Faculty Working Papers

ON THE USE OF COBB-DOUGLAS SPLINES

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1. Introduction

Since the legendary work of Cobb and Douglas [1], Cobb-Douglas production functions and (to a somewhat lesser degree) Cobb-Douglas utility functions have been popular tools of economists. This popularity can be attributed both to the simplicity and to the wide-applicability of these functions. However, these functions are of course subject to rather severe restrictions. For example, in the production function context returns to scale are non-varying, hence, U-shaped average cost curves are ruled out. Also, homotheticity and unitary elasticities of substitution are required.

This study develops the idea of continuous piecewise Cobb-Douglas functions along the spline function lines discussed in Poirier [10] - [12]. This development will permit U-shaped average cost curves and "piecewise-homotheticity" at the expense of differentiability of the functions along lines parallel to the input axes. However, the unitary elasticities of substitution requirements will remain.

Implicit in this discussion is the belief expressed in Poirier [10] - [12] that for a wide range of problems in economics, the added generality of more complicated functional forms is often best achieved by using continuous piecewise functions. Briefly, the rationale is two-fold. First,

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[1]The author is an Assistant Professor of Economics at the University of Illinois at Urbana-Champaign. The contents of this study rely heavily on Poirier [10, Chapter 4]. Thanks are due Joseph Hotz, Steven Garber, William Greene, and Diane Christensen of the University of Wisconsin at Madison for their data handling and programming assistance which contributed greatly to section 5. Of course any errors are the sole responsibility of the author.

since within each "piece" such functions have simple and familiar forms, the analysis proceeds quite straightforwardly. Indeed all economists are familiar with Cobb-Douglas functions and their properties, and so this previous knowledge can be easily applied in analyzing piecewise Cobb-Douglas functions. Second, changes in the behavior of such functions as one passes from one piece to another are often of primary concern in economic analysis (e.g., the changes in expansion paths discussed in Section 3). Indeed the testability for the existence of such "structural changes" will be an important consideration throughout this discussion.

Organizationally, we will proceed as follows. Section 2 defines a Cobb-Douglas spline. For the sake of simplicity, but not at the expense of generality, a production function context with two inputs, labor and capital, will be used. Section 3 discusses the properties of Cobb-Doublas spline production functions, leaving to section 4 a discussion of the properties of Cobb-Douglas spline utility functions. Finally, section 5 contains an empirical application of a CDS production function.

2. Definition of a Cobb-Douglas Spline

Let the sets $\Delta_L = \{\bar{L}_1 < \bar{L}_2 < \ldots < \bar{L}_{I-1}\}$ and $\Delta_K = \{\bar{K}_1 < \bar{K}_2 < \ldots < \bar{K}_{J-1}\}$ be meshes defining intervals in the labor (L) and capital (K) dimensions. The elements in Δ_L and Δ_K are called knots and they define a rectangular grid in the positive quadrant that consists of IJ rectangles (see Figure 1). A Cobb-Douglas spline (CDS) is a function Q(L,K) which can be defined as

