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Abstract

The paper introduces to the capital budgeting literature the problem of how to select the amount of capital required to exploit a stock resource. Simulation is used to examine the solutions of a discounted cash flow, pretax model of venture profit. The sensitivity of solutions to model parameters is examined with special attention to the counterintuitive result that, for this problem, an increased cost of capital, r, results in higher investment over wide ranges of the value of r. Effects of depreciation and after-tax maximization are also examined as well as the possible impact of capital investment on output demand.

Optimal Exploitation of a Stock Resource

A problem that has not received adequate attention in the capital budgeting literature is the problem of providing long-lived assets for the development of a stock resource. The mining industry provides many examples of this kind, such as extraction of marble from a quarry, gravel from a pit, oil from a reservoir. A related problem is that of outfitting with special capital assets the firm which has a one-time, fixed-quantity contract. An example of the second kind would be the production of a given number of special units in a government contract for which there is no guarantee of renewal.

In such situations, the entrepreneur must often decide simultaneously whether to engage in the venture and if so, how large the capital equipment used to produce the stock resource (or the fixed output) should be. In many capital budgeting models, the size of the capital asset is not a control variable but rather is given. B. Rapp does cite a few examples of approaches in which researchers have treated size of investment as a continuous variable [4] and Weingartner dealt with the combined problem of capital asset variability and indivisibility by his integer programming model [5]. Usually however, optimal plant size, if it is not assumed to have already been decided on, is handled by the incremental analysis of larger and larger sizes of plant [2].

In this note, the capital budgeting model integrates the decision to invest with the selection of optimal fixed asset size, dealing with assets as continuous or integer valued as each situation requires. For stock resource development, the asset size decision will depend on how much freedom the entrepreneur has in controlling the rate of production and thus the length of life of each particular venture. In the case of mineral extraction, the miner can usually decide, subject to market conditions, his rate of production; in the case of fixed quantity contracts, the buyer may impose conditions on delivery dates and eliminate many of the entrepreneur's options regarding the rate of production and thus the necessary size of capital. The capital budgeting decision may be further complicated by the fact that, by increasing the size of the capital equipment utilized in order to achieve higher rate of output, the entrepreneur may affect the demand price for his product.

We shall first consider the situation in which the entrepreneur is a pricetaker and his decision to vary output will not affect the price of that output. We assume that the businessman faced with the simultaneous decision of whether to invest or not and if so, how to decide on the optimal level of capital investment will base his decision on net present value maximization. A discounted cash flow model will be presented in its least complicated form and primary reliance made on simulation for solutions and discussions of the model. Two reasons have led to greater dependence on simulation than on analysis: first, even for the most straight-forward models, reduced form solutions quickly become mathematically complex and second, simulation is relatively easy even after simplifying assumptions are dropped to accommodate "real world" intricacies.

Model 1

A businessman owns a quarry containing an estimated stock of Y units of gravel. The price of a single unit of gravel is p. Exploitation of the quarry requires a shovel which produces y units per period; production per period can be increased proportionally by the purchase of additional shovels. The initial cost of acquiring one shovel is CC and its operating cost per period is VC per shovel. Finally, if the venture is undertaken, fixed costs per period, FC, are incurred. Letting X be the decision variable (i.e., the number of shovels to be purchased), the net present value of the venture is

$$\Pi = X \cdot p \cdot y \sum_{n=1}^{N} \frac{1}{(1+r)^n} - X \cdot VC \sum_{n=1}^{N} \frac{1}{(1+r)^n} - FC \sum_{n=1}^{N} \frac{1}{(1+r)^n} - CC \cdot X \qquad (1)$$

where r is the discount rate and N is the life of the project.

Assuming that the project meets the firm's hurdle rate for the cost of capital, i.e., Π is positive for some values of X (or N since alternative solutions can be obtained in terms of optimal N), the optimal number of shovels may be selected. One approach to solving this problem is to take the first derivative of equation (1) with respect to X (or N) and set it equal to zero and solve for X (or N). The second derivative test can be performed to check that the point is the maximum. From the point of view of maximization of a function, if a function is strictly concave over the interval (a, b), there is a single unique global maximum that may be either in the interval or at one of the end points. See Appendix for the first and second order conditions for equation (1a) below.

