation upon the final remainder in y, let us now return to the original polynomials $F(x, y)$, $f(x, y)$, and analyse the process by which we must arrive at this remainder.

(107.) The proposed functions being arranged according to the descending powers of x, will each be of the form

$$Ax^m + A_1 x^{m-1} + A_2 x^{m-2} \ldots\ldots + A_{m-1} x + A_m = 0,$$

where the first coefficient A is independent of y; the second A_1 contains no higher power of y than the first; the third A_2 contains no power higher than the second, &c.

As these coefficients may have a function of y for a common divisor, let us suppose that the greatest common divisor of the coefficients in $F(x,y)$ is $F_1(y)$, and that the greatest common divisor of the coefficients in $f(x, y)$ is $f_1(y)$; also of these two divisors let the greatest common divisor be $\phi(y)$, which will therefore be the greatest divisor common to all the coefficients of both equations. If now we represent by A the quotient of $F(x,y)$ by $F_1(y)$; by B the quotient of $f(x,y)$ by $f_1(y)$; by $F'(y)$ the quotient of $F_1(y)$ by $\phi(y)$; and by $f'(y)$ the quotient of $f_1(y)$ by $\phi(y)$, we shall obviously have

$$F(x,y)=\phi(y)\times F'(y)\times A=0$$

$$f(x,y)=\phi(y)\times f'(y)\times B=0,$$

both of which equations will be satisfied by the condition

$$\phi(y)=0,$$

as also by either of the following pairs of conditions, viz.

1. $F'(y)=0, \quad B=0.$
2. $f'(y)=0, \quad A=0.$
3. $A=0, \quad B=0.$

The conditions

$$F'(y)=0, \quad f'(y)=0,$$

it is evident cannot exist, because they involve but one unknown quantity, and their first members have no common factor.

As to the equation $\phi(y)=0$, it furnishes certain values of y for which x is indeterminate; for the proposed equations will evidently be satisfied for any value of x in conjunction with these values of y.

To find the solutions of the system $F'(y)=0$, $B=0$, we must resolve the first equation, which contains only y, and substitute the resulting values separately in B, and we shall thus have so many equations in x to determine the corresponding values. The system $f'(y)=0$, $A=0$, requires similar treatment.

The system $A=0$, $B=0$, remains to be considered, in which neither A nor B have any factor in y. To determine the solutions which satisfy this system, we must apply the process for the common measure.

(108.) 1. Suppose that the first step of this process conducts to a remainder, R, of a lower degree in x than the divisor, without our being obliged to use any preparation to render the division possible, or to avoid the occurrence of y as a denominator in the quotient Q; then, if A is the polynomial taken for the dividend, we shall have the identity

$$A=BQ+R,$$

which shows that whatever values of x and y satisfy the equations $A=0$, $B=0$, the same must also satisfy the equation $R=0$; and that whatever values satisfy the equations $B=0$, $R=0$, satisfy also the equation $A=0$; so that the solutions of the proposed equations

$$A=0, \quad B=0,$$

are exactly the same as those of the equations

$$B=0, \quad R=0,$$

which are more simple than the former system, inasmuch as they are of an inferior degree in x. The same conclusions evidently follow when the dividend A is multiplied at the outset by any numerical factor.

It is easy to prove that the consequences just deduced could not have place if the quotient Q contained y in a denominator. For suppose the form of the quotient to be $Q=\frac{H}{K}$, K being a quantity containing y; the identity above would then be

$$A=\frac{BH}{K}+R.$$

If we gave to x and y all the values which fulfil the conditions

$$A=0, \quad B=0,$$

among these values there might be some for y which, for aught we know to the contrary, might render K zero, in which case $\frac{BH}{K}$ would become $\frac{0}{0}$, which is not necessarily zero; so that $A = 0$, $B = 0$, would not necessarily imply $R = 0$, and we could not therefore assert that all the solutions of the system $A = 0$, $B = 0$, were equally given by the system $B = 0$, $R = 0$.

(109.) 2. Let us now suppose that, to avoid fractions in the quotient, it be necessary to introduce an algebraical factor into the dividend A; call it C, and let Q, R be the corresponding quotient and remainder, as before. We shall thus have the identity

$$CA = BQ + R,$$

which shows that the solutions of the equations

$$B = 0, \quad R = 0,$$

are the same as those of the equations

$$CA = 0, \quad B = 0.$$

Now this last system divides itself into two others, viz.

$$A = 0, \quad B = 0, \quad \text{and } C = 0, \quad B = 0.$$

Consequently the equations $B = 0$, $R = 0$, will give all the solutions of the proposed system $A = 0$, $B = 0$, but they will give in addition solutions to the system $C = 0$, $B = 0$.

These latter solutions we can separate from the others, for $C = 0$, containing only y, will furnish all the values of y which are doubtful, and the values of x, corresponding to these, are given by the solutions to $B = 0$, $R = 0$. Those pairs of these values which, substituted in the equation $A = 0$, satisfy its conditions are admissible, the others are to be rejected.

(110.) From the preceding discussion it appears, that the solution of the two equations proposed is reducible to the solution of the two equations

$$B = 0, \quad R = 0.$$

As the polynomial B contains no factors depending only on y, if R contain any such factors, we may of course suppress them; but then we must take account of the solutions which reduce to zero these factors, and, at the same time, the polynomial B (106.)

We shall now give an example or two of the application of this theory.

EXAMPLES.

1. Let the system of equations be

$$x^2+(8y-13)x+y^2-7y+12=0$$
$$x^2-(4y+1)x+y^2+5y=0.$$

Here the coefficients having no common measure, these equations may be regarded as the equations A = 0, B = 0, treated above; and from these we are to determine, agreeably to the general theory, the system B = 0, R = 0, which will contain all the solutions required. Dividing A by B, we have

$$x^2-(4y+1)x+y^2+5y]\,x^2+(8y-13)x+y^2-7y+12\,[1$$
$$x^2-(4y+1)x+y^2+5y$$
$$\overline{\mathrm{R}=(12y-12)x-12y+12=12(y-1)(x-1),}$$

the equations which furnish the solutions are, therefore,

$$\begin{array}{c|c} y-1=0 & x-1=0 \\ x^2-(4y+1)x+y^2+5y=0 & x^2-(4y+1)x+y^2+5y=0, \end{array}$$

and each of these systems may be solved without repeating the divisions; the solutions are

$$\begin{array}{c|c|c|c} y=1 & y=1 & y=0 & y=-1 \\ x=3 & x=2 & x=1 & x=1. \end{array}$$

2. Let the equations

$$x^3 + 2yx^2 + 2y(y-2)x + y^2 - 4 = 0$$
$$x^2 + 2yx + 2y^2 - 5y + 2 = 0$$

be proposed.

The coefficients having no common measure, we have, by dividing the first polynomial by the second, the following remainder, viz.

$$R = (y-2)x + y^2 - 4 = (y-2)(x+y+2);$$

hence the solutions to the proposed equations are those of the systems

$$\begin{array}{c|c} y - 2 = 0 & x + y + 2 = 0 \\ x^2 + 2yx + 2y^2 - 5y + 2 = 0 & x^2 + 2yx + 2y^2 - 5y + 2 = 0. \end{array}$$

The first system furnishes the solutions

$$\begin{array}{c|c} y = 2 & y = 2 \\ x = 0 & x = -4. \end{array}$$

To solve the other system, proceed as at first; that is, divide the second polynomial by the first, and there will result the remainder

$$y^2 - 5y + 6;$$

hence the system is replaced by the new system

$$y^2 - 5y + 6 = 0$$
$$x + y + 2 = 0,$$

which gives for solutions

$$\begin{array}{c|c} y = 2 & y = 3 \\ x = -4 & x = -5; \end{array}$$

so that there are, in all, four solutions to the proposed equations.

3. Let the equations

$$(y-1)x^2 + 2x - 5y + 3 = 0$$
$$yx^2 + 9x - 10y = 0$$

be proposed.

Multiplying the first polynomial by y, to render it divisible by the second, and then performing the division, we have

$$yx^2 + 9x - 10y]\ (y-1)\,yx^2 + 2yx - 5y^2 + 3y\ [y-1$$
$$(y-1)\,yx^2 + (9y-9)\,x - 10y^2 - 10y$$
$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$
$$(-7y + y)\,x + 5y^2 - 7y.$$

As we have multiplied the dividend by the factor y, the equation $y = 0$ may be a solution to test. Substitute 0 for y in the proposed equations; one, viz. the divisor, furnishes the value $x = 0$, which value however does not satisfy the other; hence the factor introduced supplies no solution. We must now proceed with the polynomials B and R, and, in order to this, must multiply the dividend B by $(-7y + 9)$, and we shall have, for the remainder arising from the division, the polynomial

$$25y^5 - 70y^4 - 126y^3 + 414y^2 - 243y.$$

The final equations are, therefore,

$$(-7y + 9)\,x + 5y^2 - 7y = 0$$
$$25y^5 - 70y^4 - 126y^3 + 414y^2 - 243y = 0;$$

the roots of the second are

$$y = 0, \quad y = 1, \quad y = 3, \quad y = \frac{-3 \pm 3\sqrt{10}}{5}.$$

and to these correspond the following values of x, deduced from the first, viz.

$$x = 0, \quad x = 1, \quad x = 2, \quad x = -5 \mp \sqrt{10}.$$

No extraneous solution has been introduced by means of the factor $-7y + 9$, by which we have multiplied the second dividend, because none of the above values of y cause it to vanish; but an inadmissible solution has been introduced by the factor y, which multiplies the first dividend, viz. the solution $y = 0$, $x = 0$; rejecting this, therefore, we have, for the entire number of true solutions, the four systems following, viz.

$$y=1 \quad \Big| \quad y=3 \quad \Big| \quad y=\frac{-3+3\sqrt{10}}{5} \quad \Big| \quad y=\frac{-3-3\sqrt{10}}{5}.$$
$$x=1 \quad \Big| \quad x=2 \quad \Big| \quad x=-5-\sqrt{10} \quad \Big| \quad x=-5+\sqrt{10}$$

4. Let the equations proposed for solution, be

$$(y^2-1)x^2+(2y^3-2y)x+y^4-2y^2+1=0$$
$$(y^2-3y+2)x^2-y^4-3y^3+7y^2+15y-18=0,$$

The coefficients of the first polynomial admit of the common divisor y^2-1; and those of the second admit of the common divisor y^2-3y+2; these two factors have themselves a common divisor, which is $y-1$; so that the proposed equations may be written thus:

$$(y-1)(y+1)(x^2+2yx+y^2-1)=0$$
$$(y-1)(y-2)(x^2-y^2-6y-9)=0.$$

These are satisfied by the value $y=1$, combined with any value of x whatever.

They are also satisfied by the values which satisfy

$$y+1=0, \quad x^2-y^2-6y-9=0;$$

which values are

$$y=-1 \quad \Big| \quad y=-1$$
$$x=2 \quad \Big| \quad x=-2.$$

They are also satisfied by the values which satisfy

$$y-2=0, \quad x^2+2yx+y^2-1=0;$$

which values are

$$y=2 \quad \Big| \quad y=2$$
$$x=-1 \quad \Big| \quad x=-2.$$

The remaining solutions are involved in the equations $A=0$, $B=0$, that is, in the equations

$$x^2 + 2yx + y^2 - 1 = 0$$
$$x^2 - \quad y^2 - 6y - 9 = 0;$$

to which we may apply the method of the common divisor; but, as it is easy to see that the second equation gives

$$x = \pm (y + 3),$$

we may substitute these values in the first equation. The first value, $y + 3$, will reduce it to

$$y^2 + 3y + 2 = 0;$$

which furnishes the values $y = -1$, $y = -2$, and, from the relation $x = y + 3$, we have, for the corresponding values of x, $x = 2$, $x = 1$. If we substitute, in the first equation, the other value, $-(y + 3)$, for x, it will be reduced to $8 = 0$; this value, therefore, furnishes no solution.

5. As a last example, let the equations

$$(y-2)\,x^2 - 2x + 5y - 2 = 0$$
$$yx^2 - 5x + 4y = 0,$$

be taken.

The coefficients having no common divisor, we at once commence the operation for finding R; but, to avoid fractions in the quotient, we must prepare the dividend by multiplying it by y.

$$yx^2 - 5x + 4y]\;(y-2)\,yx^2 - 2yx + 5y^2 - 2y\;[y-2$$
$$(y-2)\,yx^2 - (5y-10)\,x + 4y^2 - 8y$$
$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$
$$\mathrm{R} = (3y-10)\,x + y^2 + 6y.$$

It is necessary now to repeat the operation with B and R; and, for this purpose, we must multiply the dividend B by $3y - 10$; the resulting remainder will be found to be

$$y^5 + 12y^4 + 87y^3 - 200y^2 + 100y;$$

so that the final equations are

$$(3y-10)x+y^2+6y=0$$

$$y^5+12y^4+87y^3-200y^2+100y=0.$$

The last is satisfied, for $y=0$, to which corresponds $x=0$, in the second; but this is an inadmissible solution, as it does not satisfy the proposed equations. It is due to the factor y, introduced in the first division. Suppressing this factor, the final equation in y becomes

$$y^4+12y^3+87y^2-200y+100=0;$$

which cannot involve any inadmissible values of y, because the only circumstance which could cause their introduction, is the introduction of the factor, $3y-10$, in the second division, and this is reduced to zero, by the value $y=\frac{10}{3}$. But, as this value is fractional, it cannot be a root of the equation above (13). We also see, from other causes, that the factor, $3y-10$, can introduce no solution; the conditions

$$3y-10=0, \quad (3y-10)x+y^2+6y=0,$$

are incompatible.

The final equation in y has the root $y=1$, to which corresponds $x=1$, and the depressed equation in y is

$$y^3+13y^2+100y-100=0;$$

the roots of which involve interminable decimals. Hence, the remaining solutions can be obtained only by approximation.

For further particulars on the subject of elimination, the student may consult the recent publication of *Reynaud*, entitled *Théorie du plus grand Commun Diviseur et d'Elimination*, as also *Traité d'Algèbre*. by *Mayer et Choquet*, from which latter work most of the preceding examples have been taken.

In the Chapter next following, a method will be found of obtaining the final equation in y, which shall comprise all the solutions to the proposed equations, and be unembarrassed with inadmissible values.

We shall now proceed to one or two useful applications of the theory of elimination.

On Irrational Equations.

(111.) All the *direct* methods employed for the solution of equations suppose that the unknown quantities in them are not affected with any radical sign; when, therefore, the unknown is found under a radical sign, it will be necessary, before applying the process of solution, to employ some preparatory method of rendering the equation rational. Such a method is at once suggested by the theory of elimination. For, if we equate each of the irrational terms with an unknown quantity, and remove the radical from each of these new equations by involution, we shall have a series of equations (including the original one, with its irrational terms replaced by the new symbols,) without radicals, from which the quantities, temporarily introduced, may be eliminated, and thence a rational equation obtained, involving only the original unknown quantities.