$$(1) \qquad Q(L,K) = \theta_{ij} L^{\alpha_i} K^{\delta_j},$$

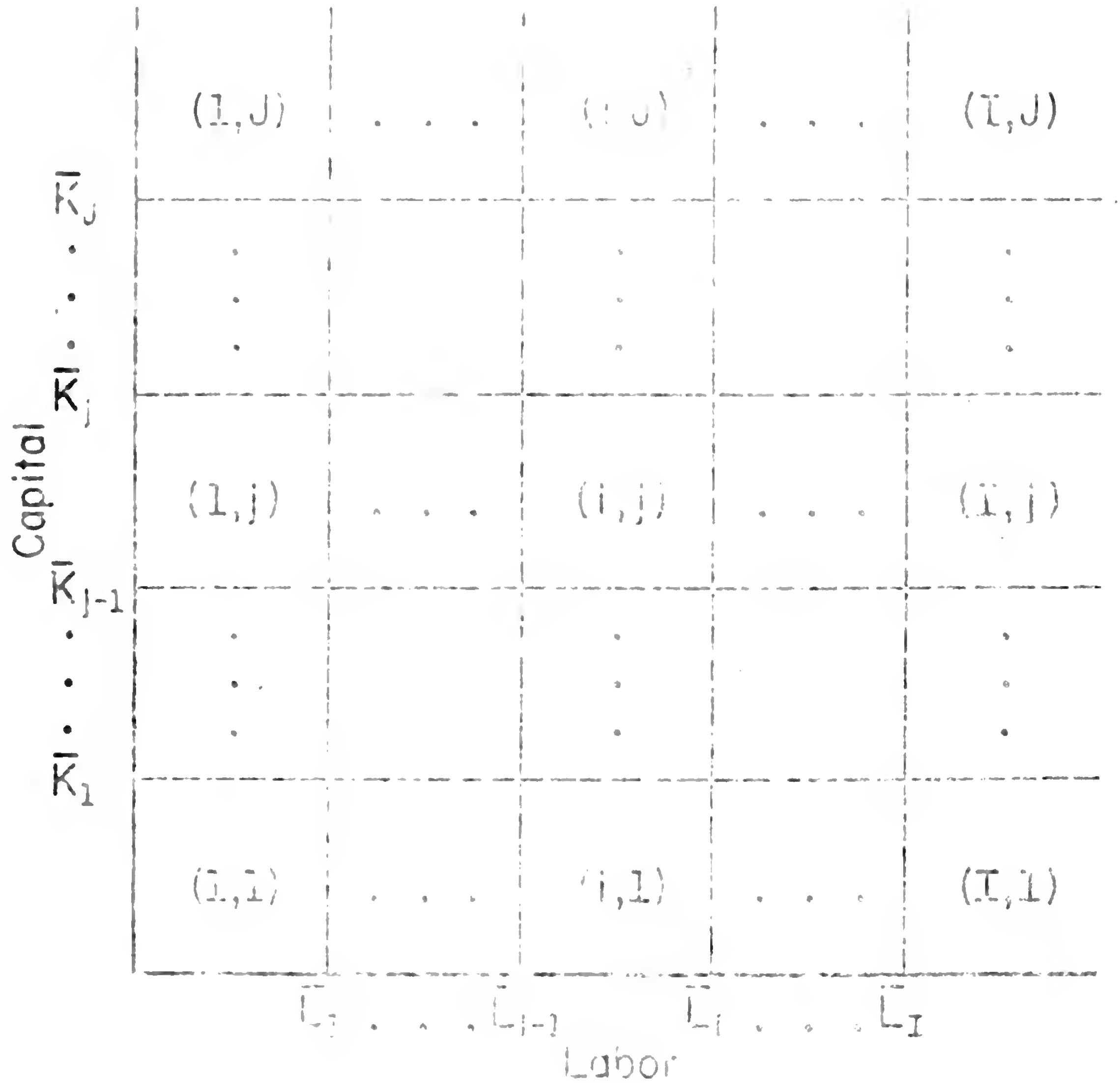

Figure 1: Labor-Capital Grid

where L and K are in rectangle (i,j), the θ_{ij}'s, α_i's, and δ_j's are positive constants, and the θ_{ij}'s are chosen so as to make Q(L,K) continuous over the positive quadrant. Given the α_i's and the δ_j's, this continuity requirement implies the continuity conditions

(2) $\ln \theta_{(i+1)j} = \ln \theta_{ij} + (\alpha_i - \alpha_{i+1}) \ln \bar{L}_i \quad i=1,2,\ldots,I-1$

for all j, and

(3) $\ln \theta_{i(j+1)} = \ln \theta_{ij} + (\delta_j - \delta_{j+1}) \ln \bar{K}_j \quad j=1,2,\ldots,J-1$

for all i. In this context the output elasticities of labor (the α_i's) and capital (the δ_j's) are step functions over the meshes Δ_L and Δ_K, respectively.