At this point, a numerical example is provided to illustrate the capital budgeting decision.

Let

Y = a total stock of 1,000,000 units of gravel
y = 50,000 units of gravel per shovel/per period
p = \$5 sales price per unit of gravel

CC = \$400,000 per shovel
VC = \$ 20,000 per shovel/per period
FC = \$ 80,000 per period
r = 10% cost of capital per period

Note that N, the life of venture, varies as a function of X, i.e., $N = Y/yX$. For the above parameter values, equation (1) becomes

$$\Pi = (230{,}000X - 80{,}000)\ \left[\frac{1 - (1+r)^{-20/X}}{r}\right] - 400{,}000X$$

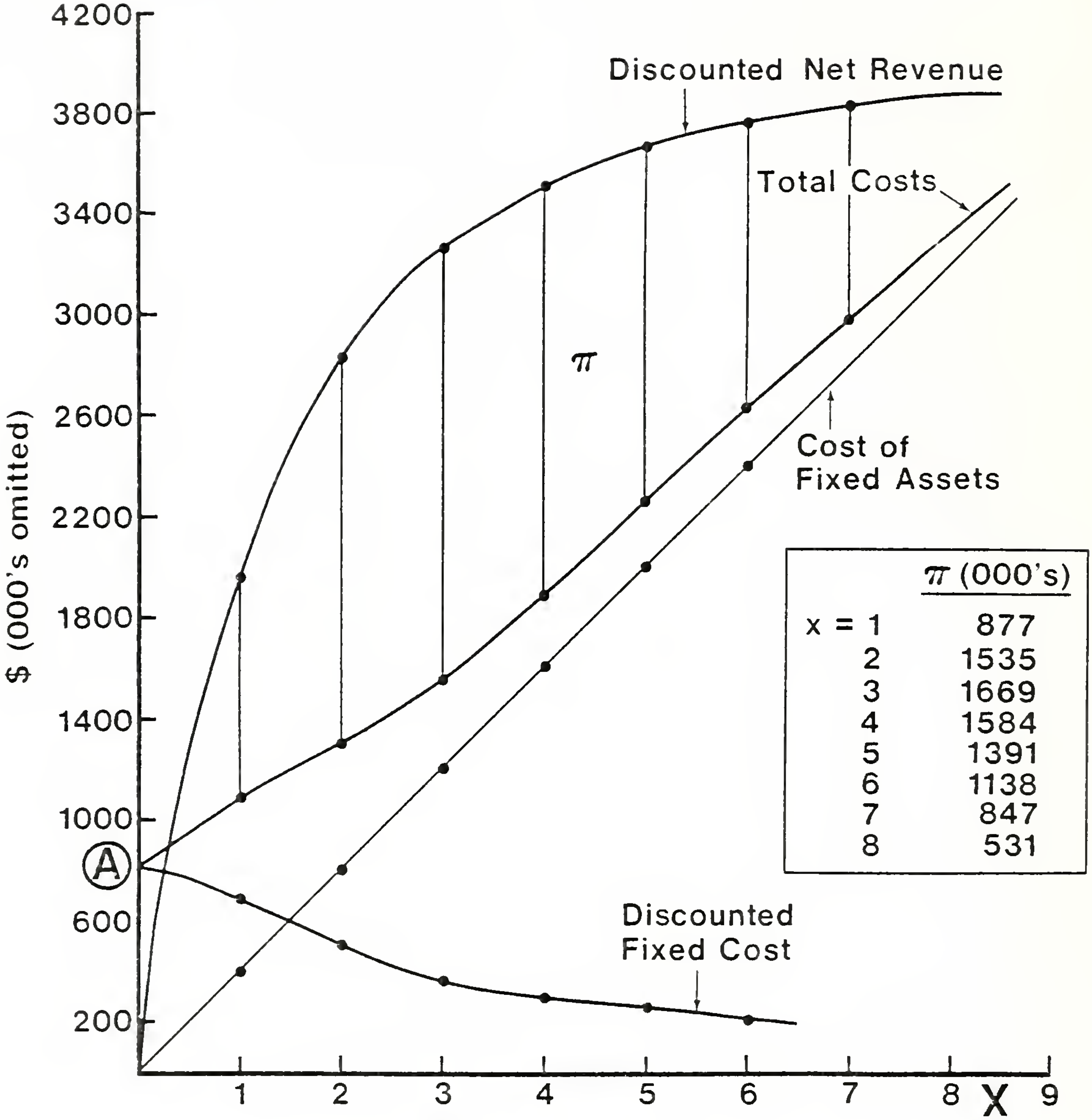
Equation (1a) takes its the maximum value when X equals 3.112 and the corresponding life of the venture is 6.426 periods. If we assume shovels (units of capital, X) to be integer values, for X rounded to 3 the net present value of Π is \$1,668,690. If X is rounded to 4, the net present value of Π is \$1,584,260. Therefore, due to the convexity of equation (1a), the optimal number of shovels is 3 for the numerical example.

