ORIGINAL EXERCISES

IN

PLANE AND SOLID GEOMETRY

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NEW YORK .:. CINCINNATI .:. CHICAGO

AMERICAN BOOK COMPANY

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PREFACE

During the last few years the custom has become quite general in high schools and academies of giving a general review of preparatory mathematics during the last year of the course; and of that review a portion now universally recognized as holding a place of great importance is the original work designed to round out a student's course in plane and solid geometry. This collection of theorems, constructions, and numerical problems is designed to supply the material for this original work.

The exercises which make up the book are arranged somewhat promiscuously, the design being that the student shall not be restricted in his method of proof; just as he is not restricted in his method of proof for any original exercise he may be asked to solve when he attempts to pass his entrance examination to college.

In the preparation of these exercises the author has sought: —

1. To obtain variety, combined with proper gradation from easy to difficult problems.

2. To generalize whenever it was possible. In other words, to gather up in one problem as many kindred problems as seemed judicious.

3. To interest the student in the history of geometry.

4. To teach from time to time new principles.

The author has solved all the problems, to test their fitness for class work.

For assistance in the preparation of the work the author desires to acknowledge his indebtedness to Mr. J. W. Thomas, formerly instructor in mathematics in the Worcester Polytechnic Institute.

LEVI L. CONANT.

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ORIGINAL EXERCISES IN PLANE AND SOLID GEOMETRY

I. TRIANGLES AND QUADRILATERALS

1. The medians drawn from the equal angles of an isosceles triangle are equal.

Is this true of any other lines except the medians? Are the medians ever equal except in the case of the isosceles triangle?

2. Any altitude of a triangle is less than half the sum of the adjacent sides; and the sum of the altitudes is less than the perimeter.

Is this true also of the medians? of the angle bisectors?

3. Any line drawn through the intersection of the diagonals of a parallelogram, and terminated by the perimeter, bisects the parallelogram and also its perimeter, and is itself bisected by the intersection of the diagonals.

NOTE. The intersection of the diagonals of a parallelogram is called its *center*. Give a general definition of the term *center of a figure*.

4. A line drawn between two sides of a triangle, parallel to the base of the triangle and equal to half the base, bisects each of the other sides.

What other lines does this line bisect?

5. The medians of a triangle meet in a point; and this point divides each median into two parts such that the greater is twice the less.

NOTE. The intersection of the medians of a triangle is called the *centroid* of the triangle. It is, also, sometimes called the *center of gravity* of the triangle.

6. The diagonals of an isosceles trapezoid are equal.

7. State and prove the converse.

If a theorem is true, is its converse necessarily true? Give an example of this.

8. If any number of points, n, are so situated that not more than two of them lie on the same straight line, the number of lines that can be drawn connecting them, pair by pair, is $\frac{n}{2}(n-1)$.

9. The maximum number of points of intersection of n straight lines is $\frac{n}{2}(n-1)$.

A comparison of the last two theorems will show that either may be transformed into the other by a slight change in the wording. The *lines* of the former are replaced by the *points* of the latter, and the *points* of the former by the *lines* of the latter. This is an illustration of what is, in mathematics, known as the *law of duality*, a principle of the greatest importance in certain branches of that science.

10. If either leg of an isosceles triangle be produced from the vertex by its own length, and the extremity joined to the extremity of the base, the joining line is perpendicular to the base.

How much larger is the right triangle than the isosceles triangle?

11. The square described on the diagonal of a given square is twice that square.

12. Divide a straight line so that the square on one of its segments shall be double the square on the other.

13. The bisector of one of the base angles of an isosceles triangle makes with the opposite leg an angle of $61° 40'$. Find all the angles of the triangle.

14. If through any point in the base of an isosceles triangle lines are drawn parallel to the legs, a parallelogram is formed whose perimeter is equal to the sum of the legs of the triangle.

15. The lines drawn from either pair of opposite vertices of a parallelogram to the middle points of the parallel sides lying opposite the vertices from which they are drawn, trisect the diagonal connecting the other pair of vertices.

16. Two straight lines from two vertices of a triangle to the opposite sides respectively, cannot bisect each other, unless — .

Let the student complete and demonstrate the above.

17. What is the area of an equilateral triangle whose side is a?

18. What is the area of an equilateral triangle whose altitude is h?

19. Prove algebraically that (i) if m be an even number, then m, $\frac{1}{4}m^2-1$, $\frac{1}{4}m^2+1$ are numerically the sides of a right triangle; (ii) if n be an odd number, then n, $\frac{1}{2}(n^2-1)$, $\frac{1}{2}(n^2+1)$ fulfill the same condition.

The first part of the above theorem is due to the Greek philosopher Plato, and the second to Pythagoras.

20. The angle formed by the bisectors of two exterior angles of a triangle is equal to a right angle minus half the third angle of the triangle.

21. How many hexagons may be made to touch at one point? How many regular hexagons? How many equal regular hexagons? State the reason in each case.

22. The perpendicular bisector of the base of an isosceles triangle passes through the vertex, and bisects the vertical angle.

Which is the easier proof for this proposition, the direct or the indirect? What is an indirect proof?

23. The bisector of any angle of a triangle lies between the altitude and the median drawn from that vertex.

What is, then, always true of the relative magnitudes of these three lines?

24. If the hypotenuse of a right triangle be twice the shorter leg, one of the acute angles is twice the other.

25. If through the vertex of an angle of a parallelogram any straight line be drawn, the perpendicular to it from the vertex of the opposite angle is equal to the sum or difference of the perpendiculars to it from the vertices of the other two angles, according as the straight line lies without the parallelogram or intersects it.

26. A line connecting the middle points of any two sides of a triangle is parallel to the third side and equal to half that side.

27. If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

Is a square a rectangle? Is a rhombus a parallelogram? Is a square a rhombus?

28. If ABC be any triangle, and AD the bisector of the exterior angle between BA and CA produced, meeting CB produced in D, then $AB \cdot AC = DB \cdot DC - AD^2$.

Can this line ever be equal to the median of the trapezoid?

29. The angle formed by the bisector of one of the base angles of a triangle and the bisector of the exterior angle at the other extremity of the base is equal to half the vertical angle of the triangle.

30. The angle formed by the altitude and the angle bisector drawn from the same vertex of the triangle is equal to half the difference of the two remaining angles of the triangle.

Is this theorem true when the altitude falls outside the triangle?

31. If from the right angle of a right triangle the altitude, the median, and the angle bisector be drawn, the bisector is also the bisector of the angle formed by the other two.

What is the value of this angle in terms of the acute angles of the original triangle?

32. The greatest of the angle bisectors of a triangle falls on the least side.

33. What is the side of a regular hexagon whose area is A?

34. The bisector of either of the base angles of an isosceles triangle whose vertical angle is 36° divides the triangle into two isosceles triangles.

35. The sum of the perpendiculars drawn from any point within a regular polygon of n sides to each of these sides is constant, and is equal to n times the apothem.

36. Into how many equal parts can a rectangular strip of paper be divided by one cut of a knife or a pair of scissors?

37. The perpendiculars from two vertices of a triangle to the opposite sides divide each other into segments which are reciprocally proportional.

Is there any modification of this theorem in the case of the isosceles or the equilateral triangle?

38. Construct a square whose area shall be three times the area of a given square.

39. The bisectors of the angles of a quadrilateral form another quadrilateral whose opposite angles are supplementary. Consider each of the six kinds of quadrilaterals.

40. ABC is an equilateral triangle in which BD is perpendicular to AC, and DE is perpendicular to BC; then is EC equal to one third BE.

41. The lines joining the middle points of the sides of any quadrilateral, taken in order, form a parallelogram whose perimeter is equal to the sum of the diagonals of the quadrilateral. Examine each kind of quadrilateral.

42. Three angles of a quadrilateral are 80°, 85°, 110°; construct a quadrilateral, none of whose angles shall equal any angle of this quadrilateral.

43. If a line be drawn from the angle at one vertex of a triangle perpendicular to the bisector of another angle, and through their intersection a line be drawn parallel to the side opposite the first angle, the line last drawn will bisect each of the other two sides.

Hint. Continue the perpendicular till it intersects the opposite side or the opposite side produced.

44. The bisector of the angle C of the triangle ABC meets AB in D, and DE is drawn parallel to AC, meeting BC in E and the bisector of the exterior angle at C in F; prove $DE = EF$.

45. If two of the medians of a triangle be produced through the respective sides to which they are drawn, each by its own length, the line joining their external extremities will pass through one of the vertices of the triangle.

46. From the vertices of a parallelogram $ABCD$ perpendiculars are drawn, meeting the diagonals AC and BD in E, F, G, H, respectively; then is $EFGH$ a parallelogram similar to $ABCD$. Define similar parallelograms.

47. If lines be drawn from the vertices of a square, bisecting the opposite sides in order, a second square will be formed whose area is $\frac{1}{5}$ the area of the first square.

Can a square be formed in a similar manner within any other kind of parallelogram?

48. How many braces does it take to stiffen a three-sided plane frame? A four-sided frame? A five-sided frame?

49. If a pavement is to be laid with blocks whose upper faces are equal, regular polygons, these faces must be triangles, squares, or hexagons.

50. If the pavement is to be laid with blocks of two different kinds at each angular point, the upper faces being regular polygons with sides of the same length, it can be laid only with: —

(i) Triangles and squares.
(ii) Triangles and hexagons.
(iii) Triangles and dodecagons.
(iv) Squares and octagons.

51. If the pavement is to be laid with blocks of three different kinds at each angular point, the upper faces being regular polygons with sides of the same length, it can be laid only with: —

(i) Triangles, squares, and hexagons.
(ii) Squares, hexagons, and dodecagons.

52. The sum of the perpendiculars let fall from any point in the base of an isosceles triangle upon the legs is constant.

What if the perpendiculars are let fall from a point in the base produced? How may this theorem be extended in the case of equilateral triangles?

53. The diagonals of a trapezoid divide each other into segments that are reciprocally proportional.

54. The cross section of a bee's cell is a regular hexagon. Could any other form be used to so good advantage? Why?

The problem of the bee is to store away a maximum amount of honey while using a minimum amount of wax. The above question must, then, be studied with reference to both these conditions.

55. The sum of the diagonals of a quadrilateral is less than its perimeter, but greater than half its perimeter.

56. Is there a point within a triangle such that the area of the figure will be bisected by every line drawn through it? By any one line? By more than one line?

57. The lines which join the middle points of the sides of a triangle form a triangle whose area is one fourth the area of the original triangle.

58. The area of a triangle having an angle of 30° is one fourth the product of the sides including that angle.

59. The altitude dropped from the vertex upon the base of any isosceles triangle whose vertical angle is 120° is greater than half of either leg.

60. A line joining the middle points of the parallel sides of a trapezoid passes through the intersection of the diagonals, and also through the intersection of the non-parallel sides produced.

61. The sum of the perpendiculars dropped from any point within an equilateral triangle upon the three sides is constant. To what is it equal?

What if the point be without the triangle?

62. A line joining the middle points of the non-parallel sides of a trapezoid bisects each of the diagonals.

Does it bisect anything else?

63. Three given straight lines meet at a point; draw another straight line so that the two portions of it intercepted between the given lines may be equal to one another.

How can a second solution to this problem be obtained?

64. A perpendicular dropped from either extremity of the base of an isosceles triangle upon the opposite leg forms with the base an angle equal to half the vertical angle of the triangle.

65. Each of the lines joining the middle points of the opposite sides of a quadrilateral bisects the line joining the middle points of the diagonals of the quadrilateral.

HINT. The line joining the opposite sides and the line joining the middle points of the diagonals are the diagonals of a parallelogram.

66. If one angle of a triangle be double or triple another, the triangle can be divided into two isosceles triangles.

67. Lines drawn through the vertices of a triangle, parallel respectively to the opposite sides, enclose a triangle four times as large as the first triangle; and the sides of the second triangle are bisected by the vertices of the first.

68. In the triangle ABC, $a = 25$, $c = 16$, and the median on b is 12; find b.

69. If the square corner of a sheet of paper be folded any number of times so that the consecutive lines of folding are all parallel, and equidistant from each other, the consecutive areas thus obtained will be to each other as $1 : 3 : 5 : 7$, etc.

70. The altitudes of a triangle are to each other in the inverse ratio of the sides on which they fall.

If two sides of a triangle are 6 in. and 8 in. respectively, how much can be determined about the altitudes upon them?

71. The sum of the four lines drawn to the vertices of any quadrilateral from any point except the intersection of the diagonals is greater than the sum of the diagonals.

72. The line joining the middle points of the diagonals of a trapezoid is equal to half the difference of the parallel sides.

73. If the line joining the middle points of the diagonals of a trapezoid be extended to meet the non-parallel sides, the sum of the two remaining segments of this line is equal to the shorter of the two parallel sides of the trapezoid.

74. If any point within a triangle be joined to the three vertices, the sum of the joining lines is less than the perimeter but greater than half the perimeter.

Is this theorem still true if the point be taken on the perimeter of the triangle?

75. If from the acute angle A of a right triangle ABC, having its right angle at C, a line be drawn cutting BC in D, will $AB^2 + CD^2 = AD^2 + CB^2$?

76. What is the area of an equilateral triangle whose center of gravity is 6 in. from the vertex?

77. The bisector of any angle of a triangle and the bisectors of the exterior angles at the other two vertices are concurrent.

78. The altitude of a trapezoid is h, and its parallel sides are a and b, with $a < b$; find the area of the triangle formed by a and the non-parallel sides produced.

What is the area if $h = 4$ in., $a = 6$ in., $b = 8$ in.?

79. If ADC be a reëntrant angle of a concave quadrilateral $ABCD$, prove that the angle ADC, exterior to the figure, is equal to the sum of the interior angles A, B, C.

80. If any point E be connected with the vertices of a rectangle $ABCD$, then is $AE^2 + CE^2 = BE^2 + DE^2$.

81. The diagonals of a trapezoid are 10 in. and 15 in. respectively, and cut each other so that the segments of the latter are 6 in. and 9 in.; what are the segments of the former?

82. On a piece of paper of given size two lines are drawn whose intersection lies off the paper; show how to draw a line through any given point on the paper so that it would, if produced, pass through the intersection of the first two lines.

83. $ABCD$ is any parallelogram, and E is any point in the diagonal AC produced; prove that the triangles EBC and EDC are equal in area.

84. Bisect a parallelogram by a line drawn through a given point, either within or without the parallelogram.

85. What is the area of a trapezoid whose parallel sides are 25 and 35, and whose non-parallel sides are 5 and $\sqrt{65}$?

86. What part of a triangle is between the base and a line parallel to the base, passing through the center of gravity of the triangle?

87. The three lines drawn through the intersection of the bisectors of the angles of an equilateral triangle, parallel to the sides of the triangle, trisect the sides.

88. Two equal triangles stand with their bases on the same straight line, but with their vertices on opposite sides of this line; prove that the line joining the vertices of the two triangles is bisected by the line on which the bases stand.

89. If two triangles have two sides of the one equal respectively to two sides of the other, and the included angles supplementary, the triangles are equal in area.

90. If equilateral triangles are described outwardly on the three sides of any triangle, the lines joining the outer vertices of the equilateral triangles to the opposite vertices respectively of the given triangle cut each other at angles of 120°.

Are these lines equal? Are they concurrent?

91. $ABCD$ is a parallelogram, and E is any point without it; prove that the sum or the difference of the triangles EAB, ECD is equal in area to half the parallelogram.

When is it the sum and when the difference?

HINT. Draw through E a line parallel to AB and CD.

92. A triangle is equal in area to the sum or the difference of two triangles on the same base or on equal bases if its altitude is equal to the sum or the difference of the altitudes of the two triangles.

93. A parallelogram is equal in area to the sum or the difference of two parallelograms of the same or of equal altitudes if its base is equal to the sum or to the difference of the bases of the two parallelograms.

94. The perpendiculars from the vertices of an acute-angled triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of these perpendiculars.

What modification does this theorem undergo in the case of the obtuse-angled triangle?

95. A triangle having two sides equal respectively to the diagonals of a quadrilateral, and the included angle equal to either angle between these diagonals, has the same area as the quadrilateral.

Hint. From two opposite vertices of the quadrilateral drop perpendiculars to the diagonal connecting the other two vertices, and find the area of each of the four triangles in the quadrilateral in terms of these perpendiculars and segments of the diagonals. The sum of these will equal the area of the given triangle.

96. $ABCD$ is a parallelogram, and E is any point without the angle BAD and its opposite vertical angle; then is the triangle EAC equal in area to the sum of the triangles EAD and EAB. (See No. 92.)

97. In the preceding theorem, if the point E be within the angle BAD or its opposite vertical angle, the triangle EAC is equal in area to the difference between the triangles EAD and EAB. (No. 92.)

HINT. Extend AE; through C draw a line parallel to AE. The altitude of AEC can then be proved equal to the difference of the altitudes of the other two triangles.

98. $ABCD$ is a parallelogram, and through E, any point within it, lines are drawn parallel to the sides of the parallelogram; then is the difference of the parallelograms BE, DE equal in area to twice the triangle ACE. (See No. 97.)

99. ABC is a triangle, and D is any point in AB; draw through D a line DE to meet BC produced in E, so that the triangles DBE and ABC shall be equal in area.

100. On the sides of the right triangle ABC, right angled at C, construct the squares $ACDE$, $BCFK$, $ABLH$, outwardly from the triangle. Draw CN parallel to BL. In the figure thus formed, prove: —

(i) $AC^2 + CL^2 = CB^2 + CH^2$.
(ii) $EH^2 = BC^2 + 4\,AC^2$.
(iii) $LK^2 + EH^2 = 5\,AB^2$.
(iv) E, C, K are collinear.
(v) AK is perpendicular to CL.

II. THE CIRCLE

101. If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.

What modification of this theorem must be made for the isosceles triangle?

102. Three consecutive sides of an inscribed quadrilateral subtend arcs of 75°, 92°, 104°, respectively; find each angle of the quadrilateral, and the angles between the diagonals.

103. A triangle is formed by joining the points of contact of a circumscribed triangle; prove that any angle of the inscribed triangle is equal to a right angle minus half the opposite angle of the circumscribed triangle.

104. What is the locus (i) of the middle points of a system of parallel chords in a circle? (ii) Of the middle points of a system of equal chords?

