From the Author

ON

THE USE OF

EQUIVALENT NUMBERS

IN THE

METHOD OF LEAST SQUARES.

BY

GEORGE P. BOND, A.M.,

ASSISTANT AT THE OBSERVATORY OF HARVARD COLLEGE.

[From the Memoirs of the American Academy of Arts and Sciences, New Series, Vol. VI.]

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Jno. B. Houck.

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One of the most important applications which has been made of mathematics to investigations in physical science has for its object to ascertain the best manner of combining data affected by unknown errors of observation, so that the probable effect of these errors shall be the least possible. The method of least squares proposes to accomplish this, by reducing to a minimum value the sum of the squares of the outstanding errors, and, by conforming to this single criterion, to fulfil the condition, so desirable in the prosecution of thorough and exact research, of reducing to its least possible amount the influence of errors in the data employed.

The investigations here presented have been entered upon with the design of determining the degree of numerical exactness proper to be observed in making use of the method of least squares, in order to secure its peculiar advantages with the least outlay of labor.

Some detail in the discussion seems to be called for from the prevalence of a practice, almost universal among computers, of adhering to the letter of the method of least squares with a strictness which implies a misapprehension of its true spirit. It is impossible to adduce any valid reasons to justify such a course when it must be followed at a serious expense of time and labor in the computations.

It has not escaped the observation of Gauss, in his original exposition of the method, that some freedom of interpretation may be allowed when its theoretical results are applied in practice, as the following passage, referring to the solution of equations by least squares, will show:—

"When the number of functions or equations proposed for solution is considerable, the computations become laborious, the more so from the circumstance that the coefficients by which the primitive equations are to be multiplied are almost always complicated decimal fractions. If it is not thought worth the trouble in such a case to calculate the products with exactness by means of logarithms, it will generally be sufficient to substitute for them (i. e. for the multiplying factors) more simple numbers differing but slightly from them." *

In his subsequent researches, it does not appear that Gauss has given any further development to the suggestion here put forth. Indeed, the introduction of modifications of a like nature, however desirable in a practical point of view, would have deprived a purely theoretical discussion of much of its elegance and symmetry. Yet the passage above quoted lends the support of the highest authority to the leading proposition which we shall have occasion most frequently to insist upon; namely, the propriety of allowing some relaxation of theory in applying the calculus of probabilities to the discussion of data affected by ordinary errors of observation, whenever the modification conduces to convenience and the saving of labor at the sacrifice of no appreciable advantages.

Even an unqualified admission of the superior probability of results which exactly fulfil the criterion proposed in the method of least squares, does not relieve us from the necessity of restricting it to examples which never actually occur, that is, if the question be made a rigorous one; † — to such, for instance, as involve the discussion of observations which are entirely free from unknown constant errors, or errors following any law of facility which does not imply the assumption that the mean error is proportional to the square root of the mean of the squares of the individual errors. But we know that this proposition, which lies at the foundation of the whole subject, is not susceptible of absolute demonstration by any process of mathematical reasoning. Further than this, we know from constant experience that the law of distribution of errors recognized in the method of least squares practically fails, in extreme cases, both for very large and for very small errors. If any illustration of the failure of the assumed law be needed, it will be found in the familiar instance of computing by it the probable error of the arithmetical mean of a very large number of observations, where common sense assures us that the theoretical probable errors of the result are invariably smaller than they should be.

Why, then, should an implicit adherence to its minutest details be required as essen-

* Theoria Motûs, § 185.

† Theor. Comb., § 17.

tial to its successful application, or to the attainment of all the advantages which its employment may confer upon the discussion of any practical problem?

It is true that no other system can be proposed which is free from similar objections, or which can be mathematically demonstrated to be exclusively the best, without qualification, and therefore the arguments above stated are of no force whatever, if employed as reasons for the rejection of the method of least squares. They nevertheless greatly weaken the position of those who would insist upon a strict compliance with its precepts, and effectually preclude all arguments of a purely theoretical character in support of such a course. Still it is desirable that the force of any objections which may be made to an attempt to modify the theoretical conditions for effecting the most favorable combination of equations should be appreciated at their true value. We therefore propose to show that the spirit of the method of least squares, rightly apprehended, in reality rather invites than discountenances a liberal construction of its rules.

Admitting that the best possible solution is attained when the sum of the squares of the outstanding errors, represented by Ω, is a minimum, it is evident that Ω is a minimum relatively to the manner in which the original equations have been treated. And since the peculiarity of the solution consists in the employment of a system of factors, α, α', &c., by which the original equations are multiplied before combination, the first differential coefficient of Ω relatively to either of these factors, in the case of the least-square solution, must have the value for each factor,

$$\frac{d\,\Omega}{d\,\alpha} = 0.$$

When, therefore, the factors are varied by small amounts, $\delta\,\alpha$, $\delta\,\alpha'$, &c., the consequent variations of Ω developed in a series, will contain only terms multiplied by the second and higher powers of $\delta\,\alpha$; or, in general terms, *if we deviate from the exact*
(1.) *precepts of the method of least squares by small variations of the first order, we shall fail to satisfy its fundamental criterion by small terms of the second order only.*

Looking thus at the most elementary principle of the method, we find a warrant for some degree of liberty in applying it, — a liberty which we can scarcely hesitate to avail ourselves of, if we further consider the peculiar circumstances attending its actual employment in the discussion of data furnished directly by observation.

Among its first requirements is the assignment of weights to the original observations; but it is one which it is not possible to fulfil correctly, for we are provided neither with a theory nor with data for the purpose. All that can be done is to accept, as indices of the relative value of the different observations, certain numbers depending either proximately or remotely upon no other authority than the mere exercise of the judgment alone. No one can pretend that this is a process susceptible of strict accuracy; yet an error here is as fatal as if we had disregarded any other of the precepts of the method.

This step being an arbitrary one, although one of fundamental importance, we may properly appeal to it as a precedent for the modification of others suggested by considerations of convenience, though they may not, like this, be justified on the plea of actual necessity. In this view of the subject, we find support for the modification suggested by Gauss, in the passage we have quoted above. Each of the complicated factors which it is there proposed to simplify is itself a product of two other factors, one of which is the weight of the equation under treatment; if one of these, that is, the number representing the weight, is erroneous, the product is of course erroneous, with whatever accuracy the other is expressed.

Again, as a matter of convenience, it is usual to express the conditional equations proposed for solution in a linear form, by reducing the indeterminates to small quantities and neglecting the terms multiplied by their second and higher powers, and to construct from them normal equations, as they are called, previously to applying the method of least squares. Both of these may be practices perfectly allowable under the circumstances, but since they are almost always theoretically incorrect, their admission is a virtual relinquishment of all pretensions to a rigorous course of computation, and cannot be compensated for by any subsequent refinements.

We will now proceed to examine the limits of accuracy appropriate to the arithmetical operations required in the combination of conditional equations by the method of least squares, and afterwards to develop in detail some proposed modifications of that method, having for their object the reduction to its minimum value of the amount of labor requisite for its successful application.

It is scarcely necessary to remark, that the subject is plainly one which is in its nature somewhat vague and insusceptible of rigorous treatment, though it is at the same time interesting from its practical bearings. If no very precise or definite rules for regulating the degree of numerical exactness suited to the discussion of any given problem can be arrived at, it may still be of service to point out the principles which ought to guide the computer in the choice of such limits as shall perfectly meet all reasonable requirements of accuracy, without imposing upon him the unprofitable labor of multiplying the extent and difficulties of calculation, to no useful purpose, and without the remotest prospect of sensibly improving the real value of the results.

Let us suppose a series of equations,

$$a\,\mathrm{x} + b\,\mathrm{y} + \ldots\ldots\ldots + m = \mathrm{e},$$
$$a'\,\mathrm{x} + b'\,\mathrm{y} + \ldots\ldots\ldots + m' = \mathrm{e}',$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

in which m is the element derived from observation, and e the unknown error of the equation, to be solved by the method of least squares, giving for x the value $\tilde{x}$, with its probable error, ε, obtained from a comparison between the observed and the computed values of m, after substituting x, y, &c. in the primitive equations.

If x_0 be the true value of x, we may represent by $\varkappa$ a quantity such that it is an even chance whether $x_0 - x$ is comprised between the limits $\varepsilon + \varkappa$ and $\varepsilon - \varkappa$. The magnitude of the limit defined by $\varkappa$ has an evident relation to the question how far the simplification of the arithmetical processes may be carried without detriment to the results.

For instance, the solution of the above equations may be repeated with small variations from the process at first applied, giving for x a new value x_1 with a probable error ε_1, differing but little from ε. If we were in entire ignorance of the relative amount of the probable errors ε_1 and ε, there would be no reason at all for giving the preference to x rather than to x_1. If only the single circumstance were known that ε_1 exceeded ε by a given small amount, we should be equally at a loss, while the value of ε remained unknown, to state the relative weight of x_1 compared with x, and should, in fact, be again obliged to resort to the hypothesis that ε and ε_1 were sensibly equal. And in general, the greater the uncertainty of ε, or, in other words, the larger the value of $\varkappa$, the less reason would there be for excluding from competition with x any other determination of x, such as x_1, of which the probable error ε_1 differed but little from ε.

In order to employ the limit $\varkappa$ as here proposed, its value must be known before the computations have reached an advanced stage. That this is not ordinarily practicable will readily appear. On the other hand, it must be left entirely to the judgment of the computer to decide as to the precise manner in which $\varkappa$ is to be applied in limiting the allowable amount of difference $\varepsilon_1 - \varepsilon$.

Objections of a similar character apply equally to other standards which might be proposed for the same object. As has before been remarked, the question must be treated, if at all, upon a somewhat arbitrary basis, and we must be content with suggestions addressed to the judgment or common sense of the computer, in cases where no fixed rule is admissible.

Viewed in this light, there will ordinarily be no difficulty in recognizing the point at which there will be danger of compromising accuracy in the attempt to simplify the computations, nearly enough at least for practical purposes, if we are prepared to admit, at least in its general spirit and tendency, the truth of the following proposition: —

(2.) *The application of the method of least squares to the discussion of observations of physical phenomena, with the exception of a few special cases of rare occurrence, requires the use of such numbers only, in the arithmetical processes peculiar to it and characteristic of the method, as may be designated by one of the numerals* 0, 1, 2 ..... 9, *or of the fractions* $\frac{1}{2}, \frac{1}{3} \ldots\ldots \frac{1}{9}$, *or by a product of one of these numbers by an integral power of* 10.

An idea may be formed of the amount of the intentional errors occasioned by these substitutions, by noticing that if by N is represented any number whatsoever, and by N' a number chosen from the proposed series which most nearly coincides with N, we shall have

(3.)
$$\text{The maximum value of } \frac{N - N'}{N} = \frac{1}{9} \text{ nearly.}$$

$$\text{The probable value of } \frac{N - N'}{N} < \frac{1}{25}.$$

Before proceeding to a detailed investigation of the consequences of the changes proposed, it will be useful to point out the degree of insecurity attaching to the values which must ordinarily be adopted to represent the probable error of x; the different sources which may be supposed to contribute to the increase of ε; and their relative importance in connection with the question of the comparative accuracy of the two results x and x_1.