Figure 1 graphs profit and shows that the advantages of increasing discounted total net revenue and decreasing discounted total fixed costs on the one hand must be balanced against the disadvantages of purchasing more shovels. Grant, et al. on page 203 [3] have a good example of a comparable situation. They show that when installing an oil pipeline the firm must consider the tradeoff between the immediate capital expenditure on larger pipe versus the increased annual expenditure incurred to run a pumping station. Economists would point out that profit is maximized when X is acquired so as to make the value of the discounted net revenue less the discounted savings in fixed costs due to an incremental shovel equal to the cost of one shovel. The point A on Figure 1 (\$800,000) can be thought of as a consol value, i.e., the present value of the fixed costs in perpetuity which would be incurred if the property were held but the project was never started. Mining companies which hold properties for future development incur such costs in the form of required land taxes and other annual fees.

Note that model (1a) assumes that the lives of the capital assets and the project are coterminous, i.e., that capital assets are committed to the project with no allowance for salvage value. The assumption of zero salvage value is reasonable for many investments in stock resource development such as soil removal for open pit mining or intangible cost of oil well drilling. If, however, some percentage of $CC \cdot X$, (αCCX) is expected to be recovered at the end of project life, a term $\alpha CC \cdot X\ (1 + r)^{-20/X}$ would have to be added to Π. It should be pointed out that, depending on the values of N, salvage values may change. The observation that salvage value can be a function of N has its parallel in the use of a bail-out factor in payback period analysis. We should also point out that variable costs (VC) have been made a function of the number of shovels rather than a function of output as is perhaps more commonly done. The same costs result, if we assume that once a shovel is acquired it is operated at capacity throughout the life of the project. If inflation is anticipated for p, VC and/or FC, the model can be expanded to take it into account.

Sensitivity Analysis

Obviously solutions depend on the particular values of the model's parameters. Most of the effects of parameter changes on optimal capital investment are predictable, at least directionally. Higher values of Y, p, and FC for example lead to higher values of X. Where the cost of capital, r, is concerned though, the effects are contrary to usual



	π (000's)
x = 1	877
2	1535
3	1669
4	1584
5	1391
6	1138
7	847
8	531

Exhibit #1

REVENUES, COSTS AND PROFIT (π)
FOR VARIOUS x
(Y = 1,000,000 and r = 10%)