Often it is more convenient to work with a CDS in its logarithmic form

(4) $\ln Q = \ln \theta_{ij} + \alpha_i \ln L + \delta_j \ln K$

for L and K in rectangle (i,j). In formulation (4) ln Q is the sum of two linear splines (see Poirier [10, Chapter 2]), one in the ln L dimension and one in the ln K dimension. In terms of Poirier [12], (4) is also a bilinear spline with no interaction terms.[2] Geometrically, (4) defines a piecewise-planar or "roof-like" surface.

Alternatively, continuity conditions (2) and (3) can be incorporated into representation (4) in the following manner. Define the variables

$$\tilde{L}_i = \max\{(\ln L - \ln \bar{L}_i), 0\} \quad i = 1, 2, \ldots, I-1$$

$$\tilde{K}_j = \max\{(\ln K - \ln \bar{K}_j), 0\} \quad j = 1, 2, \ldots, J-1$$

Then for all L and K,

(5) $\ln Q = \ln \theta_{11} + \alpha_1 \ln L + \delta_1 \ln K + \sum_{i=1}^{I-1} \tilde{\alpha}_i \tilde{L}_i + \sum_{j=1}^{J-1} \tilde{\delta}_j \tilde{K}_j$

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[2] For similar comparisons in terms of grafted polynomials, see Fuller [illegible] and Gallant and Fuller [3].

where the parameters α_i ($i=2, \ldots, I$) and β_j ($j=2, \ldots, J$) represent changes in the [illegible]. Representation [illegible] production problems in which these changes are of particular interest. For more details see [illegible] [12].

In the following sections, wherever (5) is used in a production theory context, it will be called a CDS production function, and when it is used in a utility theory context, it will be called a CDS utility function.

3. CDS Production Functions

Of foremost importance in discussing the theoretical properties of the CDS production function are its isoquants. For a fixed output level Q_0, the isoquant over rectangle (i,j) is

$$K = \left[\frac{Q_0 L^{-\alpha_i}}{\theta_{ij}} \right]^{1/\beta_j} . \tag{6}$$

Throughout the labor-capital dimensions the isoquants are continuous, however, they have "corners" along the grid lines. This first result follows from the continuity of the production function, and the latter is a result of the non-differentiability of the production function along the grid lines. The isoquants are strictly convex iff

$$\alpha_i \geq \alpha_{i+1} \qquad i = 1, \ldots, I-1$$

and

$$\beta_j \geq \beta_{j+1} \qquad j = 1, 2, \ldots, J-1 .$$

In other words the isoquants are strictly convex iff each output elasticity is a decreasing step function of its respective input.

In terms of the returns to scale (α_i [illegible]) of rectangle (i,j), the convexity conditions imply that

$$\alpha_i + \beta_j \geq \alpha_m + \beta_n \qquad m=1,2,\ldots,i; \quad n=1,2,\ldots,j$$

and

$$\gamma_i + \delta_j \geq \gamma_m + \delta_n \qquad m=i,i+1,\ldots; \quad n=j,j+1,\ldots$$

As shown in Figure 2, this means that Q(L,K) exhibits greater returns to scale over all rectangles below and to the left of rectangle (i,j), and smaller returns to scale over all rectangles above and to the right of rectangle (i,j). Comparisons to rectangles above and to the left of rectangle (i,j), and below and to the right of rectangle (i,j) are inconclusive.

It is well known that Cobb-Douglas isoquants give rise to a straight line expansion path since the production function is homothetic. The Cobb-Douglas spline production function has straight line expansion path segments over each individual rectangle, however, these paths exhibit unique behavior along the grid lines.

To see this suppose the convexity conditions hold and consider the slope of isoquant (6) in rectangle (i,j) but not along its border.

$$\frac{dK}{dL} = -\frac{\alpha_i}{\beta_j}\frac{K}{L}$$

The marginal rate of [illegible] (MRS, dK/dL) [illegible] well-defined along the borders [illegible] there as a result of the jump [illegible] in output elasticities.