ε may be referred to the combined influence of two mutually independent errors η and η', η being the probable value of $x_0 - x$ which would result from the errors of observation alone, supposing the theory of the method of least squares and its application to the data to be rigorously exact, and η' the probable amount of error in x having its origin in errors necessarily committed in the discussion of the observed data, supposing the mode of discussion, although the best practicable, to fall short of strict conformity with the theory. η'_1 represents the value of η' when the same data have been reduced, by a process made intentionally still less exact, to a small extent, both in its theory and in its arithmetic, than that which gives the error η'. η'_1 will bear to x_1 a relation similar to that which η' bears to x. η cannot be completely eliminated, so long as the errors of observation remain unknown, by any treatment, and the same may be said of η'; but η'_1 can always be reduced to its least limit, η', by suitable refinements of theory and of computation. In view of the fact that η and η' must have always sensible, but very uncertain values, it will be of but little consequence that η'_1 should be reduced to its utmost limit without regard to the labor and inconvenience which it may cost. At all events, the attempt will be ineffectual as a means of improving the substantial accuracy of the results, as we shall presently see.

Since η is independent of η' and η'_1, we have, assuming η' to be the least attainable value of η'_1,

$$\varepsilon^2 = \eta^2 + \eta'^2, \qquad \varepsilon_1^2 = \eta^2 + \eta_1'^2.$$

If η'' be used to designate the probable value of $x - x_1$ which would result from small intentional deviations from that treatment of the data which is recognized to be the best, we have

$$(4.) \qquad \eta_1'^2 = \eta'^2 + \eta''^2, \qquad \varepsilon^2 = \eta^2 + \eta'^2, \qquad \varepsilon_1^2 = \eta^2 + \eta'^2 + \eta''^2.$$

As regards the uncertainty of ε, some estimate of its extent may be obtained in the following manner.

If it is an even chance that the error of which the probable value is η is comprised somewhere between the limits $\eta + \lambda$ and $\eta - \lambda$, η having been derived from comparisons of a given system of equations with observation, the number of individual equations thus compared being represented by n, and the number of unknown quantities entering into them by n', λ may be found from the expression*

$$(5.) \qquad \lambda = 0.477 \frac{\eta}{\sqrt{n - n'}}.$$

Any value of $n - n'$ less than 100 gives

$$\lambda > \frac{1}{21}\eta.$$

The scale of substituted numbers (2) admits, as we have before stated, of representing η within the probable amount of $\frac{1}{25}\eta$; hence, for any value of $n - n'$ less than 100, the series will afford numbers representing η with a probable error less than λ. A slight examination will show that a similar remark applies still more decisively to ε.

The considerations which oblige us to attribute a sensible value to η' are too many and too obvious to require to be specified in detail. It will be sufficient to cite one or two which have already been alluded to. The existence of unknown constant errors in the data will render the application of the method of least squares, strictly speaking, inexact. From this source η' will inevitably acquire some influence. Again, the uncertainty incident to any attempt to assign to the original data their proper relative weights, will have a similar effect. No process more loose and arbitrary can well be conceived, than that by which the relative precision of the elements afforded directly by observation is graduated. Yet, imperfect as it is, improvement in this particular is scarcely to be hoped for. Exact conformity with a theory which requires a previous knowledge of the relative weight of observations is quite impossible.

* Gauss, Zeitschrift für Astr., B. I. Theor. Comb., § 40.

At the same time, then, that the existence and influence of η' are admitted, its amount is altogether uncertain, to an extent sufficient at least to make the uncertainty of ε which is dependent on that of η and η' not less in proportional amount than that of η; consequently we shall obtain from (5) the expression

(5 *a.*) $$\varkappa > 0.477 \frac{\varepsilon}{\sqrt{n - n'}}$$

by which to measure the uncertainty of ε. If $n - n' < 100$,

(5 *b.*) $$\varkappa > \frac{1}{21} \varepsilon.$$

When ε is represented by a number chosen from the series (2), the probable error of the representation is, by (3), less than $\frac{1}{25}\varepsilon$; in other words, it is more than an even chance that this number will fall within the limits $\varepsilon + \frac{1}{25}\varepsilon$ and $\varepsilon - \frac{1}{25}\varepsilon$; and since the inherent uncertainty of ε makes it more than an even chance that its actual value is outside of the limits $\varepsilon + \frac{1}{21}\varepsilon$ and $\varepsilon - \frac{1}{21}\varepsilon$, in accordance with the above determination of $\varkappa$, we conclude that ε can be represented by one of the series of numbers 0, 1, 2 ..... 9, or of the fractions $\frac{1}{2}, \frac{1}{3} \ldots\ldots \frac{1}{9}$, or by a product of one of these numbers by an integral power of 10, with more accuracy than we can determine its amount by one hundred comparisons between the observed and the computed values of m. It would be easy to show, from the probable existence of constant errors alone, that an indefinite increase of the number of comparisons with observation would not sensibly diminish the uncertainty of ε below the amount stated. The proposition (2) would thus be sustained, as far as relates to all expressions for probable errors and weights, since they must depend upon conditions similar to those limiting the accuracy of ε.

An immediate consequence of this admission will be the extension of the proposition in question, in the qualified sense, at least, in which alone it is to be understood, to all other arithmetical expressions required in the application of the method of least squares, since the peculiar province of the latter is restricted entirely to the solution of equations of the form

(5 *c.*) $$a(x - x_1) + b(y - y_1) + \ldots\ldots\ldots + (m - m_1) = e,$$

in which each separate term and factor may be defined as proposed in (2).

To illustrate this, let us suppose for the moment that x_1 has been derived from the same primitive equations, but by an essentially different process from that by which x has been obtained; x_1 would still be precisely equal to x, if it were not for the errors e, e′, &c. Any such process, not intentionally bad, must evidently lead to a determination of x_1 differing from x by an amount of an order not higher than that of ε, while one adopted expressly on account of its good qualities, though without bearing any intended resemblance to the method of least squares in its characteristic features, will diminish the difference $x - x_1$ to a value either less in absolute amount than ε, or very nearly equal to it; so that $(x - x_1)$, $(y - y_1)$, &c. may be sufficiently well expressed by the series (2) in the same sense that ε may be. The equations

$$a x + b y + \dots\dots + m = e,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a x_1 + b y_1 + \dots\dots + m_1 = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

in which e is the value of e obtained by substituting x, y, &c. in the primitive equations, give at once

(5 c.) $$a(x - x_1) + b(y - y_1) + \dots\dots + (m - m_1) = e,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The solution of which by the method of least squares gives $(x - x_1)$, $(y - y_1) \dots\dots$, and thence

$$x = x_1 + (x - x_1),$$
$$y = y_1 + (y - y_1),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

Therefore the application of that method may be confined to the equations (5 c) alone.

But by what has already been said, $(x - x_1)$, $(y - y_1) \dots\dots$, and e, are quantities sufficiently well expressed by the series (2), from which we readily infer that there can be no appreciable advantage in giving to $(m - m_1)$ or to the products $a(x - x_1)$, $b(y - y_1)$, &c., or to the factors a, b, &c., any higher exactness of expression. Hence the proposition (2) may be extended to every term and factor of the equations (5 c), and therefore to all the numerical processes peculiar to, and characteristic of, the method of least squares.

This extreme application cannot, however, be recommended even on the ground of convenience or simplicity; on the contrary, the indiscriminate use of the fractional terms of the series would often be highly inconvenient; and a form of solution like that just indicated would not always be desirable.

In considering the different sources of error from which ε and ε_1 acquire their value, it will be convenient to compare the increase given to ε by the introduction of small intentional inaccuracies, in consequence of which ε becomes ε_1, either with ε itself or with $\varepsilon - \eta$, by means of the following relations derived from (4):—

(6.) $$\frac{\varepsilon_1 - \varepsilon}{\varepsilon} < \frac{1}{2}\left(\frac{\eta''}{\varepsilon}\right)^2, \qquad \frac{\varepsilon_1 - \varepsilon}{\varepsilon - \eta} < \left(\frac{\eta''}{\eta'}\right)^2.$$

Used in connection with (3) and the limit defining the uncertainty of ε,

$$\varkappa > 0.477 \frac{\varepsilon}{\sqrt{n-n'}}, \text{ or usually } \varkappa > \frac{1}{21} \varepsilon.$$

ε_1 will thus be comprised between the limits $\varepsilon + k\varkappa$ and $\varepsilon - k\varkappa$, when η'' has such a value that

$$\left(\frac{\eta''}{\varepsilon}\right)^4 < \frac{9}{10} \frac{k^2}{n-n'}.$$

The relative accuracy of x and x_1 will now be investigated for some special examples of deviation from a strict compliance with the method of least squares.

Let the equations proposed for solution be the following: —

$$\text{(7.)} \quad \begin{array}{ll} a\,\mathrm{x} + b\,\mathrm{y} + \ldots\ldots\ldots + m = \mathrm{e}, & \text{weight} = w, \\ a'\,\mathrm{x}' + b'\,\mathrm{y} + \ldots\ldots\ldots + m' = \mathrm{e}', & \quad\text{"} \quad = w', \\ \ldots & \end{array}$$

where e is the difference between the observed and computed value of m; m being the element derived from observation.

In solving these by least squares, the final equation for x is formed by taking the sum of all the equations after multiplying the first by $a w$, the second by $a' w'$, and so on, and then making

$$a\,w\,\mathrm{e} + a'\,w'\,\mathrm{e}' + \ldots\ldots\ldots = 0,$$

and for y

$$b\,w\,\mathrm{e} + b'\,w'\,\mathrm{e}' + \ldots\ldots\ldots = 0,$$

continuing in succession to form new equations until a final equation is obtained for each unknown quantity.

We shall compare the results of two solutions of the above equations (7), *in one of which* (I.) *the factors* $a w$, $a' w'$ ..... *conform strictly to the method of least squares. In the other* (II.), *these factors are replaced respectively by* α, α' .....; α *being that one of the numbers* 0, 1, 2 ..... 9, *or of the fractions* $\frac{1}{2}$, $\frac{1}{3}$ ..... $\frac{1}{9}$, *or of their products by an integral power of* 10, *which approaches most nearly to a given ratio with* $a w$, *and* α' *that which approaches most nearly to the same ratio with* $a' w'$, *&c. In a similar manner,* β, γ ..... *are used in the place of* $b w$, $c w$ .....

The true values of x and y ..... we will indicate by x_0, y_0 ..... Those deduced by (I.) will be denoted by x, y ....., and those deduced by (II.) will be denoted by x_1, y_1 ..... For the final equation for x, we make

$$a\,w\,\mathrm{e} + a'\,w'\,\mathrm{e}' + \ldots\ldots\ldots = 0.$$

For the final equation for x_1

$$\alpha\, e_1 + \alpha'\, e_1' + \ldots\ldots\ldots = 0.$$

For the corresponding final equations for x_0, which must be rigorous, we make either

$$a\, w\, e_0 + a'\, w'\, e'_0 + \ldots\ldots\ldots = a\, w\, e_0 + a'\, w'\, e'_0 + \ldots\ldots\ldots$$

or

$$\alpha\, e_0 \quad + \alpha'\, e'_0 \quad + \ldots\ldots\ldots = \alpha\, e_0 \quad + \alpha'\, e'_0 \quad + \ldots\ldots\ldots$$

according as the first (I.) or the second (II.) form of combination is adopted.

e_0, e, and e_1 are the values of e when the indeterminates x_0, $y_0 \ldots\ldots$, x, $y \ldots\ldots$, x_1, $y_1 \ldots\ldots$, &c., replace x, y ..... in (7).