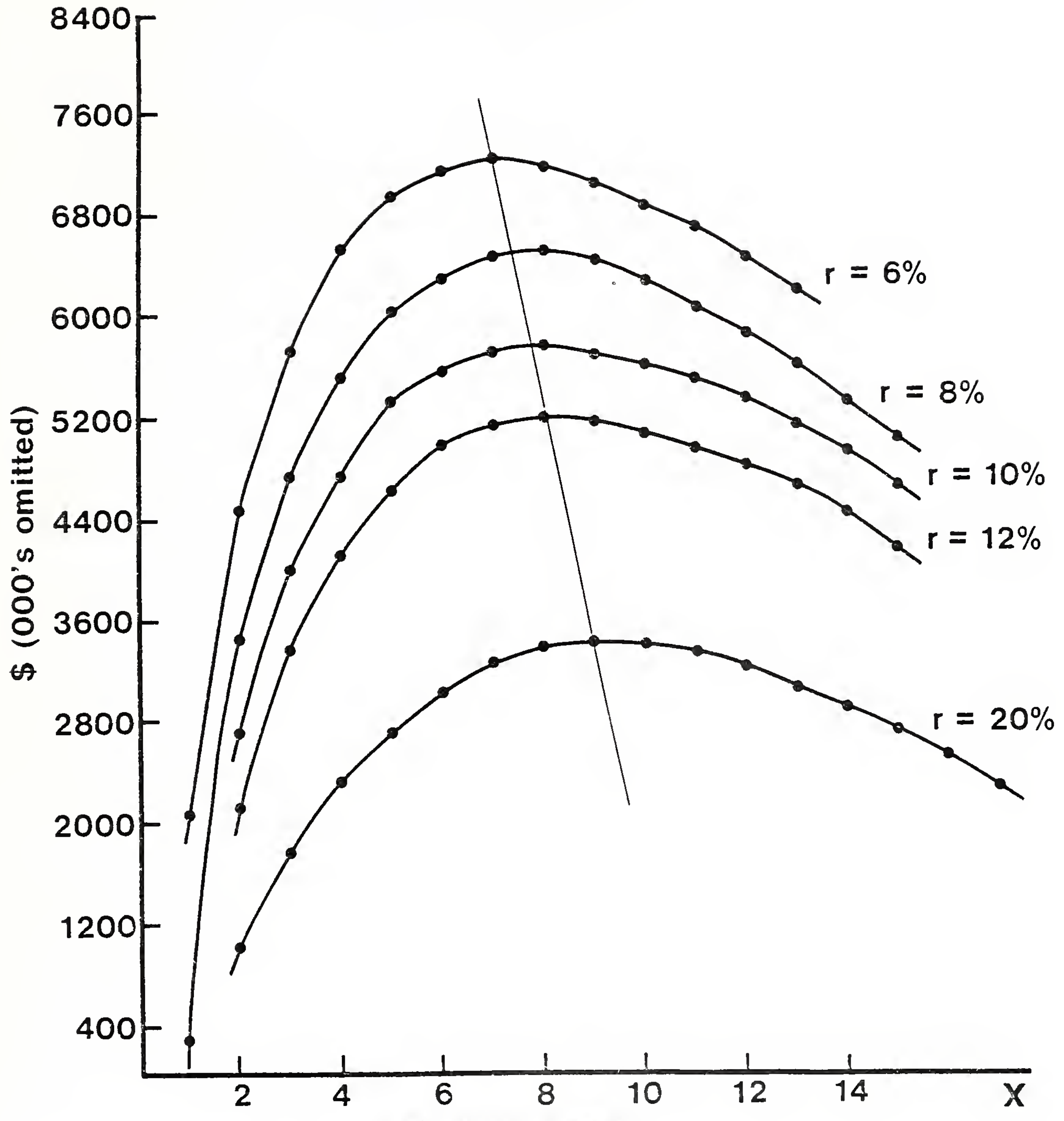


Exhibit #2

PROFIT, π, FOR VARIOUS VALUES OF r & x

(Y = 3,000,000)

capital budgeting results and somewhat couterintuitive. In order to increase the present value of net revenue and to diminish the present value of fixed cost outlays over time by shortening N, there are inducements to increase X, even when r is increasing. Figure 2 shows that for the same model [with Y = 3,000,000], optimal units of X increase as the cost of capital increases. Whereas the marginal efficiency of capital and the firm's investment opportunity schedule (Ackley [1], p. 473) are generally conceded to be downward sloping to the right (refer to line AB, in Figure 3), the firm's optimal demand for capital and for funding the development of a stock resource increases up to some level of higher interest rates and then gradually recedes, as in line CD. For example, if we examine optimal investment levels for model 1a allowing the cost of capital to vary from 0 percent to 50 percent and select for each interest rate the profit maximizing non-integer number of shovels, we see that investment in capital, line CD, increases as r increases from 0 percent to 28 percent, the point of maximum investment, after which it gradually decreases (profit itself becomes negative at 49%). As the ratio of fixed costs, FC, to capital cost CC, increases, so does the demand for capital, i.e., line CD moves directionally to C'D'.

Effect of Depreciation

Far from exhausting the subject, this note is intended primarily to open up a new direction for capital budgeting research. Two more points however will be made. The first relates to depreciation and its effect on taxes.

