ON THE

TWELFTH AXIOM

OF

THE FIRST BOOK OF EUCLID.

By LORD MAHON.

[*READ AT THE ROYAL SOCIETY, FEBRUARY* 11, 1830.]

It is observed by Dr. Simson, in the notes to his valuable edition of Euclid, that the proposition commonly called the Twelfth Axiom of the First Book, 'has given a great deal 'to do, both to ancient and modern geometers, and seems 'not to be properly placed among the axioms, as, indeed, 'it is not self-evident.' The axiom states that 'if a straight 'line meet two straight lines, so as to make the two interior 'angles on the same side of it, taken together, less than two 'right angles, these straight lines, being continually pro- 'duced, shall at length meet upon that side on which are 'the angles which are less than two right angles.' This proposition is so far from being self-evident, like the eleven axioms which precede it, that some geometers have termed it the Fourth Postulate, thereby impairing one of the principal beauties of their study, which is remarkable as rising from the clearest and plainest foundations, and which is indebted only to its own demonstrations for materials.

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Dr. Simson proceeds to prove the truth of this axiom, and for that purpose makes use, as he is undoubtedly entitled to do, of the propositions of the First Book till the Twenty-ninth, when the use of the axiom is required for the first time. His demonstration is quite satisfactory, but it extends over no less than two definitions, one axiom, and five propositions; and as shortness and simplicity are of peculiar value in geometry, I have endeavoured to afford the same proof in a single proposition.

I shall first transcribe the demonstration of Dr. Simson, and then offer my own, which, I venture to hope, may, within a much smaller compass, be found equally conclusive.

Dr. SIMSON'S DEMONSTRATION.

Definition I.

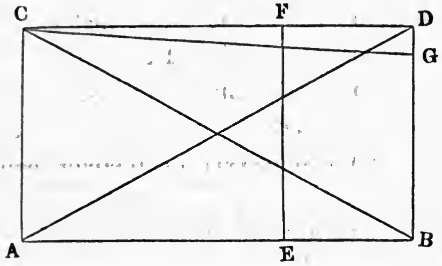
'The distance of a point from a straight line is the per-'pendicular drawn to it from the point.

Definition II.

'One straight line is said to go nearer to, or farther from 'another straight line, when the distance of the points of 'the first from the other straight line become less or greater 'than they were; and two straight lines are said to keep 'the same distance from one another, when the distance of 'the points of one of them from the other is always the 'same.

Axiom.

'A straight line cannot first come nearer to another 'straight line, and then go further from it, before it cuts 'it, and in like manner a straight line cannot go further 'from another straight line, and then come nearer to it;



' or BD; wherefore, as has been just now shown, the angle ' ACF is equal to the angle EFC. In the same manner the ' angle BDF is equal to the angle EFD; but the angles ' ACD, BDC are equal—therefore the angles EFC and ' EFD are equal and right angles*; wherefore also the ' angles ACD, BDC are right angles.

' COROLLARY.—Hence, if two straight lines AB, CD, ' be at right angles to the same straight line AC, and if ' betwixt them a straight line BD be drawn at right angles ' to either of them as to AB, then BD is equal to AC, and ' BDC is a right angle.

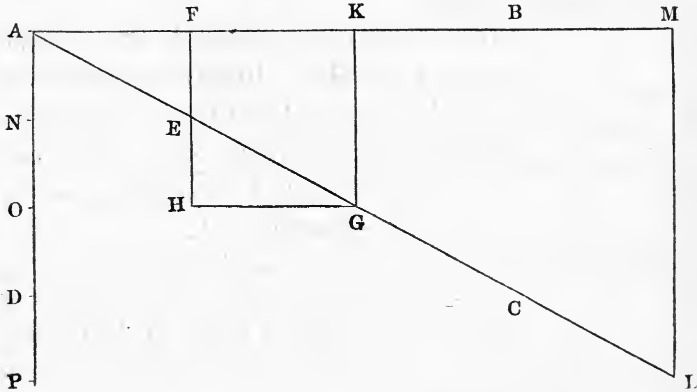
' If AC be not equal to BD, take BG equal to AC and ' join CG; therefore, by this Proposition, the angle ACG ' is a right angle, but ACD is also a right angle, wherefore ' the angles ACD, ACG, are equal to one another, which ' is impossible; therefore, BD is equal to AC, and by this ' Proposition BDC is a right angle.