The final equations for the combination (I.) are:—

$$\text{(8.)} \quad \begin{aligned} P\, x + P'\, y + P''\, z + \ldots\ldots\ldots + L &= 0, \\ P'\, x + Q\, y + Q'\, z + \ldots\ldots\ldots + M &= 0, \\ P''\, x + Q'\, y + Q''\, z + \ldots\ldots\ldots + N &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$\text{(9.)} \quad \begin{array}{lll} P = w\, a\, a + w'\, a'\, a' + \ldots\ldots, & Q = w\, b\, b + w'\, b'\, b' + \ldots\ldots, & R = w\, c\, c + w'\, c'\, c' + \ldots\ldots \\ P' = w\, a\, b + w'\, a'\, b' + \ldots\ldots, & Q' = w\, b\, c + w'\, b'\, c' + \ldots\ldots, & R' = w\, c\, d + w'\, c'\, d' + \ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \end{array}$$

From the conditions I. and II. applied to the original equations (7), if we make

$$\begin{array}{lll} A\, \alpha = a\, w + \delta\, \alpha\,, & B\, \beta = b\, w + \delta\, \beta\,, & C\, \gamma = c\, w + \delta\, \gamma\,, \ldots\ldots\ldots \\ A\, \alpha' = a'\, w' + \delta\, \alpha'\,, & B\, \beta' = b'\, w' + \delta\, \beta'\,, & C\, \gamma' = c'\, w' + \delta\, \gamma'\,, \ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots \end{array}$$

may be obtained

$$P\,(x - x_1) + P'\,(y - y_1) + \ldots\ldots\ldots + L + P\, x_1 + P'\, y_1 + \ldots\ldots\ldots = 0$$

$$L + P\, x_1 + P'\, y'_1 + \ldots\ldots\ldots = -\, e_1\, \delta\, \alpha - e'_1\, \delta\, \alpha' - \ldots\ldots\ldots$$

Hence,

$$\text{(10.)} \quad \begin{aligned} P\,(x - x_1) + P'\,(y - y_1) + \ldots\ldots\ldots &= \delta\, \alpha\, e_1 + \delta\, \alpha'\, e'_1 + \ldots\ldots\ldots \\ P'\,(x - x_1) + Q\,(y - y_1) + \ldots\ldots\ldots &= \delta\, \beta\, e_1 + \delta\, \beta'\, e'_1 + \ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

And in a similar manner,

$$\text{(11.)} \quad \begin{aligned} P\,(x_0 - x) + P'\,(y_0 - y) + \ldots\ldots\ldots &= a\, w\, e_0 + a'\, w'\, e'_0 + \ldots\ldots\ldots \\ P'\,(x_0 - x) + Q\,(y_0 - y) + \ldots\ldots\ldots &= b\, w\, e_0 + b'\, w'\, e'_0 + \ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Since ε is the probable value of $x_0 - x$, and η'' the probable value of $(x - x_1)$, to obtain the ratio $\frac{\eta''}{\varepsilon}$, we will compare the probable values of $x_0 - x$ and $x - x_1$, having, as above,

$$\text{(12.)} \qquad \frac{\eta''}{\varepsilon} = \frac{\text{Probable value of } (x - x_1)}{\text{Probable value of } (x_0 - x)}.$$

$x_0 - x$ and $x - x_1$ must be derived from a solution of equations (10) and (11), but since (II.) differs from (I.) by small variations only, we have, very nearly,

$$\text{(13.)} \qquad w\, e_1^2 + w'\, e_1'^2 + \ldots\ldots\ldots = w\, e^2 + w'\, e'^2 + \ldots\ldots\ldots$$

For the second member of (13) is a minimum relatively to the mode of solution, and, as has already been shown, (1), it differs from the first member by small terms of the second order only, those of the first order vanishing with the first differential coefficient of $\Omega = w\, e^2 + w'\, e'^2 + \ldots\ldots$

If, then, μ_0, μ, and μ_1 represent the probable values of e_0, e, and e_1 corresponding to the unit of weight of the equations (7), we may assume, for the purpose of determining $x - x_1$, that $\mu - \mu_1$ is a small quantity compared with μ, since we have

$$\frac{\mu^2}{\mu_1^2} = \frac{w\, e^2 + w'\, e'^2 + \ldots\ldots\ldots}{w\, e_1^2 + w'\, e_1'^2 + \ldots\ldots\ldots} = 1, \text{ very nearly.}$$

Moreover, in the absence of exact knowledge of the magnitude of the errors of e_0, $e'_0, \ldots\ldots$, it is necessary to admit that they are best represented by the errors $e, e', \ldots\ldots$; hence we have $\frac{\mu}{\mu_0} = 1$, and consequently $\frac{\mu_1}{\mu_0} = 1$, very nearly.

The conditions of the solution (II.) give for the probable value of either of the ratios $\frac{\delta\,\alpha}{a\,w}, \frac{\delta\,\beta}{b\,w}, \ldots\ldots, \frac{\delta\,\alpha'}{a'\,w'}, \frac{\delta\,\beta'}{b'\,w'} \ldots\ldots$

$$\text{(14.)} \qquad \frac{\delta\,\alpha}{a\,w} = g, \qquad \frac{\delta\,\beta}{b\,w} = g, \ldots\ldots\ldots \frac{\delta\,\alpha'}{a'\,w'} = g, \qquad \frac{\delta\,\beta'}{b'\,w'} = g, \ldots\ldots\ldots$$

Because α being by (II.) nearly proportional to $a\,w$, the probable value of $\delta\,\alpha$ will also be proportional to $a\,w$; and a similar remark applies equally to $\delta\,\beta$, $\delta\,\gamma$, &c.

The probable values of the second numbers of (10) and (11) are then, respectively,

$$\begin{array}{ll}
\delta\,\alpha\, e_1 + \delta\,\alpha'\, e'_1 + \ldots\ldots\ldots = g\,\mu_1 \sqrt{P}, & a\,w\,e_0 + a'\,w'\,e'_0 + \ldots\ldots\ldots = \mu_0 \sqrt{P}, \\
\delta\,\beta\, e_1 + \delta\,\beta'\, e'_1 + \ldots\ldots\ldots = g\,\mu_1 \sqrt{Q}, & b\,w\,e_0 + b'\,w'\,e'_0 + \ldots\ldots\ldots = \mu_0 \sqrt{Q}, \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{array}$$

Hence, in consequence of the identity of the coefficients $P, P' \ldots\ldots$, $Q, Q' \ldots\ldots$, &c. in the two systems (10) and (11), we obtain

$$(15.)\quad \frac{\text{Probable value of }(x - x_1)}{\text{Probable value of }(x_0 - x)} = \frac{\mu_1}{\mu_0} g = g\,; \qquad \frac{\text{Probable value of }(y - y_1)}{\text{Probable value of }(y_0 - y)} = \frac{\mu_1}{\mu_0} g = g \ldots\ldots$$

And from (12) the general expression

$$(16.)\qquad \frac{\eta''}{\varepsilon} = g.$$

Giving to g the value (3)

$$g < \frac{1}{25},$$

we shall have in the present case

$$\frac{\eta''}{\varepsilon} < \frac{1}{25},$$

and by (6)

$$\varepsilon_1 - \varepsilon < \frac{1}{1250}\,\varepsilon.$$

In other words, by using the form of solution (II.) in the place of a rigorous application of the method of least squares, the probable errors of the concluded results will not be increased by one one-thousandth part, — a difference entirely too small to be sensible. The two processes, as far as regards accuracy, therefore, may be considered as perfectly identical. On the other hand, the advantages of simplicity and convenience are altogether in favor of the second, in which all the operations of multiplication and division required in the construction of the final equations are reduced to their simplest arithmetical forms.

The necessity of distinguishing between the probable error of x_1, that is, the probable value of $(x_0 - x_1)$, and the difference between x_1 and x, or $(x_1 - x)$, deserves particular attention here. While x_1 will often differ very much from x, this fact taken by itself by no means indicates that the chances that x_1 is the true value are not sensibly as good as that x is. The discrepancy really proves that the original observations upon which the discussions have been based are so faulty, that very little confidence can be placed in either result, or in any other that can be deduced from the same data.

To give completeness to the investigation, we will compare the processes by which the probable errors of the values of the indeterminates in the two solutions (I.) and (II.) are obtained.

For the system (I.), let x, y, z, &c. be eliminated in succession from the equations (8), in the following manner.

Multiply the final equation for x by $\frac{P'}{P}$ and subtract it from the final equation for y, and again by $\frac{P''}{P}$ and subtract it from the final equation for z, forming the new equations, in which x does not enter: —

$$
(17.)\qquad
\begin{aligned}
&(Q - \frac{P'}{P} P')\,y + (Q' - \frac{P'}{P} P'')\,z + \ldots\ldots + M - \frac{P'}{P} L = 0,\\
&(Q' - \frac{P''}{P} P')\,y + (R - \frac{P''}{P} P'')\,z + \ldots\ldots + N - \frac{P''}{P} L = 0,\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

to which we shall give the following notation:

$$
(18.)\qquad
\begin{aligned}
&Q_x\, y + Q'_x z + \ldots\ldots + M_x = 0,\\
&Q'_x y + R_x\, z + \ldots\ldots + N_x = 0,\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

Let y be eliminated from these new equations (18) in the same manner that x was from the equations (8), multiplying the first by $\frac{Q'_x}{Q_x}$ and subtracting it from the second, giving

$$
(R_x - \frac{Q'_x}{Q_x} Q'_x)\,z + \ldots\ldots + N_x - \frac{Q'_x}{Q_x} M_x = 0,
$$

with the corresponding notation,

$$
(19.)\qquad R_{xy}\, z + \ldots\ldots + N_{xy} = 0.
$$

For a more complete illustration of the notation, we will collect in one view the equations obtained by the successive eliminations:

$$
(20.)\qquad
\begin{array}{lll}
(a) & P\, x + P' y + P''\, z + \ldots\ldots + L = 0 & \\
(b) & P' x + Q\, y + Q'\, z + \ldots\ldots + M = 0 & \text{Final} \\
(c) & P'' x + Q' y + R\, z + \ldots\ldots + N = 0 & \text{Equations.} \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\
(b_x) & Q_x\, y + Q'_x z + \ldots\ldots + M_x = 0 & \text{Equations formed by} \\
(c_x) & Q'_x y + R_x\, z + \ldots\ldots + N_x = 0 & \text{eliminating } x. \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\
(c_{xy}) & R_{xy} z + \ldots\ldots + N_{xy} = 0 & \text{Equations formed by} \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \text{eliminating } x \text{ and } y.
\end{array}
$$

Let μ be the probable error of one of the original equations of the unit of weight. The probable errors of the second members of the equations (20) will be,

$$
(21.)\qquad
\begin{array}{llll}
\text{Probable error of the equation} & (a) & = \mu\sqrt{P} \\
\text{"} \quad \text{"} \quad \text{"} & (b) & = \mu\sqrt{Q} \\
\text{"} \quad \text{"} \quad \text{"} & (c) & = \mu\sqrt{R} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & \\
\text{"} \quad \text{"} \quad \text{"} & (b_x) & = \mu\sqrt{Q_x} \\
\text{"} \quad \text{"} \quad \text{"} & (c_x) & = \mu\sqrt{R_x} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & \\
\text{"} \quad \text{"} \quad \text{"} & (c_{xy}) & = \mu\sqrt{R_{xy}} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & &
\end{array}
$$