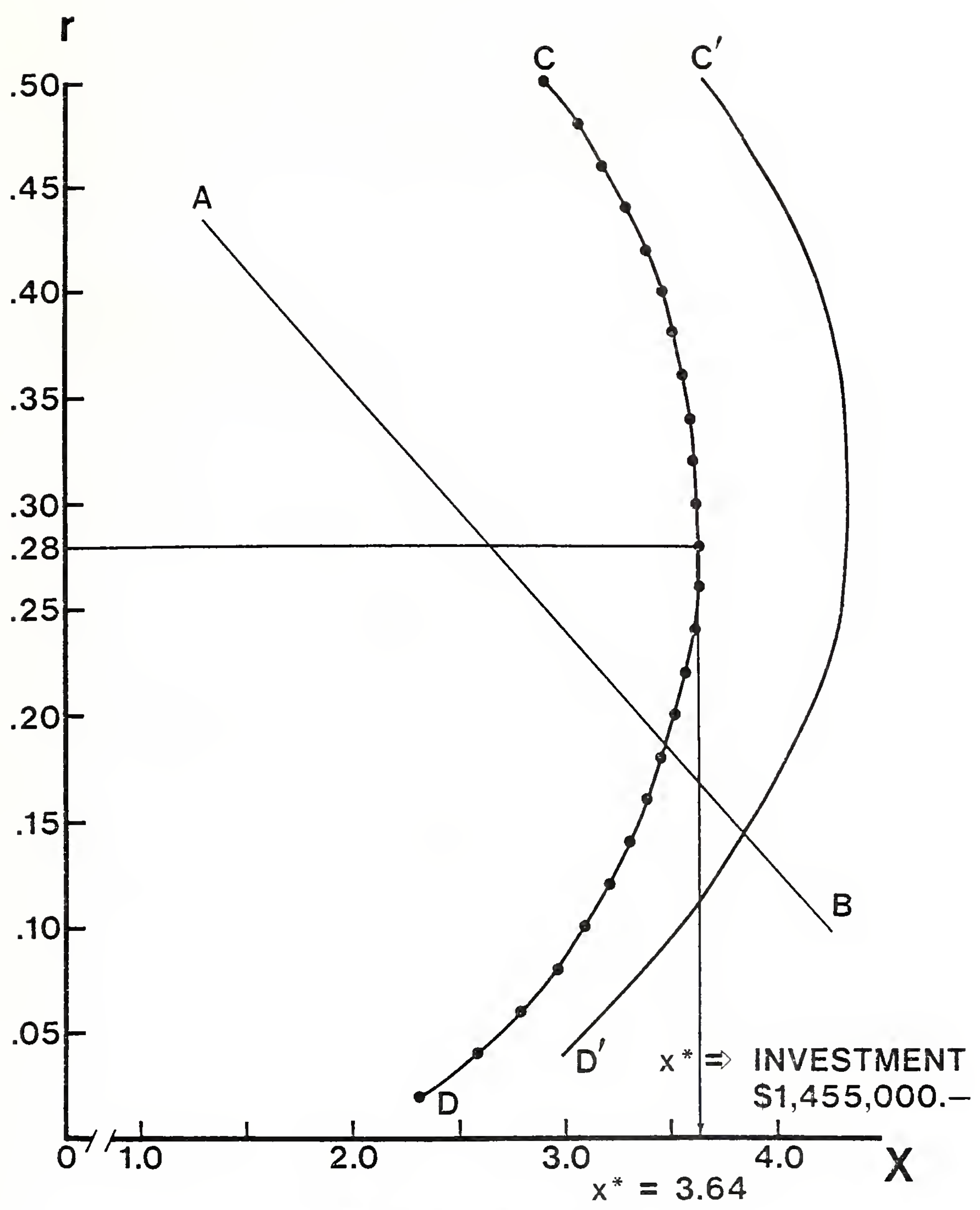


Exhibit #3

COMPARISON OF TRADITIONAL INVESTMENT OPPORTUNITY CURVE (AB) TO OPTIMAL INVESTMENT IN CAPITAL TO DEVELOP A STOCK RESOURCE (CD)

(Y = 1,000,000)

If we examine net present value after taxes, Π', equation (1) becomes:

$$\Pi' = (1 - t)(\Pi + CC \cdot X) + (t \cdot \text{depreciation})\left(\sum_{n=1}^{N} \frac{1}{(1=r)^n}\right) - CC \cdot X$$

for t equal to the tax rate.