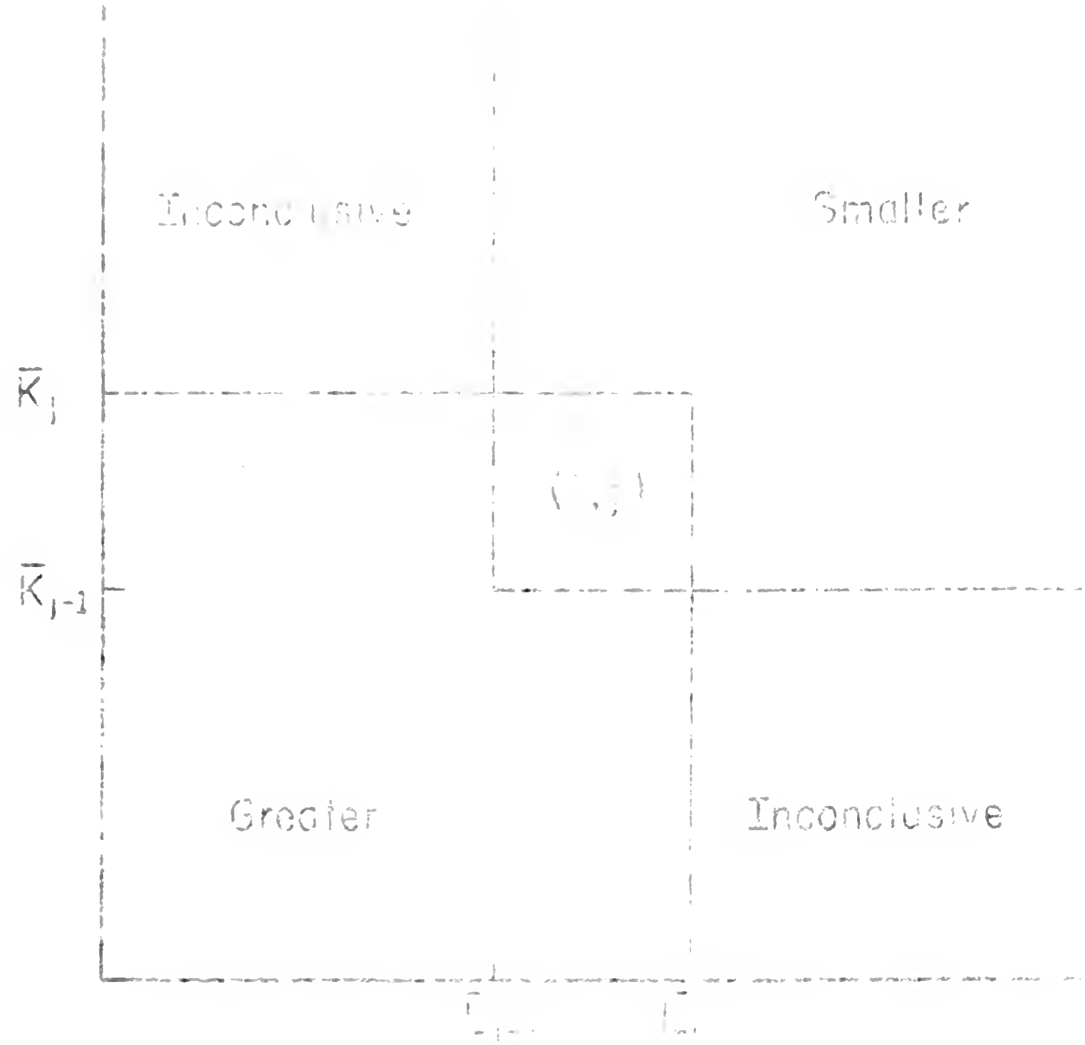

Figure 2: Return to Scale Comparisons

Considering rectangles (i,j) and (i+1,j) the one sided derivatives of the isoquant along $L = \bar{L}_i$ are

$$\frac{dK}{dL^+} = \lim_{L \to \bar{L}_i} \frac{dK}{dL} = - \frac{[illegible]}{\delta_j} \frac{K}{\bar{L}_i}$$

$$\frac{dK}{dL^-} = \lim_{L \to \bar{L}_i} \frac{dK}{dL} = - \frac{[illegible]}{\delta_j} \frac{K}{\bar{L}_i} .$$

Letting w denote the price of labor and r the price of capital, then for all price ratios w/r such that

(7) $$\left|\frac{dK}{dL^+}\right| < \left|\frac{w}{r}\right| < \left|\frac{dK}{dL^-}\right| ,$$

the first order conditions for the firm's output maximization or cost minimization problem are not satisfied, i.e., the marginal rate of technical substitution does not equal the price ratio. Hence, compensated price changes within the bounds of (7) do not change the optimal input combination for producing a fixed level of output.