105. The perpendiculars erected at the middle points of the sides of an inscribed quadrilateral are concurrent.

What if it be a pentagon? A hexagon?

106. The line joining the middle points of the non-parallel sides of a circumscribed trapezoid is equal to one fourth the perimeter.

What, then, is its value in terms of the non-parallel sides?

107. If two of the adjacent sides of an inscribed quadrilateral subtend arcs of 70° and 110°, respectively, and one of the angles formed by the diagonals is 95°, find each of the angles of the quadrilateral.

108. The line joining the middle points of the arcs subtended by the sides of an inscribed angle A intersects the sides of the angle in B, C, respectively; prove $AB = AC$.

109. The sum of the alternate angles of an inscribed octagon is equal to six right angles.

What is the generalization of the preceding theorem for any inscribed polygon having an even number of sides?

110. What is the ratio between the sides of an inscribed and of a circumscribed equilateral triangle?

Is the ratio the same in the case of the inscribed and circumscribed squares?

111. The angle formed by two tangents is equal to twice the angle between the chord of contact and the radius to the point of contact.

112. The radius of a circle is 6 in.; what is the locus of a point 4 in. from the circle? 6 in. from the circle? 8 in. from the circle?

113. Find a point equidistant from three intersecting lines.

How many solutions are there?

114. If any point on a circle be joined to the vertices of an inscribed equilateral triangle, the greatest of the joining lines is equal to the sum of the other two.

Hint. From the point on the circle, lay off on the longest joining line a segment equal to one of the other distances, and join the extremity of the segment to the extremity of that joining line to which it is equal.

115. The radius of a circle is 4 in.; find the altitudes of the inscribed and of the circumscribed equilateral triangle.

116. Two circles are tangent externally at A, and a common tangent touches the two circles in B, C, respectively; prove that the angle BAC is a right angle.

117. In the above problem, the line of centers of the two given circles is tangent to a third circle, whose diameter is BC.

118. One chord of a circle is twice as long as another; what is the ratio of their distances from the center of the circle?

119. If two circles are tangent internally, they cannot have the same center.

Find the locus of the center of a circle which has a given radius r, and which also:—

120. Passes through a given point.

121. Touches a given circle, internally or externally.

122. Cuts a given line so that the chord intercepted has a given length.

123. Cuts a given circle in a diameter.

124. Cuts a given circle orthogonally.

125. Through a given point within a circle draw a chord which shall be bisected by that point.

126. Two circles whose centers are A, B, are tangent, either internally or externally, and through their point of contact a line is drawn cutting the circles in C, D; prove that (i) the radii AC, BD, are parallel; (ii) the tangents drawn at C, D, are parallel.

127. All circles which pass through a fixed point, and have their centers on a given line, pass through a second fixed point.

128. If two equal chords of a circle intersect, the segments of the one are equal respectively to those of the other. This is a special case of what general theorem?

129. If two chords of a circle, drawn from the same point in the circumference, make equal angles with the tangent at that point, the chords are equal.

130. If two circles are tangent externally at A, the common tangent at A bisects each of the other common tangents.

131. The angle formed by the lines joining the opposite points of contact of the circumscribed quadrilateral $ABCD$ is either equal or supplementary to $\frac{1}{2}(A+C)$. When is it equal and when supplementary?

132. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

133. What is the locus of the vertex of an angle of constant magnitude, whose sides pass through the extremities of a line of constant length? Does the locus consist of one or of two lines?

134. If a tangent be drawn at any point in the convex arc included between two tangents drawn from the same exterior point, the perimeter of the triangle thus formed is constant.

135. Two circles intersect in A, B; through B a line is drawn meeting the circles in C, D, respectively; prove that the angle CAD is constant for all positions of the line CD.

HINT. It is sufficient if the sum of the angles C, D, is proved to be constant.

136. Two circles are tangent internally at A, and BC is a chord of the outer circle, tangent to the inner circle at D; prove that AD bisects the angle BAC.

HINT. Draw the common tangent.

137. Find a point equidistant from three lines, two of which are parallel.

138. Find a point equidistant from two adjacent sides of a quadrilateral, and also equidistant from the other two. Is there more than one solution?

139. The altitude of an equilateral triangle is 9 in.; find the radius of the inscribed and of the circumscribed circle.

140. Any two parallel lines drawn through the points of intersection of two circles, and terminated by the circles, are equal.

HINT. From the center of each circle, draw perpendiculars to each of the chords.

141. If any side of an inscribed quadrilateral be produced, the exterior angle thus formed is equal to the opposite interior angle.

142. Divide a given circle into two segments such that one of them shall contain an angle twice as great as the other.

143. If two chords intersect and make equal angles with the line joining their point of intersection to the center, they are equal.

144. AB is a fixed chord of a circle, and CD is any diameter; prove that the sum or difference of the perpendiculars from C, D to AB is constant.

145. Two circles intersect at A, B, and through A any two secants are drawn, one terminated by the circles in C, D, and the other in E, F, respectively; prove that the arcs CE, DF, subtend equal angles at B.

146. The bisector of any angle of an inscribed quadrilateral intersects the bisector of the opposite exterior angle, on the circle.

Is this true of all forms of inscribed quadrilaterals? Is there any kind of quadrilateral that can never, under any circumstances, become an inscribed quadrilateral?

147. Circles described on any two sides of a triangle as diameters intersect on the third side or the third side produced.

Notice what takes place when the triangle is isosceles; when it is equilateral.

148. Two circles intersect at A, B; through C, any point on one circle, lines CAD, CBE, are drawn cutting the other circle in D, E; prove that the chord DE is parallel to the tangent at C.

149. Two finite lines meet so that the product of the segments of one equals the product of the segments of the other; prove that the four extremities of the two lines are concyclic.

150. ABC is a triangle right angled at C; from any point D in BC a perpendicular DE is dropped on AB; prove that $CB \cdot DB = AB \cdot EB$.

151. Tangents drawn to two intersecting circles from any point in their common chord produced, are equal.

152. Of all lines drawn through one of the points of intersection of two circles and terminated by them, the greatest is parallel to the line of centers of the two circles.

153. Find the locus of the middle points of lines drawn from an external point to a circle.

154. The quadrilateral formed by tangents to a circle drawn at the extremities of a pair of diameters is a rhombus.

155. The bisectors of the angles of a circumscribed quadrilateral are concurrent.

156. The sides AD, BC, of the inscribed quadrilateral $ABCD$ are produced to meet at E, and a circle is circumscribed about ABE; prove that the tangent to this circle at E is parallel to CD.

157. Two circles are tangent, either internally or externally, at A; through this point two lines are drawn meeting one circle in B, C, and the other in D, E; prove that the chord DE is parallel to BC.

HINT. Draw the common tangent.

158. If the opposite angles of a quadrilateral are supplementary, the quadrilateral is inscriptible.

159. In a triangle if a perpendicular be drawn from one extremity of the base to the bisector of the opposite angle,

(i) It will make with either of the sides containing that angle an angle equal to half the sum of the angles at the base.

(ii) It will make with the base an angle equal to half the difference of the angles at the base.

160. The middle point of an arc subtended by a chord is joined to the extremities of another chord; prove that the triangles thus formed are similar, and the quadrilateral thus formed is inscriptible.

161. Find in a given line a point such that lines drawn from it to two given points are perpendicular to each other.

Note carefully the number of solutions.

162. The locus of the vertex of a triangle having a given base and a given angle at the vertex, is the arc forming with the base a segment which will contain the given angle.

163. The radii of two concentric circles are a and b, respectively; find the radius of a third circle which will be tangent to both the concentric circles, and will contain the smaller.

164. A ladder rests with one end on a horizontal floor and the other against a vertical wall; find the locus of the middle point of the ladder as the lower end slides along the floor in a direction at right angles to the wall.

165. Find the locus of the center of a circle tangent to two concentric circles. Of how many lines does the locus consist?

166. Six equal circles can be drawn about a circle of the same radius, so that each shall be tangent to the inner circle, and to two of the outer circles.

167. In the sides BC, CA, AB, of an equilateral triangle ABC the points D, E, F, are taken respectively, so that BD, CE, AF, are equal, each to each. The lines AD, BE, CF, are drawn, intersecting in G, H, K: prove that the triangle GHK is equilateral.

168. Through one of the points of intersection of two circles draw a chord of one circle which shall be bisected by the other.

169. AB, CD, are two parallel chords in a circle; prove that the points of intersection of AC, BD, and AD, BC, are collinear with the middle points of the chords.

What other point also lies in the same line with these?

170. In a given circle draw a chord equal to a given line and parallel to another given line. Between what limits in value must the former line lie?

171. In a diameter of a circle, produced indefinitely, find a point such that the tangent drawn from it to the given circle shall be of a given length.

Can this problem be solved algebraically?

172. Two circles intersect at A, B, and through A two diameters AC, AD, are drawn, one in each circle; prove that the points C, B, D, are collinear.

173. Through a given point without a circle, draw a line which shall cut off a segment capable of containing a given angle.

HINT. Reduce this problem to the problem of drawing a tangent to a circle from a given point without the circle.

174. Two intersecting circles are cut by a secant parallel to the common chord; prove that the two parts of the secant intercepted between the circumferences are equal.

175. The sides of a quadrilateral are made the diameters of circles; prove that the common chord of any two consecutive circles is parallel to the common chord of the other two.

HINT. Prove by means of No. 41.

176. A circle having for its center the middle point of one side of a triangle, and a radius equal to half the sum of the other two sides, is tangent to the circles having those sides as diameters.

177. The four common tangents to two circles external to each other intersect, pair by pair, on the line of centers of the circles.

178. Through a point without a circle, draw a secant such that the intercepted chord shall have a given value a.

179. Draw through a given point a line which shall be equidistant from two other given points. (Two solutions.)

180. Through a given point A on the circumference of the outer of two concentric circles, draw a chord that shall be trisected by the inner circle.

HINT. From A as a center, with a radius equal to the diameter of the inner circle, strike an arc intersecting the inner circle. This point of intersection will suggest a line which will give the solution.

181. If AB be a fixed chord of a circle, and C any point in either of the arcs subtended by it, the bisector of the angle ACB intersects the conjugate arc in the same point, whatever be the position of C.

182. AB, AC, are tangents to a circle from A, and D is any point on the circumference; prove that the sum of the angles ABD and ACD is constant for any point on the convex arc within the angle A, but changes as the point D passes either B or C.

183. A, B, C, D, are four points in order on a straight line, and EF is a common tangent to the circles having AC, BD, as diameters ; prove that the angle CAE is equal to the angle CEF.

184. The bisectors of the angles formed by producing the opposite sides of an inscribed quadrilateral are perpendicular to each other.

HINT. Prove by means of the arcs which measure the half angles thus formed.

185. The locus of the centers of circles inscribed in the triangles which have a given base and a given angle at the vertex is the arc forming with the base a segment capable of containing an angle equal to 90° plus half the given angle at the vertex.

186. Given any four points, A, B, C, D. Find a point E such that each of the angles AEB, CED, shall equal a right angle.

187. Given a chord AB, and a segment of a circle standing on it; find the locus of the intersection of the medians of all triangles having the chord as a base, and their vertices in the arc of the segment.

HINT. Trisect the chord. The locus is an arc on the second of the three equal portions.

188. Two parallel chords 1 in. apart are respectively 6 in. and 8 in. in length; find the radius of the circle.

Is there more than one solution?

189. A diameter of a circle is indefinitely produced; find in it a point such that two tangents to the circle shall contain a given angle.

190. On a given line as a base a system of rhombuses is constructed; find the locus of the intersection of their diagonals.

191. Through one of the points of intersection of two circles, draw a line terminated by the circles and bisected by the point of intersection.

HINT. From the middle point of the line of centers draw a line to the point of intersection of the circles. The required line can then be drawn without difficulty.

192. Through one of the points of intersection of two circles a line is drawn terminated by the circles; prove that the angle between the tangents at its extremities is equal to the angle between the tangents at the point of intersection of the circles.

193. Find a point such that the sum of the tangents drawn from it to a given circle shall equal the sum of two given lines.

194. Draw a tangent to a given circle so that the part contained within another given circle shall equal a given length a.

195. How can the distance be found between two objects on the ground, which are separated by a lake?

196. How can the distance from an accessible to an inaccessible point be found?

In connection with the preceding problem, the following anecdote is related of one of Napoleon's engineers. Coming, on one of his marches, to the bank of a river, Napoleon demanded to know the width of the stream. The engineer thus called upon said that he could not ascertain its width until his instruments arrived, which were in the rear with the baggage. The emperor insisted upon an immediate answer, but the engineer protested earnestly that it would be impossible to determine the distance without the aid of the proper instruments. "Tell me without delay the width of this river," thundered the impatient emperor. The engineer knew well the imperious temper of Napoleon, and felt his heart sink as he saw before him disgrace, and perhaps dismissal, as the result of failure. He hesitated but a single instant; then, drawing himself up erect to face the stream, he pulled his cap down over his eyes until the tip of its visor just touched the line of sight to the opposite bank. Then turning his body through a right angle, still preserving his rigid attitude, he marked with his eye the spot on the ground against which the tip of his visor now rested. Pacing off the distance thus determined, he returned to where the emperor was standing and said, "Your Majesty, the width of the river is, approximately, so many meters." The engineer was rewarded with instant promotion.

197. Draw a line parallel to a given line, and meeting the sides of an angle so that the part intercepted between the sides of the angle shall equal a given length a.

198. Between two points on the same side of a line, find the shortest path which shall touch the given line.

HINT. Solve by means of the principle of symmetry.

199. Draw a secant to two given circles so that the chords cut off may have given lengths a, b.

200. Draw a tangent to a given circle which shall be (i) parallel to a given line; (ii) perpendicular to a given line.

201. Find the locus of the centers of all circles which cut a given circle orthogonally at a given point.

202. A circle is described on one of the legs of a right triangle as diameter; prove that the tangent at the point where it cuts the hypotenuse bisects the other leg.

203. Two unequal segments of circles are described on the same side of the same chord, and the extremities of the chord are joined to any point in the outer arc; prove that the portion of the inner arc between these lines is constant.

204. Two circles intersect at A, B, and through C, any point on one of the circles, lines CA, CB, are drawn intersecting the other circle in D, E, respectively; find the locus of the intersection of AE, BD.

HINT. Prove that the angle formed by the intersecting lines is constant.

205. A straight rod AB slides between two rulers placed at right angles to each other, and from its extremities AC, BC, are drawn perpendicular to the respective rulers; find the locus of C.

206. Two equal circles intersect at A, B, and through A any straight line CAD is drawn meeting the circumferences in C, D, respectively; prove that BC is equal to BD.

207. From a point A, without a circle, a secant ABC and a tangent AD are drawn; if the bisector of the angle BDC meets BC at E, prove that $AE = AD$.

208. The bisectors of all angles inscribed in a given segment are concurrent.

Is the same theorem true of the medians?

209. Find the points of intersection of the circles whose diameters are the sides of an equilateral triangle.

210. $ABCD$ is an inscribed quadrilateral, and on AD as a chord, a circle is drawn cutting AB, DC, in E, F, respectively; prove that EF is parallel to BC.

211. If a circle be inscribed in a triangle whose sides are 8, 13, 17, respectively, find the value of each of the segments into which the sides are divided by the points of tangency.

212. If a tangent be drawn to a circle at the extremity of a chord, the middle point of the subtended arc is equidistant from the chord and the tangent.

213. Two circles whose diameters are as 1 : 2 are tangent internally; prove that the smaller circle bisects every chord of the larger drawn through the point of tangency.

214. In a circle whose center is O, the locus of the middle points of chords which pass through a given point P is the arc, comprised within the given circle, of a circle having OP for a diameter.

Notice each of the three cases, due to different positions of P.

215. If ABC be an equilateral triangle, find the locus of a point D, such that $DA = DB + DC$.

216. Find a point such that the three sides of a triangle are seen from it under equal angles.

When is this impossible?

217. Construct a circle which shall touch a given circle in a given point, and also pass through another given point.

218. C is any point on an arc whose chord is AB, and the angles CAB, CBA, are bisected by lines meeting at D; find the locus of D.

Is there a corresponding problem when the exterior angles are bisected?

219. The sides AB, BC, CD, of an inscribed quadrilateral subtend arcs of 95°, 105°, 115°, respectively; BA and CD produced meet at E, and AD, BC, at F; find the value in degrees of the angle AFB.

220. A given line revolves on a given point A as a pivot; find the locus of the foot of the perpendicular dropped on this line from a second point B.

221. If a circle be circumscribed about a triangle, the feet of the perpendiculars dropped from any point on the circle to the sides of the triangle are collinear.

HINT. Prove by means of inscriptible quadrilaterals.

NOTE. The line connecting these points is often called Simpson's line, from Robert Simpson, professor of mathematics at the University of Glasgow, to whom the discovery of this theorem is commonly attributed.

222. AB is a fixed diameter of a circle, and the chord AC is produced to D, so that $DC = BC$; find the locus of D as AC revolves on A as a pivot.

223. The feet of the medians and the feet of the perpendiculars let fall from the vertices of a triangle on the opposite sides are concyclic.

HINT. Pass a circle through the feet of the medians; then prove that the foot of any one of the three perpendiculars will lie on this circle.

224. The bisectors of the angles of an inscribed triangle ABC meet the circle in D, E, F, respectively; find each angle of the triangle DEF in terms of the angles of the triangle ABC.

225. Two circles intersect at A, B, and one of them passes through the center O of the other; prove that OA bisects the angle between the common chord and the tangent to the first circle at A.

226. Two circles intersect at A, B, and through any point in the chord AB two chords are drawn, one in each circle; prove that their four extremities are concyclic.

Is there any simple method of finding a fourth proportional except the one usually given in text-books on elementary geometry?

227. AB is a fixed diameter of a circle whose center is O; from C, any point on the circumference, a chord CD is drawn perpendicular to AB; prove that the bisector of the angle OCD cuts the circle in a fixed point as C moves over a semicircle.

228. Describe two circles of given radii to touch each other and a given line on the same side of the line.

How many solutions?

229. A series of circles touch a given line at a given point, and are cut by a line parallel to the given line; prove that all tangents to the various circles at the points where they are cut by the second line are tangent to a fixed circle whose center is the given point.