If we maintain the assumption that all assets are committed to the venture at hand, and that no salvage will be recovered, then $CC \cdot X$ must be charged off completely (as mentioned above not unrealistic for development cost such as intangible well drilling, mineshafts and excavation costs). Therefore depreciation per period is $\frac{CCX}{N}$. Since $N = \frac{Y}{Xy}$, straightline depreciation in terms of X only is $\frac{CCX^2 \cdot y}{Y}$ where y/Y is a constant.

Thus Equation (2) becomes

$$\Pi' = (1-t)(\Pi + CC\ X) + \left[\frac{tCC \cdot X^2 y}{Y}\right] \cdot \left[\sum_{n=1}^{N} \frac{1}{(1+r)^n}\right] - CC \cdot X$$

and equation (1a) becomes

$$\Pi' = [(1-t)(230{,}000X - 80{,}000) + t20{,}000X^2]\left[\frac{1 - (1+r)^{-20/X}}{r}\right] - 400{,}000X$$

In all of the cases we have examined the combined effect of taxes and the quadratic tax shield is to push the profit function down and to the right as in Figure 4 and to increase investment in X, again a counter-intuitive result.

Effect of Investment on Firms Demand

Finally, we consider the case where the level of output can affect the price of output. Now in order to make an optimal capital budgeting decision, the firm must have an estimate of demand. Demand functions are usually

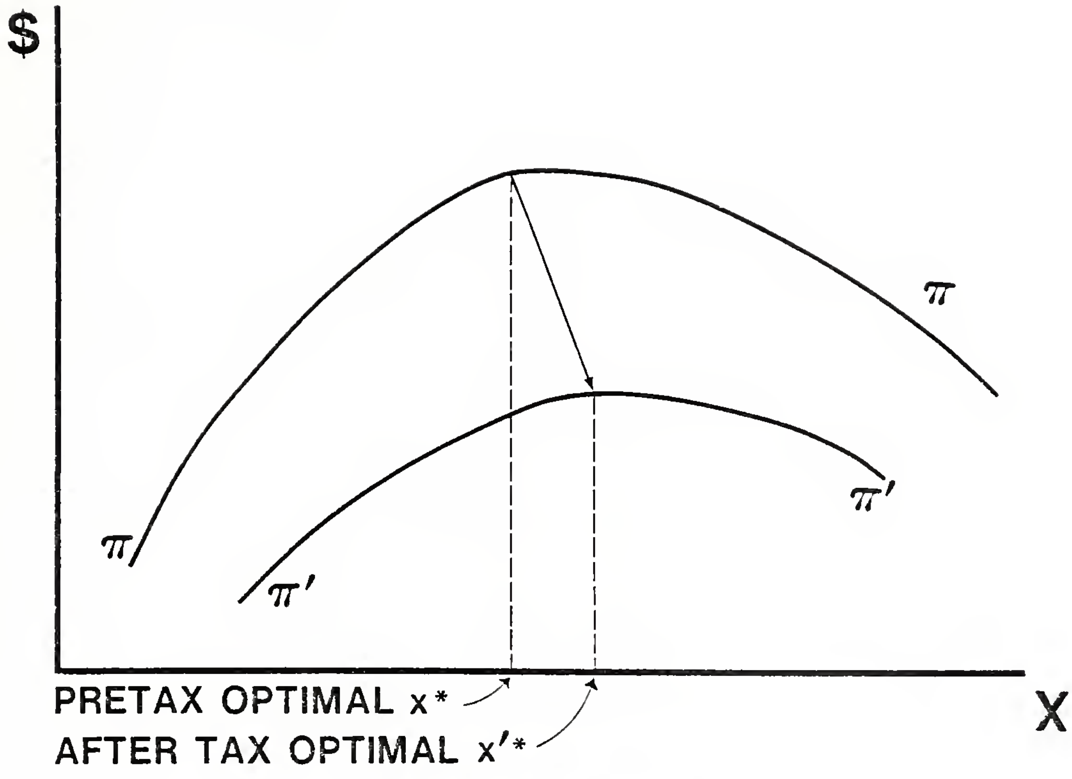


Exhibit #4

EFFECT OF TAXES AND
STRAIGHT LINE DEPRECIATION
ON PROFIT AND OPTIMAL CAPITAL

expressed in terms of output per period such as $P = a - b \cdot y \cdot X$ or $P = a - b'X$ (since y has been defined as output per period per unit of X).