PROPOSITION III.

' If two straight lines which contain an angle be produced, ' there may be found in either of them a point from which ' the perpendicular drawn to the other shall be greater than ' any given straight line.

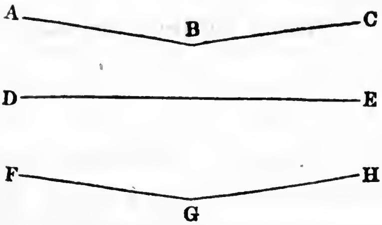
* Def. 10.

‘In AC take any point E, and draw EF perpendicular ‘to AB, and produce AE to G, so that EG be equal to ‘AE, and produce FE to H, and make EH equal to FE, ‘and join HG; because in the triangles AEF, GEH, AE, ‘EF, are equal to GE, EH, each to each, and contain ‘equal * angles; the angle GHE is therefore equal † to the ‘angle AFE, which is a right angle. Draw GK perpendi- ‘cular to AB, and because the straight lines FK, HG are



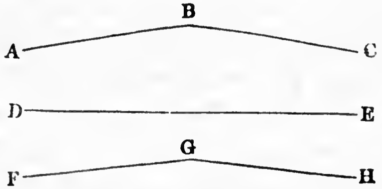
‘at right angles to FH, and KG at right angles to FK, ‘KG is equal to FH by Corol. Prop. II.; that is, to the ‘double of FE. In the same manner, if AG be produced ‘to L, so that GL be equal to AG, and LM be drawn per- ‘pendicular to AB, then LM is double of GK, and so on. ‘In AD take AN equal to FE, and AO equal to KG; ‘that is, to the double of FE, or AN; also take AP equal ‘to LM; that is, to the double of KG or AO, and let this ‘be done till the straight line taken be greater than AD. ‘Let this straight line so taken be AP, and because AP is ‘equal to LM, therefore LM is greater than AD.—Q.E.F.

* Eucl. Pr. 15. † Ib. Pr. 4.



'nor can a straight line keep the same distance from another 'straight line, and then come nearer to it or go further 'from it, for a straight line keeps always the same direction.

'For example, the straight line ABC cannot first come 'nearer to the straight line DE, as from the point A to the

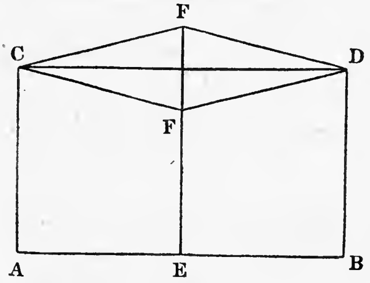


'point B, and then from the point B to the point C go 'further from the same DE; and in like manner the straight 'line FGH cannot go further from DE as from F to G, 'and then from G to H come nearer to the same DE; and 'so in the last case as in fig. 2.

Proposition I.

'If two equal straight lines, AC, BD, be each at right 'angles to the same straight line AB; if the points C, D 'be joined by the straight line CD, the straight line EF 'drawn from any point E in AB unto CD at right angles 'to AB shall be equal to AC or BD.

'If EF be not equal to AC, one of them must be greater 'than the other; let AC be the greater; then, because FE 'is less than CA, the straight line CFD is nearer to the 'straight line AB at the point F than at the point C, that 'is, CF comes nearer to AB from the point C to F. But



'because DB is greater than FE, the straight line CFD is 'further from AB at the point D than at F; that is, FD 'goes further from AB from F to D. Therefore the 'straight line CFD first comes nearer to the straight line 'AB, and then goes further from it before it cuts it, which 'is impossible; and the same thing will follow if FE be 'said to be greater than CA or DB. Therefore FE is not 'unequal to AC; that is, it is equal to it.