230. Two circles are tangent internally, and any third circle is drawn touching both; prove that the sum or the difference of the distances from the centers of the two given circles to the center of the third circle is constant.

231. Draw a line meeting the sides of the angle A in B, C, so that BC equals a given length a, and $AB = AC$.

Two circles, of any radii whatever, are tangent externally at A, and a direct common tangent touches them at B, C; if the centers of the circles are O, O', respectively, prove that: —

232. The bisectors of the angles BOO', $CO'O$, intersect at right angles on BC.

233. BC^2 is equal to the product of the diameters of the given circles.

234. Find the shortest path between two points lying within the sides of an angle, which shall touch both sides of the angle.

HINT. Prove by means of points symmetrical to the respective sides.

235. Two circles intersect, and through each point of intersection a secant is drawn terminated by the circles; prove that the chords joining their extremities are parallel.

236. The feet of the altitudes of a triangle are connected; prove that any two sides of the triangle thus formed make equal angles with that side of the original triangle on which they meet.

237. Through the extremity B of the diameter AB a tangent is drawn meeting the chords AC, AD, in E, F, respectively; prove that the triangles ACD, AEF, are similar.

238. From any point on a given arc perpendiculars are dropped upon radii drawn from the extremities of the arc; prove that the line joining the feet of these perpendiculars is constant.

239. ABC and $AB'C'$ are two triangles having a common angle A, and the circles circumscribed about them meet at D; prove that the feet of the perpendiculars from D to AB, BC, CA, $B'C'$, are collinear.

HINT. Prove by means of Simpson's line.

240. From a point A without a circle two tangents are drawn, and the chord of contact is cut at B by the line joining A to the center; prove that any circle passing through A, B, cuts the given circle orthogonally.

241. Divide a line into two parts such that their product shall be a maximum.

242. If a circle can be inscribed in an inscriptible quadrilateral, the lines joining the opposite points of contact are perpendicular to each other.

243. Two circles intersect at A, B, and at A tangents are drawn, one to each circle, to meet the circumferences at C, D; prove that the triangles ABC, DAB, are similar.

244. AB is a diameter of a circle, and AC, BD, are chords intersecting within the circle at E; prove that the circle passing through C, D, E, cuts the given circle orthogonally.

245. The tangents drawn at the points of contact of three circles that are tangent externally, pair by pair, are concurrent and equal.

246. A circle is tangent to a given circle and to a given line; prove that the two points of tangency and the extremity of the diameter of the given circle perpendicular to the given line are collinear.

247. Draw a line so that its distances from two given points shall be a and b respectively.

248. Draw a line through a given point so that the distances from this point to the feet of perpendiculars dropped on this line from two other given points shall be equal. (Two solutions.)

249. Draw a diameter to a given circle at a distance a from a given point on the circle.

NOTE. The student should, in any problem like the above, carefully observe that the construction is, in general, not possible. The conditions under which it is possible should then be determined, and the number of constructions noted.

250. Draw a chord to a given circle, equal to a and at a distance b from a given point on the circle.

251. Through two circles exterior to each other draw a secant such that the intercepted chords shall be equal.

NOTE. In many constructions like the above, the work can be greatly simplified by *reducing the problem to some other problem.* No. 251 can readily be reduced to the problem of drawing a tangent to two given circles.

252. A, B, are the middle points of the lesser arcs subtended by the chords CD, CE, respectively; if the line AB cut CD, CE, at F, G, respectively, prove $CF = CG$.

Is this true for all possible cases?

253. The opposite sides of an inscribed quadrilateral are produced to meet in the two exterior points A, B. Through the point A and the two nearest vertices of the quadrilateral a circle is drawn, and through B and the two vertices of the quadrilateral remote from B another circle is drawn. If these two circles intersect at C, then are the points A, C, B, collinear.

254. If three circles mutually intersect one another, the common chords are concurrent.

HINT. Let two of the chords intersect; from this point of intersection draw lines to the two remaining points of intersection of the circles; prove that either of these must, when produced, coincide with the other.

255. In any triangle the altitudes are produced to meet the circumscribed circle; then will each side bisect that part of the altitude perpendicular to it which lies between the orthocenter and the circumference.

256. If O be the orthocenter of the triangle ABC, the angles BOC, BAC, are supplementary.

257. AB is a fixed chord of a circle, and AC, BD, are any two parallel chords; prove that the line CD touches a fixed concentric circle.

258. If O be the orthocenter of the triangle ABC, then is any one of the four points O, A, B, C, the orthocenter of the triangle whose vertices are the other three.

259. The three circles which pass through two vertices of a triangle and its orthocenter are each equal to the circle circumscribed about the triangle.

260. On the semicircle whose diameter is AB, two points C, D, are taken, and the chords AC, BD, and AD, BC, are drawn, intersecting (produced if necessary) in F, G, respectively; prove that the line FG is perpendicular to AB.

261. ABC is an inscribed triangle, O its orthocenter, and AK a diameter; prove that $OCKB$ is a parallelogram.

262. The line joining the orthocenter of an inscribed triangle to the middle point of the base, meets the circle in the same point as the diameter drawn from the vertex of the triangle.

263. The perpendicular from the vertex of an inscribed triangle to the base, and the line joining the orthocenter to the middle point of the base are produced to meet the circumference in M, N, respectively; prove that the line MN is parallel to the base.

264. The distance from each vertex of a triangle to the orthocenter is twice the perpendicular from the center of the circumscribed circle to the opposite side.

265. Three circles are drawn, each passing through the orthocenter and two vertices of a triangle; prove that the triangle formed by joining their three centers is similar to the given triangle.

Is the triangle thus obtained equal to the given triangle?

266. The bisectors of the angles of a triangle meet at O, and are produced to meet the circumscribed circle in D, E, F; prove that O is the orthocenter of the triangle DEF.

267. The altitudes of the triangle ABC meet the sides upon which they fall in D, E, F, respectively, and the triangle DEF is drawn; prove that the triangles DEC, AEF, DBF, are similar to each other and to the original triangle.

268. Given the base and the opposite angle of a triangle; find the locus of the orthocenter.

269. Given the base and the opposite angle of a triangle; find the locus of the intersection of the angle bisectors.

270. Two circles whose centers are O, O', intersect at A, B, and the line CAD is drawn, terminated by the circles; find the locus of the intersection of CO, DO', and prove that it passes through B.

271. Two segments of circles are on the same side of the common chord AB, and C, D, are two points, one on each arc; find the locus of the intersection of the bisectors of the angles CAD, CBD.

272. Two equal circles are tangent externally to each other, and through the point of contact two chords are drawn, one in each circle, at right angles to each other; prove that the straight line joining their other extremities is equal to the diameter of either circle.

273. Given three points not in the same straight line; show how to find any number of points in the circumference of the circle passing through them, without finding the center.

274. Two equal circles intersect at A, B, and a third circle, whose center is A, and whose radius is less than AB, intersects them on the same side of AB in the points C, D; prove that the points C, D, B, are collinear.

275. Two circles intersect at A, B, and through A two lines, CAD, EAF, are drawn, terminated by the circles; if CE, DF, intersect at G, prove that C, B, D, G, are concyclic.

276. On the same side of the same chord three segments of circles are constructed, containing, respectively, a given angle, its supplement, and a right angle; prove that the intercept made by the two former arcs on any secant drawn through either extremity of the given chord is bisected by the last-named arc.

277. On the sides BC, CA, AB, in order, of the triangle ABC, any points D, E, F, are taken; prove that the circles circumscribed about the triangles AEF, BFD, CED, are concurrent.

278. ABC is an inscribed triangle, and from any point on the circle perpendiculars are drawn to BC, CA, AB, respectively, meeting the circle in A', B', C'; prove that the triangle $A'B'C'$ is equal to ABC.

279. Two tangents are drawn from a point to a circle; prove that the square of the radius of the circle is equal to the product of the line joining the external point and the center, multiplied by the inner segment of that line made by the chord of contact.

280. From the vertices A, B, of the triangle ABC, perpendiculars AD, BE, are dropped on the opposite sides respectively; prove $AC \cdot CE = BC \cdot CD$.

281. From any point in the common chord, produced, of a system of circles passing through two fixed points, tangents are drawn to all the circles; prove that the locus of the points of tangency is a circle cutting all the given circles orthogonally.

282. AB is a fixed diameter of a circle, and CD is a line perpendicular to AB, produced if necessary; if any line through A cuts CD at E, and the circle at F, then is the product $AE \cdot AF$ constant.

283. Find at what point in a given line the angle subtended by the line joining two given points on the same side of the given line is a maximum.

284. Given two lines AB, AC, and a point D between them; prove that, of all lines through D terminated by the two given lines, that which is bisected at D cuts off the minimum triangle.

HINT. Draw through D the line which is bisected at that point, and draw through D any other line terminated by AB, AC, in E, F, respectively. From one of the points E, F, draw a line lying within the sides of the angle and terminated by the line first drawn. Two equal triangles are now available, by the aid of which the theorem can be proved.

285. Find a point on a circle such that the sum of the squares of the distances to this point from two fixed points on the circle shall be a minimum.

HINT. Prove by means of the theorem which states that in any triangle the sum of the squares on two sides is equal to twice the square on half the third side plus twice the square on the median drawn to the third side.

286. Given the base and the opposite angle of a triangle; find the locus of the centroid.

287. Given the base and the opposite angle of a triangle; find the locus of the intersection of the bisectors of the exterior base angles.

288. Find the locus of the intersection of two lines intercepting on a circle an arc of constant length, and also passing through two fixed points on the circle.

289. AB is any diameter of a circle, and C, D, are two fixed points on the circle; find the locus of the intersection of AC, BD.

290. Of all triangles that can be inscribed in a given triangle, that formed by joining the feet of the altitudes has the minimum perimeter.

291. Two circles whose centers are O, O', intersect at A, B, and any line CAD is drawn, terminated by the circles; prove that the angle CBD is equal to the angle OAO'.

292. Three equal circles intersect at the point A, and their other points of intersection are B, C, D; prove that each of these four points is the orthocenter of the triangle formed by joining the other three.

293. From a given point without a circle, draw a line to the concave arc of the circle which shall be bisected by the convex arc.

294. Draw a line cutting two concentric circles so that the chord intercepted by the greater shall be twice the chord intercepted by the smaller circle.

295. From any point A on the circumference of a circle a perpendicular AB is drawn to CD, any chord of the same circle, and CE is drawn perpendicular to the tangent at A; prove that BE is parallel to DA.

296. ABC is an inscribed triangle, and from D, any point on the circle, perpendiculars are drawn to the sides BC, CA, AB, meeting the circle in A', B', C', respectively; prove that AA', BB', CC', are parallel.

297. From any point in AB, the common chord of two equal circles, a perpendicular is drawn to meet the circles on the same side of AB in C, D; prove that CD is of constant length.

298. From any point A on the circumference of a circle a perpendicular AB is drawn to CD, any diameter of the same circle, and on CB, BD, as diameters circles are described which cut CA, AD, at E, F, respectively; prove that EF is the common tangent to these two circles.

299. Two straight lines of indefinite length are tangent to a given circle, and any chord is drawn so as to be bisected by the chord of contact; if the former chord be produced, prove that the intercepts on it between the circumference and the tangents are equal.

300. D is any point on the circumscribed circle of the triangle ABC; prove that the angle between BC and Simpson's line for the point D is equal to the angle between AD and the diameter of the circle passing through A. (See No. 221.)

HINT. Prove by means of inscriptible quadrilaterals.

III. CONSTRUCTIONS

Construct a circle with its center in a given line which shall be: —

301. Tangent at a given point to another given line.

302. Tangent to two other given lines.

303. Tangent at a given point to a given circle.

When is the center of the required circle infinitely distant?

Construct a circle with its center on a given circle which shall be: —

304. Tangent to two given lines.

305. Tangent to a given circle at a given point.

306. Tangent to a given line at a given point.

307. Construct a circle which shall be tangent to a given line, and to a given circle at a given point.

308. Construct a circle which shall be tangent to a given circle, and to a given line at a given point.

309. Construct a triangle, having given the vertex, the orthocenter, and the center of the circumscribed circle.

HINT. Bisect that part of the altitude drawn from the given vertex, which lies between the orthocenter and the circumference of the circumscribed circle.

Construct a circle of given radius r, which shall: —

310. Pass through a given point and be tangent to a given line.

311. Pass through a given point and be tangent to a given circle.

312. Pass through a given point and cut a given circle in a diameter.

313. Pass through a given point and cut a given circle orthogonally.

HINT. Remember that at any given point on the circle the direction of the circle and the direction of the tangent at that point are the same.

314. Be tangent to a given line and to a given circle.

315. Be tangent to two given circles.

316. Be tangent to a given circle and cut another given circle orthogonally.

317. Be tangent to a given line and cut a given circle orthogonally.

318. Be tangent to a given line and cut a given circle in a diameter.

319. Cut from two given lines chords having given lengths a and b.

NOTE. Many problems like the last ten can be most readily solved by what is known as the "method of loci." Give a brief description of this method.

320. Construct a triangle having given the middle points of the three sides.

321. Construct a circle with its center in a given line, which shall cut this line at a given point and be tangent to another given line.

When is the number of solutions infinite?

322. Construct a circle with its center in a given line, which shall cut this line at a given point P and be tangent to a given circle.

HINT. Take PA on the given line equal to the radius of the given circle, and join the point A with the center of the circle.

323. Construct a circle which shall pass through a given point and be tangent to a given circle at a given point.

324. Construct a circle which shall be tangent to three given lines.

How many solutions are there?

325. Construct a circle which shall be tangent to two given lines, one of them at a given point.

326. The chord of a given segment is produced to a given length; on the part produced, construct a segment similar to the given segment.

Two segments are said to be similar when their arcs contain the same number of degrees, each to each.

327. Given a circle and its center; find the side of the inscribed square by means of the compasses alone.

NOTE. This problem is sometimes called "Napoleon's problem." It is said to have been proposed, and also to have been solved, by the emperor, Napoleon Bonaparte, who was especially fond of mathematics.

Through a given point P, draw a line meeting the sides of an angle A, in the points B, C, so that:—

328. AB and AC are equal.

329. PB and PC are equal.

330. BC is equal to twice AC.

331. BC is equal to BA.

Draw a line meeting the sides CA, CB, of the triangle ABC in D, E, respectively, so that:—

332. DE is parallel to AB, and equal to BE.

HINT. Bisect the angle B.

333. DE is parallel to AB, and CD is equal to BE.

334. DE is equal to a given length a, and CD is equal to BE.

HINT. Suppose the problem solved; then through C, E, draw lines parallel to DE, AC, respectively.

335. DE is parallel to AB, and DE is equal to the sum of AD and BE.

336. DE is equal to CD, and CD is equal to BE.

HINT. Suppose the problem solved; then draw BD.

337. Construct an equilateral triangle such that its sides shall pass through three given points.

338. Construct a triangle similar to a given triangle, with its sides passing through three given points.

339. Given the sum of two lines, and a square equal in area to the rectangle whose adjacent sides are the two lines respectively. Find the two lines.

340. Given the sum of two adjacent sides of a rectangle; to inscribe the rectangle in a given circle.

341. Given the sum of two lines, and the sum of the squares on them; determine each of the lines.

HINT. Let the required lines CA, CB, be the legs of a right triangle.

Construct a triangle, given:—

342. One angle, the side opposite, and the altitude on that side.

Is there more than one solution?

343. One angle, and the two segments into which the opposite side is divided by the bisector of that angle.

344. The radius of the circumscribed circle, and the segments into which one of the sides is divided by the bisector of the opposite angle.

345. One angle, the side opposite, and the angle between the median from the given angle and one of the adjacent sides.

346. A side, the median on that side, and the altitude on one of the remaining sides.

347. One of the angles, a right angle, the altitude on the hypotenuse, and one of the acute angles.

348. The triangle isosceles, the altitude on the base, and one of the base angles.

349. Two sides, and the altitude on the third side.

350. Two angles, and the altitude drawn from one of them.

351. Two sides, and the altitude on one of them.

352. The base, one of the base angles, and one of the angles between the base and the median on the base.

353. The base, the altitude on the base, and one of the angles between the base and the median on the base.

354. Two angles, and the bisector of one of them.

355. An angle, its bisector, and one side adjacent to the given angle.

356. An angle, its bisector, and one of the segments into which the given bisector divides the side on which it falls.

357. An angle, its bisector, and the altitude let fall from the vertex of this angle.

358. An altitude and an angle bisector drawn from the same vertex, and one of the segments into which the given bisector divides the side to which it is drawn.

359. A side, the altitude on one of the other sides, and one of the segments into which the last-named side is divided by the bisector of the opposite angle.

360. The difference of the two base angles, one of the segments into which the base is divided by the altitude on it, and the corresponding segment of the base made by the bisector of the opposite angle.

HINT. Using the outer extremity of the given segment as a vertex, construct on the inner side of the segment an angle equal to half the given difference of the base angles.

361. A side, the angle bisector drawn to one of the other sides, and the difference of the segments of the last-named side, which are formed by the altitude and the angle bisector, as described in No. 360.

362. Inscribe a rectangle in a circle, having given the difference of two adjacent sides of the rectangle.

HINT. Consider the problem solved. Take on the longer of the two sides of the rectangle a length equal to the shorter side, and complete the isosceles triangle. A diagonal and the given difference now form two sides of a triangle, which can be constructed.

363. Construct three equal circles such that each shall be tangent externally to the other two, and to a given circle.

364. Construct three equal circles such that each shall be tangent externally to the other two, and internally to a given circle.

365. In a square $ABCD$, construct an equilateral triangle AEF so that E, F, shall lie on sides of the square.

Construct a triangle, given: —

366. Two sides, and the median on the third side.

HINT. Consider the problem solved. Then complete the parallelogram.

367. The base angles, and the median on the included side.

368. An angle, and the median and the altitude drawn from the vertex of the given angle to the opposite side.

HINT. In the completed parallelogram, note the relation which exists between the given angle and the angles adjacent to it.

369. The median on one of the sides, and the angles made by the given median with each of the other sides.

370. An angle, the altitude on the side opposite, and the median on one of the remaining sides.

371. Construct three equal circles tangent to each other, pair by pair, so that the area inclosed within the convex arcs shall be 2 sq. in.