The following examples will fully illustrate the mode of proceeding:

1. Let the equation be

$$x - \sqrt{x-1} + \sqrt[3]{x+1} = 0.$$

Put

$$y = \sqrt{x-1}, \quad z = \sqrt[3]{x+1};$$

and we then have the three following rational equations, from which to eliminate y and z, viz.

$$y^2 = x - 1, \quad z^3 = x + 1, \quad x - y + z = 0.$$

From the last equation we get $y = x + z$, and, by substituting this value in the first, y becomes eliminated, and we have these two equations in x and z, viz.

$$z^3 - x + 1 = 0$$

$$z^2 + 2xz - x + 1 = 0;$$

and, to eliminate z from these, we apply the process explained in the preceding articles, and thus get the final equation

$$x^6 - 3x^5 + 8x^4 + x^3 + 7x^2 - 7x + 2 = 0,$$

which does not involve any superfluous roots.

2. Let the equation be

$$\sqrt[3]{4x+7} + 2\sqrt{x-4} = 1.$$

Putting

$$y = \sqrt[3]{4x+7}, \qquad z = \sqrt{x-4},$$

we have the system of equations

$$y^3 = 4x + 7, \qquad z^2 = x - 4,$$
$$y + 2z = 1.$$

From the last we find $y = 1 - 2z$, and this value of y, substituted in the first, reduces the system to the two equations

$$8z^3 - 12z^2 + 2z + 6 = 0$$
$$z^2 - x + 4 = 0,$$

from which, by the process already explained, we obtain the final equation,

$$16x^3 - 184x^2 + 801x - 1405 = 0.$$

On the Equation of the Squares of the Differences.

(112.) We have already remarked that the equation of the squares of the differences is an auxiliary equation, employed by Lagrange for the purpose of separating the real roots of any algebraical equation proposed for numerical solution.

This auxiliary equation is such as to furnish, for its roots, the squares of the differences between every two roots of the proposed equation; so that when we have ascertained the inferior and superior limits of the positive roots of an equation, if we substitute, successively, for x, in it, a series of numbers, increasing from the inferior limit, up to the superior, by differences, δ, not exceeding the least difference found to exist between the sought roots, by means of the auxiliary equation, no two roots, however close together, can exist in any interval between two consecutive substitutions; and, therefore, in thus proceeding from limit to limit, there will necessarily be presented as many successive changes of sign, in the final term, as there are positive roots between the limits, so that the situation of each root will become known. By determining the limits of the negative roots of the proposed equation, they also may be separated in a similar manner.

When the auxiliary equation, from which the value of δ is to be deduced, is found, we shall not be required actually to solve it for this purpose; it will, obviously, be sufficient to determine the inferior limit of its positive roots, which limit, being less than the square of the least difference which exists among the roots of the proposed equation, the square root of it may be taken for δ.

As the squares of the differences of all the real roots are positive, it follows that the negative roots of the auxiliary equation must arise from the imaginary roots in the proposed.

(113.) It is easy to see that the foregoing method of separating the real roots, and, consequently, of discovering the number of imaginary roots in an equation is infallible; but, as we have before observed, the great length of the calculations which are necessary to the formation of the equation of the squares of the differences, when the proposed equation is above the third or fourth degree, renders the method nearly impracticable. This is now no longer a matter of regret, as the solution of the important problem of the separation of the roots has been rendered by Sturm altogether independent of the equation of the squares of the differences; this latter problem, therefore, will henceforth be regarded with interest, only on account of its connexion with the name of Lagrange, and with the history of algebraical research.

We advert to the problem here merely to explain its meaning and object to the student, and to furnish an additional example in elimination.

(114.) Let the proposed equation be

$$f(x)=0 \quad . \; . \; . \; . \; (1),$$

and let a be any one indifferently of its n roots $a_1, a_2, a_3, \ldots a_n$;

then, in order to obtain an equation whose roots may be the differences between those of the proposed, it will be sufficient to establish the relation

$$y = x - a, \quad \text{or} \quad x = a + y;$$

which transforms the proposed into

$$f(a + y) = 0;$$

of which the development is (34)

$$f(a) + f'(a)y + f''(a)\frac{y^2}{1\cdot 2} + f'''(a)\frac{y^3}{1\cdot 2\cdot 3} + \&c. = 0;$$

but since, by hypothesis, a is a root of the proposed equation,

$$f(a) = 0;$$

hence, suppressing this term, and dividing by y, we have

$$f'(a) + f''(a)\frac{y}{1\cdot 2} + f'''(a)\frac{y^2}{1\cdot 2\cdot 3} + \&c. = 0;$$

the roots, or values of y, in this equation, are, by the condition above, the differences between any assumed root, a, and the $n - 1$ other roots of the proposed equation. By putting, in succession, all the values for a, that is, in fact, all the values of x, deduced from (1), the corresponding values of y, will, together, furnish all the possible differences between the roots of (1). In other words, all the possible differences will be obtained by substituting the values of x, deduced from the equation

$$f(x) = 0,$$

in the equation

$$f'(x) + f''(x)\frac{y}{1\cdot 2} + f'''(x)\frac{y^2}{1\cdot 2\cdot 3} + \&c. = 0;$$

and this is tantamount to saying that the differences sought arise from the solution of this system of equations.

It is easy to foresee the degree of the final equation in y arising from the elimination of x from these two equations; for, as its roots are equal to the remainders obtained, by subtracting, from each of the n roots of the proposed, all the other $n - 1$ roots in succession, there are, obviously, in the whole, $n(n - 1)$ roots, or remainders, hence the final equation, furnishing these roots, is of the $n(n - 1)$th degree.

(115.) We shall apply this process to the following equation of the third degree

$$x^3 + px + q = 0,$$

where

$$f(x) = x^3 + px + q, \quad f'(x) = 3x^2 + p$$

$$f''(x) = 6x, \quad f'''(x) = 6.$$

Hence the two equations, from which x is to be eliminated, are

$$x^3 + px + q = 0$$

$$3x^2 + 3yx + y^2 + p = 0,$$

and the operation is as follows:

$$\begin{array}{l|l} 3x^2 + 3yx + y^2 + p\,] & 3x^3 + 3px + 3q\ [x - y \\ & 3x^3 + 3yx^2 + (y^2 + p)x \\ \hline & \quad -3yx^2 - (y^2 - 2p)x + 3q \\ & \quad -3yx^2 - 3y^2x - y^3 - py \\ \hline & \quad 2(y^2 + p)x + y^3 + py + 3q \end{array}$$

This remainder is now to be taken for a new divisor, and the former divisor for a dividend, and the operation continued; but as the foregoing divisor, B, does not admit of division by the remainder, R, in its present form, we multiply it by the factor, $2\,(y^2 + p)$, in order to render the division possible, and proceed with the operation as in next page.

$$2(y^2+p)z+y^3+py+3q\,]\,6(y^2+p)z^2+6(y^2+p)yz+2(y^2+p)^2\,[\,3z+3(y^3+py-3q)$$
$$\underline{6(y^2+p)z^2+3(y^3+py+3q)z}$$
$$3(y^3+py-3q)z+2(y^2+p)^2$$

or, mult. by $2(y^2+p)$, $6(y^2+p)(y^3+py-3q)z+4(y^2+p)^3$

$$\underline{6(y^2+p)(y^3+py-3q)z+3(y^3+py+3q)(y^3+py-3q)}$$
$$4(y^2+p)^3-3(y^3+py+3q)(y^3+py-3q)$$

In the second division we have multiplied twice by y^2+p, in order to render the division possible; but this factor introduces no extraneous value of y, for the value which reduces it to zero being given by $y^2+p=0$, reduces the divisor, R, last employed, to q, and not to zero, as it ought to be a solution (106). The final equation in y, involving the values sought, and those only, is, therefore,

$$y^6+6py^4+9p^2y^2+4p^3+27q^2=0,$$

which is *the equation of the differences*; and, by putting z for y^2, we have

$$z^3+6pz^2+9p^2z+4p^3+27q^2=0$$

for *the equation of the squares of the differences.*

In the particular case of the equation

$$x^3-7x+7=0,$$

we have

$$p=-7,\quad q=7,$$

and therefore the equation in z is

$$z^3-42z^2+441z-49=0.$$

It is obvious that every equation of the differences, as well as that just deduced, will be of an even degree, and will contain only even powers of y, because every root, as $a-a_2$, is accompanied by another

$a_2 - a$, the roots being equal to the differences $a - a_2, a - a_3 \ldots\ldots$ $a - a_n$; $a_2 - a, a_2 - a_3 \ldots\ldots a_2 - a_n$, &c. of the roots of the proposed equation; so that the polynomial in y is of the form

$$(y - \alpha)(y + \alpha)(y - \beta)(y + \beta) \ldots\ldots,$$

or of the form

$$(y^2 - \alpha^2)(y^2 - \beta)^2 \ldots\ldots,$$

and therefore involves only even powers of y.

CHAPTER IV.

ON THE SYMMETRICAL FUNCTIONS OF THE ROOTS OF AN EQUATION.

(116.) A *symmetrical function* of the roots of an equation, is any expression in which all the roots are similarly involved, so that any of them may be interchanged without affecting the form or composition of the function. The coefficients, for example, of every equation are each of them symmetrical functions of its roots; for it has been shown (11) that if the roots of any equation be $a, a_2, a_3 \ldots\ldots a_n$, the successive coefficients will be the following functions of them, viz.

$$-(a + a_2 + a_3 + \ldots\ldots a_n),$$

$$aa_2 + aa_3 + a_2 a_3 + \ldots\ldots a_{n-1} a_n,$$

$$-(aa_2 a_3 + aa_2 a_4 + \ldots\ldots a_{n-2} a_{n-1} a_n),$$

$$\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots$$

$$aa_2 a_3 a_4 \ldots\ldots a_n (-1)^n;$$

and each of these is a symmetrical function, because, however we interchange the roots, the function itself will remain uuchanged.

The preceding forms are, we see, immediately given by the coefficients of the proposed equation; and it is the object of the present Chapter to show that not only these, but every other rational and symmetrical function of the roots, may always be expressed in terms of the coefficients, without the aid of the roots themselves.

Determination of the Sums of the Powers of the Roots of an Equations.

(117.) As usual, let us represent the general equation of the nth degree by

$$f(x) = x^n + A_{n-1}\, x^{n-1} + A_{n-2}\, x^{n-2} \dots \text{A}x + \text{N} = 0 \quad \dots \quad (1),$$

and its roots by

$$a, \quad a_2, \quad a_3, \quad a_4, \dots \dots a_n.$$

Then $f'(x)$ being the first derived function from the polynomial (1), we know (35) that

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-a_2} + \frac{1}{x-a_3} + \dots \frac{1}{x-a_n};$$

and, consequently,

$$f'(x) = \frac{f(x)}{x-a} + \frac{f(x)}{x-a_2} + \frac{f(x)}{x-a_3} + \dots \frac{f(x)}{x-a_n} \dots (2).$$

Performing now the actual division for any one of these fractions, as, for instance, for the fraction $\frac{f(x)}{x-a_m}$, or, which is the same thing, depressing the original equation (1), by any one of its roots, a_n, we shall get the polynomial which follows, the coefficients being formed by the rule at (10),

$$\begin{array}{r|r|r|r}
x^{n-1} + a_m & x^{n-2} + a_m^2 & x^{n-3} + a_m^3 & x^{n-4} + \ldots a_m^{n-1} \\
A_{n-1} & A_{n-1} a_m & A_{n-1} a_m^2 & A_{n-1} a_m^{n-2} \\
 & A_{n-2} & A_{n-2} a_m & A_{n-2} a_m^{n-3} \\
 & & A_{n-3} & A_{n-3} a_m^{n-4} \\
 & & & \vdots \\
 & & & N;
\end{array}$$

and this will equally represent the development of either of the other fractions, by putting the corresponding value for m. Conceiving, therefore, m to be successively 1, 2, 3, &c. to n, and, putting for abridgment,

$$\Sigma_m = a^m + a_2^m + a_3^m + \ldots . a_n^m,$$

we shall, obviously, have for the sum of all the developments that is for $f'(x)$, the polynomial

$$\begin{array}{r|r|r|r}
nx^{n-1} + \Sigma_1 & x^{n-2} + \Sigma_2 & x^{n-3} + \Sigma_3 & x^{n-4} \ldots \Sigma_{n-1} \\
nA_{n-1} & A_{n-1} \Sigma_1 & A_{n-1} \Sigma_2 & A_{n-1} \Sigma_{n-2} \\
 & nA_{n-2} & A_{n-2} \Sigma_1 & A_{n-2} \Sigma_{n-3} \\
 & & A_{n-3} & A_{n-3} \Sigma_{n-4} \\
 & & & \vdots \\
 & & & nN.
\end{array}$$

But the development of $f'(x)$ is also (33)

$$nx^{n-1} + (n-1)\, A_{n-1} x^{n-2} + (n-2)\, A_{n-2} x^{n-3} + \ldots 2A_2 x + A;$$

hence, by the method of indeterminate coefficients,

$$\begin{aligned}
&\Sigma_1 + nA_{n-1} = (n-1)A_{n-1} \\
&\Sigma_2 + A_{n-1}\Sigma_1 + nA_{n-2} = (n-2)\, A_{n-2} \\
&\Sigma_3 + A_{n-1}\Sigma_2 + A_{n-2}\Sigma_1 + A_{n-3} = (n-3)\, A_{n-3} \\
&\qquad\vdots \\
&\Sigma_{n-1} + A_{n-1}\Sigma_{n-2} + A_{n-3}\Sigma_{n-4} + \ldots . nN = A;
\end{aligned}$$

that is,

$$\Sigma_1 + A_{n-1} = 0$$
$$\Sigma_2 + A_{n-1}\Sigma_1 + 2A_{n-2} = 0$$
$$\Sigma_3 + A_{n-1}\Sigma_2 + A_{n-2}\Sigma_1 + 3A_{n-3} = 0$$
$$\vdots$$
$$\Sigma_{n-1} + A_{n-1}\Sigma_{n-2} + A_{n-2}\Sigma_{n-3} + \dots (n-1)A = 0.$$

By means of these equations the functions Σ_1, Σ_2, Σ_3, &c. may be easily calculated in succession up to the function Σ_{n-1}.