Proposition II.

'If two equal straight lines, AC, BD, be each at right 'angles to the same straight line AB, the straight line CD, 'which joins their extremities, makes right angles with AC 'and BD.

'Join AD, BC, and because in the triangles CAB, DBA, 'CA, AB are equal to DB, BA and the angle CAB equal 'to the angle DBA, the base BC is equal * to the base AD. 'And in the triangles ACD, BDC, AC, CD, are equal 'to BD, DC, and the base AD is equal to the base BC; 'therefore, the angle ACD is equal † to the angle BDC. 'From any point E in AB draw EF unto CD at right 'angles to AB; therefore by Prop. I., EF is equal to AC

* Eucl. Pr. 4. † Ib. Pr. 8.

'dicular to CF, meeting AB in L. And because AB, CF, 'contain equal angles with AC on the same side of it, by 'Prop. IV., AB and CF are at right angles to the straight 'line MNO, which bisects AC in N, and is perpendicular 'to CF. Therefore, by Coroll. Prop. II., CG and KL, 'which are at right angles to CF, are equal to one another; 'and HK is greater than CG, and, therefore, is greater 'than KL, and, consequently, the point H is in KL pro- 'duced. Wherefore the straight line CDH, drawn between 'the points C, H, which are on contrary sides of AL, must 'necessarily cut the straight line AB.'

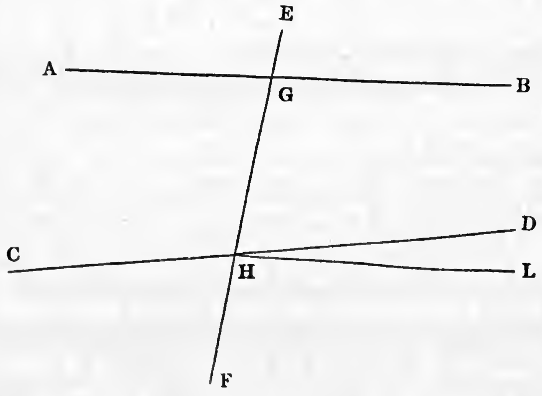
My DEMONSTRATION.

AXIOM TO BE PROVED.

If a straight line meet two straight lines, so as to make the two interior angles on the same side, taken together, less than two right angles, these straight lines, being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

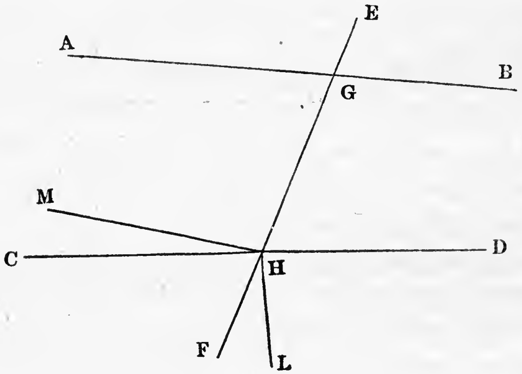
If the lines do not meet on that side, they must either be parallel or meet on the other side.

First let them, if possible, be parallel, that is, let AB and



C D be parallel, and let EF intersect them in the points G, H. Make * the angle FHL equal to the angle BGH, then the angles FHL, DHL, and DHG, are together equal † to two right angles. Therefore also the angles BGH, DHL, and DHG, are equal to two right angles, and therefore HL is ‡ parallel to AB. But two straight lines cannot have a common segment §, therefore the line HD, which in H touches HL, and then does not coincide with it, cannot also be parallel to AB.—Q. E. D.

Secondly, let the lines, if possible, meet on the other side.