372. Construct three equal circles in an equilateral triangle so that each shall be tangent to the other two and to two sides of the triangle.

373. Construct in a square four equal circles so that each shall be tangent to two others and to two sides of the square.

374. Construct in a given triangle a semicircle having its diameter on one side, and having the other two sides as tangents.

375. Inscribe a circle in a given sector.

376. Construct a right triangle, given one of the acute angles and the sum of the hypotenuse and the side adjacent to the given angle.

Construct a triangle, given: —

377. One of the base angles, the altitude on the base, and the sum of the other two sides.

378. The base, the difference between the base angles, and the sum of the other two sides.

HINT. Suppose the problem solved. With the vertex as a center and the shorter of the two sides as a radius, describe a circle cutting the longer side (produced) in two points, and join these two points to the intersection of the base and the shorter side. This forms an auxiliary triangle, for whose construction sufficient data are given.

379. One of the angles a right angle, one of the sides about the right angle, and the difference between the hypotenuse and the remaining side.

HINT. Let ABC be the required triangle, right angled at C, and BC the given side. Produce AC to E, making AE equal to AB, and draw BE.

380. The triangle isosceles, the base, and the difference between one of the legs and the altitude on the base.

381. The base, the smaller of the base angles, and the difference between the two remaining sides.

382. An angle, the difference between the adjacent sides, and the difference between the two remaining angles.

HINT. In the constructed triangle, measure off from the given vertex on the longer side the length of the shorter, and join the intersection thus formed to the extremity of the shorter side. From the auxiliary triangle thus formed at the base, the required triangle can be constructed.

383. The altitude on the base, the greater of the base angles, and the difference between the two remaining sides.

384. The perimeter, the altitude on the base, and the angle opposite the base.

HINT. Produce the base of the constructed triangle at each extremity by a length equal to the adjacent side, and join the extremities of the extended base to the vertex.

385. The perimeter, the greater of the base angles, and the shorter of the segments into which the base is divided by the altitude on the base.

386. The two base angles, and the difference between the sum of the two shorter sides and the longest side — in this case the base.

HINT. Through the vertex C of the constructed triangle ABC produce one of the shorter sides, BC, to D making $CD = AC$. On BD mark off $BE = AB$, connect AD and AE. An auxiliary triangle ADE is thus formed, which can be constructed from the given data.

387. The segments into which the base is divided by the bisector of the opposite angle, and either of the base angles.

388. The two base angles, and the greater of the segments into which the base is divided by the bisector of the opposite angle.

389. In a given square construct a square having a given side, so that its vertices shall lie on the sides of the given square.

HINT. The squares have the same center. Use the semi-diagonals.

390. Inscribe a square in the part common to two equal intersecting circles.

Construct a triangle, given: —

391. The three medians.

Hint. With double the medians as sides, construct a triangle. Complete the parallelogram, and draw the other diagonal.

392. Two sides, and the difference between the angles opposite those sides.

Hint. In the required triangle ABC let AC, BC, be the given sides. Draw CD equal to CA to meet the base in D. What is the size of the angle BCD?

393. Two sides, and the difference between the segments into which the third side is divided by the altitude on it.

394. The difference between the base angles, the greater of the two sides lying opposite the base angles, and the difference between the segments into which the base is divided by the altitude on it.

395. The smaller of the base angles, the difference between the two remaining sides, and the difference between the segments of the base made by the altitude on it.

396. One angle a right angle, the difference between the legs, and the difference between the segments of the hypotenuse made by the altitude on it.

Hint. In the completed triangle ABC let C be the right angle, and $BC > AC$. Draw the circle suggested in the hint to No. 378, and let it cut BC in E, BC produced in D, and AB in F. The triangle BFE forms an auxiliary triangle, which can be constructed if the value of the angle BFE can be ascertained.

397 The smaller of the base angles, and the segments into which the base is divided by the bisector of the opposite angle.

398. The difference between the base angles, and the segments of the base as in the preceding problem.

399. The shorter of the segments of the base made by the bisector of the opposite angle, the difference between the base angles, and the difference between the two remaining sides.

400. The smaller of the base angles, the angle opposite the base, and the difference between the segments of the base as in No. 397.

401. The radius of the circumscribed circle, the triangle isosceles, and one of the base angles.

402. The base, the larger of the base angles, and the radius of the circumscribed circle.

403. The base angles and the radius of the circumscribed circle.

404. The larger of the base angles, the difference between the other two sides, and the radius of the circumscribed circle.

405. The angle opposite the base, the sum of the sides adjacent to this angle, and the radius of the circumscribed circle.

406. The segments of the base made by the bisector of the opposite angle, and the radius of the circumscribed circle.

407. The median on the base, the difference between the segments into which the base is divided by the altitude on it, and the radius of the circumscribed circle.

HINT. The given difference may be conveniently represented by a chord parallel to the base of the triangle.

408. The altitude and the median on the base, and the radius of the circumscribed circle.

409. The base angles, and the radius of the inscribed circle.

410. The base, the greater of the base angles, and the radius of the inscribed circle.

411. The altitude on the base, either of the base angles, and the radius of the inscribed circle.

412. The altitude on the base, the greater of the segments into which that altitude divides the base, and the radius of the inscribed circle.

413. One of the angles, its bisector, and the radius of the inscribed circle.

Construct a rectangle, given: —

414. A diagonal, and the sum of two adjacent sides.

415. The sum of two adjacent sides, and the angle between a side and a diagonal.

416. The difference between a diagonal and one side, and the angle between a diagonal and the other side. (See No. 362.)

Construct a rhombus, given: —

417. A side, and the sum of the diagonals.

418. An angle, and the sum of the diagonals.

419. An angle, and the sum of a side and the longer diagonal.

Construct a rhomboid, given: —

420. The altitude, a diagonal, and the angle between the diagonals.

421. An angle, and the diagonals.

422. A side, the difference between the diagonals, and the angle between the diagonals.

423. An angle, a diagonal, and the sum of two adjacent sides.

424. An angle, the longer diagonal, and the difference between one of the shorter sides and the altitude.

425. An angle, one of the shorter sides, and the difference between the longer diagonal and one of the longer sides.

Construct an isosceles trapezoid, given: —

426. A diagonal, a leg, and the altitude.

427. A base angle, a diagonal, and the sum of the longer base and a leg.

Construct a trapezoid, given: —

428. The altitude, one of the base angles, the greater base, and the side adjacent to the greater base, remote from the given angle.

429. The greater base, the diagonals, and the angle between the diagonals.

NOTE. In constructing trapezoids, some or all the following lines will be found useful. Let $ABCD$ be the trapezoid, and AB the greater base. Draw CE parallel to DB to meet AB produced; CF parallel to DA; AH, EH, parallel to CE, CA, respectively; and HF, HB, HC.

430. The bases, one of the remaining sides, and the angle between the diagonals.

431. The altitude, the base angles, and the angle between the diagonals.

HINT. See note to No. 329. What is the size of the angle CAH?

432. The sum of the angles at the base, the legs, and the angle between the diagonals.

433. The difference between the bases, the other two sides, and the angle between the diagonals.

434. The base angles, the altitude, and the difference between the greater base and one of the adjacent sides.

435. The sum of the greater base and one of the adjacent sides, the other two sides, and the angle between the two sides whose sum is given. How many constructions?

436. The sum of the bases, the other two sides, and one of the angles at the base.

HINT. Find the difference between the bases.

437. The diagonals, the angle between them, and the difference between the greater base and one of the adjacent sides. (See Nos. 429 and 362.)

Construct an inscribed quadrilateral, given: —

438. The radius of the circumscribed circle, two adjacent sides, and the angle between the diagonals.

439. The radius of the circumscribed circle, one of the diagonals, the angle at the vertex from which that diagonal is drawn, and the angle between the diagonals.

440. The radius of the circumscribed circle, the difference between two adjacent sides, the diagonals joining the extremities of these sides, and the angle between the diagonals.

441. A side, the two adjacent angles, and a diagonal.

442. A side, two angles adjacent to each other and one of them adjacent to the given side, and the angle between the diagonals.

HINT. The diagonals divide the quadrilateral into two pairs of similar triangles. Find the angle formed by a diagonal and one of the sides adjacent to the given side.

443. The sum of a diagonal d and a side a, the side opposite a, and a third side meeting a in the vertex remote from the intersection of a and d.

444. The diagonals, an angle, and one of the angles between the diagonals.

Construct a tangent trapezoid, given: —

445. The radius of the inscribed circle, one leg, and an angle adjacent to the other leg.

446. The bases, and one of the legs.

HINT. The sum of two opposite sides of a tangent quadrilateral is equal to the sum of the other two sides.

447. The radius of the inscribed circle, the shorter base, and the difference between the longer base and a leg.

448. The sum of the bases, and the base angles.

Construct a tangent quadrilateral, given: —

449. Two adjacent angles, a diagonal, and the radius of the inscribed circle.

450. Two opposite sides, the sum of the angles adjacent to one of them, and the radius of the inscribed circle.

451. Two adjacent sides, and the angles adjacent to one of them.

452. Two adjacent angles, a side adjacent to one of them, and the diagonal drawn from the other.

453. Three sides and one of the angles included by two of those sides.

454. Construct a triangle, given the median, the altitude, and the angle bisector, all drawn from the same vertex.

HINT. Locate the center of the circumscribed circle.

455. Construct a triangle, given the feet of the three altitudes.

HINT. The altitudes of a triangle bisect the angles of the triangle formed by joining their feet in succession.

456. If the inscribed and circumscribed circles of a triangle are concentric, the diameters are in the ratio $1:2$.

457. The sum of the diameters of the inscribed and circumscribed circles of a right triangle is equal to the sum of the legs of the triangle.

458. O is the center of the inscribed circle of the triangle ABC, and O' the center of the escribed circle touching BC; prove that O, B, O', C, are concyclic.

459. In any triangle, the difference of two sides is equal to the difference of the segments of the third side formed by the point of tangency of the inscribed circle.

460. O and O' are the centers of the inscribed and circumscribed circles respectively of the triangle ABC, and AD is perpendicular to BC; prove that AO bisects the angle DAO'.

461. The diagonals of a quadrilateral $ABCD$ intersect at E; prove that the centers of the circles circumscribed about the triangles AEB, BEC, CED, DEA, are the vertices of a parallelogram.

462. Construct a circle to intercept equal chords of a given length on three given lines.

463. The radii of the circumscribed and escribed circles of an equilateral triangle are respectively double and treble the radius of the inscribed circle.

464. Three circles whose centers are A, B, C, respectively, are tangent externally, pair by pair, at D, E, F; prove that the inscribed circle of the triangle ABC is the circumscribed circle of the triangle DEF.

465. Circumscribe about a given circle a rhombus whose side is a.

What limits has a?

466. The area of a circumscribed square is twice the area of a square inscribed in the same circle.

467. $ABCD$ is an inscribed square, and E is any point on the circle; prove that $EA^2 + EB^2 + EC^2 + ED^2$ is twice the square on the diameter of the circle.

468. Circumscribe a square about a given rectangle, the vertices of the latter to lie on the sides of the former.

469. Inscribe a circle in a quadrant of a given circle.

470. Divide a right angle into five equal parts.

HINT. Divide the radius of the circle in extreme and mean ratio.

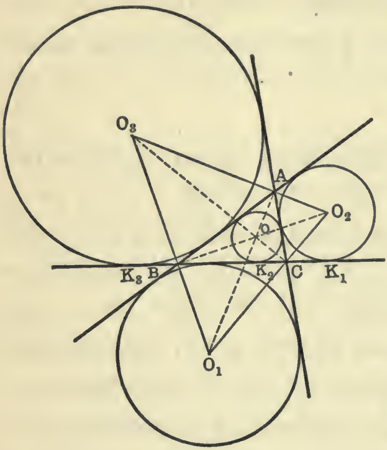
471. Two tangents, AB, AC, are drawn to a given circle from A; construct a circle tangent to AB, AC, and the convex arc BC.

472. From the vertex A of an isosceles triangle ABC a line is drawn meeting the base in D and the circumscribed circle in E; prove that AB is tangent to the circle circumscribed about the triangle BDE.

HINT. Find another angle equal to B or to C. Draw the circumscribed circle in question, and note the measures of the various angles.

473. Twice the square on a side of an inscribed equilateral triangle is equal to three times the square inscribed in the same circle.

Let O be the center of the inscribed circle of the triangle ABC, and O_1, O_2, O_3, the centers of the escribed circles tangent respectively to BC, AC, AB; also, let the three escribed circles touch BC or BC produced in K_1, K_2, K_3, as shown in the figure. *Prove the six following theorems:* —



474. The points A, O, O_1, are collinear; similarly, B, O, O_2, and C, O, O_3, are respectively collinear.

475. The points O_2, A, O_3, are collinear; similarly, O_3, B, O_1, and O_1, C, O_2, are respectively collinear.

476. The triangles BO_1C, CO_2A, AO_3B, are similar each to each.

HINT. BO_1CO is an inscriptible quadrilateral, and angle AO_2C equals angle O_1BC.

477. The triangle $O_1O_2O_3$ is similar to the triangle formed by joining the points of tangency of the inscribed circle.

478. Each of the four points O, O_1, O_2, O_3, is the orthocenter of the triangle formed by joining the other three.

479. The four circles, each of which passes through three of the points O, O_1, O_2, O_3, are equal.

480. The orthocenter and the vertices of any triangle are the centers of the inscribed and escribed circles of the triangles formed by joining the feet of the altitudes of the given triangle.

481. Given the base and the vertical angle of a triangle; find the locus of the center of the inscribed circle.

482. Given the base and the vertical angle of a triangle; find the locus of the center of the escribed circle which touches the base.

483. Given the base and the vertical angle of a triangle; prove that the locus of the center of the circumscribed circle is a point.

484. The lines joining the center of an inscribed circle of a triangle to the centers of the escribed circles, are bisected by the circumscribed circle of the triangle.

HINT. Each line is the hypotenuse of a right triangle. Join the vertex of the right angle to the given intersection, and prove the line thus drawn equal to each of the given segments.

485. With three given points as centers, construct three circles tangent to each other, pair by pair. How many solutions are there?

486. Given the centers of the three escribed circles; construct the triangle.

487. Given the center of the inscribed circle and of two of the escribed circles; construct the triangle.

488. In the triangle ABC, the center of the inscribed circle is O; prove that the centers of the circumscribed circles of the triangles AOB, BOC, COA, lie on the circumscribed circle of ABC.

HINT. This may be solved by the aid of No. 484.

489. The inscribed circle of a triangle ABC is tangent to BC at D; prove that the circles inscribed in the triangles ABD, ACD, are tangent to each other.

490. In any triangle the feet of the medians, the feet of the altitudes, and the middle points of the lines joining the orthocenter to the vertices are concyclic.

HINT. In the triangle ABC, let X, Y, Z, be the middle points of BC, AC, AB, respectively; D, E, F, the feet of the altitudes drawn from A, B, C, respectively, O the orthocenter, and P, Q, R, the middle points of AO, BO, CO, respectively. Join Y and Z to X, and each of these three points to P. Prove the angles PZX, PYX, right angles, and note that the angle PDX is a right angle.

NOTE. This circle is often called the *Nine Points Circle.* Many of its properties are to be derived from the fact that it is the circumscribed circle of the triangle whose vertices are the feet of the altitudes of the triangle.

491. The center of the nine points circle is the middle point of the line joining the orthocenter and the center of the circumscribed circle.

HINT. Letter the figure as in the preceding problem, and let S be the center of the circumscribed circle. Prove that the perpendicular bisectors of XD and EY pass through the middle point of SO.

492. The radius of the nine points circle is half the radius of the circumscribed circle.

493. The centroid of a triangle, the center of the circumscribed circle, the center of the nine points circle, and the orthocenter of the triangle are collinear.

HINT. With the figure of No. 491, let AX cut SO in G, and prove that AG is two thirds of AX.

494. The nine points circle of the triangle ABC, whose orthocenter is O, is also the nine points circle of each of the triangles AOB, BOC, COA.

495. All circles which have the same orthocenter and the same circumscribed circle have also the same nine points circle.

HINT. Prove by the aid of Nos. 491 and 492.

496. If four circles are drawn tangent to the sides of a quadrilateral, three by three, their centers are concyclic.

497. The perpendiculars drawn from the centers of the three escribed circles of a triangle to the sides they touch are concurrent.

498. Given an angle and the radii of the inscribed and circumscribed circles; construct the triangle.

499. If the bisectors of the angles of any polygon are concurrent, a circle may be inscribed in the polygon.

500. O is the orthocenter of the triangle ABC, and DE is the diameter of the circumscribed circle; prove that

$$AO^2 + BC^2 = BO^2 + CA^2 = CO^2 + AB^2 = DE^2.$$

IV. SIMILAR FIGURES

501. If three or more lines divide any number of parallels into proportional parts, these lines are concurrent.

State the converse of this theorem.

502. In a circle a chord CD is drawn perpendicular to a diameter AB, and through A any chord AE is drawn, meeting CD in F; prove that $AF \cdot AE$ is constant for all positions of AE.

A simple proof of the Pythagorean proposition can be obtained from this theorem. It is readily found by moving E along the circumference of the circle until it comes into coincidence with C.

503. Two lines, AB, AC, are divided in the points D, E, respectively, so that $AB \cdot AD = AC \cdot AE$; prove that the points B, C, D, E, are concyclic.

HINT. The proof depends on the equality of the angles DBE, ECD.

504. In a triangle ABC, $AB = 8$ in.; if a line parallel to BC divides AC in the ratio of $4:3$, what are the segments into which it divides AB?

505. The angles of a triangle are 30°, 60°, 90°; find the ratio of the segments into which the side opposite the angle of 60° is divided by the bisector of that angle.

506. Find the legs of a right triangle if their projections on the hypotenuse are 3 ft. and 6 ft. respectively.

507. In any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle and the altitude on the third side.

NOTE. This theorem enables one to find the radius of a circumscribed circle when the three sides of a triangle are given.

508. How many miles at sea is the light of a lighthouse 150 ft. high visible, reckoning the radius of the earth as 3960 miles?