(118.) The foregoing equations may be extended so as to include the functions Σ_n, Σ_{n+1}, Σ_{n+2} .... Σ_{n+p}; for, from the original equation, we have

$$a^n + A_{n-1}a^{n-1} + A_{n-2}a^{n-2} + \dots Aa + N = 0$$
$$a_2^n + A_{n-1}a_2^{n-1} + A_{n-2}a_2^{n-2} + \dots Aa_2 + N = 0$$
$$\vdots$$
$$a_n^n + A_{n-1}a_n^{n-1} + A_{n-2}a_n^{n-2} + \dots Aa_n + N = 0;$$

and, by multiplying these equations respectively by a^p, a_2^p .... a_n^p, and adding them together, there results the equation

$$\Sigma_{n+p} + A_{n-1}\Sigma_{n+p-1} + A_{n-2}\Sigma_{n+p-2} + \dots A\Sigma_{p+1} + N\Sigma_p = 0,$$

which, by putting 0, 1, 2, &c. for p, furnishes the following continuation of the foregoing relations, viz.

$$\Sigma_n + A_{n-1}\Sigma_{n-1} + A_{n-2}\Sigma_{n-2} + \dots A\Sigma_1 + nN^* = 0$$
$$\Sigma_{n+1} + A_{n-1}\Sigma_n + A_{n-2}\Sigma_{n-1} + \dots A\Sigma_2 + N\Sigma_1 = 0$$
$$\Sigma_{n+2} + A_{n-1}\Sigma_{n+1} + A_{n-2}\Sigma_n + \dots A\Sigma_3 + N\Sigma_2 = 0$$
$$\vdots$$
$$\Sigma_{n+p} + A_{n-1}\Sigma_{n+p-1} + A_{n-2}\Sigma_{n+p-2} + \dots A\Sigma_{p+1} + N\Sigma_p = 0.$$

Hence, by means of the coefficients merely, we may calculate the sums of the powers of the roots of an equation, in succession, to any extent; and it is plain, from the foregoing expressions, that the several sums will all be integral functions of the coefficients.

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* It is plain that $\Sigma_0 = a^0 + a_2^0 + a_3^0 + \dots a_n^0 = n$.

As a particular application of the preceding general formulas, let it be required to find the sum of the sixth powers of the roots of the equation

$$x^4 + x^3 - 7x^2 - x + 6 = 0.$$

$$\Sigma_1 = -A_{n-1} = -1$$

$$\Sigma_2 = -A_{n-1}\Sigma_1 - 2A_{n-2} = 1 + 14 = 15$$

$$\Sigma_3 = -A_{n-1}\Sigma_2 - A_{n-2}\Sigma_1 - 3A_{n-3} = -15 - 7 + 3 = -19$$

$$\Sigma_4 = -A_{n-1}\Sigma_3 - A_{n-2}\Sigma_2 - A_{n-3}\Sigma_1 - 4A = 19 + 105 - 1 - 24 = 99$$

$$\Sigma_5 = -A_{n-1}\Sigma_4 - A_{n-2}\Sigma_3 - A_{n-3}\Sigma_2 - A\Sigma_1 = -99 - 133 + 15 + 6 = -211$$

$$\Sigma_6 = -A_{n-1}\Sigma_5 - A_{n-2}\Sigma_4 - A_{n-3}\Sigma_3 - A\Sigma_2 = 211 + 693 - 19 - 90 = 795.$$

If the sums of the negative powers of the roots of an equation be required, we might derive suitable formulas from the general table above, by considering p to be negative; but it will be preferable in this case to transform the equation to another in $\frac{1}{x}$ by (22), and then to employ the formulas in their present state.*

(119.) By means of the general expressions in last article, we may find the values of the coefficients A_{n-1}, A_{n-2}, &c. in terms of the sums of the powers of the roots; thus:

$$A_{n-1} = -\Sigma_1$$

$$A_{n-2} = -\frac{\Sigma_2 + A_{n-1}\Sigma_1}{2}$$

$$A_{n-3} = -\frac{\Sigma_3 + A_{n-1}\Sigma_2 + A_{n-2}\Sigma_1}{4}$$

$$A_{n-4} = -\frac{\Sigma_4 + A_{n-1}\Sigma_3 + A_{n-2}\Sigma_2 + A_{n-3}\Sigma_1}{4}.$$

&c. &c.

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* For another mode of investigating the expressions for the sums of the powers of the roots of an equation, and for the use which Newton and Lagrange made of these sums for approximating to the greatest root, see the Author's *Essay on the Computation of Logarithms*, p. 97, Second Edition.

Determination of any Combination of the Powers of the Roots of an Equation.

(120.) By multiplying together the two expressions

$$\Sigma_m = a^m + a_2^m + a_3^m + a_4^m + \dots$$
$$\Sigma_p = a^p + a_2^p + a_3^p + a_4^p + \dots,$$

we have the two following series of partial products, viz.

$$\Sigma_m \times \Sigma_p = a^{m+p} + a_2^{m+p} + a_3^{m+p} + a_4^{m+p} + \dots$$
$$+ a^m a_2^p + a^m a_3^p + a^m a_4^p + a_2^m a^p + \dots$$

Each of these series is a symmetrical function of the roots; the first being the sum of their $m+p$ powers, and the second being the sum of the products of every two roots raised, the one to the power m, and the other to the power p. This latter function may be represented briefly by $S(a^m a_2^p)$; so that we shall have

$$\Sigma_m \times \Sigma_p = \Sigma_{m+p} + S(a^m a_2^p)$$
$$\therefore\ S(a^m a_2^p) = \Sigma_m \times \Sigma_p - \Sigma_{m+p}.$$

Hence the function $S(a^m a_2^p)$ is determinable in terms of the coefficients of the equation.

Again, if we multiply together the expressions

$$S(a^m a_2^p) = a^m a_2^p + a^m a_3^p + a^m a_4^p + a_2^m a^p + \dots$$
and $$\Sigma_q = a^q + a_2^q + a_3^q + a_4^q + \dots,$$

we shall have a result consisting of three series of partial products, the terms of each distinct series involving like combinations of the roots, viz. The first series will consist of the products of every two roots raised, the one to the power $m+q$, and the other to the power p, and which series may be denoted by $S(a^{m+q} a_2^p)$. The second series will be formed of the products of every two roots raised, the one to the power $p+q$, and the other to the power m, which series may be expressed by $S(a^{p+q} a_2^m)$. The third series will be the products of every three roots raised, one to the power m, one to the power p, and one to the power q; and which will be represented by $S(a^m a_2^p a_3^q)$. That is, we shall have

$$S(a^m a_2^p) \times \Sigma_q = S(a^{m+q} a_2^p) + S(a^{p+q} a_2^m) + S(a^m a_2^p a_3^q),$$

and, therefore, by transposing and replacing the functions,

$$S(a^m a_2^p), \quad S(a^{m+q} a_2^p), \quad S(a^{p+q} a_2^m),$$

by their values in last page, we have

$$S(a^m a_2^p a_3^q) = \Sigma_m \Sigma_p \Sigma_q - \Sigma_{m+p} \Sigma_q - \Sigma_{m+q} \Sigma_p - \Sigma_{p+q} \Sigma_m + 2\Sigma_{m+p+q},$$

by which equation the triple function $S(a^m a_2^p a_3^q)$ may be obtained in terms of the coefficients.

By continuing this process of deduction, we may obtain expressions for the succeeding combinations. The functions thus determined are called the elementary symmetrical functions, and it is from the union of these that every complex, rational and integral, symmetrical function is formed. We shall give a few examples of these combinations in the following article.

Before proceeding to these, however, it may be proper to show how the above general functions become modified, when the exponents m, p, q, &c. are not unequal.

The expression $S(a^m a_2^p)$ is truly the representation of

$$a^m a_2^p + a^m a_3^p + a^m a_4^p + \dots + a_2^m a^p + a_3^m a^p + \dots$$

only when m and p are unequal; for, when $m = p$, this series consists of terms which are equal two and two; so that, in that case, only half the entire sum will be expressed by $S(a^m a_2^m)$. Hence

$$S(a^m a_2^m) = \frac{(\Sigma_m)^2 - \Sigma_{2m}}{2}.$$

For similar reasons,

$$S(a^m a_2^p a_3^p) = \frac{\Sigma_m (\Sigma_p)^2 - 2\Sigma_{m+p} \Sigma_p - \Sigma_m \Sigma_{2p} + 2\Sigma_{m+2p}}{2}.$$

Lastly, when the exponents in this latter function are all three equal, the terms represented will be equal six and six; so that

$$S(a^m a_2^m a_3^m) = \frac{(\Sigma_m)^3 - 3\Sigma_{2m}\Sigma_m + 2\Sigma_{3m}}{6}.$$

Transformation of an Equation into another whose Roots shall be given Functions of those of the original Equation.

(121.) Let it be required to form the equation whose roots are the sums of the roots of the equation $f(x) = 0$, taken two and two.

If, as usual, we represent the roots of the proposed equation by

$$a, \quad a_2, \quad a_3, \quad a_4 \, \ldots \ldots \, a_n,$$

those of the transformed equation $F(y) = 0$ will be

$$a + a_2, \quad a + a_3, \quad a + a_4, \quad a_2 + a_3, \quad \&c.,$$

and will amount in number to the number of different combinations which can be formed with the roots of the proposed, taken two and two. If each were to be combined with every one of the other roots, the whole number of combinations would obviously be $n(n-1)$; but it is plain that every combination would then occur twice; so that the correct number of combinations must be $\frac{n(n-1)}{2}$.* Hence the number $\frac{n(n-1)}{2}$ denotes the degree of the transformed equation. Let us proceed to the composition of its coefficients.

The sums of the powers of the roots of the transformed equation will be expressed by the formulas

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* The doctrine of combinations and permutations is given in almost every English Treatise on Arithmetic.

$$\Sigma'_1 = (a + a_2) + (a + a_3) + (a + a_4) + (a_2 + a_3) + \&c. =$$
$$2(a + a_2 + a_3 + a_4 + \&c.) = 2S(a)$$
$$\Sigma'_2 = (a + a_2)^2 + (a + a_3)^2 + (a + a_4)^2 + (a_2 + a_3)^2 + \&c. =$$
$$2(a^2 + a_2^2 + a_3^2 + a_4^2 + \&c.) + 2(aa_2 + aa_3 + aa_4 + a_2 a_3 + \&c.)$$
$$= 2S(a^2) + 2S(aa_2)$$
$$\Sigma'_3 = (a + a_2)^3 + (a + a_3)^3 + (a + a_4)^3 + (a_2 + a_3)^3 + \&c. =$$
$$2(a^3 + a_2^3 + a_3^3 + a_4^3) + 3(a^2 a_2 + a^2 a_3 + a^2 a_4 + a_2^2 a_3) + \&c.$$
$$= 2S(a^3) + 3S(a^2 a_2)$$

&c. &c.

Hence the sums of the powers of the roots of the transformed equation may be obtained in terms of the sums of the powers of the original roots and their elementary combinations; the sums of the powers being thus known, for the transformed equations, the coefficients of this equation are found by the formulas, at (119), as in the following example:

Let it be required to transform the equation,

$$x^3 + A_2 x^2 + Ax + N = 0,$$

into another,

$$y^3 + A'_2 y^2 + A'y + N = 0,$$

whose roots shall be the sums of the roots of the former equation, taken two and two.

Let us first calculate the values of Σ_1, Σ_2, Σ_3;

$$\Sigma_1 = -A_2$$
$$\Sigma_2 = -A_2 \Sigma_1 - 2A = A_2^2 - 2A$$
$$\Sigma_3 = -A_2 \Sigma_2 - A\Sigma_1 - 3N = -A_2^3 + 3AA_2 - 3N.$$

The value of $S(aa_2)$ is, by (11), A; and for $S(a^2 a_2)$, we have (120)

$$S(a^2 a_2) = \Sigma_2 \times \Sigma_1 - \Sigma_3 = -AA_2 + 3N.$$

Consequently,

$$\Sigma'_1 = 2\Sigma_1 = -2A_2$$
$$\Sigma'_2 = 2\Sigma_2 + 2S(aa_2) = 2A_2^2 - 2A$$
$$\Sigma'_3 = -2\Sigma_3 + 3S(a^2 a_2) = -2A_2^3 + 3AA_2 + 3N.$$

Finally, by the formulas (119),

$$A'_2 = -\Sigma'_1 = 2A_2$$
$$A' = -\frac{\Sigma'_2 + A'_2 \Sigma'_1}{2} = A^2_2 + A$$
$$N' = -\frac{\Sigma'_3 + A'_2 \Sigma'_2 + A' \Sigma'_1}{3} = A_2 A - N.$$

Hence the transformed equation is

$$y^3 + 2A_2 y^2 + (A_2^2 + A) y + A_2 A - N = 0.$$

(122.) By proceeding in a similar manner, we may form an equation of which the roots are combinations of those of the original equation, of the form $a + a_2 + kaa_2$, $a + a_3 + kaa_3$, &c. k being any given number. The degree of this equation will, of course, be the same as the degree of that formed from the sums of the roots; it will, therefore, be denoted by the number $\frac{n(n-1)}{2}$. The expressions for the sums of the powers of the roots of the transformed equation are

$$\Sigma'_1 = (a + a_2 + kaa_2) + (a + a_3 + kaa_3) + \text{\&c.}$$
$$= (n-1)\,S(a) + kS(aa_2)$$
$$\Sigma'_2 = (a + a_2 + kaa_2)^2 + (a + a_3 + kaa_3)^2 + \text{\&c.}$$
$$= (n-1)\,S(a^2) + 2S(aa_2) + 2kS(a^2 a_2) + k^2 S(a^2 a_2^2)$$
$$\Sigma'_3 = (a + a_2 + kaa_2)^3 + (a + a_3 + kaa_3)^3 + \text{\&c.}$$
$$= (n-1)\,S(a^3) + 3S(a^2 a_2) + 3kS(a^3 a_2) + 6kS(a^2 a_2^2) +$$
$$3k^2 S(a^3 a_2^2) + k^3 S(a^3 a_2^3)$$

&c. &c.

from which it is evident that the coefficients of the transformed equation may be expressed in functions of the coefficients of the given equation.

(123.) As a second application, let it be required to form the equation of the squares of the differences of the roots of a given equation.

The proposed equation being $f(x) = 0$, and its n roots, as before, the roots of the transformed equation, will be

$$(a - a_2)^2, \quad (a - a_3)^2, \quad (a - a_4)^2 \ldots \ldots (a_2 - a_3)^2, \quad (a_2 - a_4)^2 \ldots \ldots$$

the number of which, obviously, amounts to the number of combinations, two and two, that can be formed with the n quantities, a, a_2, a_3, &c. Hence the degree of the required equation is $\frac{n(n-1)}{2}$, and, to find its coefficients, we must, as before, first determine the values of Σ' , Σ'_2, Σ'_3, &c. by the following formulas:

$$\Sigma'_1 = (a - a_2)^2 + (a - a_3)^2 + (a - a_4)^2 + \&c.$$
$$= (n-1)\,S(a^2) - 2S(aa_2)$$
$$\Sigma'_2 = (a - a_2)^4 + (a - a_3)^4 + (a - a_4)^4 + \&c.$$
$$= (n-1)\,S(a^4) - 4S(a^3 a_2) + 6S(a^2 a_2^2)$$
$$\Sigma'_3 = (a - a_2)^6 + (a - a_3)^6 + (a - a_4)^6 + \&c.$$
$$= (n-1)\,S(a^6) - 6S(a^5 a_2) + 15S(a^4 a_2^2) - 20S(a^3 a_2^3)$$
&c. &c.