As an example, assume that the demand function is $P = 6.5 - (\frac{1}{100})yX$ or $P = 6.5 - 0.5X$ then (equation 1a) can be restated as

$$\Pi'' = (50000X)(6.5 - 0.5X) \sum_{n=1}^{N} \frac{1}{(1+r)^n} - 20000X \sum_{n=1}^{N} \frac{1}{(1+r)^n} - FC \sum_{n=1}^{N} \frac{1}{(1+r)^n} - 400000X$$

The integer solution to equation (3a) is 2 shovels, and the resulting net present value is \$1,843,163. Recall that, for the case of a fixed price of \$5 (i.e., equation 1a), the optimal number of shovels and net present value are 3 and \$1,668,690 respectively. Based on the given demand function, the firm can increase the net present value by lowering its periodic production to 100,000 units (as opposed to 150,000 per period) thereby extending the life of venture and increasing the unit price. Extension of the life of a venture may not affect the optimal number of shovels when the discount rate is very low. For example, the optimal value of X is 2 for both equations (1a) and (3a) when the discount rate is 3% per period.

Conclusion

This note considers a problem in capital budgeting which has been largely neglected, namely the simultaneous decision to invest and to select the optimal amount of capital asset investment when a stock resource is to be developed. The relevance of this model is to extractive industries and especially to energy development is suggested. How the cost of capital affects optimal size of plant is shown as are the effects of depreciation on this particular capital budgeting situation. Finally, the note considers the possibility that the investment decision will impact on output demand and price.

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[3] Grant, E., Ireson, W. G. and Leavenworth, R. Principles of Engineering Economy, 6th Edition, New York, The Ronald Press Comp: ny, 1976.

[4] Rapp, B. Models For Optimal Investment and Maintenance Decisions, New York, John Wiley and Sons, 1974, p. I-12.

[5] Weingartner, H. M. Mathematical, Programming and the Analysis of Capital Budgeting Problems, Englewood Cliffs, Prentice Hall, 1963.

Appendix

This appendix shows the first- and second-order conditions for equation (1a). It also shows that the net present value function is strictly concave for $X \geq 1$. This is done by showing that the second derivative is negative for all $X \geq 1$. Differentiating with respect to X we get the following first-order conditions (000's omitted).

$$\Pi = (230X-80)\left[\frac{1-(1+r)^{-20/X}}{r}\right] - 400X$$

$$\frac{d\Pi}{dX} = 230\left[\frac{1-(1+r)^{-20/X}}{r}\right] - (230X-80)$$

$$\left[\frac{1}{r}(1+r)^{-20/X}\left(\frac{20}{X^2}\right)\ln(1+r)\right] - 400$$

the second-order condition is

$$\frac{d^2\Pi}{dX^2} = -230\left[\frac{1}{r}(1+r)^{-20/X}\left(\frac{20}{X^2}\right)\ln(1+r)\right]$$

$$-\frac{460}{rX}(1+r)^{-20/X}[\ln(1+r)]^2\left(\frac{20}{X^2}\right) + \frac{460}{rX^2}(1+r)^{-20/X}\ln(1+r)$$

$$+\frac{160}{rX^2}(1+r)^{-20/X}[\ln(1+r)]^2\left(\frac{20}{X^2}\right) - \frac{320}{rX^3}(1+r)^{-20/X}\ln(1+r)$$

$$= -\frac{920}{rX^3}(1+r)^{-20/X}[\ln(1+r)]^2 + \frac{320}{rX^4}(1+r)^{-20/X}[\ln(1+r)]^2$$

$$-\frac{320}{rX^3}(1+r)^{-20/X}\ln(1+r) = 0$$

The second-order condition is always negative for $X \geq 1$. The first term is obviously negative. By examining the second and the third term we can see the sum of the two are also negative.