Fixing w/r and expanding output, any expansion path in rectangle (i,j) approaching the grid line $L = \bar{L}_i$ does so at a capital level K* satisfying

$$\left|\frac{dK}{dL^-}\right| = [illegible]$$

or solving for K*,

(8) $$K^* = \frac{\delta_i \bar{L}_i}{\alpha_i} \left(\frac{w}{r}\right)$$

Since the output elasticity of labor over rectangle (i+1,j) is [illegible] than it is over rectangle (i,j), increases in output beyond $Q(\bar{L}_i, K^*)$ are met by increasing only the capital input until either the [illegible] equals the input price ratio or until the output elasticity of capital [illegible]

drops, [illegible] K^{**} [illegible]

[illegible]

or solving for K^{**},

$$K^{**} = \text{[illegible]}$$

then the expansion path [illegible] is a straight line [illegible] slope [illegible] of the expansion path [illegible] is illustrated in Figure 3(a.) for the simple case [illegible], $i = j = 1$.

However, if the grid line [illegible] is reached first, then the expansion path for outputs [illegible] $Q(\bar{L}_j, \bar{K}_i)$ continues along $L = \bar{L}_j$ [illegible]

(9) [illegible]

[illegible]

(10) [illegible]

[illegible]

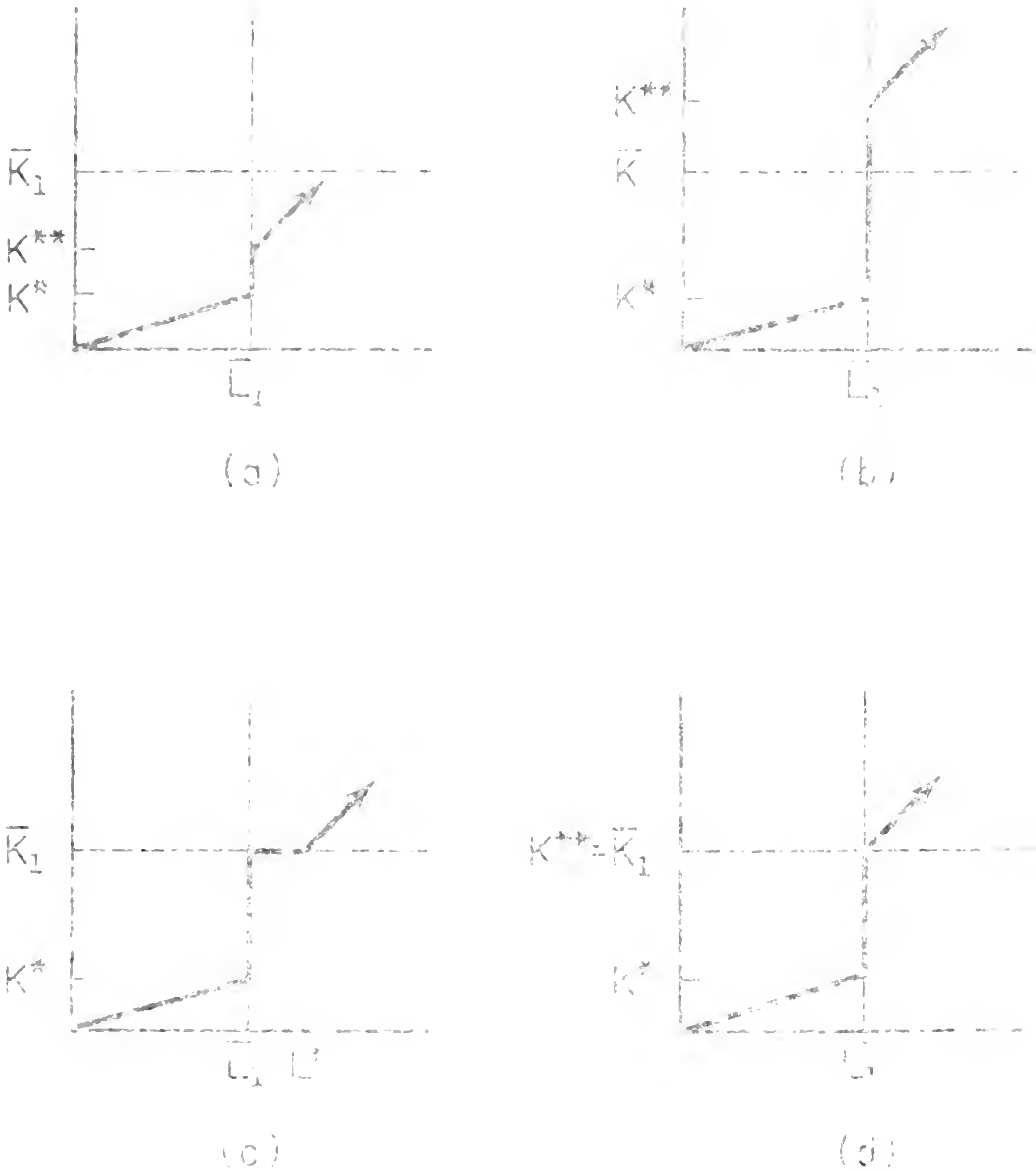

Figure 3 Expansion Paths

either [illegible]

[illegible]

illustr[illegible]

possibilities [illegible]

proceed along either [illegible]

[illegible] by [illegible]

[illegible]

If (1) holds, then the exp[illegible] rectangle [illegible]

side [illegible] from [illegible] as shown in Figure [illegible]

The [illegible] with [illegible] function [illegible]

[illegible]