As a particular example, let the equation, already considered at page 216, viz.

$$x^3 - 7x + 7 = 0,$$

be proposed, in which

$$A_2 = 0, \quad A = -7, \quad N = 7.$$

By the formulas at (117) we have

$$\Sigma_1 = 0, \quad \Sigma_2 = 14, \quad \Sigma_3 = -21, \quad \Sigma_4 = 98, \quad \Sigma_5 = -245, \quad \Sigma_6 = 833.$$

Consequently,

$$S(a^2) = \Sigma_2 = 14$$
$$S(a^4) = \Sigma_4 = 98$$
$$S(a^6) = \Sigma_6 = 833.$$

Also, from (120),

$$S(aa_2) \;\; = A = -7$$
$$S(a^3 a_2) = -\Sigma_4 = -98$$
$$S(a^5 a_2) = \Sigma_5 \Sigma_1 - \Sigma_6 = -833$$
$$S(a^2 a_2{}^2) = \frac{(\Sigma_2)^2 - \Sigma_4}{2} = \frac{196 - 98}{2} = 49$$
$$S(a^4 a_2{}^2) = \Sigma_4 \Sigma_2 - \Sigma_6 = 539$$
$$S(a^3 a_2{}^3) = \frac{(\Sigma_3)^2 - \Sigma_6}{2} = -196 \cdot$$

Consequently,

$$\Sigma'_1 = 2S(a^2) - 2S(aa_2) = 42$$
$$\Sigma'_2 = 2S(a^4) - 4S(a^3 a_2) + 6S(a^2 a_2{}^2) = 882$$
$$\Sigma'_3 = 2S(a^6) - 6S(a^5 a_2) + 15S(aa_2{}^2) - 20S(a^3 a_2{}^3) = 18669;$$

and hence, finally,

$$A'_2 = -\Sigma'_1 = -42$$
$$A' = -\frac{\Sigma'_2 + A'_2 \Sigma'_1}{2} = 141$$
$$N' = -\frac{\Sigma'_3 + A'_2 \Sigma'_2 + A' \Sigma'_1}{3} = -49,$$

so that the transformed equation is

$$y^3 - 42y^2 + 441y - 49 = 0.$$

On the Degree of the Final Equation, resulting from the Elimination of one of the Unknown Quantities from two Equations, containing two Unknowns.

(124.) Let the two equations be

$$f(x) = x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \ldots A_2x^2 + Ax + N = 0 \ldots (1)$$
$$F(x) = x^m + B_{m-1}x^{m-1} + B_{m-2}x^{m-2} + \ldots Bx^2 + Bx + M = 0 \ldots (2);$$

in which the coefficients, $A, A_2, A_3, \ldots B, B_2\ B_3$, are functions of y.

If we could resolve the first of these equations, we should obtain for x, n values, a, b, c, &c. which would be functions of y, and which, when substituted in the second, would furnish the n equations,

$$F(a) = a^m + B_{m-1}a^{m-1} + B_{m-2}a^{m-2} + \ldots B_2a^2 + Ba + M = 0$$
$$F(b) = b^m + B_{m-1}b^{m-1} + B_{m-2}b^{m-2} + \ldots B_2b^2 + Bb + M = 0$$
$$F(c) = c^m + B_{m-1}c^{m-1} + B_{m-2}c^{m-2} + \ldots B_2c^2 + Bc + M = 0$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

and these equations being solved for y, would make known the corresponding values of this quantity.

It is, however, in but few cases that we can actually solve the equation (1) for x; if we could, the determination of the corresponding values of x would not require the solution of the n separate equations, just obtained, because they may all be combined in a single equation, viz. the equation

$$F(a)\ F(b)\ F(c) \ldots = 0 \ldots (3);$$

and it is plain that the product, which forms its first member, undergoes no alteration, however we interchange a, b, c, &c. in the factors; that is, this product will contain none but rational and symmetrical functions of the roots of the equation (1). Hence, the first member of the equation (3) may be determined, by means of the coefficients of the equation (1), and the final equation in y, thus obtained.

As an example, let us take the equations

$$(y-2)x^2-2x+5y-2=0$$
$$yx^2-5x+4y=0.$$

Let us represent the values of x in terms of y, which satisfy the second equation by a and b. These, substituted in the first, furnish the two equations,

$$(y-2)a^2-2a+5y-2=0$$
$$(y-2)b^2-2b+5y-2=0;$$

of which the product is

$$(y-2)^2 S(a^2b^2)-2(y-2)S(a^2b)+$$
$$(y-2)(5y-2)\Sigma_2-2(5y-2)\Sigma_1+$$
$$4S(ab)+(5y-2)^2=0.$$

The coefficients A_2, A_1, N of the terms in the second equation, are

$$A_2=-\frac{5}{y},\quad A=4,\quad N=0;$$

consequently, we have

$$\Sigma_1=\frac{5}{y},\quad \Sigma_2=\frac{25-8y^2}{y^2},\quad \Sigma_3=\frac{125-6y^2}{y^3}$$

$$S(ab)=4,\quad S(a^2b)=\frac{20}{y},\quad S(a^2b^2)=16;$$

and, substituting these values in the preceding equation, there results

$$y^5+12y^3+87y^2-200y+100=0,$$

which is the final equation sought.

It must be confessed, however, that the foregoing method of deducing the final equation is usually very tedious, yet it has the advantage of presenting that equation unencumbered with extraneous roots. But the principal value of the foregoing investigation consists in its readily leading to the establishment of this theorem, first demonstrated by *Bezout*, viz.

The degree of the final equation, which results from the elimination of one of the unknowns, from two equations, of any degree whatever, involving two unknown quantities, can never surpass the product of the degrees of the two equations; and it is exactly equal to that product when the proposed equations are in their most general form.

In order to determine the degree in y, of the equation (3), we must consider that each term of the product (3), is formed by the multiplication of one term of the first factor, one of the second, one of the third, &c. Let, then, Ka^h, $K'b^{h'}$, $K''c^{h''}$ . . . . be terms, taken at random, in each of the n factors (3), the corresponding term of the product, will be

$$KK'K'' \ldots \times a^h b^{h'} c^{h''} \ldots;$$

moreover, the entire product is symmetrical in a, b, c, &c. so that this term forms part of one of the symmetrical functions which enter into the composition of (3), which partial function may be represented by

$$KK'K'' \ldots \times S(a^h b^{h'} c^{h''} \ldots) \ldots (4).$$

It will, therefore, be sufficient to determine the highest degree in y of this function.

Now, as by supposition, Ka^h is one of the terms in the polynomial $F(a)$ of the mth degree, it follows that the degree of y in K cannot exceed the $m - h$ degree. In like manner, the degree of y in K′ cannot exceed $m - h'$; the degree y in K″ cannot exceed $m - h''$, &c. Consequently, the product of the n polynomials $KK'K''$ . . . . cannot exceed the degree $mn - h - h' - h''$ . . . .

Let us now ascertain the degree which the polynomial $S(a^h b^{h'} c^{h''} \ldots)$ cannot surpass.

Referring to the general expressions involving Σ_1, Σ_2, Σ_3, &c. at p. 220, and recollecting that in our equation (1), page 229, the coefficient, A_{n-1}, cannot exceed the first degree in y, the coefficient, A_{n-2}, cannot exceed the second degree, and so on, we shall immediately see that the expressions for Σ_1, Σ_2, Σ_3, &c., deduced from our equation (1), cannot exceed the degree in y, denoted by the index suffixed to the symbol Σ. Referring, in like manner, to the general expressions in (120), which exhibit the double, triple, &c. functions, we there also recognize that, in $S(a^h\, b^{h'}\, c^{h''} \ldots)$, the degree in y cannot exceed $h + h' + h'' \ldots$ Hence, in the expression (4), the degree in y cannot exceed mn.

If the coefficients, A_{n-1}, A_{n-2}, A_{n-3}, &c. in the equation (1), and those in equation (2), are in their most general form, that is, if they exhibit a series of functions of y, regularly ascending, in degree, the expressions Σ_1, Σ_2, Σ_3, &c. will have the degree denoted by their suffixed indices, and hence the degree of $S(a^h\, b^{h'}\, c^{h''} \ldots)$, will be $h + h' + h'' \ldots$ It is plain, too, that in this case, K, K′, K″, &c. being, in their most general form, that their degrees in y will be exactly $m - h$, $m - h'$, $m - h''$, &c. Consequently, the degree of the final equation in y, will be exactly mn.

CHAPTER V.

ON THE FORM OF THE IMAGINARY ROOTS OF AN EQUATION.

(125.) In the second Chapter of Part II., general formulas have been investigated for the complete solution of equations, of the form $x^n \pm 1 = 0$, and it has been clearly established that every imaginary root of such equation is necessarily of the form

$$a + b\sqrt{-1}, \text{ or } a - b\sqrt{-1}.$$

We propose to demonstrate now, that this is not peculiar to binomial equations, but that it is the form assumed by the imaginary roots of every equation whatever. We have proved already (art. 14), that if an imaginary root of any equation had one of the above forms, that it must, necessarily, be accompanied by a conjugate root, having the other form. In order, therefore, to establish the fact, that every imaginary root must have one of these forms, it will be merely necessary to show that every equation of an even degree, $2n$, may be represented by

$$(x^2 + px + q)(x^2 + p_2 x + q_2) \ldots (x^2 + p_n x + q_n) = 0;$$

in which the original polynomial is replaced by n real quadratic factors. For, if we can prove this, it will immediately follow, that whatever imaginary roots enter the proposed equation, also enter one or more of the quadratic equations,

$$\begin{aligned} x^2 + px + q &= 0 \\ x^2 + p_2 x + q_2 &= 0 \\ &\vdots \\ x^2 + p_n x + q_n &= 0; \end{aligned}$$

and the roots of these we already know to be of the prescribed form.

Now *Laplace* has demonstrated the truth of the foregoing decomposition of any rational polynomial, nearly as follows.

(126.) Let $X = 0$ be any equation of an even degree, and let its roots be represented by a, a_2, a_3, &c. then the polynomial, X, will be reproduced by the multiplication of the simple factors,

$$(x - a),\quad (x - a_2),\quad (x - a_3)\ \ldots\ (x - a_n);$$

or, by the multiplication of the quadratic factors,

$$x^2 - (a + a_2)x + aa_2$$
$$x^2 - (a + a_3)x + aa_3$$
$$x^2 - (a_2 + a_3)x + a_2a_3$$
&c.

but we cannot affirm that all, or any, of these are *real* quadratic factors. We shall show that one of them, at least, is real.

1. In the first place, let n be an odd number, and let us conceive that an equation in y is formed, such that its roots may be the following symmetrical functions of those of the proposed, viz.

$$y = a + a_2 + kaa_2,\quad y = a + a_3 + kaa_3,\ \&c.$$

k being any assigned integral number; this equation, which we shall denote by $Y = 0$, being of the degree $2n\,\frac{2n-1}{2} = n(2n - 1)$, in which both n and $2n - 1$ are odd, is necessarily of an odd degree; it has, therefore, at least one real root (15), so that one, at least, of the expressions,

$$a + a_2 + kaa_2,\quad a + a_3 + kaa_3,\ \&c.$$

is real, whatever be the value of k.

These expressions are, as we have just seen, $n(2n - 1)$, in number; if therefore we put, for k, the successive values, 1, 2, 3 . . . $n(2n-1)$, we shall have exactly as many rows of different results as there are results in each row, and, as just proved, one in each row must be a real quantity; the number of rows, therefore, render it barely possible that every real result may be composed of different letters, $\alpha, \alpha_2, \alpha_3$, &c. If, however, we admit one row more, by giving to k the additional value $k = n(2n - 1) + 1$, then it is obviously impossible, that all the real results can be composed of different letters, for, if one out of m quantities is to be distinguished by any particular character, we cannot repeat those m quantities $m + 1$ times, and yet affix the distinction to a different one every time.

It follows, therefore, that among the results of our $n(2n - 1) + 1$ substitutions for K, there must occur at least two real quantities, a, a_2, composed of the same two letters as

$$a = \alpha + \alpha_2 + k\alpha\alpha_2, \quad a_2 = \alpha + \alpha_2 + k'\alpha\alpha_2$$

from which we deduce

$$\alpha\alpha_2 = \frac{a - a_2}{k - k'}, \quad \alpha + \alpha_2 = \frac{ka_2 - k'a}{k - k'};$$

which are necessarily real quantities; consequently, one at least of the quadratic factors,

$$x^2 - (\alpha + \alpha_2)x + \alpha\alpha_2, \text{ \&c.}$$

is real.

2. In the second place, let n be $2n'$ where n' is odd, and, as before, let the equation $Y = 0$, whose roots are

$$\alpha + \alpha_2 + k\alpha\alpha_2, \quad \alpha + \alpha_3 + k\alpha\alpha_3, \text{ \&c.}$$

be formed. This equation will be of the degree $2n\dfrac{2n - 1}{2} = 2n'(2n - 1)$, in which n' and $2n - 1$ being odd, the degree of the equation in y will be even, and only once divisible by 2; hence, from the former case, the equation, $Y = 0$, must have at least one real quadratic factor, which factor, equated to zero, must furnish two values of y, of the form

$$y = a \pm b\sqrt{-1},$$

in which b will be 0, if the two values of y are real. Each of these roots, then, must express the value of one or other of the combinations

$$\alpha + \alpha_2 + k\alpha\alpha_2, \quad \alpha + \alpha_3 + k\alpha\alpha_3, \text{ \&c.}$$

Let us suppose that the first root, $a + b\sqrt{-1}$, belongs to the combination $\alpha + \alpha_2 + k\alpha\alpha_2$; then, from the preceding reasoning, we know that another equation, $Y' = 0$, formed in the same manner, shall have a root, $a' + b'\sqrt{-1}$, belonging to the combination $\alpha + \alpha_2 + k'\alpha\alpha_2$, composed of the same letters, α, α_2; that is, there necessarily exists the two relations following, viz.

$$\alpha + \alpha_2 + k\alpha\alpha_2 = a + b\sqrt{-1}$$

$$\alpha + \alpha_2 + k'\alpha\alpha_2 = a' + b'\sqrt{-1};$$

from which we deduce

$$\alpha\alpha_2 = \frac{a - a' + (b - b')\sqrt{-1}}{k - k'}$$

$$\alpha + \alpha_2 = \frac{ka' - k'a + (kb' - k'b)\sqrt{-1}}{k - k'}.$$

These expressions are of the form

$$r + s\sqrt{-1} \quad \text{and} \quad r' + s'\sqrt{-1};$$

and hence the proposed equation has at least one factor of the second degree of the form

$$x^2 - (r' + s'\sqrt{-1})x + r + s\sqrt{-1} \quad \cdots\cdots \quad (1),$$

and this equated to zero, furnishes for x the two values

$$x = \frac{r' + s'\sqrt{-1}}{2} \pm \sqrt{\left\{\left(\frac{r' + s'\sqrt{-1}}{2}\right)^2 - (r + s\sqrt{-1}\right\}}.$$

The quantity under the radical being developed, will plainly give a result of the form $r'' + s''\sqrt{-1}$, and (Alg. p. 152,) the expression $\sqrt{\{r'' + s''\sqrt{-1}\}}$ is always reducible to another of the form

$r''' + s'''\sqrt{-1}$. Hence we conclude that the two values of x just deduced are of the form

$$x = p \pm q\sqrt{-1},$$

and that consequently the expression (1) which must have

$$x-(p+q\sqrt{-1}) \quad \text{and} \quad x-(p-q\sqrt{-1}),$$

for its component factors, is, notwithstanding its appearance to the contrary, a real quadratic expression equivalent to

$$x^2 - 2px + p^2 + q^2.$$

In the case where $n = 2n''$, it is therefore now demonstrated that an equation of the $2n$th degree has at least one real quadratic factor.