When the final equations obtained by (II.) are solved in an analogous manner, and the notation is changed so as to indicate the coefficients of x_1, y_1, &c., P_1, $P'_1, \ldots$. which replace P, P', &c., we have

$$
(22.)\quad
\begin{array}{lrl}
(a) & P_1 x_1 + P'_1 y_1 + P''_1 \; z_1 + \ldots\ldots\ldots + L_1 = 0 & \\
(b) & {}'Q_1 x_1 + Q_1 \; y_1 + Q'_1 \; z_1 + \ldots\ldots\ldots + M_1 = 0 & \text{Final} \\
(c) & {}''R_1 x_1 + {}'R_1 y_1 + R_1 \; z_1 + \ldots\ldots\ldots + N_1 = 0 & \text{Equations.} \\
 & \ldots\ldots\ldots\ldots & \\
(b_x) & Q_{1x} y_1 + Q'_{1x} \; z_1 + \ldots\ldots\ldots + M_{1x} = 0 & \text{Equations formed by} \\
(c_x) & {}'R_{1x} y_1 + R_{1x} \; z_1 + \ldots\ldots\ldots + N_{1x} = 0 & \text{eliminating } x_1. \\
 & \ldots\ldots\ldots\ldots & \\
(c_{xy}) & R_{1xy} \; z_1 + \ldots\ldots\ldots + N_{1xy} = 0 & \text{Equations formed by} \\
 & \ldots\ldots\ldots & \text{eliminating } x_1 \text{ and } y_1.
\end{array}
$$

$$
\begin{array}{lll}
\text{Probable error of the equation (22)} & (a) & = (1 \pm \tfrac{1}{2} g)\, \mu \sqrt{\dfrac{P_1}{A}}, \\
\text{“} \quad \text{“} \quad \text{“} & (b) & = (1 \pm \tfrac{1}{2} g)\, \mu \sqrt{\dfrac{Q_1}{B}}, \\
\text{“} \quad \text{“} \quad \text{“} & (c) & = (1 \pm \tfrac{1}{2} g)\, \mu \sqrt{\dfrac{R_1}{C}}, \\
\ldots\ldots & & \\
(23.) \quad \text{“} \quad \text{“} \quad \text{“} & (b_x) & = (1 \pm \tfrac{1}{2} g)\, \mu \sqrt{\dfrac{Q_{1x}}{B}}, \\
\text{“} \quad \text{“} \quad \text{“} & (c_x) & = (1 \pm \tfrac{1}{2} g)\, \mu \sqrt{\dfrac{R_{1x}}{C}}, \\
\ldots\ldots & & \\
\text{“} \quad \text{“} \quad \text{“} & (c_{xy}) & = (1 \pm \tfrac{1}{2} g)\, \mu \sqrt{\dfrac{R_{1xy}}{C}}, \\
\ldots\ldots & &
\end{array}
$$

To demonstrate these results, (23), it is to be observed that the probable errors of the second members of the equations (22), (a), $(b_x), \ldots$. are the probable sums of the second members of the equations

$$
\begin{aligned}
P_1 (x_0 - x_1) + P'_1 (y_0 - y_1) + \ldots\ldots\ldots &= \alpha e_0 + \alpha' e'_0 + \ldots\ldots\ldots \\
Q_{1x} (y_0 - y_1) + \ldots\ldots\ldots &= (\beta - \frac{'Q_1}{P_1} \alpha)\, e_0 + (\beta' - \frac{'Q_1}{P_1} \alpha')\, e'_0 + \ldots\ldots\ldots \\
\ldots\ldots\ldots
\end{aligned}
$$

The probable value of $\alpha^2 e_0^2$ is, by (13) and (14),

$$
\alpha^2 e_0^2 = \frac{A\,\alpha}{A}\, \alpha\, e_0^2 = w\, e_0^2 \left(1 + \frac{\delta\, \alpha}{\alpha\, w}\right) \frac{\alpha\, \alpha}{A} = \mu^2 (1 \pm g) \frac{\alpha\, \alpha}{A},
$$

and the probable value of the sum of the terms $\alpha e_0 + \alpha' e'_0 + \ldots\ldots$ is

$$\alpha e_0 + \alpha' e'_0 + \ldots\ldots\ldots = \frac{\mu}{\sqrt{A}} \sqrt{(1 \pm g)} \sqrt{\alpha a + \alpha' a' + \ldots\ldots\ldots} = \mu \left(1 \pm \tfrac{1}{2} g\right) \sqrt{\frac{P_1}{A}},$$

which is the probable error of the equation (22), (a).

Again, since

$$A \alpha = a w \left(1 + \frac{\delta \alpha}{a w}\right), \quad B \beta = b w \left(1 + \frac{\delta \beta}{b w}\right),$$

and

$$P_1 = \alpha a + \alpha' a' + \ldots\ldots\ldots, \qquad Q_1 = \beta b + \beta' b' + \ldots\ldots\ldots$$

$$P'_1 = \alpha b + \alpha' b' + \ldots\ldots\ldots, \qquad {}'Q_1 = \beta a + \beta' a' + \ldots\ldots\ldots$$

$$\left(\beta - \frac{'Q_1}{P_1} \alpha\right)^2 e_0^2 = \frac{w e_0^2}{B} \left(1 + \frac{\delta \beta}{b w}\right) \left(\beta b - \frac{'Q_1}{P_1} \alpha b\right) + \frac{w e_0^2}{A} \left(1 + \frac{\delta \alpha}{a w}\right) \left(\frac{'Q_1^2}{P_1^2} \alpha a - \frac{'Q_1}{P_1} \beta a\right);$$

or, substituting the probable values

$$e_0 \sqrt{w} = \mu, \text{ and } \left(1 + \frac{\delta \alpha}{a w}\right) = \left(1 + \frac{\delta \beta}{b w}\right) = (1 \pm g),$$

we have

$$\left(\beta - \frac{'Q_1}{P_1} \alpha\right)^2 e_0^2 = \mu^2 (1 \pm g) \left[\frac{1}{B} \left(\beta b - \frac{'Q_1}{P_1} \alpha b\right) + \frac{1}{A} \left(\frac{'Q_1^2}{P_1^2} \alpha a - \frac{'Q_1}{P_1} \beta a\right)\right].$$

Moreover,

$$\text{The sum of all the terms} \left(\frac{'Q_1^2}{P_1^2} \alpha a - \frac{'Q_1}{P_1} \beta a\right) = \left(\frac{'Q_1^2}{P_1^2} P_1 - \frac{'Q_1}{P_1} {}'Q_1\right) = 0,$$

$$\text{" \quad " \quad "} \left(\beta b - \frac{'Q_1}{P_1} \alpha b\right) = \left(Q_1 - \frac{'Q_1}{P_1} P'_1\right) = Q_{1x}.$$

Therefore the probable sum of the terms

$$\left(\beta - \frac{'Q_1}{P_1} \alpha\right) e_0 + \left(\beta' - \frac{'Q_1}{P_1} \alpha'\right) e'_0 + \ldots\ldots\ldots$$

will be

$$\mu \sqrt{1 \pm g} \sqrt{\frac{Q_{1x}}{B}} = \mu \left(1 \pm \tfrac{1}{2} g\right) \sqrt{\frac{Q_{1x}}{B}},$$

which is the probable error of the equation (22), (b_x).

The other probable errors in (23) are readily supplied by analogy.

If we neglect $\frac{1}{2} g$, of which the probable value is less than $\frac{1}{50}$, the probable errors of (22), (a), (b_x) ..... become

Probable error of equation (22) (a) $= \mu \sqrt{\frac{P_1}{A}}$; Probable error of equation (22) (b_x) $= \mu \sqrt{\frac{Q_{1x}}{B}}$.

. . . .

We shall now proceed to explain a third form of solution, (III.).

Returning to the equations (10),

$$
(24.)\qquad
\begin{aligned}
P\,(x - x_1) + P'\,(y - y_1) + \dots\dots &= e_1\,\delta\alpha + e'_1\,\delta\alpha' + \dots\dots \\
P'\,(x - x_1) + Q\,(y - y_1) + \dots\dots &= e_1\,\delta\beta + e'_1\,\delta\beta' + \dots\dots \\
\dots\quad\dots\quad\dots & \quad\dots\quad\dots
\end{aligned}
$$

we find, for the probable values of their second numbers,

$$
(25.)\qquad
\begin{aligned}
g\,\mu\,\sqrt{w\,a\,a + w'\,a'\,a' + \dots\dots} &= g\,\mu\,\sqrt{P}\,, \\
g\,\mu\,\sqrt{w\,b\,b + w'\,b'\,b' + \dots\dots} &= g\,\mu\,\sqrt{Q}\,, \\
\dots\quad\dots\quad\dots & \quad\dots
\end{aligned}
$$

It is evident that the probable sum $e_1\,\delta\,\alpha + e'_1\,\delta\,\alpha' + \dots\dots$, being proportional to the square root of the sum of the squares of the individual terms, depends mainly upon the large terms; or, since $e_1\,\delta\,\alpha = e_1\sqrt{w}\,\frac{\delta\,\alpha}{\sqrt{w}}$ and $e_1\sqrt{w} = \mu$, this sum will be

$$\mu\sqrt{\frac{\delta\,\alpha^2}{w} + \frac{\delta\,\alpha'^2}{w'} + \dots\dots}$$

If any two or more of the coefficients $\frac{\delta\,\alpha}{\sqrt{w}}$, as, for instance, $\frac{\delta\,\alpha}{\sqrt{w}}$ and $\frac{\delta\,\alpha'}{\sqrt{w'}}$, were equal, any small change increasing the former and diminishing the latter by equal amounts would not alter the coefficient of μ; but if $\frac{\delta\,\alpha'}{\sqrt{w'}}$ were much smaller than $\frac{\delta\,\alpha}{\sqrt{w}}$, we should have, very nearly,

$$\frac{\delta\,\alpha^2}{w} + \frac{\delta\,\alpha'^2}{w} = \frac{\delta\,\alpha^2}{w}\,,$$

and a small change in $\frac{\delta\,\alpha'}{\sqrt{w'}}$ would affect the coefficient of μ by an amount insensible compared with the effect of an equal change in $\frac{\delta\,\alpha}{\sqrt{w}}$.

Let (P) represent the sum of a certain number of the largest of the terms composing the series

$$P = w\,a\,a + w'\,a'\,a' + \dots\dots$$

and (p) the sum of a number of the smallest of the terms of the same series. Let also $(\delta\,P)$ be the sum of the terms $\frac{\delta\,\alpha^2}{w}$ corresponding to the series (P), and $(\delta\,p)$ the sum of the terms $\frac{\delta\,\alpha^2}{w}$, corresponding to the series (p).

Then we have the probable values

$$
(26.)\qquad
\begin{aligned}
&\text{For the large terms, } (\delta\,P) = g^2\,(P), \\
&\text{For the small terms, } (\delta\,p) \;= g^2\,(p),
\end{aligned}
$$

g representing the general probable value of $\frac{\delta\,\alpha}{a\,w}$ for all the terms, whether of large, small, or medium value.

Let us suppose the mode of solution (II.) to be itself varied by changing the factors α, α', &c., corresponding to the large and small terms, so that for the large terms, g, or the probable value of $\frac{\delta\alpha}{a w}, \frac{\delta\alpha'}{a' w'}, \ldots\ldots$, for these particular terms, becomes

$$g = H,$$

and for the small terms,

$$g = h.$$

We shall then have the probable values,

$$\text{(27.)} \qquad \begin{aligned} &\text{For the large terms } (\delta P_1) = H^2 (P), \\ &\text{For the small terms } (\delta p_1) \ = h^2\ (p). \end{aligned}$$

(δP) and (δp) becoming (δP_1) and (δp_1) when g becomes H and h.