and if it is produced [illegible] that the [illegible]

$$(14) \qquad [illegible] = w \left[\frac{[illegible]}{[illegible]} \right]^{[illegible]} [illegible] \bar{K}_i .$$

In the case [illegible] (12 [illegible] similar [illegible] as $Q \to Q(C^*, \bar{K}_i)$, (1 [illegible] continuous.

The average cost [illegible] associated with (12) – (14) is obtained by dividing each equation by (Q+θ). With regard to (12), the average cost function is increasing, constant, or decreasing, depending on whether Q is produced in a rectangle with decreasing, constant, or increasing returns to scale. Since the cost function is continuous, so is the average cost function.

The marginal cost function associated with (12) – (14) is obtained by differentiating each equation with respect to Q. At output levels corresponding to 'kinks' in the expansion path marginal cost is not defined since the left and right hand derivatives are not in general equal. Of course with regard to (12) the marginal cost segment is of the conventional type for a Cobb-Douglas production function.

4. CDS Utility Functions

[illegible] utility theory as well as in production [illegible] consider the CDS utility function

$$U = [illegible] \begin{cases} [illegible] \\ [illegible] \end{cases}$$

[illegible]

[illegible] him $60 a week [illegible] take [illegible] income [illegible] 50 per cent (t=.5) [illegible] the guarantee [illegible] earned income [illegible]. The plan has a "breakeven" [illegible] B (greater than $\bar{Y}$), corresponding to an earned income [illegible]

[illegible]