That is, since $X \geq 1$, $\frac{320}{rX^3} \geq \frac{320}{rX^4}$

and since $0 < r < 1$, $0 < \ln(1+r) < 1$

$$\ln(1+r) > [\ln(1+r)]^2$$

therefore $\frac{320}{rX^3}(1+r)^{-20/X}\ln(1+r) > \frac{320}{rX^4}(1+r)^{-20/X}[\ln(1+r)]^2$

Since the left-side term is negative and the right-side term is positive, the sum of the two are negative. Therefore the function is concave.

This result can also be derived if the assumption of continuous discounting is made. For this case equation (1a) can be rewritten as:

$$\Pi = (230X-80)\,\frac{1}{r}\,[1 - \exp(-20r/X)] - 400X \tag{4a}$$

Taking the first derivative equation 4a we get

$$\frac{d\Pi}{dX} = \frac{230}{r}[1 - \exp(-20r/X)] - \frac{230X-80}{X^2}[(20\exp(-20r/X)] - 400 = 0$$

The second-order conditions of equation 4a are

$$\frac{d^2\Pi}{dX^2} = -\frac{230}{X^2}[\exp(-20r/X)]\,(20) - \frac{230}{X}[20\exp(-20r/X)](\frac{20r}{X^2})$$
$$+\frac{230}{X^2}[20\exp(-20r/X)] = 0$$

Simplifying we get

$$-(920rX-320r+320X)\,(\frac{1}{X^4}[\exp(-20r/X)]) = 0$$

Notice that $\frac{1}{X^4}[\exp(-20r/X)] \geq 0$.

And since $0 < r < 1$ and $X \geq 1$, then $-320r < 320X$.

Thus the second-order condition is negative for all $X \geq 1$ and all r.

Q.E.D

Table 1

Discounted Costs and Net Present Value When Y = 1,000,000

X	Discounted Fixed Cost	Cost of Fixed Cost	Total Costs	Net Present Value
1	681,088	400,000	1,081,088	877,034
2	491,568	800,000	1,291,568	1,534,935
3	375.920	1,200,000	1,575,920	1,668,690
4	303,264	1,600,000	1,903,264	1,584,260
5	253,592	2,000,000	2,253,592	1,391.756
6	217,144	2,400,000	2,617,144	1,138,331
7	190,536	2,800,000	2,990,536	847,298
8	168,896	3,200,000	3,368,896	531,453
9	152,064	3,600,000	3,752,064	198,388
10	138,840	4,000,000	4,138,840	-147,107

Table 2

Net Present Value When Y = 3,000,000

X	6%	8%	10%	12%	20%
1	2024214.16	1456482.77	1095073.59	848607.39	349986.69
2	4430635.84	3477957.67	2782227.50	2260969.91	1091995.83
3	5796651.94	4789069.92	3993273.87	3356360.61	1770443.64
4	6558289.15	5589962.10	4789106.78	4121126.17	2327397.02
5	6970713.02	6063603.48	5290650.25	4627980.42	2749961.90
6	7168113.17	6323105.82	5587937.24	4945289.94	3050213.71
7	7224924.95	6436916.82	5740773.71	5123398.77	3246883.66
8	7185156.55	6447794.32	5788723.60	5197695.80	3358063.40
9	7076324.77	6383513.48	5758513.86	5193078.24	3399143.39
10	6916460.00	6262792.85	5668678.75	5127324.26	3382632.46
11	6717847.06	6698664.69	5532406.92	5013368.12	3318537.58
12	6489139.95	5900462.90	5359308.54	4860800.22	3214840.53
13	6236584.58	5675039.06	5156537.28	4676860.58	3077927.23
14	5964837.40	5427531.65	4929520.56	4467106.21	2912936.95
15	5677405.91	5161867.45	4682446.55	4235867.30	2724045.49

Table 3

Optimal Investment

Interest Rate	Optimal X
2	2.329
4	2.588
6	2.794
8	2.969
10	3.112
12	3.233
14	3.334
16	3.417
18	3.485
20	3.540
22	3.581
24	3.611
26	3.629
28	3.637
30	3.634
32	3.620
34	3.596
36	3.561
38	3.514
40	3.455
42	3.384
44	3.297
46	3.193
48	3.068
50	2.917