In order that the probable sum of the second member of the equation,

$$P(x - x_1) + P'(y - y_1) + \ldots\ldots\ldots = e_1\,\delta\alpha + e'_1\,\delta\alpha' + \ldots\ldots\ldots$$

should not be increased by the proposed changes of $\delta\alpha$, we must have

$$(\delta P) \ + (\delta p) > (\delta P_1) + (\delta p_1),$$

or, by (26) and (27),

$$H^2 (P) + h^2 (p) < g^2 (P) + g^2 (p).$$

We shall assume, for the terms corresponding to (p), that the probable value of h is

$$h = -1.$$

This condition involves only small changes in the factors $\alpha, \alpha' \ldots\ldots$, because, for the terms corresponding to (p), $a w$ being small, $\delta\alpha = a w h = - a w$, will also be small; we then have

$$H^2 (P) < g^2 (P) + (g^2 - 1)(p),$$

or, since we can put $g^2 - 1 = -1$ very nearly, g being small compared with unity, we obtain

$$\text{(28.)} \qquad g^2 - H^2 > \frac{(p)}{(P)}, \qquad H^2 < g^2 - \frac{(p)}{(P)},$$

representing the condition to be observed in order that the second members of (24) should not be increased by the changes made in the large and small values of α.

This, it will be remembered, can be applied only when the condition $h = -1$ involves only small changes in the factors $\alpha, \alpha' \ldots\ldots$ of the order of the mean value of $\delta\alpha$ for all the factors. $\frac{(p)}{(P)}$ being necessarily a positive quantity, H must always be less than g.

(28) may easily be extended to the analogous cases of the second members of the equations

$$P' (x - x_1) + Q\ (y - y_1) + \ldots\ldots\ldots = e_1\,\delta\beta + e'_1\,\delta\beta' + \ldots\ldots\ldots$$
$$P'' (x - x_1) + Q' (y - y_1) + \ldots\ldots\ldots = e_1\,\delta\gamma + e'_1\,\delta\gamma' + \ldots\ldots\ldots$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

so that

(29.) $$g^2 - H^2 > \frac{(q)}{(Q)}, \qquad g^2 - H^2 > \frac{(r)}{(R)}, \qquad \ldots\ldots\ldots$$

give the limits within which the proposed changes of the factors $\beta, \beta' \ldots\ldots \gamma, \gamma' \ldots\ldots$ will not increase the probable sums $e_1\,\delta\beta + e'_1\,\delta\beta' + \ldots\ldots$ and $e_1\,\delta\gamma + e'_1\,\delta\gamma' + \ldots\ldots$

For the factors $\alpha, \alpha' \ldots\ldots, \beta, \beta' \ldots\ldots$ corresponding to the equations most important in their influence upon the final determination of $x, y \ldots\ldots$ respectively, if we use numbers chosen from a series for which $\frac{N - N'}{N}$ is only $\frac{1}{10}$ as large as it is for the series (2), we shall have

$$H = \frac{1}{10}\,g.$$

And if at the same time we omit altogether a certain number of the unfavorable equations by making in these instances $\alpha = 0, \beta = 0 \ldots\ldots$, that is, $\delta\alpha = -a\,w$, $\delta\beta = -b\,w$, or $h = -1$, we find

$$g^2 - H^2 = \frac{1}{631}.$$

We shall therefore keep within the limits (28) and (29) as long as the coefficients in the omitted equations satisfy the conditions

$$\frac{(p)}{(P)} < \frac{1}{631}, \qquad \frac{(q)}{(Q)} < \frac{1}{631}, \qquad \ldots\ldots\ldots$$

The probable values of $x - x_1$, $y - y_1$, will not have been increased, and consequently the solution may be accepted as equivalent to II.

A general method, III., of adjusting the degree of numerical accuracy which should be observed in the expression of the factors $\alpha, \alpha' \ldots\ldots, \beta, \beta' \ldots\ldots$, may be derived from the following considerations.

In II. the adjustment is evidently not so favorable as it might be, since the limit of the intentional inaccuracies $\delta\alpha, \delta\alpha' \ldots\ldots, \delta\beta, \delta\beta' \ldots\ldots$ has been fixed by the relations

$$\delta\alpha = a\,w\,g, \qquad \delta\alpha' = a'\,w'\,g', \ldots\ldots\ldots \qquad \delta\beta = b\,w\,g, \qquad \delta\beta' = b'\,w'\,g', \ldots\ldots\ldots$$

g having the same average value whether $a\,w$, $b\,w \ldots\ldots$ be large or small; thus the largest inaccuracies are committed when $a\,w$, $b\,w$ ..... are largest, that is, when the equation has most influence upon the final result for any particular indeterminate.

In order to secure a more advantageous distribution, it will be necessary to recur again to the equations (10). It appears that, for a given limit of inaccuracy in the expression of the factors, the probable values of $x - x_1$, $y - y_1$ ..... will be least when the separate terms of the second members of these equations, irrespective of their signs, are equal to each other, or

$$e_1\,\delta\,\alpha = e'_1\,\delta\,\alpha' = \ldots\ldots\qquad e_1\,\delta\,\beta = e'_1\,\delta\,\beta' = \ldots\ldots$$

$\delta\,\alpha$, $\delta\,\beta$ ..... ought therefore to be inversely proportional to the probable errors of the primitive equations, or directly as the square roots of their weights.

We shall, then, define III. by the relations

$$(30.)\qquad \begin{array}{ll} \delta\,\alpha = \mathbf{A}\,g\,\sqrt{w}, & \delta\,\beta = \mathbf{B}\,g\,\sqrt{w}, \;\ldots\ldots \\ \delta\,\alpha' = \mathbf{A}\,g\,\sqrt{w'}, & \delta\,\beta' = \mathbf{B}\,g\,\sqrt{w'}, \;\ldots\ldots \\ \ldots & \ldots \end{array}$$

A, **B** ..... *and* g *being constant quantities.*

To secure the equivalency of II. and III., the values of **A**, **B** ..... must depend on the condition that the probable values of the second members of (10) should remain unchanged, or

$$g\,\mu\,\sqrt{\mathbf{A}^2 + \mathbf{A}^2 + \ldots\ldots} = g\,\mu\,\sqrt{P}, \qquad g\,\mu\,\sqrt{\mathbf{B}^2 + \mathbf{B}^2 + \ldots\ldots} = g\,\mu\,\sqrt{Q}, \;\ldots\ldots$$
$$\mathbf{A}\,\sqrt{n} = \sqrt{P}, \qquad \mathbf{B}\,\sqrt{n} = \sqrt{Q}, \;\ldots\ldots$$

Hence it is easy to conclude, that, if we make in (30)

$$\mathbf{A} = \text{mean value of } a\,\sqrt{w}, \qquad \mathbf{B} = \text{mean value of } b\,\sqrt{w}, \;\ldots\ldots$$

the means being in every instance taken without regard to signs, the probable values of $x - x_1$, $y - y_1$ ..... will be smaller in III. than in II., while III. in point of facility has a decided advantage over II.; since by making $\alpha = 0$, $\beta = 0$ ..... in all cases in which $a\,\sqrt{w} < \mathbf{A}\,g$, $b\,\sqrt{w} < \mathbf{B}\,g$, ..., a considerable amount of unnecessary computation may often be avoided.

The following will then be the limits of intentional numerical inaccuracies allowed in the expression of the factors α, α' ....., β, β' ..... in the three methods.

I.	II.	III.
$\delta\,\alpha = 0,\ \delta\,\beta = 0\ \ldots$	$\delta\,\alpha = a\,w\,g,\ \delta\,\beta = b\,w\,g\ \ldots$	$\delta\,\alpha = \mathbf{A}\,g\,\sqrt{w},\ \delta\,\beta = \mathbf{B}\,g\,\sqrt{w}\ \ldots$
$\delta\,\alpha' = 0,\ \delta\,\beta' = 0\ \ldots$	$\delta\,\alpha' = a'\,w'\,g,\ \delta\,\beta' = b'\,w'\,g\ \ldots$	$\delta\,\alpha' = \mathbf{A}\,g\,\sqrt{w'},\ \delta\,\beta' = \mathbf{B}\,g\,\sqrt{w'}\ \ldots$
. . . . .	. . . . . .	. . . .

The following tests of the correctness of all the numbers entering into the final equations will be found useful. They apply equally to the three methods.

Let the sums $\mathbf{S}, \mathbf{S}' \ldots\ldots, \Sigma, \Sigma' \ldots\ldots$ be formed as follows: —

$$\begin{array}{llll} (31.) & a + b + \ldots\ldots\ldots + m = \mathbf{S}\,, & & \alpha + \beta + \gamma + \ldots\ldots\ldots = \Sigma, \\ & a' + b' + \ldots\ldots\ldots + m' = \mathbf{S}'\,, & & \alpha' + \beta' + \gamma' + \ldots\ldots\ldots = \Sigma', \\ & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}$$

We shall then have the following test controlling at once all the computed quantities, $P_1, P'_1 \ldots\ldots, {}'Q_1, Q_1 \ldots\ldots, L_1, M_1 \ldots\ldots$

$$\Sigma\,(P_1, Q_1, R_1 \ldots\ldots\ldots L_1, M_1, N_1 \ldots\ldots\ldots) = \Sigma\,(\mathbf{S}\,\Sigma).$$

For we have

$$\begin{array}{ll} & P_1 + P'_1 + \ldots\ldots\ldots + L_1 = \left.\begin{array}{l} \alpha\, a + \alpha\, b + \ldots\ldots\ldots \\ + \alpha' a' + \alpha' b' + \ldots\ldots\ldots \\ + \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} = \mathbf{S}\,\alpha + \mathbf{S}'\,\alpha' + \ldots\ldots\ldots \\ (32.) & {}'Q_1 + Q_1 + \ldots\ldots\ldots + M_1 = \left.\begin{array}{l} \beta\, a + \beta\, b + \ldots\ldots\ldots \\ + \beta' a' + \beta' b' + \ldots\ldots\ldots \\ + \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} = \mathbf{S}\,\beta + \mathbf{S}'\,\beta' + \ldots\ldots\ldots \\ & {}''R_1 + {}'R_1 + \ldots\ldots\ldots + N_1 = \left.\begin{array}{l} \gamma\, a + \gamma\, b + \ldots\ldots\ldots \\ + \gamma' a' + \gamma' b' + \ldots\ldots\ldots \\ + \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} = \mathbf{S}\,\gamma + \mathbf{S}'\,\gamma' + \ldots\ldots\ldots \\ & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}$$

The sum of all the equations (32) is

$$(P_1 + P'_1 + \ldots\ldots\ldots + L_1) + ({}'Q_1 + Q_1 + \ldots\ldots\ldots + M_1) + \ldots\ldots\ldots = \mathbf{S}\,\Sigma + \mathbf{S}'\,\Sigma' + \ldots\ldots\ldots$$

or

$$(33.)\qquad \Sigma\,(P_1, Q_1, R_1 \ldots\ldots\ldots L_1, M_1, N_1 \ldots\ldots\ldots) = \Sigma\,(\mathbf{S}\,\Sigma).$$

When this is not fulfilled, the particular equation at fault may be detected by using the partial tests,

$$\begin{array}{ll} (34.) & P_1 + P'_1 + \ldots\ldots\ldots L_1 = \mathbf{S}\,\alpha + \mathbf{S}'\,\alpha' + \ldots\ldots\ldots \\ & {}'Q_1 + Q_1 + \ldots\ldots\ldots M_1 = \mathbf{S}\,\beta + \mathbf{S}'\,\beta' + \ldots\ldots\ldots \\ & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}$$

If all the numbers are not carried out to the same number of figures beyond the decimal point, it will be advisable, in order to apply the test to the best effect, to alter arbitrarily the decimal point, making, for the time being, the number of figures to the right hand of it the same in all the equations.