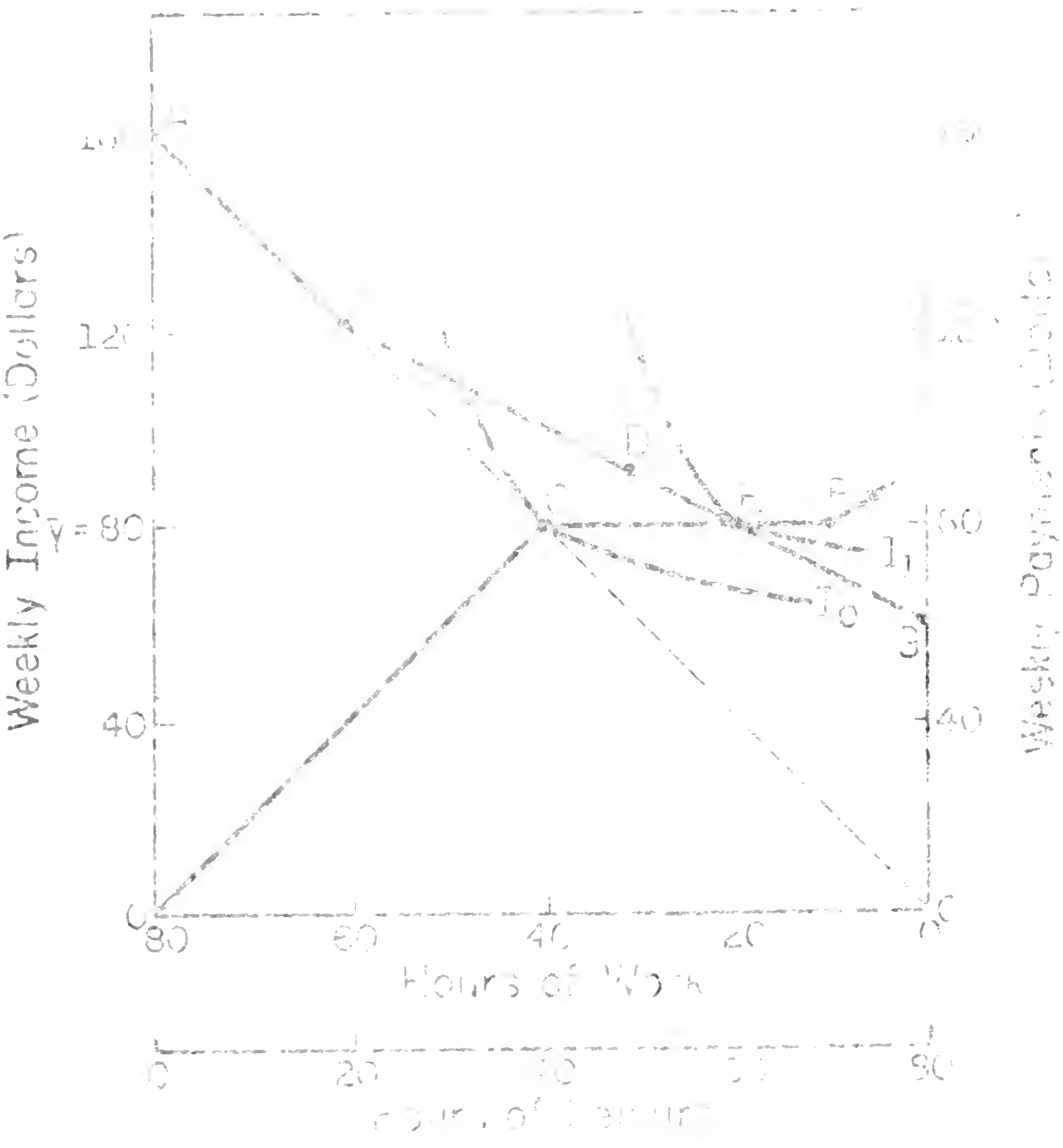

Figure 4 Response to a Negative Income Tax

where [illegible]

an [illegible]

in [illegible]

reaches [illegible]

derivative [illegible]

[illegible]

He then proceeds along an expansion path with slope [illegible] However, [illegible] before the point [illegible] is reached he encounters a budget constraint ABC [illegible] must then stop [illegible] point [illegible] on indifference curve I_1 which corresponds to no change in income with the net effect of the negative income [illegible] manifested in increased leisure time, a reduction in labor supply.)