NOTE. In ordinary problems of this kind the distance in statute miles to which an object is visible, provided there is no intervening obstacle, is approximately one and one third times the square root of the height of the object in feet. A formula often used by engineers is $K = \frac{\sqrt{h}}{.7575}$, where K is the distance in statute miles and h is the height in feet.

509. If two similar triangles have their homologous sides parallel, the lines joining their homologous vertices are concurrent.

What happens when the triangles are equal?

510. In any triangle the product of two sides is equal to the square of the bisector of the included angle plus the product of the segments into which it divides the third side.

HINT. Produce the bisector to meet the circumscribed circle, and join the point where it meets the circle to the extremities of the base.

NOTE. This theorem enables us to compute the bisectors of the angles of a triangle in terms of the sides of the triangle.

511. The legs of a right triangle are 7 ft. and 8 ft. respectively. Find their projections on the hypotenuse.

512. Find the legs of a right triangle if their ratio is 5 : 4, and the hypotenuse is 30 ft.

513. The legs of a right triangle are 1.564 ft. and 2.138 ft.; find the segments of the hypotenuse made by the bisector of the right angle.

NOTE. In connection with these and other problems involving the right triangle, it is of interest to know that the ancient Egyptians had, in an empirical way, learned how to construct a right angle, or, as it is termed in building, "a square corner." They had found that a triangle whose sides were 3, 4, 5, contained a square corner, and made use of that knowledge in squaring the corners of a building. Three stakes were driven into the ground in such positions that a rope passed around them would form a triangle of the required kind; or that an endless rope containing knots, so tied that the segments into which it was divided were as 3 : 4 : 5, would, when tightly stretched with the points of force applied at the knots, produce the same effect. From this fact builders were, in ancient Egypt, sometimes known by the curious name of "rope twisters," or "rope stretchers."

514. Find the distance from the center of a circle to a chord equal to the radius. What angle does the subtended arc measure at the center?

515. Two chords AB, CD, intersect within a circle at E; $AE = 10$ in., $BE = 12$ in., $CD = 23$ in. Find CE and DE.

516. Show how to find the height of an object situated on the opposite bank of a river from the observer.

NOTE. This problem involves finding also the distance from the observer to the object, which is not supposed to be known. In connection with problems of this nature it is interesting to note that the distance of an object can be determined at once, and with a considerable degree of accuracy, by means of an instrument known as the "range-finder," which is extensively used in naval warfare for determining the distance of a hostile ship. Engineers also use many rough methods of approximating the distance of objects, of which the following are illustrations: by ordinary eyes the windows of a large house can be counted at a distance of about 13,000 feet, or $2\frac{1}{2}$ miles; men and horses appear as points at about half that distance; a horse can be clearly distinguished at about 4000 feet; the movements of men at 2600 feet, or about half a mile; the head of a man, occasionally, at 2300 feet, and very plainly at 1300 feet, or a quarter of a mile. The Arabs of Algeria define a mile as "the distance at which you can no longer distinguish a man from a woman." These distances will, of course, vary with the condition of the atmosphere and the individual acuteness of vision.

517. In any triangle the orthocenter, the centroid, and the intersection of the perpendicular bisectors of the sides are collinear; and the distance between the first two is twice the distance between the last two.

HINT. In the triangle ABC, let CH, AK, be altitudes, CM, AN, be medians, and MF, NF, perpendicular bisectors, locating the orthocenter at D, the centroid at E, and the intersection of the perpendicular bisectors at F. The triangles ADC, MNF, are similar, and the sides of the former are double the sides of the latter. Prove that the triangles ADE, NEF, are similar.

518. The sides of a triangle are 8, 11, 13; find the segments of the sides made by the respective angle bisectors.

519. At the extremities of a line AB perpendiculars AD, BE, are drawn, and in AB, or AB produced, a point C is taken such that the angle DCA is equal to the angle EAB. If $AB = 25$ in., $AD = 13$ in., and $BE = 7$ in., find CA and CB.

520. In any inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides.

HINT. Let $ABCD$ be the quadrilateral. In the diagonal AC take a point E such that the angle EDC shall equal the angle ADB. Two pairs of similar triangles can now be found.

521. In any triangle the sum of the squares of any two sides is equal to twice the square of half the third side plus twice the square of the median drawn to the third side.

NOTE. By means of this theorem the medians can be computed when the sides are known.

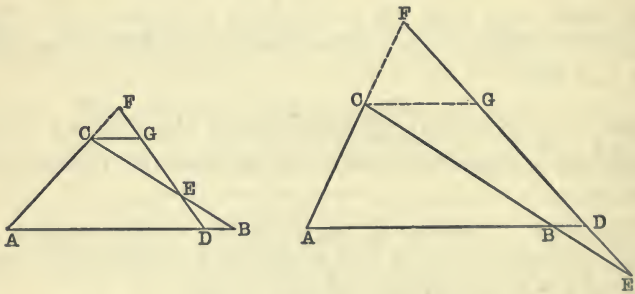
522. If E and F are the middle points of the sides AB, CD, respectively, of the parallelogram $ABCD$, and AF and CE be drawn intersecting the diagonal BD in H and L respectively, and BF and DE be drawn intersecting the diagonal AC in K and G respectively, then is $GHKL$ a parallelogram equal in area to one ninth of $ABCD$.

523. The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

524. The sum of the squares of the diagonals of a trapezoid is equal to the sum of the squares of the legs plus twice the product of the bases.

HINT. Join the middle points of the diagonals and use No. 521.

525. Every straight line cutting the sides of a triangle, or the sides produced, determines upon the sides six segments such that the product of three non-consecutive segments is equal to the product of the other three.



HINT. The cutting line DEF may cut two sides of the triangle and the third side produced, or the three sides produced. The proof is the same in either case. The segments, and the equation between them, are as follows: —

$$AD \cdot BE \cdot CF = AF \cdot BD \cdot CE.$$

The proof is obtained from the proportions existing between the sides of the similar triangles.

This theorem is due to Menelaus of Alexandria. It was discovered about 80 B.C.

526. Lines drawn through the vertices of a triangle determine, if concurrent, six segments on the sides of the triangle such that the product of three non-consecutive segments is equal to the product of the other three.

HINT. The lines may meet within or without the triangle. Use No. 525 on each of the two parts into which one of the lines divides the triangle. This theorem was invented by Ceva of Milan in 1678.

527. The hypotenuse of a right triangle is 13, and the sum of the legs is 17; find the legs.

528. Find the legs of a right triangle if their sum is 49 and the sum of their squares is 1881.

529. The radii of two circles are 7 in. and 4 in. respectively, and the distance between the centers of the circles is 12 in.; find the length of the common exterior tangent.

Can you think of a problem, similar to the above, in which the length of the tangent mentioned shall equal the diameter of one of the circles?

530. In an isosceles triangle the base is 13 in. and each leg is 11 in.; find the radius of the circumscribed circle.

HINT. See No. 507.

531. Through a point A without a circle, a tangent AB and a secant ACD are drawn; $AC = 8$ in., and $CD = 5$ in.; find AB.

532. Through a point A, 8 in. from the center of a circle whose radius is 4 in., a secant ABC is drawn such that $BC = 2$ in.; find AB.

533. The sides of a triangle are 432 in., 586 in., and 728 in. respectively; find the length of the shortest median. (See No. 521.)

To which side of a triangle is the shortest median drawn? the longest median?

534. The medians drawn from the extremities of the hypotenuse of the right triangle ABC are BE, CF; prove that $4\,BE^2 + 4\,CF^2 = 5\,BC^2$.

535. Any point E within the rectangle $ABCD$ is joined to each of the four vertices; prove that $AE^2 + CE^2 = BE^2 + DE^2$.

Examine this theorem for the case when E lies without the rectangle; on the perimeter.

536. Every quadrilateral is divided by its diagonals into four triangles whose areas are proportional to each other.

Hint. From two opposite vertices, drop perpendiculars to the same diagonal.

537. If one of the parallel sides of a trapezoid is double the other, the diagonals intersect at a point of trisection.

Hint. Make one of the diagonals the median of a triangle.

538. In a side AC of a triangle ABC any point D is taken, and the lines AD, DC, AB, BC, are bisected at the points E, F, G, H, respectively; prove that $EG = FH$.

539. The area of a ring bounded by two concentric circles is equal to the area of a circle whose diameter is a chord of the outer circle tangent to the inner circle.

How can this area be found by using the circle which lies midway between the two given circles?

540. The radius of a circle is a mean proportional between the segments of any tangent made by its point of contact and any pair of parallel tangents.

541. AB is a diameter of a circle, and through A any straight line is drawn to cut the circumference at C and the tangent at B in D; prove that AC is a third proportional to AD and AB.

Write out the proportion which results when the limiting case is reached, and state whether or not it is still true.

542. Within a regular hexagon whose side is 10 in. a second regular hexagon is inscribed by joining the middle points of the sides taken in order. What is the area of the inscribed hexagon?

Will the area of the inscribed hexagon remain the same if any other points in the outer hexagon are chosen, the inscribed hexagon still remaining regular?

543. At the extremities of a diameter of a circle tangents are drawn meeting a third tangent in A, B; if C is the point of contact of the third tangent, prove that $AC \cdot CB$ is constant for all positions of C.

HINT. Prove by means of No. 540.

544. A swimmer whose eye is on the level of the water can just see the top of a stake 8 in. high at a distance of a mile. What is the diameter of the earth?

545. If similar polygons are constructed on the three sides of a right triangle as homologous sides of the polygons, the polygon drawn on the hypotenuse is equal in area to the sum of the polygons drawn on the other two sides.

What celebrated name in mathematics and philosophy is suggested by this theorem?

546. ABC is an isosceles triangle whose vertex is A; on the base, or the base produced, any point D is taken;

prove that the circumscribed circles of the triangles ABD, ACD, are equal.

HINT. Prove by means of the equal angles of the given triangle.

547. Find the locus of a point whose distance from two intersecting lines is as $m : n$.

HINT. The locus consists of two straight lines through the intersection of the two given lines.

548. ABC and DEF are two similar triangles whose areas are respectively 245 sq. in. and 5 sq. in. If $AB =$ 21 in., find DE, the homologous side of DEF.

What answer would you obtain if the above figures were quadrilaterals instead of triangles, the given dimensions remaining the same?

549. Through any point A without a circle, secants are drawn to the circle. Find the locus of the point which divides the entire secants in the ratio of $m : n$.

HINT. Divide the line joining A with the center of the circle in the ratio of $m : n$.

550. A quarter-mile running track is to be laid out with straight parallel sides and semicircular ends. The track is to be 10 ft. wide, and the distance between the outer parallel edges is to be 220 ft. What must be the extreme length of the field in which the track is to be laid out, so that a runner may cover a quarter of a mile *by keeping in the middle of the track?* By keeping close to the inner edge of the track?

551. The radii of two intersecting circles are 10 in. and 17 in. respectively, and the distance between their centers is 21 in. Find the length of their common chord.

552. The parallel sides of a circumscribed isosceles trapezoid are 18 in. and 6 in. respectively. What is the radius of the circle?

HINT. Extend the non-parallel sides until they meet.

553. Find the locus of a point such that the sum of the squares of its distances from two given points is constant.

The locus is a circle whose center is at the middle point of the line connecting the two points. (See No. 521.)

554. Find the locus of a point such that the difference of the squares of its distances from two given points is constant.

The locus consists of two parallel straight lines, which are perpendicular to the line joining the two given points.

555. The fly wheel of an engine is connected by a belt with a smaller wheel driving the machinery of a mill. The radius of the fly wheel is 7 ft. and of the driving wheel 21 in. How many revolutions does the small wheel make to one of the large wheel? The distance between the centers is 10 ft. 6 in. What is the length of the belt connecting the two wheels?

556. Through any point A a line is drawn cutting a given circle in B, C; if A moves so that the product of the segments AB, AC, is constant, find the locus of the point A. (Two cases.)

What is the locus if $AB \cdot AC = 0$? If $AB \cdot AC = r^2$?

HINT. When A is within the circle, draw through A a chord perpendicular to the line joining A to the center of the circle.

557. Divide a line 32 in. long into three parts, which shall be to each other as $\frac{1}{2} : \frac{2}{3} : 2\frac{1}{4}$.

558. Divide a line so that the product of its segments shall equal a given quantity.

559. What is the area of a trefoil formed on an equilateral triangle whose side is 6 in.?

Note. If from each of the vertices of a regular polygon as a center, with a radius equal to half the side of the polygon, a circle be described outwardly, an ornamental figure extensively used in architecture will be obtained. If the polygon be a triangle, the figure is called a trefoil; if a square, a quatrefoil; if a pentagon, a cinquefoil. A figure of this kind formed about a polygon of many sides is often used; it is called a rose window.

560. A rose window of six lobes is to be placed in a circular opening 35 ft. in diameter. How many square feet of glass will it contain, no deduction being made for sash or leading?

561. If a circle be tangent internally to another circle of twice its radius, the path of a given point on the circumference of the smaller as it rolls about within the larger, always remaining tangent to it, is a diameter of the larger circle.

What about the velocity of the moving point, the rate of the moving circle being constant?

562. If a given square be subdivided into n^2 equal squares, n being any given number, and a circle be inscribed in each of these equal squares, the sum of these circles is equal in area to the circle inscribed in the original square.

563. Divide a line so that the product of the whole line and one of the segments shall equal a given quantity.

Hint. This problem readily reduces to the problem of finding a third proportional to two given quantities.

564. Construct two lines, given their difference and their ratio.

Note. Care should be taken to work *with lines*, not merely with algebraic values.

565. Find the locus of a point whose distances from two given points are in the ratio $m : n$.

HINT. Draw through the two points a line of indefinite length, and determine on this line two points of the locus. The entire locus is a circle passing through the two points. This problem has many important applications in certain parts of physics. Of these the following is an example: —

566. Find the locus of a point in a plane equally illuminated by two lights in the plane; given, that the intensities of the lights at a unit's distance are as $m : n$, and that the intensity of any light varies inversely as the square of the distance.

567. Find the locus of a point from which a given line is seen so as to subtend a given angle.

568. Find the locus of the vertex of a triangle, having given the base and the ratio of the other two sides.

569. Through any given point draw a line which shall cut the sides of a given angle so that the segments between the vertex and the points of intersection shall be in a given ratio. (Three cases.)

570. Find in one side of a triangle a point whose distances from the other two sides shall be in a given ratio. (See No. 547.)

571. Find within a triangle a point whose distances from the three sides of the triangle shall be as $m : n : p$.
Can such a point be found without the triangle?

572. Construct a circle having a given radius, touching a given circle, and having the distances from its center to two given lines in a given ratio.

To the above problem apply the method of loci, and discuss the number of solutions.

573. Find a point such that it shall be equidistant from two given lines, and its distances from two given points shall be in a given ratio.

574. The ground plan of a house drawn on a scale of $\frac{1}{4}$ in. to 1 ft. is represented by a rectangle $8\frac{1}{2}$ in. by 1 ft.; what are the dimensions of the foundation of the house?

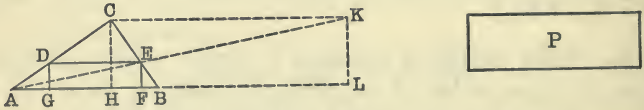
575. A field containing 9 acres is represented by a triangular plan whose sides are 12 in., 17 in., and 25 in. On what scale is the plan drawn?

576. Find a point whose distances from two given points shall be in a given ratio, and whose distances from two other given points shall also be in a given ratio.

577. Find the position of a point which is equally illuminated by three lights of unequal intensities situated in the same plane.

578. A triangle ABC is divided into three parts which are of equal area by lines drawn parallel to AB; if BC is 100 inches long, find the length of each of the segments into which it is divided by the lines parallel to AB.

579. In a given triangle inscribe a rectangle similar to a given rectangle.



HINT. Let ABC be the given triangle and P the given rectangle. On the altitude CH construct a rectangle CL similar to the given rectangle. The line AK will determine a point E which will be one of the vertices of the required rectangle. Why?

580. In the triangle ABC draw a line parallel to AB cutting AC, BC, in D, E, so that $AB : DE = m : n$.

581. Perform the above construction so that $AB : DE = DE : DC$.

582. In a given circle inscribe a triangle similar to a given triangle.

583. In a given square whose side is 16 in. a square is inscribed by joining the middle points of the sides taken in order; in this square another square is inscribed in a similar manner, and so on. Find the area of the first inscribed square; of the eighth inscribed square.

HINT. This problem can readily be converted into a problem which involves the finding of the last term of a geometrical progression.

584. Find the locus of a point such that the tangents drawn from it to two given circles shall be equal.

The locus is a line perpendicular to the line of centers, and is called the *radical axis* of the two circles; and the required point is the intersection of this locus with the given line. The point where the radical axis cuts the line of centers is a point which divides the line of centers into two segments, the difference of whose squares is equal to the difference of the squares of the radii of the circles.

If the two circles are tangent to each other, the radical axis is the common interior tangent; and if the circles intersect, it is the common chord.

The radical axis is also the locus of the centers of circles intersecting the two given circles at right angles.

585. Find in a given line a point such that the tangents drawn from it to two given circles shall be equal.

586. In a given triangle a similar triangle is inscribed by joining the middle points of the sides; in this inscribed triangle another triangle is inscribed in the same manner, and so on. What fraction of the given triangle is the area of the sixth inscribed triangle?

587. In a circle whose radius is 32 in. an equilateral triangle is inscribed, in this triangle a circle, in this circle another equilateral triangle, and so on. Find the area of the third inscribed circle. Which circle has an area of 3.14159?

588. Construct a triangle, given two of the angles, and the sum of the altitude and the median drawn from the same vertex.

In many constructions like the above, it is easy to construct a figure similar to the required figure, and then to construct the required figure by means of the familiar theorem that in similar figures homologous lines are proportional. In this example, draw a triangle similar to the required triangle, draw the altitude and median homologous to the altitude and median whose sum is given, and extend the altitude beyond the base by a length equal to the median of the triangle now drawn. The remainder of the construction can then be obtained by means of the general theorem just suggested.

589. Construct a triangle, given two angles, and the difference between the radii of the inscribed and circumscribed circles.

Are the radii of the inscribed and circumscribed circles of two similar triangles homologous lines of those triangles?