The successive stages in the computations by (II.) and (III.) will be as follows: —

a) To assign the values of the equivalent factors $\alpha, \beta \ldots\ldots$, which can readily be done by simple inspection. The factors once adopted must be used rigorously in multiplying every term of the equations to which they belong.

b) After the multiplications have been performed, and the sums taken, the numbers adopted in the final equations are to be tested by (33).

c) The solution of the final equations.

d) The determination of weights.

If changes have been made in the decimal pointing, or otherwise, by introducing the constants A, B ....., it must be remembered that, although the final equations thus formed will give the same values of x_1, y_1, &c. that would have been obtained if no such alteration had been made, the determination of the weights and probable errors of x_1, y_1, &c. requires that the correct pointing be restored in the coefficients, or else that the probable errors be computed in conformity with the formulæ (23).

When the number of indeterminates is considerable, it will be advisable, in solving the final equations, to eliminate x, y, z, &c., in succession, and then to repeat the operation, commencing the elimination in the reverse order, z, y, x, &c. One of the advantages of so doing is a complete check upon the work by the comparison of the value of that indeterminate which is obtained last by both eliminations. It is, however, mostly recommended from its facilitating the computation of the weights. In this case, the following formulæ may be used, if the number of indeterminates does not exceed six. Let these be x, y, z, ξ, η, ζ, and their weights, $W_{(x)}$, $W_{(y)}$, &c. The ordinary formulæ for computing the weights give

$$(35.) \qquad W_{(\zeta)} = U_{x\,y\,z\,\xi\,\eta}\,, \qquad W_{(x)} = P_{\zeta\,\eta\,\xi\,z\,y}\,.$$

$W_{(\zeta)}$ is the coefficient of ζ in the equation resulting from the elimination of x, y, z, ξ, and η, by the process indicated in (18), (19), and (20), and $W_{(x)}$ the coefficient of x after ζ, η, ξ, z, and y have been eliminated. We have, also,

$$(36.) \qquad W_{(\eta)} = \frac{T_{x\,y\,z\,\xi}}{U_{x\,y\,z\,\xi}}\,W_{(\zeta)}\,, \qquad W_{(y)} = \frac{Q_{\zeta\,\eta\,\xi\,x}}{P_{\zeta\,\eta\,\xi\,x}}\,W_{(x)}\,.$$

The factors and divisors required in (36) will have been already computed during the eliminations which have preceded.

From the equations containing only ξ, η, ζ, the latter is to be eliminated; and from the equations containing z, y, and x, x is to be eliminated. We then have

$$W_{(\xi)} = \frac{S_{x\,y\,z\,\zeta}}{T_{z\,y\,z\,\zeta}}\,W_{(\eta)}\,, \qquad W_{(z)} = \frac{R_{\zeta\,\eta\,\xi\,x}}{Q_{\zeta\,\eta\,\xi\,x}}\,W_{(y)}\,.$$

For the weight of x_1, y_1, &c., when (II.) or (III.) is used, we shall have, from (23),

$$(37.) \qquad \begin{aligned} \text{Weight of } x_1 &= A\,W_{(x)}\,, \\ \text{“} \qquad y_1 &= B\,W_{(y)}\,, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

The limits of effective accuracy appropriate to the numerical representation of the coefficients a, b ..... may be investigated in the following manner:—

If we determine x_1, y_1 ..... by the method of least squares from the equations,

$$(38.)\qquad (a - d\,a)\,x_1 + (b - d\,b)\,y_1 + \ldots\ldots + m - d\,m = e_1, \qquad \text{weight} = w,$$
$$\ldots\ldots\ldots\ldots$$

we shall have from (8) and (38), if $d\,a$, $d\,b$ ....., which may be employed to represent the errors of the adopted coefficients, are small,

$$(39.)\qquad P\,(x_1 - x) + P'\,(y_1 - y) + \ldots\ldots = a\,w\left(\frac{d\,a}{a}\,e + x\,d\,a + y\,d\,b + \ldots\ldots + d\,m\right) + \ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$

If μ' is the probable value of $d\,m\sqrt{w}$, the most suitable values of $x\,d\,a$, $y\,d\,b$ ..... evidently fulfil the conditions

$$(40.)\qquad x\,d\,a\sqrt{w} = y\,d\,b\sqrt{w}\ldots\ldots = d\,m\sqrt{w} = \mu', \qquad x\,d\,a'\sqrt{w'} = y\,d\,b'\sqrt{w'}\ldots\ldots = d\,m'\sqrt{w'} = \mu', \ldots\ldots$$

Observing that we may substitute in the second members of (39) the probable values $e < a\,x$, $e' < a'\,x$ ....., $e < b\,y$, $e' < b'\,y$ ....., we may conclude, by comparing (39) with (11), (12), and (6), that, if μ' is less than the limit

$$(41.)\qquad \mu' = \frac{g'\,\mu}{\sqrt{n' + 2}},$$

n' denoting the number of indeterminates in (38), the difference $\varepsilon_1 - \varepsilon$ of the probable errors of x_1, y_1 ..... derived from (38) and of x, y ..... derived from (8) will be

$$(42.)\qquad \varepsilon_1 - \varepsilon < \frac{1}{2}\,g'^2\,\varepsilon.$$

Consequently, if $g' < \frac{1}{25}$, $\varepsilon_1 - \varepsilon$ will be less than $\frac{1}{1250}\,\varepsilon$ or less than in II.

When μ' is known, the limits for admissible values of $d\,a$, $d\,b$ ..... in (38) will be

$$(43.)\qquad d\,a < \frac{\mu'}{x\sqrt{w}}, \quad d\,a' < \frac{\mu'}{x\sqrt{w'}}\ldots\ldots d\,b = \frac{x}{y}\,d\,a, \quad d\,b' = \frac{x}{y}\,d\,a'\ldots\ldots d\,m = x\,d\,a, \quad d\,m' = x\,d\,a'\ldots\ldots$$

If the mean values of $a\,x$, $b\,y$ ....., irrespective of their signs, are all of the same order of magnitude, we may substitute in (43) the *a priori* probable values

$$(44.)\qquad x < \frac{\mathbf{M}}{\mathbf{A}}\sqrt{\frac{n}{n + n' - 1}}, \qquad \frac{x}{y} = \frac{\mathbf{B}}{\mathbf{A}}, \qquad \ldots\ldots$$

where n is the number of equations (38), n' the number of indeterminates, and, taking the means without regard to signs,

$$(45.)\qquad \mathbf{A} = \text{mean value of } a\sqrt{w}, \quad \mathbf{B} = \text{mean value of } b\sqrt{w}\ldots\ldots \quad \mathbf{M} = \text{mean value of } m\sqrt{w}.$$

We will now proceed to compare the three methods of solution (I.), (II.), (III.), by applying them to the following series of equations of six indeterminates, taken from a memoir, by Gauss, on the elliptic elements of the orbit of the planet Pallas.*

Original Equations.†

1	$0 = -183''.93 + 0.79363\,dL + 143.66\,d\gamma + 0.39493\,d\pi + 0.95920\,d\varphi - 0.18856\,d\Omega + 0.17387\,di$
2	$0 = -6.81 - 0.02658\,dL + 46.71\,d\gamma + 0.02658\,d\pi - 0.20858\,d\varphi + 0.15946\,d\Omega + 1.25782\,di$
3	$0 = -0.06 + 0.58880\,dL + 358.12\,d\gamma + 0.26208\,d\pi - 0.85234\,d\varphi + 0.14912\,d\Omega + 0.17775\,di$
4	$0 = -3.09 + 0.01318\,dL + 28.39\,d\gamma - 0.01318\,d\pi - 0.07861\,d\varphi + 0.91704\,d\Omega + 0.54365\,di$
5	$0 = -0.02 + 1.73436\,dL + 1846.17\,d\gamma - 0.54603\,d\pi - 2.05662\,d\varphi - 0.18833\,d\Omega - 0.17445\,di$
6	$0 = -8.98 - 0.12606\,dL - 227.42\,d\gamma + 0.12606\,d\pi - 0.38939\,d\varphi + 0.17176\,d\Omega - 1.35441\,di$
7	$0 = -2.31 + 0.99584\,dL + 1579.03\,d\gamma + 0.06456\,d\pi + 1.99545\,d\varphi - 0.06040\,d\Omega - 0.33750\,di$
8	$0 = +2.47 - 0.08089\,dL - 67.22\,d\gamma + 0.08089\,d\pi - 0.09970\,d\varphi - 0.46359\,d\Omega + 1.22803\,di$
9	$0 = +0.01 + 0.65311\,dL + 1329.09\,d\gamma + 0.38994\,d\pi - 0.08439\,d\varphi - 0.04305\,d\Omega + 0.34268\,di$
9_a	$0 = +38.12 - 0.00218\,dL + 38.47\,d\gamma + 0.00218\,d\pi - 0.18710\,d\varphi + 0.47301\,d\Omega - 1.14371\,di$
10	$0 = -317.73 + 0.69957\,dL + 1719.32\,d\gamma + 0.12913\,d\pi - 1.38787\,d\varphi + 0.17130\,d\Omega - 0.08360\,di$
11	$0 = +117.97 - 0.01315\,dL - 43.84\,d\gamma + 0.01315\,d\pi + 0.02929\,d\varphi + 1.02138\,d\Omega - 0.27187\,di$

The probable error of one of these equations is $\mu = \pm 90''$, the weights being equal, excepting for 9_a, which has been excluded from each of the solutions.

As an illustration of the mode of applying the limits (43), we will make in (41) and (42) $g' = \frac{1}{25}$, we shall then have

$$\mu = \pm 90'', \quad n = 11, \quad \mathbf{A} = \pm 0.5, \quad \mathbf{M} = \pm 60'', \quad (x = dL) < \pm 100'', \quad da < \pm 0.013,$$
$$g' = \frac{1}{25}, \quad n' = 6, \quad \mathbf{B} = \pm 700, \quad \mu' = \pm 1''.3, \quad (y = d\gamma) < \pm 0''.07, \quad db < \pm 18.$$

C, **D**, **E**, and **F**, being of the same order of magnitude with **A**, we may conclude from (42), (43), and the above values of $d\,a$ and $d\,b$, that, if we reject in all the equations the two right-hand figures from the values of m, and the three right-hand figures from all the other numbers, writing, for instance, the first equation

$$0 = -184'' + 0.79\,dL + 140\,d\gamma + 0.39\,d\pi + 0.96\,d\varphi - 0.19\,d\Omega + 0.17\,di,$$

and the second

$$0 = -7'' - 0.03\,dL + 50\,d\gamma + 0.03\,d\pi - 0.21\,d\varphi + 0.16\,d\Omega + 1.26\,di,$$

. . . .

the probable error of the value of either of the quantities of dL, $d\gamma$ . . . . ., when the equations thus written are solved by the method of least squares, will exceed the prob-

* *Disquisitio de Elementis Ellipticis Palladis.* Comment. Soc. Reg. Gottingensis Recent. Vol. I.

† The computations necessary for the solution of these equations have been executed by Messrs. J. F. Flagg and T. H. Safford, jr.

able error of the same quantity obtained with the employment of the exact coefficients by less than $\frac{1}{1250}$ of that error.