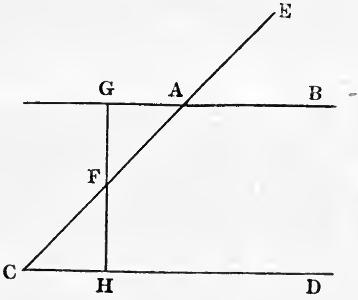
Make ‖ the angle CHL equal to the angle AGH, and then the two angles AGH, GHC, are equal to the two CHL, GHC. And because the angles AGH, CHG, HGB, and GHD, are together equal ¶ to four right angles, and as by the hypothesis BGH, GHD, are less than two right angles, the two others, or what has been proved the same, the angles GHC, CHL, are greater than two right angles, and the line HL will fall beyond HF. Make the angle CHM equal to FHL; then because the two interior angles on the same side, AGH,GHM, are equal to two right angles, the

* Eucl. Pr. 23. † Ib. Pr. 13. ‡ Ib. Pr. 28.
§ Coroll. Pr. 11. ‖ Ib. Pr. 23. ¶ Ib. Pr. 13.

Proposition IV.

'If two straight lines, AB, CD, make equal angles 'EAB, ECD, with another straight line EAC towards the 'same parts of it, AB and CD are at right angles to some 'straight line.

'Bisect AC in F, and draw FG perpendicular to AB; 'take CH in the straight line CD equal to AG, and on 'the contrary side of AC to that on which AG is, and join 'FH. Therefore, in the triangles AFG, CFH, the sides



'FA, AG are equal to FC, CH, each to each, and the 'angle FAG, that * is, EAB, is equal to the angle FCH; 'wherefore the angle AGF is † equal to CHF and AFG to 'the angle CFH. To these last add the common angle AFH, 'therefore the two angles AFG, AFH, are equal to the 'two angles CFH, HFA, which two last are equal together 'to two right angles ‡, therefore also AFG, AFH are equal 'to two right angles, and consequently § GF and FH are 'in one straight line. And because AGF is a right angle, 'CHF, which is equal to it, is also a right angle. There-'fore the straight lines AB, CD, are at right angles to 'GH.

* Eucl. Pr. 15. † Ib. Pr. 4. ‡ Ib. Pr. 13. § Ib. Pr. 14.

ON THE CONNEXION

BETWEEN

THE QUADRATURE OF THE CIRCLE

AND THE

GEOMETRICAL TRISECTION OF THE ANGLE.

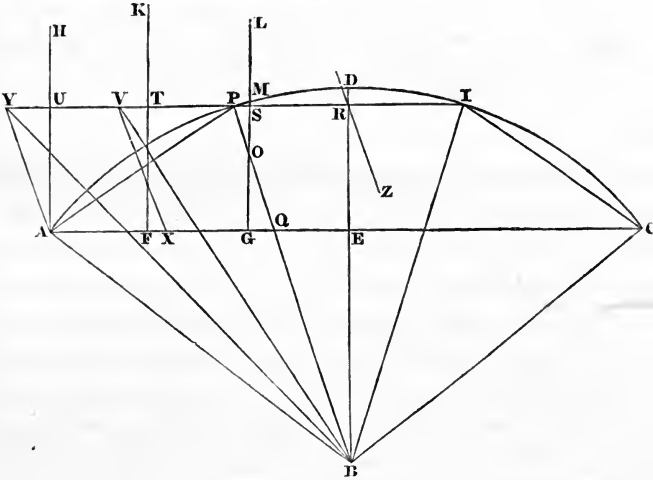
BY LORD MAHON.

READ AT THE ROYAL SOCIETY, FEB. 23, 1832.

THE trisection of an angle is a problem which has employed and eluded the ingenuity of mathematicians for the last two thousand years. Unable to solve it geometrically, the ancients had recourse to conic sections, and several demonstrations of it in this manner are given by Pappus in the fourth book of his Mathematical Collections. In modern times, the angle has been trisected by cubic equations, which may be found in Dr. Hutton, the Marquis de l'Hospital, and several other writers; but the geometrical trisection has still remained unaccomplished. In like manner the quadrature of the circle has also for the same period been attempted in vain. My object in this paper is only to show that a connexion exists between these two propositions, and that if the quadrature of the circle were once discovered, the geometrical trisection of the angle might be deduced from it as follows :—

PROBLEM.