3. In the third place, let $n = 2^2 n''$ where n'' is odd; and, as before, let us conceive the equation $Y = 0$, analogous to the preceding, to be formed. The degree of this equation will be $2n \frac{2n-1}{2} = 2^2 n''$ $(2n-1)$; so that the exponent will be divisible by 2^2, and the quotient $n''(2n-1)$ will be an odd number.

Hence, from what has just been proved, the equation $Y = 0$ must have one real quadratic factor at least; and, consequently, by repeating the foregoing reasoning, we arrive at the conclusion that the proposed equation itself must have at least one real quadratic factor. Hence, generally, *every equation of an even degree has at least one real quadratic factor.*

(127.) It is now easy to show further that every equation of an even degree is decomposable into as many real quadratic factors as there are units in half the exponent of its degree.

For, since an equation of an even degree has at least one real quadratic factor, we can divide its first member by that factor, and thus obtain a depressed equation of a degree two units lower. This also must have a real quadratic factor, and may therefore be depressed two units lower in degree, and so on, till the polynomial is exhausted; consequently, *the first member of every equation of an even degree may always be regarded as the product of as many real quadratic factors as there are units in half the exponent of its degree.*

The imaginary roots, therefore, necessarily take the form hitherto assumed for them, viz. the form $a \pm b\sqrt{-1}$.

(128.) The form of the imaginary roots being thus known, they may be determined as follows:

Let

$$x^n + A_{n-1} x^{n-1} + \ldots . A x + N = 0$$

be an equation containing imaginary roots; then, by substituting $a + b\sqrt{-1}$ for x, we have

$$(a + b\sqrt{-1})^n + A_{n-1}(a + b\sqrt{-1})^{n-1} + \ldots . A(a + b\sqrt{-1}) + N = 0;$$

or, by developing the terms by the binomial theorem, and collecting the real and imaginary quantities separately, we have the form

$$M + N\sqrt{-1} = 0,$$

an equation which cannot exist except under the conditions

$$M = 0, \quad N = 0.$$

From these two equations, therefore, in which M, N contain only the quantities a, b, combined with the given coefficients, all the systems of values of a and b may be determined; and these, substituted in the expression $a + b\sqrt{-1}$, will make known all the imaginary roots of the proposed equation.

CHAPTER VI.

ON THE SOLUTION OF CUBIC EQUATIONS BY THE METHOD OF CARDAN, AND ON THE SOLUTION OF EQUATIONS OF THE FOURTH DEGREE BY THE METHODS OF EULER AND FERRARI.

(129.) In the First Part of the present treatise ample instructions have been given for the complete solution of every algebraical equation whose coefficients are expressed in known numbers.

It still remains for us to give a concise account of the labours of mathematicians, as far as they have been successful in the solution of equations with literal coefficients. The problem we now propose to consider is therefore this, viz. to determine finite expressions for the roots of an equation in functions of the coefficients; a problem long regarded as the most important in Algebra, because of its involving the complete solution of numerical equations. But the recent discoveries of Budan, Horner, and Sturm, as unfolded in the First Part of the present work, has reduced this celebrated problem to one of comparative insignificance; and have removed that regret which was so long and so universally felt on account of the failure of every attempt to extend the solution of literal equations beyond the four first degrees. We shall, therefore, content ourselves with briefly explaining the principal formulas which have been proposed for the solution of cubic and biquadratic equations.

Solution of a Cubic Equation by the Method of Cardan.

(130.) Let the proposed equation be first deprived of its second term by the rule at (24), it will then have the form

$$x^3 + px + q = 0.$$

Assume x equal to the sum of two other unknown quantities; that is, put

$$x = y + z,$$

we shall then have

$$x^3 = y^3 + z^3 + 3yz(y+z);$$

that is, replacing $y + z$ by x, and transposing,

$$x^3 - 3yzx - y^3 - z^3 = 0,$$

and, in order that this may be identical with the proposed equation, we must determine y and z so as to satisfy these conditions, viz.

$$yz = -\frac{p}{3}, \quad y^3 + z^3 = -q.$$

The problem is therefore reduced to the determination of y and z from these two equations.

From the first we have

$$y^3 z^3 = -\frac{p^3}{27};$$

hence, combining this with the second, we have the sum of two quantities, $y^3 + z^3$, and their product, $y^3 z^3$, given to determine the quantities: a problem which we know may be solved by help of a quadratic equation (Alg. p. 129), viz. the equation

$$v^2 + qv - \frac{p^3}{27} = 0,$$

of which the two roots or values of v will be the expressions for y^3 and z^3. Hence, solving the equation, and separating the two roots, we have

$$y^3 = -\frac{q}{2} + \sqrt{\left\{\frac{q^2}{4} + \frac{p^3}{27}\right\}}$$

$$z^3 = -\frac{q}{2} - \sqrt{\left\{\frac{q^2}{4} + \frac{p^3}{27}\right\}};$$

and consequently, since $x = y + z$, there results the following general expression for the roots of the proposed equation, viz.

$$x = \sqrt{\left\{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right\}^{\frac{1}{3}}} + \sqrt{\left\{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right\}^{\frac{1}{3}}},$$

which is the formula of *Cardan*.

Since the cube root of y^3 may be represented indifferently by either of the three expressions (page 191)

$$y, \quad \frac{-1+\sqrt{-3}}{2}y, \quad \frac{-1-\sqrt{-3}}{2}y,$$

and the cube root of z^3 by either of the expressions

$$z, \quad \frac{-1+\sqrt{-3}}{2}z, \quad \frac{-1-\sqrt{-3}}{2}z,$$

it would seem that $x = y + z$ admits of nine values, or that the proposed equation has nine roots. It must be remembered, however, that in all cases when we assign the root of any expression, a tacit reference is made to the generation of the proposed power, the root being in fact assumed to be the expression from which the given power has been actually produced. When we speak of any proposed power having a multiplicity of roots, we merely refer to the various expressions from each of which that power *might* be generated; and as many of these as prove inconsistent with the conditions involved in the production of the power, are of course to be rejected. Now one of the conditions in virtue of which y^3 and z^3 have been produced, is

$$yz = -\frac{p}{3};$$

that is to say, the product of the roots y, z, must be possible; but of the nine products which the preceding expressions for y and z are competent to furnish, six will be found to be imaginary: such a combination of values must therefore be rejected, as inconsistent with the conditions to be fulfilled; the other three products are possible.

Hence the only admissible solutions are the three following, where y and z represent the arithmetic values of the two terms in the second member of the expression for x, at the bottom of last page,

$$y + z,$$

$$\frac{-1+\sqrt{-3}}{2}y + \frac{-1-\sqrt{-3}}{2}z,$$

$$\frac{-1-\sqrt{-3}}{2}y + \frac{-1+\sqrt{-3}}{2}z.$$

If the relation between p and q be such that

$$\frac{q^2}{4} + \frac{p^3}{27} < 0,$$

the values of y and z will be imaginary, and the expression for x will then consist of the cube roots of two imaginary quantities. This expression, in such a case, is altogether incompetent to furnish the true values of x. They would seem to be imaginary, but we know, from the theorem of Sturm, (see page 156,) that the case supposed is the only one in which the roots are all real. This case is known by the name of the *irreducible case*, because, although the roots are then all real, yet algebra furnishes no means of reducing the complicated imaginary forms, under which they occur in Cardan's rule, to real finite expressions. We may hence infer that this rule is limited to equations of the third degree, which contain two imaginary roots.

Euler's Method of solving a Biquadratic Equation.

(131.) Let the proposed biquadratic, when deprived of its second term, be

$$x^4 + qx^2 + rx + s = 0.$$

Assume x equal to the sum of three other unknown quantities; that is, put

$$x = u + v + w \quad \ldots \ldots (1),$$

then

$$x^2 = u^2 + v^2 + w^2 + 2\,(uv + uw + vw).$$

Put P for $u^2 + v^2 + w^2$, and we shall have

$$(x^2 - \mathrm{P})^2 = 4\,(uv + uw + vw)^2$$
$$= 4\,(u^2v^2 + u^2w^2 + v^2w^2) +$$
$$8uvw\,(u + v + w);$$

that is, putting Q for $u^2v^2 + u^2w^2 + v^2w^2$, and replacing $u + v + w$ by x,

$$x^4 - 2Px^2 + P^2 = 4Q + 8uvwx$$

$$\therefore\ x^4 - 2Px^2 - 8uvwx + P^2 - 4Q = 0,$$

and, in order that this may be identical with the proposed equation, we must have these conditions, viz.

$$P = u^2 + v^2 + w^2 = -\frac{q}{2}$$

$$Q = u^2v^2 + u^2w^2 + v^2w^2 = \frac{P^2 - s}{4} = \frac{q^2 - 4s}{16}$$

$$uvw = -\frac{r}{8} \quad \text{or} \quad u^2v^2w^2 = \frac{r^2}{64}.$$

The conditions show that the quantities u^2, v^2, w^2, must be the roots of the cubic equation

$$y^3 + \frac{q}{2}y^2 + \frac{q^2 - 4s}{16}y - \frac{r^2}{64} = 0 \quad \dots \quad (2);$$

or, putting

$$y = \frac{z}{4},$$

the roots of the equation

$$z^3 + 2qz^2 + (q^2 - 4s)z - r^2 = 0 \quad \dots \quad (3).$$

Call these roots z', z'', z''', then the roots of (2) will be

$$\frac{z'}{4},\ \frac{z''}{4},\ \frac{z'''}{4},$$

and hence the expression (1) for x takes the following forms, viz.

$$\frac{\sqrt{z'}}{2} + \frac{\sqrt{z''}}{2} + \frac{\sqrt{z'''}}{2}, \quad -\frac{\sqrt{z'}}{2} - \frac{\sqrt{z''}}{2} - \frac{\sqrt{z'''}}{2},$$

$$\frac{\sqrt{z'}}{2} + \frac{\sqrt{z''}}{2} - \frac{\sqrt{z'''}}{2}, \quad \frac{\sqrt{z'}}{2} - \frac{\sqrt{z''}}{2} + \frac{\sqrt{z'''}}{2},$$

$$-\frac{\sqrt{z'}}{2} + \frac{\sqrt{z''}}{2} + \frac{\sqrt{z'''}}{2}, \quad -\frac{\sqrt{z'}}{2} - \frac{\sqrt{z''}}{2} - \frac{\sqrt{z'''}}{2},$$

$$-\frac{\sqrt{z'}}{2} + \frac{\sqrt{z''}}{2} - \frac{\sqrt{z'''}}{2}, \quad \frac{\sqrt{z'}}{2} - \frac{\sqrt{z''}}{2} - \frac{\sqrt{z'''}}{2}.$$

But some of these values are inadmissible, since a necessary condition is, that $uvw = -\frac{r}{8}$; hence, we must preserve only those of the foregoing trinomial expressions of which the product gives always a sign contrary to that of r, and these are, when r is positive,

$$x = \frac{-\sqrt{z'} - \sqrt{z''} - \sqrt{z'''}}{2}$$

$$x = \frac{-\sqrt{z'} + \sqrt{z''} + \sqrt{z'''}}{2}$$

$$x = \frac{-\sqrt{z'} + \sqrt{z''} + \sqrt{z'''}}{2}$$

$$x = \frac{\sqrt{z'} - \sqrt{z''} + \sqrt{z'''}}{2};$$

when r is negative

$$x = \frac{\sqrt{z'} + \sqrt{z''} + \sqrt{z'''}}{2}$$

$$x = \frac{-\sqrt{z'} + \sqrt{z''} - \sqrt{z'''}}{2}$$

$$x = \frac{\sqrt{z'} + \sqrt{z''} + \sqrt{z'''}}{2}$$

$$x = \frac{\sqrt{z'} + \sqrt{z''} - \sqrt{z'''}}{2};$$

and these formulas exhibit the four roots of the proposed equation.

Solution of an Equation of the Fourth Degree, by the method of Lewis Ferrari.

(132.) Taking the same general form as before, viz.

$$x^4 + qx^2 + rx + s = 0;$$

we have, by transposition,

$$x^4 = -qx^2 - rx - s.$$

Add the quantity

$$2kx^2 + k^2,$$

to both sides, and we shall then have

$$(x^2 + k)^2 = (2k - q)\,x^2 - rx + (k^2 - s);$$

and it remains to determine k, so that the second member of this equation may be a complete square. In order, this k must fulfil the condition

$$(2k - q)\,(k^2 - s) = \frac{r^2}{4},$$

since, in every perfect square, four times the product of the extreme terms is equal to the square of the middle one.

Actually multiplying the two factors, and dividing by the coefficient, 2, of k^3, we have, finally, the cubic equation,

$$k^3 - \frac{q}{2}k^2 - sk + \frac{qs}{2} - \frac{r^2}{8};$$

and the real root of this being determined by the rule of Cardan, before given, the solution of the proposed biquadratic is reduced to that of the two quadratics following, viz.

$$x^2 + k = \sqrt{2k - q}\; x - \sqrt{k^2 - s}$$

$$x^2 + k = -\sqrt{2k - q}\; x - \sqrt{k^2 - s}.$$

Another method of solving biquadratic equations, by means of particular artifices, was given by Descartes, in the third book of his Géométrie, and another by Mr. Thomas Simpson, in the second edition of his Algebra. This latter method is very similar to that of Ferrari, given above. Besides these investigations, others have been prosecuted with more general views, and by more uniform processes, in the hope of extending the powers of the analysis beyond equations of the fourth degree. But the great difficulties with which the calculation is encumbered, beyond this stage of the enquiry, has led to the entire abandonment of the undertaking. We shall, however, give in conclusion, a short Chapter, exhibiting the character of these researches as far as they have been extended.