590. Construct a triangle, given two angles, and the sum of the bisector of the third angle and the side opposite that angle.

591. Construct a triangle, given one angle, the ratio between the adjacent sides, and the sum of the third side and the altitude let fall upon that side.

592. Construct a triangle, given the ratio of one side to each of the other sides, and the radius of the inscribed circle.

593. Construct a rectangle, given the ratio between one of the longer sides and a diagonal, and the sum of a diagonal and one of the shorter sides.

594. Construct a triangle, having given the three altitudes.

HINT. Any two altitudes of a triangle are inversely proportional to the sides upon which they are let fall.

595. Construct a parallelogram, having given an angle, the sum of the diagonals, and the ratio between two adjacent sides.

596. In a given circle, draw a chord which shall be trisected by two given radii.

597. Construct a chord quadrilateral, having given two opposite sides, one angle, and the angle between the diagonals.

598. Construct a quadrilateral, given one diagonal, and the four angles which the other diagonal makes with the sides.

599. Construct a quadrilateral, given the sum of the diagonals, and the four angles which one of the diagonals makes with the sides.

600. Construct a triangle, given two angles, and the sum of the included side and the angle bisector drawn upon that side.

V. AREAS OF FIGURES. EQUAL AREAS

601. Two triangles or two parallelograms having two adjacent sides respectively equal and the included angles supplementary have equal areas.

Compare the area of a triangle and that of a parallelogram under the conditions of the problem.

602. The areas of two mutually equiangular parallelograms are to each other as the products of two adjacent sides.

603. Find the side of a square whose area is equal to that of an equilateral triangle whose side is 6 in.

Which will have the greater perimeter?

604. The area of a rectangular field is $\frac{3}{5}$ of an acre, and its length is double its breadth; if a horse is picketed at the middle point of one of the longer sides, find the length of the picket rope which will enable him to graze over half the field.

605. A circle whose radius is 8 in. has half its area removed by cutting a ring away around the outside; find the width of the ring.

606. The perimeter of a square is 72 ft., and the perimeter of a rectangle whose length is twice its breadth is also 72 ft. Find the difference between their areas.

If two figures, square and rectangle, have equal areas, which has the lesser perimeter? What other figures of equal area can be mentioned which have a still smaller perimeter? Which of them has the least?

607. Find the area of a rectangle if its diagonal is 50 m. and its sides are in the ratio 3 : 5.

608. The dimensions of a rectangle are 64 ft. and 58 ft. respectively. If the length is diminished by 10 ft., how much must the breadth be increased in order that the area may remain the same?

609. The perimeter of a rhombus is 20 in. and its shorter diagonal is 6 in.; find the side of a square of the same area.

Is the side of the required square greater or less than the diameter of an equivalent circle?

610. Each of the acute angles of a rhombus is 60°; find the ratio of its area to that of a square whose perimeter is equal to the perimeter of the rhombus.

As the acute angles of the rhombus decrease, its perimeter remaining the same, how does its area change?

611. If two parallelograms have equal areas, and the altitude of one is four times the altitude of the other, what is the ratio of their bases?

What would the ratio be if the figures were triangles instead of parallelograms?

612. The bases of two triangles are equal and their altitudes are 6 in. and 8 in. respectively; what is the ratio of their areas?

What would the ratio be if the above figures were parallelograms instead of triangles?

613. The altitudes of two triangles are equal, and their bases are 2 in. and 3 in. respectively; what is the base of a triangle whose area is equal to the sum of their areas, and whose altitude is half as great as their common altitude?

If in the above problem the last-named figure were a parallelogram, what would be its base?

614. One leg of a right triangle is 12 m. and the altitude on its hypotenuse is 10 m.; find the area of the triangle.

In what two ways can the hypotenuse be found?

615. The area of a triangle is 68 sq. yd.; find its base and altitude if they are as 12 : 7.

616. What is the area of a triangle if the perimeter is 21 in. and the radius of the inscribed circle is 1.6 in.?

617. Find the area of the greatest circle that can be cut out of a triangle whose sides are 6 in., 8 in., 10 in. long respectively.

Is the perimeter of a triangle any index of the area of the inscribed circle? of the circumscribed circle? Why?

618. What is the area of a trapezoid if its altitude is 8 ft. and its median is 18 ft.?

Can a circle be inscribed in a trapezoid? circumscribed about a trapezoid?

619. A trapezoid contains 450 sq. ft. and its altitude is 20 ft.; find the bases of the trapezoid (i) if one is 4 ft. longer than the other; (ii) if they are in the ratio 3 : 4.

620. The value of a field in the shape of a trapezoid is $6000. The bases are 120 yd. and 220 yd. respectively, and the altitude is 206 yd.; what is the value of the land per acre?

621. The base of a triangle is 150 ft. and its altitude is 180 ft.; find the base of a parallelogram of equal area whose altitude is 52 ft.

622. Upon the diagonal of a rectangle 20 ft. long and 18 ft. wide a triangle is constructed whose area is equal to the area of the rectangle; calling this line the base of the triangle, what must be its altitude?

623. Find the perimeter of a rhombus whose shorter diagonal is 6 in., if its area is equal to that of a triangle whose base is 12 in. and whose altitude is 14 in.

624. Prove that the area of a regular dodecagon is three times the square of the radius of the circumscribed circle.

625. Find the side of a regular hexagon whose area is equal to the area of an equilateral triangle whose side is 12 ft.

If the areas of two rhombuses are equal, are their perimeters necessarily equal?

626. Find a point in one side of a triangle such that a line drawn through it parallel to one of the other sides shall bisect the triangle.

Having found such a point, does it make any difference which of the remaining sides is chosen as the one to which the required line shall be drawn parallel?

627. Find the side of an equilateral triangle whose area is equal to the area of a square having a diagonal 6 in. in length.

628. Find the side of a square equal in area to a quadrilateral whose sides are 3 ft., 4 ft., 5 ft., 6 ft., respectively, and one of whose diagonals is 7 ft.

In the arrangement of the sides of this quadrilateral, place the sides opposite each other which are 3 ft. and 5 ft. respectively. What other arrangement is possible? Is the area the same in both cases?

629. How must a line parallel to the base of a triangle be drawn so as to cut off, in that part of the triangle next the vertex, (i) two fifths of the area of the triangle? (ii) three eighths of the area?

630. Find the side of a rhombus composed of two equilateral triangles, and equal in area to a rhombus whose diagonals are, respectively, 1 ft. 3 in. and 1 ft. 8 in.

631. Through the vertex of a triangle whose area is 100 sq. ft., a line is drawn dividing it into two parts, one containing 12 sq. ft. more than the other; what must be the length of each of the segments into which the base is divided by this line, if the entire length of the base is 14 ft.?

632. In a triangle ABC the side $AB = 7$ in., and the side $AC = 9$ in. On AB a point D is taken 4 in. from A, and DE is drawn, cutting the side AC in E so that the triangle ADE is two sevenths of ABC. Find the length of the segment AE.

If two sides of a triangle are known, how much do we know about its area?

633. How must lines be drawn through the middle point of one of the sides of a triangle so as to divide its area into three parts which are to each other as $2:3:7$?

634. In the base AB of the triangle ABC, the point D is 50 ft. from A and 175 ft. from B. The triangle is to be divided into three parts, which are as $2:3:5$, by lines DE, DF, drawn from D to cut AC, BC, respectively. Find AE, BF, in terms of AC, BC, respectively.

Hint. Apply the theorem which proves that triangles having equal altitudes are to each other as their bases.

635. The base AB of a triangle is 50 ft. long, and D, the foot of the altitude from the vertex C, is 40 ft. from A; find a point E in AD such that the perpendicular erected at E will divide the triangle into (i) equivalent parts; (ii) parts in the ratio of $3:4$.

HINT. Obtain three proportions involving three unknown quantities, by means of the theorem that triangles are to each other as the products of their bases and altitudes.

636. The area of a parallelogram is 80 sq. ft.; how must a line from one vertex divide one of the opposite sides in order that the triangle cut off may contain 30 sq. ft.?

637. What part of a parallelogram is contained between one half of one side and one third of one of the adjacent sides?

Does it make any difference which of the adjacent sides is chosen?

638. In a trapezoid $ABCD$, the bases are $AB = 6$ ft., $CD = 4$ ft.; find a point E in the diagonal DB, such that a line through E, parallel to AD, will divide the trapezoid into two parts, which shall be to each other as $3:4$.

639. The base and altitude of a triangle are 28 ft. and 22 ft. respectively; find the area of a triangle formed by drawing a line parallel to the base and 6 ft. from the vertex.

640. The sides of two equilateral triangles are 3 ft. and 4 ft. respectively; find the side of an equilateral triangle equivalent to their sum.

641. Find the area of a square if the sum of a diagonal and a side is 14 in.

642. The altitude of a triangle is 8 in.; what is the homologous altitude of a similar triangle ten times as large? Of a similar triangle n times as large?

643. One side of a triangle is a; find the homologous side of a similar triangle one third as large. One nth as large.

644. One side of a polygon is 3 in.; find the homologous side of a similar polygon having to the given polygon a ratio of 3 : 5.

645. If the side of one equilateral triangle is equal to the altitude of another, find the ratio of their areas.

646. Transform a given triangle into an equivalent rhombus with a given length a for one of its diagonals.

Does it make any difference which diagonal is chosen equal to a?

647. Trisect a parallelogram by lines drawn from one of the vertices.

648. The diagonals of two squares are 3 ft. and 4 ft. respectively; find the diagonal of a square equal in area to their sum.

649. The sides of two equilateral triangles are 2 ft. and 5 ft. respectively; find the side of an equilateral triangle whose area is equal to their difference.

650. The sides of a triangle are 10 in., 16 in., and 20 in. respectively; find the area of each of the two parts into which the triangle is divided by the bisector of the angle between the first two sides.

651. The base of a triangle is 6 in. and its altitude is 8 in.; find the change in area if these dimensions are: (i) increased by 3 in. and 2 in. respectively; (ii) decreased by 3 in. and 2 in. respectively; (iii) one increased by 3 in. and the other diminished by the same amount.

652. Transform a given triangle into an equivalent isosceles triangle with a given length a for one of its legs.

653. Construct a triangle five times as large as a given triangle; one fifth as large.

How many ways are there of performing this construction?

654. Construct a triangle equivalent to (i) the sum of two triangles with equal altitudes; (ii) the difference of two triangles with equal bases.

655. Find the change in area of a triangle of base 16 in. and altitude 18 in.: (i) if the base is increased by 5 in. and the altitude is diminished by 4 in.; (ii) if the base is decreased by 5 in. and the altitude is increased by 4 in.; (iii) if the base is increased by 5 in. and the altitude is diminished by 5 in.

656. Transform a given triangle into an isosceles triangle of equal area, whose altitude shall equal a.

657. Construct a square equivalent to three sevenths of a given square.

658. Transform a triangle into an equivalent triangle with its altitude equal to a, and the angle from which this altitude is drawn equal to a given angle θ.

659. Transform a given triangle into an equivalent triangle with base and altitude equal to each other, and a given length a for the median drawn to the base.

660. Transform a given triangle into an equivalent triangle similar to another given triangle.

Hint. The base of the required triangle is a mean proportional between the base of the first triangle and a triangle which is similar to the second given triangle, and which has an altitude equal to the altitude of the first triangle.

661. Divide a circle by one or more concentric circumferences into

(i) Two equivalent parts.

(ii) Four equivalent parts.

(iii) Two parts of which the inner shall be three times the outer.

662. Transform a square into an equivalent rectangle having a given perimeter.

HINT. Form two equations containing two unknown quantities.

663. Transform a square into an equivalent rectangle having a given diagonal.

664. Transform a rectangle into an equivalent rectangle such that the difference between two of its adjacent sides shall equal a given quantity a.

665. Transform a parallelogram into an equivalent rhombus having for one of its diagonals a side of the parallelogram.

666. Inscribe in a given circle a rectangle whose area is equal to the area of a given square.

HINT. Form two equations containing two unknown quantities.

667. Transform a given polygon into an equivalent polygon similar to another given polygon.

HINT. First transform the two given polygons into squares.

668. Construct a polygon equivalent to the sum of two given polygons and similar to another given polygon.

669. Divide a trapezoid into five parts whose areas shall be equal, each to each.

670. Divide a trapezium into seven parts whose areas shall be equal, each to each.

HINT. Draw one of the diagonals and divide it into seven equal parts.

671. Divide a triangle into three parts which shall be to each other as $2:3:4$.

Can this division be effected by more than one method?

672. Divide a triangle into two equivalent parts by a line drawn through a point in one side. Divide the triangle into five equivalent parts by lines drawn through a point in the longest side.

HINT. To bisect the triangle ABC, take any point D in the side AB as the point through which the bisecting line is to be drawn. From C draw a line to M, the middle point of AB, and connect CD. Then the required line DE can be found by drawing ME parallel to CD.

673. Find a point within a triangle such that the lines drawn from this point to the three vertices shall divide the triangle into three parts whose areas shall be equal, each to each.

HINT. Draw the medians of the triangle.

674. Divide a triangle into five equivalent parts by lines drawn from a given point within the triangle.

HINT. Connect the given point with one of the vertices of the triangle, and begin by cutting off on either side of this line a triangle or a quadrilateral as the case may be, whose area is equal to one fifth the area of the original triangle. This is a general method, and is applicable where no special method of solution can be found.

675. Divide a triangle into two equivalent parts by a line perpendicular to one of its sides.

676. Divide a triangle into four equivalent parts by lines parallel to one of the angle bisectors.

677. Divide a triangle into three equivalent parts by lines parallel to one of the medians.

678. Divide a parallelogram into five equivalent parts by lines drawn from one of its vertices.

679. Divide a trapezoid into three equivalent parts by lines drawn from one of its vertices.

HINT. First divide the trapezoid into two equivalent parts.

680. Divide a parallelogram into two parts which shall be to each other as 2 : 3 by a line drawn from a given point in one side.

681. Bisect a parallelogram by a line drawn through any given point.

HINT. What point is the center of a parallelogram? Has a triangle a center? A trapezoid?

682. Bisect a parallelogram by a line parallel to any given line.

683. Bisect a trapezoid by a line parallel to the bases. (See hint to No. 662.)

684. Bisect a trapezoid by a line drawn from any given point in one of the bases of the trapezoid.

685. Bisect a trapezoid by a line parallel to any given line.

686. Divide a hexagon into two parts which shall be in the ratio of 2 : 3 by a line drawn through a given point in one of the sides. (See hint to No. 674.)

687. Inscribe in a triangle a rectangle which shall have a given area. (See hint to No. 662.)

688. Inscribe in a given parallelogram a rhombus which shall have a given area.

HINT. Using the values of the diagonals of the rhombus, form two equations as in No. 662. Note the relation of the center of the rhombus to the center of the parallelogram.

689. ABC is any triangle, and D is any point in AB; draw through D a line DE to meet BC produced in E, so that the triangle DBE shall be equivalent to the triangle ABC.

690. On a base of given length construct a triangle equivalent to any given triangle, and having its vertex on a given line.

691. Construct a parallelogram having the same area and the same perimeter as a given triangle.

Is there more than one construction? Is the construction always possible? Will the required figure ever be a square?

692. Bisect a trapezium by a line drawn through one of its vertices.

693. On the base AD of any quadrilateral $ABCD$ construct a triangle whose area shall be equal to the area of the quadrilateral, and having the base and the angle A coincide with the base and the angle A of the quadrilateral.

694. Cut off from a given quadrilateral a third, a fourth, a fifth, or any given fraction of its area by a straight line drawn through one of its vertices.

695. Construct a circle equivalent to a given triangle.

696. On a given base construct a triangle whose area is equal to the area of a given circle.

697. Construct a circle whose area is equal to the area of a given quadrilateral.

698. Construct a quadrilateral whose area is equal to the area of a given circle.

Can more than one quadrilateral be constructed that will fulfill the given conditions?

699. A circular field is 300 ft. in diameter, and a horse is picketed by a rope 20 ft. long, the picket pin being driven exactly on the boundary of the field; over how many square feet can the horse graze

(i) Inside the field?

(ii) Outside the field?

700. Find the center of gravity of any convex quadrilateral.

VI. MISCELLANEOUS THEOREMS AND PROBLEMS

701. Divide a line internally and externally so that the segments may be in a given ratio.

DEF. A straight line divided internally and externally into segments having the same ratio is said to be divided *harmonically.*

702. If AB be divided harmonically at P, Q, where Q is the external point, then is AB the harmonic mean between AQ and AP.

703. If AB be divided harmonically at P, Q, where Q is the external point, and C is the middle point of AB, then is $CP \cdot CQ = CA^2$.

704. Two villages are on opposite sides of a river. Show by construction how a bridge should be located in order that the distance from a selected point in each village to the nearer end of the bridge shall be the same in each case.

For convenience suppose the banks of the river to be straight and parallel, and the ground level. Examine the problem for different locations of the villages.

DEFINITIONS

If AB be divided harmonically at P, Q, then are P, Q, the *harmonic conjugates* of A, B. In this case the points A, B, are also harmonic conjugates of P, Q.

Any series of points in a straight line is called a *range.*

Four points so placed that two of them are harmonic conjugates of the other two are said to form a *harmonic range.*

If any number of straight lines intersect at a common point, these lines are said to form a *pencil.*

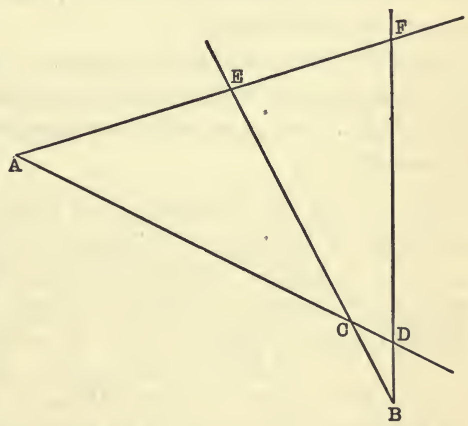
Each of the lines which form a pencil is called a *ray.*

The point of divergence of the rays of a pencil is called the *vertex of the pencil.*

A pencil of four rays which pass through the four points of a harmonic range respectively is called a *harmonic pencil.*

A system of four straight lines, none of which are concurrent, intersect with each other to form a figure which is called a *complete quadrilateral.* Such a quadrilateral has six vertices and three diagonals.