To trisect a given angle, the Quadrature of the Circl being assumed as geometrically proved.



Let ABC be the given angle. Make AB equal to BC and describe the circle ADC. Bisect the circumferenc ADC at D and join BD and AC. Let AC meet BD in E Trisect the line AE in F and G, and through A, F, and G draw AH, FK, and GL parallel to BD. Let GL meet th circumference in M. Then, through some point in the lin GM it is required to draw a radius from B, so that the par intersected between AE and GM may be double of the pa intersected between GM and the circumference.

It is obvious, in the first place, that such a line exists; fo in proportion as any radius intersecting the line GM approx imates to the point G, the part intersected between GM an AE diminishes, till at G it ceases altogether; while the par intersected between GM and the circumference proportion ably increases. On the other hand, in proportion as any radius intersecting the line GM approximates to the point M, the part intersected between GM and the circumference continually diminishes, till at M it ceases altogether, and the part intersected between GM and AE proportionably increases. Thus, as they mutually become evanescent in proportion as the radius approaches the opposite extremities of GM, it is evident that there is some point in that line, through which a radius being drawn, the part between GM and GE may be double of that between GM and the circumference. Now, if the Quadrature of the Circle were within the limits of a problem, then the geometrical placing of this point, required in the line GM, might be deduced from it as one of its corollaries, since we should then have no greater difficulty in dealing with the curve of the circle ADC, than with the straight lines GM or GE. If I assume, therefore, the Quadrature of the Circle to be geometrically proved, I may also, I conceive, assume that I can draw the radius POB in the manner I have mentioned.

Let, then, OQ be double of OP in the radius POB, the point Q being in the line AE. If, by the same process, the radius BI be drawn in the semi-angle DBC, then, by these two radii, the given angle ABC is trisected.

Through P draw PR parallel to AC, meeting GL in S and DB in R, and produce PR from P, meeting FK in T and AH in U. Now because POS and GOQ are vertical angles, and because the bases PS and GQ are parallel, the triangles PSO and GQO are similar. But OQ is double of OP, therefore GO is also double of SO, and the base GQ double of the base PS. Make VT equal to GQ and FX to PS, and make UY equal to GQ and PS together, or to three times PS, and join VX and YA. Now, because FX is equal to half GQ, the line XQ is equal to the lines FG and PS. In like manner, because VT is double of PS, the line VP, in which VT is added to, and PS sub tracted from, TS, is equal to the lines TS and PS. There fore, since TS is equal to FG, the line VP is equal to th line XQ, and they are parallel; therefore, the lines VX an PQ which join them are also parallel. But because YU i equal to PS and VT, the line YV, in which YU is added and VT subtracted from UT, is equal to the lines UT an PS. Therefore, the line YV is equal to the line AX (sinc AF is equal to UT, and FX to PS) and YV and AX ar parallel, therefore YA and VX are also parallel. Throug R draw RZ parallel to PQ, then the four lines RZ, PQ VX, and YA are parallel to one another. But they ar also equidistant, because the distance from each to each a measured by the same line YR, amount in each case to PS together with one of the trisections of AE, as has bee already shown.

Join VB, YB, and PA; then the triangle VBP is equal t the triangle PBR, being on equal bases and between the sam parallels. For the same reason the triangle YBV is equal t the triangle VBP, and the whole YBP is double of the triangl PBR. But the triangle APB is equal to the triangle YBI as being between the same parallels and on the same bas Therefore the triangle APB is double the triangle PBI Thus also in the other semi-angle DBC, it may be shown tha the triangle IBC is double of the triangle IBR, and the tri angle IBR equal to the triangle PBR, and therefore the tr angle APB is equal to the whole triangle PBI, and also t the triangle IBC; that is, the three triangles APB, PBI and IBC are equal to one another. But they must also b similar, their sides being radii of the same circle, therefor the angles ABP, PBI, and IBC are equal to one anothe and the given angle ABC is trisected. Q. E. F.

LONDON:
Printed by William Clowes,
Stamford-Street.