We shall, however, for the present confine our attention to a direct comparison between the results of the solutions I., II., and III., and retain in each the exact coefficients of the original equations, adopting the constants A, C, D, E, $F = 1.0$, and $B = 1000.0$, and for the factors $\alpha\,\beta\,\ldots\ldots$ a somewhat ruder system of representation, especially in II., than that upon which the previous discussions have been based.

If we employ the limits (28), (29), we find that the most favorable equations are (1), (5), (7), and (10), and the least favorable (2), (4), (8), (11). Applying (III.) in the particular form to which it is limited in (28) and (29), we shall omit (2), (4), (8), and (11), as ineffective in the final equations for $d\,L$. (1), (5), (7), and (10) give

$$(P) = 1 \times 0.8^2 + 1 \times 1.7^2 + 1 \times 1.0^2 + 1 \times 0.7^2 = 5.0000,$$

and (2), (4), (8), and (11),

$$(p) = 1 \times 0.03^2 + 1 \times 0.01^2 + 1 \times 0.08^2 + 1 \times 0.01^2 = 0.0075.$$

$$\frac{(p)}{(P)} = \frac{\cdot 1}{667}.$$

According to (28), we must have, in this case,

$$H < \frac{1}{100},$$

in order to preserve the equivalency of (II.) and (III.), if the original equations (2), (4), (8), and (11) are omitted in forming the final equation for $d\,L$. We may, then, adopt the following system of factors.

d L.

No. of Original Equation.	Factors in the Method of Least Squares. I.	Equivalent Factors. II.	Equivalent Factors. III.
1	+ 0.79363	+ 0.8	+ 0.8
2	— 0.02658	— 0.03	0.0
3	+ 0.58880	+ 0.6	+ 0.6
4	+ 0.01318	+ 0.01	0.0
5	+ 1.73436	+ 1.7	$+\left(2-\frac{1}{4}\right)$
6	— 0.12606	— 0.1	— 0.1
7	+ 0.99584	+ 1.0	+ 1.0
8	— 0.08089	— 0.08	0.0
9	+ 0.65311	+ 0.7	+ 0.7
10	+ 0.69957	+ 0.7	+ 0.7
11	— 0.01315	— 0.01	0.0

The mean value of $H = \frac{\delta\alpha}{a w}$ for the equations (1), (5), (7), and (10) is

$$H = \frac{1}{200} \text{ or } < \frac{1}{100},$$

as required by (28).

For the second indeterminate $d\gamma$, the favorable equations are (5), (7), (9), (10). The unfavorable ones are (2), (4), (8), and (11).

$$(Q) = 10600000, \qquad (q) = 9400,$$

$$\frac{(q)}{(Q)} = \frac{1}{1130}, \qquad H \text{ must be } < \frac{1}{37}.$$

$d\gamma$.

No. of Original Equation.	Factors in the Method of Least Squares. I.	Equivalent Factors. II.	Eqnivalent Factors. III.
1	+ 143.66	+ 0.1	+ 0.1
2	+ 46.71	+ 0.05	0.0
3	+ 358.12	+ 0.4	+ 0.4
4	+ 28.39	+ 0.03	0.0
5	+ 1846.17	+ 2.0	$+\left(2-\frac{1}{7}\right)$
6	− 227.42	− 0.2	− 0.2
7	+ 1579.03	+ 2.0	+ 1.6
8	− 67.22	− 0.07	0.0
9	+ 1329.09	+ 1.0	$+\left(1+\frac{1}{3}\right)$
10	+ 1719.32	+ 2.0	+ 1.7
11	− 43.84	− 0.04	0.0

The mean value of $H = \frac{\delta\beta}{b w}$ for the equations (5), (7), (9), (10), is

$$H = \frac{1}{125} \text{ or } < \frac{1}{37},$$

the limit given by (29).

For $d\pi$ the favorable equations are (1), (3), (5), (9); the unfavorable, (2), (4), (11).

$$\frac{(r)}{(R)} = \frac{1}{675}, \qquad H \text{ must be } < \frac{1}{92}.$$

$d\pi$.

No. of Original Equation.	Factors in the Method of Least Squares. I.	Equivalent Facters. II.	Equivalent Factors. III.
1	+ 0.39493	+ 4.0	+ 4.0
2	+ 0.02658	+ 0.3	0.0
3	+ 0.26208	+ 3.0	+ 2.6
4	— 0.01318	— 0.1	0.0
5	— 0.54603	— 5.0	— 5.5
6	+ 0.12606	+ 1.0	+ 1.0
7	+ 0.06456	+ 0.6	+ 0.6
8	+ 0.08089	+ 0.8	+ 1.0
9	+ 0.38994	+ 4.0	+ 4.0
10	+ 0.12913	+ 1.0	+ 1.0
11	+ 0.01315	+ 0.1	0.0

The mean value of H is $\frac{1}{80}$, which exceeds the limit $H < \frac{1}{92}$.

For $d\varphi$ the favorable equations are (1), (5), 7), (10); the unfavorable, (4), (8), (9), (11).

$$\frac{(s)}{(S)} = \frac{1}{465}.$$

The value of H is imaginary, or $< \sqrt{\frac{1}{631} - \frac{1}{465}}$.

The rejection of the unfavorable equations cannot in this instance be compensated for by increasing the accuracy of treatment of the favorable ones.

$d\varphi$.

No. of Original Equation.	Factors in the Method of Least Squares. I.	Equivalent Factors. II.	Equivalent Factors. III.
1	+ 0.95920	+ 1.0	$+\left(1 - \frac{6}{100}\right)$
2	— 0.20858	— 0.2	— 0.2
3	— 0.85234	— 0.9	$-\left(1 - \frac{1}{7}\right)$
4	— 0.07861	— 0.08	0.0
5	— 2.05662	— 2.0	— 2.0
6	— 0.38939	— 0.4	— 0.4
7	+ 1.99545	+ 2.0	+ 2.0
8	— 0.09970	— 0.1	0.0
9	— 0.08439	— 0.08	0.0
10	— 1.38787	— 1.0	$-\left(1 + \frac{1}{3}\right)$
11	+ 0.02929	+ 0.03	0.0

The mean value of H is $\frac{1}{56}$; it should be as above < 0.

For $d\Omega$ the favorable equations are (4) and (11); the unfavorable ones, (7) and (9).

$$\frac{(t)}{(T)} = \frac{1}{344}.$$

H is again imaginary. In rejecting (7) and (9), we have passed the prescribed limits.

$d\,\Omega$.

No. of Original Equation.	Factors in the Method of Least Squares. I.	Equivalent Factors. II.	Equivalent Factors. III.
1	-0.18856	-0.2	-0.2
2	$+0.15946$	$+0.2$	$+\frac{1}{6}$
3	$+0.14912$	$+0.1$	$+\frac{1}{7}$
4	$+0.91704$	$+0.9$	$+(1-0.08)$
5	-0.18833	-0.2	-0.2
6	$+0.17176$	$+0.2$	$+\frac{1}{6}$
7	-0.06040	-0.1	0.0
8	-0.46359	-0.5	$-\frac{1}{2}$
9	-0.04305	-0.04	0.0
10	$+0.17130$	$+0.2$	$+\frac{1}{6}$
11	$+1.02138$	$+1.0$	$+(1+0.02)$

For $d\,i$ the favorable equations are (2), (6), (8), and the unfavorable (10).

$$\frac{(u)}{(U)} = \frac{1}{769}, \qquad H < \frac{1}{58}.$$

$d\,i$.

No. of Original Equation.	Factors in the Method of Least Squares. I.	Equivalent Factors. II.	Equivalent Factors. III.
1	$+0.17387$	$+0.2$	$+\frac{1}{6}$
2	$+1.25782$	$+1.0$	$+\left(1+\frac{1}{4}\right)$
3	$+0.17775$	$+0.2$	$+\frac{1}{6}$
4	$+0.54365$	$+0.5$	$+\frac{1}{2}$
5	-0.17445	-0.2	$-\frac{1}{6}$
6	-1.35441	-1.0	$-\left(1+\frac{1}{3}\right)$
7	-0.33750	-0.3	$-\frac{1}{3}$
8	$+1.22803$	$+1.0$	$+\left(1+\frac{1}{4}\right)$
9	$+0.34268$	$+0.3$	$+\frac{1}{3}$
10	-0.08360	-0.08	0
11	-0.27187	-0.3	$-\frac{1}{4}$

The mean value of H is $\frac{1}{59}$, the computed limit being $H < \frac{1}{58}$. Applying these factors to the original equations, we obtain, —

By the Method of Least Squares. I.

	Coef. of dL.	Coef. of $d\gamma$.	Coef. of $d\pi$.	Coef. of $d\varphi$.	Coef. of $d\Omega$.	Coef. of di.
0 = − 371″.09	+ 5.91569	+ 7203.91	− 0.09344	− 2.28516	− 0.34664	− 0.18194
0 = − 580104.00	+ 7203.91	+ 10834225.00	− 49.06	− 3229.77	− 198.64	− 143.05
0 = − 113.45	− 0.09344	− 49.06	+ 0.71917	+ 1.13382	+ 0.06400	+ 0.26341
0 = + 268.53	− 2.28516	− 3229.77	+ 1.13382	+ 12.00340	− 0.37137	− 0.11762
0 = + 94.26	− 0.34664	− 198.64	+ 0.06400	− 0.37137	+ 2.28215	− 0.36136
0 = − 31.81	− 0.18194	− 143.05	+ 0.26341	− 0.11762	− 0.36136	+ 5.62456

By the Equivalent Factors. II.

	Coef. of dL.	Coef. of $d\gamma$.	Coef. of $d\pi$.	Coef. of $d\varphi$.	Coef. of $d\Omega$.	Coef. of di.
0 = − 372″.23	+ 5.89945	+ 7208.64	− 0.04729	− 2.22332	− 0.33807	− 0.19932
0 = − 662.04	+ 7.85800	+ 11830.88	− 0.20089	− 3.15658	− 0.16440	− 0.48494
0 = − 1051.81	− 0.02274	+ 126.84	+ 7.02511	+ 10.51379	+ 0.45680	+ 3.10941
0 = + 137.76	− 1.90276	− 2453.82	+ 1.15752	+ 11.43969	− 0.33172	− 0.14026
0 = + 84.26	− 0.42389	− 250.17	+ 0.05158	− 0.45069	+ 2.27706	− 0.39471
0 = − 42.98	− 0.20005	− 247.16	+ 0.29878	− 0.04721	− 0.30249	+ 4.50959

By the Equivalent Factors. III.

	Coef. of dL.	Coef. of $d\gamma$.	Coef. of $d\pi$.	Coef. of $d\varphi$.	Coef. of $d\Omega$.	Coef. of di.
0 = − 371″.03	+ 5.97864	+ 7296.25	− 0.06706	− 2.33932	− 0.37876	− 0.08024
0 = − 560.48	+ 7.21446	+ 10853.11	− 0.05221	− 3.26575	− 0.20615	− 0.18982
0 = − 1061.36	− 1.13102	− 959.72	+ 7.19887	+ 11.91488	+ 0.34031	+ 3.07535
0 = + 251.17	− 2.11274	− 2917.00	+ 1.13986	+ 11.78434	− 0.37820	+ 0.08665
0 = + 97.46	− 0.29118	− 75.36	+ 0.07546	− 0.22584	+ 2.29772	− 0.39567
0 = − 54.37	− 0.12928	− 4.62	+ 0.26533	− 0.24548	− 0.37541	+ 5.56738

In order to test the numerical values in these final equations by (33), the decimal points for L, M, N, &c., and for the coefficients of $d\gamma$, should be changed three places to the left. This being done, we have for II.

$$\Sigma (P_1, Q_1, \ldots\ldots L_1, M_1, \ldots\ldots) = +\,57.87362,$$
$$\Sigma (\mathrm{S}\,\Sigma) = +\,57.87354.$$

And for III.

$$\Sigma (P_1, Q_1, \ldots\ldots L_1, M_1, \ldots\ldots) = +\,57.56917,$$
$$\Sigma (\mathrm{S}\,\Sigma) = +\,57.56916.$$

In both cases, the agreement is as near as could be desired.