CHAPTER VII.

ON THE SOLUTION OF EQUATIONS OF THE THIRD AND FOURTH DEGREE, BY MEANS OF SYMMETRICAL FUNCTIONS.

(133.) We shall now explain the methods which Lagrange has employed for the general solution of equations of the third and fourth degrees, by means of equations of inferior degrees. These methods, which are founded upon the theory of symmetrical functions, were first developed by Lagrange, in the Berlin Memoirs for 1770 and 1771, and are also given with some modifications in the Traité de la Resolution des Equations Numériques, Note XIII.

Equation of the Third Degree.

Let the proposed equation be

$$x^3 + px + q = 0,$$

in which the second term is absent. Call the roots a, a_2, a_3, then we immediately have the relation

$$a + a_2 + a_3 = 0;$$

and, if we could discover two other equations of the first degree in a, a_2, a_3, the values of these quantities might be easily determined by elimination.

Let us assume the relation

$$la + ma_2 + na_3 = z;$$

then, as there is nothing to distinguish one root from either of the others, the relation which we have just assumed may be indifferently one or the other of the six following, viz.

$$la + ma_2 + na_3 = z$$
$$la + ma_3 + na_2 = z$$
$$la_2 + ma + na_3 = z$$
$$la_2 + ma_3 + na = z$$
$$la_3 + ma + na_2 = z$$
$$la_3 + ma_2 + na = z;$$

and these could all be given by the solution of an equation of the sixth degree, in z. But, in order that such an equation might be solved as a quadratic, it must be of the form

$$z^6 + Az^3 + B = 0 \quad . \; . \; . \; . \quad (1);$$

which, if we put u for z^3, becomes

$$u^2 + Au + B = 0$$

$$\therefore \; u = -\frac{A}{2} \pm \sqrt{\frac{A^2}{4} - B};$$

hence, putting

$$z'^3 = -\frac{A}{2} + \sqrt{\frac{A^2}{2} - B}$$

$$z''^3 = -\frac{A}{2} - \sqrt{\frac{A^2}{4} - B};$$

and, recollecting that the three cube roots of unity are 1, α, α^2, (see Article 92, and p. 191), we have, for the six values of z, the following expressions, viz.

$$z', \quad \alpha z', \quad \alpha^2 z', \quad z'', \quad \alpha z'', \quad \alpha^2 z'';$$

taking, then, any two of the six expressions above, for z' and z'', as, for instance,

$$la + ma_2 + na_3 = z', \quad la + ma_3 + na_2 = z'';$$

the four others must fulfil the following conditions, viz.

$$la_2 + ma + na_3 = a\,(la + ma_3 + na_2)$$
$$la_2 + ma_3 + na = a^2(la + ma_2 + na_3)$$
$$la_3 + ma + na_2 = a\,(la + ma_2 + na_3)$$
$$la_3 + ma_2 + na = a^2(la + ma_3 + na_2);$$

which must be formed so that the coefficients of a, a_2, a_3, in one member of each, shall be different from those in other members, in order to avoid contradictory conditions.

The four equations, just deduced, are transformable into the following, viz.

$$(l - an)a_2 \;+ (m - al)a \;+ (n - am)a_3 = 0$$
$$(l - a^2m)a_2 + (m - a^2n)a_3 + (n - a^2l)a = 0$$
$$(l - an)a_3 \;+ (m - al)a \;+ (n - am)a_2 = 0$$
$$(l - a^2m)a_3 + (m - a^2n)a_2 + (n - a^2l)a = 0;$$

which will evidently be satisfied if we can fulfil the conditions

$$l = an, \quad m = al, \quad n = am$$
$$l = a^2m, \quad m = a^2n, \quad n = a^2l,$$

which are reducible to the two following, viz.

$$m = al, \quad \text{and} \quad n = a^2l;$$

for, from $a^3 = 1$, we have $a = \frac{1}{a^2}$, and $a^2 = \frac{1}{a}$, so that $m = al$ is the same as $m = \frac{1}{a^2}\,l$, whence $l = a^2m$. In like manner, $n = a^2l$ is the same as $n = \frac{1}{a}\,l$; whence $l = an$. Lastly, the relations, $m = al$, $n = a^2l$, divided the one by the other, give $\frac{m}{n} = \frac{1}{a}$ $\therefore m = \frac{1}{a}\,n =$

$\alpha^2 n$, and $n = \alpha m$; hence it will be sufficient to consider the two relations,

$$m = \alpha l, \quad n = \alpha^2 l;$$

from which, as we have just seen, all the others are deducible. We thus have m and n expressed in terms of l, which, being arbitrary, put it for simplicity equal to unity; then we shall have

$$m = \alpha, \quad n = \alpha^2,$$

and thus the three values, l, m, n, are no other than the three cube roots of unity.

Substituting these values in the expressions

$$la + ma_2 + na_3 = z', \quad la + ma_3 + na_2 = z'';$$

they become

$$a + \alpha a_2 + \alpha^2 a_3 = z', \quad a + \alpha a_3 + \alpha^2 a_2 = z''.$$

We may, in like manner, substitute the same values in the four remaining equations, and afterwards form, by multiplication, the equation in z; as, however, we know that this equation is to be of the form (1), its six roots must be comprised in the two equations,

$$z^3 = z'^3 = (a + \alpha a + \alpha^2 a_3)^3$$

$$z^3 = z''^3 = (a + \alpha a_3 + \alpha^2 a_2)^3;$$

or, which is the same thing, in the single equation,

$$\{z^3 - (a + \alpha a + \alpha^2 a_3)^3\}\{z^3 - (a + \alpha a_3 + \alpha^2 a_2)^3\} = 0.$$

By actually performing the multiplication here indicated, and comparing the coefficients of the resulting terms with those of the corresponding terms in (1), we have these conditions, viz.

$$A = -\{(a + \alpha a_2 + \alpha^2 a_3)^3 + (a + \alpha a_3 + \alpha^2 a_2)^3\}$$

$$B = \quad (a + \alpha a_2 + \alpha^2 a_3)^3 \cdot (a + \alpha a_3 + \alpha^2 a_2)^3.$$

Hence the coefficients of the equation,

$$z^6 + Az^3 + B = 0,$$

are symmetrical functions of the roots proposed.

If we develope these values of A and B, and keep in mind that

$$\alpha^3 = 1, \quad \alpha^4 = \alpha, \quad \alpha^5 = \alpha^2, \quad \alpha^6 = 1, \text{ \&c.}$$

and, because the coefficient of the second term in $y^3 - 1 = 0$ is zer that

$$1 + \alpha + \alpha^2 = 0, \text{ and, therefore, } \alpha + \alpha^2 = -1,$$

we shall have these values, viz. (see page 222),

$$A = -\{2S(a^3) + 3(\alpha + \alpha^2)\,S(a^2 a_2) + 12 a a_2 a_3$$
$$= -\{2S(a^3) - 3S(a^2 a_2) + 12 a a_2 a_3$$
$$B = \{S(a^2) + (\alpha + \alpha^2)\,S(a a_2)\}^3$$
$$= \{S(a^2) - S(a a_2)\}^3.$$

Now, from the original equation,

$$x^3 + px + q = 0,$$

we obtain, in terms of the coefficients, p, q, the following values these symmetrical functions, viz.

$$S(a^2) = \Sigma_2 = -2p, \quad S(a^3) = \Sigma_3 = -3q, \quad S(a a_2) = p,$$
$$S(a^2 a_2) = \Sigma_2\,\Sigma_1 - \Sigma_3 = -\Sigma_3 = 3q.$$

Moreover,

$$a a_2 a_3 = -q;$$

hence, substituting these values in the preceding expressions fo and B, we have

$$A = -\{-6q - 9q - 12q\} = 27q$$
$$B = \{-2p - p\}^3 = (-3p)^3 = -27p^3;$$

consequently, the equation in z is now fully determined; it is

$$z^6 + 27qz^3 - 27p^3 = 0;$$

from which we get

$$z^3 = -\frac{27q}{2} \pm 27\sqrt{\frac{q^2}{4}+\frac{p^3}{27}};$$

and thence

$$z' = 3\left\{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right\}^{\frac{1}{3}}$$

$$z'' = 3\left\{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right\}^{\frac{1}{3}}.$$

These values being now known, we have, for the determination of the roots, a, a_2, a_3, the three simple equations,

$$a + a_2 + a_3 = 0$$

$$a + \alpha a_2 + \alpha^2 a_3 = z'$$

$$a + \alpha^2 a_2 + \alpha a_3 = z''.$$

By adding these equations together, taking account of the property,

$$1 + \alpha + \alpha^2 = 0,$$

we have

$$3a = z' + z'';$$

which gives, for the root a, the value

$$a = \frac{z' + z''}{3};$$

that is, substituting for z', z'', their values in terms of p and q, as expressed above,

$$a = \left\{-\frac{q}{2}+\sqrt{\frac{q}{4}+\frac{p^3}{27}}\right\}^{\frac{1}{3}} + \left\{-\frac{q}{2}-\sqrt{\frac{q}{4}+\frac{p^3}{27}}\right\}^{\frac{1}{3}};$$

which agrees with the formula, before found, by the method of Cardan, (p. 240).

To obtain the other two roots, multiply the second of the three equations above by α, the third by α^2, and add the results to the first, we shall thus have

$$3a_3 = \alpha z' + \alpha^2 z'' \therefore a_3 = \frac{\alpha z' + \alpha^2 z''}{3}.$$

Lastly, multiply the second by α^2, the third by α, add as before, and we shall have

$$3a_2 = \alpha^2 z' + \alpha z'' \therefore a_2 = \frac{\alpha^2 z' + \alpha z''}{3};$$

and thus all three of the roots are determined.

Equation of the Fourth Degree.

(134.) Let the equation,

$$x^4 + px^2 + qx + r = 0,$$

be proposed for solution.

As the second term is absent, one relation among the roots is

$$a + a_2 + a_3 + a_4 = 0.$$

Let us endeavour to obtain three other relations of the first degree, in a, a_2, a_3, a_4. For this purpose, assume

$$ka + la_2 + ma_3 + na_4 = z;$$

then, as there is no distinction between the roots expressed in this relation, it may represent indifferently any one of the 24 equations which arise from permuting the letters a, a_2, a_3, a_4, in all the ways possible. Hence the equation in z, which would be satisfied for any one of these 24 values, indifferently, must be of the 24th degree, and, in order that it may be resolvable by the formula for equations of the third degree, it must take the form

$$z^{24} + Az^{16} + Bz^8 + C = 0.$$

It is possible to reduce the degree of this equation; for, since k, l, m, n, are indeterminate, we may suppose $k = l$, and thus reduce the number of distinct equations to twelve. By supposing, moreover, $m = n$, the equations are further reduced to six, which are as follows

$$l(a + a_2) + m(a_3 + a_4) = z$$
$$l(a + a_3) + m(a_2 + a_4) = z$$
$$l(a + a_4) + m(a_2 + a_3) = z$$
$$l(a_3 + a_4) + m(a + a_2) = z$$
$$l(a_2 + a_4) + m(a + a_3) = z$$
$$l(a_2 + a_3) + m(a + a_4) = z.$$

The equation in z will, therefore, under these restrictions, be only of the sixth degree, and, in order to solve it, it must be of the form

$$z^6 + Az^4 + Bz^2 + C = 0,$$
$$\text{or } (z^2)^3 + A\,(z^2)^2 + Bz^2 + C = 0,$$

and whatever value of z satisfies this equation, the same value, with contrary sign, will also satisfy it. The roots are therefore equal in magnitude two and two, but of contrary signs; and it is plain that the six values of z exhibited above will represent these relations by putting $l = -m = 1$; in fact, we shall then have for the values of z the expressions,

$$\begin{cases} a + a_2 - (a_3 + a_4) = z \\ a + a_3 - (a_2 + a_4) = z \\ a + a_4 - (a_2 + a_3) = z \end{cases}$$
$$\begin{cases} a_3 + a_4 - (a + a_2) = z \\ a_2 + a_4 - (a + a_3) = z \\ a_2 + a_3 - (a + a_4) = z \end{cases}$$

where the last three values of z are in magnitude, the same as the first three, but with opposite signs. Hence, by transposing, and multiplying the several pairs of factors together, we have the following single equation in z, involving all the six values above, viz.,

$$\{z^2 - (a + a_2 - a_3 - a_4)^2\} \times$$
$$\{z^2 - (a + a_3 - a_2 - a_4)^2\} \times$$
$$\{z^2 - (a + a_4 - a_2 - a_3)^2\} = 0;$$

and as this involves none but symmetrical functions of a, a_2, a_3, a_4, its coefficients may be expressed by means of the coefficients of the proposed equation; but the following considerations will facilitate their determination. By actually squaring the quantities within the brackets, we have

$$(a + a_2 - a_3 - a_4)^2 =$$
$$(a + a_2 + a_3 + a_4)^2 - 4(aa_3 + aa_4 + a_2 a_3 + a_2 a_4);$$

but

$$a + a_2 + a_3 + a_4 = 0$$
$$aa_2 + aa_3 + aa_4 + a_2 a_3 + a_2 a_4 + a_3 a_4 = p;$$

therefore,

$$-(a + a_2 - a_3 - a_4)^2 = 4p - 4(aa_2 + a_3 a_4).$$

In like manner,

$$-(a + a_3 - a_2 - a_4)^2 = 4p - 4(aa_3 + a_2 a_4)$$
$$-(a + a_4 - a_2 - a_3)^2 = 4p - 4(aa_4 + a_2 a_3);$$

putting, therefore, for abridgment,

$$z^2 + 4p = 4u,$$

the equation in z will be transformed into the following, viz.

$$\{u - (aa_2 + a_3 a_4)\}\ \{u - (aa_3 + a_2 a_4)\}\ \{u - (aa_4 + a_2 a_3)\} = 0:$$

which is of the form,

$$u^3 + A'u^2 + B'u + C' = 0;$$

its coefficients being

$$A' = -(aa_2 + aa_3 + aa_4 + a_2 a_3 + a_2 a_4 + a_3 a_4) = -p$$
$$B' = (aa_2 + a_3 a_4)(aa_3 + a_2 a_4) +$$
$$(aa_2 + a_3 a_4)(aa_4 + a_2 a_3) +$$
$$(aa_3 + a_2 a_4)(aa_4 + a_2 a_3) = S\,(a^2 a_2 a_3)$$
$$C' = -(aa_2 + a_3 a_4)(aa_3 + a_2 a_4)(aa_4 + a_2 a_3).$$