In the following figure we have a system of four straight lines, none of which are concurrent.



These lines form by their intersection the complete quadrilateral *ABCDEF.* The six vertices are indicated by the six letters, and the diagonals, which are not drawn in the figure, are the lines *AB*, *ED*, *CF*. Any ordinary quadrilateral can be transformed into a complete quadrilateral by extending the opposite pairs of sides until they meet. If either pair of opposite sides consists of parallel lines, these lines are said to meet and to form a vertex at infinity, or at an infinite distance from the other vertices of the quadrilateral.

705. If a line be drawn parallel to any ray of a harmonic pencil, the other three rays intercept equal parts on it.

706. Any transversal is cut harmonically by the rays of a harmonic pencil.

707. The bisector of the angle formed by a pair of conjugate rays is perpendicular to its own conjugate ray.

708. If the homologous sides of any two similar, unequal figures are parallel, each to each, the lines joining corresponding vertices are concurrent; and the distances from any pair of homologous vertices to the point of concurrence are in the same ratio as any pair of homologous sides.

Does this theorem undergo any modification when the two figures are equal?

Def. The point of concurrence of the lines connecting the homologous vertices of two similar figures so placed that their homologous sides are parallel is called the *center of similitude.*

709. A secant intersects two given circles, and meets the line of centers at its point of intersection with the common tangent. Prove that the radii drawn to the points of intersection of the secant and the circle are parallel, pair by pair.

710. In the above construction, let A, A', be the points of tangency, B the point of intersection of the tangent and the line of centers, and C, D, C', D', the points of intersection of the secant and the circles, taken in order from B; to prove that $BA \cdot BA' = BC \cdot BD' = BD \cdot BC'$.

DEF. The points which divide externally and internally the line of centers of two circles in the ratio of their radii are called the *direct* and the *inverse centers of similitude* respectively.

The definition just given is readily seen to cover the special case, that which applies to circles only, of the more general definition of center of similitude given above.

711. If a variable circle touches two fixed circles, the line which passes through the two points of contact passes also through a center of similitude. How many cases?

712. The two radii of a circle drawn to its points of intersection with any line passing through a center of similitude are parallel, pair by pair, to the two radii of the other circle drawn to its intersections with the same line.

713. The common exterior tangents of two circles pass through the direct center of similitude, and the common interior tangents pass through the inverse center of similitude.

What method of drawing the common tangents to two circles does this theorem suggest? Is there any other simple method of performing this construction?

714. In a given sector of a circle, inscribe a square so that two of the vertices of the square shall be on the arc, and the other two on the radii of the sector.

POLES AND POLARS

DEF. If on any line drawn from the center of a circle two points are taken such that the product of their distances from the center is equal to the square of the radius, each of these points is called the *inverse* of the other.

DEF. If through either of two inverse points a line is drawn perpendicular to the line joining the inverse points with the center, the line thus drawn is called the *polar* of the other point with respect to the circle; and the latter point is called the *pole* of the polar thus drawn.

A simple construction will show that the polar of an external point cuts the circle; the polar of an internal point lies wholly without the circle; and the polar of a point on the circle is tangent to the circle at that point.

715. The polar with respect to a circle of an external point is the chord of contact of the tangents drawn from that point to the circle.

716. If A, B, are any two points such that the polar of A with respect to a given circle passes through B, the polar of B with respect to the same circle passes through A.

717. The locus of the intersection of tangents drawn through the extremities of all chords passing through a given point is the polar of that point with respect to the circle.

718. Any straight line drawn through a point is cut harmonically by the point, its polar with respect to any circle, and the circumference of that circle.

The preceding theorems show that

(i) The polar of an external point with respect to a circle is the chord of contact of tangents from that point.

(ii) The polar of an internal point is the locus of the intersection of tangents drawn at the extremities of all chords passing through it.

(iii) The polar of a point on a circle is the tangent at that point.

719. The point of intersection of the polars of two given points with respect to any circle is the pole of the line passing through the given points.

720. Find the locus of the poles with respect to a given circle of all straight lines passing through a given point.

721. Find the locus of poles with respect to a given circle of tangents drawn to a concentric circle.

722. Any two points subtend at the center of a circle an angle equal to one of the angles formed by the polars of the given points.

723. If two circles intersect orthogonally, the center of each is the pole of their common chord with respect to the other circle.

724. Any two points subtend at the center of a circle an angle equal to one of the angles formed by the polars of the two given points.

725. Given any point C on a fixed straight line AB; find the locus of the point inverse to C with respect to the circle.

726. The polars with respect to a given circle of the four points of a harmonic range form a harmonic pencil.

DEF. *The radical axis* of two circles is the locus of a point so situated that tangents drawn from any point of the locus to the two circles are equal.

727. Prove that the radical axis of two circles is a line perpendicular to the line of centers, such that the difference of the squares of the segments of the line of centers is equal to the difference of the squares of the radii of the given circles.

728. Draw the radical axis of two given circles. If the circles intersect, what does the radical axis become?

729. The radical axes of three circles taken pair by pair are concurrent.

DEF. The point of intersection of the radical axes of three or more circles is called *the radical center.*

730. The radical axis of any pair of circles bisects any one of their four common tangents.

731. If tangents are drawn to two circles from any point in their radical axis, a circle with this point as a center and either tangent as radius cuts both the given circles orthogonally.

732. If tangents to three circles are drawn from their radical center, a circle whose center is the radical center of the three circles and whose radius is any tangent thus drawn cuts all the circles orthogonally.

733. If three circles are tangent to each other, pair by pair, their common tangents at their points of contact are concurrent.

734. If circles are drawn on the three sides of a triangle as diameters, their radical center is the orthocenter of the triangle.

735. All circles passing through a fixed point and cutting a given circle orthogonally pass through a second fixed point.

736. Find the locus of the centers of all circles which pass through a given point and cut a given circle orthogonally.

737. Find the locus of the centers of all circles which cut two given circles orthogonally.

738. Given a line and a point without the line. Draw a line passing through the point and perpendicular to the given line

(i) When the point is accessible and the line is not.

(ii) When the line is accessible and the point is not.

739. If two triangles have one angle of one equal to one angle of the other, and a second angle of one supplementary to a second angle of the other, the sides about the third angles are proportional.

740. AD bisects the angle A of the triangle ABC, and meets the base at D; prove that if circles are circumscribed about the triangles ABD, ACD, their diameters are to each other as the segments of the base.

741. AD and AE bisect internally and externally the vertical angle A of a triangle, meeting the base at D, E, respectively; if F is the middle point of BC, prove that FB is a mean proportional between FD and FE.

742. In the triangle ABC, $AC = 2BC$; if CD, CE, bisect the angle C internally and externally, meeting AB in D, E, respectively, prove that the areas of the triangles CBD, ACD, ABC, CDE are as $1 : 2 : 3 : 4$.

743. How can the area of a triangular field be found when one side of the field cannot be measured? Give more than one method.

744. If through the middle point of the base of a triangle any line be drawn intersecting one side of the triangle, the other side produced, and the line drawn from the vertex parallel to the base, it will be divided harmonically.

745. If from either base angle of a triangle a line be drawn intersecting the median from the vertex, the opposite side, and the line drawn from the vertex parallel to the base, it will be divided harmonically.

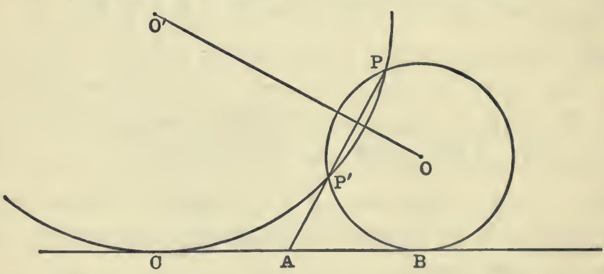
746. Any transversal is divided harmonically by two sides of a triangle and the internal and external bisectors of the angle included by these sides.

THE PROBLEM OF APOLLONIUS

The ten following problems are closely related, and form a series of which the last is called the problem of Apollonius. It was first solved by the famous Greek geometer of that name about 200 B.C.

747. Construct a circle which shall pass through three given points.

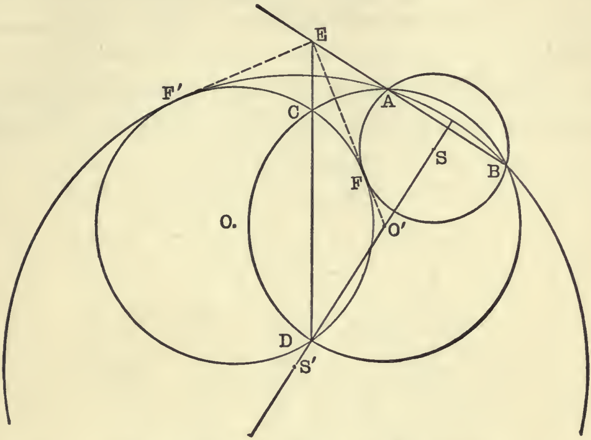
748. Construct a circle which shall pass through two given points and be tangent to a given line.



HINT. Since $AB^2 = AP \times AP'$, the point B can be determined as soon as A has been found. As the figure shows, there are two solutions.

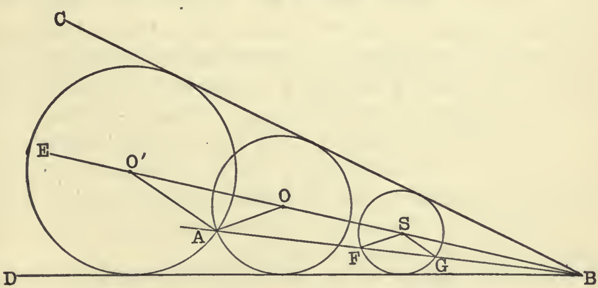
749. Construct a circle which shall pass through two given points and be tangent to a given circle.

HINT. Let A, B, be the given points and the circle whose center is O be the given circle. Draw any circle, with its center at O', which shall pass through A, B, and intersect the given circle in two points, as C, D. Draw a line connecting CD and extend it to intersect the line joining



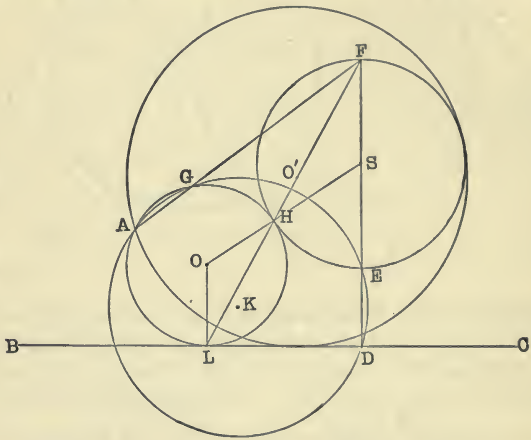
A, B, in some point E. Draw tangents from E to the circle O, touching at FF'. The required circle is the circle passing through the points A, B, F, or through the points A, B, F'. There are two solutions. Can there ever be more than two? Can there ever be less than two?

750. Construct a circle which shall pass through a given point and be tangent to two given lines.



Hint. Let A be the given point and BC, BD, be the given lines. Construct any circle tangent to both lines and lying within that angle formed by the two lines, in which A lies. Let S be the center of the circle thus drawn. Draw BA, SF, SG. Then the center of the required circle is determined by a line AO or AO', drawn from A parallel to FS or to GS, intersecting the bisector of the angle CBD in O or in O'. There are two solutions. Can there ever be more or less than two?

751. Construct a circle which shall pass through a given point and be tangent to a given line and to a given circle.

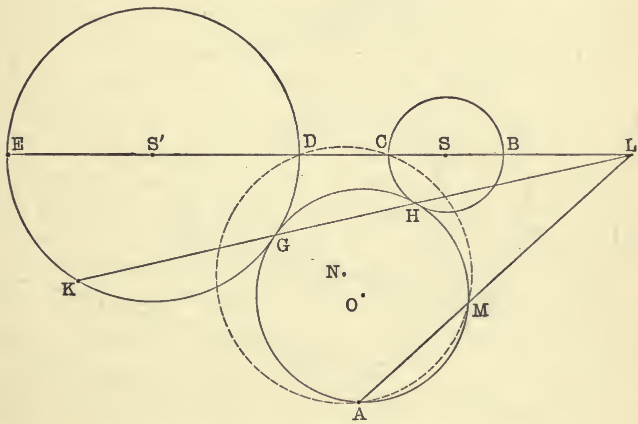


Hint. Ket A be the given point and BC the given line, and the circle whose center is S the given circle. Suppose the problem solved, and let the circle whose center is O be the required circle. Draw $SD \perp BC$ and SO connecting the centers of the given and of the required circles. Draw FH from F through the point of tangency of the two circles, and prolong it to L. Draw the line AGF, and connect HE and OL. It can now be shown that OL and FD are parallel, that $FA \times FG = FH \times FL = FE \times FD$, and that the auxiliary circle whose center is K, drawn through the points D, E, A, will also pass through G. This determines a second point on the required circle, and reduces the problem to No. 748.

The maximum number of solutions is four; two are obtained by connecting A with F and two by connecting A with E. The center of one of the latter is O'. Are there ever less than four solutions? Is the problem ever impossible?

752. Construct a circle which shall pass through a given point and be tangent to two given circles.

HINT. Let the circles whose centers are S and S' be the given circles, and A the given point. Suppose the problem solved, and let the circle whose center is O be the required circle. Draw a line through S and S', and also a line through G, H, the points where the required circle touches the given circles. Let these lines intersect at L. This point is the center of similitude of the given circles. Pass a circle through the three points D, C, A, and it will intersect the



required circle at a second point M. A line drawn through A, M, will pass through L. Hence the problem can be reduced to No. 749 by drawing the auxiliary circle through the points D, C, A, thus determining a second point on the required circle, namely, the point of intersection of the auxiliary circle and the line connecting A with the center of similitude of the given circles.

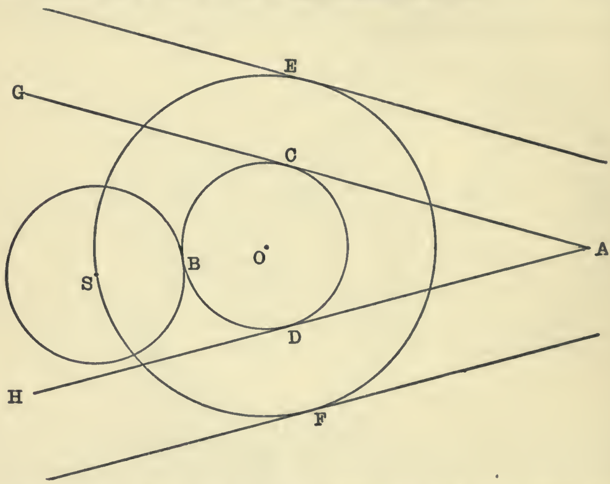
Is the problem ever impossible? How many solutions are there? Is the number of solutions always the same?

753. Construct a circle which shall be tangent to three given lines.

Is the number of solutions to this problem always the same?

754. Construct a circle which shall be tangent to two given lines and to a given circle.

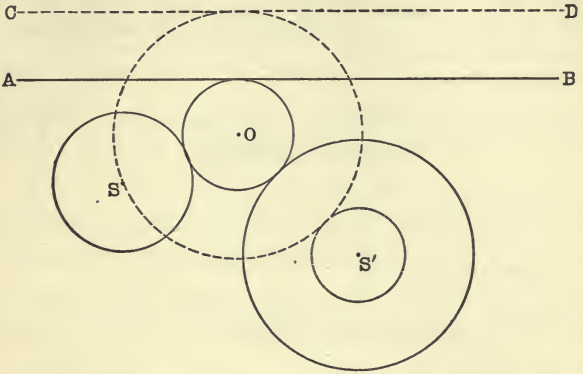
HINT. Let AG, AH, be the given lines, and the circle whose center is S the given circle. Draw lines parallel to AG, AH, as indicated in the figure, at a distance from them equal to the radius of the given circle. Then by No. 750 construct a circle which shall pass through S and be tangent to the two parallels thus drawn. The center of this circle O is also the center of the required circle.



This problem should be fully discussed for different cases, arising from different relative positions of the circle and the lines, and the number of solutions noted. The maximum number of solutions is eight.

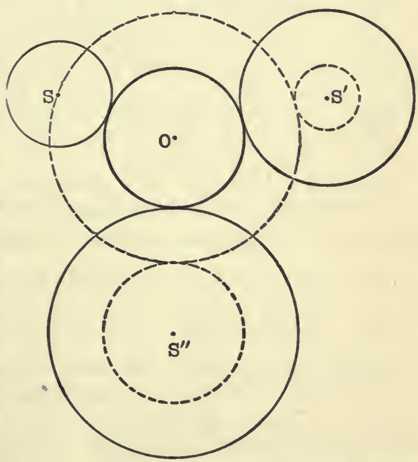
755. Construct a circle which shall be tangent to a given line and to two given circles.

HINT. Let AB be the given line, and the circles whose centers are S and S', the given circles. Draw a line parallel to AB at a distance from it equal to the radius of the smaller circle, letting S be the center of that circle, and with S' as a center and a radius equal to the difference of the radii of the two given circles, construct a circle concentric with the greater of the two given circles. By No. 751 construct a circle which shall pass through S and be tangent to CD, and to the auxiliary circle whose center is S'. The center of the circle thus found, O, is also the center of the required circle. This



gives a single construction under a special set of conditions. The problem should be discussed for different relative positions of the line and circles, and also for the use of the sum instead of the difference of the radii of the given circles as the radius of the auxiliary circle. The maximum number of solutions is eight.

756. Construct a circle which shall be tangent to three given circles. (The problem of Apollonius.)



HINT. Let the circles whose centers are S, S', S'', be the given circles, and let their radii be r, r', r'', respectively, with $r'' > r' > r$. Construct a circle whose center is S' and whose radius is $r \doteq r'$, and a circle whose center is S'' and whose radius is $r'' - r$. Then by No. 752 construct a circle which shall pass through S and be tangent to these two auxiliary circles. The center of the circle O is also the center of the required circle.