We have, then, the following equations by successive eliminations: —

I.

		Coef. of dL.	Coef. of $d\gamma$.	Coef. of $d\pi$.	Coef. of $d\varphi$.	Coef. of $d\Omega$.	Coef. of di.
0 =	− 371″.09	+ 5.91569	+ 7203.91	− 0.09344	− 2.28516	− 0.34664	− 0.18194
0 =	−138534.00		+ 2458225.00	+ 62.13	− 510.58	+ 213.84	+ 73.45
0 =	− 115.81			+ 0.71612	+ 1.11063	− 0.06392	+ 0.25868
0 =	+ 25.66				+ 9.29213	− 0.36175	− 0.57384
0 =	+ 75.23					+ 2.22346	− 0.37766
0 =	+ 17.11						+ 5.42383

II.

		Coef. of dL.	Coef. of $d\gamma$.	Coef. of $d\pi$.	Coef. of $d\varphi$.	Coef. of $d\Omega$.	Coef. of di.
0 =	− 372″.23	+ 5.89945	+ 7208.64	− 0.04729	− 2.22332	− 0.33807	− 0.19932
0 =	− 166.23		+ 2229.06	− 0.13790	− 0.19504	+ 0.28590	− 0.21945
0 =	− 1041.71			+ 7.03449	+ 10.51875	+ 0.43566	+ 3.12386
0 =	+ 176.07				+ 9.01520	− 0.49449	− 0.72094
0 =	+ 100.43					+ 2.17690	− 0.46611
0 =	+ 16.00						+ 4.24758

III.

		Coef. of dL.	Coef. of $d\gamma$.	Coef. of $d\pi$.	Coef. of $d\varphi$.	Coef. of $d\Omega$.	Coef. of di.
0 =	− 371″.03	+ 5.97864	+ 7296.25	− 0.06706	− 2.33932	− 0.37876	− 0.08024
0 =	− 112.76		+ 2048.68	+ 0.02871	− 0.44288	+ 0.25090	− 0.09299
0 =	− 1108.40			+ 7.18029	+ 11.56325	+ 0.21715	+ 3.07926
0 =	+ 274.45				+ 9.07933	− 0.50448	− 0.43778
0 =	+ 117.10					+ 2.22130	− 0.43492
0 =	+ 30.76						+ 5.33957

From which are finally deduced the values of the six unknown quantities: —

	I.	II.	III.
di =	− 3″.15	− 3″.77	− 5″.76
$d\Omega$ =	− 34.37	− 46.94	− 53.85
$d\varphi$ =	− 4.29	− 22.41	− 33.50
$d\pi$ =	+ 166.44	+ 186.17	+ 212.41
$d\gamma$ =	+ 0.054335	+ 0.08963	+ 0.05115
dL =	− 3.06	− 56.32	− 14.58

The following are the outstanding errors of the original equations: —

	I.	II.	III.
1	− 111″.00	− 155″.51	− 127″.23
2	− 8.31	− 3.70	− 7.29
3	+ 59.18	+ 59.08	+ 84.73
4	− 36.67	− 47.02	− 54.55
5	+ 19.92	+ 21.77	+ 33.12
6	+ 0.07	+ 6.92	+ 19.63
7	+ 85.77	+ 54.52	+ 16.00
8	+ 25.01	+ 35.37	+ 38.66
9	+ 135.88	+ 157.61	+ 144.46
10	− 204.64	− 155.64	− 174.86
11	+ 83.44	+ 69.63	+ 64.28

The sums of the squares of these errors are

I.	II.	III.
$\Omega = 92919$	88556	85205

It is obvious that the solution I., as given by Gauss in the memoir above quoted, is incorrect, since the sum of the squares of the errors should be less by the method of least squares than by any other mode of combination.* A revised solution gives the following equations for I. Those for II. and III. are repeated for the sake of comparison.

I. (Revised solution.)

		Coef. of dL.	Coef. of $d\gamma$.	Coef. of $d\pi$.	Coef. of $d\varphi$.	Coef. of $d\Omega$.	Coef. of di.
0 =	− 371″.09	+ 5.91569	+ 7203.91	− 0.09344	− 2.28516	− 0.34664	− 0.18194
0 =	− 128204.00		+ 2061567.00	+ 64.72780	− 446.986	+ 223.4856	+ 78.50985
0 =	− 115.28			+ 0.71566	+ 1.11176	+ 0.05150	+ 0.25807
0 =	+ 276.46				+ 9.29666	− 0.53681	− 0.57179
0 =	+ 110.68					+ 2.20290	− 0.43212
0 =	+ 41.94						+ 5.40298

II.

		Coef. of dL.	Coef. of $d\gamma$.	Coef. of $d\pi$.	Coef. of $d\varphi$.	Coef. of $d\Omega$.	Coef. of di.
0 =	− 372″.23	+ 5.89945	+ 7208.64	− 0.04729	− 2.22332	− 0.33807	− 0.19932
0 =	− 166.23		+ 2229.06	− 0.13790	− 0.19504	+ 0.28590	− 0.21945
0 =	− 1041.71			+ 7.03449	+ 10.51875	+ 0.43566	+ 3.12386
0 =	+ 176.07				+ 9.01520	− 0.49449	− 0.72094
0 =	+ 100.43					+ 2.17690	− 0.46611
0 =	+ 16.00						+ 4.24758

* The errors of equations (10) and (11) given in the *Disq. de Elementis Palladis* are −216″.54 and +83″.01; the joint effect of these would increase still further the discrepancy in Ω.

III.

		Coef. of dL.	Coef. of $d\gamma$.	Coef. of $d\pi$.	Coef. of $d\varphi$.	Coef. of $d\Omega$.	Coef. of di.
$0 = -$	$371.03''$	$+\ 5.97864$	$+\ 7296.25$	$-\ 0.06706$	$-\ 2.33932$	$-\ 0.37876$	$-\ 0.08024$
$0 = -$	112.76		$+\ 2048.68$	$+\ 0.02871$	$-\ 0.44288$	$+\ 0.25090$	$-\ 0.09299$
$0 = -$	1108.40			$+\ 7.18029$	$+\ 11.56325$	$+\ 0.21715$	$+\ 3.07926$
$0 = +$	274.45				$+\ 9.07933$	$-\ 0.50448$	$-\ 0.43778$
$0 = +$	117.10					$+\ 2.22130$	$-\ 0.43492$
$0 = +$	30.76						$+\ 5.33957$

Giving, for the unknown quantities,

		I. (Revised solution.)	II.	III.
(46.)	$di =$	$-\ 7.76''$	$-\ 3.77''$	$-\ 5.76''$
	$d\Omega =$	$-\ 51.76$	$-\ 46.94$	$-\ 53.85$
	$d\varphi =$	$-\ 33.20$	$-\ 22.41$	$-\ 33.50$
	$d\pi =$	$+\ 219.19$	$+\ 186.17$	$+\ 212.41$
	$d\gamma =$	$+\ 0.05401$	$+\ 0.08963$	$-\ 0.05115$
	$dL =$	$-\ 15.68$	$-\ 56.32$	$-\ 14.58$

These results will form the basis for a comparison of the relative probability of the three systems.

In view of the apparently gross discrepancies of the solutions II. and III. from I., and with the admitted fact before us that the latter reduces perfectly to a minimum the sum of the squares of the residual errors, we should at first sight scarcely be disposed to admit that there is in reality no sensible difference in the accuracy of the three systems. This unfavorable impression must have its origin in the erroneous inference, that, inasmuch as I. is theoretically the best of all solutions, the errors of the results of other systems must be proportional to their deviation from I.

For instance, in I. we have $dL = -16''$, but in II. $dL = -56''$; from which is inferred that in II. the error of dL is proportional, or nearly so, to $-16'' + 56'' = +40''$; and in a similar way, $d\Omega$, $d\varphi$, &c. will be condemned or approved by the standard of their disagreement or accordance with the corresponding values in I.

Such a mistake may, we repeat, be easily committed, in a cursory glance at the numbers in question. That it is a violation of the fundamental principles of the method of least squares will appear on a very little consideration.

To apply the criterion proposed in that method for determining the relative probability of the three systems, we obtain first the residual errors of the original equations as follows:—

Residual Errors of Original Equations.

	I. (Revised solution.)	II.	III.
1	$-125''.45$	$-155''.51$	$-127''.23$
2	-9.14	-3.70	-7.29
3	$+86.62$	$+59.08$	$+84.73$
4	-53.78	-47.02	-54.55
5	$+32.12$	$+21.77$	$+33.12$
6	$+22.92$	$+6.92$	$+19.63$
7	$+21.00$	$+54.52$	$+16.00$
8	$+35.51$	$+35.37$	$+38.66$
9	$+149.37$	$+157.61$	$+144.46$
10	-169.77	-155.64	-174.86
11	$+66.95$	$+69.63$	$+64.28$

The sums of the squares of these errors are

I.	II.	III.
$\Omega = 85091$	88556	85205

The probable value of a residual error for one of the original equations — since there are eleven equations* and six indeterminates — will be obtained from the expression

$$\mu = 0.67459 \sqrt{\frac{\Omega}{11-6}},$$

giving the values

	I.	II.	III.
(47.)	$\mu = \pm 88''.00$	$\pm 89''.77$	$\pm 88''.06.$

The agreement of μ in I. and III. is a conclusive proof of the equivalency of the two solutions, notwithstanding the freedom which has been exercised in the choice of factors for the latter. In the case of II., the discrepancy amounts to $\frac{1}{50}$ of μ; but it will be noticed that the factors actually employed were based upon a system of representation considerably less exact than the series (2). It might easily be shown that this difference would have been reduced to less than one fifth of its present amount if the numbers for the factors had been selected from (2), agreeably to the conditions by which II. has been defined.

* Equation 9_a having been excluded from each of the solutions.

If we compute the limits of these values of μ, we find that it is only an even chance that μ, for the best solution, is comprised within the limits

$$\mu = \pm 69''.3, \qquad \text{and} \qquad \mu = \pm 106''.7.$$

Such is the extreme uncertainty of the only element by which the question of preference between I., II., and III. can be decided. Compared with it, the inconsiderable differences which we find between the values of μ in I., II., and III. will admit of but the single inference, that either of the systems of values of $d\,i$, $d\,\Omega$, $d\,\varphi$, &c., presented in (46), notwithstanding their great disagreements with each other, actually fulfils the criterion of accuracy proposed in the method of least squares so nearly, that it is impossible to give a decisive reason for adopting one rather than another as the most probable solution. It is worthy of notice, moreover, that the arithmetical mean of the above residual errors, irrespective of their signs, is *less* in II. and III. than in I. Thus in this instance the latter would rank lowest of the three, if we were to compute the relative probabilities according to a process recommended by the highest authorities* as the most suitable for ordinary use, in which the probable error is directly proportional to the arithmetical mean of the errors irrespective of their signs.

* Laplace, *Théorie Analytique des Probabilités ;* Gauss, *Zeitschrift für Astronomie*, B. I. ; Peters, *Astr. Nach.*, No. 1034.