Now, from the formula at page 223, we have,

$$S(a^2 a_2 a_3) = \frac{\Sigma_2 (\Sigma_1)^2 - 2\Sigma_3\Sigma_1 - (\Sigma_2)^2 + 2\Sigma_4}{2},$$

$$S(a^2 a_2^2 a_3^2) = \frac{(\Sigma_2)^3 - 3\Sigma_4\Sigma_2 + 2\Sigma_6}{6};$$

in which,

$$\Sigma_1 = 0,\ \Sigma_2 = -2p,\ \Sigma_3 = -3q$$
$$\Sigma_4 = -p\,\Sigma_2 - 4r = 2p^2 - 4r,$$
$$\Sigma_6 = -2p^3 + 4pr + 3q^2 + 2pr = -2p^3 + 6pr + 3q^2.$$

Hence, substituting these values in the expressions for B′ and C′, we have

$$B' = \frac{-4p^2 + 4p^2 - 8r}{2} = -4r,$$

$$C' = -q^2 + 2pr + 2pr = 4pr - q^2;$$

and thus the equation in u is

$$u^3 - pu^2 - 4ru + 4pr - q^2 = 0;$$

and replacing u by its value $\frac{z^2 + 4p}{4} = \frac{z^2}{4} + p$, or, for simplicity, by $z_, + p$, we have, for the final cubic, the equation

$$z_,^3 + 2pz_,^2 + (p^2 - 4r)\,z_, - q^2 = 0.$$

Calling the roots of this equation z', z'', z''', we shall have $4z'$, $4z''$, $4z'''$ for the squares of the expressions $a + a_2 - a_3 - a_4$, $a + a_3 - a_2 - a_4$, $a + a_4 - a_2 - a_3$; that is, these expressions are

$$a + a_2 - a_3 - a_4 = \pm 2\sqrt{z'}$$
$$a + a_3 - a_2 - a_4 = \pm 2\sqrt{z''}$$
$$a + a_4 - a_2 - a_4 = \pm 2\sqrt{z'''}$$

also, $$a + a_2 + a_3 + a_4 = 0.$$

By adding these, we find

$$4a = \pm 2\sqrt{z'} \pm 2\sqrt{z''} \pm 2\sqrt{z'''};$$

$$\therefore a = \pm \tfrac{1}{2}\sqrt{z'} \pm \tfrac{1}{2}\sqrt{z''} \pm \tfrac{1}{2}\sqrt{z'''}.$$

Again, adding the first and fourth, and subtracting the sum of the other two from the result, we have

$$4a_2 = \pm 2\sqrt{z'} \mp 2\sqrt{z''} \mp 2\sqrt{z'''}$$

$$\therefore a_2 = \pm \tfrac{1}{2}\sqrt{z'} \mp \tfrac{1}{2}\sqrt{z''} \mp \tfrac{1}{2}\sqrt{z'''}.$$

Similarly,

$$a_3 = \mp \tfrac{1}{2}\sqrt{z'} \pm \tfrac{1}{2}\sqrt{z''} \mp \tfrac{1}{2}\sqrt{z'''}$$

$$a_4 = \mp \tfrac{1}{2}\sqrt{z'} \pm \tfrac{1}{2}\sqrt{z''} \mp \tfrac{1}{2}\sqrt{z'''}.$$

As to the proper signs of the radicals $\sqrt{z'}$, $\sqrt{z''}$, $\sqrt{z'''}$, we must observe that since

$$(a + a_2 - a_3 - a_4)(a + a_3 - a_2 - a_4)(a + a_4 - a_2 - a_3) = -8q,$$

these signs ought to be such as to render their product positive if q is negative, and negative if q is positive. The values thus deduced are the same as those otherwise determined in the preceding chapter.

The above method of investigating analytical expressions for the roots of equations by means of symmetrical functions, may be extended to equations of higher degrees than the fourth; but the auxiliary equation in z, to which the investigation leads, is, after the fourth degree, of a higher order than the proposed. In equations of the fifth degree the auxiliary one rises to the 120th degree, which, by means of certain artifices, is, however, capable of depression. But no method has yet been devised, whereby an equation of the fifth degree can be solved by help of an auxiliary equation below the sixth (See Note XIII. of Lagrange's Treatise).

NOTES.

Note A.—*Page 5.*

To prove that every Equation has a Root.

Cauchy's proof that every equation has a root, whether its coefficients are real or imaginary, is to be found in that author's Cours d'Analyse, 1re partie, Chap. X; it is given also with more or less modification in most of the recent publications on the theory of equations, which have appeared on the Continent. The following version of it is from the *Algebré* of *M. Lefebure de Fourcy.* It will be necessary, before entering upon the investigation itself, first to explain a new term that we shall have occasion to employ, and then to establish two preliminary propositions concerning imaginary quantities.

Definition. The expression $\sqrt{a^2 + b^2}$ formed from the real parts of the imaginary quantity, $a + b\sqrt{-1}$ is called the *modulus* of that quantity; for example, $\sqrt{9 + 16}$ or 5 is the modulus of $3 - 4\sqrt{-1}$. Hence two *conjugate* expressions (see p. 19), have the same modulus.

It may be remarked here, that, although we have called $a + b\sqrt{-1}$ an imaginary quantity; yet it may fitly represent any quantity, real or imaginary; for, by putting $b = 0$, it is reduced to a; in this case the modulus is the quantity itself. The lemmas we have adverted to are these.

Lemma I. The sum or difference of any two quantities has a modulus comprised between the sum and difference of the moduli of the proposed quantities. Let the two quantities be

$$a + b\sqrt{-1}, \quad a' + b'\sqrt{-1},$$

and let r and r' represent their moduli; that is, let

$$r^2 = a^2 + b^2, \quad r'^2 = a'^2 + b'^2.$$

Let also R be the modulus of their sum; we shall then evidently have

$$R^2 = (a + a')^2 + (b + b')^2$$
$$= a^2 + a'^2 + b^2 + b'^2 + 2(aa' + bb')$$
$$= r^2 + r'^2 + 2(aa' + bb').$$

But by multiplying r^2 by r'^2, it is easy to see that

$$r^2r'^2 = a^2a'^2 + b^2b'^2 + a^2b'^2 + a'^2b^2$$
$$= (aa' + bb')^2 + (ab' - ba')^2;$$

hence the numerical value of $aa' + bb'$ is below, or at most equal to rr'; and consequently, R^2 must be comprised between the two quantities

$$r^2 + r'^2 + 2rr', \text{ and } r^2 + r'^2 - 2rr',$$

or which is the same thing, between

$$(r + r')^2 \text{ and } (r - r')^2.$$

Hence the modulus R is comprised between the sum and difference of the moduli r and r'. The demonstration is exactly similar when, instead of the sum of the imaginary quantities, we take their difference.

Lemma II. The product of two quantities has for modulus the product of their moduli. For the multiplication gives

$$(a + b\sqrt{-1})(a' + b'\sqrt{-1}) = aa' - bb' + (ab' + ba')\sqrt{-1},$$

and taking the modulus of this product, we have

$$\sqrt{(aa' - bb')^2 + (ab' + ba')^2} = \sqrt{a^2a'^2 + b^2b'^2 + a^2b'^2 + b^2a'^2}$$
$$= \sqrt{(a^2 + b^2)(a'^2 + b'^2)},$$

as announced.

Corollary. The product of any number of factors must have for modulus the product of the moduli of all the factors; hence, when all the factors are equal, and in number n, we conclude that the nth power of an imaginary quantity has for modulus the nth power of the modulus of the root.

We may now proceed to investigate the proposition announced.

I. First, let us consider equations of the form

$$x^m = \alpha + \beta\sqrt{-1},$$

where α and β denote any real quantities whatever, and may be zero, one or both. We shall show that there always exists a value of x of the form $a + b\sqrt{-1}$, which will satisfy the equation.

When $m = 2$, the determination of x merely requires the extraction of several square roots in succession, and then the extraction of a root of an odd degree.

The square root of every expression of the form $\alpha + \beta\sqrt{-1}$ is always itself of the same form $a + b\sqrt{-1}$* and hence, however often the process of extracting the square root is repeated, the final result will still be of the same form.

When m is a power of 2, multiplied by an odd number, the determination of x requires that after the successive extractions of the square root, we extract a root of an odd degree; it is, therefore, only necessary to prove that the extraction of this root may be effected, and that the result will be of the assigned form. We have then to confine our attention to the case of m = an odd number.

If one of the quantities α, β, is zero, the truth of the proposition is at once seen. For if $\beta = 0$, the equation is

$$x^m = \alpha;$$

and whatever be the sign of α, $\sqrt[m]{\alpha}$ will have a real value, which is the root of the equation.

If $\alpha = 0$, the equation is

$$x^m = \beta\sqrt{-1};$$

or, putting $x'\sqrt{-1}$ for x,

$$x^m = \pm x'^m\sqrt{-1}$$

$$\therefore x'^m = \pm\beta.$$

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* See Algebra, page 152.

Now this equation has a real root, b; hence the equation, $x^m = \beta\sqrt{-1}$, must have the root $x = b\sqrt{-1}$.

Let us now examine the case in which neither α nor β is zero. Transposing the terms to the first member, the proposed equation is

$$x^m - (\alpha + \beta\sqrt{-1} = 0,$$

or, if for abridgment, we represent the first member by X, we shall have

$$X = x^m - (\alpha + \beta\sqrt{-1});$$

and, if for x, we put any expression of the form,

$$a + b\sqrt{-1},$$

X will be transformed into a similar expression,

$$A + B\sqrt{-1} \quad . \; . \; . \; . \; (1);$$

where A and B are functions of a, b, without $\sqrt{-1}$, and we shall now prove that there exists real values for a and b, which will cause the expression (1) to vanish, or that will cause A and B to vanish simultaneously, or, which is the same thing, that will satisfy the condition

$$A^2 + B^2 = 0;$$

for the sum of two squares cannot vanish unless each one vanishes separately. Put

$$\rho = \sqrt{\alpha^2 + \beta^2}, \quad v = \sqrt{a^2 + b^2}, \quad V = \sqrt{A^2 + B^2},$$

then, if we take $b = 0$, and $a^m = \alpha$, it is evident that x^m will become a^m or α, so that

$$X = \alpha - (\alpha + \beta\sqrt{-1}) = -\beta\sqrt{-1}$$

$$\therefore\ V^2 = \beta^2 \therefore\ V^2 < (\alpha + \beta^2) \therefore\ V < \rho.$$

Since we thus find a value of V less than ρ, we may conclude, with certainty, that in making a and b to vary in all possible ways, the least value that V could take would be $< \rho$. It must be remarked, however, that this minimum value cannot correspond to a value of x, in which a and b are zero; for else we should have $V = \sqrt{\alpha^2 + \beta^2} = \rho$; neither can it correspond to a value of x, in which a or b is infinite, for then we should have $v = \infty$, and, consequently, $V = \infty$, because $V > v^m - \rho$, by the Lemmas.

Let c be any value of x, real or imaginary, but subject to these last restrictions; and let us suppose that the corresponding value of V is not zero; let V′ represent this value, and C the corresponding value of X, which must also be different from zero. If we put

$$x = c + z,$$

and take account of the condition

$$C = c^m - \alpha - \beta\sqrt{-1},$$

the polynomial X will become

$$X = (c + z)^m - \alpha - \beta\sqrt{-1}$$
$$= C + mc^{m-1}z + \frac{m(m-1)}{2}c^{m-2}z^2 \ldots . + z^m.$$

In this development, the sum of the two first terms vanishes when

$$z = -\frac{C}{mc^{m-1}}.$$

But, if ε denote a positive quantity, which we may make as small as we please, and we put

$$z = -\frac{C}{mc^{m-1}}\varepsilon;$$

the first two terms of the development will become $C(1 - \varepsilon)$, and the entire development may be written

$$X = C(1 - \varepsilon + f\varepsilon^2 + f'\varepsilon^3 + \&c.)$$

when f, f', &c. are quantities of the form $a + b\sqrt{-1}$, so that the factor which multiplies C may be represented by $a' + b'\sqrt{-1}$. Put Φ for the modulus, $\sqrt{a'^2 + b'^2}$, then (lemma 2),

$$V = V'\Phi.$$

Moreover, since ε is a real quantity, if we call ϕ, ϕ', &c. the moduli of f, f', &c. we shall have $1 - \varepsilon$, $\phi\varepsilon^2$, $\phi'\varepsilon^3$, &c. for those of the quantities, $1 - \varepsilon$, $f\varepsilon^2$, $f'\varepsilon^3$, &c. and, by the first lemma, the modulus, Φ, cannot surpass

$$1 - \varepsilon + \phi\varepsilon^2 + \phi'\varepsilon^3 + \&c.$$

$$\text{or } 1 - \varepsilon(1 - \phi\varepsilon - \phi'\varepsilon^2 - \&c.)$$

Now, for very small values of ε, the quantity within the parentheses is < 1; consequently, the whole expression is < 1, therefore $\Phi < 1$, and $V < V'$.

Hence, when V' is not zero, we may so choose x that the modulus V, of X, may be $< V'$, therefore the minimum value of this modulus cannot differ from zero. But the value of x, which makes $V = 0$, is a root of the equation $X = 0$; hence the binomial equation always admits of a root of the form $a + b\sqrt{-1}$.

II. Let now the general equation,

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} + \&c. = 0 \quad \ldots \quad (1),$$

be examined, in which P, Q, R, &c. are any quantities, real or imaginary, of the form $a + b\sqrt{-1}$; and, for abridgment, put X for the first member of the equation. Replace x by the value

$$x = a + b\sqrt{-1},$$

and we shall have

$$X = A + B\sqrt{-1};$$

where A and B are functions of a and b, without $\sqrt{-1}$; and, in order that the equation (1) may be satisfied, for the assumed value of x, it is sufficient that the modulus, $\sqrt{A^2 + B^2}$ of X, be zero. Now it is the object of the following reasoning to show that real values do exist for a and b, which cause this modulus to vanish.