By using all possible combinations of the sums, as well as the differences of the radii of the given circles, the maximum number of solutions is found to be eight.

SOLUTION OF GEOMETRIC PROBLEMS BY THE ALGEBRAIC METHOD

757. Construct x, when $x = a + b - c$.

758. Construct $x = \dfrac{ab}{c}$.

HINT. Make x a fourth proportional.

759. Construct $x = a\sqrt{2}$.

760. Construct $x = a\sqrt{5}$.

761. Construct $x = \dfrac{ab}{\sqrt{a^2 + b^2}}$.

762. Construct $x = \sqrt{a^2 + b^2 - ab}$.

HINT. Construct a triangle two of whose sides are a and b, and whose included angle is 60°; x is the altitude of this triangle.

763. Construct the roots of the equation $x^2 - ax + b^2 = 0$.

HINT. Since the sum of the roots of any quadratic equation, $x^2 + px + q = 0$, is equal to $-p$, and the product of the roots is equal to q, the problem is at once reduced to the following problem: To construct two lines, given their sum a and their product b^2.

764. Construct the roots of the equation $x^2 + ax + b^2 = 0$.

765. Construct the roots of the equation $x^2 - ax - b^2 = 0$.

HINT. In this case the roots must have contrary signs, since their product, b^2, is negative. Hence it can readily be seen that this problem reduces to the following: To construct two lines, given their difference a and their product b^2. To find these lines remember that if a tangent and a secant are drawn to a circle from any external point, the tangent is a mean proportional between the whole secant and its external segment.

766. Construct the roots of the equation $x^2 + ax - b^2 = 0$.

NOTE. Every complete quadratic equation can be reduced to one of the four forms given in Nos. 762–766, if it be an equation containing linear factors.

767. Divide a triangle into two parts having equal perimeters by a line from one vertex.

768. Divide a parallelogram, by a line from one vertex, into parts whose perimeters shall differ by a.

769. Take equal lengths from two sides of a triangle, such that the sum of the remainders shall equal the third side.

770. Draw parallels at equal distances, respectively, from the sides of a given rectangle, so that they shall form another rectangle of given perimeter.

771. Divide one side of a triangle into two parts such that their difference shall be equal to one third the sum of the other two sides.

772. Find a point E in the same line with the four collinear points whose order is A, B, C, D, such that $AE : BE = DE : CE$.

773. Produce a given chord of a circle, so that the tangent drawn from its extremity shall have a given length.

774. Construct a rectangle, having given one side and the sum of the diagonal and the other side.

775. Divide a given line into two segments such that the difference of their squares shall equal a^2.

776. Divide a given line into two segments such that their ratio shall be equal to that of two given squares.

777. In the triangle ABC draw a line DE parallel to BC, so that $DE^2 = AC \cdot CE$.

778. In the triangle ABC draw a line DE parallel to BC, so that $DE^2 = AD \cdot CE$.

779. In the triangle ABC draw a line DE parallel to BC, so that the perimeter of the triangle ADE shall equal half the perimeter of the trapezoid $BCED$.

780. Inscribe in a given triangle a rectangle with a given perimeter.

781. Construct a right triangle, given the perimeter and the altitude on the hypotenuse.

782. In a given square inscribe five equal circles, so that the middle circle shall touch the other four, and each of these four shall touch two adjacent sides of the square.

783. Given a rectangle; construct a square such that the perimeters of the two figures shall have the same ratio as their areas.

784. In a triangle ABC draw a line DE between AC and BC, so that the triangle CDE shall be isosceles and equal in area to half the triangle ABC.

785. Within a given circle construct five equal squares, so that each of the four outer squares shall have two vertices in the circumference, and one side in common with the fifth square.

786. Construct a square which shall be to a given square as 3 : 2.

787. Construct two lines, given their ratio and their product.

788. Construct two lines, given their ratio and the difference of their squares.

789. Divide a given trapezoid into two equivalent parts by a line parallel to the bases.

790. Divide a given line into two parts, such that the smaller shall be a mean proportional between the greater and the difference of the two.

791. Given a circle and a chord; produce the chord to a point A, such that the tangent from A shall be equal to the chord.

792. In the triangle ABC draw DE parallel to AB, so that $AB \cdot AD = DE \cdot CD$.

793. Through a given point within a given circle draw a chord such that the difference of the two segments shall equal a given line.

794. Divide a given line into two parts so that the square of one part shall equal half the square of the other.

795. Divide a given line into two parts whose product shall equal the square of the given line.

796. Inscribe in a given equilateral triangle another equilateral triangle half as large.

797. Given the point C in AB; find a point D in AB between B and C, such that $AD^2 = BD \cdot CD$.

798. Construct a rectangle equal in area to a given rectangle, having a perimeter equal to the perimeter of another given rectangle.

799. Construct a right triangle, given the hypotenuse and the sum of the legs.

800. Transform a given square into an isosceles triangle in which the sum of the base and the altitude is given.

VII. THEOREMS AND PROBLEMS RELATING TO THE COMMON GEOMETRICAL SOLIDS

801. What is the weight of a ball of lead whose diameter is 1.68 m., the specific gravity of lead being 11.36?

802. How many square meters of surface must be cemented in constructing a cubical reservoir which will hold 10,000 Kg. of water?

803. How many square meters of surface must be cemented in constructing a reservoir in the shape of a cylinder 15 m. deep, capable of holding 10,000,000 liters of water?

804. A cubical vessel requires 245 sq. ft. of lead for lining the bottom and sides; how many gallons of water will it hold?

805. What is the volume of a right hexagonal prism whose height is 8 ft., each side of the hexagon being 6 ft.?

806. How many cubic inches of mahogany will it take to veneer the top of a circular table whose diameter is 2 ft. 6 in., the veneer being $\frac{1}{16}$ of an inch thick?

807. How many cubic yards of stone are needed to build a dam 1200 ft. long, 40 ft. high, 16 ft. wide at the bottom and 4 ft. wide at the top?

808. If the atmosphere extends to a height of 45 miles above the surface of the earth, what is the ratio of its volume to the volume of the earth, assuming the latter to be a sphere whose diameter is 7912 miles?

809. How much will it cost to dig a well 6 ft. in diameter and 28 ft. deep, at $3.50 per cubic yard of earth removed?

810. How many cubic yards of earth must be removed in constructing a semicircular tunnel 500 yd. long, the radius being 8 ft.?

811. A cylindrical glass jar holds 1500 cu. cm.; find its dimensions if its depth is three times the radius of its base.

812. An iron cylinder 6 in. in diameter and 20 ft. long is reduced in diameter in a lathe half an inch; what is the loss in weight, the specific gravity of iron being 7.2?

813. The piston of a pump is 8 cm. in diameter, and the stroke of the pump is 50 cm.; how many liters of water are pumped out by 500 strokes?

814. How many feet of wire $\frac{1}{32}$ of an inch in diameter can be drawn out of 2 cu. in. of brass?

815. Given the circumference c of the base of a right cylinder, and the total surface T; find the volume.

816. Given the volume V of a right cylinder, and the altitude h; find the total surface.

817. A cone whose altitude is 8 ft. has a base containing 64 sq. ft.; at what distance from the vertex must a plane parallel to the base be passed to contain 45 sq. ft.?

818. A pyramid 6 ft. high has a square base 2.5 ft. on a side; find the area of a section made by a plane parallel to the base and 2 ft. from the vertex.

819. The base of a regular pyramid is a hexagon which measures 2 ft. on a side; find the altitude of the pyramid if the lateral area is six times the area of the base.

820. Find the volume of a regular triangular pyramid whose altitude is 18 in. and whose base edges are each 5 in.

821. Find the total surface of a regular triangular pyramid if each side of its base is 4 ft., and its slant height is 8 ft.

822. Find the altitude of a triangular pyramid if its volume is 26 cu. ft., and the sides of its base are 3, 4, and 5 ft.

823. A regular pyramid with a square base contains 122 cu. yd., and its altitude is 41 ft.; what is the area of its base?

824. The total surface of a regular quadrangular pyramid is T, and its eight edges are equal, each to each; find the length of an edge.

825. The radius of the base of a cone of revolution is 30 in., and its altitude is 6 ft.; how far from its base must a parallel plane be passed to cut a section of the cone whose radius shall be 20 in.?

826. The slant height of a cone is 3 in.; how must the slant height be divided in order that the lateral surface may be divided into two equal parts? into three equal parts?

827. The volume of a cone is V; what does the volume become if the altitude is doubled? if the radius of the base is doubled? if both are doubled? if both are trebled?

828. A conical mound of earth measures 148 yd. around the base, and its slope measures 52 yd.; how many cubic yards are there in the mound?

829. The weight of a cone of revolution of silver is 5 Kg., and its altitude is twice the diameter of its base; find the dimensions of the cone, the specific gravity of silver being 10.47.

830. Find the volume of a cone of revolution whose slant height is equal to the diameter of its base, the total surface being T.

831. Find the lateral surface of the frustum of a right cone whose altitude is 2 m., and the radii of whose bases are 6 m. and 3 m. respectively.

832. The altitude of a cone of revolution is 8 in. and the radius of its base is 2 in.; find the lateral surface of the frustum made by a plane parallel to the base 3 in. from the vertex.

833. A bucket is 14 in. in diameter at the top and 10 in. in diameter at the bottom, and its depth is 12 in.; how many gallons will it hold?

834. The bases of a frustum of a regular pyramid are hexagons whose sides are 1 ft. and 2 ft. respectively, and its volume is 14 cu. ft.; find its altitude.

835. The radii of the bases of a right frustum are 3 m. and 5 m. respectively, and its volume is 30 cu. m.; find the volume of the cone from which the frustum was cut.

836. The frustum of a cone of revolution is 12 ft. high and its volume is 228 cu. ft.; find the radii of the bases if their sum is 6 ft.

837. A square tower is built 14 m. high, each side of the base being 15 m., and each side of the top being 11 m. Within the tower is a circular shaft 2 m. in diameter, for a spiral staircase. How many cubic meters of material were used in building the tower?

838. What is the locus of tangents drawn from a point to a sphere?

839. What is the locus of all points 3 in. from the surface of a sphere whose radius is 5 in.? 6 in. from the surface? 5 in. from the surface?

840. The distances from the poles of a small circle of a sphere to the circumference of the circle are 4 m. and 7 m. respectively; find the area of the circle.

841. Find the surface of a lune if its angle is 31° 30′, and the radius of the sphere is 2 ft.

842. The angles of a spherical triangle are 38° 41′, 84° 50′, 115° 52′, and the radius of the sphere is 8 in.; find the area of the triangle.

843. The sides of a spherical triangle are 52°, 83°, 116°, and the radius of the sphere is 7 in.; find the area of its polar triangle.

844. The diameter of a sphere is 16 in.; find the area of a zone whose altitude is 3 in.

845. In a sphere of radius r, find the altitude of a zone whose area is equal to that of a great circle. If the area is doubled, is the altitude doubled?

846. The radius of a sphere is 6 in. The sphere is cut by two parallel planes respectively 2 ft. and 4 ft. from the center. Find the area of the zone thus formed. (Two solutions.)

847. Find the area of a zone on a sphere of radius r illuminated by a lamp placed at a distance a from the sphere.

848. How far from the surface of a sphere must the eye be placed to see one sixth of the surface?

849. Find the volume of a sphere if the section made by a plane 3 cm. from the center contains 12 sq. cm.

850. How many cubic inches of iron are there in a shell $\frac{3}{4}$ in. thick, if the inside diameter of the shell is $7\frac{1}{4}$ in.?

851. If an iron ball 4 in. in diameter weighs 9 lb., what is the weight of a hollow iron shell $\frac{1}{2}$ in. thick, its outside diameter being 12 in.?

852. Find the volume of a triangular spherical pyramid if the angles at its base are each 110°, and the radius of the sphere is 8 in.

853. The radius of the base of a segment of a sphere is 14 in., and its height is 8 in.; find its volume.

854. A sphere of radius 8 ft. is cut by parallel planes on the same side of the center, distant from the center 3 ft. and 5 ft. respectively; find the volume comprised within the planes.

855. A spherical segment is half as large as the spherical sector to which it belongs; if the radius of the sphere is 4 in., find the altitude of the segment.

856. Dry sand is poured over a sphere of 2 ft. radius until it forms a conical pile whose height is 8 ft., and the circumference of whose base is 50 ft.; how many cubic feet of sand are used?

857. Find the dimensions of a right circular cone 1 in. high that can be made from a cubic inch of material.

858. A regular hexagonal pyramid is inscribed in a cone of revolution; the slant height of the pyramid is 12 ft. and its lateral edges are each 13 ft.; how many cubic feet of the cone are not comprised in the pyramid?

859. Find the area of a zone generated by an arc of 20° revolving about a diameter passing through one of its extremities, the radius being a.

860. A sphere of radius 4 in. is cut by two parallel planes equally distant from the center, so that the area of the zone between them is equal to the sum of the areas of the sections made by the planes ; find the distance of either plane from the center.

861. Prove that a line and a plane which are perpendicular to the same straight line are parallel.

862. If two planes are parallel, a line parallel to one of them through a point in the other lies in the other.

863. The alternate interior dihedral angles formed by the intersection of two parallel planes with a third plane are equal.

864. If a plane be drawn through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

865. If each of two intersecting planes be cut by two parallel planes not parallel to the intersection of the first two planes, the intersections of these planes with the parallel planes are the edges of equal dihedral angles.

866. The three planes bisecting the dihedrals of a trihedral meet in a common straight line.

867. If from any point in either face of a dihedral a line be drawn perpendicular to the edge of the dihedral, and from the same point a perpendicular be dropped upon the other face of the dihedral, the plane of these two lines is perpendicular to the plane containing the point from which they are drawn.

868. The volume of a regular pyramid is equal to its lateral area multiplied by one third the distance from the center of its base to any lateral face.

869. Lines joining the middle points of two pairs of opposite edges of a tetrahedron inclose a parallelogram.

870. A plane passed through the center of a parallelopiped divides the parallelopiped into two solids which are equal in volume. Are the solids equal in all respects? Are they symmetrical?

871. How many cubic feet of earth and rock are taken out by a boring 3 in. in diameter sunk 2400 ft. into the ground?

872. Two tetrahedrons are equal in all respects if three faces of the one are equal respectively to three faces of the other, and are similarly placed.

873. The volume of a triangular prism is equal to a lateral face multiplied by one half the distance from any point in the opposite lateral edge to that face.

874. The volume of a truncated right parallelopiped is equal to the area of its lower base multiplied by the perpendicular drawn to the lower base from the center of the upper base.

875. The perpendicular drawn to the lower base of a truncated right triangular prism from the intersection of the medians of the upper base is equal to one third the sum of the lateral edges.

876. The three planes passing through the lateral edges of a triangular pyramid, bisecting the sides of the base, intersect in one straight line.

877. Any section of a tetrahedron parallel to two opposite edges of the tetrahedron is a parallelogram.

878. The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of the twelve edges.

879. A section of a tetrahedron $ABCD$ containing CD and perpendicular to AB intersects the faces ABC and ABD in CE and ED respectively. If the bisector of the angle CED meets CD in F, prove that $CF : DF =$ area ABC : area ABD.

880. The radius of the base of a right circular cone of copper is 4 in. and its altitude is 10 in.; a core 1 in. in diameter is bored out, whose axis coincides with the axis of the cone; what is the weight of the copper removed, its specific gravity being 8.75?

881. If the four diagonals of a quadrangular prism are concurrent, the prism is a parallelopiped.

882. ABC and $A'B'C'$ are two polar triangles on a sphere whose center is O; prove that OA' is perpendicular to the plane of OBC.

883. Any angle of a spherical triangle is greater than the difference between 180° and the sum of the other two angles.

884. The distance between the centers of two spheres whose radii are 25 in. and 17 in. respectively, is 28 in.; what is the diameter of the circle of intersection of the two spheres?

885. The volume of a quadrangular spherical pyramid whose base angles are 110°, 122°, 135°, and 146° respectively, is 15 cu. ft.; what is the volume of the entire sphere?

886. A spherical cannon ball 8 in. in diameter is dropped into a cubical tank full of water. If the inside edges of the tank are each 8 in., how many cubic inches of water will remain in it after the cannon ball has been dropped in?

887. How many spherical bullets, each $\frac{5}{8}$ in. in diameter, can be run from a cylindrical block of lead 7 in. in diameter and 18 in. long?

888. The volume of a cylinder of revolution is equal to the area of its generating rectangle multiplied by the circumference of a circle whose radius is the distance from the center of the rectangle to the axis of the cylinder.

889. The volume of a cone of revolution is equal to its lateral area multiplied by one third the distance of any element of its lateral surface from the center of the base.

890. A cone of revolution is circumscribed about a sphere whose diameter is two thirds the altitude of the cone; prove that the lateral surface and the volume of the cone are respectively three halves and nine fourths the surface and the volume of the sphere.

891. A square whose area is m revolves about one of its diagonals as an axis; find the volume and the convex surface of the solid thus generated.

892. A right triangle whose legs are 6 in. and 8 in. respectively, revolves about its hypotenuse as an axis; find the volume and the convex surface of the solid thus generated.

893. An equilateral triangle whose side is a revolves about one of its sides as an axis; find the volume and area of the surface of the solid generated.

894. Find the lateral area and the volume of a cylinder of revolution whose altitude is equal to the diameter of its base, inscribed in a sphere whose radius is r.

895. An equilateral triangle whose side is a revolves about a line through one of its vertices and parallel to the opposite side; find the lateral area and the volume of the solid thus generated.

896. The cross section of a tunnel $2\frac{1}{2}$ mi. long is a rectangle 6 yd. long and 2 yd. in height, surmounted by a semicircle whose diameter equals the length of the rectangle; how many cubic yd. of material were removed in constructing it?

897. The volume of a cone of revolution equals the area of its generating triangle multiplied by the circumference of a circle whose radius is the distance from the orthocenter of the triangle to the axis of the cone.

898. If the earth be a sphere of radius r, what is the area of a zone visible from a point whose height above the surface of the earth is h?

899. A projectile consists of a cylinder of revolution 18 in. long and a cone of revolution 12 in. long; if the diameter of the projectile be 13 in., what is its volume?

900. Find the surface of a sphere circumscribing a regular tetrahedron, one of whose edges is 8 in.