[illegible] the explanation is as follows. The reduction of the effective wage as a result of the conditions that the price of leisure has relatively speaking dropped [illegible] more [illegible] [illegible]

[illegible]

[illegible]

[illegible] be[illegible] est[illegible] [illegible] utilities [illegible] study [illegible] [illegible] function [illegible] function, another equ[illegible] returns to [illegible]

In light of [illegible] developments [illegible] consider the question of whether there exist [illegible] to scale in the output of electrical [illegible] relevant range [illegible] in terms of public subsidies and investment policies [illegible] in order to [illegible] scale, [illegible] suggested that public utility firms [illegible] exogenous, [illegible] Cobb-Douglas [illegible] [illegible] source [illegible]

[illegible] produc[illegible]

[illegible]

[illegible] accuracy of [illegible] questions which [illegible] production [illegible] results presented [illegible] only [illegible] example.

where [illegible] output [illegible] and $P_{[illegible]}$ [illegible] the [illegible] expression [illegible] firms, and [illegible] and [illegible] are the [illegible] Unlike Nerlove it will be [illegible] equals [illegible] $F \leq \overline{F}_1$, or [illegible] if $F > \overline{F}_1$. [illegible] that $\overline{F}_1$ = 61?5.4 (millions of Btu's), is admittedly somewhat arbitrary [illegible] in more elaborate applications the choice of $\overline{F}_1$, [illegible] for that fact labor and capital knots, will reflect technological considerations of the production process. The choice here is no more arbitrary than the five group breakdown used by Nerlove and it corresponds to approximately the 33rd percentile of the firms' fuel input level.[5]

Following Nerlove [9, pp. 171-175] it can be showed that cost minimization subject to (15) and exogenous output and input prices implies the cost function

$$\text{(16)} \quad \ln C = [illegible]$$

where w_1 [illegible] is the [illegible] $P_{[illegible]}$ is the price of labor, [illegible] fuel, and

[illegible]

Proceeding [illegible] firms, (16) implies

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[5] [illegible]

[illegible]

[illegible]

Coefficient	Estimated Value	Estimated Standard Error
a	[illegible]	[illegible]
δ_1	.5905**	.1240
δ_2	.5591**	.1244
η	.01574	[illegible]294
[illegible]	1823.	3498.

"*" denotes significance at the 5% level.

"**" denotes significance at the 1% level.

$$(17) \qquad w_i \ln C = \beta_i + \ln Q + \alpha \ln P_L + \gamma \ln P_F - \ln \epsilon$$

where $\beta_i = w_i \ln \theta_i + \eta \ln P_K$. Note that in order for the disturbance term in (17) to be homoscedastic when returns to scale vary (i.e., $\delta_1 \neq \delta_2$), (16) was first multiplied through by w_i.

Under the assumption that the residuals in (17) are independent, homoscedastic, and normally distributed, the likelihood function corresponding to (17) can be maximized subject to the continuity constraint $\ln \theta_2 = \ln \theta_1 + (\delta_1 - \delta_2) \ln \overline{F}_1$. The resulting maximum likelihood estimators $\hat{\alpha}$, $\hat{\delta}_1$, $\hat{\delta}_2$, $\hat{\eta}$, and $\hat{\theta}_1$ will asymptotically be consistent, normally distributed, and efficient. Since the likelihood function is nonlinear in these parameters, it is necessary to employ a technique such as Marquardt's method of steepest descent to obtain the maximum.

The actual maximum likelihood estimators (together with their standard errors) are given in Table 1. The results are similiar to those of Nerlove: the output elasticities of labor and fuel highly significant, and the output elasticity of capital is insignificant (as it often was in Nerlove's study). Of course the result of principal concern is the estimated change $\hat{\delta}_1 - \hat{\delta}_2 =$ .03140 in the output elasticity of fuel at $\overline{F}_1$. The asymptotically normal test statistic for testing the significance of this change is

$$\frac{\hat{\delta}_1 - \hat{\delta}_2}{\left[\text{Est. Var}(\hat{\delta}_1) + \text{Est. Var}(\hat{\delta}_2) - 2\ \text{Est. Cov}(\hat{\delta}_1, \hat{\delta}_2)\right]^{1/2}} = 3.112$$

which is highly significant. Thus, while there are increasing returns to scale everywhere, they appear to undergo a significant drop along $\overline{F}_1$.

[illegible]

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