Put ρ, ρ', ρ'', &c. for the moduli of the coefficients P, Q, R, &c., v for the modulus of the expression $x = a + b\sqrt{-1}$, and V for that of X. Then, by Lemma II., when we substitute $a + b\sqrt{-1}$ for x, the powers x^m, x^{m-1}, x^{m-2}, &c. have for moduli, v^m, v^{m-1}, v^{m-2}, &c.; and the different terms

$$x^m, \quad Px^{m-1}, \quad Qx^{m-2}, \quad Rx^{m-3}, \quad \&c.,$$

which compose X, have for moduli

$$v^m, \quad \rho v^{m-1}, \quad \rho' v^{m-2}, \quad \rho'' v^{m-3}, \quad \&c.$$

Hence, by Lemma I., we conclude that the modulus of the polynomial

$$Px^{m-1} + Qx^{m-2} + Rx^{m-3} + \&c.$$

cannot surpass the sum

$$\rho v^{m-1} + \rho' v^{m-2} + \rho'' v^{m-3} + \&c.,$$

and that consequently the modulus V, of the polynomial X, cannot be less than the difference

$$v^m - \rho v^{m-1} - \rho' v^{m-2} - \rho'' v^{m-3} - \&c. \quad . \; . \; . \; . \quad (2),$$

which we shall suppose positive, or else take it with a contrary sign.

If now we give to v values continually increasing in magnitude, commencing at a certain limit, the expression (2) will be constantly positive, and will go on increasing to infinity. Hence the modulus V, which can never be below this difference, will itself acquire values surpassing any limit.

If we make a or b infinite, the modulus v of x will be infinite; and, from what has just been shown, the expression (2) will be infinite also, and consequently the modulus V. But so long as a and b are not infinite, it is evident, from the nature of the polynomials A and B, that this modulus cannot become infinite. Whence it follows, that if it does not become zero for any value of x, we may be sure that there exists one value formed with finite values of a and b, that gives for V a value below which there cannot fall any other value of this modulus. The whole is then reduced to proving that this minimum modulus is no other than zero.

Let c be any particular value of x, real or imaginary; let C be the corresponding value of X, which is supposed different from zero; and let V′ be the modulus of C. If we take for x a value, $x = c + z$, different from c, there will result for X a development which may be represented by

$$X = C + C'z + \tfrac{1}{2} C'' z^2 + \&c. \; . \; . \; . \; . \; (3).$$

Assume first that C′ is not zero, and put

$$z = -\frac{C}{C'}\varepsilon,$$

ε being a positive quantity, that we can make as small as we please. The expression (3) may be then thus written:

$$X = C(1 - \varepsilon + f\varepsilon^2 + f'\varepsilon^3 + \&c.),$$

where f, f' are still quantities of the form $a + b\sqrt{-1}$; and we see here, as at page 262, that values of ε may be taken small enough to render $V < V'$.

Suppose now that C′ is zero, and that, proceeding from this coefficient, the first which differs from zero is that of z^n. Call it C_1, and those which follow C_2, C_3, &c.; the expression (3) will then be simply

$$X = C + C_1 z^n + C_2 z^{n+1} + \&c. \; . \; . \; . \; . \; (4).$$

If we assume

$$z^n = -\frac{C}{C_1},$$

then, since the quantity $-\frac{C}{C_1}$ is of the form $a + b\sqrt{-1}$, we know, from what has been said of the binomial equation, that there exists a value of z which will satisfy the assumption. Call z' this value, and take

$$z = z'\varepsilon,$$

ε being as small as we please; then, since $z' = -\frac{C}{C_1}$, the expression (4) is changed into

$$X = C(1 - \varepsilon^n + f_1\varepsilon^{n+1} + f_2\varepsilon^{n+2} + \&c.),$$

in which f_1, f_2, &c. are always of the form $a + b\sqrt{-1}$; and, reasoning as at page 262, already cited, we see that there are small values of ε which will render the modulus of the expression above less than that of C; that is to say, we shall have $V < V'$.

Hence, when V' is not zero, we may always choose x so that the modulus V of the polynomial X may be $< V'$; so that the minimum value of V will not differ from zero; and, consequently, the value of x, to which this minimum corresponds, is a root of the equation (1). Without assigning any value to this root, and without examining if there exists several values of x which satisfies $V = 0$, we may nevertheless conclude with certainty that an equation of any degree, whose coefficients are either real or imaginary quantities of the form $a + b\sqrt{-1}$, has always at least one root of the form $a + b\sqrt{-1}$.

Note B.—*Page* 174.

To prove that the Continued Fraction, which is the Development of an Irrational Quantity, is Periodical.

Let A be any number whatever, a^2 being the greatest square contained in it, and b the remainder, so that $A = a^2 + b$; the develop ment of $\sqrt{A}$, in a continued fraction, will give

$$x = \sqrt{A} = a + \frac{\sqrt{A} - a}{1}$$

$$x' = \frac{1}{\sqrt{A} - a} = \frac{\sqrt{A} + a}{b} = \&c.$$

Suppose that by prolonging the operation indefinitely we arrive at the complete quotient* $x^{(n)}$ or $y=\frac{\sqrt{A}+I}{D}$; let μ be the whole number contained in y, the remainder will be $\frac{\sqrt{A}+I-\mu D}{D}$; calling this remainder $\frac{1}{y'}$, we shall have $y'=\frac{D}{\sqrt{A}+I-\mu D}$; and, since also the analogy of the forms require that we should have $y'=\frac{\sqrt{A}+I'}{D'}$, we shall thence obtain the following equation, in order to determine I′ and D′, viz.

$$\frac{D}{\sqrt{A}+I-\mu D}=\frac{\sqrt{A}+I'}{D'}.$$

This equation, in which we must equalize separately the rational part to the rational, and the irrational part to the irrational, will give

$$I'=\mu D-I$$
$$D'=\frac{A-I'I'}{D}.$$

Such is the very simple law by which, from any complete quotient whatever, $\frac{\sqrt{A}+I}{D}$, we may deduce the following complete quotient $\frac{\sqrt{A}+I'}{D'}$. It must not be supposed that the numbers I′ and D′ are fractional, for if we substitute the value of I′ in that of D′, we shall have $D'=\frac{A-(\mu D-I)}{D}=\frac{A-I^2}{D}+2\mu I-\mu^2 D$. But, having $A-I'^2=D'D$, if we represent by $\frac{\sqrt{A}+I^0}{D^0}$ the complete quotient

* The expressions which we have denoted by α, β, γ, &c. p. 173, Legendre calls *complete quotients*, to distinguish them from the *quotients a, b, c*, &c.

which precedes $\frac{\sqrt{A}+I}{D}$, we shall have, in like manner, $A - I^2 = D D^0$, therefore

$$D' = D^0 + 2\mu I - \mu^2 D.$$

Whence we perceive that since the numbers D and I are integral in the first two complete quotients $\frac{\sqrt{A}+0}{1}$, $\frac{\sqrt{A}+a}{b}$, they will necessarily be so in all the others to infinity.

The value we have just found for D', may also be put under the form $D' = D^0 + \mu(I - I')$; hence, from the two consecutive complete quotients

$$\frac{\sqrt{A}+I^0}{D^0} = \mu^0 +$$

$$\frac{\sqrt{A}+I}{D} = \mu +$$

we deduce the following complete quotient $\frac{\sqrt{A}+I'}{D'}$, by means of the formulas $I' = \mu D - I$, $D' = D^0 + \mu(I - I')$; which reduces the law of continuation to the greatest degree of simplicity.

Suppose, now, that $\frac{p^0}{q^0}$, $\frac{p}{q}$ be two consecutive fractions converging towards $\sqrt{A}$; let $\frac{\sqrt{A}+I}{D}$ be the complete quotient which answers to the fraction $\frac{p}{q}$ we shall have, according to the known principle,

$$\sqrt{A} = \frac{p\left(\frac{\sqrt{A}+I}{D}\right) + p^0}{q\left(\frac{\sqrt{A}+I}{D}\right) + q^0} = \frac{p\sqrt{A} + pI + p^0 D}{q\sqrt{A} + qI + q^0 D},$$

whence we deduce the two equations

$$pI + p^0 D = qA$$

$$qI + q^0 D = p,$$

which give

$$(pq^0 - p^0 q)\,I = qq^0 A - pp^0$$
$$(pq^0 - p^0 q)\,D = pp - Aqq.$$

Now, by virtue of the property of the continued fractions (Art. 87), we have $pq^0 - p^0 q = +1$, if $\frac{p}{q}$ be $> \sqrt{A}$, and $pq^0 - p^0 q = -1$, if $\frac{p}{q}$ be $< \sqrt{A}$; whence we perceive that $pq^0 - p^0 q$ has always the same sign as $pp - A\,qq$, and that thus D is always positive. These values also prove immediately that D and I are always whole numbers; I say, moreover, that I is always positive; for, on the one hand, the equation $qI + q^0 D = p$ gives $\frac{q^0}{q} = (\frac{p}{q} - I) \div D$, and since q^0 is $< q$, we must have $D > \frac{p}{q} - I$, where $D > \sqrt{A} - I$; on the other hand, we have $\frac{\sqrt{A} + I}{D} > \mu$, therefore $D < \sqrt{A} + 1$, which two conditions would be incompatible, if I were negative.

This being established, it is easy to find limits which the numbers I and D cannot surpass; the equation $A - I^2 = DD^0$ gives $I < \sqrt{A}$; hence I cannot exceed the whole number a contained in $\sqrt{A}$; and, since we have also $I' + I = \mu\,D$, it follows that $2a$ is the limit of D, and at the same time that of the quotient μ.

But since the continued fraction which represents the value of an irrational quantity must extend to infinity, and as there can only be a certain number of different values as well for I as for D, the same value of I must be found an infinity of times, with the same value of D; now, when we again find for the complete quotient $\frac{\sqrt{A} + I}{D}$, a value already found, it is obvious that the following quotients of the continued fraction must be the same, and in the same order as those which we have already obtained; therefore, the continued fraction which expresses

$\sqrt{A}$ will be composed (at least after a certain number of terms) of a constant period, which will be repeated to infinity, as we have already seen in a particular case (Art. 89.)

We have now to determine the precise point where the period commences. We will suppose that this period is $\mu, \mu', \mu'' \ldots \omega$, and we will represent as usual the series of quotients, and that of the converging fractions which correspond to them up to the commencement of the second period, as follows:

Quotients, $a, \alpha, \beta, \gamma \ldots \lambda, \mu, \mu', \mu'' \ldots \omega, \mu, \mu', \mu'' \ldots \omega$, &c.

converg. frac., $\frac{1}{0}, \frac{a}{1} \ldots\ldots \frac{p^0}{q^0}, \frac{p}{q} \ldots\ldots \frac{p_1{}^0}{q_1{}^0}, \frac{p_1}{q_1} \ldots\ldots$

Let also the corresponding values of the complete quotient be

$$\frac{\sqrt{A}}{1}, \frac{\sqrt{A}+a}{b} \ldots \frac{\sqrt{A}+I^0}{D^0}, \frac{\sqrt{A}+I}{D} \ldots \frac{\sqrt{A}+I_1{}^0}{D}, \frac{\sqrt{A}+I}{D} \ldots$$

we shall have, from what has been demonstrated, $A - I^2 = DD_1{}^0$, and and $A - I^2 = DD_1{}^0$, which gives $D_1{}^0 = D^0$; we shall also have $I = \lambda D^0 - I^0$, and $I = \omega D_1{}^0 - I_1{}^0$, whence we deduce $\frac{I^0 - I_1{}^0}{D^0} = \lambda - \omega$. But on the other hand, the equation $q I + q^0 D = p$,

gives $I = \frac{p}{q} - \frac{q^0 D}{q}$; and since $\frac{p}{q}$ is an approximate value of $\sqrt{A}$,

we must have $\frac{p}{q} = a$ + a fraction $\frac{r}{q}$, whence results

$$a - I = \frac{q^0 D - r}{q};$$

therefore by reason of $q^0 < q$, we shall have $a - I < D$; we shall have, in like manner, $a - I^0 < D^0$, $a - I_1{}^0 < D_1{}^0$; much more, therefore, $I^0 - I_1{}^0 < D^0$. But we have found $\frac{I^0 - I_1{}^0}{D^0}$ = the whole number, $\lambda - \omega$; therefore, this whole number must be zero; therefore, we shall have $I^0 = I_1^0$, and $\lambda = \omega$.

We may demonstrate, in like manner, that the quotient which precedes λ is equal to that which precedes ω, and so on to the quotient α; so that the quotient α is that which first returns, and which must commence the period.

This being established, we may thus represent the series of the quotients, and that of the converging fractions, which correspond to them in the development of $\sqrt{A}$, viz.

quotients, a; $\alpha, \beta, \ldots \lambda, \mu$; $\alpha, \beta, \ldots \lambda, \mu$; $\alpha, \beta, \ldots \lambda, \mu$, &c.

conver. frac. $\frac{1}{0}, \quad \frac{a}{1}, \ldots \frac{p^0}{q^0}, \quad \frac{p}{q}, \quad \frac{p'}{q'}, \ldots \frac{p^0_1}{q^0_1}, \quad \frac{p_1}{q_1}, \quad \frac{p'_1}{q'_1}, \ldots$

In this disposition $\frac{p}{q}$ is the converging fraction which answers to the last quotient, μ, of the first period, $\alpha, \beta \ldots \lambda, \mu$; let z be the complete quotient corresponding, we shall have $z - \mu = \sqrt{A} - a$, or $z = \mu - a + \sqrt{A}$, and there will result, according to the general principle,

$$\sqrt{A} = \frac{pz + p^0}{qz + q^0} = \frac{p\sqrt{A} + p(\mu - a) + p^0}{q\sqrt{A} + q(\mu - a) + q^0};$$

which furnishes the two equations,

$$p(\mu - a) + p^0 = Aq$$

$$q(\mu - a) + q^0 = p.$$

The second equation gives $\mu - a + \frac{q^0}{q} = \frac{p}{q}$, whence it follows that $\mu - a$ is the greatest whole number contained in $\frac{p}{q}$; this whole number is equal to a, thus we have $\mu - a = a$, or $\mu = 2a$. At the same time, since $q^0 = p - aq$, it follows that the series of quotients, $\alpha, \beta \ldots \theta, \lambda$, which precede μ is, symmetrical, that is, the same whether taken in direct or inverse order, for, $\frac{p - aq}{q}$ is one of the fractions converging towards $\sqrt{A} - a$, a quantity equal to the continued fraction $\frac{1}{\alpha + \frac{1}{\beta + \&c.}}$, and this converging fraction is preceded by $\frac{p^0 - aq^0}{q^0}$; therefore, since we have $q^0 = p - aq$, the period $\alpha, \beta, \ldots . \theta, \lambda$, must be identical with its inverse $\lambda, \theta, \ldots . \beta, \alpha$. And, from all these observations, it follows that the quotients, resulting from the development of $\sqrt{A}$, proceed according to this law.

$$a;\ \alpha, \beta, \gamma, \ldots . \gamma, \beta, \alpha, 2a;\ \alpha, \beta, \gamma, \ldots . \gamma, \beta, \alpha, 2a, \&c.$$

a law which would become still more regular if the first quotient were $2a$ or zero; that is, if we were considering the development of $\sqrt{A} \pm a$. *Legendre Théorie des Nombres*, page 43—47.

FINIS.

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